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A Logic for Intersection and Union Types

PhD Thesis

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Athens, October 2015

To Smara and Thanassis

Acknowledgements

I would like to express my gratitude to my supervisor, Assistant Professor Yannis Stephanou, for his constructive guidance and elaborate remarks over the entire length and depth of my scientific work, as well as his extremely supportive advice and attitude throughout my struggle to manage the ups and downs of the PhD-adventure. Thanks to him, this mathematical thesis enjoys the special treats of philosophy's verbal accuracy.

I would also like to thank the external collaborator of MPLA (graduate program in Logic and Algorithms) Yiorgos Stavrinos for acquainting me with the subjects of Lambda Calculus, Proof Theory, and Linear Logic through his excellent teaching, for guiding my first research steps into the topic of the logical interpretation of intersection (and union) types, and for being an instructive co-author of several papers.

I deeply thank Professor George Koletsos, member of the Three Member Committee, and the external collaborator of MPLA Nikos Rigas for inspiring me with fruitful discussions and novel ideas on the wider field of my research during study courses and private seminars.

I also thank Professor Simona Ronchi Della Rocca, member of the Three Member Committee, who provided me with the scientific "starting material" and shared her expertise and ideas with me during conferences and, especially, during her visit to Athens in 2009. Having the honour to work with her and her colleague Alexis Saurin on a research paper, I learned a great deal about scientific team work and interaction.

I would finally like to thank for their precious cooperation in completing this huge endeavour the other members of the Seven Member Exam Committee: Professor Costas Dimitracopoulos, Professor Emeritus Stathis Zachos, Associate Professor Panos Rondogiannis, and Lecturer Petros Stephaneas.

My PhD was co-financed by the European Union (ESF) and Greek national funds through the funding program "Heracleitus II". I owe special thanks to Prof. Dimitracopoulos for helping me complete and submit my funding application.

I thank my friend, my "mikri" Maria Fasouli, for generously offering me her sweet, warm, and witty company, running by my side in the PhD-marathon for five years. I thank my mother, Smara Veneti, for endlessly enduring and unconditionally loving me. It was *she* who sponsored my studies, when Heracleitus was burdened with far more years than it could bear. My beloved father, Thanassis Venetis, already knows how blessed I feel that it was *he* who held my hand for forty one years...

Contents

Introduction

This thesis aims to define a logical system corresponding to the type system with intersection and union types in the perspective of the Curry-Howard isomorphism. The type system with intersection and union types [2] assigns types built by implication, intersection, and union to terms of the untyped λ-calculus; it is a type system à la Curry. We initially consider a natural deduction presentation for systems in logic or type theory, i.e. a presentation with introduction and elimination rules for every logical connective or type constructor, respectively.

The Curry-Howard isomorphism [19] states a correspondence between systems of formal logic as encountered in *proof theory* and computational calculi as found in *type theory.* For instance, the implicative fragment of intuitionistic propositional logic corresponds to the simply typed λ-calculus à la Church, in the sense that any *proof* in the logic corresponds to a typable *term* à la Church, which thoroughly encodes the implicational structure of the proof, or to a proof in the $\lambda_{\to}$ Church type system typing this very term. More precisely, any proof in the logic gives a proof in the type system, if "decorated" with simply typed terms and, conversely, any proof in the type system gives back a proof in the logic, if terms are erased. In the direction from the logic to the type system, this is meant modulo the conversion of *formulas* to *types* and the elimination of structural rules; in the direction from the type system to the logic, it is meant modulo the conversion of types to formulas and the addition of structural rules. In the same manner, the implicative, conjunctive, and disjunctive fragment of intuitionistic propositional logic corresponds to the $\lambda^{\wedge\vee}_{\to}$ Church type system through decoration and erasure procedures. In particular, any proof in the logic provides a proof in the type system, if decorated with typed terms with pairs and injections and, conversely, any proof in the type system returns a proof in the logic, if terms are erased[1]; corresponding proofs in the logic and the type system are such that the term typed by the latter proof records the implicative, conjunctive, and disjunctive structure of the former. *Provability* of a certain formula translates to *inhabitation* of the corresponding type, while *normalization* of a certain proof translates to *reduction* of the corresponding term to normal form. In higher levels of the isomorphism, first-order logic corresponds to dependent types and second-order logic corresponds to polymorphic types.

As far as the type system with intersection and union types is concerned, we say that we seek a logic corresponding to it in the *perspective* of the Curry-Howard isomorphism, since it is a Curry type system and the isomorphism actually applies to Church type systems. However, adjusting the isomorphism's main idea to its case, we seek a logic corresponding to it through a decoration with untyped terms. Such a logic needs to have logical connectives corresponding to the type constructors of intersection and union, which implies an interpretation of intersection and union in logical terms.

The literature so far has offered logics corresponding to the type system with intersection types in the Curry-Howard perspective. The natural question whether intersection is logically interpreted as

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[1] Both directions hold modulo the conversions already mentioned for the correspondence between the implicative logic and the $\lambda_{\to}$ type system.

conjunction motivates the investigation whether the implicative and conjunctive fragment of intuitionistic propositional logic corresponds to the type system with intersection types through a decoration with untyped terms. Such a correspondence is proved unfeasible in [18, 15]; the decoration on the logic needs to simulate the terms in the type system and therefore to ignore, i.e. not to encode, conjunction introduction, but such a decoration is impossible on proofs containing conjunction introductions on conjuncts which are not identically decorated.

$$\frac{\vdash t : \sigma \qquad \vdash t : \tau}{\vdash t : \sigma \cap \tau}\,(\cap\mathbf{I}) \qquad\qquad \frac{\vdash t : \sigma \qquad \vdash u : \tau \qquad t = u}{\vdash t : \sigma \wedge \tau}\,(\wedge\mathbf{I})$$

It is only a proper subset of this logic that corresponds to intersection types through decoration, namely the part that admits a decoration which ignores conjunction introduction. Since conjunction introduction in this part involves identically decorated conjuncts, called “synchronous” conjuncts, we can roughly say that *intersection* is logically interpreted as a kind of *synchronous conjunction*. The logics offered in the literature for intersection types attempt to express this specific part of the implicative and conjunctive fragment of intuitionistic logic as an autonomous logical system by internalizing the metatheoretical condition that conjuncts be identically decorated. The logics in question [18, 15], introduced by S. Ronchi Della Rocca and her colleagues in the early 2000s, employ intersection (synchronous conjunction) as a logical connective together with implication. The logic in [18] is called “Intersection Logic” and uses the structure of full binary trees, called “kits”, to internalize the condition mentioned above. A refinement of this logic is the system “Intersection Synchronous Logic”, proposed in [15], which linearizes kits into multisets of statements, called “molecules”.

We aim to offer a logic corresponding to the type system with intersection and union types in the Curry-Howard perspective, i.e. to study an extended-with-union version of the setup described above. Besides the type system with intersection and union types, such a version involves the implicative, conjunctive, and disjunctive fragment of intuitionistic propositional logic, which is the natural candidate for a logic corresponding to intersection and union types through decoration. As expected, though, this correspondence is unfeasible; the decoration on the logic needs to simulate the terms in the type system and therefore to induce a substitution term on disjunction elimination, but such a decoration is impossible on proofs containing disjunction eliminations with minor premises which are not identically decorated[2].

$$\frac{\vdash t : \sigma \cup \tau \qquad x : \sigma \vdash u : \rho \qquad x : \tau \vdash u : \rho}{\vdash u[t/x] : \rho}\,(\cup\mathbf{E}) \qquad \frac{\vdash t : \sigma \vee \tau \qquad x : \sigma \vdash u : \rho \qquad x : \tau \vdash v : \rho \qquad u = v}{\vdash u[t/x] : \rho}\,(\vee\mathbf{E})$$

The extended version, therefore, includes the proper subset of the implicative, conjunctive, and disjunctive fragment of intuitionistic logic that indeed corresponds to intersection and union types through decoration, namely the part that admits a decoration which induces a substitution on disjunction elimination. Since disjunction elimination in this part involves synchronous minor premises, the logical interpretation of *union* is a kind of *synchronous disjunction*. We aim to complete the picture in the extended setup with the logic that expresses this specific part of the implicative, conjunctive, and disjunctive fragment of intuitionistic logic as an autonomous logical system by internalizing the condition that minor premises in disjunction elimination be identically decorated. The obvious way to achieve this is to extend the logics offered by the team of Ronchi with union (synchronous disjunction) as an additional logical connective.

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[2] The decoration on conjunction introduction still needs to be as already described in the restricted, i.e. the union-free, version of the setup.

Chapter 1 outlines the research results established before the start of this thesis and familiarizes the reader with the basic argument modes for the topic. Working in natural deduction style, we present the type system with intersection types IT and explain why the implicative and conjunctive fragment of intuitionistic logic, denoted LJ, does not correspond to it through a decoration with untyped terms. Spotting the proper subset LJns of LJ that indeed corresponds to IT through decoration, we then present the logics "Intersection Logic" IL and "Intersection Synchronous Logic" ISL, which both aim to express LJns as an autonomous system. We demonstrate the correspondence between each of these logics and IT through decoration; in both cases, such a correspondence interrelates a decorated derivation in the logic with a finite number of derivations in the type system. This chapter summarizes the work in [18, 15].

Chapter 2 illustrates in detail the type system with intersection and union types IUT and its rule or style variants, as well as its basic properties. First, a natural deduction and a sequent calculus formulation of the system are presented and proved equivalent, the former being additive and the latter multiplicative. A sequent calculus formulation is one with left and right introduction rules for every type constructor, and a cut rule. Then, while the usual subject reduction is shown to fail, a more elaborate kind of reduction, called *parallel* reduction, is defined and shown to hold. Further, a cut elimination proof is given for the sequent calculus formulation of the system, when contraction is explicitly included. Finally, certain typings in IUT or its rule variants are examined with respect to the properties the typable terms display; among others, it is deduced that the terms typable in IUT are all and only the strongly normalizing ones. This chapter combines results in [2] and original work.

Chapter 3 exposes an early stage attempt to define a logic corresponding to intersection and union types in the Curry-Howard perspective. Working in natural deduction style, we first show that the implicative, conjunctive, and disjunctive fragment of intuitionistic logic, denoted ML, does not correspond to the type system IUT through a decoration with untyped terms. We then identify the proper subset MLns of ML that indeed corresponds to IUT through decoration and aim to represent it as an independent logic. Toward this end, we extend the logics IL and ISL with union rules to define the logics IUL_k and IUL_m, respectively. We show that the extended logics are equivalent and examine whether the correspondence between the restricted logic (IL or ISL) and IT through decoration can be extended to a correspondence between the extended logic (IUL_k or IUL_m) and IUT through decoration. We demonstrate how the substitution terms in union eliminations hinder the extended correspondence. Finally, we discuss the advantages of the formalism of molecules over the formalism of kits that arise from comparing the union elimination rules in the extended logics. This chapter is a revised version of the work in [20].

Chapter 4 introduces a modification[3] of the logic IUL_m with respect to the definition of "molecule" and the definition of rules, but still with introduction and elimination rules for implication, intersection, and union. First, we present the modified structure and rules, drawing attention to the crucial distinction between *global* and *local* rules and to the additiveness of the connectives. Then, we state and prove certain derivable rules and properties of the logic. We also elaborate on derivable rules and properties of the type system IUT in natural deduction style. Finally, we define a decoration of the logic with terms that "copy" the ones in the type system and we interrelate the decorated logic with the type system, so as to explain how the former is meant to use its structure to depict the latter on a logical level.

Chapter 5 resolves the correspondence between the decorated logic IUL_m and the type system IUT in natural deduction style. We first define the notion of *tree of implications and union eliminations with terms* for both the decorated logic and the type system. In the decorated logic, such trees record the inferences of rules that are global and have a counterpart in the type system, which are the inferences

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[3] The use of this modification, besides providing a more convenient system, will become clear in the next chapter, where we exploit it to settle the correspondence between IUL_m and IUT through decoration.

of implications and union elimination, as well as the decoration terms on these inferences. In the type system, such trees record the inferences of rules that have a global counterpart in the logic, which are again the inferences of implications and union elimination, as well as the terms in these inferences. While every derivation in the decorated logic has such a tree, there are derivations in the type system which do not have such a tree, as the procedure for such trees in the type system is algorithmic and does not always terminate. We then state and prove correspondence theorems between the decorated logic and the type system, i.e. from the decorated IUL$_m$ to IUT and conversely, which interrelate a decorated derivation in the logic with a finite number of derivations in the type system via restrictions that involve the trees described above. A derivation in the decorated logic gives finitely many derivations in the type system, whose trees all exist and are identical and also identical to the tree of the derivation in the decorated logic. Conversely, finitely many derivations in the type system whose trees all exist and are identical give back a derivation in the decorated logic with a tree identical to the tree of the derivations in the type system. We also give a detailed counterexample against the position that the restrictions could be removed and that we could thus have a correspondence in the manner of the correspondence given in the first chapter between the decorated IL (or ISL) and IT. Finally, we explicate the definitional factors in the decorated logic that necessitate the restrictions.

Chapter 6 examines how the method of trees, employed in the previous chapter to describe the correspondence between the decorated logic IUL$_m$ and the type system IUT, can be adjusted to the correspondence between the decorated logic IL$_m$ and the type system IT, where the logic IL$_m$ is the restriction of the logic IUL$_m$ to implication and intersection. As IL$_m$ is a modification of ISL, the examination of the correspondence in question with the method of trees is actually a re-examination of the correspondence between the decorated ISL and IT with the method of trees. Adjusting the method leads to the definition of the notion of *tree of implications with terms* for both the decorated logic and the type system. The procedure to attain the trees in the type system is still algorithmic, but we prove that it always terminates. We then state and prove correspondence theorems between the decorated IL$_m$ and IT, which revise the correspondence theorems between the decorated ISL and IT in that they add the fact that each of the trees of the derivations in the type system is identical to the tree of the derivation in the decorated logic. We finally compare and contrast the two correspondences, i.e. between the decorated IUL$_m$ and IUT and between the decorated IL$_m$ and IT, to decide whether IUL$_m$ is indeed a logic for IUT in the manner that IL$_m$ (or ISL) is a logic for IT.

Chapter 7 presents a sequent calculus formulation of the modified logic IUL$_m$, which retains the additive character of the natural deduction formulation. First, we display the sequent calculus rules of the logic, focusing on the distinction between global and local rules. Then, we prove the equivalence between the sequent calculus and natural deduction presentations of the logic. We also prove derivable rules and properties of the sequent calculus logic, which are roughly the same as the ones of the natural deduction logic. Moreover, we present an additive account of the sequent calculus formulation of the type system IUT. We prove the equivalence between the sequent calculus and natural deduction formulations of the type system and also the equivalence between the additive and multiplicative accounts of the sequent calculus formulation of the type system. We elaborate on derivable rules and properties of the newly introduced type system, which are similar to the ones of the natural deduction type system. Finally, working with the sequent calculus logic and type system, we translate into the sequent calculus language the intended interrelation between the logic and the type system through decoration and the actual correspondence between the decorated logic and the type system through the notion of trees. Chapters 4 to 7 contain exclusively original work.

CHAPTER 1

A Logic for Intersection Types

The type assignment system with intersection types, denoted IT [18, 15] or D [13], was introduced in the early eighties by M. Coppo and M. Dezani-Ciancaglini [7, 8] to enhance the typability power of Curry's type assignment system $\lambda_{\rightarrow}$. It is very useful as a tool for investigating pure λ-calculus, since it has nice syntactical properties. In particular, we can prove that it assigns types to all and only the strongly normalizing terms [13].

Due to the peculiar nature of the intersection, IT cannot be used as a model for a programming language; however, intersection types have been particularly useful in studying the semantics of various kinds of λ-calculi. This can be done by extending the system with suitable sub-typing relations, so that the type assignment acts as a finitary tool to reason about the interpretation of λ-terms in topological models of λ-calculus, like Scott domains, DI-domains and coherence spaces [1, 5, 10, 11].

Definition 1.1 (IT) *(i)* Terms *of the untyped λ-calculus Λ are defined by the grammar: $t ::= x \,|\, \lambda x.t \,|\, tt$.*

(ii) The set $\mathcal{T}_{\mathrm{IT}}$ of intersection types *is generated by the grammar $\mathcal{T}_{\mathrm{IT}} \ni \sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma$, where α belongs to a countable set of type variables. We use α, β, γ, etc. to denote type variables and σ, τ, ρ, etc. to denote types. In omitting parentheses, we assume associativity to the right for implication, associativity to the left for intersection, and precedence of intersection over implication.*

(iii) A basis *B is a finite set $\{ x_1 : \sigma_1, \ldots, x_m : \sigma_m \}$ of assignments of intersection types to distinct variables. We define $dom(B)$ as the set $\{ x_1, \ldots, x_m \}$. We write $B, x : \sigma$ for a basis $B \cup \{ x : \sigma \}$, i.e. for a $x \notin dom(B)$.*

(iv) The type system IT *proves statements of the form $B \vdash t : \sigma$, where B is a basis, $t \in \Lambda$ and σ is an intersection type. Its rules are shown in Figure 1.1. We write $\pi :: B \vdash t : \sigma$ to denote a particular derivation π proving $B \vdash t : \sigma$.*

Proposition 1.2 *(i) (Renaming) If $\pi :: B, x : \sigma \vdash t : \tau$ and y is fresh with respect to π, then there exists a $\pi' :: B, y : \sigma \vdash t[y/x] : \tau$.*

(ii) (Weakening) If $B \vdash t : \sigma$ and $B \subseteq B'$, where B' is a basis, then $B' \vdash t : \sigma$.

(iii) (Strengthening) If $B \vdash t : \sigma$, then $FV(t) \subseteq dom(B)$ and $B \supseteq B' \vdash t : \sigma$, where $dom(B') = FV(t)$.

Proof. By induction on the given derivation in each case. Proposition (i) is used to show (ii), while (ii) is used to show (iii). ⊣

By adding the constant ω to $\mathcal{T}_{\mathrm{IT}}$ and the so-called (ω)-rule to the rules of IT, we get the type system IT_{ω}, denoted DΩ in [13]. The (ω)-rule is actually an axiom stating that, for any basis B and any term t, it is $B \vdash t : \omega$. The following proposition holds for both IT and IT_{ω}.

$$\frac{}{B,\, x:\sigma \vdash x:\sigma}\;\textbf{(ax)}$$

$$\frac{B,\, x:\sigma \vdash t:\tau}{B \vdash \lambda x.t:\sigma \to \tau}\;(\to\textbf{I}) \qquad \frac{B \vdash t:\sigma \to \tau \qquad B \vdash u:\sigma}{B \vdash tu:\tau}\;(\to\textbf{E})$$

$$\frac{B \vdash t:\sigma \qquad B \vdash t:\tau}{B \vdash t:\sigma \cap \tau}\;(\cap\textbf{I}) \qquad \frac{B \vdash t:\sigma \cap \tau}{B \vdash t:\sigma}\;(\cap\textbf{E}_1) \qquad \frac{B \vdash t:\sigma \cap \tau}{B \vdash t:\tau}\;(\cap\textbf{E}_2)$$

Figure 1.1: The type system IT.

Proposition 1.3 (Subject reduction) *If* $B \vdash t:\sigma$ *and* $t \to_\beta t'$*, then* $B \vdash t':\sigma$*.*

Proof. A proof can be found in [13]. $\dashv$

Subject expansion does not hold in IT. For instance, it is $\lambda y.(\lambda x.y)(yy) \to_\beta \lambda y.y$ and $\vdash \lambda y.y : \alpha \to \alpha$, but $\nvdash \lambda y.(\lambda x.y)(yy) : \alpha \to \alpha$. An explanation of this fact can be found in [13]. On the other hand, subject expansion does hold in IT_ω and is proved in [13]. The most important property of IT, though, is stated in the following theorem.

Theorem 1.4 *A term* $t \in \Lambda$ *is typable in* IT *if and only if it is strongly normalizing.*

Proof. Given in [13] by the reducibility method. $\dashv$

Remark 1.5 *The result of Theorem 1.4 breaks down in* IT_ω*, which may assign the type* ω *to* any $t \in \Lambda$.

For a proof-theoretical justification of intersection types, we may leave ω aside and consider the minimal type system with intersection types IT. A first attempt to find a logic corresponding to intersection types consisted in investigating if and how the implicative and conjunctive fragment of intuitionistic logic, denoted LJ in [18], could be associated with IT.

In [17, 9] it is argued that intersection types do not correspond to provable formulas of LJ. In particular, it is shown that the set of all intersection types which are inhabited by a closed term does not coincide with the set of all provable formulas of LJ, if the type constructor of intersection is converted to the logical connective of conjunction. A simple counter-example is the type $\sigma \to \tau \to \sigma \cap \tau$ which is not inhabited, while its corresponding formula $\sigma \to \tau \to \sigma \wedge \tau$ is provable in LJ. The result holds, though, for the set of all inhabited Curry types and the set of all provable formulas of implicational intuitionistic logic.

In [18, 15] it is argued that LJ does not correspond to IT through a *standard decoration* of its derivations with untyped λ-terms. A standard decoration of LJ is one that encodes all logical rules, i.e. both implication and conjunction. In fact, such a decoration delivers the Curry type system $\lambda^{\wedge}_{\to}$. At this point, we may recall LJ and $\lambda^{\wedge}_{\to}$, and define the decoration which serves as a "bridge" between the two in the Curry-Howard perspective.

$$\frac{}{\sigma \vdash \sigma}\ (\mathbf{ax})$$

$$\frac{\Gamma \vdash \tau}{\Gamma, \sigma \vdash \tau}\ (\mathbf{W}) \qquad \frac{\Gamma, \sigma, \tau, \Delta \vdash \rho}{\Gamma, \tau, \sigma, \Delta \vdash \rho}\ (\mathbf{X})$$

$$\frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \to \tau}\ (\to\mathbf{I}) \qquad \frac{\Gamma \vdash \sigma \to \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau}\ (\to\mathbf{E})$$

$$\frac{\Gamma \vdash \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma \wedge \tau}\ (\wedge\mathbf{I}) \qquad \frac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \sigma}\ (\wedge\mathbf{E}_1) \qquad \frac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \tau}\ (\wedge\mathbf{E}_2)$$

Figure 1.2: The logic LJ.

$$\frac{}{B,\ x : \sigma \vdash x : \sigma}\ (\mathbf{ax})$$

$$\frac{B,\ x : \sigma \vdash t : \tau}{B \vdash \lambda x.t : \sigma \to \tau}\ (\to\mathbf{I}) \qquad \frac{B \vdash t : \sigma \to \tau \qquad B \vdash u : \sigma}{B \vdash tu : \tau}\ (\to\mathbf{E})$$

$$\frac{B \vdash t : \sigma \qquad B \vdash u : \tau}{B \vdash (t,u) : \sigma \wedge \tau}\ (\wedge\mathbf{I}) \qquad \frac{B \vdash t : \sigma \wedge \tau}{B \vdash \pi_1(t) : \sigma}\ (\wedge\mathbf{E}_1) \qquad \frac{B \vdash t : \sigma \wedge \tau}{B \vdash \pi_2(t) : \tau}\ (\wedge\mathbf{E}_2)$$

Figure 1.3: The type system $\lambda^{\wedge}_{\to}$.

Definition 1.6 (LJ) *Considering* formulas *generated by the grammar* $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \wedge \sigma$, *where* α *belongs to a countable set of atomic formulas, the logical system* LJ *proves statements* $\Gamma \vdash \sigma$, *where the* context Γ *is a finite sequence of formulas and* σ *is a formula. Its rules are displayed in Figure 1.2. Implication is right associative, while conjunction is left associative and precedes over implication.*

Definition 1.7 ($\lambda^{\wedge}_{\to}$) *Considering types built by implication and conjunction, also known as* simple types, *the type system* $\lambda^{\wedge}_{\to}$ *proves statements* $B \vdash t : \sigma$, *where* B *is a basis,* t *belongs to the set* Λ_p *of terms with pairs, i.e.* $t ::= x \mid \lambda x.t \mid tt \mid (t,t) \mid \pi_1(t), \pi_2(t)$, *and* σ *is a simple type. Its rules are shown in Figure 1.3.*

Definition 1.8 (Standard decoration of LJ) *Let* $\pi :: \Gamma = \sigma_1, \ldots, \sigma_m \vdash \tau$ *be a derivation in* LJ. *By decorating contexts bottom-up with distinct variables starting with the sequence* $p = x_1, \ldots, x_m$ *and then decorating formulas to the right of "*$\vdash$*" top-down with terms in* Λ_p, *we get a decorated derivation* $\pi^* :: \Gamma^p = x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$. *The decoration rules are depicted in Figure 1.4. When decorating*

$$\frac{}{x:\sigma \vdash x:\sigma}\;\textbf{(ax)}$$

$$\frac{\Gamma^p \vdash t:\tau}{\Gamma^p,\, x:\sigma \vdash t:\tau}\;\textbf{(W)} \qquad \frac{\Gamma^p,\, y:\sigma,\, x:\tau,\, \Delta^q \vdash t:\rho}{\Gamma^p,\, x:\tau,\, y:\sigma,\, \Delta^q \vdash t:\rho}\;\textbf{(X)}$$

$$\frac{\Gamma^p,\, x:\sigma \vdash t:\tau}{\Gamma^p \vdash \lambda x.\, t:\sigma \to \tau}\;(\to\!\textbf{I}) \qquad \frac{\Gamma^p \vdash t:\sigma \to \tau \qquad \Gamma^p \vdash u:\sigma}{\Gamma^p \vdash tu:\tau}\;(\to\!\textbf{E})$$

$$\frac{\Gamma^p \vdash t:\sigma \qquad \Gamma^p \vdash u:\tau}{\Gamma^p \vdash (t,u):\sigma \wedge \tau}\;(\wedge\textbf{I}) \qquad \frac{\Gamma^p \vdash t:\sigma \wedge \tau}{\Gamma^p \vdash \pi_1(t):\sigma}\;(\wedge\textbf{E}_1) \qquad \frac{\Gamma^p \vdash t:\sigma \wedge \tau}{\Gamma^p \vdash \pi_2(t):\tau}\;(\wedge\textbf{E}_2)$$

Figure 1.4: Standard decoration of LJ.

contexts bottom-up, the new variable in a $(\to\!\textbf{I})$*-premise is fresh with respect to the variables in the branch connecting the* $(\to\!\textbf{I})$*-conclusion to the root.*

Any derivation of LJ can be standardly decorated to provide a derivation of $\lambda^{\wedge}_{\to}$, if decorated contexts are seen as sets, formulas are seen as types, and structural rules are ignored. Conversely, any derivation of $\lambda^{\wedge}_{\to}$ can be converted to one of LJ, if terms are erased, variable-free bases are seen as sequences, types are seen as formulas, and structural rules are added, where necessary. The following example shows the decoration and erasure directions between LJ and $\lambda^{\wedge}_{\to}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \beta \vdash \alpha \to \beta}{\alpha \to \beta, \alpha \vdash \alpha \to \beta}\;\textbf{(W)}}{\alpha, \alpha \to \beta \vdash \alpha \to \beta}\;\textbf{(X)} \qquad \dfrac{\alpha \vdash \alpha}{\alpha, \alpha \to \beta \vdash \alpha}\;\textbf{(W)}}{\alpha, \alpha \to \beta \vdash \beta}\;(\to\!\textbf{E}) \qquad \dfrac{\alpha \vdash \alpha}{\alpha, \alpha \to \beta \vdash \alpha}\;\textbf{(W)}}{\alpha, \alpha \to \beta \vdash_{\mathrm{LJ}} \beta \wedge \alpha}\;(\wedge\textbf{I}) \qquad \begin{array}{c}\text{decoration}\\ \longrightarrow\\ \longleftarrow\\ \text{erasure}\end{array}$$

$$\dfrac{\dfrac{x:\alpha,\, y:\alpha \to \beta \vdash y:\alpha \to \beta \qquad x:\alpha,\, y:\alpha \to \beta \vdash x:\alpha}{x:\alpha,\, y:\alpha \to \beta \vdash yx:\beta}\;(\to\!\textbf{E}) \qquad x:\alpha,\, y:\alpha \to \beta \vdash x:\alpha}{x:\alpha,\, y:\alpha \to \beta \vdash_{\lambda^{\wedge}_{\to}} (yx,x):\beta \wedge \alpha}\;(\wedge\textbf{I})$$

Such a connection through decoration and erasure also holds between the implicative fragment of intuitionistic logic and Curry's type assignment system $\lambda_{\to}$.

It is further argued in [18, 15] that even if a so-called *non-standard decoration* is employed, LJ does not correspond to IT. The idea for a non-standard decoration that encodes the implication, but ignores the conjunction, derives from the intersection rules of IT, in which premise and conclusion terms are identical, and from the fact that we would like a decorated derivation of LJ to provide a derivation of IT, if conjunction were converted to intersection. The rules for such a decoration are shown in Figure 1.5.

$$\frac{}{x : \sigma \vdash x : \sigma}\,\textbf{(ax)}$$

$$\frac{\Gamma^p \vdash t : \tau}{\Gamma^p,\, x : \sigma \vdash t : \tau}\,\textbf{(W)} \qquad \frac{\Gamma^p,\, y : \sigma,\, x : \tau,\, \Delta^q \vdash t : \rho}{\Gamma^p,\, x : \tau,\, y : \sigma,\, \Delta^q \vdash t : \rho}\,\textbf{(X)}$$

$$\frac{\Gamma^p,\, x : \sigma \vdash t : \tau}{\Gamma^p \vdash \lambda x.\, t : \sigma \to \tau}\,(\to\mathbf{I}) \qquad \frac{\Gamma^p \vdash t : \sigma \to \tau \qquad \Gamma^p \vdash u : \sigma}{\Gamma^p \vdash tu : \tau}\,(\to\mathbf{E})$$

$$\frac{\Gamma^p \vdash t : \sigma \qquad \Gamma^p \vdash t : \tau}{\Gamma^p \vdash t : \sigma \wedge \tau}\,(\wedge\mathbf{I}) \qquad \frac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash t : \sigma}\,(\wedge\mathbf{E}_1) \qquad \frac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash t : \tau}\,(\wedge\mathbf{E}_2)$$

Figure 1.5: Non-standard decoration of LJ.

It is clear that the decoration terminates only in derivations of LJ in which all $(\wedge\mathbf{I})$'s are applied to identically decorated[1] premises; otherwise, the decoration fails. Identically decorated (sub)derivations are called *isomorphic* in [18]. Isomorphic derivations share the same implicative structure.

Consequently, only a proper subset of LJ, denoted LJns, admits a non-standard decoration and it is this subset that corresponds to IT through decoration and erasure. As when relating the whole of LJ to $\lambda^{\wedge}_{\to}$, a derivation of LJns can be non-standardly decorated to provide a derivation of IT, if decorated contexts are seen as sets, conjunction is converted to intersection, and structural rules are ignored. Conversely, a derivation of IT can be converted to one of LJns, if terms are erased, variable-free bases are seen as sequences, intersection is converted to conjunction, and structural rules are added, if necessary. An example of derivations in LJns and IT with such a connection follows.

$$\dfrac{\dfrac{\alpha \vdash \alpha}{\vdash \alpha \to \alpha}\,(\to\mathbf{I}) \qquad \dfrac{\dfrac{\beta \vdash \beta}{\vdash \beta \to \beta}\,(\to\mathbf{I}) \qquad \dfrac{\gamma \vdash \gamma}{\vdash \gamma \to \gamma}\,(\to\mathbf{I})}{\vdash (\beta \to \beta) \wedge (\gamma \to \gamma)}\,(\wedge\mathbf{I})}{\vdash_{\mathrm{LJns}} (\alpha \to \alpha) \wedge ((\beta \to \beta) \wedge (\gamma \to \gamma))}\,(\wedge\mathbf{I}) \qquad \begin{array}{c}\text{decoration}\\ \longrightarrow \\ \longleftarrow \\ \text{erasure}\end{array}$$

$$\dfrac{\dfrac{x : \alpha \vdash x : \alpha}{\vdash \lambda x.\, x : \alpha \to \alpha}\,(\to\mathbf{I}) \qquad \dfrac{\dfrac{x : \beta \vdash x : \beta}{\vdash \lambda x.\, x : \beta \to \beta}\,(\to\mathbf{I}) \qquad \dfrac{x : \gamma \vdash x : \gamma}{\vdash \lambda x.\, x : \gamma \to \gamma}\,(\to\mathbf{I})}{\vdash \lambda x.\, x : (\beta \to \beta) \cap (\gamma \to \gamma)}\,(\cap\mathbf{I})}{\vdash_{\mathrm{IT}} \lambda x.\, x : (\alpha \to \alpha) \cap ((\beta \to \beta) \cap (\gamma \to \gamma))}\,(\cap\mathbf{I})$$

Derivations in LJ\LJns do not admit a non-standard decoration. Such a derivation is the one proving $\alpha, \alpha \to \beta \vdash \beta \wedge \alpha$, shown on the previous page. The left and right premises of $(\wedge\mathbf{I})$ are decorated by yx and x, respectively, if contexts are decorated by x, y, which means that a non-standard decoration cannot proceed to the conclusion.

---

[1] We mean decorated *non-standardly*.

The above discussion testifies that LJ restricted to implication offers a logical foundation to the type system $\lambda_{\rightarrow}$, while the whole of LJ offers a logical foundation to $\lambda_{\rightarrow}^{\wedge}$. It is *not* the case that LJ is the logic behind IT through a correspondence by decoration and erasure. If we employ the standard decoration, we end up corresponding with another type system, namely $\lambda_{\rightarrow}^{\wedge}$, while, if we employ the non-standard decoration, only a proper subset of LJ corresponds to IT. This means that intersection cannot be logically interpreted as conjunction. It is rather a special kind of conjunction between isomorphic or *synchronous* conjuncts; it is referred to as *synchronous conjunction* in [15], while the standard intuitionistic conjunction is referred to as *asynchronous conjunction.* The notion of "isomorphism" or "synchronicity" of conjuncts is a metatheoretical restriction on LJ, as noted in [18], brought to light only by means of the non-standard decoration. The subset LJns expresses this special kind of conjunction and would serve as a logic for IT, if it were somehow autonomized and described as a logic by itself. This is exactly what is attempted in [18] and [15] by introducing the logical systems "Intersection Logic" and "Intersection Synchronous Logic", respectively.

1.1 Intersection Logic

Intersection Logic IL works with full binary trees[2], called *kits*, whose leaves are formulas generated by implication and intersection. It is a natural deduction system which proves judgements in sequent style. Judgements include kits of the same structure, which are called *overlapping.* Since IL is intended to realize the part of LJ where ($\wedge$**I**) is applied to isomorphic premises, namely LJns, the rule introducing the intersection in IL should embody this isomorphism or *sameness* of premises. This is achieved by binary trees; in particular, the premises become leaves originating from the *same* parent-node in a kit, so that intersection introduction in IL has a single premise. Its conclusion gives a kit where the intersection of the two leaves is a leaf on the parent-node. As a result, a non-standard decoration of kits, encoding the implication solely, is free to terminate in any derivation of IL.

$$\frac{\vdash t:\sigma \qquad \vdash t:\tau}{\vdash t:\sigma\wedge\tau}\,(\wedge\mathbf{I})\text{ in LJns} \qquad\qquad \frac{\vdash t:[\sigma,\tau]}{\vdash t:\sigma\cap\tau}\,(\cap\mathbf{I})\text{ in IL}$$

A concise definition of IL and its accompanying notions follows.

Definition 1.9 (IL) *(i) A* kit *is a full binary tree $K ::= \sigma \mid [K,K]$ whose leaves are formulas generated by the grammar $\sigma ::= \alpha \mid \sigma \rightarrow \sigma \mid \sigma\cap\sigma$, where α belongs to a countable set of atomic formulas. We use K,H,L to denote kits and σ,τ,ρ, etc. to denote leaves.*

(ii) Two kits H,K overlap*, denoted $H\simeq K$, if they share the same tree structure, but possibly differ on their leaves.*

(iii) A path *of length n in a kit is a string of n letters from the set $\{l,r\}$, where l stands for "left" and r for "right", that corresponds to the part of the kit which starts at the root and ends at the node reached after n left or right steps. We use the letters p and q with subscripts, primes, etc. to denote paths. The* subtree of a kit K at path p*, denoted K^p, is the subtree of K rooted at the end of p in K. A* terminal path *is one that ends at a leaf; the set of terminal paths of a kit K is denoted $P_T(K)$. Two paths p and q of K are* different*, if they split at a node of K.*

---

[2] A *full binary tree* is a tree in which every node other than the leaves has two child-nodes.

$$\frac{}{K \vdash K}\;(\mathbf{ax}) \qquad \frac{H_1, \ldots, H_m \vdash K \qquad H \simeq H_j \;\; (1 \leqslant j \leqslant m)}{H_1, \ldots, H_m, H \vdash K}\;(\mathbf{W})$$

$$\frac{\Gamma, H_1, H_2, \Delta \vdash K}{\Gamma, H_2, H_1, \Delta \vdash K}\;(\mathbf{X}) \qquad \frac{\Gamma = H_1, \ldots, H_m \vdash K}{\Gamma\backslash^{ps} = H_1\backslash^{ps}, \ldots, H_m\backslash^{ps} \vdash K\backslash^{ps}}\;(\mathbf{P})$$

$$\frac{\Gamma, H \vdash K}{\Gamma \vdash H \to K}\;(\to\mathbf{I}) \qquad \frac{\Gamma \vdash H \to K \qquad \Gamma \vdash H}{\Gamma \vdash K}\;(\to\mathbf{E})$$

$$\frac{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_m[p := [\sigma_m, \sigma_m]] \vdash K[p := [\sigma, \tau]]}{H_1[p := \sigma_1], \ldots, H_m[p := \sigma_m] \vdash K[p := \sigma \cap \tau]}\;(\cap\mathbf{I})$$

$$\frac{\Gamma \vdash K[p := \sigma \cap \tau]}{\Gamma \vdash K[p := \sigma]}\;(\cap\mathbf{E}_1) \qquad \frac{\Gamma \vdash K[p := \sigma \cap \tau]}{\Gamma \vdash K[p := \tau]}\;(\cap\mathbf{E}_2)$$

Figure 1.6: The logic IL.

(iv) If $H \simeq K$*, then* $H \to K$ *is a kit overlapping with* H, K *and such that* $(H \to K)^p = H^p \to K^p$*, for every* $p \in P_T(H \to K)[= P_T(H) = P_T(K)]$.

(v) The notation $H[p := K]$ *stands for the kit resulting from the substitution of* H^p *by* K *in* H*. If* ps *is a path in* H*, where* $s \in \{l, r\}$*, the* pruning *of* H *at path* ps*, denoted* $H\backslash^{ps}$*, is defined as* $H[p := H^{ps}]$.

(vi) The deductive system IL *derives judgements* $\Gamma \vdash K$*, where the context* Γ *is a sequence of kits and* K *is a kit. It consists of the rules in Figure 1.6.*

Remark 1.10 *The inclusion of the structural rule of pruning, rule* (**P**) *in Figure 1.6, is motivated by purely technical reasons, i.e. reasons concerning the manipulation of the tree structure.*

It is easy to show that all judgements derived in IL include overlapping kits, i.e. if $H_1, \ldots, H_m \vdash K$, then $H_j \simeq K$ $(1 \leqslant j \leqslant m)$.

The implicative rules affect all terminal paths (or leaves) of (some of) the kits involved and are called *global.* On the other hand, the notation "$\_\,[p := \_]$" used in the intersection rules shows that these rules act on a specific path p. Rules affecting a specific path are called *local.* Pruning also acts locally on kits.

The system just defined as "Intersection Logic" is actually called "pre-Intersection Logic", denoted pIL, in [18]. Then, a derivation of IL proving $\Gamma \vdash K$ is defined as an equivalence class of derivations of pIL, all proving $\Gamma \vdash K$. The equivalence relation between derivations of pIL is introduced to eliminate unnecessary differentiations resulting from differences in the order of application of consecutive intersection rules concerning different paths. In practice, though, a derivation of IL is identified with a derivation of pIL in the specified equivalence class.

To give a correspondence between IL and LJns and also between IL and IT, a non-standard decoration of IL is defined in [18]. The decoration employs untyped λ-terms to keep track of the implicative structure of derivations.

$$\frac{}{x : K \vdash x : K}\,\textbf{(ax)} \qquad \frac{x_1 : H_1, \ldots, x_m : H_m \vdash t : K}{x_1 : H_1, \ldots, x_m : H_m, x : H \vdash t : K}\,\textbf{(W)}$$

$$\frac{\Gamma^r, y : H_1, x : H_2, \Delta^{r'} \vdash t : K}{\Gamma^r, x : H_2, y : H_1, \Delta^{r'} \vdash t : K}\,\textbf{(X)} \qquad \frac{x_1 : H_1, \ldots, x_m : H_m \vdash t : K}{x_1 : H_1\backslash^{ps}, \ldots, x_m : H_m\backslash^{ps} \vdash t : K\backslash^{ps}}\,\textbf{(P)}$$

$$\frac{\Gamma^r, x : H \vdash t : K}{\Gamma^r \vdash \lambda x.t : H \to K}\,(\to\textbf{I}) \qquad \frac{\Gamma^r \vdash t : H \to K \qquad \Gamma^r \vdash u : H}{\Gamma^r \vdash tu : K}\,(\to\textbf{E})$$

$$\frac{x_1 : H_1[p := [\sigma_1, \sigma_1]], \ldots, x_m : H_m[p := [\sigma_m, \sigma_m]] \vdash t : K[p := [\sigma, \tau]]}{x_1 : H_1[p := \sigma_1], \ldots, x_m : H_m[p := \sigma_m] \vdash t : K[p := \sigma \cap \tau]}\,(\cap\textbf{I})$$

$$\frac{\Gamma^r \vdash t : K[p := \sigma \cap \tau]}{\Gamma^r \vdash t : K[p := \sigma]}\,(\cap\textbf{E}_1) \qquad \frac{\Gamma^r \vdash t : K[p := \sigma \cap \tau]}{\Gamma^r \vdash t : K[p := \tau]}\,(\cap\textbf{E}_2)$$

Figure 1.7: Non-standard decoration of IL.

Definition 1.11 (Non-standard decoration of IL) *Let* $\pi :: \Gamma = H_1, \ldots, H_m \vdash K$ *be a derivation in* IL*. By decorating contexts bottom-up with distinct variables starting with the sequence* $r = x_1, \ldots, x_m$ *and then decorating kits to the right of "*$\vdash$*" top-down with terms in* Λ*, we get a decorated derivation* $\pi^\star :: \Gamma^r = x_1 : H_1, \ldots, x_m : H_m \vdash t : K$*. The decoration rules are shown in Figure 1.7. When decorating contexts bottom-up, the new variable in a* $(\to\textbf{I})$*-premise is fresh with respect to the variables in the branch connecting the* $(\to\textbf{I})$*-conclusion to the root.*

The following theorem connects IL to LJns modulo the conversion of intersection to conjunction. It states that a derivation in IL projects to a finite number of derivations in LJns that all admit the same non-standard decoration, namely the non-standard decoration of the IL-derivation.

Theorem 1.12 *Let* $\pi :: H_1, \ldots, H_m \vdash K$ *be a derivation in* IL*, s.t.* $\pi^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$*. For every* $p \in P_T(K)$*, there exists a derivation* $\pi^p :: (H_1)^p, \ldots, (H_m)^p \vdash K^p$ *in* LJns*, such that it admits the same (non-standard) decoration as* π*, i.e. such that* $(\pi^p)^\star :: x_1 : (H_1)^p, \ldots, x_m : (H_m)^p \vdash t : K^p$*.*

Proof. Given in [21] by induction on π. ⊣

1.1.1 Strong normalization of IL

Derivations in IL are shown to be strongly normalizing in [18, 21]. A *normal* derivation is one which is free of the pruning rule and also free of implication and intersection redexes. The pruning rule can be easily eliminated, since it commutes with every other rule and can thus be shifted up just below axioms, where it can be ignored. Then, implication and intersection redexes can be reduced as shown below.

$$\dfrac{\dfrac{\pi_0 :: \Gamma, H \vdash K}{\Gamma \vdash H \to K}\,(\to\mathbf{I}) \qquad \pi_1 :: \Gamma \vdash H}{\Gamma \vdash K}\,(\to\mathbf{E}) \qquad \hookrightarrow_{\to} \qquad S(\pi_1, \pi_0) :: \Gamma \vdash K$$

The notation $S(\pi_1, \pi_0)$ stands for the derivation obtained from π_0 by substituting specific[3] instances of axioms $H \vdash H$ by π_1 and then possibly eliminating some instances of weakening and exchange.

$$\dfrac{\dfrac{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_m[p := [\sigma_m, \sigma_m]] \vdash K[p := [\sigma, \tau]]}{H_1[p := \sigma_1], \ldots, H_m[p := \sigma_m] \vdash K[p := \sigma \cap \tau]}\,(\cap\mathbf{I})}{H_1[p := \sigma_1], \ldots, H_m[p := \sigma_m] \vdash K[p := \sigma]}\,(\cap\mathbf{E}_1) \qquad \hookrightarrow_{\cap}$$

$$\dfrac{H_1[p := [\sigma_1, \sigma_1]], \ldots, H_m[p := [\sigma_m, \sigma_m]] \vdash K[p := [\sigma, \tau]]}{H_1[p := \sigma_1], \ldots, H_m[p := \sigma_m] \vdash K[p := \sigma]}\,(\mathbf{P}) \text{ on } pl$$

To show that IL is strongly normalizing, Theorem 1.12 and the strong normalization of LJ are used.

Theorem 1.13 *A derivation in* IL *is strongly normalizing.*

Proof. A detailed proof is given in [21]. ⊣

1.1.2 Correspondence between IL and IT

The following two theorems are stated and proved in [18]. The first one relates a derivation of IL to a finite number of derivations in IT through the non-standard decoration of IL. The second one relates a single derivation of IT to a derivation in IL, whose non-standard decoration are the terms in the derivation of IT.

Theorem 1.14 *Let* $\pi :: H_1, \ldots, H_m \vdash K$ *be a derivation in* IL, *s.t.* $\pi^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$. *For every* $p \in P_T(K)$, *there exists a derivation* $\pi^p :: \{\, x_1 : (H_1)^p, \ldots, x_m : (H_m)^p \,\} \vdash t : K^p$ *in* IT.

Proof. By induction on the IL-derivation π. ⊣

Theorem 1.15 *If* $\pi :: \{\, x_1 : \sigma_1, \ldots, x_m : \sigma_m \,\} \vdash t : \tau$ *is a derivation in* IT, *there exists a derivation* $\pi' :: \sigma_1, \ldots, \sigma_m \vdash \tau$ *in* IL, *where* $\sigma_1, \ldots, \sigma_m$, *and* τ *are kits consisting of a single node, such that* $(\pi')^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$.

Proof. By induction on the IT-derivation π. ⊣

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[3] Instances such that the kit H to the left of "⊢" does not move to the right of "⊢" by an $(\to\mathbf{I})$ rule.

1.2 Intersection Synchronous Logic

Intersection Synchronous Logic ISL is a natural deduction system proving multisets, called *molecules*, whose members are *atoms*. Roughly speaking, atoms are intuitionistic statements, where conjunction is converted to intersection. This logic is also intended to realize LJns, where $(\wedge\mathbf{I})$ is applied to isomorphic premises, so the rule introducing the intersection embodies this isomorphism, as was the case in IL. This is achieved by "gathering" isomorphic premises as atoms of the *same* molecule, so that intersection introduction has again a single premise. Its conclusion gives a molecule where the two atoms, corresponding to the isomorphic premises, have merged into one atom that contains the intersection of the premises. As was the case with kits, a non-standard decoration of molecules, encoding the implication solely, is free to terminate in any derivation of ISL.

$$\frac{\vdash t:\sigma \qquad\quad \vdash t:\tau}{\vdash t:\sigma\wedge\tau}\,(\wedge\mathbf{I})\text{ in LJns} \qquad\qquad \frac{t:[(\ \ ;\sigma),(\ \ ;\tau)]}{t:[(\ \ ;\sigma\cap\tau)]}\,(\cap\mathbf{I})\text{ in ISL}$$

The structural components and the rules of ISL are defined as follows.

Definition 1.16 (ISL) *(i)* Formulas *are generated by the grammar* $\sigma ::= \alpha \mid \sigma\to\sigma \mid \sigma\cap\sigma$*, where* α *belongs to a countable set of atomic formulas.*

(ii) An atom *is a pair* $(\Gamma\,;\,\sigma)$*, where the context* Γ *is a finite sequence of formulas and* σ *is a formula. We use* $\mathcal{A},\mathcal{B},\mathcal{C}$ *to denote atoms.*

(iii) A molecule $[\mathcal{A}_1,\ldots,\mathcal{A}_n]$ *is a finite multiset of atoms that all share the same context cardinality. We use* $\mathcal{M},\mathcal{N}$ *to denote molecules.*

(iv) The deductive system ISL *proves molecules by the rules depicted in Figure 1.8. We use the notation* $[(\Gamma_i\,;\,\tau_i)_i]$ *for a molecule* $[(\sigma^i_1,\ldots,\sigma^i_m\,;\,\tau_i)\mid 1\leqslant i\leqslant n]$ *and the symbol "*$\cup$*" for multiset union.*

It is explained in detail in [15, 21] why it is necessary to define atom-contexts as sequences and have explicit weakening and exchange in order for ISL to correctly capture the behavior of the intersection connective.

The rules of ISL can be categorized as *global* or *local* according to whether they affect *all* or *some* atoms of the premise molecule(s), respectively. The structural rules of weakening and exchange and the implication rules are global, while the structural rule of pruning and the intersection rules are local.

A non-standard decoration of ISL is defined in [15]. This decoration is used in [21] to establish a correspondence between ISL and LJns and is also used in [15] to establish a correspondence between ISL and IT.

Definition 1.17 (Non-standard decoration of ISL) *Let* $\pi :: \mathcal{M} = [(\Gamma_i\,;\,\tau_i)_i] = [(\sigma^i_1,\ldots,\sigma^i_m\,;\,\tau_i)_i]$ *be a derivation in* ISL*. By decorating contexts bottom-up with distinct variables starting with the sequence* $p = x_1,\ldots,x_m$ *and then decorating molecules top-down with terms in* Λ*, we get a decorated derivation* $\pi^\star :: t:\mathcal{M}_p = [(\Gamma_i\,;\,\tau_i)_i]_p = [(x_1:\sigma^i_1,\ldots,x_m:\sigma^i_m\,;\,\tau_i)_i]$*. The decoration rules are shown in Figure 1.9. When decorating contexts bottom-up, the new variable in an* $(\to\mathbf{I})$*-premise is fresh with respect to the variables in the branch connecting the* $(\to\mathbf{I})$*-conclusion to the root.*

The following theorem is analogous to Theorem 1.12 for IL. It is stated and proved in [21] and connects ISL to LJns modulo the conversion of intersection to conjunction.

$$\frac{}{[(\sigma_i\,;\,\sigma_i)_i]}\ \textbf{(ax)} \qquad \frac{[(\Gamma_i\,;\,\tau_i)_i]}{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}\ \textbf{(W)}$$

$$\frac{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i)_i]}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\rho_i)_i]}\ \textbf{(X)} \qquad \frac{\mathcal{M}\cup\mathcal{N}}{\mathcal{M}}\ \textbf{(P)}$$

$$\frac{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}{[\Gamma_i\,;\,\sigma_i\to\tau_i)_i]}\ \textbf{(}{\to}\textbf{I)} \qquad \frac{[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i] \qquad [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\ \textbf{(}{\to}\textbf{E)}$$

$$\frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma),(\Gamma\,;\,\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}\ \textbf{(}{\cap}\textbf{I)}$$

$$\frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma)]}\ \textbf{(}{\cap}\textbf{E}_1\textbf{)} \qquad \frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\tau)]}\ \textbf{(}{\cap}\textbf{E}_2\textbf{)}$$

Figure 1.8: The logic ISL.

$$\frac{}{x:[(\sigma_i\,;\,\sigma_i)_i]_x}\ \textbf{(ax)} \qquad \frac{t:[(\Gamma_i\,;\,\tau_i)_i]_p}{t:[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]_{p,\,x}}\ \textbf{(W)}$$

$$\frac{t:[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i)_i]_{p,\,y,\,x,\,q}}{t:[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\rho_i)_i]_{p,\,x,\,y,\,q}}\ \textbf{(X)} \qquad \frac{t:\mathcal{M}_p\cup\mathcal{N}_p}{t:\mathcal{M}_p}\ \textbf{(P)}$$

$$\frac{t:[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]_{p,\,x}}{\lambda x.t:[\Gamma_i\,;\,\sigma_i\to\tau_i)_i]_p}\ \textbf{(}{\to}\textbf{I)} \qquad \frac{t:[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i]_p \qquad u:[(\Gamma_i\,;\,\sigma_i)_i]_p}{tu:[(\Gamma_i\,;\,\tau_i)_i]_p}\ \textbf{(}{\to}\textbf{E)}$$

$$\frac{t:\mathcal{M}_p\cup[(\Gamma\,;\,\sigma),(\Gamma\,;\,\tau)]_p}{t:\mathcal{M}_p\cup[(\Gamma\,;\,\sigma\cap\tau)]_p}\ \textbf{(}{\cap}\textbf{I)}$$

$$\frac{t:\mathcal{M}_p\cup[(\Gamma\,;\,\sigma\cap\tau)]_p}{t:\mathcal{M}_p\cup[(\Gamma\,;\,\sigma)]_p}\ \textbf{(}{\cap}\textbf{E}_1\textbf{)} \qquad \frac{t:\mathcal{M}_p\cup[(\Gamma\,;\,\sigma\cap\tau)]_p}{t:\mathcal{M}_p\cup[(\Gamma\,;\,\tau)]_p}\ \textbf{(}{\cap}\textbf{E}_2\textbf{)}$$

Figure 1.9: Non-standard decoration of ISL.

Theorem 1.18 *Let* $\pi :: [(\Gamma_i\,;\,\tau_i)_i]$ *be a derivation in* ISL*, such that* $\pi^\star :: t : [(\Gamma_i\,;\,\tau_i)_i]_p$*. For every* i*, there exists a derivation* $\pi^i :: \Gamma_i \vdash \tau_i$ *in* LJns*, such that it admits the same (non-standard) decoration as* π*, i.e. such that* $(\pi^i)^\star :: (\Gamma_i)^p \vdash t : \tau_i$*.*

Proof. By induction on π. ⊣

1.2.1 Strong normalization of ISL

Derivations of ISL are shown to be strongly normalizing in [15, 21], using the notion of "normal derivation" as defined for IL. Pruning is eliminated by commuting conversions as in IL, and redexes of logical connectives are reduced as shown below. The substitution notation $S(\pi_1, \pi_0)$ bears the usual meaning[4].

$$\dfrac{\dfrac{\pi_0 :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]}{[(\Gamma_i\,;\,\sigma_i \to \tau_i)_i]}\,(\to\mathbf{I}) \qquad \pi_1 :: [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\,(\to\mathbf{E}) \qquad \hookrightarrow_{\to} \qquad S(\pi_1, \pi_0) :: [(\Gamma_i\,;\,\tau_i)_i]$$

$$\dfrac{\dfrac{\mathcal{M} \cup [(\Gamma\,;\,\sigma), (\Gamma\,;\,\tau)]}{\mathcal{M} \cup [(\Gamma\,;\,\sigma \cap \tau)]}\,(\cap\mathbf{I})}{\mathcal{M} \cup [(\Gamma\,;\,\sigma)]}\,(\cap\mathbf{E}_1) \qquad \hookrightarrow_{\cap} \qquad \dfrac{\mathcal{M} \cup [(\Gamma\,;\,\sigma), (\Gamma\,;\,\tau)]}{\mathcal{M} \cup [(\Gamma\,;\,\sigma)]}\,(\mathbf{P})$$

In [15] it is also noted that, in a **(P)**-free derivation, the structural rules of weakening and exchange can all be moved up above the logical rules, so that an axiom is followed by a sequence of weakenings, which is followed by a sequence of exchanges, which is followed by logical rules. Such derivations are called *canonical*. It may be necessary to bring a derivation to canonical form for redexes to be properly revealed. Nonetheless, reduction steps preserve canonical forms, provided that any pruning generated by reduction is eliminated.

To show the strong normalization of ISL, we use Theorem 1.18 and the strong normalization of LJ.

Theorem 1.19 *A derivation in* ISL *is strongly normalizing.*

Proof. A detailed proof is given in [21]. ⊣

1.2.2 Correspondence between ISL and IT

A theorem which gives a correspondence between ISL and IT through the decoration of ISL is stated and proved in [15].

Theorem 1.20 *If* $\pi :: [(\sigma^i_1, \ldots, \sigma^i_m\,;\,\tau_i)_i]$ *is in* ISL*, then* $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m\,;\,\tau_i)_i]_{x_1, \ldots, x_m}$ *if and only if* $\pi_i :: \{x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m\} \vdash t : \tau_i$ *in* IT*, for every* i*.*

Proof. The "only if" direction is shown by induction on π, while the "if" direction is shown by induction on t. ⊣

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[4] Here it stands for the derivation obtained from π_0 by substituting specific instances of axioms $[(\sigma_i\,;\,\sigma_i)_i]$ by π_1 and then possibly eliminating some instances of weakening and exchange.

CHAPTER 2

Union Types

We start by presenting a type system with intersection and union types in natural deduction style. The system assigns intersection and union types to terms of the untyped λ-calculus Λ. It is introduced in [2], where it is denoted $\mathfrak{N}$, as an extension of IT_ω with rules for union.

Definition 2.1 ($\mathbf{IUT}_\omega$) *(i) The set* $\mathcal{T}_{\mathrm{IUT}_\omega}$ *of* intersection and union types *is generated by the grammar* $\mathcal{T}_{\mathrm{IUT}_\omega} \ni \sigma ::= \alpha \,|\, \omega \,|\, \sigma \to \sigma \,|\, \sigma \cap \sigma \,|\, \sigma \cup \sigma$, *where* α *belongs to a countable set of type variables. As usual, we use* α, β, γ, *etc. to denote type variables and* σ, τ, ρ, *etc. to denote types. Implication is right associative, while intersection and union are left associative and precede over implication.*

(ii) A typing statement *is an expression* $t : \sigma$, *where* $t \in \Lambda$ *and* $\sigma \in \mathcal{T}_{\mathrm{IUT}_\omega}$. *Term* t *is called the* subject *and type* σ *the* predicate *of the typing statement. A* basic typing statement $x : \sigma$ *is a typing statement whose subject is a variable. A* basis *is a set of basic typing statements such that the subjects are pairwise distinct. If* B *is a basis, then* $dom(B)$ *denotes the set of term variables which are subjects of basic typing statements in* B.

(iii) The type system IUT_ω *proves* statements $B \vdash t : \sigma$, *where* B *is a basis and* $t : \sigma$ *a typing statement. We call* B *the* assumptions *and* $t : \sigma$ *the* succedent *of* $B \vdash t : \sigma$. *The rules of the system are shown in Figure 2.1.*

Remark 2.2 *(i) The system is* additive, *i.e. all rules with more than one premise are context-sharing.*

(ii) Contraction **(C)** *is derivable and equivalent to a union redex* ($\cup$**IE**), *as shown below.*

$$\frac{B,\,x:\sigma,\,y:\sigma \vdash t:\tau}{B,\,x:\sigma \vdash t[x/y]:\tau}\,\textbf{(C)} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{}{B,\,x:\sigma\vdash x:\sigma}\,\textbf{(ax)}}{B,\,x:\sigma\vdash x:\sigma\cup\sigma}\,(\cup\textbf{I}) \qquad B,\,x:\sigma,\,y:\sigma\vdash t:\tau \qquad B,\,x:\sigma,\,y:\sigma\vdash t:\tau}{B,\,x:\sigma\vdash t[x/y]:\tau}\,(\cup\textbf{E})$$

The next lemma shows that a cut rule is also derivable.

Lemma 2.3 (Substitution lemma) *If* $B \vdash t : \sigma$ *and* $B, x : \sigma \vdash u : \tau$, *then* $B \vdash u[t/x] : \tau$.

Proof. Shown from hypotheses in [2] by employing a union redex. $\dashv$

As noted in [2], if one is interested in the proof-theoretical properties of the system, it can be useful to reformulate it in a sequent calculus style, i.e. to present it with left and right introduction rules and

$$\frac{}{B,\, x:\sigma \vdash x:\sigma}\;(\mathbf{ax}) \qquad \frac{}{B \vdash t:\omega}\;(\boldsymbol{\omega})$$

$$\frac{B,\, x:\sigma \vdash t:\tau}{B \vdash \lambda x.t:\sigma \to \tau}\;(\to\mathbf{I}) \qquad \frac{B \vdash t:\sigma \to \tau \qquad B \vdash u:\sigma}{B \vdash tu:\tau}\;(\to\mathbf{E})$$

$$\frac{B \vdash t:\sigma \qquad B \vdash t:\tau}{B \vdash t:\sigma \cap \tau}\;(\cap\mathbf{I}) \qquad \frac{B \vdash t:\sigma \cap \tau}{B \vdash t:\sigma}\;(\cap\mathbf{E}_1) \qquad \frac{B \vdash t:\sigma \cap \tau}{B \vdash t:\tau}\;(\cap\mathbf{E}_2)$$

$$\frac{B \vdash t:\sigma}{B \vdash t:\sigma \cup \tau}\;(\cup\mathbf{I}_1) \qquad \frac{B \vdash t:\tau}{B \vdash t:\sigma \cup \tau}\;(\cup\mathbf{I}_2)$$

$$\frac{B \vdash t:\sigma \cup \tau \qquad B,\, x:\sigma \vdash u:\rho \qquad B,\, x:\tau \vdash u:\rho}{B \vdash u[t/x]:\rho}\;(\cup\mathbf{E})$$

Figure 2.1: The type system IUT_ω in natural deduction style.

in a multiplicative manner. The sequent calculus version is presented in Figure 2.2. Statements $B \vdash t:\sigma$ are now called *sequents*. We write B, B' to mean that $B \cup B'$ is still a basis, i.e. if $x \in dom(B) \cap dom(B')$, then there is a unique σ, such that $x:\sigma \in B$ and $x:\sigma \in B'$. In $(\to\mathbf{L})$, variable y in the conclusion sequent is fresh with respect to the derivations proving the premise sequents.

$$\frac{}{B,\, x:\sigma \vdash x:\sigma}\;(\mathbf{ax}) \qquad \frac{}{B \vdash t:\omega}\;(\boldsymbol{\omega})$$

$$\frac{B \vdash t:\sigma \qquad B',\, x:\tau \vdash u:\rho}{B,\, B',\, y:\sigma \to \tau \vdash u[yt/x]:\rho}\;(\to\mathbf{L}) \qquad \frac{B,\, x:\sigma \vdash t:\tau}{B \vdash \lambda x.t:\sigma \to \tau}\;(\to\mathbf{R})$$

$$\frac{B,\, x:\sigma \vdash t:\rho}{B,\, x:\sigma \cap \tau \vdash t:\rho}\;(\cap\mathbf{L}_1) \qquad \frac{B,\, x:\tau \vdash t:\rho}{B,\, x:\sigma \cap \tau \vdash t:\rho}\;(\cap\mathbf{L}_2) \qquad \frac{B \vdash t:\sigma \qquad B' \vdash t:\tau}{B,\, B' \vdash t:\sigma \cap \tau}\;(\cap\mathbf{R})$$

$$\frac{B,\, x:\sigma \vdash t:\rho \qquad B',\, x:\tau \vdash t:\rho}{B,\, B',\, x:\sigma \cup \tau \vdash t:\rho}\;(\cup\mathbf{L}) \qquad \frac{B \vdash t:\sigma}{B \vdash t:\sigma \cup \tau}\;(\cup\mathbf{R}_1) \qquad \frac{B \vdash t:\tau}{B \vdash t:\sigma \cup \tau}\;(\cup\mathbf{R}_2)$$

$$\frac{B \vdash t:\sigma \qquad B',\, x:\sigma \vdash u:\tau}{B,\, B' \vdash u[t/x]:\tau}\;(\mathbf{cut})$$

Figure 2.2: The type system IUT_ω in sequent calculus style.

Remark 2.4 *(i) In the sequent calculus formulation, the system is* multiplicative*, i.e. all the rules with more than one premise are context-free.*

(ii) Contraction **(C)** *is still derivable and equivalent to a cut rule.*

$$\frac{B,\, x:\sigma,\, y:\sigma \vdash t:\tau}{B,\, x:\sigma \vdash t[x/y]:\tau}\,\textbf{(C)} \quad\rightsquigarrow\quad \frac{\dfrac{}{x:\sigma\vdash x:\sigma}\,\textbf{(ax)} \qquad B,\, x:\sigma,\, y:\sigma\vdash t:\tau}{B,\, x:\sigma\vdash t[x/y]:\tau}\,\textbf{(cut)}$$

The following remark, definition, and proposition hold for both formulations of the system.

Remark 2.5 *The proposition "if $B \vdash t:\sigma$ is provable, then $FV(t) \subseteq dom(B)$" is* not *valid due to the* (ω)*-rule. Removing the* (ω)*-rule, though, retrieves the validity of the proposition.*

Definition 2.6 (Similar derivations) *A derivation π' is* similar *to a derivation π if and only if π' can be obtained from π by adding basic typing statements to the bases or renaming term variables.*

Similar derivations share the same derivation tree, i.e. the same sequence of rules, and differ only in the bases and the term variables.

Proposition 2.7 *(i) (Renaming) If $\pi :: B,\, x:\sigma \vdash t:\tau$ and y is fresh with respect to π, then there exists a $\pi' :: B,\, y:\sigma \vdash t[y/x]:\tau$ similar to π.*

(ii) (Weakening) If $\pi :: B \vdash t:\sigma$ and $B \subseteq B'$, where B' is a basis, then there exists a $\pi' :: B' \vdash t:\sigma$ similar to π.

Proof. Either by induction on π (for both (i) and (ii)) or as explained in [2]. ⊣

It is shown in detail in [2] that the two formulations of the system are equivalent, i.e. that $B \vdash t:\sigma$ is proved in natural deduction if and only if $B \vdash t:\sigma$ is proved in sequent calculus. It is interesting to notice that, in order to derive the cut rule in natural deduction, a union redex is employed. If y is fresh with respect to the derivation proving $B',\, x:\sigma \vdash u:\tau$ and $y \notin dom(B)$, the following is a derivation of cut in natural deduction.

$$\frac{\dfrac{\dfrac{B\vdash t:\sigma}{B,\, B'\vdash t:\sigma}\,\textbf{(W)}}{B,\, B'\vdash t:\sigma\cup\sigma}\,\textbf{(}\cup\textbf{I)} \qquad \dfrac{B',\, x:\sigma\vdash u:\tau}{B,\, B',\, y:\sigma\vdash u[y/x]:\tau}\,\textbf{(Ren)+(W)} \qquad \dfrac{B',\, x:\sigma\vdash u:\tau}{B,\, B',\, y:\sigma\vdash u[y/x]:\tau}\,\textbf{(Ren)+(W)}}{B,\, B'\vdash u[y/x][t/y] = u[t/x]:\tau}\,\textbf{(}\cup\textbf{E)}$$

The dashed lines refer to Proposition 2.7 and the terms $u[y/x][t/y]$ and $u[t/x]$ in the final statement are identical, since $y \notin FV(u)$.

2.1 Subject reduction

As argued in [2], the type system IUT_ω is *not* invariant under β-reduction of subjects, meaning that from $B \vdash t : \sigma$ and $t \twoheadrightarrow_\beta u$ we cannot infer $B \vdash u : \sigma$. It is the union elimination rule that is blamed for this lack of invariance; the substitution that it contains causes the loss of correspondence between subterms and subderivations. In fact, many occurrences of the same subterm t in the term typed by the conclusion correspond to a unique subderivation (premise) typing t.

$$\frac{B \vdash t : \sigma \cup \tau \qquad B,\, x : \sigma \vdash (\ldots x \ldots x \ldots) = u : \rho \qquad B,\, x : \tau \vdash (\ldots x \ldots x \ldots) = u : \rho}{B \vdash (\ldots t \ldots t \ldots) = u[t/x] : \rho}\;(\cup\mathbf{E})$$

If one attempted to show subject reduction in IUT_ω by induction on $B \vdash t : \sigma$, as is done for IT_ω in [13], the many-to-one correspondence discussed above would induce a problem. For, supposing a redex in t were reduced, so that $t \rightarrow_\beta t'$ and $u[t/x] \rightarrow_\beta (\ldots t' \ldots t \ldots)$, the induction hypothesis would give $B \vdash t' : \sigma \cup \tau$ and then an application of union elimination with $B \vdash t' : \sigma \cup \tau$ as major premise and the same minor premises as before would derive $B \vdash (\ldots t' \ldots t' \ldots) = u[t'/x] : \rho$ which is obviously not the required conclusion.

The example given in [2] is that one can assign the type

$$(\sigma \rightarrow \sigma \rightarrow \tau) \cap (\rho \rightarrow \rho \rightarrow \tau) \rightarrow (\alpha \rightarrow \sigma \cup \rho) \rightarrow \alpha \rightarrow \tau$$

to both $\lambda xyz.\, x\,((\lambda w.\, w)\, yz)((\lambda w.\, w)\, yz)$ and $\lambda xyz.\, x\,(yz)(yz)$, but neither to $\lambda xyz.\, x\,(yz)((\lambda w.\, w)\, yz)$, nor to $\lambda xyz.\, x\,((\lambda w.\, w)\, yz)(yz)$. Hence, the system IUT_ω is not invariant under β-expansion of subjects, either.

The solution proposed in [2] is a different notion of β-reduction, called *parallel β-reduction*, which, roughly speaking, allows reductions performed simultaneously on all the occurrences of t in $u[t/x]$. In other words, a contraction step in now defined in such a way that $u[t/x] \rightarrow u[t'/x]$. The system is proved to be invariant under parallel β-reduction.

For the precise definition of "parallel reduction", which is somewhat stronger than the informal description given above, we need some preliminary definitions.

1. A non-empty set $\mathcal{F}$ of redex occurrences in a term t is called *uniform*, if, for every redex R of t, either all occurrences of R in t are in $\mathcal{F}$ or no occurrence of R in t is in $\mathcal{F}$.
2. If $t \rightarrow_\beta u$ and R is a redex occurrence in t, the set of *residuals* of R in u is the (possibly empty) set of redexes which are either instances of R or copies of it generated by the reduction.
3. A *complete development* of $(t, \mathcal{F})$, where $\mathcal{F}$ is a set of redex occurrences in t, is a reduction such that all and only residuals of redexes in $\mathcal{F}$ are reduced.

Formal definitions of these notions can be found in [4].

Definition 2.8 (Parallel Reduction) *The reduction relation $\Rightarrow_p$ over λ-terms is defined as follows: $t \Rightarrow_p u$ if and only if there exists a uniform set $\mathcal{F}$ of redex occurrences in t, such that $(t, \mathcal{F}) \twoheadrightarrow_{\mathrm{cpl}} u$, where $(t, \mathcal{F}) \twoheadrightarrow_{\mathrm{cpl}} u$ is the complete development of $(t, \mathcal{F})$.*

Invariance of typing under parallel reduction is then proved in [2] for IUT_ω.

Theorem 2.9 *If $B \vdash t : \sigma$ and $t \Rightarrow_p u$, then $B \vdash u : \sigma$.*

The proof is given for the sequent calculus formulation of the system, but the theorem holds for both formulations, since they are equivalent.

2.2 Cut elimination

In this section we consider the system in Figure 2.2 with the (ω)-rule excluded and contraction explicitly included; let us denote it $\mathrm{IUT_C}$. We will show cut elimination in $\mathrm{IUT_C}$ by means of Gentzen's method [12]. The need to remove the (ω)-rule and admit the contraction rule will be justified after the details of the proof have been provided. The cut elimination proof will be derived as a consequence of a multicut elimination proof in the type system $\mathrm{IUT'_C}$, which is defined below.

Definition 2.10 ($\mathbf{IUT'_C}$) *The type system* $\mathrm{IUT'_C}$ *is defined by the rules in Figure 2.2, if we exclude the (ω)-rule and include contraction and also substitute the cut rule by a multicut rule, called "*mix rule*".*

$$\frac{B,\, x:\sigma,\, y:\sigma \vdash t:\tau}{B,\, x:\sigma \vdash t[x/y]:\tau}\,\textbf{(C)} \qquad \frac{B \vdash t:\sigma \qquad B',\, x_1:\sigma, \ldots, x_m:\sigma \vdash u:\tau}{B,\, B' \vdash u[t/x_1, \ldots, t/x_m]:\tau}\,\textbf{(mix)}$$

In the mix rule we are allowed to eliminate any number of basic typing statements with predicate σ from the basis of the right premise and not just a single such typing statement as in the cut rule. The set B' may also contain basic typing statements with predicate σ. Type σ is called the *mix-type.*

Theorem 2.11 *The systems* $\mathrm{IUT_C}$ *and* $\mathrm{IUT'_C}$ *are equivalent.*

Proof. It suffices to show that (i) the cut rule can be derived in $\mathrm{IUT'_C}$ and (ii) the mix rule can be derived in $\mathrm{IUT_C}$. Since a cut can be seen as a special case of mix, (i) is obvious. For (ii) we show that a mix can be simulated in $\mathrm{IUT_C}$ by consecutive contractions followed by a cut.

$$\cfrac{\cfrac{B',\, x_1:\sigma,\, x_2:\sigma,\, x_3:\sigma, \ldots, x_m:\sigma \vdash u:\tau}{B',\, x_2:\sigma,\, x_3:\sigma, \ldots, x_m:\sigma \vdash u[x_2/x_1]:\tau}\,\textbf{(C)}}{B',\, x_3:\sigma, \ldots, x_m:\sigma \vdash u[x_2/x_1][x_3/x_2]:\tau}\,\textbf{(C)}$$

$$\vdots$$

$$\frac{B \vdash t:\sigma \qquad B',\, x_m:\sigma \vdash u[x_2/x_1][x_3/x_2]\ldots[x_m/x_{m-1}]:\tau}{B, B' \vdash (u[x_2/x_1][x_3/x_2]\ldots[x_m/x_{m-1}])[t/x_m] = u[t/x_1, \ldots, t/x_m]:\tau}\,\textbf{(cut)}$$

When the mix involves m eliminations of basic typing statements, the number of consecutive contractions is $m-1$. ⊣

Remark 2.12 *(i) In* $\mathrm{IUT'_C}$ *(resp.* $\mathrm{IUT_C}$*) the only rule that can generate redexes in the term typed by its conclusion is the mix-rule (resp. the cut-rule). So, a derivation in* $\mathrm{IUT'_C}$ *without mix (resp. in* $\mathrm{IUT_C}$ *without cut) types a normal term.*

(ii) A derivation in $\mathrm{IUT}'_{\omega\mathrm{C}} = \mathrm{IUT}'_{\mathrm{C}} + (\omega)$ *without mix (resp. in* $\mathrm{IUT}_{\omega\mathrm{C}} = \mathrm{IUT}_{\mathrm{C}} + (\omega)$ *without cut) does not necessarily type a normal term, since the* (ω)*-rule may introduce a term with redexes which are transfered, modulo substitutions*[1]*, to the root-term.*

It is easy to check that Proposition 2.7 holds for $\mathrm{IUT}'_{\mathrm{C}}$, as well.

Proposition 2.13 *(i) (Renaming) If* $\pi :: B, x:\sigma \vdash t:\tau$ *and* y *is fresh with respect to* π*, then there exists a* $\pi' :: B, y:\sigma \vdash t[y/x]:\tau$ *similar to* π*.*

(ii) (Weakening) If $\pi :: B \vdash t:\sigma$ *and* $B \subseteq B'$*, where* B' *is a basis, then there exists a* $\pi' :: B' \vdash t:\sigma$ *similar to* π*.*

Proof. By induction on π for both (i) and (ii). ⊣

Remark 2.14 *The similarity of* π *and* π' *in Proposition 2.13 implies that, if* π *is mix-free, then* π' *is mix-free, too.*

Definition 2.15 (Degree of type) *The* degree $d(\sigma)$ *of a type* $\sigma \in \mathcal{T}_{\mathrm{IUT}_\omega} \setminus \{\omega\}$ *is defined inductively as follows: (i)* $d(\alpha) = 0$*, for every type variable* α*, and (ii)* $d(\sigma * \tau) = d(\sigma) + d(\tau) + 1$*, where* $* \in \{\rightarrow, \cap, \cup\}$*.*

Definition 2.16 (Degree, rank, and measure of mix) *Consider a mix with mix-type* σ*.*

$$\dfrac{B \vdash t:\sigma \qquad B', x_1:\sigma, \ldots, x_m:\sigma \vdash u:\tau}{B, B' \vdash u[t/x_1, \ldots, t/x_m]:\tau}\,(\mathbf{mix})$$

(i) The degree d *of the mix is the degree* $d(\sigma)$ *of* σ*.*

(ii) The left rank lr *of the mix is the largest number of consecutive sequents rooted at the left premise, such that each has predicate* σ *in the succedent.*

(iii) The right rank rr *of the mix is the largest number of consecutive sequents rooted at the right premise, such that each has at least one basic typing statement from* $x_1:\sigma, \ldots, x_m:\sigma$ *in the assumptions.*

(iv) The rank r *of the mix is the sum* $lr + rr$ *of the left and right ranks of the mix.*

(v) The measure *of the mix is the ordered pair* (d, r)*, where* d *is the degree and* r *the rank of the mix.*

We note that the smallest possible degree of a mix is 0, while the smallest possible rank is 2.

Example 2.17 *Let* $\tau = \alpha \rightarrow \alpha$*,* $\sigma = \tau \rightarrow \tau$*, and* π *be the following derivation in* $\mathrm{IUT}'_{\mathrm{C}}$*.*

$$\dfrac{\dfrac{\dfrac{x:\tau \vdash x:\tau}{\emptyset \vdash \lambda x.x:\sigma}\,(\rightarrow\mathbf{R}) \qquad \dfrac{x:\alpha \vdash x:\alpha}{\emptyset \vdash \lambda x.x:\tau}\,(\rightarrow\mathbf{R})}{\emptyset \vdash \lambda x.x:\sigma \cap \tau}\,(\cap\mathbf{R}) \qquad \dfrac{\dfrac{\dfrac{y:\tau \vdash y:\tau \qquad w:\tau \vdash w:\tau}{y:\tau,\, z:\sigma \vdash zy:\tau}\,(\rightarrow\mathbf{L})}{y:\sigma \cap \tau,\, z:\sigma \vdash zy:\tau}\,(\cap\mathbf{L}_2)}{y:\sigma \cap \tau,\, z:\sigma \cap \tau \vdash zy:\tau}\,(\cap\mathbf{L}_1)}{\pi :: \emptyset \vdash (\lambda x.x)(\lambda x.x):\tau}\,(\mathbf{mix})$$

The mix has degree $d = d(\sigma \cap \tau) = 5$ *and rank* $r = lr + rr = 1 + 2 = 3$*. So, its measure is* $m = (5, 3)$*.*

---

[1] These substitutions come from the rules $(\mathbf{C})$ or $(\rightarrow\mathbf{L})$, but do not generate new redexes.

The next lemma is the main tool for eliminating the mix in $\mathrm{IUT}'_{\mathrm{C}}$.

Lemma 2.18 *If* $\pi :: B \vdash t : \sigma$ *is a derivation in* $\mathrm{IUT}'_{\mathrm{C}}$ *with a mix as final rule and no other mix contained, then there is a mix-free derivation* $\pi' :: B \vdash t' : \sigma$ *in* $\mathrm{IUT}'_{\mathrm{C}}$*, where* $t \twoheadrightarrow_\beta t'$*. (Remark 2.12(i) implies that* t' *is normal.)*

Proof. In Appendix A. $\dashv$

Definition 2.19 (Topmost mix or cut) *A mix (resp. cut) in a derivation* π *of* $\mathrm{IUT}'_{\mathrm{C}}$ *(resp.* $\mathrm{IUT}_{\mathrm{C}}$*) is called* topmost*, if there is no other mix (resp. cut) above it in the tree of* π*.*

Theorem 2.20 (Mix elimination in $\mathbf{IUT}'_{\mathbf{C}}$) *For every derivation* $\pi :: B \vdash t : \sigma$ *in* $\mathrm{IUT}'_{\mathrm{C}}$*, there is a mix-free derivation* $\pi' :: B \vdash t' : \sigma$*, where* $t \twoheadrightarrow_\beta t'$*.*

Proof. Using Lemma 2.18, we successively eliminate topmost mixes in π. In every elimination of a topmost mix with subderivation π_h the term typed by the root-sequent of π_h reduces to a normal term, while the basis and type remain unchanged. Rules with a single mix-free premise "pass on" the reduction to their conclusion. [Rule $(\mathbf{C})$: If $t \twoheadrightarrow_\beta t'$, then $t[x/y] \twoheadrightarrow_\beta t'[x/y]$. Rule $(\to\mathbf{R})$: If $t \twoheadrightarrow_\beta t'$, then $\lambda x.t \twoheadrightarrow_\beta \lambda x.t'$. Rules $(\cap\mathbf{L})$,$(\cup\mathbf{R})$: If $t \twoheadrightarrow_\beta t'$ in the premise, then $t \twoheadrightarrow_\beta t'$ in the conclusion.] Rules with two mix-free premises also "pass on" the reduction to their conclusion. [Rule $(\to\mathbf{L})$: If $t \twoheadrightarrow_\beta t'$ and $u \twoheadrightarrow_\beta u'$, then $u[yt/x] \twoheadrightarrow_\beta u'[yt'/x]$. Rules $(\cap\mathbf{R})$,$(\cup\mathbf{L})$: If $t \twoheadrightarrow_\beta t_0$ in the left premise and $t \twoheadrightarrow_\beta t_1$ in the right premise, then $t_0 = t_1 = t'$, since mix-free derivations type normal terms and the normal form is unique; so, we have $t \twoheadrightarrow_\beta t'$ in the conclusion.] If we run this procedure top-down in π, we eliminate all mixes in a finite number of steps and obtain a mix-free $\pi' :: B \vdash t' : \sigma$, where $t \twoheadrightarrow_\beta t'$. $\dashv$

Theorem 2.21 (Cut elimination in $\mathbf{IUT}_{\mathbf{C}}$) *For every derivation* $\pi :: B \vdash t : \sigma$ *in* $\mathrm{IUT}_{\mathrm{C}}$*, there is a cut-free derivation* $\pi' :: B \vdash t' : \sigma$*, where* $t \twoheadrightarrow_\beta t'$*.*

Proof. If $(\mathrm{IUT}_{\mathrm{C}})_{\mathrm{cf}}$ is the system $\mathrm{IUT}_{\mathrm{C}}$ without the cut-rule (cut-free) and $(\mathrm{IUT}'_{\mathrm{C}})_{\mathrm{mf}}$ is the system $\mathrm{IUT}'_{\mathrm{C}}$ without the mix-rule (mix-free), then $(\mathrm{IUT}_{\mathrm{C}})_{\mathrm{cf}} = (\mathrm{IUT}'_{\mathrm{C}})_{\mathrm{mf}}$. If $\pi :: B \vdash t : \sigma$ in $\mathrm{IUT}_{\mathrm{C}}$, then, by Theorem 2.11, there is a $\pi_0 :: B \vdash t : \sigma$ in $\mathrm{IUT}'_{\mathrm{C}}$. So, by Theorem 2.20, there is a $\pi'_0 :: B \vdash t' : \sigma$, where $t \twoheadrightarrow_\beta t'$, in $(\mathrm{IUT}'_{\mathrm{C}})_{\mathrm{mf}}$. Since $(\mathrm{IUT}'_{\mathrm{C}})_{\mathrm{mf}} = (\mathrm{IUT}_{\mathrm{C}})_{\mathrm{cf}}$, there is a $\pi' = \pi'_0 :: B \vdash t' : \sigma$ in $(\mathrm{IUT}_{\mathrm{C}})_{\mathrm{cf}}$. $\dashv$

Remark 2.22 *The inclusion of contraction is necessary for the proof of cut elimination. For, if we attempt to eliminate the cut shown below in the system* IUT*, which is* $\mathrm{IUT}_{\mathrm{C}}$ *without* $(\mathbf{C})$*, we see that the tree with root-sequent* $x : (\alpha \to \beta) \cap \alpha \vdash xx : \beta$ *fails to complete bottom-up without cut and without contraction. The boxes mark further failures.*

$$\dfrac{\dfrac{x:\alpha \vdash x:\alpha}{x:(\alpha\to\beta)\cap\alpha \vdash x:\alpha}\,(\cap\mathbf{L}_2) \qquad \dfrac{\dfrac{y:\alpha\vdash y:\alpha \qquad z:\beta\vdash z:\beta}{x:\alpha\to\beta,\, y:\alpha \vdash xy:\beta}\,(\to\mathbf{L})}{x:(\alpha\to\beta)\cap\alpha,\, y:\alpha \vdash xy:\beta}\,(\cap\mathbf{L}_1)}{x:(\alpha\to\beta)\cap\alpha \vdash xx:\beta}\,(\mathbf{cut})$$

not an axiom

$$\dfrac{\dfrac{\emptyset \vdash \boxed{y} : \alpha \qquad z:\beta\vdash z:\beta}{x:\alpha\to\beta \vdash x\,\boxed{x} : \beta}\,(\to\mathbf{L})}{x:(\alpha\to\beta)\cap\alpha \vdash xx:\beta}\,(\cap\mathbf{L}_1)$$

not an axiom

$$\dfrac{\dfrac{\emptyset \vdash \boxed{x} : \alpha \qquad z:\beta\vdash z:\beta}{\boxed{x} : \alpha\to\beta \vdash xx : \beta}\,(\to\mathbf{L})}{x:(\alpha\to\beta)\cap\alpha \vdash xx:\beta}\,(\cap\mathbf{L}_1)$$

cannot proceed bottom-up

$$\dfrac{x:\alpha \vdash xx:\beta}{x:(\alpha\to\beta)\cap\alpha \vdash xx:\beta}\,(\cap\mathbf{L}_2)$$

In IUT$_{\mathrm{C}}$ *we can prove the sequent* $x : (\alpha \to \beta) \cap \alpha \vdash xx : \beta$ *without cut, using the contraction-rule.*

$$\cfrac{\cfrac{\cfrac{\cfrac{y : \alpha \vdash y : \alpha \qquad z : \beta \vdash z : \beta}{x : \alpha \to \beta,\ y : \alpha \vdash xy : \beta}\,(\to\mathbf{L})}{x : (\alpha \to \beta) \cap \alpha,\ y : \alpha \vdash xy : \beta}\,(\cap\mathbf{L}_1)}{x : (\alpha \to \beta) \cap \alpha,\ y : (\alpha \to \beta) \cap \alpha \vdash xy : \beta}\,(\cap\mathbf{L}_2)}{x : (\alpha \to \beta) \cap \alpha \vdash xx : \beta}\,\mathbf{(C)}$$

We can establish this derivation-tree, if we consider the cut as a special case of mix that eliminates a single basic typing statement from the right premise and follow the method shown in the proof of Lemma 2.18. The contraction-rule appears in case A : $(\cap\mathbf{L})$.

Since we can derive the contraction-rule in IUT *using the cut-rule, the cuts that cannot be eliminated in this system are essentially the ones that embody contractions.*

$$\cfrac{x : \sigma \vdash x : \sigma \qquad B,\ x : \sigma,\ y : \sigma \vdash t : \tau}{B,\ x : \sigma \vdash t[x/y] : \tau}\,\mathbf{(cut)}$$

These cuts introduce substitutions of variables by variables, which do not create redexes, so they are clearly "good" cuts. A derivation in IUT *that contains solely "good" cuts types a normal term. Nonetheless, we choose to show a* total *cut elimination in* IUT$_{\mathrm{C}}$ *than a* partial *cut elimination in* IUT.

Given the necessity of contraction for the elimination of all cuts, we can now justify the definition of IUT$'_{\mathrm{C}}$ and explain why cut elimination in IUT$_{\mathrm{C}}$ was shown through mix elimination in IUT$'_{\mathrm{C}}$. Lemma 2.18 cannot be proved for IUT$_{\mathrm{C}}$. In particular, with cut in place of mix, case B : **(C)** : a does not work, since the cut-rule eliminates exactly one basic typing statement from the right premise.

$$\cfrac{\pi_0 :: B \vdash t : \sigma \qquad \cfrac{B',\ x : \sigma,\ y : \sigma \vdash u : \rho}{\pi_1 :: B',\ x : \sigma \vdash u[x/y] : \rho}\,\mathbf{(C)}}{\pi :: B, B' \vdash u[x/y][t/x] : \rho}\,\mathbf{(cut)} \qquad \hookrightarrow$$

$$\begin{array}{c} \pi_0 :: B \vdash t : \sigma \qquad B',\ \underline{x : \sigma,\ y : \sigma} \vdash u : \rho \\ \hdashline \text{would need a sequent: } B, B' \vdash \text{a term } v = u[x/y][t/x] : \rho \end{array} \ \text{would need a } \mathbf{(cut)}'$$

On the other hand, trying to eliminate $x : \sigma$, $y : \sigma$ by two consecutive cuts, we wouldn't end up with two cuts of less measure than the initial cut. A schematic counterexample is shown below.

$$\cfrac{\pi_0 :: B \vdash t : \sigma \qquad \cfrac{\begin{array}{c} y : \sigma \\ x : \sigma,\ y : \sigma \\ B',\ x : \sigma,\ y : \sigma \vdash u : \rho \end{array}}{\pi_1 :: B',\ x : \sigma \vdash u[x/y] : \rho}\,\mathbf{(C)}}{\pi :: B, B' \vdash u[x/y][t/x] : \rho}\,\mathbf{(cut)},\ m = (d(\sigma), lr + 3) \qquad \hookrightarrow$$

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad \dfrac{\pi_0 :: B \vdash t : \sigma \qquad \overset{\overset{y : \sigma}{x : \sigma,\ y : \sigma}}{B',\ x : \sigma,\ y : \sigma \vdash u : \rho}}{B, B',\ x : \sigma \vdash u[t/y] : \rho}\ (\mathbf{cut})',\ m' = (d(\sigma), lr + 3) = m}{B, B' \vdash u[t/y][t/x] : \rho}\ (\mathbf{cut})'',\ m'' = (d(\sigma), lr + 3) = m$$

<u>or</u>

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad \dfrac{\pi_0 :: B \vdash t : \sigma \qquad \overset{\overset{y : \sigma}{x : \sigma,\ y : \sigma}}{B',\ x : \sigma,\ y : \sigma \vdash u : \rho}}{B, B',\ y : \sigma \vdash u[t/x] : \rho}\ (\mathbf{cut})',\ m' = (d(\sigma), lr + 2) < m}{B, B' \vdash u[t/x][t/y] : \rho}\ (\mathbf{cut})'',\ m'' = (d(\sigma), lr + 4) > m$$

The next remark sustains the necessity for exclusion of the $(\boldsymbol{\omega})$-rule in order to gain cut elimination.

Remark 2.23 *Cut elimination is not valid in* $\mathrm{IUT}_{\omega\mathrm{C}}$*, since mix elimination is not valid in* $\mathrm{IUT}'_{\omega\mathrm{C}}$*. Lemma 2.18 cannot be proved for* $\mathrm{IUT}'_{\omega\mathrm{C}}$ *because a mix-free derivation in* $\mathrm{IUT}'_{\omega\mathrm{C}}$ *does not necessarily type a normal term, as explained in Remark 2.12(ii). For example, in case* $\mathrm{A}:(\cup\mathbf{L}):\mathrm{a}$ *we would have that* $t_0\,{}_\beta\!\twoheadleftarrow u[z/y][t/x_j] \twoheadrightarrow_\beta t_1$*, but without the restriction that* t_0 *and* t_1 *are normal and consequently identical. So, we wouldn't be able to apply* $(\cup\mathbf{L})$ *to* π'_0 *and* π'_1*, as they would (possibly) type different terms. This problem would also arise in cases* $\mathrm{A}:(\cup\mathbf{L}):\mathrm{b}$*,* $\mathrm{B}:(\cup\mathbf{L}):\mathrm{a}$*,* $\mathrm{B}:(\cup\mathbf{L}):\mathrm{b}$*, and* $\mathrm{B}:(\cap\mathbf{R})$*.*

2.3 Term characterizations

In this section we show three theorems which characterize λ-terms according to their typings in $\mathrm{IUT}_{\omega\mathrm{C}}$ and one theorem which characterizes terms that are typable in $\mathrm{IUT}_{\mathrm{C}}$. The general schema of these theorems is the following: "t is typable in $\mathrm{IUT}_{\omega\mathrm{C}}$ (resp. $\mathrm{IUT}_{\mathrm{C}}$) in such and such a way if and only if t has such and such a property". The theorems for $\mathrm{IUT}_{\omega\mathrm{C}}$ also hold for the systems IUT_{ω}, $\mathrm{IT}_{\omega\mathrm{C}} = \mathrm{IT} + (\boldsymbol{\omega}) + (\mathbf{C})$, and $\mathrm{IT}_{\omega} = \mathrm{IT} + (\boldsymbol{\omega})$. The theorem for $\mathrm{IUT}_{\mathrm{C}}$ also holds for $\mathrm{IUT} = \mathrm{IUT}_{\mathrm{C}} - (\mathbf{C})$, $\mathrm{IT}_{\mathrm{C}} = \mathrm{IT} + (\mathbf{C})$, and IT. The theorems for IT_{ω} and IT have already been proved in [13], where the systems are denoted $\mathrm{D}\Omega$ and D, respectively. Combining the theorems for $\mathrm{IUT}_{\omega\mathrm{C}}$ and IT_{ω} (resp. $\mathrm{IUT}_{\mathrm{C}}$ and IT), we deduce conclusions of the form "t is typable in $\mathrm{IUT}_{\omega\mathrm{C}}$ (resp. in $\mathrm{IUT}_{\mathrm{C}}$) in a certain way if and only if t is typable in IT_{ω} (resp. in IT) in exactly the same way if and only if t belongs to a set of λ-terms defined by a certain characteristic property".

All the type systems are considered in natural deduction style. They can be gathered into two groups: the type systems $\mathrm{IUT}_{\omega\mathrm{C}}, \mathrm{IUT}_{\mathrm{C}}, \mathrm{IUT}_{\omega}, \mathrm{IUT}$ with intersection and union types and the type systems $\mathrm{IT}_{\omega\mathrm{C}}, \mathrm{IT}_{\mathrm{C}}, \mathrm{IT}_{\omega}, \mathrm{IT}$ with intersection types. Figure 2.3 displays the two rectangles of type systems where downward arrows remove contraction and rightward arrows remove the $(\boldsymbol{\omega})$-rule.

We start by recalling basic definitions and properties concerning λ-terms and sets of λ-terms.

Proposition 2.24 *Every* $t \in \Lambda$ *can be uniquely written in the form*

$$\lambda x_1 \ldots \lambda x_m . (\kappa)\, t_1 \ldots t_n \qquad (m, n \geqslant 0)$$

where $t_1, \ldots, t_n \in \Lambda$ *and* κ *is either a variable or a redex.*

$$\begin{tikzcd}
\mathrm{IUT}_{\omega\mathrm{C}} \arrow[r] \arrow[d] & \mathrm{IUT}_{\mathrm{C}} \arrow[d] & & \mathrm{IT}_{\omega\mathrm{C}} \arrow[r] \arrow[d] & \mathrm{IT}_{\mathrm{C}} \arrow[d] \\
\mathrm{IUT}_{\omega} \arrow[r] & \mathrm{IUT} & & \mathrm{IT}_{\omega} \arrow[r] & \mathrm{IT} \\
\mathfrak{N} \text{ in [2]} & & & \mathrm{D}\Omega \text{ in [13]} & \mathrm{D} \text{ in [13]}
\end{tikzcd}$$

Figure 2.3: The type systems.

Proof. In [13]. ⊣

Definition 2.25 (Head reduction) *(i) If* $t = \lambda x_1 \ldots \lambda x_m.(y)\, t_1 \ldots t_n$*, for some variable* y*, i.e. if the* κ *in Proposition 2.24 is a variable, then* t *is in* head normal form*.*

(ii) If $t = \lambda x_1 \ldots \lambda x_m.(\lambda x.u)\, v\, t_1 \ldots t_n$*, i.e. if the* κ *in Proposition 2.24 is a redex, then the redex* $(\lambda x.u)\, v$ *is called the* head redex *of* t*.*

(iii) The head reduction *of a term* t *is the (finite or infinite) sequence* $t_0, t_1, \ldots, t_n, \ldots$*, such that* $t_0 = t$ *and* t_{n+1} *is obtained from* t_n *by contraction of the head redex of* t_n*, if such a redex exists. If* t_n *does not have a head redex, then* t_n *is in head normal form and the sequence ends with* t_n*. We write* $t \to_h t'$ *for a head contraction and* $t \twoheadrightarrow_h t'$ *for a head reduction.*

By the above definition, a finite head reduction ends in head normal form.

Definition 2.26 (Leftmost reduction) *The* leftmost reduction *of a term* t *is the (finite or infinite) sequence* $t_0, t_1, \ldots, t_n, \ldots$*, such that* $t_0 = t$ *and* t_{n+1} *is obtained from* t_n *by contraction of the leftmost redex of* t_n*, if such a redex exists. If* t_n *does not have a leftmost redex, then* t_n *is normal and the sequence ends with* t_n*. We write* $t \to_l t'$ *for a leftmost contraction and* $t \twoheadrightarrow_l t'$ *for a leftmost reduction.*

Definition 2.27 (Quasi leftmost reduction) *An infinite* quasi leftmost reduction *of a term* t *is a sequence* $t = t_0, t_1, \ldots, t_n, \ldots$*, such that* $(\forall n \geqslant 0)[t_n \to_\beta t_{n+1}]$ *and* $(\forall n \geqslant 0)(\exists p \geqslant n)[t_p \to_l t_{p+1}]$*.*

If $\mathcal{X}, \mathcal{Y} \subseteq \Lambda$, then $\Lambda \supseteq \mathcal{X} \to \mathcal{Y}$ is defined as follows: $(\forall t \in \Lambda)[\, t \in \mathcal{X} \to \mathcal{Y} \;\Leftrightarrow\; (\forall u \in \mathcal{X})[tu \in \mathcal{Y}]\,]$. It is easily proved that, if $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{Y}'$, then $\mathcal{X} \to \mathcal{Y} \subseteq \mathcal{X}' \to \mathcal{Y}'$.

