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Applications of Baire's Category Theorem in Complex Analysis in One and Several complex variables

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to Thea
to Vassili
to Kostas "the other one"
and to Mitsos

thank you

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1 Introduction

In this work we present several results concerning mostly applications of Baire's Category theorem in Complex Analysis both in one and in several complex variables, combined with known approximation results. Roughly speaking, we consider various complete metric spaces (or Fréchet spaces) of complex functions and we examine particular subsets of them that are usually defined as all elements of the space that satisfy a specific property, most of the times concerning universal approximation. Each such class is proved to be topologically generic, in the sense that it contains a G_δ - dense subset. This is mainly achieved by using Baire's theorem, among other arguments.

In some of our results we do not make use of Baire's theorem at all and we work from the perspective of Functional Analysis in order to obtain results of different nature. We consider various Fréchet spaces of functions and we examine whether they contain the translation of a dense vector subspace (affine genericity), whether they contain a dense vector subspace except 0 (algebraic genericity) and whether they contain a closed vector subspace, except 0, of infinite dimension (spaceability). See also the relevant definitions (Definitions 4.22, 4.29 and 4.34 for affine genericity, algebraic genericity and spaceability respectively) for further details. We will now give a brief overview of the entire work per chapter.

In chapter 2 we present a generic result in infinitely (denumerably) many complex variables concerning Hypercyclicity (Theorem 2.4). We consider the following class of functions

$$\mathcal{A} = \{f : \ell_\infty \to \mathbb{C} : \text{ for every } n \in \mathbb{N}, \text{ there exists a sequence of polynomials}$$

$$\text{converging uniformly on } B_n \text{ to the restriction } f_{|B_n}\}.$$

where $B_n = \overline{B(0,n)}^{\mathbb{N}}$ for every $n \geq 1$ and $\ell_\infty \equiv \ell_\infty(\mathbb{C})$ is the set of all bounded complex sequences. The class $\mathcal{A}$ is endowed with the seminorms

$$\rho_n(f) = ||f||_{B_n} = \sup\{|f(z)| : z \in B_n\}$$

where $n \geq 1$. In this way, if we set

$$\rho(f,g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f-g)}{1+\rho_n(f-g)} \text{ for every } f,g \in \mathcal{A}$$

then, it is known that $(\mathcal{A}, \rho)$ becomes a Fréchet space. Our main result in this chapter (Theorem 2.4) states that for for ever element $a \in \ell_\infty \setminus \{0\}$, the corresponding translation operator $T_a : \mathcal{A} \to \mathcal{A}$ with $T_a(f) = f(z+a)$ is hypercyclic. Moreover, this result is proved to be generic in the space $(\mathcal{A}, \rho)$.

At this point, it is important to highlight the following difference: although the previous result is an extension of Birkhoff's theorem in infinitely (denumerable) many variables, we are working on a subspace of holomorphic functions which comes from a different point of view of infinite dimensional holomorphy ([26]). However, if we work in the whole space of entire functions in infinitely many variables, this result fails ([18]). For other extensions of Birkhoff's theorem in $\mathbb{C}^N$ $(N \geq 1)$ we refer to [10].

In chapter 3 we deal with complex functions which are continuous and nowhere differentiable. Initially, the main idea was to complexify the Weierstrass function, a well known explicit example of a 2π - periodic function $u_0 : \mathbb{R} \to \mathbb{R}$ which is continuous and nowhere differentiable, and to obtain a relevant result (Theorem 3.2) concerning functions of the disc algebra $A(\mathbb{D})$, where both their real and imaginary part satisfy the main properties as the Weierstrass function u_0 on the boundary $\mathbb{T}$ of the unit circle $\mathbb{D}$. Moreover, this result (Theorem 3.2) is generic in the disc algebra $A(\mathbb{D})$. We remind that the class $A(\mathbb{D})$ consists precisely of all functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ which are continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$ (where, clearly, $\mathbb{D}$ is the open unit disc centered at 0). This class of functions endowed with the topology induced by the supremum norm $||\cdot||_\infty$ is a complete metric space.

A similar idea was also developed in infinitely (denumerable) many complex variables in [17], but in the present work we do not insist towards this direction.

However, in Theorem 3.2 the notion of a nowhere differentiable function is with respect to the parameter, since the set $\mathbb{T} = \partial\mathbb{D}$ is a Jordan curve.

This work was later extended to more general Jordan domains in [28], presenting many interesting results in one and in several complex variable. Once more, the notion of nowhere differentiability in [28] was also considered with respect to the parametrization of the boundary.

A relevant approach was later developed (in one variable) in [23] in simply connected domains, but this time, the notion of nowhere differentiability was considered with respect to the position, since a parametrization of the boundary was no longer required. Most of the work in [23] is also presented in the second half og this chapter, where the main relevant results (Theorems 3.6 and 3.9) express a dichotomy principle: the classes of functions studied are either void or G_δ - dense in suitable metric spaces. The chapter ends with a few examples concerning the previous dichotomy results.

In chapter 4 we present some generic results concerning Padé approximants of several types (see Definitions 4.6 and 4.7). The Padé approximants are rational functions that satisfy specific properties (see Definition 4.1 for further details) and we used them in order to obtain generic results of simultaneous approximation with the same indices. However, the approximation is not necessarily meant only with the usual Euclidean distance $|\cdot|$ in $\mathbb{C}$, but also with the chordal metric χ in $\mathbb{C} \cup \{\infty\}$. Such results are presented in the first half of the chapter.

All results concerning Padé approximants are proved to be generic by using Baire's theorem and are derived mainly from [27]. Moreover, they can be altered in order to achieve simultaneous Padé - Taylor approximation with the same indices ([24]). However, many results of approximation presented in [27] are omitted, mainly those of Seleznev type (concerning formal power series) and those referring to the space $X^\infty(\Omega)$ which is a closed subspace of $A^\infty(\Omega)$ (where $\Omega \subseteq \mathbb{C}$ is an open set).

At this point we remind that given an open set $\Omega \subseteq \mathbb{C}$, a holomorphic function $f \in H(\Omega)$ belongs to the $A^\infty(\Omega)$ if and only if for every $\ell \in \{0, 1, 2, \cdots\}$ the ℓ - th derivative $f^{(\ell)}$ of f can be continuously extended to $\overline{\Omega}$. In $A^\infty(\Omega)$ we consider the seminorms

$$\rho_{n,\ell}(f, g) = \sup_{z \in K_n} |f^{(\ell)}(z) - g^{(\ell)}(z)|$$

for every $n, \ell \in \mathbb{N}$, where the family $\{K_n\}_{n\in\mathbb{N}}$ consists of compact subsets of $\overline{\Omega}$ such that for every compact set $L \subseteq \overline{\Omega}$, there exists an index $n_0 \in \mathbb{N}$ such that $L \subseteq K_{n_0}$. For instance, it suffices to set $K_n = \overline{\Omega} \cap \overline{B(0,n)}$ for every $n \in \mathbb{N}^*$. It is known that with these seminorms $A^\infty(\Omega)$ becomes a Fréchet space.

We also set $X^\infty(\Omega)$ to be the closure in $A^\infty(\Omega)$ of all rational functions with poles off $\overline{\Omega}$. Thus, $X^\infty(\Omega)$ is a closed subset of a complete metric space and therefore is a complete metric space itself. We refer to [27] and [37] for the further results and definitions.

In the second half of chapter 4 we present results of different nature concerning algebraic and affine genericity, as well as spaceability of certain classes of Padé approximants. Our results do not make use of Baire's theorem this time and are mostly constructive.

