SYMPLECTIC REALIZATIONS

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Les études géométriques m'ont incité, entre 1896 et 1871, á traiter les groupes finis et continus. Certes, je me suis d'abord limité á transformer certains groupes continus importants par des transformations analytiques convenables (transformations de contact algébriques ou transendantes) en d'autres groupes connus ; sous ce rapport, les travaux qui s'y rapportaient avaient un caractére special[...]. Commencées également en 1869, les études sur les équations différentielles admettant un groupe continu étaient de nature plus générale. J'ai remarqué que la plupart des équations différentielles, dont l'intégration ne réussit pas par les anciennes méthodes d'intégration, restent invariantes par certaines transformations, et que ces méthodes d'intégration consistent dans l'application de cette propriété á une équation différentielle appropriée [...]. Ayant ainsi représenté du point du vue général plusieurs anciennes méthodes d'intégration, je me suis posé un probléme naturel: développer la th'eorie d'intégration générale pour toutes les équations différentielles ordinaires admettant des transformations finies ou infinitésimales.

Sophus Lie [1]

[1] Theorie der Transformationsgruppen, unter Mitwirkung von Friedrich Engel, 3 vol., Leipzig, B.G. Teubner, 1888-1893, t. 1, p. iv-v.

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Περίληψη[2]

Το 1983, ο Alan Weinstein δημοσίευσε μια πρωτοποριακή εργασία, η οποία έθεσε τα θεμέλια της σύγχρονης αντιμετώσης της Γεωμετρίας Poisson. Στην εν λόγω εργασία, υπάρχουν τρείς σημαντικές ιδέες.

- Το Θεώρημα Splitting, το οποίο δίνει την τοπική δομή μιας πολλαπλότητας Poisson. Στην πραγματικότητα, το Splitting αποτελεί μια πρώτη υπόδειξη ότι οι πολλαπλότητες Poisson είναι ουσιαστικά φυλλώδεις δομές με συμπλεκτικά φύλλα που συχνά παρουσιάζουν ιδιομορφίες. Επίσης, το ίδιο θεώρημα αποτελεί την απαρχή της Τοπολογίας Poisson, ενός πεδίου που αυξάνεται με ταχείς ρυθμούς στις μέρες μας.

- Μέσω του Splitting, καταλαβαίνει κανείς ότι οι πολλαπλότητες Poisson είναι αρκετά περίπλοκες δομές. Στην προσπάθεια για απλοποίηση, και στηριζόμενος στις ιδέες του Sophus Lie, ο A. Weinstein διατύπωσε το πρόβλημα της συμπλεκτικής υλοποίησης μιας δομής Poisson, (P,Π). Δηλαδή, υλοποίηση αυτής ως πηλίκο μιας συμπλεκτικής πολλαπλότητας (S,ω) με μια απεικόνιση $S \to P$, η οποία είναι επί και Poisson.

  Εφόσον έχουμε μια τέτοια συμπλεκτική υλοποίηση, μπορούμε να αντιμετωπίζουμε τα διάφορα προβλήματα που έχουμε στο επίπεδο της (P,Π) ως προβλήματα στο επίπεδο της (S,ω), όπου η αντιμετώπισή τους είναι σημαντικά απλούστερη. Έχοντας την απεικόνιση $S \to P$, η ελπίδα είναι ότι οι λύσεις στο επίπεδο της S, θα 'μεταφέρονται' προς τα κάτω σε λύσεις στο επίπεδο της P. Επιπλέον, γνωρίζουμε ότι, τοπικά, υπάρχουν πάντα τέτοιου είδους συμπλεκτικές υλοποιήσεις.

- Ψάχνοντας για ολικές συμπλεκτικές υλοποίησεις, ο A.Weinstein εξετάζει σαν παράδειγμα την περίπτωση των δομών Lie-Poisson. Έδειξε ότι μια τέτοια υλοποίηση είναι T^*G, η οποία, άλλωστε, φέρει με φυσικό τρόπο, μια δομή ομαδοειδούς Lie.

  Με άλλα λόγια, για να βρούμε μια ολική συμπλεκτική υλοποίηση, θα πρέπει να ψάχνουμε για ένα κατάλληλο ομαδοειδές Lie.

Όλα τα παραπάνω μας οδηγούν στην μελέτη της Γεωμετρίας Poisson στα πλαίσια της θεωρίας των αλγεβροείδων και ομαδοειδών Lie. Πράγματι, αποδεικνύεται ότι μια δομή Poisson P είναι το ίδιο με μια δομή αλγεβροειδούςLie στη συνεφαπτομένη δέσμη T^*P.

Αποδεικνύεται, λοιπόν, ότι μια ολική συμπλεκτική υλοποίηση αντιστοιχεί σε μια ολοκλήρωση του αλγεβροειδούςLie T^*P. Έτσι, στην πραγματικότητα, το πρόβλημα της ολικής συμπλεκτικής υλοποίησης είναι ουσιαστικά ένα πρόβλημα ολοκληρωσιμότητας.

Σκοπός της παρούσας μεταπτυχιακής εργασίας, είναι να κάνει μια γρήγορη εισαγωγή στη Γεωμετρία Poisson και να παρουσιάσει το ρόλο των αλγεβροείδων Lie καθώς, επίσης, και της Θεωρίας για Φυλλώδεις Δομές στην εν λόγω θεωρία, μελετώντας τα προαναφερθέντα αποτελέσματα.

[2] Abstract in Greek language

Contents

1 Introduction

Historical overview. Galileo Galilei's principle of relativity and Newton's differential equation constitute the foundations of Classical mechanics.

Sir Isaac Newton, in his attempt to explain the laws of Kepler in celestial mechanics, introduced what is, nowadays, known as Newtonian mechanics.

Newtonian mechanics studies the motion of a system of a point masses in three-dimensional euclidean space.

Some years later, Joseph-Louis Lagrange described motion in a mechanical system by means of the configuration space. Since the configuration space has the structure of a differentiable manifold, a Lagrangian mechanical system is given by a manifold (configuration space) together with a function on the tangent bundle. Here, the newtonian potential system is a particular case of a lagrangian system (Hamilton's principle of least action).

The Legendre transformation of the lagrangian function gives the hamiltonian function. Therefore, Hamiltonian mechanics arises naturally, since it is geometry in the cotangent bundle of the configuration space. The basis of this concept is the Legendre transformation, mentioned previously, between the tangent and the cotangent bundles.

A Hamiltonian mechanical system is given by a symplectic manifold (phase space) and a function on it (Hamiltonian function H). In this way, Lagrangian mechanics is contained in Hamiltonian mechanics as a special case, and Lagrange's equation of motion are, now, translated into Hamilton's equations:

$$\dot{q} = \frac{\partial H}{\partial p}$$

and

$$\dot{p} = \frac{\partial H}{\partial q}$$

where $(q(t), p(t))$ are the configuration coordinates of the mechanical system.

The description of motion in Mechanics is the origin for Poisson geometry. Simeon Denis Poisson, in 1809, introduced the notion of the Poisson bracket between any two smooth functions f and g, by setting

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right)$$

Once a Hamiltonian function is fixed, Hamilton's equations can be written with the help of the Poisson bracket:

$$\dot{p} = \{H, p\}$$

and,

$$\dot{q} = \{H, q\}$$

Many properties of Hamilton's equations can be rephrased via the Poisson bracket. Carl Jacobi, around 1842, showed that the Poisson bracket satisfies the famous Jacobi identity. Moreover, a Poisson bracket satisfies the following properties: skew-symmetry, $\mathbb{R}$-bilinearity and Leibniz rule. The axiomatization of these properties leads to the abstract definition of Poisson bracket, and, consequently, in Poisson geometry.

The history of Poisson manifolds is complicated by the fact that the notion was rediscovered many times under different names; they occur in the works of Lie (1890) [40], Dirac (1930 [25], 1964 [26]), Pauli (1953) [51], Martin (1959) [46], Jost (1964) [33], Arens (1970) [6], Hermann (1973) [31], Sudarshan and Mukunda (1974) [58], Vinogradov and Krasilshchik (1975) [62], and Lichnerowicz (1975) [42]. The name Poisson manifold was coined by Lichnerowicz.

The geometry of Poisson structures, which began as an outgrowth of symplectic geometry, has seen rapid growth in the last decades, and has now become a very large theory with interactions with many other domains of mathematics, including Hamiltonian dynamics, integrable systems, representation theory, quantum groups, quantization, noncommutative geometry, singularity theory and so on.

Nowadays, a Poisson structure on a manifold P is a bivector field Π such that the Poisson bracket is defined by:

$$\{f,g\} = < df \wedge dg, \Pi >$$

This notion of Poisson manifolds generalizes both symplectic manifolds and Lie algebras. For instance, every symplectic manifold has a natural Poisson bracket and every Poisson bracket determines a foliation of the manifold by symplectic submanifolds. On the other hand, every finite-dimensional Lie algebra gives rise to a linear Poisson tensor on its dual space and vice versa.

Scope of this dissertation. In 1983, Alan Weinstein published a groundbreaking paper [64], which set the foundations of the modern treatment of the theory. In that paper, there are three important ideas.

- Splitting theorem, which gives the local structure of Poisson manifolds. Actually, the Splitting theorem is the first instance of the fact that Poisson manifolds are foliations with symplectic leaves, often presenting singularities. Also, the Splitting theorem is the beginning of Poisson Topology, a field which is growing rapidly these days.

- The Splitting theorem shows that the Poisson manifolds are quite complicated structures. In the effort to simplify them, and based in the ideas by Shopus Lie, A.Weinstein postulated the problem of symplectic realizations of Poisson manifolds. That is realize a Poisson manifold (P,Π), as a quotient of a symplectic manifold (S,ω) under a surjective submersion $S \to P$, which is a Poisson map.

  Once we have a symplectic realization as such, we may be able to lift problems from (P,Π) to (S,ω), which is considerably simpler. Since the submersion $S \to P$ is a Poisson map, the hope is that the solutions at that level of S, will pushed down to solutions at the level of P. He also proved that locally, symplectic realizations as such always exist.

- Looking for global symplectic realizations A.Weinstein examined the example of the case of Lie-Poisson structures. He showed that such a realization is T^*G, which, moreover, carries a natural Lie groupoid structure. This relates the symplectic realization problem with the external symmetries of the structure (see [66]).

In other words, to find a global symplectic realization, one should be looking for an appropriate Lie groupoid.

All the above points put Poisson geometry at the context of Lie algebroids and Lie groupoids. Indeed, it turns out that a Poisson structure on P is the same as a Lie algebroid structure on the cotangent bundle T^*P (see [32] for a detailed history).

It turns out that global symplectic realizations correspond to an integration of the Lie algebroid T^*P. So, in fact, the global symplectic realization problem is really a problem of integrability.

The scope of this dissertation is to present a quick introduction to Poisson geometry and the role of Lie algebroids and foliations in the theory, and to present the above results.

Structure of this dissertation. This dissertation consists of six chapters and one appendix. Here, is a brief summary of them, where Chapter 1 is the introduction of the dissertation.

In Chapter 2, we, firstly, recall basic notions and results concerning symplectic structures, such as Hamiltonian vector fields, local forms, the Liouville form of the cotangent bundle, symplectomorphisms and Lagrangian submanifolds. Then we introduce and study classical topics in Poisson geometry, including Hamiltonian vector fields, Poisson brackets, singular foliations. We also express Poisson structures, in terms of bivector fields and we study the Lie-Poisson structure on the dual of a Lie algebra. We conclude Chapter 2 by showing that attached to each Poisson manifold $(P.\Pi)$ there exists a natural Lie algebroid structure on the cotangent bundle of P.

Chapter 3 is about local structure of Poisson manifolds. Here, we prove A.Weinstein's Splitting theorem and we show that coadjoint orbits of a Lie group are symplectic. In fact, these symplectic manifolds are the symplectic leaves of the Lie-Poisson bracket.

In Chapter 4 we explain the links among S.Lie's original ideas on function groups, local and global symplectic realization problem. We then prove the local existence of such realizations.

Chapter 5 is about global symplectic realizations. In particular, we examine the example of the case of the Lie-Poisson structures.

Chapter 6 is a discussion on the problem of integrability of Lie algebroids, and in particular, Poisson structures.

Finally, we conclude this dissertation with Appendix A. Here, we give an introduction in the theory of foliations as it evolves through the centuries. Starting from regular foliations and Frobenius theorem, we extend the theory to the singular case, and study the latter from a different perspective, presented by I.Androulidakis and G.Skandalis in [4].

2 Background on Poisson manifolds

2.1 Symplectic structures

The starting point of the theory of Poisson manifolds is symplectic geometry. Not only do the symplectic manifolds offer the most basic Poisson bracket, but the geometry of these manifolds, is the source of ideas on which the new theory, that of Poisson manifolds is based. We follow closely [10], [14] and [15].

2.1.1 Symplectic structures

In this section we will present basic definitions, properties and results concerning symplectic structures.

Definition 2.1. On a manifold $\mathscr{M}$ a closed non-degenerate 2-form ω is called a *symplectic form*, that is an $\omega \in \Omega^2(\mathscr{M})$ such that:

a) $d\omega = 0$ (closedness),

b) on each tangent space $T_p\mathscr{M}, p \in \mathscr{M}$, if $\omega_p(X,Y) = 0$ for all $Y \in T_p\mathscr{M}$ then $X = 0$ (nondegeracy).

So a *symplectic structure* is a pair $(\mathscr{M}, \omega)$, where $\mathscr{M}$ is a manifold and ω is a symplectic form. In this case, we call $\mathscr{M}$ a *symplectic manifold*.

Remark 2.2. Saying that ω is non-degenerate means that the bundle map $\omega^\flat : T\mathscr{M} \to T^*\mathscr{M}$ defined, for each $X \in TM$, by $\omega^\flat(X) = i_X\omega$ or, equivalently, by $\langle \omega^\flat(X), Y\rangle = \omega(X,Y)$, is an isomorphism.

The fact that ω is non-degenerate also implies that $\mathscr{M}$ must be even-dimensional. Indeed, let $d = dim\mathscr{M}$, then the bundle map $\omega^\flat$ is represented by a skew-symmetric $n \times n$-matrix, denoted by Ω. Since ω is non-degenerate, $det\Omega \neq 0$. However, Ω is skew-symmetric, so $\Omega = -\Omega^T$. This means that $det\Omega = det(\Omega^T) = det(-\Omega) = (-1)^n det\Omega$, which implies $d = 2k$.

Examples 2.3. a) A simple example is the 2-sphere with its standard area 2-form ω given by the formula $\omega_x(u,v) = \langle x, u \times v\rangle$ for $u, v \in T_xS^2$ and $x \in S^2$, where $\langle .,. \rangle$ is the inner product and $\times$ is the exterior product. This form is closed because it is of top degree, and it is nondegenerate because $\langle x, u \times v\rangle \neq 0$ when $u \neq 0$.

b) Let's generalize this class of examples by considering an oriented surfase $M \subset \mathbb{R}^3$. The Gauss map $N : M \to S^2$ associates to every $x \in M$ the outward unit normal vector $N(x) \perp T_xM$. Then, as in the case of S^2, the formula $\omega_x(u,v) = \langle N(x), u \times v\rangle$ for $u, v \in T_xM$ defines a symplectic 2-form on M.

c) For every positive integer n, the space $\mathbb{R}^{2n}$ is a symplectic manifold, by considering on each tangent space $T_m\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ the symplectic vector space structure. If

$$dq^1, dq^2, ..., dq^n, dp_1, dp_2, ..., dp_n$$

are the basic differential 1-forms on $\mathbb{R}^{2n}$, then the symplectic structure is defined by the 2-form

$$\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

Let us show that ω_0 is a symplectic 2-form. Recall that for coordinates $dx_1, dx_2, ..., dx_n$, we calculate the differential of a 2-form $\alpha = \sum_{i,j} f dx_i \wedge dx_j$, to be $d\alpha = \sum_{i,j} df \wedge dx_i \wedge dx_j$. Here, the coefficients of dq^i, dp_i are constants, so that ω_0 is a closed 2-form is obvious.

It remains to prove the nondegeracy. In order to so, we will show that if $X \neq 0$, then $\omega(X,Y) \neq 0$ for all $Y \in T\mathbb{R}^{2n}$. We consider a non zero vector field $X \in T\mathbb{R}^{2n}$, this means that

$$X = \sum_{i=1}^{n} (a_i dq^i + b_i dp_i)$$

where $a_i \neq 0$ or $b_i \neq 0$. So

$$\omega_0(X,.) = \sum_{i=1}^{n} (b_i \frac{\partial}{\partial q^i} - a_i \frac{\partial}{\partial p_i}).$$

Concluding that

$$\omega_0(X, \frac{\partial}{\partial q^i}) = \sum_{i=1}^{n} (-a_i)$$

and, similarly,

$$\omega_0(X, \frac{\partial}{\partial p_i}) = \sum_{i=1}^{n} (b_i)$$

which completes the proof.

Now let V be a finite-dimensional, real vector space, and V^* its dual. The space $\bigwedge V^*$ denotes the exterior product of copies of the space V^* and can be identified with the space of skew-symmetric bilinear forms ω.

Definition 2.4. Let V a finite-dimensional, real vector space equipped with symplectic structure ω, then the pair (V, ω) is called *symplectic vector space.*

The next theorem states that there is a (canonical) basis, by a skew symmetric version of the Gram-Schmidt process, for which a skew-symmetric bilinear form can be written in the standard form for skew-symmetric bilinear maps.

Theorem 2.5. ***(Standard Form for Skew-symmetric Bilinear Maps)***

Let (V, ω) be a symplectic vector space. Then V admits a basis $e_1, ..., e_n, f_1, ..., f_n$ satisfying

$$\omega(e_i, f_j) = \delta_{ij} \text{ and}$$
$$\omega(e_i, e_j) = 0 = \omega(f_i, f_j) = 0.$$

Moreover, if $e^1,...,e^n,f^1,...,f^n$ *is the dual basis. Then*

$$\omega = e^1 \wedge f^1 + ... + e^n \wedge f^n$$

Such a basis is then called a symplectic basis *of* (V,ω).

Proof. Let $e_1,...,e_n$ be a basis of V and $e^1,...,e^n$ be the corresponding dual basis of V^*. If $\alpha_{ij} = \omega(e_i, e_j)$ for $i < j$, then

$$\omega = \sum_{i<j} \alpha_{ij} e^i \wedge e^j$$

We assume that $\omega \neq 0$, because if $\omega = 0$ then it is trivial. Since $\omega \neq 0$, there are some $1 \leq i \leq j \leq n$ such that $\alpha_{ij} \neq 0$. We may assume that $\alpha_{12} \neq 0$, changing the numbering if necessary. Thus, we have that

$$\omega = \left(e^1 - \frac{\alpha_{23}}{\alpha_{12}} e^3 - ... - \frac{\alpha_{2n}}{\alpha_{12}} e^n\right) \wedge \left(\alpha_{12} e^2 + ... + \alpha_{1n} e^n\right) + \omega_1$$

Let

$$f_1 = e^1 - \frac{\alpha_{23}}{\alpha_{12}} e^3 - ... - \frac{\alpha_{2n}}{\alpha_{12}} e^n$$
$$f_2 = \alpha_{12} e^2 + ... + \alpha_{1n} e^n$$

the set $f_1, f_2, e^3, ..., e^n$ is now a new basis of V^*. If $\omega_1 = 0$, we are done. Otherwise, we repeat the above taking ω_1 in the place of ω. So, inductively, we arrive at the conclusion, since V has finite dimension. □

Example 2.6. Let the symplectic manifold $(\mathscr{M}, \omega_0) = (\mathbb{R}^{2n}, \sum_{i=1}^n dq^i \wedge dp_i)$. By example 2.3, it is an easy consequence that the set

$$\left\{\left(\frac{\partial}{\partial q^1}\right)_m, ..., \left(\frac{\partial}{\partial q^n}\right)_m, \left(\frac{\partial}{\partial p_1}\right)_m, ..., \left(\frac{\partial}{\partial p_n}\right)_m\right\}$$

is a symplectic basis of $T_m\mathscr{M}$.

Inspired by this theorem, we will describe normal neighborhoods of a point with Darboux's theorem and generalize this result to Poisson manifolds.

2.1.2 Hamiltonian vector fields

In remark 2.2 we estabished an isomorphism $\omega^\flat : T\mathscr{M} \to T^*\mathscr{M}$ between the spaces of tangent vectors and 1-forms. Now, we consider the inverse isomorphism $(\omega^\flat)^{-1} : T^*\mathscr{M} \to T\mathscr{M}$, and let f be a smooth function on a symplectic manifold $\mathscr{M}$. Then the differential df is a smooth section of $\Gamma(T^*\mathscr{M}) = \Omega^1(\mathscr{M})$, and via the bundle map $(\omega^\flat)^{-1} : \Gamma(T^*\mathscr{M}) \equiv \Omega^1(\mathscr{M}) \to \Gamma(T\mathscr{M}) \equiv \mathfrak{X}(\mathscr{M})$ we obtain the following definition of a vector field $(\omega^\flat)^{-1}(df)$ on $\mathscr{M}$.

Definition 2.7. Let (M,ω) be a symplectic manifold. To each $f \in C^\infty(M)$ we associate a vector field X_f, defined by

$$X_f = (\omega^\flat)^{-1}(df)$$

called the *Hamiltonian vector field* associated to f. We say that the function f is a *Hamiltonian* of the field X_f.

Remarks 2.8. a) Equivalently, the symplectic form ω makes possible the identification of

$$\omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$$
$$X \mapsto \omega^\flat(X)$$

with

$$\omega^\flat(X)(Y) = \omega(X,Y)$$

for $X, Y \in T\mathcal{M}$. So we have that

$$i_{X_f}\omega = \omega(X_f) = \omega^\flat(X_f) = \omega^\flat((\omega^\flat)^{-1}(df)) = df$$

and we obtain $i_{X_f}\omega = df$, where this can be rewritten as $(i_{X_f}\omega)(Y) = \omega(X_f, Y) = df(Y) = Y(f)$.

b) Basically, we observe that the existence of the Hamiltonian vector fields is guaranteed by the nondegeneracy of the symplectic structure ω. Thus, we can define a vector field ξ to be *Hamiltonian*, if there exists $f \in C^\infty(\mathcal{M})$ such that

$$i_\xi\omega = df.$$

We note that this is the definition we find in the literature.

Example 2.9. If $\mathcal{M} = \mathbb{R}^{2n}$ with the symplectic 2-form

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i$$

in any local coordinate system $(\mathcal{U}, (q^1, ..., q^n, p_1, ..., p_n))$ of $(\mathcal{M}, \omega)$, we calculate the Hamiltonian vector field X_f, of a function $f \in C^\infty(\mathcal{M})$ as follows.

By definition of vector fields we have

$$X_f = \sum_{i=1}^{n} \Big(a_i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}\Big)$$

where a_i, b_i are smooth coefficient functions on $\mathcal{U}$, so we compute

$$i_{X_f}\omega = \sum_{i=1}^{n} \Big(dq^i(X_f)dp_i - dp_i(X_f)dq^i\Big) = \sum_{i=1}^{n} \Big(a_i dp_i - b_i dq^i\Big).$$

On the other hand, we know that the differential of the function f is

$$df = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i\Big).$$

Consequently, a Hamiltonian vector field is written as

$$X_f = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}\Big)$$

and we note that $X_f(g) = \omega(X_f, X_g)$.

Remark 2.10. Since the map $\omega^\flat : TM \to T^*M$ is an isomorphism, *every* $f \in C^\infty(M)$ corresponds to a Hamiltonian vector field. This provides symplectic manifolds with extremely rich dynamics. Note that Riemannian structures, albeit they are similar to symplectic, do not have this property. In the next section we discuss a much deeper, topological property of symplectic manifolds.