Definition 2.28 (Saturated and $\mathcal{N}$-saturated sets) *Let* $\mathcal{X}, \mathcal{N} \subseteq \Lambda$*.*

(i) The set $\mathcal{X}$ *is called* saturated*, if for every* $u, t, x, t_1, \ldots, t_n \in \Lambda$*:*

$$(u[t/x])\, t_1 \ldots t_n \in \mathcal{X} \;\Rightarrow\; (\lambda x.u)\, t\, t_1 \ldots t_n \in \mathcal{X}$$

(ii) The set $\mathcal{X}$ *is called* $\mathcal{N}$-saturated*, if for every* $u, x, t_1, \ldots, t_n \in \Lambda$ *and* $t \in \mathcal{N}$*:*

$$(u[t/x])\, t_1 \ldots t_n \in \mathcal{X} \;\Rightarrow\; (\lambda x.u)\, t\, t_1 \ldots t_n \in \mathcal{X}$$

Proposition 2.29 *Let* $\mathcal{X}, \mathcal{Y}, \mathcal{N} \subseteq \Lambda$*.*

(i) If $\mathcal{Y}$ *is saturated (resp.* $\mathcal{N}$*-saturated), then* $\mathcal{X} \to \mathcal{Y}$ *is saturated (resp.* $\mathcal{N}$*-saturated).*

(ii) If $\mathcal{X}, \mathcal{Y}$ *are saturated (resp.* $\mathcal{N}$*-saturated), then* $\mathcal{X} \cap \mathcal{Y}$ *and* $\mathcal{X} \cup \mathcal{Y}$ *are saturated (resp.* $\mathcal{N}$*-saturated).*

Proof. Easy to show using Definition 2.28. ⊣

Definition 2.30 (Interpretation and $\mathcal{N}$-interpretation) *(i) An* interpretation *$\mathcal{I}$ is a function which associates with each type variable α a saturated $|\alpha|_{\mathcal{I}} \subseteq \Lambda$.*

*(ii) If $\mathcal{N} \subseteq \Lambda$, an $\mathcal{N}$-*interpretation *$\mathcal{I}$ is a function which associates with each type variable α an $\mathcal{N}$-saturated $|\alpha|_{\mathcal{I}} \subseteq \Lambda$.*

An interpretation (resp. $\mathcal{N}$-interpretation) $\mathcal{I}$ can be extended, so that it associates with each type σ a saturated (resp. $\mathcal{N}$-saturated) subset of Λ. Given the images of type variables by definition and letting $|\omega|_{\mathcal{I}} = \Lambda$, we extend $\mathcal{I}$ inductively as follows: $|\sigma \to \tau|_{\mathcal{I}} = |\sigma|_{\mathcal{I}} \to |\tau|_{\mathcal{I}}$, $|\sigma \cap \tau|_{\mathcal{I}} = |\sigma|_{\mathcal{I}} \cap |\tau|_{\mathcal{I}}$, and $|\sigma \cup \tau|_{\mathcal{I}} = |\sigma|_{\mathcal{I}} \cup |\tau|_{\mathcal{I}}$. The soundness of this extension is ensured by Proposition 2.29. From now on, given an interpretation (resp. $\mathcal{N}$-interpretation) $\mathcal{I}$, we will write $|\sigma|$ instead of $|\sigma|_{\mathcal{I}}$.

The next two lemmas play a key role in proving the four central theorems of this section.

Lemma 2.31 (Adequacy lemma 1) *Let $\pi :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash u : \tau$ be a derivation in* $\mathrm{IUT}_{\omega\mathrm{C}}$ *and $\mathcal{I}$ be an interpretation. If $t_1 \in |\sigma_1|, \ldots, t_m \in |\sigma_m|$, then $u[t_1/x_1, \ldots, t_m/x_m] \in |\tau|$.*

Proof. By induction on π. For the base case and the cases of the implication and intersection rules, we refer to [13]. We show the rest of the cases, writing B for $x_1 : \sigma_1, \ldots, x_m : \sigma_m$.

$$\triangleright \quad \frac{B, y : \sigma_m \vdash u : \tau}{B \vdash u[x_m/y] : \tau} \ \textbf{(C)}$$

The IH gives that $u[t_j/x_j, t_m/y] \in |\tau|$, where "$t_j/x_j$" stands for the substitutions "$t_1/x_1, \ldots, t_m/x_m$". It is $u[x_m/y][t_j/x_j] = u[t_j/x_j, t_m/y] \in |\tau|$.

$$\triangleright \quad \frac{B \vdash u : \tau}{B \vdash u : \tau \cup \rho} \ \textbf{(}\cup\textbf{I)}$$

By the IH, we have that $u[t_j/x_j] \in |\tau| \subseteq |\tau \cup \rho|$.

$$\triangleright \quad \frac{B \vdash t : \tau \cup \rho \qquad B, y : \tau \vdash u : \phi \qquad B, y : \rho \vdash u : \phi}{B \vdash u[t/y] : \phi} \ \textbf{(}\cup\textbf{E)}$$

By the IH, we have that $t[t_j/x_j] \in |\tau \cup \rho|$. If $t[t_j/x_j] \in |\tau|$, the IH gives that $u[t_j/x_j, t[t_j/x_j]/y] \in |\phi|$. It is then $u[t/y][t_j/x_j] = u[t_j/x_j, t[t_j/x_j]/y] \in |\phi|$. If $t[t_j/x_j] \in |\rho|$, we proceed in a similar manner. ⊣

Lemma 2.32 (Adequacy lemma 2) *Let $\pi :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash u : \tau$ be a derivation in* $\mathrm{IUT}_{\mathrm{C}}$*, $\mathcal{N}$ be a subset of Λ, and $\mathcal{I}$ be an $\mathcal{N}$-interpretation, such that $|\phi| \subseteq \mathcal{N}$, for every type $\phi \neq \omega$. If $t_1 \in |\sigma_1|, \ldots, t_m \in |\sigma_m|$, then $u[t_1/x_1, \ldots, t_m/x_m] \in |\tau|$.*

Proof. By induction on π. The most interesting case is the **(→I)** case where the hypothesis "$|\phi| \subseteq \mathcal{N}$, for every type ϕ of $\mathrm{IUT}_{\mathrm{C}}$" is used (see [13]). The rest cases work as in the proof of Lemma 2.31. ⊣

We continue with some basic definitions that concern intersection and union types.

Definition 2.33 (Positive and negative occurrences) *The* positive and negative occurrences *of a type variable or of ω in a type σ are defined by induction on σ as follows:*

1. If $\sigma = \alpha$ (or ω), then the occurrence of α (or ω) in σ is positive.

2. If $\sigma = \tau \to \rho$, then the positive (resp. negative) occurrences of a type variable or ω in ρ are positive (resp. negative) occurrences in σ, while the positive (resp. negative) occurrences of a type variable or ω in τ are negative (resp. positive) occurrences in σ.

3. If $\sigma = \tau * \rho$, where $*$ is intersection or union, then the positive (resp. negative) occurrences of a type variable or ω in τ or ρ are positive (resp. negative) occurrences in σ.

Definition 2.34 (Final occurrences) *The* final occurrences *of a type variable or of ω in a type σ are defined by induction on σ as follows:*

1. If $\sigma = \alpha$ (or ω), then the occurrence of α (or ω) in σ is final.

2. If $\sigma = \tau \to \rho$, then the final occurrences of a type variable or ω in ρ are final occurrences in σ.

3. If $\sigma = \tau \cap \rho$, then the final occurrences of a type variable or ω in τ or ρ are final occurrences in σ.

4. If $\sigma = \tau \cup \rho$, then no occurrence of a type variable or ω in σ is final.

Definition 2.35 (Non-trivial types) *A type is called* non-trivial*, if it contains a final occurrence of some type variable; otherwise, it is called trivial.*

According to the above definitions, the non-trivial types can be defined inductively as follows: (i) all type variables are non-trivial, (ii) if τ is non-trivial, then $\sigma \to \tau$ is non-trivial, for every σ, and (iii) if σ or τ are non-trivial, then $\sigma \cap \tau$ is non-trivial. Similarly, the trivial types can be defined inductively as follows: (i) ω is trivial, (ii) if τ is trivial, then $\sigma \to \tau$ is trivial, for every σ, (iii) if σ and τ are both trivial, then $\sigma \cap \tau$ is trivial, and (iv) $\sigma \cup \tau$ is trivial, for every σ and τ.

We can now state the first of the four theorems.

Theorem 2.36 (Head normal form theorem) *A term t admits a non-trivial type in* $\mathrm{IUT}_{\omega\mathrm{C}}$ *if and only if its head reduction is finite.*

To prove the "only if" direction of the this theorem, we need the following lemma.

Lemma 2.37 *Let $\mathcal{N}_0, \mathcal{N} \subseteq \Lambda$ be such that: 1. $\mathcal{N}_0 \subseteq \mathcal{N}$, 2. $\mathcal{N}_0 \subseteq \Lambda \to \mathcal{N}_0$, 3. $\mathcal{N}_0 \to \mathcal{N} \subseteq \mathcal{N}$, and 4. $\mathcal{N}$ is saturated. If $\mathcal{I}$ is an interpretation, such that $|\alpha| = \mathcal{N}$, for every type variable α, then: (i) $\mathcal{N}_0 \subseteq |\sigma|$, for every type σ, and (ii) $|\sigma| \subseteq \mathcal{N}$, for every non-trivial type σ.*

Proof. (i) By induction on σ. We only show the union case and for the other cases we refer to [13]. If $\sigma = \tau \cup \rho$, then, using the IH, we have $\mathcal{N}_0 \subseteq |\tau| \subseteq |\sigma|$.

(ii) By induction on the non-trivial σ. Since union types are trivial, we refer to [13] for the whole induction. $\dashv$

For the "if" direction of the head normal form theorem we will use the next two results.

Proposition 2.38 *Every term in head normal form admits a non-trivial type in* IT_ω.

Proof. We denote $\mathcal{T}_{\mathrm{IT}_\omega}$ the set of types $\sigma ::= \alpha \mid \omega \mid \sigma \to \sigma \mid \sigma \cap \sigma$. Let $u = \lambda x_1 \ldots \lambda x_m.(y)\, t_1 \ldots t_n$ be a term in head normal form and $\mathcal{T}_{\mathrm{IT}_\omega} \ni \tau = \omega \to \ldots \to \omega \to \alpha = \omega^{(n)} \to \alpha$. If $B = x_1 : \sigma_1, \ldots, x_m : \sigma_m$, for some $\sigma_1, \ldots, \sigma_m \in \mathcal{T}_{\mathrm{IT}_\omega}$, we can apply implication elimination n times and then implication introduction m times, as shown below, to type u in IT_ω by the non-trivial type $\sigma_1 \to \ldots \to \sigma_m \to \alpha$.

$$\cfrac{\cfrac{B,\,y:\tau\vdash y:\tau \qquad B,\,y:\tau\vdash t_1:\omega}{B,\,y:\tau\vdash (y)\,t_1:\omega^{\,(n-1)}\to\alpha}\;(\to\mathbf{E}) \qquad B,\,y:\tau\vdash t_2:\omega}{B,\,y:\tau\vdash (y)\,t_1t_2:\omega^{\,(n-2)}\to\alpha}\;(\to\mathbf{E})$$

$$\vdots$$

$$\begin{array}{c} B,\,y:\tau\vdash (y)\,t_1\ldots t_n:\alpha \\ \hdashline y:\tau\vdash u:\sigma_1\to\ldots\to\sigma_m\to\alpha \end{array}\;(\to\mathbf{I})$$

$\dashv$

Theorem 2.39 *If* $B \vdash t : \sigma$ *in* IT_ω *and* $t =_\beta t'$, *then* $B \vdash t' : \sigma$ *in* IT_ω.

Proof. In [13]. $\dashv$

We can now provide the proof of Theorem 2.36.

Proof of Theorem 2.36. ($\Rightarrow$): Let $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ be a typing of t in $\mathrm{IUT}_{\omega\mathrm{C}}$ with τ non-trivial. Also, let $\mathcal{N}_0$ and $\mathcal{N}$ be the following subsets of Λ.

$$\mathcal{N}_0 = \{\, (x)\,t_1 \ldots t_n \mid n \geqslant 0 \text{ and } t_1, \ldots, t_n \in \Lambda \,\}$$

$$\mathcal{N} = \{\, t \in \Lambda \mid \text{the head reduction of } t \text{ is finite} \,\}$$

These $\mathcal{N}_0$,$\mathcal{N}$ satisfy conditions 1-4 of Lemma 2.37 (proof in [13]). So, if we consider an interpretation $\mathcal{I}$, such that $|\alpha| = \mathcal{N}$, for every type variable α, we have that $\mathcal{N}_0 \subseteq |\sigma_j|$, for every j from 1 to m, and $|\tau| \subseteq \mathcal{N}$. Since $x_j \in \mathcal{N}_0 \subseteq |\sigma_j|$, Lemma 2.31 (Adequacy lemma 1) implies that $t[x_j/x_j] = t \in |\tau| \subseteq \mathcal{N}$, i.e. the head reduction of t is finite.

($\Leftarrow$): If the head reduction of t is finite, then $t \twoheadrightarrow_h t'$, for some t' in head normal form. By Proposition 2.38 we infer that t' admits a non-trivial type in IT_ω, i.e. that $B \vdash t' : \sigma$ in IT_ω, for some basis B and some non-trivial type $\sigma \in \mathcal{T}_{\mathrm{IT}_\omega}$. Theorem 2.39 then implies that $B \vdash t : \sigma$ in IT_ω, so finally $B \vdash t : \sigma$ in $\mathrm{IUT}_{\omega\mathrm{C}}$. $\dashv$

The head normal form theorem also holds for systems IT_ω, $\mathrm{IT}_{\omega\mathrm{C}}$, and IUT_ω, as the following theorem states.

Theorem 2.40 *A term* t *admits a non-trivial type in* IT_ω *(resp.* $\mathrm{IT}_{\omega\mathrm{C}}$,$\mathrm{IUT}_\omega$*) if and only if its head reduction is finite.*

Proof. If t admits a non-trivial type in IT_ω (resp. $\mathrm{IT}_{\omega\mathrm{C}}$,$\mathrm{IUT}_\omega$), then it also does in the "bigger" system $\mathrm{IUT}_{\omega\mathrm{C}}$, so, by Theorem 2.36, its head reduction is finite. Conversely, if the head reduction of t is finite, then t admits a non-trivial type in IT_ω, as already shown in the proof of Theorem 2.36, so it also does in the "bigger" systems $\mathrm{IT}_{\omega\mathrm{C}}$ and IUT_ω. $\dashv$

Theorems 2.36 and 2.40 imply that $\mathrm{IUT}_{\omega\mathrm{C}}$ and IT_ω assign non-trivial types to exactly the same set of terms, namely to the ones whose head reduction is finite. Although $\mathrm{IUT}_{\omega\mathrm{C}}$ is enriched with union rules and contraction compared to IT_ω, it cannot assign non-trivial types to a larger set of terms than IT_ω. This is in a way expected, since union types are themselves trivial. Nonetheless, as the next example shows, a term with a finite head reduction can have additional non-trivial types assigned to it in $\mathrm{IUT}_{\omega\mathrm{C}}$ besides the non-trivial types assigned to it in IT_ω.

Example 2.41 *We consider the term* $t = \lambda x.\, (\lambda y.\, y)zx$ *whose head reduction is finite, since* $t \to_h \lambda x.\, zx$ *and* $\lambda x.\, zx$ *is in head normal form. If* $B = x : \alpha,\ z : \alpha \to \beta$*, term* t *admits the non-trivial type* $\alpha \to \beta$ *in* IT_ω*, as shown below.*

$$\dfrac{\dfrac{\dfrac{\dfrac{B, y : \alpha \to \beta \vdash y : \alpha \to \beta}{B \vdash \lambda y.\, y : (\alpha \to \beta) \to \alpha \to \beta}\,(\to\mathbf{I}) \qquad B \vdash z : \alpha \to \beta}{B \vdash (\lambda y.\, y)z : \alpha \to \beta}\,(\to\mathbf{E}) \qquad B \vdash x : \alpha}{B \vdash (\lambda y.\, y)zx : \beta}\,(\to\mathbf{E})}{z : \alpha \to \beta \vdash t : \alpha \to \beta}\,(\to\mathbf{I})$$

This typing is also valid in $\mathrm{IUT}_{\omega\mathrm{C}}$*. But in* $\mathrm{IUT}_{\omega\mathrm{C}}$ *we can get a second non-trivial typing, as well, if we substitute* α *by a union type* $\alpha_1 \cup \alpha_2$ *in the above derivation.*

The next basic theorem of this section is the following.

Theorem 2.42 (Leftmost reduction theorem) *A term* t *admits a type* τ *in* $\mathrm{IUT}_{\omega\mathrm{C}}$ *in a context* $x_1 : \sigma_1, \ldots, x_m : \sigma_m$*, where* $\sigma_1, \ldots, \sigma_m$ *contain no negative occurrences of* ω *and* τ *contains no positive occurrences of* ω*, if and only if its leftmost reduction is finite.*

For the "only if" direction of this theorem we will need the next lemma.

Lemma 2.43 *Let* $\mathcal{N}_0, \mathcal{N}$ *be subsets of* Λ *such that: 1.* $\mathcal{N}_0 \subseteq \mathcal{N}$*, 2.* $\mathcal{N}_0 \subseteq \mathcal{N} \to \mathcal{N}_0$*, 3.* $\mathcal{N}_0 \to \mathcal{N} \subseteq \mathcal{N}$*, and 4.* $\mathcal{N}$ *is saturated. If* $\mathcal{I}$ *is an interpretation, such that* $\mathcal{N}_0 \subseteq |\alpha| \subseteq \mathcal{N}$*, for every type variable* α*, then: (i)* $\mathcal{N}_0 \subseteq |\sigma|$*, for every type* σ *that contains no negative occurrences of* ω*, and (ii)* $|\sigma| \subseteq \mathcal{N}$*, for every type* σ *that contains no positive occurrences of* ω*.*

Proof. We show (i) and (ii) simultaneously by induction on σ. We only give the case of union and for the other cases we refer to [13]. If $\sigma = \tau \cup \rho$ and σ contains no negative occurrences of ω, then τ contains no negative occurrences of ω and, using the IH for τ, we get $\mathcal{N}_0 \subseteq |\tau| \subseteq |\sigma|$. If $\sigma = \tau \cup \rho$ and σ contains no positive occurrences of ω, then neither τ nor ρ contain positive occurrences of ω and, by the IH for τ and ρ, we have that $|\tau| \subseteq \mathcal{N}$ and $|\rho| \subseteq \mathcal{N}$, respectively. So, we get that $|\sigma| = |\tau| \cup |\rho| \subseteq \mathcal{N}$. ⊣

Corollary 2.44 *If* $\mathcal{N}_0, \mathcal{N}$ *are subsets of* Λ *that satisfy conditions 1-4 of Lemma 2.43 and* $\mathcal{I}$ *is an interpretation, such that* $\mathcal{N}_0 \subseteq |\alpha| \subseteq \mathcal{N}$*, for every type variable* α*, then* $\mathcal{N}_0 \subseteq |\sigma| \subseteq \mathcal{N}$*, for every type* σ *that contains no occurrences of* ω*.*

For the "if" direction of the leftmost reduction theorem we will use the fact that every normal term is typable in IT.

Proposition 2.45 *If* t *is normal, then* $B \vdash t : \sigma$ *in* IT*, for some type* σ *and basis* B*.*

Proof. In [13]. ⊣

The proof of Theorem 2.42 can now be supplied.

Proof of Theorem 2.42. ($\Rightarrow$): Let $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ be a typing of t in $\mathrm{IUT}_{\omega\mathrm{C}}$, such that $\sigma_1, \ldots, \sigma_m$ contain no negative occurrences of ω and τ contains no positive occurrences of ω. Also, let $\mathcal{N}$ and $\mathcal{N}_0$ be the following subsets of Λ.

$$\mathcal{N} = \{\, t \in \Lambda \mid \text{the leftmost reduction of } t \text{ is finite} \,\}$$

$$\mathcal{N}_0 = \{\, (x)\, t_1 \ldots t_n \mid n \geqslant 0 \text{ and } t_1, \ldots, t_n \in \mathcal{N} \,\}$$

These $\mathcal{N}$,$\mathcal{N}_0$ satisfy conditions 1-4 of Lemma 2.43 (proof in [13]). So, if we consider an interpretation $\mathcal{I}$ such that $|\alpha| = \mathcal{N}$, for every type variable α, we have that $\mathcal{N}_0 \subseteq |\sigma_j|$, for every j from 1 to m, and $|\tau| \subseteq \mathcal{N}$. Since $x_j \in \mathcal{N}_0 \subseteq |\sigma_j|$, Lemma 2.31 (Adequacy lemma 1) implies that $t[x_j/x_j] = t \in |\tau| \subseteq \mathcal{N}$, i.e. the leftmost reduction of t is finite.

($\Leftarrow$): If the leftmost reduction of t is finite, then $t =_\beta t'$, for some normal term t'. By Proposition 2.45 we have that $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t' : \tau$ in IT, for some $\sigma_1, \ldots, \sigma_m$, and τ in $\mathcal{T}_{\mathrm{IT}}$. Consequently, we have that $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t' : \tau$ in IT_ω, for some $\sigma_1, \ldots, \sigma_m$ with no negative occurrences of ω and some τ with no positive occurrences of ω. Theorem 2.39 implies that $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ in IT_ω, which, in turn, implies that $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ in the "bigger" system $\mathrm{IUT}_{\omega\mathrm{C}}$. ⊣

The leftmost reduction theorem also holds for the systems IT_ω, $\mathrm{IT}_{\omega\mathrm{C}}$, and IUT_ω, as was the case with the head normal form theorem.

Theorem 2.46 *A term t admits a type τ in IT_ω (resp. $\mathrm{IT}_{\omega\mathrm{C}}$,$\mathrm{IUT}_\omega$) in a context $x_1 : \sigma_1, \ldots, x_m : \sigma_m$, where $\sigma_1, \ldots, \sigma_m$ contain no negative occurrences of ω and τ contains no positive occurrences of ω, if and only if its leftmost reduction is finite.*

Proof. If t admits such a typing in IT_ω (resp. $\mathrm{IT}_{\omega\mathrm{C}}$,$\mathrm{IUT}_\omega$), then it also admits such a typing in the "bigger" system $\mathrm{IUT}_{\omega\mathrm{C}}$, so, by Theorem 2.42, its leftmost reduction is finite. Conversely, if the leftmost reduction of t is finite, then t admits such a typing in IT_ω, as already shown in the proof of Theorem 2.42, so it also does in the "bigger" systems $\mathrm{IT}_{\omega\mathrm{C}}$ and IUT_ω. ⊣

Obviously, the systems $\mathrm{IUT}_{\omega\mathrm{C}}$ and IT_ω type exactly the same terms in this specific way, i.e. in a context with types that contain no negative occurrences of ω and with a succedent type that contains no positive occurrences of ω. These terms are the ones whose leftmost reduction is finite.

The third basic theorem follows.

Theorem 2.47 (Quasi leftmost reduction theorem) *A term t admits a type τ in $\mathrm{IUT}_{\omega\mathrm{C}}$ in a context $x_1 : \sigma_1, \ldots, x_m : \sigma_m$, where $\sigma_1, \ldots, \sigma_m, \tau$ contain no occurrences of ω, if and only there is no infinite quasi leftmost reduction starting with t.*

Proof. ($\Rightarrow$): Let $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ be a typing of t in $\mathrm{IUT}_{\omega\mathrm{C}}$, such that $\sigma_1, \ldots, \sigma_m, \tau$ contain no occurrences of ω. Also, let $\mathcal{N}$ and $\mathcal{N}_0$ be the following subsets of Λ.

$$\mathcal{N} = \{\, t \in \Lambda \mid \text{there is no infinite quasi leftmost reduction of } t \,\}$$

$$\mathcal{N}_0 = \{\, (x)\, t_1 \ldots t_n \mid n \geqslant 0 \text{ and } t_1, \ldots, t_n \in \mathcal{N}\}$$

These $\mathcal{N}$,$\mathcal{N}_0$ satisfy conditions 1-4 of Lemma 2.43 (proof in [13]). So, if we consider an interpretation $\mathcal{I}$, such that $|\alpha| = \mathcal{N}$, for every type variable α, we have, by Corollary 2.44, that the interpretations of

$\sigma_1, \ldots, \sigma_m, \tau$ all contain $\mathcal{N}_0$ and are contained in $\mathcal{N}$. Since $x_j \in \mathcal{N}_0 \subseteq |\sigma_j|$, Lemma 2.31 implies that $t[x_j/x_j] = t \in |\tau| \subseteq \mathcal{N}$, i.e. there is no infinite quasi leftmost reduction of t.

($\Leftarrow$): If there is no infinite quasi leftmost reduction of t, then the leftmost reduction of t is finite. So, we have that $t =_\beta t'$, for some normal term t', and $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t' : \tau$ in IT, for some $\sigma_1, \ldots, \sigma_m, \tau \in \mathcal{T}_{\mathrm{IT}}$. Therefore, it is $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t' : \tau$ in IT_ω with $\sigma_1, \ldots, \sigma_m, \tau$ free of occurrences of ω. Theorem 2.39 implies that $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ in IT_ω with $\sigma_1, \ldots, \sigma_m, \tau$ free of occurrences of ω. Therefore, it is also $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ in $\mathrm{IUT}_{\omega\mathrm{C}}$ with $\sigma_1, \ldots, \sigma_m, \tau$ free of occurrences of ω. $\dashv$

The quasi leftmost reduction theorem holds for IT_ω, $\mathrm{IT}_{\omega\mathrm{C}}$, and IUT_ω, as well.

Theorem 2.48 *A term t admits a type τ in IT_ω (resp. $\mathrm{IT}_{\omega\mathrm{C}}, \mathrm{IUT}_\omega$) in a context $x_1 : \sigma_1, \ldots, x_m : \sigma_m$, where $\sigma_1, \ldots, \sigma_m, \tau$ contain no occurrences of ω, if and only there is no infinite quasi leftmost reduction starting with t.*

Proof. Similar to the proofs of Theorems 2.40 and 2.46. $\dashv$

By Theorems 2.47 and 2.48, the systems $\mathrm{IUT}_{\omega\mathrm{C}}$ and IT_ω type the same set of λ-terms in a way such that the types in the root-statement contain no occurrences of ω; namely, the terms with no infinite quasi leftmost reduction. Here again the “bigger” system does not “widen” the set of terms typable in the specific way in question.

The last and most important theorem of this section is the following.

Theorem 2.49 (Strong normalization theorem) *A term t is typable in IUT_C if and only if it is strongly normalizing.*

For the “only if” direction of this theorem we will use the next lemma.

Lemma 2.50 *Let $\mathcal{N}_0, \mathcal{N}$ be subsets of Λ such that: 1. $\mathcal{N}_0 \subseteq \mathcal{N}$, 2. $\mathcal{N}_0 \subseteq \mathcal{N} \to \mathcal{N}_0$, 3. $\mathcal{N}_0 \to \mathcal{N} \subseteq \mathcal{N}$, and 4. $\mathcal{N}$ is $\mathcal{N}$-saturated. If $\mathcal{I}$ is an $\mathcal{N}$-interpretation, such that $\mathcal{N}_0 \subseteq |\alpha| \subseteq \mathcal{N}$, for every type variable α, then $\mathcal{N}_0 \subseteq |\sigma| \subseteq \mathcal{N}$, for every type $\mathcal{T}_{\mathrm{IUT}_\mathrm{C}} \ni \sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma \mid \sigma \cup \sigma$.*

Proof. By induction on σ. We only show the union case and refer to [13] for the other cases. If $\sigma = \tau \cup \rho$, then, using the IH for τ and ρ, we get that $\mathcal{N}_0 \subseteq |\tau| \subseteq |\sigma| = |\tau| \cup |\rho| \subseteq \mathcal{N}$. $\dashv$

Proof of Theorem 2.49. ($\Rightarrow$): Let $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ be a typing of t in IUT_C. Also, let $\mathcal{N}$ and $\mathcal{N}_0$ be the following subsets of Λ.

$$\mathcal{N} = \{\, t \in \Lambda \mid t \text{ is strongly normalizing} \,\}$$

$$\mathcal{N}_0 = \{\, (x)\, t_1 \ldots t_n \mid n \geqslant 0 \text{ and } t_1, \ldots, t_n \in \mathcal{N} \,\}$$

These $\mathcal{N}, \mathcal{N}_0$ satisfy conditions 1-4 of Lemma 2.50 (proof in [13]). So, if we consider an $\mathcal{N}$-interpretation $\mathcal{I}$, such that $|\alpha| = \mathcal{N}$, for every type variable α, we have that $\mathcal{N}_0 \subseteq |\sigma_j|$, for all j from 1 to m, and $|\tau| \subseteq \mathcal{N}$. Since $x_j \in \mathcal{N}_0 \subseteq |\sigma_j|$, Lemma 2.32 (Adequacy lemma 2) implies that $t[x_j/x_j] = t \in |\tau| \subseteq \mathcal{N}$, i.e. t is strongly normalizing.

($\Leftarrow$): If t is strongly normalizing, then it is typable in IT (see proof in [13]), so it is also typable in IUT_C. $\dashv$

The strong normalization theorem holds for IT, IT_C, and IUT, as well.

Theorem 2.51 *A term t is typable in* IT *(resp.* $\mathrm{IT_C}$, IUT*) if and only if it is strongly normalizing.*

Proof. If t is typable in IT (resp. $\mathrm{IT_C}$, IUT), then it is also typable in the "bigger" system $\mathrm{IUT_C}$, so, by Theorem 2.49, it is strongly normalizing. Conversely, if t is strongly normalizing, then it is typable in IT, so it is also typable in the "bigger" systems $\mathrm{IT_C}$ and IUT. $\dashv$

The systems $\mathrm{IUT_C}$, IUT on one hand and $\mathrm{IT_C}$, IT on the other are all equivalent with respect to the set of terms they type, as they all exclusively type the strongly normalizing terms. It is worth noting that union types in $\mathrm{IUT_C}$ and IUT do not type a larger set of terms than intersection types in $\mathrm{IT_C}$ and IT. We can, therefore, say that type systems with intersection and union types are *conservative extensions* of corresponding type systems with intersection types as far as typable terms are concerned.

CHAPTER 3

Toward a Logic for Union Types

Working in natural deduction style, the aim of this chapter is to find a logic corresponding to the minimal type system with intersection and union types IUT in the manner that the logics IL and ISL correspond to the type system IT. Toward this end, we may start by examining whether *minimal intuitionistic logic*, denoted ML, would be suitable as such a logic, although the failure in correlating LJ with IT in Chapter 1 forces us to expect a negative result. The logic ML is the implicative, conjunctive, and disjunctive fragment of intuitionistic logic; actually, it is the extension of LJ with rules for disjunction.

Definition 3.1 (ML) *Considering formulas generated by the grammar* $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \wedge \sigma \mid \sigma \vee \sigma$*, where* α *belongs to a countable set of atomic formulas, the logical system* ML *proves statements* $\Gamma \vdash \sigma$*, where* Γ *is a sequence of formulas. Its rules are shown in Figure 3.1. Implication is, as usual, right associative, while conjunction and disjunction are left associative and precede over implication.*

$$\frac{}{\sigma \vdash \sigma}\ (\mathbf{ax})$$

$$\frac{\Gamma \vdash \tau}{\Gamma, \sigma \vdash \tau}\ (\mathbf{W}) \qquad \frac{\Gamma, \sigma, \tau, \Delta \vdash \rho}{\Gamma, \tau, \sigma, \Delta \vdash \rho}\ (\mathbf{X})$$

$$\frac{\Gamma, \sigma \vdash \tau}{\Gamma \vdash \sigma \to \tau}\ (\to\mathbf{I}) \qquad \frac{\Gamma \vdash \sigma \to \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau}\ (\to\mathbf{E})$$

$$\frac{\Gamma \vdash \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma \wedge \tau}\ (\wedge\mathbf{I}) \qquad \frac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \sigma}\ (\wedge\mathbf{E}_1) \qquad \frac{\Gamma \vdash \sigma \wedge \tau}{\Gamma \vdash \tau}\ (\wedge\mathbf{E}_2)$$

$$\frac{\Gamma \vdash \sigma}{\Gamma \vdash \sigma \vee \tau}\ (\vee\mathbf{I}_1) \qquad \frac{\Gamma \vdash \tau}{\Gamma \vdash \sigma \vee \tau}\ (\vee\mathbf{I}_2) \qquad \frac{\Gamma \vdash \sigma \vee \tau \qquad \Gamma, \sigma \vdash \rho \qquad \Gamma, \tau \vdash \rho}{\Gamma \vdash \rho}\ (\vee\mathbf{E})$$

Figure 3.1: The logic ML.

$$\frac{}{x : \sigma \vdash x : \sigma}\ (\mathbf{ax})$$

$$\frac{\Gamma^p \vdash t : \tau}{\Gamma^p,\, x : \sigma \vdash t : \tau}\ (\mathbf{W}) \qquad \frac{\Gamma^p,\, y : \sigma,\, x : \tau,\, \Delta^q \vdash t : \rho}{\Gamma^p,\, x : \tau,\, y : \sigma,\, \Delta^q \vdash t : \rho}\ (\mathbf{X})$$

$$\frac{\Gamma^p,\, x : \sigma \vdash t : \tau}{\Gamma^p \vdash \lambda x.\, t : \sigma \to \tau}\ (\to\!\mathbf{I}) \qquad \frac{\Gamma^p \vdash t : \sigma \to \tau \qquad \Gamma^p \vdash u : \sigma}{\Gamma^p \vdash tu : \tau}\ (\to\!\mathbf{E})$$

$$\frac{\Gamma^p \vdash t : \sigma \qquad \Gamma^p \vdash u : \tau}{\Gamma^p \vdash (t, u) : \sigma \wedge \tau}\ (\wedge\mathbf{I}) \qquad \frac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash \pi_1(t) : \sigma}\ (\wedge\mathbf{E}_1) \qquad \frac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash \pi_2(t) : \tau}\ (\wedge\mathbf{E}_2)$$

$$\frac{\Gamma^p \vdash t : \sigma}{\Gamma^p \vdash i_1(t) : \sigma \vee \tau}\ (\vee\mathbf{I}_1) \qquad \frac{\Gamma^p \vdash t : \tau}{\Gamma^p \vdash i_2(t) : \sigma \vee \tau}\ (\vee\mathbf{I}_2)$$

$$\frac{\Gamma^p \vdash t : \sigma \vee \tau \qquad \Gamma^p,\, x : \sigma \vdash u : \rho \qquad \Gamma^p,\, y : \tau \vdash v : \rho}{\Gamma^p \vdash \text{case } t \text{ of } i_1(x) \to u\, ;\, i_2(y) \to v : \rho}\ (\vee\mathbf{E})$$

Figure 3.2: Standard decoration of ML.

Obviously, a standard decoration of ML with untyped λ-terms does not deliver IUT. Such a decoration encodes all the logical connectives and delivers the Curry type system $\lambda^{\wedge\vee}_{\to}$ in the Curry-Howard perspective.

Definition 3.2 (Standard decoration of ML) *Let $\pi :: \Gamma = \sigma_1, \ldots, \sigma_m \vdash \tau$ be a derivation in* ML. *By decorating contexts bottom-up with distinct variables starting with the sequence $p = x_1, \ldots, x_m$ and then decorating formulas to the right of "$\vdash$" top-down with terms generated by the grammar*

$$t ::= x \mid \lambda x.t \mid tt \mid (t,t) \mid \pi_1(t), \pi_2(t) \mid i_1(t), i_2(t) \mid \text{case } t \text{ of } i_1(x) \to t\, ;\, i_2(x) \to t$$

we get a decorated derivation $\pi^* :: \Gamma^p = x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$. The decoration rules are presented in Figure 3.2. When decorating contexts bottom-up, the new variable in a $(\to\!\mathbf{I})$ *premise or in a* $(\vee\mathbf{E})$ *minor premise is fresh with respect to the variables in the branch connecting the conclusion to the root. In addition, the fresh variables in two* $(\vee\mathbf{E})$ *minor premises are distinct.*

Definition 3.3 ($\lambda^{\wedge\vee}_{\to}$) *Considering types built by implication, conjunction, and disjunction, i.e. simple types extended with disjunction, the type system $\lambda^{\wedge\vee}_{\to}$ proves statements $B \vdash t : \sigma$, where B is a basis, t belongs to the set of terms generated by the grammar in Definition 3.2, and σ is a type. Its rules are displayed in Figure 3.3.*

The logic ML relates to the type system $\lambda^{\wedge\vee}_{\to}$ through (standard) decoration and erasure in the same way that LJ relates to $\lambda^{\wedge}_{\to}$.

$$\frac{}{B, x : \sigma \vdash x : \sigma}\ (\mathbf{ax})$$

$$\frac{B, x : \sigma \vdash t : \tau}{B \vdash \lambda x.t : \sigma \to \tau}\ (\to\mathbf{I}) \qquad \frac{B \vdash t : \sigma \to \tau \qquad B \vdash u : \sigma}{B \vdash tu : \tau}\ (\to\mathbf{E})$$

$$\frac{B \vdash t : \sigma \qquad B \vdash u : \tau}{B \vdash (t, u) : \sigma \wedge \tau}\ (\wedge\mathbf{I}) \qquad \frac{B \vdash t : \sigma \wedge \tau}{B \vdash \pi_1(t) : \sigma}\ (\wedge\mathbf{E}_1) \qquad \frac{B \vdash t : \sigma \wedge \tau}{B \vdash \pi_2(t) : \tau}\ (\wedge\mathbf{E}_2)$$

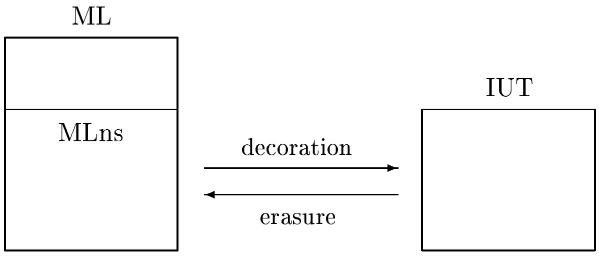
$$\frac{B \vdash t : \sigma}{B \vdash i_1(t) : \sigma \vee \tau}\ (\vee\mathbf{I}_1) \qquad \frac{B \vdash t : \tau}{B \vdash i_2(t) : \sigma \vee \tau}\ (\vee\mathbf{I}_2)$$

$$\frac{B \vdash t : \sigma \vee \tau \qquad B, x : \sigma \vdash u : \rho \qquad B, y : \tau \vdash v : \rho}{B \vdash \text{case } t \text{ of } i_1(x) \to u\, ;\, i_2(y) \to v : \rho}\ (\vee\mathbf{E})$$

Figure 3.3: The type system $\lambda^{\wedge\vee}_{\to}$.

The next step is to attempt a correspondence between ML and IUT through a *non-standard* decoration of ML. The aim is to define a decoration of ML that transforms a derivation of ML to one of IUT, provided the additional conversion of conjunction and disjunction to intersection and union, respectively. The very rules of IUT dictate that we introduce a decoration which encodes the implication, ignores the conjunction and the introduction of disjunction, and induces a substitution operation in the case of the elimination of disjunction. The rules for such a decoration are shown in Figure 3.4. As in the case of the non-standard decoration of LJ, the decoration terminates only in derivations of ML in which the ($\wedge$**I**) rule is applied to isomorphic premises and the ($\vee$**E**) rule is applied to isomorphic minor premises; otherwise, the decoration fails.

Obviously, it is only a proper subset of ML, denoted MLns, that admits a non-standard decoration and this subset corresponds to IUT through decoration and erasure.



In particular, a derivation of MLns can be non-standardly decorated to provide a derivation of IUT, if decorated contexts are seen as sets, conjunction and disjunction are converted to intersection and union,

$$\dfrac{}{x : \sigma \vdash x : \sigma}\ \textbf{(ax)}$$

$$\dfrac{\Gamma^p \vdash t : \tau}{\Gamma^p,\ x : \sigma \vdash t : \tau}\ \textbf{(W)} \qquad \dfrac{\Gamma^p,\ y : \sigma,\ x : \tau,\ \Delta^q \vdash t : \rho}{\Gamma^p,\ x : \tau,\ y : \sigma,\ \Delta^q \vdash t : \rho}\ \textbf{(X)}$$

$$\dfrac{\Gamma^p,\ x : \sigma \vdash t : \tau}{\Gamma^p \vdash \lambda x.\, t : \sigma \to \tau}\ (\to\!\textbf{I}) \qquad \dfrac{\Gamma^p \vdash t : \sigma \to \tau \qquad \Gamma^p \vdash u : \sigma}{\Gamma^p \vdash tu : \tau}\ (\to\!\textbf{E})$$

$$\dfrac{\Gamma^p \vdash t : \sigma \qquad \Gamma^p \vdash t : \tau}{\Gamma^p \vdash t : \sigma \wedge \tau}\ (\wedge\textbf{I}) \qquad \dfrac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash t : \sigma}\ (\wedge\textbf{E}_1) \qquad \dfrac{\Gamma^p \vdash t : \sigma \wedge \tau}{\Gamma^p \vdash t : \tau}\ (\wedge\textbf{E}_2)$$

$$\dfrac{\Gamma^p \vdash t : \sigma}{\Gamma^p \vdash t : \sigma \vee \tau}\ (\vee\textbf{I}_1) \qquad \dfrac{\Gamma^p \vdash t : \tau}{\Gamma^p \vdash t : \sigma \vee \tau}\ (\vee\textbf{I}_2)$$

$$\dfrac{\Gamma^p \vdash t : \sigma \vee \tau \qquad \Gamma^p,\ x : \sigma \vdash u : \rho \qquad \Gamma^p,\ x : \tau \vdash u : \rho}{\Gamma^p \vdash u[t/x] : \rho}\ (\vee\textbf{E})$$

Figure 3.4: Non-standard decoration of ML.

respectively, and structural rules are ignored. Conversely, a derivation of IUT can be converted to one of MLns, if terms are erased, variable-free bases are seen as sequences, intersection and union are restored to conjunction and disjunction, respectively, and structural rules are added, if necessary. The example below depicts this back and forth between MLns and IUT. Dashed lines denote consecutive structural rules and $\Gamma = (\alpha \to \beta) \wedge (\gamma \to \beta),\ \alpha \vee \gamma$, while $B = x : (\alpha \to \beta) \cap (\gamma \to \beta),\ y : \alpha \cup \gamma$.

$$\dfrac{\begin{array}{c}\alpha \vee \gamma \vdash \alpha \vee \gamma \\ \hdashline \Gamma \vdash \alpha \vee \gamma\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}(\alpha \to \beta) \wedge (\gamma \to \beta) \vdash (\alpha \to \beta) \wedge (\gamma \to \beta) \\ \hdashline \Gamma, \alpha \vdash (\alpha \to \beta) \wedge (\gamma \to \beta)\end{array}}{\Gamma, \alpha \vdash \alpha \to \beta}\ (\wedge\textbf{E}) \quad \begin{array}{c}\alpha \vdash \alpha \\ \hdashline \Gamma, \alpha \vdash \alpha\end{array}}{\Gamma, \alpha \vdash \beta}\ (\to\!\textbf{E}) \qquad \dfrac{\dfrac{\begin{array}{c}(\alpha \to \beta) \wedge (\gamma \to \beta) \vdash (\alpha \to \beta) \wedge (\gamma \to \beta) \\ \hdashline \Gamma, \gamma \vdash (\alpha \to \beta) \wedge (\gamma \to \beta)\end{array}}{\Gamma, \gamma \vdash \gamma \to \beta}\ (\wedge\textbf{E}) \quad \begin{array}{c}\gamma \vdash \gamma \\ \hdashline \Gamma, \gamma \vdash \gamma\end{array}}{\Gamma, \gamma \vdash \beta}\ (\to\!\textbf{E})}{\Gamma \vdash_{\text{MLns}} \beta}\ (\vee\textbf{E})$$

$$\begin{array}{c}\text{decoration} \\ \longrightarrow \\ \longleftarrow \\ \text{erasure}\end{array}$$

$$\dfrac{B \vdash y : \alpha \cup \gamma \qquad \dfrac{\dfrac{B,\ z : \alpha \vdash x : (\alpha \to \beta) \cap (\gamma \to \beta)}{B,\ z : \alpha \vdash x : \alpha \to \beta}\ (\cap\textbf{E}) \quad B,\ z : \alpha \vdash z : \alpha}{B,\ z : \alpha \vdash \boxed{xz} : \beta}\ (\to\!\textbf{E}) \qquad \dfrac{\dfrac{B,\ z : \gamma \vdash x : (\alpha \to \beta) \cap (\gamma \to \beta)}{B,\ z : \gamma \vdash x : \gamma \to \beta}\ (\cap\textbf{E}) \quad B,\ z : \gamma \vdash z : \gamma}{B,\ z : \gamma \vdash \boxed{xz} : \beta}\ (\to\!\textbf{E})}{B \vdash_{\text{IUT}} xz[y/z] = xy : \beta}\ (\cup\textbf{E})$$

Derivations in ML\MLns do not admit a non-standard decoration. An example of such a derivation is shown below, where $\Gamma = \alpha \to \beta, \gamma \to \beta, \alpha \vee \gamma$ and $\Gamma^\star = x : \alpha \to \beta,\ y : \gamma \to \beta,\ w : \alpha \vee \gamma$.

$$\dfrac{\begin{array}{c}\alpha \vee \gamma \vdash \alpha \vee \gamma \\ \hdashline \Gamma \vdash \alpha \vee \gamma\end{array} \quad \dfrac{\begin{array}{c}\alpha \to \beta \vdash \alpha \to \beta \\ \hdashline \Gamma, \alpha \vdash \alpha \to \beta\end{array} \quad \begin{array}{c}\alpha \vdash \alpha \\ \hdashline \Gamma, \alpha \vdash \alpha\end{array}}{\Gamma, \alpha \vdash \beta}\,(\to\mathbf{E}) \quad \dfrac{\begin{array}{c}\gamma \to \beta \vdash \gamma \to \beta \\ \hdashline \Gamma, \gamma \vdash \gamma \to \beta\end{array} \quad \begin{array}{c}\gamma \vdash \gamma \\ \hdashline \Gamma, \gamma \vdash \gamma\end{array}}{\Gamma, \gamma \vdash \beta}\,(\to\mathbf{E})}{\Gamma \vdash \beta}\,(\vee\mathbf{E}) \qquad \xrightarrow{\text{decoration}}$$

$$\dfrac{\begin{array}{c}w : \alpha \vee \gamma \vdash w : \alpha \vee \gamma \\ \hdashline \Gamma^\star \vdash w : \alpha \vee \gamma\end{array} \quad \dfrac{\begin{array}{c}x : \alpha \to \beta \vdash x : \alpha \to \beta \\ \hdashline \Gamma^\star,\ z : \alpha \vdash x : \alpha \to \beta\end{array} \quad \begin{array}{c}z : \alpha \vdash z : \alpha \\ \hdashline \Gamma^\star,\ z : \alpha \vdash z : \alpha\end{array}}{\Gamma^\star,\ z : \alpha \vdash \boxed{xz} : \beta}\,(\to\mathbf{E}) \quad \dfrac{\begin{array}{c}y : \gamma \to \beta \vdash y : \gamma \to \beta \\ \hdashline \Gamma^\star,\ z : \gamma \vdash y : \gamma \to \beta\end{array} \quad \begin{array}{c}z : \gamma \vdash z : \gamma \\ \hdashline \Gamma^\star,\ z : \gamma \vdash z : \gamma\end{array}}{\Gamma^\star,\ z : \gamma \vdash \boxed{yz} : \beta}\,(\to\mathbf{E})}{\Gamma^\star \vdash \boxed{?} : \beta}\,(\vee\mathbf{E})$$

We conclude by the above that ML is not a logic for IUT via a decoration-erasure correspondence. Actually, a standard decoration of ML renders a correspondence between ML and $\lambda_{\to}^{\wedge\vee}$ and a non-standard decoration of ML renders a correspondence between MLns and IUT. This non-standard decoration marks out the synchronous aspect of conjunction and disjunction by presupposing identically decorated premises in $(\wedge\mathbf{I})$ and identically decorated minor premises in $(\vee\mathbf{E})$, respectively. The correspondence between MLns and IUT manifests that intersection and union correspond to synchronous conjunction and disjunction, respectively. It remains to examine synchronous conjunction (or intersection) and synchronous disjunction (or union) as logical connectives. Toward this end, we aim to express MLns as a logic of its own by introducing extensions with union of the logical systems IL and ISL.

3.1 Intersection and Union Logic IUL$_k$

We define Intersection and Union Logic IUL$_k$ as an extension with union of Intersection Logic IL. The goal is to achieve a correspondence between IUL$_k$ and MLns. Since MLns corresponds to IUT, this is equivalent to showing a correspondence between IUL$_k$ and IUT.

The following definition assumes the notions of *overlapping* kits and implication between such kits, of *paths*, *subtrees* at certain paths, *terminal* paths, *different* paths, and of *pruning*, as given in 1.9.

Definition 3.4 (IUL$_k$) *(i) A* kit *is a binary tree* $K ::= \sigma \,|\, [K, K]$ *with leaves* $\sigma ::= \alpha \,|\, \sigma \to \sigma \,|\, \sigma \cap \sigma \,|\, \sigma \cup \sigma$*, where* α *belongs to a countable set of atomic formulas. We use* K, H, L *to denote kits and* σ, τ, ρ*, etc. to denote leaves.*

(ii) The notation $H[p := K]$ *stands for the kit resulting from the substitution of subtree* H^p *by* K *in* H*. If* q *and* p *are paths in* H *and* q *is terminal, the* left doubling *of leaf* H^q *at path* p*, denoted* $H^q/_{pl}$*, is defined as* $H[p := [H^q, H^p]]$*, while the* right doubling *of leaf* H^q *at path* p*, denoted* $H^q/_{pr}$*, is defined as* $H[p := [H^p, H^q]]$*.*

(iii) The deductive system IUL$_k$ *derives judgements* $\Gamma \vdash K$*, where the context* Γ *is a sequence of kits and* K *is a kit. It extends* IL *with rules for doubling and union, as shown in Figure 3.5. The letter* s *stands for either path* l *or path* r *and the index* j *in contexts runs from 1 to* m*.*

$$\frac{}{K \vdash K}\ \textbf{(ax)}$$

$$\frac{\Gamma \vdash K}{\Gamma, H \vdash K}\ \textbf{(W)} \qquad \frac{\Gamma, H_1, H_2, \Delta \vdash K}{\Gamma, H_2, H_1, \Delta \vdash K}\ \textbf{(X)}$$

$$\frac{\Gamma \vdash K}{\Gamma\backslash^{ps} \vdash K\backslash^{ps}}\ \textbf{(P)} \qquad \frac{\Gamma \vdash K}{\Gamma^{q}/_{ps} \vdash K^{q}/_{ps}}\ \textbf{(D)}$$

$$\frac{\Gamma, H \vdash K}{\Gamma \vdash H \to K}\ (\to\textbf{I}) \qquad \frac{\Gamma \vdash H \to K \qquad \Gamma \vdash H}{\Gamma \vdash K}\ (\to\textbf{E})$$

$$\frac{H_j[p := [\sigma_j, \sigma_j]] \vdash K[p := [\sigma, \tau]]}{H_j[p := \sigma_j] \vdash K[p := \sigma \cap \tau]}\ (\cap\textbf{I})$$

$$\frac{\Gamma \vdash K[p := \sigma \cap \tau]}{\Gamma \vdash K[p := \sigma]}\ (\cap\textbf{E}_1) \qquad \frac{\Gamma \vdash K[p := \sigma \cap \tau]}{\Gamma \vdash K[p := \tau]}\ (\cap\textbf{E}_2)$$

$$\frac{\Gamma \vdash K[p := \sigma]}{\Gamma \vdash K[p := \sigma \cup \tau]}\ (\cup\textbf{I}_1) \qquad \frac{\Gamma \vdash K[p := \tau]}{\Gamma \vdash K[p := \sigma \cup \tau]}\ (\cup\textbf{I}_2)$$

$$\frac{H_j[p := \sigma_j] \vdash K[p := \sigma \cup \tau] \qquad H_j[p := [\sigma_j, \sigma_j]], K[p := [\sigma, \tau]] \vdash L[p := [\rho, \rho]]}{H_j[p := \sigma_j] \vdash L[p := \rho]}\ (\cup\textbf{E})$$

Figure 3.5: The logic IUL_k.

Remark 3.5 *(i) The inclusion of the rule of doubling* **(D)** *is motivated by technical reasons, as was the case with the inclusion of pruning in the first place. If $q = p$, then the (left or right) doubling of leaf $H^q = H^p = \sigma$ at path p is $H^p/_{pl} = H^p/_{pr} = H[p := [\sigma, \sigma]]$. This gives the following special case of the rule.*

$$\frac{H_j[p := \sigma_j] \vdash K[p := \tau]}{H_j[p := [\sigma_j, \sigma_j]] \vdash K[p := [\tau, \tau]]}\ \mathbf{(D)}$$

(ii) If $s, s' \in \{l, r\}$, the following equalities hold.
1. For any context Γ where $p \neq q$, it is $(\Gamma \backslash^{ps}) \backslash^{qs'} = (\Gamma \backslash^{qs'}) \backslash^{ps}$.
2. For any context Γ where $p \notin \{q, q'\}$ and q is terminal, it is $(\Gamma \backslash^{ps})^q/_{q's'} = (\Gamma^q/_{q's'}) \backslash^{ps}$.
3. For any context Γ where $p, p' \notin \{q, q'\}$ and p, q are terminal, it is $(\Gamma^p/_{p's})^q/_{q's'} = (\Gamma^q/_{q's'})^p/_{p's}$.
4. For any context Γ where p is terminal, it is $(\Gamma^p/_{ps}) \backslash^{ps} = \Gamma$.

Since IUL_k is intended to realize MLns, where disjunction elimination is applied to isomorphic minor premises, i.e. intended to express the synchronous aspect of disjunction as union, the union elimination rule in IUL_k incorporates this isomorphism of minor premises by joining them together in the kit structure. As was the case with intersection introduction, isomorphic or *same* premises occupy terminal paths in the *same* kit, paths which differ only in the last letter. Therefore, union elimination has a single minor premise and a non-standard decoration in IUL_k always terminates.

$$\frac{\ldots \vdash t : \sigma \vee \tau \qquad \ldots, x : \sigma \vdash u : \rho \qquad \ldots, x : \tau \vdash u : \rho}{\ldots \vdash u[t/x] : \rho}\ (\vee\mathbf{E}) \text{ in MLns}$$

$$\frac{\ldots \vdash t : \sigma \cup \tau \qquad \ldots, x : [\sigma, \tau] \vdash u : [\rho, \rho]}{\ldots \vdash u[t/x] : \rho}\ (\cup\mathbf{E}) \text{ in } \mathrm{IUL}_k$$

As already noted in the discussion of IL, the implicative rules affect all terminal paths of certain kits and are called *global.* Doubling alters the part of a kit rooted at the end of a specific path, so it can be categorized as *local* together with pruning. As far as union rules are concerned, the notation "$\_\_\,[p := \_\_]$" used in their presentation urges a packaging with intersection rules which are presented likewise. We are inclined to say that union rules, as well, act on specific paths and are therefore local. However, a more thorough investigation of rule globality and locality will later show that such a classification is not accurate in the case of union elimination.

We next define a non-standard decoration of IUL_k which encodes the implication, brings about a substitution in the case of union elimination, and ignores all other rules. This decoration actually extends the non-standard decoration of IL (see Definition 1.11) to doubling and the union rules.

Definition 3.6 (Non-standard decoration of $\mathbf{IUL}_k$) *Suppose that $\pi :: \Gamma = H_1, \ldots, H_m \vdash K$ is a derivation in IUL_k. By decorating contexts bottom-up with distinct variables starting with $r = x_1, \ldots, x_m$ and then decorating kits to the right of "$\vdash$" top-down with terms in Λ, we get a decorated derivation $\pi^\star :: \Gamma^r = x_1 : H_1, \ldots, x_m : H_m \vdash t : K$. The decoration rules are demonstrated in Figure 3.6. When decorating contexts bottom-up, the new variable in an* $(\to\mathbf{I})$ *premise or in a* $(\cup\mathbf{E})$ *minor premise is fresh with respect to the variables in the branch connecting the conclusion to the root.*

$$\frac{}{x : K \vdash x : K}\;(\mathbf{ax})$$

$$\frac{\Gamma^{r} \vdash t : K}{\Gamma^{r},\, x : H \vdash t : K}\;(\mathbf{W}) \qquad \frac{\Gamma^{r},\, y : H_1,\, x : H_2,\, \Delta^{r'} \vdash t : K}{\Gamma^{r},\, x : H_2,\, y : H_1,\, \Delta^{r'} \vdash t : K}\;(\mathbf{X})$$

$$\frac{\Gamma^{r} \vdash t : K}{(\Gamma\backslash^{ps})^{r} \vdash t : K\backslash^{ps}}\;(\mathbf{P}) \qquad \frac{\Gamma^{r} \vdash t : K}{(\Gamma^{q}/_{ps})^{r} \vdash t : K^{q}/_{ps}}\;(\mathbf{D})$$

$$\frac{\Gamma^{r},\, x : H \vdash t : K}{\Gamma^{r} \vdash \lambda x.t : H \to K}\;(\to\mathbf{I}) \qquad \frac{\Gamma^{r} \vdash t : H \to K \qquad \Gamma^{r} \vdash u : H}{\Gamma^{r} \vdash tu : K}\;(\to\mathbf{E})$$

$$\frac{x_j : H_j[p := [\sigma_j, \sigma_j]] \vdash t : K[p := [\sigma, \tau]]}{x_j : H_j[p := \sigma_j] \vdash t : K[p := \sigma \cap \tau]}\;(\cap\mathbf{I})$$

$$\frac{\Gamma^{r} \vdash t : K[p := \sigma \cap \tau]}{\Gamma^{r} \vdash t : K[p := \sigma]}\;(\cap\mathbf{E}_1) \qquad \frac{\Gamma^{r} \vdash t : K[p := \sigma \cap \tau]}{\Gamma^{r} \vdash t : K[p := \tau]}\;(\cap\mathbf{E}_2)$$

$$\frac{\Gamma^{r} \vdash t : K[p := \sigma]}{\Gamma^{r} \vdash t : K[p := \sigma \cup \tau]}\;(\cup\mathbf{I}_1) \qquad \frac{\Gamma^{r} \vdash t : K[p := \tau]}{\Gamma^{r} \vdash t : K[p := \sigma \cup \tau]}\;(\cup\mathbf{I}_2)$$

$$\frac{x_j : H_j[p := \sigma_j] \vdash t : K[p := \sigma \cup \tau] \qquad x_j : H_j[p := [\sigma_j, \sigma_j]],\, x : K[p := [\sigma, \tau]] \vdash u : L[p := [\rho, \rho]]}{x_j : H_j[p := \sigma_j] \vdash u[t/x] : L[p := \rho]}\;(\cup\mathbf{E})$$

Figure 3.6: Non-standard decoration of IUL_k.

Remark 3.7 *We can easily show that, if* $\pi^{\star} :: \Gamma^{r} \vdash t : K$*, then* $FV(t) \subseteq \{r\}$.

We stress the fact that *every* derivation of IUL_k admits a non-standard decoration. This is because the kit structure has been used to unite the isomorphic premises of ($\wedge$**I**), so that ($\cap$**I**) has a single premise, and also to unite the isomorphic minor premises of ($\vee$**E**), so that ($\cup$**E**) has a single minor premise.

3.1.1 Commutations of local rules

As already mentioned in Chapter 1, a derivation of IL is defined in [18] as an equivalence class of derivations of pIL which differ only in the order of application of consecutive local rules concerning *different* paths. A derivation of IUL_k can be formally defined in a similar manner provided that ($\cup$**E**) is *not* considered local. Thus, if the system introduced by Definition 3.4 is called "pre-Intersection and Union Logic with kits", denoted pIUL_k, a more rigorous definition of IUL_k can be pursued as follows.

Definition 3.8 ($\mathbf{IUL}_k$ formal) *Intersection and Union Logic* IUL_k *is the quotient set* $\mathrm{pIUL}_k/\sim$ *of* pIUL_k *by the equivalence relation "$\sim$" defined below*[1]*. Paths p and q are different in commutations that involve only p and q, whereas $p \notin \{q,q'\}$ in commuting* (P,D)*, $p,p' \notin \{q,q'\}$ in commuting* (D,D)*, and $q \notin \{p,p'\}$ in commuting* $(\mathrm{D},\cap\mathrm{I}),(\mathrm{D},\cap\mathrm{E}),(\mathrm{D},\cup\mathrm{I})$.

$$\dfrac{\dfrac{\Gamma \vdash K}{\Gamma\backslash^{ps} \vdash K\backslash^{ps}}\,\mathbf{(P)}_{ps}}{(\Gamma\backslash^{ps})\backslash^{qs'} \vdash (K\backslash^{ps})\backslash^{qs'}}\,\mathbf{(P)}_{qs'} \qquad \underset{3.5(\mathrm{ii},1)}{\sim} \qquad \dfrac{\dfrac{\Gamma \vdash K}{\Gamma\backslash^{qs'} \vdash K\backslash^{qs'}}\,\mathbf{(P)}_{qs'}}{(\Gamma\backslash^{qs'})\backslash^{ps} \vdash (K\backslash^{qs'})\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K}{\Gamma\backslash^{ps} \vdash K\backslash^{ps}}\,\mathbf{(P)}_{ps}}{(\Gamma\backslash^{ps})^{q}/_{q's'} \vdash (K\backslash^{ps})^{q}/_{q's'}}\,\mathbf{(D)}_{q'} \qquad \underset{3.5(\mathrm{ii},2)}{\sim} \qquad \dfrac{\dfrac{\Gamma \vdash K}{\Gamma^{q}/_{q's'} \vdash K^{q}/_{q's'}}\,\mathbf{(D)}_{q'}}{(\Gamma^{q}/_{q's'})\backslash^{ps} \vdash (K^{q}/_{q's'})\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := [\sigma,\tau]]}{\Gamma\backslash^{ps} \vdash (K[q := [\sigma,\tau]])\backslash^{ps}}\,\mathbf{(P)}_{ps}}{(\Gamma\backslash^{ps})\backslash^{qs} \vdash (K[q := \sigma\cap\tau])\backslash^{ps}}\,(\cap\mathbf{I})_{q} \qquad \underset{3.5(\mathrm{ii},1)}{\sim} \qquad \dfrac{\dfrac{\Gamma \vdash K[q := [\sigma,\tau]]}{\Gamma\backslash^{qs} \vdash K[q := \sigma\cap\tau]}\,(\cap\mathbf{I})_{q}}{(\Gamma\backslash^{qs})\backslash^{ps} \vdash (K[q := \sigma\cap\tau])\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma\cap\tau]}{\Gamma\backslash^{ps} \vdash (K[q := \sigma\cap\tau])\backslash^{ps}}\,\mathbf{(P)}_{ps}}{\Gamma\backslash^{ps} \vdash (K[q := \sigma])\backslash^{ps}}\,(\cap\mathbf{E})_{q} \qquad \sim \qquad \dfrac{\dfrac{\Gamma \vdash K[q := \sigma\cap\tau]}{\Gamma \vdash K[q := \sigma]}\,(\cap\mathbf{E})_{q}}{\Gamma\backslash^{ps} \vdash (K[q := \sigma])\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma]}{\Gamma\backslash^{ps} \vdash (K[q := \sigma])\backslash^{ps}}\,\mathbf{(P)}_{ps}}{\Gamma\backslash^{ps} \vdash (K[q := \sigma\cup\tau])\backslash^{ps}}\,(\cup\mathbf{I})_{q} \qquad \sim \qquad \dfrac{\dfrac{\Gamma \vdash K[q := \sigma]}{\Gamma \vdash K[q := \sigma\cup\tau]}\,(\cup\mathbf{I})_{q}}{\Gamma\backslash^{ps} \vdash (K[q := \sigma\cup\tau])\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K}{\Gamma^{p}/_{p's} \vdash K^{p}/_{p's}}\,\mathbf{(D)}_{p'}}{(\Gamma^{p}/_{p's})^{q}/_{q's'} \vdash (K^{p}/_{p's})^{q}/_{q's'}}\,\mathbf{(D)}_{q'} \qquad \underset{3.5(\mathrm{ii},3)}{\sim} \qquad \dfrac{\dfrac{\Gamma \vdash K}{\Gamma^{q}/_{q's'} \vdash K^{q}/_{q's'}}\,\mathbf{(D)}_{q'}}{(\Gamma^{q}/_{q's'})^{p}/_{p's} \vdash (K^{q}/_{q's'})^{p}/_{p's}}\,\mathbf{(D)}_{p'}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := [\sigma,\tau]]}{\Gamma^{p}/_{p's} \vdash (K[q := [\sigma,\tau]])^{p}/_{p's}}\,\mathbf{(D)}_{p'}}{(\Gamma^{p}/_{p's})\backslash^{qs} \vdash (K[q := \sigma\cap\tau])^{p}/_{p's}}\,(\cap\mathbf{I})_{q} \qquad \underset{3.5(\mathrm{ii},2)}{\sim} \qquad \dfrac{\dfrac{\Gamma \vdash K[q := [\sigma,\tau]]}{\Gamma\backslash^{qs} \vdash K[q := \sigma\cap\tau]}\,(\cap\mathbf{I})_{q}}{(\Gamma\backslash^{qs})^{p}/_{p's} \vdash (K[q := \sigma\cap\tau])^{p}/_{p's}}\,\mathbf{(D)}_{p'}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma\cap\tau]}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma\cap\tau])^{p}/_{p's}}\,\mathbf{(D)}_{p'}}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma])^{p}/_{p's}}\,(\cap\mathbf{E})_{q} \qquad \sim \qquad \dfrac{\dfrac{\Gamma \vdash K[q := \sigma\cap\tau]}{\Gamma \vdash K[q := \sigma]}\,(\cap\mathbf{E})_{q}}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma])^{p}/_{p's}}\,\mathbf{(D)}_{p'}$$

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[1] Strictly speaking, the equivalence relation "$\sim$" is the reflexive and transitive closure of the relation given in 3.8.

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma]}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma])^{p}/_{p's}}\,(\mathbf{D})_{p'}}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma \cup \tau])^{p}/_{p's}}\,(\cup\mathbf{I})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[q := \sigma]}{\Gamma \vdash K[q := \sigma \cup \tau]}\,(\cup\mathbf{I})_{q}}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma \cup \tau])^{p}/_{p's}}\,(\mathbf{D})_{p'}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := [\rho, \upsilon]]}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := [\rho, \upsilon]]}\,(\cap\mathbf{I})_{p}}{(\Gamma\backslash^{ps})\backslash^{qs} \vdash K[p := \sigma \cap \tau][q := \rho \cap \upsilon]}\,(\cap\mathbf{I})_{q} \quad \underset{3.5(\mathrm{ii},1)}{\sim} \quad \dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := [\rho, \upsilon]]}{\Gamma\backslash^{qs} \vdash K[p : [\sigma, \tau]][q := \rho \cap \upsilon]}\,(\cap\mathbf{I})_{q}}{(\Gamma\backslash^{qs})\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho \cap \upsilon]}\,(\cap\mathbf{I})_{p}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho \cap \upsilon]}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho \cap \upsilon]}\,(\cap\mathbf{I})_{p}}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho]}\,(\cap\mathbf{E})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho \cap \upsilon]}{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho]}\,(\cap\mathbf{E})_{q}}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho]}\,(\cap\mathbf{I})_{p}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho]}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho]}\,(\cap\mathbf{I})_{p}}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho]}{\Gamma \vdash K[p := [\sigma, \tau]][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q}}{\Gamma\backslash^{ps} \vdash K[p := \sigma \cap \tau][q := \rho \cup \upsilon]}\,(\cap\mathbf{I})_{p}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho \cap \upsilon]}{\Gamma \vdash K[p := \sigma][q := \rho \cap \upsilon]}\,(\cap\mathbf{E})_{p}}{\Gamma \vdash K[p := \sigma][q := \rho]}\,(\cap\mathbf{E})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho \cap \upsilon]}{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho]}\,(\cap\mathbf{E})_{q}}{\Gamma \vdash K[p := \sigma][q := \rho]}\,(\cap\mathbf{E})_{p}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho]}{\Gamma \vdash K[p := \sigma][q := \rho]}\,(\cap\mathbf{E})_{p}}{\Gamma \vdash K[p := \sigma][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho]}{\Gamma \vdash K[p := \sigma \cap \tau][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q}}{\Gamma \vdash K[p := \sigma][q := \rho \cup \upsilon]}\,(\cap\mathbf{E})_{p}$$

$$\dfrac{\dfrac{\Gamma \vdash K[p : \sigma][q := \rho]}{\Gamma \vdash K[p : \sigma \cup \tau][q := \rho]}\,(\cup\mathbf{I})_{p}}{\Gamma \vdash K[p : \sigma \cup \tau][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q} \quad \sim \quad \dfrac{\dfrac{\Gamma \vdash K[p : \sigma][q := \rho]}{\Gamma \vdash K[p : \sigma][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{q}}{\Gamma \vdash K[p : \sigma \cup \tau][q := \rho \cup \upsilon]}\,(\cup\mathbf{I})_{p}$$

A derivation $\pi :: \Gamma \vdash K$ *in* IUL_k *formally denotes an equivalence class of derivations in* pIUL_k*, all proving* $\Gamma \vdash K$*.*

Remark 3.9 *(i) Since the local rules* $(\mathbf{P}), (\mathbf{D}), (\cap\mathbf{I}), (\cap\mathbf{E})$*, and* $(\cup\mathbf{I})$ *are not impressed on the decoration of a derivation, we can safely say that derivations of* pIUL_k *in the same equivalence class admit the same decoration provided that contexts are identically decorated. This decoration is also the one for the* IUL_k*-derivation representing the equivalence class in question.*

(ii) As already remarked for the case of pIL *and* IL *in Chapter 1, in practice an equivalence class of* pIUL_k*-derivations, i.e. an* IUL_k*-derivation, is identified with a specific member of the class, i.e. a specific* pIUL_k*-derivation. Thereupon, we can actually ignore Definition 3.8 and confine ourselves to Definition 3.4.*

Had we considered **(∪E)** local, we would have also had to examine the commutations of the pairs (P, ∪E), (D, ∪E), (∩E, ∪E), (∪I, ∪E), (∩I, ∪E), and (∪E, ∪E).

The first four pairs commute symmetrically, though not without minor restrictions which stem from the fact that **(∪E)** is a two-premise rule. In particular, for the pair (P, ∪E), the only case that works is when both premises of **(∪E)** derive from **(P)** and this is because this structural rule messes up with the tree-structure. Cases where only one premise of **(∪E)** derives from **(P)** do not work. The same holds for the pair (D, ∪E).

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma \cup \tau]}{\Gamma\backslash^{ps} \vdash (K[q := \sigma \cup \tau])\backslash^{ps}}\,\mathbf{(P)}_{ps} \qquad \dfrac{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\rho, \rho]]}{(\Gamma^{q}/_{qs})\backslash^{ps}, (K[q := [\sigma, \tau]])\backslash^{ps} \vdash (L[q := [\rho, \rho]])\backslash^{ps}}\,\mathbf{(P)}_{ps}}{\Gamma\backslash^{ps} \vdash (L[q := \rho])\backslash^{ps}}\,\mathbf{(\cup E)}_{q} \qquad \overset{\sim}{3.5(\text{ii},2)}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma \cup \tau] \qquad \Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\rho, \rho]]}{\Gamma \vdash L[q := \rho]}\,\mathbf{(\cup E)}_{q}}{\Gamma\backslash^{ps} \vdash (L[q := \rho])\backslash^{ps}}\,\mathbf{(P)}_{ps}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma \cup \tau]}{\Gamma^{p}/_{p's} \vdash (K[q := \sigma \cup \tau])^{p}/_{p's}}\,\mathbf{(D)}_{p'} \qquad \dfrac{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\rho, \rho]]}{(\Gamma^{q}/_{qs})^{p}/_{p's}, (K[q := [\sigma, \tau]])^{p}/_{p's} \vdash (L[q := [\rho, \rho]])^{p}/_{p's}}\,\mathbf{(D)}_{p'}}{\Gamma^{p}/_{p's} \vdash (L[q := \rho])^{p}/_{p's}}\,\mathbf{(\cup E)}_{q} \qquad \overset{\sim}{3.5(\text{ii},3)}$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma \cup \tau] \qquad \Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\rho, \rho]]}{\Gamma \vdash L[q := \rho]}\,\mathbf{(\cup E)}_{q}}{\Gamma^{p}/_{p's} \vdash (L[q := \rho])^{p}/_{p's}}\,\mathbf{(D)}_{p'}$$

On the other hand, for the pair (∩E, ∪E), the only case that works is when the minor premise of **(∪E)** derives from **(∩E)**. The same holds for the pair (∪I, ∪E).

$$\dfrac{\Gamma \vdash K[q := \sigma \cup \tau] \qquad \dfrac{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\eta, \eta]][p := \rho \cap \upsilon]}{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\eta, \eta]][p := \rho]}\,\mathbf{(\cap E)}_{p}}{\Gamma \vdash L[q := \eta][p := \rho]}\,\mathbf{(\cup E)}_{q} \qquad \sim$$

$$\dfrac{\dfrac{\Gamma \vdash K[q := \sigma \cup \tau] \qquad \Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\eta, \eta]][p := \rho \cap \upsilon]}{\Gamma \vdash L[q := \eta][p := \rho \cap \upsilon]}\,\mathbf{(\cup E)}_{q}}{\Gamma \vdash L[q := \eta][p := \rho]}\,\mathbf{(\cap E)}_{p}$$

$$\dfrac{\Gamma \vdash K[q := \sigma \cup \tau] \qquad \dfrac{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\eta, \eta]][p := \rho]}{\Gamma^{q}/_{qs}, K[q := [\sigma, \tau]] \vdash L[q := [\eta, \eta]][p := \rho \cup \upsilon]}\,\mathbf{(\cup I)}_{p}}{\Gamma \vdash L[q := \eta][p := \rho \cup \upsilon]}\,\mathbf{(\cup E)}_{q} \qquad \sim$$

$$\dfrac{\dfrac{\Gamma\vdash K[q:=\sigma\cup\tau] \qquad \Gamma^q/_{qs}, K[q:=[\sigma,\tau]]\vdash L[q:=[\eta,\eta]][p:=\rho]}{\Gamma\vdash L[q:=\eta][p:=\rho]}\,(\cup\mathbf{E})_q}{\Gamma\vdash L[q:=\eta][p:=\rho\cup\upsilon]}\,(\cup\mathbf{I})_p$$

For the last two pairs the interchange relation is not exactly symmetrical, since the cases that work involve additional structural rules or restrictions on certain leaves. As examples, we show the pair (∩I, ∪E) in the case where the *minor* premise of (∪**E**) derives from (∩**I**), which is actually the only case that works for this pair, and the pair (∪E, ∪E) in the case where again the *minor* premise of the lower (∪**E**) derives from the upper (∪**E**). We present the latter pair using a simple kit-structure to avoid heavy formalism.

$$\dfrac{\Gamma\vdash K[q:=\sigma\cup\tau][p:=\phi] \qquad \dfrac{(\Gamma^q/_{qs})^p/_{ps}, K[q:=[\sigma,\tau]][p:=[\phi,\phi]]\vdash L[q:=[\eta,\eta]][p:=[\rho,\upsilon]]}{\Gamma^q/_{qs}, K[q:=[\sigma,\tau]][p:=\phi]\vdash L[q:=[\eta,\eta]][p:=\rho\cap\upsilon]}\,(\cap\mathbf{I})_p}{\Gamma\vdash L[q:=\eta][p:=\rho\cap\upsilon]}\,(\cup\mathbf{E})_q \qquad \underset{3.5(\text{ii},3)}{\overset{?}{\sim}}$$

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash K[q:=\sigma\cup\tau][p:=\phi]}{\Gamma^p/_{ps}\vdash K[q:=\sigma\cup\tau][p:=[\phi,\phi]]}\,(\mathbf{D})_p \qquad (\Gamma^p/_{ps})^q/_{qs}, K[q:=[\sigma,\tau]][p:=[\phi,\phi]]\vdash L[q:=[\eta,\eta]][p:=[\rho,\upsilon]]}{\Gamma^p/_{ps}\vdash L[q:=\eta][p:=[\rho,\upsilon]]}\,(\cup\mathbf{E})_q}{\Gamma\vdash L[q:=\eta][p:=\rho\cap\upsilon]}\,(\cap\mathbf{I})_p$$

$$\dfrac{\Gamma\vdash[\chi,\rho\cup\upsilon] \qquad \dfrac{\Gamma^r/_{rr},[\chi,[\rho,\upsilon]]\vdash[\sigma\cup\tau,[\boldsymbol{\phi},\boldsymbol{\phi}]] \qquad (\Gamma^r/_{rr})^l/_{ll},[[\chi,\chi],[\rho,\upsilon]],[[\sigma,\tau],[\boldsymbol{\phi},\boldsymbol{\phi}]]\vdash[[\eta,\eta],[\theta,\theta]]}{\Gamma^r/_{rr},[\chi,[\rho,\upsilon]]\vdash[\eta,[\theta,\theta]]}\,(\cup\mathbf{E})_l}{\Gamma\vdash[\eta,\theta]}\,(\cup\mathbf{E})_r \qquad \underset{3.5(\text{ii},3)}{\overset{?}{\sim}}$$

$$\dfrac{\dfrac{\Gamma\vdash[\chi,\rho\cup\upsilon] \qquad \Gamma^r/_{rr},[\chi,[\rho,\upsilon]]\vdash[\sigma\cup\tau,[\boldsymbol{\phi},\boldsymbol{\phi}]]}{\Gamma\vdash[\sigma\cup\tau,\phi]}\,(\cup\mathbf{E})_r \qquad \begin{matrix}\text{see right below}\\ \pi::\Gamma^l/_{ll},[[\sigma,\tau],\phi]\vdash[[\eta,\eta],\theta]\end{matrix}}{\Gamma\vdash[\eta,\theta]}\,(\cup\mathbf{E})_l$$

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash[\chi,\rho\cup\upsilon]}{\Gamma^l/_{ll}\vdash[[\chi,\chi],\rho\cup\upsilon]}\,(\mathbf{D})_l}{\Gamma^l/_{ll},[[\sigma,\tau],\phi]\vdash[[\chi,\chi],\rho\cup\upsilon]}\,(\mathbf{W}) \qquad \dfrac{(\Gamma^r/_{rr})^l/_{ll},[[\chi,\chi],[\rho,\upsilon]],[[\sigma,\tau],[\boldsymbol{\phi},\boldsymbol{\phi}]]\vdash[[\eta,\eta],[\theta,\theta]]}{(\Gamma^r/_{rr})^l/_{ll},[[\sigma,\tau],[\boldsymbol{\phi},\boldsymbol{\phi}]],[[\chi,\chi],[\rho,\upsilon]]\vdash[[\eta,\eta],[\theta,\theta]]}\,(\mathbf{X})}{\pi::\Gamma^l/_{ll},[[\sigma,\tau],\phi]\vdash[[\eta,\eta],\theta]}\,(\cup\mathbf{E})_r$$

In the case of (∪E, ∪E), the leaves of subtree $[\phi,\phi]$ must be identical, so that $(\cup\mathbf{E})_r$ can be applied, and even twice, in the derivation to the right of "∼". This means that a restriction is posed on leaves of the premises of $(\cup\mathbf{E})_l$ in the derivation to the left of "∼", since, in its general case, this rule would be applied with different such leaves.

The above discussion highlights the peculiar nature of (∪**E**), when compared to the (other) local rules (**P**), (**D**), (∩**I**), (∩**E**), (∪**I**). Besides the fact that union elimination is a two-premise rule, while all the others are one-premise rules, there are significant abnormalities in commuting union elimination with the others, while the others commute with each other quite smoothly. The formalism of molecules will later reveal a certain kind of globality inherent in the union elimination rule which is as yet concealed by the complex notation of kits. So, fortunately, union elimination will prove to differ from the rules categorized as "local", retaining the validity of Definition 3.8.

3.1.2 Relating IUL$_k$ to MLns

Using the non-standard decorations of ML and IUL$_k$, we will attain a connection between a single IUL$_k$ derivation and a finite set of MLns derivations modulo the conversion of intersection and union to conjunction and disjunction, respectively. We will show that any derivation π in IUL$_k$ provides a finite number of derivations in MLns which all share the decoration of π. The next theorem is an extension of Theorem 1.12.

Theorem 3.10 (From IUL$_k$ to MLns) *Let* $\pi :: H_1, \ldots, H_m \vdash K$ *be a derivation in* IUL$_k$*, such that* $\pi^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$*. For every terminal path* p *in* $P_T(K)$*, there exists a derivation* $\pi^p :: (H_1)^p, \ldots, (H_m)^p \vdash K^p$ *in* MLns*, such that* $(\pi^p)^\star :: x_1 : (H_1)^p, \ldots, x_m : (H_m)^p \vdash t : K^p$*.*

Proof. By induction on $\pi^\star$.

Base: If $\pi^\star :: x : K \vdash x : K$ is an IUL$_k^\star$-axiom and $p \in P_T(K)$, there is an axiom $\pi^p :: K^p \vdash K^p$ in MLns, such that $(\pi^p)^\star :: x : K^p \vdash x : K^p$.

Induction step: We show the most interesting cases.

$$\triangleright \quad \frac{\pi_0^\star :: x_j : H_j[p := [\sigma_j, \sigma_j]] \vdash t : K[p := [\sigma, \tau]]}{\pi^\star :: x_j : H_j[p : \sigma_j] \vdash t : K[p := \sigma \cap \tau]} \ (\cap\mathbf{I})$$

Let $q \in P_T(K[p := [\sigma \cap \tau])$. We distinguish two subcases.

1. If $q \neq p$, then $q \in P_T(K[p := [\sigma, \tau]])$. So, by the IH, there is a

$$\pi_0^q :: (H_j[p := [\sigma_j, \sigma_j]])^q \vdash (K[p := [\sigma, \tau]])^q$$

in MLns, such that $(\pi_0^q)^\star :: x_j : (H_j[p := [\sigma_j, \sigma_j]])^q \vdash t : (K[p := [\sigma, \tau]])^q$. Since $(H_j[p := [\sigma_j, \sigma_j]])^q = (H_j[p := \sigma_j])^q$ and $(K[p := [\sigma, \tau]])^q = (K[p := \sigma \cap \tau])^q$, it is $\pi_0^q = \pi^q$.

2. If $q = p$, then $pl, pr \in P_T(K[p := [\sigma, \tau]])$. So, by the IH, there exist $\pi_0^{pl} :: \sigma_j \vdash \sigma$ and $\pi_0^{pr} :: \sigma_j \vdash \tau$ in MLns, such that $(\pi_0^{pl})^\star :: x_j : \sigma_j \vdash t : \sigma$ and $(\pi_0^{pr})^\star :: x_j : \sigma_j \vdash t : \tau$. Applying **(**$\wedge$**I)** to π_0^{pl}, π_0^{pr}, we get a $\pi^p :: \sigma_j \vdash \sigma \wedge \tau$ which is in MLns, since both π_0^{pl} and π_0^{pr} are in MLns and they are isomorphic. Moreover, it is $(\pi^p)^\star :: x_j : \sigma_j \vdash t : \sigma \wedge \tau$.

$$\triangleright \quad \frac{\pi_0^\star :: x_j : H_j[p := \sigma_j] \vdash t : K[p := \sigma \cup \tau] \qquad \pi_1^\star :: x_j : H_j[p := [\sigma_j, \sigma_j]], x : K[p := [\sigma, \tau]] \vdash u : L[p := [\rho, \rho]]}{\pi^\star :: x_j : H_j[p := \sigma_j] \vdash u[t/x] : L[p := \rho]} \ (\cup\mathbf{E})$$

Let $q \in P_T(L[p := \rho])$, then $q \in P_T(K[p := \sigma \cup \tau])$. We distinguish two subcases.

1. If $q \neq p$, then $q \in P_T(L[p := [\rho, \rho]])$. We have that $(H_j[p := [\sigma_j, \sigma_j]])^q = (H_j[p := \sigma_j])^q = \phi_j$, $(K[p := [\sigma, \tau]])^q = (K[p := \sigma \cup \tau])^q = \zeta$, and $(L[p := [\rho, \rho]])^q = (L[p := \rho])^q = \xi$. By the IH, there exist $\pi_0^q :: \phi_j \vdash \zeta$ and $\pi_1^q :: \phi_j, \zeta \vdash \xi$ in MLns, such that $(\pi_0^q)^\star :: x_j : \phi_j \vdash t : \zeta$ and $(\pi_1^q)^\star :: x_j : \phi_j, x : \zeta \vdash u : \xi$. It is $\pi^q = S(\pi_0^q, \pi_1^q) :: \phi_j \vdash \xi$, where $S(\pi_0^q, \pi_1^q)$ stands for the derivation obtained from π_1^q by substituting specific instances of axioms $\zeta \vdash \zeta$ by π_0^q and then possibly eliminating some structural rules. The (non-standard) decoration of the substitution derivation π^q gives $(\pi^q)^\star :: x_j : \phi_j \vdash u[t/x] : \xi$.

2. If $q = p$, then $pl, pr \in P_T(L[p := [\rho, \rho]])$. So, by the IH, there exist $\pi_0^p :: \sigma_j \vdash \sigma \vee \tau$, $\pi_1^{pl} :: \sigma_j, \sigma \vdash \rho$, and $\pi_1^{pr} :: \sigma_j, \tau \vdash \rho$ in MLns, such that $(\pi_0^p)^\star :: x_j : \sigma_j \vdash t : \sigma \vee \tau$, $(\pi_1^{pl})^\star :: x_j : \sigma_j, x : \sigma \vdash u : \rho$, and $(\pi_1^{pr})^\star :: x_j : \sigma_j, x : \tau \vdash u : \rho$. Applying **(**$\vee$**E)** to $\pi_0^p, \pi_1^{pl}, \pi_1^{pr}$, we get a $\pi^p :: \sigma_j \vdash \rho$ which is in MLns, since each of $\pi_0^p, \pi_1^{pl}, \pi_1^{pr}$ is in MLns and π_1^{pl}, π_1^{pr} are isomorphic. Moreover, it is $(\pi^p)^\star :: x_j : \sigma_j \vdash u[t/x] : \rho$. $\dashv$

Definition 3.11 *Let $\pi :: \Gamma \vdash K$ be a derivation in* IUL_k *and* $\mathrm{ML}(\pi) = \{\pi^p \mid p \in P_T(K)\}$*. A derivation π^p in* $\mathrm{ML}(\pi)$ *will be called a* projection *of π in* ML*.*

Example 3.12 *Let $\sigma = \alpha \cap \beta$, $\tau = \gamma \cap \delta$, and $\rho = (\delta \to \eta) \cap (\zeta \to \eta)$. If $\Gamma_0 = [\sigma, \tau], [\alpha \to \theta, \rho]$ and $\Gamma_1 = [\sigma, [\tau, \tau]], [\alpha \to \theta, [\rho, \rho]], [\alpha, [\delta, \zeta]]$, consider the following derivation π in* IUL_k*.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\sigma,\tau]\vdash[\sigma,\tau]}{\Gamma_0\vdash[\sigma,\tau]}\,(\mathbf{w})}{\Gamma_0\vdash[\alpha,\delta]}\,(\cap\mathbf{E})}{\Gamma_0\vdash[\alpha,\delta\cup\zeta]}\,(\cup\mathbf{I}) \qquad \dfrac{\dfrac{\dfrac{[\alpha\to\theta,[\rho,\rho]]\vdash[\alpha\to\theta,[\rho,\rho]]}{\Gamma_1\vdash[\alpha\to\theta,[\rho,\rho]]}\,(\mathbf{wx})}{\Gamma_1\vdash[\alpha\to\theta,[\delta\to\eta,\zeta\to\eta]]}\,(\cap\mathbf{E}) \qquad \dfrac{[\alpha,[\delta,\zeta]]\vdash[\alpha,[\delta,\zeta]]}{\Gamma_1\vdash[\alpha,[\delta,\zeta]]}\,(\mathbf{wx})}{\Gamma_1\vdash[\theta,[\eta,\eta]]}\,(\to\mathbf{E})}{\Gamma_0\vdash[\theta,\eta]}\,(\cup\mathbf{E})_r}{\pi :: [\sigma,\tau]\vdash[(\alpha\to\theta)\to\theta,\rho\to\eta]}\,(\to\mathbf{I})$$

There are two projections π^l and π^r of π in ML*. Abstracting the left paths in π, we arrive at a substitution operation which is carried out to give π^l.*

$$\dfrac{\dfrac{\dfrac{\sigma\vdash\sigma}{\sigma,\alpha\to\theta\vdash\sigma}\,(\mathbf{w})}{\pi_0^l :: \sigma,\alpha\to\theta\vdash\alpha}\,(\wedge\mathbf{E}) \qquad \dfrac{\dfrac{\dfrac{\alpha\to\theta\vdash\alpha\to\theta}{\sigma,\alpha\to\theta\vdash\alpha\to\theta}\,(\mathbf{wx})}{\sigma,\alpha\to\theta,\alpha\vdash\alpha\to\theta}\,(\mathbf{w}) \qquad \dfrac{\alpha\vdash\alpha}{\sigma,\alpha\to\theta,\alpha\vdash\alpha}\,(\mathbf{wx})}{\pi_1^l :: \sigma,\alpha\to\theta,\alpha\vdash\theta}\,(\to\mathbf{E})}{S(\pi_0^l,\pi_1^l) :: \sigma,\alpha\to\theta\vdash\theta}\,[\text{substitution}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\to\theta\vdash\alpha\to\theta}{\sigma,\alpha\to\theta\vdash\alpha\to\theta}\,(\mathbf{wx}) \qquad \pi_0^l :: \sigma,\alpha\to\theta\vdash\alpha}{S(\pi_0^l,\pi_1^l) :: \sigma,\alpha\to\theta\vdash\theta}\,(\to\mathbf{E})}{\pi^l :: \sigma\vdash(\alpha\to\theta)\to\theta}\,(\to\mathbf{I})}{}$$

Abstracting the right paths in π, or, more precisely, the terminal paths whose string starts with r, we arrive at a $(\vee\mathbf{E})$ *inference in π^r.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\tau\vdash\tau}{\tau,\rho\vdash\tau}\,(\mathbf{w})}{\tau,\rho\vdash\delta}\,(\wedge\mathbf{E})}{\tau,\rho\vdash\delta\vee\zeta}\,(\vee\mathbf{I}) \qquad \dfrac{\dfrac{\dfrac{\rho\vdash\rho}{\tau,\rho,\delta\vdash\rho}\,(\mathbf{wx})}{\tau,\rho,\delta\vdash\delta\to\eta}\,(\wedge\mathbf{E}) \qquad \dfrac{\delta\vdash\delta}{\tau,\rho,\delta\vdash\delta}\,(\mathbf{wx})}{\tau,\rho,\delta\vdash\eta}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{\dfrac{\rho\vdash\rho}{\tau,\rho,\zeta\vdash\rho}\,(\mathbf{wx})}{\tau,\rho,\zeta\vdash\zeta\to\eta}\,(\wedge\mathbf{E}) \qquad \dfrac{\zeta\vdash\zeta}{\tau,\rho,\zeta\vdash\zeta}\,(\mathbf{wx})}{\tau,\rho,\zeta\vdash\eta}\,(\to\mathbf{E})}{\tau,\rho\vdash\eta}\,(\vee\mathbf{E})}{\pi^r :: \tau\vdash\rho\to\eta}\,(\to\mathbf{I})$$

So, the $(\cup\mathbf{E})$ *inference at path r in π is translated to a* $(\vee\mathbf{E})$ *inference in π^r.*

Given that contexts are decorated by x, derivations π, π^l, and π^r are all (non-standardly) decorated by $\lambda y.yx$.

It is worth noting that the conclusive judgement $[\sigma, \tau] \vdash [(\alpha \to \theta) \to \theta, \rho \to \eta]$ of π, which is in the language of IL*, i.e. it does not involve union, is already provable in* IL*.*

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha \to \theta, \rho] \vdash [\alpha \to \theta, \rho]}{[\sigma,\tau],[\alpha \to \theta, \rho] \vdash [\alpha \to \theta, \rho]}\,(\mathbf{wx})}{[\sigma,\tau],[\alpha \to \theta, \rho] \vdash [\alpha \to \theta, \delta \to \eta]}\,(\cap\mathbf{E}) \qquad \dfrac{\dfrac{[\sigma,\tau] \vdash [\sigma,\tau]}{[\sigma,\tau],[\alpha \to \theta, \rho] \vdash [\sigma,\tau]}\,(\mathbf{w})}{[\sigma,\tau],[\alpha \to \theta, \rho] \vdash [\alpha, \delta]}\,(\cap\mathbf{E})}{[\sigma,\tau],[\alpha \to \theta, \rho] \vdash [\theta, \eta]}\,(\to\mathbf{E})}{\pi' :: [\sigma,\tau] \vdash [(\alpha \to \theta) \to \theta, \rho \to \eta]}\,(\to\mathbf{I})$$

This is an instance of the fact that IUL$_k$ *is a conservative extension of* IL. *Finally, derivation π' is also (non-standardly) decorated by $\lambda y.yx$, if the context is decorated by x.*

From MLns to IUL$_k$?

The aim of this paragraph is to spotlight the problems evolving in the attempt to prove the inverse of Theorem 3.10. We will study the simple case where we start off with a single derivation in MLns and try to attain its corresponding derivation in IUL$_k$.

If $\pi :: \sigma_1, \ldots, \sigma_m \vdash \rho$ is in MLns with a non-standard decoration $\pi^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \rho$, we would like to show that there exists a derivation $\pi' :: \sigma_1, \ldots, \sigma_m \vdash \rho$ in IUL$_k$, where $\sigma_1, \ldots, \sigma_m, \rho$ are single-node kits modulo the conversion of connectives, such that $(\pi')^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \rho$. Supposing we proceed by induction on π, let us consider the case of $(\wedge\mathbf{I})$.

$$\dfrac{\pi_0^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \sigma \qquad \pi_1^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau}{\pi^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \sigma \wedge \tau}\,(\wedge\mathbf{I})$$

By the IH, we would get derivations $\pi_0' :: \sigma_1, \ldots, \sigma_m \vdash \sigma$ and $\pi_1' :: \sigma_1, \ldots, \sigma_m \vdash \tau$ in IUL$_k$, such that $(\pi_0')^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \sigma$ and $(\pi_1')^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$. So, we would have two identically decorated derivations in IUL$_k$. We would like to be able to join together these two derivations with the same decoration, so as to get a single derivation with this very decoration. That is to say, we would like to be able to merge π_0' and π_1' into a single $\pi_{01}' :: [\sigma_1, \sigma_1], \ldots, [\sigma_m, \sigma_m] \vdash [\sigma, \tau]$, such that $(\pi_{01}')^\star :: x_1 : [\sigma_1, \sigma_1], \ldots, x_m : [\sigma_m, \sigma_m] \vdash t : [\sigma, \tau]$. Then, by $(\cap\mathbf{I})$ on π_{01}', we would get the required $\pi' :: \sigma_1, \ldots, \sigma_m \vdash \sigma \cap \tau$ with $(\pi')^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \sigma \cap \tau$.

The crucial step is the unification of two identically decorated derivations of IUL$_k$ into a single derivation of IUL$_k$ with this very decoration. Formally, we would like to prove the following claim.

> **Claim:** *Two identically decorated* IUL$_k$*-derivations* $\pi_0^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : H$ *and* $\pi_1^\star :: x_1 : K_1, \ldots, x_m : K_m \vdash t : K$ *can be joined together into a single* IUL$_k$*-derivation* $\pi^\star :: x_1 : [H_1, K_1], \ldots, x_m : [H_m, K_m] \vdash t : [H, K]$ *with this very decoration.*

However, as the next example demonstrates, the substitution term in the decoration of $(\cup\mathbf{E})$ poses a serious problem to this unification task.

Example 3.13 *Let $\phi = (\sigma \cup \tau) \cap \alpha$, $\sigma = \rho \cap \sigma_2$, $\tau = \tau_1 \cap \rho$, and $\chi = (\zeta \cup \xi) \cap \beta$. Consider the identically decorated* IUL$_k$*-derivations π_0 and π_1, shown below.*

$$\dfrac{\dfrac{\dfrac{x:\phi\vdash x:\phi}{x:\phi,\,y:\psi\vdash x:\phi}\,\textbf{(w)}}{x:\phi,\,y:\psi\vdash x:\sigma\cup\tau}\,(\cap\mathbf{E}_1)\qquad\dfrac{\dfrac{z:[\sigma,\tau]\vdash z:[\sigma,\tau]}{x:[\phi,\phi],\,y:[\psi,\psi],\,z:[\sigma,\tau]\vdash z:[\sigma,\tau]}\,\textbf{(wx)}}{x:[\phi,\phi],\,y:[\psi,\psi],\,z:[\sigma,\tau]\vdash z:[\rho,\rho]}\,(\cap\mathbf{E})}{\pi_0^\star :: \;x:\phi,\,y:\psi\vdash z[x/z]=x:\rho}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{x:\chi\vdash x:\chi}{x:\chi,\,y:\upsilon\vdash x:\chi}\,\textbf{(w)}}{x:\chi,\,y:\upsilon\vdash x:\zeta\cup\xi}\,(\cap\mathbf{E}_1)\qquad\dfrac{\dfrac{x:[\chi,\chi]\vdash x:[\chi,\chi]}{x:[\chi,\chi],\,y:[\upsilon,\upsilon],\,z:[\zeta,\xi]\vdash x:[\chi,\chi]}\,\textbf{(w)}}{x:[\chi,\chi],\,y:[\upsilon,\upsilon],\,z:[\zeta,\xi]\vdash x:[\beta,\beta]}\,(\cap\mathbf{E})}{\pi_1^\star :: \;x:\chi,\,y:\upsilon\vdash x[x/z]=x:\beta}\,(\cup\mathbf{E})$$

An attempt to construct a derivation $\pi^\star :: x:[\phi,\chi],\,y:[\psi,\upsilon]\vdash x:[\rho,\beta]$ *in a bottom-up manner fails, as shown below.*

$$\dfrac{\dfrac{\dfrac{[\phi,\chi]^x\vdash x:[\phi,\chi]}{([\phi,\chi],[\psi,\upsilon])^{x,y}\vdash x:[\phi,\chi]}\,\textbf{(w)}}{([\phi,\chi],[\psi,\upsilon])^{x,y}\vdash x:[\sigma\cup\tau,\zeta\cup\xi]}\,(\cap\mathbf{E})\quad\dfrac{\dfrac{\text{axiom ?}}{([[\phi,\phi],[\chi,\chi]],[[\psi,\psi],[\upsilon,\upsilon]],[[\sigma,\tau],[\zeta,\xi]])^{x,y,z}\vdash\ ?:[[\sigma,\tau],[\chi,\chi]]}\,\text{(struct.)}}{([[\phi,\phi],[\chi,\chi]],[[\psi,\psi],[\upsilon,\upsilon]],[[\sigma,\tau],[\zeta,\xi]])^{x,y,z}\vdash\ ?:[[\rho,\rho],[\beta,\beta]]}\,(\cap\mathbf{E})}{\pi^\star :: \;x:[\phi,\chi],\,y:[\psi,\upsilon]\vdash\ ?:[\rho,\beta]}\,(\cup\mathbf{E})^\dagger$$

For such an attempt to work, we would, at first, need to have a notion of union elimination allowing to apply the rule to different paths in parallel. In this example, the variant rule $(\cup\mathbf{E})^\dagger$ *applies union elimination to paths l and r simultaneously. However, even with* $(\cup\mathbf{E})^\dagger$*, we cannot reach an axiom of* IUL$_k$ *in the right branch. This is because the judgement obtained after having applied the intersection eliminations does not contain the succedent-kit in the context, i.e. the kit* $[[\sigma,\tau],[\chi,\chi]]$ *is not in the context* $[[\phi,\phi],[\chi,\chi]],[[\psi,\psi],[\upsilon,\upsilon]],[[\sigma,\tau],[\zeta,\xi]]$*. So, any further attempt to apply structural rules to reach an axiom fails. This problem derives from the fact that, in the right branch of* π_0*, the kit-sequence* $[\phi,\phi],[\psi,\psi],[\sigma,\tau]$ *entails the kit* $[\sigma,\tau]$*, which is the third member of the sequence, while, in the right branch of* π_1*, the kit-sequence* $[\chi,\chi],[\upsilon,\upsilon],[\zeta,\xi]$ *entails the kit* $[\chi,\chi]$*, which is the first member of the sequence. Termwise, given that the contexts in the right premises of* $(\cup\mathbf{E})$ *in* π_0 *and* π_1 *are decorated by the same sequence of variables* x,y,z*, the kit-situation just described reflects on different terms* z *and* x *decorating the succedent-kits in these premises in* π_0 *and* π_1*, respectively. Since* z *(trivially) contains a free occurrence of* z*, while* x *doesn't, this translates to two different kinds of substitution in the decorations of* π_0 *and* π_1*: a* proper *substitution* $z[x/z]$ *in* π_0 *and a* phony *substitution* $x[x/z]$ *in* π_1*. Hence, the incompatibility of* $\pi_0^\star$ *and* $\pi_1^\star$ *essentially reduces to these two different ways of expressing a term, namely* x*, as a substitution.*

The problem of the twofold decomposition of substitution, depicted in the above example for the case of the logic IUL$_k$, is a problem already spotted in the literature for the case of union types (see [2, 22]).