Finally, in chapter 5 we present a result concerning universal Laurent series on domains of infinite connectivity. Our main result (Theorem 5.11) is proved to be generic in a specific space of functions, once again by using Baire's theorem. However, by applying Theorem 5.11 in different cases we obtain significantly different results. The most striking application gives a generic result where universal approximation holds on a part of the boundary of an open set, while on another disjoint part, the universal function is smooth. Most of this chapter is contained in [25].

Each chapter of this work is carefully presented so that the reader could study each chapter independently, apart, of course, from the necessary cross - references between chapters. Naturally, many results were omitted, although we present most of the results in [17], [23], [25] and [27] and we refer to these works (and also in [24]) for further results, examples and information in general.

Last but not least, one could naturally ask for further relevant problems related to this work. We only mention the following.

Problem 1.1. In Theorem 3.2 we mention the class S which is precisely the set of all elements $g \in A(\mathbb{D})$ such that both functions u_g and v_g (the real and the imaginary part of g respectively) satisfy the following property

$$\limsup_{t \to t_0^+} \left| \frac{u_g(e^{it}) - u_g(e^{it_0})}{t - t_0} \right| = \limsup_{t \to t_0^+} \left| \frac{v_g(e^{it}) - v_g(e^{it_0})}{t - t_0^+} \right| = +\infty$$

for every $t_0 \in [0, 2\pi]$. Is the class S affinely and / or algebraically generic? Is it spaceable?

Problem 1.2. Consider the set of frequently universal martingales on trees in the sense of [1]. Is this class algebraically generic and / or spaceable?

2 Hypercyclicity of the translation operator in infinitely many variables

2.1 A few things about Hypercyclicity

Birkhoff in [6] showed that there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that its translates $f_a^{[n]}(z) = f(z + na)$, $n \in \mathbb{N}$ are dense in the set of entire functions $H(\mathbb{C})$ endowed with the topology of uniform convergence on compacta, for $a = 1$. That is, the translation operator $T_1 : H(\mathbb{C}) \to H(\mathbb{C})$, $T_1(f) = f_1^{[1]}$ is hypercyclic. We remind the following definition.

Definition 2.1 (Hypercyclic operators and Hypercyclic vectors)**.** Let X be a topological vector space and $T : X \to X$ a continuous linear operator. The operator T is called hypercyclic if there exists a vector $x \in X$ such that the set $\{T^{(n)}(x) : n \geq 1\}$ is dense in X. Obviously

$$T^{(n)}(x) = \underbrace{T \circ T \cdots \circ T(x)}_{n \text{ times}} \tag{2.1}$$

In addition, if such a vector exists, then it is called hypercyclic vector for the operator T. It is also known ([21]) that the set of hypercyclic vectors of a hypercyclic operator is a G_δ - dense set, provided that it is not void.

Thus, the set of all functions $f \in H(\mathbb{C})$ such that the set $\{f_1^{[n]} : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$ is G_δ - dense subset of $H(\mathbb{C})$. The previous results remain valid if 1 is replaced by any $a \in \mathbb{C} \setminus \{0\}$.

Baire's theorem implies that if $S \subseteq \mathbb{C} \setminus \{0\}$ is denumerable, then there exists a common hypercyclic function (vector) $f \in H(\mathbb{C})$ for all operators T_a, $a \in S$ and that their set is G_δ - dense set in $H(\mathbb{C})$.

Costakis and Sambarino in [10] proved that the set of common hypercyclic functions $f \in H(\mathbb{C})$ for all operators T_a, $a \in \mathbb{C} \setminus \{0\}$ is still residual in $H(\mathbb{C})$.

Extensions of Birkhoff's result are known in $\mathbb{C}^N$, $N \in \mathbb{N}$. In infinite many variables similar extensions fail ([19]). However, considering holomorphic functions on $\ell_\infty(\mathbb{N})$ compatible with the notion of a holomorphic function in [26], it turns out that Birkhoff's result can be extended in this case, provided we consider a smaller subspace of entire functions. A similar result has been obtained earlier in [2] by using different methods. It would be interesting to examine if the result of Costakis and Sambarino in [10] extends as well or not.

2.2 Notations and preliminaries

We start with a few basic notations and preliminaries which will be used in order to prove our main result concerning Hypercyclicity (Theorem (2.4)).

For every $n \in \mathbb{N}$, we set $B_n = \overline{B(0,n)}^{\mathbb{N}}$; that is a closed polydisc in the space $\mathbb{C}^{\mathbb{N}}$ of infinitely many complex variables. We also consider the set of all bounded complex sequences, i.e. the set

$$\ell_\infty(\mathbb{C}) \equiv \ell_\infty := \{z \in \mathbb{C}^{\mathbb{N}} : \exists\, n \in \mathbb{N} \text{ such that } z \in B_n\} = \bigcup_{n \geq 0} B_n. \tag{2.2}$$

Moreover, ℓ_∞ is endowed with the relevant (cartesian) topology, as a subspace of $\mathbb{C}^{\mathbb{N}}$. Next, we consider the following set

$$\mathcal{A} = \{f : \ell_\infty \to \mathbb{C} : \text{ for every } n \in \mathbb{N}, \text{ there exists a sequence of polynomials}$$
$$\text{converging uniformly on } B_n \text{ to the restriction } f_{|B_n}\}. \tag{2.3}$$

At this point we recall that every polynomial depends only on a finite number of variables. It is also known ([26]) that for every $f : \ell_\infty \to \mathbb{C}$ and for every $n \in \mathbb{N}$, $(i) \iff (ii)$ & (iii), where

(i) There exists a sequence of polynomials converging uniformly on B_n to the restriction $f_{|B_n}$.

(ii) The restriction $f_{|B_n}$ is a continuous function with respect to the topology of pointwise convergence.

(iii) The restriction $f_{|B_n}$ belongs separately, as a function of a single variable, to the disc algebra $A(B(0,n))$.

By $A(B(0,n))$ we denote the algebra of the disc $\overline{B(0,n)}$, i.e. the set of all functions $h : \overline{B(0,n)} \to \mathbb{C}$ which are continuous on $\overline{B(0,n)}$ and holomorphic in $B(0,n)$.

Remark 2.2. Obviously, every function $f : \ell_\infty \to \mathbb{C}$ which is separately holomorphic and continuous on ℓ_∞ with respect to the topology of pointwise convergence belongs to $\mathcal{A}$. The converse does not hold. Indeed, consider the following example.

Let $f : \ell_\infty \to \mathbb{C}$ with

$$f(z_1, z_2, \cdots) = \sum_{j=1}^{+\infty} \frac{z_j}{j^2} \tag{2.4}$$

for every $z \equiv (z_1, z_2, \cdots) \in \ell_\infty$. In order to prove that $f \in \mathcal{A}$, we consider the polynomials

$$p_N(z) = \sum_{j=1}^{N} \frac{z_j}{j^2} \tag{2.5}$$

for $N \geq 0$. It is easy to verify that the sequence $\{P_N\}_{N\geq 0}$ converges uniformly on $B_n = \overline{B(0,n)}^{\mathbb{N}}$ to f for every $n \in \mathbb{N}$. It follows that $f \in \mathcal{A}$.

We will show that f is not continuous on $0 \in \ell_\infty$, where ℓ_∞ is consider endowed with the relevant (cartesian) topology. Suppose that f is continuous on 0. Then, for $\varepsilon = 1$, there exists an index $N \in \mathbb{N}$ and $\delta > 0$ such that if $|z_j| < \delta$ for $j = 1, \cdots, N$ and $z_j \in \mathbb{C}$ for $j \geq N+1$ it holds

$$|f((z_j)_{j\geq 1})| < \varepsilon = 1 \Leftrightarrow \Big| \sum_{j=1}^{N} \frac{z_j}{j^2} + \sum_{j=N+1}^{+\infty} \frac{z_j}{j^2} \Big| < 1. \tag{2.6}$$

We set $z_j = 0$ for $j = 1, \cdots, N$ and $z_j = c \in \mathbb{C}$ for $j \geq N+1$. Then

$$|c(\sum_{j=N+1}^{+\infty} \frac{1}{j^2})| < 1. \tag{2.7}$$

If

$$c_1 = \sum_{j=N+1}^{+\infty} \frac{1}{j^2}, \tag{2.8}$$

then Relation (2.7) is equivalent to $|cc_1| < 1$ for every $c \in \mathbb{C}$, which is clearly false for $c \to \infty$. Thus, f is not continuous on $0 \in \ell_\infty$.