2.1.3 Local structure

We would like to classify symplectic manifolds up to symplectomorphism. A global realization of this goal is very hard, but the Darboux theorem takes care of this classification locally: the dimension is the only local invariant of symplectic manifolds up to symplectomorphisms. The main tool is Moser's trick, which leads to Moser's theorems, which are extremely useful for many arguments in symplectic geometry (see [49]).

Definition 2.11. Let $\mathscr{M}$ be a manifold and consider a map $\rho : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$. Denote $\rho_t(p) := \rho(p,t)$. We say that ρ is an *isotopy* if each $\rho_t : \mathscr{M} \to \mathscr{M}$ is a diffeomorphism, and $\rho_0 = id_{\mathscr{M}}$.

Definition 2.12. Let $(\mathscr{M},\omega_1)$ and $(\mathscr{M},\omega_2)$ be $2n$-dimensional symplectic manifolds. We say that

a) $(\mathscr{M},\omega_1)$ and $(\mathscr{M},\omega_2)$ are *symplectomorphic* if there is a diffeomorphism $g : \mathscr{M} \to \mathscr{M}$ with $g^*\omega_2 = \omega_1$. Such g is called a *symplectomorphism.*

b) $(\mathscr{M},\omega_1)$ and $(\mathscr{M},\omega_2)$ are *strongly isotopic* if there is an isotopy $\rho_t : \mathscr{M} \to \mathscr{M}$ such that $\rho_1^*\omega_2 = \omega_1$.

Remark 2.13. Clearly, the notion of strongly isotopy is more powerful than the notion of symplectomorphism. Hence, if the symplectic forms ω_1 and ω_2 are strongly isotopic, then obviously they are symplectomorphic.

Lemma 2.14. *Let $\mathscr{M}$ be a compact manifold then the isotopies of $\mathscr{M}$ are in one-to-one correspondence with the time-dependent vector fields on $\mathscr{M}$.*

Proof. (sketch, a more detailed approach can be found in [14]) Given an isotopy ρ, we obtain a time-dependent vector field, that is, a family of vector fields X_t, $t \in \mathbb{R}$ which at $p \in \mathscr{M}$ satisfy:

$$X_t(p) = \frac{d}{ds}\rho_s(q)\Big|_{s=t}$$

where $q = \rho_t^{-1}(p)$. This means that

$$\frac{d\rho_t}{dt} = X_t \circ \rho_t.$$

That is, the velocity vector of the curve $t \mapsto \rho_t(q)$ at time t, which is a tangent vector to $\mathscr{M}$ at the point $p = \rho_t(q)$.

Conversely, given a time-dependent vector field X_t, $t \in \mathbb{R}$, we consider that $X_t \circ \rho_t = \dfrac{d\rho_t}{dt}$ and $\rho_0 = id_{\mathscr{M}}$. Since $\mathscr{M}$ is compact, then by solving the previous ordinary differential equation, there exists an isotopy ρ. $\square$

Definition 2.15. The *Lie derivative* by a time-dependent vector field X_t is

$$\mathscr{L}_{X_t} : \Omega^k(\mathscr{M}) \to \Omega^k(\mathscr{M})$$

defined by

$$\mathscr{L}_{X_t}\omega = \frac{d}{dt}(\rho_t)^*\omega\Big|_{t=0}.$$

Theorem 2.16. ***(Moser,1965)*** *Let $\mathscr{M}$ be a compact manifold and ω_1, ω_2 two symplectic forms on the manifold $\mathscr{M}$. Suppose that $\omega_2 - \omega_1$ is exact and that the 2-form $\omega_t = (1-t)\omega_1 + t\omega_2$ is symplectic for each $t \in \mathbb{R}$ on $\mathscr{M}$. Then there exists an isotopy $\rho : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$ such that $\rho^*\omega_t = \omega_1$, for all $t \in \mathbb{R}$. In particular, $(\mathscr{M}, \omega_1)$ is strongly isotopic to $(\mathscr{M}, \omega_2)$.*

Proof. First, we reformulate the problem using time-dependent vector fields instead of isotopies. Suppose that there exists an isotopy $\rho : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$ such that $\rho_t^*\omega_t = \omega_1$, $t \in \mathbb{R}$. Let

$$X_t = \frac{d\rho_t}{dt} \circ \rho_t^{-1}$$

be the time-dependent vector field, for $t \in \mathbb{R}$. Then since ω_1 is closed we have

$$\begin{aligned}
0 = \frac{d}{dt}\omega_1 &= \frac{d}{dt}(\rho_t^*\omega_t) \\
&= \frac{d}{dx}(\rho_x^*\omega_t)\Big|_{x=t} + \frac{d}{dy}(\rho_t^*\omega_y)\Big|_{y=t} \\
&= \rho_x^*(\mathscr{L}_{X_x}\omega_t)\Big|_{x=t} + \rho_t^*(\frac{d}{dt}\omega_y)\Big|_{y=t} \\
&= \rho_t^*(\mathscr{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t))
\end{aligned}$$

This is true if and only if

$$\mathscr{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0$$

since ρ_t is a diffeomorphism. Equivalently, we have by hypothesis that $\frac{d}{dt}\omega_t = \omega_2 - \omega_1$ so we conclude that

$$\mathscr{L}_{X_t}\omega_t + \omega_2 - \omega_1 = 0, \forall\, t \in \mathbb{R}$$

Suppose conversely that that we have a time-dependent vector field X_t , $t \in \mathbb{R}$ which satisfies the above equation $\mathscr{L}_{X_t}\omega_t + \omega_2 - \omega_1 = 0, \forall\, t \in \mathbb{R}$. Since $\mathscr{M}$ is compact, we can integrate X_t to an isotopy $\rho : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$ with

$$0 = \frac{d}{dt}(\rho_t^*\omega_t)$$

so we obtain $\rho_t^*\omega_t = \rho_1^*\omega_1 = \omega_1$.

We have shown that the existence of an isotopy $\rho : \mathscr{M} \times \mathbb{R} \to \mathscr{M}$ such that $\rho^*\omega_t = \omega_1$, for all $t \in \mathbb{R}$, is equivalent to the existence of a time-dependent vector field X_t , $t \in \mathbb{R}$ which satisfies $\mathscr{L}_{X_t}\omega_t + \omega_2 - \omega_1 =$

$0, \forall t \in \mathbb{R}$. Therefore we end up that it suffices to solve the equation $\mathscr{L}_{X_t}\omega_t + \omega_2 - \omega_1 = 0$ for X_t. The technique presented is known as the *Moser trick*.

Since $\omega_2 - \omega_1$ is exact, there exists a 1-form θ such that

$$\omega_2 - \omega_1 = d\theta.$$

Furthermore, we have the Cartan magic formula

$$\mathscr{L}_{X_t}\omega_t = d i_{X_t}\omega_t + i_{X_t} d\omega_t$$

where $d\omega_t = (1-t)d\omega_1 + t d\omega_2 = 0$. Thus we have

$$\begin{aligned} \mathscr{L}_{X_t}\omega_t + \omega_2 - \omega_1 &= 0 \\ \Leftrightarrow d i_{X_t}\omega_t + d\theta &= 0 \\ \Leftrightarrow i_{X_t}\omega_t + \theta &= 0 \\ \Leftrightarrow i_{X_t}\omega_t &= -\theta \end{aligned}$$

The existence and uniqueness of such vector field X_t is guaranteed by the nondegeneracy of ω_t, since it is symplectic by the hypothesis. □

Theorem 2.17. ***(Moser local theorem)*** *Let $\mathscr{M}$ be a manifold, $\mathscr{N} \subseteq \mathscr{M}$ a submanifold, and ω_1, ω_2 symplectic forms of $\mathscr{M}$ with $\omega_1|_p = \omega_2|_p$, $\forall p \in \mathscr{N}$. Then there exists neighborhoods $\mathscr{U}_1, \mathscr{U}_2$ of $\mathscr{N}$ in $\mathscr{M}$ and a diffeomorphism $\rho : \mathscr{U}_1 \mapsto \mathscr{U}_2$ such that $\rho|_{\mathscr{N}} = id$ and $\rho^*\omega_2 = \omega_1$.*

Proof. Since the 2-form $\omega_2 - \omega_1$ is closed, i.e. $d(\omega_2 - \omega_1) = 0$ and $\omega_2 - \omega_1|_{\mathscr{N}} = 0$, there exists a neighborhood $\mathscr{U}_1$ of $\mathscr{N}$ in $\mathscr{M}$ and a 1-form θ on $\mathscr{U}_1$ such that

$$\theta|_{\mathscr{N}} = 0 \text{ and } \omega_2 - \omega_1 = d\theta$$

The argument involves the Poincaré lemma for compactly-supported forms see [54].

We consider the family

$$\omega_t = (1-t)\omega_1 + t\omega_2 = \omega_1 + t d\theta$$

of 2-forms on $\mathscr{U}_1$, $t \in \mathbb{R}$. Obviously, ω_t is closed because ω_1 is closed. Moreover, we have that $\omega_t|_{\mathscr{N}} = \omega_1|_{\mathscr{N}}$ where ω_1 is nondegenerate. So by shrinking $\mathscr{U}_1$ we can assume that ω_t is symplectic $\forall t$. This is true, because, that ω_1 is nondegenerate means that there exists an isomorphism

$$(\omega_t)^\flat|_{\mathscr{N}} = (\omega_1 + d\theta)^\flat|_{\mathscr{N}} : T_{\mathscr{N}}\mathscr{U}_1 \to T^*_{\mathscr{N}}\mathscr{U}_1$$

From the Tubular Neighborhood Theorem (see [14]), there exists

$$\mathscr{U} \simeq \frac{T_{\mathscr{N}}\mathscr{U}_1}{T\mathscr{N}}.$$

Thus, we have that

$$(\omega_1 + d\theta)^\flat : T\Big(\frac{T_{\mathscr{N}}\mathscr{U}_1}{T\mathscr{N}}\Big) \to T^*\Big(\frac{T^*_{\mathscr{N}}\mathscr{U}_1}{T^*\mathscr{N}}\Big)$$

is an isomorphism. Concluding that

$$(\omega_t)^\flat : T\mathscr{U} \to T^*\mathscr{U}$$

is an isomorphism and ω_t is symplectic $\forall\, t$.

Now applying Moser's trick and solving the equation

$$i_{X_t}\omega_t = -\theta$$

we take a vector field X_t on $\mathscr{U}_1$. Since ω_t is nondegenerate and $\theta|_{\mathscr{N}} = 0$, we notice that $X_t|_{\mathscr{N}} = 0$.

Thus by shrinking $\mathscr{U}_1$ again, there exists from theorem 2.16 an isotopy $\phi : \mathscr{U}_1 \times \mathbb{R} \mapsto \mathscr{U}_1$, with $\phi_t^*\omega_t = \omega_1$ and $\phi_t|_{\mathscr{N}} = id_{\mathscr{N}}$.

Finally, we set $\mathscr{U}_2 = \phi_1(\mathscr{U}_1 \times \mathbb{R})$ and $\rho = \phi_1$ to complete the proof. □

Theorem 2.18. ***(Darboux,1882)*** *Let $(\mathscr{M}, \omega)$ be a $2n$-dimensional symplectic manifold, and let y_0 be any point in $\mathscr{M}$.*

There exists a coordinate chart $(\mathscr{U}, q^1, ..., q^n, p_1, ..., p_n)$ centered at y_0 such that on $\mathscr{U}$:

$$\omega = \textstyle\sum_{i=1}^n dq^i \wedge dp_i.$$

Coordinate charts that have this property are called Darboux's coordinate charts.

The classical proof of Dardoux's theorem is by induction on the dimension of the manifold (see a detailed proof in [7]). Here our proof was first provided by Weinstein in [63] and uses Moser's theorem 2.16.

Proof. Let $(\mathscr{M}, \omega)$ be a $2n$-dimensional symplectic manifold and $y_0 \in \mathscr{M}$. Then ω_{y_0} is a symplectic form. More precisely, from the theorem 2.5 there exists a symplectic basis $(e^1, ..., e^n, f^1, ..., f^n)$ for $(T^*_{y_0}\mathscr{M}, \omega_{y_0})$, such that $\omega_{y_0} = \sum_{i=1}^n e^i \wedge f^i$. Now, we consider coordinates $(\mathscr{U}, q^1, ..., q^n, p_1, ..., p_n)$ centered at y_0, such that $dq^i = e^i$ and $dp_i = f^i$ so that

$$\omega_{y_0} = \sum_{i=1}^n dq^i \wedge dp_i\Big|_{y_0}$$

We set $\omega_0 = \omega$ and $\omega_1 = \omega_{y_0} = \sum_{i=1}^n dq^i \wedge dp_i\Big|_{y_0}$. So there are two symplectic forms on $\mathscr{U}$, such that $\omega_0|_{y_0} = \omega_1|_{y_0}$. By theorem 2.16, there are neighborhoods $\mathscr{U}_0$ and $\mathscr{U}_1$ of y_0, and a diffeomorphism $\phi : \mathscr{U}_0 \to \mathscr{U}_1$ such that $\phi(y_0) = y_0$ and $\phi^*\omega_1 = \omega_0$. Thus, we conclude that

$$\begin{aligned}
\omega = \omega_0 &= \phi^*\Big(\sum_{i=1}^n dq^i \wedge dp_i\Big) \\
&= \sum_{i=1}^n \phi^* dq^i \wedge \phi^* dp_i \\
&= \sum_{i=1}^n d(q^i \circ \phi) \wedge d(p_i \circ \phi))
\end{aligned}$$

By an abuse of notation we set new coordinates $q^i = q^i \circ \phi$ and $p_i = p_i \circ \phi$ to complete the proof. □

Remark 2.19. Theorem 2.18 shows that in symplectic geometry there are no local invariants, in contrast to Riemannian geometry, where there are highly non-trivial local invariants. In other words, the study of symplectic manifolds or more generally Poisson manifolds is of global nature.

2.1.4 Cotangent bundle

We will present the symplectic form on the cotangent bundle. First, recall that for a smooth manifold M its cotangent bundle is T^*M, and any point ξ of T^*M may be denoted as an ordered pair $p = (x, \xi)$, with $x \in M$ as well as a single element $\xi \in T^*_x M$.

We take $\pi : T^*M \to M$ with $p = (x, \xi) \mapsto x$, the canonical fiber bundle projection, which assigns to each covector p its base point x. We will now define the *Liouville 1-form* (or *tautological 1-form*) α on T^*M. Let

$$d\pi_p : T_p T^*M \to T_x M$$

be the induced tangent map. We consider the pullback of $d\pi_p$,

$$(d\pi_p)^* : T^*_x M \to T^*_p T^*M$$

that is, $(d\pi_p)^*\xi = \xi \circ d\pi_p$.

Thus the Liouville 1-form may be defined point-wise by

$$\alpha_p = (d\pi_p)^*\xi.$$

Equivalently, for $u \in T_p T^*M$ we have

$$\alpha_p(u) = \xi(d\pi_p(u)).$$

The canonical symplectic 2-form ω on T^*M is defined as

$$\omega = -d\alpha.$$

We will prove that ω is a closed nondenerate 2-form. It is, clearly, a closed 2-form because it is exact.

Now, let $(U, x^1, ..., x^n)$ be a smooth local system of coordinates with $x^i : U \to \mathbb{R}$, then at any $p \in U$, the differentials $(dx^1)_p, ..., (dx^n)_p$ form a basis of $T^*_p M$. Namely, if $\xi \in T^*_p M$ then $\xi = \sum_{i=1}^n \xi_i dx^i$, for some real coefficients $\xi_1, ..., \xi_n$. This induces a map

$$T^*U \to \mathbb{R}^{2n}$$

which maps (x, ξ) to $(x^1, ..., x^n, \xi_1, ..., \xi_n)$. The canonical fiber bundle projection π, in terms of these coordinates, is expressed

$$\pi(x^1, ..., x^n, \xi_1, ..., \xi_n) = (x^1, ..., x^n).$$

Clearly, $d\pi_p$ is represented by the matrix

$$\begin{pmatrix} I_n & O_{n\times n} \end{pmatrix}$$

and its pullback is represented by its transpose. Therefore,

$$(d\pi^*)\xi = \begin{pmatrix} I_n \\ O_{n\times n} \end{pmatrix} \begin{pmatrix} \xi_1 \\ . \\ . \\ . \\ \xi_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ . \\ . \\ . \\ \xi_n \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}.$$

Consequently, we express locally the Liouville 1-form in terms of these coordinates

$$\alpha = \xi_1 dx^1 + ... + \xi_n dx^n + 0d\xi_1 + ... + 0dx^n$$

So,

$$\omega = d\alpha = \sum_{i=1}^{n} d\xi_i \wedge dx^i$$

and by example 2.3we have similarly that ω is nondegenerate. Finally, ω is the canonical symplectic form for the cotangent bundle since $(x^1, ..., x^n, \xi_1, ..., \xi_n)$ is a Darboux's coordinate system for ω.

2.1.5 Lagrangian Submanifolds

Here, we first, define immersions, submersions and embeddings. We use these notions to define submanifolds of a manifold M. In particular, we define Lagrangian submanifolds and study some results we need in section 5.2, in order to prove that the cotangent bundle has the structure of a symplectic groupoid.

Let M, N be manifolds with $dimN < dimM$.

Definition 2.20. An *immersion* is a smooth map $i : N \to M$ with the property that $i_* : T_pN \to T_{i(p)}M$ is injective at each point. In this case, N is called *immersed submanifold* of M. In a similar way, a *submersion* is a smooth map $i : N \to M$ such that $i_* : T_pN \to T_{i(p)}M$ is surjective at each point.

One special kind of immersion is particularly important.

Definition 2.21. A smooth *embedding* is an injective immersion $i : N \to M$, that is also a topological embedding, i.e. a homomorphism onto its image $i(N) \subseteq M$ in the subspace topology. In this case, N is called *embedded submanifold* of M.

Definition 2.22. Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold N of M is a *Lagrangian submanifold* if, at each point $p \in N$, T_pN is a Lagrangian subspace of T_pM, i.e.

$$\omega_p|_{T_pN} \equiv 0$$

and $dim(T_pN) = \frac{1}{2}dim(T_pM)$. Equivalently, if $i : N \hookrightarrow M$ is the inclusion map, then N is a Lagrangian submanifold, if and only if

$$i^*\omega = 0$$

and $dimN = \frac{1}{2}dimM$.

Definition 2.23. The *conormal space* at $x \in N$ is

$$\nu_x^* N = \{\xi \in T_x^* M : \xi(u) = 0, \forall u \in T_x N\}$$

Accordingly, the *conormal bundle* of N is

$$\nu^* N = \{(x,\xi) \in T^* M : x \in N, \xi \in \nu_x^* N\}$$

Proposition 2.24. *Let $i : \nu^* N \hookrightarrow T^* M$ be the inclusion, and let α be the tautological 1-form on $T^* M$. Then $i^*\alpha = 0$.*

Proof. We consider

$$(U, x^1, ..., x^n)$$

to be local coordinates on M, centered at $x \in N$, such that, N is described by $x^{k+1} = ... = x^n = 0$. Let us take

$$(T^* U, x^1, ..., x^n, \xi_1, ..., \xi_n)$$

be the associated cotangent coordinate system. Then the submanifold $\nu^* N$ is described by

$$x^{k+1} = ... = x^n = 0 \qquad \text{and} \qquad \xi_1 = ... = \xi_k = 0$$

Since $\alpha = \sum \xi_i dx^i$ on $T^* U$, at $p \in \nu^* N$, we obtain that

$$(i^*\alpha)_p = (\alpha_p)|_{T_p(\nu^* N)} = \sum_{i>k} \xi_i dx^i \Big|_{\text{span}\left\{\frac{\partial}{\partial x^i}, i \le k\right\}} = 0$$

$\square$

Corollary 2.25. *For every submanifold N of a differential manifold M, the conormal bundle $\nu^* N$ is a Lagrangian sumbanifold of $(T^* M, d\alpha)$.*

2.2 Poisson structures

This section aims to offer a quick introduction to Poisson structures. We will define the Poisson structure on a manifold to be a Lie algebra structure on its space of smooth functions (i.e. a bilinear skew symmetric operation of Poisson bracket on functions, satisfying the Jacobi identity) such that the operator $\{f,.\}$ is an operator of differentiation by some vector field X_f. The vector field X_f is called the Hamiltonian vector field and the smooth function f hamiltonian function. We then express Poisson structures in terms of bivector fields, and we will study the Lie-Poisson structure on the dual of a Lie algebra. We follow closely [27] and [61].

2.2.1 Poisson structures

Definition 2.26. A C^∞-smooth *Poisson structure* on a C^∞-smooth finite-dimensional manifold $\mathscr{M}$ is an $\mathbb{R}$-bilinear, antisymmetric operation

$$C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M}), (f,g) \mapsto \{f,g\}$$

on the space of C^∞-smooth functions on $\mathscr{M}$, which satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

and the Leibniz rule

$$\{f,gh\} = \{f.g\}h + g\{f,h\}, \forall f,g,h \in C^\infty(\mathscr{M}).$$

This bracket $\{.,.\}$ is called *Poisson bracket.* A manifold $\mathscr{M}$ equipped with such a bracket is called *Poisson manifold.*

Examples 2.27. a) On a manifold $\mathscr{M}$ we consider the bracket $\{f,g\} = 0$ for all functions f and g in $C^\infty(\mathscr{M})$, then we can easily see that this is a Poisson bracket and the manifold $\mathscr{M}$ equipped with this structure is a Poisson manifold. So on any manifold we can define a trivial Poisson structure.

b) Every symplectic manifold $(\mathscr{M},\omega)$ is Poisson. We define on the manifold $\mathscr{M}$ the bracket

$$\{f,g\} = \omega(X_f, X_g)$$

for every smooth functions f and g.

By definition 2.7 we have that $i_{X_f}\omega = df$ and $X_g = (\omega)^{-1}(dg)$, thus $\{f,g\} = i_{X_f}(dg)$. We need this formula to prove that this bracket is bilinear, skew symmetric and satisfies the Leibniz rule.

This bracket $\{.,.\}$ is bilinear and skew symmetric, since $\{f,g\} = i_{X_f}(dg)$, $d(g_1+g_2) = dg_1 + dg_2$, $\langle X, dg_1+dg_2\rangle = \langle X, dg_1\rangle + \langle X, dg_2\rangle$, $X_{f+g} = X_f + X_g$ and ω is both $\mathbb{R}$-bilinear and skew symmetric $\omega(X_{f+g}, -) = \omega(X_f,-) + \omega(X_g,-)$. Furthermore, $\{.,.\}$ satisfies the Leibniz rule:

$$\begin{aligned}\{f,gh\} &= i_{X_f} d(gh)\\ &= i_{X_f}(gdh + hdg)\\ &= g i_{X_f} dh + h i_{X_f} dg\\ &= g\{f,h\} + h\{f,g\}.\end{aligned}$$

It remains to verify the Jacobi identity of the above bracket. Since ω is closed we obtain

$$
\begin{aligned}
0 &= d\omega(X_f, X_g, X_h) \\
&= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) \\
&- \omega([X_f, X_g], X_h) - \omega([X_f, X_g], X_h) - \omega([X_f, X_g], X_h) \\
&= X_f\{g, h\} + X_g\{h, f\} + X_h\{f, g\} - [X_f, X_g](h) \\
&- [X_g, X_h](f) - [X_h, X_f](g) \\
&= \omega(X_f, X_{\{g,h\}}) + \omega(X_g, X_{\{h,f\}}) + \omega(X_h, X_{\{f,g\}}) - (X_f X_g(h) - X_g X_f(h)) \\
&- (X_g X_h(f) - X_h X_g(f)) - (X_h X_f(g) - X_f X_h(g)) \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} - \{f, \{g, h\}\} + \{g, \{f, h\}\} \\
&- \{g, \{h, f\}\} + \{h, \{g, f\}\} - \{h, \{f, g\}\} + \{f, \{h, g\}\} \\
&= -\{f, \{g, h\}\} - \{g, \{h, f\}\} - \{h, \{f, g\}\}
\end{aligned}
$$

where we used the Cartan's formula:

$$
\begin{aligned}
d\eta(X_1, ..., X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i \eta(X_1, ..., \widehat{X_i}, ..., X_{k+1}) \\
&+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta([X_i, X_j], X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_{k+1})
\end{aligned}
$$

with the usual hat notation to denote missing terms.

c) Applying the above result to the manifold $\mathscr{M} = \mathbb{R}^{2n}$, with coordinates

$$(q, p) = (q_1, ..., q_n, p_1, ..., p_n),$$

one can define, in a similar way, a smooth Poisson structure on $\mathbb{R}^{2n}$ for every f and g in $C^\infty(\mathscr{M})$ by putting

$$\{f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \Big).$$

d) Symplectic manifolds inherit a natural Poisson structure. However, let us give an example of a Poisson manifold that is not symplectic. We take $\mathscr{M} = \mathbb{R}^2$ and put $\{f, g\}(q, p) = q\Big(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\Big)$, we will see that not all Poisson brackets enamate from a symplectic structure on a manifold.