3.2 Intersection and Union Logic IUL$_m$

We define Intersection and Union Logic IUL$_m$ as an extension with union of Intersection Synchronous Logic ISL. This system is also intended as a logical foundation for IUT, i.e. as a logic corresponding

$$\frac{}{[(\sigma_i\,;\,\sigma_i)_i]}\ (\mathbf{ax})$$

$$\frac{[(\Gamma_i\,;\,\tau_i)_i]}{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}\ (\mathbf{W}) \qquad \frac{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i)_i]}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\rho_i)_i]}\ (\mathbf{X})$$

$$\frac{\mathcal{M}\cup\mathcal{N}}{\mathcal{M}}\ (\mathbf{P}) \qquad \frac{\mathcal{M}\cup[\mathcal{A}]}{\mathcal{M}\cup[\mathcal{A},\mathcal{A}]}\ (\mathbf{D})$$

$$\frac{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}{[\Gamma_i\,;\,\sigma_i\to\tau_i)_i]}\ (\to\!\mathbf{I}) \qquad \frac{[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i] \qquad [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\ (\to\!\mathbf{E})$$

$$\frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma),(\Gamma\,;\,\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}\ (\cap\mathbf{I})$$

$$\frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma)]}\ (\cap\mathbf{E}_1) \qquad \frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cap\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\tau)]}\ (\cap\mathbf{E}_2)$$

$$\frac{\mathcal{M}\cup[(\Gamma\,;\,\sigma)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cup\tau)]}\ (\cup\mathbf{I}_1) \qquad \frac{\mathcal{M}\cup[(\Gamma\,;\,\tau)]}{\mathcal{M}\cup[(\Gamma\,;\,\sigma\cup\tau)]}\ (\cup\mathbf{I}_2)$$

$$\frac{[(\Gamma_i\,;\,\phi_i)_i]\cup[(\Gamma\,;\,\sigma\cup\tau)] \qquad [(\Gamma_i,\phi_i\,;\,\psi_i)_i]\cup[(\Gamma,\sigma\,;\,\rho),(\Gamma,\tau\,;\,\rho)]}{[(\Gamma_i\,;\,\psi_i)_i]\cup[(\Gamma\,;\,\rho)]}\ (\cup\mathbf{E})$$

Figure 3.7: The logic IUL$_m$.

to IUT through a non-standard decoration of its derivations. Since IUT has been shown to correspond to MLns through decoration and erasure, we may restrict our study to the relation between IUL$_m$ and MLns, as was done in the case of IUL$_k$.

Presuming the notions of *atom* and *molecule* as given in 1.16, we can define IUL$_m$ as follows.

Definition 3.14 (IUL$_m$) *(i) Formulas are generated by the grammar* $\sigma ::= \alpha \mid \sigma\to\sigma \mid \sigma\cap\sigma \mid \sigma\cup\sigma$, *where* α *belongs to a countable set of atomic formulas.*

(ii) The logic IUL$_m$ *derives molecules* $[(\sigma_1^i,\ldots,\sigma_m^i\,;\,\tau_i) \mid 1\leqslant i\leqslant n] = [(\Gamma_i\,;\,\tau_i)_i]$ *by the rules displayed in Figure 3.7.*

A rule in IUL$_m$ can be derived from the corresponding rule in IUL$_k$ by using the following method for transforming a judgement in IUL$_k$ to a molecule in IUL$_m$. If $H_1,\ldots,H_m \vdash K$ is a judgement in IUL$_k$ and there are n terminal paths $p_1,\ldots,p_n$ in $H_1,\ldots,H_m,K$, then the corresponding molecule in IUL$_m$ is $[(H_1^{p_1},\ldots,H_m^{p_1}\,;\,K^{p_1}),\ldots,(H_1^{p_n},\ldots,H_m^{p_n}\,;\,K^{p_n})]$. In particular, each *terminal path* in the kits produces an *atom* in the molecule. This is illustrated by the following example of corresponding union elimination instances in the two logics.

$$\text{IUL}_k\text{:}\qquad \frac{[\alpha,\sigma_1],[\beta,\sigma_2]\vdash[\gamma,\sigma\cup\tau] \qquad [\alpha,[\sigma_1,\sigma_1]],[\beta,[\sigma_2,\sigma_2]],[\gamma,[\sigma,\tau]]\vdash[\delta,[\rho,\rho]]}{[\alpha,\sigma_1],[\beta,\sigma_2]\vdash[\delta,\rho]}\ (\cup\mathbf{E})_r$$

$$\text{IUL}_m\text{:}\qquad \frac{[(\alpha,\beta\,;\,\gamma),(\sigma_1,\sigma_2\,;\,\sigma\cup\tau)] \qquad [(\alpha,\beta,\gamma\,;\,\delta),(\sigma_1,\sigma_2,\sigma\,;\,\rho),(\sigma_1,\sigma_2,\tau\,;\,\rho)]}{[(\alpha,\beta\,;\,\delta),(\sigma_1,\sigma_2\,;\,\rho)]}\ (\cup\mathbf{E})$$

Using the notation "$\_\,[p := \_]$" of kits, though, the above IUL_k-instance is written as follows.

$$\frac{H_1[r:=\sigma_1],H_2[r:=\sigma_2]\vdash K[r:=\sigma\cup\tau] \qquad H_1[r:=[\sigma_1,\sigma_1]],H_2[r:=[\sigma_2,\sigma_2]],K[r:=[\sigma,\tau]]\vdash L[r:=[\rho,\rho]]}{H_1[r:=\sigma_1],H_2[r:=\sigma_2]\vdash L[r:=\rho]}\ (\cup\mathbf{E})_r$$

This kit-notation focuses on the path where union elimination is performed, which is path r in the specific example. So, the substitution operation (cut) that takes place at path l is ignored. On the other hand, this substitution is explicitly shown in the notation of molecules where each terminal path is "represented" by an atom. It is now more than obvious that union elimination cannot be considered local, at least not in the sense that local rules leave certain atoms completely unchanged.

As pointed out for $(\cup\mathbf{E})$ in IUL_k, $(\cup\mathbf{E})$ in IUL_m also aims to join together the isomorphic minor premises of $(\vee\mathbf{E})$ in MLns. This is achieved by placing them both in the same molecule, so that $(\cup\mathbf{E})$ has a single minor premise and a non-standard decoration in IUL_m always terminates.

$$\frac{\ldots\vdash t:\sigma\vee\tau \qquad \ldots,x:\sigma\vdash u:\rho \qquad \ldots,x:\tau\vdash u:\rho}{\ldots\vdash u[t/x]:\rho}\ (\vee\mathbf{E})$$

$$\frac{t:[\ldots,(\ldots\,;\,\sigma\cup\tau)] \qquad u:[\ldots,(\ldots,x:\sigma\,;\,\rho),(\ldots,x:\tau\,;\,\rho)]}{u[t/x]:[\ldots,(\ldots\,;\,\rho)]}\ (\cup\mathbf{E})$$

The non-standard decoration of IUL_m is dictated by the very rules of IUT, as was the case with the non-standard decoration of ML, and actually extends the non-standard decoration of ISL (see 1.17) to doubling and the union rules. It will be used in the theorems proving the equivalence of IUL_k and IUL_m (Theorems 3.18 and 3.21) and also in the theorem relating IUL_m to MLns (Theorem 3.22).

Definition 3.15 (Non-standard decoration of IUL_m) *Let* $\pi :: \mathcal{M} = [(\Gamma_i\,;\,\tau_i)_i] = [(\sigma^i_1,\ldots,\sigma^i_m\,;\,\tau_i)_i]$ *be a derivation in* IUL_m*. By decorating contexts bottom-up with distinct variables, starting with the sequence* $p = x_1,\ldots,x_m$*, and then decorating molecules top-down with terms in* Λ*, we get a decorated derivation* $\pi^\star :: t:\mathcal{M}_p = [(\Gamma_i\,;\,\tau_i)_i]_p = [(x_1:\sigma^i_1,\ldots,x_m:\sigma^i_m\,;\,\tau_i)_i]$*. The decoration rules are presented in Figure 3.8. When decorating contexts bottom-up, the new variable in an* $(\to\mathbf{I})$ *premise or in a* $(\cup\mathbf{E})$ *minor premise is fresh with respect to the variables in the branch connecting the conclusion to the root.*

$$\frac{}{x : [(\sigma_i \,;\, \sigma_i)_i]_{\,x}}\;\textbf{(ax)}$$

$$\frac{t : [(\Gamma_i \,;\, \tau_i)_i]_p}{t : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}\;\textbf{(W)} \qquad \frac{t : [(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]_{p,\,y,\,x,\,q}}{t : [(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]_{p,\,x,\,y,\,q}}\;\textbf{(X)}$$

$$\frac{t : \mathcal{M}_p \cup \mathcal{N}_p}{t : \mathcal{M}_p}\;\textbf{(P)} \qquad \frac{t : \mathcal{M}_p \cup [\mathcal{A}]_p}{t : \mathcal{M}_p \cup [\mathcal{A}, \mathcal{A}]_p}\;\textbf{(D)}$$

$$\frac{t : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}{\lambda x.t : [\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p}\;(\to\!\textbf{I}) \qquad \frac{t : [(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p \qquad u : [(\Gamma_i \,;\, \sigma_i)_i]_p}{tu : [(\Gamma_i \,;\, \tau_i)_i]_p}\;(\to\!\textbf{E})$$

$$\frac{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma), (\Gamma \,;\, \tau)]_p}{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma \cap \tau)]_p}\;(\cap\textbf{I})$$

$$\frac{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma \cap \tau)]_p}{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma)]_p}\;(\cap\textbf{E}_1) \qquad \frac{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma \cap \tau)]_p}{t : \mathcal{M}_p \cup [(\Gamma \,;\, \tau)]_p}\;(\cap\textbf{E}_2)$$

$$\frac{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma)]_p}{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma \cup \tau)]_p}\;(\cup\textbf{I}_1) \qquad \frac{t : \mathcal{M}_p \cup [(\Gamma \,;\, \tau)]_p}{t : \mathcal{M}_p \cup [(\Gamma \,;\, \sigma \cup \tau)]_p}\;(\cup\textbf{I}_2)$$

$$\frac{t : [(\Gamma_i \,;\, \phi_i)_i]_p \cup [(\Gamma \,;\, \sigma \cup \tau)]_p \qquad u : [(\Gamma_i, \phi_i \,;\, \psi_i)_i]_{p,\,x} \cup [(\Gamma, \sigma \,;\, \rho), (\Gamma, \tau \,;\, \rho)]_{p,\,x}}{u[t/x] : [(\Gamma_i \,;\, \psi_i)_i]_p \cup [(\Gamma \,;\, \rho)]_p}\;(\cup\textbf{E})$$

Figure 3.8: Non-standard decoration of IUL_m.

Remark 3.16 *Obviously, if* $\pi^\star :: t : \mathcal{M}_p$, *then* $FV(t) \subseteq \{p\}$.

As was the case with IUL_k, *every* derivation in IUL_m admits a decoration, since $(\cap\textbf{I})$ has a single premise and $(\cup\textbf{E})$ has a single minor premise.

Remark 3.17 *The logic* IUL_m *is formally defined as a quotient set of equivalence classes of derivations, in the manner of the formal definition of* IUL_k *(see 3.8). The equivalence relation is between derivations that disagree only in the order of consecutive local rules concerning different atoms. The commutations of the local rules* $\textbf{(P)}, \textbf{(D)}, (\cap\textbf{I}), (\cap\textbf{E}), (\cup\textbf{I})$ *follow the pattern in 3.8, only in the molecule setup. Derivations in the same equivalence class admit the same (non-standard) decoration.*

3.2.1 Equivalence of IUL_k and IUL_m

The logics IUL_k and IUL_m are equivalent. This is a desired result, since they were both designed to do the same job, namely to express MLns as an independent logic. We show a transformation of a decorated

IUL$_k$-derivation into an identically decorated IUL$_m$-derivation and conversely. In fact, the following theorem formalizes the method already described for converting a kit-judgement to a molecule.

Theorem 3.18 *Let* $\pi^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$ *be in* $\mathrm{IUL}_k^\star$ *and* $P_T(K) = \{p_1, \ldots, p_n\}$*. Then, there exists a* $(\pi')^\star :: t : [(H_1^{p_1}, \ldots, H_m^{p_1} \mathbin{;} K^{p_1}), \ldots, (H_1^{p_n}, \ldots, H_m^{p_n} \mathbin{;} K^{p_n})]_{x_1, \ldots, x_m}$ *in* $\mathrm{IUL}_m^\star$*.*

Proof. By induction on $\pi^\star$.

Base: If $\pi^\star :: x : K \vdash x : K$ is an $\mathrm{IUL}_k^\star$-axiom, then $(\pi')^\star :: x : [(K^{p_1} \mathbin{;} K^{p_1}), \ldots, (K^{p_n} \mathbin{;} K^{p_n})]_x$ is an $\mathrm{IUL}_m^\star$-axiom.

Induction step: We show three characteristic cases.

$$\triangleright \quad \frac{\pi_0^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K}{\pi^\star :: x_1 : H_1\backslash^{pl}, \ldots, x_m : H_m\backslash^{pl} \vdash t : K\backslash^{pl}}\ \textbf{(P)}$$

If[2] $P_T(K) = \{q_1, \ldots, q_\nu, plt_1, \ldots, plt_\mu, pru_1, \ldots, pru_\kappa\}$, then $P_T(K\backslash^{pl}) = \{q_1, \ldots, q_\nu, pt_1, \ldots, pt_\mu\}$. The following equalities hold.

1. $(H_j)^{q_i} = (H_j\backslash^{pl})^{q_i}$ and $K^{q_i} = (K\backslash^{pl})^{q_i}$, for $i \in \{1, \ldots, \nu\}$
2. $(H_j)^{plt_i} = (H_j\backslash^{pl})^{pt_i}$ and $K^{plt_i} = (K\backslash^{pl})^{pt_i}$, for $i \in \{1, \ldots, \mu\}$

By the IH, there exists a $(\pi_0')^\star :: t : (\mathcal{M} \cup \mathcal{N})_{x_1, \ldots, x_m}$ in IUL$_m$, where

$$\begin{aligned} \mathcal{M} &= [(H_1^{q_1}, \ldots, H_m^{q_1} \mathbin{;} K^{q_1}), \ldots, (H_1^{q_\nu}, \ldots, H_m^{q_\nu} \mathbin{;} K^{q_\nu}), \\ &\quad\ (H_1^{plt_1}, \ldots, H_m^{plt_1} \mathbin{;} K^{plt_1}), \ldots, (H_1^{plt_\mu}, \ldots, H_m^{plt_\mu} \mathbin{;} K^{plt_\mu})] \\ \mathcal{N} &= [(H_1^{pru_1}, \ldots, H_m^{pru_1} \mathbin{;} K^{pru_1}), \ldots, (H_1^{pru_\kappa}, \ldots, H_m^{pru_\kappa} \mathbin{;} K^{pru_\kappa})] \end{aligned}$$

Applying **(P)** to $(\pi_0')^\star$, we get a $(\pi')^\star :: t : \mathcal{M}_{x_1, \ldots, x_m}$, where 1 and 2 give

$$\begin{aligned} \mathcal{M} &= [((H_1\backslash^{pl})^{q_1}, \ldots, (H_m\backslash^{pl})^{q_1} \mathbin{;} (K\backslash^{pl})^{q_1}), \ldots, ((H_1\backslash^{pl})^{q_\nu}, \ldots, (H_m\backslash^{pl})^{q_\nu} \mathbin{;} (K\backslash^{pl})^{q_\nu}), \\ &\quad\ ((H_1\backslash^{pl})^{pt_1}, \ldots, (H_m\backslash^{pl})^{pt_1} \mathbin{;} (K\backslash^{pl})^{pt_1}), \ldots, ((H_1\backslash^{pl})^{pt_\mu}, \ldots, (H_m\backslash^{pl})^{pt_\mu} \mathbin{;} (K\backslash^{pl})^{pt_\mu})] \end{aligned}$$

$$\triangleright \quad \frac{\pi_0^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K}{\pi^\star :: x_1 : H_1\,{}^{q}/_{pl}, \ldots, x_m : H_m\,{}^{q}/_{pl} \vdash t : K^{q}/_{pl}}\ \textbf{(D)}$$

We consider two subcases.

i) $p \neq q$: If $P_T(K) = \{q, q_1, \ldots, q_\nu, pt_1, \ldots, pt_\mu\}$, then $P_T(K^{q}/_{pl}) = \{q, q_1, \ldots, q_\nu, pl, prt_1, \ldots, prt_\mu\}$. The following equalities hold.

1. $(H_j)^{q_i} = (H_j\,{}^{q}/_{pl})^{q_i}$ and $K^{q_i} = (K^{q}/_{pl})^{q_i}$, for $i \in \{1, \ldots, \nu\}$
2. $(H_j)^{pt_i} = (H_j\,{}^{q}/_{pl})^{prt_i}$ and $K^{pt_i} = (K^{q}/_{pl})^{prt_i}$, for $i \in \{1, \ldots, \mu\}$
3. $(H_j)^{q} = (H_j\,{}^{q}/_{pl})^{q} = (H_j\,{}^{q}/_{pl})^{pl}$ and $K^{q} = (K^{q}/_{pl})^{q} = (K^{q}/_{pl})^{pl}$

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[2] In this proof, we exceptionally use the letters t and u to denote paths, so as to avoid heavy notation caused by extra insignia on p or q.

By the IH, there exists a $(\pi_0')^\star :: \ t : (\mathcal{M} \cup [\mathcal{A}])_{x_1,\ldots,x_m}$ in IUL_m, where

$$\begin{array}{rcl}
\mathcal{M} & = & [(H_1^{q_1},\ldots,H_m^{q_1} \,;\, K^{q_1}),\ldots,(H_1^{q_\nu},\ldots,H_m^{q_\nu} \,;\, K^{q_\nu}),\\
 & & (H_1^{pt_1},\ldots,H_m^{pt_1} \,;\, K^{pt_1}),\ldots,(H_1^{pt_\mu},\ldots,H_m^{pt_\mu} \,;\, K^{pt_\mu})]\\
\mathcal{A} & = & (H_1^{q},\ldots,H_m^{q} \,;\, K^{q})
\end{array}$$

Applying **(D)** to $(\pi_0')^\star$, we get a $(\pi')^\star :: \ t : (\mathcal{M} \cup [\mathcal{A},\mathcal{A}])_{x_1,\ldots,x_m}$, where 1-3 give

$$\begin{array}{rcl}
\mathcal{M} & = & [((H_1\,{}^{q}/_{pl})^{q_1},\ldots,(H_m\,{}^{q}/_{pl})^{q_1} \,;\, (K^{q}/_{pl})^{q_1}),\ldots,((H_1\,{}^{q}/_{pl})^{q_\nu},\ldots,(H_m\,{}^{q}/_{pl})^{q_\nu} \,;\, (K^{q}/_{pl})^{q_\nu}),\\
 & & ((H_1\,{}^{q}/_{pl})^{prt_1},\ldots,(H_m\,{}^{q}/_{pl})^{prt_1} \,;\, (K^{q}/_{pl})^{prt_1}),\ldots,((H_1\,{}^{q}/_{pl})^{prt_\mu},\ldots,(H_m\,{}^{q}/_{pl})^{prt_\mu} \,;\, (K^{q}/_{pl})^{prt_\mu})]\\
\mathcal{A},\mathcal{A} & = & ((H_1\,{}^{q}/_{pl})^{q},\ldots,(H_m\,{}^{q}/_{pl})^{q} \,;\, (K^{q}/_{pl})^{q}),((H_1\,{}^{q}/_{pl})^{pl},\ldots,(H_m\,{}^{q}/_{pl})^{pl} \,;\, (K^{q}/_{pl})^{pl})
\end{array}$$

ii) $p \subseteq q$: Without loss of generality, we may assume that $P_T(K) = \{q_1,\ldots,q_\nu, q = pt_1,\ldots,pt_\mu\}$. Then, we have that $P_T(K^{q}/_{pl}) = \{q_1,\ldots,q_\nu, pl, prt_1,\ldots,prt_\mu\}$ and get the following equalities.

1. $(H_j)^{q_i} = (H_j\,{}^{q}/_{pl})^{q_i}$ and $K^{q_i} = (K^{q}/_{pl})^{q_i}$, for $i \in \{1,\ldots,\nu\}$
2. $(H_j)^{pt_i} = (H_j\,{}^{q}/_{pl})^{prt_i}$ and $K^{pt_i} = (K^{q}/_{pl})^{prt_i}$, for $i \in \{2,\ldots,\mu\}$
3. $(H_j)^{pt_1} = (H_j\,{}^{q}/_{pl})^{prt_1} = (H_j\,{}^{q}/_{pl})^{pl}$ and $K^{pt_1} = (K^{q}/_{pl})^{prt_1} = (K^{q}/_{pl})^{pl}$

By the IH, there exists a $(\pi_0')^\star :: \ t : (\mathcal{M} \cup [\mathcal{A}])_{x_1,\ldots,x_m}$ in IUL_m, where

$$\begin{array}{rcl}
\mathcal{M} & = & [(H_1^{q_1},\ldots,H_m^{q_1} \,;\, K^{q_1}),\ldots,(H_1^{q_\nu},\ldots,H_m^{q_\nu} \,;\, K^{q_\nu}),\\
 & & (H_1^{pt_2},\ldots,H_m^{pt_2} \,;\, K^{pt_2}),\ldots,(H_1^{pt_\mu},\ldots,H_m^{pt_\mu} \,;\, K^{pt_\mu})]\\
\mathcal{A} & = & (H_1^{pt_1},\ldots,H_m^{pt_1} \,;\, K^{pt_1})
\end{array}$$

Applying **(D)** to $(\pi_0')^\star$, we get a $(\pi')^\star :: \ t : (\mathcal{M} \cup [\mathcal{A},\mathcal{A}])_{x_1,\ldots,x_m}$, where 1-3 give

$$\begin{array}{rcl}
\mathcal{M} & = & [((H_1\,{}^{q}/_{pl})^{q_1},\ldots,(H_m\,{}^{q}/_{pl})^{q_1} \,;\, (K^{q}/_{pl})^{q_1}),\ldots,((H_1\,{}^{q}/_{pl})^{q_\nu},\ldots,(H_m\,{}^{q}/_{pl})^{q_\nu} \,;\, (K^{q}/_{pl})^{q_\nu}),\\
 & & ((H_1\,{}^{q}/_{pl})^{prt_2},\ldots,(H_m\,{}^{q}/_{pl})^{prt_2} \,;\, (K^{q}/_{pl})^{prt_2}),\ldots,((H_1\,{}^{q}/_{pl})^{prt_\mu},\ldots,(H_m\,{}^{q}/_{pl})^{prt_\mu} \,;\, (K^{q}/_{pl})^{prt_\mu})]\\
\mathcal{A},\mathcal{A} & = & ((H_1\,{}^{q}/_{pl})^{prt_1},\ldots,(H_m\,{}^{q}/_{pl})^{prt_1} \,;\, (K^{q}/_{pl})^{prt_1}),((H_1\,{}^{q}/_{pl})^{pl},\ldots,(H_m\,{}^{q}/_{pl})^{pl} \,;\, (K^{q}/_{pl})^{pl})
\end{array}$$

$$\triangleright \quad \frac{\pi_0^\star :: \ x_j : H_j \vdash t : K[p := \sigma \cup \tau] \qquad \pi_1^\star :: \ x_j : H_j\,{}^{p}/_{pl},\, x : K[p := [\sigma,\tau]] \vdash u : L[p := [\rho,\rho]]}{\pi^\star :: \ x_j : H_j \vdash u[t/x] : L[p := \rho]}\ (\cup\mathbf{E})$$

If $P_T(K[p := \sigma \cup \tau]) = P_T(L[p := \rho]) = \{q_1,\ldots,q_\nu,p\}$, then $P_T(L[p := [\rho,\rho]]) = \{q_1,\ldots,q_\nu,pl,pr\}$. The following equalities hold.

1. $(H_j)^{q_i} = (H_j\,{}^{p}/_{pl})^{q_i}$, $(K[p := \sigma \cup \tau])^{q_i} = (K[p := [\sigma,\tau]])^{q_i}$,
   and $(L[p := \rho])^{q_i} = (L[p := [\rho,\rho]])^{q_i}$, for $i \in \{1,\ldots,\nu\}$
2. $H_j^{p} = (H_j\,{}^{p}/_{pl})^{pl} = (H_j\,{}^{p}/_{pl})^{pr}$

By the IH, there is a $(\pi'_0)^\star :: t : (\mathcal{M} \cup [(H_1^p, \ldots, H_m^p ; \sigma \cup \tau)])_{x_1,\ldots,x_m}$, where

$$\mathcal{M} \quad = \quad [(H_1^{q_1}, \ldots, H_m^{q_1} ; K[p := \sigma \cup \tau]^{q_1}), \ldots, (H_1^{q_\nu}, \ldots, H_m^{q_\nu} ; K[p := \sigma \cup \tau]^{q_\nu})]$$

and also, using 2, a $(\pi'_1)^\star :: u : (\mathcal{N} \cup [(H_1^p, \ldots, H_m^p, \sigma ; \rho), (H_1^p, \ldots, H_m^p, \tau ; \rho)])_{x_1,\ldots,x_m,x}$, where

$$\begin{aligned}
\mathcal{N} \quad &= \quad [((H_1 {}^p/_{pl})^{q_1}, \ldots, (H_m {}^p/_{pl})^{q_1}, (K[p := [\sigma, \tau]])^{q_1} ; (L[p := [\rho, \rho]])^{q_1}), \ldots, \\
&\qquad ((H_1 {}^p/_{pl})^{q_\nu}, \ldots, (H_m {}^p/_{pl})^{q_\nu}, (K[p := [\sigma, \tau]])^{q_\nu} ; (L[p := [\rho, \rho]])^{q_\nu})] \\
&\overset{1}{=} \quad [(H_1^{q_1}, \ldots, H_m^{q_1}, K[p := \sigma \cup \tau]^{q_1} ; L[p := \rho]^{q_1}), \ldots, \\
&\qquad (H_1^{q_\nu}, \ldots, H_m^{q_\nu}, K[p := \sigma \cup \tau]^{q_\nu} ; L[p := \rho]^{q_\nu})]
\end{aligned}$$

Applying **(∪E)** to $(\pi'_0)^\star$ and $(\pi'_1)^\star$, we get a $(\pi')^\star :: u[t/x] : (\mathcal{M}' \cup [(H_1^p, \ldots, H_m^p ; \rho)])_{x_1,\ldots,x_m}$, where

$$\mathcal{M}' \quad = \quad [(H_1^{q_1}, \ldots, H_m^{q_1} ; L[p := \rho]^{q_1}), \ldots, (H_1^{q_\nu}, \ldots, H_m^{q_\nu} ; L[p := \rho]^{q_\nu})] \qquad \dashv$$

To transform a decorated IUL$_m$-derivation to an identically decorated IUL$_k$-derivation, we need the following proposition.

Proposition 3.19 *Let* $\mathcal{M} = [(\sigma_1^1, \ldots, \sigma_m^1 ; \tau_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n ; \tau_n)]$ *be a molecule of* $n \geqslant 1$ *atoms of context-cardinality* $m \geqslant 0$. *Then, there exists a sequence* $H_1, \ldots, H_m, K$ *of* $m+1$ *overlapping kits with* n *terminal paths* $p_1, \ldots, p_n$, *such that* $H_j^{p_i} = \sigma_j^i$ *and* $K^{p_i} = \tau_i$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$.

Proof. By induction on n. The index j runs from 1 to m.

Base: If $\mathcal{M} = [(\sigma_1, \ldots, \sigma_m ; \tau)]$, then the $m+1$ overlapping kits are the single-node kits $\sigma_1, \ldots, \sigma_m, \tau$ with one terminal path, namely the empty path ϵ. It is $\sigma_j^\epsilon = \sigma_j$ and $\tau^\epsilon = \tau$.

Induction step: Let $\mathcal{M} = [(\sigma_1^i, \ldots, \sigma_m^i ; \tau_i) \mid 1 \leqslant i \leqslant n] \cup [(\sigma_1^{n+1}, \ldots, \sigma_m^{n+1} ; \tau_{n+1})]$. By the IH, there is a sequence $H_1, \ldots, H_m, K$ of $m+1$ overlapping kits with n terminal paths $p_1, \ldots, p_n$, such that $H_j^{p_i} = \sigma_j^i$ and $K^{p_i} = \tau_i$. In addition, there is a sequence $\sigma_1^{n+1}, \ldots \sigma_m^{n+1}, \tau_{n+1}$ of $m+1$ single-node kits. We consider the sequence $[H_1, \sigma_1^{n+1}], \ldots, [H_m, \sigma_m^{n+1}], [K, \tau_{n+1}]$ of $m+1$ overlapping kits with $n+1$ terminal paths $q_1 = lp_1, \ldots,\ q_n = lp_n,\ q_{n+1} = r$. For $1 \leqslant i \leqslant n$, it is $[H_j, \sigma_j^{n+1}]^{q_i} = H_j^{p_i} = \sigma_j^i$ and $[K, \tau_{n+1}]^{q_i} = K^{p_i} = \tau_i$. Also, it is $[H_j, \sigma_j^{n+1}]^{q_{n+1}} = \sigma_j^{n+1}$ and $[K, \tau_{n+1}]^{q_{n+1}} = \tau_{n+1}$. ⊣

Definition 3.20 *The sequence* $H_1, \ldots, H_m, K$ *of overlapping kits in Proposition 3.19 will be called a* kit-representation *of* $\mathcal{M}$.

It is obvious that a kit-representation of a molecule $\mathcal{M}$ is not unique; different kit-representations of $\mathcal{M}$ may have different tree structures or the same tree structure, but different leaves in corresponding kits.

Theorem 3.21 *Let* $\pi^\star :: t : (\mathcal{M} = [(\sigma_1^i, \ldots, \sigma_m^i ; \tau_i) \mid 1 \leqslant i \leqslant n])_{x_1,\ldots,x_m}$ *be in* IUL$_m^\star$. *Then, for every kit-representation* $H_1, \ldots, H_m, K$ *of* $\mathcal{M}$, *there is a* $(\pi')^\star :: x_1 : H_1, \ldots,\ x_m : H_m \vdash t : K$ *in* IUL$_k^\star$.

Proof. By induction on $\pi^\star$.

Base: If $\pi^\star :: x : (\mathcal{M} = [(\sigma_i \,;\, \sigma_i) \mid 1 \leqslant i \leqslant n])_{\,x}$ is an $\mathrm{IUL}_m^\star$-axiom and H, K is a kit-representation of $\mathcal{M}$, then the kits H, K have n terminal paths $p_1, \ldots, p_n$ and, for $1 \leqslant i \leqslant n$, it is $H^{p_i} = \sigma_i$ and $K^{p_i} = \sigma_i$. Therefore, it is $H = K$ and there is an $\mathrm{IUL}_k^\star$-axiom $(\pi')^\star :: x : K \vdash x : K$.

Induction step: We display the most interesting cases, letting j run from 1 to m.

$$\rhd \quad \frac{\pi_0^\star :: t : (\mathcal{M} \cup \mathcal{N})_{\,x_1, \ldots, x_m}}{\pi^\star :: t : \mathcal{M}_{\,x_1, \ldots, x_m}}\ \mathbf{(P)}$$

where $\mathcal{M} = [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i) \mid 1 \leqslant i \leqslant n]$ and $\mathcal{N} = [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i) \mid n+1 \leqslant i \leqslant n+k]$.

If $H_1, \ldots, H_m, K$ is a kit-representation of $\mathcal{M}$ and $H_1', \ldots, H_m', K'$ is a kit-representation of $\mathcal{N}$, then $[H_1, H_1'], \ldots, [H_m, H_m'], [K, K']$ is a kit-representation of $\mathcal{M} \cup \mathcal{N}$. [Justification: The kits $H_1, \ldots, H_m, K$ have n terminal paths $p_1, \ldots, p_n$ and, for $1 \leqslant i \leqslant n$, it is $H_j^{p_i} = \sigma_j^i$ and $K^{p_i} = \tau_i$. The kits $H_1', \ldots, H_m', K'$ have k terminal paths $p_{n+1}, \ldots, p_{n+k}$ and, for $n+1 \leqslant i \leqslant n+k$, it is $(H_j')^{p_i} = \sigma_j^i$ and $(K')^{p_i} = \tau_i$. Therefore, the kits $[H_1, H_1'], \ldots, [H_m, H_m'], [K, K']$ have $n+k$ terminal paths $q_1 = lp_1, \ldots, q_n = lp_n$, $q_{n+1} = rp_{n+1}, \ldots, q_{n+k} = rp_{n+k}$. For $1 \leqslant i \leqslant n$, it is $[H_j, H_j']^{q_i} = H_j^{p_i} = \sigma_j^i$ and $[K, K']^{q_i} = K^{p_i} = \tau_i$, while, for $n+1 \leqslant i \leqslant n+k$, it is $[H_j, H_j']^{q_i} = (H_j')^{p_i} = \sigma_j^i$ and $[K, K']^{q_i} = (K')^{p_i} = \tau_i$.] Hence, the IH gives a $(\pi_0')^\star :: x_1 : [H_1, H_1'], \ldots, x_m : [H_m, H_m'] \vdash t : [K, K']$ in $\mathrm{IUL}_k^\star$. Applying $\mathbf{(P)}_l$ to $(\pi_0')^\star$, we get a $(\pi')^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$ in $\mathrm{IUL}_k^\star$.

$$\rhd \quad \frac{\pi_0^\star :: t : (\mathcal{M} \cup [\mathcal{A}])_{\,x_1, \ldots, x_m}}{\pi^\star :: t : (\mathcal{M} \cup [\mathcal{A}, \mathcal{A}])_{\,x_1, \ldots, x_m}}\ \mathbf{(D)}$$

where $\mathcal{M} = [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i) \mid 1 \leqslant i \leqslant n]$ and $\mathcal{A} = (\sigma_1^{n+1}, \ldots, \sigma_m^{n+1} \,;\, \tau_{n+1})$.

If $H_1, \ldots, H_m, K$ is a kit-representation of $\mathcal{M} \cup [\mathcal{A}, \mathcal{A}]$, the kits $H_1, \ldots, H_m, K$ have $n+2$ terminal paths $p_1, \ldots, p_n, p_{n+1}, p_{n+2}$ and, for $1 \leqslant i \leqslant n$, it is $H_j^{p_i} = \sigma_j^i$ and $K^{p_i} = \tau_i$, while $H_j^{p_{n+1}} = H_j^{p_{n+2}} = \sigma_j^{n+1}$ and $K^{p_{n+1}} = K^{p_{n+2}} = \tau_{n+1}$. We may prune all kits in $H_1, \ldots, H_m, K$ at such a path, so as to get a sequence $H_1', \ldots, H_m', K'$ of overlapping kits that have $n+1$ terminal paths $q_1, \ldots, q_n, q_{n+1}$ and, for $1 \leqslant i \leqslant n$, it is $(H_j')^{q_i} = \sigma_j^i$ and $(K')^{q_i} = \tau_i$, while $(H_j')^{q_{n+1}} = \sigma_j^{n+1}$ and $(K')^{q_{n+1}} = \tau_{n+1}$. The sequence $H_1', \ldots, H_m', K'$ is a kit-representation of $\mathcal{M} \cup [\mathcal{A}]$, so the IH gives a $(\pi_0')^\star :: x_1 : H_1', \ldots, x_m : H_m' \vdash t : K'$ in $\mathrm{IUL}_k^\star$. Applying an appropriate (i.e. left or right) doubling at an appropriate path to $(\pi_0')^\star$, so as to iterate the leaf at the end of q_{n+1}, we get a $(\pi')^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K$ in $\mathrm{IUL}_k^\star$.

$$\rhd \quad \frac{\pi_0^\star :: t : (\mathcal{M}_0)_{\,x_1, \ldots, x_m} \qquad \pi_1^\star :: u : (\mathcal{M}_1)_{\,x_1, \ldots, x_m, x}}{\pi^\star :: u[t/x] : \mathcal{M}_{\,x_1, \ldots, x_m}}\ (\cup\mathbf{E})$$

where $\mathcal{M}_0 = [(\chi_1^i, \ldots, \chi_m^i \,;\, \phi_i) \mid 1 \leqslant i \leqslant n] \cup [(\upsilon_1, \ldots, \upsilon_m \,;\, \sigma \cup \tau)]$,
$\mathcal{M}_1 = [(\chi_1^i, \ldots, \chi_m^i, \phi_i \,;\, \psi_i) \mid 1 \leqslant i \leqslant n] \cup [(\upsilon_1, \ldots, \upsilon_m, \sigma \,;\, \rho), (\upsilon_1, \ldots, \upsilon_m, \tau \,;\, \rho)]$, and
$\mathcal{M} = [(\chi_1^i, \ldots, \chi_m^i \,;\, \psi_i) \mid 1 \leqslant i \leqslant n] \cup [(\upsilon_1, \ldots, \upsilon_m \,;\, \rho)]$.

If $H_1, \ldots, H_m, L$ is a kit-representation of $\mathcal{M}$, the kits $H_1, \ldots, H_m, L$ have $n+1$ terminal paths $p_1, \ldots, p_n, q$ and, for $1 \leqslant i \leqslant n$, it is $H_j^{p_i} = \chi_j^i$ and $L^{p_i} = \psi_i$, while $H_j^q = \upsilon_j$ and $L^q = \rho$. Then, the sequence $H_1, \ldots, H_m, K[q := \sigma \cup \tau]$, where $K = L[p_i := \phi_i]$, is a kit-representation of $\mathcal{M}_0$ and the sequence $H_1\, {}^{q}\!/_{ql}, \ldots, H_m\, {}^{q}\!/_{ql}, K[q := [\sigma, \tau]], L[q := [\rho, \rho]]$ is a kit-representation of $\mathcal{M}_1$. The IH yields a $(\pi_0')^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash t : K[q := \sigma \cup \tau]$ in $\mathrm{IUL}_k^\star$ and also a

$$(\pi_1')^\star :: x_1 : H_1\, {}^{q}\!/_{ql}, \ldots, x_m : H_m\, {}^{q}\!/_{ql}, x : K[q := [\sigma, \tau]] \vdash u : L[q := [\rho, \rho]]$$

in $\mathrm{IUL}_k^\star$. By $(\cup\mathbf{E})_q$, we then obtain a $(\pi')^\star :: x_1 : H_1, \ldots, x_m : H_m \vdash u[t/x] : L[q := \rho] = L$ in $\mathrm{IUL}_k^\star$. $\dashv$

As already noted in describing the method to attain a molecule from a kit-judgement, Theorem 3.18 indicates that each terminal path of the kits in the conclusion of π gives rise to an atom in the molecule proved by π'. Conversely, Proposition 3.19 indicates that all formulas in an atom of $\mathcal{M}$ are leaves at the same terminal path in a kit-representation of $\mathcal{M}$. Therefore, terminal paths in IUL_k correspond to atoms in IUL_m. In addition, it is easy to see, in both theorems 3.18 and 3.21, that the context-cardinality of the judgement proved by an IUL_k-derivation coincides with the atom context-cardinality of the molecule proved by its corresponding IUL_m-derivation.

3.2.2 Relating IUL_m to MLns

We can restate Theorem 3.10 in the molecule framework and prove it via the equivalence of IUL_k and IUL_m.

Theorem 3.22 (From IUL_m to MLns) *Let $\pi :: [(\sigma_1^i, \ldots, \sigma_m^i\,;\, \tau_i) | 1 \leqslant i \leqslant n]$ be a derivation in* IUL_m*, such that $\pi^\star :: t : [(\sigma_1^i, \ldots, \sigma_m^i\,;\, \tau_i) \mid 1 \leqslant i \leqslant n]_{x_1,\ldots,x_m}$. For every $i \in \{1, \ldots, n\}$, there is a derivation $\pi^i :: \sigma_1^i, \ldots, \sigma_m^i \vdash \tau_i$ in* MLns*, such that $(\pi^i)^\star :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$.*

Proof. Either by induction on π or by theorems 3.21 and 3.10. $\dashv$

Example 3.23 *The* IUL_k*-derivation $\pi :: [\sigma, \tau] \vdash [(\alpha \to \theta) \to \theta, \rho \to \eta]$ given in Example 3.12, where $\sigma = \alpha \cap \beta$, $\tau = \gamma \cap \delta$, and $\rho = (\delta \to \eta) \cap (\zeta \to \eta)$, corresponds to the* IUL_m*-derivation*

$$\tilde{\pi} :: [(\sigma\,;\, (\alpha \to \theta) \to \theta), (\tau\,;\, \rho \to \eta)]$$

according to Theorem 3.18. We denote $\Gamma = \sigma, \alpha \to \theta$ and $\Delta = \tau, \rho$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\sigma\,;\,\sigma),(\tau\,;\,\tau)]}{[(\Gamma\,;\,\sigma),(\Delta\,;\,\tau)]}\,\textbf{(W)}}{[(\Gamma\,;\,\alpha),(\Delta\,;\,\delta)]}\,(\cap\textbf{E})}{[(\Gamma\,;\,\alpha),(\Delta\,;\,\delta\cup\zeta)]}\,(\cup\textbf{I}) \qquad \dfrac{\dfrac{\dfrac{[(\alpha\to\theta\,;\,\alpha\to\theta),(\rho\,;\,\rho),(\rho\,;\,\rho)]}{[(\Gamma,\alpha\,;\,\alpha\to\theta),(\Delta,\delta\,;\,\rho),(\Delta,\zeta\,;\,\rho)]}\,\textbf{(WX)}}{[(\Gamma,\alpha\,;\,\alpha\to\theta),(\Delta,\delta\,;\,\delta\to\eta),(\Delta,\zeta\,;\,\zeta\to\eta)]}\,(\cap\textbf{E}) \quad \dfrac{[(\alpha\,;\,\alpha),(\delta\,;\,\delta),(\zeta\,;\,\zeta)]}{[(\Gamma,\alpha\,;\,\alpha),(\Delta,\delta\,;\,\delta),(\Delta,\zeta\,;\,\zeta)]}\,\textbf{(WX)}}{[(\Gamma,\alpha\,;\,\theta),(\Delta,\delta\,;\,\eta),(\Delta,\zeta\,;\,\eta)]}\,(\to\textbf{E})}{[(\Gamma\,;\theta),(\Delta\,;\,\eta)]}\,(\cup\textbf{E})}{\tilde{\pi} :: [(\sigma\,;\,(\alpha\to\theta)\to\theta),(\tau\,;\,\rho\to\eta)]}\,(\to\textbf{I})$$

The decoration of $\tilde{\pi}$ is identical to that of π, i.e. it is $(\tilde{\pi})^\star :: \lambda y.\, yx : [(\sigma\,;\, (\alpha \to \theta) \to \theta), (\tau\,;\, \rho \to \eta)]_x$.

This $\tilde{\pi}$ gives two derivations in MLns*, namely $\tilde{\pi}^1 = \pi^l :: \sigma \vdash (\alpha \to \theta) \to \theta$ and $\tilde{\pi}^2 = \pi^r :: \tau \vdash \rho \to \eta$. The substitution operation carried out to generate $\tilde{\pi}^1$ is now in full accordance to the substitution (cut) performed on the atoms $(\Gamma; \alpha)$ and $(\Gamma, \alpha; \theta)$ in the premises of* $(\cup\mathbf{E})$.

From MLns to IUL_m?

The problem of the decomposition of substitution, discussed in the subsection "From MLns to IUL_k?", is also met in the attempt to prove the inverse of Theorem 3.22.

If $\pi^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \rho$ is in $\mathrm{MLns}^\star$, we would like to show, modulo the conversion of connectives, that there exists a $(\pi')^\star :: t : [(\sigma_1, \ldots, \sigma_m\,;\, \rho)]_{x_1,\ldots,x_m}$ in $\mathrm{IUL}_m^\star$. An induction on $\pi^\star$, though, would hit a problem in the $(\wedge\mathbf{I})$ case and also in the $(\vee\mathbf{E})$ case.

$$\dfrac{\overset{\pi_0^\star}{x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \sigma \vee \tau} \qquad \overset{\pi_1^\star}{x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \sigma \vdash u : \rho} \qquad \overset{\pi_2^\star}{x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \tau \vdash u : \rho}}{\pi^\star :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash u[t/x] : \rho}\ (\vee\mathbf{E})$$

The IH would give derivations $(\pi_0')^\star, (\pi_1')^\star$, and $(\pi_2')^\star$ in $\mathrm{IUL}_m^\star$, as shown below.

$$\begin{aligned}
(\pi_0')^\star &:: \quad t : [(\sigma_1, \ldots, \sigma_m \,;\, \sigma \cup \tau)]_{x_1, \ldots, x_m}\\
(\pi_1')^\star &:: \quad u : [(\sigma_1, \ldots, \sigma_m, \sigma \,;\, \rho)]_{x_1, \ldots, x_m, x}\\
(\pi_2')^\star &:: \quad u : [(\sigma_1, \ldots, \sigma_m, \tau \,;\, \rho)]_{x_1, \ldots, x_m, x}
\end{aligned}$$

We would like to be able to merge the identically decorated π_1' and π_2' into a single

$$(\pi_{12}')^\star :: \quad u : [(\sigma_1, \ldots, \sigma_m, \sigma \,;\, \rho), (\sigma_1, \ldots, \sigma_m, \tau \,;\, \rho)]_{x_1, \ldots, x_m, x}$$

so that applying **($\cup$E)** to $(\pi_0')^\star$ and $(\pi_{12}')^\star$ would give a $(\pi')^\star :: u[t/x] : [(\sigma_1, \ldots, \sigma_m \,;\, \rho)]_{x_1, \ldots, x_m}$. The claim that two identically decorated derivations can be unified to give a single derivation with this very decoration is rephrased in the molecule setup as follows.

> **Claim:** *Two identically decorated* IUL_m*-derivations* $\pi_0^\star :: t : [(\Gamma_i \,;\, \tau_i) \mid 1 \leqslant i \leqslant n]_{x_1, \ldots, x_m}$ *and* $\pi_1^\star :: t : [(\Gamma_i \,;\, \tau_i) \mid n+1 \leqslant i \leqslant k]_{x_1, \ldots, x_m}$ *can be combined into a single* IUL_m*-derivation* $\pi^\star :: t : [(\Gamma_i \,;\, \tau_i) \mid 1 \leqslant i \leqslant k]_{x_1, \ldots, x_m}$ *with this very decoration.*

However, as in the case of IUL_k, there is no natural way to join together two derivations whose decorating term derives from two different kinds of substitution (see Example 3.13).

3.3 Discussion of kits and molecules

As already explained, the use of molecules reveals the globality inherent in union elimination. Terminal paths in IUL_k correspond to atoms in IUL_m and actually an atom in a molecule is constructed by abstracting a specific terminal path from a kit-judgement. Thus, the union elimination rule in IUL_m brings to light the "action" at every terminal path in the corresponding rule in IUL_k. This is made clear in the following corresponding instances of the rule in IUL_k and IUL_m.

$$\dfrac{\underbrace{H_j[p := \sigma_j] \vdash K[p := \sigma \cup \tau]}_{\text{terminal paths } q_1, \ldots, q_n, p} \qquad \underbrace{H_j[p := [\sigma_j, \sigma_j]], K[p := [\sigma, \tau]] \vdash L[p := [\rho, \rho]]}_{\text{terminal paths } q_1, \ldots, q_n, pl, pr}}{\underbrace{H_j[p := \sigma_j] \vdash L[p := \rho]}_{\text{terminal paths } q_1, \ldots, q_n, p}}\ (\cup\mathbf{E})_p$$

$$\dfrac{[\underbrace{(\gamma_j^i \,;\, \phi_i) \mid 1 \leqslant i \leqslant n}_{\text{atoms } \mathcal{B}_1^0, \ldots, \mathcal{B}_n^0}] \cup [\underbrace{(\sigma_j \,;\, \sigma \cup \tau)}_{\text{atom } \mathcal{A}^0}] \qquad [\underbrace{(\gamma_j^i, \phi_i \,;\, \psi_i) \mid 1 \leqslant i \leqslant n}_{\text{atoms } \mathcal{B}_1^1, \ldots, \mathcal{B}_n^1}] \cup [\underbrace{(\sigma_j, \sigma \,;\, \rho), (\sigma_j, \tau \,;\, \rho)}_{\text{atoms } \mathcal{A}_1^1, \mathcal{A}_2^1}]}{[\underbrace{(\gamma_j^i \,;\, \psi_i) \mid 1 \leqslant i \leqslant n}_{\text{atoms } \mathcal{B}_1, \ldots, \mathcal{B}_n}] \cup [\underbrace{(\sigma_j \,;\, \rho)}_{\text{atom } \mathcal{A}}]}\ (\cup\mathbf{E})$$

In the major premises, leaves at terminal paths $q_1, \ldots, q_n$ translate to atoms $\mathcal{B}_1^0, \ldots, \mathcal{B}_n^0$, respectively, while leaves at p translate to $\mathcal{A}^0$; similar correspondences hold for the minor premises and the conclusions. The action at paths $q_1, \ldots, q_n$, which is hidden in the IUL$_k$-instance, is brought to light in the IUL$_m$-instance. The latter works locally on atoms $\mathcal{A}^0, \mathcal{A}_1^1, \mathcal{A}_2^1$, where it performs a proper union elimination to render atom $\mathcal{A}$, and globally on the $\mathcal{B}_i^0$'s and $\mathcal{B}_i^1$'s, where it performs substitutions (cuts) on corresponding atoms to provide the $\mathcal{B}_i$'s. Therefore, union elimination in IUL$_m$ displays both local and global characteristics.

In fact, union elimination in IUL$_k$ enjoys both characteristics, as well. The rule can be rewritten as follows, if we aim to unfold what happens at a path q_i besides p.

$$\frac{H_j[p := \sigma_j][q_i := \gamma_j^i] \vdash K[p := \sigma \cup \tau][q_i := \phi_i] \qquad H_j[p := [\sigma_j, \sigma_j]][q_i := \gamma_j^i], K[p := [\sigma, \tau]][q_i := \phi_i] \vdash L[p := [\rho, \rho]][q_i := \psi_i]}{H_j[p := \sigma_j][q_i := \gamma_j^i] \vdash L[p := \rho][q_i := \psi_i]}$$

The substitution carried out at q_i is now designated by the rule. This hidden aspect of union elimination in IUL$_k$ has been actually demonstrated in the proof of Theorem 3.10 (case ($\cup$**E**), subcase 1) and also in Example 3.12, where a substitution operation was required for the formation of π^l.

The benefit of unveiling locality and globality issues is only one aspect of the more general benefit of adopting a notation for Intersection and Union Logic that is simpler and easier to handle. The formalism of kits, which seeks to recreate the geometric structures of trees, can be awkward and vague, as it has so far been verified. On the other hand, the formalism of molecules, which has arisen from the flattening of kits by converting (leaves at) terminal paths to atoms, is more clean-cut and explicit.

A different formalism for a logic corresponding to intersection (and union) types is that of *hyperformulas*, proposed in [6]. Hyperformulas also linearize the kit-structure, as molecules do, but are nonetheless harder to manipulate than molecules. Very roughly speaking, the syntax of hyperformulas is easier than that of kits, but more complicated than that of molecules. Consequently, hyperformulas also encounter the problem that molecules (and kits) encounter in corresponding with MLns. We have focused on the comparison of kits with molecules, leaving hyperformulas aside, so as to better indicate the advantages of molecules, which bear the most concise formalism among the three.

CHAPTER 4

Natural Deduction $\mathbf{IUL}_m$ and $\mathbf{IUT}^{\oplus}$

We present a new version of the logic IUL_m in natural deduction style. This new version involves a modification of the definition of "molecule", as well as modifications of rules. In particular, a molecule is no longer a *multiset* of atoms, but a *sequence* of atoms, while the rules of the system undergo the following changes: (i) the axiom is allowed to contain enriched atom-contexts, (ii) the structural rules of weakening, pruning, and doubling are eliminated, but are still valid as derivable rules, (iii) the local rules of intersection (introduction and elimination) and union introduction are allowed to act on several atoms (or sequences of atoms) of a molecule in one step, and (iv) the union elimination rule is modified to an explicitly global version. We also present the type system $\mathrm{IUT}^{\oplus}$ in natural deduction style. This system is actually the natural deduction type system IUT_{ω} of Chapter 2 without the $(\boldsymbol{\omega})$-rule. The "$\oplus$" sign emphasizes its additive character. We finally interrelate the new natural deduction logic with the natural deduction type system to show how the former attempts to capture the latter on a logical level.

The changes that the new version of the logic bears, with respect to the version presented in the previous chapter, can be briefly justified as follows. Change (i) allows the derivability of weakening (observe the base case in the inductive proof of Proposition 4.5), while change (ii) provides a more economical, elegant, and handy system. Change (iii) serves the derivability of doubling (see footnote 6 in case 1 of $(\cap\mathbf{I})$ in the inductive proof of Proposition 4.11(ii)), while change (iv) provides a system with an explicit categorization of rules as global or local, which lies at the core of the method that will be used in the next chapter to show correspondence theorems between the logic and the type system (see Section 5.4 for a detailed justification of this method).

4.1 The logic $\mathbf{IUL}_m$ in natural deduction

We redefine the natural deduction logic IUL_m, first introduced in Chapter 3, as follows.

Definition 4.1 ($\mathbf{IUL}_m$) *(i)* Formulas *are generated by the grammar* $\sigma ::= \alpha \,|\, \sigma \to \sigma \,|\, \sigma \cap \sigma \,|\, \sigma \cup \sigma$, *where* α *belongs to a countable set of atomic formulas. An* atom *is a pair* $(\Gamma\,;\,\sigma)$, *where the* context Γ *is a finite sequence of formulas.*

(ii) Molecules *are finite* sequences *of atoms, such that all atoms share the same context cardinality. A molecule* $\mathcal{M} = [(\Gamma_1\,;\,\sigma_1), \ldots, (\Gamma_n\,;\,\sigma_n)]$ *is also denoted* $[(\Gamma_i\,;\,\sigma_i)_{i=1}^{n}]$ *or* $[(\Gamma_i\,;\,\sigma_i)_{1}^{n}]$ *or just* $[(\Gamma_i\,;\,\sigma_i)_i]$. *Sequences of atoms which are subsequences of molecules are denoted by* $\mathcal{U}, \mathcal{V}$.

(iii) The logical system IUL_m *proves molecules in natural deduction style by the rules displayed in Figure 4.1. The index i in molecules runs from 1 to n.*

$$\frac{}{[(\Gamma_i,\sigma_i\,;\,\sigma_i)_i]}\;(\mathbf{ax}) \qquad \frac{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i)_i]}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\rho_i)_i]}\;(\mathbf{X})$$

$$\frac{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}{[(\Gamma_i\,;\,\sigma_i\rightarrow\tau_i)_i]}\;(\rightarrow\!\mathbf{I}) \qquad \frac{[(\Gamma_i\,;\,\sigma_i\rightarrow\tau_i)_i] \qquad [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\;(\rightarrow\!\mathbf{E})$$

$$\frac{[\mathcal{U},((\Gamma_i\,;\,\sigma_i),(\Gamma_i\,;\,\tau_i))_i,\mathcal{V}]}{[\mathcal{U},(\Gamma_i\,;\,\sigma_i\cap\tau_i)_i,\mathcal{V}]}\;(\cap\mathbf{I}) \qquad \frac{[\mathcal{U},(\Gamma_i\,;\,\sigma_i\cap\tau_i)_i,\mathcal{V}]}{[\mathcal{U},(\Gamma_i\,;\,\sigma_i)_i,\mathcal{V}]}\;(\cap\mathbf{E}_1) \qquad \frac{[\mathcal{U},(\Gamma_i\,;\,\sigma_i\cap\tau_i)_i,\mathcal{V}]}{[\mathcal{U},(\Gamma_i\,;\,\tau_i)_i,\mathcal{V}]}\;(\cap\mathbf{E}_2)$$

$$\frac{[\mathcal{U},(\Gamma_i\,;\,\sigma_i)_i,\mathcal{V}]}{[\mathcal{U},(\Gamma_i\,;\,\sigma_i\cup\tau_i)_i,\mathcal{V}]}\;(\cup\mathbf{I}_1) \qquad \frac{[\mathcal{U},(\Gamma_i\,;\,\tau_i)_i,\mathcal{V}]}{[\mathcal{U},(\Gamma_i\,;\,\sigma_i\cup\tau_i)_i,\mathcal{V}]}\;(\cup\mathbf{I}_2)$$

$$\frac{[(\Gamma_i\,;\,\sigma_i\cup\tau_i)_i] \qquad [((\Gamma_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\tau_i\,;\,\rho_i))_i]}{[(\Gamma_i\,;\,\rho_i)_i]}\;(\cup\mathbf{E})$$

Figure 4.1: The logic IUL_m in natural deduction style.

Remark 4.2 *(i) In the exchange rule* (**X**)*, the* Γ_i*'s have the same cardinality.*

(ii) The intersection (introduction and elimination) and union introduction rules presented in Figure 4.1 are, in fact, special versions of the actual intersection (introduction and elimination) and union introduction rules; this is done for simplicity and space economy. The actual ($\cap$**I**) *rule is meant as shown below.*

$$\frac{[\mathcal{U}_1,(\Gamma_1\,;\,\sigma_1),(\Gamma_1\,;\,\tau_1),\mathcal{U}_2,(\Gamma_2\,;\,\sigma_2),(\Gamma_2\,;\,\tau_2),\ldots,\mathcal{U}_n,(\Gamma_n\,;\,\sigma_n),(\Gamma_n\,;\,\tau_n),\mathcal{U}_{n+1}]}{[\mathcal{U}_1,(\Gamma_1\,;\,\sigma_1\cap\tau_1),\mathcal{U}_2,(\Gamma_2\,;\,\sigma_2\cap\tau_2),\ldots,\mathcal{U}_n,(\Gamma_n\,;\,\sigma_n\cap\tau_n),\mathcal{U}_{n+1}]}\;(\cap\mathbf{I})$$

The actual ($\cap$**E**$_1$)*,*($\cap$**E**$_2$)*,*($\cup$**I**$_1$)*, and* ($\cup$**I**$_2$) *rules can be figured from their special cases in a similar manner.*

The categorization of rules as *global* or *local* is according to whether they affect *all* or *some* atoms in premise level, respectively. The exchange rule, the implication rules, and the union elimination rule are global, while the intersection rules and the union introduction rules are local[1]. Unlike the case of IUL_m as presented in Chapter 3, where union elimination assembled both global and local characteristics, the classification of rules as global or local is here very clear and definite.

The connectives of the grammar are all *additive*. This is done by *necessity* in the cases of intersection introduction and union introduction. The claim that atoms in the same molecule should have the same context cardinality forbids a multiplicative presentation of the intersection introduction rule; a multiplicative premise $[(E_i\,;\,\phi_i)_1^k,((\Gamma_i\,;\,\sigma_i),(\Delta_i\,;\,\tau_i))_1^n]$ with $|E_i| = |\Gamma_i| = |\Delta_i| = m$ would give a conclusion $[(E_i\,;\,\phi_i)_1^k,(\Gamma_i,\Delta_i\,;\,\sigma_i\cap\tau_i)_1^n]$ with $|E_i| = m$, but $|\Gamma_i,\Delta_i| = 2m$. Moreover, the intuitionistic claim that atoms should contain exactly one formula to the right of ";" forbids a multiplicative presentation of

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[1] Local rules become global to the limit where $\mathcal{U}$ and $\mathcal{V}$ are empty.

the union introduction rule; a multiplicative premise $[\mathcal{U}, (\Gamma_i\,;\,\sigma_i, \tau_i)_i, \mathcal{V}]$ would no longer belong to an intuitionistic system. On the other hand, the additive style is picked by *choice* in the cases of implication elimination and union elimination. Indeed, the implication elimination rule can also be presented in a multiplicative manner, that is with premises $[(\Gamma_i\,;\,\sigma_i \to \tau_i)_i], [(\Delta_i\,;\,\sigma_i)_i]$ and conclusion $[(\Gamma_i, \Delta_i\,;\,\tau_i)_i]$. As far as the union elimination rule is concerned, the choice of additive style refers to both i) the right-premise "twin" atoms $(\Gamma_i, \sigma_i\,;\,\rho_i)$ and $(\Gamma_i, \tau_i\,;\,\rho_i)$ and ii) the left-premise atom $(\Gamma_i\,;\,\sigma_i \cup \tau_i)$ and its corresponding right-premise twin atoms $(\Gamma_i, \sigma_i\,;\,\rho_i), (\Gamma_i, \tau_i\,;\,\rho_i)$. Abolishing the additiveness with respect to (ii) still yields an acceptable union elimination rule with a mixed multiplicative-additive character (see $(\cup\mathbf{E})^2$ below), while further abolishing the additiveness with respect to (i) also provides an acceptable union elimination rule with a purely multiplicative character (see $(\cup\mathbf{E})^3$ below).

$$\frac{[(\Gamma_i\,;\,\sigma_i \cup \tau_i)_i] \qquad [((\Delta_i, \sigma_i\,;\,\rho_i), (\Delta_i, \tau_i\,;\,\rho_i))_i]}{[(\Gamma_i, \Delta_i\,;\,\rho_i)_i]}\,(\cup\mathbf{E})^2 \qquad \frac{[(\Gamma_i\,;\,\sigma_i \cup \tau_i)_i] \qquad [((\Delta_i, \sigma_i\,;\,\rho_i), (E_i, \tau_i\,;\,\rho_i))_i]}{[(\Gamma_i, \Delta_i, E_i\,;\,\rho_i)_i]}\,(\cup\mathbf{E})^3$$

In an IUL_m-derivation, an exchange inference can be moved upward above all the inferences of logical rules[2], so that only an axiom and possibly some other exchange inferences may appear above it. This is formalized by the next definition and proposition.

Definition 4.3 (Canonical derivation) *An* IUL_m*-derivation* π *is* canonical[3]*, if every exchange inference in* π *appears just below an axiom or another exchange inference.*

The definition implies that, roughly speaking, a branch in the tree of a canonical derivation consists of an axiom, which is followed by a (possibly empty) sequence of exchange inferences, which is, in turn, followed by a (possibly empty) sequence of inferences of logical rules.

Proposition 4.4 *For every* $\pi :: \mathcal{M}$*, there is a canonical* $\pi' :: \mathcal{M}$*.*

Proof. This is formally proved by induction on π. In practice, it suffices to show that the exchange rule commutes with any logical rule. We show two characteristic cases.

$\triangleright$ A local logical rule: $(\cap\mathbf{I})$

$$\dfrac{\dfrac{[(E_i, \phi_i, \psi_i, Z_i\,;\,\chi_i)_1^k, ((\Gamma_i, \sigma_i, \tau_i, \Delta_i\,;\,\rho_i), (\Gamma_i, \sigma_i, \tau_i, \Delta_i\,;\,\upsilon_i))_1^n]}{[(E_i, \phi_i, \psi_i, Z_i\,;\,\chi_i)_1^k, (\Gamma_i, \sigma_i, \tau_i, \Delta_i\,;\,\rho_i \cap \upsilon_i)_1^n]}\,(\cap\mathbf{I})}{[(E_i, \psi_i, \phi_i, Z_i\,;\,\chi_i)_1^k, (\Gamma_i, \tau_i, \sigma_i, \Delta_i\,;\,\rho_i \cap \upsilon_i)_1^n]}\,(\mathbf{X}) \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{[(E_i, \phi_i, \psi_i, Z_i\,;\,\chi_i)_1^k, ((\Gamma_i, \sigma_i, \tau_i, \Delta_i\,;\,\rho_i), (\Gamma_i, \sigma_i, \tau_i, \Delta_i\,;\,\upsilon_i))_1^n]}{[(E_i, \psi_i, \phi_i, Z_i\,;\,\chi_i)_1^k, ((\Gamma_i, \tau_i, \sigma_i, \Delta_i\,;\,\rho_i), (\Gamma_i, \tau_i, \sigma_i, \Delta_i\,;\,\upsilon_i))_1^n]}\,(\mathbf{X})}{[(E_i, \psi_i, \phi_i, Z_i\,;\,\chi_i)_1^k, (\Gamma_i, \tau_i, \sigma_i, \Delta_i\,;\,\rho_i \cap \upsilon_i)_1^n]}\,(\cap\mathbf{I})$$

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[2] A *logical rule* is a rule introducing or eliminating a logical connective.

[3] The term "canonical" is borrowed from [15].

$\triangleright$ A global logical rule: **(∪E)**

$$\dfrac{\dfrac{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i\cup\upsilon_i)_i] \qquad [((\Gamma_i,\sigma_i,\tau_i,\Delta_i,\rho_i\,;\,\phi_i),(\Gamma_i,\sigma_i,\tau_i,\Delta_i,\upsilon_i\,;\,\phi_i))_i]}{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\phi_i)_i]}\,\mathbf{(\cup E)}}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\phi_i)_i]}\,\mathbf{(X)} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{[(\Gamma_i,\sigma_i,\tau_i,\Delta_i\,;\,\rho_i\cup\upsilon_i)_i]}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\rho_i\cup\upsilon_i)_i]}\,\mathbf{(X)} \qquad \dfrac{[((\Gamma_i,\sigma_i,\tau_i,\Delta_i,\rho_i\,;\,\phi_i),(\Gamma_i,\sigma_i,\tau_i,\Delta_i,\upsilon_i\,;\,\phi_i))_i]}{[((\Gamma_i,\tau_i,\sigma_i,\Delta_i,\rho_i\,;\,\phi_i),(\Gamma_i,\tau_i,\sigma_i,\Delta_i,\upsilon_i\,;\,\phi_i))_i]}\,\mathbf{(X)}}{[(\Gamma_i,\tau_i,\sigma_i,\Delta_i\,;\,\phi_i)_i]}\,\mathbf{(\cup E)}$$

$\dashv$

The structural rules of weakening and contraction are derivable, as the next two propositions show.

Proposition 4.5 *Weakening is derivable: if* $\pi :: [(\Gamma_i\,;\,\tau_i)_i]$, *there exists a* $\pi' :: [(\Gamma_i,\sigma_i\,;\,\tau_i)_i]$.

Proof. By induction on π.

Base: If $\pi :: [(\Gamma_i,\tau_i\,;\,\tau_i)_i]$ is an axiom, then a $\pi' :: [(\Gamma_i,\tau_i,\sigma_i\,;\,\tau_i)_i]$ contains an axiom $[(\Gamma_i,\sigma_i,\tau_i\,;\,\tau_i)_i]$ and an application of exchange.

Induction step: We show three characteristic cases, denoting [h] the induction hypothesis.

$$\triangleright \quad \dfrac{\pi_0 :: [(\Gamma_i,\tau_i\,;\,\rho_i)_i]}{\pi :: [(\Gamma_i\,;\,\tau_i\to\rho_i)_i]}\,\mathbf{(\to I)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\pi_0' :: [(\Gamma_i,\tau_i,\sigma_i\,;\,\rho_i)_i]\ \ [\mathrm{h}]}{[(\Gamma_i,\sigma_i,\tau_i\,;\,\rho_i)_i]}\,\mathbf{(X)}}{\pi' :: [(\Gamma_i,\sigma_i\,;\,\tau_i\to\rho_i)_i]}\,\mathbf{(\to I)}$$

$$\triangleright \quad \dfrac{\pi_0 :: [(\Delta_i\,;\,\phi_i)_1^k,((\Gamma_i\,;\,\tau_i),(\Gamma_i\,;\,\rho_i))_1^n]}{\pi :: [(\Delta_i\,;\,\phi_i)_1^k,(\Gamma_i\,;\,\tau_i\cap\rho_i)_1^n]}\,\mathbf{(\cap I)} \quad \rightsquigarrow \quad \dfrac{\pi_0' :: [(\Delta_i,\psi_i\,;\,\phi_i)_1^k,((\Gamma_i,\sigma_i\,;\,\tau_i),(\Gamma_i,\sigma_i\,;\,\rho_i))_1^n]\ \ [\mathrm{h}]}{\pi' :: [(\Delta_i,\psi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\,;\,\tau_i\cap\rho_i)_1^n]}\,\mathbf{(\cap I)}$$

$$\triangleright \quad \dfrac{\pi_0 :: [(\Gamma_i\,;\,\tau_i\cup\rho_i)_i] \qquad \pi_1 :: [((\Gamma_i,\tau_i\,;\,\upsilon_i),(\Gamma_i,\rho_i\,;\,\upsilon_i))_i]}{\pi : [(\Gamma_i\,;\,\upsilon_i)_i]}\,\mathbf{(\cup E)} \quad \rightsquigarrow$$

$$\dfrac{\pi_0' :: [(\Gamma_i,\sigma_i\,;\,\tau_i\cup\rho_i)_i]\ \ [\mathrm{h}] \qquad \dfrac{\pi_1' :: [((\Gamma_i,\tau_i,\sigma_i\,;\,\upsilon_i),(\Gamma_i,\rho_i,\sigma_i\,;\,\upsilon_i))_i]\ \ [\mathrm{h}]}{[((\Gamma_i,\sigma_i,\tau_i\,;\,\upsilon_i),(\Gamma_i,\sigma_i,\rho_i\,;\,\upsilon_i))_i]}\,\mathbf{(X)}}{\pi' :: [(\Gamma_i,\sigma_i\,;\,\upsilon_i)_i]}\,\mathbf{(\cup E)}$$

$\dashv$

Proposition 4.6 *Contraction is derivable: if* $\pi :: [(\Gamma_i,\sigma_i,\sigma_i\,;\,\tau_i)_i]$, *there exists a* $\pi' :: [(\Gamma_i,\sigma_i\,;\,\tau_i)_i]$.

Proof. We derive contraction through an implication redex.

$$\dfrac{\dfrac{\pi :: [(\Gamma_i,\sigma_i,\sigma_i\,;\,\tau_i)_i]}{[(\Gamma_i,\sigma_i\,;\,\sigma_i\to\tau_i)_i]}\,\mathbf{(\to I)} \qquad \dfrac{}{[(\Gamma_i,\sigma_i\,;\,\sigma_i)_i]}\,\mathbf{(ax)}}{\pi' :: [(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}\,\mathbf{(\to E)}$$

We can check that, if we chose a multiplicative implication elimination rule, the derivability of contraction through an implication redex would fail. A proof by induction on π would also fail. $\dashv$

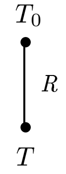
We next define the notions of *tree* (of a derivation) and of *derivation height*, which will be used in subsequent propositions.

Definition 4.7 (Tree) *The* tree T *(or* T_π*) of a derivation* π *is defined inductively as follows.*

▷ *If* π *is an axiom, the tree* T *consists of a single node.*

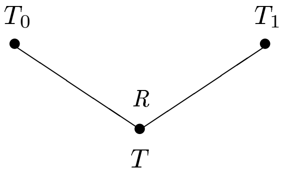
▷ *If* π *derives from* π_0 *with tree* T_0 *by a one-premise rule* R*, then the root of tree* T *has a single child-node, namely the root of* T_0*.*

$$\frac{\pi_0}{\pi}\,R \quad \rightsquigarrow$$



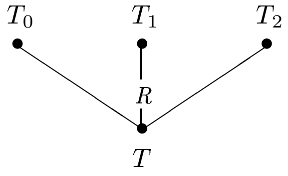
▷ *If* π *derives from* π_0 *and* π_1 *with trees* T_0 *and* T_1*, respectively, by a two-premise rule* R*, then the root of tree* T *has two child-nodes, namely the roots of* T_0 *and* T_1*.*

$$\frac{\pi_0 \qquad \pi_1}{\pi}\,R \quad \rightsquigarrow$$



▷ *If*[4] π *derives from* π_0, π_1*, and* π_2 *with trees* T_0, T_1*, and* T_2*, respectively, by a three-premise rule* R*, then the root of tree* T *has three child-nodes, namely the roots of* T_0, T_1*, and* T_2*.*

$$\frac{\pi_0 \qquad \pi_1 \qquad \pi_2}{\pi}\,R \quad \rightsquigarrow$$



In the induction cases, the node associated to the rule R *is the root of* T*.*

Definition 4.8 (Derivation height) *The* derivation height h *(or* h_π*) of a derivation* π *is the height of the tree of* π*, i.e. the maximal length of the branches in the tree, where the length of a branch is the number of nodes in the branch minus 1.*

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[4] We include the case of a three-premise rule in preparation for the presentation of the type system $\mathrm{IUT}^{\oplus}$, whose ($\cup$**E**) rule has three premises.

Remark 4.9 *For any derivations* π *and* π'*, we have that* $T = T' \Rightarrow h = h'$*, but* $h = h' \nRightarrow T = T'$*.*

Before we establish the derivability of the structural rules of pruning and doubling, we need to show that atoms can be exchanged in provable molecules.

Proposition 4.10 *If* $\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]$*, there exists a* $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ *with* $T' = T$*.*

Proof. By induction on π.

Base: If $\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]$ is an axiom, then $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ is an axiom, as well. Both T and T' consist of a single node.

Induction step: We present two characteristic cases.

▷ A local rule: $(\cap\mathbf{I})$

Case 1: $$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]}\,(\cap\mathbf{I})$$

where $|\mathcal{U}| \leqslant |\mathcal{U}_0|$ and $|\mathcal{V}| \leqslant |\mathcal{V}_0|$

Applying the IH four times[5], we get a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}_0, \mathcal{B}_1, \mathcal{A}_0, \mathcal{A}_1, \mathcal{V}_0]$ with $T_0' = T_0$. By $(\cap\mathbf{I})$, we then get a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$.

Case 2: $$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \mathcal{B}, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]}\,(\cap\mathbf{I})$$

where $|\mathcal{U}| \leqslant |\mathcal{U}_0|$ and $|\mathcal{V}| \leqslant |\mathcal{V}_0|$

Applying the IH twice, we obtain a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{V}_0]$ with $T_0' = T_0$. By $(\cap\mathbf{I})$, we then get a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$.

Case 3: $$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}, \mathcal{B}, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]}\,(\cap\mathbf{I})$$

where either $(|\mathcal{U}| < |\mathcal{U}_0|$ and $|\mathcal{V}| \leqslant |\mathcal{V}_0|)$ or $(|\mathcal{U}| \leqslant |\mathcal{U}_0|$ and $|\mathcal{V}| < |\mathcal{V}_0|)$

The IH gives a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}, \mathcal{A}, \mathcal{V}_0]$ with $T_0' = T_0$. By $(\cap\mathbf{I})$, we then get a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$.

▷ A global rule: $(\cup\mathbf{E})$

$$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{V}_0] \qquad \pi_1 :: [\mathcal{U}_1, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{B}_{10}, \mathcal{B}_{11}, \mathcal{V}_1]}{\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]}\,(\cup\mathbf{E})$$

where $|\mathcal{U}_1| = 2|\mathcal{U}_0|$ and $|\mathcal{U}| = |\mathcal{U}_0|$

The IH on π_0 gives a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}_0, \mathcal{A}_0, \mathcal{V}_0]$ with $T_0' = T_0$. Starting with π_1 and applying the IH four times, we get a $\pi_1' :: [\mathcal{U}_1, \mathcal{B}_{10}, \mathcal{B}_{11}, \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{V}_1]$ with $T_1' = T_1$. Then, applying $(\cup\mathbf{E})$ to π_0' and π_1', we obtain a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$. ⊣

Proposition 4.11 *(i) Pruning is derivable: if* $\pi :: [\mathcal{U}, \mathcal{V}]$*, there exists a* $\pi' :: [\mathcal{U}]$ *with* $h' \leqslant h$*.*

(ii) Doubling is derivable: if $\pi :: [\mathcal{U}, \mathcal{A}]$*, there exists a* $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ *with* $T' = T$*, where* $2\mathcal{A} = \mathcal{A}, \mathcal{A}$*.*

---

[5] We can have multiple applications of the IH, as the exchange of atoms leaves the tree, and hence the height, unaltered.

Proof. (i) By induction on π.

Base: If $\pi :: [\mathcal{U}, \mathcal{V}]$ is an axiom, then $\pi' :: [\mathcal{U}]$ is an axiom, as well, and both heights equal 0.

Induction step: We demonstrate two characteristic cases.

▷ A global rule: **(→E)**

$$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{V}_0] \qquad \pi_1 :: [\mathcal{U}_1, \mathcal{V}_1]}{\pi :: [\mathcal{U}, \mathcal{V}]}\ (\rightarrow\mathbf{E})$$

where $|\mathcal{U}_0| = |\mathcal{U}_1| = |\mathcal{U}|$

The IH gives a $\pi_0' :: [\mathcal{U}_0]$ with $h_0' \leqslant h_0$ and a $\pi_1' :: [\mathcal{U}_1]$ with $h_1' \leqslant h_1$. By **(→E)**, we then get a $\pi' :: [\mathcal{U}]$ with $h' = \max(h_0', h_1') + 1 \leqslant \max(h_0, h_1) + 1 = h$.

▷ A local rule: **(∩I)**

Case 1: $\dfrac{\pi_0 :: [\mathcal{U}_0, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{V}]}\ (\cap\mathbf{I})$

where $|\mathcal{U}| < |\mathcal{U}_0|$

The IH gives a $\pi_0' :: [\mathcal{U}_0]$ with $h_0' \leqslant h_0$. By **(∩I)**, we then get a $\pi' :: [\mathcal{U}]$ with $h' = h_0' + 1 \leqslant h_0 + 1 = h$.

Case 2: $\dfrac{\pi_0 :: [\mathcal{U}, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{V}]}\ (\cap\mathbf{I})$

where $|\mathcal{V}| < |\mathcal{V}_0|$

The IH gives a $\pi_0' :: [\mathcal{U}]$ with $h_0' \leqslant h_0$. It is $\pi' = \pi_0'$ and $h' = h_0' < h$.

(ii) By induction on π.

Base: If $\pi :: [\mathcal{U}, \mathcal{A}]$ is an axiom, then $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ is an axiom, as well, and both trees consist of a single node.

Induction step: We expose two characteristic cases.

▷ A local rule: **(∩I)**

Case 1: $\dfrac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1]}{\pi :: [\mathcal{U}, \mathcal{A}]}\ (\cap\mathbf{I})$

where $|\mathcal{U}| \leqslant |\mathcal{U}_0|$

The IH gives a $\pi_0' :: [\mathcal{U}_0, \mathcal{A}_0, 2\mathcal{A}_1]$ with $T_0' = T_0$. Then, by two applications of 4.10, we obtain a $\pi_0^2 :: [\mathcal{U}_0, 2\mathcal{A}_1, \mathcal{A}_0]$ with $T_0^2 = T_0$. By the IH once again[6], we get a $\pi_0^3 :: [\mathcal{U}_0, 2\mathcal{A}_1, 2\mathcal{A}_0]$ with $T_0^3 = T_0$. Starting with π_0^3 and applying 4.10 three times, we derive a $\pi_0^4 :: [\mathcal{U}_0, 2(\mathcal{A}_0, \mathcal{A}_1)]$ with $T_0^4 = T_0$. Finally, applying **(∩I)** to π_0^4, we get a $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ with $T' = T$.

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[6] To apply the IH once again and double $\mathcal{A}_0$ after having doubled $\mathcal{A}_1$, which is an important step for the derivability of doubling in this case, we need to have the conclusion that $T' = T$ in the statement of the derivability of doubling and also in the statement of the derivability of atom exchange. In the case of the derivability of doubling, though, this conclusion is not be maintained, if the local rules of intersection (introduction and elimination) and union introduction are not allowed to act on more than one atom (or sequence of atoms) in one step. The reader may easily verify this by attempting the current case of **(∩I)** with a version of the rule acting solely on one sequence of atoms $\mathcal{A}_0, \mathcal{A}_1$ or the corresponding case of **(∩E)** (resp. **(∪I)**) with a version of the rule acting solely on one atom $\mathcal{A}_0$.

Case 2: $\dfrac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}]}{\pi :: [\mathcal{U}, \mathcal{A}]}\ (\cap\mathbf{I})$

where $|\mathcal{U}| < |\mathcal{U}_0|$

The IH yields a $\pi_0' :: [\mathcal{U}_0, 2\mathcal{A}]$ with $T_0' = T_0$. By $(\cap\mathbf{I})$, we then get a $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ with $T' = T$.

$\triangleright$ A global rule: $(\cup\mathbf{E})$

$$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0] \qquad \pi_1 :: [\mathcal{U}_1, \mathcal{A}_{10}, \mathcal{A}_{11}]}{\pi :: [\mathcal{U}, \mathcal{A}]}\ (\cup\mathbf{E})$$

The IH on π_0 gives a $\pi_0' :: [\mathcal{U}_0, 2\mathcal{A}_0]$ with $T_0' = T_0$, while the IH on π_1 yields a $\pi_1' :: [\mathcal{U}_1, \mathcal{A}_{10}, 2\mathcal{A}_{11}]$ with $T_1' = T_1$. Starting with π_1' and applying 4.10 twice, we get a $\pi_1^2 :: [\mathcal{U}_1, 2\mathcal{A}_{11}, \mathcal{A}_{10}]$ with $T_1^2 = T_1$. The IH on π_1^2 gives a $\pi_1^3 :: [\mathcal{U}_1, 2\mathcal{A}_{11}, 2\mathcal{A}_{10}]$ with $T_1^3 = T_1$. Starting with π_1^3 and applying 4.10 three times, we derive a $\pi_1^4 :: [\mathcal{U}_1, 2(\mathcal{A}_{10}, \mathcal{A}_{11})]$ with $T_1^4 = T_1$. Finally, applying $(\cup\mathbf{E})$ to π_0' and π_1^4, we get a $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ with $T' = T$. $\dashv$

Remark 4.12 *An alternative phrasing for the derivability of weakening and contraction, which includes the notion of "tree", is the following.*

(i) Weakening is derivable: if $\pi :: [(\Gamma_i, \Delta_i\,;\, \tau_i)_i]$, *where the* Γ_i*'s have the same cardinality and the* Δ_i*'s are non-empty, there exists a* $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$ *with* $T' = T$.

(ii) Contraction is derivable: if $\pi :: [(\Gamma_i, \sigma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$, *where the* Γ_i*'s have the same cardinality and the* Δ_i*'s are non-empty, there exists a* $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$ *with* $T' = T$.

For both (i) and (ii), the proof is by induction on π. *If the* Δ_i*'s are empty in (i), the induction works, only if the conclusion* $T' = T$ *is removed (see Proposition 4.5). If the* Δ_i*'s are empty in (ii), the induction does not work. We can only derive contraction through an implication redex (see Proposition 4.6), in which case the conclusion* $T' = T$ *does not hold.*

If we consider a union elimination rule $(\cup\mathbf{E})'$ that resembles the union elimination rule of the presentation of IUL_m given in Chapter 3, we can show that it is derivable in the current presentation of IUL_m.

$$\frac{[(\Delta_i\,;\,\phi_i)_1^k, (\Gamma_i\,;\,\sigma_i \cup \tau_i)_1^n] \qquad [(\Delta_i, \phi_i\,;\,\psi_i)_1^k, ((\Gamma_i, \sigma_i\,;\,\rho_i), (\Gamma_i, \tau_i\,;\,\rho_i))_1^n]}{[(\Delta_i\,;\,\psi_i)_1^k, (\Gamma_i\,;\,\rho_i)_1^n]}\ (\cup\mathbf{E})'$$

We use the derivable rule $(\cup\mathbf{E})'$ in Chapter 7, where we introduce a sequent calculus presentation of IUL_m, to facilitate the proof of equivalence between the natural deduction and sequent calculus presentations of IUL_m (see Theorem 7.2).

Proposition 4.13 *The rule* $(\cup\mathbf{E})'$ *is derivable: if*

$$\pi_0 :: [(\Delta_i\,;\,\phi_i)_1^k, (\Gamma_i\,;\,\sigma_i \cup \tau_i)_1^n] \text{ and } \pi_1 :: [(\Delta_i, \phi_i\,;\,\psi_i)_1^k, ((\Gamma_i, \sigma_i\,;\,\rho_i), (\Gamma_i, \tau_i\,;\,\rho_i))_1^n]$$

there exists a $\pi :: [(\Delta_i\,;\,\psi_i)_1^k, (\Gamma_i\,;\,\rho_i)_1^n]$.

Proof. We derive $(\cup\mathbf{E})'$ through a union redex, with the aid of Propositions 4.10 and 4.11(ii).

$$\dfrac{\dfrac{\pi_0 :: [(\Delta_i\,;\,\phi_i)_1^k, (\Gamma_i\,;\,\sigma_i \cup \tau_i)_1^n]}{[(\Delta_i\,;\,\phi_i \cup \phi_i)_1^k, (\Gamma_i\,;\,\sigma_i \cup \tau_i)_1^n]}\,(\cup\mathbf{I}) \qquad \begin{array}{c}\pi_1 :: [(\Delta_i, \phi_i\,;\,\psi_i)_1^k, ((\Gamma_i, \sigma_i\,;\,\rho_i), (\Gamma_i, \tau_i\,;\,\rho_i))_1^n] \\ \hdashline [((\Delta_i, \phi_i\,;\,\psi_i), (\Delta_i, \phi_i\,;\,\psi_i))_1^k, ((\Gamma_i, \sigma_i\,;\,\rho_i), (\Gamma_i, \tau_i\,;\,\rho_i))_1^n]\end{array}\,[4.10,\ 4.11(ii)]}{\pi :: [(\Delta_i\,;\,\psi_i)_1^k, (\Gamma_i\,;\,\rho_i)_1^n]}\,(\cup\mathbf{E})$$

$\dashv$

Having redefined the logic and established its basic properties, we move on to present the type system and demonstrate some (new) properties of it.

4.2 The type system IUT$^\oplus$ in natural deduction

As already mentioned, the type system IUT$^\oplus$ in natural deduction style is the natural deduction type system IUT$_\omega$ of Chapter 2 without the $(\boldsymbol{\omega})$-rule. It assigns types $\sigma ::= \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma \mid \sigma \cup \sigma$ to terms $t \in \Lambda$ according to the rules in Figure 4.2.

$$\dfrac{}{B,\, x : \sigma \vdash x : \sigma}\,(\mathbf{ax})$$

$$\dfrac{B,\, x : \sigma \vdash t : \tau}{B \vdash \lambda x.t : \sigma \to \tau}\,(\to\mathbf{I}) \qquad \dfrac{B \vdash t : \sigma \to \tau \qquad B \vdash u : \sigma}{B \vdash tu : \tau}\,(\to\mathbf{E})$$

$$\dfrac{B \vdash t : \sigma \qquad B \vdash t : \tau}{B \vdash t : \sigma \cap \tau}\,(\cap\mathbf{I}) \qquad \dfrac{B \vdash t : \sigma \cap \tau}{B \vdash t : \sigma}\,(\cap\mathbf{E}_1) \qquad \dfrac{B \vdash t : \sigma \cap \tau}{B \vdash t : \tau}\,(\cap\mathbf{E}_2)$$

$$\dfrac{B \vdash t : \sigma}{B \vdash t : \sigma \cup \tau}\,(\cup\mathbf{I}_1) \qquad \dfrac{B \vdash t : \tau}{B \vdash t : \sigma \cup \tau}\,(\cup\mathbf{I}_2)$$

$$\dfrac{B \vdash t : \sigma \cup \tau \qquad B,\, x : \sigma \vdash u : \rho \qquad B,\, x : \tau \vdash u : \rho}{B \vdash u[t/x] : \rho}\,(\cup\mathbf{E})$$

Figure 4.2: The type system IUT$^\oplus$ in natural deduction style.

Let us denote V_π (or just V) the set of all term variables appearing in a derivation π of IUT$^\oplus$. The next proposition establishes that *renaming*[7] of a term variable, *weakening* and *strengthening* of the assumptions, and *contraction* of basic typing statements are all admissible in IUT$^\oplus$.

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[7] The term "renaming" is very common in the literature, when speaking of a variable change in the assumptions (e.g. see [2]). Although we use this terminology to be in accordance with the majority of authors, it is important to stress that the change in question does not actually concern *the name* of the variable, but *the variable itself.*

Proposition 4.14 *(i) (Renaming) If* $\pi :: B,\, x : \sigma \vdash t : \tau$ *and* y *is fresh with respect to* π*, there exists a* $\pi' :: B,\, y : \sigma \vdash t[y/x] : \tau$*, such that* $V' = (V \setminus \{x\}) \cup \{y\}$ *and* $T' = T$*.*

(ii) (Weakening) If $\pi :: B \vdash t : \tau$ *and* x *is fresh with respect to* π*, there exists a* $\pi' :: B,\, x : \sigma \vdash t : \tau$*, such that* $V' = V \cup \{x\}$ *and* $T' = T$*.*

(iii) (Strengthening) If $\pi :: B,\, x : \sigma \vdash t : \tau$ *and* $x \notin FV(t)$*, there exists a* $\pi' :: B \vdash t : \tau$*, such that* $x \notin V' \subsetneq V$ *and* $h' \leqslant h$*.*

(iv) (Contraction) If $\pi :: B,\, x : \sigma,\, y : \sigma \vdash t : \tau$*, there exists a* $\pi' :: B,\, x : \sigma \vdash t[x/y] : \tau$*, such that* $V' = V \setminus \{y\}$ *and* $T' = T$*.*

Proof. (i) By induction on π.

Base: If π is an axiom, we distinguish two cases.

Case 1: If $\pi :: B,\, x : \sigma \vdash x : \sigma$ with $V = dom(B) \cup \{x\}$, there is an axiom $\pi' :: B,\, y : \sigma \vdash y : \sigma$, such that $V' = dom(B) \cup \{y\} = (V \backslash \{x\}) \cup \{y\}$ and $T' = T$.

Case 2: If $\pi :: B',\, z : \tau,\, x : \sigma \vdash z : \tau$ with $V = dom(B') \cup \{z, x\}$, there is an axiom

$$\pi' :: B',\, z : \tau,\, y : \sigma \vdash z : \tau$$

such that $V' = dom(B') \cup \{z, y\} = (V \backslash \{x\}) \cup \{y\}$ and $T' = T$.

Induction step: We demonstrate two typical cases.

$$\triangleright \quad \frac{\pi_0 :: B,\, x : \sigma \vdash t : \tau \to \rho \qquad \pi_1 :: B,\, x : \sigma \vdash u : \tau}{\pi :: B,\, x : \sigma \vdash tu : \rho}\ (\to\mathbf{E})$$

Supposing that $V_{\pi_0} = V_0 \cup \{x\}$ and $V_{\pi_1} = V_1 \cup \{x\}$, we get that $V = V_0 \cup V_1 \cup \{x\}$. The IH gives a $\pi'_0 :: B,\, y : \sigma \vdash t[y/x] : \tau \to \rho$, such that $V'_0 = V_0 \cup \{y\}$ and $T'_0 = T_0$, and a $\pi'_1 :: B,\, y : \sigma \vdash u[y/x] : \tau$, such that $V'_1 = V_1 \cup \{y\}$ and $T'_1 = T_1$. By $(\to\mathbf{E})$, we then get a $\pi' :: B,\, y : \sigma \vdash (t[y/x])(u[y/x]) = (tu)[y/x] : \rho$, such that $V' = V'_0 \cup V'_1 = V_0 \cup V_1 \cup \{y\} = (V \backslash \{x\}) \cup \{y\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B,\, x : \sigma \vdash t : \tau \cup \rho \qquad \pi_1 :: B,\, x : \sigma,\, z : \tau \vdash u : \phi \qquad \pi_2 :: B,\, x : \sigma,\, z : \rho \vdash u : \phi}{\pi :: B,\, x : \sigma \vdash u[t/z] : \phi}\ (\cup\mathbf{E})$$

Supposing that $V_{\pi_i} = V_i \cup \{x\}$ $(i = 0, 1, 2)$, we have that $V = \bigcup_i V_{\pi_i} = (\bigcup_i V_i) \cup \{x\}$. The IH gives a $\pi'_0 :: B,\, y : \sigma \vdash t[y/x] : \tau \cup \rho$, a $\pi'_1 :: B,\, y : \sigma,\, z : \tau \vdash u[y/x] : \phi$, and a $\pi'_2 :: B,\, y : \sigma,\, z : \rho \vdash u[y/x] : \phi$, such that $V'_i = V_i \cup \{y\}$ and $T'_i = T_i$. Applying $(\cup\mathbf{E})$ to π'_0, π'_1, and π'_2, we then obtain a

$$\pi' :: B,\, y : \sigma \vdash (u[y/x])[t[y/x]/z] = (u[t/z])[y/x] : \phi$$

such that $V' = \bigcup_i V'_i = (\bigcup_i V_i) \cup \{y\} = (V \backslash \{x\}) \cup \{y\}$ and $T' = T$.

For the rest of the proof, it is $V_i = V_{\pi_i}$ $(i = 0, 1, 2)$.

(ii) By induction on π.

Base: If $\pi :: B',\, y : \tau \vdash y : \tau$ is an axiom, there is an axiom $\pi' :: B',\, y : \tau,\, x : \sigma \vdash y : \tau$, such that $V' = dom(B') \cup \{y, x\} = V \cup \{x\}$ and $T' = T$.

Induction step: We once more demonstrate the cases of $(\to\mathbf{E})$ and $(\cup\mathbf{E})$.

$$\triangleright\ \frac{\pi_0 :: B \vdash t : \tau \to \rho \qquad \pi_1 :: B \vdash u : \tau}{\pi :: B \vdash tu : \rho}\ (\to\mathbf{E})$$

The IH yields a $\pi_0' :: B,\, x : \sigma \vdash t : \tau \to \rho$, such that $V_0' = V_0 \cup \{x\}$ and $T_0' = T_0$, and also a $\pi_1' :: B,\, x : \sigma \vdash u : \tau$, such that $V_1' = V_1 \cup \{x\}$ and $T_1' = T_1$. Applying $(\to\mathbf{E})$ to π_0' and π_1', we then get a $\pi' :: B,\, x : \sigma \vdash tu : \rho$, such that $V' = V_0' \cup V_1' = V_0 \cup V_1 \cup \{x\} = V \cup \{x\}$ and $T' = T$.

$$\triangleright\ \frac{\pi_0 :: B \vdash t : \tau \cup \rho \qquad \pi_1 :: B,\, y : \tau \vdash u : \phi \qquad \pi_2 :: B,\, y : \rho \vdash u : \phi}{\pi :: B \vdash u[t/y] : \phi}\ (\cup\mathbf{E})$$

The IH gives a $\pi_0' :: B,\, x : \sigma \vdash t : \tau \cup \rho$, a $\pi_1' :: B,\, y : \tau,\, x : \sigma \vdash u : \phi$, and a $\pi_2' :: B,\, y : \rho,\, x : \sigma \vdash u : \phi$, such that $V_i' = V_i \cup \{x\}$ and $T_i' = T_i$ $(i = 0, 1, 2)$. Applying $(\cup\mathbf{E})$ to π_0', π_1', and π_2', we obtain a $\pi' :: B,\, x : \sigma \vdash u[t/y] : \phi$, such that $V' = \bigcup_i V_i' = (\bigcup_i V_i) \cup \{x\} = V \cup \{x\}$ and $T' = T$.

(iii) By induction on π.

Base: If $\pi :: B',\, y : \tau,\, x : \sigma \vdash y : \tau$ is an axiom, there is an axiom $\pi' :: B',\, y : \tau \vdash y : \tau$, such that $x \notin V' = \mathit{dom}(B') \cup \{y\} \subsetneq \mathit{dom}(B') \cup \{y, x\} = V$ and $h' = h = 0$.

Induction step: We show two distinctive cases.

$$\triangleright\ \frac{\pi_0 :: B,\, x : \sigma,\, y : \tau \vdash t : \rho}{\pi :: B,\, x : \sigma \vdash \lambda y.\, t : \tau \to \rho}\ (\to\mathbf{I})$$

Since $x \notin FV(\lambda y.\, t)$ and $x \neq y$, we have that $x \notin FV(\lambda y.\, t) \cup \{y\} = FV(t)$. Hence, the IH yields a $\pi_0' :: B,\, y : \tau \vdash t : \rho$, such that $x \notin V_0' \subsetneq V_0$ and $h_0' \leqslant h_0$. By $(\to\mathbf{I})$, we then get a $\pi' :: B \vdash \lambda y.\, t : \tau \to \rho$, such that $x \notin V' = V_0' \subsetneq V_0 = V$ and $h' = h_0' + 1 \leqslant h_0 + 1 = h$.

$$\triangleright\ \frac{\pi_0 :: B,\, x : \sigma \vdash t : \tau \cup \rho \qquad \pi_1 :: B,\, x : \sigma,\, y : \tau \vdash u : \phi \qquad \pi_2 :: B,\, x : \sigma,\, y : \rho \vdash u : \phi}{\pi :: B,\, x : \sigma \vdash u[t/y] : \phi}\ (\cup\mathbf{E})$$

We suppose that $x \notin FV(u[t/y])$ and distinguish two cases.

Case 1: $y \notin FV(u) \Rightarrow u[t/y] = u$. The IH on π_1 gives a $\pi_1' :: B,\, x : \sigma \vdash u : \phi$, such that $y \notin V_1' \subsetneq V_1$ and $h_1' \leqslant h_1$. Since $h_1' \leqslant h_1 < h$ and $x \notin FV(u[t/y] = u)$, the IH on π_1' yields a $\pi' :: B \vdash u = u[t/y] : \phi$, such that $x \notin V' \subsetneq V_1' \subsetneq V_1 \subseteq V_0 \cup V_1 \cup V_2 = V$ and $h' \leqslant h_1' < h$.

Case 2: $y \in FV(u) \Rightarrow x \notin FV(t)$ and $x \notin FV(u)$. The IH gives derivations

$$\pi_0' :: B \vdash t : \tau \cup \rho,\ \pi_1' :: B,\, y : \tau \vdash u : \phi,\ \text{and } \pi_2' :: B,\, y : \rho \vdash u : \phi$$

such that $x \notin V_i' \subsetneq V_i$ and $h_i' \leqslant h_i$ $(i = 0, 1, 2)$. By $(\cup\mathbf{E})$, we obtain a $\pi' :: B \vdash u[t/y] : \phi$, such that $x \notin V' = \bigcup_i V_i' \subsetneq \bigcup_i V_i = V$ and $h' = \max_i(h_i') + 1 \leqslant \max_i(h_i) + 1 = h$.

(iv) By induction on π.

Base: If π is an axiom, we distinguish three cases.

Case 1: If $\pi :: B,\, x : \sigma,\, y : \sigma \vdash x : \sigma$ with $V = \mathit{dom}(B) \cup \{x, y\}$, there is an axiom

$$\pi' :: B,\, x : \sigma \vdash x[x/y] = x : \sigma$$

such that $V' = \mathit{dom}(B) \cup \{x\} = V \setminus \{y\}$ and $T' = T$.

Case 2: If $\pi :: B,\, x : \sigma,\, y : \sigma \vdash y : \sigma$ with $V = \mathit{dom}(B) \cup \{x, y\}$, there is an axiom

$$\pi' :: B,\, x : \sigma \vdash y[x/y] = x : \sigma$$

such that $V' = dom(B) \cup \{x\} = V \setminus \{y\}$ and $T' = T$.

Case 3: If $\pi :: B', z : \tau, x : \sigma, y : \sigma \vdash z : \tau$ with $V = dom(B') \cup \{z, x, y\}$, there is an axiom

$$\pi' :: B', z : \tau, x : \sigma \vdash z[x/y] = z : \tau$$

such that $V' = dom(B') \cup \{z, x\} = V \setminus \{y\}$ and $T' = T$.