Since f is a linear function, it follows that it is not continuous on any $a \in \ell_\infty$.

Now, for every $n \in \mathbb{N}$ we consider the seminorms $\rho_n : \mathcal{A} \to [0, +\infty)$, defined for every $f \in \mathcal{A}$ as follows

$$\rho_n(f) = ||f||_{B_n} = \sup\{|f(z)| : z \in B_n\}. \tag{2.9}$$

We know that if we set

$$\rho(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \frac{\rho_n(f-g)}{1+\rho_n(f-g)} \text{ for every } f, g \in \mathcal{A} \tag{2.10}$$

then ρ is a metric on $\mathcal{A}$. In this way, $(\mathcal{A}, \rho)$ becomes a Fréchet space. Indeed, if $\{f_n\}_{n\geq 0} \in \mathcal{A}$ is a ρ - basic sequence of functions, then it is easy to see that the sequence $\{f_n\}_{n\geq 0}$ is also ρ_k - basic and that this holds for every $k \in \mathbb{N}$. Thus, for every $\varepsilon > 0$, there exists an index $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$ it holds $||f_n - f_m||_{B_k} < \varepsilon$. It follows that for every $z \in B_k$ and for every $n, m \geq n_0$ it holds

$$|f_n(z) - f_m(z)| \leq ||f_n - f_m||_{B_k} < \varepsilon. \tag{2.11}$$

It is now clear that for every $z \in B_k$ the sequence $\{f_n(z)\}_{n\geq 0}$ is basic in $\mathbb{C}$ and since $\mathbb{C}$ is a complete metric space, it converges. We set

$$f(z) = \lim_{n\to+\infty} f_n(z) \tag{2.12}$$

for every $z \in B_k$. In this way we have defined a function $f : \ell_\infty \to \mathbb{C}$ which is the pointwise limit of the sequence $\{f_n\}_{n\geq 0}$. By taking limits in Relation (2.11) for $m \to +\infty$, we obtain that $|f_n(z) - f(z)| \leq \varepsilon$ for every $z \in B_k$, which yields the relation $||f_n - f||_{B_k} \leq \varepsilon$ for every $n \geq n_0$. Since $f_n \in \mathcal{A}$, it follows that there exists a polynomial P_n such that $\rho_n(f_n, P_n) < \frac{1}{n}$ for all n. It follows easily that $P_n \to f$ uniformly on each B_k. Hence $f \in \mathcal{A}$. Therefore, $\rho(f_n, f) \to 0$ and thus $(\mathcal{A}, \rho)$ is a complete metric space.

Let $a \in \ell_\infty \setminus \{0\}$ and $f : \ell_\infty \to \mathbb{C}$ be a function. For every $n \in \mathbb{N}$ we use the following notation

$$f_a^{[n]}(z) = f(z + na) \text{ for every } z \in \ell_\infty. \tag{2.13}$$

In the particular case where $a = (1, 1, \cdots) \in \ell_\infty$, we use the notation $f^{[n]}(z)$ instead.

Finally, we will use the following well known result.

Proposition 2.3. (See Lemmas 1.1 and 1.2 of [13]) Let K_1 and K_2 two disjoint, compact and convex subsets of $\mathbb{C}^N$ $(N \in \mathbb{N})$. Then the set $K_1 \cup K_2$ is polynomially convex. It follows that if p_1 and p_2 are two polynomials of N complex variables and $\varepsilon > 0$, then there exists a polynomial p of N complex variables such that $||p - p_1||_{K_1} < \varepsilon$ and $||p - p_2||_{K_2} < \varepsilon$.

2.3 A generic result about Hypercyclicity

In this section we prove that for every $a \in \ell_\infty \setminus \{0\}$ the corresponding translation operator $T_a : \mathcal{A} \to \mathcal{A}$ is hypercyclic.

Theorem 2.4. There exists a function $f \in \mathcal{A}$ such that the translations of f, i.e. the set $\{f^{[n]} : n \geq 0\}$ is dense in $(\mathcal{A}, \rho)$. Moreover, the set of all such functions f is G_δ - dense in $(\mathcal{A}, \rho)$.

Proof. We consider an enumeration $\{f_j\}_{j\geq 0}$ of polynomials with coefficients in $\mathbb{Q}+i\mathbb{Q}$. Of course, every polynomial is a function that depends only on a finite number of variables. One can verify that the set $\{f_j : j \geq 0\}$ is dense in $(\mathcal{A}, \rho)$.

Next, for every $m, n, j, s \in \mathbb{N}$ we consider the following sets

$$\mathcal{B}_{m,n,j,s} = \Big\{ f \in \mathcal{A} : ||f^{[n]} - f_j||_{B_m} < \frac{1}{s} \Big\}. \tag{2.14}$$

If $\mathcal{B}$ is the set of all functions satisfying Theorem 2.4, one can verify that the following holds

$$\mathcal{B} = \bigcap_{m,j,s} \bigcup_{n\geq 0} \mathcal{B}_{m,n,j,s}. \tag{2.15}$$

In order to use Baire's theorem and prove that $\mathcal{B}$ is non void (in fact, a G_δ - dense set), it suffices to prove the following.

Claim 2.5. For every $m, n, j \in \mathbb{N}$ and $s \geq 1$ the sets $\mathcal{B}_{m,n,j,s}$ are open in $\mathcal{A}$.

Proof of Claim 2.5 Let $m, n, j, s \in \mathbb{C}$ and $f \in \mathcal{B}_{m,n,j,s}$ be fixed. We need to find an $\varepsilon > 0$ such that for every $g \in \mathcal{A}$ with $\rho(f, g) < \varepsilon$ it follows that $g \in \mathcal{B}_{m,n,j,s}$, or, equivalently

$$||g^{[n]} - f_j||_{B_m} < \frac{1}{s} \tag{2.16}$$

The triangle inequality implies that

$$||g^{[n]} - f_j||_{B_m} \leq ||g^{[n]} - f^{[n]}||_{B_m} + ||f^{[n]} - f_j||_{B_m}. \tag{2.17}$$

Thus, it suffices to prove the following

$$||g^{[n]} - f^{[n]}||_{B_m} < \frac{1}{s} - ||f^{[n]} - f_j||_{B_m}. \tag{2.18}$$

We know that it holds

$$||g^{[n]} - f^{[n]}||_{B_m} = \sup_{z\in B_m} |g(z+n) - f(z+n)| \leq ||f - g||_{B_{m+n}}. \tag{2.19}$$

The quantity $||f - g||_{B_{m+n}}$ can become arbitrary small, provided that $\rho(f, g) < \varepsilon$. Thus, $\mathcal{B}_{m,n,j,s}$ is open in $\mathcal{A}$.

■ ■

Claim 2.6. For every $m, j \in \mathbb{N}$ and $s \geq 1$ the set

$$\bigcup_{n\geq 0} \mathcal{B}_{m,n,j,s} \tag{2.20}$$

is dense in $\mathcal{A}$.

Proof of Claim 2.6 We fix the parameters $m, j \in \mathbb{N}$ and $s \geq 1$ and we want to prove that the set in Relation (2.20) is dense in $\mathcal{A}$.