First of all, it is obvious that the above bracket is $\mathbb{R}$-bilinear and antisymmetric. For the Leibniz rule we have:

$$\begin{aligned}
\{f, gh\}(q,p) &= q\Big(\frac{\partial f}{\partial p}\frac{\partial gh}{\partial q} - \frac{\partial gh}{\partial p}\frac{\partial f}{\partial q}\Big) \\
&= q\Big(\frac{\partial f}{\partial p}\Big(\frac{\partial g}{\partial q}h + g\frac{\partial h}{\partial q}\Big) - \Big(\frac{\partial g}{\partial p}h + g\frac{\partial h}{\partial p}\Big)\frac{\partial f}{\partial q}\Big) \\
&= q\Big(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}h - \frac{\partial g}{\partial p}h\frac{\partial f}{\partial q}\Big) + q\Big(\frac{\partial f}{\partial p}g\frac{\partial h}{\partial q} - g\frac{\partial h}{\partial p}\frac{\partial f}{\partial q}\Big) \\
&= (\{f.g\}h + g\{f,h\})(q,p)
\end{aligned}$$

Similarly, via calculations we can verify the Jacobi identity. Thus, we conclude that the bracket $\{f,g\}(q,p) = q\Big(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\Big)$ is a Poisson bracket.

However, this a Poisson structure on $\mathbb{R}^2$ which is not symplectic. This is true because the bracket vanishes when $q = 0$ and therefore it is not non-degenerate.

2.2.2 Hamiltonian vector fields and Poisson bracket

Proposition 2.28. *Let $\mathcal{M}$ be a Poisson manifold, then $\forall f \in C^\infty(\mathcal{M})$ there exists a unique, differentiable vector field X_f on $\mathcal{M}$ such that, for every function $g \in C^\infty(\mathcal{M})$,*

$$X_f(g) = \{f, g\}.$$

Proof. Let us first assume that $f \in C^\infty(\mathcal{M})$ is fixed. By the Leibniz property and linearity of the Poisson bracket, the endomorphism of $C^\infty(\mathcal{M})$: $g \mapsto \{f,g\}$ is a derivation. Since we can identify derivations of $C^\infty(\mathcal{M})$ with smooth vector fields on $\mathcal{M}$, as $C^\infty(\mathcal{M})$-modules, there exists a unique differentiable vector field X_f on $\mathcal{M}$ which satisfies property above for every $g \in C^\infty(\mathcal{M})$. □

Since symplectic manifolds are special cases of Poisson manifolds, it is legitimate to give a more general definition for Hamiltonian vector fields.

Definition 2.29. A vector field $X \in \mathfrak{X}(\mathcal{M})$ is called *Hamiltonian* iff there exists $f \in C^\infty(\mathcal{M})$ such that for every $g \in C^\infty(\mathcal{M})$ we have $\{f,g\} = X_f(g) = -df(X_g)$. We write $X \equiv X_f \in \mathfrak{X}(\mathcal{M})$.

Proposition 2.30. *For functions $f, g \in C^\infty(\mathcal{M})$ the Hamiltonian vector fields X_f, X_g satisfy the following identity $[X_f, X_g] = X_{\{f,g\}}$.*

Proof. If f, g and h are functions in $C^\infty(\mathcal{M})$, then by definition of the Lie bracket and the Jacobi identity we have

$$\begin{aligned}
[X_f.X_g] &= X_f(X_g)(h) - X_g(X_f)(h) \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&= \{\{f,g\},h\} \\
&= X_{\{f,g\}}
\end{aligned}$$

Since h is arbitrary, it follows that $[X_f.X_g] = X_{\{f,g\}}$. □

There are many ways to introduce the Poisson brackets, which then naturally lead to the same results for the Poisson bracket of two functions in canonical coordinates.

Definition 2.31. Let $(\mathscr{M},\omega)$ be a symplectic manifold, f and g two smooth functions and X_f, X_g their associated Hamiltonian vector fields. The *Poisson bracket* of the ordered pair $(f.g)$ of smooth functions defined on $(\mathscr{M},\omega)$ is the smooth function $\{f,g\}$ defined by the formulae:

$$\{f,g\} = \omega(X_f, X_g) = -\langle df, X_g\rangle = -X_g(f) = X_f(g).$$

In any Darboux coordinates of the symplectic manifold $(\mathscr{M},\omega)$ we can compute the Poisson bracket $\{f,g\}$ explicitly

$$\{f,g\} = \omega(X_f, X_g) = X_f(g) = \sum_{i=1}^{n}\Big(\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}\Big)(g) = \sum_{i=1}^{n}\Big(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\Big).$$

The following classical theorem of Poisson is an immediate consequence of the definition of Poisson brackets.

Theorem 2.32. *(Poisson) If g and h are functions such that $X_f(g) = 0$ and $X_f(h) = 0$ respectively, where X_f is a Hamiltonian vector field on a Poisson manifold $\mathscr{M}$, then $X_f(\{g,h\}) = 0$*

Proof. This a corollary of the Jacobi identity. □

2.2.3 Poisson bivector fields

In this section we will express Poisson structures in terms of 2-vector fields (bivector fields).

Definition 2.33. Let M be a manifold and V a vector space. A *bivector* is a vector bundle over M, whose fiber over each point $x \in M$ is the space $\Lambda^2 TM$, where we denote by $\Lambda^2 TM$ the exterior product of two copies of the tangent space TM. In particular, $\Lambda^2 T_x M = T_x M \wedge T_x M$.

A smooth *bivector field* Π on M is, by definition, a smooth section of $\Lambda^2 TM$, i.e. a map Π from V to $\Lambda^2 TM$, which associates to each point x of M a bivector (2-vector) $\Pi(x) \in \Lambda^2 TM$, in a smooth way. Therefore, we conclude that $\Pi \in \Gamma(\Lambda^2 TM)$.

Proposition 2.34. *On every Poisson manifold $\mathscr{M}$ there exists a unique differentiable bivector field Π such that:*

$$\{f,g\} = \langle \Pi, df \wedge dg\rangle.$$

We call the bivector field Π the Poisson tensor *of the Poisson structure and we denote the Poisson manifold equipped with its Poisson structure by $(\mathscr{M},\Pi)$.*

Proof. We need to show that $\{f,g\}(x)$ depends only on $d_x f$ and $d_x g$ in order to prove the existence and unicity of Π.

Suppose that the function f is fixed, then we have

$$\{f,g\}(x) = (X_f(g))(x) = \langle d_x g, X_f(x)\rangle.$$

Consequently, when g varies, $\{f,g\}(x)$ only depends on $d_x g$.

Similarly, for g fixed when f varies, $\{f,g\}(x)$ only depends on $d_x f$, since

$$\{f,g\}(x) = (X_f(g))(x) = -\langle d_x f, X_g(x)\rangle.$$

Furthermore, we observe that the map $C^\infty(\mathscr{M}) \ni f \mapsto d_x f \in T_x^*\mathscr{M}$ is surjective and the Poisson bracket is bilinear and skew-symmetric, therefore there exists a bilinear and skew-symmetric form $\Pi(x)$ on the vector space $T_x^*\mathscr{M}$ such that:

$$\{f,g\}(x) = \Pi(x)(d_x f, d_x g).$$

The map $x \mapsto \Pi(x)$ is a differential bivector field on $\mathscr{M}$, since $\Pi \in \Gamma(\Lambda^2 T\mathscr{M})$.

Locally, in a coordinate system $(x^1,\ldots,x^n)$ where $n = dim\mathscr{M}$, Π is written as

$$\Pi(x) = \textstyle\sum_{i<j} \Pi^{ij}(x)\dfrac{\partial}{\partial x^i} \wedge \dfrac{\partial}{\partial x^j}.$$

This means that $\Pi^{ij}(x) = \{x^i, x^j\}$, which are smooth functions of x, hence Π is smooth as well. □

Remark 2.35. The converse also holds i.e. a manifold $\mathscr{M}$ equipped with a bivector field Π is a Poisson manifold, if and only if the Schouten bracket of the tensor field Π vanishes, $[\Pi,\Pi]_{SN} = 0$, so the Jacobi identity is satisfied.

Examples 2.36. a) We will calculate the Poisson tensor corresponding to the standard symplectic structure $\omega_0 = \sum_{i=1}^n dq^i \wedge dp_i$ on $\mathbb{R}^{2n}$. We know that $\Pi : T^*M \wedge T^*M \to TM \wedge TM$, so Π is

$$\Pi = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

since every dq^i and dp_i maps to $\dfrac{\partial}{\partial q^i}$ and $\dfrac{\partial}{\partial p_i}$ respectively, via the isomorphism $(\omega^\flat)^{-1} : T^*M \to TM$, which can be extended to $(\omega^{\flat,2})^{-1} \equiv \Pi : T^*M \wedge T^*M \to TM \wedge TM$ in an obvious way.

The Poisson tensor can be expressed as :

$$\Pi = \sum_{i=1}^n \Big(\frac{\partial}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial}{\partial q_i}\Big).$$

and, in addition, we have seen in examples 2.27 that

$$\{f,g\} = \sum_{i=1}^n \Big(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\Big)$$

Now, it is an easy consequence that $\{f,g\} = \Pi(df, dg)$.

b) Let us take the manifold $\mathcal{M} = \mathbb{R}^2$, we define the bivector $\Pi = q\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$. Then, we calculate that

$$\begin{aligned}\Pi_{(q,p)}(d_{(q,p)}f, d_{(q,p)}g) &= q\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}(f,g)\\ &= q\Big(\frac{\partial}{\partial q}\frac{\partial}{\partial p} - \frac{\partial}{\partial p}\frac{\partial}{\partial q}\Big)(f,g)\\ &= q\Big(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\Big)\\ &= \{f,g\}(q,p)\end{aligned}$$

and by example 2.27 it is a Poisson tensor that is not symplectic.

Definition 2.37. Let $\mathcal{M}$ be a manifold and Π a Poisson bivector, then for every Poisson manifold $(\mathcal{M},\Pi)$ we define its sharp map

$$\sharp_\Pi : T^*M \to TM$$

so $\alpha \mapsto \sharp_\Pi(\alpha) = i_\alpha\Pi$, where $\sharp_\Pi(\alpha)(\beta) = (i_\alpha\Pi)(\beta) = \langle \Pi, \alpha \wedge \beta\rangle = \Pi(\alpha.\beta)$.

Remarks 2.38. a) $\sharp_\Pi$ is a bundle map on $\mathcal{M}$, which maps each covector $\alpha \in T^*M$ over a point x to a unique vector $\sharp_\Pi(\alpha) \in T_x\mathcal{M}$.

b) Being a bundle map it induces a map on sections

$$\sharp_\Pi : \Omega^1(\mathcal{M}) \mapsto \mathfrak{X}(\mathcal{M}),$$

$$\alpha \mapsto i_\alpha \Pi.$$

c) In particular, on exact 1-forms one easily has $\sharp_\Pi(df) = X_f$. Indeed,

$$\langle \sharp_\Pi(df), dg\rangle = \Pi(df, dg) = \{f,g\} = \langle X_f, dg\rangle.$$

Remark then, that a vector field is uniquely determined by its contractions with exact 1-forms. Consequently, $im\sharp_\Pi = Ham(\mathcal{M})$- vector subspace of TM.

As illustrated in example 2.27 we know that any symplectic manifold $(\mathcal{M},\omega)$ is a Poisson manifold. The Poisson bivector field Π is related to the symplectic form ω by

$$\{f,g\} = \Pi(df,dg) = \omega(X_f, X_g)$$

for $f, g \in C^\infty(\mathcal{M})$.

In this case the map $\Pi : T^*M \to TM$ is the inverse of the map $\omega^\flat : TM \to T^*M$ such that

$$X \mapsto \omega^\flat(X)$$

and for a vector $Y \in TM$,

$$\langle \omega^\flat(X), Y\rangle = -\omega(X,Y).$$

The question is when a Poisson manifold is symplectic? Only in the case where Π is a nondegenerate Poisson structure on an even dimensional smooth manifold $\mathscr{M}$ does it follow that is symplectic as well. Its symplectic form is then

$$\omega(X,Y) = \langle (\sharp_\Pi)^{-1}(X), Y\rangle, X, Y \in TM.$$

Indeed, Π is nondegenerate, i.e., $\sharp_\Pi : T^*M \to TM$ is an isomorphism, so the inverse map $\sharp_\Pi^{-1} : TM \to T^*M$ is defined. If we set

$$\omega^\flat = \sharp_\Pi^{-1} : TM \to T^*M$$

it follows that

$$\omega(X,Y) = -\langle \omega^\flat(X), Y\rangle = -\langle \sharp_\Pi^{-1}(X), Y\rangle$$

is a symplectic form. By definition, it is nondegenerate, skew-symmetric and bilinear. It remains to prove that ω is closed. In order to do so, let x be a point in $\mathscr{M}$, and let X_x, Y_x, Z_x be vectors in T_xM. Since the Poisson tensor Π is nondegenerate there exist differentiable functions f, g and h defined on $\mathscr{M}$ which satisfy:

$$\begin{aligned} \sharp_{\Pi_x}(d_x f) &= X_x \\ \sharp_{\Pi_x}(d_x g) &= Y_x \\ \sharp_{\Pi_x}(d_x h) &= Z_x. \end{aligned}$$

Let $\sharp_\Pi(df)$, $\sharp_\Pi(dg)$ and $\sharp_\Pi(dh)$ be the Hamiltonian vector fields associated to the Hamiltonians f, g and h respectively, then we obtain $d\omega(\sharp_\Pi(df), \sharp_\Pi(dg), \sharp_\Pi(dh)) = -\{g,\{h,f\}\} - \{f,\{g,h\}\} - \{h,\{f,g\}\} = 0$. (Jacobi identity)

Evaluating the above expression at x shows that the 2-form ω is closed. It is thus a symplectic form on the manifold $\mathscr{M}$. Finally, by $\langle \omega^\flat(X), Y\rangle = -\omega(X,Y)$, it follows the associated Poisson structure to this symplectic structure, coincides with the one defined by Π.

2.2.4 Lie - Poisson structure

The phrase "Lie-Poisson structure" was introduced by Marsden and Weinstein (1983), but it can be traced back to S.Lie around 1880 in the chapter 17, pages 294 – 298, where Lie defines a linear Poisson structure on the dual of a Lie algebra, today called the Lie-Poisson structure.

Definition 2.39. Let $\mathfrak{g}, [.,.]$ be a finite dimensional real Lie algebra and $\mathfrak{g}^*$ its dual space. Then a Lie Poisson structure on $\mathfrak{g}^*$ is a Poisson structure on $\mathfrak{g}^*$ i.e. $C^\infty(\mathfrak{g}^*)$ equipped with the Poisson bracket:

$$\{f,g\}(x) = \langle x, [d_x f, d_x g]\rangle = x([d_x f, d_x g]).$$

For $\mathfrak{g}$ a Lie algebra the underlying dual vector space $\mathfrak{g}^*$ canonically inherits the structure of a Poisson manifold whose Poisson Lie bracket reduces on linear functions $\mathfrak{g} \hookrightarrow C^\infty(\mathfrak{g}^*)$ to the original Lie bracket on $\mathfrak{g}$. This is the Lie-Poisson structure on $\mathfrak{g}^*$.

In particular, consider $f \in C^\infty(\mathfrak{g}^*)$ as smooth function on the dual of a Lie algebra, then its de Rham differential 1-form at some $x \in \mathfrak{g}^*$, being a linear map

$$df|_x : T_x\mathfrak{g}^* \to \mathbb{R}$$

is canonically identified with a Lie algebra element itself, since $T_x\mathfrak{g}^* \equiv \mathfrak{g}^*$ and obviously $T_x\mathbb{R} \equiv \mathbb{R}$. Therefore, $df|_x : \mathfrak{g}^* \to \mathbb{R}$ is a linear map on $\mathfrak{g}^*$, so $df \in (\mathfrak{g}^*)^* \equiv \mathfrak{g}$. Similarly, $dg \in \mathfrak{g}$ for $g \in C^\infty(\mathfrak{g}^*)$. Consequently, we we have that $[d_x f, d_x g] \in \mathfrak{g}$.

Conversely, let V be a finite dimensional vector space on $\mathbb{R}$. A linear Poisson structure on V is a Poisson structure on V for which the Poisson bracket of two linear functions is again a linear function. Equivalently, in linear coordinates, the components of the corresponding Poisson tensor (bivector) are linear functions. In this case, by restriction to linear functions, the operation $(f,g) \mapsto \{f,g\}$ gives rise to an operation $[,] : V^* \times V^* \to V^*$, which is a Lie algebra structure on V^*.

Example 2.40. Consider the Lie group $G = SU(2)$, where $U(2) = \{A \in M_2(\mathbb{C}) : A^*A = I = AA^*\}$, so we have that $SU(2) = \{A \in U(2) : detA = 1\}$. We can easily see that $SU(2) = \{\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} : a, b \in \mathbb{C}, and, |a|^2 + |b|^2 = 1\}$ since $A^* = A^{-1}$ and $detA = 1$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, we can calculate the Lie algebra of $SU(2)$ to be

$$\mathfrak{g} = Lie(SU(2)) = T_I SU(2) = \{X \in M_2(\mathbb{C}) : X^* + X = 0, trX = 0\}$$

Indeed, let a function $\gamma : (-\epsilon, \epsilon) \to SU(2)$ with $\gamma(0) = I$, $X = \frac{d}{dt}|_{t=0}\gamma(t) \in \mathfrak{g}$ and take $det(\gamma(t)) = 0$, $\gamma^*(t)\gamma(t) = 1$, then we have the Lie algebra of $SU(2)$.

Observe that $\mathfrak{g} \equiv \mathbb{R}^3$.

We equip $\mathfrak{g}$ with the usual bracket on the space of matrices and we have

$$[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$$

, where e_1, e_2, e_3 are elements of a basis of the space $\mathfrak{g}$ and

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This Poisson bracket denotes the exterior product of the space $\mathbb{R}^3$.

We remark that $[e_i, e_j] = c_{ij}^k e_k$, so $c_{12}^3 = c_{23}^1 = c_{31}^2 = 1$ and all the other constants are zero.

In order to define the Poisson structure Π on $\mathfrak{g}^*$ we use the relation

$$\{f, g\}(\mu) = \langle \mu, [df, dg] \rangle = \mu([df, dg]).$$

We associate to (e_1, e_2, e_3) the linear coordinate system (x^1, x^2, x^3) for $\mathfrak{g}^*$ and now it sufficies to determine the components $\Pi^{ij}(x) = \{x^i, x^j\}$. Therefore,

$$\begin{aligned}\Pi^{ij}(x) &= \{x^i, x^j\}(\mu)\\ &= \langle \mu, [dx^i, dx^j]\rangle\\ &= \langle \mu, [e_i, e_j]\rangle\\ &= \mu([e_i, e_j])\\ &= \mu(c^k_{ij} e_k)\\ &= c^k_{ij} x^k\end{aligned}$$

Finally, we defined the Poisson bracket on the dual space $(\mathfrak{g}^*, \{.,.\})$ to be again the exterior product of the space $\mathbb{R}^3$, since $\mathfrak{g} \equiv \mathfrak{g}^*$ and $c^3_{12} = c^1_{23} = c^2_{31} = 1$ and all the other constants are zero.

2.2.5 Poisson morphisms

Definition 2.41. If $(\mathcal{M}_1, \{.,.\}_1)$ and $(\mathcal{M}_2, \{.,.\}_2)$ are two Poisson manifolds then a map $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ between the Poisson manifolds is called a *Poisson map* or a *Poisson morphism* if

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

$\forall f, g \in C^\infty(\mathcal{M}_2)$. In other words ϕ is a Poisson map or Poisson morphism if the associated pull-back map $\phi^* : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ is a Lie algebra homomorphism with respect to the corresponding Poisson brackets.

Example 2.42. Let $(\mathcal{M}_1, \{.,.\}_1)$ and $(\mathcal{M}_2, \{.,.\}_2)$ be two Poisson manifolds. Then their direct product $\mathcal{M}_1 \times \mathcal{M}_2$ is also a Poisson manifold in an obvious way. So we are looking for a bracket satisfying:

$$\{.,.\} : C^\infty(\mathcal{M}_1 \times \mathcal{M}_2) \times C^\infty(\mathcal{M}_1 \times \mathcal{M}_2) \to C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$$

Equivalently, we can define its Poisson tensor

$$\Pi : T^*(\mathcal{M}_1 \times \mathcal{M}_2) \to T(\mathcal{M}_1 \times \mathcal{M}_2)$$

or,

$$\Pi : T^*(\mathcal{M}_1) \times T^*(\mathcal{M}_2) \to T(\mathcal{M}_1) \times T(\mathcal{M}_2)$$

to be $\Pi := \Pi_1 \times \Pi_2$, where Π_1, Π_2 are the Poisson tensors for the manifolds $\mathcal{M}_1, \mathcal{M}_2$ respectively. This true because $C^\infty(\mathcal{M}_1) \times C^\infty(\mathcal{M}_2) \subseteq C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$. Now, we consider $f, g \in C^\infty(\mathcal{M}_1 \times \mathcal{M}_2)$ and define

$$\Pi(df, dg) := \Pi_1(df_2, dg_2) \times \Pi_2(df_1, dg_1)$$

where we use the notation $f_1(x_2) = f_2(x_1) = f(x_1, x_2)$, similarly, for the smooth function g in $\mathcal{M}_1 \times \mathcal{M}_2$, $x_1 \in \mathcal{M}_2$ and $x_2 \in \mathcal{M}_1$.

Thus, the direct product $\mathcal{M}_1 \times \mathcal{M}_2$ can be equipped with the following natural bracket

$$\{f, g\} = (\{f_2, g_2\}, \{f_1, g_1\})$$

and it is called the *product Poisson structure*. Finally, with respect to this product Poisson structure, the projection maps $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_1$ and $\mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_2$ are Poisson maps.

2.3 The Lie algebroid structure of a Poisson manifold

In this section, we will prove that attached to each Poisson manifold $(\mathscr{M},\Pi)$, there exists a natural Lie algebroid structure $(T^*\mathscr{M},[.,.],\sharp)$ on the cotangent bundle of $\mathscr{M}$. In this view, we will discuss how the Hamiltonian vector fields span a completely integrable singular foliation, in the sense of Stefan-Sussmann. Our presentation is mainly based on [43], [27], [61] and [44].