Induction step: We show two characteristic cases.

$$\triangleright \quad \frac{\pi_0 :: B, x : \sigma, y : \sigma \vdash t : \tau \to \rho \qquad \pi_1 :: B, x : \sigma, y : \sigma \vdash u : \tau}{\pi :: B, x : \sigma, y : \sigma \vdash tu : \rho}\ (\to\mathbf{E})$$

The IH yields a $\pi_0' :: B, x : \sigma \vdash t[x/y] : \tau \to \rho$, such that $V_0' = V_0 \setminus \{y\}$ and $T_0' = T_0$, and also a $\pi_1' :: B, x : \sigma \vdash u[x/y] : \tau$, such that $V_1' = V_1 \setminus \{y\}$ and $T_1' = T_1$. Applying $(\to\mathbf{E})$ to π_0' and π_1', we obtain a $\pi' :: B, x : \sigma \vdash (t[x/y])(u[x/y]) = (tu)[x/y] : \rho$, such that $V' = V_0' \cup V_1' = (V_0 \setminus \{y\}) \cup (V_1 \setminus \{y\}) = (V_0 \cup V_1) \setminus \{y\} = V \setminus \{y\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B, x : \sigma, y : \sigma \vdash t : \tau \cup \rho \qquad \pi_1 :: B, x : \sigma, y : \sigma, z : \tau \vdash u : \phi \qquad \pi_2 :: B, x : \sigma, y : \sigma, z : \rho \vdash u : \phi}{\pi :: B, x : \sigma, y : \sigma \vdash u[t/z] : \phi}\ (\cup\mathbf{E})$$

The IH gives derivations

$$\pi_0' :: B, x : \sigma \vdash t[x/y] : \tau \cup \rho,\ \pi_1' :: B, x : \sigma, z : \tau \vdash u[x/y] : \phi, \text{ and } \pi_2' :: B, x : \sigma, z : \rho \vdash u[x/y] : \phi$$

such that $V_i' = V_i \setminus \{y\}$ and $T_i' = T_i$ $(i = 0, 1, 2)$. By $(\cup\mathbf{E})$, we then get a

$$\pi' :: B, x : \sigma \vdash (u[x/y])[t[x/y]/z] = (u[t/z])[x/y] : \phi$$

such that $V' = \bigcup_i V_i' = \bigcup_i (V_i \setminus \{y\}) = (\bigcup_i V_i) \setminus \{y\} = V \setminus \{y\}$ and $T' = T$. ⊣

Remark 4.15 *Contrary to* IUL$_m$*, where contraction is derivable through an implication redex, we cannot derive contraction in* IUT$^\oplus$ *through an implication redex.*

$$\frac{\dfrac{\pi :: B, x : \sigma, y : \sigma \vdash t : \tau}{B, x : \sigma \vdash \lambda y.\, t : \sigma \to \tau}\ (\to\mathbf{I}) \qquad \dfrac{}{B, x : \sigma \vdash x : \sigma}\ (\mathbf{ax})}{\pi' :: B, x : \sigma \vdash (\lambda y.\, t)\, x : \tau}\ (\to\mathbf{E})$$

As shown above, such an attempt provides a π' typing the redex $(\lambda y.\, t)\, x$ instead of the contractum $t[x/y]$ and, as argued in Section 2.1, the type system is not invariant under β-reduction of subjects. On the other hand, as already shown in Remark 2.2(ii), we can derive contraction in IUT$^\oplus$ *through a union redex.*

The following proposition declares that the sets of free and bound variables of a term typable in IUT$^\oplus$ are disjoint.

Proposition 4.16 *If $B \vdash t : \sigma$, then $dom(B) \cap BV(t) = \emptyset$, Consequently, since*[8] *$FV(t) \subseteq dom(B)$, it is $FV(t) \cap BV(t) = \emptyset$.*

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[8] See Remark 2.5.

Proof. By induction on $B \vdash t : \sigma$.

Base: If $B', x : \sigma \vdash x : \sigma$, then $(dom(B') \cup \{x\}) \cap BV(x) = (dom(B') \cup \{x\}) \cap \emptyset = \emptyset$.

Induction step: We show the most notable cases.

$$\triangleright \quad \frac{B, x : \sigma \vdash t : \tau}{B \vdash \lambda x.t : \sigma \to \tau}\ (\to\mathbf{I})$$

We have that $x \notin dom(B)$ and also, by the IH, that $(dom(B) \cup \{x\}) \cap BV(t) = \emptyset$. Therefore, the sets $dom(B), \{x\}$, and $BV(t)$ are pairwise disjoint, which implies that $dom(B) \cap (BV(t) \cup \{x\}) = \emptyset$, i.e. that $dom(B) \cap BV(\lambda x.t) = \emptyset$.

$$\triangleright \quad \frac{B \vdash t : \sigma \to \tau \qquad B \vdash u : \sigma}{B \vdash tu : \tau}\ (\to\mathbf{E})$$

The IH gives that $dom(B) \cap BV(t) = \emptyset$ and that $dom(B) \cap BV(u) = \emptyset$. Therefore, we have that $dom(B) \cap (BV(t) \cup BV(u)) = \emptyset$, i.e. that $dom(B) \cap BV(tu) = \emptyset$.

$$\triangleright \quad \frac{B \vdash t : \sigma \cup \tau \qquad B, x : \sigma \vdash u : \rho \qquad B, x : \tau \vdash u : \rho}{B \vdash u[t/x] : \rho}\ (\cup\mathbf{E})$$

The IH gives that $dom(B) \cap BV(t) = \emptyset$ and that $(dom(B) \cup \{x\}) \cap BV(u) = \emptyset$. The latter implies that $dom(B) \cap BV(u) = \emptyset$. Therefore, it is $dom(B) \cap (BV(u) \cup BV(t)) = \emptyset$, i.e. $dom(B) \cap BV(u[t/x]) = \emptyset$. $\dashv$

The next proposition concerns the top-down development of certain variables in a derivation.

Proposition 4.17 *Let π be a derivation in* IUT$^\oplus$*, R be a rule in π, and $B_1, \ldots, B_n$ be the bases in the branch connecting the conclusion of R to the root of π.*

(i) If R is $(\to\mathbf{I})$ *and x is the variable bounded in the course of R, then $x \notin \bigcup_{i=1}^{n} dom(B_i)$.*

(ii) If R is $(\cup\mathbf{E})$ *and x is the variable substituted in the course of R, then $x \notin \bigcup_{i=1}^{n} dom(B_i)$.*

Proof. We use induction on n for both (i) and (ii). We show (ii) below, noting that (i) is dealt with in a similar manner.

Base: If $n = 1$, we have the following picture.

$$\frac{B \vdash t : \sigma \cup \tau \qquad B, x : \sigma \vdash u : \rho \qquad B, x : \tau \vdash u : \rho}{\pi :: B_1 = B \vdash u[t/x] : \rho}\ R = (\cup\mathbf{E})$$

By the definition of "basis", we have that $x \notin dom(B) = dom(B_1)$.

Induction step: We suppose that $x \notin \bigcup_{i=1}^{n} dom(B_i)$ and seek to show that $x \notin \bigcup_{i=1}^{n+1} dom(B_i)$.

If a one-premise rule among $(\to\mathbf{I})$,$(\cap\mathbf{E})$, or $(\cup\mathbf{I})$ intervenes between B_n and B_{n+1} with B_n being the basis of the premise, it is $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$. If a two-premise rule among $(\to\mathbf{E})$ or $(\cap\mathbf{I})$ intervenes between B_n and B_{n+1} with B_n being the basis of either the left or the right premise, it is once again $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$. In all these cases, the result follows from the IH.

We examine the case of the three-premise $(\cup\mathbf{E})$ rule between B_n and B_{n+1} a bit more closely. If a $(\cup\mathbf{E})$ intervenes between B_n and B_{n+1} with B_n being the basis of the major premise, we have the following picture.

$$\frac{B \vdash t : \sigma \cup \tau \qquad B, x : \sigma \vdash u : \rho \qquad B, x : \tau \vdash u : \rho}{B_1 = B \vdash u[t/x] : \rho}\, R = (\cup\mathbf{E})$$

$$\vdots$$

$$\frac{\pi_0 :: B_n \vdash t' : \phi \cup \psi \qquad \pi_1 :: B_n,\, y : \phi \vdash u' : \upsilon \qquad \pi_2 :: B_n,\, y : \psi \vdash u' : \upsilon}{\pi :: B_{n+1} = B_n \vdash u'[t'/y] : \upsilon}\, (\cup\mathbf{E})$$

Since $B_{n+1} = B_n$, we have that $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$. Hence, the IH that $x \notin \bigcup_{i=1}^{n} dom(B_i)$ actually says that $x \notin \bigcup_{i=1}^{n+1} dom(B_i)$. [We note that the IH entails that $x \notin dom(B_n)$, so that it may be $y = x$.] If a $(\cup\mathbf{E})$ intervenes between B_n and B_{n+1} with B_n being the basis of a minor premise, the picture is reformed as follows.

$$\frac{B \vdash t : \sigma \cup \tau \qquad B, x : \sigma \vdash u : \rho \qquad B, x : \tau \vdash u : \rho}{B_1 = B \vdash u[t/x] : \rho}\, R = (\cup\mathbf{E})$$

$$\vdots$$

$$\frac{\pi_0 :: B' \vdash t' : \phi \cup \psi \qquad \pi_1 :: B_n = B' \cup \{\, y : \phi \,\} \vdash u' : \upsilon \qquad \pi_2 :: B',\, y : \psi \vdash u' : \upsilon}{\pi :: B_{n+1} = B' \vdash u'[t'/y] : \upsilon}\, (\cup\mathbf{E})$$

Since $B_{n+1} = B' \subsetneq B_n$, we once more have that $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$, which implies the result. [We note that the IH entails that $x \notin dom(B_n) = dom(B') \cup \{y\}$, so that $y \neq x$.] $\dashv$

4.3 Relating IUL_m to $\mathrm{IUT}^{\oplus}$ in natural deduction

Having completed the presentation of both the logic IUL_m and the type system $\mathrm{IUT}^{\oplus}$ in natural deduction style, we describe how the logic sets about accomplishing its definitional goal, which is the depiction of the type system on a logical level. To do this, we need the definitions of *non-standard decoration* for derivations in the logic and of *term-statement* for statements in the type system.

The so-called "non-standard" decoration of the logic is a decoration that does not encode every logical rule; it is actually dictated by the very rules of the type system[9] and hence encodes the implication, ignores the intersection (introduction and elimination) and the union introduction, and induces a substitution in the case of union elimination. Its formal definition is along the line given in 3.15 and its rules are shown in Figure 4.3.

Definition 4.18 (Term-statement) *Given a statement* $B = \{x_1 : \sigma_1, \ldots, x_m : \sigma_m\} \vdash t : \tau$ *in* $\mathrm{IUT}^{\oplus}$*, we define the* term-statement *deriving from it to be* $\{x_1, \ldots, x_m\} \vdash t$*, abbreviated* $x_1, \ldots, x_m \vdash t$*.*

To depict the type system $\mathrm{IUT}^{\oplus}$ on a logical level, we needed to define a logic with implication, intersection, and union, such that it admits a decoration encoding the implication, ignoring the intersection (introduction and elimination) and the union introduction, and inducing a substitution in the case of

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[9] This is because this decoration is in essence defined to achieve a correspondence between the logic and the type system in the perspective of a Curry-Howard correspondence. This correspondence is examined in detail in Chapter 5.

$$\frac{}{x : [(\Gamma_i, \sigma_i \,;\, \sigma_i)_i]_{p,\,x}}\ (\mathbf{ax}) \qquad \frac{t : [(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]_{p,\,y,\,x,\,q}}{t : [(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]_{p,\,x,\,y,\,q}}\ (\mathbf{X})$$

$$\frac{t : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}{\lambda x.t : [(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p}\ (\to\mathbf{I}) \qquad \frac{t : [(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p \qquad u : [(\Gamma_i \,;\, \sigma_i)_i]_p}{tu : [(\Gamma_i \,;\, \tau_i)_i]_p}\ (\to\mathbf{E})$$

$$\frac{t : [\mathcal{U}, ((\Gamma_i \,;\, \sigma_i), (\Gamma_i \,;\, \tau_i))_i, \mathcal{V}]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}]_p}\ (\cap\mathbf{I}) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}]_p}\ (\cap\mathbf{E}_1) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}]_p}\ (\cap\mathbf{E}_2)$$

$$\frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}]_p}\ (\cup\mathbf{I}_1) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}]_p}\ (\cup\mathbf{I}_2)$$

$$\frac{t : [(\Gamma_i \,;\, \sigma_i \cup \tau_i)_i]_p \qquad u : [((\Gamma_i, \sigma_i \,;\, \rho_i), (\Gamma_i, \tau_i \,;\, \rho_i))_i]_{p,\,x}}{u[t/x] : [(\Gamma_i \,;\, \rho_i)_i]_p}\ (\cup\mathbf{E})$$

Figure 4.3: Non-standard decoration of natural deduction IUL_m.

union elimination. For such a decoration to be feasible, the logic needed to have an $(\cap\mathbf{I})$ rule with a single premise and a $(\cup\mathbf{E})$ rule with a single minor-premise[10]. Indeed, the logic IUL_m, as defined in 4.1 and decorated in 4.3, uses the molecule structure to join together statements in the type system that share the same term-statement[11]. In the case of intersection introduction, the (decorated) logic merges into the same (decorated) molecule the left and right $\mathrm{IUT}^{\oplus}$-premises, in parallel for multiple rule instances that share the same term-statement[12].

$$\begin{array}{c} \dfrac{x_1 : \sigma_1^1, \ldots, x_m : \sigma_m^1 \vdash t : \tau_1 \qquad x_1 : \sigma_1^1, \ldots, x_m : \sigma_m^1 \vdash t : \rho_1}{x_1 : \sigma_1^1, \ldots, x_m : \sigma_m^1 \vdash t : \tau_1 \cap \rho_1}\ (\cap\mathbf{I})_1 \\ \vdots \\ \dfrac{x_1 : \sigma_1^n, \ldots, x_m : \sigma_m^n \vdash t : \tau_n \qquad x_1 : \sigma_1^n, \ldots, x_m : \sigma_m^n \vdash t : \rho_n}{x_1 : \sigma_1^n, \ldots, x_m : \sigma_m^n \vdash t : \tau_n \cap \rho_n}\ (\cap\mathbf{I})_n \end{array} \qquad \rightsquigarrow$$

---

[10] If the logic had an $(\cap\mathbf{I})$ with two premises, a decoration ignoring it would proceed only under the metatheoretical condition that the two premises are identically decorated. A similar remark holds for a $(\cup\mathbf{E})$ with two minor premises.

[11] This should be kept in mind with a small asterisk, as in the following two chapters we establish that it is *not every* set of (derivations proving) statements sharing the same term-statement that can be joined into a single (derivation proving a) decorated molecule, which actually renders IUL_m inappropriate as a logic for $\mathrm{IUT}^{\oplus}$ (see Section 6.3). It would be more accurate at this point to say that we *assume* that, as in the case of the intersection molecule-logic with respect to the intersection type system, the intersection-and-union molecule-logic IUL_m uses the molecule structure to join together statements in the intersection-and-union type system $\mathrm{IUT}^{\oplus}$ that share the same term-statement.

[12] Obviously, the term-statement of an $(\cap\mathbf{I})$ instance with premises $B \vdash t : \tau$, $B \vdash t : \rho$ and conclusion $B \vdash t : \tau \cap \rho$, where $dom(B) = \{x_1, \ldots, x_m\}$, is meant to be $x_1, \ldots, x_m \vdash t$.

$$\frac{t : [\mathcal{U}, (\sigma_1^1, \ldots, \sigma_m^1 \,;\, \tau_1), (\sigma_1^1, \ldots, \sigma_m^1 \,;\, \rho_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n \,;\, \tau_n), (\sigma_1^n, \ldots, \sigma_m^n \,;\, \rho_n), \mathcal{V}\,]_{\,x_1, \ldots, x_m}}{t : [\mathcal{U}, (\sigma_1^1, \ldots, \sigma_m^1 \,;\, \tau_1 \cap \rho_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n \,;\, \tau_n \cap \rho_n), \mathcal{V}\,]_{\,x_1, \ldots, x_m}}\ (\cap\mathbf{I})$$

Likewise, in the case of union elimination, the (decorated) logic merges into the same (decorated) molecule the left and right minor IUT$^\oplus$-premises, in parallel for multiple rule instances whose corresponding statements share the same term-statement.

$$\begin{array}{cc}
\dfrac{B_1 \vdash t : \tau_1 \cup \rho_1 \qquad B_1,\, x : \tau_1 \vdash u : \upsilon_1 \qquad B_1,\, x : \rho_1 \vdash u : \upsilon_1}{B_1 = \{\, x_1 : \sigma_1^1, \ldots,\, x_m : \sigma_m^1 \,\} \vdash u[t/x] : \upsilon_1}\ (\cup\mathbf{E})_1 & \\
\vdots & \rightsquigarrow \\
\dfrac{B_n \vdash t : \tau_n \cup \rho_n \qquad B_n,\, x : \tau_n \vdash u : \upsilon_n \qquad B_n,\, x : \rho_n \vdash u : \upsilon_n}{B_n = \{\, x_1 : \sigma_1^n, \ldots,\, x_m : \sigma_m^n \,\} \vdash u[t/x] : \upsilon_n}\ (\cup\mathbf{E})_n &
\end{array}$$

$$\frac{t : [(\Gamma_1 \,;\, \tau_1 \cup \rho_1), \ldots, (\Gamma_n \,;\, \tau_n \cup \rho_n)]_p \qquad u : [(\Gamma_1, \tau_1 \,;\, \upsilon_1), (\Gamma_1, \rho_1 \,;\, \upsilon_1), \ldots, (\Gamma_n, \tau_n \,;\, \upsilon_n), (\Gamma_n, \rho_n \,;\, \upsilon_n)]_{p,\, x}}{u[t/x] : [(\Gamma_1 = \sigma_1^1, \ldots, \sigma_m^1 \,;\, \upsilon_1), \ldots, (\Gamma_n = \sigma_1^n, \ldots, \sigma_m^n \,;\, \upsilon_n)]_{p\, =\, x_1, \ldots,\, x_m}}\ (\cup\mathbf{E})$$

A similar note is given in Chapter 3 to explain how the **(**$\cup$**E)** in IUL$_m$, as IUL$_m$ is presented there, uses the molecule structure to join together the isomorphic minor premises of the **(**$\vee$**E)** in MLns (see p. 52).

Considering the logic and the type system as presented in this chapter, we (re)examine their correspondence in the following chapter. We there reconsider the handling of substitution terms, an issue that blocked a complete solution to the correspondence problem back in Chapter 3 (see subsections 3.1.2 and 3.2.2).

CHAPTER 5

Correspondence between IUL$_m$ and IUT$^\oplus$

We aim to achieve a correspondence between the natural deduction logic IUL$_m$ and the natural deduction type system IUT$^\oplus$ through the non-standard decoration of the logic, given in the previous chapter. Toward this end, we first define the notions "tree with terms" and "tree of implications and union eliminations with terms" for both the decorated logic and the type system. We then state and prove theorems of correspondence, which strongly depend on restrictions involving the latter notion. We finally examine if and to what extent we can get rid of these restrictions.

5.1 Trees of iue with terms

To obtain some kind of correspondence between the decorated logic IUL$_m^\star$ and the type system IUT$^\oplus$, we will need the auxiliary notion of *tree of implications and union eliminations with terms*, defined for both IUL$_m^\star$ and IUT$^\oplus$. The definition of this notion is based on the definition of the notion of *tree with terms*, for both systems.

Definition 5.1 (IUL$_m^\star$: Tree with terms $T^{\,t}$) *(i) Given a decorated molecule* $t : \mathcal{M}_p$ *in* IUL$_m^\star$*, we define the* decoration-statement *deriving from it to be the statement* $\{p\} \vdash t$ *with set-context* $\{p\}$*. We may abbreviate the decoration-statement as* $p \vdash t$*.*

(ii) Given the tree T *of a derivation* $\pi^\star$ *in* IUL$_m^\star$ *and the fact that each node of the tree represents a decorated molecule in* $\pi^\star$*, the* tree with terms $T^{\,t}$ *of* $\pi^\star$ *is* T *with each node decorated by the decoration-statement deriving from the node's decorated molecule.*

Definition 5.2 (IUL$_m^\star$: Tree of implics and union elimins with terms $T^{\,t}_{\text{iue}}$) *We derive the* tree of implications and union eliminations with terms $T^{\,t}_{\text{iue}}$ *of a derivation* $\pi^\star$ *in* IUL$_m^\star$ *from the tree with terms* $T^{\,t}$ *of* $\pi^\star$ *by erasing all nodes and corresponding decoration-statements associated to the rules* **(X)***,***(∩IE)***, and* **(∪I)***.*

Remark 5.3 *The procedure of erasing nodes and corresponding decoration-statements associated to the rules* **(X)***,***(∩IE)***, and* **(∪I)** *is well-defined, since these rules provide, when decorated, the same decoration-statement in premise and conclusion. This fact also implies that the tree* $T^{\,t}_{\text{iue}}$ *displays at the root the same decoration-statement as the tree* $T^{\,t}$*.*

Example 5.4 (IUL$_m^\star$: $T^{\,t}$ and $T^{\,t}_{\text{iue}}$) *If* $\sigma = (\alpha \cup \beta) \cap (\alpha \cup \gamma)$ *and* $\tau = (\alpha \to \delta \cap \varepsilon) \cap (\beta \to \delta) \cap (\gamma \to \varepsilon)$*, we consider the* IUL$_m^\star$*-derivation* $\pi^\star :: \lambda y.yx : [(\sigma\,;\,(\tau \to \delta \cap \varepsilon) \cup \zeta)]_x$*, exhibited below, and present its* *trees* $T^{\,t}$ *and* $T^{\,t}_{\mathrm{iue}}$*. For space economy, we denote* τ_1 *the type* $(\alpha \to \delta \cap \varepsilon) \cap (\beta \to \delta)$ *and* $\pi^{\star}_{11}$ *the decorated axiom* $z : [(\sigma,\tau,\alpha\,;\,\alpha),(\sigma,\tau,\beta\,;\,\beta),(\sigma,\tau,\alpha\,;\,\alpha),(\sigma,\tau,\gamma\,;\,\gamma)]_{\,x,\,y,\,z}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{x : [(\tau,\sigma\,;\,\sigma),(\tau,\sigma\,;\,\sigma)]_{\,y,\,x}}{x : [(\tau,\sigma\,;\,\alpha\cup\beta),(\tau,\sigma\,;\,\sigma)]_{\,y,\,x}}\,(\cap\mathbf{E}_1)}{x : [(\tau,\sigma\,;\,\alpha\cup\beta),(\tau,\sigma\,;\,\alpha\cup\gamma)]_{\,y,\,x}}\,(\cap\mathbf{E}_2)}{\pi^{\star}_0 :: x : [(\sigma,\tau\,;\,\alpha\cup\beta),(\sigma,\tau\,;\,\alpha\cup\gamma)]_{\,x,\,y}}\,(\mathbf{X}) \qquad \overset{\text{see below}}{\pi^{\star}_1 :: yz : [(\sigma,\tau,\alpha\,;\,\delta),(\sigma,\tau,\beta\,;\,\delta),(\sigma,\tau,\alpha\,;\,\varepsilon),(\sigma,\tau,\gamma\,;\,\varepsilon)]_{\,x,\,y,\,z}}}{\dfrac{\dfrac{\dfrac{yz[x/z] = yx : [(\sigma,\tau\,;\,\delta),(\sigma,\tau\,;\,\varepsilon)]_{\,x,\,y}}{yx : [(\sigma,\tau\,;\,\delta\cap\varepsilon)]_{\,x,\,y}}\,(\cap\mathbf{I})}{\lambda y.\,yx : [(\sigma\,;\,\tau\to\delta\cap\varepsilon)]_{\,x}}\,(\to\mathbf{I})}{\pi^{\star} :: \lambda y.\,yx : [(\sigma\,;\,(\tau\to\delta\cap\varepsilon)\cup\zeta)]_{\,x}}\,(\cup\mathbf{I})}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : [(\sigma,\alpha,\tau\,;\,\tau),(\sigma,\beta,\tau\,;\,\tau),(\sigma,\alpha,\tau\,;\,\tau),(\sigma,\gamma,\tau\,;\,\tau)]_{\,x,\,z,\,y}}{y : [(\sigma,\alpha,\tau\,;\,\tau_1),(\sigma,\beta,\tau\,;\,\tau_1),(\sigma,\alpha,\tau\,;\,\tau_1),(\sigma,\gamma,\tau\,;\,\tau)]_{\,x,\,z,\,y}}\,(\cap\mathbf{E}_1)}{y : [(\sigma,\alpha,\tau\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\beta,\tau\,;\,\tau_1),(\sigma,\alpha,\tau\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\gamma,\tau\,;\,\tau)]_{\,x,\,z,\,y}}\,(\cap\mathbf{E}_1)}{y : [(\sigma,\alpha,\tau\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\beta,\tau\,;\,\beta\to\delta),(\sigma,\alpha,\tau\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\gamma,\tau\,;\,\gamma\to\varepsilon)]_{\,x,\,z,\,y}}\,(\cap\mathbf{E}_2)}{y : [(\sigma,\tau,\alpha\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\tau,\beta\,;\,\beta\to\delta),(\sigma,\tau,\alpha\,;\,\alpha\to\delta\cap\varepsilon),(\sigma,\tau,\gamma\,;\,\gamma\to\varepsilon)]_{\,x,\,y,\,z}}\,(\mathbf{X}) \qquad \text{axiom } \pi^{\star}_{11}}{\dfrac{\dfrac{yz : [(\sigma,\tau,\alpha\,;\,\delta\cap\varepsilon),(\sigma,\tau,\beta\,;\,\delta),(\sigma,\tau,\alpha\,;\,\delta\cap\varepsilon),(\sigma,\tau,\gamma\,;\,\varepsilon)]_{\,x,\,y,\,z}}{yz : [(\sigma,\tau,\alpha\,;\,\delta),(\sigma,\tau,\beta\,;\,\delta),(\sigma,\tau,\alpha\,;\,\delta\cap\varepsilon),(\sigma,\tau,\gamma\,;\,\varepsilon)]_{\,x,\,y,\,z}}\,(\cap\mathbf{E}_1)}{\pi^{\star}_1 :: yz : [(\sigma,\tau,\alpha\,;\,\delta),(\sigma,\tau,\beta\,;\,\delta),(\sigma,\tau,\alpha\,;\,\varepsilon),(\sigma,\tau,\gamma\,;\,\varepsilon)]_{\,x,\,y,\,z}}\,(\cap\mathbf{E}_2)}\,(\to\mathbf{E})$$

To facilitate the layout, the trees $T^{\,t}$ *and* $T^{\,t}_{\mathrm{iue}}$ *of* $\pi^{\star}$ *are displayed on the next page in Figure 5.1, where* S *denotes the set* $\{x,y\}$ *and* S,z *the set* $\{x,y,z\}$.

We next define the tree with terms of a derivation in $\mathrm{IUT}^{\oplus}$ and then provide an algorithm for constructing the tree of implications and union eliminations with terms of such a derivation, given its tree with terms.

Definition 5.5 ($\mathrm{IUT}^{\oplus}$: Tree with terms $T^{\,t}$) *Given the tree* T *of a derivation* π *in* $\mathrm{IUT}^{\oplus}$ *and the fact that each node of the tree represents a statement in* π*, the* tree with terms $T^{\,t}$ *of* π *is* T *with each node decorated by the term-statement deriving from the node's statement.*

Definition 5.6 ($\mathrm{IUT}^{\oplus}$: Tree of implics and union elimins with terms $T^{\,t}_{\mathrm{iue}}$) *We derive the* tree of implications and union eliminations with terms $T^{\,t}_{\mathrm{iue}}$ *of a derivation* π *in* $\mathrm{IUT}^{\oplus}$ *from the tree with terms* $T^{\,t}$ *of* π *by the following algorithm.*

▷ *We choose a topmost* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$ *in the tree with terms of* π*, i.e. an* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$ *that has no other* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$ *above it. Then, we erase all nodes and corresponding term-statements associated to* $(\cap\mathbf{E})$ *or* $(\cup\mathbf{I})$ *in the trees with terms of all premises. If the topmost rule-inference chosen is an* $(\cap\mathbf{I})$ *and the resulting premise trees of implications with terms are identical, i.e. if they share the same rule structure and the same term-statements at corresponding nodes, we identify them and erase the node and corresponding term-statement associated to the* $(\cap\mathbf{I})$*. If the topmost rule-inference chosen is a* $(\cup\mathbf{E})$ *and the resulting minor-premise trees of implications with terms are identical, we identify them and keep a single minor-premise tree of implications with terms, so that the node associated to the* $(\cup\mathbf{E})$ *becomes a two-children node.*

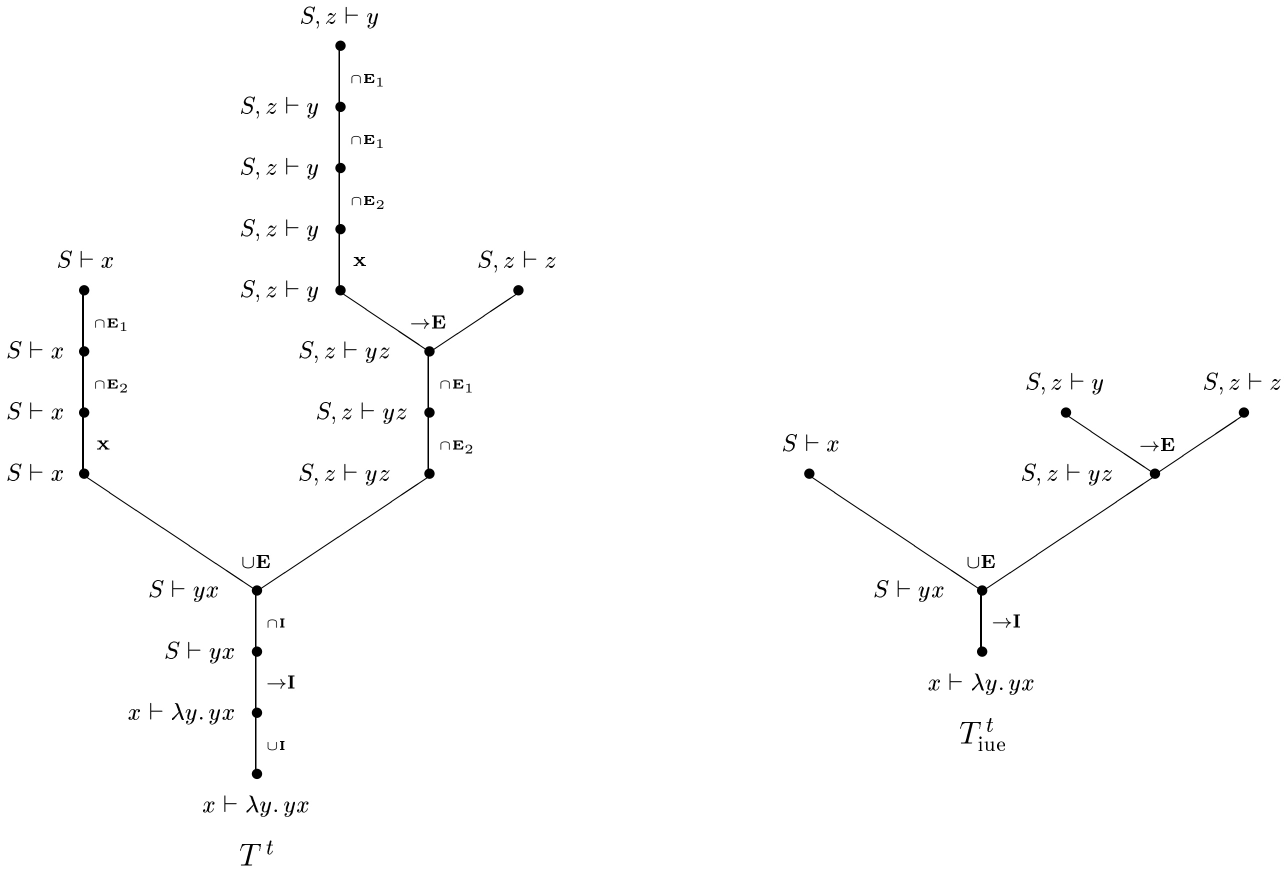


Figure 5.1: The trees $T^{\,t}$ and T^{t}_{iue} of $\pi^{\star}$ in Example 5.4.

▷ *We iterate the above procedure for the tree with terms resulting from the previous step. At any step $n > 1$, we ignore any two-children* $(\cup\mathbf{E})$*'s, when choosing the step's topmost* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$*, and the trees with terms resulting from the premises of the topmost* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$ *chosen—after erasing nodes and corresponding term-statements associated to* $(\cap\mathbf{E})$ *or* $(\cup\mathbf{I})$*—are, in general, trees of implications and union eliminations with terms, not merely trees of implications with terms, as they were at step 1.*

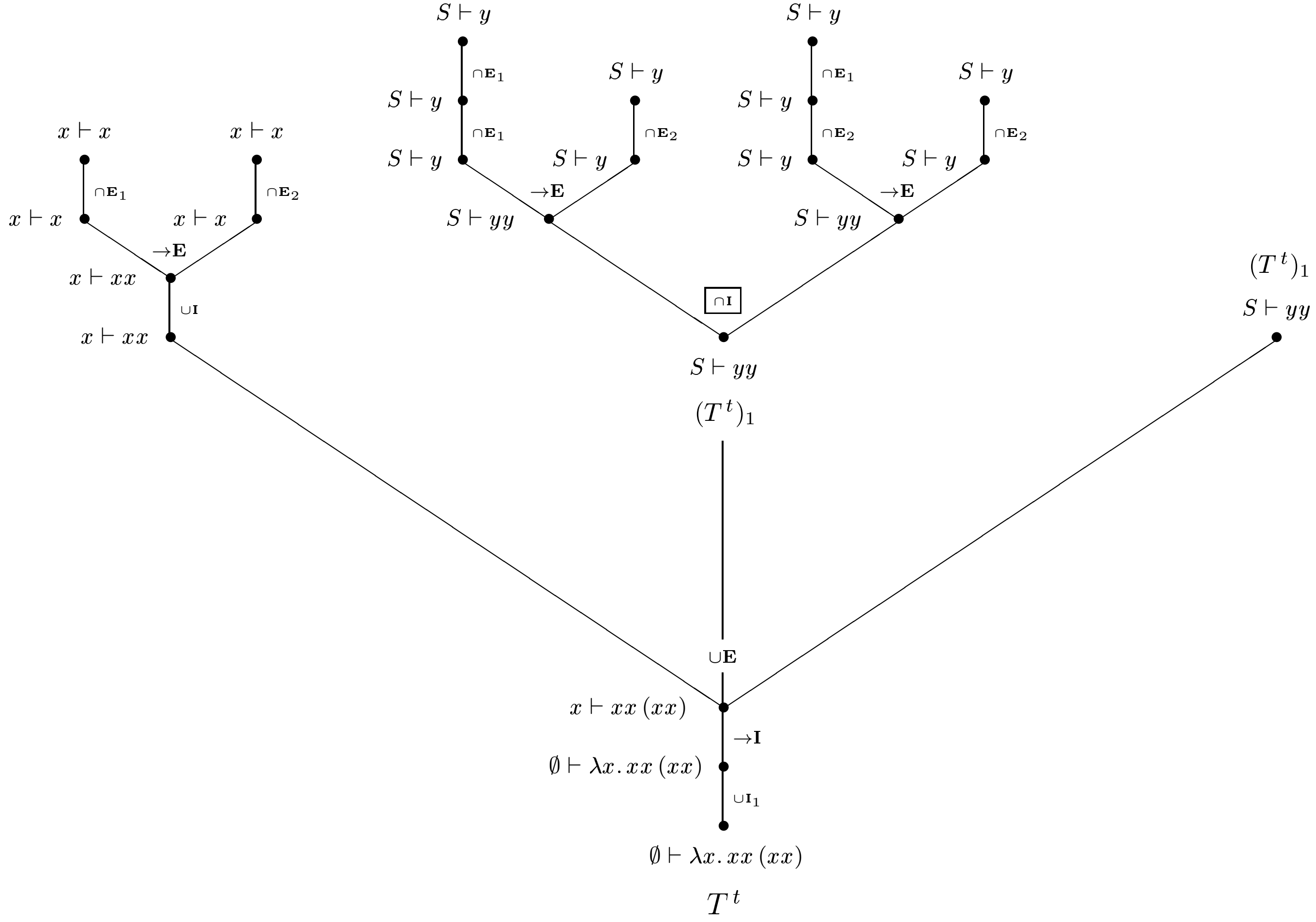
▷ *When all the* $(\cap\mathbf{I})$*'s and* $(\cup\mathbf{E})$*'s have been dealt with, we make a final step to erase any remaining nodes and corresponding term-statements associated to* $(\cap\mathbf{E})$ *or* $(\cup\mathbf{I})$*.*

Remark 5.7 *Since the rules* $(\cap\mathbf{E})$ *and* $(\cup\mathbf{I})$ *display the same term-statement in premise and conclusion, a tree of implications and union eliminations with terms attained from a topmost-*$(\cap\mathbf{I})$ *or a topmost-*$(\cup\mathbf{E})$ *premise, after erasing nodes and corresponding term-statements associated to* $(\cap\mathbf{E})$ *or* $(\cup\mathbf{I})$*, is well-defined and has a term-statement at the root which is identical to the term-statement at the root of the premise's tree with terms. Moreover, since the* $(\cap\mathbf{I})$ *rule displays the same term-statement in premises and conclusion, a tree of implications and union eliminations with terms attained from a topmost-*$(\cap\mathbf{I})$ *tree with terms, after identifying matching premise trees of implications and union eliminations with terms and erasing the* $(\cap\mathbf{I})$ *node and its corresponding term-statement, has a term-statement at the root which is identical to the term-statement at the root of the topmost-*$(\cap\mathbf{I})$ *tree with terms in question. Given a topmost-*$(\cup\mathbf{E})$ *tree with terms, there is obviously no alteration in the term-statement at the root, after identifying matching minor-premise trees of implications and union eliminations with terms. The fact that* $(\cap\mathbf{E})$ *and* $(\cup\mathbf{I})$ *display the same term-statement in premise and conclusion is once more used to argue that a final algorithmic step concerning such rule-inferences does not alter the term-statement at the root or anywhere else. So, in conclusion, the procedure described by the algorithm in 5.6 is well-defined and the final tree* T^t_{iue} *attained, if the algorithm terminates, has a term-statement at the root identical to the term-statement at the root of the original tree* T^t*.*

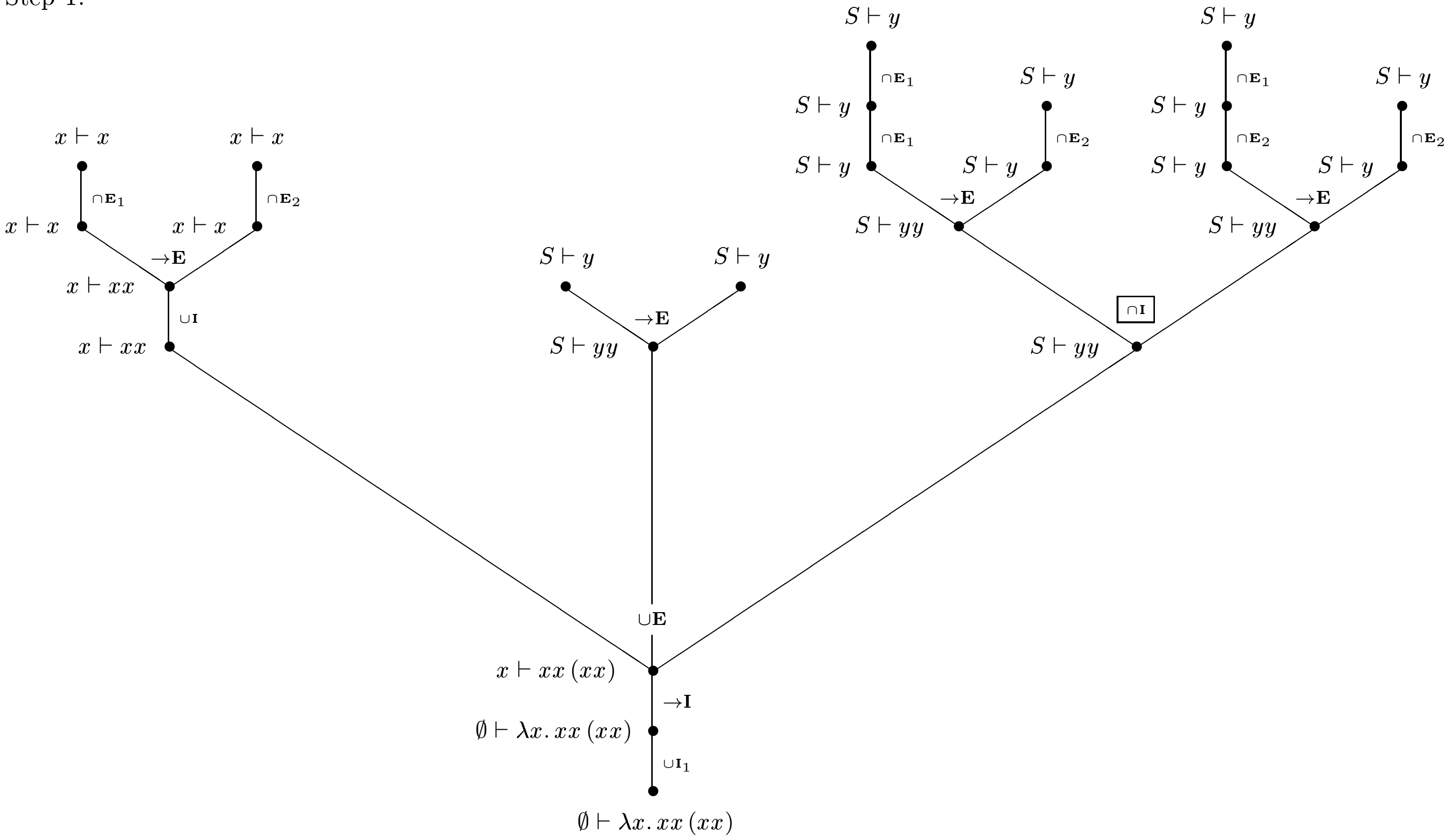
Example 5.8 ($\mathbf{IUT}^{\oplus}$: T^t and T^t_{iue}) *If* $\sigma = (\gamma \to \alpha) \cap (\gamma \to \beta) \cap \gamma$ *and* $\tau = (\delta \to \sigma) \cap \delta$*, we consider the* $\mathrm{IUT}^{\oplus}$*-derivation* $\pi :: \emptyset \vdash \lambda x.\, xx\,(xx) : (\tau \to \alpha \cap \beta) \cup \varepsilon$*, as shown below. We denote* σ_1 *the type* $(\gamma \to \alpha) \cap (\gamma \to \beta)$ *and* B *the basis* $\{\, x : \tau,\, y : \sigma \,\}$*. We then demonstrate the tree* T^t *of* π *and the procedure to attain the tree* T^t_{iue} *of* π *from it in four steps. In trees, the letter* S *stands for the set* $\{x, y\}$*, while the topmost* $(\cap\mathbf{I})$ *or* $(\cup\mathbf{E})$ *chosen is enclosed in a box.*

$$
\dfrac{
\overset{\text{see below}}{\pi_0 :: x : \tau \vdash xx : \sigma \cup \sigma}
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{B \vdash y : \sigma}{B \vdash y : \sigma_1}\,(\cap\mathbf{E}_1)}{B \vdash y : \gamma \to \alpha}\,(\cap\mathbf{E}_1)
\quad
\dfrac{B \vdash y : \sigma}{B \vdash y : \gamma}\,(\cap\mathbf{E}_2)
}{B \vdash yy : \alpha}\,(\to\mathbf{E})
\quad
\dfrac{
\dfrac{\dfrac{B \vdash y : \sigma}{B \vdash y : \sigma_1}\,(\cap\mathbf{E}_1)}{B \vdash y : \gamma \to \beta}\,(\cap\mathbf{E}_2)
\quad
\dfrac{B \vdash y : \sigma}{B \vdash y : \gamma}\,(\cap\mathbf{E}_2)
}{B \vdash yy : \beta}\,(\to\mathbf{E})
}{\pi_1 :: B = \{\, x : \tau,\, y : \sigma \,\} \vdash yy : \alpha \cap \beta}\,(\cap\mathbf{I})
\qquad
\pi_1
}{
\dfrac{
\dfrac{x : \tau \vdash yy[xx/y] = xx\,(xx) : \alpha \cap \beta}{\emptyset \vdash \lambda x.\, xx\,(xx) : \tau \to \alpha \cap \beta}\,(\to\mathbf{I})
}{\pi :: \emptyset \vdash \lambda x.\, xx\,(xx) : (\tau \to \alpha \cap \beta) \cup \varepsilon}\,(\cup\mathbf{I}_1)
}\,(\cup\mathbf{E})
$$

$$
\dfrac{
\dfrac{
\dfrac{x : \tau \vdash x : \tau}{x : \tau \vdash x : \delta \to \sigma}\,(\cap\mathbf{E}_1)
\quad
\dfrac{x : \tau \vdash x : \tau}{x : \tau \vdash x : \delta}\,(\cap\mathbf{E}_2)
}{x : \tau \vdash xx : \sigma}\,(\to\mathbf{E})
}{\pi_0 :: x : \tau \vdash xx : \sigma \cup \sigma}\,(\cup\mathbf{I})
$$


$x \vdash x$
$\cap\mathbf{E}_1$
$x \vdash x$
$x \vdash x$
$\cap\mathbf{E}_2$
$x \vdash x$
$\to\mathbf{E}$
$x \vdash xx$
$\cup\mathbf{I}$
$x \vdash xx$
$S \vdash y$
$\cap\mathbf{E}_1$
$S \vdash y$
$\cap\mathbf{E}_1$
$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_2$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$S \vdash y$
$\cap\mathbf{E}_1$
$S \vdash y$
$\cap\mathbf{E}_2$
$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_2$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$\boxed{\cap\mathbf{I}}$
$S \vdash yy$
$(T^t)_1$
$(T^t)_1$
$S \vdash yy$
$\cup\mathbf{E}$
$x \vdash xx\,(xx)$
$\to\mathbf{I}$
$\emptyset \vdash \lambda x.\, xx\,(xx)$
$\cup\mathbf{I}_1$
$\emptyset \vdash \lambda x.\, xx\,(xx)$
T^t

Step 1:


$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_1$
$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_1$
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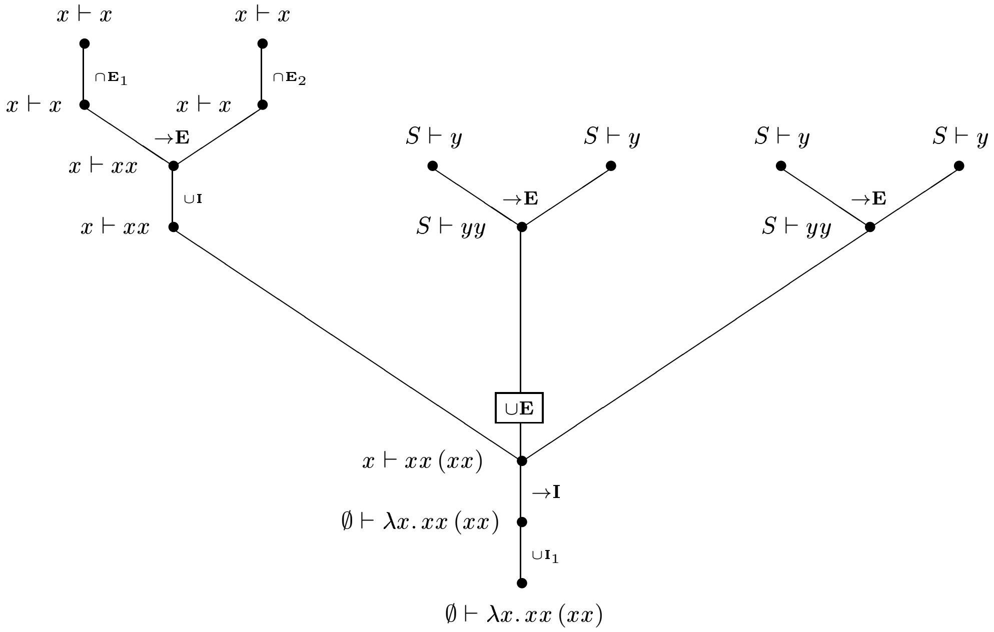
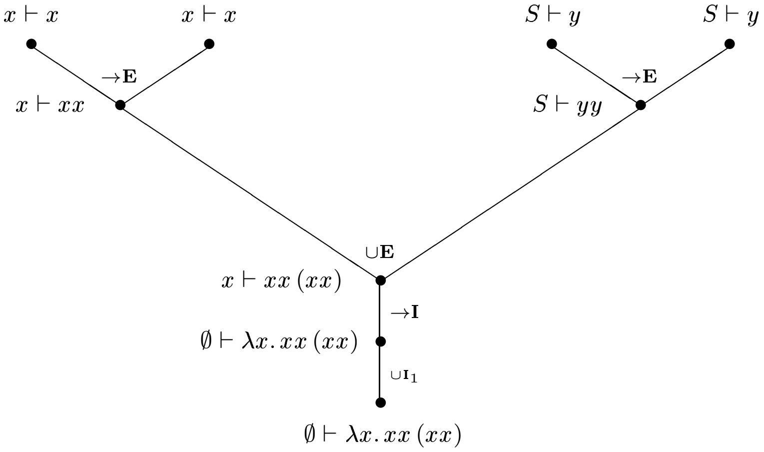
$S \vdash y$
$\cap\mathbf{E}_1$
$\cap\mathbf{E}_2$
$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_1$
$\cap\mathbf{E}_2$
$\cap\mathbf{E}_2$
$S \vdash y$
$S \vdash y$
$\cap\mathbf{E}_2$
$\to\mathbf{E}$
$\to\mathbf{E}$
$S \vdash yy$
$S \vdash yy$
$x \vdash x$
$x \vdash x$
$\cap\mathbf{E}_1$
$\cap\mathbf{E}_2$
$x \vdash x$
$x \vdash x$
$\to\mathbf{E}$
$x \vdash xx$
$\cup\mathbf{I}$
$x \vdash xx$
$S \vdash y$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$\boxed{\cap\mathbf{I}}$
$S \vdash yy$
$\cup\mathbf{E}$
$x \vdash xx\,(xx)$
$\to\mathbf{I}$
$\emptyset \vdash \lambda x.\, xx\,(xx)$
$\cup\mathbf{I}_1$
$\emptyset \vdash \lambda x.\, xx\,(xx)$

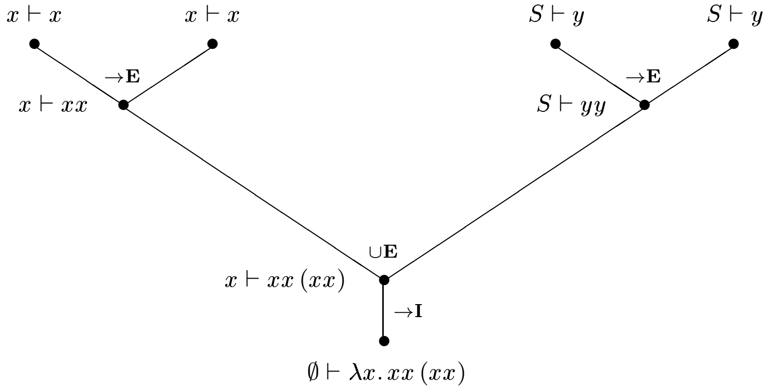
Step 2:


$x \vdash x$
$x \vdash x$
$\cap\mathbf{E}_1$
$\cap\mathbf{E}_2$
$x \vdash x$
$x \vdash x$
$\to\mathbf{E}$
$x \vdash xx$
$\cup\mathbf{I}$
$x \vdash xx$
$S \vdash y$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$S \vdash y$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$\boxed{\cup\mathbf{E}}$
$x \vdash xx\,(xx)$
$\to\mathbf{I}$
$\emptyset \vdash \lambda x.\, xx\,(xx)$
$\cup\mathbf{I}_1$
$\emptyset \vdash \lambda x.\, xx\,(xx)$

Step 3:


$x \vdash x$
$x \vdash x$
$\to\mathbf{E}$
$x \vdash xx$
$S \vdash y$
$S \vdash y$
$\to\mathbf{E}$
$S \vdash yy$
$\cup\mathbf{E}$
$x \vdash xx\,(xx)$
$\to\mathbf{I}$
$\emptyset \vdash \lambda x.\, xx\,(xx)$
$\cup\mathbf{I}_1$
$\emptyset \vdash \lambda x.\, xx\,(xx)$

Step 4:



T^t_{iue}

The algorithm in 5.6 stops in case the trees of implications and union eliminations with terms attained from the premises of a topmost $(\cap\mathbf{I})$ or from the minor premises of a topmost $(\cup\mathbf{E})$—after erasing nodes and corresponding term-statements associated to $(\cap\mathbf{E})$ or $(\cup\mathbf{I})$—do not coincide. The next example puts up an $\mathrm{IUT}^{\oplus}$-derivation for which the algorithm does not terminate.

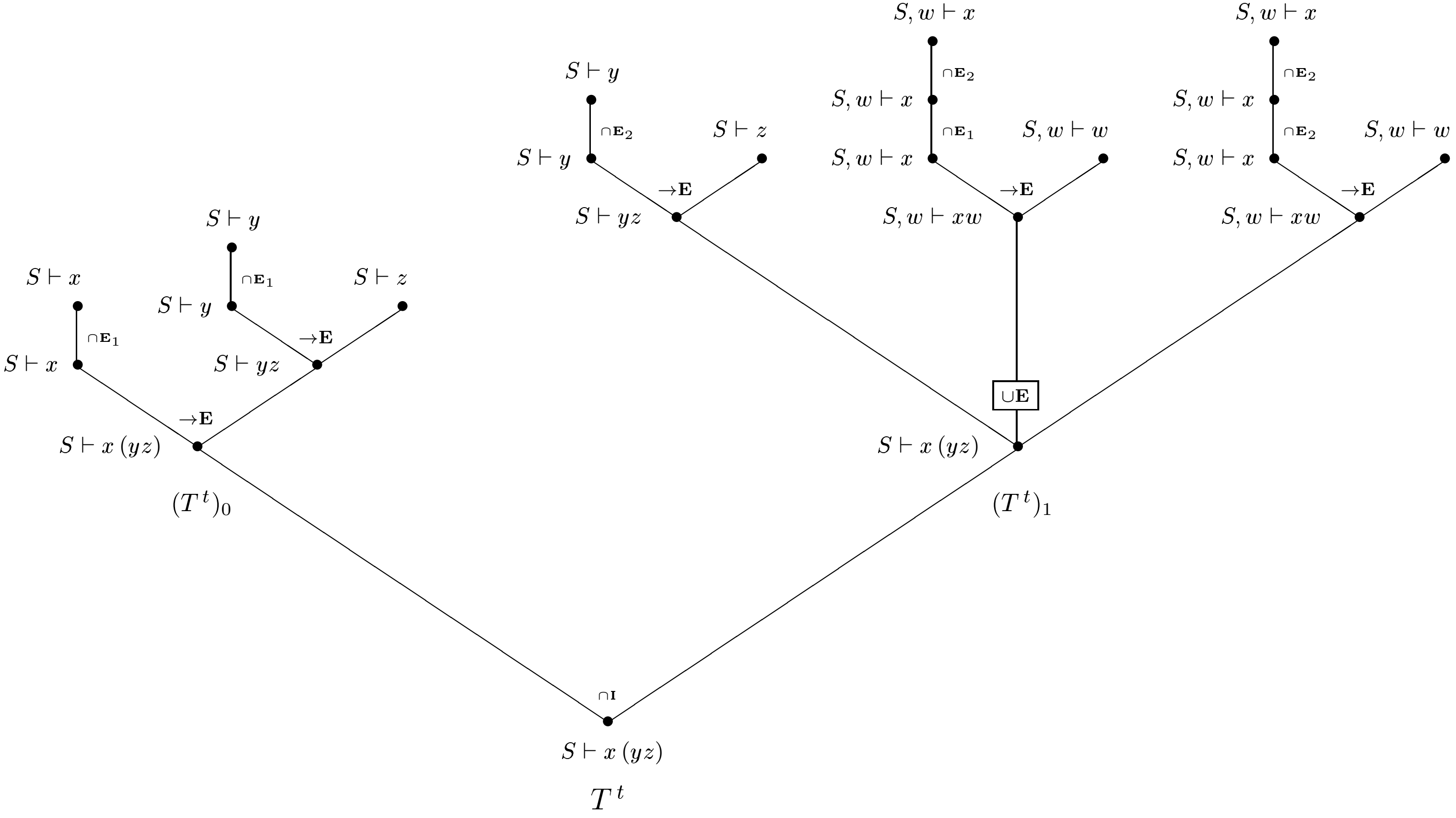
Example 5.9 ($\mathbf{IUT}^{\oplus}$: no T^t_{iue}) *If $\sigma = (\gamma \to \alpha) \cap ((\delta \to \beta) \cap (\varepsilon \to \beta))$, $\tau = (\zeta \to \gamma) \cap (\zeta \to \delta \cup \varepsilon)$, and $B = \{\, x : \sigma,\ y : \tau,\ z : \zeta \,\}$, we consider the $\mathrm{IUT}^{\oplus}$-derivation $\pi :: B \vdash x\,(yz) : \alpha \cap \beta$, as shown below. We denote σ_2 the type $(\delta \to \beta) \cap (\varepsilon \to \beta)$.*

$$\dfrac{\overset{\text{see below}}{\pi_0 :: B \vdash x\,(yz) : \alpha} \qquad \overset{\text{see below}}{\pi_1 :: B \vdash x\,(yz) : \beta}}{\pi :: B \vdash x\,(yz) : \alpha \cap \beta}\,(\cap\mathbf{I})$$

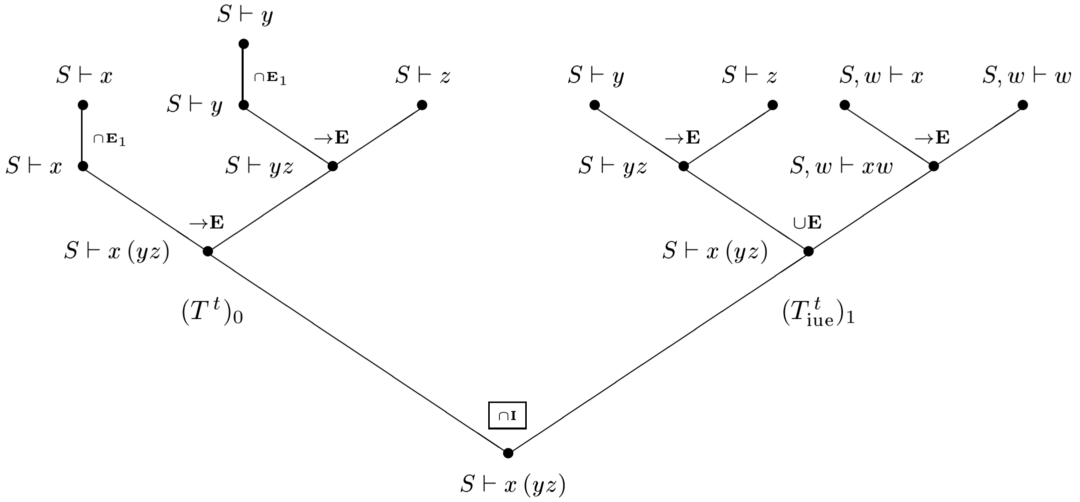
$$\dfrac{\dfrac{B \vdash x : \sigma}{B \vdash x : \gamma \to \alpha}\,(\cap\mathbf{E}_1) \qquad \dfrac{\dfrac{B \vdash y : \tau}{B \vdash y : \zeta \to \gamma}\,(\cap\mathbf{E}_1) \qquad B \vdash z : \zeta}{B \vdash yz : \gamma}\,(\to\mathbf{E})}{\pi_0 :: B \vdash x\,(yz) : \alpha}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B \vdash y : \tau}{B \vdash y : \zeta \to \delta \cup \varepsilon}\,(\cap\mathbf{E}_2) \quad B \vdash z : \zeta}{B \vdash yz : \delta \cup \varepsilon}\,(\to\mathbf{E}) \quad \dfrac{\dfrac{\dfrac{B,\, w : \delta \vdash x : \sigma}{B,\, w : \delta \vdash x : \sigma_2}\,(\cap\mathbf{E}_2)}{B,\, w : \delta \vdash x : \delta \to \beta}\,(\cap\mathbf{E}_1) \quad B,\, w : \delta \vdash w : \delta}{B,\, w : \delta \vdash xw : \beta}\,(\to\mathbf{E}) \quad \dfrac{\dfrac{\dfrac{B,\, w : \varepsilon \vdash x : \sigma}{B,\, w : \varepsilon \vdash x : \sigma_2}\,(\cap\mathbf{E}_2)}{B,\, w : \varepsilon \vdash x : \varepsilon \to \beta}\,(\cap\mathbf{E}_2) \quad B,\, w : \varepsilon \vdash w : \varepsilon}{B,\, w : \varepsilon \vdash xw : \beta}\,(\to\mathbf{E})}{\pi_1 :: B \vdash x\,(yz) : \beta}\,(\cup\mathbf{E})$$

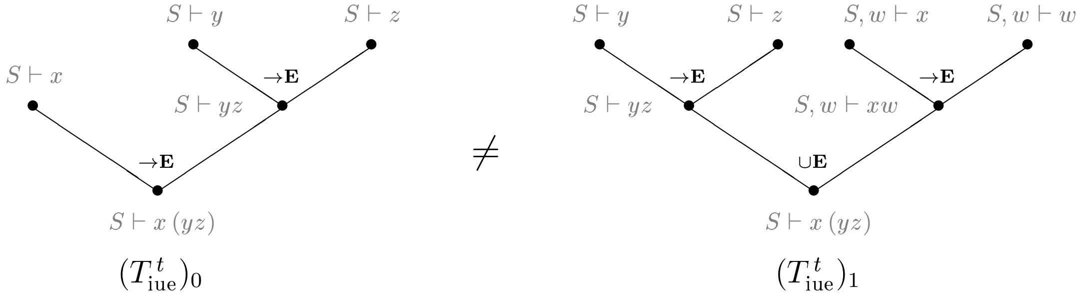
The tree T^t of π is displayed on the next page, where S denotes the set $\{x, y, z\}$. We then elaborate on the steps of the algorithm in 5.6 in order to spot the problem in obtaining a tree T^t_{iue} of π.


$S, w \vdash x$
$S, w \vdash x$
$\cap\mathbf{E}_2$
$\cap\mathbf{E}_2$
$S \vdash y$
$S, w \vdash x$
$S, w \vdash x$
$\cap\mathbf{E}_2$
$\cap\mathbf{E}_1$
$\cap\mathbf{E}_2$
$S \vdash z$
$S, w \vdash w$
$S, w \vdash w$
$S \vdash y$
$S, w \vdash x$
$S, w \vdash x$
$\to\mathbf{E}$
$\to\mathbf{E}$
$\to\mathbf{E}$
$S \vdash yz$
$S, w \vdash xw$
$S, w \vdash xw$
$S \vdash y$
$S \vdash x$
$\cap\mathbf{E}_1$
$S \vdash z$
$S \vdash y$
$\cap\mathbf{E}_1$
$\to\mathbf{E}$
$S \vdash x$
$S \vdash yz$
$\cup\mathbf{E}$
$\to\mathbf{E}$
$S \vdash x\,(yz)$
$S \vdash x\,(yz)$
$(T^{\,t})_0$
$(T^{\,t})_1$
$\cap\mathbf{I}$
$S \vdash x\,(yz)$
$T^{\,t}$

Step 1:



Step 2:



Step 2 cannot be completed, as the trees of implications and union eliminations with terms obtained from the premises of $(\cap\mathbf{I})$ *are not identical, i.e. it is* $(T^t_{\text{iue}})_0 \neq (T^t_{\text{iue}})_1$. *Therefore, the algorithm stops and there is no tree* T^t_{iue} *of* π.

5.2 Restricted correspondence theorems

Having defined the notion "tree of implications and union eliminations with terms" for both the decorated logic and the type system, we can now use it to state and prove theorems of correspondence between the two systems. The inevitable restriction[1] which the use of this notion[2] poses on the correspondence forces us to call these theorems "*restricted* correspondence theorems".

---

[1] The restriction is meant in comparison to the correspondence achieved in Chapter 1 between the decorated logic ISL and the type system IT (see Theorem 1.20).

[2] A detailed justification of this notion's necessity in securing the correspondence is offered in Section 5.4.

Theorem 5.10 (From IUL$_m$ to IUT$^\oplus$) *If* $\pi^\star :: t : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i)_{i=1}^n]_{x_1,\ldots,x_m}$ *is a decorated derivation in* IUL$_m$*, there are derivations* $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *in* IUT$^\oplus$*, such that 1.* $(T_{\text{iue}}^t)_i$ *exists, 2.* $(T_{\text{iue}}^t)_i = (T_{\text{iue}}^t)_j$ $(1 \leqslant i \neq j \leqslant n)$*, and 3.* $(T_{\text{iue}}^t)_i = (T_{\text{iue}}^t)_{\pi^\star}$*.*

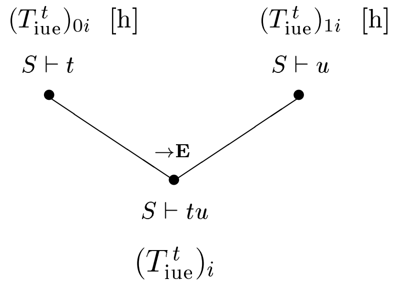
Proof. We proceed by induction on $\pi^\star$, denoting S the set $\{x_1, \ldots, x_m\}$.

Base: If $\pi^\star :: x : [(\sigma_1^i, \ldots, \sigma_m^i, \tau_i \,;\, \tau_i)_{i=1}^n]_{x_1,\ldots,x_m,x}$ is a decorated axiom, then there exist axioms $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i, x : \tau_i \vdash x : \tau_i$ $(1 \leqslant i \leqslant n)$ in IUT$^\oplus$. The tree $(T_{\text{iue}}^t)_i$ is a single node with the term-statement $S, x \vdash x$, so that conclusions 1 and 2 hold. The tree $(T_{\text{iue}}^t)_{\pi^\star}$ is a single node with the decoration-statement $S, x \vdash x$, so that conclusion 3 holds, too.

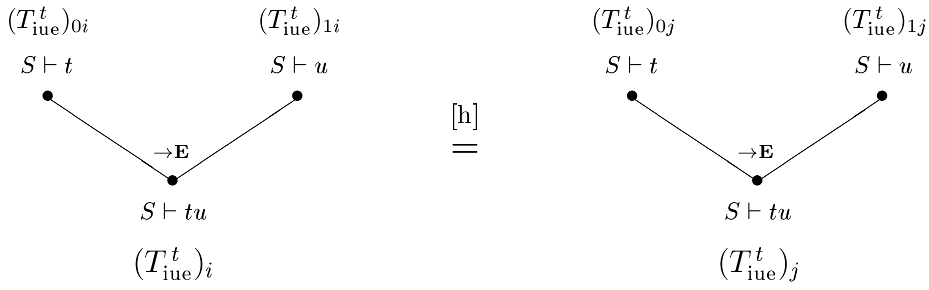
Induction step: We show the most demanding cases, abbreviating [h] the induction hypothesis.

$$\triangleright \quad \frac{\pi_0^\star :: t : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i \to \rho_i)_{i=1}^n]_{x_1,\ldots,x_m} \qquad \pi_1^\star :: u : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i)_{i=1}^n]_{x_1,\ldots,x_m}}{\pi^\star :: tu : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \rho_i)_{i=1}^n]_{x_1,\ldots,x_m}} \ (\to\mathbf{E})$$

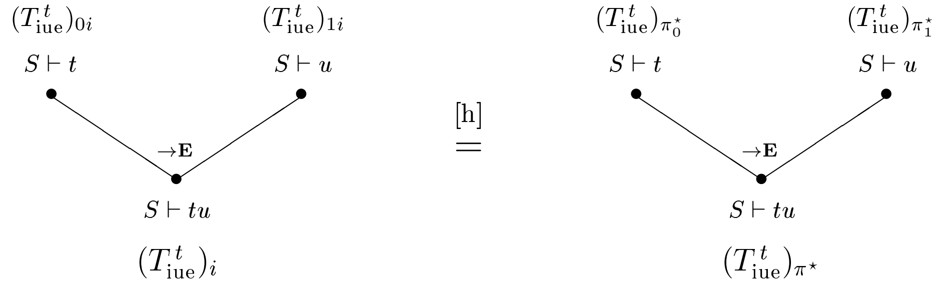
The [h] gives derivations $\pi_{0i} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i \to \rho_i$ $(1 \leqslant i \leqslant n)$, such that $(T_{\text{iue}}^t)_{0i}$ exists, $(T_{\text{iue}}^t)_{0i} = (T_{\text{iue}}^t)_{0j}$, and $(T_{\text{iue}}^t)_{0i} = (T_{\text{iue}}^t)_{\pi_0^\star}$. It also gives $\pi_{1i} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash u : \tau_i$ $(1 \leqslant i \leqslant n)$, such that $(T_{\text{iue}}^t)_{1i}$ exists, $(T_{\text{iue}}^t)_{1i} = (T_{\text{iue}}^t)_{1j}$, and $(T_{\text{iue}}^t)_{1i} = (T_{\text{iue}}^t)_{\pi_1^\star}$. Applying **(→E)** to π_{0i} and π_{1i}, we obtain $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash tu : \rho_i$ $(1 \leqslant i \leqslant n)$. Since the trees $(T_{\text{iue}}^t)_{0i}$ and $(T_{\text{iue}}^t)_{1i}$ exist, the tree $(T_{\text{iue}}^t)_i$ also exists, as shown below.



Since $(T_{\text{iue}}^t)_{0i} = (T_{\text{iue}}^t)_{0j}$ and $(T_{\text{iue}}^t)_{1i} = (T_{\text{iue}}^t)_{1j}$, we get that $(T_{\text{iue}}^t)_i = (T_{\text{iue}}^t)_j$, as displayed below.



Finally, since $(T_{\text{iue}}^t)_{0i} = (T_{\text{iue}}^t)_{\pi_0^\star}$ and $(T_{\text{iue}}^t)_{1i} = (T_{\text{iue}}^t)_{\pi_1^\star}$, we obtain that $(T_{\text{iue}}^t)_i = (T_{\text{iue}}^t)_{\pi^\star}$, as shown below.



$$\triangleright\quad \frac{\pi^\star_0 :: \ t : [(\phi^i_1, \ldots, \phi^i_m \,;\, \psi_i)\,^k_{i=1}, ((\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i), (\sigma^i_1, \ldots, \sigma^i_m \,;\, \rho_i))\,^n_{i=k+1}]\,_{x_1, \ldots, x_m}}{\pi^\star :: \ t : [(\phi^i_1, \ldots, \phi^i_m \,;\, \psi_i)\,^k_{i=1}, (\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i \cap \rho_i)\,^n_{i=k+1}]\,_{x_1, \ldots, x_m}}\ (\cap\mathbf{I})$$

For $1 \leqslant i \leqslant k$, the [h] yields derivations $\pi_{0i} :: x_1 : \phi^i_1, \ldots, x_m : \phi^i_m \vdash t : \psi_i$, such that the trees $(T^t_{\text{iue}})_{0i}$ exist and are identical and $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{\pi^\star_0}$. It is $\pi_i = \pi_{0i}$, so the trees $(T^t_{\text{iue}})_i[= (T^t_{\text{iue}})_{0i}]$ exist and are identical. Moreover, it is $(T^t_{\text{iue}})_i = (T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{\pi^\star_0} = (T^t_{\text{iue}})_{\pi^\star}$. For $k+1 \leqslant i \leqslant n$, the [h] gives

$$\pi_{0i0} :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i \quad \text{and} \quad \pi_{0i1} :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \rho_i$$

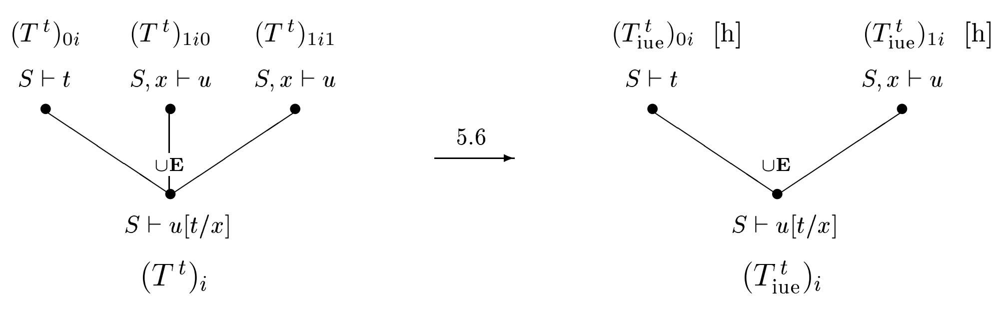
such that the trees $(T^t_{\text{iue}})_{0i0}, (T^t_{\text{iue}})_{0i1}$ exist and are identical. Applying **(**$\cap$**I)** to π_{0i0} and π_{0i1}, we get $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i \cap \rho_i$. Since $(T^t_{\text{iue}})_{0i0} = (T^t_{\text{iue}})_{0i1}$, the tree $(T^t_{\text{iue}})_i$ exists and is identical to $(T^t_{\text{iue}})_{0i0}$. Hence, the trees $(T^t_{\text{iue}})_i$ are identical. For $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$, the [h] yields that $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{0j0}$, which implies that $(T^t_{\text{iue}})_i = (T^t_{\text{iue}})_j$. Therefore, we altogether have that, for $1 \leqslant i \leqslant n$, the trees $(T^t_{\text{iue}})_i$ exist and are identical. Consequently, the already established equality $(T^t_{\text{iue}})_i = (T^t_{\text{iue}})_{\pi^\star}$, where $1 \leqslant i \leqslant k$, also holds for $1 \leqslant i \leqslant n$.

$$\triangleright\quad \frac{\pi^\star_0 :: \ t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i \cup \rho_i)\,^n_{i=1}]_p \qquad \pi^\star_1 :: \ u : [((\sigma^i_1, \ldots, \sigma^i_m, \tau_i \,;\, \upsilon_i), (\sigma^i_1, \ldots, \sigma^i_m, \rho_i \,;\, \upsilon_i))\,^n_{i=1}]_{p,\, x}}{\pi^\star :: \ u[t/x] : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \upsilon_i)\,^n_{i=1}]_{p\, =\, x_1, \ldots, x_m}}\ (\cup\mathbf{E})$$

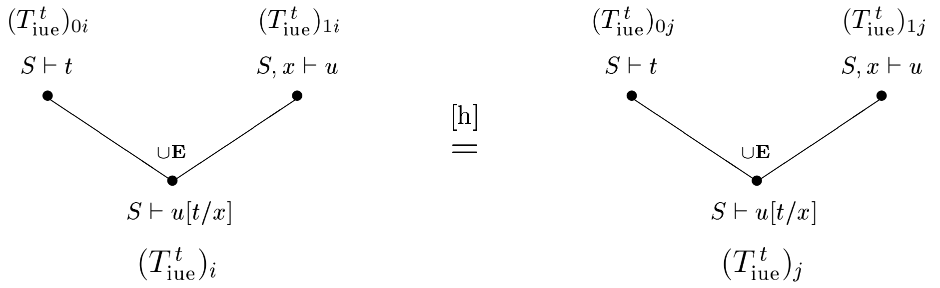
The [h] gives derivations $\pi_{0i} :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i \cup \rho_i$ $(1 \leqslant i \leqslant n)$, such that $(T^t_{\text{iue}})_{0i}$ exists, $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{0j}$, and $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{\pi^\star_0}$. It also gives

$$\pi_{1i0} :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m,\, x : \tau_i \vdash u : \upsilon_i \quad \text{and} \quad \pi_{1i1} :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m,\, x : \rho_i \vdash u : \upsilon_i$$

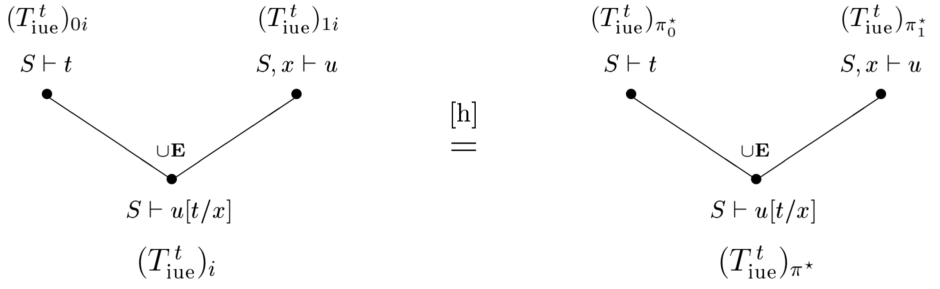
for $1 \leqslant i \leqslant n$, such that $(T^t_{\text{iue}})_{1i0}, (T^t_{\text{iue}})_{1i1}$ exist, $(T^t_{\text{iue}})_{1j0} = (T^t_{\text{iue}})_{1i0} = (T^t_{\text{iue}})_{1i1}$, and $(T^t_{\text{iue}})_{1i0} = (T^t_{\text{iue}})_{\pi^\star_1}$. Applying **(**$\cup$**E)** to π_{0i}, π_{1i0}, and π_{1i1}, we obtain $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash u[t/x] : \upsilon_i$ $(1 \leqslant i \leqslant n)$. Since the tree $(T^t_{\text{iue}})_{0i}$ exists and the trees $(T^t_{\text{iue}})_{1i0}, (T^t_{\text{iue}})_{1i1}$ exist and are identical $[(T^t_{\text{iue}})_{1i0} = (T^t_{\text{iue}})_{1i1} = (T^t_{\text{iue}})_{1i}]$, the tree $(T^t_{\text{iue}})_i$ also exists according to the algorithm in 5.6.



Since $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{0j}$ and $(T^t_{\text{iue}})_{1i} = (T^t_{\text{iue}})_{1i0} = (T^t_{\text{iue}})_{1j0} = (T^t_{\text{iue}})_{1j}$, we get that $(T^t_{\text{iue}})_i = (T^t_{\text{iue}})_j$, as displayed below.



Finally, since $(T^t_{\text{iue}})_{0i} = (T^t_{\text{iue}})_{\pi^\star_0}$ and $(T^t_{\text{iue}})_{1i} = (T^t_{\text{iue}})_{1i0} = (T^t_{\text{iue}})_{\pi^\star_1}$, we obtain that $(T^t_{\text{iue}})_i = (T^t_{\text{iue}})_{\pi^\star}$, as shown below.



The (→**I**) case is similar to the (→**E**) case, while the cases of (∩**E**) and (∪**I**) are similar to the (∩**I**) case. ⊣

Corollary 5.11 *If* $\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m \,;\, \tau)]_{x_1, \ldots, x_m}$ *is a derivation in* $\text{IUL}^\star_m$*, there exists a derivation* $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ *in* $\text{IUT}^\oplus$*, such that 1.* $(T^t_{\text{iue}})_1$ *exists and 2.* $(T^t_{\text{iue}})_1 = (T^t_{\text{iue}})_{\pi^\star}$*.*

Proof. By Theorem 5.10, for $n = 1$. ⊣

The next example illustrates the formalities in Theorem 5.10.

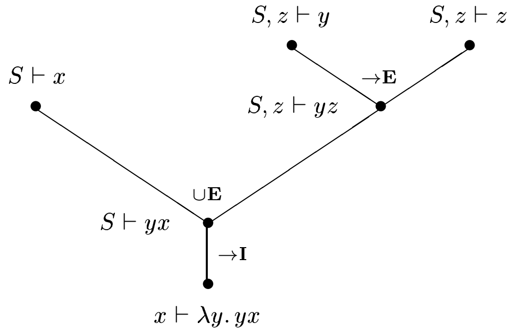
Example 5.12 *We consider* $\pi^\star :: \lambda y.\, yx : [(\sigma \,;\, \tau \to \delta), (\sigma \,;\, \tau \to \varepsilon)]_x$*, as displayed below, where* $\sigma, \tau, \pi^\star_0$*, and* $\pi^\star_1$ *are as in Example 5.4.*

$$\dfrac{\dfrac{\pi^\star_0 :: x : [(\sigma, \tau \,;\, \alpha \cup \beta), (\sigma, \tau \,;\, \alpha \cup \gamma)]_{x,\,y} \qquad \pi^\star_1 :: yz : [(\sigma, \tau, \alpha \,;\, \delta), (\sigma, \tau, \beta \,;\, \delta), (\sigma, \tau, \alpha \,;\, \varepsilon), (\sigma, \tau, \gamma \,;\, \varepsilon)]_{x,\,y,\,z}}{yx : [(\sigma, \tau \,;\, \delta), (\sigma, \tau \,;\, \varepsilon)]_{x,\,y}}\,(\cup\mathbf{E})}{\pi^\star :: \lambda y.\, yx : [(\sigma \,;\, \tau \to \delta), (\sigma \,;\, \tau \to \varepsilon)]_x}\,(\to\mathbf{I})$$

There are two derivations $\pi_1 :: x : \sigma \vdash \lambda y.\, yx : \tau \to \delta$ *and* $\pi_2 :: x : \sigma \vdash \lambda y.\, yx : \tau \to \varepsilon$ *in* $\mathrm{IUT}^{\oplus}$*, such that the trees* $(T^t_{\mathrm{iue}})_1$ *and* $(T^t_{\mathrm{iue}})_2$ *both exist and are identical and also identical to the tree* $(T^t_{\mathrm{iue}})_{\pi^\star}$*. Roughly speaking, we derive* π_1 *and* π_2 *from* $\pi^\star$ *by tracing the decorated "ancestors" of the 1st and 2nd decorated atoms in the conclusion of* $\pi^\star$*, respectively. We denote* B *the basis* $\{\, x : \sigma,\, y : \tau \,\}$ *and* S *the set* $dom(B) = \{x, y\}$*.*

$$\dfrac{\dfrac{\dfrac{B \vdash x : \sigma}{B \vdash x : \alpha \cup \beta}\,(\cap\mathbf{E}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{B,\, z : \alpha \vdash y : \tau}{B,\, z : \alpha \vdash y : \tau_1}\,(\cap\mathbf{E}_1)}{B,\, z : \alpha \vdash y : \alpha \to \delta \cap \varepsilon}\,(\cap\mathbf{E}_1) \quad B,\, z : \alpha \vdash z : \alpha}{B,\, z : \alpha \vdash yz : \delta \cap \varepsilon}\,(\to\mathbf{E})}{B,\, z : \alpha \vdash yz : \delta}\,(\cap\mathbf{E}_1) \qquad \dfrac{\dfrac{\dfrac{B,\, z : \beta \vdash y : \tau}{B,\, z : \beta \vdash y : \tau_1}\,(\cap\mathbf{E}_1)}{B,\, z : \beta \vdash y : \beta \to \delta}\,(\cap\mathbf{E}_2) \quad B,\, z : \beta \vdash z : \beta}{B,\, z : \beta \vdash yz : \delta}\,(\to\mathbf{E})}{B = \{\, x : \sigma,\, y : \tau \,\} \vdash yx : \delta}\,(\cup\mathbf{E})}{\pi_1 :: x : \sigma \vdash \lambda y.\, yx : \tau \to \delta}\,(\to\mathbf{I})$$

$$\dfrac{\dfrac{\dfrac{B \vdash x : \sigma}{B \vdash x : \alpha \cup \gamma}\,(\cap\mathbf{E}_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{B,\, z : \alpha \vdash y : \tau}{B,\, z : \alpha \vdash y : \tau_1}\,(\cap\mathbf{E}_1)}{B,\, z : \alpha \vdash y : \alpha \to \delta \cap \varepsilon}\,(\cap\mathbf{E}_1) \quad B,\, z : \alpha \vdash z : \alpha}{B,\, z : \alpha \vdash yz : \delta \cap \varepsilon}\,(\to\mathbf{E})}{B,\, z : \alpha \vdash yz : \varepsilon}\,(\cap\mathbf{E}_2) \qquad \dfrac{\dfrac{B,\, z : \gamma \vdash y : \tau}{B,\, z : \gamma \vdash y : \gamma \to \varepsilon}\,(\cap\mathbf{E}_2) \quad B,\, z : \gamma \vdash z : \gamma}{B,\, z : \gamma \vdash yz : \varepsilon}\,(\to\mathbf{E})}{B = \{\, x : \sigma,\, y : \tau \,\} \vdash yx : \varepsilon}\,(\cup\mathbf{E})}{\pi_2 :: x : \sigma \vdash \lambda y.\, yx : \tau \to \varepsilon}\,(\to\mathbf{I})$$



$$(T^t_{\mathrm{iue}})_1 = (T^t_{\mathrm{iue}})_2 = (T^t_{\mathrm{iue}})_{\pi^\star}$$

The inverse of 5.10 can now be phrased and proved as follows.

Theorem 5.13 (From $\mathbf{IUT}^{\oplus}$ to $\mathbf{IUL}_m$) *If* $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *are derivations in* $\mathrm{IUT}^{\oplus}$*, such that 1.* $(T^t_{\mathrm{iue}})_i$ *exists and 2.* $(T^t_{\mathrm{iue}})_i = (T^t_{\mathrm{iue}})_j$ $(1 \leqslant i \neq j \leqslant n)$*, then there is a decorated derivation* $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m\, ;\, \tau_i)\,^n_{i=1}]_{x_1, \ldots, x_m}$ *in* IUL_m*, such that* $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_i$*.*

Proof. For the sake of simplicity, we consider two derivations $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ and $\pi_2 :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash t : \psi$, and we proceed by induction on π_1. Nonetheless, we still consider that the [h] can be applied to any finite number of derivations. We denote S the set $\{x_1, \ldots, x_m\}$.

Base: If $\pi_1 :: x : \tau, x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash x : \tau$ is an axiom, then, since $(T^t_{\mathrm{iue}})_2 = (T^t_{\mathrm{iue}})_1$, derivation π_2 may only contain rule inferences among $(\cap\mathbf{I})$,$(\cap\mathbf{E})$, and $(\cup\mathbf{I})$.

$$\bullet$$

$$S, x \vdash x$$

$$(T^t_{\mathrm{iue}})_1 = (T^t_{\mathrm{iue}})_2$$

$$\pi_{21} :: x : \phi,\, x_1 : \rho_1, \ldots, x_m : \rho_m \vdash x : \phi \qquad \cdots \qquad \pi_{2k} :: x : \phi,\, x_1 : \rho_1, \ldots, x_m : \rho_m \vdash x : \phi$$

$$\ddots \qquad (\cap\mathbf{IE}), (\cup\mathbf{I}) \qquad \iddots$$

$$\pi_2 :: x : \phi,\, x_1 : \rho_1, \ldots, x_m : \rho_m \vdash x : \psi$$

We achieve a $\pi^\star :: x : [(\tau, \sigma_1, \ldots, \sigma_m \,;\, \tau), (\phi, \rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x,\, x_1, \ldots, x_m}$ by merging $\pi_1, \pi_{21}, \ldots, \pi_{2k}$ into an axiom of the (decorated) logic and then applying exchanges[3] and the logical $(\cap\mathbf{IE})$,$(\cup\mathbf{I})$ inferences that correspond[4] to the $(\cap\mathbf{IE})$,$(\cup\mathbf{I})$ inferences in π_2.

$$x : [\underbrace{(\sigma_1, \ldots, \sigma_m, \tau \,;\, \tau), (\rho_1, \ldots, \rho_m, \phi \,;\, \phi)^k_{i=1}}_{}]_{\,x_1, \ldots, x_m,\, x}$$

$$\vdots$$

$$(\mathbf{X})\text{'s}$$

$$\vdots$$

$$x : [(\tau, \sigma_1, \ldots, \sigma_m \,;\, \tau), \underbrace{(\phi, \rho_1, \ldots, \rho_m \,;\, \phi)^k_{i=1}}_{}]_{\,x,\, x_1, \ldots, x_m}$$

$$\vdots$$

$$(\cap\mathbf{IE}),(\cup\mathbf{I})$$

$$\vdots$$

$$\pi^\star :: x : [(\tau, \sigma_1, \ldots, \sigma_m \,;\, \tau), (\phi, \rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x,\, x_1, \ldots, x_m}$$

Since $\pi^\star$ does not contain implications or union eliminations, the tree $(T^t_{\mathrm{iue}})_{\pi^\star}$ is a single node with the decoration-statement $S, x \vdash x$, i.e. it is $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_1$.

---

[3] The number of these exchanges is the least possible, as we choose not to interfere with the 1-to-m order in axiom level.

[4] It may be the case that a number of inferences of the same kind in the type-system level are translated as a single inference of this very kind in the logical level; e.g. a number of $(\cap\mathbf{E}_1)$'s in π_2 may render a single $(\cap\mathbf{E}_1)$ in $\pi^\star$. This is because the local rules of the logic, i.e. $(\cap\mathbf{IE})$ and $(\cup\mathbf{I})$, are allowed to act on several atoms (or sequences of atoms) in one step.

Induction step: We show the most important cases.

$$\triangleright \quad \dfrac{\pi_{10} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \chi \to \tau \qquad \pi_{11} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash u : \chi}{\pi_1 :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash tu : \tau}\ (\to\mathbf{E})$$

The tree $(T^t_{\mathrm{iue}})_1$ with root-node accompanied by the term-statement $S \vdash tu$ derives by $(\to\mathbf{E})$ from the trees $(T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{11}$ with root-nodes accompanied by $S \vdash t$ and $S \vdash u$, respectively. Since the tree $(T^t_{\mathrm{iue}})_2$ exists and is identical to the tree $(T^t_{\mathrm{iue}})_1$, derivation π_2 has the form shown below, where, for $1 \leqslant i \leqslant k$, the trees $(T^t_{\mathrm{iue}})_{2i0}, (T^t_{\mathrm{iue}})_{2i1}$ all exist and it is $(T^t_{\mathrm{iue}})_{2i0} = (T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{2i1} = (T^t_{\mathrm{iue}})_{11}$.

$$\begin{array}{ccc}
\dfrac{\pi_{210} :: B_2 \vdash t : \phi_1 \to \psi_1 \qquad \pi_{211} :: B_2 \vdash u : \phi_1}{\pi_{21} :: B_2 \vdash tu : \psi_1}\ (\to\mathbf{E}) & \cdots & \dfrac{\pi_{2k0} :: B_2 \vdash t : \phi_k \to \psi_k \qquad \pi_{2k1} :: B_2 \vdash u : \phi_k}{\pi_{2k} :: B_2 \vdash tu : \psi_k}\ (\to\mathbf{E}) \\
\ddots & (\cap\mathbf{IE}), (\cup\mathbf{I}) & \iddots \\
& \pi_2 :: B_2 = \{\, x_1 : \rho_1, \dots, x_m : \rho_m \,\} \vdash tu : \psi &
\end{array}$$

The [h] on $\pi_{10}, \pi_{210}, \dots, \pi_{2k0}$ gives a

$$\pi^\star_0 :: t : [(\sigma_1, \dots, \sigma_m \,;\, \chi \to \tau), (\rho_1, \dots, \rho_m \,;\, \phi_i \to \psi_i)^k_{i=1}]_{x_1, \dots, x_m}$$

such that $(T^t_{\mathrm{iue}})_{\pi^\star_0} = (T^t_{\mathrm{iue}})_{10}$. In addition, the [h] on $\pi_{11}, \pi_{211}, \dots, \pi_{2k1}$ gives a

$$\pi^\star_1 :: u : [(\sigma_1, \dots, \sigma_m \,;\, \chi), (\rho_1, \dots, \rho_m \,;\, \phi_i)^k_{i=1}]_{x_1, \dots, x_m}$$

with $(T^t_{\mathrm{iue}})_{\pi^\star_1} = (T^t_{\mathrm{iue}})_{11}$. We then derive a $\pi^\star :: tu : [(\sigma_1, \dots, \sigma_m \,;\, \tau), (\rho_1, \dots, \rho_m \,;\, \psi)]_{x_1, \dots, x_m}$ as follows.

$$\begin{array}{c}
\dfrac{\pi^\star_0 \qquad\qquad \pi^\star_1}{tu : [(\sigma_1, \dots, \sigma_m \,;\, \tau), \underbrace{(\rho_1, \dots, \rho_m \,;\, \psi_i)^k_{i=1}}]_{x_1, \dots, x_m}}\ (\to\mathbf{E}) \\
\vdots \\
(\cap\mathbf{IE}), (\cup\mathbf{I}) \\
\vdots \\
\pi^\star :: tu : [(\sigma_1, \dots, \sigma_m \,;\, \tau), (\rho_1, \dots, \rho_m \,;\, \psi)]_{x_1, \dots, x_m}
\end{array}$$

Since $(T^t_{\mathrm{iue}})_{\pi^\star_0} = (T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{\pi^\star_1} = (T^t_{\mathrm{iue}})_{11}$, we infer that $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_1$.

$$\triangleright \quad \dfrac{\pi_{10} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \tau \qquad \pi_{11} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \chi}{\pi_1 :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \tau \cap \chi}\ (\cap\mathbf{I})$$

Since the tree $(T^t_{\mathrm{iue}})_1$ exists, the trees $(T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{11}$ both exist and are identical, so that $(T^t_{\mathrm{iue}})_1 = (T^t_{\mathrm{iue}})_{10} = (T^t_{\mathrm{iue}})_{11}$. Moreover, since $(T^t_{\mathrm{iue}})_1 = (T^t_{\mathrm{iue}})_2$, we have that $(T^t_{\mathrm{iue}})_{10} = (T^t_{\mathrm{iue}})_{11} = (T^t_{\mathrm{iue}})_2$.

We can therefore apply the [h] on $\pi_{10}, \pi_{11}, \pi_2$ to get a

$$\pi_0^\star :: t : [(\sigma_1, \ldots, \sigma_m ; \tau), (\sigma_1, \ldots, \sigma_m ; \chi), (\rho_1, \ldots, \rho_m ; \psi)]_{x_1, \ldots, x_m}$$

such that $(T^t_{\mathrm{iue}})_{\pi_0^\star} = (T^t_{\mathrm{iue}})_{10}$. By **$(\cap$I$)$**, we then obtain a

$$\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m ; \tau \cap \chi), (\rho_1, \ldots, \rho_m ; \psi)]_{x_1, \ldots, x_m}$$

such that $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_{\pi_0^\star} = (T^t_{\mathrm{iue}})_{10} = (T^t_{\mathrm{iue}})_1$.

$$\triangleright \quad \dfrac{\pi_{10} :: B_1 \vdash t : \tau \cup \chi \qquad \pi_{110} :: B_1,\, x : \tau \vdash u : v \qquad \pi_{111} :: B_1,\, x : \chi \vdash u : v}{\pi_1 :: B_1 = \{\, x_1 : \sigma_1, \ldots,\, x_m : \sigma_m \,\} \vdash u[t/x] : v}\ (\cup\mathbf{E})$$

The tree $(T^t_{\mathrm{iue}})_1$ with root-node accompanied by the term-statement $S \vdash u[t/x]$ derives by **$(\cup$E$)$** from the trees $(T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{11}$, where $(T^t_{\mathrm{iue}})_{11} = (T^t_{\mathrm{iue}})_{110} = (T^t_{\mathrm{iue}})_{111}$, with root-nodes accompanied by $S \vdash t$ and $S, x \vdash u$, respectively. The hypothesis that the tree $(T^t_{\mathrm{iue}})_2$ exists and is identical to the tree $(T^t_{\mathrm{iue}})_1$ implies the following. Derivation π_2 has the form depicted below, where, for $1 \leqslant i \leqslant k$, it is $\pi_{2i0} :: B_2 \vdash t : \phi_{i0} \cup \phi_{i1}$, $\pi_{2i10} :: B_2, x : \phi_{i0} \vdash u : \psi_i$, and $\pi_{2i11} :: B_2, x : \phi_{i1} \vdash u : \psi_i$. The trees $(T^t_{\mathrm{iue}})_{2i0}, (T^t_{\mathrm{iue}})_{2i10}, (T^t_{\mathrm{iue}})_{2i11}$ all exist and it is $(T^t_{\mathrm{iue}})_{2i10} = (T^t_{\mathrm{iue}})_{2i11}$ $[= (T^t_{\mathrm{iue}})_{2i1}]$, $(T^t_{\mathrm{iue}})_{2i0} = (T^t_{\mathrm{iue}})_{10}$, and $(T^t_{\mathrm{iue}})_{2i1} = (T^t_{\mathrm{iue}})_{11}$.

$$\dfrac{\pi_{210} \qquad \pi_{2110} \qquad \pi_{2111}}{\pi_{21} :: B_2 \vdash u[t/x] : \psi_1}\ (\cup\mathbf{E}) \qquad \cdots \qquad \dfrac{\pi_{2k0} \qquad \pi_{2k10} \qquad \pi_{2k11}}{\pi_{2k} :: B_2 \vdash u[t/x] : \psi_k}\ (\cup\mathbf{E})$$

$$\ddots \qquad (\cap\mathbf{IE}), (\cup\mathbf{I}) \qquad \therefore$$

$$\pi_2 :: B_2 = \{\, x_1 : \rho_1, \ldots,\, x_m : \rho_m \,\} \vdash u[t/x] : \psi$$

If $\Gamma = \sigma_1, \ldots, \sigma_m$ and $\Delta = \rho_1, \ldots, \rho_m$, the [h] on $\pi_{10}, \pi_{210}, \ldots, \pi_{2k0}$ gives a

$$\pi_0^\star :: t : [(\Gamma ; \tau \cup \chi), (\Delta ; \phi_{i0} \cup \phi_{i1})_{i=1}^k]_{x_1, \ldots, x_m}$$

such that $(T^t_{\mathrm{iue}})_{\pi_0^\star} = (T^t_{\mathrm{iue}})_{10}$, while the [h] on $\pi_{110}, \pi_{111}, \pi_{2110}, \pi_{2111}, \ldots, \pi_{2k10}, \pi_{2k11}$ gives a

$$\pi_1^\star :: u : [(\Gamma, \tau ; v), (\Gamma, \chi ; v), ((\Delta, \phi_{i0} ; \psi_i), (\Delta, \phi_{i1} ; \psi_i))_{i=1}^k]_{x_1, \ldots, x_m, x}$$

such that $(T^t_{\mathrm{iue}})_{\pi_1^\star} = (T^t_{\mathrm{iue}})_{11}$. We then derive a $\pi^\star :: u[t/x] : [(\Gamma ; v), (\Delta ; \psi)]_{x_1, \ldots, x_m}$ as follows.

$$\dfrac{\pi_0^\star \qquad \pi_1^\star}{u[t/x] : [(\Gamma ; v), \underbrace{(\Delta ; \psi_i)_{i=1}^k}]_{x_1, \ldots, x_m}}\ (\cup\mathbf{E})$$

$$\vdots$$

$$(\cap\mathbf{IE}), (\cup\mathbf{I})$$

$$\vdots$$

$$\pi^\star :: u[t/x] : [(\Gamma ; v), (\Delta ; \psi)]_{x_1, \ldots, x_m}$$

The identities $(T^t_{\mathrm{iue}})_{\pi^\star_0} = (T^t_{\mathrm{iue}})_{10}$ and $(T^t_{\mathrm{iue}})_{\pi^\star_1} = (T^t_{\mathrm{iue}})_{11}$ imply that $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_1$. ⊣

Corollary 5.14 *If $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ is a derivation in $\mathrm{IUT}^\oplus$, such that $(T^t_{\mathrm{iue}})_1$ exists, there is a decorated derivation $\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m\,;\,\tau)]_{x_1,\ldots,x_m}$ in IUL_m, such that $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_1$.*

Proof. By Theorem 5.13, for $n = 1$. ⊣

Remark 5.15 *(i) A more accurate phrasing of Theorem 5.13 would be the following.*

> *If $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ are derivations in $\mathrm{IUT}^\oplus$, s.t. 1. $(T^t_{\mathrm{iue}})_i$ exists and 2. $(T^t_{\mathrm{iue}})_i = (T^t_{\mathrm{iue}})_j$ $(1 \leqslant i \neq j \leqslant n)$, then, for every bijection $b : \{1, \ldots, m\} \to \{1, \ldots, m\}$, there is a decorated derivation $\pi^\star :: t : [(\sigma^i_{b(1)}, \ldots, \sigma^i_{b(m)}\,;\,\tau_i)^n_{i=1}]_{x_{b(1)},\ldots,x_{b(m)}}$ in IUL_m with $(T^t_{\mathrm{iue}})_{\pi^\star} = (T^t_{\mathrm{iue}})_i$.*

In 5.13 we consider the identity bijection for simplicity.