Let $g \in \mathcal{A}$, $\varepsilon > 0$ and $M \in \mathbb{N}$. We want to find an index $n_0 \in \mathbb{N}$ and a function $f \in \mathcal{B}_{m,n_0,j,s}$ such that $||f-g||_{B_M} < \varepsilon$. There is no problem if we assume that it holds $\varepsilon < \frac{1}{s}$. The function f must satisfy the following properties.

(i) $||f-g||_{B_M} < \varepsilon$.

(ii) $||f^{[n_0]} - f_j||_{B_m} < \dfrac{1}{s}$ or, equivalently

$$\sup_{z \in \overline{B(n_0,m)}^{\mathbb{N}}} |f(z) - f_j(z - n_0 \cdot a)| < \frac{1}{s}. \tag{2.21}$$

Since $g \in \mathcal{A}$, there exists a polynomial p_1 such that $||p_1 - g||_{B_M} < \frac{\varepsilon}{2}$. Suppose that the polynomial p_1 depends on n_1 variables, while the polynomial f_j depends on n_2 variables. Then we can consider both polynomials as functions from $\mathbb{C}^{n_1+n_2}$ to $\mathbb{C}$. This allows us to use Proposition 2.3, since the closed polydiscs $\overline{B(0,M)}^{n_1+n_2}$ and $\overline{B(n_0,m)}^{n_1+n_2}$ in $\mathbb{C}^{n_1+n_2}$ are compact, convex and disjoint sets, provided that n_0 is large enough. We set $B = \overline{B(0,M)}^{n_1+n_2} \cup \overline{B(n_0,m)}^{n_1+n_2}$. We consider the function $h : B \to \mathbb{C}$ defined as follows

$$h(z) = \begin{cases} p_1(z), \text{ for } z \in \overline{B(0,M)}^{n_1+n_2} \\ f_j(z - n_0 \cdot a), \text{ for } z \in \overline{B(n_0,M)}^{n_1+n_2}. \end{cases} \tag{2.22}$$

Since B is polynomially convex, according to Proposition 2.3 there exists a polynomial p depending at most $n_1 + n_2$ variables such that $||p - h||_B < \frac{\varepsilon}{2}$. It follows that

$$||p - p_1||_{\overline{B(0,M)}^{n_1+n_2}} < \frac{\varepsilon}{2} \tag{2.23}$$

and also

$$||p - f_j||_{\overline{B(n_0,m)}^{n_1+n_2}} < \frac{\varepsilon}{2}. \tag{2.24}$$

We set $f = p$ and we are done by the triangle inequality.

■■

We apply Baire's theorem and that completes the proof.

■

Remark 2.7. The previous proof is valid for $a = (1, 1, \cdots) \in \ell_\infty$. More generally, if $a \equiv (a_j)_{j \geq 0} \in \ell_\infty \setminus \{0\}$, then the same proof works. It suffices to consider n_0 large enough such that for some fixed j with $a_j \neq 0$ we have $\overline{B(n_0 a_j, m)} \cap \overline{B(0,M)} = \emptyset$. For instance, it suffices to choose

$$n_0 > \frac{M+m}{a_j}. \tag{2.25}$$

Thus, we obtain the following theorem.

Theorem 2.8. For every $a \in \ell_\infty \setminus \{0\}$ there exists a function $f \in \mathcal{A}$ such that the set $\{f_a^{[n]} : n \geq 0\}$ is dense in $(\mathcal{A}, \rho)$. Moreover, the set of all such functions is G_δ - dense in $(\mathcal{A}, \rho)$.

Question 2.9. Is it possible to find a function $f \in \mathcal{A}$ such that the set $\{f_a^{[n]} : n \geq 0\}$ is dense in $(\mathcal{A}, \rho)$ simultaneously for all $a \in \mathbb{C}^{\mathbb{N}} \setminus \{0\}$? If this is true, is it possible that the set of all such functions is residual in $(\mathcal{A}, \rho)$? This question relates to [10].

3 Nowhere differentiable functions

3.1 Nowhere differentiable functions in the disc algebra

In 1872, Weierstrass gave an explicit example of a function $u_0 : \mathbb{R} \to \mathbb{R}$ which is continuous but nowhere differentiable. Although this is considered to be the first such known result, as E. Maestre informed us, Bolzano has obtained a similar result before 1831 but it was not published until 1930.

Theorem 3.1 (Weierstrass' explicit example). Let $0 < \alpha < 1$ and b be an odd integer satisfying $ab > 1 + \frac{3\pi}{2}$. Then, the function $u_0 : \mathbb{R} \to \mathbb{R}$ defined as

$$u_0(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n x) \tag{3.1}$$

for every $x \in \mathbb{R}$ is continuous, 2π - periodic and nowhere differentiable. In fact, something even stronger holds ([31]); for every $x_0 \in \mathbb{R}$ we have that

$$\limsup_{x \to x_0^+} \left| \frac{u_0(x) - u_0(x_0)}{x - x_0} \right| = +\infty. \tag{3.2}$$

In this section we prove that for almost every function f in the disc algebra $A(\mathbb{D})$, both functions $u_f(\theta) = Re\big(f(e^{i\theta})\big)$ and $v_f(\theta) = Im\big(f(e^{i\theta})\big)$ are continuous and nowhere differentiable with respect to the real parameter θ. This result clearly indicates that almost every function $f \in A(\mathbb{D})$ is nowhere differentiable on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In addition, the set of all functions satisfying this property is residual in $A(\mathbb{D})$; that is, it contains a G_δ - dense set.

We remind that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a function f belongs to the disc algebra $A(\mathbb{D})$ if and only if f is continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. The space $A(\mathbb{D})$ is endowed with the topology induced by the supremum norm on $\overline{\mathbb{D}}$

$$||f||_\infty = \sup_{|z| \leq 1} |f(z)|. \tag{3.3}$$

Moreover, it is known that $\big(A(\mathbb{D}), ||\cdot||_\infty\big)$ is a Banach space. We now return to Theorem 3.1.

Let $0 < \alpha < 1$ and b be an odd integer satisfying $ab > 1 + \frac{3\pi}{2}$. We consider the function $f_0 : \overline{\mathbb{D}} \to \mathbb{C}$ defined as

$$f_0(z) = \sum_{n=0}^{+\infty} a^n z^{b^n} \tag{3.4}$$

for every $z \in \overline{\mathbb{D}}$. The function f_0 is well defined and it holds

$$u_0(x) = Re\big(f_0(e^{ix})\big) \tag{3.5}$$

for every $x \in \mathbb{R}$. In addition, if $\widetilde{u_0} : \mathbb{R} \to \mathbb{R}$ is the following function

$$\widetilde{u_0}(x) = \sum_{n=0}^{+\infty} a^n \sin(b^n x) \tag{3.6}$$

for every $x \in \mathbb{R}$, then clearly it holds

$$\widetilde{u_0}(x) = Im\big(f_0(e^{ix})\big) \tag{3.7}$$

for every $x \in \mathbb{R}$. The following relation explains the link between the functions $u_0, \widetilde{u_0}$ and f_0 (restricted to $\mathbb{T}$)

$$f_0(e^{ix}) = Re\big(f_0(e^{ix})\big) + iIm\big(f_0(e^{ix})\big) = u_0(x) + i\widetilde{u_0}(x) \tag{3.8}$$

and that holds for every $x \in \mathbb{R}$. The function f_0 is continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$ because $|a^n z^{b^n}| \leq a^n$ for every $|z| \leq 1$ and the series $\sum_{n=0}^{+\infty} a^n$ is convergent; in other words $f \in A(\mathbb{D})$.