2.3.1 Basic definitions and properties

In section 2.2.3 we defined the bundle map $\sharp\Pi : T^*\mathscr{M} \to T\mathscr{M}$ on a Poisson manifold $(\mathscr{M},\Pi)$, with $\sharp_\Pi(\alpha)(\beta) = (i_\alpha\Pi)(\beta) = \langle \Pi, \alpha\wedge\beta\rangle = \Pi(\alpha,\beta)$. We will abbreviate $\sharp_\Pi(\alpha)$, for $\alpha\in\Omega^1(\mathscr{M})$, to $\alpha^\sharp$.

Definition 2.43. Let $(\mathscr{M},\Pi)$ be a Poisson manifold and $\alpha,\beta\in\Omega^1(\mathscr{M})$. The Poisson bracket of α and β is the 1-form:

$$\{\alpha,\beta\}_1 = [\alpha^\sharp,\beta^\sharp]^\flat.$$

Theorem 2.44. *For $\alpha,\beta\in\Omega^1(\mathscr{M})$, we have*

$$\{\alpha,\beta\}_1 = -\mathfrak{L}_{\alpha^\sharp}\beta + \mathfrak{L}_{\beta^\sharp}\alpha + d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega)$$

where $\mathfrak{L}$ is the Lie derivative.

Proof. We will use the calculus of the Lie derivative.

$$d\omega(X,Y,Z) = \mathfrak{L}_X(\omega(Y,Z)) + \mathfrak{L}_Y(\omega(Z,X)) + \mathfrak{L}_Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)$$

for $X,Y,Z\in\mathfrak{X}$.

Let us replace $X=\alpha^\sharp$ and $Y=\beta^\sharp$. Moreover, we observe that $\omega(\alpha^\sharp,Z)=\alpha(Z)$. Indeed,

$$\omega(\alpha^\sharp,Z) = \omega^\flat(\alpha^\sharp)(Z) = (\alpha^\sharp)^\flat = \alpha(Z).$$

Now, it follows that

$$0 = \mathfrak{L}_{\alpha^\sharp}(\beta(Z)) - \mathfrak{L}_{\beta^\sharp}(\alpha(Z)) - \mathfrak{L}_Z(i(\alpha^\sharp)i(\beta^\sharp)\omega) + \{\alpha,\beta\}(Z) + \alpha(\mathfrak{L}_{\beta^\sharp}Z) - \beta(\mathfrak{L}_{\alpha^\sharp}Z).$$

So,

$$0 = \{\alpha,\beta\}_1 + \mathfrak{L}_{\alpha^\sharp}\beta - \mathfrak{L}_{\beta^\sharp}\alpha - d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega).$$

□

The connection between the Poisson bracket of 1-forms and that of functions is now at hand.

Theorem 2.45. *For $f,g\in C^\infty(\mathscr{M})$, we have*

$$d\{f,g\} = \{df,dg\}_1.$$

Proof. $\{df, dg\}_1 = -\mathfrak{L}_{X_f} dg + \mathfrak{L}_{X_g} df + d(i_{X_f} i_{X_g}\omega) = d(\mathfrak{L}_{X_f} g + \mathfrak{L}_{X_g} f + (i_{X_f} i_{X_g}\omega)) = d(i_{X_f} i_{X_g}\omega) = d\{f, g\}$. □

Theorem 2.46. *Let $(\mathscr{M}, \Pi)$ be a Poisson manifold. The cotangent bundle of $\mathscr{M}$, $T^*\mathscr{M}$ has a Lie algebroid structure, the Lie bracket of which is given by*

$$[\alpha, \beta] = \{\alpha, \beta\}_1$$

with $\alpha, \beta \in \Omega^1(\mathscr{M}) \equiv \Gamma(T^*\mathscr{M})$, and whose anchor map $\sharp\Pi : T^*\mathscr{M} \to T\mathscr{M}$ is the usual anchor map of Π.

Proof. It is immediate that the bracket as defined, satisfies the Jacobi identity. We consider $f, g, h \in C^\infty(\mathscr{M})$, then by theorem 2.45 we have that

$$d\{f, g\} = \{df, dg\}_1$$

and so on. Thus, if $\alpha = df, \beta = dg, \gamma = dh$ are exact 1-forms, then the Jacobi identity for the triple (df, dg, dh) follows from the Jacobi identity for the triple (f, g, h) with respect to the Poisson bracket.

It remains to verify that this bracket satisfies Leibniz rule

$$\begin{aligned}
[\alpha, u\beta] &= \{\alpha, u\beta\}_1 \\
&= -L_{\alpha^\sharp}(u\beta) + L_{u\beta^\sharp}\alpha + d(i_{\alpha^\sharp} i_{u\beta^\sharp}\omega) \\
&= d(u(\Pi(\alpha, \beta))) + i_{\alpha^\sharp} d(u\beta) - u i_{\beta^\sharp} d\alpha \\
&= u[\alpha, \beta] + \Pi(\alpha, \beta) du + i_{\alpha^\sharp}(du \wedge \beta) + i_{\alpha^\sharp} d\beta - u i_{\beta^\sharp} d\alpha \\
&= u[\alpha, \beta] + i_{\alpha^\sharp}(du \wedge \beta) \\
&= u[\alpha, \beta] + ((\alpha^\sharp)(f))\beta.
\end{aligned}$$

□

Remark 2.47. The Lie algebroid structure $(T^*\mathscr{M}, [.,.], \sharp)$ on the cotangent bundle of $\mathscr{M}$ is called *cotangent algebroid* of the Poisson manifold $(\mathscr{M}, \Pi)$.

In the following proposition, we will prove that a Lie algebroid structure $([.,.], \sharp)$ on $T^*\mathscr{M}$ comes from a Poisson structure on $\mathscr{M}$ if and only if $\sharp$ is antisymmetric and the bracket of two arbitrary closed 1-forms is again a closed 1-form. This is a necessary condition for a Lie algebroid structure on a cotangent bundle to correspond to a Poisson structure. So the next criterion is now at hand.

Proposition 2.48. *Let $\mathscr{M}$ be a manifold. Suppose that $T^*\mathscr{M}$ has a Lie algebroid structure $([.,.], \sharp)$ such that*

$$\sharp^t = -\sharp$$

and such that

$$[df, dg] = d(\sharp(df)(g))$$

, for all $f, g \in C^\infty(\mathscr{M})$. Then

$$\{f, g\} = \sharp(df)(g)$$

defines a Poisson structure on $\mathscr{M}$ for which the Lie algebroid structure on $T^*\mathscr{M}$ is the given one.

Proof. The bracket $\{.,.\}$ defined on $C^\infty(\mathscr{M})$ is skew-symmetric. This is true because

$$\{g,f\} = \sharp(dg)(f) = \langle df, \sharp(dg)\rangle = \langle dg, \sharp^t(df)\rangle = -\langle dg, \sharp(df)\rangle = -\{f,g\}.$$

It satisfies the Leibniz rule

$$\begin{aligned}
\{f,gh\} &= \sharp(df)(gh) \\
&= \langle d(gh), \sharp(df)\rangle \\
&= \langle gdh + hdg, \sharp(df)\rangle \\
&= g\langle dh, \sharp(df)\rangle + h\langle dg, \sharp(df)\rangle \\
&= g\{f,h\} + h\{f,g\}.
\end{aligned}$$

Furthermore, it satisfies the Jacobi identity. We have,

$$\begin{aligned}
\{\{f,g\},h\} &= \sharp(d\{f,g\})(h) \\
&= \langle dh, \sharp(d\{f,g\})\rangle \\
&= \langle dh, \sharp(d(\sharp(df)(g)))\rangle \\
&= \langle dh, \sharp([df,dg])\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\{\{g,h\},f\} + \{\{h,f\},g\} &= \langle df, \sharp(d(\sharp(dg)(h)))\rangle + \langle dg, -\sharp(d(\sharp(df)(h)))\rangle \\
&= \langle \sharp^t(df), d(\sharp(dg)(h))\rangle - \langle \sharp^t(dg), d(\sharp(df)(h))\rangle \\
&= -\langle \sharp(df), d(\sharp(dg)(h))\rangle + \sharp(dg), d(\sharp(df)(h))\rangle \\
&= -\sharp(df)(\sharp(dg)h) + \sharp(dg)(\sharp(df)h) \\
&= -(\sharp(df)\sharp(dg) - \sharp(dg)\sharp(df))h \\
&= -[\sharp(df), \sharp(dg)]h \\
&= -\langle dh, [\sharp(df), \sharp(dg)]\rangle.
\end{aligned}$$

Adding these terms up, we get

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \langle dh, \sharp([df,dg])\rangle - \langle dh, [\sharp(df), \sharp(dg)]\rangle = 0$$

since $\sharp$ is the anchor map, so $\sharp([df,dg]) = [\sharp(df), \sharp(dg)]$.

It remains to show that the Lie algebroid structure on the cotangent bundle $T^*\mathscr{M}$ of $\mathscr{M}$, defined by this Poisson structure, as follows by theorem 2.46, is the given one. It suffices to show that $\sharp = \Pi^\sharp$, where $\Pi^\sharp$ is the Poisson bivector field on $\mathscr{M}$, defined by the Poisson bracket $\{.,.\}$.

By hypothesis, $\{f,g\} = \sharp(df)(g)$, and since we just proved that $\{.,.\}$ is a Poisson bracket, it follows that

$$\{f,g\} = \Pi^\sharp(df)(g)$$

, whence we conclude that $\sharp(df) = \Pi^\sharp(df)$, for any $f \in C^\infty(\mathscr{M})$. Since both $\sharp$ and $\Pi^\sharp$ are bundle maps and they coincide on exact forms, we have that $\sharp = \Pi^\sharp$. □

2.3.2 Singular foliation of a Poisson manifold

In appendix A.4 we proved that on every Lie algebroid corresponds a singular foliation, which is called the characteristic foliation of a Lie algebroid. Here we discuss, in a similar way, the case of Poisson manifolds.

We analyzed how the cotangent bundle $T^*\mathscr{M}$ of a Poisson manifold has a canonical Lie algebroid structure (cotangent algebroid). In this case, the anchor map $\rho : A \to TM$ defined by

$$\Pi^\sharp : T^*\mathscr{M} \to T\mathscr{M}$$

induces a morphism of $C^\infty(\mathscr{M})$-modules

$$\Pi^\sharp : \Omega^1(\mathscr{M}) \to \mathfrak{X}(\mathscr{M})$$

with $\Omega^1(\mathscr{M}) \equiv \Gamma(T^*\mathscr{M})$ and equivalently $\mathfrak{X}(\mathscr{M}) \equiv \Gamma(T\mathscr{M})$.

We denote the set that is spanned by the Hamiltonian vector fields by $\mathscr{F}$, namely,

$$\mathscr{F} = span_{C^\infty(\mathscr{M})}\langle X_f : f \in C^\infty(\mathscr{M})\rangle.$$

This is a singular foliation as we showed in appendix A.4. In the next section, we will show that the leaves of this foliation have a canonical symplectic structure.

3 The splitting theorem and the symplectic foliation

Here we discuss the generalization of the Darboux theorem given by Weinstein in [64]. In this paper, it was shown that a neighborhood of a point x, in a Poisson manifold can be written as the product of a symplectic submanifold with a transverse submanifold endowed with a Poisson tensor which vanishes at the point x.

3.1 Preliminaries

Before we prove the splitting theorem and study some of the basic consequences and results arising from it, we first recall some important notions and results needed. For a more detailed exposition of the following, one should read [38] and [39].

Definition 3.1. Let M and N be smooth manifolds and $f : M \to N$ be a smooth map. A point $p \in M$ is called a *regular point* of the map f, if the differential

$$df_p : T_pM \to T_{f(p)}N$$

is a surjective linear map. A point $q \in N$ is a *regular value* of f if all points p in the pre-image $f^{-1}(q)$ are regular points.

The implicit function theorem gives conditions under which a level set of a smooth map is locally a smooth embedded submanifold.

Theorem 3.2. ***Implicit Function theorem*** *Let M and N be smooth manifolds, $f : M \to N$ be a smooth map and $q \in N$ a regular value of f, then $f^{-1}(q) \subseteq M$ is a smooth embedded submanifold of M, such that*

$$T_pf^{-1}(q) = ker(df_p)$$

Remark 3.3. The implicit function theorem asserts that $C = f^{-1}(q)$ is a smooth embedded submanifold of M.

Another useful result is the Flow-Box theorem.

Theorem 3.4. ***Flow-Box theorem*** *Let M be a smooth and X a smooth vector field on M. If $X(p) \neq 0$ for a point $p \in M$, then there exists a local coordinate system*

$$(U, (y_1, ..., y_n))$$

on an open neighborhood U of p so that, on U

$$X = \frac{\partial}{\partial y_1}$$

Remark 3.5. This theorem can be interpreted as follows. After a change of coordinates, i.e., in the new coordinates, the vector field is very simple. Its solutions are horizontal straight lines. This means that in a small neighborhood the dynamics is just monotonic evolution in time along parallel flow lines. This is the reason why theorem 3.4 in the literature is also called the Straightening-Out theorem.

3.2 The Splitting Theorem

Theorem 3.6. *Let (P^n, Π) be a Poisson manifold and x_0 be a point in P of rank $2s = dim\mathscr{C}_{x_0}$, where $\mathscr{C}_{x_0}$ is the leaf at x_0. Let N be an arbitrary $(n-2s)$-dimensional manifold of P which contains x_0 and is transversal to $\mathscr{C}_{x_0}$ at x_0. We denote N_{x_0} to be a small neighborhood of x_0 in N. Then there is a system of coordinates*

$$(N_{x_0}, (p_1, ..., p_s, q_1, ..., q_s, z_1, ..., z_{n-2s})) \tag{3.1}$$

which satisfies the following conditions:

a) $p_i(N_{x_0}) = q_i(N_{x_0}) = 0$.

b) $\{q_i, q_j\} = \{p_i, p_j\} = 0$ *if* $i \neq j$ *and* $\{p_i, q_i\} = 1$, $\forall i$.

c) $\{z_i, p_j\} = \{z_i, q_j\} = 0$, $\forall i, j$.

d) $\{z_i, z_j\}(x_0) = 0$, $\forall i, j$.

The coordinates 3.1 are called canonical coordinates. *In such canonical coordinates the Poisson structure* Π *can be expressed as*

$$\Pi = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i,j} \{z_i, z_j\} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

Remark 3.7. Geometrically, theorem 3.6 is called splitting because locally the Poisson manifold (P^n, Π) can be splitted into the product of a $2s$-dimensional symplectic manifold, with the standard symplectic structure:

$$\Pi_S = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$$

and a $(n-2s)$-dimensional Poisson manifold, with Poisson structure defined by:

$$\Pi_N = \sum_{i,j} \{z_i, z_j\} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$$

on a neighborhood of x_0 in N. Since $\{z_i, p_j\} = \{z_i, q_j\} = 0$, $\forall i, j$, the functions $\{z_i, z_j\}$ do not depend on the variables $(p_1, ..., p_s, q_1, ..., q_s)$. To better understand this, it suffices to show that $X_f(\{z_i, z_j\}) = 0$. Let us suppose that the converse is true, i.e. $X_f \neq 0$, for $f = p_i$ or q_j. By the Flow-Box theorem 3.4, there exists a local function ϕ such that $X_f = \dfrac{\partial}{\partial \phi}$. Now, the Jacobi identity gives that

$$X_f(\{z_i, z_j\}) = 0$$

so we get the result required.

The equality $\{z_i, z_j\}(x_0) = 0$, $\forall i, j$ means that the Poisson structure Π_N vanishes at x_0. So locally, we can split a Poisson structure into two parts i.e. a regular part and a singular part which vanishes at a point.

Proof. We will prove the splitting theorem using the coordinate-by-coordinate construction method. We will contruct these coordinates by induction on s.

If Π has rank zero at x_0, then $X_f(x_0) = 0$ for all $f \in C^\infty(P)$. This means that $\mathscr{C}_{x_0} = \{x_0\}$. So N is diffeomorphic to P , $\{x_0\}$ is the symplectic manifold, and we are done.

We suppose that $\Pi(x_0) \neq 0$. First of all, we will construct the coordinates p_1 and q_1. We know that $\mathscr{C}_{x_0}$ is a submanifold of the Poisson manifold P, so by the Implicit function theorem 3.2 there exists a local function p_1 such that $p_1 : \mathscr{U} \to \mathbb{R}$ which vanishes on N, where $\mathscr{U}$ is a small neighborhood of x in P, and such that $dp_1(x_0) \neq 0$. Since $\mathscr{C}_{x_0}$ is transversal to N, there is a vector $X_g(x_0) \in \mathscr{C}_{x_0}$ such that $X_{p_1}(g)(x_0) \neq 0$, where X_{p_1} denotes the Hamiltonian vector field of p_1. This true because let us consider

$$\Pi\Big|_{\mathscr{U}} : T^*\mathscr{U} \to T\mathscr{U}$$

We know that $X_{p_1} \in \mathfrak{X}(\mathscr{U})$ equals $\Pi|_{\mathscr{U}}(dp_1)$. Moreover, we have that $X_{p_1}(x_0) \in T_{x_0}\mathscr{C}_{x_0}$ and that $\Pi|_{\mathscr{C}_{x_0}}$ is an isomorphism. Thus, if we assume that $X_{p_1}(x_0) = 0$, then we obtain that $(\Pi|_{\mathscr{C}_{x_0}})^{-1}(X_{p_1}(x_0)) = 0$, which is false because $dp_1(x_0) \neq 0$.

Therefore $X_{p_1}(x_0) \neq 0$. By the Flow-Box theorem 3.4, there exists a local function q_1, such that $q_1 : \mathscr{U} \to \mathbb{R}$ with $X_{p_1} = \dfrac{\partial}{\partial q_1}$. In a neighborhood of x_0 we have

$$\{p_1, q_1\} = X_{p_1} q_1 = \frac{\partial q_1}{\partial q_1} = 1 \neq 0.$$

In addition, X_{q_1} and X_{p_1} are linearly independent because $X_{q_1} = \lambda X_{p_1}$ implies that $\{q_1, p_1\} = X_{q_1} p_1 = -\lambda X_{p_1} p_1 = 0$. From the Jacobi identity for the Poisson bracket, we have that X_{q_1} and X_{p_1} commute

$$[X_{q_1}, X_{p_1}] = X_{\{q_1, p_1\}} = 0.$$

By the Frobenius theorem , these vector fields can be integrated to define a regular two dimensional foliation in an neighborhood of x_0. As a consequence, we can find a local system of coordinates $(y_1, ..., y_n)$ such that

$$X_{q_1} = \frac{\partial}{\partial y_1} \text{ and } X_{p_1} = \frac{\partial}{\partial y_2}.$$

With these coordinates we $\{q_1, y_i\} = X_{q_1}(y_i) = 0$ and $\{p_1, y_i\} = X_{p_1}(y_i) = 0$, for $i = 3, 4, ..., n$. Poisson's theorem 2.32 then implies that $\{q_1, \{y_i, y_j\}\} = \{p_1, \{y_i, y_j\}\} = 0$ for $i, j \geq 3$. We conclude that $\{y_i, y_j\}$ must be a function of y_i's.

We consider $(p_1, q_1, y_3, ..., y_n)$ as new local system of coordinates and we have

$$\Pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \sum_{i,j \geq 3} \Pi'_{i,j}(y_3, ..., y_n) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

The above formula implies that our Poisson structure is locally the product of a standard symplectic structure on the plane $S_1 = \{(p_1, q_1)\}$ with a Poisson structure on a $(n-2)$-dimensional manifold $U_1 =$

$\{(y_3,...,y_n)\}$. In this product, $N_1 = S_1 \times U_1$ is also the direct product of a point of the plane $\{(p_1,q_1)\}$ with a local submanifold in the Poisson manifold $\{(y_3,...,y_n)\}$.

Now, we apply the same procedure as presented above to U_1 and so on, going through the procedure s-times after which we have a resulting neighborhood N of x_0 such that $N = S_1 \times S_s \times N_n$, with local system of coordinates

$$(p_1,...,p_s,q_1,...,q_s,z_1,...,z_{n-2s})$$

in a neighborhood of x_0 satisfying

$$\{q_i,q_j\} = \{p_i,p_j\} = \{z_i,p_j\} = \{z_i,q_j\} = 0$$

for all i,j, and,

$$\{p_i,q_j\} = \delta_{ij}.$$

We conclude that $N = S \times U$, where $S = S_1 \times ... \times S_s$ is the symplectic manifold, and $U = U_n$ is the Poisson manifold with Poisson bracket determined by $\Pi'_{ij}(y_i,y_j)$, which has zero rank at $\Pi_N(x_0)$ for large enough n. To justify this, we need only to show that an n exists such that the rank of the Poisson bracket of U is zero at $\Pi_N(x_0)$. This is easy because if we consider $n = s$, then the Poisson bracket becomes trivial. This completes the existence proof. □

Remark 3.8. a) The manifolds S and U as described in the above theorem are in fact unique up to local Poisson diffeomorphism (the proof of which shall be omitted, see [64]).

b) Moreover, in the view of the splitting theorem, for $x_0 \in U \cap S$ we have that $\Pi(x_0) = 0$.

Corollary 3.9. *Let us take a symplectic manifold, this is a Poisson manifold* $(\mathcal{M},\Pi)$ *where rank*Π = *dim*$\mathcal{M}$ *everywhere. In this case, the splitting theorem gives canonical coordinates*

$$(p_1,...,p_s,q_1,...,q_s)$$

such that

$$\Pi = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}.$$

Equivalently, since $\Pi^\sharp : T^*\mathcal{M} \to T\mathcal{M}$ *is an isomorphism, we can define a symplectic 2-form on* $\mathcal{M}$

$$\omega = \sum dp_i \wedge dq_i.$$

In other words, we recover Darboux's theorem which gives local canonical coordinates for symplectic manifolds. This explains why Weinstein's splitting theorem is a generalization of Darboux's theorem.

Examining the proof of the splitting theorem 3.6 more closely, we can see that it is a direct consequence that the leaves of the singular foliation defined in section 2.3 are symplectic.

Proposition 3.10. *Let* $(\mathcal{M},\Pi)$ *a Poisson manifold. On each leaf* $\mathscr{C}_x$ *there is a well defined symplectic structure.*

Proof. Let a point $x \in \mathscr{F}$. We consider a local canonical coordinate neighborhood

$$(U, p_1, ..., p_n, q_1, ..., q_n, z_1, ..., z_{n-2s})$$

for $\mathscr{C}_x$. Therefore, $\mathscr{C}_x$ has a natural symplectic structure with Darboux's coordinates

$$(p_1, ..., p_n, q_1, ..., q_n).$$

We define

$$\omega_x = \sum_{i=1}^n d_x p_i \wedge d_x q_i$$

for $x \in U$. This is the symplectic structure on each leaf $\mathscr{C}_x$ of $\mathscr{F}$. □

Remark 3.11. The Poisson structure is completely determined by the symplectic leaves of $\mathscr{F}$. (see [27])

In the next section, we give one example of the Splitting theorem and the singular symplectic foliation.

3.3 Coadjoint Orbits

Here, we will discuss a result developed by Kirillov, Konstant and Souriau. We will prove that the coadjoint orbits of a Lie group are symplectic. Moreover, these symplectic manifolds are the symplectic leaves of the Lie-Poisson bracket. We follow [36], [35] and [45].

First, let us recall that the *adjoint representation* of a Lie group G is defined by

$$Ad_g = T_e I_g : \mathfrak{g} \to \mathfrak{g}$$

where $g \in G$, $\mathfrak{g} = Lie(G)$ and $I_g : G \to G$ is the inner automorphism $I_g(h) = ghg^{-1}$. Now we consider $\mathfrak{g}^*$, the vector space dual to $\mathfrak{g}$. Let $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$, then the *coadjoint representation*

$$Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$$

of G in $\mathfrak{g}^*$ is defined by

$$Ad^*(g) : \mathfrak{g}^* \to \mathfrak{g}^*$$

where $Ad^*(g) \equiv Ad(-g)^*$. Thus

$$\langle Ad(-g)^* F, X\rangle = \langle F, Ad(-g)X\rangle$$

by $\langle ., .\rangle$ we denote the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

We can now define the notion of a *coadjoint orbit*.