(ii) In the base case of the inductive proof of 5.13, we present the axiom π_1 somewhat awkwardly as $x : \tau$, $B \vdash x : \tau$, where $B = \{\, x_1 : \sigma_1, \ldots, x_m : \sigma_m \,\}$, in order to demonstrate that there might be need for some exchange inferences in $\pi^\star$. Had we chosen the usual presentation[5] B, $x : \tau \vdash x : \tau$, this fact would not have been illustrated. The need for exchanges becomes explicit in Example 5.16 right below.

The next example is a concrete instance of the $(\cup\mathbf{E})$ case displayed in the proof of 5.13.

Example 5.16 *If $\sigma = (\alpha \to \gamma) \cup (\beta \to \gamma)$, $\tau = (\alpha \to \rho_1 \cap \rho_2) \cup (\beta \to \rho_1 \cap \rho_2)$, where $\rho_1 = \delta \cap \varepsilon$ and $\rho_2 = \zeta \cap \eta$, and $\rho = (\varepsilon \cap \eta) \cup \theta$, we consider the $\mathrm{IUT}^\oplus$-derivations $\pi_1 :: B = \{\, x : \sigma,\ y : \alpha \cap \beta \,\} \vdash xy : \gamma$ and $\pi_2 :: B' = \{\, y : \alpha \cap \beta,\ x : \tau \,\} \vdash xy : \rho$, as shown below.*

$$\dfrac{B \vdash x : \sigma \qquad \dfrac{B,\ z : \alpha \to \gamma \vdash z : \alpha \to \gamma \qquad \dfrac{B,\ z : \alpha \to \gamma \vdash y : \alpha \cap \beta}{B,\ z : \alpha \to \gamma \vdash y : \alpha}\,(\cap\mathbf{E}_1)}{B,\ z : \alpha \to \gamma \vdash zy : \gamma}\,(\to\mathbf{E}) \qquad \dfrac{B,\ z : \beta \to \gamma \vdash z : \beta \to \gamma \qquad \dfrac{B,\ z : \beta \to \gamma \vdash y : \alpha \cap \beta}{B,\ z : \beta \to \gamma \vdash y : \beta}\,(\cap\mathbf{E}_2)}{B,\ z : \beta \to \gamma \vdash zy : \gamma}\,(\to\mathbf{E})}{\pi_1 :: B = \{\, x : \sigma,\ y : \alpha \cap \beta \,\} \vdash xy : \gamma}\,(\cup\mathbf{E})$$

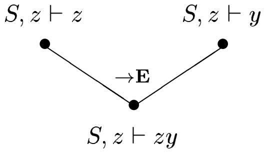
$$\dfrac{\dfrac{\dfrac{\overset{\text{see } \pi_{2i}\ (i = 1, 2) \text{ below}}{\pi_{21} :: B' \vdash xy : \rho_1}}{B' \vdash xy : \varepsilon}\,(\cap\mathbf{E}_2) \qquad \dfrac{\overset{\text{see } \pi_{2i}\ (i = 1, 2) \text{ below}}{\pi_{22} :: B' \vdash xy : \rho_2}}{B' \vdash xy : \eta}\,(\cap\mathbf{E}_2)}{B' \vdash xy : \varepsilon \cap \eta}\,(\cap\mathbf{I})}{\pi_2 :: B' = \{\, y : \alpha \cap \beta,\ x : \tau \,\} \vdash xy : \rho}\,(\cup\mathbf{I}_1)$$

$$\dfrac{B' \vdash x : \tau \qquad \dfrac{\dfrac{B'_1 \vdash z : \alpha \to \rho_1 \cap \rho_2 \qquad \dfrac{B'_1 \vdash y : \alpha \cap \beta}{B'_1 \vdash y : \alpha}\,(\cap\mathbf{E}_1)}{B'_1 \vdash zy : \rho_1 \cap \rho_2}\,(\to\mathbf{E})}{B'_1 = B' \cup \{\, z : \alpha \to \rho_1 \cap \rho_2 \,\} \vdash zy : \rho_i}\,(\cap\mathbf{E}_i) \qquad \dfrac{\dfrac{B'_2 \vdash z : \beta \to \rho_1 \cap \rho_2 \qquad \dfrac{B'_2 \vdash y : \alpha \cap \beta}{B'_2 \vdash y : \beta}\,(\cap\mathbf{E}_2)}{B'_2 \vdash zy : \rho_1 \cap \rho_2}\,(\to\mathbf{E})}{B'_2 = B' \cup \{\, z : \beta \to \rho_1 \cap \rho_2 \,\} \vdash zy : \rho_i}\,(\cap\mathbf{E}_i)}{\pi_{2i} :: B' = \{\, y : \alpha \cap \beta,\ x : \tau \,\} \vdash xy : \rho_i}\,(\cup\mathbf{E})$$

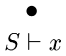
---

[5] There is no actual difference between the two presentations, as bases are sets, but 5.13 tacitly presumes an order in bases, the same in all n of them, which is the order aimed at in (the conclusion of) $\pi^\star$.

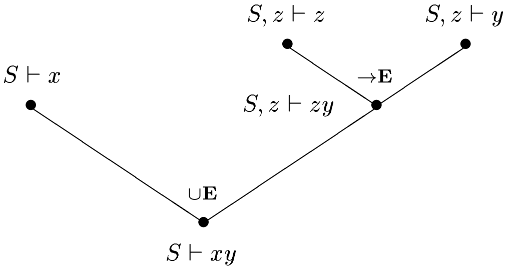
We attain the tree $(T^t_{\text{iue}})_1$ *from the tree* $(T^{\,t})_1$ *in one step, which is achieved by the fact that* $(T^t_{\text{iue}})_{110} = (T^t_{\text{iue}})_{111}$ $[= (T^t_{\text{iue}})_{11}]$. *We then attain the* $(T^t_{\text{iue}})_2$ *from the tree* $(T^{\,t})_2$ *in four steps: the 1st and 2nd steps are accomplished by the tree-identity* $(T^t_{\text{iue}})_{2i10} = (T^t_{\text{iue}})_{2i11}$ $[= (T^t_{\text{iue}})_{2i1}]$ *for* $i = 1$ *and* $i = 2$, *respectively, while the 3rd step is accomplished by the tree-identities* $(T^t_{\text{iue}})_{210} = (T^t_{\text{iue}})_{220}$ $[= (T^t_{\text{iue}})_{2i0}]$ *and* $(T^t_{\text{iue}})_{211} = (T^t_{\text{iue}})_{221}$ $[= (T^t_{\text{iue}})_{2i1}]$. *Finally, we see that* $(T^t_{\text{iue}})_1 = (T^t_{\text{iue}})_2$ *from the facts that* $(T^t_{\text{iue}})_{10} = (T^t_{\text{iue}})_{2i0}$ *and* $(T^t_{\text{iue}})_{11} = (T^t_{\text{iue}})_{2i1}$. *In the trees below, the letter* S *denotes the set* $\{x,y\}$.



$(T^t_{\text{iue}})_{110} = (T^t_{\text{iue}})_{111}$
$(T^t_{\text{iue}})_{2i10} = (T^t_{\text{iue}})_{2i11}$
$(T^t_{\text{iue}})_{211} = (T^t_{\text{iue}})_{221}$
$(T^t_{\text{iue}})_{11} = (T^t_{\text{iue}})_{2i1}$



$(T^t_{\text{iue}})_{210} = (T^t_{\text{iue}})_{220}$
$(T^t_{\text{iue}})_{10} = (T^t_{\text{iue}})_{2i0}$



$(T^t_{\text{iue}})_1 = (T^t_{\text{iue}})_2$

Derivations π_1 *and* π_2 *satisfy the hypotheses of 5.13; so, for* $x = x_1$ *and* $y = x_2$, *there is a decorated derivation* $\pi^\star :: xy : [(\sigma, \alpha \cap \beta \,;\, \gamma), (\tau, \alpha \cap \beta \,;\, \rho)]_{x,y}$ *with* $(T^t_{\text{iue}})_{\pi^\star} = (T^t_{\text{iue}})_1$. *If* $\Gamma = \sigma, \alpha \cap \beta$ *and* $\Delta = \tau, \alpha \cap \beta$, *we write* $\Gamma_\alpha = \Gamma, \alpha \to \gamma$, $\Gamma_\beta = \Gamma, \beta \to \gamma$, $\Delta_\alpha = \Delta, \alpha \to \rho_1 \cap \rho_2$, *and* $\Delta_\beta = \Delta, \beta \to \rho_1 \cap \rho_2$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x : [(\alpha\cap\beta, \sigma \,;\, \sigma), 2(\alpha\cap\beta, \tau \,;\, \tau)]_{y,x}}{x : [(\Gamma \,;\, \sigma), 2(\Delta \,;\, \tau)]_{x,y}}\,(\mathbf{X}) \qquad \dfrac{\dfrac{\pi^\star_{10}\ \text{(see below)} \qquad \pi^\star_{11}\ \text{(see below)}}{zy : [(\Gamma_\alpha \,;\, \gamma), (\Gamma_\beta \,;\, \gamma), 2((\Delta_\alpha \,;\, \rho_1\cap\rho_2), (\Delta_\beta \,;\, \rho_1\cap\rho_2))]_{x,y,z}}\,(\to\!\mathbf{E})}{zy : [(\Gamma_\alpha \,;\, \gamma), (\Gamma_\beta \,;\, \gamma), ((\Delta_\alpha \,;\, \rho_i), (\Delta_\beta \,;\, \rho_i))^2_{i=1}]_{x,y,z}}\,(\cap\mathbf{E}_{1,2})}{xy : [(\Gamma \,;\, \gamma), (\Delta \,;\, \rho_1), (\Delta \,;\, \rho_2)]_{x,y}}\,(\cup\mathbf{E})}{xy : [(\Gamma \,;\, \gamma), (\Delta \,;\, \varepsilon), (\Delta \,;\, \eta)]_{x,y}}\,(\cap\mathbf{E}_2)}{xy : [(\Gamma \,;\, \gamma), (\Delta \,;\, \varepsilon\cap\eta)]_{x,y}}\,(\cap\mathbf{I})}{\pi^\star :: xy : [(\Gamma \,;\, \gamma), (\Delta \,;\, \rho)]_{x,y}}\,(\cup\mathbf{I}_1)$$

$$\pi^\star_{10} :: z : [(\Gamma_\alpha \,;\, \alpha\to\gamma), (\Gamma_\beta \,;\, \beta\to\gamma), 2((\Delta_\alpha \,;\, \alpha\to\rho_1\cap\rho_2), (\Delta_\beta \,;\, \beta\to\rho_1\cap\rho_2))]_{x,y,z}$$

$$\dfrac{\dfrac{y : [(\sigma, \alpha\to\gamma, \alpha\cap\beta \,;\, \alpha\cap\beta), (\sigma, \beta\to\gamma, \alpha\cap\beta \,;\, \alpha\cap\beta), 2((\tau, \alpha\to\rho_1\cap\rho_2, \alpha\cap\beta \,;\, \alpha\cap\beta), (\tau, \beta\to\rho_1\cap\rho_2, \alpha\cap\beta \,;\, \alpha\cap\beta))]_{x,z,y}}{y : [(\Gamma_\alpha \,;\, \alpha\cap\beta), (\Gamma_\beta \,;\, \alpha\cap\beta), 2((\Delta_\alpha \,;\, \alpha\cap\beta), (\Delta_\beta \,;\, \alpha\cap\beta))]_{x,y,z}}\,(\mathbf{X})}{\pi^\star_{11} :: y : [(\Gamma_\alpha \,;\, \alpha), (\Gamma_\beta \,;\, \beta), 2((\Delta_\alpha \,;\, \alpha), (\Delta_\beta \,;\, \beta))]_{x,y,z}}\,(\cap\mathbf{E}_{1,2})$$

Two exchange inferences, just below two axioms, are necessary in $\pi^\star$. *If we chose to name* $y = x_1$ *and* $x = x_2$, *we would get a* $\pi^\star :: xy : [(\alpha\cap\beta, \sigma \,;\, \gamma), (\alpha\cap\beta, \tau \,;\, \rho)]_{y,x}$. *This is actually in accordance with Remark 5.15(i). In this case, we would also need two exchange inferences, but both (consecutively) below the same axiom. For the* $\pi^\star$ *shown above, it is easy to verify that the tree* $(T^t_{\text{iue}})_{\pi^\star}$ *is identical to the tree* $(T^t_{\text{iue}})_1$.

The question that now arises is the following. Can a finite number of $\mathrm{IUT}^{\oplus}$-derivations that share the same term-statement at the root, but are such that the conjunction of hypotheses 1 and 2 in 5.13 fails, be transformed to derivations that prove the same statements and are such that 1 and 2 both hold? To simplify the situation, let us consider two $\mathrm{IUT}^{\oplus}$-derivations $\pi_1 :: B_1 = \{\, x_1 : \sigma_1, \ldots, x_m : \sigma_m \,\} \vdash t : \tau$ and $\pi_2 :: B_2 = \{\, x_1 : \rho_1, \ldots, x_m : \rho_m \,\} \vdash t : \psi$ that share the term-statement $\{x_1, \ldots, x_m\} \vdash t$ at the root and are such that the conjunction of 1 and 2 fails [notation: $\neg(1 \wedge 2)_{\pi_1,\pi_2}$], i.e. it is not the case that the trees $(T^{\,t}_{\mathrm{iue}})_1$ and $(T^{\,t}_{\mathrm{iue}})_2$ both exist and are identical. Can we find transformed derivations $\pi_1' :: B_1 \vdash t : \tau$ and $\pi_2' :: B_2 \vdash t : \psi$ for which 1 and 2 hold [notation: $(1 \wedge 2)_{\pi_1',\pi_2'}$], i.e. for which the trees $(T^{\,t}_{\mathrm{iue}})_1'$ and $(T^{\,t}_{\mathrm{iue}})_2'$ both exist and are identical? As the next section illustrates, this is *not* always possible.

5.3 A transformation counterexample

Consider the following λ-terms.

$$\begin{array}{ll} u' = xx_1 & v' = x_1x \\ u'' = x_2yy & v'' = y\,(x_2x_1) \\ u = x_2x_1x_1 & v = x_1(x_2x_1) \end{array}$$

If $s = x_2x_1$ and $r = x_1$, it is $u = u'[s/x] = u''[r/y]$ and $v = v'[s/x] = v''[r/y]$. Moreover, if $s' = x_2y$, the following λ-term relations hold.

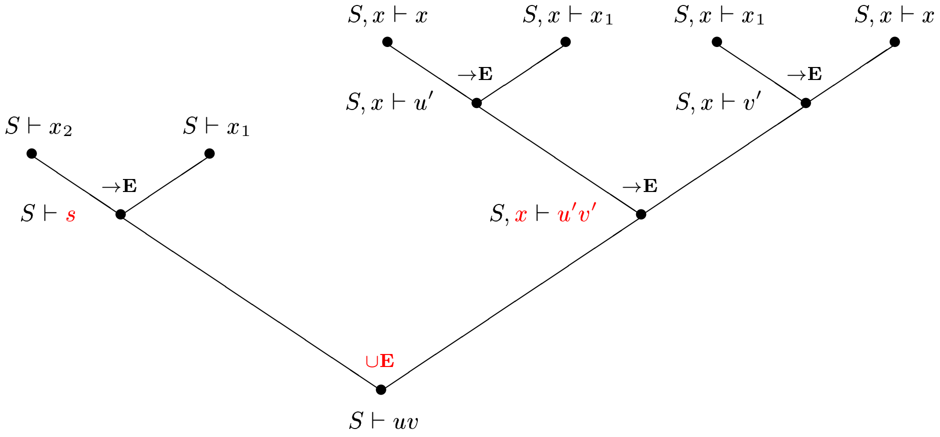
$$\begin{array}{ll} u' = xr & v' = rx \\ u'' = s'y & v'' = ys \\ u = sr & v = rs \end{array}$$

If $\sigma = (\beta \to \gamma \to \alpha) \cap \delta$, $\tau = (\varepsilon \to \zeta \to \alpha) \cap \eta$, and $\rho = (\delta \to \gamma) \cap (\eta \to \zeta) \cap \beta \cap \varepsilon$, consider the $\mathrm{IUT}^{\oplus}$-derivation $\pi_1 :: B_1 = \{\, x_1 : \rho,\, x_2 : \beta \to \sigma \cup \tau \,\} \vdash uv : \alpha$ and its tree $(T^{\,t}_{\mathrm{iue}})_1$, as shown below. The letter S denotes the set $\{x_1, x_2\}$.

$$\dfrac{\dfrac{B_1 \vdash x_2 : \beta \to \sigma \cup \tau \qquad \dfrac{B_1 \vdash x_1 : \rho}{B_1 \vdash x_1 : \beta}\,(\cap\mathbf{E})}{\pi_{10} :: B_1 \vdash x_2x_1 = s : \sigma \cup \tau}\,(\to\mathbf{E}) \qquad \begin{array}{c}\text{see below}\\ \pi_{11} :: B_1,\, x : \sigma \vdash xr\,(rx) = u'v' : \alpha\end{array} \qquad \begin{array}{c}\text{see below}\\ \pi_{12} :: B_1,\, x : \tau \vdash xr\,(rx) = u'v' : \alpha\end{array}}{\pi_1 :: B_1 \vdash sr\,(rs) = uv : \alpha}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B_1,\, x : \sigma \vdash x : \sigma}{B_1,\, x : \sigma \vdash x : \beta \to \gamma \to \alpha}\,(\cap\mathbf{E}_1) \quad \dfrac{B_1,\, x : \sigma \vdash x_1 : \rho}{B_1,\, x : \sigma \vdash x_1 : \beta}\,(\cap\mathbf{E})}{\pi_{110} :: B_1,\, x : \sigma \vdash xx_1 : \gamma \to \alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_1,\, x : \sigma \vdash x_1 : \rho}{B_1,\, x : \sigma \vdash x_1 : \delta \to \gamma}\,(\cap\mathbf{E}) \quad \dfrac{B_1,\, x : \sigma \vdash x : \sigma}{B_1,\, x : \sigma \vdash x : \delta}\,(\cap\mathbf{E}_2)}{\pi_{111} :: B_1,\, x : \sigma \vdash x_1x : \gamma}\,(\to\mathbf{E})}{\pi_{11} :: B_1,\, x : \sigma \vdash xx_1(x_1x) = u'v' : \alpha}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B_1,\, x : \tau \vdash x : \tau}{B_1,\, x : \tau \vdash x : \varepsilon \to \zeta \to \alpha}\,(\cap\mathbf{E}_1) \quad \dfrac{B_1,\, x : \tau \vdash x_1 : \rho}{B_1,\, x : \tau \vdash x_1 : \varepsilon}\,(\cap\mathbf{E}_2)}{\pi_{120} :: B_1,\, x : \tau \vdash xx_1 : \zeta \to \alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_1,\, x : \tau \vdash x_1 : \rho}{B_1,\, x : \tau \vdash x_1 : \eta \to \zeta}\,(\cap\mathbf{E}) \quad \dfrac{B_1,\, x : \tau \vdash x : \tau}{B_1,\, x : \tau \vdash x : \eta}\,(\cap\mathbf{E}_2)}{\pi_{121} :: B_1,\, x : \tau \vdash x_1x : \zeta}\,(\to\mathbf{E})}{\pi_{12} :: B_1,\, x : \tau \vdash xx_1(x_1x) = u'v' : \alpha}\,(\to\mathbf{E})$$



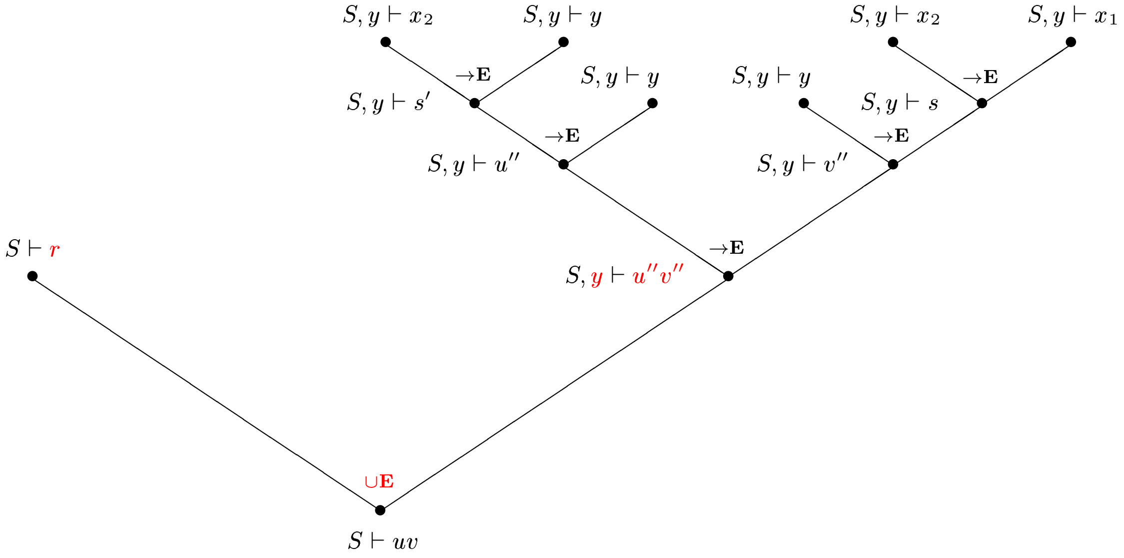
$(T^t_{\text{iue}})_1$

If $\phi = \zeta \to \alpha$, $\psi = \zeta \to \gamma$, $\chi = (\phi \cup \psi) \cap \varepsilon$, and $\upsilon = (\phi \to \phi \to \alpha \to \beta) \cap (\psi \to \psi \to \gamma \to \beta) \cap (\varepsilon \to \zeta)$, consider the IUT$^{\oplus}$-derivation $\pi_2 :: B_2 = \{\, x_1 : \chi,\ x_2 : \upsilon \,\} \vdash uv : \beta$ and its tree $(T^t_{\text{iue}})_2$, as demonstrated below. For space economy, we denote B_ϕ and B_ψ the bases $B_2,\ y : \phi$ and $B_2,\ y : \psi$, respectively.

$$\dfrac{\dfrac{B_2 \vdash x_1 = r : \chi}{\pi_{20} :: B_2 \vdash r : \phi \cup \psi}\,(\cap\mathbf{E}) \qquad \overset{\text{see below}}{\pi_{21} :: B_2,\, y : \phi \vdash x_2 yy\,(ys) = u''v'' : \beta} \qquad \overset{\text{see below}}{\pi_{22} :: B_2,\, y : \psi \vdash x_2 yy\,(ys) = u''v'' : \beta}}{\pi_2 :: B_2 \vdash sr\,(rs) = uv : \beta}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_\phi \vdash x_2 : \upsilon}{B_\phi \vdash x_2 : \phi \to \phi \to \alpha \to \beta}\,(\cap\mathbf{E}) \quad B_\phi \vdash y : \phi}{B_\phi \vdash x_2 y : \phi \to \alpha \to \beta}\,(\to\mathbf{E}) \quad B_\phi \vdash y : \phi}{\pi_{210} :: B_2,\, y : \phi \vdash x_2 yy : \alpha \to \beta}\,(\to\mathbf{E}) \quad \dfrac{B_\phi \vdash y : \phi \quad \dfrac{\dfrac{B_\phi \vdash x_2 : \upsilon}{B_\phi \vdash x_2 : \varepsilon \to \zeta}\,(\cap\mathbf{E}) \quad \dfrac{B_\phi \vdash x_1 : \chi}{B_\phi \vdash x_1 : \varepsilon}\,(\cap\mathbf{E})}{B_\phi \vdash x_2 x_1 = s : \zeta}\,(\to\mathbf{E})}{\pi_{211} :: B_2,\, y : \phi \vdash ys : \alpha}\,(\to\mathbf{E})}{\pi_{21} :: B_2,\, y : \phi \vdash x_2 yy\,(ys) = u''v'' : \beta}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_\psi \vdash x_2 : \upsilon}{B_\psi \vdash x_2 : \psi \to \psi \to \gamma \to \beta}\,(\cap\mathbf{E}) \quad B_\psi \vdash y : \psi}{B_\psi \vdash x_2 y : \psi \to \gamma \to \beta}\,(\to\mathbf{E}) \quad B_\psi \vdash y : \psi}{\pi_{220} :: B_2,\, y : \psi \vdash x_2 yy : \gamma \to \beta}\,(\to\mathbf{E}) \quad \dfrac{B_\psi \vdash y : \psi \quad \dfrac{\dfrac{B_\psi \vdash x_2 : \upsilon}{B_\psi \vdash x_2 : \varepsilon \to \zeta}\,(\cap\mathbf{E}) \quad \dfrac{B_\psi \vdash x_1 : \chi}{B_\psi \vdash x_1 : \varepsilon}\,(\cap\mathbf{E})}{B_\psi \vdash x_2 x_1 = s : \zeta}\,(\to\mathbf{E})}{\pi_{221} :: B_2,\, y : \psi \vdash ys : \gamma}\,(\to\mathbf{E})}{\pi_{22} :: B_2,\, y : \psi \vdash x_2 yy\,(ys) = u''v'' : \beta}\,(\to\mathbf{E})$$



$(T^t_{\mathrm{iue}})_2$

It is obvious that $(T^t_{\mathrm{iue}})_1 \neq (T^t_{\mathrm{iue}})_2$, so that $\neg(1 \wedge 2)_{\pi_1,\pi_2}$. Before attempting to transform π_1 and π_2 to $\pi_1' :: B_1 \vdash uv : \alpha$ and $\pi_2' :: B_2 \vdash uv : \beta$, respectively, so that $(1 \wedge 2)_{\pi_1',\pi_2'}$, some preliminary notes are in order.

Note 1. The *complexity* $c(t)$ *of a* λ*-term* t is defined inductively as follows.

$$c(x) = 1 \qquad c(\lambda x.\, t) = c(t) + 1 \qquad c(tu) = c(t) + c(u)$$

We write $\leqslant_c$ $(<_c, =_c)$ to mean $\leqslant$ $(<, =)$ with respect to complexity. Obviously, for any term t, it is $t \geqslant_c 1$, and, for any non-variable term t, it is $t >_c 1$. The next lemma states term-complexity relations and properties we will be using later on.

Lemma 5.17 *For any terms t, u, v, and any variable x free in u, we have that: (i) $t <_c tu$ and $u <_c tu$, (ii) if $t <_c u$, then $tv <_c uv$ and $vt <_c vu$, and (iii) if $x <_c t$, then $u = u(x) <_c u(t) = u[t/x]$.*

Proof. (i) It is $tu =_c t + u \geqslant_c t + 1 >_c t$ and $tu =_c t + u \geqslant_c 1 + u >_c u$.

(ii) It is $tv =_c t + v \overset{[t<_c u]}{<_c} u + v =_c uv$ and $vt =_c v + t \overset{[t<_c u]}{<_c} v + u =_c vu$.

(iii) By induction on $u(x)$. Base: If $u(x) = x$, then $u(t) = t$, so that $u(x) <_c u(t)$ by hypothesis. Induction step: If $u(x) = (\lambda y.\, u_1)(x) = \lambda y.\, u_1(x)$, then $u_1(x) \overset{[\mathrm{h}]}{<_c} u_1(t)$, so that $u(x) = \lambda y.\, u_1(x) =_c u_1(x) + 1 <_c u_1(t) + 1 =_c \lambda y.\, u_1(t) = u(t)$. If $u(x) = (u_1 u_2)(x)$, then x is free in u_1 or free in u_2, so we need to consider three cases: a) x free in u_1, but not free in u_2, b) x free in u_2, but not free in u_1, and c) x free in both u_1 and u_2. For case a), it is $u(x) = (u_1(x))u_2$, so that $u_1(x) \overset{[\mathrm{h}]}{<_c} u_1(t)$, which implies that $u(x) = (u_1(x))u_2 <_c (u_1(t))u_2 = u(t)$ by (ii). The other two cases are dealt with in a similar manner. $\dashv$

Note 2. In the attempted transformations, we only consider $(\cup\mathbf{E})$'s where a *proper substitution* occurs, as a $(\cup\mathbf{E})$ where a *phony substitution* occurs is eliminable.

$$\dfrac{\pi_0 :: B \vdash t : \sigma \cup \tau \qquad \pi_1 :: B,\, x : \sigma \vdash u : \rho \qquad \pi_2 :: B,\, x : \tau \vdash u : \rho}{\pi :: B \vdash u[t/x] \neq u : \rho}\ (\cup\mathbf{E})_{\text{proper}} \quad [x \in FV(u)]$$

$$\dfrac{\pi_0 :: B \vdash t : \sigma \cup \tau \qquad \pi_1 :: B,\, x : \sigma \vdash u : \rho \qquad \pi_2 :: B,\, x : \tau \vdash u : \rho}{\pi :: B \vdash u[t/x] = u : \rho}\ (\cup\mathbf{E})_{\text{phony}} \quad [x \notin FV(u)]$$

Considering $\pi_1 :: B,\, x : \sigma \vdash u : \rho$ in $(\cup\mathbf{E})_{\text{phony}}$ and using Proposition 4.14(iii), we get that there exists a $\pi_1' :: B \vdash u : \rho$ with $x \notin V_1' \subsetneq V_1$ and $h_1' \leqslant h_1$. Actually, as can be determined from the proof of 4.14(iii), derivation π_1' derives from π_1 by eliminating some (possibly none) rules in π_1. Therefore, the set of rules proving $B \vdash u : \rho$ in π_1' is a subset of the set of rules in π_1, which implies that we can prove $B \vdash u : \rho$ without the phony $(\cup\mathbf{E})$ in question.

It can further be shown that, if the transformed derivations $\pi_1' :: B_1 \vdash uv : \alpha$ and $\pi_2' :: B_2 \vdash uv : \beta$ we are looking for contain phony $(\cup\mathbf{E})$'s, then, eliminating the phony $(\cup\mathbf{E})$'s from π_1' and π_2' and obtaining $\pi_1'' :: B_1 \vdash uv : \alpha$ and $\pi_2'' :: B_2 \vdash uv : \beta$, respectively, we still have that $(1 \wedge 2)_{\pi_1'',\pi_2''}$. Hence, if there are transformed π_1' and π_2' with $(1 \wedge 2)_{\pi_1',\pi_2'}$, which include phony $(\cup\mathbf{E})$'s, then there are also transformed π_1'' and π_2'' with $(1 \wedge 2)_{\pi_1'',\pi_2''}$, which exclude phony $(\cup\mathbf{E})$'s. Consequently, if there are *not* transformed π_1'' and π_2'' with $(1 \wedge 2)_{\pi_1'',\pi_2''}$, which *exclude* phony $(\cup\mathbf{E})$'s, then there are *not* transformed π_1' and π_2' with $(1 \wedge 2)_{\pi_1',\pi_2'}$, which *include* phony $(\cup\mathbf{E})$'s. In other words, including phony $(\cup\mathbf{E})$'s would not alter a negative outcome in the search for transformations.

In the following notes, unless otherwise indicated, we consider an arbitrary term uv built from variables by applications.

Note 3. A derivation that proves a statement typing uv and that contains only proper[6] $(\cup\mathbf{E})$'s cannot contain an $(\to\mathbf{I})$. Since all the rules, except phony $(\cup\mathbf{E})$'s, carry a λ-abstraction from premise-level to conclusion, if the derivation contained an $(\to\mathbf{I})$, then the λ-abstraction formed by it would have to appear in uv, which is a contradiction.

$$\dfrac{B \vdash \lambda y.t : \sigma \cup \tau \qquad B,\, x : \sigma \vdash u(x) : \rho \qquad B,\, x : \tau \vdash u(x) : \rho}{B \vdash u(\lambda y.t) : \rho}\ (\cup\mathbf{E})_{\text{proper}}$$

$$\dfrac{B \vdash \lambda y.t : \sigma \cup \tau \qquad B,\, x : \sigma \vdash u : \rho \qquad B,\, x : \tau \vdash u : \rho}{B \vdash u : \rho}\ (\cup\mathbf{E})_{\text{phony}}$$

Hence, we will be trying to construct derivations π_1' and π_2' that contain only $(\to\mathbf{E})$'s and proper $(\cup\mathbf{E})$'s, as far as rules recorded in a tree T^t_{iue} are concerned.

Note 4. Supposing that the first bottom-up rule-inference among inferences of $(\to\mathbf{E})$ and of proper $(\cup\mathbf{E})$ in a derivation proving $B \vdash uv : \alpha$, where B is an appropriate[7] context and α is a type variable, is an $(\to\mathbf{E})$, then this $(\to\mathbf{E})$ is the first bottom-up rule-inference at all in a derivation proving $B \vdash uv : \omega$,

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[6] We mean that all the $(\cup\mathbf{E})$'s that appear in it are proper. It may, of course, contain other rules besides $(\cup\mathbf{E})$'s.

[7] The context B is "appropriate" in the sense that its domain contains the free variables of uv.

where[8] ω is either α or an intersection type with a factor[9] α. The type ω cannot be an implication type, e.g. of the form $\omega' \to \alpha$, because then an ($\to$**E**), lying below the lowest ($\to$**E**) in $B \vdash uv : \alpha$, would be required to "extract" α from ω. The type ω can neither be a union type, e.g. of the form $(\omega' \cap \alpha) \cup (\alpha \cap \omega'')$, because then a proper ($\cup$**E**), lying below the lowest proper ($\cup$**E**) in $B \vdash uv : \alpha$, would be required to eliminate the union and deliver α at the root. If $\omega \neq \alpha$, we need only consider ($\cap$**E**)'s in between the ($\to$**E**) in question and the root $B \vdash uv : \alpha$. This is because α is a type variable, so rules like ($\cap$**I**) or ($\cup$**I**), which increase a type's complexity, are not appropriate[10].

$$\dfrac{\dfrac{B \vdash u : \omega_1 \to \omega \qquad B \vdash v : \omega_1}{B \vdash uv : \omega}\,(\to\mathbf{E})}{B \vdash uv : \alpha}\,(\cap\mathbf{E})$$

Supposing that the first bottom-up rule-inference among inferences of ($\to$**E**) and of proper ($\cup$**E**) in a derivation proving $B \vdash uv : \alpha$, where B is an appropriate context and α is a type variable, is a proper ($\cup$**E**), then this proper ($\cup$**E**) can be considered as the first bottom-up rule-inference at all in $B \vdash uv : \alpha$. The first step is to argue, as in the case of an ($\to$**E**) above, that this proper ($\cup$**E**) is the first bottom-up rule-inference at all in a derivation proving $B \vdash uv : \omega$, where ω is either α or an intersection type with a factor α. However, in this case, any ($\cap$**E**) in between the proper ($\cup$**E**) in question and the root $B \vdash uv : \alpha$ can be shifted above the proper ($\cup$**E**) in question.

$$\dfrac{\dfrac{B \vdash t : \omega_1 \cup \omega_2 \qquad B,\, x : \omega_1 \vdash s(x) : \omega \qquad B,\, x : \omega_2 \vdash s(x) : \omega}{B \vdash uv = s(t) : \omega}\,(\cup\mathbf{E})}{B \vdash uv = s(t) : \alpha}\,(\cap\mathbf{E}) \qquad \rightsquigarrow$$

$$\dfrac{B \vdash t : \omega_1 \cup \omega_2 \qquad \dfrac{B,\, x : \omega_1 \vdash s(x) : \omega}{B,\, x : \omega_1 \vdash s(x) : \alpha}\,(\cap\mathbf{E}) \qquad \dfrac{B,\, x : \omega_2 \vdash s(x) : \omega}{B,\, x : \omega_2 \vdash s(x) : \alpha}\,(\cap\mathbf{E})}{B \vdash uv = s(t) : \alpha}\,(\cup\mathbf{E})$$

Note 5. Examining bottom-up whether uv is typable in an appropriate context $B = \{\, \ldots,\, x_i : \sigma_i\,, \ldots \}$ by some type[11] ω, i.e. examining whether ***bottom-up completion of a potential typing*** $B \vdash uv : \omega$ is possible, not all the rules from the set $\{(\to\mathbf{E}), (\cup\mathbf{E})_{\text{proper}}, (\cap\mathbf{I}), (\cap\mathbf{E}), (\cup\mathbf{I})\}$ have the same status, when considered at the first bottom-up position. The essence of bottom-up completion of a potential typing $B \vdash uv : \omega$ lies in the decomposition of uv to terms of smaller complexity in succedents higher up, so that we eventually reach variables in the succedents of axioms, and also in the decomposition of union

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[8] The letter ω here bears no connection to the type constant ω of Chapter 2.

[9] Saying that ω is an intersection type with a *factor* α, we roughly mean that ω has the form $\mathrm{f}_1 \cap \mathrm{f}_2$, where f_1 and f_2 are the factors of the intersection and ($\mathrm{f}_1 = \alpha$ or $\mathrm{f}_2 = \alpha$). The word "roughly" implies the fact that a factor of an intersection type may itself be an intersection type with factors which are intersection types and so forth. That is to say, the intersection $\alpha \cap \mathrm{f}_2$ (or $\mathrm{f}_1 \cap \alpha$), mentioned above, may be nested into a "bigger" intersection type.

[10] If there was a ($\cap$**I**) in between the ($\to$**E**) and the root, it would have to be followed by an ($\cap$**E**), so it would be eliminable. On the other hand, there couldn't be a ($\cup$**I**) in between the ($\to$**E**) and the root, as it would have to be followed by a proper ($\cup$**E**), which would lie below the lowest proper ($\cup$**E**).

[11] The type ω may be either a specific type, e.g. a certain type variable α, or a type which is loosely specified by a certain description, e.g. an intersection type with a factor α or an implication type, or just an arbitrary type.

types assigned to variables in B to their components[12] in contexts higher up and the decomposition of intersection types in $uv : \omega$ to their factors in succedents higher up. There are two categories of rules from the above set; one with rules that meaningfully contribute to the bottom-up completion of a potential typing of a term uv and another one with rules that just shift a potential typing of a term uv (or a version of it that is harder to bottom-up complete) upward. Before elaborating on the two categories of rules, let us first define four categories of proper $(\cup\mathbf{E})$. Distinguishing between the various kinds of proper $(\cup\mathbf{E})$ is necessary in order to distinguish the two categories of rules.

A category-1 proper $(\cup\mathbf{E})$ is one whose major premise types a proper, non-variable subterm t of uv, denoted $(\cup\mathbf{E})[1, t]$. A category-2 proper $(\cup\mathbf{E})$ is one whose major premise assigns to a variable subterm x_i of uv a union type $\omega_1 \cup \omega_2$, such that $\sigma_i = \omega_1 \cup \omega_2$ or σ_i is an intersection type with a factor $\omega_1 \cup \omega_2$; we denote it $(\cup\mathbf{E})[2, x_i]$. A category-3 proper $(\cup\mathbf{E})$ is one whose major premise types uv itself, denoted $(\cup\mathbf{E})[3]$. Finally, a category-4 proper $(\cup\mathbf{E})$ is one whose major premise assigns to a variable subterm x_i of uv a union type $\omega_1 \cup \omega_1$, such that σ_i is *not* a union type or an intersection type with a union factor and $\omega_1 = \sigma_i$; we denote it $(\cup\mathbf{E})[4, x_i]$. Taking $uv = x_2x_1x_1(x_1(x_2x_1)) = sr\,(rs)$ and $B = B_2 = \{\, x_1 : (\phi \cup \psi) \cap \varepsilon,\ x_2 : \upsilon \,\}$, we give some examples in each of the categories 1-4. The word "same" in place of the right minor premise of a union elimination indicates a recurrence of the left minor premise.

$$\dfrac{B_2 \vdash s : \omega_1 \cup \omega_2 \qquad B_2,\, x : \omega_1 \vdash xr\,(rs) : \omega \qquad B_2,\, x : \omega_2 \vdash xr\,(rs) : \omega}{B_2 \vdash uv = sr\,(rs) : \omega}\;(\cup\mathbf{E})[1, s]\,(i)$$

$$\dfrac{B_2 \vdash s : \omega_1 \cup \omega_2 \qquad B_2,\, x : \omega_1 \vdash xr\,(rx) : \omega \qquad B_2,\, x : \omega_2 \vdash xr\,(rx) : \omega}{B_2 \vdash uv = sr\,(rs) : \omega}\;(\cup\mathbf{E})[1, s]\,(ii)$$

$$\dfrac{B_2 \vdash x_1 : \phi \cup \psi \qquad B_2,\, x : \phi \vdash x_2xx_1(x_1(x_2x)) : \omega \qquad B_2,\, x : \psi \vdash x_2xx_1(x_1(x_2x)) : \omega}{B_2 \vdash uv = x_2x_1x_1(x_1(x_2x_1)) : \omega}\;(\cup\mathbf{E})[2, x_1]\,(i)$$

$$\dfrac{B_2 \vdash x_1 : \phi \cup \psi \qquad B_2,\, x : \phi \vdash x_2xx\,(x\,(x_2x)) : \omega \qquad B_2,\, x : \psi \vdash x_2xx\,(x\,(x_2x)) : \omega}{B_2 \vdash uv = x_2x_1x_1(x_1(x_2x_1)) : \omega}\;(\cup\mathbf{E})[2, x_1]\,(ii)$$

$$\dfrac{B_2 \vdash uv : \omega_1 \cup \omega_2 \qquad B_2,\, x : \omega_1 \vdash x : \omega \qquad B_2,\, x : \omega_2 \vdash x : \omega}{B_2 \vdash uv : \omega}\;(\cup\mathbf{E})[3]$$

$$\dfrac{B_2 \vdash x_2 : \upsilon \cup \upsilon \qquad B_2,\, x : \upsilon \vdash x_2x_1x_1(x_1(xx_1)) : \omega \qquad \text{same}}{B_2 \vdash uv = x_2x_1x_1(x_1(x_2x_1)) : \omega}\;(\cup\mathbf{E})[4, x_2]$$

Since s has two occurrences in $uv = sr\,(rs)$, there are three possible $(\cup\mathbf{E})[1, s]$'s according to which occurrences of s in uv are substituted by x to form the subject in the minor premises. This subject may be either $xr\,(rs)$ (see the $(\cup\mathbf{E})[1, s]\,(i)$ above) or $sr\,(rx)$ or $xr\,(rx)$ (see the $(\cup\mathbf{E})[1, s]\,(ii)$ above). A similar

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[12] The *components* of a union type $\mathrm{c}_1 \cup \mathrm{c}_2$ are the types c_1 and c_2. We use the word "factor" exclusively for intersections and the word "component" exclusively for unions.

argument holds for the $(\cup\mathbf{E})[2, x_1]$, which has fifteen different instances, and for the $(\cup\mathbf{E})[4, x_2]$, which has three different instances. Obviously, a category-1 union elimination may only be considered, if there exists a proper, non-variable subterm t of uv. To consider a category-2 union elimination, there must exist a variable subterm x_i of uv, such that σ_i is a union type or an intersection type with a union factor; on the other hand, to consider a category-4 union elimination, there must exist a variable subterm x_i of uv, such that σ_i is a type variable or an implication type or an intersection type with no union factor.

Before presenting the two categories of rules, we also need some notes on comparable potential typings of uv. We say that (i) σ is a *subtype* of τ, denoted $\sigma \leqslant \tau$, if and only if $x : \sigma \vdash x : \tau$, (ii) σ is *equal* to τ, denoted $\sigma = \tau$, if and only if ($\sigma \leqslant \tau$ and $\tau \leqslant \sigma$), and (iii) σ is a *proper* subtype of τ, denoted $\sigma < \tau$, if and only if ($\sigma \leqslant \tau$ and $\sigma \neq \tau$). Adopting a set-theoretical view for types, which roughly means considering a type as a set of terms with this type, if $\sigma < \tau$, then the property defining σ is more specific than the one defining τ, i.e. σ carries more information than τ. Let us now consider two potential typings $x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash uv : \omega$ (typing A) and $x_1 : \tau_1, \ldots, x_m : \tau_m \vdash uv : \omega'$ (typing B) of uv and an index i from 1 to m. We distinguish three cases. Case a: if $[\,\forall i\,(\tau_i = \sigma_i)$ and $(\omega' = \omega)\,]$, then the two typings are *equal.* Case b: if either 1. $[\,\forall i\,(\tau_i \leqslant \sigma_i)$ and $\exists i\,(\tau_i < \sigma_i)$ and $(\omega' = \omega)\,]$ or 2. $[\,\forall i\,(\tau_i = \sigma_i)$ and $(\omega' > \omega)\,]$, then typing B is *easier* than typing A. If 1 holds, typing B displays stronger assumptions and an equal succedent with respect to typing A, i.e. it provides more information in the assumptions to derive the information in the succedent. If 2 holds, typing B displays equal assumptions and a weaker succedent with respect to typing A, i.e. it is called to derive less information in the succedent from the information in the assumptions. Obviously, in either case, typing B is an easier version of typing A. Case c: if either 1. $[\,\forall i\,(\tau_i \geqslant \sigma_i)$ and $\exists i\,(\tau_i > \sigma_i)$ and $(\omega' = \omega)\,]$ or 2. $[\,\forall i\,(\tau_i = \sigma_i)$ and $(\omega' < \omega)\,]$, then typing B is *harder* than typing A. If 1 holds, typing B displays weaker assumptions and an equal succedent with respect to typing A, i.e. it provides less information in the assumptions to derive the information in the succedent. If 2 holds, typing B displays equal assumptions and a stronger succedent with respect to typing A, i.e. it is called to derive more information in the succedent from the information in the assumptions. This time, in either case, typing B is a harder version of typing A. A bottom-up rule which advances from a potential typing of uv at the conclusion to an easier version of it at the premise-level certainly promotes the bottom-up search. On the other hand, a bottom-up rule which advances from a potential typing of uv at the conclusion to a harder version of it at the premise-level hinders the bottom-up search. Finally, let us consider two potential typings $x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \sigma \vdash uv = (uv)(x) = (\ldots x \ldots x \ldots x \ldots) : \omega$ and $x_1 : \tau_1, \ldots, x_m : \tau_m, x : \tau, y : \rho \vdash s(y,x) = (\ldots y \ldots x \ldots y \ldots) : \omega'$ of uv and $s(y,x)$, respectively, where all the free occurrences of x in uv are marked and $s(y,x)$ derives from $(uv)(x)$ by substituting some (possibly none or all) free occurrence of x by y. If $[\,\forall i\,(\tau_i = \sigma_i)$ and $(\tau = \sigma)$ and $(\rho = \sigma)$ and $(\omega' = \omega)\,]$, the two typings are *equivalent.* Equal typings are equivalent, but the inverse is not true.

The first rule-category is the set $\{(\to\mathbf{E}), (\cup\mathbf{E})[1], (\cup\mathbf{E})[2], (\cap\mathbf{I})\}$. These rules meaningfully contribute to the bottom-up completion of a potential typing $B \vdash uv : \omega$, when considered at the first bottom-up position. An implication elimination decomposes uv to the smaller-complexity terms u and v in the left and right premise, respectively. A category-1 union elimination decomposes uv to smaller-complexity terms t and $s(x)$ in the major and minor premises, respectively.

$$\dfrac{B \vdash t : \omega_1 \cup \omega_2 \qquad B,\, x : \omega_1 \vdash s(x) : \omega \qquad B,\, x : \omega_2 \vdash s(x) : \omega}{B \vdash uv = s(t) : \omega}\ (\cup\mathbf{E})[1, t]$$

Since t is a proper subterm of uv, it is $t <_c uv$. Moreover, since t is not a variable, it is $x =_c 1 <_c t$, which implies that $s(x) <_c s(t) = uv$ by 5.17(iii). A category-2 union elimination decomposes a union type $\omega_1 \cup \omega_2$ in (the context of) the conclusion to its components ω_1 and ω_2 in (the context of) the left minor premise and (the context of) the right minor premise, respectively.

$$\frac{B \vdash x_i : \omega_1 \cup \omega_2 \qquad B,\, x : \omega_1 \vdash s(x, x_i) : \omega \qquad B,\, x : \omega_2 \vdash s(x, x_i) : \omega}{B = \{\, \ldots,\, x_i : \omega_1 \cup \omega_2\,, \ldots \} \vdash uv = (uv)(x_i) : \omega}\ (\cup\mathbf{E})[2, x_i]$$

This decomposition actually conveys the very purpose of a union elimination rule, which is the elimination of union, in a bottom-up manner. If ω_1 and ω_2 are not comparable, we have that $\omega_1 < \omega_1 \cup \omega_2$ and $\omega_2 < \omega_1 \cup \omega_2$. This implies that each of the minor-premise typings is easier than the conclusion typing[13], which promotes the bottom-up search. If $\omega_1 < \omega_2$, then $\omega_2 = \omega_1 \cup \omega_2$, which implies that the typing at the right minor-premise is equivalent to the conclusion typing, and $\omega_1 < \omega_1 \cup \omega_2$, which implies that the typing at the left minor-premise is easier than the conclusion typing. If $\omega_1 = \omega_2$, then $\omega_1 = \omega_1 \cup \omega_2 = \omega_2$, which implies that each of the minor-premise typings is equivalent to the conclusion typing. If $\omega_1 > \omega_2$, then $\omega_1 = \omega_1 \cup \omega_2$, which implies that the typing at the left minor-premise is equivalent to the conclusion typing, and $\omega_2 < \omega_1 \cup \omega_2$, which implies that the typing at the right minor-premise is easier than the conclusion typing. In any case, what is important for the bottom-up completion in a category-2 union elimination is the decomposition of a union context-type to its components. An intersection introduction decomposes an intersection type $\omega_1 \cap \omega_2$ in (the succedent of) the conclusion[14] to its factors ω_1 and ω_2 in (the succedent of) the left premise and (the succedent of) the right premise, respectively. If ω_1 and ω_2 are not comparable, we have that $\omega_1 > \omega_1 \cap \omega_2$ and $\omega_2 > \omega_1 \cap \omega_2$. This implies that each of the premise typings is easier than the conclusion typing, which promotes the bottom-up search. If $\omega_1 < \omega_2$, then $\omega_1 = \omega_1 \cap \omega_2$, which implies that the left-premise typing is equivalent to the conclusion typing, and $\omega_2 > \omega_1 \cap \omega_2$, which implies that the right-premise typing is easier than the conclusion typing. If $\omega_1 = \omega_2$, then $\omega_1 = \omega_1 \cap \omega_2 = \omega_2$, which implies than each of the premise typings is equivalent to the conclusion typing. If $\omega_1 > \omega_2$, then $\omega_2 = \omega_1 \cap \omega_2$, which implies that the right-premise typing is equivalent to the conclusion typing, and $\omega_1 > \omega_1 \cap \omega_2$, which implies that the left-premise typing is easier than the conclusion typing. In any case, though, what is important for the bottom-up completion in an intersection introduction is the decomposition of an intersection succedent-type to its factors.

The second rule-category is the set $\{(\cup\mathbf{E})[3], (\cup\mathbf{E})[4], (\cap\mathbf{E}), (\cup\mathbf{I})\}$. These rules just shift a potential typing $B \vdash uv : \omega$ (or a harder version of it) one level up, when considered at the first bottom-up position. A category-3 union elimination displays an equivalent or harder version of $B \vdash uv : \omega$, namely $B \vdash uv : \omega_1 \cup \omega_2$, at the major premise.

$$\frac{B \vdash uv : \omega_1 \cup \omega_2 \qquad B,\, x : \omega_1 \vdash x : \omega \qquad B,\, x : \omega_2 \vdash x : \omega}{B \vdash uv : \omega}\ (\cup\mathbf{E})[3]$$

The type $\omega_1 \cup \omega_2$ is such that $x : \omega_1 \vdash x : \omega$ and $x : \omega_2 \vdash x : \omega$, from which, by an appropriate union elimination application, we get that $y : \omega_1 \cup \omega_2 \vdash y : \omega$, i.e. that $\omega_1 \cup \omega_2 \leqslant \omega$. If $\omega_1 \cup \omega_2 = \omega$, then $B \vdash uv : \omega_1 \cup \omega_2$ is equivalent to $B \vdash uv : \omega$; if $\omega_1 \cup \omega_2 < \omega$, then $B \vdash uv : \omega_1 \cup \omega_2$ is harder than $B \vdash uv : \omega$. It is easy to check that a category-4 union elimination displays minor premises which are equivalent to the conclusion.

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[13] For example, $B,\, x : \omega_1 \vdash s(x, x_i) : \omega$ is easier than $B,\, x : \omega_1 \cup \omega_2 \vdash s(x, x_i) : \omega$, which is equivalent to $B \vdash (uv)(x_i) : \omega$. Therefore, $B,\, x : \omega_1 \vdash s(x, x_i) : \omega$ is easier than $B \vdash (uv)(x_i) : \omega$. This is a natural extension of the concept "easier", defined on the preceding page for comparing potential typings.

[14] We should note that the necessary and sufficient condition to consider an intersection introduction at the first bottom-up position of a potential typing $B \vdash uv : \omega$ is that ω is specified as an intersection type $\omega_1 \cap \omega_2$.

$$\dfrac{B \vdash x_i : \sigma_i \cup \sigma_i \qquad B,\, x : \sigma_i \vdash s(x, x_i) : \omega \qquad \text{same}}{B = \{\,\ldots,\, x_i : \sigma_i\,,\ldots\,\} \vdash uv = (uv)(x_i) : \omega}\,(\cup\mathbf{E})[4, x_i]$$

An intersection elimination displays an equivalent or harder version of $B \vdash uv : \omega$, namely $B \vdash uv : \omega \cap \omega'$, at the premise. In general, we have that $\omega \cap \omega' \leqslant \omega$. If $\omega \cap \omega' = \omega$, then $B \vdash uv : \omega \cap \omega'$ is equivalent to $B \vdash uv : \omega$; if $\omega \cap \omega' < \omega$, then $B \vdash uv : \omega \cap \omega'$ is harder than $B \vdash uv : \omega$. A union introduction also displays an equivalent or harder version of $B \vdash uv : \omega = \omega_1 \cup \omega_2$, namely $B \vdash uv : \omega_1$, at the premise. In general, we have that $\omega_1 \leqslant \omega_1 \cup \omega_2$. If $\omega_1 = \omega_1 \cup \omega_2$, then $B \vdash uv : \omega_1$ is equivalent to $B \vdash uv : \omega$; if $\omega_1 < \omega_1 \cup \omega_2$, then $B \vdash uv : \omega_1$ is harder than $B \vdash uv : \omega$.

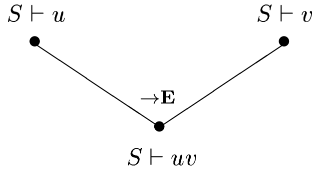
We conclude that in order to decide whether bottom-up completion of a potential typing $B \vdash uv : \omega$ is possible, we only need to examine rules from the first set at the first bottom-up position. Rules from the second set do not meaningfully contribute to the bottom-up search and can be ignored in making this decision; shifting the typing upward just defers the decision to a later bottom-up step, while shifting a harder version of the typing upward may even mislead to a negative decision. However, if the typing is indeed possible, it may be the case that the actual first bottom-up rule belongs to the second set, e.g. is an $(\cap\mathbf{E})$ (see note 4 where $(\cap\mathbf{E})$'s cannot be shifted above an $(\to\mathbf{E})$), but this can be easily settled at the end, i.e. after a positive decision has been made. If all the rules from the first set fail at the first bottom-up position, which may require to further bottom-up examine rules from the first set at first bottom-up positions, then the typing is not possible.

Note 6. If $uv = x_2x_1x_1(x_1(x_2x_1))$, the transformed derivations $\pi_1' :: B_1 \vdash uv : \alpha$ and $\pi_2' :: B_2 \vdash uv : \beta$ with identical trees T^t_{iue} that we are looking for must i) *type* uv in contexts B_1 and B_2, respectively, by α and β, respectively, and ii) *resemble* each other with respect to the structure of $(\to\mathbf{E})$'s and proper $(\cup\mathbf{E})$'s and their term-statements. Working the trees $(T^t_{\mathrm{iue}})_1'$ and $(T^t_{\mathrm{iue}})_2'$ bottom-up, a first bottom-up step of a shared $(\to\mathbf{E})$ or a shared proper $(\cup\mathbf{E})$ must prove progress with respect to the typing in at least one of π_1' and π_2'; there is no point in trying a step where the typing (or a harder version of it) is shifted upward in both π_1' and π_2'. Among $(\to\mathbf{E})$'s and proper $(\cup\mathbf{E})$'s, the set $\{(\to\mathbf{E}), (\cup\mathbf{E})[1], (\cup\mathbf{E})[2]\}$ proves progress with respect to the typing, while the set $\{(\cup\mathbf{E})[3], (\cup\mathbf{E})[4]\}$ does not (see note 5). So, a first bottom-up step of a shared $(\cup\mathbf{E})[3]$ or a shared $(\cup\mathbf{E})[4]$ is excluded; a first bottom-up step of a shared proper $(\cup\mathbf{E})$ where one of the derivations displays a $(\cup\mathbf{E})[3]$ and the other one displays a $(\cup\mathbf{E})[4]$ does not even deliver matching term-statements, so it is excluded anyway. If there is progress with respect to the typing in both π_1' and π_2', then the step involves a shared $(\to\mathbf{E})$ (see case 1 below) or a shared $(\cup\mathbf{E})[1]$ (see case 2 below) or a shared $(\cup\mathbf{E})[2]$ (see case 3 below). We cannot consider a step of a shared proper $(\cup\mathbf{E})$ where one of the derivations displays a $(\cup\mathbf{E})[1]$ and the other one displays a $(\cup\mathbf{E})[2]$, as this combination does not deliver matching term-statements. If there is progress with respect to the typing in either π_1' or π_2', then the step involves a shared proper $(\cup\mathbf{E})$ and the derivation in which progress is made displays a $(\cup\mathbf{E})[2]$, while the other one displays a $(\cup\mathbf{E})[4]$ (see case 4 below); this is the only combination between the progress-set $\{(\cup\mathbf{E})[1], (\cup\mathbf{E})[2]\}$ and the non-progress-set $\{(\cup\mathbf{E})[3], (\cup\mathbf{E})[4]\}$ which delivers matching term-statements.

In constructing π_1' and π_2', the general idea is to make a first bottom-up transformation step which gives an identical bottom-part in trees of implications and union eliminations with terms and also makes enough bottom-up progress with respect to the typing, so that the remaining transformation needs to be done on finite sets of derivations, each of which contains derivations proving statements that type a term of smaller complexity than uv.

Having in mind the preliminary notes 1-6 given above, we need to examine the following cases of a first bottom-up step for the trees $(T^t_{\mathrm{iue}})_1'$ and $(T^t_{\mathrm{iue}})_2'$.

1. Can we construct π_1' and π_2', such that the trees $(T^{t}_{\text{iue}})_1'$ and $(T^{t}_{\text{iue}})_2'$ both exist and share a bottom $(\to\mathbf{E})$, as shown below?



We want a $\pi_1' :: B_1 \vdash uv : \alpha$ with the following bottom part.

$$\cfrac{\cfrac{B_1 \vdash u : \omega_1 \to \omega \qquad B_1 \vdash v : \omega_1}{B_1 \vdash uv : \omega}\,(\to\mathbf{E})}{\pi_1' :: B_1 \vdash uv : \alpha}\,(\cap\mathbf{E})$$

However, the term u is not typable in context B_1 by an implication type. We outline below the bottom-up search with root $B_1 \vdash u : \omega_1 \to \omega$. In this bottom-up search and others to follow, we only consider rules from the set $\{(\to\mathbf{E}), (\cup\mathbf{E})[1], (\cup\mathbf{E})[2], (\cap\mathbf{I})\}$ at the first bottom-up position (recall note 5). The symbol "×" next to a rule-sign indicates that such a rule-application at the first bottom-up position cannot deliver the required root-typing, in which case we use a dotted horizontal line in-between the premise and conclusion levels. Further, the shorthand "not" next to a rule-sign indicates that such a rule-application cannot be considered at the first bottom-up position due to inappropriate form of the context-types or the subject or the predicate of the required root-typing. We also use the gray color for succedent-types which are initially desirable in a bottom-up search, but finally prove impossible.

i) Considering an $(\to\mathbf{E})$ at the first bottom-up position of a potential typing $B_1 \vdash u : \omega_1 \to \omega$, we see that it does not work.

$$\cfrac{\overset{\substack{\text{by } (\to\mathbf{E}),\text{ see } \pi_{10} \\ [(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}]}}{B_1 \vdash s : \sigma \cup \tau \neq \text{implication type}^{15}} \qquad \text{right premise}}{B_1 \vdash u = sr : \omega_1 \to \omega}\,(\to\mathbf{E})\ \times$$

ii) Considering the only possible $(\cup\mathbf{E})[1]$, which is the $(\cup\mathbf{E})[1, s]$ shown below, at the first bottom-up position of a potential typing $B_1 \vdash u : \omega_1 \to \omega$, we see that it does not work.

$$\cfrac{B_1 \vdash s : \sigma \cup \tau \qquad \overset{\substack{\text{by } (\to\mathbf{E}),\text{ see } \pi_{110} \\ [(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}]}}{B_1, x : \sigma \vdash xx_1 : \omega_1 \to \omega = \gamma \to \alpha} \qquad \overset{\substack{\text{by } (\to\mathbf{E}),\text{ see } \pi_{120} \\ [(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}]}}{B_1, x : \tau \vdash xx_1 : \omega_1 \to \omega = \zeta \to \alpha}}{B_1 \vdash u = sx_1 : \omega_1 \to \omega}\,(\cup\mathbf{E})[1, s]\ \times$$

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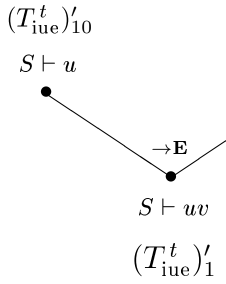
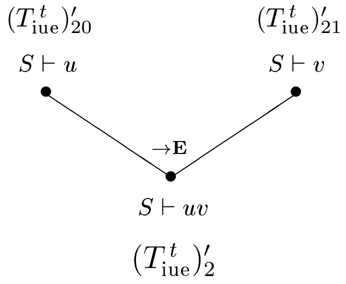
[15] We cannot extract σ (and then the implication type $\beta \to \gamma \to \alpha$) from $\sigma \cup \tau$ by a $(\cup\mathbf{E})$-application with major premise $B_1 \vdash s : \sigma \cup \tau$ and conclusion $B_1 \vdash s : \sigma$, as such an application would require a right minor premise $B_1, x : \tau \vdash x : \sigma$, which is not possible. A similar argument shows that we can neither extract τ.

iii) We cannot consider a $(\cup\mathbf{E})[2]$ at the first bottom-up position of a potential typing $B_1 \vdash u : \omega_1 \rightarrow \omega$, as the only variable subterms x_1 and x_2 of u are assigned $\rho = (\delta \rightarrow \gamma) \cap (\eta \rightarrow \zeta) \cap \beta \cap \varepsilon$ and $\beta \rightarrow \sigma \cup \tau$, respectively, in B_1. We can neither consider an $(\cap\mathbf{I})$ at the first bottom-up position of a potential typing $B_1 \vdash u : \omega_1 \rightarrow \omega$, as the succedent-type is not specified as an intersection type.

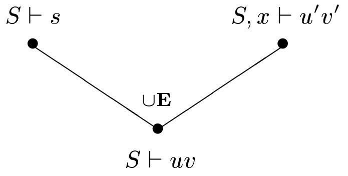
We gather that a π_1' with an $(\rightarrow\mathbf{E})$ bottom part is not feasible; so, there is no need to examine if such a π_2' is doable. Still, if we achieved π_1' and π_2' whose trees $(T^t_{\mathrm{iue}})_1'$ and $(T^t_{\mathrm{iue}})_2'$ existed and shared a bottom $(\rightarrow\mathbf{E})$, the transformation would reduce to further transforming π_{10}' and π_{20}', which would type $u <_c uv$, and also to further transforming π_{11}' and π_{21}', which would type $v <_c uv$.





2. Can we construct a π_1' and π_2', such that the trees $(T^t_{\mathrm{iue}})_1'$ and $(T^t_{\mathrm{iue}})_2'$ both exist and share a bottom $(\cup\mathbf{E})[1]$? We distinguish three cases.

2a. A bottom $(\cup\mathbf{E})[1, s]$. Since s has two occurrences in $uv = sr\,(rs)$, there are three possible $(\cup\mathbf{E})[1, s]$'s. We examine the case with subject $xr\,(rx) = u'v'$ in the minor premises, which, since π_1 already displays such a bottom part, amounts to examining if we can construct a π_2', such that the tree $(T^t_{\mathrm{iue}})_2'$ exists and has the following bottom part.



We want a $\pi_2' :: B_2 \vdash uv : \beta$ with the following bottom part.

$$\begin{array}{l} \text{by } (\rightarrow\mathbf{E})\text{: } \omega_1 \cup \omega_2 = \zeta \cup \zeta \\ \text{by } (\cup\mathbf{E})[2, x_1]\text{: } \omega_1 \cup \omega_2 = \phi_{\alpha\beta} \cup \psi_{\gamma\beta} \\ [(\cup\mathbf{E})[1]\text{ not}, (\cap\mathbf{I})\text{ not}] \end{array}$$

$$\dfrac{B_2 \vdash s : \omega_1 \cup \omega_2 \qquad B_2,\, x : \omega_1 \vdash xr\,(rx) = u'v' : \beta \qquad B_2,\, x : \omega_2 \vdash xr\,(rx) = u'v' : \beta}{\pi_2' :: B_2 \vdash uv = sr\,(rs) : \beta}\,(\cup\mathbf{E})[1, s]$$

The type $\omega_1 \cup \omega_2$ may be either $\phi_{\alpha\beta} \cup \psi_{\gamma\beta}$, where $\phi_{\alpha\beta} = \phi \rightarrow \alpha \rightarrow \beta$ and $\psi_{\gamma\beta} = \psi \rightarrow \gamma \rightarrow \beta$, or $\zeta \cup \zeta$. We outline below how the $\phi_{\alpha\beta} \cup \psi_{\gamma\beta}$ case fails. The $\zeta \cup \zeta$ case fails, as well.

$$\dfrac{\overset{\text{by }(\cup\mathbf{E})[2, x_1]\text{, see right below}}{B_2 \vdash s : \phi_{\alpha\beta} \cup \psi_{\gamma\beta}} \qquad \overset{\overset{\text{see i), ii), and iii) below}}{[(\cap\mathbf{I})\text{ not}]}}{B_2,\, x : \phi_{\alpha\beta} \vdash xr\,(rx) = u'v' : \beta} \qquad B_2,\, x : \psi_{\gamma\beta} \vdash xr\,(rx) = u'v' : \beta}{\pi'_2 :: B_2 \vdash uv = sr\,(rs) : \beta}\,(\cup\mathbf{E})[1, s]$$

$$\dfrac{\dfrac{B_2 \vdash x_1 : \chi}{B_2 \vdash x_1 : \phi \cup \psi}\,(\cap\mathbf{E}_1) \qquad \dfrac{\dfrac{\dfrac{\ldots \vdash x_2 : v}{\ldots \vdash x_2 : \phi \to \phi_{\alpha\beta}}\,(\cap\mathbf{E}) \qquad \ldots \vdash y : \phi}{B_2,\, y : \phi \vdash x_2 y : \phi_{\alpha\beta}}\,(\to\mathbf{E})}{B_2,\, y : \phi \vdash x_2 y : \phi_{\alpha\beta} \cup \psi_{\gamma\beta}}\,(\cup\mathbf{I}_1) \qquad \dfrac{\dfrac{\dfrac{\ldots \vdash x_2 : v}{\ldots \vdash x_2 : \psi \to \psi_{\gamma\beta}}\,(\cap\mathbf{E}) \qquad \ldots \vdash y : \psi}{B_2,\, y : \psi \vdash x_2 y : \psi_{\gamma\beta}}\,(\to\mathbf{E})}{B_2,\, y : \psi \vdash x_2 y : \phi_{\alpha\beta} \cup \psi_{\gamma\beta}}\,(\cup\mathbf{I}_2)}{B_2 \vdash x_2 x_1 = s : \phi_{\alpha\beta} \cup \psi_{\gamma\beta}}\,(\cup\mathbf{E})[2, x_1]$$

i) Considering an **(→E)** at the first bottom-up position of a potential typing $B_2,\, x : \phi_{\alpha\beta} \vdash u'v' : \beta$, we see that it does not work. The abbreviations "lmp" and "rmp" stand for "left minor premise" and "right minor premise", respectively. Likewise, the abbreviations "lp" and "rp" stand for "left premise" and "right premise", respectively.

$$\dfrac{\dfrac{\dfrac{B_2,\, x : \phi_{\alpha\beta} \vdash x_1 : \chi}{B_2,\, x : \phi_{\alpha\beta} \vdash x_1 : \phi \cup \psi}\,(\cap\mathbf{E}_1) \qquad \text{lmp} \qquad \dfrac{\cdots \vdash x : \phi \to \alpha \to \beta \qquad \cdots \vdash y : \psi}{B_2,\, x : \phi_{\alpha\beta},\, y : \psi \vdash xy : \omega \to \beta}\,(\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, 2]\text{ not}, (\cap\mathbf{I})\text{ not}]}{B_2,\, x : \phi_{\alpha\beta} \vdash u' = xx_1 : \omega \to \beta}\,(\cup\mathbf{E})[2, x_1]\times\ [(\to\mathbf{E})\times, (\cup\mathbf{E})[1]\text{ not}, (\cap\mathbf{I})\text{ not}] \qquad \text{rp}}{B_2,\, x : \phi_{\alpha\beta} \vdash u'v' : \beta}\,(\to\mathbf{E})\times$$

ii) Considering a **(∪E)**[1] at the first bottom-up position of a potential typing $B_2,\, x : \phi_{\alpha\beta} \vdash u'v' : \beta$, we find that it does not work. The $(\cup\mathbf{E})[1, u']$ does not work, since a typing $B,\, x : \phi_{\alpha\beta} \vdash u' : \omega_1 \cup \omega_2$ is not possible; the bottom-up search for such a typing is similar to the one shown in i) above for a typing $B,\, x : \phi_{\alpha\beta} \vdash u' : \omega \to \beta$. We present the failure of the $(\cup\mathbf{E})[1, v']$ below.

$$\dfrac{\overset{\overset{(\cup\mathbf{E})[2, x_1]\times\text{, see right below}}{[(\to\mathbf{E})\times, (\cup\mathbf{E})[1]\text{ not}, (\cap\mathbf{I})\text{ not}]}}{B_2,\, x : \phi_{\alpha\beta} \vdash v' = x_1 x : \omega_1 \cup \omega_2} \qquad \text{left minor premise} \qquad \text{right minor premise}}{B_2,\, x : \phi_{\alpha\beta} \vdash u'v' : \beta}\,(\cup\mathbf{E})[1, v']\times$$

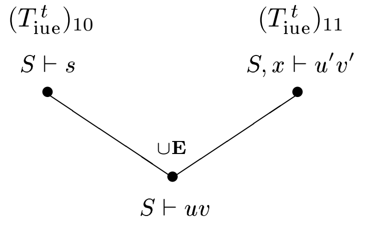
$$\dfrac{B_2,\, x : \phi_{\alpha\beta} \vdash x_1 : \phi \cup \psi \qquad \dfrac{\cdots \vdash y : \phi \qquad \cdots \vdash x : \phi_{\alpha\beta}}{B_2,\, x : \phi_{\alpha\beta},\, y : \phi \vdash yx : \omega_1 \cup \omega_2}\,(\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, 2]\text{ not}, (\cap\mathbf{I})\text{ not}] \qquad \text{rmp}}{B_2,\, x : \phi_{\alpha\beta} \vdash v' = x_1 x : \omega_1 \cup \omega_2}\,(\cup\mathbf{E})[2, x_1]\times$$

iii) Considering a **(∪E)**[2] at the first bottom-up position of a potential typing $B_2,\, x : \phi_{\alpha\beta} \vdash u'v' : \beta$, we also find that it does not work. We illustrate the failure of one of the three possible $(\cup\mathbf{E})[2, x_1]$'s below. The other two fail, as well.

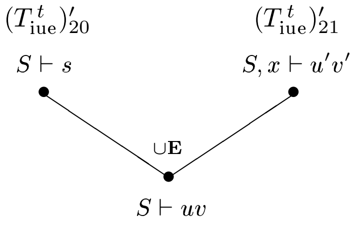
$$\dfrac{B_2,\, x : \phi_{\alpha\beta} \vdash x_1 : \phi \cup \psi \qquad \text{left minor premise} \qquad \begin{array}{c}(\cup\mathbf{E})[2, x_1]\times\text{, see right below}\\ [(\to\mathbf{E})\times, (\cup\mathbf{E})[1]\times, (\cap\mathbf{I})\text{ not}]\\ B_2,\, x : \phi_{\alpha\beta},\, y : \psi \vdash xy\,(x_1x) : \beta\end{array}}{B_2,\, x : \phi_{\alpha\beta} \vdash u'v' = xx_1(x_1x) : \beta}\,(\cup\mathbf{E})[2, x_1]\ \times$$

$$\dfrac{B_2,\, x : \phi_{\alpha\beta},\, y : \psi \vdash x_1 : \phi \cup \psi \qquad \text{lmp} \qquad \dfrac{}{B_2,\, x : \phi_{\alpha\beta},\, y : \psi,\, z : \psi \vdash xy\,(zx) : \beta}\,(\to\mathbf{E})\times, (\cup\mathbf{E})[1]\times\ [(\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}]}{B_2,\, x : \phi_{\alpha\beta},\, y : \psi \vdash xy\,(x_1x) : \beta}\,(\cup\mathbf{E})[2, x_1]\ \times$$

We conclude that a π_2' with a bottom $(\cup\mathbf{E})[1, s]$, which is identical (with respect to term-statements) to the bottom $(\cup\mathbf{E})[1, s]$ in π_1, is not possible. Yet, if we achieved a π_2' with a tree $(T^t_{\text{iue}})_2'$ bottom-identical to the tree $(T^t_{\text{iue}})_1$, the transformation would reduce to further transforming π_{10} and π_{20}', which would type $s <_c sr = u <_c uv$, and also to further transforming $\pi_{11}, \pi_{12}, \pi_{21}'$, and π_{22}', which would type $u'v' <_c uv$.



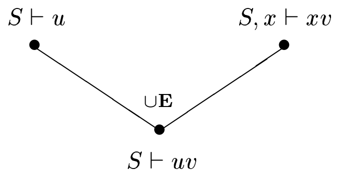
$(T^t_{\text{iue}})_1$



$(T^t_{\text{iue}})_2'$

For each of the other two possible $(\cup\mathbf{E})[1, s]$'s, at least one of π_1' and π_2' fails.

2b. A bottom $(\cup\mathbf{E})[1, u]$. We seek derivations π_1' and π_2' whose trees $(T^t_{\text{iue}})_1'$ and $(T^t_{\text{iue}})_2'$ both exist and share the following bottom part.



So, we seek a $\pi_1' :: B_1 \vdash uv : \alpha$ with the following bottom part.

$$\dfrac{\begin{array}{c}\text{by }(\cup\mathbf{E})[1, s]\text{, see right below}\\ [(\to\mathbf{E})\times, (\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}]\\ B_1 \vdash u : (\gamma \to \alpha) \cup (\zeta \to \alpha)\end{array} \qquad \begin{array}{c}\text{see i) and ii) below}\\ [(\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}]\\ B_1,\, x : \gamma \to \alpha \vdash xv : \alpha\end{array} \qquad B_1,\, x : \zeta \to \alpha \vdash xv : \alpha}{\pi_1' :: B_1 \vdash uv : \alpha}\,(\cup\mathbf{E})[1, u]$$

$$\begin{prooftree}
\AxiomC{$B_1 \vdash s : \sigma \cup \tau$}
\AxiomC{$\ldots \vdash y : \sigma$}
\RightLabel{$(\cap\mathbf{E}_1)$}
\UnaryInfC{$\ldots \vdash y : \beta \to \gamma \to \alpha$}
\AxiomC{$\ldots \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E})$}
\dashedLine
\UnaryInfC{$\ldots \vdash x_1 : \beta$}
\RightLabel{$(\to\mathbf{E})$}
\BinaryInfC{$B_1,\, y : \sigma \vdash yx_1 : \gamma \to \alpha$}
\RightLabel{$(\cup\mathbf{I}_1)$}
\UnaryInfC{$B_1,\, y : \sigma \vdash yx_1 : (\gamma \to \alpha) \cup (\zeta \to \alpha)$}
\AxiomC{$\ldots \vdash y : \tau$}
\RightLabel{$(\cap\mathbf{E}_1)$}
\UnaryInfC{$\ldots \vdash y : \varepsilon \to \zeta \to \alpha$}
\AxiomC{$\ldots \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E}_2)$}
\UnaryInfC{$\ldots \vdash x_1 : \varepsilon$}
\RightLabel{$(\to\mathbf{E})$}
\BinaryInfC{$B_1,\, y : \tau \vdash yx_1 : \zeta \to \alpha$}
\RightLabel{$(\cup\mathbf{I}_2)$}
\UnaryInfC{$B_1,\, y : \tau \vdash yx_1 : (\gamma \to \alpha) \cup (\zeta \to \alpha)$}
\RightLabel{$(\cup\mathbf{E})[1,\, s]$}
\TrinaryInfC{$B_1 \vdash sx_1 = u : (\gamma \to \alpha) \cup (\zeta \to \alpha)$}
\end{prooftree}$$

i) Considering an $(\to\mathbf{E})$ at the first bottom-up position of a potential typing $B_1,\, x : \gamma \to \alpha \vdash xv : \alpha$, we see that it does not work.

$(\cup\mathbf{E})[1, s]\times$, see right below

$[(\to\mathbf{E})\times, (\cup\mathbf{E})[2]$ not, $(\cap\mathbf{I})$ not$]$

$$\begin{prooftree}
\AxiomC{$B_1,\, x : \gamma \to \alpha \vdash x : \gamma \to \alpha$}
\AxiomC{$B_1,\, x : \gamma \to \alpha \vdash v = x_1 s : \gamma$}
\RightLabel{$(\to\mathbf{E})\ \times$}
\dottedLine
\BinaryInfC{$B_1,\, x : \gamma \to \alpha \vdash xv : \alpha$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$B'_1 \vdash s : \sigma \cup \tau$}
\AxiomC{$\ldots \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E})$}
\dashedLine
\UnaryInfC{$\ldots \vdash x_1 : \delta \to \gamma$}
\AxiomC{$\ldots \vdash y : \sigma$}
\RightLabel{$(\cap\mathbf{E}_2)$}
\UnaryInfC{$\ldots \vdash y : \delta$}
\RightLabel{$(\to\mathbf{E})$}
\BinaryInfC{$B'_1,\, y : \sigma \vdash x_1 y : \gamma = \gamma$}
\AxiomC{$\ldots \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E})$}
\dashedLine
\UnaryInfC{$\ldots \vdash x_1 : \eta \to \zeta$}
\AxiomC{$\ldots \vdash y : \tau$}
\RightLabel{$(\cap\mathbf{E}_2)$}
\UnaryInfC{$\ldots \vdash y : \eta$}
\RightLabel{$(\to\mathbf{E})\ [(\cup\mathbf{E})[1,\, 2]\text{ not}, (\cap\mathbf{I})\text{ not}]$}
\BinaryInfC{$B'_1,\, y : \tau \vdash x_1 y : \gamma \neq \zeta$}
\RightLabel{$(\cup\mathbf{E})[1,\, s]\ \times$}
\dottedLine
\TrinaryInfC{$B'_1 = B_1 \cup \{\, x : \gamma \to \alpha \,\} \vdash v = x_1 s : \gamma$}
\end{prooftree}$$

ii) Considering an $(\cup\mathbf{E})[1]$ at the first bottom-up position of a potential typing $B_1,\, x : \gamma \to \alpha \vdash xv : \alpha$, we find that it does not work. We examine the two possible $(\cup\mathbf{E})[1]$'s, the $(\cup\mathbf{E})[1, v]$ and the $(\cup\mathbf{E})[1, s]$.

by $(\cup\mathbf{E})[1, s]$, see right below

$[(\to\mathbf{E})\times, (\cup\mathbf{E})[2]$ not, $(\cap\mathbf{I})$ not$]$

$$\begin{prooftree}
\AxiomC{$B_1,\, x : \gamma \to \alpha \vdash v : \gamma \cup \zeta$}
\AxiomC{lmp}
\AxiomC{$\cdots \vdash x : \gamma \to \alpha$}
\AxiomC{$\cdots \vdash y : \gamma \neq \zeta$}
\RightLabel{$(\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, 2]\text{ not}, (\cap\mathbf{I})\text{ not}]$}
\dottedLine
\BinaryInfC{$B_1,\, x : \gamma \to \alpha,\, y : \zeta \vdash xy : \alpha$}
\RightLabel{$(\cup\mathbf{E})[1, v]\ \times$}
\dottedLine
\TrinaryInfC{$B_1,\, x : \gamma \to \alpha \vdash xv : \alpha$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$B'_1 \vdash s : \sigma \cup \tau$}
\AxiomC{$B'_1,\, y : \sigma \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E})$}
\dashedLine
\UnaryInfC{$B'_1,\, y : \sigma \vdash x_1 : \delta \to \gamma$}
\AxiomC{$B'_1,\, y : \sigma \vdash y : \sigma$}
\RightLabel{$(\cap\mathbf{E}_2)$}
\UnaryInfC{$B'_1,\, y : \sigma \vdash y : \delta$}
\RightLabel{$(\to\mathbf{E})$}
\BinaryInfC{$B'_1,\, y : \sigma \vdash x_1 y : \gamma$}
\RightLabel{$(\cup\mathbf{I}_1)$}
\UnaryInfC{$B'_1,\, y : \sigma \vdash x_1 y : \gamma \cup \zeta$}
\AxiomC{$B'_1,\, y : \tau \vdash x_1 : \rho$}
\RightLabel{$(\cap\mathbf{E})$}
\dashedLine
\UnaryInfC{$B'_1,\, y : \tau \vdash x_1 : \eta \to \zeta$}
\AxiomC{$B'_1,\, y : \tau \vdash y : \tau$}
\RightLabel{$(\cap\mathbf{E}_2)$}
\UnaryInfC{$B'_1,\, y : \tau \vdash y : \eta$}
\RightLabel{$(\to\mathbf{E})$}
\BinaryInfC{$B'_1,\, y : \tau \vdash x_1 y : \zeta$}
\RightLabel{$(\cup\mathbf{I}_2)$}
\UnaryInfC{$B'_1,\, y : \tau \vdash x_1 y : \gamma \cup \zeta$}
\RightLabel{$(\cup\mathbf{E})[1,\, s]$}
\TrinaryInfC{$B'_1 = B_1 \cup \{\, x : \gamma \to \alpha \,\} \vdash x_1 s = v : \gamma \cup \zeta$}
\end{prooftree}$$

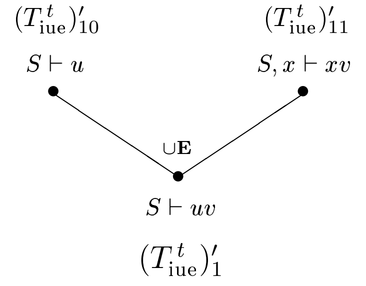
$(\cup\mathbf{E})[1, x_1 y]\times$, see right below

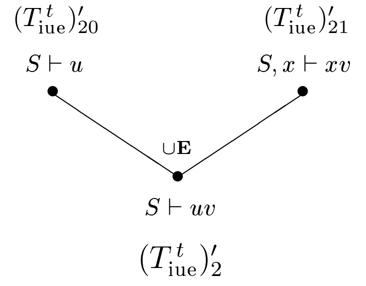
$[(\to\mathbf{E})\times, (\cup\mathbf{E})[2]$ not, $(\cap\mathbf{I})$ not$]$

$$\begin{prooftree}
\AxiomC{$B_1,\, x : \gamma \to \alpha \vdash s : \sigma \cup \tau$}
\AxiomC{left minor premise}
\AxiomC{$B_1,\, x : \gamma \to \alpha,\, y : \tau \vdash x\,(x_1 y) : \alpha$}
\RightLabel{$(\cup\mathbf{E})[1, s]\ \times$}
\dottedLine
\TrinaryInfC{$B_1,\, x : \gamma \to \alpha \vdash xv = x\,(x_1 s) : \alpha$}
\end{prooftree}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dots\vdash x_1:\rho}{\dots\vdash x_1:\eta\to\zeta}\,(\cap\mathbf{E}) \qquad \dfrac{\dots\vdash y:\tau}{\dots\vdash y:\eta}\,(\cap\mathbf{E}_2)}{B_1''\vdash x_1y:\zeta}\,(\to\mathbf{E})}{B_1''\vdash x_1y:\zeta\cup\zeta}\,(\cup\mathbf{I}) \qquad \dfrac{\cdots\vdash x:\gamma\to\alpha \qquad \cdots\vdash z:\gamma\neq\zeta}{B_1'',\,z:\zeta\vdash xz:\alpha}\,(\to\mathbf{E})\times\ [(\cup\mathbf{E})[1,2]\text{ not},\,(\cap\mathbf{I})\text{ not}] \qquad \text{same}}{B_1''=B_1\cup\{\,x:\gamma\to\alpha,\,y:\tau\,\}\vdash x\,(x_1y):\alpha}\,(\cup\mathbf{E})[1,\,x_1y]\ \times$$

Since such a π_1' is not possible, there is no need to look for such a π_2'. Still, if we achieved π_1' and π_2' whose trees $(T_{\text{iue}}^{t})_1'$ and $(T_{\text{iue}}^{t})_2'$ existed and shared a bottom $(\cup\mathbf{E})[1,u]$, the transformation would reduce to transforming π_{10}' and π_{20}', which would type $u <_c uv$, and also to transforming $\pi_{11}', \pi_{12}', \pi_{21}'$, and π_{22}', which would type $xv <_c uv$.





2c. A bottom $(\cup\mathbf{E})[1,v]$. This case also fails.

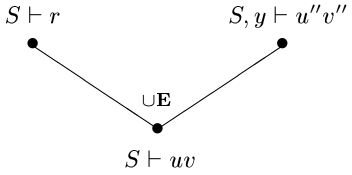
3. Can we construct π_1' and π_2', such that the trees $(T_{\text{iue}}^{t})_1'$ and $(T_{\text{iue}}^{t})_2'$ both exist and share a bottom $(\cup\mathbf{E})[2]$? This case is not possible, as the types assigned to x_1 and x_2 in B_1 do not permit the consideration of a first bottom-up $(\cup\mathbf{E})[2]$ in a $\pi_1' :: B_1\vdash uv:\alpha$.

4. Can we construct π_1' and π_2', such that the trees $(T_{\text{iue}}^{t})_1'$ and $(T_{\text{iue}}^{t})_2'$ both exist and share a bottom proper $(\cup\mathbf{E})$, which is a first bottom-up $(\cup\mathbf{E})[2]$ in one of the derivations and a first bottom-up $(\cup\mathbf{E})[4]$ in the other? We distinguish two cases.

4a. A bottom $(\cup\mathbf{E})[2]$ in π_1' and a bottom $(\cup\mathbf{E})[4]$ in π_2'. Such a case is not possible because, as already explained in 3, we cannot consider a first bottom-up $(\cup\mathbf{E})[2]$ in π_1'.

4b. A bottom $(\cup\mathbf{E})[4]$ in π_1' and a bottom $(\cup\mathbf{E})[2]$ in π_2'. Starting from a root $B_2\vdash uv:\beta$ and working bottom-up, there are fifteen different cases of a $(\cup\mathbf{E})[2,x_1]$, according to which occurrences of x_1 in uv are substituted by a variable $y\notin\{x_1,x_2\}$ to form the subject in the minor premises, and no case of a $(\cup\mathbf{E})[2,x_2]$. So, there are fifteen different cases of a first bottom-up $(\cup\mathbf{E})[4,x_1]$ in π_1' and a first bottom-up $(\cup\mathbf{E})[2,x_1]$ in π_2' with matching corresponding term-statements. We examine two such cases 4b_1 and 4b_2, showing the failure of π_1' in the former and the failure of π_2' in the latter.

4b_1. The case with subject $x_2yy\,(y\,(x_2x_1)) = s'y\,(ys) = u''v''$ in the minor premises. Since π_2 already displays such a bottom part, the case reduces to examining if we can construct a π_1', such that the tree $(T_{\text{iue}}^{t})_1'$ exists and has the following bottom part.



We seek a $\pi'_1 :: B_1 \vdash uv : \alpha$ with the following bottom part.

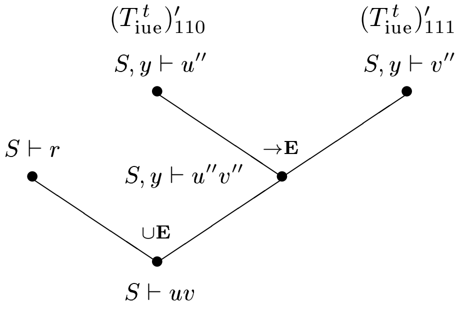
$$\dfrac{B_1 \vdash x_1 : \rho \cup \rho \qquad \begin{array}{c}\text{see i) and ii) below}\\ \big[(\cup\mathbf{E})[2]\,\text{not}, (\cap\mathbf{I})\,\text{not}\big]\\ B_1,\, y : \rho \vdash x_2yy\,(y\,(x_2x_1)) = u''v'' : \alpha\end{array} \qquad \text{same}}{\pi'_1 :: B_1 \vdash uv = x_2x_1x_1(x_1(x_2x_1)) : \alpha}\,(\cup\mathbf{E})[4, x_1]$$

i) Considering an $(\to\mathbf{E})$ at the first bottom-up position of a potential typing $B_1,\, y : \rho \vdash u''v'' : \alpha$, we see that it does not work.

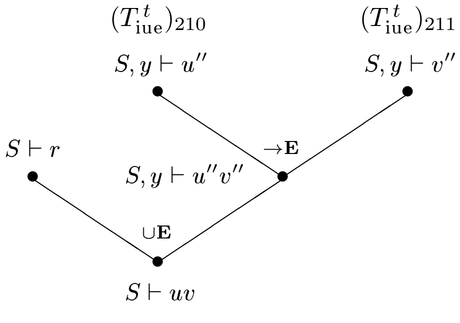
$$\dfrac{\begin{array}{c}(\cup\mathbf{E})[1, s']\times,\ \text{see right below}\\ \big[(\to\mathbf{E})\times, (\cup\mathbf{E})[2]\,\text{not}, (\cap\mathbf{I})\,\text{not}\big]\\ B_1,\, y : \rho \vdash u'' = s'y : \omega \to \alpha\end{array} \qquad \text{right premise}}{B_1,\, y : \rho \vdash u''v'' : \alpha}\,(\to\mathbf{E})\ \times$$

$$\dfrac{\begin{array}{c}\text{by } (\to\mathbf{E})\\ \big[(\cup\mathbf{E})[1,2]\,\text{not}, (\cap\mathbf{I})\,\text{not}\big]\\ B_1,\, y : \rho \vdash s' = x_2y : \sigma \cup \tau\end{array} \quad \begin{array}{c}\text{by } (\to\mathbf{E})\\ \big[(\cup\mathbf{E})[1,2]\,\text{not}, (\cap\mathbf{I})\,\text{not}\big]\\ B_1,\, y : \rho,\, x : \sigma \vdash xy : \omega \to \alpha = \gamma \to \alpha\end{array} \quad \begin{array}{c}\text{by } (\to\mathbf{E})\\ \big[(\cup\mathbf{E})[1,2]\,\text{not}, (\cap\mathbf{I})\,\text{not}\big]\\ B_1,\, y : \rho,\, x : \tau \vdash xy : \omega \to \alpha = \zeta \to \alpha\end{array}}{B_1,\, y : \rho \vdash u'' = s'y : \omega \to \alpha}\,(\cup\mathbf{E})[1, s']\ \times$$

If the $(\to\mathbf{E})$ worked and the tree $(T^t_{\text{iue}})'_1$ existed, we would have the following trees $(T^t_{\text{iue}})'_1$ and $(T^t_{\text{iue}})_2$. The transformation would then reduce to transforming $\pi'_{110}, \pi'_{120}, \pi_{210}$, and π_{220}, which would type $u'' =_c u <_c uv$, and also to transforming $\pi'_{111}, \pi'_{121}, \pi_{211}$, and π_{221}, which would type $v'' =_c v <_c uv$.



$(T^t_{\text{iue}})'_1$



$(T^t_{\text{iue}})_2$

ii) Considering a $(\cup\mathbf{E})[1]$ at the first bottom-up position of a potential typing $B_1,\, y:\rho \vdash u''v'' : \alpha$, we find that it does not work. We present the failure of the $(\cup\mathbf{E})[1, s']$ below. The other three possible $(\cup\mathbf{E})[1]$'s fail, as well.

$$\dfrac{B_1,\, y:\rho\vdash s':\sigma\cup\tau \qquad \overset{\substack{\text{see a), b), c), and d) below}\\ [(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1,\, y:\rho,\, x:\sigma\vdash xy\,(ys):\alpha} \qquad \text{right minor premise}}{B_1,\, y:\rho\vdash u''v''=s'y\,(ys):\alpha}\;(\cup\mathbf{E})[1,s']\;\times$$

$$\text{a)}\;\dfrac{B_1,\, y:\rho,\, x:\sigma\vdash xy:\gamma\to\alpha \qquad \overset{\substack{(\cup\mathbf{E})[1,s]\times,\ \text{see right below}\\ [(\to\mathbf{E})\times,\,(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1,\, y:\rho,\, x:\sigma\vdash ys:\gamma}}{B_1,\, y:\rho,\, x:\sigma\vdash xy\,(ys):\alpha}\;(\to\mathbf{E})\;\times$$

$$\dfrac{B_1,\, y:\rho,\, x:\sigma\vdash s:\sigma\cup\tau \qquad \overset{\substack{\text{by }(\to\mathbf{E})\\ [(\cup\mathbf{E})[1,2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1,\, y:\rho,\, x:\sigma,\, z:\sigma\vdash yz:\gamma=\gamma} \qquad \overset{\substack{\text{by }(\to\mathbf{E})\\ [(\cup\mathbf{E})[1,2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1,\, y:\rho,\, x:\sigma,\, z:\tau\vdash yz:\gamma\neq\zeta}}{B_1,\, y:\rho,\, x:\sigma\vdash ys:\gamma}\;(\cup\mathbf{E})[1,s]\;\times$$

$$\text{b)}\;\dfrac{B_1,\, y:\rho,\, x:\sigma\vdash xy:(\gamma\to\alpha)\cup(\gamma\to\alpha) \qquad \overset{\substack{(\cup\mathbf{E})[1,s]\times,\ \text{see right below}\\ [(\to\mathbf{E})\times,\,(\cup\mathbf{E})[1,ys]\times,\,(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1,\, y:\rho,\, x:\sigma,\, z:\gamma\to\alpha\vdash z\,(ys):\alpha} \qquad \text{same}}{B_1,\, y:\rho,\, x:\sigma\vdash xy\,(ys):\alpha}\;(\cup\mathbf{E})[1,xy]\;\times$$

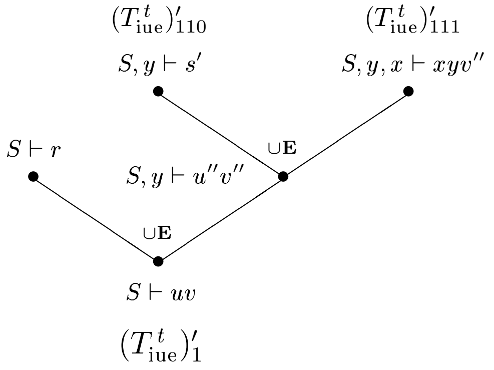
$$\dfrac{B_1'\vdash s:\sigma\cup\tau \qquad \text{lmp} \qquad \dfrac{B_1',\, w:\tau\vdash z:\gamma\to\alpha \qquad \overset{\substack{\text{by }(\to\mathbf{E})\\ [(\cup\mathbf{E})[1,2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1',\, w:\tau\vdash yw:\gamma\neq\zeta}}{B_1',\, w:\tau\vdash z\,(yw):\alpha}\;(\to\mathbf{E})\times\;[(\cup\mathbf{E})[1,yw]\times,\,(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}{B_1'=B_1\cup\{\,y:\rho,\, x:\sigma,\, z:\gamma\to\alpha\,\}\vdash z\,(ys):\alpha}\;(\cup\mathbf{E})[1,s]\;\times$$

$$\text{c)}\;\dfrac{\overset{\substack{\text{by }(\cup\mathbf{E})[1,s],\ \text{see right below}\\ [(\to\mathbf{E})\times,\,(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_1''\vdash ys:\gamma\cup\zeta} \qquad \text{lmp} \qquad \dfrac{B_1'',\, z:\zeta\vdash xy:\gamma\to\alpha \qquad B_1'',\, z:\zeta\vdash z:\gamma\neq\zeta}{B_1'',\, z:\zeta\vdash xyz:\alpha}\;(\to\mathbf{E})\times\;[(\cup\mathbf{E})[1,xy]\times,\,(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}{B_1''=B_1\cup\{\,y:\rho,\, x:\sigma\,\}\vdash xy\,(ys):\alpha}\;(\cup\mathbf{E})[1,ys]\;\times$$

$$\dfrac{B_1''\vdash s:\sigma\cup\tau \qquad \dfrac{\dfrac{\dfrac{B_1'',\, z:\sigma\vdash y:\rho}{B_1'',\, z:\sigma\vdash y:\delta\to\gamma}\,(\cap\mathbf{E}) \qquad \dfrac{B_1'',\, z:\sigma\vdash z:\sigma}{B_1'',\, z:\sigma\vdash z:\delta}\,(\cap\mathbf{E}_2)}{B_1'',\, z:\sigma\vdash yz:\gamma}\,(\to\mathbf{E})}{B_1'',\, z:\sigma\vdash yz:\gamma\cup\zeta}\,(\cup\mathbf{I}_1) \qquad \dfrac{\dfrac{\dfrac{B_1'',\, z:\tau\vdash y:\rho}{B_1'',\, z:\tau\vdash y:\eta\to\zeta}\,(\cap\mathbf{E}) \qquad \dfrac{B_1'',\, z:\tau\vdash z:\tau}{B_1'',\, z:\tau\vdash z:\eta}\,(\cap\mathbf{E}_2)}{B_1'',\, z:\tau\vdash yz:\zeta}\,(\to\mathbf{E})}{B_1'',\, z:\tau\vdash yz:\gamma\cup\zeta}\,(\cup\mathbf{I}_2)}{B_1''=B_1\cup\{\,y:\rho,\, x:\sigma\,\}\vdash ys:\gamma\cup\zeta}\;(\cup\mathbf{E})[1,s]$$

$$\text{d)}\quad \dfrac{B_1'' \vdash s : \sigma \cup \tau \qquad \text{lmp} \qquad \dfrac{\overset{\text{by } (\to\mathbf{E})}{\overset{[(\cup\mathbf{E})[1,\,2]\ \text{not},\ (\cap\mathbf{I})\ \text{not}]}{B_1'',\, z : \tau \vdash xy : \gamma \to \alpha \qquad B_1'',\, z : \tau \vdash yz : \gamma \neq \zeta}}}{B_1'',\, z : \tau \vdash xy\,(yz) : \alpha}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1,\,xy]\times,\ (\cup\mathbf{E})[1,\,yz]\times,\ (\cup\mathbf{E})[2]\ \text{not},\ (\cap\mathbf{I})\ \text{not}]}{B_1'' = B_1 \cup \{\, y : \rho,\, x : \sigma \,\} \vdash xy\,(ys) : \alpha}\ (\cup\mathbf{E})[1,\,s]\ \times$$

If the $(\cup\mathbf{E})[1, s']$ worked and the tree $(T^{\,t}_{\text{iue}})'_1$ existed, we would have the following tree $(T^{\,t}_{\text{iue}})'_1$.