We will also use the following notation: for every function $g \in A(\mathbb{D})$ we consider the functions $u_g, v_g : \mathbb{R} \to \mathbb{R}$ satisfying

$$u_g(x) = Re\big(g(e^{ix})\big) \tag{3.9}$$

and

$$v_g(x) = Im\big(g(e^{ix})\big) \tag{3.10}$$

for every $x \in \mathbb{R}$. In addition, if $h : \mathbb{R} \to \mathbb{R}$, we say that h satisfies Property (3.11) below if it holds

$$\limsup_{x \to x_0^+} \left| \frac{h(x) - h(x_0)}{x - x_0} \right| = +\infty \tag{3.11}$$

for every $x_0 \in \mathbb{R}$. It is known that both functions u_0 and $\widetilde{u_0}$ satisfy Property (3.11) (see [31] for the function u_0; for the function $\widetilde{u_0}$ a similar calculation works, at least if $\lambda \equiv 1 mod 4$).

Theorem 3.2. ([[17], [16]) Let $S \subseteq A(\mathbb{D})$ be the class of all functions $g \in A(\mathbb{D})$ such that both functions u_g and v_g satisfy Property (3.11). Then S is G_δ - dense in $A(\mathbb{D})$.

Proof. We know that $S \neq \emptyset$ since $f_0 \in S$. For every $n \geq 1$ we consider the following sets

$$D_n = \Big\{ h : \mathbb{R} \to \mathbb{R} : \text{ for every } \theta \in \mathbb{R} \text{ there exists a } \theta_0 \in \Big(\theta, \theta + \frac{1}{n}\Big) \\ \text{such that } |h(\theta) - h(\theta_0)| > n|\theta - \theta_0| \Big\}. \tag{3.12}$$

Also, for every $n \geq 1$, let

$$E_n = \{ f \in A(\mathbb{D}) : u_f \ \& \ v_f \in D_n \}. \tag{3.13}$$

One can easily verify that it holds

$$S = \bigcap_{n=1}^{+\infty} E_n. \tag{3.14}$$

In order to use Baire's theorem we have to prove that each E_n is an open and dense set in $A(\mathbb{D})$.

Claim 3.3. For every $n \geq 1$ the sets E_n are open in $A(\mathbb{D})$.

Proof of Claim 3.3 Let $n \geq 1$ be a fixed natural number. In order to prove that the set E_n is open in $A(\mathbb{D})$ we will prove equivalently that the set $A(\mathbb{D}) \setminus E_n$ is closed in $A(\mathbb{D})$.

We consider a sequence of functions $\{g_m\}_{m\geq 1} \subseteq A(\mathbb{D}) \setminus E_n$ and let $g \in A(\mathbb{D})$ such that $g_m \to g$ in $A(\mathbb{D})$. Now, since $g_m \notin E_n$, for every $m \geq 1$ there exist $\theta_m \in \mathbb{R}$ such that

$$|u_{g_m}(\theta) - u_{g_m}(\theta_m)| \leq n|\theta - \theta_m| \tag{3.15}$$

for every $\theta \in \left(\theta_m, \theta_m + \frac{1}{n}\right)$ or there exist $\theta'_m \in \mathbb{R}$ such that

$$|v_{g_m}(\theta) - v_{g_m}(\theta'_m)| \leq n|\theta - \theta'_m| \tag{3.16}$$

for every $\theta \in \left(\theta'_m, \theta'_m + \frac{1}{n}\right)$.

According to the definition of the functions u_{g_m} and v_{g_m}, since both of them are 2π - periodic, we may assume that the sequences $\{\theta_m\}_{m\geq 1}$ and $\{\theta'_m\}_{m\geq 1}$ are bounded ones; thus, without loss of generality, we may assume that the former converges to a single $\theta \in \mathbb{R}$ and the latter converges to a single $\theta' \in \mathbb{R}$.

Suppose that we are in the first case and Relation (3.15) holds for infinitely many $m \geq 1$. Let $x \in \left(\theta, \theta + \frac{1}{n}\right)$. Then, it is easy to see that there exists an index $m_0 \geq 1$ such that $x \in \left(\theta_m, \theta_m + \frac{1}{n}\right)$ for every $m \geq m_0$. The triangle inequality implies that

$$\begin{aligned} |u_g(x) - u_g(\theta_m)| &\leq |u_g(x) - u_{g_m}(x)| + |u_{g_m}(x) - u_{g_m}(\theta_m)| \\ &\qquad + |u_{g_m}(\theta_m) - u_g(\theta_m)| \\ &\leq 2||u_{g_m} - u_g||_\infty + n|x - \theta_m| \end{aligned} \tag{3.17}$$

By taking limits as $m \to +\infty$ in Relation (3.17) we obtain that it holds

$$|u_g(x) - u_g(\theta)| \leq n|x - \theta| \tag{3.18}$$

for every $x \in \left(\theta, \theta + \frac{1}{n}\right)$. It follows that $u_g \notin D_n$ and thus, in this case $g \in A(\mathbb{D}) \setminus E_n$.

We consider now the second case, where Relation (3.16) holds for infinitely many $m \geq 1$. Let $y \in \left(\theta', \theta' + \frac{1}{n}\right)$. Then, it is easy to see that there exists an index $m_1 \geq 1$ such that $y \in \left(\theta'_m, \theta'_m + \frac{1}{n}\right)$ for every $m \geq m_1$. The triangle inequality implies that

$$\begin{aligned} |v_g(y) - v_g(\theta'_m)| &\leq |v_g(y) - v_{g_m}(y)| + |v_{g_m}(y) - v_{g_m}(\theta'_m)| \\ &\qquad + |v_{g_m}(\theta'_m) - v_g(\theta'_m)| \\ &\leq 2||v_{g_m} - v_g||_\infty + n|y - \theta'_m| \end{aligned} \tag{3.19}$$

By taking limits as $m \to +\infty$ in Relation (3.19) we obtain that it holds

$$|u_g(y) - u_g(\theta')| \leq n|y - \theta'| \tag{3.20}$$

for every $y \in \left(\theta', \theta' + \frac{1}{n}\right)$. It follows that $v_g \notin D_n$ and thus, in this case $g \in A(\mathbb{D}) \setminus E_n$.

In any case, the set $A(\mathbb{D}) \setminus E_n$ is closed in $A(\mathbb{D})$, or equivalently, the set E_n is open in $A(\mathbb{D})$. This part of the proof is complete.

■ ■

Claim 3.4. For every $n \geq 1$ the sets E_n are dense in $A(\mathbb{D})$.

Proof of Claim 3.4 Let $n \geq 1$ be a fixed number and f_0 be the function defined in Relation (3.4). Our aim is to prove that if p is a polynomial (restricted to $\overline{\mathbb{D}}$) then it holds $f_0 + p \in E_n$.

Indeed, let p be a polynomial (restricted to $\overline{\mathbb{D}}$). Since p' is bounded in $\overline{\mathbb{D}}$, from the Mean Value theorem, there exists a $M > 0$ such that

$$|u_p(y) - u_p(\theta)| \leq M|y - \theta| \tag{3.21}$$

and also

$$|v_p(y) - v_p(\theta)| \leq M|y - \theta| \tag{3.22}$$

for every $y, \theta \in \mathbb{R}$ with $y \neq \theta$. Let $\theta \in \mathbb{R}$. Since $f_0 \in E_s$ for every $s \geq 1$, there exist $x, y \in (\theta, \theta + \frac{1}{s}) \subseteq (\theta, \theta + \frac{1}{n})$ such that

$$|u_{f_0}(x) - u_{f_0}(\theta)| > s|x - \theta| \tag{3.23}$$

and

$$|v_{f_0}(y) - v_{f_0}(\theta)| > s|y - \theta|. \tag{3.24}$$

Let $k > M + n$. By using the triangle inequality, we obtain

$$\begin{aligned}
|u_{f_0+p}(x) - u_{f_0+p}(\theta)| &= |\big(u_{f_0}(x) - u_{f_0}(\theta)\big) + \big(u_p(x) - u_p(\theta)\big)| \\
&\geq |u_{f_0}(x) - u_{f_0}(\theta)| - |u_p(x) - u_p(\theta)| \\
&> k|x - \theta| - M|x - \theta| \\
&= (k - M)|x - \theta| \\
&> n|x - \theta|
\end{aligned} \tag{3.25}$$

In the same way, we can simultaneously prove that it holds

$$|v_{f_0+p}(y) - v_{f_0+p}(\theta)| > n|y - \theta|. \tag{3.26}$$

Therefore, according to Relations (3.25) and (3.26) we conclude that

$$u_{f_0+p}, v_{f_0+p} \in D_n \tag{3.27}$$

or equivalently, $f_0 + p \in E_n$ and this part of the proof is complete.