Definition 3.12. Given $F \in \mathfrak{g}^*$. The *coadjoint orbit* $\mathscr{O}_F$ is the subset of $\mathfrak{g}^*$ defined by

$$\mathscr{O}_F = \{Ad(-g)^* F : g \in G\}$$

Remark 3.13. Like the orbit of any group action, $\mathscr{O}_F$ is a submanifold of $\mathfrak{g}^*$.

Example 3.14. Here we will calculate the coadjoints orbits of $G = SU(2)$. Recall that the Lie algebra of $SU(2)$ is

$$\mathfrak{g} = Lie(SU(2)) = \{X \in M_2(\mathbb{C}) : X^* + X = 0, tr X = 0\}$$

equivalently,

$$\mathfrak{g} = \left\{ \begin{pmatrix} ib & c+id \\ -c+id & -ib \end{pmatrix} : b, b, d \in \mathbb{R} \right\}$$

and that $\mathfrak{g} \simeq \mathfrak{g}^* \simeq \mathbb{R}^3$.

Let $F \in \mathfrak{g}^*$, then, by definition 3.12 it suffices to find all $F' \in \mathfrak{g}^*$ such that $F' = gFg^{-1}$ for all $g \in SU(2)$. Equivalently, it suffices to find a function $Q : \mathfrak{g} \to \mathbb{C}$ which is invariant on every orbit, this is $Q(gFg^{-1}) = Q(F)$ i.e. $Q = ct$ on every orbit.

Consider $Q(F) = tr(F^2)$, where $F \in \mathfrak{g}$ means that

$$F = \begin{pmatrix} ib & c+id \\ -c+id & -ib \end{pmatrix}$$

and

$$F^2 = \begin{pmatrix} -b^2-c^2-d^2 & 0 \\ 0 & -b^2-c^2-d^2 \end{pmatrix}$$

so $tr(F^2) = -2(b^2+c^2+d^2) = ct$. Without loss of generality, we can write $r^2 = -\dfrac{ct}{2}$, for $r \in \mathbb{R}$. Hence, we obtain $b^2+c^2+d^2 = r^2$ i.e. the two-dimensional concentric spheres and the origin.

Remark 3.15. Notice that the coadjoint orbits of $SU(2)$ are always even-dimensional.

The next theorem explains how the coadjoint orbits are endowed with symplectic structure (the proof of which we omit, for a detailed proof see [45]) .

Theorem 3.16. ***Kirillov-Konstant-Souriau*** *Let G be a Lie group and $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. Then on every coadjoint orbit there exists a symplectic form Ω, defined by*

$$\Omega(F)(X,Y) = \langle F, [X,Y] \rangle$$

for $X, Y \in \mathfrak{g}$ and $F \in \mathfrak{g}^*$. This symplectic form is also called the Kirillov form *or* Kirillov-Konstant-Souriau form *(KKS-form).*

Example 3.17. We will calculate explicitly the symplectic form of the coadjoint orbits of example 3.14 in the spherical coordinate system. In particular, in order to find the Kirillov form, we consider $\mathbb{R}^3$ with Poisson bracket defined in 2.40, then, we will restrict this bracket on the coadjoint orbits, and finally, show that this bracket is non degenerate.

Let $\{x_1, x_2, x_3\}$ be the standard coordinates in $\mathbb{R}^3$. The spheres of radius R in$\mathbb{R}^3$ are given by the equation

$$x_1^2 + x_2^2 + x_3^2 = R^2$$

We can cover the sphere by two charts. One chart being the whole sphere minus the northpole ($\theta = 0$), which we denote by U_+, the other chart being the whole sphere minus the southpole ($\theta = \pi$), which we denote by U_-. Let us calculate the symplectic form on U_-.

The parametrization of U_- is in terms of the parameters ϕ , θ and is given by

$$x_1 = Rsin(\theta)cos(\phi)$$

$$x_2 = Rsin(\theta)sin(\phi)$$

$$x_3 = Rcos(\theta)$$

for $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$.

In example 2.40 we calculated the Poisson bracket in $\mathbb{R}^3 \equiv \mathfrak{g} \equiv \mathfrak{g}^*$. Thus $\mathbb{R}^3$ is a Poisson manifold. We know that in this case the structure constants of the Poisson bracket relations and the Lie bracket relations between the generators of $\mathfrak{g} = SU(2)$ are the same.

As we saw in 2.40, it holds that

$$\{x_1, x_2\} = 2x_3 \text{ , } \{x_2, x_3\} = 2x_1 \text{ and } \{x_3, x_1\} = 2x_2$$

Now we will calculate $\{\phi,\theta\}$. First, let us calculate $\left\{\frac{x_1}{x_2}, x_3\right\}$, because we have

$$\{f(\phi), g(\theta)\} = \frac{\partial f}{\partial \phi}\frac{\partial g}{\partial \theta}\{\phi,\theta\}$$

Thus, we obtain

$$\left\{\frac{x_1}{x_2}, x_3\right\} = \{tan(\phi), Rcos(\theta)\} = \Big(\frac{-1}{cos^2(\phi)}\Big)Rsin(\theta)\{\phi,\theta\}$$

The Leibniz rule gives

$$\begin{aligned}
\left\{\frac{x_1}{x_2}, x_3\right\} &= \frac{1}{x_2}\{x_1, x_3\} - \frac{x_1}{x_2^2}\{x_2, x_3\} \\
&= \frac{1}{x_2}(-2x_2) - \frac{x_1}{x_2^2}(2x_1) \\
&= 2\Big(-1 - \frac{x_1}{x_2^2}\Big) \\
&= -2\Big(1 + tan^2(\phi)\Big) \\
&= \frac{-2}{cos^2(\phi)}
\end{aligned}$$

Consequently, it follows that

$$\{\phi,\theta\} = \frac{2}{Rsin(\theta)}$$

So we obtain the matrix

$$\omega = \begin{pmatrix} \{\phi,\phi\} & \{\phi,\theta\} \\ \{\theta,\phi\} & \{\theta,\theta\} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{2}{Rsin(\theta)} \\ -\dfrac{2}{Rsin(\theta)} & 0 \end{pmatrix}$$

It clearly has a nonvanishing determinant, implying that it is invertible. The inverse is given by

$$\omega = \begin{pmatrix} 0 & \dfrac{1}{2}Rsin(\theta) \\ -\dfrac{1}{2}Rsin(\theta) & 0 \end{pmatrix}$$

So we obtain the nondegenerate 2-form ω (notational abuse):

$$\begin{aligned} \omega &= 0d\phi \wedge d\phi + 0d\theta \wedge d\theta - \frac{1}{2}Rsin(\theta)d\phi \wedge d\theta + \frac{1}{2}Rsin(\theta)d\theta \wedge d\phi \\ &= Rsin(\theta)d\theta \wedge d\phi \end{aligned}$$

It remains to show that ω is closed. Indeed, $d\omega = Rsin(\theta)d\theta \wedge d\theta \wedge d\phi = 0$. Similarly, we can calculate the Kirillov form on U_+.

Remark 3.18. In the above example we calculated the Kirilov form on the coadjoint orbits of $SU(2)$. This symplectic form can define a bracket on every coadjoint orbit, in an obvious way. The bracket is sometimes called *orbit bracket.* It can be defined via restriction of the Lie-Poisson bracket as illustrated in the above example. The next theorem 3.19 summarizes all this (see [45]).

Theorem 3.19. *The Lie-Poisson bracket and the coadjoint orbit symplectic structure (Kirillov form) are consistent in the following sense. For $F, H : \mathfrak{g}^* \to \mathbb{R}$ and $\mathcal{O}$ a coadjoint orbit in $\mathfrak{g}^*$, we have*

$$\{F, H\}|_{\mathcal{O}} = \{F|_{\mathcal{O}}, H|_{\mathcal{O}}\}$$

Here, the first bracket is the Lie-Poisson bracket, while the bracket on the right-hand side of is the Poisson bracket defined by the coadjoint orbit symplectic structure on $\mathcal{O}$.

Another way to calculate the coadjoint orbits of $SU(2)$ is to look the Lie-Poisson structure $\mathbb{R}^3 \equiv \mathfrak{su}^*(2)$ as a foliated manifold and then calculate the leaves of this foliation.

So we consider, again, the set

$$\mathscr{F} = span_{C_c^\infty(\mathscr{M})}\langle X_f : f \in C^\infty(\mathscr{M})\rangle,$$

which is a foliation on $\mathbb{R}^3$.

We will calculate the leaves of this foliation as follows. A basic property of the Casimir functions is that they are constant along the integral curves of the Hamiltonian vector fields. In other words, the integral curves of the Hamiltonian vector fields reside on the level sets of the Casimir functions. So the symplectic leaves, in this case, are the connected components of the level sets of the Casimir functions.

In example 2.40 we showed that the components Π^{ij} of the Poisson structure Π on $(\mathfrak{su}(2))^*$ are: $\Pi^{12}_x = x^3, \Pi^{23}_x = x^1, \Pi^{13}_x = -x^2$.

Consequently, the Poisson structure is

$$\Pi_x = x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$$

By solving the system of pde's $\Pi^\sharp(dC) = 0$, where C is the Casimir function of Π, we have that $C = (x^1) + (x^2) + (x^3)$ i.e. the symplectic leaves of the $\mathfrak{g}^*$ are again the concentric spheres and the origin $\{0\}$ which is itself a singular symplectic leaf.

Example 3.20. Let us consider the Lie group $G = SU(2)$, with Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ and dual space $\mathfrak{g}^*$. In example 2.40, we saw that $\mathfrak{g}^*$ is endowed with the Lie-Poisson structure

$$\Pi = x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$$

As we saw previously the symplectic leaves of $\mathfrak{g}^*$ are the concentric spheres and the origin $\{0\}$.

Now we take the leaf $\mathscr{C}_{x_0} = \{0\}$, which is a manifold equipped with the trivial symplectic structure. In the view of the Splitting theorem 3.6, the transversal Poisson manifold N is $\mathfrak{g}^* = \mathbb{R}^3$.

4 Local symplectic Realizations of Poisson manifolds

As we saw in the splitting theorem, Poisson structures are quite complicated. Alan Weinstein's idea ([64]) is to search for lifts which have simpler structure. Namely, given a Poisson manifold (P,Π) find a symplectic manifold (S,ω) together with a surjective submersion $r: S \to P$, which is a Poisson map. This is called a symplectic realization and the idea is that we can work at (S,ω) and push our results down to (P,Π) via r.

In the next section, we overview the origins of this idea.

4.1 Function Groups and Realizations

Sophus Lie in his treatise on transformation groups written around 1890, considers functions $F_1,\ldots,F_r$ on a symplectic manifold and states the next definition for function groups.

Definition 4.1. Let (S,ω) be a symplectic manifold, and (q_i, p_i) be the canonical variables defined by the Darboux theorem. Then, a *function group* is a collection $\mathfrak{F}$ of functions of the canonical variables such that

a) $\mathfrak{F}$ is a Lie algebra under Poisson bracket,

b) if $F_1,\ldots,F_r \in \mathfrak{F}$ and $G:\mathbb{R}^r \to \mathbb{R}$, then $G(F_1,\ldots,F_r) \in \mathfrak{F}$.

In what follows, we will explain the definition of a function group 4.1 and analyze how the symplectic realization problem arises from it, in global terms.

Remark 4.2. Let (S,ω) be a symplectic manifold and Φ a foliation on the manifold S such that the quotient space S/Φ is a manifold. We may define a *global function group*, which we will denote with $\mathfrak{F}_\Phi$, to be the space of functions constant on the leaves L of Φ closed under Poisson bracket. Thus,

$$\mathfrak{F}_\Phi = C^\infty_\Phi(S) = \{f \in C^\infty(S) : f|_L = ct, \forall L\}.$$

Firstly, let us explain what the condition that $\mathfrak{F}_\Phi$ be closed under Poisson bracket means geometrically.

By definition A.28 a foliation Φ on S is a locally finitely generated submodule of the $C^\infty(S)$-module of compactly supported vector fields $\mathfrak{X}_c(S)$ which is involutive. Let $\Phi^\perp = span_{C^\infty_c(S)}\{Z \in \mathfrak{X}(S) : \omega(X,Z) = 0, \forall X \in \Phi\}$ be its orthogonal complement under the symplectic structure ω. We consider the set

$$X_{\mathfrak{F}_\Phi} = span_{C^\infty_c(S)}\langle \xi_f : f \in \mathfrak{F}_\Phi\rangle \subseteq \mathfrak{X}_c(S)$$

where ξ_f are the hamiltonian vector fields of functions along the leaves of Φ. If $Z \in \Phi^\perp$ and $\xi_f \in X_{\mathfrak{F}_\Phi}$, then we have that $\omega(Z,\xi_f) = i_{\xi_f}\omega(Z) = Z(f) = 0$, because f is a function in $\mathfrak{F}_\Phi$. So the hamiltonian vector field ξ_f lies in $\Phi^\perp$, i.e. $X_{\mathfrak{F}_\Phi} \subseteq \Phi^\perp$. On the other hand, we take $Z \in \Phi^\perp$ and we have that $\omega(X,Z) = 0$, forall $X \in \Phi$. But we also know that $\omega(Z,\xi_f) = 0$. Hence, $\omega(X,Z) = \omega(Z,\xi_f)$ and by dimension count, since ω is an isomorphism, we can easily see that $\Phi^{\perp}$ is filled at each point by the hamiltonian vector fields of functions in $\mathfrak{F}_{\Phi}$. Finally, we obtain that $X_{\mathfrak{F}_{\Phi}} = \Phi^{\perp}$.

If $\mathfrak{F}_{\Phi}$ is a Lie algebra, then the identity $[X_f, X_g] = X_{\{f,g\}}$ holds 2.30 and the set $X_{\mathfrak{F}_{\Phi}}$ is involutive, since $[X_{\mathfrak{F}_{\Phi}}, X_{\mathfrak{F}_{\Phi}}] \subseteq X_{\mathfrak{F}_{\Phi}}$.

Futhermore, by Lie's definition, the set $\mathfrak{F}_{\Phi}$ must be finitely generated in order to be a function group. Thus $X_{\mathfrak{F}_{\Phi}}$ is finitely generated.

Therefore, by Stefan-Sussmann theorem A.33 there is another foliation, which we may call $\overline{\Phi}$, and we obtain the following result $X_{\mathfrak{F}_{\Phi}} \subseteq \overline{\Phi}$. Conversely, if the foliation $\overline{\Phi}$ satisfies the conditions above,then it is obviously that $\overline{\Phi} \subseteq X_{\mathfrak{F}_{\Phi}}$.

Therefore, we have proven the following result.

Proposition 4.3. *Let Φ be a singular foliation on a symplectic manifold S. Then the space $\mathfrak{F}_{\Phi}$ of functions along the leaves of Φ is a function group if and only if $X_{\mathfrak{F}_{\Phi}}$ is involutive.*

Remark 4.4.

a) If the hypotheses of proposition 4.3 are satisfied the functions along the leaves of $\overline{\Phi}$ form another global function group $\mathfrak{F}_{\Phi^{\perp}}$ called its *polar*.

b) The quotient spaces $S/\overline{\Phi}$ and $S/\overline{\Phi^{\perp}}$ are Poisson manifolds. Indeed, the following bracket obviously defines a Poisson structure on $C^{\infty}(S/\overline{\Phi})$:

$$\{f, g\}_{S/\overline{\Phi}} \circ \pi = \{f \circ \pi, g \circ \pi\}_S$$

where $f, g \in C^{\infty}(S/\overline{\Phi})$ and the canonical projection $\pi : S \to S/\overline{\Phi}$ is a Poisson map.

Definition 4.5. A *symplectic realization* of a Poisson manifold $(P, \{.,.\})$ is a symplectic manifold (S, ω) together with a submersion $r : S \to P$, which is a Poisson map.

Example 4.6. Let the circle S^1 acting differentially on the sphere S^2. The orbits of the action are the parallels and the poles of the sphere and form a singular foliation. We consider the function $h : S^2 \to \mathbb{R}$, with $h(p) = z$, where $p = (x, y, z)$ is a point on the sphere.

Now, we observe that h is a surjective submersion. Moreover, we can identify $\mathbb{R}$ with the quotient S^2/S^1, so we have $h : S^2 \to S^2/S^1 \equiv \mathbb{R}$. The quotient S^2/S^1 is a manifold (with boundary), which we will write C. In order to prove that the map h is a symplectic realization it remains to show it is also a Poisson map, since the manifold S^2 is obviously symplectic. This is true in a trivial way, because if we equip $\mathbb{R} \equiv lie^*(S^1)$ with the Lie -Poisson bracket, then the bracket defined is trivial.

4.2 Existence of Local Symplectic Realizations

In this section we will prove the local existence of symplectic realization of a Poisson manifold.

The next lemma is also called Perturbation theorem and it is a result we need in order to prove the local existence of symplectic realizations.

Lemma 4.7. *Let ξ and η be vector fields, with compact support on a differentiable manifold M. Denote ϕ^ξ and ϕ^η the flows of ξ and η respectively. Then, for all $x \in M$ and $t \in \mathbb{R}$ one has*

$$\frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_s^{\xi+\lambda\eta}(x)) = \int_0^s (\phi^\xi)_*(\eta(\phi_{s-t}^\xi(x)))dt$$

Proof. With respect to λ, $\phi_s^{\xi+\lambda\eta}$ is a path of local diffeomorphisms. Thus $\frac{d}{d\lambda}\Big|_{\lambda=0}\phi_s^{\xi+\lambda\eta}(x)$ is the tangent vector of this path at $y = \phi_s^\xi(x)$. The point y as defined gives

$$\frac{d}{d\lambda}\Big|_{\lambda=0}\phi_s^{\xi+\lambda\eta}(x) = \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_s^{\xi+\lambda\eta}(\phi_{-s}^\xi(y)) = \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_s^{\xi+\lambda\eta}\circ\phi_{-s}^\xi(y))$$

We denote this tangent vector by

$$\zeta_s(y) = \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_s^{\xi+\lambda\eta}\circ\phi_{-s}^\xi(y))$$

and obviously we have $\zeta_s(y) \in T_yM$. This expression suggests that we consider the path given by

$$\zeta_\tau(y) = \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_\tau^{\xi+\lambda\eta}\circ\phi_{-\tau}^\xi(y))$$

with $0 \le \tau \le s$, in the tangent space T_yM. Equivalently,

$$\zeta_s(y) = \int_0^s \frac{d}{d\tau}\Big|_{\tau=t}(\zeta_\tau(y))dt = \int_0^s \frac{d}{d\tau}\Big|_{\tau=0}(\zeta_{\tau+t}(y))dt$$

So we should calculate $\zeta_{\tau+t}(y)$

$$\begin{aligned}\zeta_{\tau+t}(y) &= \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_{\tau+t}^{\xi+\lambda\eta}\circ\phi_{-\tau-t}^\xi(y))\\ &= \frac{d}{d\lambda}\Big|_{\lambda=0}(\phi_\tau^{\xi+\lambda\eta}\circ\phi_t^{\xi+\lambda\eta}\circ\phi_{-\tau}^\xi\circ\phi_{-t}^\xi(y))\\ &= \zeta_t(y) + (\phi_t^\xi)_*(\zeta_\tau(\phi_{-t}^\xi(y)))\end{aligned}$$

where we derivated the two appearances of λ (see also `Posilicano` [52]). Applying $\frac{d}{d\tau}$ we get

$$\begin{aligned}\frac{d}{d\tau}\Big|_{\tau=0}(\zeta_{\tau+t}(y)) &= \frac{d}{d\tau}\Big|_{\tau=0}(\zeta_t(y) + (\phi_t^\xi)_*(\zeta_\tau(\phi_{-t}^\xi(y))))\\ &= \frac{d}{d\tau}\Big|_{\tau=0}(\phi_t^\xi)_*(\zeta_\tau(\phi_{-t}^\xi(y)))\\ &= (\phi_t^\xi)_*\frac{d}{d\tau}\Big|_{\tau=0}(\zeta_\tau(\phi_{-t}^\xi(y)))\\ &= (\phi_t^\xi)_*(\frac{\partial}{\partial\tau}\Big|_{\tau=0}\frac{\partial}{\partial\lambda}\Big|_{\lambda=0}(\phi_\tau^{\xi+\lambda\eta}(\phi_{-\tau-t}^\xi(y))))\\ &= (\phi_t^\xi)_*(\frac{d}{d\lambda}\Big|_{\lambda=0}((\xi+\lambda\eta)(\phi_{-t}^\xi(y) + (\phi_0^{\xi+\lambda\eta})_*\xi(\phi_{-t}^\xi(y))))\end{aligned}$$

Moreover, since $\phi_0^{\xi+\lambda\eta} = id$, we calculate

$$\frac{d}{d\tau}\Big|_{\tau=0}(\zeta_{\tau+t}(y)) = (\phi_t^{\xi})_*(\frac{d}{d\lambda}\Big|_{\lambda=0}((\xi+\lambda\eta)(\phi_{-t}^{\xi}(y)) + \xi(\phi_{-t}^{\xi}(y))))$$
$$= (\phi_t^{\xi})_*(\eta(\phi_{-t}^{\xi}(y)))$$

Now we put back $y = \phi_s^{\xi+\lambda\eta}(x)$ and we are done. □

Theorem 4.8. ***Weinstein,1983*** *Any point x of a Poisson manifold (P,Π) has an open neighborhood U such that $(U,\Pi|_U)$ admits a realization by a symplectic manifold of dimension* $2(dimP-(1/2)rank_x\Pi)$.

Proof. By the splitting theorem 3.6, it suffices to discuss only the transversal part of Π. So we assume that $rank_x\Pi = 0$.

We can, now, think of our problem as that of finding a symplectic structure on $\mathbb{R}^{2n}$, with coordinates

$$(x^1,...,x^n,y^1,...,y^n)$$

together with the projection

$$(x^i,y^i)\mapsto(x^i)$$

to be a local symplectic realization of $(\mathbb{R}^n,\{.,.\})$ with $\{x^i,x^j\} = \Pi_{ij}$.

One may think that the canonical structure

$$\sum_{i=1}^n dx^i\wedge dy^i$$

yields such a realization. However, this is true only for the trivial Poisson structure $\Pi_{ij}\equiv 0$.

This leads us to reformulate the problem and look for new coordinates $\overline{x}_i = \phi^i(x,y)$ instead of the x_i. There is no need in changing the y_i-coordinates, because the Poisson structure depends only by the x_i's via the projection map.