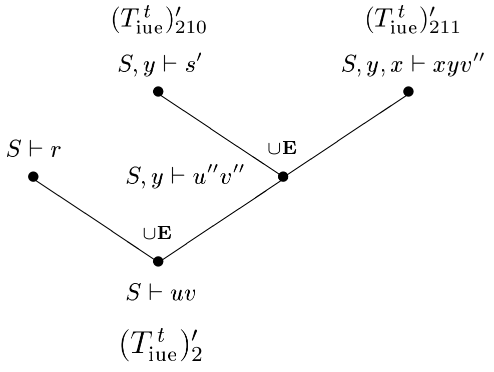


We would then transform π_2 to a $\pi_2' :: B_2 \vdash uv : \beta$, such that the tree $(T^{\,t}_{\text{iue}})'_2$ exists and is bottom-identical to the tree $(T^{\,t}_{\text{iue}})'_1$. We denote "$\pi$(w)" a weakened version of a derivation π.

$$\dfrac{\dfrac{B_2 \vdash x_1 = r : \chi}{B_2 \vdash r : \phi \cup \psi}\ (\cap\mathbf{E}_1) \qquad \overset{\text{see below}}{\pi_{21}' :: B_2,\, y : \phi \vdash x_2yy\,(ys) = u''v'' : \beta} \qquad \overset{\text{see below}}{\pi_{22}' :: B_2,\, y : \psi \vdash x_2yy\,(ys) = u''v'' : \beta}}{\pi_2' :: B_2 \vdash sr\,(rs) = uv : \beta}\ (\cup\mathbf{E})$$

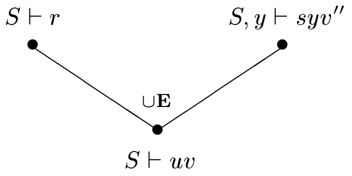
$$\dfrac{\dfrac{\dfrac{\dfrac{B_\phi \vdash x_2 : \upsilon}{B_\phi \vdash x_2 : \phi \to \phi_{\alpha\beta}}\ (\cap\mathbf{E}) \qquad B_\phi \vdash y : \phi}{B_\phi \vdash x_2y : \phi_{\alpha\beta}}\ (\to\mathbf{E})}{\pi_{210}' :: B_\phi \vdash x_2y = s' : \phi_{\alpha\beta} \cup \phi_{\alpha\beta}}\ (\cup\mathbf{I}) \qquad \dfrac{\dfrac{\ldots \vdash x : \phi_{\alpha\beta} \qquad \ldots \vdash y : \phi}{B_\phi,\, x : \phi_{\alpha\beta} \vdash xy : \alpha \to \beta}\ (\to\mathbf{E}) \qquad \overset{\pi_{211}(\text{w})}{B_\phi,\, x : \phi_{\alpha\beta} \vdash ys : \alpha}}{\pi_{211}' :: B_\phi,\, x : \phi_{\alpha\beta} \vdash xy\,(ys) : \beta}\ (\to\mathbf{E}) \qquad \text{same}}{\pi_{21}' :: B_\phi = B_2 \cup \{\, y : \phi \,\} \vdash x_2yy\,(ys) = u''v'' : \beta}\ (\cup\mathbf{E})[1,\,s']$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_\psi \vdash x_2 : \upsilon}{B_\psi \vdash x_2 : \psi \to \psi_{\gamma\beta}}\ (\cap\mathbf{E}) \qquad B_\psi \vdash y : \psi}{B_\psi \vdash x_2y : \psi_{\gamma\beta}}\ (\to\mathbf{E})}{\pi_{220}' :: B_\psi \vdash x_2y = s' : \psi_{\gamma\beta} \cup \psi_{\gamma\beta}}\ (\cup\mathbf{I}) \qquad \dfrac{\dfrac{\ldots \vdash x : \psi_{\gamma\beta} \qquad \ldots \vdash y : \psi}{B_\psi,\, x : \psi_{\gamma\beta} \vdash xy : \gamma \to \beta}\ (\to\mathbf{E}) \qquad \overset{\pi_{221}(\text{w})}{B_\psi,\, x : \psi_{\gamma\beta} \vdash ys : \gamma}}{\pi_{221}' :: B_\psi,\, x : \psi_{\gamma\beta} \vdash xy\,(ys) : \beta}\ (\to\mathbf{E}) \qquad \text{same}}{\pi_{22}' :: B_\psi = B_2 \cup \{\, y : \psi \,\} \vdash x_2yy\,(ys) = u''v'' : \beta}\ (\cup\mathbf{E})[1,\,s']$$



The transformation would thus reduce to $\pi'_{110}, \pi'_{120}, \pi'_{210}$, and π'_{220}, which would type $s' <_c u <_c uv$, and also to $\pi'_{111}, \pi'_{112}, \pi'_{121}, \pi'_{122}, \pi'_{211}, \pi'_{212}, \pi'_{221}$, and π'_{222}, which would type $xyv'' <_c u''v'' =_c uv$.

4b$_2$. The case with subject $x_2x_1y\,(y\,(x_2x_1)) = sy\,(ys) = syv''$ in the minor premises. We seek to construct π'_1 and π'_2, such that the trees $(T^t_{\mathrm{iue}})'_1$ and $(T^t_{\mathrm{iue}})'_2$ both exist and share a bottom $(\cup\mathbf{E})$, as shown below.



We want a $\pi'_2 :: B_2 \vdash uv : \beta$ with the following bottom part.

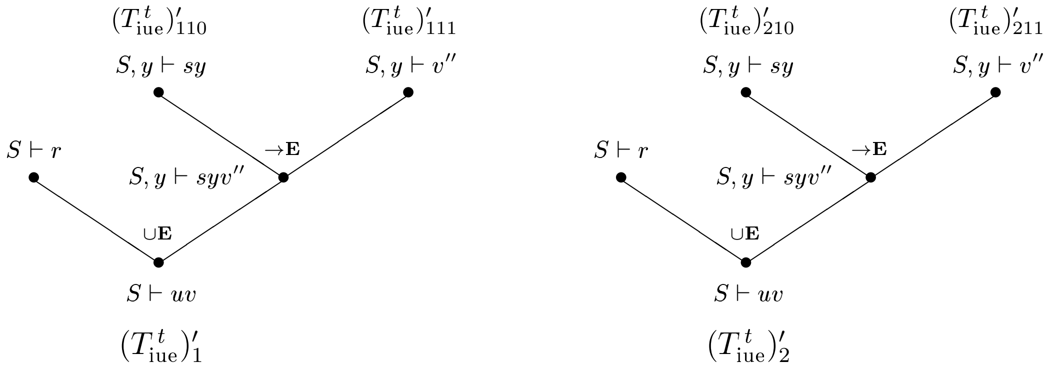
$$\begin{array}{c} \text{see i), ii), and iii) below} \\ [\mathbf{(\cap I)}\ \text{not}] \\ \dfrac{B_2 \vdash x_1 : \phi \cup \psi \qquad B_2,\ y : \phi \vdash x_2x_1y\,(y\,(x_2x_1)) = sy\,(ys) : \beta \qquad \text{right minor premise}}{\pi'_2 :: B_2 \vdash uv = x_2x_1x_1(x_1(x_2x_1)) : \beta}\ \mathbf{(\cup E)}[2, x_1] \end{array}$$

i) Considering an $\mathbf{(\to E)}$ at the first bottom-up position of a potential typing $B_2,\ y : \phi \vdash sy\,(ys) : \beta$, we see that it does not work.

$$\begin{array}{c} \mathbf{(\cup E)}[2, x_1]\times,\ \text{see right below} \\ [\mathbf{(\to E)}\times, \mathbf{(\cup E)}[1, s]\times, \mathbf{(\cap I)}\ \text{not}] \\ B_2,\ y : \phi \vdash sy : \omega \to \beta \qquad \text{right premise} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ \mathbf{(\to E)}\ \times \\ B_2,\ y : \phi \vdash sy\,(ys) : \beta \end{array}$$

$$\dfrac{B_2,\, y:\phi \vdash x_1 : \phi \cup \psi \qquad \text{lmp} \qquad \dfrac{\begin{array}{c}\text{by } (\rightarrow\mathbf{E}) \\ \left[(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}\right] \\ \cdots \vdash x_2 x : \psi \rightarrow \gamma \rightarrow \beta \qquad \cdots \vdash y : \psi \neq \phi\end{array}}{B_2,\, y:\phi,\, x:\psi \vdash x_2 x y : \omega \rightarrow \beta}\;(\rightarrow\mathbf{E})\times\ [(\cup\mathbf{E})[1, x_2 x]\times, (\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}]}{B_2,\, y:\phi \vdash sy = x_2 x_1 y : \omega \rightarrow \beta}\;(\cup\mathbf{E})[2, x_1]\ \times$$

If the $(\rightarrow\mathbf{E})$ worked in π'_2 right above the $(\cup\mathbf{E})[2, x_1]$ and also in π'_1 right above the[16] $(\cup\mathbf{E})[4, x_1]$ and the trees $(T^t_{\text{iue}})'_1$ and $(T^t_{\text{iue}})'_2$ existed, the transformation would reduce to $\pi'_{110}, \pi'_{120}, \pi'_{210}$, and π'_{220}, which would type $sy =_c u <_c uv$, and also to $\pi'_{111}, \pi'_{121}, \pi'_{211}$, and π'_{221}, which would type $v'' =_c v <_c uv$.



ii) Considering a $(\cup\mathbf{E})[1]$ at the first bottom-up position of a potential typing $B_2,\, y : \phi \vdash sy\,(ys) : \beta$, we find that it does not work. We show the failure of one of the six[17] possible $(\cup\mathbf{E})[1, s]$'s. The other five, as well as the $(\cup\mathbf{E})[1, sy]$ and the $(\cup\mathbf{E})[1, ys]$, also fail.

$$\dfrac{B_2,\, y:\phi \vdash s : \phi_{\alpha\beta} \cup \psi_{\gamma\beta} \qquad \begin{array}{c}\text{see a), b), and c) below} \\ \left[(\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}\right] \\ B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash xy\,(yx) : \beta\end{array} \qquad \text{right minor premise}}{B_2,\, y:\phi \vdash sy\,(ys) : \beta}\;(\cup\mathbf{E})[1, s]\ \times$$

$$\text{a)}\quad \dfrac{\begin{array}{c}\text{by } (\rightarrow\mathbf{E}) \\ \left[(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}\right] \\ B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash xy : \omega \rightarrow \beta = \alpha \rightarrow \beta\end{array} \qquad \dfrac{\cdots \vdash y : \phi = \zeta \rightarrow \alpha \qquad \cdots \vdash x : \zeta \neq \phi_{\alpha\beta}}{B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash yx : \alpha}\;(\rightarrow\mathbf{E})\times\ [(\cup\mathbf{E})[1,2]\text{ not}, (\cap\mathbf{I})\text{ not}]}{B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash xy\,(yx) : \beta}\;(\rightarrow\mathbf{E})\ \times$$

$$\text{b)}\quad \dfrac{B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash xy : (\alpha \rightarrow \beta) \cup (\alpha \rightarrow \beta) \qquad \begin{array}{c}(\rightarrow\mathbf{E})\times\text{, see right below} \\ \left[(\cup\mathbf{E})[1, yx]\times, (\cup\mathbf{E})[2]\text{ not}, (\cap\mathbf{I})\text{ not}\right] \\ B_2,\, y:\phi,\, x:\phi_{\alpha\beta},\, z:\alpha \rightarrow \beta \vdash z\,(yx) : \beta\end{array} \qquad \text{same}}{B_2,\, y:\phi,\, x:\phi_{\alpha\beta} \vdash xy\,(yx) : \beta}\;(\cup\mathbf{E})[1, xy]\ \times$$

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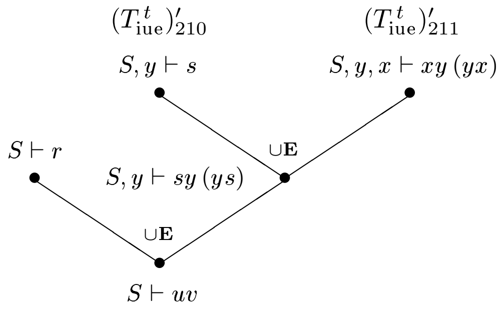
[16] It can be shown that in a π'_1 with a first bottom-up $(\cup\mathbf{E})[4, x_1]$, the $(\rightarrow\mathbf{E})$ right above the $(\cup\mathbf{E})[4, x_1]$ does not work.

[17] There are six possible $(\cup\mathbf{E})[1, s]$'s, as there are three choices for the subject in the minor premises $[xy\,(ys)$ or $sy\,(yx)$ or $xy\,(yx)]$ and two choices for the union predicate in the major premise $[\phi_{\alpha\beta} \cup \psi_{\gamma\beta}$ or $\zeta \cup \zeta]$.

$$\dfrac{B_2' \vdash z : \alpha \to \beta \qquad \dfrac{B_2' \vdash y : \phi = \zeta \to \alpha \qquad B_2' \vdash x : \zeta \neq \phi_{\alpha\beta}}{B_2' \vdash yx : \alpha}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1,2]\,\text{not}, (\cap\mathbf{I})\,\text{not}]}{B_2' = B_2 \cup \{\, y : \phi,\, x : \phi_{\alpha\beta},\, z : \alpha \to \beta \,\} \vdash z\,(yx) : \beta}\ (\to\mathbf{E})\ \times$$

$$\text{c)}\quad \dfrac{\dfrac{\cdots\ \vdash y : \phi = \zeta \to \alpha \qquad \cdots\ \vdash x : \zeta \neq \phi_{\alpha\beta}}{B_2,\, y : \phi,\, x : \phi_{\alpha\beta} \vdash yx : \omega_1 \cup \omega_2}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1,2]\,\text{not}, (\cap\mathbf{I})\,\text{not}] \qquad \text{lmp} \qquad \text{rmp}}{B_2,\, y : \phi,\, x : \phi_{\alpha\beta} \vdash xy\,(yx) : \beta}\ (\cup\mathbf{E})[1, yx]\ \times$$

If this $(\cup\mathbf{E})[1, s]$ worked and the tree $(T^{t}_{\text{iue}})'_2$ existed, we would have the following tree $(T^{t}_{\text{iue}})'_2$.



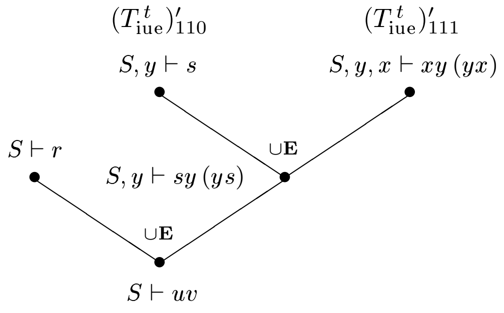
$(T^{t}_{\text{iue}})'_2$

We would then construct a $\pi_1' :: B_1 \vdash uv : \alpha$, such that the tree $(T^{t}_{\text{iue}})'_1$ exists and is bottom-identical to the tree $(T^{t}_{\text{iue}})'_2$.

$$\dfrac{\dfrac{B_1 \vdash x_1 = r : \rho}{B_1 \vdash r : \rho \cup \rho}\ (\cup\mathbf{I}) \qquad \overset{\text{see right below}}{\pi_{11}' :: B_1,\, y : \rho \vdash sy\,(ys) : \alpha} \qquad \text{same}}{\pi_1' :: B_1 \vdash sr\,(rs) = uv : \alpha}\ (\cup\mathbf{E})[4, x_1]$$

$$\dfrac{\overset{\text{by }(\to\mathbf{E})}{\pi_{110}' :: B_1,\, y : \rho \vdash x_2 x_1 = s : \sigma \cup \tau} \qquad \overset{\text{see below}}{\pi_{111}' :: B_1,\, y : \rho,\, x : \sigma \vdash xy\,(yx) : \alpha} \qquad \overset{\text{see below}}{\pi_{112}' :: B_1,\, y : \rho,\, x : \tau \vdash xy\,(yx) : \alpha}}{\pi_{11}' :: B_1,\, y : \rho \vdash sy\,(ys) : \alpha}\ (\cup\mathbf{E})[1, s]$$

$$\dfrac{\overset{\text{by }(\to\mathbf{E})}{B_1,\, y : \rho,\, x : \sigma \vdash xy : \gamma \to \alpha} \qquad \overset{\text{by }(\to\mathbf{E})}{B_1,\, y : \rho,\, x : \sigma \vdash yx : \gamma}}{\pi_{111}' :: B_1,\, y : \rho,\, x : \sigma \vdash xy\,(yx) : \alpha}\ (\to\mathbf{E}) \qquad \dfrac{\overset{\text{by }(\to\mathbf{E})}{B_1,\, y : \rho,\, x : \tau \vdash xy : \zeta \to \alpha} \qquad \overset{\text{by }(\to\mathbf{E})}{B_1,\, y : \rho,\, x : \tau \vdash yx : \zeta}}{\pi_{112}' :: B_1,\, y : \rho,\, x : \tau \vdash xy\,(yx) : \alpha}\ (\to\mathbf{E})$$



$(T^{\,t}_{\text{iue}})'_1$

The transformation would thus reduce to $\pi'_{110}, \pi'_{120}, \pi'_{210}$, and π'_{220}, which would type $s <_c uv$, and also to $\pi'_{111}, \pi'_{112}, \pi'_{121}, \pi'_{122}, \pi'_{211}, \pi'_{212}, \pi'_{221}$, and π'_{222}, which would type $xy\,(yx) <_c uv$.

iii) Considering a $(\cup\mathbf{E})[2]$ at the first bottom-up position of a potential typing $B_2,\, y : \phi \vdash sy\,(ys) : \beta$, we also find that it does not work. We lay out the failure of one of the three possible $(\cup\mathbf{E})[2, x_1]$'s. The other two fail, as well.

$$\frac{B_2,\, y : \phi \vdash x_1 : \phi \cup \psi \qquad \text{left minor premise} \qquad \overset{\substack{\text{see a), b), c), and d) below} \\ [(\cup\mathbf{E})[2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B_2,\, y : \phi,\, x : \psi \vdash x_2xy\,(y\,(x_2x)) : \beta}}{B_2,\, y : \phi \vdash sy\,(ys) = x_2x_1y\,(y\,(x_2x_1)) : \beta}\ (\cup\mathbf{E})[2, x_1]\ \times$$

$$\text{a)}\ \frac{\dfrac{\overset{\substack{\text{by } (\to\mathbf{E}) \\ [(\cup\mathbf{E})[1,2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}}{B'_2 \vdash x_2x : \psi \to \gamma \to \beta} \qquad B'_2 \vdash y : \psi \neq \phi}{B'_2 \vdash x_2xy : \omega \to \beta}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, x_2x]\times, (\cup\mathbf{E})[2]\,\text{not}, (\cap\mathbf{I})\,\text{not}] \qquad \text{right premise}}{B'_2 = B_2 \cup \{\, y : \phi,\, x : \psi \,\} \vdash x_2xy\,(y\,(x_2x)) : \beta}\ (\to\mathbf{E})\ \times$$

If the $(\to\mathbf{E})$ worked, so that the tree $(T^{\,t}_{\text{iue}})'_2$ existed and displayed a bottom part of two $(\cup\mathbf{E})[2, r]$'s and one $(\to\mathbf{E})$, and if there was a π'_1 with an identical bottom part in its tree $(T^{\,t}_{\text{iue}})'_1$, the transformation would reduce to transforming eight derivations typing $x_2xy <_c uv$ and another eight derivations typing $y\,(x_2x) <_c uv$.

$$\text{b)}\ \frac{\dfrac{B'_2 \vdash x_2x : \psi_{\gamma\beta} \cup \psi_{\gamma\beta} \qquad \overset{[(\to\mathbf{E})\times,\,(\cup\mathbf{E})[1,\,2]\,\text{not},\,(\cap\mathbf{I})\,\text{not}]}{B'_2,\, z : \psi_{\gamma\beta} \vdash zy : \omega_1 \cup \omega_2} \qquad \text{same}}{B'_2 \vdash x_2xy : \omega_1 \cup \omega_2}\ (\cup\mathbf{E})[1,\, x_2x]\times\ [(\to\mathbf{E})\times, (\cup\mathbf{E})[2]\,\text{not}, (\cap\mathbf{I})\,\text{not}] \qquad \text{lmp} \quad \text{rmp}}{B'_2 = B_2 \cup \{\, y : \phi,\, x : \psi \,\} \vdash x_2xy\,(y\,(x_2x)) : \beta}\ (\cup\mathbf{E})[1,\, x_2xy]\ \times$$

If the $(\cup\mathbf{E})[1, x_2xy]$ worked, so that the tree $(T^{\,t}_{\text{iue}})'_2$ existed and displayed a bottom part of two $(\cup\mathbf{E})[2, r]$'s and one $(\cup\mathbf{E})[1, x_2xy]$, and if there was a π'_1 with an identical bottom part in its tree $(T^{\,t}_{\text{iue}})'_1$, the transformation would reduce to transforming eight derivations typing $x_2xy <_c uv$ and sixteen derivations typing $z\,(y\,(x_2x)) <_c uv$.

$$\text{c)}\ \dfrac{\dfrac{B'_2 \vdash y : \phi = \zeta \to \alpha \qquad B'_2 \vdash x_2x : \zeta \neq \psi_{\gamma\beta}}{B'_2 \vdash y\,(x_2x) : \omega_1 \cup \omega_2}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, x_2x]\times, (\cup\mathbf{E})[2]\,\mathrm{not}, (\cap\mathbf{I})\,\mathrm{not}] \qquad \mathrm{lmp} \quad \mathrm{rmp}}{B'_2 = B_2 \cup \{\, y : \phi,\ x : \psi \,\} \vdash x_2xy\,(y\,(x_2x)) : \beta}\ (\cup\mathbf{E})[1, y\,(x_2x)]\ \times$$

If the $(\cup\mathbf{E})[1, y\,(x_2x)]$ worked, so that the tree $(T^{t}_{\mathrm{iue}})'_2$ existed and displayed a bottom part of two $(\cup\mathbf{E})[2, r]$'s and one $(\cup\mathbf{E})[1, y\,(x_2x)]$, and if there was a π'_1 with an identical bottom part in its tree $(T^{t}_{\mathrm{iue}})'_1$, the transformation would reduce to transforming eight derivations typing $y\,(x_2x) <_c uv$ and sixteen derivations typing $x_2xyz <_c uv$.

$$\text{d)[18]}\ \dfrac{B'_2 \vdash x_2x : \psi_{\gamma\beta} \cup \psi_{\gamma\beta} \qquad \overset{\substack{(\cup\mathbf{E})[1, zy]\times,\ \text{see right below} \\ [(\to\mathbf{E})\times, (\cup\mathbf{E})[1, yz]\times, (\cup\mathbf{E})[2]\,\mathrm{not}, (\cap\mathbf{I})\,\mathrm{not}]}}{B'_2,\ z : \psi_{\gamma\beta} \vdash zy\,(yz) : \beta} \qquad \text{same}}{B'_2 = B_2 \cup \{\, y : \phi,\ x : \psi \,\} \vdash x_2xy\,(y\,(x_2x)) : \beta}\ (\cup\mathbf{E})[1, x_2x]\ \times$$

$$\dfrac{\dfrac{B'_2,\ z : \psi_{\gamma\beta} \vdash z : \psi \to \gamma \to \beta \qquad B'_2,\ z : \psi_{\gamma\beta} \vdash y : \psi \neq \phi}{B'_2,\ z : \psi_{\gamma\beta} \vdash zy : \omega_1 \cup \omega_2}\ (\to\mathbf{E})\times\ [(\cup\mathbf{E})[1, 2]\,\mathrm{not}, (\cap\mathbf{I})\,\mathrm{not}] \qquad \mathrm{lmp} \quad \mathrm{rmp}}{B'_2,\ z : \psi_{\gamma\beta} \vdash zy\,(yz) : \beta}\ (\cup\mathbf{E})[1, zy]\ \times$$

If this $(\cup\mathbf{E})[1, x_2x]$ worked, so that the tree $(T^{t}_{\mathrm{iue}})'_2$ existed and displayed a bottom part of two $(\cup\mathbf{E})[2, r]$'s and one $(\cup\mathbf{E})[1, x_2x]$, and if there was a π'_1 with an identical bottom part in its tree $(T^{t}_{\mathrm{iue}})'_1$, the transformation would reduce to transforming eight derivations typing $x_2x <_c uv$ and sixteen derivations typing $zy\,(yz) <_c uv$.

Besides cases 4b$_1$ and 4b$_2$, the other thirteen possible cases of a first bottom-up $(\cup\mathbf{E})[4, x_1]$ in π'_1 and a first bottom-up $(\cup\mathbf{E})[2, x_1]$ in π'_2 also fail[19].

Cases 1 to 4 all fail. There seems to be no other meaningful first bottom-up step to "equalize" π_1 and π_2 with respect to trees of implications and union eliminations with terms. We therefore conclude that we *cannot* transform π_1 and π_2 to $\pi'_1 :: B_1 \vdash uv : \alpha$ and $\pi'_2 :: B_2 \vdash uv : \beta$, respectively, such that $(1 \wedge 2)_{\pi'_1, \pi'_2}$. Yet, we urge the reader to further examine the two derivations and propose any transformation we may have missed.

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[18] This is one of the three possible $(\cup\mathbf{E})[1, x_2x]$'s. The other two do not work, either.

[19] This is an appropriate point to elaborate a bit on the intention of a category-4 proper $(\cup\mathbf{E})$ and how it is actually realized in its definition in Note 5. We intend to define a bottom-up $(\cup\mathbf{E})[4, x_i]$ as a union elimination that does *not* decompose a union type $\omega_1 \cup \omega_2$ in σ_i, that could become the predicate in the major premise, to its components ω_1 and ω_2 in the (context of the) left and the (context of the) right minor premise, respectively. This is because we want to have a union elimination that matches a $(\cup\mathbf{E})[2, x_i]$ termwise without offering progress with respect to the typing. However, the definition of a $(\cup\mathbf{E})[4, x_i]$ in Note 5 only covers the cases where σ_i is *not* a union type or an intersection type with a union factor, i.e. the cases where σ_i contains no union type that could become the predicate in the major premise. This is because these cases suffice to handle the bottom-up search for transforming the specific derivations $\pi_1 :: B_1 \vdash uv : \alpha$ and $\pi_2 :: B_2 \vdash uv : \beta$ of this section. As explained in case 4 of the bottom-up search, starting from a root $B_1 \vdash uv : \alpha$ and a root $B_2 \vdash uv : \beta$ and examining whether we can have a first bottom-up step involving a $(\cup\mathbf{E})[2]$ and a $(\cup\mathbf{E})[4]$, the only possible case is a first bottom-up $(\cup\mathbf{E})[4, x_1]$ concluding $B_1 \vdash uv : \alpha$ and a first bottom-up $(\cup\mathbf{E})[2, x_1]$ concluding $B_2 \vdash uv : \beta$. As the type ρ assigned to x_1 in B_1 is *not* a union type or an intersection type with a union factor, it suffices to define a $(\cup\mathbf{E})[4, x_i]$ for a σ_i which is *not* a union type or an intersection type with a union factor. If we had a different pair of roots to start from, e.g. roots which would both admit a first bottom-up $(\cup\mathbf{E})[2, x_i]$, we would need to extend the definition of a $(\cup\mathbf{E})[4, x_i]$ to cover all the cases of σ_i.

In contrast to the transformation counterexample given so far, there are quite many transformation examples, i.e. examples of derivations $\pi_1 :: B_1 \vdash t : \tau$ and $\pi_2 :: B_2 \vdash t : \psi$, where $dom(B_1) = dom(B_2)$, such that $\neg(1 \wedge 2)_{\pi_1,\pi_2}$, which are transformable to $\pi'_1 :: B_1 \vdash t : \tau$ and $\pi'_2 :: B_2 \vdash t : \psi$, respectively, so that $(1 \wedge 2)_{\pi'_1,\pi'_2}$. These examples range from very simple ones, i.e. involving simple derivations π_1 and π_2, to significantly complex ones. A complex one, which is actually a variation of the counterexample, can be found in Appendix B.

5.4 Non-restricted correspondence theorems?

It remains to examine whether the correspondence between $\mathrm{IUL}^{\star}_{m}$ and $\mathrm{IUT}^{\oplus}$ can be sustained, if the auxiliary notion "T^{t}_{iue}" is removed. This amounts to examining 1. whether Theorem 5.10 can be reformulated[20] to just saying "*if* $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)^n_{i=1}]_{x_1,\ldots,x_m}$ *is a decorated derivation in* IUL_m*, there are derivations* $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *in* $\mathrm{IUT}^{\oplus}$" and 2. whether Theorem 5.13 can be reformulated to just saying "*if* $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *are derivations in* $\mathrm{IUT}^{\oplus}$*, there is a decorated derivation* $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)^n_{i=1}]_{x_1,\ldots,x_m}$ *in* IUL_m", so that the correspondence between $\mathrm{IUL}^{\star}_{m}$ and $\mathrm{IUT}^{\oplus}$ is in accordance to the correspondence between $\mathrm{ISL}^\star$ and IT, introduced in Chapter 1 (see Theorem 1.20). Obviously, given a derivation $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)^n_{i=1}]_{x_1,\ldots,x_m}$ in $\mathrm{IUL}^{\star}_{m}$, there are derivations $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ in $\mathrm{IUT}^{\oplus}$; this is already proved in 5.10. But what about the inverse? Given derivations $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ in $\mathrm{IUT}^{\oplus}$, without any additional information about their potential trees T^{t}_{iue}, is there always a derivation $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)^n_{i=1}]_{x_1,\ldots,x_m}$ in $\mathrm{IUL}^{\star}_{m}$? To answer this question, we should reflect on the features the π_i's need to share, so that their "merging" into a single $\pi^\star$ is secured. Is the common term-statement $x_1, \ldots, x_m \vdash t$ at the root a sufficient condition for merging, besides being a necessary one? The answer is negative, as the following example[21] indicates.

Example 5.18 *Let* $\phi = (\sigma \cup \tau) \cap \alpha$*,* $\sigma = \rho \cap \sigma_2$*,* $\tau = \tau_1 \cap \rho$*, and* $\chi = (\zeta \cup \xi) \cap \beta$*. Consider the* $\mathrm{IUT}^{\oplus}$*-derivations* $\pi_1 :: \{\, x : \phi,\ y : \psi \,\} \vdash x : \rho$ *and* $\pi_2 :: \{\, x : \chi,\ y : \upsilon \,\} \vdash x : \beta$*, as shown below.*

$$\cfrac{\cfrac{x : \phi,\ y : \psi \vdash x : \phi}{x : \phi,\ y : \psi \vdash x : \sigma \cup \tau}\,(\cap\mathbf{E}_1) \qquad \cfrac{x : \phi,\ y : \psi,\ z : \sigma \vdash z : \sigma}{x : \phi,\ y : \psi,\ z : \sigma \vdash z : \rho}\,(\cap\mathbf{E}_1) \qquad \cfrac{x : \phi,\ y : \psi,\ z : \tau \vdash z : \tau}{x : \phi,\ y : \psi,\ z : \tau \vdash z : \rho}\,(\cap\mathbf{E}_2)}{\pi_1 :: \{\, x : \phi,\ y : \psi \,\} \vdash z[x/z] = x : \rho}\,(\cup\mathbf{E})_{\mathrm{proper}}$$

$$\cfrac{\cfrac{x : \chi,\ y : \upsilon \vdash x : \chi}{x : \chi,\ y : \upsilon \vdash x : \zeta \cup \xi}\,(\cap\mathbf{E}_1) \qquad \cfrac{x : \chi,\ y : \upsilon,\ z : \zeta \vdash x : \chi}{x : \chi,\ y : \upsilon,\ z : \zeta \vdash x : \beta}\,(\cap\mathbf{E}_2) \qquad \cfrac{x : \chi,\ y : \upsilon,\ z : \xi \vdash x : \chi}{x : \chi,\ y : \upsilon,\ z : \xi \vdash x : \beta}\,(\cap\mathbf{E}_2)}{\pi_2 :: \{\, x : \chi,\ y : \upsilon \,\} \vdash x[x/z] = x : \beta}\,(\cup\mathbf{E})_{\mathrm{phony}}$$

Derivations π_1 *and* π_2 *share the term-statement* $x, y \vdash x$ *at the root. However, they cannot be* naturally *merged*[22] *into a single* $\pi^\star :: x : [(\phi, \psi \,;\, \rho), (\chi, \upsilon \,;\, \beta)]_{x,\,y}$*. Any bottom-up attempt*[23] *for such a merging fails, as displayed below.*

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[20] Such a reformulated theorem is qualitatively same to Theorems 3.10 and 3.22.

[21] This is actually Example 3.13 customized to the current context.

[22] Nonetheless, as we will later explicate, we may transform π_2 to a $\pi'_2 :: \{\, x : \chi,\ y : \upsilon \,\} \vdash x : \beta$ containing a proper $(\cup\mathbf{E})$, so that π_1 and π'_2 are compatible and mergeable into a $(\pi')^\star :: x : [(\phi, \psi \,;\, \rho), (\chi, \upsilon \,;\, \beta)]_{x,\,y}$.

[23] There are two different bottom-up ways to (naturally) merge π_1 and π_2 into a (canonical) $\pi^\star$ depending on the order of application of $(\cap\mathbf{E}_1)$ and $(\cap\mathbf{E}_2)$ in the right branch.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x : [(\psi,\phi\,;\,\phi),(\upsilon,\chi\,;\,\chi)]_{\,y,\,x}}\;\textbf{(ax)}}{x : [(\phi,\psi\,;\,\phi),(\chi,\upsilon\,;\,\chi)]_{\,x,\,y}}\;\textbf{(X)}}{x : [(\phi,\psi\,;\,\sigma\cup\tau),(\chi,\upsilon\,;\,\zeta\cup\xi)]_{\,x,\,y}}\;(\cap\mathbf{E}_1) \qquad \dfrac{\overset{\text{cannot reach an axiom}}{\overset{\uparrow}{?: [(\phi,\psi,\sigma\,;\,\sigma),(\phi,\psi,\tau\,;\,\tau),(\chi,\upsilon,\zeta\,;\,\chi),(\chi,\upsilon,\xi\,;\,\chi)]_{\,x,\,y,\,z}}}}{?: [(\phi,\psi,\sigma\,;\,\rho),(\phi,\psi,\tau\,;\,\rho),(\chi,\upsilon,\zeta\,;\,\beta),(\chi,\upsilon,\xi\,;\,\beta)]_{\,x,\,y,\,z}}\;(\cap\mathbf{E}_1),(\cap\mathbf{E}_2)}{\pi^\star :: \boxed{?} : [(\phi,\psi\,;\,\rho),(\chi,\upsilon\,;\,\beta)]_{\,x,\,y}}\;(\cup\mathbf{E})$$

As already noted in Example 3.13, the failure is due to the incompatibility of the proper $(\cup\mathbf{E})$ *in* π_1 *and the* phony $(\cup\mathbf{E})$ *in* π_2.

The above example suggests that the π_i's need to share more than the term-statement at the root, if they are to be merged into a single $\pi^\star$. The additional common features required are actually dictated by features of the decorated logic and are the following.

I. The π_i's should have a common structure of rules that are global in the logic's level, i.e. they should have a *common structure of implications and union eliminations.* Roughly speaking, the root-statement S_i of π_i is meant to correspond to the (decorated) atom $\mathcal{A}_i$ at the root of $\pi^\star$ and, moreover, the rule structure of π_i is meant to impress upon the ancestor-atoms of $\mathcal{A}_i$ in $\pi^\star$. But, since the global rule-inferences in $\pi^\star$, i.e. the implications and the union eliminations, "scan" all the atoms in the premise molecule(s), it follows that, for $i \neq j$, the structure of implications and union eliminations read off from the ancestors of $\mathcal{A}_i$ should be the same as the structure of implications and union eliminations read off from the ancestors of $\mathcal{A}_j$, i.e. that π_i and π_j should have a common structure of implications and union eliminations. On the other hand, the π_i's may differ with respect to rules that are local in the logic's level, i.e. with respect to intersections and union introductions, as these rules may impress upon ancestors of $\mathcal{A}_i$ without at the same time impressing upon ancestors of $\mathcal{A}_j$.

$$\ldots\,, \quad \dfrac{\dfrac{\dfrac{\mathrm{S}^{1}_{i0}}{\mathrm{S}^{20}_{i0}}\;\mathrm{R}_1{=}(\cap\mathbf{E}) \qquad \mathrm{S}^{21}_{i0}}{\dfrac{\mathrm{S}^{3}_{i0}}{\mathrm{S}^{4}_{i0}}\;\mathrm{R}_3{=}(\rightarrow\mathbf{I})}\;\mathrm{R}_2{=}(\rightarrow\mathbf{E}) \qquad \dfrac{\mathrm{S}^{20}_{i1} \qquad \mathrm{S}^{21}_{i1}}{\dfrac{\mathrm{S}^{3}_{i1}}{\mathrm{S}^{4}_{i1}}\;\mathrm{R}_3{=}(\rightarrow\mathbf{I})}\;\mathrm{R}_2{=}(\rightarrow\mathbf{E})}{\pi_i :: \mathrm{S}_i}\;\mathrm{R}_4{=}(\cap\mathbf{I}) \quad , \ldots , \quad \dfrac{\dfrac{\mathrm{S}^{20}_{j} \qquad \mathrm{S}^{21}_{j}}{\mathrm{S}^{3}_{j}}\;\mathrm{R}_2{=}(\rightarrow\mathbf{E})}{\pi_j :: \mathrm{S}_j}\;\mathrm{R}_3{=}(\rightarrow\mathbf{I}) \quad , \ldots \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\,\ldots, \mathcal{A}^{1}_{i0}, \mathcal{A}^{1}_{i1}, \ldots, \mathcal{A}^{1}_{j}, \ldots\,]}{[\,\ldots, \mathcal{A}^{20}_{i0}, \mathcal{A}^{20}_{i1} = \mathcal{A}^{1}_{i1}, \ldots, \mathcal{A}^{20}_{j} = \mathcal{A}^{1}_{j}, \ldots\,]}\;\mathrm{R}_1{=}(\cap\mathbf{E}) \qquad [\,\ldots, \mathcal{A}^{21}_{i0}, \mathcal{A}^{21}_{i1}, \ldots, \mathcal{A}^{21}_{j}, \ldots\,]}{[\,\ldots, \mathcal{A}^{3}_{i0}, \mathcal{A}^{3}_{i1}, \ldots, \mathcal{A}^{3}_{j}, \ldots\,]}\;\mathrm{R}_2{=}(\rightarrow\mathbf{E})}{[\,\ldots, \mathcal{A}^{4}_{i0}, \mathcal{A}^{4}_{i1}, \ldots, \mathcal{A}^{4}_{j}, \ldots\,]}\;\mathrm{R}_3{=}(\rightarrow\mathbf{I})}{\pi^\star :: \text{term} : [\,\ldots, \mathcal{A}_i, \ldots, \mathcal{A}_j = \mathcal{A}^{4}_{j}, \ldots\,]_{\text{sequence}}}\;\mathrm{R}_4{=}(\cap\mathbf{I})$$

Is a common structure of implications and union eliminations enough, though? Derivations π_1 and π_2 of Example 5.18 have such a common structure, which consists of a single union elimination, but cannot be naturally integrated into a $\pi^\star$. Studying the example carefully, we see that the term-statements in the $(\cup\mathbf{E})$ of π_1 do not match the corresponding term-statements in the $(\cup\mathbf{E})$ of π_2. In particular, if

$S = \{x, y\}$, the term-statement $S, z \vdash z$ in the minor premises of π_1 does not match the term-statement $S, z \vdash x$ in the minor premises of π_2; this is what the incompatibility of the proper ($\cup$**E**) in π_1 and the phony ($\cup$**E**) in π_2 reduces to. Going back to the π_i's, we reason that a second common feature is required for a natural merging to be possible.

II. Corresponding implications or union eliminations in the common (with respect to implications and union eliminations) structure of the π_i's should have *matching corresponding term-statements.* Roughly speaking, the term-statements in π_i are meant to become the decoration in $\pi^\star$. But, since the decoration "scans" all atoms in a molecule and the only rules in the logic—among the ones that have a counterpart in the type system, i.e. among the introduction and elimination rules—where the decoration is modified are the implications and the union elimination, it follows that, for $i \neq j$, the modification of decoration (by an implication or a union elimination) in ancestors of $\mathcal{A}_i$ should be the same as the modification of decoration in ancestors of $\mathcal{A}_j$, i.e. that corresponding implications or union eliminations in π_i and π_j should have matching corresponding term-statements.

$$\ldots\,, \quad \frac{\dfrac{x : \sigma_1^i,\, y : \sigma_2^i \vdash t : \rho_i \to \upsilon_i \qquad x : \sigma_1^i,\, y : \sigma_2^i \vdash u : \rho_i}{x : \sigma_1^i,\, y : \sigma_2^i \vdash tu : \upsilon_i}\,\textbf{(}{\to}\textbf{E)} \qquad \dfrac{x : \sigma_1^i,\, y : \sigma_2^i \vdash t : \phi_i \to \psi_i \qquad x : \sigma_1^i,\, y : \sigma_2^i \vdash u : \phi_i}{x : \sigma_1^i,\, y : \sigma_2^i \vdash tu : \psi_i}\,\textbf{(}{\to}\textbf{E)}}{\pi_i :: x : \sigma_1^i,\, y : \sigma_2^i \vdash tu : \upsilon_i \cap \psi_i = \tau_i}\,\textbf{(}{\cap}\textbf{I)} \quad , \ldots$$

$$\ldots\,, \quad \frac{x : \sigma_1^j,\, y : \sigma_2^j \vdash t : \rho_j \to \tau_j \qquad x : \sigma_1^j,\, y : \sigma_2^j \vdash u : \rho_j}{\pi_j :: x : \sigma_1^j,\, y : \sigma_2^j \vdash tu : \tau_j}\,\textbf{(}{\to}\textbf{E)} \quad , \ldots \qquad \rightsquigarrow$$

$$\frac{\dfrac{t : [\ \ldots, (\Gamma_i\,;\,\rho_i \to \upsilon_i), (\Gamma_i\,;\,\phi_i \to \psi_i), \ldots, (\Gamma_j\,;\,\rho_j \to \tau_j), \ldots\ ]_{x,\,y} \qquad u : [\ \ldots, (\Gamma_i\,;\,\rho_i), (\Gamma_i\,;\,\phi_i), \ldots, (\Gamma_j\,;\,\rho_j), \ldots\ ]_{x,\,y}}{tu : [\ \ldots, (\Gamma_i\,;\,\upsilon_i), (\Gamma_i\,;\,\psi_i), \ldots, (\Gamma_j\,;\,\tau_j), \ldots\ ]_{x,\,y}}\,\textbf{(}{\to}\textbf{E)}}{\pi^\star :: tu : [\ \ldots, \mathcal{A}_i = (\Gamma_i = (\sigma_1^i, \sigma_2^i)\,;\,\tau_i), \ldots, \mathcal{A}_j = (\Gamma_j = (\sigma_1^j, \sigma_2^j)\,;\,\tau_j), \ldots\ ]_{x,\,y}}\,\textbf{(}{\cap}\textbf{I)}$$

As the above two sketches of π_i reveal, features I and II should hold not only for two distinct π_i's, but also for premises of an ($\cap$**I**) (and minor premises of a ($\cup$**E**)) within a single π_i. This is because such premises, which share the same term-statement, are also merged into the same molecule in $\pi^\star$, exactly as the root-statements of the π_i's are merged into the root-molecule of $\pi^\star$. In general, the merging of statements into the same molecule goes through ($\cap$**I**) or ($\cup$**E**) inferences within each of the π_i's, creating nesting phenomena.

Putting features I and II together, we conclude that the π_i's should have a common structure of implications and union eliminations, in which corresponding implications or union eliminations should have matching corresponding term-statements; this should, of course, hold modulo multiple nestings due to ($\cap$**I**) or ($\cup$**E**) inferences within each of the π_i's. The definition of *trees of implications and union eliminations with terms* for derivations in IUT$^\oplus$ (Definition 5.6) and the demand that the π_i's have *existing*[24] *and identical* such trees in order to be compatible for merging into a single $\pi^\star$ (hypotheses 1 and 2 in Theorem 5.13) put in formal status the conclusion just stated.

The "restriction" that the π_i's have existing and identical trees T^t_{iue} in order to be compatible for merging into a single $\pi^\star$ could serve as a means for checking if the π_i's, for which the only common feature given is the term-statement at the root, are indeed compatible or if they could be made compatible. In

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[24] The "existing" part takes care of the nestings.

particular, if the trees $(T^t_{\mathrm{iue}})_1, \ldots, (T^t_{\mathrm{iue}})_n$ all exist and are identical, then the π_i's are *naturally compatible* for merging into a single $\pi^\star$. If not, we could check if there are transformed π_i''s, where π_i transforms to π_i' which proves the same statement as π_i, such that the trees $(T^t_{\mathrm{iue}})_1', \ldots, (T^t_{\mathrm{iue}})_n'$ all exist and are identical. If so, then the π_i's can be made *compatible through transformations* to the π_i''s, which are themselves naturally compatible for merging into a single $(\pi')^\star$, proving the desired decorated molecule. Derivations π_1 and π_2 of Example 5.18 are not naturally compatible, as it is $(T^t_{\mathrm{iue}})_1 \neq (T^t_{\mathrm{iue}})_2$, but can, nonetheless, be made compatible by transforming π_2 to a $\pi_2' :: \{\, x : \chi,\, y : \upsilon \,\} \vdash x : \beta$, such that $(T^t_{\mathrm{iue}})_2' = (T^t_{\mathrm{iue}})_1$.

$$\dfrac{\dfrac{\dfrac{x:\chi,\; y:\upsilon \vdash x:\chi}{x:\chi,\; y:\upsilon \vdash x:\beta}\,(\cap\mathbf{E}_2)}{x:\chi,\; y:\upsilon \vdash x:\beta\cup\beta}\,(\cup\mathbf{I}) \qquad x:\chi,\; y:\upsilon,\; z:\beta \vdash z:\beta \qquad x:\chi,\; y:\upsilon,\; z:\beta \vdash z:\beta}{\pi_2' :: \{\, x:\chi,\; y:\upsilon \,\} \vdash z[x/z] = x:\beta}\,(\cup\mathbf{E})_{\mathrm{proper}}$$

If there were appropriate transformations for every case of π_i's which are not naturally compatible, we would have a *non-restricted* (i.e. without any reference to trees T^t_{iue}) *inverse theorem modulo transformations*, i.e. a theorem from $\mathrm{IUT}^{\oplus}$ to $\mathrm{IUL}^\star_m$ saying "*if* $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *are derivations in* $\mathrm{IUT}^{\oplus}$*, there is a decorated derivation* $\pi^\star :: t : [(\sigma^i_1, \ldots, \sigma^i_m\, ; \tau_i)^n_{i=1}]_{x_1,\ldots,x_m}$ *in* IUL_m *modulo appropriate transformations of the* π_i*'s*". The proof of such a theorem would use the notion of trees T^t_{iue}—the so-called "restriction"—to consider two cases: (i) the case where the π_i's are naturally compatible, which would point to the proof of 5.13 and (ii) the case where the π_i's are not naturally compatible, which would need a proof that there is *always* a transformation to π_i''s, which are naturally compatible. However, as the counterexample section clearly shows, it is *not* always possible to perform transformations which adjust the compatibility. Therefore, we *cannot* have a non-restricted inverse theorem modulo transformations.

Removing restrictions, imposed through the notion of trees T^t_{iue}, from the direct theorem, i.e. the theorem from $\mathrm{IUL}^\star_m$ to $\mathrm{IUT}^{\oplus}$, we see that, although a non-restricted direct theorem is possible, it does not offer a complete account of the *projection*—if we may call it so—of $\mathrm{IUL}^\star_m$ into $\mathrm{IUT}^{\oplus}$. This is because it does not document the features of $\pi^\star$ that impress upon each of the π_i's and constitute their common attributes. The notion of trees T^t_{iue}, employed for both $\mathrm{IUL}^\star_m$ and $\mathrm{IUT}^{\oplus}$ in conclusions 1-3 of the restricted theorem 5.10, serves exactly the purpose of describing these features[25] of $\pi^\star$, thus formalizing the projection to its full extent.

Conclusively, it is *preferred* to stick to a restricted direct theorem, while it is *necessary* to stick to a restricted inverse theorem.

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[25] Inverting the analysis about common features of compatible π_i's, the features of $\pi^\star$ impressed upon each of the derived π_i's are I. the structure of implications and union eliminations and II. the decoration (of implications and union eliminations). The trace of I and II on the π_i's should, of course, be considered modulo nestings due to $(\cap\mathbf{I})$'s or $(\cup\mathbf{E})$'s within each of them.

CHAPTER 6

Correspondence between $\mathbf{IL}_m$ and $\mathbf{IT}^{\oplus}$

We examine how the *method of trees*, i.e. the method employed in Chapter 5 to describe the correspondence between $\mathrm{IUL}_m^{\star}$ and $\mathrm{IUT}^{\oplus}$ with the aid of trees T_{iue}^{t}, applies to the correspondence between the union-excluded systems $\mathrm{IL}_m^{\star}$ and $\mathrm{IT}^{\oplus}$. Toward this end, we first define the notion "tree of implications with terms", denoted $T_{\mathrm{i}}{}^{t}$, for both the decorated logic $\mathrm{IL}_m^{\star}$ and the type system $\mathrm{IT}^{\oplus}$. We then state and prove theorems of correspondence between $\mathrm{IL}_m^{\star}$ and $\mathrm{IT}^{\oplus}$ that revise, with the aid of trees $T_{\mathrm{i}}{}^{t}$, the correspondence between $\mathrm{ISL}^{\star}$ and IT, given in Chapter 1. We finally discuss the correspondences $\mathrm{IUL}_m^{\star}\leftrightarrow\mathrm{IUT}^{\oplus}$ and $\mathrm{IL}_m^{\star}\leftrightarrow\mathrm{IT}^{\oplus}$ to decide to what extent the logics IUL_m and IL_m indeed correspond, through decoration, to the type systems $\mathrm{IUT}^{\oplus}$ and $\mathrm{IT}^{\oplus}$, respectively.

6.1 Trees of i with terms

We start by defining the logic IL_m and its decoration and also the type system $\mathrm{IT}^{\oplus}$, all as restrictions of definitions given in Chapter 4. We then adjust the method of trees to the restricted systems by defining the notion of *tree of implications with terms* for both the decorated logic and the type system.

The natural deduction logic IL_m, exposed in Figure 6.1, derives from the natural deduction logic IUL_m, if we exclude the union rules. The exchange rule and the implication rules are *global*, while the intersection rules are *local*. The system is *additive*, which is *necessitated* in the case of intersection introduction, but *chosen* in the case of implication elimination. It is easy to check that Propositions 4.4-4.6, 4.10, and 4.11, which are all shown for IUL_m in Chapter 4, also hold for the "smaller" system IL_m. The decoration of IL_m, shown in Figure 6.2, is the restriction of the decoration of IUL_m to the rules of IL_m.

The natural deduction type system $\mathrm{IT}^{\oplus}$, depicted in Figure 6.3, derives from the natural deduction type system $\mathrm{IUT}^{\oplus}$, if we exclude the union rules. It coincides with the system IT of Chapter 1 and also with the system deriving from the natural deduction system IUT_{ω} of Chapter 2, if we exclude the (ω)-rule and the union rules. It is easy to verify that Propositions 4.14, 4.16, and 4.17(i), which are all shown for $\mathrm{IUT}^{\oplus}$ in Chapter 4, also hold for the "smaller" system $\mathrm{IT}^{\oplus}$.

Remark 6.1 *(i) Since subject reduction is valid in* $\mathrm{IT}^{\oplus}$ *(recall Proposition 1.3), contraction can be derived in* $\mathrm{IT}^{\oplus}$ *through an implication redex along with subject reduction.*

$$\dfrac{\dfrac{B,\, x:\sigma,\, y:\sigma \vdash t:\tau}{B,\, x:\sigma \vdash \lambda y.\, t:\sigma\to\tau}\,(\to\mathbf{I}) \qquad \dfrac{}{B,\, x:\sigma \vdash x:\sigma}\,(\mathbf{ax})}{B,\, x:\sigma \vdash (\lambda y.\, t)x:\tau}\,(\to\mathbf{E}) \qquad \overset{1.3}{\Longrightarrow} \qquad B,\, x:\sigma \vdash t[x/y]:\tau$$

$$\frac{}{[(\Gamma_i, \sigma_i \,;\, \sigma_i)_i]}\ \textbf{(ax)} \qquad \frac{[(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]}{[(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]}\ \textbf{(X)}$$

$$\frac{[(\Gamma_i, \sigma_i \,;\, \tau_i)_i]}{[\Gamma_i \,;\, \sigma_i \to \tau_i)_i]}\ (\to\!\textbf{I}) \qquad \frac{[(\Gamma_i \,;\, \sigma_i \to \tau_i)_i] \qquad [(\Gamma_i \,;\, \sigma_i)_i]}{[(\Gamma_i \,;\, \tau_i)_i]}\ (\to\!\textbf{E})$$

$$\frac{[\mathcal{U}, ((\Gamma_i \,;\, \sigma_i), (\Gamma_i \,;\, \tau_i))_i, \mathcal{V}\,]}{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]}\ (\cap\textbf{I}) \qquad \frac{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]}{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}\,]}\ (\cap\textbf{E}_1) \qquad \frac{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]}{[\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}\,]}\ (\cap\textbf{E}_2)$$

Figure 6.1: The logic IL_m in natural deduction style.

$$\frac{}{x : [(\Gamma_i, \sigma_i \,;\, \sigma_i)_i]_{p,\,x}}\ \textbf{(ax)} \qquad \frac{t : [(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]_{p,\,y,\,x,\,q}}{t : [(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]_{p,\,x,\,y,\,q}}\ \textbf{(X)}$$

$$\frac{t : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}{\lambda x.t : [\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p}\ (\to\!\textbf{I}) \qquad \frac{t : [(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p \qquad u : [(\Gamma_i \,;\, \sigma_i)_i]_p}{tu : [(\Gamma_i \,;\, \tau_i)_i]_p}\ (\to\!\textbf{E})$$

$$\frac{t : [\mathcal{U}, ((\Gamma_i \,;\, \sigma_i), (\Gamma_i \,;\, \tau_i))_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]_p}\ (\cap\textbf{I}) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}\,]_p}\ (\cap\textbf{E}_1) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}\,]_p}\ (\cap\textbf{E}_2)$$

Figure 6.2: Non-standard decoration of natural deduction IL_m.

$$\frac{}{B,\, x : \sigma \vdash x : \sigma}\ \textbf{(ax)}$$

$$\frac{B,\, x : \sigma \vdash t : \tau}{B \vdash \lambda x.t : \sigma \to \tau}\ (\to\!\textbf{I}) \qquad \frac{B \vdash t : \sigma \to \tau \qquad B \vdash u : \sigma}{B \vdash tu : \tau}\ (\to\!\textbf{E})$$

$$\frac{B \vdash t : \sigma \qquad B \vdash t : \tau}{B \vdash t : \sigma \cap \tau}\ (\cap\textbf{I}) \qquad \frac{B \vdash t : \sigma \cap \tau}{B \vdash t : \sigma}\ (\cap\textbf{E}_1) \qquad \frac{B \vdash t : \sigma \cap \tau}{B \vdash t : \tau}\ (\cap\textbf{E}_2)$$

Figure 6.3: The type system $\mathrm{IT}^{\oplus}$ in natural deduction style.

(ii) An implication redex along with subject reduction can also derive a cut-like rule in IT$^{\oplus}$.

$$\dfrac{\dfrac{B,\, x:\sigma \vdash u:\tau}{B \vdash \lambda x.\, u : \sigma \to \tau}\,(\to\!\mathbf{I}) \qquad B \vdash t:\sigma}{B \vdash (\lambda x.\, u)t : \tau}\,(\to\!\mathbf{E}) \qquad \overset{1.3}{\Longrightarrow} \qquad B \vdash u[t/x] : \tau$$

In the natural deduction IUT$^{\oplus}$, *where subject reduction is not valid, a cut-like rule can be derived through a union redex; this will be shown in the next chapter (see Theorem 7.9(i)).*

The method of trees in Chapter 5 uses trees with terms that encode only the implications and the union eliminations, i.e. these logical rules that are global and have a counterpart in the type system. In the current context, the logical rules that are global and have a counterpart in the type system are the implications solely, so we need to define trees with terms that encode only the implications.

As far as IL$_m^{\star}$ is concerned, considering the “tree with terms” as expected[1], we define the “tree of implications with terms” as follows.

Definition 6.2 (IL$_m^{\star}$: Tree of implics with terms $T_{\mathrm{i}}^{\,t}$) *The* tree of implications with terms $T_{\mathrm{i}}^{\,t}$ *of a derivation* $\pi^{\star}$ *in* IL$_m^{\star}$ *derives from the tree with terms* $T^{\,t}$ *of* $\pi^{\star}$, *if we erase all nodes and corresponding decoration-statements associated to the rules* **(X)** *and* **(∩IE)**.

As in the case of IUL$_m^{\star}$, the procedure of erasing nodes and corresponding decoration-statements associated to the rules **(X)** and **(∩IE)** is well-defined, and the tree $T_{\mathrm{i}}^{\,t}$ displays at the root the same decoration-statement as the tree $T^{\,t}$.

As far as IT$^{\oplus}$ is concerned, considering the “tree with terms” as expected[2], we define the “tree of implications with terms” as follows.

Definition 6.3 (IT$^{\oplus}$: Tree of implics with terms $T_{\mathrm{i}}^{\,t}$) *We derive the* tree of implications with terms $T_{\mathrm{i}}^{\,t}$ *of a derivation* π *in* IT$^{\oplus}$ *from the tree with terms* $T^{\,t}$ *of* π *by the following algorithm.*

▹ *We choose a topmost* **(∩I)** *in the tree with terms of* π *and erase all nodes and corresponding term-statements associated to* **(∩E)** *in the trees with terms of both premises. If the resulting premise trees of implications with terms are identical, we identify them and erase the node and corresponding term-statement associated to the* **(∩I)**.

▹ *We iterate the above procedure for the tree with terms resulting from the previous step.*

▹ *When all the* **(∩I)***'s are eliminated, we make a final step to erase any remaining nodes and corresponding term-statements associated to* **(∩E)**.

As in the case of IUT$^{\oplus}$, the procedure described by the above algorithm is well-defined, and the final tree $T_{\mathrm{i}}^{\,t}$ attained has a term-statement at the root which is identical to the term-statement at the root of the original tree $T^{\,t}$. However, unlike the algorithm in 5.6, the algorithm in 6.3 always terminates. To show this, we need the following lemma.

Lemma 6.4 *If* $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *are derivations in* IT$^{\oplus}$ *that share the same term-statement* $x_1, \ldots, x_m \vdash t$ *at the root, then the trees* $(T_{\mathrm{i}}^{\,t})_1, \ldots, (T_{\mathrm{i}}^{\,t})_n$ *all exist and are identical.*

---

[1] This is as given in 5.1, but with IL$_m^{\star}$ in place of IUL$_m^{\star}$.

[2] This is as given in 5.5, but with IT$^{\oplus}$ in place of IUT$^{\oplus}$.

Proof. We take two derivations $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ and $\pi_2 :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash t : \psi$, and we proceed by induction on π_1. We allow the [h] apply to any finite number of derivations and denote S the set $\{x_1, \ldots, x_m\}$.

Base: If $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \tau \vdash x : \tau$ is an axiom, then π_2 contains only intersections.

$$\pi_{21} :: x_1 : \rho_1, \ldots, x_m : \rho_m, x : \phi \vdash x : \phi \qquad \ldots \qquad \pi_{2k} :: x_1 : \rho_1, \ldots, x_m : \rho_m, x : \phi \vdash x : \phi$$

$$\ddots \qquad (\cap\mathbf{IE}) \qquad \iddots$$

$$\pi_2 :: x_1 : \rho_1, \ldots, x_m : \rho_m, x : \phi \vdash x : \psi$$

The tree $({T_{\mathrm{i}}}^{t})_1$ is a single node with term-statement $S, x \vdash x$. The algorithm for the tree $({T_{\mathrm{i}}}^{t})_2$ goes as follows. At any step where a topmost $(\cap\mathbf{I})$ is chosen, after erasing nodes and corresponding term-statements associated to $(\cap\mathbf{E})$, we get identical premise-trees ${T_{\mathrm{i}}}^{t}$, which consist of a single node with term-statement $S, x \vdash x$. Identifying them and erasing the node and corresponding term-statement associated to the $(\cap\mathbf{I})$ results to a single node with term-statement $S, x \vdash x$ in place of the tree with terms rooted at the topmost $(\cap\mathbf{I})$. When all the $(\cap\mathbf{I})$'s are eliminated, we are left with a tree with terms which is a branch of $(\cap\mathbf{E})$'s with all nodes "carrying" the term-statement $S, x \vdash x$. Erasing the nodes and corresponding term-statements associated to the $(\cap\mathbf{E})$'s yields the tree $({T_{\mathrm{i}}}^{t})_2$, which is a single node with term-statement $S, x \vdash x$. Since both trees $({T_{\mathrm{i}}}^{t})_1$ and $({T_{\mathrm{i}}}^{t})_2$ are a single node with term-statement $S, x \vdash x$, they are identical.

Induction step: We show the most important cases.

$$\triangleright \quad \frac{\pi_{10} :: x_1 : \sigma_1, \ldots, x_m : \sigma_m, x : \tau_1 \vdash t : \tau_2}{\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash \lambda x.t : \tau_1 \to \tau_2} \ (\to\mathbf{I})$$

Since a λ-abstraction can be generated only by an $(\to\mathbf{I})$, derivation π_2 has the following form.

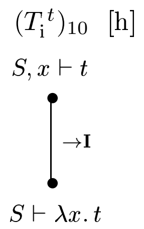
$$\frac{\pi_{210} :: x_1 : \rho_1, \ldots, x_m : \rho_m, x : \phi_1 \vdash t : \psi_1}{\pi_{21} :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash \lambda x.t : \phi_1 \to \psi_1} \ (\to\mathbf{I}) \quad \ldots \quad \frac{\pi_{2k0} :: x_1 : \rho_1, \ldots, x_m : \rho_m, x : \phi_k \vdash t : \psi_k}{\pi_{2k} :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash \lambda x.t : \phi_k \to \psi_k} \ (\to\mathbf{I})$$

$$\ddots \qquad (\cap\mathbf{I}) \qquad \iddots$$

$$\pi_2 :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash \lambda x.t : \psi$$

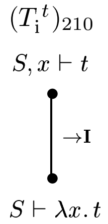
We take that there are no $(\cap\mathbf{E})$'s in the part of π_2 below $\pi_{21}, \ldots, \pi_{2k}$, as we cannot apply an $(\cap\mathbf{E})$ to a statement whose predicate is an implication type $\phi_i \to \psi_i$ $(1 \leqslant i \leqslant k)$, so that any $(\cap\mathbf{E})$ must be roughly following an $(\cap\mathbf{I})$, in which case it can be eliminated.

The [h] on $\pi_{10}, \pi_{210}, \ldots, \pi_{2k0}$ implies that the trees $({T_{\mathrm{i}}}^{t})_{10}, ({T_{\mathrm{i}}}^{t})_{210}, \ldots, ({T_{\mathrm{i}}}^{t})_{2k0}$ all exist and are identical. The existence of the tree $({T_{\mathrm{i}}}^{t})_{10}$ entails the existence of the tree $({T_{\mathrm{i}}}^{t})_1$, which has the form shown below.

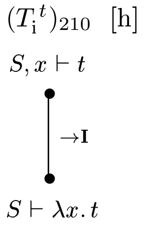


$(T_{\mathrm{i}}{}^{t})_{1}$

Denoting $(T_{\mathrm{i}}{}^{t})_{210}$ the common tree of implications with terms of $\pi_{210}, \dots, \pi_{2k0}$, the algorithm for the tree $(T_{\mathrm{i}}{}^{t})_{2}$ goes as follows. At any step where a topmost **(∩I)** is chosen, we get identical premise-trees $T_{\mathrm{i}}{}^{t}$ of the form displayed below.



Identifying them and erasing the node and corresponding term-statement associated to the **(∩I)** results to a tree $T_{\mathrm{i}}{}^{t}$ of the above form in place of the tree with terms rooted at the topmost **(∩I)**. Therefore, when all the **(∩I)**'s are eliminated, we are left with a tree $(T_{\mathrm{i}}{}^{t})_{2}$, as shown below.



$(T_{\mathrm{i}}{}^{t})_{2}$

Since $(T_{\mathrm{i}}{}^{t})_{10} = (T_{\mathrm{i}}{}^{t})_{210}$, we get that $(T_{\mathrm{i}}{}^{t})_{1} = (T_{\mathrm{i}}{}^{t})_{2}$.

$$\triangleright \quad \frac{\pi_{10} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \tau_1 \to \tau \qquad \pi_{11} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash u : \tau_1}{\pi_1 :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash tu : \tau} \ (\to\mathbf{E})$$

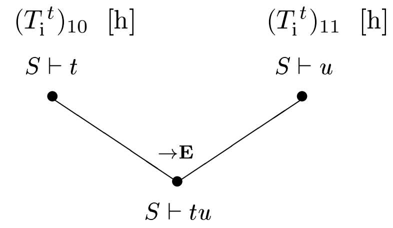
Since an application-term can only arise from an **(→E)**, derivation π_2 is schematically depicted as shown below.

$$\dfrac{\pi_{210} :: B_2 \vdash t : \phi_1 \to \psi_1 \qquad \pi_{211} :: B_2 \vdash u : \phi_1}{\pi_{21} :: B_2 \vdash tu : \psi_1}\,(\to\mathbf{E}) \qquad \cdots \qquad \dfrac{\pi_{2k0} :: B_2 \vdash t : \phi_k \to \psi_k \qquad \pi_{2k1} :: B_2 \vdash u : \phi_k}{\pi_{2k} :: B_2 \vdash tu : \psi_k}\,(\to\mathbf{E})$$

$$\ddots \qquad (\cap\mathbf{IE}) \qquad \iddots$$

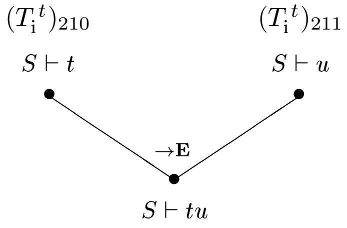
$$\pi_2 :: B_2 = \{\, x_1 : \rho_1, \ldots,\, x_m : \rho_m \,\} \vdash tu : \psi$$

The [h] on $\pi_{10}, \pi_{210}, \ldots, \pi_{2k0}$ implies that the trees $(T_{\mathrm{i}}{}^{t})_{10}, (T_{\mathrm{i}}{}^{t})_{210}, \ldots, (T_{\mathrm{i}}{}^{t})_{2k0}$ all exist and are identical, while the [h] on $\pi_{11}, \pi_{211}, \ldots, \pi_{2k1}$ gives that the trees $(T_{\mathrm{i}}{}^{t})_{11}, (T_{\mathrm{i}}{}^{t})_{211}, \ldots, (T_{\mathrm{i}}{}^{t})_{2k1}$ all exist and are identical. The existence of the trees $(T_{\mathrm{i}}{}^{t})_{10}$ and $(T_{\mathrm{i}}{}^{t})_{11}$ entails the existence of the tree $(T_{\mathrm{i}}{}^{t})_1$, which has the following form.

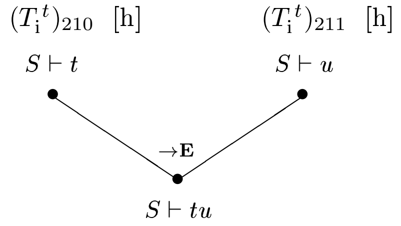


$(T_{\mathrm{i}}{}^{t})_1$

Denoting $(T_{\mathrm{i}}{}^{t})_{210}$ the common tree of implications with terms of $\pi_{210}, \ldots, \pi_{2k0}$ and $(T_{\mathrm{i}}{}^{t})_{211}$ the common tree of implications with terms of $\pi_{211}, \ldots, \pi_{2k1}$, the algorithm for the tree $(T_{\mathrm{i}}{}^{t})_2$ proceeds as follows. At any step where a topmost $(\cap\mathbf{I})$ is chosen, after erasing nodes and corresponding term-statements associated to $(\cap\mathbf{E})$, we get identical premise-trees $T_{\mathrm{i}}{}^{t}$ of the following form.



Identifying them and erasing the node and corresponding term-statement associated to the $(\cap\mathbf{I})$ results to a tree $T_{\mathrm{i}}{}^{t}$ of the above form in place of the tree with terms rooted at the topmost $(\cap\mathbf{I})$. When all the $(\cap\mathbf{I})$'s are eliminated, we are left with a tree with terms which is the tree $T_{\mathrm{i}}{}^{t}$ shown above with a branch of $(\cap\mathbf{E})$'s pasted on its root. Erasing the nodes and corresponding term-statements associated to the $(\cap\mathbf{E})$'s, we obtain the following tree $(T_{\mathrm{i}}{}^{t})_2$.



$(T_{\mathrm{i}}^{t})_2$

Since $(T_{\mathrm{i}}^{t})_{10} = (T_{\mathrm{i}}^{t})_{210}$ and $(T_{\mathrm{i}}^{t})_{11} = (T_{\mathrm{i}}^{t})_{211}$, we get that $(T_{\mathrm{i}}^{t})_1 = (T_{\mathrm{i}}^{t})_2$.

$$\triangleright \quad \frac{\pi_{10} :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau_1 \qquad \pi_{11} :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau_2}{\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau_1 \cap \tau_2}\ (\cap\mathbf{I})$$

The [h] on $\pi_{10}, \pi_{11}, \pi_2$ implies that the trees $(T_{\mathrm{i}}^{t})_{10}, (T_{\mathrm{i}}^{t})_{11}$, and $(T_{\mathrm{i}}^{t})_2$ exist and are identical. Since $(T_{\mathrm{i}}^{t})_{10} = (T_{\mathrm{i}}^{t})_{11}$, the algorithm for the tree $(T_{\mathrm{i}}^{t})_1$ terminates and gives $(T_{\mathrm{i}}^{t})_1 = (T_{\mathrm{i}}^{t})_{10}$. Therefore, it is $(T_{\mathrm{i}}^{t})_1 = (T_{\mathrm{i}}^{t})_2$. ⊣

Corollary 6.5 *The algorithm in 6.3 always terminates, i.e. any derivation in* $\mathrm{IT}^{\oplus}$ *has a tree* T_{i}^{t}.

Proof. By Lemma 6.4, for $n = 1$. If $\pi :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ is a derivation in $\mathrm{IT}^{\oplus}$, then the tree $(T_{\mathrm{i}}^{t})_{\pi}$ exists. ⊣

The notion of *tree of implications with terms* for derivations in $\mathrm{IT}^{\oplus}$ is actually a revision of the notion of *skeleton*, introduced in [15] for derivations of an extended natural deduction type system, called NJR. In [15], derivations displaying the same skeleton are called *synchronous* and it is shown that two derivations proving statements that type the same term are synchronous. In the current context, synchronicity refers to derivations proving statements that share the same term-statement, which are shown to display the same tree T_{i}^{t} by Lemma 6.4.

6.2 Revised correspondence theorems

Having done the preliminary work, i.e. having introduced the trees T_{i}^{t} for derivations in the decorated logic $\mathrm{IL}_m^{\star}$ and in the type system $\mathrm{IT}^{\oplus}$, we can now relate $\mathrm{IL}_m^{\star}$ to $\mathrm{IT}^{\oplus}$ in a way that is compatible with the way $\mathrm{IUL}_m^{\star}$ is related to $\mathrm{IUT}^{\oplus}$ in Chapter 5 and, furthermore, revises the theorem relating $\mathrm{ISL}^{\star}$ to IT in Chapter 1.

Theorem 6.6 (From $\mathbf{IL}_m$ to $\mathbf{IT}^{\oplus}$) *If* $\pi^{\star} :: t : [(\sigma_1^i, \ldots, \sigma_m^i ; \tau_i)_{i=1}^{n}]_{x_1,\ldots,x_m}$ *is a decorated derivation in* IL_m*, then there exist derivations* $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ *in* $\mathrm{IT}^{\oplus}$*, such that* $(T_{\mathrm{i}}^{t})_i = (T_{\mathrm{i}}^{t})_{\pi^{\star}}$.

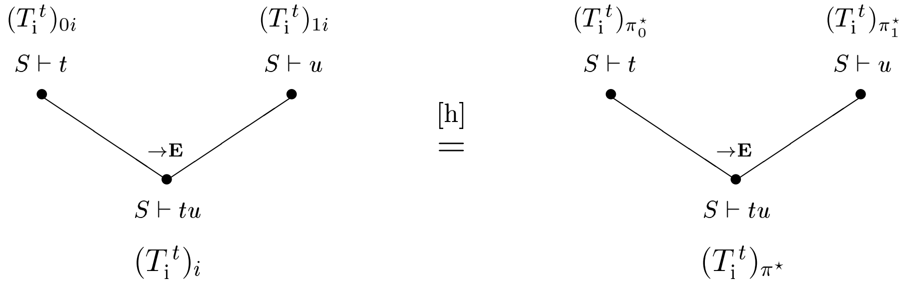
Proof. Given the π_i's $(1 \leqslant i \leqslant n)$ in $\mathrm{IT}^{\oplus}$, Lemma 6.4 guarantees that the trees $(T_{\mathrm{i}}^{t})_1, \ldots, (T_{\mathrm{i}}^{t})_n$ all exist and are identical, so that the identity $(T_{\mathrm{i}}^{t})_i = (T_{\mathrm{i}}^{t})_{\pi^{\star}}$ is meaningful. The proof is by induction on $\pi^{\star}$, letting S denote the set $\{x_1, \ldots, x_m\}$.

Base: If $\pi^\star :: x : [(\sigma_1^i, \ldots, \sigma_m^i, \tau_i \,;\, \tau_i)_{i=1}^{n}]_{x_1,\ldots,x_m,x}$ is a decorated axiom, then there exist axioms $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i, x : \tau_i \vdash x : \tau_i$ $(1 \leqslant i \leqslant n)$ in $\mathrm{IT}^{\oplus}$. It is $(T_\mathrm{i}^{t})_i = (T_\mathrm{i}^{t})_{\pi^\star}$, since both trees are a single node with $S, x \vdash x$.

Induction step: We show two characteristic cases.

$$\triangleright \quad \frac{\pi_0^\star :: t : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i \to \rho_i)_{i=1}^{n}]_{x_1,\ldots,x_m} \qquad \pi_1^\star :: u : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i)_{i=1}^{n}]_{x_1,\ldots,x_m}}{\pi^\star :: tu : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \rho_i)_{i=1}^{n}]_{x_1,\ldots,x_m}}\ (\mathbf{\to E})$$

The [h] gives $\pi_{0i} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_1 \to \rho_i$ $(1 \leqslant i \leqslant n)$, such that $(T_\mathrm{i}^{t})_{0i} = (T_\mathrm{i}^{t})_{\pi_0^\star}$, and also $\pi_{1i} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash u : \tau_i$ $(1 \leqslant i \leqslant n)$, such that $(T_\mathrm{i}^{t})_{1i} = (T_\mathrm{i}^{t})_{\pi_1^\star}$. Applying **(→E)** to π_{0i} and π_{1i}, we obtain $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash tu : \rho_i$ $(1 \leqslant i \leqslant n)$. Since $(T_\mathrm{i}^{t})_{0i} = (T_\mathrm{i}^{t})_{\pi_0^\star}$ and $(T_\mathrm{i}^{t})_{1i} = (T_\mathrm{i}^{t})_{\pi_1^\star}$, we get that $(T_\mathrm{i}^{t})_i = (T_\mathrm{i}^{t})_{\pi^\star}$.



$$\triangleright \quad \frac{\pi_0^\star :: t : [(\phi_1^i, \ldots, \phi_m^i \,;\, \psi_i)_{i=1}^{k}, ((\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i), (\sigma_1^i, \ldots, \sigma_m^i \,;\, \rho_i))_{i=k+1}^{n}]_{x_1,\ldots,x_m}}{\pi^\star :: t : [(\phi_1^i, \ldots, \phi_m^i \,;\, \psi_i)_{i=1}^{k}, (\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i \cap \rho_i)_{i=k+1}^{n}]_{x_1,\ldots,x_m}}\ (\cap\mathbf{I})$$

For $1 \leqslant i \leqslant k$, the [h] yields $\pi_{0i} :: x_1 : \phi_1^i, \ldots, x_m : \phi_m^i \vdash t : \psi_i$, such that $(T_\mathrm{i}^{t})_{0i} = (T_\mathrm{i}^{t})_{\pi_0^\star}$. It is $\pi_i = \pi_{0i}$, so that $(T_\mathrm{i}^{t})_{1 \leqslant i \leqslant k} = (T_\mathrm{i}^{t})_{0i} = (T_\mathrm{i}^{t})_{\pi_0^\star} = (T_\mathrm{i}^{t})_{\pi^\star}$. For $k+1 \leqslant i \leqslant n$, the [h] gives derivations $\pi_{0i0} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$ and $\pi_{0i1} :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \rho_i$ from which, by **(∩I)**, we derive $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i \cap \rho_i$. The trees $(T_\mathrm{i}^{t})_1, \ldots, (T_\mathrm{i}^{t})_k, (T_\mathrm{i}^{t})_{k+1}, \ldots, (T_\mathrm{i}^{t})_n$ are all identical (Lemma 6.4), so it is $(T_\mathrm{i}^{t})_{1 \leqslant i \leqslant n} = (T_\mathrm{i}^{t})_{1 \leqslant i \leqslant k} = (T_\mathrm{i}^{t})_{\pi^\star}$. ⊣

Corollary 6.7 *If $\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m \,;\, \tau)]_{x_1,\ldots,x_m}$ is a decorated derivation in IL_m, there is a derivation $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ in $\mathrm{IT}^{\oplus}$, such that $(T_\mathrm{i}^{t})_1 = (T_\mathrm{i}^{t})_{\pi^\star}$.*

Proof. By Theorem 6.6, for $n = 1$. ⊣

Theorem 6.8 (From IT$^{\oplus}$ to IL$_m$) *If $\pi_i :: x_1 : \sigma_1^i, \ldots, x_m : \sigma_m^i \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ are derivations in $\mathrm{IT}^{\oplus}$, there is a decorated derivation $\pi^\star :: t : [(\sigma_1^i, \ldots, \sigma_m^i \,;\, \tau_i)_{i=1}^{n}]_{x_1,\ldots,x_m}$ in IL_m, such that $(T_\mathrm{i}^{t})_{\pi^\star} = (T_\mathrm{i}^{t})_i$.*

Proof. Lemma 6.4 guarantees that the trees $(T_\mathrm{i}^{t})_1, \ldots, (T_\mathrm{i}^{t})_n$ all exist and are identical, so that the identity $(T_\mathrm{i}^{t})_{\pi^\star} = (T_\mathrm{i}^{t})_i$ is meaningful. We consider two derivations $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ and $\pi_2 :: x_1 : \rho_1, \ldots, x_m : \rho_m \vdash t : \psi$ and proceed by induction on π_1, allowing the [h] apply to any finite number of derivations. The letter S stands once more for the set $\{x_1, \ldots, x_m\}$.

Base: If $\pi_1 :: x_1 : \sigma_1, \dots, x_m : \sigma_m, x : \tau \vdash x : \tau$ is an axiom, then π_2 may only contain intersections.

$$\pi_{21} :: x_1 : \rho_1, \dots, x_m : \rho_m, x : \phi \vdash x : \phi \qquad \dots \qquad \pi_{2k} :: x_1 : \rho_1, \dots, x_m : \rho_m, x : \phi \vdash x : \phi$$

$$\ddots \qquad (\cap\mathbf{IE}) \qquad \iddots$$

$$\pi_2 :: x_1 : \rho_1, \dots, x_m : \rho_m, x : \phi \vdash x : \psi$$

We obtain a $\pi^\star :: x : [(\sigma_1, \dots, \sigma_m, \tau\,;\, \tau), (\rho_1, \dots, \rho_m, \phi\,;\, \psi)]_{x_1, \dots, x_m, x}$ by merging $\pi_1, \pi_{21}, \dots, \pi_{2k}$ into an axiom of the (decorated) logic and then applying exchanges, if necessary, and the **(**$\cap$**IE)** inferences in the logic that correspond to the **(**$\cap$**IE)** inferences in π_2.

$$x : [(\sigma_1, \dots, \sigma_m, \tau\,;\, \tau), \underbrace{(\rho_1, \dots, \rho_m, \phi\,;\, \phi)_{i=1}^{k}}]_{x_1, \dots, x_m, x}$$

$$\vdots$$

$$(\cap\mathbf{IE})$$

$$\vdots$$

$$\pi^\star :: x : [(\sigma_1, \dots, \sigma_m, \tau\,;\, \tau), (\rho_1, \dots, \rho_m, \phi\,;\, \psi)]_{x_1, \dots, x_m, x}$$

The tree $({T_{\mathrm{i}}}^{t})_{\pi^\star}$ is a single node with decoration-statement $S, x \vdash x$, i.e. it is $({T_{\mathrm{i}}}^{t})_{\pi^\star} = ({T_{\mathrm{i}}}^{t})_1$.

Induction step: We show the most typical cases.

$$\rhd \quad \frac{\pi_{10} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash t : \chi \to \tau \qquad \pi_{11} :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash u : \chi}{\pi_1 :: x_1 : \sigma_1, \dots, x_m : \sigma_m \vdash tu : \tau}\,(\to\mathbf{E})$$

Since tu can only be generated by an **(**$\to$**E)** in $\mathrm{IT}^{\oplus}$, derivation π_2 has the form shown below.

$$\frac{\pi_{210} :: B_2 \vdash t : \phi_1 \to \psi_1 \qquad \pi_{211} :: B_2 \vdash u : \phi_1}{\pi_{21} :: B_2 \vdash tu : \psi_1}\,(\to\mathbf{E}) \qquad \dots \qquad \frac{\pi_{2k0} :: B_2 \vdash t : \phi_k \to \psi_k \qquad \pi_{2k1} :: B_2 \vdash u : \phi_k}{\pi_{2k} :: B_2 \vdash tu : \psi_k}\,(\to\mathbf{E})$$

$$\ddots \qquad (\cap\mathbf{IE}) \qquad \iddots$$

$$\pi_2 :: B_2 = \{x_1 : \rho_1, \dots, x_m : \rho_m\} \vdash tu : \psi$$

The [h] on $\pi_{10}, \pi_{210}, \dots, \pi_{2k0}$ gives a

$$\pi_0^\star :: t : [(\sigma_1, \dots, \sigma_m\,;\, \chi \to \tau), (\rho_1, \dots, \rho_m\,;\, \phi_i \to \psi_i)_{i=1}^{k}]_{x_1, \dots, x_m}$$

such that $(T_{\mathrm{i}}{}^{t})_{\pi_0^\star} = (T_{\mathrm{i}}{}^{t})_{10}$, while the [h] on $\pi_{11}, \pi_{211}, \ldots, \pi_{2k1}$ yields a

$$\pi_1^\star :: u : [(\sigma_1, \ldots, \sigma_m \,;\, \chi), (\rho_1, \ldots, \rho_m \,;\, \phi_i)_{i=1}^{k}]_{\,x_1,\ldots,\,x_m}$$

such that $(T_{\mathrm{i}}{}^{t})_{\pi_1^\star} = (T_{\mathrm{i}}{}^{t})_{11}$. We then get a $\pi^\star :: tu : [(\sigma_1, \ldots, \sigma_m \,;\, \tau), (\rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x_1,\ldots,\,x_m}$ as follows.

$$\begin{array}{c}
\dfrac{\pi_0^\star \qquad\qquad\qquad \pi_1^\star}{tu : [(\sigma_1, \ldots, \sigma_m \,;\, \tau), \underbrace{(\rho_1, \ldots, \rho_m \,;\, \psi_i)_{i=1}^{k}}]_{\,x_1,\ldots,\,x_m}}\ (\mathbf{\rightarrow E}) \\
\vdots \\
(\mathbf{\cap IE}) \\
\vdots \\
\pi^\star :: tu : [(\sigma_1, \ldots, \sigma_m \,;\, \tau), (\rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x_1,\ldots,\,x_m}
\end{array}$$

Since $(T_{\mathrm{i}}{}^{t})_{\pi_0^\star} = (T_{\mathrm{i}}{}^{t})_{10}$ and $(T_{\mathrm{i}}{}^{t})_{\pi_1^\star} = (T_{\mathrm{i}}{}^{t})_{11}$, we infer that $(T_{\mathrm{i}}{}^{t})_{\pi^\star} = (T_{\mathrm{i}}{}^{t})_{1}$.

$$\triangleright \quad \dfrac{\pi_{10} :: x_1 : \sigma_1, \ldots,\, x_m : \sigma_m \vdash t : \tau \qquad\qquad \pi_{11} :: x_1 : \sigma_1, \ldots,\, x_m : \sigma_m \vdash t : \chi}{\pi_1 :: x_1 : \sigma_1, \ldots,\, x_m : \sigma_m \vdash t : \tau \cap \chi}\ (\mathbf{\cap I})$$

The [h] on $\pi_{10}, \pi_{11}, \pi_2$ gives a $\pi_0^\star :: t : [(\sigma_1, \ldots, \sigma_m \,;\, \tau), (\sigma_1, \ldots, \sigma_m \,;\, \chi), (\rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x_1,\ldots,\,x_m}$, such that $(T_{\mathrm{i}}{}^{t})_{\pi_0^\star} = (T_{\mathrm{i}}{}^{t})_{10}$. By **(∩I)**, we then get a $\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m \,;\, \tau \cap \chi), (\rho_1, \ldots, \rho_m \,;\, \psi)]_{\,x_1,\ldots,\,x_m}$, such that $(T_{\mathrm{i}}{}^{t})_{\pi^\star} = (T_{\mathrm{i}}{}^{t})_{\pi_0^\star} = (T_{\mathrm{i}}{}^{t})_{10} \overset{6.4}{=} (T_{\mathrm{i}}{}^{t})_{1}$. ⊣

Corollary 6.9 *If* $\pi_1 :: x_1 : \sigma_1, \ldots, x_m : \sigma_m \vdash t : \tau$ *is a derivation in* $\mathrm{IT}^{\oplus}$*, then there is a decorated derivation* $\pi^\star :: t : [(\sigma_1, \ldots, \sigma_m \,;\, \tau)]_{\,x_1,\ldots,\,x_m}$ *in* IL_m*, such that* $(T_{\mathrm{i}}{}^{t})_{\pi^\star} = (T_{\mathrm{i}}{}^{t})_{1}$.

Proof. By Theorem 6.8, for $n = 1$. ⊣

Putting aside the small dissimilarities between the (decorated) logics IL_m and ISL, Theorem 6.6 revises the "only if" direction of Theorem 1.20 in that it puts forth the additional fact that the π_i's and $\pi^\star$ share the same implicative structure (with terms), which is expressed by the identity $(T_{\mathrm{i}}{}^{t})_{i} = (T_{\mathrm{i}}{}^{t})_{\pi^\star}$. Moreover, Theorem 6.8 revises the "if" direction of Theorem 1.20 by adding the fact that $\pi^\star$ displays the same implicative structure (with terms) as the π_i's, which is expressed by the identity $(T_{\mathrm{i}}{}^{t})_{\pi^\star} = (T_{\mathrm{i}}{}^{t})_{i}$.

Comparing Theorem 6.6 (from IL_m to $\mathrm{IT}^{\oplus}$) to Theorem 5.10 (from IUL_m to $\mathrm{IUT}^{\oplus}$), we see that, due to Lemma 6.4, there is no need for conclusions of the form "$(T_{\mathrm{i}}{}^{t})_{i}$ exists and $(T_{\mathrm{i}}{}^{t})_{i} = (T_{\mathrm{i}}{}^{t})_{j}$ $(i \neq j)$" in the former, as there are in the latter[3]. Furthermore, comparing Theorem 6.8 (from $\mathrm{IT}^{\oplus}$ to IL_m) to Theorem 5.13 (from $\mathrm{IUT}^{\oplus}$ to IUL_m), we find that, due to the same lemma, there is no need for hypotheses of the form "$(T_{\mathrm{i}}{}^{t})_{i}$ exists and $(T_{\mathrm{i}}{}^{t})_{i} = (T_{\mathrm{i}}{}^{t})_{j}$ $(i \neq j)$" in the former, as there are in the latter[4].

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[3] This is meant modulo the differentiation in the rules documented by the trees in the latter.

[4] See footnote 3.

6.3 Discussion of the correspondences

Looking at the correspondence between $\mathrm{IL}^{\star}_{m}$ and $\mathrm{IT}^{\oplus}$, let $S_{\mathrm{IT}^{\oplus}}$ be the set of finite sets of $\mathrm{IT}^{\oplus}$-derivations that share the same term-statement at the root. Obviously, the set $S_{\mathrm{IT}^{\oplus}}$ is a proper subset of the powerset $\mathscr{P}(\mathrm{IT}^{\oplus})$ of $\mathrm{IT}^{\oplus}$. Lemma 6.4 implies that a member $\{\pi_1, \ldots, \pi_n\}$ of $S_{\mathrm{IT}^{\oplus}}$ is such that the trees $({T_{\mathrm{i}}}^{t})_1, \ldots, ({T_{\mathrm{i}}}^{t})_n$ all exist and are identical. Theorems 6.6 and 6.8 establish a one-to-one correspondence between $\mathrm{IL}^{\star}_{m}$ and $S_{\mathrm{IT}^{\oplus}}$. In particular, Theorem 6.6 matches a $\pi^{\star}$ in $\mathrm{IL}^{\star}_{m}$, considered modulo the number and position of exchange inferences and also modulo the number and order of application of consecutive local rule-inferences, to a set $\{\pi_1, \ldots, \pi_n\}$ in $S_{\mathrm{IT}^{\oplus}}$, such that $({T_{\mathrm{i}}}^{t})_{1\leqslant i\leqslant n} = ({T_{\mathrm{i}}}^{t})_{\pi^{\star}}$. Conversely, Theorem 6.8 matches a set $\{\pi_1, \ldots, \pi_n\}$ in $S_{\mathrm{IT}^{\oplus}}$ to a $\pi^{\star}$ in $\mathrm{IL}^{\star}_{m}$, considered modulo the things mentioned above, such that $({T_{\mathrm{i}}}^{t})_{\pi^{\star}} = ({T_{\mathrm{i}}}^{t})_{1\leqslant i\leqslant n}$.

The question we now have to tackle is if we also have a one-to-one correspondence between $\mathrm{IUL}^{\star}_{m}$ and $S_{\mathrm{IUT}^{\oplus}}$, where $S_{\mathrm{IUT}^{\oplus}}$ is the set of finite sets of $\mathrm{IUT}^{\oplus}$-derivations that share the same term-statement at the root. The set $S_{\mathrm{IUT}^{\oplus}}$ is a proper subset of the powerset $\mathscr{P}(\mathrm{IUT}^{\oplus})$ of $\mathrm{IUT}^{\oplus}$. The situation here is a bit more complex and we need to also define two subsets C_1 and C_2 of $S_{\mathrm{IUT}^{\oplus}}$ to get the picture. Let $C_1 \subseteq S_{\mathrm{IUT}^{\oplus}}$ be such that, for any set $A = \{\pi_1, \ldots, \pi_n\}$ in C_1, the trees $(T^{t}_{\mathrm{iue}})_1, \ldots, (T^{t}_{\mathrm{iue}})_n$ all exist and are identical, i.e. hypotheses 1 and 2 of Theorem 5.13 hold for the members of A [notation: $(1 \wedge 2)_A$]. Further, let $C_2 \subseteq S_{\mathrm{IUT}^{\oplus}}$ be such that, for any set $B = \{\pi_1, \ldots, \pi_n\}$ in C_2, it is not the case that the trees $(T^{t}_{\mathrm{iue}})_1, \ldots, (T^{t}_{\mathrm{iue}})_n$ all exist and are identical [notation: $\neg(1 \wedge 2)_B$], but there is a transformation to a set $A = \{\pi'_1, \ldots, \pi'_n\}$ in C_1, where π_i transforms to π'_i which proves the same statement as π_i $(1 \leqslant i \leqslant n)$. To use the terminology introduced in Chapter 5, the members of a set in C_1 are "naturally compatible", while the members of a set in C_2 are "compatible through transformations"; the choice of the letter "C" for the subsets of $S_{\mathrm{IUT}^{\oplus}}$ derives from the word "**c**ompatible". The facts that $(1 \wedge 2)_A$ and $\neg(1 \wedge 2)_B$, for any A in C_1 and B in C_2, imply that $C_1 \cap C_2 = \emptyset$. Moreover, if $C = C_1 \cup C_2$, the transformation counterexample in Section 5.3 shows that there is a set $\{\pi_1, \pi_2\}$ in $S_{\mathrm{IUT}^{\oplus}} \setminus C$, i.e. that $C \subsetneq S_{\mathrm{IUT}^{\oplus}}$.

What we have shown in Chapter 5 is a one-to-one correspondence between $\mathrm{IUL}^{\star}_{m}$ and C_1, which matches a $\pi^{\star}$ in $\mathrm{IUL}^{\star}_{m}$, considered modulo the number and position of exchange inferences and also modulo the number and order of application of consecutive local rule-inferences, to a set $\{\pi_1, \ldots, \pi_n\}$ in C_1, such that $(T^{t}_{\mathrm{iue}})_{1\leqslant i\leqslant n} = (T^{t}_{\mathrm{iue}})_{\pi^{\star}}$. Theorem 5.10 states the direction from $\pi^{\star}$ to $\{\pi_1, \ldots, \pi_n\}$, while Theorem 5.13 states the inverse. However, we can also consider one-to-many correspondences from $\mathrm{IUL}^{\star}_{m}$ to C and from C to $\mathrm{IUL}^{\star}_{m}$. A one-to-many correspondence from $\mathrm{IUL}^{\star}_{m}$ to C matches a $\pi^{\star}$ in $\mathrm{IUL}^{\star}_{m}$, considered modulo the usual, not only to its one-to-one equivalent set $\{\pi_1, \ldots, \pi_n\}$ in C_1, but also to all the sets $\{\pi'_1, \ldots, \pi'_n\}$ in C_2, such that π'_i proves the same statement as π_i $(1 \leqslant i \leqslant n)$. Two distinct $\mathrm{IUL}^{\star}_{m}$-derivations $\pi^{\star}$ and $(\pi')^{\star}$ are not necessarily matched to distinct subsets of C. This is the case when $\pi^{\star}$ and $(\pi')^{\star}$ prove the same decorated molecule[5]. A one-to-many correspondence from C to $\mathrm{IUL}^{\star}_{m}$ matches a $\{\pi_1, \ldots, \pi_n\}$ in C_1 to its one-to-one equivalent derivation $\pi^{\star}$ in $\mathrm{IUL}^{\star}_{m}$ and a $\{\pi_1, \ldots, \pi_n\}$ in C_2 to the subset of $\mathrm{IUL}^{\star}_{m}$ including all $(\pi')^{\star}$ whose one-to-one equivalent set $\{\pi'_1, \ldots, \pi'_n\}$ in C_1 is such that π'_i proves the same statement as π_i $(1 \leqslant i \leqslant n)$. Obviously, distinct sets in C_1 are matched to distinct derivations in $\mathrm{IUL}^{\star}_{m}$, but distinct sets in C_2 are not necessarily matched to distinct subsets of $\mathrm{IUL}^{\star}_{m}$. We can specify the latter case, if we consider two sets $A = \{\pi_1, \pi_2\}$ and $A' = \{\pi'_1, \pi'_2\}$ in C_2, such that π'_1

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[5] Derivations $\pi^{\star}$ and $(\pi')^{\star}$ correspond—via the one-to-one correspondence between $\mathrm{IUL}^{\star}_{m}$ and C_1—to two distinct sets $A = \{\pi_1, \ldots, \pi_n\}$ and $A' = \{\pi'_1, \ldots, \pi'_n\}$ in C_1, respectively. The fact that $\pi^{\star}$ and $(\pi')^{\star}$ prove the same decorated molecule implies that π'_i proves the same statement as π_i $(1 \leqslant i \leqslant n)$. The π_i's all display the same tree T^{t}_{iue} as $\pi^{\star}$, while the π'_i's all display the same tree T^{t}_{iue} as $(\pi')^{\star}$; these trees are distinct, since $\pi^{\star}$ and $(\pi')^{\star}$ are distinct. Therefore, there exists a set $B = \{\pi_1, \ldots, \pi_k, \pi'_{k+1}, \ldots, \pi'_n\}$ in C_2, where $1 \leqslant k < n$. This set B belongs to both the subset of C matched to $\pi^{\star}$ and the subset of C matched to $(\pi')^{\star}$.

and π_2' prove the same statements as π_1 and π_2, respectively, the trees $(T^t_{\text{iue}})_1, (T^t_{\text{iue}})_2, (T^t_{\text{iue}})_1', (T^t_{\text{iue}})_2'$ all exist, and it is[6] $(T^t_{\text{iue}})_1 = (T^t_{\text{iue}})_2' \neq (T^t_{\text{iue}})_2 = (T^t_{\text{iue}})_1'$. It is clear that the correspondences just described differ from the intended one, i.e. from a one-to-one correspondence between $\text{IUL}_m^\star$ and $S_{\text{IUT}^\oplus}$. Figure 6.4 illustrates the one-to-one correspondences in the intersection and intersection-and-union contexts. In addition, Figure 6.5 demonstrates the subsets of $S_{\text{IUT}^\oplus}$ with respect to hypotheses 1 and 2 of 5.13 and shows the paths from a member of $S_{\text{IUT}^\oplus}$ to a member of $\text{IUL}_m^\star$.