■■

Since $A(\mathbb{D})$ is a complete metric space and the set of all polynomials (restricted to $\mathbb{D}$) is dense in $A(\mathbb{D})$, we apply Baire's theorem and that completes the proof.

■

Remark 3.5. The previous result (Theorem3.2) implies that the class $S_0 \subseteq A(\mathbb{D})$ containing all $f \in A(\mathbb{D})$ such that both functions u_f and v_f are nowhere differentiable is residual in $A(\mathbb{D})$, since it holds $S \subseteq S_0$.

3.2 Nowhere differentiable functions with respect to the position

In this section we present a few results concerning nowhere differentiable functions with respect to the position. The main idea is the following: we consider a domain $\Omega \subseteq \mathbb{C}$ (bounded, or even unbounded) a compact set $J \subseteq \partial\Omega$ with no isolated points and we study particular classes of functions f that satisfy the following property

$$\limsup_{\substack{z\to z_0\\ z\in J\setminus\{z_0\}}} \Big|\frac{f(z)-f(z_0)}{z-z_0}\Big| = +\infty \tag{3.28}$$

for every $z_0 \in J$. These classes are proved to be either generic in suitable (complete) metric spaces or simply void. Apparently, the respective classes of functions that are nowhere differentiable on J are residual, since they contain a G_δ - dense sets. We also mention that in every case no parametrization of the boundary is required.

At the end of this section we also give a few examples relevant to the previous results. In some of them (Examples 3.12 and 3.13) the respective classes are G_δ - dense, while in Examples 3.14 and 3.15 the respective classes are void.

3.2.1 The case of bounded domains

Let $K \subseteq \mathbb{C}$ be a compact set. We denote with $R(K)$ is the set of uniform limits on K of rational functions with poles off K. Naturally, the space $R(K)$ is endowed with the topology induced by the supremum norm on K

$$||f||_\infty = \sup_{z\in K}|f(z)|. \tag{3.29}$$

It is known that the space $\big(R(K), ||\cdot||_\infty\big)$ is a Banach space. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain and $J \subseteq \partial\Omega$ be a compact set without isolated points. We denote with $S(\Omega, J)$ the following class of functions

$$S(\Omega, J) = \{f \in R(\overline{\Omega}) : \limsup_{\substack{z\to z_0\\ z\in J\setminus\{z_0\}}} \Big|\frac{f(z)-f(z_0)}{z-z_0}\Big| = +\infty \text{ for every } z_0 \in J\}. \tag{3.30}$$

Theorem 3.6. ([23]) Under the above assumptions and notations, the class $S(\Omega, J)$ is either void or G_δ - dense in $R(\overline{\Omega})$.

Proof. We suppose that it holds $S(\Omega, J) \neq \emptyset$ and let $f \in S(\Omega, J)$. We denote with E_n the following sets

$$E_n = \Big\{g \in R(\overline{\Omega}) : \text{ for every } z_0 \in J \text{ there exists a } z \in (J\setminus\{z_0\}) \cap D\Big(z_0, \frac{1}{n}\Big)$$
$$\text{such that } \Big|\frac{f(z)-f(z_0)}{z-z_0}\Big| > n\Big\}. \tag{3.31}$$

The reader can verify that it holds

$$S(\Omega, J) = \bigcap_{n=1}^{+\infty} E_n. \tag{3.32}$$

In order to use Baire's theorem we have to prove that each E_n is an open and dense set in $R(\overline{\Omega})$.

Claim 3.7. For every $n \geq 1$ the sets E_n are open in $R(\overline{\Omega})$.

Proof of Claim 3.7 Let $n \geq 1$ be a fixed natural number. In order to prove that the set E_n is open in $R(\overline{\Omega})$ we will prove equivalently that the set $R(\overline{\Omega}) \setminus E_n$ is closed in $R(\overline{\Omega})$.

Indeed, let $\{g_m\}_{m\geq 1} \subseteq R(\overline{\Omega}) \setminus E_n$ and $g \in R(\overline{\Omega})$ such that $g_m \to g$ in $R(\overline{\Omega})$. Then, for every $m \geq 1$, there exists a $z_m \in J$ satisfying

$$\Big|\frac{g_m(z) - g_m(z_m)}{z - z_m}\Big| \leq n \tag{3.33}$$

for every $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$. Since J is a compact set, there exists a subsequence of $\{z_m\}_{m\geq 1}$ which converges to a single point $z_0 \in J$. Without loss of generality, we may assume that $\{z_m\}_{m\geq 1}$ converges to z_0. Let $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$ be a fixed point. Then, there exists an index $m_0 \geq 1$ satisfying $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$ for every $m \geq m_0$. Consequently, for every $m \geq m_0$, the triangle inequality implies that

$$\begin{aligned}
|g(z) - g(z_0)| &\leq |g(z) - g_m(z)| + |g_m(z) - g_m(z_m)| + |g_m(z_m) - g(z_0)| \\
&\leq |g(z) - g_m(z)| + n|z_m - z| + |g_m(z_m) - g(z_0)| \\
&\leq |g(z) - g_m(z)| + |g_m(z_m) - g(z_m)| + |g(z_m) - g(z_0)| + n|z_m - z| \\
&\leq 2||g_m - g||_\infty + |g(z_m) - g(z_0)| + n|z_m - z|.
\end{aligned} \tag{3.34}$$

Therefore, by taking limits as $m \to +\infty$ in Relation (3.34) we obtain that it holds

$$\Big|\frac{g(z) - g(z_0)}{z - z_0}\Big| \leq n \tag{3.35}$$

for every $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$. Relation (3.35) implies that $g \in R(\overline{\Omega}) \setminus E_n$ and as a result, E_n is a closed set. This part of the proof is complete.

■■

Claim 3.8. For every $n \geq 1$ the sets E_n are dense in $R(\overline{\Omega})$.

Proof of Claim 3.8 Let $n \geq 1$ be a fixed natural number, $g \in R(\overline{\Omega})$ and $\varepsilon > 0$. According to the definition of the class $R(\overline{\Omega})$, since $g - f \in R(\overline{\Omega})$, there exists a rational function $q \equiv q_\varepsilon$ with poles off $\overline{\Omega}$ such that

$$||(g - f) - q||_\infty < \varepsilon. \tag{3.36}$$

Since q' is continuous on $\overline{\Omega}$, there exists a $M > 0$ satisfying $||q'||_\infty \leq M$. Let $z_0 \in J$ be a fixed point and a sequence $\{z_m\}_{m\geq 1}$ in $J \setminus \{z_0\}$ such that $z_m \to z_0$ satisfying

$$\lim_{m\to+\infty} \Big|\frac{f(z_m) - f(z_0)}{z_m - z_0}\Big| = +\infty. \tag{3.37}$$

The triangle inequality implies that

$$\begin{aligned}
\Big|\frac{(f+q)(z_m) - (f+q)(z_0)}{z_m - z_0}\Big| &= \Big|\frac{f(z_m) - f(z_0)}{z_m - z_0} + \frac{q_m(z_m) - q_m(z_0)}{z_m - z_0}\Big| \\
&\geq \Big|\frac{f(z_m) - f(z_0)}{z_m - z_0}\Big| - \Big|\frac{q(z_m) - q(z_0)}{z_m - z_0}\Big|.
\end{aligned} \tag{3.38}$$

At the same time, it also holds

$$\lim_{m\to+\infty}\left|\frac{q(z_m)-q(z_0)}{z_m-z_0}\right| = |q'(z_0)| \leq M. \tag{3.39}$$

Therefore, by combining Relations (3.38) and (3.39), we obtain that it holds

$$\left|\frac{(f+q)(z_m)-(f+q)(z_0)}{z_m}\right| > n \tag{3.40}$$

for m large enough. Consequently, we deduce that $(f+q) \in E_n$. Since $||g-(f+q)||_\infty < \varepsilon$ and $\varepsilon > 0$ is arbitrary, it follows that $g \in E_n$. Thus, the set E_n is dense in $R(\overline{\Omega})$ and this part of the proof is complete.