By Darboux theorem 2.18, the requested symplectic form is expressed as

$$\sigma = \sum_{i=1}^n d\phi^i\wedge dy^i$$

hence,

$$\sigma = \sum_{i,j=1}^n \frac{\partial\phi^i}{\partial x^j}dx^j\wedge dy^i - \frac{1}{2}\sum_{i,j=1}^n\Big(\frac{\partial\phi^i}{\partial y^j} - \frac{\partial\phi^j}{\partial y^i}\Big)dy^i\wedge dy^j$$

To find σ it suffices to solve the well-known equation

$$i(X_i)\sigma = -dx_i$$

where X_i is the Hamiltonian vector field of x^i with respect to σ, given by the formula

$$X_i = \xi_i^j\frac{\partial}{\partial x^j} + \eta_i^j\frac{\partial}{\partial y^i}$$

with $\xi_i^j = \{x^i, x^j\}$. Then, we get

$$\sum_h \eta_i^j \frac{\partial \phi^h}{\partial x^k} = -\delta_i^k$$

and,

$$\sum_h \Big\{ \Pi_{ik} \frac{\partial \phi^h}{\partial x^k} - \Big(\frac{\partial \phi^k}{\partial y^h} - \frac{\partial \phi^k}{\partial y^k} \Big) \eta_i^k \Big\} = 0$$

So, $\eta_i^j = -\dfrac{\partial x^i}{\partial \overline{x}^j}$, and we remain with the system of equations

$$\Pi_{ik} \frac{\partial \phi^h}{\partial x^k} - \Big(\frac{\partial \phi^k}{\partial y^h} - \frac{\partial \phi^h}{\partial y^k} \Big) \frac{\partial x^i}{\partial \overline{x}^j} = 0$$

or,

$$\frac{\partial \phi^k}{\partial y^h} - \frac{\partial \phi^h}{\partial y^k} = -\{\phi_k, \phi_h\}_\Pi$$

Let us take the function $f^y(x) = \sum_{i=1}^n x^i y^i$, and let X_{f^y} be the Hamiltonian vector field where the function f^y is seen as a function in x. We denote by ϕ_s^y the flow of X_{f^y}, and define

$$\overline{x}_i = \phi^i(x, y) = \int_0^s x^i \circ \phi_s^y ds$$

Now we differentiate ϕ^i with respect to y^i, and we have

$$\frac{\partial \phi^i}{\partial y^j} = \frac{\partial \phi^i(y^j)}{\partial y^j}\Big|_{y^j} = \frac{\partial \phi^i(y^j + \lambda)}{\partial (y^j + \lambda)}\Big|_{\lambda=0} = \frac{\partial \phi^i(y^j + \lambda)}{\partial \lambda}\Big|_{\lambda=0}$$

We know, by the definition of ϕ^i's above, that they are integrals of the coordinates of the points along the flow of X_{f^y}. In this view, $\phi^i(y^j + \lambda)$ are the integrals of the coordinates of the points along the flow of $X_{f^{y^j+\lambda}}$. This is

$$\begin{aligned} X_{f^{y^j+\lambda}} &= \sum_{i,k} \Pi_{ik} y^i \frac{\partial}{\partial x^k} + \lambda \sum_k \Pi_{jk} \frac{\partial}{\partial x^k} \\ &= X_{f^{y^j}} + \lambda X_{x^j} \end{aligned}$$

So by lemma 4.7 the required derivative is given by a straightforward calculation. Hence,

$$
\begin{aligned}
\frac{\partial \phi^i}{\partial y^j} &= \frac{\partial}{\partial y^j}\Big(\int_0^1 x^i(\phi_s^{X_{f^y}+\lambda X_{x^j}}(x))\Big)ds \\
&= \int_0^1 \frac{\partial}{\partial \lambda}\Big|_{\lambda=0}\big(x^i(\phi_s^{X_{f^y}+\lambda X_{x^j}}(x))\big)ds \\
&= \int_0^1 (dx^i)\big(\frac{\partial}{\partial \lambda}\Big|_{\lambda=0}\phi_s^{X_{f^y}+\lambda X_{x^j}}(x))\big)ds \\
&\overset{4.7}{=} \int_0^1 (dx^i)\Big((\phi^{X_{f^y}})_*(X_{x^j}(\phi_{s-t}^{X_{f^y}}))dt\Big)ds \\
&= \int_0^1\int_0^1 \Big(\{x^j, x^i\circ\phi_t^{X_{f^y}}\}\circ\phi_{s-t}^{X_{f^y}}\Big)dtds \\
&= \int_0^1\int_0^1 \{x^j\circ\phi_{s-t}^{X_{f^y}}, x^i\circ\phi_t^{X_{f^y}}\}dtds
\end{aligned}
$$

By the change of variables $(s,t)\mapsto(s,\tau=s-t)$ we have

$$
\frac{\partial \phi^i}{\partial y^j} = \int_0^1\int_0^1 \{x^j\circ\phi_\tau^{X_{f^y}}, x^i\circ\phi_t^{X_{f^y}}\}d\tau ds
$$

Similarly, interchanging i and j we get

$$
\frac{\partial \phi^j}{\partial y^i} = -\int_0^1\int_0^1 \{x^j\circ\phi_\tau^{X_{f^y}}, x^i\circ\phi_t^{X_{f^y}}\}d\tau ds
$$

Thus, we combine the last two results to get

$$
\begin{aligned}
\frac{\partial \phi^i}{\partial y^j} - \frac{\partial \phi^j}{\partial y^i} &= \int_0^1\int_0^1 \{x^j\circ\phi_\tau^{X_{f^y}}, x^i\circ\phi_t^{X_{f^y}}\}d\tau ds \\
&= \Big\{\int_0^1 x^j\circ\phi_\tau^{X_{f^y}}d\tau, \int_0^1 x^i\circ\phi_s^{X_{f^y}}ds\Big\} \\
&= \{\phi^j,\phi^i\}
\end{aligned}
$$

which completes the proof of existence. □

5 Global symplectic realizations of Lie-Poisson structures

In theorem 4.8 we showed that every Poisson manifold (M,Π) has local symplectic realizations. In this section we examine the problem of existence of global realizations as such. We start with the case of a Lie group G and the Lie-Poisson structure on $\mathfrak{g}^*$ and show that it admits T^*G as a global symplectic realization. We observe that in fact T^*G has extra structure, namely it has the structure of a Lie groupoid and moreover its standard symplectic structure is compatible with the groupoid structure. In other words, it is a *symplectic groupoid*.

This understanding of the Lie-Poisson case gives rise to the idea that global symplectic realizations of an arbitrary Poisson manifold (M,Π), if they exist, might be found among the symplectic groupoids over M. On the other hand, as we saw in section 2.3, a Poisson structure on M is really a Lie algebroid structure on the cotangent bundle T^*M. In view of this, in chapter 6 we discuss how the search for groupoids as such can be cast in the integrability of Lie algebroids. In particular, we prove that, given a Poisson manifold (M,Π) such that the Lie algebroid T^*M integrates to a Lie groupoid Σ over M, then Σ is a global symplectic realization in a canonical way. This is a result by Karasev and Weinstein.

5.1 Global realizations of Lie-Poisson structures

Let G be a Lie group and $(\mathcal{M},\Pi)$ be a Poisson manifold. We consider the (smooth) left action of G on $\mathcal{M}$. Namely,

$$\Phi : G\times\mathcal{M}\to\mathcal{M}$$

with the action of a group element g on the point m, written as $\Phi_g(m)$. In terms of this notation, a group action Φ satisfies:

a) $\Phi_{g_1}\circ\Phi_{g_2}=\Phi_{g_1g_2}$, and

b) $\Phi_e = Id_{\mathcal{M}}$.

Since the action is smooth, for each $g\in G$, the map

$$\Phi_g : \mathcal{M}\to\mathcal{M}$$

is a diffeomorhism with inverse $\Phi_{g^{-1}}$. So, locally, we get

$$\Phi : G\to Diff_{loc}(\mathcal{M})$$

where,

$$Diff_{loc}(\mathcal{M}) = \{f : U_f\to V_f \text{diffeomorphisms} : U_f, V_f\subseteq\mathcal{M}\text{open}\}$$

We, also, have that $Diff(\mathcal{M})\subseteq Diff_{loc}(\mathcal{M})$, which is a Lie pseudogroup. So a smooth action of a Lie group G is, in fact, a homomorphism of Lie groups.

Differentiating this map ($Diff_{loc}(\mathscr{M})$ is an infinite-dimensional manifold which is smooth), we obtain the *infinitesimal action* associated to $X \in \mathfrak{g}$:

$$T_e\Phi : T_eG \to T_{Id_{\mathscr{M}}}(Diff_{loc}(\mathscr{M}))$$

simplifying the notation, we have equivalently,

$$\Phi_* : \mathfrak{g} \to \mathfrak{X}(\mathscr{M})$$

defined by $X \mapsto X_{\mathscr{M}}$. In detail,

$$\begin{aligned} X_{\mathscr{M}}(m) &:= \frac{d}{dt}\Big|_{t=0} \Phi(exp(tX))(m) \\ &= \frac{d}{dt}\Big|_{t=0} \Phi_{exp(tX)}(m) \\ &= -\Phi^m_*(X)(e) \end{aligned}$$

The induced vector field $X_{\mathscr{M}}$ is called *infinitesimal generator*.

Definition 5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\Phi : G \times \mathscr{M} \to \mathscr{M}$ the group action of G on $\mathscr{M}$. We call Φ a *Poisson action* if, $\forall g \in G$, the map $\Phi_g : \mathscr{M} \to \mathscr{M}$ is a Poisson map. Furthermore, if $\forall X \in \mathfrak{g}$ there is a function $f_X \in C^\infty(\mathscr{M})$ such that $X_{\mathscr{M}}$ is precisely the Hamiltonian vector field of f_X, then Φ is called *Hamiltonian action*. In this case, we have

$$df_X = i_{X_{\mathscr{M}}}\Pi$$

Let us give an example of a Hamiltonian group action.

Example 5.2. Let us take the manifold $\mathbb{R}^2$, equipped with Poisson structure given by

$$\Pi = dx \wedge dy$$

Now we consider the one-dimensional torus S^1 acting on $\mathbb{R}^2$ by rotations

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} = exp(\theta X_{\mathscr{M}})$$

where

$$X_{\mathscr{M}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then the corresponding vector field on $\mathbb{R}^2$ is

$$X_{\mathscr{M}} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Hence, for $f_X = \frac{1}{2}(x^2 + y^2)$ we have an example of a Hamiltonian group action. Indeed,

$$i_{X_{\mathscr{M}}}\Pi = ydy + xdx = df_X$$

Proposition 5.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^*$ its dual space. An action of a connected Lie group G on the Poisson manifold $(\mathcal{M},\Pi)$ is Hamiltonian if and only if there exists a differentiable mapping $J:\mathcal{M}\to\mathfrak{g}^*$ such that for all $X\in\mathfrak{g}$ the function $J(X)\in C^\infty(\mathcal{M})$ defined by*

$$J(X)(x) = J(x)(X) \tag{5.1}$$

satisfies

$$X_{\mathcal{M}} = X_{J(X)} \tag{5.2}$$

Proof. If J exists we obviously have a Hamiltonian action. Conversely, We assume that for all $X\in\mathfrak{g}$, we have some $f_X\in C^\infty(\mathcal{M})$, such that

$$X_{\mathcal{M}} = X_{f_X}$$

We consider a basis $X_1,...,X_s$ on $\mathfrak{g}$ and we define $J(x)$ by 5.1 where if

$$X=\sum_{i=1}^{s} c^i X_i$$

we take

$$J(x)=\sum_{i=1}^{s} c^i f_{X_i}$$

Then $J:\mathcal{M}\to\mathfrak{g}^*$, and it satisfies the properties requested. □

Definition 5.4. A mapping $J:\mathcal{M}\to\mathfrak{g}^*$ that satisfies 5.1 and 5.2 is called a *momentum map* of the Hamiltonian action of G on $(\mathcal{M},\Pi)$.

Definition 5.5. A momentum map is called *equivariant* if $J(g(x)) = Ad^*_g(J(x))$, for $g\in G$.

Proposition 5.6. *An equivariant momentum map $J:\mathcal{M}\to\mathfrak{g}^*$ is a Poisson morphism, if $\mathfrak{g}^*$ is endowed with its Lie-Poisson structure.*

Proof. Let $X,Y\in\mathfrak{g}$. Firstly, we need to prove that an equivariant momentum map satisfies

$$\{J_X,J_Y\} = J_{[X,Y]}$$

For $x\in\mathcal{M}$, we have

$$\begin{aligned}
\{J_X,J_Y\}(x) &= X_{J(X)}(x)(J_Y)\\
&= X_{\mathcal{M}}(x)(J_Y)\\
&= -\frac{d}{dt}\Big|_{t=0} J_Y(exp(tX)(x))\\
&= -\frac{d}{dt}\Big|_{t=0} (Ad^*(exp(tX))(J(x))(Y)\\
&= \frac{d}{dt}\Big|_{t=0} (J(x))(Ad(exp(tX))(Y)\\
&= (J(x))([X,Y])\\
&= J_{[X,Y]}(x)
\end{aligned}$$

Now, we can prove that J is a Poisson morphism. Let us take functions $l_1, l_2 \in C^\infty(\mathfrak{g}^*)$ and check that

$$\{l_1 \circ J, l_2 \circ J\} = \{l_1, l_2\} \circ J$$

where the bracket of the right hand side is the Lie - Poisson bracket of $\mathfrak{g}^*$. So

$$\begin{aligned}(\{l_1, l_2\} \circ J)(x) &= \{l_1, l_2\} \circ (J(x)) \\ &\overset{2.2.4}{=} J(x)([l_1, l_2]) \\ &\overset{5.4}{=} J_{[l_1, l_2]}(x) \\ &= \{J_{l_1}, J_{l_2}\}(x) \\ &= \{l_1 \circ J, l_2 \circ J\}(x)\end{aligned}$$

since x is an arbitrary element of $\mathscr{M}$, we are done. $\square$

Now we consider the group actions of G on itself

a) *Left action* $L_g : G \to G$ defined by $L_g(h) = gh$

b) *Right action* $R_g : G \to G$ defined by $R_g(h) = hg^{-1}$.

We denote by Φ_g and Ψ_g the lift of these actions to T^*G defined by

a) $L^*_{g^{-1}} : T^*_{L_{g^{-1}}(h)}G \to T^*_h G$ with $L^*_{g^{-1}}(\xi)(\sigma) = \xi((L_g)_*(\sigma))$, and

b) $R^*_{g^{-1}} : T^*_{L_{g^{-1}}(h)}G \to T^*_h G$ with $R^*_{g^{-1}}(\xi)(\sigma) = \xi((R_g)_*(\sigma))$, respectively.

Then we will show that Φ and Ψ are Hamiltonian actions that have equivariant momentum maps

$$J^\Phi, J^\Psi : T^*G \to \mathfrak{g}^*$$

defined by

a) $J^\Phi(\xi) = -R^*_g(\xi)$, and

b) $J^\Psi(\xi) = L^*_g(\xi)$, for $\xi \in T^*_g G$.

As a first step, we notice that we have a natural identification

$$T^*G = G \times \mathfrak{g}^*$$

given by the natural projection $\pi : T^*G \to G$, and by the projection $pr : T^*G \to \mathfrak{g}^*$ defined by $pr(\xi) = L^*_{\pi(\xi)}(\xi)$. This also induces an identification

$$T_\xi T^*G = \mathfrak{g} \times \mathfrak{g}^*$$

given by the projections $\pi' = L_{g*} \circ \pi$, and $pr' = pr_*$. Thus we express the actions Φ and Ψ by

$$\pi \circ \Phi_g = L_g \circ \pi \ , \ pr \circ \Phi_g = pr \ , \ \pi \circ \Psi_g = R_g \circ \pi \text{ and } pr \circ \Psi_g = Ad^*_g \circ pr.$$

We will compute $X^{\Phi}_{T^*G}$ and $X^{\Psi}_{T^*G}$, which are the corresponding infinitesimal actions, and we will show that $X^{\Phi}_{T^*G}$ and $X^{\Psi}_{T^*G}$ are precisely the Hamiltonian vector fields of J^{Φ}_X and J^{Ψ}_X respectively. Hence, J^{Φ} and J^{Ψ} are indeed momentum maps and the actions Φ and Ψ are Hamiltonian actions. So

$$\begin{aligned}
\pi'(X^{\Phi}_{T^*G}(\xi)) &= L^{-1}_{g*}\frac{d}{dt}\Big|_{t=0}(exp(-tX)(g)) \\
&= -L_{g*}R_{g*}(X) \\
&= -(Adg^{-1})(X),
\end{aligned}$$

$$pr'(X^{\Phi}_{T^*G}(\xi)) = 0,$$

$$\pi'(X^{\Psi}_{T^*G}(\xi)) = L^{-1}_{g*}\frac{d}{dt}\Big|_{t=0}(gexp(tX)) = X$$

and

$$\begin{aligned}
pr'(X^{\Psi}_{T^*G}) &= -\frac{d}{dt}\Big|_{t=0}(Ad^*(exp(tX))L^*_g(\xi)) \\
&= (Ad^* X)(pr(\xi))
\end{aligned}$$

Now, let α be the Liouville form, and $\omega = d\alpha$ be the canonical symplectic form of T^*G defined in 2.1.4. Let $\xi \in T^*_g G$, $\Xi, \Xi_1, \Xi_2 \in T_\xi T^*G$, and $\overline{\Xi}_1, \overline{\Xi}_2$ be the values at $\overline{\xi}$ in T^*G of the vector fields given by the cross sections of $TT^*G = G \times \mathfrak{g} \times \mathfrak{g}^*$ over G that have the same (constant) projections on $\mathfrak{g}$ and $\mathfrak{g}^*$ as Ξ_1, Ξ_2. Then the definition of α yields

$$\alpha_\xi(\Xi) = \xi(\pi_*\Xi) = pr(\xi)(\pi'(\Xi))$$

and

$$\begin{aligned}
(d\alpha)_\xi(\Xi_1, \Xi_2) &= \Xi_1(\alpha(\overline{\Xi}_2)) - \Xi_2(\alpha(\overline{\Xi}_1))\alpha_\xi([\overline{\Xi}_1, \overline{\Xi}_2]) \\
&= \Xi_1(pr(\overline{\xi})(\pi'(\overline{\Xi}_2)) - \Xi_2(pr(\overline{\xi})(\pi'(\overline{\Xi}_1)) - pr(\xi)([\pi'\overline{\Xi}_1, \pi'\overline{\Xi}_2]) \\
&= pr'(\Xi_1)(\pi'(\Xi_2)) - pr'(\Xi_2)(\pi'(\Xi_1)) - pr(\xi)([\pi'\overline{\Xi}_1, \pi'\overline{\Xi}_2]).
\end{aligned}$$

Now we have to compute $i(X^{\Phi,\Psi}_{T^*G})d\alpha$, for $X \in \mathfrak{g}$, and at a point $\xi \in T^*G$. Let $y \in T_\xi T^*G$ be extended by a field $\overline{Y}$ as we did above. Then we get

$$(i(X^{\Phi}_{T^*G})d\alpha)(Y) = (d\alpha)_\xi(X^{\Phi}_{T^*G}, Y) = pr'(Y)((Ad_{g^{-1}})(X)) + pr(\xi)([(Ad_{g^{-1}})(X), \pi'Y])$$

and

$$\begin{aligned}
(i(X^{\Phi}_{T^*G})d\alpha)(Y) = (d\alpha)_\xi(X^{\Phi}_{T^*G}, Y) = -((Ad^*X)(pr(\xi)))(\pi'Y) - (pr'Y)(X) - pr(\xi)([X, \pi'Y]) = \\
-(pr'(Y))(X)
\end{aligned}$$

On the other hand, for $X \in \mathfrak{g}$, we take J_X^Φ and J_X^Ψ as defined in 5.4 and we will compute their derivatives in the direction of the field $\overline{Y}$, at the point $\xi \in T_g^*G$. So we have

$$J_X^\Phi(\xi) = -(R_g^*\xi)(X) = -(R_g^* L_{g^{-1}}^* pr(\xi)(X)) = -(Ad_g^*)(pr(\xi))(X)$$

whence

$$\begin{aligned}
(dJ_X^\Phi)_\xi(\overline{Y}) &= \frac{d}{dt}\Big|_{t=0}(J_X^\Phi(exp(t\overline{Y})(\xi))) \\
&= -\frac{d}{dt}\Big|_{t=0}(Ad^*(exp(t\pi_*\overline{Y})(\xi))) \\
&= -\frac{d}{dt}\Big|_{t=0}(Ad^*(g(exp(t\pi' Y))(pr(\xi) + tpr'Y))(X) \\
&= Ad_g^*((ad^*(\pi' Y)(pr(\xi))))(X) - ((Ad_g^*)(pr'Y))(X) \\
&= -pr(\xi)([(Ad_{g^{-1}})X, \pi' Y]) - (pr'Y)(Ad_{^{-1}}(X))
\end{aligned}$$

In a similar way we obtain,

$$J_X^\Psi(\xi) = L_g^*(\xi)(X) = pr(\xi)(X)$$

and

$$\begin{aligned}
(dJ_X^\Psi)_\xi(\overline{Y}) = YJ_X^\Psi &= \frac{d}{dt}\Big|_{t=0}(J_X^\Psi(expt\overline{Y}(\xi)) \\
&= \frac{d}{dt}\Big|_{t=0}(expt(pr'\overline{Y})(pr(\xi)))(X) \\
&= (pr'(Y))(X)
\end{aligned}$$

Summarizing all the above, we conclude that $X_{T^*G}^\Psi$ are precisely the Hamiltonian vector fields of J_X^Φ and J_X^Ψ, respectively, and that J^Φ , J^Ψ are momentum maps as mentioned previously. Moreover, these momentum maps are equivariant.

Therefore, by proposition 5.6 we can easily, see that J_X^Φ and J_X^Ψ are symplectic realizations of $\mathfrak{g}^*$ of the symplectic manifold T^*G.

5.2 The Lie groupoid structure of T*G

In this section we, briefly, introduce the notion of a Lie groupoid and study some basic results. Our goal is to define a symplectic groupoid structure on the cotangent bundle T^*G.

Definition 5.7. A *groupoid* over a set M is a set Σ together with the following structure maps:

1. Two maps s and t from Σ to M, called respectively the *source* map and the *target* map.
2. A *product* map, which is a partial multiplication $m : \Sigma_{(2)} \to \Sigma$ with $(g,h) \mapsto gh$, where

   $$\Sigma_{(2)} = \{(g,h) \in \Sigma \times \Sigma : s(h) = t(g)\}$$

   subject to the following condition:

(a) compatibility with s and t:

$$s(gh) = s(h) \text{ and } t(gh) = t(g)\ ,\ \forall (g,h) \in \Sigma_{(2)}$$

(b) associativity:

$$(gh)j = g(hj)$$

for all $g, h, j \in \Sigma$ such that $s(g) = t(h)$ and $s(h) = t(j)$.

3. A map $1 : M \to \Sigma$, $m \mapsto 1_m$ called the object *inclusion* such that:

$$m1_{s(m)} = m = 1_{t(m)}m$$

In particular, $s(1_m) = t(1_m) = m$ is the identity map on M.

4. An *inversion* $i(g)$ of an element $g \in \Sigma$ is denoted by g^{-1}.

Remark 5.8. M is also denoted by $\Sigma_{(0)}$ and is called the *set of objects*, or *base points* and is often identified with the set 1_M of identity elements of Σ. Σ is also denoted by $\Sigma_{(1)}$. An element of Σ may be called an *arrow*. We often indicate a groupoid and its base by $\Sigma \rightrightarrows M$.

Definition 5.9. A *Lie groupoid* $\Sigma \rightrightarrows M$, is a groupoid Σ on base M together with smooth structures on G and M such that the maps $s, t : \Sigma \to M$ are surjective submersions, the inclusion map $1 : M \to \Sigma$ is smooth, and the partial multiplication $\Sigma_{(2)} \to \Sigma$ is smooth.

Remark 5.10. The fact that s, t are surjective submersions implies that $\Sigma_{(2)}$ is a closed embedded submanifold. Whence, it makes sense for the partial multiplication to be smooth. (see `MacKenzie` [43])

The following examples are basic.