The failure of a one-to-one correspondence between $\text{IUL}_m^\star$ and $S_{\text{IUT}^\oplus}$ confutes the very definition of IUL_m as a logic for $\text{IUT}^\oplus$. As explained at the end of Section 4.2, in defining IUL_m we have assumed—following the pattern in the definition of IL_m (or ISL) as a logic for $\text{IT}^\oplus$—that the molecule structure serves the purpose of "joining together" statements in $\text{IUT}^\oplus$ that share the same term-statement, so that the premises of an ($\cap$**I**) in $\text{IUT}^\oplus$ provide a single ($\cap$**I**)-premise in IUL_m and the minor premises of a ($\cup$**E**) in $\text{IUT}^\oplus$ provide a single ($\cup$**E**)-minor-premise in IUL_m, thus allowing a decoration for IUL_m that simulates the terms in $\text{IUT}^\oplus$. However, this amounts to assuming a one-to-one correspondence between $\text{IUL}_m^\star$ and $S_{\text{IUT}^\oplus}$, which is *not* the case. As shown so far, statements in $\text{IUT}^\oplus$ sharing the same term-statement, e.g. the premises of an ($\cap$**I**) in $\text{IUT}^\oplus$, must either be naturally compatible, i.e. in C_1, or, at most, compatible through transformations, i.e. in C_2, in order to be mergeable into the same decorated molecule in $\text{IUL}_m^\star$. Premises[7] of an ($\cap$**I**) in $S_{\text{IUT}^\oplus} \setminus C$ cannot be joined together in $\text{IUL}_m^\star$, which means that we have assumed more than is actually the case in defining IUL_m. On the other hand, the one-to-one correspondence between $\text{IL}_m^\star$ and $S_{\text{IT}^\oplus}$ confirms the definition of IL_m as a logic for $\text{IT}^\oplus$; Lemma 6.4 ensures that the premises of any ($\cap$**I**) in $\text{IT}^\oplus$ are naturally compatible for merging into the same decorated molecule in $\text{IL}_m^\star$. So, unfortunately, although the logic IL_m indeed expresses the type system $\text{IT}^\oplus$ on a logical level, its extension with union IUL_m is not appropriate to express (the whole of) $\text{IUT}^\oplus$ on a logical level. It actually expresses the proper subset of $\text{IUT}^\oplus$ where premises of an ($\cap$**I**) and minor premises of a ($\cup$**E**) belong to C, i.e. where premises of an ($\cap$**I**) and minor premises of a ($\cup$**E**) display, modulo transformations, the same tree T^t_{iue}.

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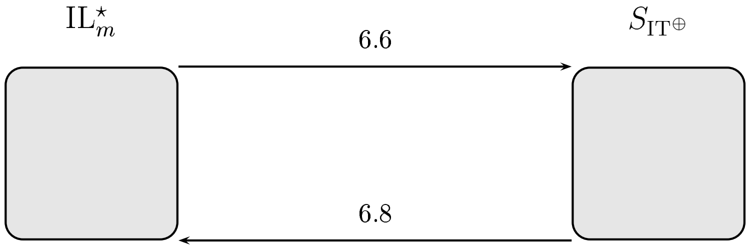
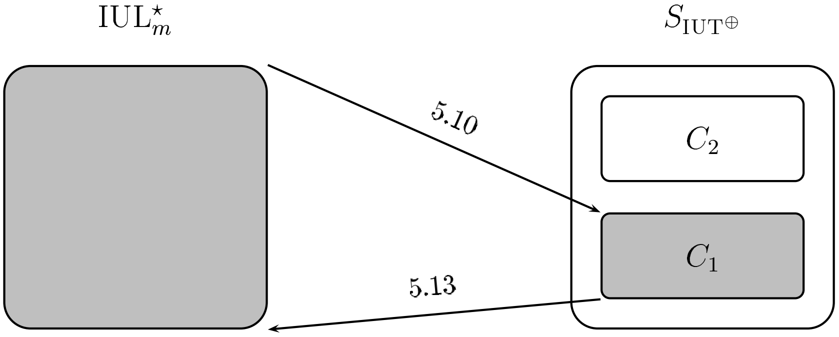
[6] The set $B = \{\pi_1, \pi_2'\}$ is in C_1 and is such that π_1 and π_2' prove the same statements as π_1 and π_2, respectively, and also the same statements as π_1' and π_2', respectively. If $\pi^\star$ is the one-to-one equivalent derivation of B in $\text{IUL}_m^\star$, then $\pi^\star$ belongs to both the subset of $\text{IUL}_m^\star$ matched to A and the subset of $\text{IUL}_m^\star$ matched to A'.

[7] The counterexample derivations $\pi_1 :: x_1 : \rho, x_2 : \beta \to \sigma \cup \tau \vdash uv : \alpha$ and $\pi_2 :: x_1 : \chi, x_2 : v \vdash uv : \beta$ (see Section 5.3), which are in $S_{\text{IUT}^\oplus} \setminus C$, are not eligible for premises of an ($\cap$**I**). So, one might wonder if there actually exist premises of an ($\cap$**I**) in $S_{\text{IUT}^\oplus} \setminus C$. However, we believe that modifying π_1 to $\tilde{\pi_1} :: x_1 : \rho \cap \chi, x_2 : (\beta \to \sigma \cup \tau) \cap v \vdash uv : \alpha$ and π_2 to $\tilde{\pi_2} :: x_1 : \rho \cap \chi, x_2 : (\beta \to \sigma \cup \tau) \cap v \vdash uv : \beta$, so that we get derivations which are eligible for premises of an ($\cap$**I**), we still have a pair in $S_{\text{IUT}^\oplus} \setminus C$. Derivations $\tilde{\pi_1}$ and $\tilde{\pi_2}$ differ from π_1 and π_2, respectively, only in additional ($\cap$**E**) inferences at the top, which implies that $(T^t_{\text{iue}})\tilde{}_1 = (T^t_{\text{iue}})_1$ and $(T^t_{\text{iue}})\tilde{}_2 = (T^t_{\text{iue}})_2$, which, in turn, implies that $\{\tilde{\pi_1}, \tilde{\pi_2}\} \notin C_1$. To justify that $\{\tilde{\pi_1}, \tilde{\pi_2}\} \notin C_2$, we follow the pattern given in 5.3 to justify that $\{\pi_1, \pi_2\} \notin C_2$, although the work required is considerably increased.



Intersection



Intersection and Union

Figure 6.4: One-to-one correspondences.

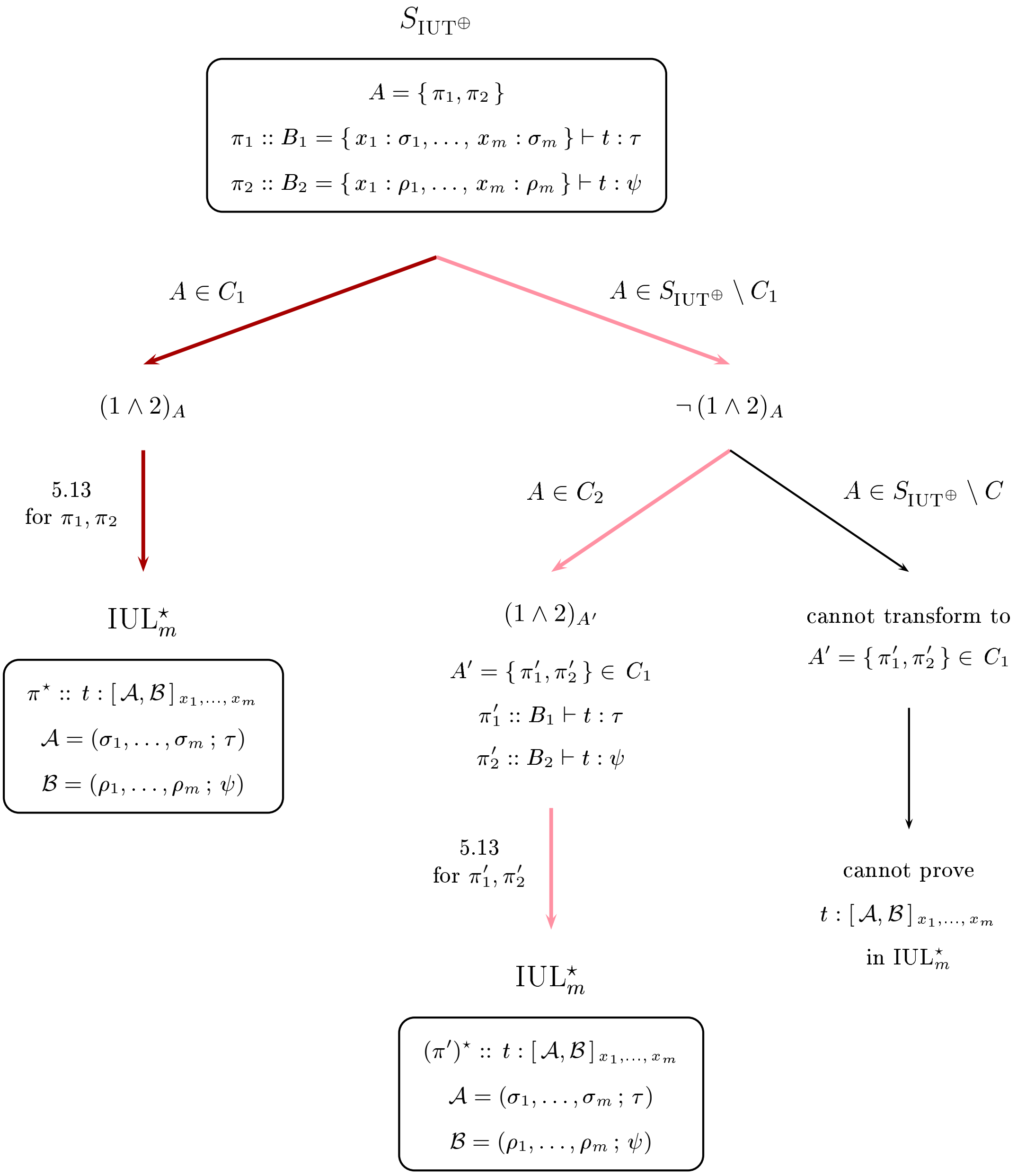


Figure 6.5: Subsets of $S_{\mathrm{IUT}^{\oplus}}$ and paths from $S_{\mathrm{IUT}^{\oplus}}$ to $\mathrm{IUL}_m^{\star}$.

CHAPTER 7

Sequent Calculus $\mathbf{IUL}_m$ and $\mathbf{IUT}^{\oplus}$

We present the logic IUL_m and the type system $\mathrm{IUT}^{\oplus}$ in sequent calculus style, retaining the additive character of their natural deduction presentations. For both the logic and the type system, we show that the two styles of presentation are equivalent and that the basic natural deduction properties (derivability properties, etc.) hold in the sequent calculus context, as well. We also prove that the additive and multiplicative[1] sequent calculus presentations of the type system are equivalent. We finally elaborate on how the sequent calculus logic attempts to represent the sequent calculus type system on a logical level and sketch how the sequent calculus correspondence between the logic and the type system can be studied with tools analogous to the ones used to study the natural deduction correspondence between the logic and the type system.

7.1 The logic $\mathbf{IUL}_m$ in sequent calculus

Keeping (i) and (ii) of Definition 4.1 as it is, the sequent calculus logic IUL_m proves molecules by the rules displayed in Figure 7.1.

Remark 7.1 *(i) In the exchange rule* $(\mathbf{X})$*, the* Γ_i*'s have the same cardinality.*

(ii) As was the case in the natural deduction presentation, the (left and right) intersection and (left and right) union rules demonstrated in Figure 7.1 are only special versions of the actual (left and right) intersection and (left and right) union rules. The actual $(\cup\mathbf{L})$ *is meant as follows.*

$$\frac{[\,\mathcal{U}_1,(\Gamma_1,\sigma_1\,;\,\rho_1),(\Gamma_1,\tau_1\,;\,\rho_1),\mathcal{U}_2,(\Gamma_2,\sigma_2\,;\,\rho_2),(\Gamma_2,\tau_2\,;\,\rho_2),\ldots,\mathcal{U}_n,(\Gamma_n,\sigma_n\,;\,\rho_n),(\Gamma_n,\tau_n\,;\,\rho_n),\mathcal{U}_{n+1}\,]}{[\,\mathcal{U}_1,(\Gamma_1,\sigma_1\cup\tau_1\,;\,\rho_1),\mathcal{U}_2,(\Gamma_2,\sigma_2\cup\tau_2\,;\,\rho_2),\ldots,\mathcal{U}_n,(\Gamma_n,\sigma_n\cup\tau_n\,;\,\rho_n),\mathcal{U}_{n+1}\,]}\,(\cup\mathbf{L})$$

The actual $(\cap\mathbf{L}_1)$*,*$(\cap\mathbf{L}_2)$*,*$(\cap\mathbf{R})$*,*$(\cup\mathbf{R}_1)$*, and* $(\cup\mathbf{R}_2)$ *can be figured from their special versions in a similar manner.*

The categorization of rules as *global* or *local* is still according to whether they affect *all* or *some* atoms in premise level, respectively. The exchange rule, the implication rules, and the cut rule are global, while the intersection and union rules are local.

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[1] We remind the reader that the multiplicative sequent calculus version of the type system is studied in Chapter 2.

$$\frac{}{[(\Gamma_i, \sigma_i \,;\, \sigma_i)_i]}\ \textbf{(ax)} \qquad \frac{[(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]}{[(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]}\ \textbf{(X)}$$

$$\frac{[(\Gamma_i \,;\, \sigma_i)_i] \qquad [(\Gamma_i, \tau_i \,;\, \rho_i)_i]}{[\Gamma_i, \sigma_i \to \tau_i \,;\, \rho_i)_i]}\ (\to\!\mathbf{L}) \qquad \frac{[(\Gamma_i, \sigma_i \,;\, \tau_i)_i]}{[(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]}\ (\to\!\mathbf{R})$$

$$\frac{[\mathcal{U}, (\Gamma_i, \sigma_i \,;\, \rho_i)_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i, \sigma_i \cap \tau_i \,;\, \rho_i)_i, \mathcal{V}]}\ (\cap\mathbf{L}_1) \qquad \frac{[\mathcal{U}, (\Gamma_i, \tau_i \,;\, \rho_i)_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i, \sigma_i \cap \tau_i \,;\, \rho_i)_i, \mathcal{V}]}\ (\cap\mathbf{L}_2) \qquad \frac{[\mathcal{U}, ((\Gamma_i \,;\, \sigma_i), (\Gamma_i \,;\, \tau_i))_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}]}\ (\cap\mathbf{R})$$

$$\frac{[\mathcal{U}, ((\Gamma_i, \sigma_i \,;\, \rho_i), (\Gamma_i, \tau_i \,;\, \rho_i))_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i, \sigma_i \cup \tau_i \,;\, \rho_i)_i, \mathcal{V}]}\ (\cup\mathbf{L}) \qquad \frac{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}]}\ (\cup\mathbf{R}_1) \qquad \frac{[\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}]}{[\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}]}\ (\cup\mathbf{R}_2)$$

$$\frac{[(\Gamma_i \,;\, \sigma_i)_i] \qquad [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]}{[(\Gamma_i \,;\, \tau_i)_i]}\ \textbf{(cut)}$$

Figure 7.1: The logic IUL_m in sequent calculus style.

The connectives of the grammar are all *additive*. This is done by *necessity* in the cases of intersection and union. The claim that atoms in the same molecule should have the same context cardinality forbids a multiplicative presentation of the intersection rules and the left union rule. Considering the left intersection, a multiplicative premise $[(\Delta_i \,;\, \phi_i)_1^k, (\Gamma_i, \sigma_i, \tau_i \,;\, \rho_i)_1^n]$ with $|\Delta_i| = |\Gamma_i, \sigma_i, \tau_i| = m + 2$ would give a conclusion $[(\Delta_i \,;\, \phi_i)_1^k, (\Gamma_i, \sigma_i \cap \tau_i \,;\, \rho_i)_1^n]$ with $|\Delta_i| = m + 2$, but $|\Gamma_i, \sigma_i \cap \tau_i| = m + 1$. Similar arguments hold for the right intersection and the left union. Moreover, the intuitionistic claim that atoms should contain exactly one formula to the right of ";" forbids a multiplicative presentation of the right union; a multiplicative premise $[\mathcal{U}, (\Gamma_i \,;\, \sigma_i, \tau_i)_i, \mathcal{V}]$ would no longer belong to an intuitionistic system. On the other hand, the additive presentation is picked by *choice* in the case of implication. This is because the left implication can also be given multiplicatively with premises $[(\Gamma_i \,;\, \sigma_i)_i], [(\Delta_i, \tau_i \,;\, \rho_i)_i]$ and conclusion $[(\Gamma_i, \Delta_i, \sigma_i \to \tau_i \,;\, \rho_i)_i]$. The cut rule is additive by choice, as well.

The sequent calculus presentation of IUL_m is equivalent to the natural deduction presentation of IUL_m, given in Chapter 4.

Theorem 7.2 *(i) If $\pi :: \mathcal{M}$ in sequent calculus style, there is a $\pi' :: \mathcal{M}$ in natural deduction style.*
(ii) If $\pi :: \mathcal{M}$ in natural deduction style, there is a $\pi' :: \mathcal{M}$ in sequent calculus style.

Proof. For both (i) and (ii), the formal proof is by induction on π.

(i) In practice, the inductive proof reduces to showing that the sequent calculus rules are derivable in the natural deduction system. The axiom and the exchange rule are common in both presentations, while the sequent calculus right rules correspond to the natural deduction introduction rules. Hence, it remains to show the derivability of the left rules and the cut rule in natural deduction.

$$\triangleright\ \frac{[(\Gamma_i \,;\, \sigma_i)_i] \qquad [(\Gamma_i, \tau_i \,;\, \rho_i)_i]}{[(\Gamma_i, \sigma_i \to \tau_i \,;\, \rho_i)_i]}\ (\to\!\mathbf{L}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{[(\Gamma_i,\tau_i\,;\,\rho_i)_i]}{[(\Gamma_i\,;\,\tau_i\to\rho_i)_i]}\,(\to\mathbf{I})}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\tau_i\to\rho_i)_i]}\,[4.5] \qquad \dfrac{\dfrac{}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\sigma_i\to\tau_i)_i]}\,(\mathbf{ax}) \quad \dfrac{[(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\sigma_i)_i]}\,[4.5]}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\tau_i)_i]}\,(\to\mathbf{E})}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\rho_i)_i]}\,(\to\mathbf{E})$$

$$\triangleright\quad \dfrac{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\,;\,\rho_i)_1^n]}{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\,;\,\rho_i)_1^n]}\,(\cap\mathbf{L}_1) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\,;\,\rho_i)_1^n]}{[(\Delta_i\,;\,\phi_i\to\psi_i)_1^k,(\Gamma_i\,;\,\sigma_i\to\rho_i)_1^n]}\,(\to\mathbf{I})}{[(\Delta_i,\phi_i\,;\,\phi_i\to\psi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\,;\,\sigma_i\to\rho_i)_1^n]}\,[4.5] \qquad \dfrac{\dfrac{}{[(\Delta_i,\phi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\,;\,\sigma_i\cap\tau_i)_1^n]}\,(\mathbf{ax})}{[(\Delta_i,\phi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\,;\,\sigma_i)_1^n]}\,(\cap\mathbf{E}_1)}{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\,;\,\rho_i)_1^n]}\,(\to\mathbf{E})$$

$$\triangleright\quad \dfrac{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,((\Gamma_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\tau_i\,;\,\rho_i))_1^n]}{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\cup\tau_i\,;\,\rho_i)_1^n]}\,(\cup\mathbf{L}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{}{[(\Delta_i,\phi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\cup\tau_i\,;\,\sigma_i\cup\tau_i)_1^n]}\,(\mathbf{ax}) \quad \dfrac{\dfrac{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,((\Gamma_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\tau_i\,;\,\rho_i))_1^n]}{[(\Delta_i,\phi_i,\phi_i\,;\,\psi_i)_1^k,((\Gamma_i,\sigma_i,\sigma_i\cup\tau_i\,;\,\rho_i),(\Gamma_i,\tau_i,\sigma_i\cup\tau_i\,;\,\rho_i))_1^n]}\,[4.5]}{[(\Delta_i,\phi_i,\phi_i\,;\,\psi_i)_1^k,((\Gamma_i,\sigma_i\cup\tau_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\sigma_i\cup\tau_i,\tau_i\,;\,\rho_i))_1^n]}\,(\mathbf{X})}{[(\Delta_i,\phi_i\,;\,\psi_i)_1^k,(\Gamma_i,\sigma_i\cup\tau_i\,;\,\rho_i)_1^n]}\,(\cup\mathbf{E})'$$

$$\triangleright\quad \dfrac{[(\Gamma_i\,;\,\sigma_i)_i] \qquad [(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\,(\mathbf{cut}) \quad \rightsquigarrow \quad \dfrac{\dfrac{[(\Gamma_i,\sigma_i\,;\,\tau_i)_i]}{[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i]}\,(\to\mathbf{I}) \qquad [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\,(\to\mathbf{E})$$

(ii) The inductive proof reduces to showing that the natural deduction rules are derivable in the sequent calculus system. Since the introduction rules translate to the corresponding right rules, it remains to show the derivability of the elimination rules in sequent calculus.

$$\triangleright\quad \dfrac{[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i] \qquad [(\Gamma_i\,;\,\sigma_i)_i]}{[(\Gamma_i\,;\,\tau_i)_i]}\,(\to\mathbf{E}) \quad \rightsquigarrow \quad \dfrac{[(\Gamma_i\,;\,\sigma_i\to\tau_i)_i] \qquad \dfrac{[(\Gamma_i\,;\,\sigma_i)_i] \qquad \dfrac{}{[(\Gamma_i,\tau_i\,;\,\tau_i)_i]}\,(\mathbf{ax})}{[(\Gamma_i,\sigma_i\to\tau_i\,;\,\tau_i)_i]}\,(\to\mathbf{L})}{[(\Gamma_i\,;\,\tau_i)_i]}\,(\mathbf{cut})$$

$$\triangleright\quad \dfrac{[(\Delta_i\,;\,\phi_i)_1^k,(\Gamma_i\,;\,\sigma_i\cap\tau_i)_1^n]}{[(\Delta_i\,;\,\phi_i)_1^k,(\Gamma_i\,;\,\sigma_i)_1^n]}\,(\cap\mathbf{E}_1) \quad \rightsquigarrow \quad \dfrac{[(\Delta_i\,;\,\phi_i)_1^k,(\Gamma_i\,;\,\sigma_i\cap\tau_i)_1^n] \qquad \dfrac{\dfrac{}{[(\Delta_i,\phi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\ \ \sigma_i)_1^n]}\,(\mathbf{ax})}{[(\Delta_i,\phi_i\,;\,\phi_i)_1^k,(\Gamma_i,\sigma_i\cap\tau_i\ \ \sigma_i)_1^n]}\,(\cap\mathbf{L}_1)}{[(\Delta_i\,;\,\phi_i)_1^k,(\Gamma_i\,;\,\sigma_i)_1^n]}\,(\mathbf{cut})$$

$$\triangleright\quad \dfrac{[(\Gamma_i\,;\,\sigma_i\cup\tau_i)_i] \qquad [((\Gamma_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\tau_i\,;\,\rho_i))_i]}{[(\Gamma_i\,;\,\rho_i)_i]}\,(\cup\mathbf{E}) \quad \rightsquigarrow \quad \dfrac{[(\Gamma_i\,;\,\sigma_i\cup\tau_i)_i] \qquad \dfrac{[((\Gamma_i,\sigma_i\,;\,\rho_i),(\Gamma_i,\tau_i\,;\,\rho_i))_i]}{[(\Gamma_i,\sigma_i\cup\tau_i\,;\,\rho_i)_i]}\,(\cup\mathbf{L})}{[(\Gamma_i\,;\,\rho_i)_i]}\,(\mathbf{cut})$$

$\dashv$

Following the equivalence of the two presentations of the logic, we expect that the propositions on derivability, shown in Chapter 4 for the natural deduction presentation (Propositions 4.5, 4.6, 4.10, and 4.11), also hold for the sequent calculus presentation. The next two propositions show that weakening and contraction are derivable.

Proposition 7.3 *Weakening is derivable: if* $\pi :: [(\Gamma_i\,;\,\tau_i)_i]$*, there exists a* $\pi' :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]$.

Proof. Either by Theorem 7.2 and Proposition 4.5 or directly by induction on π. We show three inductive cases of the direct proof.

$$\triangleright\quad \dfrac{\pi_0 :: [(\Gamma_i\,;\,\tau_i)_i] \qquad \pi_1 :: [(\Gamma_i, \rho_i\,;\,\upsilon_i)_i]}{\pi :: [(\Gamma_i, \tau_i \to \rho_i\,;\,\upsilon_i)_i]}\,(\to\mathbf{L}) \quad\rightsquigarrow\quad \dfrac{\dfrac{\pi_0' :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]\ [\mathrm{h}] \qquad \dfrac{\pi_1' :: [(\Gamma_i, \rho_i, \sigma_i\,;\,\upsilon_i)_i]\ [\mathrm{h}]}{[(\Gamma_i, \sigma_i, \rho_i\,;\,\upsilon_i)_i]}\,(\mathbf{X})}{[(\Gamma_i, \sigma_i, \tau_i \to \rho_i\,;\,\upsilon_i)_i]}\,(\to\mathbf{L})}{\pi' :: [(\Gamma_i, \tau_i \to \rho_i, \sigma_i\,;\,\upsilon_i)_i]}\,(\mathbf{X})$$

$$\triangleright\quad \dfrac{\pi_0 :: [(\Delta_i\,;\,\phi_i)_1^k, (\Gamma_i\,;\,\tau_i)_1^n]}{\pi :: [(\Delta_i\,;\,\phi_i)_1^k, (\Gamma_i\,;\,\tau_i \cup \rho_i)_1^n]}\,(\cup\mathbf{R}_1) \quad\rightsquigarrow\quad \dfrac{\pi_0' :: [(\Delta_i, \psi_i\,;\,\phi_i)_1^k, (\Gamma_i, \sigma_i\,;\,\tau_i)_1^n]\ [\mathrm{h}]}{\pi' :: [(\Delta_i, \psi_i\,;\,\phi_i)_1^k, (\Gamma_i, \sigma_i\,;\,\tau_i \cup \rho_i)_1^n]}\,(\cup\mathbf{R}_1)$$

$$\triangleright\quad \dfrac{\pi_0 :: [(\Gamma_i\,;\,\tau_i)_i] \qquad \pi_1 :: [(\Gamma_i, \tau_i\,;\,\rho_i)_i]}{\pi :: [(\Gamma_i\,;\,\rho_i)_i]}\,(\mathbf{cut}) \quad\rightsquigarrow\quad \dfrac{\pi_0' :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]\ [\mathrm{h}] \qquad \dfrac{\pi_1' :: [(\Gamma_i, \tau_i, \sigma_i\,;\,\rho_i)_i]\ [\mathrm{h}]}{[(\Gamma_i, \sigma_i, \tau_i\,;\,\rho_i)_i]}\,(\mathbf{X})}{\pi' :: [(\Gamma_i, \sigma_i\,;\,\rho_i)_i]}\,(\mathbf{cut})$$

$\dashv$

Proposition 7.4 *Contraction is derivable: if* $\pi :: [(\Gamma_i, \sigma_i, \sigma_i\,;\,\tau_i)_i]$*, there exists a* $\pi' :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]$.

Proof. Either by Theorem 7.2 and Proposition 4.6 or directly through the cut rule.

$$\dfrac{\dfrac{}{[(\Gamma_i, \sigma_i\,;\,\sigma_i)_i]}\,(\mathbf{ax}) \qquad \pi :: [(\Gamma_i, \sigma_i, \sigma_i\,;\,\tau_i)_i]}{\pi' :: [(\Gamma_i, \sigma_i\,;\,\tau_i)_i]}\,(\mathbf{cut})$$

It is easy to check that, if we chose a multiplicative cut rule, the derivability of contraction through cut would fail. A proof by induction on π would also fail. $\dashv$

Before showing that pruning and doubling are derivable, we need to establish the exchange of atoms within provable molecules. The definitions of *tree* and *derivation height* remain as given in 4.7 and 4.8, respectively.

Proposition 7.5 *If* $\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]$*, there exists a* $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ *with* $T' = T$.

Proof. By induction on π. We show two characteristic general cases of the induction step.

▷ A global rule (R): e.g. **(→L)** or **(cut)**

$$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}_0, \mathcal{B}_0, \mathcal{V}_0] \qquad \pi_1 :: [\mathcal{U}_1, \mathcal{A}_1, \mathcal{B}_1, \mathcal{V}_1]}{\pi :: [\mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{V}]} \;(\mathrm{R})$$

where $|\mathcal{U}_0| = |\mathcal{U}_1| = |\mathcal{U}|$

The IH gives a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}_0, \mathcal{A}_0, \mathcal{V}_0]$ with $T_0' = T_0$ and a $\pi_1' :: [\mathcal{U}_1, \mathcal{B}_1, \mathcal{A}_1, \mathcal{V}_1]$ with $T_1' = T_1$. Applying (R) to π_0' and π_1', we get a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$.

▷ A local rule (R): e.g. **(∩L)** or **(∪R)**

Case 1: $$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}, \mathcal{B}, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{A}_{\mathrm{R}}, \mathcal{B}_{\mathrm{R}}, \mathcal{V}]} \;(\mathrm{R})$$

where $|\mathcal{U}_0| = |\mathcal{U}|$, and $\mathcal{A}_{\mathrm{R}}$ and $\mathcal{B}_{\mathrm{R}}$ derive from $\mathcal{A}$ and $\mathcal{B}$, respectively, by (R)

The IH gives a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}, \mathcal{A}, \mathcal{V}_0]$ with $T_0' = T_0$. Applying (R) to π_0', we get a $\pi' :: [\mathcal{U}, \mathcal{B}_{\mathrm{R}}, \mathcal{A}_{\mathrm{R}}, \mathcal{V}]$ with $T' = T$.

Case 2: $$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{A}, \mathcal{B}, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{A}_{\mathrm{R}}, \mathcal{B}, \mathcal{V}]} \;(\mathrm{R})$$

where $|\mathcal{U}_0| = |\mathcal{U}|$

The IH yields a $\pi_0' :: [\mathcal{U}_0, \mathcal{B}, \mathcal{A}, \mathcal{V}_0]$ with $T_0' = T_0$. Applying (R) to π_0', we obtain a $\pi' :: [\mathcal{U}, \mathcal{B}, \mathcal{A}_{\mathrm{R}}, \mathcal{V}]$ with $T' = T$.

Case 3: $$\frac{\pi_0 :: [\mathcal{U}_0(n, \mathcal{C}^k), \mathcal{A}, \mathcal{B}, \mathcal{V}_0]}{\pi :: [\mathcal{U}(n, \mathcal{C}^k_{\mathrm{R}}), \mathcal{A}, \mathcal{B}, \mathcal{V}]} \;(\mathrm{R})$$

where $\mathcal{U}_0(n, \mathcal{C}^k)$ denotes a sequence $\mathcal{U}_0$ of n atoms, which contains an atom $\mathcal{C}$ at position $k \leqslant n$ and $\mathcal{U}(n, \mathcal{C}^k_{\mathrm{R}})$ denotes a sequence $\mathcal{U}$ of n atoms, which contains an atom $\mathcal{C}_{\mathrm{R}}$ at position k

The IH gives a $\pi_0' :: [\mathcal{U}_0(n, \mathcal{C}^k), \mathcal{B}, \mathcal{A}, \mathcal{V}_0]$ with $T_0' = T_0$. By (R), we then get a $\pi' :: [\mathcal{U}(n, \mathcal{C}^k_{\mathrm{R}}), \mathcal{B}, \mathcal{A}, \mathcal{V}]$ with $T' = T$.

The local rules of **(∩R)** and **(∪L)** are dealt with as **(∩I)** in the proof of 4.10. ⊣

Proposition 7.6 *(i) Pruning is derivable: if* $\pi :: [\mathcal{U}, \mathcal{V}]$, *there exists a* $\pi' :: [\mathcal{U}]$ *with* $h' \leqslant h$.
(ii) Doubling is derivable: if $\pi :: [\mathcal{U}, \mathcal{A}]$, *there exists a* $\pi' :: [\mathcal{U}, 2\mathcal{A}]$ *with* $T' = T$.

Proof. (i) By induction on π. We demonstrate two typical general cases of the induction step.

▷ A global rule (R): e.g. **(X)** or **(→R)**

$$\frac{\pi_0 :: [\mathcal{U}_0, \mathcal{V}_0]}{\pi :: [\mathcal{U}, \mathcal{V}]} \;(\mathrm{R})$$

where $|\mathcal{U}_0| = |\mathcal{U}|$

The IH gives a $\pi_0' :: [\mathcal{U}_0]$ with $h_0' \leqslant h_0$. By (R), we obtain a $\pi' :: [\mathcal{U}]$ with $h' = h_0' + 1 \leqslant h_0 + 1 = h$.

The global rules of **(→L)** and **(cut)** are dealt with as **(→E)** in the proof of 4.11(i).

▷ A local rule (R): e.g. **(∩L)** or **(∪R)**

Case 1: $$\frac{\pi_0 :: [\mathcal{U}_0(n,\mathcal{A}^k),\mathcal{V}_0]}{\pi :: [\mathcal{U}(n,\mathcal{A}^k_{\mathrm{R}}),\mathcal{V}\,]}\ (\mathrm{R})$$

The IH gives a $\pi'_0 :: [\mathcal{U}_0(n,\mathcal{A}^k)]$ with $h'_0 \leqslant h_0$. Applying (R) to π'_0, we obtain a $\pi' :: [\mathcal{U}(n,\mathcal{A}^k_{\mathrm{R}})]$ with $h' = h'_0 + 1 \leqslant h_0 + 1 = h$.

Case 2: $$\frac{\pi_0 :: [\mathcal{U},\mathcal{V}_0(n,\mathcal{A}^k)]}{\pi :: [\mathcal{U},\mathcal{V}(n,\mathcal{A}^k_{\mathrm{R}})]}\ (\mathrm{R})$$

The IH gives a $\pi'_0 :: [\mathcal{U}\,]$ with $h'_0 \leqslant h_0$. It is $\pi' = \pi'_0$ and $h' = h'_0 < h$.

The local rules of **(∩R)** and **(∪L)** are dealt with as **(∩I)** in the proof of 4.11(i).

(ii) By induction on π. We exhibit two typical general cases of the induction step.

▷ A global rule (R): e.g. **(→L)** or **(cut)**

$$\frac{\pi_0 :: [\mathcal{U}_0,\mathcal{A}_0] \qquad \pi_1 :: [\mathcal{U}_1,\mathcal{A}_1]}{\pi :: [\mathcal{U},\mathcal{A}\,]}\ (\mathrm{R})$$

The IH gives a $\pi'_0 :: [\mathcal{U}_0, 2\mathcal{A}_0]$ with $T'_0 = T_0$ and a $\pi'_1 :: [\mathcal{U}_1, 2\mathcal{A}_1]$ with $T'_1 = T_1$. Applying (R) to π'_0 and π'_1, we obtain a $\pi' :: [\mathcal{U}, 2\mathcal{A}\,]$ with $T' = T$.

▷ A local rule (R): e.g. **(∩L)** or **(∪R)**

Case 1: $$\frac{\pi_0 :: [\mathcal{U}_0,\mathcal{A}\,]}{\pi :: [\mathcal{U},\mathcal{A}_{\mathrm{R}}]}\ (\mathrm{R})$$

The IH gives a $\pi'_0 :: [\mathcal{U}_0, 2\mathcal{A}\,]$ with $T'_0 = T_0$. By (R), we then get a $\pi' :: [\mathcal{U}, 2\mathcal{A}_{\mathrm{R}}]$ with $T' = T$.

Case 2: $$\frac{\pi_0 :: [\mathcal{U}_0(n,\mathcal{B}^k),\mathcal{A}\,]}{\pi :: [\mathcal{U}(n,\mathcal{B}^k_{\mathrm{R}}),\mathcal{A}\,]}\ (\mathrm{R})$$

The IH yields a $\pi'_0 :: [\mathcal{U}_0(n,\mathcal{B}^k), 2\mathcal{A}\,]$ with $T'_0 = T_0$. Applying (R) to π'_0, we get a $\pi' :: [\mathcal{U}(n,\mathcal{B}^k_{\mathrm{R}}), 2\mathcal{A}\,]$ with $T' = T$.

The local rules of **(∩R)** and **(∪L)** are dealt with as **(∩I)** in the proof of 4.11(ii). In these two cases, we need to use Proposition 7.5. ⊣

Remark 7.7 *In the sequent calculus context, the following alternative phrasings for the derivability of weakening and contraction are provable.*

(i) Weakening is derivable: if $\pi :: [(\Gamma_i, \Delta_i\,;\, \tau_i)_i]$*, where the* Γ_i*'s have the same cardinality and the* Δ_i*'s are non-empty, there exists a* $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$.

(ii) Contraction is derivable: if $\pi :: [(\Gamma_i, \sigma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$*, where the* Γ_i*'s have the same cardinality and the* Δ_i*'s are non-empty, there exists a* $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i\,;\, \tau_i)_i]$.

Compared to the natural deduction alternative phrasings in Remark 4.12, the conclusion that $T' = T$ *has been removed from both (i) and (ii).*

For (i), the proof is by induction on π. A subcase of the $(\rightarrow\mathbf{L})$ *case, shown below, illustrates why a conclusion that $T' = T$ is no longer attainable.*

$$\triangleright \quad \dfrac{\pi_0 :: [(\Gamma'_i, \phi_i \,;\, \tau_i)_i] \qquad \pi_1 :: [(\Gamma'_i, \phi_i, \rho_i \,;\, \upsilon_i)_i]}{\pi :: [(\Gamma_i = (\Gamma'_i, \phi_i), \Delta_i = \tau_i \rightarrow \rho_i \,;\, \upsilon_i)_i]} \, (\rightarrow\mathbf{L})$$

The induction hypothesis gives a $\pi'_0 :: [(\Gamma'_i, \sigma_i, \phi_i \,;\, \tau_i)_i]$ and a $\pi'_1 :: [(\Gamma'_i, \phi_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i]$. We then obtain a $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i \,;\, \upsilon_i)_i]$, as follows.

$$\dfrac{\dfrac{\pi'_0 :: [(\Gamma'_i, \sigma_i, \phi_i \,;\, \tau_i)_i]}{[(\Gamma'_i, \phi_i, \sigma_i \,;\, \tau_i)_i]} \, \mathbf{(X)} \qquad \pi'_1 :: [(\Gamma'_i, \phi_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i]}{\pi' :: [(\Gamma_i = (\Gamma'_i, \phi_i), \sigma_i, \Delta_i = \tau_i \rightarrow \rho_i \,;\, \upsilon_i)_i]} \, (\rightarrow\mathbf{L})$$

Even if we assume that $T'_0 = T_0$ and $T'_1 = T_1$, the exchange inference forbids a conclusion that $T' = T$.

For (ii), the proof is by induction on π, with the aid of Proposition 7.4. We show the same subcase of the $(\rightarrow\mathbf{L})$ *case below.*

$$\triangleright \quad \dfrac{\pi_0 :: [(\Gamma_i, \sigma_i, \sigma_i \,;\, \tau_i)_i] \qquad \pi_1 :: [(\Gamma_i, \sigma_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i]}{\pi :: [(\Gamma_i, \sigma_i, \sigma_i, \Delta_i = \tau_i \rightarrow \rho_i \,;\, \upsilon_i)_i]} \, (\rightarrow\mathbf{L})$$

By 7.4, there is a $\pi'_0 :: [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]$, while the induction hypothesis gives a $\pi'_1 :: [(\Gamma_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i]$. Applying $(\rightarrow\mathbf{L})$ *to π'_0 and π'_1, we obtain a $\pi' :: [(\Gamma_i, \sigma_i, \Delta_i \,;\, \upsilon_i)_i]$. Even if we assume that $T'_1 = T_1$, the fact that $T'_0 \neq T_0$ (see the proof of 7.4) forbids a conclusion that $T' = T$.*

If the Δ_i's are empty in (i) and (ii), we fall back to Propositions 7.3 and 7.4, respectively.

Remark 7.8 *Proposition 4.4 does not hold in the sequent calculus context, i.e. not every sequent calculus* IUL$_m$*-derivation has a canonical form. This is because the exchange rule does not always commute with a left rule, as shown below.*

$$\dfrac{\dfrac{[(\Gamma_i, \sigma_i \,;\, \tau_i)_i] \qquad [(\Gamma_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i]}{[(\Gamma_i, \sigma_i, \tau_i \rightarrow \rho_i \,;\, \upsilon_i)_i]} \, (\rightarrow\mathbf{L})}{[(\Gamma_i, \tau_i \rightarrow \rho_i, \sigma_i \,;\, \upsilon_i)_i]} \, \mathbf{(X)} \qquad \rightsquigarrow \qquad \begin{array}{c} \begin{array}{c}[(\Gamma_i, \sigma_i \,;\, \tau_i)_i] \\ \cdots\cdots\cdots\cdots \; \mathbf{(X)} \times\end{array} \quad \begin{array}{c}[(\Gamma_i, \sigma_i, \rho_i \,;\, \upsilon_i)_i] \\ \cdots\cdots\cdots\cdots\cdots \; \mathbf{(X)} \times\end{array} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \; (\rightarrow\mathbf{L}) \times \end{array}$$

The formula $\tau_i \rightarrow \rho_i$, which is to be exchanged with σ_i, is not yet formed in the premises of $(\rightarrow\mathbf{L})$*; therefore, an* $\mathbf{(X)}$*-application involving σ_i and $\tau_i \rightarrow \rho_i$ cannot be performed before the* $(\rightarrow\mathbf{L})$*-application introducing $\tau_i \rightarrow \rho_i$.*

Having completed the sequent calculus presentation of the logic, we move on to the additive sequent calculus presentation of the type system.

$$\frac{}{B,\, x:\sigma \vdash x:\sigma}\,(\mathbf{ax})$$

$$\frac{B \vdash t:\sigma \qquad B,\, x:\tau \vdash u:\rho}{B,\, y:\sigma \to \tau \vdash u[yt/x]:\rho}\,(\to\mathbf{L}) \qquad \frac{B,\, x:\sigma \vdash t:\tau}{B \vdash \lambda x.t:\sigma \to \tau}\,(\to\mathbf{R})$$

$$\frac{B,\, x:\sigma \vdash t:\rho}{B,\, x:\sigma \cap \tau \vdash t:\rho}\,(\cap\mathbf{L}_1) \qquad \frac{B,\, x:\tau \vdash t:\rho}{B,\, x:\sigma \cap \tau \vdash t:\rho}\,(\cap\mathbf{L}_2) \qquad \frac{B \vdash t:\sigma \qquad B \vdash t:\tau}{B \vdash t:\sigma \cap \tau}\,(\cap\mathbf{R})$$

$$\frac{B,\, x:\sigma \vdash t:\rho \qquad B,\, x:\tau \vdash t:\rho}{B,\, x:\sigma \cup \tau \vdash t:\rho}\,(\cup\mathbf{L}) \qquad \frac{B \vdash t:\sigma}{B \vdash t:\sigma \cup \tau}\,(\cup\mathbf{R}_1) \qquad \frac{B \vdash t:\tau}{B \vdash t:\sigma \cup \tau}\,(\cup\mathbf{R}_2)$$

$$\frac{B \vdash t:\sigma \qquad B,\, x:\sigma \vdash u:\tau}{B \vdash u[t/x]:\tau}\,(\mathbf{cut})$$

Figure 7.2: The type system $\mathrm{IUT}^{\oplus}$ in sequent calculus style.

7.2 The type system $\mathrm{IUT}^{\oplus}$ in sequent calculus

The type system $\mathrm{IUT}^{\oplus}$ in sequent calculus style is the sequent calculus type system IUT_{ω} of Chapter 2, presented additively and without the $(\boldsymbol{\omega})$-rule. The additive presentation serves the proof of (restricted) correspondence theorems between it and the additive sequent calculus logic (see Section 7.3). It assigns types built by implication, intersection, and union to terms of the untyped λ-calculus according to the rules in Figure 7.2. As was emphasized for IUT_{ω} in Chapter 2, the new variable in the conclusion of an $(\to\mathbf{L})$ inference is fresh with respect to the derivations proving the premises.

The additive *sequent calculus* $\mathrm{IUT}^{\oplus}$ of the current section is equivalent to the additive *natural deduction* $\mathrm{IUT}^{\oplus}$ of Chapter 4. We remind the reader that we denote V_π (or just V) the set of all term variables appearing in a derivation π of the type system.

Theorem 7.9 *(i) If $\pi :: B \vdash t:\sigma$ in sequent calculus and $x_1, \ldots, x_n \notin V$, there is a $\pi' :: B \vdash t:\sigma$ in natural deduction, such that $x_1, \ldots, x_n \notin V' \supseteq V$.*

(ii) If $\pi :: B \vdash t:\sigma$ in natural deduction, there is a $\pi' :: B \vdash t:\sigma$ in sequent calculus, such that $V' \supseteq V$.

Proof. (i) By induction on π.

Base: If $\pi :: B',\, x:\sigma \vdash x:\sigma$ is an axiom, then $\pi' = \pi$ and $x_1, \ldots, x_n \notin V' = V$.

Induction step: Since the right rules translate to the corresponding introduction rules, we demonstrate the cut case and the cases of left rules.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t:\sigma \qquad \pi_1 :: B,\, x:\sigma \vdash u:\tau}{\pi :: B \vdash u[t/x]:\tau}\,(\mathbf{cut}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\pi'_0 :: B \vdash t : \sigma \quad [\mathrm{h}]}{B \vdash t : \sigma \cup \sigma}\,(\cup\mathbf{I}) \qquad \pi'_1 :: B,\, x : \sigma \vdash u : \tau \quad [\mathrm{h}] \qquad \pi'_1 :: B,\, x : \sigma \vdash u : \tau \quad [\mathrm{h}]}{\pi' :: B \vdash u[t/x] : \tau}\,(\cup\mathbf{E})$$

If $x_1, \ldots, x_n \notin V = V_0 \cup V_1$, the IH yields that $x_1, \ldots, x_n \notin V' = V'_0 \cup V'_1 \supseteq V_0 \cup V_1 = V$.

$$\triangleright \quad \dfrac{\pi_0 :: B \vdash t : \sigma \qquad \pi_1 :: B,\, x : \tau \vdash u : \rho}{\pi :: B,\, y : \sigma \to \tau \vdash u[yt/x] : \rho}\,(\to\mathbf{L}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{}{B,\, y : \sigma \to \tau \vdash y : \sigma \to \tau}\,(\mathbf{ax}) \qquad \dfrac{\pi'_0 :: B \vdash t : \sigma \quad [\mathrm{h}]}{\pi''_0 :: B,\, y : \sigma \to \tau \vdash t : \sigma}\,[4.14(\mathrm{ii})]}{B,\, y : \sigma \to \tau \vdash yt : \tau}\,(\to\mathbf{E}) \qquad \dfrac{\pi'_1 :: B,\, x : \tau \vdash u : \rho \quad [\mathrm{h}]}{\pi''_1 :: B,\, y : \sigma \to \tau,\, x : \tau \vdash u : \rho}\,[4.14(\mathrm{ii})]}{\pi' :: B,\, y : \sigma \to \tau \vdash u[yt/x] : \rho}\,(\cup\mathbf{IE})$$

If $x_1, \ldots, x_n \notin V$, then $x_1, \ldots, x_n, y \notin V_0 \cup V_1$. The IH gives that $x_1, \ldots, x_n, y \notin V'_0 \cup V'_1 \supseteq V_0 \cup V_1$. Since $y \notin V'_0 \cup V'_1$, we can apply 4.14(ii) to π'_0 and π'_1 to get π''_0 and π''_1, respectively, such that

$$x_1, \ldots, x_n \notin V' = V''_0 \cup V''_1 = V'_0 \cup V'_1 \cup \{y\} \supseteq V_0 \cup V_1 \cup \{y\} = V$$

$$\triangleright \quad \dfrac{\pi_0 :: B,\, x : \sigma \vdash t : \rho}{\pi :: B,\, x : \sigma \cap \tau \vdash t : \rho}\,(\cap\mathbf{L}_1) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{}{B,\, x : \sigma \cap \tau \vdash x : \sigma \cap \tau}\,(\mathbf{ax})}{B,\, x : \sigma \cap \tau \vdash x : \sigma}\,(\cap\mathbf{E}_1) \qquad \dfrac{\dfrac{\pi'_0 :: B,\, x : \sigma \vdash t : \rho \quad [\mathrm{h}]}{\pi''_0 :: B,\, y : \sigma \vdash t[y/x] : \rho}\,[4.14(\mathrm{i})]}{\pi'''_0 :: B,\, x : \sigma \cap \tau,\, y : \sigma \vdash t[y/x] : \rho}\,[4.14(\mathrm{ii})]}{\pi' :: B,\, x : \sigma \cap \tau \vdash (t[y/x])[x/y] \overset{y \notin FV(t)}{=} t : \rho}\,(\cup\mathbf{IE})$$

If $x_1, \ldots, x_n \notin V = V_0$, the IH gives that $x_1, \ldots, x_n \notin V'_0 \supseteq V_0$. If y is such that $x_1, \ldots, x_n \neq y \notin V'_0$, we can apply 4.14(i) to π'_0 to get π''_0 with $x_1, \ldots, x_n, x \notin V''_0 = (V'_0 \setminus \{x\}) \cup \{y\}$. Since $x \notin V''_0$, we can further apply 4.14(ii) to π''_0 to get π'''_0 with $x_1, \ldots, x_n \notin V'''_0 = V''_0 \cup \{x\} = V'_0 \cup \{y\}$. Since $y \notin V'_0 \supseteq V_0$, we finally get that $x_1, \ldots, x_n \notin V' = V'''_0 \supseteq V_0 \cup \{y\} \supsetneq V_0 = V$.

$$\triangleright \quad \dfrac{\pi_0 :: B,\, x : \sigma \vdash t : \rho \qquad \pi_1 :: B,\, x : \tau \vdash t : \rho}{\pi :: B,\, x : \sigma \cup \tau \vdash t : \rho}\,(\cup\mathbf{L}) \quad \rightsquigarrow$$

$$\dfrac{\dfrac{}{B,\, x : \sigma \cup \tau \vdash x : \sigma \cup \tau}\,(\mathbf{ax}) \qquad \dfrac{\dfrac{\pi'_0 :: B,\, x : \sigma \vdash t : \rho \quad [\mathrm{h}]}{\pi''_0 :: B,\, y : \sigma \vdash t[y/x] : \rho}\,[4.14(\mathrm{i})]}{\pi'''_0 :: B,\, x : \sigma \cup \tau,\, y : \sigma \vdash t[y/x] : \rho}\,[4.14(\mathrm{ii})] \qquad \dfrac{\dfrac{\pi'_1 :: B,\, x : \tau \vdash t : \rho \quad [\mathrm{h}]}{\pi''_1 :: B,\, y : \tau \vdash t[y/x] : \rho}\,[4.14(\mathrm{i})]}{\pi'''_1 :: B,\, x : \sigma \cup \tau,\, y : \tau \vdash t[y/x] : \rho}\,[4.14(\mathrm{ii})]}{\pi' :: B,\, x : \sigma \cup \tau \vdash (t[y/x])[x/y] \overset{y \notin FV(t)}{=} t : \rho}\,(\cup\mathbf{E})$$

If $x_1, \ldots, x_n \notin V = V_0 \cup V_1$, the IH gives that $x_1, \ldots, x_n \notin V'_0 \cup V'_1 \supseteq V_0 \cup V_1$. If y is such that $x_1, \ldots, x_n \neq y \notin V'_0 \cup V'_1$, we can apply 4.14(i) to π'_0 and π'_1 to get π''_0 and π''_1, respectively, such that $x_1, \ldots, x_n, x \notin V''_0 \cup V''_1 = ((V'_0 \cup V'_1) \setminus \{x\}) \cup \{y\}$. Since $x \notin V''_0 \cup V''_1$, we can apply 4.14(ii) to π''_0 and π''_1 to get π'''_0 and π'''_1, respectively, such that $x_1, \ldots, x_n \notin V'''_0 \cup V'''_1 = V''_0 \cup V''_1 \cup \{x\} = V'_0 \cup V'_1 \cup \{y\}$. Since $y \notin V'_0 \cup V'_1 \supseteq V_0 \cup V_1$, we finally get that $x_1, \ldots, x_n \notin V' = V'''_0 \cup V'''_1 \supseteq V_0 \cup V_1 \cup \{y\} \supsetneq V_0 \cup V_1 = V$.

(ii) By induction on π.

Base: If $\pi :: B', x : \sigma \vdash x : \sigma$ is an axiom, then $\pi' = \pi$ and $V' = V$.

Induction step: Since the introduction rules correspond to the right rules, we show the cases of elimination rules.

$$\triangleright\quad \dfrac{\pi_0 :: B \vdash t : \sigma \to \tau \qquad \pi_1 :: B \vdash u : \sigma}{\pi :: B \vdash tu : \tau}\,(\mathbf{\to E}) \quad \rightsquigarrow$$

$$\dfrac{\pi'_0 :: B \vdash t : \sigma \to \tau\ \ [\text{h}] \qquad \dfrac{\pi'_1 :: B \vdash u : \sigma\ \ [\text{h}] \qquad \dfrac{}{B,\ x : \tau \vdash x : \tau}\,(\mathbf{ax})}{B,\ y : \sigma \to \tau \vdash yu : \tau}\,(\mathbf{\to L})}{\pi' :: B \vdash tu : \tau}\,(\mathbf{cut})$$

It is $V' = V'_0 \cup V'_1 \cup \{x, y\} \overset{[\text{h}]}{\supseteq} V_0 \cup V_1 \cup \{x, y\} \supseteq V_0 \cup V_1 = V$. [Example 7.10 illustrates one case where $V' \supsetneq V$ and another case where $V' = V$.]

$$\triangleright\quad \dfrac{\pi_0 :: B \vdash t : \sigma \cap \tau}{\pi :: B \vdash t : \sigma}\,(\mathbf{\cap E_1}) \quad \rightsquigarrow \quad \dfrac{\pi'_0 :: B \vdash t : \sigma \cap \tau\ \ [\text{h}] \qquad \dfrac{\dfrac{}{B,\ x : \sigma \vdash x : \sigma}\,(\mathbf{ax})}{B,\ x : \sigma \cap \tau \vdash x : \sigma}\,(\mathbf{\cap L_1})}{\pi' :: B \vdash t : \sigma}\,(\mathbf{cut})$$

It is $V' = V'_0 \cup \{x\} \overset{[\text{h}]}{\supseteq} V_0 \cup \{x\} \supseteq V_0 = V$.

$$\triangleright\quad \dfrac{\pi_0 :: B \vdash t : \sigma \cup \tau \qquad \pi_1 :: B,\ x : \sigma \vdash u : \rho \qquad \pi_2 :: B,\ x : \tau \vdash u : \rho}{\pi :: B \vdash u[t/x] : \rho}\,(\mathbf{\cup E}) \quad \rightsquigarrow$$

$$\dfrac{\pi'_0 :: B \vdash t : \sigma \cup \tau\ \ [\text{h}] \qquad \dfrac{\pi'_1 :: B,\ x : \sigma \vdash u : \rho\ \ [\text{h}] \qquad \pi'_2 :: B,\ x : \tau \vdash u : \rho\ \ [\text{h}]}{B,\ x : \sigma \cup \tau \vdash u : \rho}\,(\mathbf{\cup L})}{\pi' :: B \vdash u[t/x] : \rho}\,(\mathbf{cut})$$

It is $V' = V'_0 \cup V'_1 \cup V'_2 \overset{[\text{h}]}{\supseteq} V_0 \cup V_1 \cup V_2 = V$. $\dashv$

Example 7.10 *(i) Consider the following natural deduction derivation* $\pi ::\ z : \sigma \to \tau,\ w : \sigma \vdash zw : \tau$.

$$\dfrac{\pi_0 ::\ z : \sigma \to \tau,\ w : \sigma \vdash z : \sigma \to \tau \qquad \pi_1 ::\ z : \sigma \to \tau,\ w : \sigma \vdash w : \sigma}{\pi ::\ z : \sigma \to \tau,\ w : \sigma \vdash zw : \tau}\,(\mathbf{\to E})$$

Following the method in the proof of 7.9(ii), derivation π *transforms to the following sequent calculus derivation* $\pi' ::\ z : \sigma \to \tau,\ w : \sigma \vdash zw : \tau$.

$$\dfrac{\pi'_0 ::\ z : \sigma \to \tau,\ w : \sigma \vdash z : \sigma \to \tau \qquad \dfrac{\pi'_1 ::\ z : \sigma \to \tau,\ w : \sigma \vdash w : \sigma \qquad z : \sigma \to \tau,\ w : \sigma,\ x : \tau \vdash x : \tau}{z : \sigma \to \tau,\ w : \sigma,\ y : \sigma \to \tau \vdash yw : \tau}\,(\mathbf{\to L})}{\pi' ::\ z : \sigma \to \tau,\ w : \sigma \vdash zw : \tau}\,(\mathbf{cut})$$

The definition of "basis" implies that $x \neq z, w$ *and the definition of* $(\rightarrow\mathbf{L})$ *implies that* $y \neq z, w, x$. *Hence, it is* $V' = \{z, w, x, y\} \supsetneq \{z, w\} = V$.

(ii) Consider the following natural deduction derivation $\pi :: B = \{\, z : (\sigma \cap \rho) \cup (\rho \cap \tau)\,\} \vdash z\,(\lambda x.\, x) : \beta$, *where* $\rho = (\alpha \rightarrow \alpha) \rightarrow \beta$.

$$\dfrac{\dfrac{B \vdash z : (\sigma \cap \rho) \cup (\rho \cap \tau) \qquad \dfrac{B,\, y : \sigma \cap \rho \vdash y : \sigma \cap \rho}{B,\, y : \sigma \cap \rho \vdash y : \rho}\,{\scriptstyle(\cap\mathbf{E}_2)} \qquad \dfrac{B,\, y : \rho \cap \tau \vdash y : \rho \cap \tau}{B,\, y : \rho \cap \tau \vdash y : \rho}\,{\scriptstyle(\cap\mathbf{E}_1)}}{\pi_0 :: B \vdash z : \rho}\,{\scriptstyle(\cup\mathbf{E})} \qquad \dfrac{B,\, x : \alpha \vdash x : \alpha}{\pi_1 :: B \vdash \lambda x.\, x : \alpha \rightarrow \alpha}\,{\scriptstyle(\rightarrow\mathbf{I})}}{\pi :: B = \{\, z : (\sigma \cap \rho) \cup (\rho \cap \tau)\,\} \vdash z\,(\lambda x.\, x) : \beta}\,{\scriptstyle(\rightarrow\mathbf{E})}$$

Following the method in the proof of 7.9(ii), derivation π *transforms to the following sequent calculus derivation* $\pi' :: B = \{\, z : (\sigma \cap \rho) \cup (\rho \cap \tau)\,\} \vdash z\,(\lambda x.\, x) : \beta$.

$$\dfrac{\dfrac{B \vdash z : (\sigma \cap \rho) \cup (\rho \cap \tau) \qquad \overset{\text{see below}}{\pi'_{01} :: B,\, y : (\sigma \cap \rho) \cup (\rho \cap \tau) \vdash y : \rho}}{\pi'_0 :: B \vdash z : \rho}\,{\scriptstyle(\mathbf{cut})} \qquad \dfrac{\dfrac{B,\, x : \alpha \vdash x : \alpha}{\pi'_1 :: B \vdash \lambda x.\, x : \alpha \rightarrow \alpha}\,{\scriptstyle(\rightarrow\mathbf{R})} \qquad B,\, x : \beta \vdash x : \beta}{B,\, y : \rho \vdash y\,(\lambda x.\, x) : \beta}\,{\scriptstyle(\rightarrow\mathbf{L})}}{\pi' :: B = \{\, z : (\sigma \cap \rho) \cup (\rho \cap \tau)\,\} \vdash z\,(\lambda x.\, x) : \beta}\,{\scriptstyle(\mathbf{cut})}$$

$$\dfrac{\dfrac{B,\, y : \sigma \cap \rho \vdash y : \sigma \cap \rho \qquad \dfrac{B,\, y : \sigma \cap \rho,\, x : \rho \vdash x : \rho}{B,\, y : \sigma \cap \rho,\, x : \sigma \cap \rho \vdash x : \rho}\,{\scriptstyle(\cap\mathbf{L}_2)}}{B,\, y : \sigma \cap \rho \vdash y : \rho}\,{\scriptstyle(\mathbf{cut})} \qquad \dfrac{B,\, y : \rho \cap \tau \vdash y : \rho \cap \tau \qquad \dfrac{B,\, y : \rho \cap \tau,\, x : \rho \vdash x : \rho}{B,\, y : \rho \cap \tau,\, x : \rho \cap \tau \vdash x : \rho}\,{\scriptstyle(\cap\mathbf{L}_1)}}{B,\, y : \rho \cap \tau \vdash y : \rho}\,{\scriptstyle(\mathbf{cut})}}{\pi'_{01} :: B,\, y : (\sigma \cap \rho) \cup (\rho \cap \tau) \vdash y : \rho}\,{\scriptstyle(\cup\mathbf{L})}$$

The premises $B,\, y : \sigma \cap \rho \vdash y : \rho$ *and* $B,\, y : \rho \cap \tau \vdash y : \rho$ *of* $(\cup\mathbf{L})$ *can also be derived from the axiom* $B,\, y : \rho \vdash y : \rho$ *by* $(\cap\mathbf{L}_2)$ *and* $(\cap\mathbf{L}_1)$, *respectively; in fact, this is the easiest way to derive them in sequent calculus. However, we choose to stick to the method of 7.9(ii) in obtaining* π' *from* π. *We observe that it is* $V' = \{z, y, x\} = V$.

The equivalence of the two presentations of IUT$^{\oplus}$ implies that the derivability of renaming, weakening, strengthening, and contraction, shown in Chapter 4 for the natural deduction presentation, must also hold for the sequent calculus presentation. We next elaborate on these derivabilities and explain how the derivability of contraction in sequent calculus differs qualitatively and quantitatively from the derivability of contraction in natural deduction.

Proposition 7.11 *(i) (Renaming) If* $\pi :: B,\, x : \sigma \vdash t : \tau$ *and* y *is fresh with respect to* π, *there exists a* $\pi' :: B,\, y : \sigma \vdash t[y/x] : \tau$, *such that* $V' = (V \setminus \{x\}) \cup \{y\}$ *and* $T' = T$.

(ii) (Weakening) If $\pi :: B \vdash t : \tau$ *and* x *is fresh with respect to* π, *there exists a* $\pi' :: B,\, x : \sigma \vdash t : \tau$, *such that* $V' = V \cup \{x\}$ *and* $T' = T$.

(iii) (Strengthening) If $\pi :: B,\, x : \sigma \vdash t : \tau$ *and* $x \notin FV(t)$, *there exists a* $\pi' :: B \vdash t : \tau$, *such that* $x \notin V' \subsetneq V$ *and* $h' \leqslant h$.

(iv) (Contraction) If $\pi :: B,\, x : \sigma,\, y : \sigma \vdash t : \tau$, *there exists a* $\pi' :: B,\, x : \sigma \vdash t[x/y] : \tau$.

Proof. Throughout the proof, unless otherwise stated, it is $V_0 = V_{\pi_0}$ and $V_1 = V_{\pi_1}$.

(i) By induction on π. We demonstrate three cases of the induction step.

$$\triangleright \quad \frac{\pi_0 :: B,\, z : \upsilon \vdash t : \sigma \qquad \pi_1 :: B,\, z : \upsilon,\, w : \tau \vdash u : \rho}{\pi :: B,\, z : \upsilon,\, x : \sigma \to \tau \vdash u[xt/w] : \rho}\,(\to\mathbf{L})$$

Case 1: rename x to y. We have that $V = V_0 \cup V_1 \cup \{x\}$. Since y is fresh with respect to π, it is also fresh with respect to π_0 and π_1; hence, we can apply $(\to\mathbf{L})$ to π_0 and π_1 with y in place of x to get a $\pi' :: B,\, z : \upsilon,\, y : \sigma \to \tau \vdash u[yt/w] : \rho$. Since $x \notin FV(t) \cup FV(u)$, it is $u[yt/w] = (u[xt/w])[y/x]$. Moreover, it it $V' = V_0 \cup V_1 \cup \{y\} = (V \setminus \{x\}) \cup \{y\}$ and $T' = T$.

Case 2: rename z to y. If $V_{\pi_0} = V_0 \cup \{z\}$ and $V_{\pi_1} = V_1 \cup \{z\}$, then $V = V_0 \cup V_1 \cup \{z, x\}$. The IH gives a $\pi_0' :: B,\, y : \upsilon \vdash t[y/z] : \sigma$, such that $V_0' = V_0 \cup \{y\}$ and $T_0' = T_0$, and a $\pi_1' :: B,\, y : \upsilon,\, w : \tau \vdash u[y/z] : \rho$, such that $V_1' = V_1 \cup \{y\}$ and $T_1' = T_1$. Since $x \notin V_0 \cup V_1$ [by definition of the $(\to\mathbf{L})$ which yields π] and $x \neq y$ [by hypothesis], we have that $x \notin V_0 \cup V_1 \cup \{y\} = V_0' \cup V_1'$ and we can apply an x-introducing $(\to\mathbf{L})$ to π_0' and π_1' to get a $\pi' :: B,\, y : \upsilon,\, x : \sigma \to \tau \vdash (u[y/z])[x(t[y/z])/w] = (u[xt/w])[y/z] : \rho$. It is $V' = V_0' \cup V_1' \cup \{x\} = V_0 \cup V_1 \cup \{y, x\} = (V \setminus \{z\}) \cup \{y\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B,\, z : \upsilon,\, x : \sigma \vdash t : \rho \qquad \pi_1 :: B,\, z : \upsilon,\, x : \tau \vdash t : \rho}{\pi :: B,\, z : \upsilon,\, x : \sigma \cup \tau \vdash t : \rho}\,(\cup\mathbf{L})$$

Case 1: rename x to y. If $V_{\pi_0} = V_0 \cup \{x\}$ and $V_{\pi_1} = V_1 \cup \{x\}$, then $V = V_0 \cup V_1 \cup \{x\}$. The IH gives a $\pi_0' :: B,\, z : \upsilon,\, y : \sigma \vdash t[y/x] : \rho$, such that $V_0' = V_0 \cup \{y\}$ and $T_0' = T_0$, and a $\pi_1' :: B,\, z : \upsilon,\, y : \tau \vdash t[y/x] : \rho$, such that $V_1' = V_1 \cup \{y\}$ and $T_1' = T_1$. By $(\cup\mathbf{L})$, we then obtain a $\pi' :: B,\, z : \upsilon,\, y : \sigma \cup \tau \vdash t[y/x] : \rho$, such that $V' = V_0' \cup V_1' = V_0 \cup V_1 \cup \{y\} = (V \setminus \{x\}) \cup \{y\}$ and $T' = T$.

Case 2: rename z to y. If $V_{\pi_0} = V_0 \cup \{z\}$ and $V_{\pi_1} = V_1 \cup \{z\}$, then $V = V_0 \cup V_1 \cup \{z\}$. The IH gives a $\pi_0' :: B,\, y : \upsilon,\, x : \sigma \vdash t[y/z] : \rho$, such that $V_0' = V_0 \cup \{y\}$ and $T_0' = T_0$, and a $\pi_1' :: B,\, y : \upsilon,\, x : \tau \vdash t[y/z] : \rho$, such that $V_1' = V_1 \cup \{y\}$ and $T_1' = T_1$. By $(\cup\mathbf{L})$, we then get a $\pi' :: B,\, y : \upsilon,\, x : \sigma \cup \tau \vdash t[y/z] : \rho$, such that $V' = V_0' \cup V_1' = V_0 \cup V_1 \cup \{y\} = (V \setminus \{z\}) \cup \{y\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B,\, x : \sigma \vdash t : \tau \qquad \pi_1 :: B,\, x : \sigma,\, z : \tau \vdash u : \rho}{\pi :: B,\, x : \sigma \vdash u[t/z] : \rho}\,(\mathbf{cut})$$

If $V_{\pi_0} = V_0 \cup \{x\}$ and $V_{\pi_1} = V_1 \cup \{x\}$, then $V = V_0 \cup V_1 \cup \{x\}$. The IH gives a $\pi_0' :: B,\, y : \sigma \vdash t[y/x] : \tau$, such that $V_0' = V_0 \cup \{y\}$ and $T_0' = T_0$, and a $\pi_1' :: B,\, y : \sigma,\, z : \tau \vdash u[y/x] : \rho$, such that $V_1' = V_1 \cup \{y\}$ and $T_1' = T_1$. Applying $(\mathbf{cut})$ to π_0' and π_1', we get a $\pi' :: B,\, y : \sigma \vdash (u[y/x])[t[y/x]/z] = (u[t/z])[y/x] : \rho$, such that $V' = V_0' \cup V_1' = V_0 \cup V_1 \cup \{y\} = (V \setminus \{x\}) \cup \{y\}$ and $T' = T$.

(ii) By induction on π. We develop the most notable cases of the induction step.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t : \tau \qquad \pi_1 :: B,\, z : \rho \vdash u : \upsilon}{\pi :: B,\, y : \tau \to \rho \vdash u[yt/z] : \upsilon}\,(\to\mathbf{L})$$

It is $V = V_0 \cup V_1 \cup \{y\}$. The IH provides a $\pi_0' :: B,\, x : \sigma \vdash t : \tau$, such that $V_0' = V_0 \cup \{x\}$ and $T_0' = T_0$, and a $\pi_1' :: B,\, z : \rho,\, x : \sigma \vdash u : \upsilon$, such that $V_1' = V_1 \cup \{x\}$ and $T_1' = T_1$. Since $y \notin V_0 \cup V_1$ [by definition of the $(\to\mathbf{L})$ which yields π] and $y \neq x$ [by hypothesis], we have that $y \notin V_0 \cup V_1 \cup \{x\} = V_0' \cup V_1'$ and we can apply a y-introducing $(\to\mathbf{L})$ to π_0' and π_1' to get a $\pi' :: B,\, x : \sigma,\, y : \tau \to \rho \vdash u[yt/z] : \upsilon$, such that $V' = V_0' \cup V_1' \cup \{y\} = V_0 \cup V_1 \cup \{x, y\} = V \cup \{x\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B,\, y:\tau \vdash t:\upsilon \qquad \pi_1 :: B,\, y:\rho \vdash t:\upsilon}{\pi :: B,\, y:\tau\cup\rho \vdash t:\upsilon}\,(\cup\mathbf{L})$$

It is $V = V_0 \cup V_1$. The IH gives a $\pi'_0 :: B,\, y:\tau,\, x:\sigma \vdash t:\upsilon$, such that $V'_0 = V_0 \cup \{x\}$ and $T'_0 = T_0$, and a $\pi'_1 :: B,\, y:\rho,\, x:\sigma \vdash t:\upsilon$, such that $V'_1 = V_1 \cup \{x\}$ and $T'_1 = T_1$. Applying $(\cup\mathbf{L})$ to π'_0 and π'_1, we get a $\pi' :: B,\, y:\tau\cup\rho,\, x:\sigma \vdash t:\upsilon$, such that $V' = V'_0 \cup V'_1 = V_0 \cup V_1 \cup \{x\} = V \cup \{x\}$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t:\tau \qquad \pi_1 :: B,\, y:\tau \vdash u:\rho}{\pi :: B \vdash u[t/y]:\rho}\,(\mathbf{cut})$$

It is $V = V_0 \cup V_1$. The IH yields a $\pi'_0 :: B,\, x:\sigma \vdash t:\tau$, such that $V'_0 = V_0 \cup \{x\}$ and $T'_0 = T_0$, and a $\pi'_1 :: B,\, y:\tau,\, x:\sigma \vdash u:\rho$, such that $V'_1 = V_1 \cup \{x\}$ and $T'_1 = T_1$. By **(cut)**, we then obtain a $\pi' :: B,\, x:\sigma \vdash u[t/y]:\rho$, such that $V' = V'_0 \cup V'_1 = V_0 \cup V_1 \cup \{x\} = V \cup \{x\}$ and $T' = T$.

(iii) By induction on π. We show three characteristic cases of the induction step.

$$\triangleright \quad \frac{\pi_0 :: B,\, x:\sigma \vdash t:\tau \qquad \pi_1 :: B,\, x:\sigma,\, z:\rho \vdash u:\upsilon}{\pi :: B,\, x:\sigma,\, y:\tau\to\rho \vdash u[yt/z]:\upsilon}\,(\to\!\mathbf{L})$$

Case 1: $y \notin FV(u[yt/z]) \Rightarrow z \notin FV(u) \Rightarrow u[yt/z] = u$. Applying the IH to π_1, we obtain a derivation $\pi' :: B,\, x:\sigma \vdash u[yt/z]:\upsilon$, such that $V' \subsetneq V_1 \subsetneq V_0 \cup V_1 \cup \{y\} = V$ and $h' \leqslant h_1 < h$. Since $y \notin V_1 \supsetneq V'$, we have that $y \notin V' \subsetneq V$.

Case 2: $x \notin FV(u[yt/z])$. We distinguish two subcases.

Subcase 2i: $z \notin FV(u) \Rightarrow u[yt/z] = u$. The IH on π_1 yields a $\pi'_1 :: B,\, x:\sigma \vdash u:\upsilon$, such that $V'_1 \subsetneq V_1$ and $h'_1 \leqslant h_1$. Since $h'_1 \leqslant h_1 < h$ and $x \notin FV(u[yt/z] = u)$, the IH on π'_1 gives a $\pi''_1 :: B \vdash u[yt/z]:\upsilon$, such that $x \notin V''_1 \subsetneq V'_1$ and $h''_1 \leqslant h'_1$. Since $y \notin V_1 \supsetneq V''_1$, we have that $y \notin V''_1$, i.e. that y is fresh with respect to π''_1, so that (ii) gives a $\pi' :: B,\, y:\tau\to\rho \vdash u[yt/z]:\upsilon$, such that $V' = V''_1 \cup \{y\}$ and $T' = T''_1$. It is $x \notin V''_1$ and $x \neq y$, so that $x \notin V' = V''_1 \cup \{y\} \subsetneq V_1 \cup \{y\} \subseteq V_0 \cup V_1 \cup \{y\} = V$. Moreover, since $T' = T''_1$, it is $h' = h''_1 < h$.

Subcase 2ii: $z \in FV(u) \Rightarrow x \notin FV(t)$ and $x \notin FV(u)$. The IH on π_0 gives a $\pi'_0 :: B \vdash t:\tau$, such that $x \notin V'_0 \subsetneq V_0$ and $h'_0 \leqslant h_0$, while the IH on π_1 gives a $\pi'_1 :: B,\, z:\rho \vdash u:\upsilon$, such that $x \notin V'_1 \subsetneq V_1$ and $h'_1 \leqslant h_1$. Since $y \notin V_0 \cup V_1 \supsetneq V'_0 \cup V'_1$, we have that $y \notin V'_0 \cup V'_1$ and we can apply a y-introducing $(\to\!\mathbf{L})$ to π'_0 and π'_1 to get a $\pi' :: B,\, y:\tau\to\rho \vdash u[yt/z]:\upsilon$, such that $x \notin V' = V'_0 \cup V'_1 \cup \{y\} \subsetneq V_0 \cup V_1 \cup \{y\} = V$ and $h' = \max(h'_0, h'_1) + 1 \leqslant \max(h_0, h_1) + 1 = h$.

$$\triangleright \quad \frac{\pi_0 :: B,\, x:\sigma,\, y:\tau \vdash t:\upsilon \qquad \pi_1 :: B,\, x:\sigma,\, y:\rho \vdash t:\upsilon}{\pi :: B,\, x:\sigma,\, y:\tau\cup\rho \vdash t:\upsilon}\,(\cup\mathbf{L})$$

Case 1: $y \notin FV(t)$. The IH on π_0 gives a $\pi' :: B,\, x:\sigma \vdash t:\upsilon$, such that $y \notin V' \subsetneq V_0 \subseteq V_0 \cup V_1 = V$ and $h' \leqslant h_0 < h$.

Case 2: $x \notin FV(t)$. The IH gives a $\pi'_0 :: B,\, y:\tau \vdash t:\upsilon$, such that $x \notin V'_0 \subsetneq V_0$ and $h'_0 \leqslant h_0$, and a $\pi'_1 :: B,\, y:\rho \vdash t:\upsilon$, such that $x \notin V'_1 \subsetneq V_1$ and $h'_1 \leqslant h_1$. By $(\cup\mathbf{L})$, we then get a $\pi' :: B,\, y:\tau\cup\rho \vdash t:\upsilon$, such that $x \notin V' = V'_0 \cup V'_1 \subsetneq V_0 \cup V_1 = V$ and $h' = \max(h'_0, h'_1) + 1 \leqslant \max(h_0, h_1) + 1 = h$.

$$\triangleright \quad \frac{\pi_0 :: B,\, x:\sigma \vdash t:\tau \qquad \pi_1 :: B,\, x:\sigma,\, z:\tau \vdash u:\rho}{\pi :: B,\, x:\sigma \vdash u[t/z]:\rho}\,(\mathbf{cut})$$

If $x \notin FV(u[t/z])$, we distinguish two cases.

Case 1: $z \notin FV(u) \Rightarrow u[t/z] = u$. The IH on π_1 yields a $\pi_1' :: B,\, x : \sigma \vdash u : \rho$, such that $V_1' \subsetneq V_1$ and $h_1' \leqslant h_1$. Since $h_1' \leqslant h_1 < h$ and $x \notin FV(u[t/z] = u)$, the IH on π_1' gives a $\pi' :: B \vdash u[t/z] : \rho$, such that $x \notin V' \subsetneq V_1' \subsetneq V_1 \subseteq V_0 \cup V_1 = V$ and $h' \leqslant h_1' < h$.

Case 2: $z \in FV(u) \Rightarrow x \notin FV(t)$ and $x \notin FV(u)$. The IH yields a $\pi_0' :: B \vdash t : \tau$, such that $x \notin V_0' \subsetneq V_0$ and $h_0' \leqslant h_0$, and a $\pi_1' :: B,\, z : \tau \vdash u : \rho$, such that $x \notin V_1' \subsetneq V_1$ and $h_1' \leqslant h_1$. Applying **(cut)** to π_0' and π_1', we obtain a $\pi' :: B \vdash u[t/z] : \rho$, such that $x \notin V' = V_0' \cup V_1' \subsetneq V_0 \cup V_1 = V$ and $h' = \max(h_0', h_1') + 1 \leqslant \max(h_0, h_1) + 1 = h$.

(iv) We distinguish two cases.

Case 1: $y \notin FV(t) \Rightarrow t[x/y] = t$. Applying (iii) to π, we get a $\pi' :: B,\, x : \sigma \vdash t[x/y] : \tau$, such that $y \notin V' \subsetneq V$ and $h' \leqslant h$.

Case 2: $y \in FV(t)$. In this case, we derive contraction through the cut rule.

$$\dfrac{\dfrac{}{B,\, x : \sigma \vdash x : \sigma}\,\textbf{(ax)} \qquad \pi :: B,\, x : \sigma,\, y : \sigma \vdash t : \tau}{\pi' :: B,\, x : \sigma \vdash t[x/y] : \tau}\,\textbf{(cut)}$$

It is $V' = V_{\text{ax}} \cup V = V$ and $h' = h + 1 > h$. $\dashv$

Remark 7.12 *(i) Contrary to* IUL$_m$*, where contraction is derivable only through an additive cut, contraction is still derivable in case 2 of 7.11(iv), if we consider a multiplicative cut (recall Remark 2.4).*

(ii) The derivability of contraction in sequent calculus differs qualitatively *from the derivability of contraction in natural deduction. This is because, in sequent calculus, we cannot prove it by induction on π, as we do in natural deduction. If we attempt an induction on π in sequent calculus, there are certain subcases of the induction step that cannot proceed, e.g. the following* ($\cup$**L**) *subcase.*

$$\dfrac{\pi_0 :: B,\, x : \sigma_1 \cup \sigma_2,\, y : \sigma_1 \vdash t : \tau \qquad \pi_1 :: B,\, x : \sigma_1 \cup \sigma_2,\, y : \sigma_2 \vdash t : \tau}{\pi :: B,\, x : \sigma_1 \cup \sigma_2,\, y : \sigma_1 \cup \sigma_2 \vdash t : \tau}\,(\cup\textbf{L})$$

This subcase cannot proceed, as we cannot apply the induction hypothesis to the premises, where x and y are not assigned the same type.

(iii) The derivability of contraction in sequent calculus also differs quantitatively *from the derivability of contraction in natural deduction. This is because, in sequent calculus, we cannot prove that $V' = V \backslash \{y\}$ and $T' = T$, as we do in natural deduction. Case 2 of 7.11(iv), where $V' \neq V \setminus \{y\}$ and $T' \neq T$, justifies this claim.*

(iv) As far as renaming, weakening, and strengthening are concerned, the derivability in sequent calculus displays no qualitative or quantitative difference from the derivability in natural deduction.

It is easy to check that, if $B \vdash t : \sigma$ is provable in the sequent calculus IUT$^{\oplus}$, then $FV(t) \subseteq dom(B)$. We can thus show that Proposition 4.16 still holds in the sequent calculus context.

Proposition 7.13 *If $B \vdash t : \sigma$, then $dom(B) \cap BV(t) = \emptyset$, Consequently, since $FV(t) \subseteq dom(B)$, it is $FV(t) \cap BV(t) = \emptyset$.*

Proof. By induction on $B \vdash t : \sigma$. We show the most remarkable cases of the induction step.

$$\triangleright \quad \frac{B \vdash t : \sigma \qquad B, x : \tau \vdash u : \rho}{B, y : \sigma \to \tau \vdash u[yt/x] : \rho}\ (\to\mathbf{L})$$

The IH implies that $dom(B) \cap BV(t) = \emptyset$ and that $dom(B) \cap BV(u) = \emptyset$. Therefore, we get that $dom(B) \cap (BV(u) \cup BV(t)) = \emptyset$. Since $y \notin BV(u) \cup BV(t)$ by definition of the $(\to\mathbf{L})$, we further get that $(dom(B) \cup \{y\}) \cap (BV(u) \cup BV(t)) = \emptyset$. This is the required result, as $BV(u) \cup BV(t) = BV(u[yt/x])$.

$$\triangleright \quad \frac{B \vdash t : \sigma \qquad B, x : \sigma \vdash u : \tau}{B \vdash u[t/x] : \tau}\ (\mathbf{cut})$$

The IH implies that $dom(B) \cap BV(t) = \emptyset$ and that $dom(B) \cap BV(u) = \emptyset$. Therefore, we get that $dom(B) \cap (BV(u) \cup BV(t)) = \emptyset$. This is the required result, as $BV(u) \cup BV(t) = BV(u[t/x])$. $\dashv$

The sequent calculus counterpart of Proposition 4.17 is stated and proved as follows.

Proposition 7.14 *Let π be a derivation in* IUT$^\oplus$*, R be a rule in π, and $B_1, \ldots, B_n$ be the bases in the branch connecting the conclusion of R to the root of π.*
(i) If R is $(\to\mathbf{L})$ *or* $(\mathbf{cut})$ *and x is the variable substituted in the course of R, then $x \notin \bigcup_{i=1}^{n} dom(B_i)$.*
(ii) If R is $(\to\mathbf{R})$ *and x is the variable bounded in the course of R, then $x \notin \bigcup_{i=1}^{n} dom(B_i)$.*

Proof. We use induction on n for both (i) and (ii). We show the $(\to\mathbf{L})$ case, noting that the other two cases are dealt with in a similar manner.

Base: If $n = 1$, the picture is as shown below.

$$\frac{B \vdash t : \sigma \qquad B, x : \tau \vdash u : \rho}{\pi :: B_1 = B \cup \{\, y : \sigma \to \tau \,\} \vdash u[yt/x] : \rho}\ \mathrm{R}{=}(\to\mathbf{L})$$

By the definition of "basis", we have that $x \notin dom(B)$; moreover, by the definition of $(\to\mathbf{L})$, we have that $x \neq y$. Therefore, we get that $x \notin dom(B) \cup \{y\} = dom(B_1)$.

Induction step: We suppose that $x \notin \bigcup_{i=1}^{n} dom(B_i)$ and seek to show that $x \notin \bigcup_{i=1}^{n+1} dom(B_i)$.

If a one-premise rule among $(\to\mathbf{R})$,$(\cap\mathbf{L})$, or $(\cup\mathbf{R})$ intervenes between B_n and B_{n+1} with B_n being the basis of the premise, it is $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$. If a two-premise rule among $(\cap\mathbf{R})$,$(\cup\mathbf{L})$, or $(\mathbf{cut})$ intervenes between B_n and B_{n+1} with B_n being the basis of either the left or the right premise, it is once again $\bigcup_{i=1}^{n+1} dom(B_i) = \bigcup_{i=1}^{n} dom(B_i)$. In all these cases, the result follows from the IH.

We elaborate on the case of an $(\to\mathbf{L})$ between B_n and B_{n+1}. If an $(\to\mathbf{L})$ intervenes between B_n and B_{n+1} with B_n being the basis of the left premise, we have the following picture.

$$\begin{array}{c}
\dfrac{B \vdash t : \sigma \qquad B, x : \tau \vdash u : \rho}{B_1 = B \cup \{\, y : \sigma \to \tau \,\} \vdash u[yt/x] : \rho}\ \mathrm{R}{=}(\to\mathbf{L}) \\
\vdots \\
\dfrac{\pi_0 :: B_n \vdash t' : \upsilon \qquad\qquad \pi_1 :: B_n,\, z : \phi \vdash u' : \psi}{\pi :: B_{n+1} = B_n \cup \{\, w : \upsilon \to \phi \,\} \vdash u'[wt'/z] : \psi}\ (\to\mathbf{L})
\end{array}$$

By the definition of $\mathbf{(\to L)}$, variable w is fresh with respect to π_0 and therefore $w \neq x$. Hence, we have that $x \notin (\bigcup_{i=1}^{n} dom(B_i)) \cup \{w\} = \bigcup_{i=1}^{n+1} dom(B_i)$. [We note that the IH entails that $x \notin dom(B_n)$, so that we may have $z = x$.] If an $\mathbf{(\to L)}$ intervenes between B_n and B_{n+1} with B_n being the basis of the right premise, the picture is reformed as follows.

$$\dfrac{B \vdash t : \sigma \qquad B,\, x : \tau \vdash u : \rho}{B_1 = B \cup \{\, y : \sigma \to \tau \,\} \vdash u[yt/x] : \rho}\ \text{R=}\mathbf{(\to L)}$$

$$\vdots$$

$$\dfrac{\pi_0 :: B' \vdash t' : \upsilon \qquad \pi_1 : B_n = B' \cup \{\, z : \phi \,\} \vdash u' : \psi}{\pi :: B_{n+1} = B' \cup \{\, w : \upsilon \to \phi \,\} \vdash u'[wt'/z] : \psi}\ \mathbf{(\to L)}$$

By the definition of $\mathbf{(\to L)}$, variable w is fresh with respect to π_1 and therefore $w \neq x$. Hence, we have that $x \notin (\bigcup_{i=1}^{n} dom(B_i)) \cup \{w\} = \bigcup_{i=1}^{n+1} dom(B_i)$. [The IH entails that $x \notin dom(B_n) = dom(B') \cup \{z\}$, so that $z \neq x$.] $\dashv$

Remark 7.15 *Propositions 7.13 and 7.14 do not hold in the multiplicative sequent calculus* IUT *of Chapter 2. The following derivation is a counterexample for both.*

$$\dfrac{x : \tau \vdash x : \tau \qquad \dfrac{\dfrac{x : \sigma \vdash x : \sigma}{\emptyset \vdash \lambda x.x : \sigma \to \sigma}\ \mathbf{(\to R)} \qquad \dfrac{x : \sigma \vdash x : \sigma \qquad x : \sigma \vdash x : \sigma}{B_1 = \{\, x : \sigma,\, y : \sigma \to \sigma \,\} \vdash yx : \sigma}\ \mathbf{(\to L)}_1}{B_2 = \{\, x : \sigma \,\} \vdash (\lambda x.x)x : \sigma}\ \mathbf{(cut)}}{\pi :: B = B_3 = \{\, x : \tau,\, z : \tau \to \sigma \,\} \vdash_{\mathrm{IUT}} t = (\lambda x.x)(zx) : \sigma}\ \mathbf{(\to L)}_2$$

Proposition 7.13 is contradicted, as it is $dom(B) \cap BV(t) = FV(t) \cap BV(t) = \{x, z\} \cap \{x\} \neq \emptyset$. *Proposition 7.14 is contradicted in two instances: i) the variable substituted in the course of* $\mathbf{(\to L)}_1$*, namely* x*, belongs to* $\bigcap_{i=1}^{3} dom(B_i) \subseteq \bigcup_{i=1}^{3} dom(B_i)$*, and ii) the variable substituted in the course of* $\mathbf{(\to L)}_2$*, which is* x *again, belongs to* $dom(B)$*.*

The *additive* sequent calculus $\mathrm{IUT}^{\oplus}$ is equivalent to the *multiplicative* sequent calculus IUT, as the next theorem shows.

Theorem 7.16 *(i) If* $\pi :: B \vdash t : \sigma$ *in* $\mathrm{IUT}^{\oplus}$*, there exists a* $\pi' :: B \vdash t : \sigma$ *in* IUT*, such that* $V' = V$ *and* $T' = T$*.*

(ii) If $\pi :: B \vdash t : \sigma$ *in* IUT *and* $x_1, \ldots, x_n \notin V$*, there exists a* $\pi' :: B \vdash t' =_\alpha t : \sigma$ *in* $\mathrm{IUT}^{\oplus}$*, such that* $x_1, \ldots, x_n \notin V' \supseteq V$ *and* $T' = T$*.*

Proof. (i) By induction on the $\mathrm{IUT}^{\oplus}$-derivation π.

Base: Since an $\mathrm{IUT}^{\oplus}$-axiom is also an IUT-axiom, if π is an axiom, then $\pi' = \pi$.

Induction step: We show two representative cases.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t : \sigma \qquad \pi_1 :: B,\, x : \tau \vdash u : \rho}{\pi :: B,\, y : \sigma \to \tau \vdash u[yt/x] : \rho}\ (\to\mathbf{L})$$

The IH gives a $\pi_0' :: B \vdash t : \sigma$ in IUT, such that $V_0' = V_0$ and $T_0' = T_0$, and also a $\pi_1' :: B,\, x : \tau \vdash u : \rho$ in IUT, such that $V_1' = V_1$ and $T_1' = T_1$. Since $y \notin V_0 \cup V_1 = V_0' \cup V_1'$, we can apply a y-introducing, multiplicative $(\to\mathbf{L})$ to π_0' and π_1' to get a $\pi' :: B,\, y : \sigma \to \tau \vdash u[yt/x] : \rho$ in IUT, s.t. $V' = V_0' \cup V_1' \cup \{y\} = V_0 \cup V_1 \cup \{y\} = V$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t : \sigma \qquad \pi_1 :: B,\, x : \sigma \vdash u : \tau}{\pi :: B \vdash u[t/x] : \tau}\ (\mathbf{cut})$$

The IH yields a $\pi_0' :: B \vdash t : \sigma$ in IUT, such that $V_0' = V_0$ and $T_0' = T_0$, and also a $\pi_1' :: B,\, x : \sigma \vdash u : \tau$ in IUT, such that $V_1' = V_1$ and $T_1' = T_1$. Applying a multiplicative **(cut)** to π_0' and π_1', we obtain a $\pi' :: B \vdash u[t/x] : \tau$ in IUT, such that $V' = V_0' \cup V_1' = V_0 \cup V_1 = V$ and $T' = T$.

(ii) By induction on the IUT-derivation π.

Base: Since an IUT-axiom is also an IUT$^{\oplus}$-axiom, if π is an axiom, then $\pi' = \pi$.

Induction step: We elaborate on two characteristic cases, assuming that $dom(B) \cap dom(B') = \emptyset$.

$$\triangleright \quad \frac{\pi_0 :: B \vdash t : \sigma \qquad \pi_1 :: B',\, z : \tau \vdash u : \rho}{\pi :: B,\, B',\, y : \sigma \to \tau \vdash u[yt/z] : \rho}\ (\to\mathbf{L})$$

We suppose that $x_1, \ldots, x_n \notin V = V_0 \cup V_1 \cup \{y\}$, so that $x_1, \ldots, x_n \notin V_0 \cup V_1$ and $y \neq x_1, \ldots, x_n$. Since $y \notin V_0 \cup V_1$ [by definition of the $(\to\mathbf{L})$], we have that $x_1, \ldots, x_n, y \notin V_0$ and $x_1, \ldots, x_n, y \notin V_1$. The IH gives a $\pi_0' :: B \vdash t' =_\alpha t : \sigma$ in IUT$^{\oplus}$, such that $x_1, \ldots, x_n, y \notin V_0' \supseteq V_0$ and $T_0' = T_0$, and a $\pi_1' :: B',\, z : \tau \vdash u' =_\alpha u : \rho$ in IUT$^{\oplus}$, such that $x_1, \ldots, x_n, y \notin V_1' \supseteq V_1$ and $T_1' = T_1$. If

$$V_0' \cap dom(B') = S_0'$$

we rename the set[2] S_0' in π_0' to a fresh-with-respect-to-$(V_0' \cup dom(B') \cup \{x_1, \ldots, x_n, y\})$ set to attain a $\pi_0^2 :: B \vdash t'' =_\alpha t' : \sigma$, such that the sets $V_0^2, dom(B')$, and $\{x_1, \ldots, x_n, y\}$ are pairwise disjoint and $T_0^2 = T_0$. Successive applications of weakening to π_0^2 by elements in B' provide a $\pi_0^3 :: B,\, B' \vdash t'' =_\alpha t : \sigma$, such that $x_1, \ldots, x_n, y \notin V_0^3 = V_0^2 \cup dom(B') \not\supseteq V_0' \cup dom(B')$ and $T_0^3 = T_0$. If

$$V_1' \cap dom(B) = S_1' \ni z$$

we rename the set[3] S_1' in π_1' to a fresh-with-respect-to-$(V_1' \cup dom(B) \cup \{x_1, \ldots, x_n, y\})$ set to attain a $\pi_1^2 :: B',\, w : \tau \vdash u'' =_\alpha u'[w/z] : \rho$, such that the sets $V_1^2, dom(B)$, and $\{x_1, \ldots, x_n, y\}$ are pairwise disjoint and $T_1^2 = T_1$. Weakening π_1^2 by elements in B, we get a $\pi_1^3 :: B,\, B',\, w : \tau \vdash u'' =_\alpha u[w/z] : \rho$, such that $x_1, \ldots, x_n, y \notin V_1^3 = V_1^2 \cup dom(B) \not\supseteq V_1' \cup dom(B)$ and $T_1^3 = T_1$. Since $y \notin V_0^3 \cup V_1^3$, we can apply a y-introducing, additive $(\to\mathbf{L})$ to π_0^3 and π_1^3 to obtain a

$$\pi' :: B,\, B',\, y : \sigma \to \tau \vdash u''[yt''/w] =_\alpha (u[w/z])[yt/w] = u[yt/z] : \rho$$

---

[2] Since $dom(B) \cap dom(B') = \emptyset$, a variable of $dom(B')$ which is in V_0' may appear bound in t' or elsewhere in the body of π_0', where the "body" of a derivation consists of all sequents in the derivation besides the conclusion.

[3] Since $dom(B) \cap dom(B') = \emptyset$, a variable of $dom(B)$ which is in V_1' may appear either (in the place of z) or (bound in u' or elsewhere in the body of π_1').

in $\mathrm{IUT}^{\oplus}$, where the term-equality $(u[w/z])[yt/w] = u[yt/z]$ is justified by the fact that $w \notin V(u')$, which implies that $w \notin FV(u)$. It is $x_1, \ldots, x_n \notin V' = V_0^3 \cup V_1^3 \cup \{y\} \supsetneq (V_0' \cup dom(B')) \cup (V_1' \cup dom(B)) \cup \{y\} = V_0' \cup V_1' \cup \{y\} \supseteq V_0 \cup V_1 \cup \{y\} = V$ and $T' = T$.

$$\triangleright \quad \frac{\pi_0 :: B,\, x : \sigma \vdash t : \rho \qquad \pi_1 :: B',\, x : \tau \vdash t : \rho}{\pi :: B,\, B',\, x : \sigma \cup \tau \vdash t : \rho}\;(\cup\mathbf{L})$$

We suppose that $x_1, \ldots, x_n \notin V = V_0 \cup V_1$. The IH yields a $\pi_0' :: B,\, x : \sigma \vdash t_0' =_\alpha t : \rho$ in $\mathrm{IUT}^{\oplus}$, such that $x_1, \ldots, x_n \notin V_0' \supseteq V_0$ and $T_0' = T_0$, and a $\pi_1' :: B',\, x : \tau \vdash t_1' =_\alpha t : \rho$ in $\mathrm{IUT}^{\oplus}$, such that $x_1, \ldots, x_n \notin V_1' \supseteq V_1$ and $T_1' = T_1$. We can actually have $t_0' = t_1' = t'$ (see Example 7.17 below), so we assume that $\pi_0' :: B,\, x : \sigma \vdash t' =_\alpha t : \rho$ and $\pi_1' :: B',\, x : \tau \vdash t' =_\alpha t : \rho$. If $V_0' \cap dom(B') = S_0'$, we rename the set[4] S_0' in π_0' to a fresh-with-respect-to-$(V_0' \cup dom(B') \cup \{x_1, \ldots, x_n\})$ set to attain a $\pi_0^2 :: B,\, x : \sigma \vdash t' =_\alpha t : \rho$, such that the sets $V_0^2, dom(B')$, and $\{x_1, \ldots, x_n\}$ are pairwise disjoint and $T_0^2 = T_0$. Weakening π_0^2 by B', we get a $\pi_0^3 :: B,\, B',\, x : \sigma \vdash t' =_\alpha t : \rho$, such that $x_1, \ldots, x_n \notin V_0^3 = V_0^2 \cup dom(B') \supsetneq V_0' \cup dom(B')$ and $T_0^3 = T_0$. If $V_1' \cap dom(B) = S_1'$, we rename the set S_1' in π_1' to a fresh-with-respect-to-$(V_1' \cup dom(B) \cup \{x_1, \ldots, x_n\})$ set to attain a $\pi_1^2 :: B',\, x : \tau \vdash t' =_\alpha t : \rho$, such that $V_1^2, dom(B)$, and $\{x_1, \ldots, x_n\}$ are pairwise disjoint and $T_1^2 = T_1$. Weakening π_1^2 by elements in B, we obtain a $\pi_1^3 :: B,\, B',\, x : \tau \vdash t' =_\alpha t : \rho$, such that $x_1, \ldots, x_n \notin V_1^3 = V_1^2 \cup dom(B) \supsetneq V_1' \cup dom(B)$ and $T_1^3 = T_1$. Applying an additive **(∪L)** to π_0^3 and π_1^3, we then obtain a $\pi' :: B,\, B',\, x : \sigma \cup \tau \vdash t' =_\alpha t : \rho$ in $\mathrm{IUT}^{\oplus}$, such that $x_1, \ldots, x_n \notin V' = V_0^3 \cup V_1^3 \supsetneq (V_0' \cup dom(B')) \cup (V_1' \cup dom(B)) = V_0' \cup V_1' \supseteq V_0 \cup V_1 = V$ and $T' = T$. ⊣

The next example illustrates the transition from the multiplicative IUT to the additive $\mathrm{IUT}^{\oplus}$ in sequent calculus.