■■

The result follows now from Baire's theorem.

■

3.2.2 The case of unbounded domains

Let $E \subseteq \mathbb{C}$ be an unbounded open set. We denote with $\widetilde{R}(E)$ the set of all functions which are uniform limits on each compact subset of $\overline{E}$ of rational functions with poles off $\overline{E}$. The natural topology of $\widetilde{R}(E)$ is the topology of uniform convergence on each compact subset of $\overline{E}$. Equivalently, the topology of $\widetilde{R}(E)$ can be defined by the sequence of seminorms

$$\rho_k(f,g) = \sup\left\{|f(z)-g(z)| : z \in \overline{E} \cap \overline{B(0,k)}\right\} \tag{3.41}$$

for every $f, g \in \widetilde{R}(E)$ and for every $k \geq 1$. Moreover, the space $\widetilde{R}(E)$ endowed with these seminorms is a Fréchet space.

Theorem 3.9. ([23]) Let $\Omega \subseteq \mathbb{C}$ be an unbounded domain and $J \subseteq \partial\Omega$ be a compact set without isolated points. Then, the class of functions

$$S(\Omega, J) = \{f \in \widetilde{R}(\Omega) : \limsup_{\substack{z\to z_0 \\ z\in J\setminus\{z_0\}}} \left|\frac{f(z)-f(z_0)}{z-z_0}\right| = +\infty \text{ for every } z_0 \in J\} \tag{3.42}$$

is either void or G_δ - dense in $\widetilde{R}(\Omega)$.

Proof. We suppose that it holds $S(\Omega, J) \neq \emptyset$ and let $f \in S(\Omega, J)$. We denote with E_n the following sets

$$E_n = \Big\{g \in \widetilde{R}(\Omega) : \text{ for every } z_0 \in J \text{ there exists a } z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$$
$$\text{such that } \left|\frac{g(z)-g(z_0)}{z-z_0}\right| > n\Big\}. \tag{3.43}$$

The reader could verify that it holds

$$S(\Omega, J) = \bigcap_{n=1}^{+\infty} E_n. \tag{3.44}$$

Thus, in order to apply Baire's theorem we have to prove the following.

Claim 3.10. For every $n \geq 1$ the sets E_n are open in $\widetilde{R}(\Omega)$.

Proof of Claim 3.10 Let $n \geq 1$ be a fixed natural number. In order to prove that the set E_n is open in $\widetilde{R}(\Omega)$ we will prove equivalently that the set $\widetilde{R}(\Omega) \setminus E_n$ is closed in $\widetilde{R}(\Omega)$.

Indeed, let $\{g_m\}_{m\geq 1}$ be a sequence of functions in $\widetilde{R}(\Omega) \setminus E_n$ which converges uniformly on the compact subsets of $\overline{\Omega}$ to a function $g \in \widetilde{R}(\Omega)$. Then, for every $m \geq 1$, there exists a $z_m \in J$ satisfying

$$\Big|\frac{g_m(z) - g_m(z_m)}{z - z_m}\Big| \leq n \tag{3.45}$$

for every $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$. Since J is a compact set, there exists a subsequence of $\{z_m\}_{m\geq 1}$ which converges to a single point $z_0 \in J$. Without loss of generality, we may assume that $\{z_m\}_{m\geq 1}$ converges to z_0. Let $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$ be a fixed point. Then, there exists an index $m_0 \geq 1$ satisfying $z \in (J \setminus \{z_m\}) \cap D(z_m, \frac{1}{n})$ for every $m \geq m_0$. Consequently, for every $m \geq m_0$, the triangle inequality implies that

$$\begin{aligned}
|g(z) - g(z_0)| &\leq |g(z) - g_m(z)| + |g_m(z) - g_m(z_m)| + |g_m(z_m) - g(z_0)| \\
&\leq |g(z) - g_m(z)| + n|z_m - z| + |g_m(z_m) - g(z_0)| \\
&\leq |g(z) - g_m(z)| + |g_m(z_m) - g(z_m)| + |g(z_m) - g(z_0)| + n|z_m - z| \\
&\leq 2\rho(f, g) + |g(z_m) - g(z_0)| + n|z_m - z|
\end{aligned} \tag{3.46}$$

where, of course, ρ is the metric of $\widetilde{R}(\Omega)$. Furthermore, the sequence $\{g_m\}_{m\geq 1}$ converges uniformly on J to g, since $J \subseteq \overline{\Omega}$ is a compact set. Therefore, by taking limits as $m \to +\infty$ in Relation (3.46) we obtain that it holds

$$\Big|\frac{g(z) - g(z_0)}{z - z_0}\Big| \leq n \tag{3.47}$$

for every $z \in (J \setminus \{z_0\}) \cap D(z_0, \frac{1}{n})$. Thus $g \in \widetilde{R}(\Omega) \setminus E_n$ and as a result, E_n is a closed set. This part of the proof is complete.

■■

Claim 3.11. For every $n \geq 1$ the sets E_n are dense in $\widetilde{R}(\Omega)$

Proof of Claim 3.11 Let $n \geq 1$ be a fixed natural number, $g \in \widetilde{R}(\Omega)$. According to the definition of the class $\widetilde{R}(\Omega)$, there exists a sequence of rational functions $\{q_m\}_{m\geq 1}$ with poles off $\overline{\Omega}$ which converges uniformly to the function $g - f$ on each compact subset of $\overline{\Omega}$. Obviously, the sequence $\{f + q_m\}_{m\geq 1}$ converges uniformly to g on the compacts subsets of $\overline{\Omega}$. Our aim is to show that $(f + q_k) \in E_n$ for every $k \geq 1$. Let also $k \geq 1$ be fixed. Since q_k' is continuous on the compact set $J \subseteq \partial\Omega$, there exists a $M > 0$ such that $|q_k'(z)| \leq M$ for every $z \in J$. Let $z_0 \in J$; since it holds

$$\limsup_{\substack{z \to z_0 \\ z \in J \setminus \{z_0\}}} \Big|\frac{f(z) - f(z_0)}{z - z_0}\Big| = +\infty \tag{3.48}$$

there exists a sequence $\{z_m\}_{m\geq 1}$ in $(J\setminus\{z_0\})\cap B(z_0,\frac{1}{n})$ such that $z_m \to z_0$ satisfying

$$\lim_{m\to+\infty}\Big|\frac{f(z_m)-f(z_0)}{z_m-z_0}\Big| = +\infty. \tag{3.49}$$

The triangle inequality implies that

$$\begin{aligned}\Big|\frac{(f+q_k)(z_m)-(f+q_k)(z_0)}{z_m-z_0}\Big| &= \Big|\frac{f(z_m)-f(z_0)}{z_m-z_0}+\frac{q_k(z_m)-q_k(z_0)}{z_m-z_0}\Big| \\ &\geq \Big|\frac{f(z_m)-f(z_0)}{z_m-z_0}\Big| - \Big|\frac{q_k(z_m)-q_k(z_0)}{z_m-z_0}\Big|.\end{aligned} \tag{3.50}$$

At the same time, it also holds

$$\lim_{m\to+\infty}\Big|\frac{q_k(z_m)-q_k(z_0)}{z_m-z_0}\Big| = |q_k'(z_0)| \leq M. \tag{3.51}$$

Therefore, by combining Relations (3.50) and (3.51) we obtain that

$$\Big|\frac{(f+q_k)(z_m)-(f+q_k)(z_0)}{z_m-z_0}\Big| > n \tag{3.52}$$

for m large enough. Therefore, we deduce that $(f+q_k)\in E_n$ for every $k\geq 1$. This part of the proof is complete.