Examples 5.11. a) A *group* is a groupoid over a point.

b) Let M be an arbitrary manifold, and we will show that the cartesian square $\Sigma = M \times M$ is a Lie groupoid on M. So we define the structure maps in the following way:

1. source map, $s : \Sigma \to M$ with $s(x, y) = x$ and target map, $t : \Sigma \to M$ with $t(x, y) = y$.
2. product map $m : \Sigma_{(2)} \to \Sigma$ with

$$\begin{aligned}\Sigma_{(2)} &= \{(x, y, z, h) \in M^4 : s(z, h) = t(x, y)\} \\ &= \{(x, y, z, h) \in M^4 : z = y\}\end{aligned}$$

so, we get

$$m((x, y)(y, h)) = (x, y) \cdot (y, h) = (x, h)$$

3. inclusion map $1 : M \to \Sigma$ with $1_x = (x, x)$, and
4. inversion map $i : \Sigma \to \Sigma$ with

$$i(x, y) = (x, y)^{-1} = (y, x)$$

The Lie groupoid $\sigma = M \times M$ is called the *pair groupoid* on M.

c) We consider $\delta : G \times M \to M$ the left action of a group G on a manifold M, and we denote the product manifold $G \times M$ by $\Sigma = G \times M$. This the *action groupoid* defined as follows:

$$s(g,x) = x$$

$$t(x,y) = \delta(g,x)$$

$$1_x = (e,x)$$

$$(g,x)^{-1} = (g^{-1}, \delta(g,x))$$

where $g \in G$, $x \in M$ and $e \in G$ is the neutral element of G. Moreover, the multiplication

$$(g_2, y) \cdot (g_1, x) = (g_2 g_1, x)$$

is defined if and only if $y = g_1 x$. In the literature, the Lie groupoid $\Sigma = G \times M$ is also called the *transformation groupoid* (see `Dufour` [27]).

Proposition 5.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then T^*G has a natural structure of a Lie groupoid with base $\mathfrak{g}^*$, source map $s = J^\Psi$ and target map $t = -J^\Phi$.*

Proof. We consider the multiplication of the group G

$$m : G \times G \to G$$

and, we take the differential

$$Tm : T(G \times G) \equiv TG \times TG \to TG$$

Now, we suppose that $X \in T_{g_1}G$ and $Y \in T_{g_2}G$ for $g_1, g_2 \in G$. Applying the Leibniz rule in the usual way to the map

$$T_{(g_1,g_2)}m : T_{g_1}G \times T_{g_2}G \to T_{g_1 g_2}G$$

we define an operation on the tangent bundle TG

$$\bullet : T_{g_1}G \times T_{g_2}G \to T_{g_1 g_2}G$$

given by

$$X \bullet Y = R_{g_2 *}(X) + L_{g_1 *}(Y)$$

Therefore, the formula

$$(\xi_1 \bullet \xi_2)(X \bullet Y) = \xi_1(X) + \xi_2(Y)$$

yields a well defined operation between $\xi_1 \in T^*_{g_1}G$ and $\xi_2 \in T^*_{g_2}G$ with the result in $T_{g_1 g_2}G$ if and only if the above formula vanishes on $ker \bullet = \{(X,Y) : (R_{g_2})_*(X) = -(L_{g_1})_*(Y)\}$.

Equivalently, $(X,Y) \in ker \bullet$ means that $X = -(R_{g_2^{-1}})_*(L_{g_1})_*(Y)$. Thus we get

$$L^*_{g_1} R^*_{g_2} \xi_1 = \xi_2.$$

Since left and right translations commute, we obtain the following result

$$J^\Psi(\xi_1) = -J^\Phi(\xi_2).$$

In other words, the operation $\bullet$ is the product map. So $\bullet$ satisfies the second axiom of definition 5.7. It is easy to verify the rest axioms, if we take into account the formula

$$\xi_1 \bullet \xi_2 = \frac{1}{2}(R^*_{g_2}\xi_1 + L^*_{g_1}\xi_2)$$

with inverse map defined by

$$\xi^{-1} = L^*_{g_1} R^*_{g_2}\xi.$$

These formulas, also, define differentiable operators. Moreover, J^Ψ and J^Φ are submersions. Thus, we conclude that the cotangent bundle T^*G is equipped naturally with a Lie groupoid structure. □

At this point, let us recall some classical results of symplectic geometry presented in section 2.1.5. So let N be a k-dimensional submanifold of an n-dimensional manifold M, then we have the following:

For every submanifold N of a differential manifold M, the conormal bundle $\nu^* N$ is a Lagrangian sumbanifold of $(T^*M, d\alpha)$. (see 2.24), where the conormal bundle of N is

$$\nu^* N = \{(x,\xi) \in T^*M : x \in N, \xi \in \nu^*_x N\}$$

Proposition 5.13. *The graph of the multiplication $T^*_{(2)}G \to T^*G$ is a Lagrangian submanifold of $T^*G \times T^*G \times T^*G$, endowed with the symplectic structure $d\alpha \times d\alpha \times (-d\alpha)$, where α is the Liouville form.*

Proof. The multiplication space $G_3 = \{(g_1, g_2, g_1 g_2) : g_1, g_2 \in G\}$ is a submanifold of $G \times G \times G$. Then, using proposition 2.24, we have that the conormal bundle $\nu^* G_3$ given by $\{(\xi_1, \xi_2, -\xi_1 \bullet \xi_2)\}$, is Lagrangian in $T^*(G \times G \times G) = T^*G \times T^*G \times T^*G$, where T^*G has the canonical symplectic structure $\omega = d\alpha$.

We observe that the graph of the multiplication mapping $T^*_{(2)}G \to T^*G$ differs from $\nu^* G_3$ only by the sign of the last factor. Thus, multiplying the last component by -1 we get a Lagrangian submanifold of $T^*G \times T^*G \times T^*G, d\alpha \times d\alpha \times (-d\alpha)$. This completes the proof. □

6 Global Symplectic Realization and Integrability (discussion)

In sections 5.1 and 5.2 we proved explicitly that given a Lie group G with Lie algebra $\mathfrak{g}$ and the Lie-Poisson structure on $\mathfrak{g}^*$, we can find global symplectic realization by the cotangent bundle T^*G, which has the algebraic structure of a Lie groupoid. But what happens for an arbitrary Poisson manifold (P,Π)? Is there a global symplectic realization in this case?

There are two ways to approach the global symplectic realization problem.

6.1 Glueing

As we saw in section 4.2 local symplectic realizations always exist. One might try to glue them to a global symplectic realization. This was performed first by Karasev in [34].

Another approach was given by Crainic and Marcut in [22]. This approach uses the formulation of Poisson geometry using the language of Lie algebroids, namely the Lie algebroid T^*P. Very roughly, using the contravariant geometry of T^*P together with an averaging process, these authors showed that there exists a neighborhood of the zero section of T^*P which has a certain symplectic structure so that the restriction of the projection map $\pi : T^*P \to P$ is a Poisson map. This symplectic structure coincides locally with the one constructed by Weinstein.

6.2 Integrating

The case of the Lie-Poisson structure leads to the following notion which was introduced independently by Karasev [34], Weinstein [65], Coste,Dazord and Weinstein [21], and by Zakrzewski [67],[68].

Definition 6.1. Let P be a smooth manifold. A *symplectic groupoid* over P is a Lie groupoid $\Sigma \rightrightarrows P$, equipped with a symplectic form ω on Σ, such that the graph of the multiplication map is a Lagrangian submanifold of $\Sigma \times \Sigma \times (-\Sigma)$, where $-\Sigma$ means the manifold Σ with the opposite symplectic form $-\omega$.

Note that, given a Lie groupoid $G \rightrightarrows M$, there is a Lie functor which associates to G a Lie algebroid $AG \to M$. Roughly, if s, t are the source and target maps of G respectively, the vector bundle AG is $\ker ds|_M$ and the anchor map is the restriction $dt|_{AG} : AG \to TM$. The full details of this construction can be found in [43], [27] and [47].

For a symplectic groupoid $\Sigma \rightrightarrows P$ in particular, we find in [43, Prop. 11.5.3] the following properties:

a) The manifold P has a canonical Poisson structure Π.

b) The source map $s : \Sigma \to P$ is a Poisson map and the target map $t : \Sigma \to P$ is anti-Poisson.

c) The source map induces an isomorphism of Lie algebroids between $A\Sigma$ and the Lie algebroid structure of T^*P induced by the Poisson structure Π.

On the other hand, in [43] we find the following result:

Proposition 6.2. *Let (P, Π) be a Poisson manifold and consider the associated Lie algebroid structure on T^*P. If there exists a Lie groupoid $\Sigma \rightrightarrows P$ such that $T^*P = A\Sigma$ then the canonical symplectic structure of T^*P gives rise to a symplectic structure on the manifold Σ which makes $\Sigma \rightrightarrows P$ a symplectic groupoid.*

The above results cast the problem of existence of global realizations to the problem of intgrability for Lie algebroids. Namely, given a Lie algebroid $A \to M$, is there a Lie groupoid $G \rightrightarrows M$ such that $AG = A$? In other words, does Sophus Lie's third theorem (Lie III) apply for Lie algebroids?

To discuss this, first recall that Lie's third theorem produces a connected and simply connected Lie group. Given a Lie groupoid $G \rightrightarrows M$, the fibers of AG are nothing else than tangent spaces at identity elements of the s-fibers of G. Whence, the correct formulation of Lie III in the context of Lie algebroids is that the integrating groupoid G should have connected and simply connected s-fibers.

As it happens, Lie's third theorem does not hold in the context of Lie algebroids. The specific integrability obstructions were given by Crainic and Fernandes in [23], following the work of Cattaneo and Felder [16]. Note that the ideas involved can be traced back to the proof of Lie III given by Duistermaat and Kolk in [28].

Due to the lack of time we do not discuss these obstructions here. However, it is worth mentioning that, the integrability of a Poisson manifold (P, Π) is really controlled by the topology of its associated symplectic foliation.

A Foliations

The study of foliations has a long history in mathematics, even though it did not emerge as a distinct field, until the 1940's, when the concept of a foliation first appeared explicitly in the work of Ehresmann and Reeb. They were motivated by the question of existence of completely integrable vector fields on three dimensional manifolds. Since that time the subject has enjoyed a rapid development and the theory of foliations has now become a rich and exiting geometric subject by itself as illustrated by the famous results of Reeb (1952) [53], Haefliger (1956) [30], Novikov (1964) [50], Thurston (1974) [60], Molino (1988) [48], Connes (1994) [20] and many others. At the moment it is the focus of a great deal of research activity.

The purpose of this chapter is to provide an introduction to the subject and present the field as it is currently evolving.

A.1 Partitions to leaves and foliations

Definition A.1. Let $\mathscr{M}$ be a finite-dimensional manifold and $\mathscr{F}$ be a decomposition of $\mathscr{M}$ into immersed submanifolds, called *leaves*. Then $\mathscr{F}$ is called a *smooth partition to leaves* (possibly of different dimension, hence the singularities).

Remark A.2. In the definition above A.1 we consider smooth foliations. By smooth we mean that for every $x \in \mathscr{M}$ and every $u \in T_x L_x$, then there is a vector field $X \in \mathscr{M}$ such that it satisfies the following:

a) $u = X(x)$, and

b) $X(y) \in TyLy$, $\forall y \in \mathscr{M}$.

Such partition to leaves occur naturally in various geometric contexts.

Examples A.3. a) Let a submersion $f : \mathscr{M} \to \mathscr{N}$, from a manifold $\mathscr{M}$ of dimension n to a manifold $\mathscr{N}$ of dimension d. Any submersion defines a partition to leaves $\mathscr{F}(f)$ of $\mathscr{M}$ whose leaves are the connected components of the fibers of $\mathscr{N}$.

The Submersion Theorem asserts that for each $p \in \mathscr{M}$, there is a coordinate neighborhood $(\mathscr{U}, y^1, ..., y^n)$ of p in $\mathscr{M}$ and a coordinate neighborhood $(\mathscr{V}, x^1, ..., x^d)$ of $f(p)$ in $\mathscr{N}$, relative to which the formula for $f|_{\mathscr{U}}$ becomes:

$$f(y^1, ..., y^n) = (y^1, ..., y^d).$$

It follows via the surjective form of the implicit function theorem that the level sets:

$$f^{-1}(x) = \{p \in \mathscr{M} | f(p) = x\}$$

are properly embedded submanifolds of $\mathscr{M}$, of dimension $k = n - d$, for every p $\in \mathscr{M}$, since $df_p : T_p\mathscr{M} \to T_{f(p)}\mathscr{N}$ is surjective at every point of $\mathscr{M}$. Locally these submanifolds fit together exactly like parallel copies of $\mathbb{R}^k$ in $\mathbb{R}^n$. An atlas representing $\mathscr{F}(f)$ is also constructed by the implicit function theorem and is called atlas of submersions.

The leaves of the partition $\mathscr{F}(f)$ are the level sets of f (the fibers of $\mathscr{N}$).

If $\mathscr{N}$ is connected and each of these level sets is compact then $f : \mathscr{M} \to \mathscr{N}$ is actually a fiber bundle.

A fiber bundle is always a submersion, but the inverse is not true. Indeed, consider

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$f(x, y) = (x^2 - 1)e^y$$

Here,

$$f_x(x, y) = 2xe^y$$
$$f_y(x, y) = (x^2 - 1)e^y$$

so f is a submersion because the derivative f_x vanishes when $x = 0$, while f_y only vanishes along the lines $x = \pm 1$. As mention above, the level sets of f give a partition to leaves $\mathscr{F}$ on $\mathbb{R}^2$. In this case, the leaves are of the form: $f^{-1}(p) = \{(x, y) \in \mathbb{R}^2 | f(x, y) = p\}$. For $p = 0$ the leaves are the vertical lines $x = \pm 1$, for $p < 0$ the leaves are asymptotic curves between the lines $x = \pm 1$ and for $p > 0$ each leave falls into two components, one lying in the region $x < -1$ and asymptotic to $x = -1$ and one lying in the region $x > 1$ and asymptotic to $x = 1$.

The leaf space $\mathbb{R}^2/\mathscr{F}$ formed by collapsing each leaf to a point equipped with the quotient topology, is locally Euclidean of dimension 1, but is not Hausdorff , so it cannot be base manifold of a bundle.

b) Let $X \in \mathfrak{X}(\mathscr{M})$ a nonsingular complete vector field for a manifold $\mathscr{M}$, then the local flow lines defined by X patched together define a partition to leaves of dimension 1.

The fact that X is nonsingular allows us to utilize the Flow Box Theorem for an arbitrary point $x \in \mathscr{M}$ to find a coordinate neighborhood $(\mathscr{U}, x^1, \ldots, x^n)$ about x such that $-\varepsilon < x^i < \varepsilon, 1 < i < n$, and $\dfrac{\partial}{\partial x^1} = X|U$.

Geometrically, the flowlines (integral curves) are the level sets $x^i = c^i, 2 \le i \le n$ where all $|c^i| < \varepsilon$.

In order to better understand these class of examples, consider the partition to leaves of the torus $\mathscr{T}^2$:

Given $X = \partial_x + \theta\partial_y$ a vector field on $\mathbb{R}^2$. The partition to leaves $\mathscr{F}'$ on $\mathbb{R}^2$ has as leaves the parallel lines of slope θ, which are of the form: $L' = \{(x_0 + t, y_0 + \theta t)\}_{t \in \mathbb{R}}$. This partition to leaves is invariant under translations and passes to a partition $\mathscr{F}$ on the torus $\mathscr{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Now, we consider two different cases for the slope to be rational and irrational.

When the slope is rational, the corresponding leaves of $\mathscr{F}$ are closed curves homeomorphic to the circle. Indeed, for fixed $t_0 \in \mathbb{R}$, the points of L' corresponding to values of $t \in t_0 + \mathbb{Z}$ all project to the same point of $\mathscr{T}^2$. Since L is arbitrary, $\mathscr{F}$ is a partition to leaves $\mathscr{T}^2$ by circles.

In the irrational case, the partition to leaves $\mathscr{F}$ on the torus $\mathscr{T}^2$ is totally different. The leaves of $\mathscr{F}$ are noncompact, homeomorphic to the real line $\mathbb{R}$ and are everywhere dense in $\mathscr{T}^2$ (Kronecker's Theorem).

If we restrict the plane $\mathbb{R}^2$ to a unitary 2-cube (square), we visualize the example by observing a starting point which is moved by the flow in the direction θ at constant speed and when it reaches the border of the unitary square it jumps to the opposite face of the square.

c) Let G be a Lie group and $H \hookrightarrow G$ a connected Lie subgroup then the partition to leaves $\mathscr{H}$ of G is defined by the collection $\{gH\}_{g\in H}$ of the left cosets of H(set of leaves).

If H is closed subgroup, then G/H is a manifold and the fiber bundle $\pi : G \to G/H$ defines the leaves of the partition to leaves $\mathscr{H}$ as illustrated in example 2.2.a. If not, we choose instead of H the closure $\overline{H}$ which is also a Lie subgroup so the leaves of the partition to leaves in these case, are the fibers of the bundle $\pi : G \to G/\overline{H}$.

The linear partitions to leaves of torus $\mathscr{T}^2$ are special cases of this, where $\mathscr{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

d) Let G be a Lie group acting differentiably on a manifold $\mathscr{M}$. For every point $m \in M$ the orbit is defined by $\mathscr{O}_m = Gm = \{g * m : g \in G\}$ and the isotropy subgroup by $H_m = Stab(m) = \{g \in G : g * m = m\}$. Then H is a Lie subgroup in G, and $g \mapsto g * m$ is an injective immersion $G/H \hookrightarrow \mathscr{M}$ whose image coincides with the orbit $\mathscr{O}_m$. Whence, the orbits of G form a partition to leaves of $\mathscr{M}$ as illustrated in the previous example.

To make these class of examples more meaningful, we consider the Lie group action of the Special Orthogonal Group $SO(3)$ on the Sphere $S^2 \subseteq \mathbb{R}^3$. In this case the orbits of the action are the parallels of the sphere and the poles.

Remark A.4. We observe that in the last two examples the dimension of the leaves is not the same everywhere. For instance, the parallels of the sphere, in the last example, are of dimension 1 , while the poles are of dimension zero 0, hence the singularities we will present and study later on.

A.2 Regular Foliations

Before setting out the theory of foliations in the singular case, the regular case is required. So we assume these leaves to be of the same dimension and fit together nicely. We follow closely [9], [17], [37] and [47].

A.2.1 Foliation atlas

A way to define a regular foliation is to give only the *foliation atlas*. Let $\mathscr{M}$ be a smooth manifold of dimension n. A foliation atlas of codimension q ($0 \le q \le n$) is an atlas $(\varphi_i : U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i\in I}$ of $\mathscr{M}$. $\mathbb{R}^{n-q}$ coordinate is the longitudinal direction and the $\mathbb{R}^n$ coordinate is the transversal direction. The change of charts diffeomorphisms φ_{ij} are locally of the form:

$$\varphi_{ij}(l,t) = (g_{ij}(l,t), h_{ij}(t))$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$. This means that the change of charts diffeomorphisms depends only on the transversal direction in the second variable. The charts of a foliation atlas are called foliation charts.

Thus each U_i is divided into plaques, which are the connected components of the submanifolds $\varphi_i^{-1}(\mathbb{R}^{n-q}\times\{t\})$, $t\in\mathbb{R}^q$ (Indeed, for $id_{\mathscr{M}}:\varphi_i^{-1}(\mathbb{R}^{n-q}\times\{t\})\longrightarrow\mathscr{M}$).

The plaques globally amalgamate into leaves. These leaves are immersed submanifolds of dimension $n-q$.

A foliation of dimension $n-q$ of $\mathscr{M}$ is a maximal foliation atlas of $\mathscr{M}$ of dimension $n-q$. Each foliation atlas determines a foliation, since it is included in a unique maximal foliation atlas. Two foliation atlases define the same foliation of $\mathscr{M}$ precisely if they induce the same partition of $\mathscr{M}$ into leaves. A (smooth) foliated manifold is a pair $(\mathscr{M},\mathscr{F})$, where $\mathscr{M}$ is a smooth manifold and $\mathscr{F}$ a foliation of $\mathscr{M}$.

Finally, we can obtain the space of leaves by defining an equivalence relation on $\mathscr{M}$. Let $x\backsim y$ iff $x,y\in\mathscr{M}$ if they lie on the same leaf $\mathscr{F}$. Then the space of leaves $\mathscr{M}/\mathscr{F}$ is the quotient space of $\mathscr{M}$.

A.2.2 Distributions and Frobenius Theorem

In the previous section, we defined foliations given by a suitable foliation atlas on manifold $\mathscr{M}$ and we saw that, in general, a foliation on $\mathscr{M}$ is a decomposition of $\mathscr{M}$ into leaves which are locally given by the fibers of a submersion. In this section, we present an equivalent way of defining a foliation by an integrable subbundle of the tangent bundle of $\mathscr{M}$. The equivalence of all these descriptions is a consequence of the Frobenius integrability theorem (see [11] and [39] for any proofs omitted).

Definition A.5. Let $\mathscr{M}$ be a smooth manifold, and suppose for each $p\in\mathscr{M}$ we are given a linear subspace $\Delta_p\subset T_p\mathscr{M}$, whose dimension is $k(p)$. Then $\Delta=\sqcup_{p\in\mathscr{M}}\Delta_p\subseteq T\mathscr{M}$ is a *smooth distribution* if the following condition is satisfied:

> "Each point $p\in\mathscr{M}$ has a neighborhood U on which there are smooth vector fields $Y_1,\dots,Y_{k(p)}:U\to T\mathscr{M}$ such that $Y_1|_p,\dots,Y_{k(p)}|_p$ form a basis for Δ_p for every $p\in U$."

Remark A.6. a) The function

$$\mathscr{M}\to\mathbb{Z}$$

defined by

$$p\mapsto k(p)$$

is assumed to be lower semi-continuous.

b) In case $k(p)$ is constant, the distribution is called regular.

Definition A.7. A *vector subbundle* of the tangent bundle TM is a bundle $F$with a vector bundle morphism $i:F\to TM$, which is everywhere injective.

Remark A.8. The dimension of each fiber of the subbundle Δ is called dimension of the distribution. In case where the dimension is constant everywhere, then we call the distribution regular, otherwise singular. So in what follows we work on the regular case, unless otherwise specified.

Proposition A.9. *A regular distribution Δ is a vector subbundle of TM.*

Proof. A subbundle may be defined, equivalently, as follows: Given a smooth vector bundle $\pi : E \to \mathscr{M}$, a (smooth) subbundle of E is a subset $\Delta \subseteq E$ with the following properties:

a) Δ is an embedded submanifold of E.

b) For each $p \in \mathscr{M}$, the fiber $\Delta_p = \Delta \cap \pi^{-1}(p)$ is a linear subspace of $E_p = \pi^{-1}(p)$.

c) With the vector space structure on each Δ_p inherited from E_p and the projection $\pi|_\Delta : \Delta \to \mathscr{M}$, Δ is a smooth vector bundle over $\mathscr{M}$.

In this view, if Δ is a distribution, then by definition A.5, obviously satisfies b. Thus, it remains to show that Δ satisfies the other two conditions, in order to be a subbundle.

To prove that Δ is an embedded submanifold, it suffices to show that each point $p \in \mathscr{M}$ has a neighborhood U such that $\Delta \cap \pi^{-1}(U)$ is an embedded submanifold of $\pi^{-1}(U) \subseteq T\mathscr{M}$. Given $p \in \mathscr{M}$, let

$$Y_1, \dots, Y_{k(p)}$$

be vector fields defined on a neighborhood of p and satisfying the hypothesis of definition A.5. The independent vectors

$$Y_1|_p, \dots, Y_{k(p)}|_p$$

can be extended to a basis

$$Y_1|_p, \dots, Y_n|_p$$

for $T_p\mathscr{M}$, and then

$$Y_{k+1}|_p, \dots, Y_{k(p)}|_n$$

can be extended to vector fields in a neighborhood of p. By continuity, they will still be independent in some neighborhood U of p. Hence, they form a local frame for $T\mathscr{M}$ over U. This yields a local trivialization

$$\pi^{-1}(U) : U \to \mathbb{R}^n$$

defined by

$$y^i Y_i|_p \mapsto (p, (y^1, \dots, y^n))$$

In terms of this trivialization, $\Delta \cap \pi^{-1}(U)$ corresponds to

$$U \times \mathbb{R}^k = \{(p, (y^1, \dots, y^k, 0, \dots, 0)) \subset U \times \mathbb{R}^n$$

which is obviously a regular submanifold. Moreover, the map

$$\Phi|_{\Delta\cap\pi^{-1}(U)} : \Delta \cap \pi^{-1}(U) \to U \times \mathbb{R}^k$$

is obviously a local trivialization of Δ, showing that Δ is itself a vector bundle. $\square$

Definition A.10. Suppose that $\Delta \subset T\mathscr{M}$ is a distribution. An immersed submanifold $\mathscr{N} \subset \mathscr{M}$ is called an *integral manifold* of Δ if $T_p\mathscr{N} = \Delta_p$ at each point $p \in \mathscr{N}$.