Example 7.17 *Let* $\phi = (\sigma \to \sigma) \to \alpha,\ \psi = (\tau \to \tau) \to \alpha$ *and consider*

$$\pi ::\ x : \phi \cup \psi,\ y : \alpha \to \beta \vdash t = y\,(x\,(\lambda y.\, y)) : \beta$$

in IUT*, as shown below.*

$$\frac{\overset{\text{see below}}{\pi_0 ::\ x : \phi,\ y : \alpha \to \beta \vdash y\,(x\,(\lambda y.\, y)) : \beta} \qquad \overset{\text{see below}}{\pi_1 ::\ x : \psi,\ y : \alpha \to \beta \vdash y\,(x\,(\lambda y.\, y)) : \beta}}{\pi ::\ x : \phi \cup \psi,\ y : \alpha \to \beta \vdash_{\mathrm{IUT}} t = y\,(x\,(\lambda y.\, y)) : \beta}\;(\cup\mathbf{L})$$

$$\frac{x : \phi \vdash x : \phi \qquad \dfrac{\dfrac{y : \sigma \vdash y : \sigma}{\pi_{010} ::\ \emptyset \vdash \lambda y.\, y : \sigma \to \sigma}\;(\to\mathbf{R}) \qquad \dfrac{x : \alpha \vdash x : \alpha \qquad x : \beta \vdash x : \beta}{\pi_{011} ::\ x : \alpha,\ y : \alpha \to \beta \vdash yx : \beta \quad [z \notin V_{011}]}\;(\to\mathbf{L})}{\pi_{01} ::\ y : \alpha \to \beta,\ z : \phi \vdash y\,(z\,(\lambda y.\, y)) : \beta}\;(\to\mathbf{L})}{\pi_0 ::\ x : \phi,\ y : \alpha \to \beta \vdash_{\mathrm{IUT}} y\,(x\,(\lambda y.\, y)) : \beta}\;(\mathbf{cut})$$

---

[4] Since $dom(B') \cap (dom(B) \cup \{x\} \cup BV(t')) = \emptyset$, a variable of $dom(B')$ which is in V_0' may only appear in the body of π_0'. A similar note holds for a variable of $dom(B)$ which is in V_1'.

$$\dfrac{y:\alpha\to\beta\vdash y:\alpha\to\beta \qquad \dfrac{\dfrac{\dfrac{y:\tau\vdash y:\tau}{\emptyset\vdash\lambda y.\,y:\tau\to\tau}\,\mathbf{(\to R)} \qquad y:\alpha\vdash y:\alpha}{\pi_{110} :: x:\psi\vdash x\,(\lambda y.\,y):\alpha}\,\mathbf{(\to L)} \qquad x:\beta\vdash x:\beta}{\pi_{11} :: x:\psi,\, z:\alpha\to\beta\vdash z\,(x\,(\lambda y.\,y)):\beta}\,\mathbf{(\to L)}}{\pi_1 :: x:\psi,\, y:\alpha\to\beta\vdash_{\mathrm{IUT}} y\,(x\,(\lambda y.\,y)):\beta}\,\mathbf{(cut)}$$

To transform π *to a* $\pi' :: x:\phi\cup\psi,\, y:\alpha\to\beta\vdash t' =_\alpha t:\beta$ *in* $\mathrm{IUT}^{\oplus}$*, we need to transform* π_0 *to a* $\pi'_0 :: x:\phi,\, y:\alpha\to\beta\vdash t'_0 =_\alpha t:\beta$ *in* $\mathrm{IUT}^{\oplus}$ *and* π_1 *to a* $\pi'_1 :: x:\psi,\, y:\alpha\to\beta\vdash t'_1 =_\alpha t:\beta$ *in* $\mathrm{IUT}^{\oplus}$*, so that* $t'_0 = t'_1 = t'$*. The transformation of* π_0 *to* π'_0 *proceeds top-down as follows. We first transform* π_{011} *to a* $\pi'_{011} :: x:\alpha,\, y:\alpha\to\beta\vdash yx:\beta$ *in* $\mathrm{IUT}^{\oplus}$*, such that* $z\notin V'_{011}$*. To do this, we need to rename* x *in* $x:\beta\vdash x:\beta$ *to a fresh-wrt-*$\{x,z,y\}$ *variable* w *and weaken by* $x:\alpha$*.*

$$\dfrac{x:\alpha\vdash x:\alpha \qquad x:\alpha,\, w:\beta\vdash w:\beta}{\pi'_{011} :: x:\alpha,\, y:\alpha\to\beta\vdash yx:\beta \quad [z\notin V'_{011}]}\,\mathbf{(\to L)}^{\oplus}$$

We then transform π_{01} *to a* $\pi'_{01} :: y:\alpha\to\beta,\, z:\phi\vdash t'_{01} =_\alpha y\,(z\,(\lambda y.\,y)):\beta$ *in* $\mathrm{IUT}^{\oplus}$*. To do this, we need to rename* y *in* π_{010} *to a fresh-wrt-*$\{y,z\}$ *variable* x *and weaken by* $y:\alpha\to\beta$*.*

$$\dfrac{\dfrac{y:\alpha\to\beta,\, x:\sigma\vdash x:\sigma}{y:\alpha\to\beta\vdash\lambda x.\,x:\sigma\to\sigma}\,\mathbf{(\to R)} \qquad \pi'_{011} :: x:\alpha,\, y:\alpha\to\beta\vdash yx:\beta}{\pi'_{01} :: y:\alpha\to\beta,\, z:\phi\vdash y\,(z\,(\lambda x.\,x)):\beta}\,\mathbf{(\to L)}^{\oplus}$$

To attain π'_0*, we further need to rename* x *in* π'_{01} *to a fresh-wrt-*$\{x,y,z,w\}$ *variable* v *and weaken by* $x:\phi$ *and also to weaken* $x:\phi\vdash x:\phi$ *by* $y:\alpha\to\beta$*.*

$$\dfrac{x:\phi,\, y:\alpha\to\beta\vdash x:\phi \qquad \dfrac{\dfrac{x:\phi,\, y:\alpha\to\beta,\, v:\sigma\vdash v:\sigma}{x:\phi,\, y:\alpha\to\beta\vdash\lambda v.\,v:\sigma\to\sigma}\,\mathbf{(\to R)} \qquad \dfrac{x:\phi,\, v:\alpha\vdash v:\alpha \qquad x:\phi,\, v:\alpha,\, w:\beta\vdash w:\beta}{x:\phi,\, y:\alpha\to\beta,\, v:\alpha\vdash yv:\beta}\,\mathbf{(\to L)}}{x:\phi,\, y:\alpha\to\beta,\, z:\phi\vdash y\,(z\,(\lambda v.\,v)):\beta}\,\mathbf{(\to L)}}{\pi'_0 :: x:\phi,\, y:\alpha\to\beta\vdash_{\mathrm{IUT}^{\oplus}} t' = y\,(x\,(\lambda v.\,v)) =_\alpha t:\beta}\,\mathbf{(cut)}^{\oplus}$$

To top-down transform π_1 *to* π'_1*, we observe that* π_{110} *is already in* $\mathrm{IUT}^{\oplus}$ *and we proceed to transform* π_{11} *to a* $\pi'_{11} :: x:\psi,\, z:\alpha\to\beta\vdash t'_{11} =_\alpha z\,(x\,(\lambda y.\,y)):\beta$ *in* $\mathrm{IUT}^{\oplus}$*. To do this, we need to rename* x *in* $x:\beta\vdash x:\beta$ *to a fresh-wrt-*$\{x,z\}$ *variable* y *and weaken by* $x:\psi$*.*

$$\dfrac{\pi_{110} :: x:\psi\vdash x\,(\lambda y.\,y):\alpha \qquad x:\psi,\, y:\beta\vdash y:\beta}{\pi'_{11} :: x:\psi,\, z:\alpha\to\beta\vdash z\,(x\,(\lambda y.\,y)):\beta}\,\mathbf{(\to L)}^{\oplus}$$

To attain π'_1*, we then need to rename* y *in* π'_{11} *to a fresh-wrt-*$\{x,y,z\}$ *variable* v *and weaken by* $y:\alpha\to\beta$ *and also to weaken* $y:\alpha\to\beta\vdash y:\alpha\to\beta$ *by* $x:\psi$*.*

$$\dfrac{x : \psi,\, y : \alpha \to \beta \vdash y : \alpha \to \beta \qquad \dfrac{\dfrac{\dfrac{y : \alpha \to \beta,\, v : \tau \vdash v : \tau}{y : \alpha \to \beta \vdash \lambda v.\, v : \tau \to \tau}\,\scriptstyle(\to\mathbf{R}) \qquad y : \alpha \to \beta,\, v : \alpha \vdash v : \alpha}{y : \alpha \to \beta,\, x : \psi \vdash x\,(\lambda v.\, v) : \alpha}\,\scriptstyle(\to\mathbf{L}) \qquad y : \alpha \to \beta,\, x : \psi,\, v : \beta \vdash v : \beta}{x : \psi,\, y : \alpha \to \beta,\, z : \alpha \to \beta \vdash z\,(x\,(\lambda v.\, v)) : \beta}\,\scriptstyle(\to\mathbf{L})}{\pi'_1 :: x : \psi,\, y : \alpha \to \beta \vdash_{\mathrm{IUT}^{\oplus}} t' = y\,(x\,(\lambda v.\, v)) =_{\alpha} t : \beta}\,\scriptstyle(\mathbf{cut})^{\oplus}$$

We finally obtain π' by applying an additive $(\cup\mathbf{L})$ *to π'_0 and π'_1.*

$$\dfrac{\pi'_0 :: x : \phi,\, y : \alpha \to \beta \vdash t' : \beta \qquad \pi'_1 :: x : \psi,\, y : \alpha \to \beta \vdash t' : \beta}{\pi' :: x : \phi \cup \psi,\, y : \alpha \to \beta \vdash_{\mathrm{IUT}^{\oplus}} t' =_{\alpha} t : \beta}\,(\cup\mathbf{L})^{\oplus}$$

It is $V' = \{x, y, z, w, v\} \supsetneq \{x, y, z\} = V$ and $T' = T$. In transforming π_0 and π_1 to π'_0 and π'_1, respectively, we choose the new names (new variables), so that we have i) the least possible number of new variables in V' and ii) $t'_0 = t'_1 = t'$.

Combining Theorems 7.16 and 7.9, we see that the three different presentations of the type system with intersection and union types are equivalent. We abbreviate "nd" and "sc" the natural deduction style and the sequent calculus style, respectively.

$$\mathrm{nd\,IUT}^{\oplus} \overset{7.9}{\Longleftrightarrow} \mathrm{sc\,IUT}^{\oplus} \overset{7.16}{\Longleftrightarrow} \mathrm{sc\,IUT}$$

The sequent calculus $\mathrm{IUT}^{\oplus}$ does *not* enjoy cut elimination, at least not a total cut elimination, as it does not contain an explicit contraction rule. Remark 2.22 for the sequent calculus IUT holds for the sequent calculus $\mathrm{IUT}^{\oplus}$, as well, if modified appropriately.

7.3 Relating IUL_m to $\mathrm{IUT}^{\oplus}$ in sequent calculus

As in the natural deduction case, the sequent calculus logic IUL_m is intended to capture the sequent calculus type system $\mathrm{IUT}^{\oplus}$ on a logical level. In order to elaborate on how the logic attempts to accomplish this goal, we need the notions of *non-standard decoration* of the logic and of *term-sequent* of a sequent.

A decoration of the logic dictated by the very rules of the type system encodes the implication, but does not embody the intersection or the union; it is therefore a "non-standard" decoration. Its formal definition is once more along the line given in 3.15 and its rules are displayed in Figure 7.3. When decorating contexts bottom-up, the new variable in an $(\to\mathbf{L})$ right premise or an $(\to\mathbf{R})$ premise or a **(cut)** right premise is fresh with respect to the variables in the branch connecting the conclusion to the root. The term-sequent of a given sequent derives from the given sequent exactly as the term-statement of a given statement derives from the given statement in natural deduction (recall Definition 4.18).

For a decoration dictated by the type system to be possible, which is essential in examining a correspondence between the logic and the type system, the logic needs to have a single-premise $(\cap\mathbf{R})$ and a single-premise $(\cup\mathbf{L})$. This is achieved by the molecule structure, which joins together in the same (decorated) molecule sequents that share the same term-sequent[5]. The right intersection case coincides with

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[5] As in the natural deduction case, this should only be kept in mind as a wishful intention. It can be shown in the sequent calculus context, as well, that *not every* set of (derivations proving) sequents sharing the same term-sequent can be joined into a single (derivation proving a) decorated molecule.

$$\frac{}{x : [(\Gamma_i, \sigma_i \,;\, \sigma_i)_i]_{p,\,x}}\,(\mathbf{ax}) \qquad \frac{t : [(\Gamma_i, \sigma_i, \tau_i, \Delta_i \,;\, \rho_i)_i]_{p,\,y,\,x,\,q}}{t : [(\Gamma_i, \tau_i, \sigma_i, \Delta_i \,;\, \rho_i)_i]_{p,\,x,\,y,\,q}}\,(\mathbf{X})$$

$$\frac{t : [(\Gamma_i \,;\, \sigma_i)_i]_p \qquad u : [(\Gamma_i, \tau_i \,;\, \rho_i)_i]_{p,\,x}}{u[yt/x] : [(\Gamma_i, \sigma_i \to \tau_i \,;\, \rho_i)_i]_{p,\,y}}\,(\to\!\mathbf{L}) \qquad \frac{t : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}{\lambda x.\,t : [(\Gamma_i \,;\, \sigma_i \to \tau_i)_i]_p}\,(\to\!\mathbf{R})$$

$$\frac{t : [\mathcal{U}, (\Gamma_i, \sigma_i \,;\, \rho_i)_i, \mathcal{V}\,]_{p,\,x}}{t : [\mathcal{U}, (\Gamma_i, \sigma_i \cap \tau_i \,;\, \rho_i)_i, \mathcal{V}\,]_{p,\,x}}\,(\cap\mathbf{L}_1) \qquad \frac{t : [\mathcal{U}, (\Gamma_i, \tau_i \,;\, \rho_i)_i, \mathcal{V}\,]_{p,\,x}}{t : [\mathcal{U}, (\Gamma_i, \sigma_i \cap \tau_i \,;\, \rho_i)_i, \mathcal{V}\,]_{p,\,x}}\,(\cap\mathbf{L}_2)$$

$$\frac{t : [\mathcal{U}, ((\Gamma_i \,;\, \sigma_i), (\Gamma_i \,;\, \tau_i))_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cap \tau_i)_i, \mathcal{V}\,]_p}\,(\cap\mathbf{R}) \qquad \frac{t : [\mathcal{U}, ((\Gamma_i, \sigma_i \,;\, \rho_i), (\Gamma_i, \tau_i \,;\, \rho_i))_i, \mathcal{V}\,]_{p,\,x}}{t : [\mathcal{U}, (\Gamma_i, \sigma_i \cup \tau_i \,;\, \rho_i)_i, \mathcal{V}\,]_{p,\,x}}\,(\cup\mathbf{L})$$

$$\frac{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i)_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}\,]_p}\,(\cup\mathbf{R}_1) \qquad \frac{t : [\mathcal{U}, (\Gamma_i \,;\, \tau_i)_i, \mathcal{V}\,]_p}{t : [\mathcal{U}, (\Gamma_i \,;\, \sigma_i \cup \tau_i)_i, \mathcal{V}\,]_p}\,(\cup\mathbf{R}_2)$$

$$\frac{t : [(\Gamma_i \,;\, \sigma_i)_i]_p \qquad u : [(\Gamma_i, \sigma_i \,;\, \tau_i)_i]_{p,\,x}}{u[t/x] : [(\Gamma_i \,;\, \tau_i)_i]_p}\,(\mathbf{cut})$$

Figure 7.3: Non-standard decoration of sequent calculus IUL$_m$.

the intersection introduction case in natural deduction. In the case of left union, the (decorated) logic merges into the same (decorated) molecule the left and right IUT$^\oplus$-premises, in parallel for multiple rule instances that share the same term-sequent[6].

$$\begin{array}{c}
\dfrac{x_1 : \sigma_1^1, \ldots,\, x_m : \sigma_m^1,\, x : \tau_1 \vdash t : \upsilon_1 \qquad x_1 : \sigma_1^1, \ldots,\, x_m : \sigma_m^1,\, x : \rho_1 \vdash t : \upsilon_1}{x_1 : \sigma_1^1, \ldots,\, x_m : \sigma_m^1,\, x : \tau_1 \cup \rho_1 \vdash t : \upsilon_1}\,(\cup\mathbf{L})_1 \\
\vdots \\
\dfrac{x_1 : \sigma_1^n, \ldots,\, x_m : \sigma_m^n,\, x : \tau_n \vdash t : \upsilon_n \qquad x_1 : \sigma_1^n, \ldots,\, x_m : \sigma_m^n,\, x : \rho_n \vdash t : \upsilon_n}{x_1 : \sigma_1^n, \ldots,\, x_m : \sigma_m^n,\, x : \tau_n \cup \rho_n \vdash t : \upsilon_n}\,(\cup\mathbf{L})_n
\end{array} \qquad \rightsquigarrow$$

$$\frac{t : [\mathcal{U}, (\sigma_1^1, \ldots, \sigma_m^1, \tau_1 \,;\, \upsilon_1), (\sigma_1^1, \ldots, \sigma_m^1, \rho_1 \,;\, \upsilon_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n, \tau_n \,;\, \upsilon_n), (\sigma_1^n, \ldots, \sigma_m^n, \rho_n \,;\, \upsilon_n), \mathcal{V}\,]_{x_1, \ldots,\, x_m,\, x}}{t : [\mathcal{U}, (\sigma_1^1, \ldots, \sigma_m^1, \tau_1 \cup \rho_1 \,;\, \upsilon_1), \ldots, (\sigma_1^n, \ldots, \sigma_m^n, \tau_n \cup \rho_n \,;\, \upsilon_n), \mathcal{V}\,]_{x_1, \ldots,\, x_m,\, x}}\,(\cup\mathbf{L})$$

It should be obvious by now that the sequent calculus presentation of the logic and the type system is susceptible to remarks, concerning the relation of the two systems, which are completely analogous to

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[6] The term-sequent of a ($\cup$**L**) instance with premises $B,\, x : \tau \vdash t : \upsilon$, $B,\, x : \rho \vdash t : \upsilon$ and conclusion $B,\, x : \tau \cup \rho \vdash t : \upsilon$, where $dom(B) = \{x_1, \ldots, x_m\}$ is meant to be $x_1, \ldots, x_m, x \vdash t$.

the ones given for the natural deduction presentation. Taking this argument further, we expect that a sequent calculus notion analogous to the natural deduction notion of tree $T^{\,t}_{\mathrm{iue}}$ assists the sequent calculus IUL_m-$\mathrm{IUT}^{\oplus}$ correspondence.

In natural deduction, we stated and proved correspondence theorems between IUL_m and $\mathrm{IUT}^{\oplus}$, using the restrictive notion of trees $T^{\,t}_{\mathrm{iue}}$. Looking at the logic, the implications and the union elimination are the global rules which have a counterpart in the type system. In sequent calculus, the global rules which have a counterpart in the type system are the implications and the cut. Defining *trees of implications and cuts with terms*, denoted $T^{\,t}_{\mathrm{ic}}$, for both the decorated logic $\mathrm{IUL}^{\star}_m$ and the type system $\mathrm{IUT}^{\oplus}$, we can state and prove restricted correspondence theorems in sequent calculus, as well. We outline the basic points below.

Definition 7.18 ($\mathrm{IUL}^{\star}_m$: $T^{\,t}$ and $T^{\,t}_{\mathrm{ic}}$) *(i) Given a decorated molecule $t : \mathcal{M}_p$ in $\mathrm{IUL}^{\star}_m$, the* decoration-sequent *deriving from it is the sequent $\{p\} \vdash t$, abbreviated $p \vdash t$.*

(ii) Given the tree T of a derivation $\pi^{\star}$ in $\mathrm{IUL}^{\star}_m$, the tree with terms *$T^{\,t}$ of $\pi^{\star}$ is T with each node decorated by the decoration-sequent deriving from the decorated molecule that corresponds to it.*

(iii) Given the tree $T^{\,t}$ of a derivation $\pi^{\star}$ in $\mathrm{IUL}^{\star}_m$, we derive the tree of implications and cuts with terms *$T^{\,t}_{\mathrm{ic}}$ of $\pi^{\star}$ from it by erasing all nodes and corresponding decoration-sequents associated to the rules* **(X)**,**(∩LR)**, *and* **(∪LR)**.

Definition 7.19 ($\mathrm{IUT}^{\oplus}$: $T^{\,t}$ and $T^{\,t}_{\mathrm{ic}}$) *(i) Given the tree T of a derivation π in $\mathrm{IUT}^{\oplus}$, the* tree with terms *$T^{\,t}$ of π is T with each node decorated by the term-sequent deriving from the sequent that corresponds to it.*

(ii) Given the tree $T^{\,t}$ of a derivation π in $\mathrm{IUT}^{\oplus}$, we derive the tree of implications and cuts with terms *$T^{\,t}_{\mathrm{ic}}$ of π from it by the following algorithm.*

▹ *We choose a topmost* **(∩R)** *or* **(∪L)** *in the tree with terms of π and erase all nodes and corresponding term-sequents associated to* **(∩L)** *or* **(∪R)** *in the trees with terms of the premises. If the resulting premise trees of implications and cuts with terms are identical, we identify them and erase the node and corresponding term-sequent associated to the* **(∩R)** *or* **(∪L)**.

▹ *We iterate the above procedure for the tree with terms resulting from the previous step.*

▹ *When all the* **(∩R)***'s and* **(∪L)***'s have been dealt with, we make a final step to erase any remaining nodes and corresponding term-sequents associated to* **(∩L)** *or* **(∪R)**.

As in the natural deduction case, the algorithm in 7.19(ii) does not always terminate.

Theorem 7.20 (From IUL_m to $\mathrm{IUT}^{\oplus}$) *If $\pi^{\star} :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)\,{}^n_{i=1}]_{\,x_1,\ldots,x_m}$ is a decorated derivation in IUL_m, there are derivations $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ in $\mathrm{IUT}^{\oplus}$, such that 1. $(T^{\,t}_{\mathrm{ic}})_i$ exists, 2. $(T^{\,t}_{\mathrm{ic}})_i = (T^{\,t}_{\mathrm{ic}})_j$ $(1 \leqslant i \neq j \leqslant n)$, and 3. $(T^{\,t}_{\mathrm{ic}})_i = (T^{\,t}_{\mathrm{ic}})_{\pi^{\star}}$.*

Theorem 7.21 (From $\mathrm{IUT}^{\oplus}$ to IUL_m) *If $\pi_i :: x_1 : \sigma^i_1, \ldots, x_m : \sigma^i_m \vdash t : \tau_i$ $(1 \leqslant i \leqslant n)$ are derivations in $\mathrm{IUT}^{\oplus}$, such that 1. $(T^{\,t}_{\mathrm{ic}})_i$ exists and 2. $(T^{\,t}_{\mathrm{ic}})_i = (T^{\,t}_{\mathrm{ic}})_j$ $(1 \leqslant i \neq j \leqslant n)$, then there is a decorated derivation $\pi^{\star} :: t : [(\sigma^i_1, \ldots, \sigma^i_m \,;\, \tau_i)\,{}^n_{i=1}]_{\,x_1,\ldots,x_m}$ in IUL_m, such that $(T^{\,t}_{\mathrm{ic}})_{\pi^{\star}} = (T^{\,t}_{\mathrm{ic}})_i$.*

The proofs of 7.20 and 7.21 are the sequent calculus counterparts of the proofs of 5.10 and 5.13, respectively. They have been checked, but are not exposed here. The **(→L)** case in 7.20 is quite demanding, while 7.21 requires a quite different handling of the exchange inferences compared to 5.13 (Remark 7.8 is relevant).

At this point, we can explain why an *additive* presentation of the sequent calculus type system is chosen. If we attempted the (restricted) correspondence theorems, stated above, with the multiplicative (sequent calculus) type system instead of the additive (sequent calculus) type system, we would discover the following. The theorem from the logic to the type system would work fine, as the additive logic would project additively to the type-system level and the multiplicative type system behaves exactly as the additive one, given additive premises. On the other hand, the theorem from the type system to the logic would not work. Although the hypothesis that the trees $(T^t_{\mathrm{ic}})_1, \ldots, (T^t_{\mathrm{ic}})_n$ all exist would restrict the $(\cap\mathbf{R})$ and $(\cup\mathbf{L})$ rule-inferences in $\pi_1, \ldots \pi_n$ to additive versions, the (still) multiplicative $(\rightarrow\mathbf{L})$ and $(\mathbf{cut})$ rule-inferences in $\pi_1, \ldots, \pi_n$ would only return to multiplicative $(\rightarrow\mathbf{L})$ and $(\mathbf{cut})$ rule-inferences in the logic. For both theorems to work, we either need a logic with multiplicative versions of $(\rightarrow\mathbf{L})$ and $(\mathbf{cut})$ opposite the multiplicative type system or the additive logic introduced in Section 7.1 opposite the additive type system.

Following the natural deduction case, we estimate[7] that a set of derivations $\pi_1, \ldots, \pi_n$, sharing the same term-sequent at the root and such that it is not the case that the trees $(T^t_{\mathrm{ic}})_1, \ldots, (T^t_{\mathrm{ic}})_n$ all exist and are identical, is *not* always transformable to a set of derivations $\pi'_1, \ldots, \pi'_n$, proving the same sequents as $\pi_1, \ldots, \pi_n$, respectively, and such that the trees $(T^t_{\mathrm{ic}})'_1, \ldots, (T^t_{\mathrm{ic}})'_n$ all exist and are identical. Given this estimate, the claims in Sections 5.4 and 6.3 about non-restricted correspondence theorems and the actual success of IUL_m as a logic for $\mathrm{IUT}^{\oplus}$, respectively, can also be sustained in sequent calculus, modulo the conversion of natural deduction notions or rules to the corresponding sequent calculus notions or rules.

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[7] We use the verb "estimate", as we have not attempted to establish a transformation counterexample in sequent calculus. It would be interesting to translate the natural deduction derivations π_1 and π_2 of Section 5.3 in sequent calculus style, examine their compatibility with respect to trees T^t_{ic}, and decide whether they constitute a transformation counterexample in sequent calculus, as well.

Conclusions and Future Work

The main aim of this thesis was to offer a logic corresponding to the type system with intersection and union types IUT through decoration, in the manner that the logics offered in [18, 15] correspond to the type system with intersection types IT through decoration. We modified and extended with union the logic ISL in [15] to define the logic IUL_m as a logic intended to correspond to IUT through decoration. Decorating IUL_m with untyped terms that simulate the terms in IUT, we proved restricted correspondence theorems between the decorated IUL_m and IUT. The restrictions involve the trees of implications and union eliminations with terms T^t_{iue}, which are defined for both the decorated IUL_m and IUT. A decorated derivation $\pi^\star$ in IUL_m with decoration-statement $x_1, \ldots, x_m \vdash t$ at the root corresponds to a finite number of derivations $\pi_1, \ldots, \pi_n$ in IUT that share the term-statement $x_1, \ldots, x_m \vdash t$ at the root, and the trees T^t_{iue} of all these derivations $\pi^\star, \pi_1, \ldots, \pi_n$ are identical (recall Theorems 5.10 and 5.13). More precisely, in the direction from IUT to the decorated IUL_m, it is only under the condition that the trees T^t_{iue} of $\pi_1, \ldots, \pi_n$ all exist and are identical that we can merge $\pi_1, \ldots, \pi_n$ into a single $\pi^\star$ with this very tree T^t_{iue} (recall the intuitive justification of this fact in Section 5.4). Since it is *not* always the case that derivations $\pi_1, \ldots, \pi_n$ that share the same term-statement at the root have existing and identical trees T^t_{iue} or, at least, can be transformed into derivations $\pi'_1, \ldots, \pi'_n$ that prove the same statements as $\pi_1, \ldots, \pi_n$, respectively, and have existing and identical trees T^t_{iue} (recall the transformation counterexample in Section 5.3), the condition that secures that $\pi_1, \ldots, \pi_n$ can be merged into a single $\pi^\star$ is indeed a restriction. This restriction does not agree with the original definition of IUL_m as a logic meant to correspond to IUT through decoration; this is because the definition assumed that *any* statements in IUT that share the same term-statement can be merged into a single decorated molecule in IUL_m, so that the two-premise ($\cap$**I**) and the two-minor-premise ($\cup$**E**) in IUT translate into a single-premise ($\cap$**I**) and a single-minor-premise ($\cup$**E**) in the decorated IUL_m, respectively, allowing the decoration to simulate the terms in IUT without the inclusion of metatheoretical conditions (recall Section 4.3). Therefore, the logic IUL_m does not actually meet the expectations of its definition as a logic for IUT in the manner that the logic ISL (or its modified version IL_m) meets the expectations of its definition as a logic for IT (recall the discussion in Section 6.3). This is a negative result that raises questions about the adequacy of structures like kits or molecules to describe logics that correspond to intersection (and union) types, in the sense that an adequate logic would need to retain its good properties under extension. It may be the case that the logical foundation of intersection (and union) types requires a drastically different treatment than what is studied in this thesis.

However, besides the interrelation between IUL_m and IUT, we studied IUT in itself, both in natural deduction and sequent calculus styles, and provided many interesting results about it. We proved cut elimination in the sequent calculus $\mathrm{IUT}_{\mathrm{C}}$ and emphasized the necessity of an explicit contraction rule for the elimination of *all* cuts (recall Theorem 2.21 and Remark 2.22). We extended the theorems in [13] that characterize λ-terms according to their typings in IT_ω and IT to theorems that characterize λ-terms according to their typings in $\mathrm{IUT}_{\omega\mathrm{C}}$ and $\mathrm{IUT}_{\mathrm{C}}$, respectively, to conclude that the correspondences between typings and characterizations remain unchanged under the extension of the type systems with contraction and union (recall Theorems 2.36, 2.42, 2.47, and 2.49). We also elaborated on properties of IUT, enriching already established ones with new information and also proving additional ones; this was done for both the natural deduction and sequent calculus formulations of the system (recall Propositions 4.14, 4.16, and 4.17 in natural deduction and their counterparts 7.11, 7.13, and 7.14, respectively, in sequent calculus).

Thoughts for future work include the examination of cut elimination in the sequent calculus IUL_m. Some work has already been done in this direction, although it is not incorporated in this thesis. In particular, we have shown cut elimination in the sequent calculus IL_m by means of Gentzen's method [12] and, together with S. Ronchi Della Rocca, Y. Stavrinos, and A. Saurin, have recorded some serious evidence that the property breaks down in IUL_m. If we turn this evidence into proof, we will have another argument against the adequacy of the molecule structure to describe logics for intersection (and union) types.

Another interesting related study, which is actually a work in progress with Stavrinos, is the study of a new version $\mathrm{IUL}_m^{\wedge}$ of IUL_m with rules for conjunction and with $(\cup\mathbf{E})'$ in place of $(\cup\mathbf{E})$ (recall Proposition 4.13) in juxtaposition with intuitionistic linear logic ILL [14], so that the relation between intersection (or *synchronous* conjunction) and conjunction (or *asynchronous* conjunction) in the former logic is investigated under the light of the relation between *additive* and *multiplicative* conjunction in the latter. The extended logic $\mathrm{IUL}_m^{\wedge}$ contains an introduction rule and a *general elimination rule* [16] for conjunction, which are asynchronous and multiplicative, whereas the rules for intersection and union remain synchronous and therefore additive. We have defined a translation of formulas of $\mathrm{IUL}_m^{\wedge}$ into formulas of ILL by interpreting conjunction $\wedge$, intersection $\cap$, and union $\cup$ in the former logic as multiplicative conjunction $\otimes$, additive conjunction &, and additive disjunction $\oplus$ in the latter, respectively. We have further noted that intersection implies conjunction in $\mathrm{IUL}_m^{\wedge}$ and not conversely, while the translation of conjunction implies the translation of intersection in ILL and not conversely; this non-monotonicity of the translation reveals a duality of the $\cap$-$\wedge$ relation to the &-$\otimes$ relation. Decorating $\mathrm{IUL}_m^{\wedge}$ and ILL with untyped terms, so that implication and conjunction are the only connectives encoded in the former logic and their corresponding connectives through the translation are the only connectives encoded in the latter, we have then proved a full embedding of $\mathrm{IUL}_m^{\wedge}$ into ILL. Future work may include i) examining the faithfulness of the embedding through an inverse translation from ILL into $\mathrm{IUL}_m^{\wedge}$, ii) further examining interpretations, properties, and relations of the connectives in $\mathrm{IUL}_m^{\wedge}$ through interpretations, properties, and relations of their corresponding connectives in ILL, iii) investigating normalization in $\mathrm{IUL}_m^{\wedge}$ through normalization in ILL, iv) a categorical study of the embedding, viewing the two logics as categories and the translation as a contravariant functor, and v) a semantical comparative study of the two logics.

APPENDIX A

Proof of Lemma 2.18

A fully detailed proof of Lemma 2.18 follows.

Lemma A.1 (Lemma 2.18) *If* $\pi :: B \vdash t : \sigma$ *is a derivation in* $\mathrm{IUT}'_{\mathrm{C}}$ *with a mix as final rule and no other mix contained, then there is a mix-free derivation* $\pi' :: B \vdash t' : \sigma$ *in* $\mathrm{IUT}'_{\mathrm{C}}$*, where* $t \twoheadrightarrow_\beta t'$*.*

Proof. Writing "mf" for "mix-free" and "t/x_j" for the substitutions in parallel "$t/x_1, \dots, t/x_m$", we can display the final mix of π as follows.

$$\dfrac{\overset{\text{mf}}{\pi_0 :: B \vdash t : \sigma} \qquad \overset{\text{mf}}{\pi_1 :: B',\, x_1 : \sigma, \dots,\, x_m : \sigma \vdash u : \tau}}{\pi :: B,\, B' \vdash u[t/x_j] : \tau}\ \textbf{(mix)},\ m = (d, r)$$

We proceed by transfinite induction on the measure m of the mix, considering the lexicographic order for measures.

Base: If $m = (0, 2)$, then: (i) $d = 0 \;\Rightarrow\; \sigma = \alpha$, for some type variable $\alpha \;\Rightarrow$ the final rule of π_1 is *not* a left rule introducing σ and (ii) $r = 2 \;\Rightarrow\; rr = 1 \;\Rightarrow$ the final rule of π_1 is *not* a right rule or a left rule introducing some type in B' or contraction in B' or contraction of σ. So, π_1 must be an axiom and we distinguish the following cases.

Case 1: The term typed by π_1 belongs to $\{x_1, \dots, x_m\}$.

$$\dfrac{\overset{\text{mf}}{\pi_0 :: B \vdash t : \sigma} \qquad \dfrac{}{\pi_1 :: B',\, x_1 : \sigma, \dots,\, x_m : \sigma \vdash x_j : \sigma}\ \textbf{(ax)}}{\pi :: B, B' \vdash t : \sigma}\ \textbf{(mix)} \quad \hookrightarrow_{\pi_0 + \text{Lemma 2.13(ii)}} \quad \overset{\text{mf}}{\pi' :: B, B' \vdash t : \sigma}$$

Case 2: The term typed by π_1 does *not* belong to $\{x_1, \dots, x_m\}$.

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad \dfrac{}{\pi_1 :: B',\, y : \tau,\, x_1 : \sigma, \dots,\, x_m : \sigma \vdash y : \tau}\ \textbf{(ax)}}{\pi :: B, B',\, y : \tau \vdash y : \tau}\ \textbf{(mix)} \quad \hookrightarrow \quad \dfrac{}{\pi' :: B, B',\, y : \tau \vdash y : \tau}\ \textbf{(ax)}$$

Induction step for limit points: If $m = (d, 2)$ with $d > 0$, then: (i) $lr = 1 \Rightarrow \pi_0$ is an axiom or its final rule is a right rule and (ii) $rr = 1 \Rightarrow \pi_1$ is an axiom or its final rule is a left rule introducing σ with $m = 1$. From (i) and (ii) we have the following cases.

Case 1: If π_1 is an axiom, we refer to the base case.

Case 2: If π_0 is an axiom, it suffices to show the case where the final rule of π_1 is a left rule introducing σ with $m = 1$.

$$\dfrac{\dfrac{}{\pi_0 :: B,\, y:\sigma \vdash y:\sigma}\,(\mathbf{ax}) \qquad \begin{array}{c}\text{mf}\\ \pi_1 :: B',\, x:\sigma \vdash u:\tau\end{array}}{\pi :: B, B',\, y:\sigma \vdash u[y/x]:\tau}\,(\mathbf{mix}) \quad \hookrightarrow_{\pi_1 + \text{Lemma 2.13}} \quad \begin{array}{c}\text{mf}\\ \pi' :: B, B',\, y:\sigma \vdash u[y/x]:\tau\end{array}$$

Case 3: Derivations π_0, π_1 have $(\rightarrow\mathbf{R})$,$(\rightarrow\mathbf{L})$ as final rules, respectively.

$$\dfrac{\dfrac{B,\, y:\sigma \vdash v:\tau}{\pi_0 :: B \vdash \lambda y.v : \sigma \rightarrow \tau}\,(\rightarrow\mathbf{R}) \qquad \dfrac{B' \vdash t:\sigma \qquad B'',\, z:\tau \vdash u:\rho}{\pi_1 :: B', B'',\, x:\sigma \rightarrow \tau \vdash u[xt/z]:\rho}\,(\rightarrow\mathbf{L})}{\pi :: B, B', B'' \vdash u[xt/z][\lambda y.v/x]:\rho}\,(\mathbf{mix}),\ m = (d(\sigma \rightarrow \tau), 2) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{B' \vdash t:\sigma \qquad B,\, y:\sigma \vdash v:\tau}{B, B' \vdash v[t/y]:\tau}\,(\mathbf{mix})',\ m' = (d(\sigma), r')}{B, B' \vdash t_0:\tau \ \ (\text{mf})}\,[\text{IH: } m' < m] \qquad B'',\, z:\tau \vdash u:\rho}{\dfrac{B, B', B'' \vdash u[t_0/z]:\rho}{\pi' :: B, B', B'' \vdash t_1:\rho \ \ (\text{mf})}\,[\text{IH: } m'' < m]}\,(\mathbf{mix})'',\ m'' = (d(\tau), r'')$$

By the IH, we have $v[t/y] \twoheadrightarrow_\beta t_0$ and $u[t_0/z] \twoheadrightarrow_\beta t_1$. Since x is not free in u, we get $u[xt/z][\lambda y.v/x] = u[(\lambda y.v)t/z] \twoheadrightarrow_\beta u[v[t/y]/z] \twoheadrightarrow_\beta u[t_0/z] \twoheadrightarrow_\beta t_1$.

Case 4: Derivations π_0, π_1 have $(\cap\mathbf{R})$,$(\cap\mathbf{L})$ as final rules, respectively.

$$\dfrac{\dfrac{B \vdash v:\sigma \qquad B' \vdash v:\tau}{\pi_0 :: B, B' \vdash v:\sigma \cap \tau}\,(\cap\mathbf{R}) \qquad \dfrac{B'',\, x:\sigma \vdash u:\rho}{\pi_1 :: B'',\, x:\sigma \cap \tau \vdash u:\rho}\,(\cap\mathbf{L})}{\pi :: B, B', B'' \vdash u[v/x]:\rho}\,(\mathbf{mix}),\ m = (d(\sigma \cap \tau), 2) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{B \vdash v:\sigma \qquad B'',\, x:\sigma \vdash u:\rho}{B, B'' \vdash u[v/x]:\rho}\,(\mathbf{mix})',\ m' = (d(\sigma), r')}{B, B'' \vdash t_0:\rho \ \ (\text{mf})}\,[\text{IH: } m' < m]}{\pi' :: B, B', B'' \vdash t_0:\rho \ \ (\text{mf})}\,[\text{Lemma 2.13(ii)}]$$

By the IH, we have $u[v/x] \twoheadrightarrow_\beta t_0$.

Case 5: If π_0, π_1 have $(\cup\mathbf{R})$,$(\cup\mathbf{L})$ as final rules, respectively, the case is very similar to case 4.

Induction step for successor points: If $m = (d,r)$ with $r > 2$, then: A) $lr > 1$ or B) $rr > 1$.

Case A: $lr > 1 \Rightarrow$ the final rule of π_0 is a contraction or a left rule.

Case **(C)**: In what follows, we consider z fresh with respect to π_1; otherwise, we substitute it by a fresh (wrt π_1) w, using Lemma 2.13(i).

$$\dfrac{\dfrac{B,\, y:\tau,\, z:\tau \vdash t:\sigma}{\pi_0 :: B,\, y:\tau \vdash t[y/z]:\sigma}\,\textbf{(C)} \qquad \pi_1 :: B',\, x_1:\sigma,\ldots,x_m:\sigma \vdash u:\rho}{\pi :: B,B',\, y:\tau \vdash u[t[y/z]/x_j]:\rho}\,\textbf{(mix)},\ m=(d(\sigma),r) \quad \hookrightarrow$$

$$\begin{array}{cl} B,\, y:\tau,\, z:\tau \vdash t:\sigma \qquad \pi_1 :: B',\, x_1:\sigma,\ldots,x_m:\sigma \vdash u:\rho & \\ \hline B,B',\, y:\tau,\, z:\tau \vdash u[t/x_j]:\rho & \textbf{(mix)}',\ m'=(d(\sigma),r-1) \\ \hdashline B,B',\, y:\tau,\, z:\tau \vdash t_0:\rho\ \ \text{(mf)} & [\text{IH: } m' < m] \\ \hline \pi' :: B,B',\, y:\tau \vdash t_0[y/z]:\rho & \textbf{(C)} \end{array}$$

By the IH, we have $u[t/x_j] \twoheadrightarrow_\beta t_0$. Since z is not free in u, we get $u[t[y/z]/x_j] = u[t/x_j][y/z] \twoheadrightarrow_\beta t_0[y/z]$.

Case **($\rightarrow$L)**: In what follows, we consider z, y fresh with respect to π_1 and $\tau \neq \sigma$.

$$\dfrac{\dfrac{B \vdash t:\tau \qquad B',\, z:\phi \vdash v:\sigma}{\pi_0 :: B,B',\, y:\tau\rightarrow\phi \vdash v[yt/z]:\sigma}\,\textbf{($\rightarrow$L)} \qquad \pi_1 :: B'',\, x_1:\sigma,\ldots,x_m:\sigma \vdash u:\rho}{\pi :: B,B',B'',\, y:\tau\rightarrow\phi \vdash u[v[yt/z]/x_j]:\rho}\,\textbf{(mix)},\ m=(d(\sigma),r) \quad \hookrightarrow$$

$$\dfrac{B \vdash t:\tau \qquad \begin{array}{cl} B',\, z:\phi \vdash v:\sigma \qquad \pi_1 :: B'',\, x_1:\sigma,\ldots,x_m:\sigma \vdash u:\rho & \\ \hline B',B'',\, z:\phi \vdash u[v/x_j]:\rho & \textbf{(mix)}',\ m'=(d(\sigma),r-1) \\ \hdashline B',B'',\, z:\phi \vdash t_0:\rho\ \ \text{(mf)} & [\text{IH: } m' < m] \end{array}}{\pi' :: B,B',B'',\, y:\tau\rightarrow\phi \vdash t_0[yt/z]:\rho}\,\textbf{($\rightarrow$L)}$$

By the IH, we have $u[v/x_j] \twoheadrightarrow_\beta t_0$. Since z is not free in u, we get $u[v[yt/z]/x_j] = u[v/x_j][yt/z] \twoheadrightarrow_\beta t_0[yt/z]$.

Case **($\cap$L)**: If the final rule of π_0 is a left intersection

$$\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma}{\pi_0 :: B,\, y:\tau\cap\phi \vdash t:\sigma}\,\textbf{($\cap$L)} \qquad \pi_1 :: B',\, x_1:\sigma,\ldots,x_m:\sigma \vdash u:\rho}{\pi :: B,B',\, y:\tau\cap\phi \vdash u[t/x_j]:\rho}\,\textbf{(mix)},\ m=(d(\sigma),r)$$

we distinguish two subcases according to whether $y : \tau\cap\phi$ belongs to B' or not.

Subcase a: Suppose that $B' = B'',\, y:\tau\cap\phi$. In what follows, we consider z fresh with respect to both π_1 and π_0.

$$\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma}{\pi_0 :: B,\, y:\tau\cap\phi \vdash t:\sigma}\,(\cap\mathbf{L}) \qquad \pi_1 :: B'',\, y:\tau\cap\phi,\, x_1:\sigma,\dots,x_m:\sigma \vdash u:\rho}{\pi :: B, B'',\, y:\tau\cap\phi \vdash u[t/x_j]:\rho}\,\mathbf{(mix)},\ m=(d(\sigma),r) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma \qquad \dfrac{\pi_1 :: B'',\, y:\tau\cap\phi,\, x_1:\sigma,\dots,x_m:\sigma \vdash u:\rho\ \text{(mf)}}{B'',\, z:\tau\cap\phi,\, x_1:\sigma,\dots,x_m:\sigma \vdash u[z/y]:\rho\ \text{(mf)}}\,[\text{Lemma 2.13(i)}]}{B, B'',\, y:\tau,\, z:\tau\cap\phi \vdash u[z/y][t/x_j]:\rho}\,\mathbf{(mix)}',\ m'=(d(\sigma),r-1)}{B, B'',\, y:\tau,\, z:\tau\cap\phi \vdash t_0:\rho\ \text{(mf)}}\,[\text{IH: } m'<m]}{B, B'',\, y:\tau\cap\phi,\, z:\tau\cap\phi \vdash t_0:\rho}\,(\cap\mathbf{L})}{\pi' :: B, B'',\, y:\tau\cap\phi \vdash t_0[y/z]:\rho}\,\mathbf{(C)}$$

By the IH, we have $u[z/y][t/x_j] \twoheadrightarrow_\beta t_0$. As z is not free in u or t, we get $u[t/x_j] = u[z/y][t/x_j][y/z] \twoheadrightarrow_\beta t_0[y/z]$.

Subcase b: Suppose that $y:\tau\cap\phi \notin B'$.

$$\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma}{\pi_0 :: B,\, y:\tau\cap\phi \vdash t:\sigma}\,(\cap\mathbf{L}) \qquad \pi_1 :: B',\, x_1:\sigma,\dots,x_m:\sigma \vdash u:\rho}{\pi :: B, B',\, y:\tau\cap\phi \vdash u[t/x_j]:\rho}\,\mathbf{(mix)},\ m=(d(\sigma),r) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma \qquad \pi_1 :: B',\, x_1:\sigma,\dots,x_m:\sigma \vdash u:\rho}{B, B',\, y:\tau \vdash u[t/x_j]:\rho}\,\mathbf{(mix)}',\ m'=(d(\sigma),r-1)}{B, B',\, y:\tau \vdash t_0:\rho\ \text{(mf)}}\,[\text{IH: } m'<m]}{\pi' :: B, B',\, y:\tau\cap\phi \vdash t_0:\rho}\,(\cap\mathbf{L})$$

By the IH, we have $u[t/x_j] \twoheadrightarrow_\beta t_0$.

Case $(\cup\mathbf{L})$: If the final rule of π_0 is a left union

$$\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma \qquad B',\, y:\phi \vdash t:\sigma}{\pi_0 :: B, B',\, y:\tau\cup\phi \vdash t:\sigma}\,(\cup\mathbf{L}) \qquad \pi_1 :: B'',\, x_1:\sigma,\dots,x_m:\sigma \vdash u:\rho}{\pi :: B, B', B'',\, y:\tau\cup\phi \vdash u[t/x_j]:\rho}\,\mathbf{(mix)},\ m=(d(\sigma),r)$$

we again distinguish two subcases according to whether $y:\tau\cup\phi$ belongs to B'' or not.

Subcase a: Suppose that $B'' = B''',\, y:\tau\cup\phi$. In what follows, we write "$x_j:\sigma$" for $x_1:\sigma,\dots,x_m:\sigma$ and consider z fresh with respect to π_1 and π_0.

$$\dfrac{\dfrac{B,\, y:\tau \vdash t:\sigma \qquad B',\, y:\phi \vdash t:\sigma}{\pi_0 :: B, B',\, y:\tau\cup\phi \vdash t:\sigma}\,(\cup\mathbf{L}) \qquad \pi_1 :: B''',\, y:\tau\cup\phi,\, x_j:\sigma \vdash u:\rho}{\pi :: B, B', B''',\, y:\tau\cup\phi \vdash u[t/x_j]:\rho}\,\mathbf{(mix)},\ m=(d(\sigma),r) \quad \hookrightarrow$$

$$\cfrac{\cfrac{\pi'_0 :: B, B''',\, y:\tau,\, z:\tau\cup\phi \vdash t_0:\rho \;\;(\mathrm{mf}) \qquad \pi'_1 :: B', B''',\, y:\phi,\, z:\tau\cup\phi \vdash t_1:\rho \;\;(\mathrm{mf})}{B, B', B''',\, y:\tau\cup\phi,\, z:\tau\cup\phi \vdash t'(=t_0=t_1):\rho}\,(\cup\mathbf{L})}{\pi' :: B, B', B''',\, y:\tau\cup\phi \vdash t'[y/z]:\rho}\,(\mathbf{C})$$

We derive π'_0, π'_1 as shown below.

$$\cfrac{\cfrac{B,\, y:\tau\vdash t:\sigma \qquad \cfrac{\pi_1 :: B''',\, y:\tau\cup\phi,\, x_j:\sigma\vdash u:\rho \;\;(\mathrm{mf})}{B''',\, z:\tau\cup\phi,\, x_j:\sigma\vdash u[z/y]:\rho\;\;(\mathrm{mf})}\,[\text{Lemma 2.13(i)}]}{B, B''',\, y:\tau,\, z:\tau\cup\phi\vdash u[z/y][t/x_j]:\rho}\,(\mathbf{mix})',\ m'=(d(\sigma),r')}{\pi'_0 :: B, B''',\, y:\tau,\, z:\tau\cup\phi\vdash t_0:\rho\;\;(\mathrm{mf})}\,[\text{IH: } r'<r\Rightarrow m'<m]$$

$$\cfrac{\cfrac{B',\, y:\phi\vdash t:\sigma \qquad \cfrac{\pi_1 :: B''',\, y:\tau\cup\phi,\, x_j:\sigma\vdash u:\rho \;\;(\mathrm{mf})}{B''',\, z:\tau\cup\phi,\, x_j:\sigma\vdash u[z/y]:\rho\;\;(\mathrm{mf})}\,[\text{Lemma 2.13(i)}]}{B', B''',\, y:\phi,\, z:\tau\cup\phi\vdash u[z/y][t/x_j]:\rho}\,(\mathbf{mix})'',\ m''=(d(\sigma),r'')}{\pi'_1 :: B', B''',\, y:\phi,\, z:\tau\cup\phi\vdash t_1:\rho\;\;(\mathrm{mf})}\,[\text{IH: } r''<r\Rightarrow m''<m]$$

By the IH, we have $t_0 \;{}_\beta\!\twoheadleftarrow u[z/y][t/x_j] \twoheadrightarrow_\beta t_1$. But t_0 and t_1 are normal terms (Remark 2.12(i)), so, by uniqueness of the normal form, we get $t_0 = t_1 = t'$. Finally, since z is not free in u or t, we have $u[t/x_j] = u[z/y][t/x_j][y/z] \twoheadrightarrow_\beta t'[y/z]$.

Subcase b: Suppose that $y:\tau\cup\phi\notin B''$.

$$\cfrac{\cfrac{\pi_{00} :: B,\, y:\tau\vdash t:\sigma \qquad \pi_{01} :: B',\, y:\phi\vdash t:\sigma}{\pi_0 :: B, B',\, y:\tau\cup\phi\vdash t:\sigma}\,(\cup\mathbf{L}) \qquad \pi_1 :: B'',\, x_j:\sigma\vdash u:\rho}{\pi :: B, B', B'',\, y:\tau\cup\phi\vdash u[t/x_j]:\rho}\,(\mathbf{mix}),\ m=(d(\sigma),r) \quad\hookrightarrow$$

$$\cfrac{\cfrac{\cfrac{\pi_{00}\qquad\pi_1}{B, B'',\, y:\tau\vdash u[t/x_j]:\rho}\,(\mathbf{mix})',\ m'=(d(\sigma),r')}{B, B'',\, y:\tau\vdash t_0:\rho\;\;(\mathrm{mf})}\,[\text{IH: } m'<m] \qquad \cfrac{\cfrac{\pi_{01}\qquad\pi_1}{B', B'',\, y:\phi\vdash u[t/x_j]:\rho}\,(\mathbf{mix})'',\ m''=(d(\sigma),r'')}{B', B'',\, y:\phi\vdash t_1:\rho\;\;(\mathrm{mf})}\,[\text{IH: } m''<m]}{\pi' :: B, B', B'',\, y:\tau\cup\phi\vdash t'(=t_0=t_1):\rho}\,(\cup\mathbf{L})$$

By the IH and using the uniqueness of normal form, we get $u[t/x_j]\twoheadrightarrow_\beta t'$.

Case B: $rr>1 \Rightarrow$ the final rule of π_1 is a contraction or a left rule or a right rule.

Case $(\mathbf{C})$: We distinguish two subceses.

Subcase a: The mix-type is contracted.

$$\cfrac{\pi_0 :: B\vdash t:\sigma \qquad \cfrac{B',\, x_0:\sigma,\, x_1:\sigma,\ldots,\, x_m:\sigma\vdash u:\rho}{\pi_1 :: B',\, x_1:\sigma,\ldots,\, x_m:\sigma\vdash u[x_1/x_0]:\rho}\,(\mathbf{C})}{\pi :: B, B'\vdash u[x_1/x_0][t/x_j]:\rho}\,(\mathbf{mix}),\ m=(d(\sigma),r) \quad\hookrightarrow$$

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad B',\, x_0 : \sigma,\, x_1 : \sigma, \ldots,\, x_m : \sigma \vdash u : \rho}{\begin{array}{c} B, B' \vdash u[t/x_0, t/x_j] : \rho \\ \hdashline \pi' :: B, B' \vdash t_0 : \rho \ \ (\mathrm{mf}) \end{array} \ [\text{IH: } m' < m]} \ \textbf{(mix)}',\ m' = (d(\sigma), r-1)$$

It is $u[x_1/x_0][t/x_j] = u[t/x_0, t/x_j] \overset{[\text{IH}]}{\twoheadrightarrow_\beta} t_0$.

Subcase b: A type different from the mix-type is contracted. In what follows, we consider z fresh with respect to π_0.

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad \dfrac{B',\, y : \tau,\, z : \tau,\, x_1 : \sigma, \ldots,\, x_m : \sigma \vdash u : \rho}{\pi_1 :: B',\, y : \tau,\, x_1 : \sigma, \ldots,\, x_m : \sigma \vdash u[y/z] : \rho} \ \textbf{(C)}}{\pi :: B, B',\, y : \tau \vdash u[y/z][t/x_j] : \rho} \ \textbf{(mix)},\ m = (d(\sigma), r) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\pi_0 :: B \vdash t : \sigma \qquad B',\, y : \tau,\, z : \tau,\, x_1 : \sigma, \ldots,\, x_m : \sigma \vdash u : \rho}{\begin{array}{c} B, B',\, y : \tau,\, z : \tau \vdash u[t/x_j] : \rho \\ \hdashline B, B',\, y : \tau,\, z : \tau \vdash t_0 : \rho \ \ (\mathrm{mf}) \end{array} \ [\text{IH: } m' < m]} \ \textbf{(mix)}',\ m' = (d(\sigma), r-1)}{\pi' :: B, B',\, y : \tau \vdash t_0[y/z] : \rho} \ \textbf{(C)}$$

Since z is not free in t, we have $u[y/z][t/x_j] = u[t/x_j][y/z] \overset{[\text{IH}]}{\twoheadrightarrow_\beta} t_0[y/z]$.

Case **(→L)**: We distinguish two subcases.

Subcase a: The mix-type is introduced by **(→L)**. In what follows, it is $1 \leqslant g \leqslant k,\ k+1 \leqslant h \leqslant m-1$, and z, x_m fresh with respect to π_0.

$$\dfrac{\pi_0 :: B \vdash t : \sigma \qquad \dfrac{\pi_{10} :: B',\, x_1 : \sigma, \ldots,\, x_k : \sigma \vdash v : \sigma_1 \qquad \pi_{11} :: B'',\, x_{k+1} : \sigma, \ldots,\, x_{m-1} : \sigma,\, z : \sigma_2 \vdash u : \rho}{\pi_1 :: B', B'',\, x_1 : \sigma, \ldots,\, x_{m-1} : \sigma,\, x_m : \sigma \vdash u[x_m v/z] : \rho} \ \textbf{(→L)}}{\pi :: B, B', B'' \vdash u[x_m v/z][t/x_j] : \rho} \ \textbf{(mix)},\ m = (d(\sigma), r)$$

$$\hookrightarrow \ \dfrac{\pi_0 \qquad \dfrac{\dfrac{\pi_0 \qquad \pi_{10}}{\begin{array}{c} B, B' \vdash v[t/x_g] : \sigma_1 \\ \hdashline B, B' \vdash t_0 : \sigma_1 \ \ (\mathrm{mf}) \end{array} \ [\text{IH: } r' < r \Rightarrow m' < m]} \ \textbf{(mix)}',\ m' = (d(\sigma), r') \qquad \dfrac{\pi_0 \qquad \pi_{11}}{\begin{array}{c} B, B'',\, z : \sigma_2 \vdash u[t/x_h] : \rho \\ \hdashline B, B'',\, z : \sigma_2 \vdash t_1 : \rho \ \ (\mathrm{mf}) \end{array} \ [\text{IH: } r'' < r \Rightarrow m'' < m]} \ \textbf{(mix)}'',\ m'' = (d(\sigma), r'')}{B, B', B'',\, x_m : \sigma \vdash t_1[x_m t_0/z] : \rho} \ \textbf{(→L)}}{\begin{array}{c} B, B', B'' \vdash t_1[x_m t_0/z][t/x_m] : \rho \\ \hdashline \pi' :: B, B', B'' \vdash t_2 : \rho \ \ (\mathrm{mf}) \end{array} \ [\text{IH: } r''' < r \Rightarrow m''' < m]} \ \textbf{(mix)}''',\ m''' = (d(\sigma), r''')$$

It is $r''' = lr''' + rr''' = lr + 1 < lr + rr = r$. By the IH, we have $v[t/x_g] \twoheadrightarrow_\beta t_0,\ u[t/x_h] \twoheadrightarrow_\beta t_1$, and $t_1[x_m t_0/z][t/x_m] \twoheadrightarrow_\beta t_2$. Since z, x_m are not free in t, we get

$$u[x_m v/z][t/x_j] = u[t/x_h][x_m(v[t/x_g])/z][t/x_m] \twoheadrightarrow_\beta t_1[x_m t_0/z][t/x_m] \twoheadrightarrow_\beta t_2$$

Subcase b: A type different from the mix-type is introduced by **(→L)**. In what follows, it is $1 \leqslant g \leqslant k$, $k+1 \leqslant h \leqslant m$, and z, y fresh with respect to π_0.

$$
\begin{prooftree}
\AxiomC{$\pi_0 :: B \vdash t : \sigma$}
\AxiomC{$\pi_{10} :: B',\, x_1 : \sigma, \dots, x_k : \sigma \vdash v : \tau_1$}
\AxiomC{$\pi_{11} :: B'',\, x_{k+1} : \sigma, \dots, x_m : \sigma,\, z : \tau_2 \vdash u : \rho$}
\RightLabel{$(\to\mathbf{L})$}
\BinaryInfC{$\pi_1 :: B', B'',\, x_1 : \sigma, \dots, x_m : \sigma,\, y : \tau \vdash u[yv/z] : \rho$}
\RightLabel{$\mathbf{(mix)},\ m = (d(\sigma), r)$}
\BinaryInfC{$\pi :: B, B', B'',\, y : \tau \vdash u[yv/z][t/x_j] : \rho$}
\end{prooftree}
$$

$$
\hookrightarrow
\begin{prooftree}
\AxiomC{π_0}
\AxiomC{π_{10}}
\RightLabel{$\mathbf{(mix)}',\ m' = (d(\sigma), r')$}
\BinaryInfC{$B, B' \vdash v[t/x_g] : \tau_1$}
\dashedLine
\RightLabel{[IH: $r' < r \Rightarrow m' < m$]}
\UnaryInfC{$B, B' \vdash t_0 : \tau_1 \ \ (\mathrm{mf})$}
\AxiomC{π_0}
\AxiomC{π_{11}}
\RightLabel{$\mathbf{(mix)}'',\ m'' = (d(\sigma), r'')$}
\BinaryInfC{$B, B'',\, z : \tau_2 \vdash u[t/x_h] : \rho$}
\dashedLine
\RightLabel{[IH: $r'' < r \Rightarrow m'' < m$]}
\UnaryInfC{$B, B'',\, z : \tau_2 \vdash t_1 : \rho \ \ (\mathrm{mf})$}
\RightLabel{$(\to\mathbf{L})$}
\BinaryInfC{$\pi' :: B, B', B'',\, y : \tau \vdash t_1[yt_0/z] : \rho$}
\end{prooftree}
$$

By the IH, we have $v[t/x_g] \twoheadrightarrow_\beta t_0$ and $u[t/x_h] \twoheadrightarrow_\beta t_1$. As z is not free in t, we get $u[yv/z][t/x_j] = u[t/x_h][y(v[t/x_g])/z] \twoheadrightarrow_\beta t_1[yt_0/z]$.

Case $(\cup\mathbf{L})$: We distinguish two subcases.

Subcase a: The mix-type is introduced by $(\cup\mathbf{L})$. In what follows, it is $1 \leqslant g \leqslant m-1$ and we consider $\{x_1, \dots, x_m\} \subseteq FV(u)$ and x_m fresh with respect to π_0.

$$
\begin{prooftree}
\AxiomC{$\pi_0 :: B \vdash t : \sigma$}
\AxiomC{π_{10}}
\noLine
\UnaryInfC{$B',\, x_1 : \sigma, \dots, x_{m-1} : \sigma,\, x_m : \sigma_1 \vdash u : \rho$}
\AxiomC{π_{11}}
\noLine
\UnaryInfC{$B'',\, x_1 : \sigma, \dots, x_{m-1} : \sigma,\, x_m : \sigma_2 \vdash u : \rho$}
\RightLabel{$(\cup\mathbf{L})$}
\BinaryInfC{$\pi_1 :: B', B'',\, x_1 : \sigma, \dots, x_{m-1} : \sigma,\, x_m : \sigma \vdash u : \rho$}
\RightLabel{$\mathbf{(mix)},\ m = (d(\sigma), r) \quad \hookrightarrow$}
\BinaryInfC{$\pi :: B, B', B'' \vdash u[t/x_j] : \rho$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{π_0}
\AxiomC{π_0}
\AxiomC{π_{10}}
\RightLabel{$\mathbf{(mix)}',\ m' = (d(\sigma), r')$}
\BinaryInfC{$B, B',\, x_m : \sigma_1 \vdash u[t/x_g] : \rho$}
\dashedLine
\RightLabel{[IH: $r' < r \Rightarrow m' < m$]}
\UnaryInfC{$B, B',\, x_m : \sigma_1 \vdash t_0 : \rho \ \ (\mathrm{mf})$}
\AxiomC{π_0}
\AxiomC{π_{11}}
\RightLabel{$\mathbf{(mix)}'',\ m'' = (d(\sigma), r'')$}
\BinaryInfC{$B, B'',\, x_m : \sigma_2 \vdash u[t/x_g] : \rho$}
\dashedLine
\RightLabel{[IH: $r'' < r \Rightarrow m'' < m$]}
\UnaryInfC{$B, B'',\, x_m : \sigma_2 \vdash t_1 : \rho \ \ (\mathrm{mf})$}
\RightLabel{$(\cup\mathbf{L})$}
\BinaryInfC{$B, B', B'',\, x_m : \sigma \vdash t'(= t_0 = t_1) : \rho$}
\RightLabel{$\mathbf{(mix)}''',\ m''' = (d(\sigma), r''')$}
\BinaryInfC{$B, B', B'' \vdash t'[t/x_m] : \rho$}
\dashedLine
\RightLabel{[IH: $r''' < r \Rightarrow m''' < m$]}
\UnaryInfC{$\pi' :: B, B', B'' \vdash t_2 : \rho \ \ (\mathrm{mf})$}
\end{prooftree}
$$

It is $r''' = lr''' + rr''' = lr + 1 < lr + rr = r$. By the IH, we have $t_0 \ {}_\beta\!\twoheadleftarrow u[t/x_g] \twoheadrightarrow_\beta t_1$ and $t'[t/x_m] \twoheadrightarrow_\beta t_2$. The terms t_0, t_1 are normal (Remark 2.12(i)) and by uniqueness of normal form, we get $t_0 = t_1 = t'$. Finally, since x_m is not free in t, we get $u[t/x_j] = u[t/x_g][t/x_m] \twoheadrightarrow_\beta t'[t/x_m] \twoheadrightarrow_\beta t_2$.

Subcase b: A type different from the mix-type is introduced by $(\cup\mathbf{L})$. In what follows, we write "$x_j : \sigma$" for $x_1 : \sigma, \dots, x_m : \sigma$ and consider $\{x_1, \dots, x_m\} \subseteq FV(u)$ and z fresh with respect to π_{10}, π_{11}, and π_0.

$$
\begin{prooftree}
\AxiomC{$\pi_0 :: B,\, y : \tau \vdash t : \sigma$}
\AxiomC{$\pi_{10} :: B',\, x_j : \sigma,\, y : \tau_1 \vdash u : \rho$}
\AxiomC{$\pi_{11} :: B'',\, x_j : \sigma,\, y : \tau_2 \vdash u : \rho$}
\RightLabel{$(\cup\mathbf{L})$}
\BinaryInfC{$\pi_1 :: B', B'',\, x_j : \sigma,\, y : \tau \vdash u : \rho$}
\RightLabel{$\mathbf{(mix)},\ m = (d(\sigma), r) \quad \hookrightarrow$}
\BinaryInfC{$\pi :: B, B', B'',\, y : \tau \vdash u[t/x_j] : \rho$}
\end{prooftree}
$$

$$\dfrac{\dfrac{\dfrac{\pi_0 \qquad \dfrac{\pi_{10} :: B',\, x_j:\sigma,\, y:\tau_1 \vdash u:\rho \ \ (\mathrm{mf})}{B',\, x_j:\sigma,\, z:\tau_1 \vdash u[z/y]:\rho \ \ (\mathrm{mf})}\,[2.13(\mathrm{i})]}{B, B',\, y:\tau,\, z:\tau_1 \vdash u[z/y][t/x_j]:\rho}\,\mathbf{(mix)}',\, m'}{B, B',\, y:\tau,\, z:\tau_1 \vdash t_0:\rho \ \ (\mathrm{mf})}\,[\text{IH: } m' < m] \qquad \dfrac{\dfrac{\pi_0 \qquad \dfrac{\pi_{11} :: B'',\, x_j:\sigma,\, y:\tau_2 \vdash u:\rho \ \ (\mathrm{mf})}{B'',\, x_j:\sigma,\, z:\tau_2 \vdash u[z/y]:\rho \ \ (\mathrm{mf})}\,[2.13(\mathrm{i})]}{B, B'',\, y:\tau,\, z:\tau_2 \vdash u[z/y][t/x_j]:\rho}\,\mathbf{(mix)}'',\, m''}{B, B'',\, y:\tau,\, z:\tau_2 \vdash t_1:\rho \ \ (\mathrm{mf})}\,[\text{IH: } m'' < m]}{\dfrac{B, B', B'',\, y:\tau,\, z:\tau \vdash t'(=t_0=t_1):\rho}{\pi' :: B, B', B'',\, y:\tau \vdash t'[y/z]:\rho \ \ (\mathrm{mf})}\,\mathbf{(C)}}\,\mathbf{(\cup L)}$$

By the IH, we have $t_0 \ {}_\beta\!\twoheadleftarrow u[z/y][t/x_j] \twoheadrightarrow_\beta t_1$. As z is not free in t and t_0, t_1 are identical, since they are both normal, we get $u[t/x_j] = u[z/y][t/x_j][y/z] \twoheadrightarrow_\beta t'[y/z]$.

Case ($\cap$**L**): This case is handled in a manner similar to the two left-rule cases shown above. It is even easier, since the rule in question has a single premise.

Case ($\to$**R**): We consider y fresh with respect to π_0.

$$\dfrac{\pi_0 :: B \vdash t:\sigma \qquad \dfrac{B',\, x_1:\sigma, \ldots, x_m:\sigma,\, y:\tau \vdash u:\rho}{\pi_1 :: B',\, x_1:\sigma, \ldots, x_m:\sigma \vdash \lambda y.\, u:\tau \to \rho}\,\mathbf{(\to R)}}{\pi :: B, B' \vdash (\lambda y.\, u)[t/x_j]:\tau \to \rho}\,\mathbf{(mix)},\ m = (d(\sigma), r) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{\pi_0 :: B \vdash t:\sigma \qquad B',\, x_1:\sigma, \ldots, x_m:\sigma,\, y:\tau \vdash u:\rho}{B, B',\, y:\tau \vdash u[t/x_j]:\rho}\,\mathbf{(mix)}',\ m' = (d(\sigma), r-1)}{B, B',\, y:\tau \vdash t_0:\rho \ \ (\mathrm{mf})}\,[\text{IH: } m' < m]}{\pi' :: B, B' \vdash \lambda y.\, t_0:\tau \to \rho}\,\mathbf{(\to R)}$$

By the IH, we have $u[t/x_j] \twoheadrightarrow_\beta t_0$, so $(\lambda y.\, u)[t/x_j] = \lambda y.\, u[t/x_j] \twoheadrightarrow_\beta \lambda y.\, t_0$.

Case ($\cap$**R**): We consider $\{x_1, \ldots, x_m\} \subseteq FV(u)$ and write "$x_j:\sigma$" for $x_1:\sigma, \ldots, x_m:\sigma$.

$$\dfrac{\pi_0 :: B \vdash t:\sigma \qquad \dfrac{\pi_{10} :: B',\, x_j:\sigma \vdash u:\tau \qquad \pi_{11} :: B'',\, x_j:\sigma \vdash u:\rho}{\pi_1 :: B', B'',\, x_j:\sigma \vdash u:\tau \cap \rho}\,\mathbf{(\cap R)}}{\pi :: B, B', B'' \vdash u[t/x_j]:\tau \cap \rho}\,\mathbf{(mix)},\ m = (d(\sigma), r) \quad \hookrightarrow$$

$$\dfrac{\dfrac{\dfrac{\pi_0 \qquad \pi_{10}}{B, B' \vdash u[t/x_j]:\tau}\,\mathbf{(mix)}',\ m' = (d(\sigma), r')}{B, B' \vdash t_0:\tau \ \ (\mathrm{mf})}\,[\text{IH: } r' < r \Rightarrow m' < m] \qquad \dfrac{\dfrac{\pi_0 \qquad \pi_{11}}{B, B'' \vdash u[t/x_j]:\rho}\,\mathbf{(mix)}'',\ m'' = (d(\sigma), r'')}{B, B'' \vdash t_1:\rho \ \ (\mathrm{mf})}\,[\text{IH: } r'' < r \Rightarrow m'' < m]}{\pi' :: B, B', B'' \vdash t'(=t_0=t_1):\tau \cap \rho}\,\mathbf{(\cap R)}$$

By the IH, we have $t_0 \ {}_\beta\!\twoheadleftarrow u[t/x_j] \twoheadrightarrow_\beta t_1$. But t_0, t_1 are normal and the normal form is unique, so $t_0 = t_1 = t'$ and $u[t/x_j] \twoheadrightarrow_\beta t'$.

Case ($\cup$**R**): Very straightforward, even easier than the two right-rule cases shown above. $\dashv$

Remark A.2 *In Lemma A.1 we could have also included the fact that π' does not contain any fresh-with-respect-to-π variables. This fact is tacitly used in the proof, in cases* A: ($\to$**L**) *and* B: ($\to$**L**).

APPENDIX B

A Transformation Example

Consider the following λ-terms.

$$\begin{array}{ll} u' = xx_1 & v' = x_1 x \\ u'' = x_2 yy & v'' = y(x_2 y) \\ u = x_2 x_1 x_1 & v = x_1(x_2 x_1) \end{array}$$

If $s = x_2 x_1$ and $r = x_1$, it is $u = u'[s/x] = u''[r/y]$ and $v = v'[s/x] = v''[r/y]$. Moreover, if $s' = x_2 y$, the following λ-term relations hold.

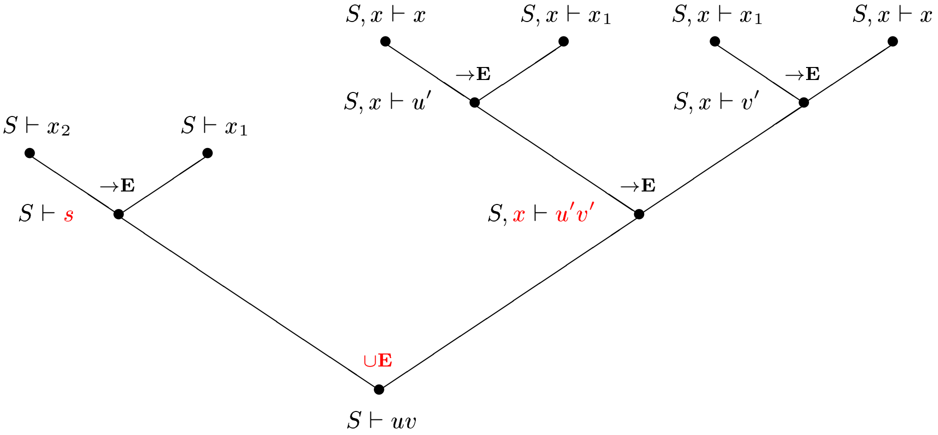
$$\begin{array}{ll} u' = xr & v' = rx \\ u'' = s'y & v'' = ys' \\ u = sr & v = rs \end{array}$$

If $\sigma = (\beta \to \gamma \to \alpha) \cap \delta$, $\tau = (\varepsilon \to \zeta \to \alpha) \cap \eta$, and $\rho = (\delta \to \gamma) \cap (\eta \to \zeta) \cap \beta \cap \varepsilon$, consider the IUT$^{\oplus}$-derivation $\pi_1 :: B_1 = \{\, x_1 : \rho,\, x_2 : \beta \to \sigma \cup \tau \,\} \vdash uv : \alpha$ and its tree $(T^{t}_{\text{iue}})_1$, exactly as given in the transformation counterexample of Chapter 6. The letter S denotes the set $\{x_1, x_2\}$.

$$\dfrac{\dfrac{B_1 \vdash x_2 : \beta \to \sigma \cup \tau \qquad \dfrac{B_1 \vdash x_1 : \rho}{B_1 \vdash x_1 : \beta}\,(\cap\mathbf{E})}{B_1 \vdash x_2 x_1 = s : \sigma \cup \tau}\,(\to\mathbf{E}) \qquad \overset{\text{see below}}{\pi_{11} :: B_1,\, x : \sigma \vdash xr\,(rx) = u'v' : \alpha} \qquad \overset{\text{see below}}{\pi_{12} :: B_1,\, x : \tau \vdash xr\,(rx) = u'v' : \alpha}}{\pi_1 :: B_1 \vdash sr\,(rs) = uv : \alpha}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B_1,\, x : \sigma \vdash x : \sigma}{B_1,\, x : \sigma \vdash x : \beta \to \gamma \to \alpha}\,(\cap\mathbf{E}_1) \quad \dfrac{B_1,\, x : \sigma \vdash x_1 : \rho}{B_1,\, x : \sigma \vdash x_1 : \beta}\,(\cap\mathbf{E})}{B_1,\, x : \sigma \vdash xx_1 : \gamma \to \alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_1,\, x : \sigma \vdash x_1 : \rho}{B_1,\, x : \sigma \vdash x_1 : \delta \to \gamma}\,(\cap\mathbf{E}) \quad \dfrac{B_1,\, x : \sigma \vdash x : \sigma}{B_1,\, x : \sigma \vdash x : \delta}\,(\cap\mathbf{E}_2)}{B_1,\, x : \sigma \vdash x_1 x : \gamma}\,(\to\mathbf{E})}{\pi_{11} :: B_1,\, x : \sigma \vdash xx_1(x_1 x) = u'v' : \alpha}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B_1,\, x:\tau \vdash x:\tau}{B_1,\, x:\tau \vdash x:\varepsilon\to\zeta\to\alpha}\,(\cap\mathbf{E}_1) \quad \dfrac{B_1,\, x:\tau \vdash x_1:\rho}{B_1,\, x:\tau \vdash x_1:\varepsilon}\,(\cap\mathbf{E}_2)}{B_1,\, x:\tau \vdash xx_1:\zeta\to\alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_1,\, x:\tau \vdash x_1:\rho}{B_1,\, x:\tau \vdash x_1:\eta\to\zeta}\,(\cap\mathbf{E}) \quad \dfrac{B_1,\, x:\tau \vdash x:\tau}{B_1,\, x:\tau \vdash x:\eta}\,(\cap\mathbf{E}_2)}{B_1,\, x:\tau \vdash x_1x:\zeta}\,(\to\mathbf{E})}{\pi_{12} :: B_1,\, x:\tau \vdash xx_1(x_1x) = u'v' : \alpha}\,(\to\mathbf{E})$$



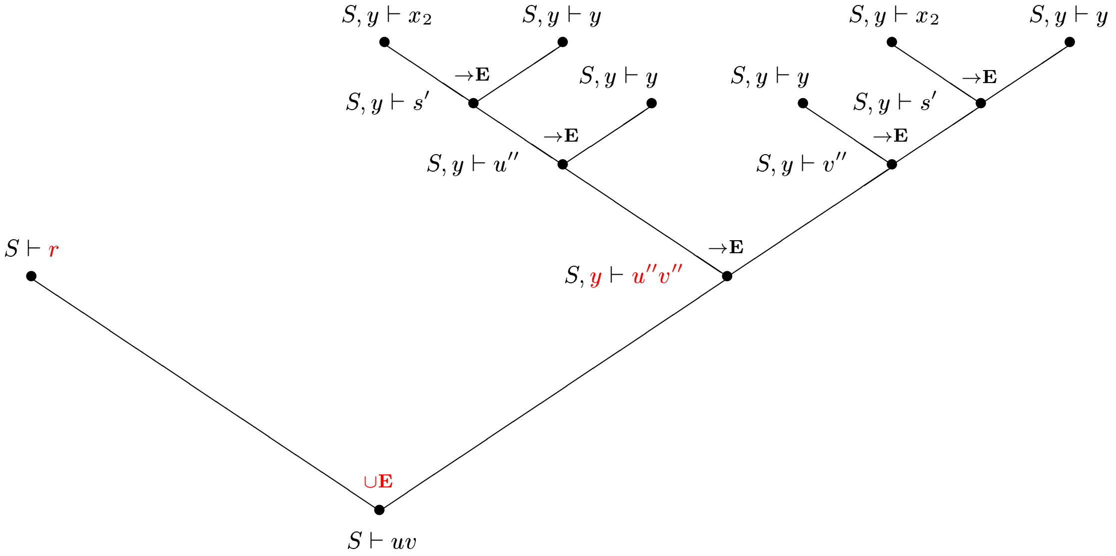
$(T^t_{\mathrm{iue}})_1$

If $\phi = (\zeta\to\alpha)\cap\varepsilon$, $\psi = (\zeta\to\gamma)\cap\varepsilon$, $\chi = \phi\cup\psi$, and $\upsilon = (\phi\to\phi_{\alpha\beta})\cap(\psi\to\psi_{\gamma\beta})\cap(\varepsilon\to\zeta)$, where $\phi_{\alpha\beta} = \phi\to\alpha\to\beta$ and $\psi_{\gamma\beta} = \psi\to\gamma\to\beta$, consider also $\pi_2 :: B_2 = \{\, x_1 : \chi,\, x_2 : \upsilon \,\} \vdash uv : \beta$ and its tree $(T^t_{\mathrm{iue}})_2$, as demonstrated below. For space economy, we denote B_ϕ and B_ψ the bases $B_2,\, y:\phi$ and $B_2,\, y:\psi$, respectively.

$$\dfrac{B_2 \vdash x_1 = r : \phi\cup\psi \qquad \overset{\text{see below}}{\pi_{21} :: B_2,\, y:\phi \vdash x_2yy\,(y\,(x_2y)) = u''v'' : \beta} \qquad \overset{\text{see below}}{\pi_{22} :: B_2,\, y:\psi \vdash x_2yy\,(y\,(x_2y)) = u''v'' : \beta}}{\pi_2 :: B_2 \vdash x_2rr\,(r\,(x_2r)) = uv : \beta}\,(\cup\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_\phi \vdash x_2 : \upsilon}{B_\phi \vdash x_2 : \phi\to\phi_{\alpha\beta}}\,(\cap\mathbf{E}) \quad B_\phi \vdash y:\phi}{B_\phi \vdash x_2y : \phi_{\alpha\beta}}\,(\to\mathbf{E}) \quad B_\phi \vdash y:\phi}{B_2,\, y:\phi \vdash x_2yy : \alpha\to\beta}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_\phi \vdash y:\phi}{B_\phi \vdash y:\zeta\to\alpha}\,(\cap\mathbf{E}_1) \quad \dfrac{\dfrac{B_\phi \vdash x_2:\upsilon}{B_\phi \vdash x_2:\varepsilon\to\zeta}\,(\cap\mathbf{E}_2) \quad \dfrac{B_\phi \vdash y:\phi}{B_\phi \vdash y:\varepsilon}\,(\cap\mathbf{E}_2)}{B_\phi \vdash x_2y:\zeta}\,(\to\mathbf{E})}{B_2,\, y:\phi \vdash y\,(x_2y) : \alpha}\,(\to\mathbf{E})}{\pi_{21} :: B_2,\, y:\phi \vdash x_2yy\,(y\,(x_2y)) = u''v'' : \beta}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_\psi \vdash x_2 : \upsilon}{B_\psi \vdash x_2 : \psi\to\psi_{\gamma\beta}}\,(\cap\mathbf{E}) \quad B_\psi \vdash y:\psi}{B_\psi \vdash x_2y : \psi_{\gamma\beta}}\,(\to\mathbf{E}) \quad B_\psi \vdash y:\psi}{B_2,\, y:\psi \vdash x_2yy : \gamma\to\beta}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_\psi \vdash y:\psi}{B_\psi \vdash y:\zeta\to\gamma}\,(\cap\mathbf{E}_1) \quad \dfrac{\dfrac{B_\psi \vdash x_2:\upsilon}{B_\psi \vdash x_2:\varepsilon\to\zeta}\,(\cap\mathbf{E}_2) \quad \dfrac{B_\psi \vdash y:\psi}{B_\psi \vdash y:\varepsilon}\,(\cap\mathbf{E}_2)}{B_\psi \vdash x_2y:\zeta}\,(\to\mathbf{E})}{B_2,\, y:\psi \vdash y\,(x_2y) : \gamma}\,(\to\mathbf{E})}{\pi_{22} :: B_2,\, y:\psi \vdash x_2yy\,(y\,(x_2y)) = u''v'' : \beta}\,(\to\mathbf{E})$$



$(T^{t}_{\text{iue}})_2$

Trying to *bottom-up* transform π_1, so that its bottom $(\cup\mathbf{E})$ is like the one in π_2, i.e. with term-statements $S \vdash r$ and $S, y \vdash u''v''$ at the major and minor premises, respectively, we end up with the following π'_1.

$$\dfrac{\dfrac{B_1 \vdash x_1 : \rho}{B_1 \vdash r = x_1 : \rho \cup \rho}\,(\cup\mathbf{I}) \qquad \dfrac{\begin{matrix}\pi'_{110}\ \text{(see below)}\\ B'_1 \vdash s' = x_2y : \sigma \cup \tau\end{matrix} \qquad \begin{matrix}\pi'_{111}\ \text{(see below)}\\ B'_1,\, x : \sigma \vdash xy\,(yx) : \alpha\end{matrix} \qquad \begin{matrix}\pi'_{112}\ \text{(see below)}\\ B'_1,\, x : \tau \vdash xy\,(yx) : \alpha\end{matrix}}{B'_1 = B_1 \cup \{\, y : \rho \,\} \vdash x_2yy\,(y\,(x_2y)) = s'y\,(ys') = u''v'' : \alpha}\,(\cup\mathbf{E})[1,\, s'] \qquad \text{same}}{\pi'_1 :: B_1 \vdash uv = x_2rr\,(r\,(x_2r)) : \alpha}\,(\cup\mathbf{E})[4,\, r]$$

$$\dfrac{B_1,\, y : \rho \vdash x_2 : \beta \to \sigma \cup \tau \qquad \dfrac{B_1,\, y : \rho \vdash y : \rho}{B_1,\, y : \rho \vdash y : \beta}\,(\cap\mathbf{E})}{\pi'_{110} :: B_1,\, y : \rho \vdash s' = x_2y : \sigma \cup \tau}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B'_1,\, x : \sigma \vdash x : \sigma}{B'_1,\, x : \sigma \vdash x : \beta \to \gamma \to \alpha}\,(\cap\mathbf{E}_1) \qquad \dfrac{B'_1,\, x : \sigma \vdash y : \rho}{B'_1,\, x : \sigma \vdash y : \beta}\,(\cap\mathbf{E})}{B'_1,\, x : \sigma \vdash xy : \gamma \to \alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B'_1,\, x : \sigma \vdash y : \rho}{B'_1,\, x : \sigma \vdash y : \delta \to \gamma}\,(\cap\mathbf{E}) \qquad \dfrac{B'_1,\, x : \sigma \vdash x : \sigma}{B'_1,\, x : \sigma \vdash x : \delta}\,(\cap\mathbf{E}_2)}{B'_1,\, x : \sigma \vdash yx : \gamma}\,(\to\mathbf{E})}{\pi'_{111} :: B'_1,\, x : \sigma \vdash xy\,(yx) : \alpha}\,(\to\mathbf{E})$$

$$\dfrac{\dfrac{\dfrac{B'_1,\, x : \tau \vdash x : \tau}{B'_1,\, x : \tau \vdash x : \varepsilon \to \zeta \to \alpha}\,(\cap\mathbf{E}_1) \qquad \dfrac{B'_1,\, x : \tau \vdash y : \rho}{B'_1,\, x : \tau \vdash y : \varepsilon}\,(\cap\mathbf{E}_2)}{B'_1,\, x : \tau \vdash xy : \zeta \to \alpha}\,(\to\mathbf{E}) \qquad \dfrac{\dfrac{B'_1,\, x : \tau \vdash y : \rho}{B'_1,\, x : \tau \vdash y : \eta \to \zeta}\,(\cap\mathbf{E}) \qquad \dfrac{B'_1,\, x : \tau \vdash x : \tau}{B'_1,\, x : \tau \vdash x : \eta}\,(\cap\mathbf{E}_2)}{B'_1,\, x : \tau \vdash yx : \zeta}\,(\to\mathbf{E})}{\pi'_{112} :: B'_1,\, x : \tau \vdash xy\,(yx) : \alpha}\,(\to\mathbf{E})$$

It is worth noting that the $(\cup\mathbf{E})[1, s']$ considered right above the $(\cup\mathbf{E})[4, r]$ is the only rule-application that works at that point. The $(\rightarrow\mathbf{E})$, the $(\cup\mathbf{E})[1, u'']$, the $(\cup\mathbf{E})[1, v'']$, and the other two possible $(\cup\mathbf{E})[1, s']$'s all fail. We cannot consider a $(\cup\mathbf{E})[2]$ or an $(\cap\mathbf{I})$. Comparing this transformation of π_1 with its counterpart in the transformation counterexample of Chapter 6 (see case $4b_1$), we observe the following.

counterexample		example	
$u''v'' = s'y\,(ys)$		$u''v'' = s'y\,(ys')$	
rule	outcome	rule	outcome
$(\rightarrow\mathbf{E})$	×	$(\rightarrow\mathbf{E})$	×
$(\cup\mathbf{E})[1, u'']$	×	$(\cup\mathbf{E})[1, u'']$	×
$(\cup\mathbf{E})[1, v'']$	×	$(\cup\mathbf{E})[1, v'']$	×
$(\cup\mathbf{E})[1, s']$	×	$(\cup\mathbf{E})[1, s']\,(i)$ $[\,xy\,(ys')\,]$	×
		$(\cup\mathbf{E})[1, s']\,(ii)$ $[\,s'y\,(yx)\,]$	×
		$(\cup\mathbf{E})[1, s']\,(iii)$ $[\,xy\,(yx)\,]$	✓
$(\cup\mathbf{E})[1, s]$	×	$(\cup\mathbf{E})[1, s]$	not
$(\cup\mathbf{E})[2]$	not	$(\cup\mathbf{E})[2]$	not
$(\cap\mathbf{I})$	not	$(\cap\mathbf{I})$	not

We then accordingly transform π_2 to π'_2, still working bottom-up.

$$\dfrac{B_2 \vdash r = x_1 : \phi \cup \psi \qquad \overset{\pi'_{21}\ \text{(see below)}}{B_2,\, y : \phi \vdash x_2yy\,(y\,(x_2y)) = s'y\,(ys') : \beta} \qquad \overset{\pi'_{22}\ \text{(see below)}}{B_2,\, y : \psi \vdash x_2yy\,(y\,(x_2y)) = s'y\,(ys') : \beta}}{\pi'_2 :: B_2 \vdash uv = x_2rr\,(r\,(x_2r)) : \beta}\ (\cup\mathbf{E})[2, r]$$

$$\dfrac{\overset{\text{see below}}{\pi'_{210} :: B_2,\, y : \phi \vdash s' : (\phi_{\alpha\beta} \cap \zeta) \cup (\phi_{\alpha\beta} \cap \zeta)} \qquad \overset{\text{see below}}{\pi'_{211} :: B_2,\, y : \phi,\, x : \phi_{\alpha\beta} \cap \zeta \vdash xy\,(yx) : \beta} \qquad \text{same}}{\pi'_{21} :: B_2,\, y : \phi \vdash s'y\,(ys') : \beta}\ (\cup\mathbf{E})[1, s']$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_2,\, y : \phi \vdash x_2 : \upsilon}{B_2,\, y : \phi \vdash x_2 : \phi \rightarrow \phi_{\alpha\beta}}\ (\cap\mathbf{E}) \qquad B_2,\, y : \phi \vdash y : \phi}{B_2,\, y : \phi \vdash x_2y : \phi_{\alpha\beta}}\ (\rightarrow\mathbf{E}) \qquad \dfrac{\dfrac{B_2,\, y : \phi \vdash x_2 : \upsilon}{B_2,\, y : \phi \vdash x_2 : \varepsilon \rightarrow \zeta}\ (\cap\mathbf{E}_2) \qquad \dfrac{B_2,\, y : \phi \vdash y : \phi}{B_2,\, y : \phi \vdash y : \varepsilon}\ (\cap\mathbf{E}_2)}{B_2,\, y : \phi \vdash x_2y : \zeta}\ (\rightarrow\mathbf{E})}{B_2,\, y : \phi \vdash x_2y : \phi_{\alpha\beta} \cap \zeta}\ (\cap\mathbf{I})}{\pi'_{210} :: B_2,\, y : \phi \vdash x_2y = s' : (\phi_{\alpha\beta} \cap \zeta) \cup (\phi_{\alpha\beta} \cap \zeta)}\ (\cup\mathbf{I})$$

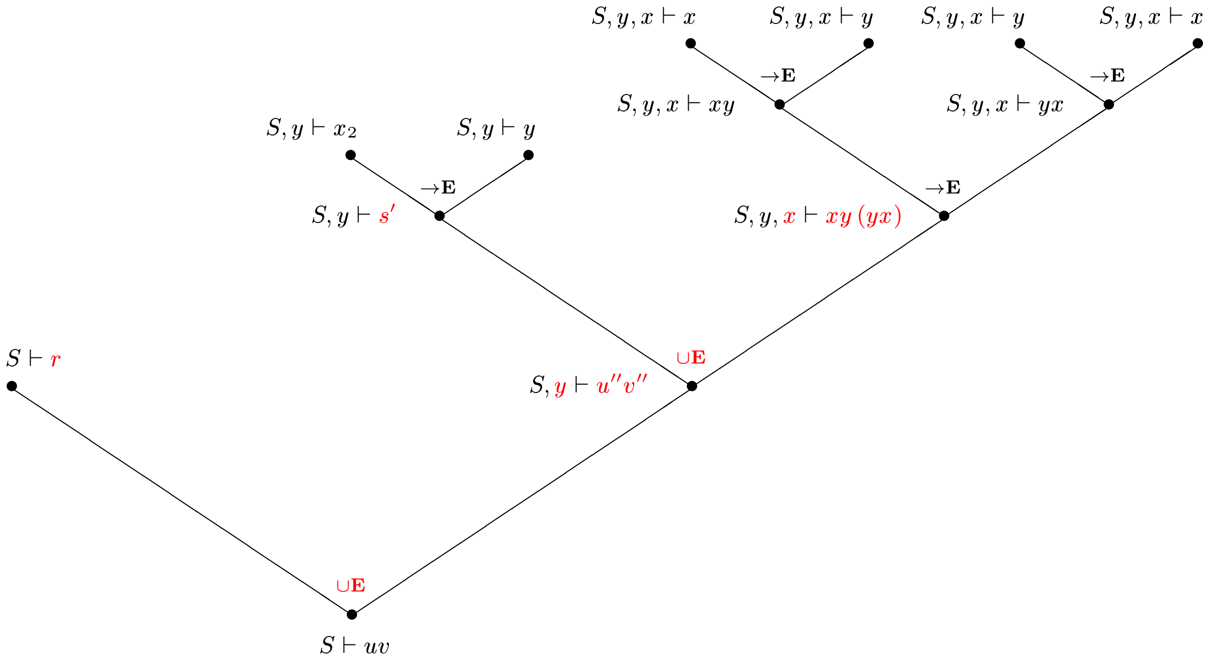
$$\dfrac{\dfrac{\dfrac{B_2' \vdash x : \phi_{\alpha\beta} \cap \zeta}{B_2' \vdash x : \phi_{\alpha\beta}}\;(\cap\mathbf{E}_1) \qquad B_2' \vdash y : \phi}{B_2' \vdash xy : \alpha \to \beta}\;(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_2' \vdash y : \phi}{B_2' \vdash y : \zeta \to \alpha}\;(\cap\mathbf{E}_1) \qquad \dfrac{B_2' \vdash x : \phi_{\alpha\beta} \cap \zeta}{B_2' \vdash x : \zeta}\;(\cap\mathbf{E}_2)}{B_2' \vdash yx : \alpha}\;(\to\mathbf{E})}{\pi'_{211} :: B_2' = B_2 \cup \{\, y : \phi,\, x : \phi_{\alpha\beta} \cap \zeta \,\} \vdash xy\,(yx) : \beta}\;(\to\mathbf{E})$$

$$\dfrac{\overset{\text{see below}}{\pi'_{220} :: B_2,\, y : \psi \vdash s' : (\psi_{\gamma\beta} \cap \zeta) \cup (\psi_{\gamma\beta} \cap \zeta)} \qquad \overset{\text{see below}}{\pi'_{221} :: B_2,\, y : \psi,\, x : \psi_{\gamma\beta} \cap \zeta \vdash xy\,(yx) : \beta} \qquad \text{same}}{\pi'_{22} :: B_2,\, y : \psi \vdash s'y\,(ys') : \beta}\;(\cup\mathbf{E})[1, s']$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B_2,\, y : \psi \vdash x_2 : \upsilon}{B_2,\, y : \psi \vdash x_2 : \psi \to \psi_{\gamma\beta}}\;(\cap\mathbf{E}) \qquad B_2,\, y : \psi \vdash y : \psi}{B_2,\, y : \psi \vdash x_2 y : \psi_{\gamma\beta}}\;(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_2,\, y : \psi \vdash x_2 : \upsilon}{B_2,\, y : \psi \vdash x_2 : \varepsilon \to \zeta}\;(\cap\mathbf{E}_2) \qquad \dfrac{B_2,\, y : \psi \vdash y : \psi}{B_2,\, y : \psi \vdash y : \varepsilon}\;(\cap\mathbf{E}_2)}{B_2,\, y : \psi \vdash x_2 y : \zeta}\;(\to\mathbf{E})}{B_2,\, y : \psi \vdash x_2 y : \psi_{\gamma\beta} \cap \zeta}\;(\cap\mathbf{I})}{\pi'_{220} :: B_2,\, y : \psi \vdash x_2 y = s' : (\psi_{\gamma\beta} \cap \zeta) \cup (\psi_{\gamma\beta} \cap \zeta)}\;(\cup\mathbf{I})$$

$$\dfrac{\dfrac{\dfrac{B_2'' \vdash x : \psi_{\gamma\beta} \cap \zeta}{B_2'' \vdash x : \psi_{\gamma\beta}}\;(\cap\mathbf{E}_1) \qquad B_2'' \vdash y : \psi}{B_2'' \vdash xy : \gamma \to \beta}\;(\to\mathbf{E}) \qquad \dfrac{\dfrac{B_2'' \vdash y : \psi}{B_2'' \vdash y : \zeta \to \gamma}\;(\cap\mathbf{E}_1) \qquad \dfrac{B_2'' \vdash x : \psi_{\gamma\beta} \cap \zeta}{B_2'' \vdash x : \zeta}\;(\cap\mathbf{E}_2)}{B_2'' \vdash yx : \gamma}\;(\to\mathbf{E})}{\pi'_{221} :: B_2'' = B_2 \cup \{\, y : \psi,\, x : \psi_{\gamma\beta} \cap \zeta \,\} \vdash xy\,(yx) : \beta}\;(\to\mathbf{E})$$

The trees $(T^{\,t}_{\mathrm{iue}})'_1$ and $(T^{\,t}_{\mathrm{iue}})'_2$ both exist and are identical, as required.



$(T^{\,t}_{\mathrm{iue}})'_1 = (T^{\,t}_{\mathrm{iue}})'_2$

Investigating closely the transformation counterexample in Chapter 6 and the transformation example given here, we note the following. In the counterexample, the terms u' and v' are symmetric with respect to application ($u' = xr,\ v' = rx$), while u'' and v'' are not ($u'' = s'y,\ v'' = ys$). On the contrary, in the example, both u', v' and u'', v'' are symmetric with respect to application ($u' = xr,\ v' = rx$ and $u'' = s'y,\ v'' = ys'$). If $(u'v')[s/x] = sr\,(rs) = uv = x_2rr\,(r\,(x_2r)) = (u''v'')[r/y]$, there are three different choices for $u'v'$, one of which employs symmetric-with-respect-to-application u' and v', and fifteen different choices for $u''v''$, three of which employ symmetric u'' and v''.

	$u'v'$	symmetry
1	$xr\,(rs)$	no
2	$sr\,(rx)$	no
3	$xr\,(rx)$	✓

	$u''v''$	symmetry
1	$x_2yr\,(r\,(x_2r)) = s'r\,(rs)$	no
2	$x_2ry\,(r\,(x_2r)) = sy\,(rs)$	no
3	$x_2rr\,(y\,(x_2r)) = sr\,(ys)$	no
4	$x_2rr\,(r\,(x_2y)) = sr\,(rs')$	no
5	$x_2yy\,(r\,(x_2r)) = s'y\,(rs)$	no
6	$x_2yr\,(y\,(x_2r)) = s'r\,(ys)$	no
7	$x_2yr\,(r\,(x_2y)) = s'r\,(rs')$	✓
8	$x_2ry\,(y\,(x_2r)) = sy\,(ys)$	✓
9	$x_2ry\,(r\,(x_2y)) = sy\,(rs')$	no
10	$x_2rr\,(y\,(x_2y)) = sr\,(ys')$	no
11	$x_2ry\,(y\,(x_2y)) = sy\,(ys')$	no
12	$x_2yr\,(y\,(x_2y)) = s'r\,(ys')$	no
13	$x_2yy\,(r\,(x_2y)) = s'y\,(rs')$	no
14	$x_2yy\,(y\,(x_2r)) = s'y\,(ys)$	no
15	$x_2yy\,(y\,(x_2y)) = s'y\,(ys')$	✓

It would be interesting to further examine if all the combinations which involve symmetry for both $u'v'$ and $u''v''$ can provide transformation examples, i.e. if, besides combination 3-15, which is met in the example presented here, combinations 3-7 and 3-8 can also provide transformation examples. It would also be interesting to test if all the rest combinations can deliver transformation counterexamples; the counterexample in Chapter 6 uses combination 3-14. These conjectures and their likely consequences are left open for future study.

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