Before we present the general theory, let us describe an example of distributions and integral manifolds.

Example A.11. If $0 \neq V : \mathcal{M} \to T\mathcal{M}$ is any nowhere-vanishing vector field on a manifold $\mathcal{M}$. Then for each $p \in \mathcal{M}$ take

$$\Delta_p = span < V_p >$$

so V spans a 1-dimensional distribution on $\mathcal{M}$. Now, consider an integral curve $\gamma : J \to \mathcal{M}$ and take $\mathcal{N} \leq \mathcal{M}$ a submanifold of $\mathcal{M}$, where $\mathcal{N} = Im\gamma$ then $T_p\mathcal{N} = \Delta_p$. So the image of any integral curve of V is an integral manifold of Δ.

Definition A.12. We call a distribution $\Delta \leq T\mathcal{M}$ *involutive* if there exists a local basis $X_1, ..., X_n$ in a neighborhood of each point such that:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

$1 \leq i, j \leq n$. (The c_{ij}^k will not in general be constants, but will be C^∞ functions on the neighborhood.)

Remark A.13. In other words, we say that a tangent distribution Δ is involutive if given any pair of local sections of Δ (i.e., vector fields X, Y defined on an open subset of $\mathcal{M}$, such that $X_p, Y_p \in \Delta_p$ for each p), their Lie bracket is also a section of Δ.

Example A.14. An important example of an involutive distribution is furnished by the Lie algebra $\mathfrak{h}$ of a subgroup H of a Lie group G. Here $\mathfrak{h}$ consists of left-invariant vector fields on G which are tangent to H at the identity. We know that this determines a subalgebra, the image of the Lie algebra of H under the inclusion map. These give a (left-invariant) distribution Δ on G such that $\Delta_q = T_q(H)$ for every $q \in H$. The cosets gH are the integral manifolds of this distribution, which is evidently involutive since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

Definition A.15. A distribution $\Delta \leq T\mathcal{M}$ is *integrable*, if through each point of $\mathcal{M}$, there exists an integral manifold of Δ.

Proposition A.16. *Every integrable distribution is involutive.*

Proof. Suppose X and Y are local sections of Δ defined on some open subset $U \subseteq M$. Let p be any point in U, and let N be an integral manifold of Δ passing through p. The fact that X and Y are sections of Δ means that X and Y are tangent to N. This implies that $[X, Y]$ is also tangent to N, and therefore $[X, Y]_p \in \Delta_p$. □

Theorem A.17. ***(Local Frobenius Theorem)*** *Every involutive distribution is integrable.*

Remark A.18. The local form of Frobenius Theorem says that a neighborhood of every point on a manifold is filled up with integral manifolds, fitting together nicely like parallel subspaces of $\mathbb{R}^n$.

The main fact about foliations is that they are in one-to-one correspondence with involutive distributions. One direction, expressed in the next lemma, is an easy consequence of the definitions.

Lemma A.19. *Let $\mathscr{F}$ be a foliation of a smooth manifold $\mathscr{M}$. Then the collection of tangent spaces to the leaves of $\mathscr{F}$ forms an involutive distribution on $\mathscr{M}$.*

Proof. The tangent spaces to the leaves clearly give a distribution on $\mathscr{M}$, because for each point we have identified a subspace of the tangent space at that point.

We must verify that this distribution is involutive. Integrability implies involutivity, so we can see through the leaves that the distribution is integrable, because $T_pL_p = \Delta_p$, for every $p \in \mathscr{M}$. $\square$

Theorem A.20. ***Global Frobenius Theorem)*** *Let Δ be an involutive distribution on a manifold $\mathscr{M}$. Then there is a partition of immersed submanifolds L_x on $\mathscr{M}$, such that $T_xL_x = \Delta_x$ for every $x \in \mathscr{M}$.*

A.3 Singular Foliations

A.3.1 Stefan-Sussmann Theorem

In the regular case, the classical Frobenius Theorem yields a necessary and sufficient condition of integrability. However, the following example due to Sussmann [57] shows that this theorem may not hold in the singular case.

Example A.21. Let $\mathscr{M} = \mathbb{R}^2$ and the function $\phi(x) = \{^{0, x\leq 0}_{e^{-(\frac{1}{x})}, x>0}$. Consider, now, the vector fields $X = \frac{\partial}{\partial x}$, $Y = \phi(x)\frac{\partial}{\partial y}$ and take the $C_c^\infty(\mathbb{R}^2)$-module

$$\mathscr{F} = span_{C_c^\infty(\mathbb{R}^2)} < X, Y >$$

Here we have $[X, Y] = 0$ for $x \leq 0$ and

$$[X,Y] = \frac{\partial}{\partial x}\phi(x)\frac{\partial}{\partial y} - \phi(x)\frac{\partial}{\partial y}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}(e^{-(\frac{1}{x})})\frac{\partial}{\partial y} = e^{-(\frac{1}{x})}\frac{\partial}{\partial x}(-(\frac{1}{x}))\frac{\partial}{\partial y} = \phi(x)\frac{\partial}{\partial x}ln\phi\frac{\partial}{\partial y} = \frac{\partial ln\phi}{\partial x}\phi(x)\frac{\partial}{\partial y}$$

for $x > 0$. So the Frobenius involutivity condition holds, but the integrability condition is obviously violated for translations parallel to the x-axis and which cross the y-axis. Thus, there are no leaves through the points of the y-axis.

Sussmann and Stefan have found another condition which ensures complete integrability in all cases.

Definition A.22. A distribution Δ is called *invariant* with respect to a family of smooth vector fields C if it is invariant with respect to every element of C: if $X \in C$ and ϕ_X^t denotes the local flow of X, then we have $(\phi_X^t)_*\Delta_x = \Delta_{\phi_X^t(x)}$ wherever $\phi_X^t(x)$ is well defined.

The following result, due to Stefan and Sussmann (see [56] and [57]) gives an answer to the following question: what are the conditions for a smooth singular distribution to be the tangent distribution of a singular foliation?

Theorem A.23. ***(Stefan - Sussmann)*** *Let Δ be a distribution on a smooth manifold $\mathscr{M}$. Then the following three conditions are equivalent:*

a) Δ *is integrable,*

b) Δ *is generated by a family C of smooth vector fields and is invariant with respect to C,*

c) Δ *is the tangent distribution $\Delta^{\mathscr{F}}$ of a smooth singular foliation $\mathscr{F}$.*

Remark A.24. It is clear that if a singular distribution is integrable, then it is involutive. Conversely, for regular distributions we have the Frobenius Theorem as presented in section 2.1, but what happens in the singular case?

Definition A.25. A smooth distribution Δ on a manifold $\mathscr{M}$ is called *locally finitely generated* if for any $x \in \mathscr{M}$ there is a finite number of smooth vector fields $X_1, ..., X_n$ in a neighborhood U of x, which are tangent to Δ, such that any smooth vector field Y in U which is tangent to Δ can be written as: $Y = \sum_{i=1}^{n} f_i X_i$ with $f_i \in C^\infty(U)$.

Theorem A.26. ***(Hermann,1963)*** *Any locally finitely generated smooth involutive distribution on a smooth manifold is integrable.*

A.3.2 Modules and Serre-Swan Theorem

In the previous sections we analysed the term foliation on a manifold $\mathscr{M}$ in either of the following ways:

a) A partition of $\mathscr{M}$ to disjoint submanifolds (leaves), possibly of different dimension (hence the singularities), or

b) A distribution $\mathscr{F}$ on the tangent bundle $T\mathscr{M}$ which is locally finitely generated by (globally defined) vector fields and involutive (satisfying the conditions given by Stefan and Sussmann).

If a foliation is regular, then the two notions coincide (Frobenius Theorem), namely the leaves determine the vector fields which define the distribution. Another way to see this is that in this case $\mathscr{F}$ is a (constant rank) vector subbundle of $T\mathscr{M}$, so locally its module of sections does not depend on the choice of vector fields which generate it.

In the singular case though, this is no longer true. One can get the same leaves from different choices of vector fields as we can see from the examples below.

Examples A.27. a) Consider the real line and the partition of $\mathbb{R}$ into three leaves $L_1 = \mathbb{R}^*_-$, $L_2 = \{0\}$ and $L_3 = \mathbb{R}^*_+$. These may be considered integral submanifolds to any of the submodules $\mathscr{F}_n =< x^n \frac{d}{dt} >$ of $\mathfrak{X}(\mathscr{M})$ for a positive integer n. Although, $\mathscr{F}_{n+1}$ lies inside $\mathscr{F}_n$ the converse does not hold. In this example we have a preferred choice of module, say $\mathscr{F}_1$, but in several other cases no such choice is possible.

b) Suppose that $\mathbb{R}$ is foliated by the leaves $\mathbb{R}_+$ and $\{x\}$ for any $x \leq 0$. Then we can take $\mathscr{F} =< f \frac{\partial}{\partial x} >$ where the function $f \equiv 0$ vanishes for every non-positive real. Observe that we cannot consider the module of all vector fields which vanish on $\mathbb{R}_-$, as it is not locally finitely generated.

So in the singular case one needs to determine a priori the module of vector fields which gives the distribution. We therefore need to postulate the following definition given by I.Androulidakis and G.Skandalis:(see [4])

Definition A.28. Let $\mathscr{M}$ be a smooth manifold. A (Stefan-Sussmann) foliation on $\mathscr{M}$ is a locally finitely generated submodule of the $C^\infty(\mathscr{M})$-module of compactly supported vector fields $\mathfrak{X}_c(\mathscr{M})$, stable under Lie brackets.

Remark A.29. It was shown by Stefan and Sussmann that such a module induces a partition of $\mathscr{M}$ to (immersed) submanifolds, called leaves. The leaf at $x \in \mathscr{M}$ of a singular foliation $\mathscr{F}$ is the set of points in $\mathscr{M}$ which can be connected to x following integral curves of vector fields in $\mathscr{F}$.

Now, we consider a manifold $\mathscr{M}$, $\mathscr{F}$ a foliation on $\mathscr{M}$ and $x \in \mathscr{M}$. We take $I_x = \{f \in C^\infty(\mathscr{M}) : f(x) = 0\}$ and the space $\mathscr{F}(x) = \{X \in \mathscr{F} : X_x = 0\}$, which is a Lie subalgebra of $\mathscr{F}$. Then $I_x\mathscr{F} \subset \mathscr{F}(x)$.

Moreover, we notice that the evaluation map $\overline{e}_x : \mathscr{F} \to T_x\mathscr{M}$ vanishes on $I_x\mathscr{F}$. Thus, we obtain a surjective homomorphism $e_x : \mathscr{F}_x \to F_x$, where F_x is the tangent space of the leaf L, i.e. $F_x = T_xL$ and it is, obviously, the image of the evaluation map. They are both finite-dimensional linear spaces.

We denote the kernel of the evaluation map by $\mathfrak{g}_x$ and we get that $\mathfrak{g}_x = \mathscr{F}(x)/I_x\mathscr{F}$. We have the short exact sequence

$$0 \to \mathfrak{g}_x \to \mathscr{F}_x \to F_x \to 0$$

Proposition A.30. *Let L be a leaf of $\mathscr{F}$. Then for any $x \in L$, $\mathfrak{g}_x$ vanishes if and only if L is a regular leaf.*

Proof. If L is a regular leaf, then nearby the point x we can find generators of $\mathscr{F}$ which are linearly independent at x, implying that $\mathscr{F}(x) = I_x\mathscr{F}$. If L is a singular leaf, pick a neighborhood W in $\mathscr{M}$ of some $x \in L$, and pick a set of generators $X_1, .., X_n$ of $\mathscr{F}$ defined on W. We may assume that this is a minimal set of generators, i.e. we may assume that none of the X_i can be written as a $C^\infty(W)$ linear combination of the others. Further, as $\{X_i(x)\}$ spans T_xL and W contains leaves of dimension $> dim(L)$, we may assume that $X_1(x)$ is a linear combination of the remaining $X_i(x)$, in other words $X_1 \in \mathscr{F}(x)$. However $X_1 \notin I_xF$ because if we could write $X_1 = \sum_{i=1}^n f_i X_i$ with $fi \in I_x$ then we would have $X_1 = \sum_{i\neq 1} f_i/(1-f_1)X_i$, which contradicts the minimality assumption. □

Theorem A.31. ***(Serre-Swan)*** *The category of vector bundles over a compact Haussdorf space X is equivalent to the category of finitely generated projective modules over the algebra of continuous functions on X.*

Remark A.32. Consider a vector bundle $\pi : E \to \mathscr{M}$ and take the set of sections $\Gamma E = \{\sigma : \mathscr{M} \to E, \pi\circ\sigma = id\}$ i.e. for $E = T\mathscr{M}$ we have $\Gamma E = \mathfrak{X}(\mathscr{M})$, then one can easily see that ΓE is $C^\infty(\mathscr{M})$-module.

On the other hand, we have the Serre-Swan theorem, which relates the category of vector bundles over a compact smooth manifold $\mathscr{M}$ to the category of finite rank projective modules, over the algebra of smooth functions $C^\infty(M)$ of $\mathscr{M}$. It relates geometric and algebraic notions and is, in particular, the starting point for the definition of vector bundles in non-commutative geometry.

Therefore if the submodule of the definition A.28 is projective we have the regular case (where $\mathscr{F}_x \equiv F_x$, since $\Gamma E/I_x\Gamma E \equiv E_x = \pi^{-1}(x)$), otherwise we have the singular case.

Theorem A.33. ***(Stefan - Sussmann)*** *Let $\mathscr{F}$ be a $C^\infty(M)$ - submodule of compactly supported vector fields $\mathfrak{X}_c(\mathscr{M})$ (not necessarily projective) which is locally finitely generated and involutive ($[\mathscr{F},\mathscr{F}] \subseteq \mathscr{F}$). Then there is a decomposition of $\mathscr{M}$ into immersed submanifolds (the leaves of the foliation $\mathscr{F}$).*

Example A.34. Take $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi\Big|_{(-\frac{1}{2},\frac{1}{2})} = 0$ and nonzero elsewhere. Consider now, the vector field $X = \varphi(x)\dfrac{\partial}{\partial x}$ and for $\mathscr{F} = \langle X \rangle$ we will prove that this is a singular foliation of the space $\mathbb{R}$.

Observe that $\mathscr{F}$ is obviously, locally finitely generated submodule of compactly supported vector fields $\mathfrak{X}_c(\mathbb{R})$.

Moreover, the involutivity condition is satisfied. Indeed, for $\xi,\eta \in \mathscr{F}$, we have that $\xi = fX$ and $\eta = gX$, where f,g are $C^\infty(\mathbb{R})$- functions. Then, we calculate their Lie bracket

$$\begin{aligned}
[\xi,\eta] &= [fX.gX] \\
&= f[X.gX] + X(f)gX \\
&= -f[gX,X] + X(f)gX \\
&= -fg[X,X] - fX(g)X + X(f)gX \\
&= 0 + (-fX(g) + X(f)g)X \in \mathscr{F}
\end{aligned}$$

Now, in order to complete our proof, we will show that the Serre-Swan theorem A.31 does not hold in this case. In this way, we show that $\mathscr{F}$ is, in fact, a singular foliation. So we need to examine the following cases:

x=1: Let us consider the function

$$\mathscr{F}/I_1\mathscr{F} \to \mathbb{R}$$

which maps $[\xi]$ to $\xi(1)$. In this case, we can easily verify that

$$\mathscr{F}/I_1\mathscr{F} \cong \mathbb{R}$$

For the one-to-one correspondence (1-1), it suffices to show that

$$ker(\mathscr{F}/I_1\mathscr{F} \to \mathbb{R}) = \{0\}$$

Taking an arbitrary $\xi \in \mathscr{F}$, we have that $\xi = gX$, where $X = \phi(x)\dfrac{\partial}{\partial x}$ and $g \in C^\infty(\mathscr{F}/I_1\mathscr{F})$. Hence, $\xi = g\phi\dfrac{\partial}{\partial x}$. Now we take $[\xi] \in ker(\mathscr{F}/I_1\mathscr{F} \to \mathbb{R})$, then

$$0 = \xi(1) = g(1)\phi(1)\frac{\partial}{\partial x}\Big|_1$$

and we obtain that $g(1) = 0$, because for $x = 1$, we know that $\phi(1) \neq 0$. Therefore, we get that $g \in I_1$, and $\xi = gX \in I_1\mathscr{F}$. Consequently, $ker(\mathscr{F}/I_1\mathscr{F} \to \mathbb{R}) = \{0\}$, since $[\xi] = 0$. The surjectivity of $\mathscr{F}/I_1\mathscr{F} \to \mathbb{R}$ is obvious.

x=0: On the other hand, we consider

$$\mathscr{F}/I_0\mathscr{F} \to \{0\}_{\mathscr{F}/I_0\mathscr{F}}$$

where $[\xi] \mapsto \xi(0)$. The function as defined is surjective. Indeed, for every $\xi \in \mathscr{F}$, we have $\xi(0) = 0$, since $\xi \in \mathscr{F} \Rightarrow \xi = g\phi\frac{\partial}{\partial x}$. We calculate

$$\xi(0) = g(0)\phi(0)\frac{\partial}{\partial x}\Big|_0 = 0$$

because $\phi(0) = 0$, for every $x \in (-\frac{1}{2}, \frac{1}{2})$.

It remains to show the one-to-one correspondence. In a similar way as previously, we will show that the kernel is trivial, i.e.

$$ker(\mathscr{F}/I_0\mathscr{F} \to \{0\}_{\mathscr{F}/I_0\mathscr{F}}) = \{0\}$$

Taking an arbitrary $\xi \in \mathscr{F}$, we will show that this belongs in $I_0\mathscr{F}$. This is true, because we have

$$\xi = g\phi\frac{\partial}{\partial x} = (g - g(0))\phi\frac{\partial}{\partial x} + g(0)\phi\frac{\partial}{\partial x}$$

- If $g(0) = 0$, then obviously $g \in I_0$,
- If $g(0) \neq 0$, then $[\xi] = g(0)\left[\phi\frac{\partial}{\partial x}\right] = g(0)[X] = g(0)\mathrm{ev}_0(X) = 0$.

A.4 Foliations arising from Lie algebroids

In this section we will show that every Lie algebroid has an associated foliation, which in general will be a foliation with singularities.

Firstly, let us introduce the concept of a Lie algebroid and study some basic consequences coming up.

Definition A.35. Let M be a smooth manifold. A *Lie algebroid* over M is a vector bundle $A \to M$ equipped with a smooth vector bundle map $\rho : A \to TM$, called the *anchor* of A, and a Lie bracket $[.,.]$ on the space $\Gamma(A)$ of smooth sections of A

$$\Gamma(A) \times \Gamma(A) \to \Gamma(A)$$
$$(X, Y) \mapsto [X, Y]$$

such that, we have the following Leibniz-type formula:

$$[X, fY] = f[X, Y] + (\rho X(f))Y$$

for allX, Y smooth sections of A and $f \in C^\infty(M)$.

Remark A.36. We will denote a Lie algebroid by $(A, [.,.], \rho)$, or only by the letter A.

Next lemma describes a fundamental property of Lie algebroids, and is often considered as a part of the definition A.35. However, we will see that it is a consequence of the other conditions.

Lemma A.37. *Let* $(A, [.,.], \rho)$ *be a Lie algebroid, then the anchor map is a Lie algebra homomorphism*

$$\rho[\sigma,\eta] = [\rho\sigma, \rho\eta]$$

for all $\sigma, \eta \in \Gamma(A)$.

Proof. By the Jacobi identity and the Leibniz rule, we have

$$\begin{aligned}
0 &= [[\sigma,\eta], f\theta] + [[\eta, f\theta], \sigma] + [[f\theta, \sigma], \eta] \\
&= f[[\sigma,\eta],\theta] + (\rho[\sigma,\eta](f))\theta + f[[\eta,\theta],\sigma] \\
&- (\rho\sigma(f))[\eta,\theta] + (\rho\eta(f))[\theta,\sigma] - (\rho\sigma(\rho\eta(f)))\theta + f[[\theta,\sigma],\eta] \\
&- (\rho\eta(f))[\theta,\sigma] - (\rho\sigma(f))[\theta,\eta] + (\rho\eta(\rho\sigma(f)))\theta \\
&= ((\rho[\sigma,\eta] - [\rho\sigma,\rho\eta])(f))\theta.
\end{aligned}$$

Since $\sigma, \eta \in \Gamma(A)$ and function f are arbitrary, we conclude that

$$\rho[\sigma,\eta] = [\rho\sigma, \rho\eta].$$

□

The anchor map $\rho : A \to TM$ induces a morphism of $C^\infty(M)$-modules,

$$\rho : \Gamma(A) \to \mathfrak{X}(M)$$

(we abuse the notation and denote this map ρ as well). Now we put

$$\mathscr{F} = span_{C^\infty(M)}(Im\rho).$$

In other words, elements of $\mathscr{F}$ are $C^\infty(M)$-linear combinations

$$\sum_{i=1}^{n} f_i \rho(\sigma_i)$$

where $f_i \in C^\infty(M)$, $\sigma_i \in \Gamma(A)$. In addition, $\mathscr{F}$ satisfies the following properties:

a) $\mathscr{F}$ is locally finitely generated $C^\infty(M)$-submodule of $\mathfrak{X}(M)$, because $\Gamma(A)$ is so.

b) $\mathscr{F}$ is involutive because for $\sigma, \eta \in \Gamma(A)$ we have the following fundamental property $\rho[\sigma,\eta] = [\rho\sigma, \rho\eta]$.

This means that the set $\mathscr{F}$ is completely integrable in the sense of Stefan-Sussmann. Thus every Lie algebroid corresponds to a singular foliation.

Remark A.38. The singular foliation of a Lie algebroid $(A, [.,.], \rho)$ is also called the *characteristic foliation* of A.

Example A.39. Consider $A = T^*\mathscr{M}$ and $\rho = \sharp$ for $(\mathscr{M}, \Pi)$ a Poisson manifold (see 2.3). Note that in this case

$$\mathscr{F} = span_{C^\infty(\mathscr{M})}\langle X_f : f \in C^\infty(\mathscr{M})\rangle.$$

We showed how foliations arise naturally from Lie algebroids. Now we consider the singular foliation $\mathscr{F}$ and take a leaf L. For a point $x \in L$, we will denote by A_x the fiber over x, and by $ker\rho_x$ the kernel of the anchor map

$$\rho_x : A_x \to T_x M$$

The kernel $ker\rho_x$ has a natural Lie algebra structure, defined as follows. For any $a_x, b_x \in ker\rho_x$, denote by a , b arbitrary sections of A whose value at x is a_x and b_x respectively, and put

$$[a_x, b_x] = [a, b](x).$$

Hence $\mathfrak{g}_x = ker\rho_x$ is a Lie algebra, called the *isotropy algebra* of A at x. We have that

$$0 \to \mathfrak{g}_x \to A_x \xrightarrow{\rho_x} T_x M \to 0$$

We rewrite $\mathfrak{g}_x = \mathscr{F}(x)/I_x\mathscr{F}$, where the space $\mathscr{F}(x) = \{X \in \mathscr{F} : X_x = 0\}$ is a Lie subalgebra of $\mathscr{F}$ as illustrated in section A.3.2.

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