■■

The result follows now from Baire's Theorem.

■

3.2.3 A few examples

We now present a few examples concerning Theorems 3.6 and 3.9. In Example 3.12 we deal with an unbounded domain $\Omega \subseteq \mathbb{C}$ (the open right half plane) and a closed set $J \subseteq \partial\Omega$ of its boundary (the y - axis) and we prove that the respective class $S(\Omega, J)$ is G_δ - dense in $\widetilde{R}(\Omega)$. Notice that in this specific example the set J is not a compact one.

In Example 3.13 we deal with a bounded domain $\Omega \subseteq \mathbb{C}$ (an angular sector) and the compact set $J = \partial\Omega$ and we prove that the respective class $S(\Omega, J)$ is G_δ - dense in $R(\overline{\Omega})$.

In Example 3.14 we deal with a bounded domain $\Omega \subseteq \mathbb{C}$ (an open disc minus a line segment), where the respective class $S(\Omega, \partial\Omega)$ is void in $R(\overline{\Omega})$.

Finally, in Example 3.15 we deal with an ubounded domain $\Omega \subseteq \mathbb{C}$ (an open half strip minus a half-line), where the respective class $S(\Omega, \partial\Omega)$ is void in $\widetilde{R}(\Omega)$.

Example 3.12. ([23]) We consider the (open) right half plane $\Omega = \{z \in \mathbb{C} : Re(z) > 0\}$.

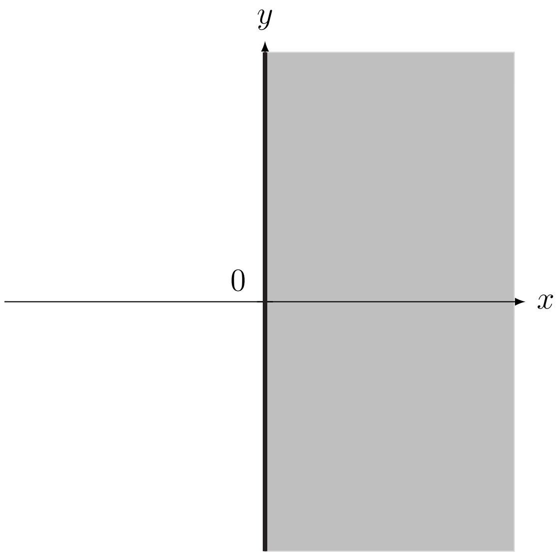

Figure 1: The unbounded domain Ω of Example 3.12.

For every $n \geq 1$ we consider the following classes of functions

$$S(\Omega, J_n) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in J_n \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J_n\} \quad (3.53)$$

where $J_n = [-in, +in]$. In addition, let

$$S(\Omega, J) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in J \setminus \{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = +\infty \text{ for every } z_0 \in J\} \quad (3.54)$$

where

$$J = \bigcup_{n=1}^{+\infty} J_n = i\mathbb{R}. \quad (3.55)$$

Then, the class $S(\Omega, J)$ is G_δ - dense in $A(\Omega)$, where the space $A(\Omega)$ is endowed with the topology of uniform convergence on the compact subsets of $\overline{\Omega}$.

Proof. We prove that each $S(\Omega, J_n)$ is G_δ - dense in $A(\Omega) = \widetilde{R}(\Omega)$. Since $(\mathbb{C} \cup \{\infty\}) \setminus (\overline{\Omega} \cap \overline{D(0,n)})$ is a connected set, it follows that $A((\overline{\Omega})^\circ \cap D(0,n)) = \widetilde{R}(\overline{\Omega} \cap \overline{D(0,n)})$ for every $n \geq 1$ and thus, according to Theorem 3.9, it is enough to prove that each $S(\Omega, J_n)$ is non - void.

We consider the entire function $\phi : \mathbb{C} \to \mathbb{C}$ satisfying $\phi(w) = e^{-w}$ for every $w \in \mathbb{C}$. Obviously, $\phi'(w) = -e^{-w} \neq 0$ for every $w \in \mathbb{C}$. We also consider the function $f : \Omega \to \mathbb{C}$ with $f = f_0 \circ \phi$, where f_0 is the function defined in Relation (3.4). Obviously, it holds $\phi(\Omega) \subseteq D(0,1)$ and $\phi(J_n) \subseteq \mathbb{T}$. The reader can easily verify that it holds $f \in S(\Omega, J_n)$ for every $n \geq 1$. Thus, the class $S(\Omega, J_n)$ is G_δ - dense in $A(\Omega)$ and since it holds

$$S = \bigcap_{n=1}^{+\infty} S(\Omega, J_n) \tag{3.56}$$

Baire's Theorem implies that the class $S(\Omega, J)$ is G_δ - dense in $A(\Omega)$.

■

Example 3.13. ([23]) We consider the following sets

$$A = \{re^{i\frac{3\pi}{4}} : 0 \leq r \leq 1\} \tag{3.57}$$

$$B = \left\{e^{i\theta} : \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\right\} \tag{3.58}$$

and

$$C = \{re^{i\frac{\pi}{4}} : 0 \leq r \leq 1\}. \tag{3.59}$$

Let Ω be the Jordan domain bounded by $A \cup B \cup C$. It clearly holds $\partial\Omega = A \cup B \cup C$. Then, the class of functions

$$S(\Omega, \partial\Omega) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in \partial\Omega \setminus \{z_0\}}} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| = +\infty \text{ for every } z_0 \in \partial\Omega\} \tag{3.60}$$

is G_δ - dense in $(A(\Omega), ||\cdot||_\infty)$.

Proof. We consider the following classes of functions

$$S(\Omega, A) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in A \setminus \{z_0\}}} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| = +\infty \text{ for every } z_0 \in A\} \tag{3.61}$$

$$S(\Omega, B) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in B \setminus \{z_0\}}} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| = +\infty \text{ for every } z_0 \in B\} \tag{3.62}$$

$$S(\Omega, C) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in C \setminus \{z_0\}}} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| = +\infty \text{ for every } z_0 \in C\}. \tag{3.63}$$

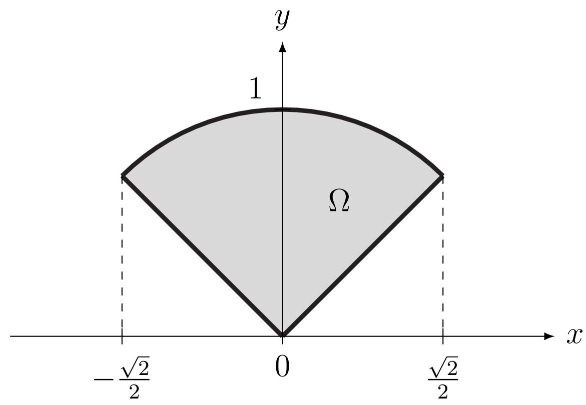

Figure 2: The bounded (Jordan) domain Ω of Example 3.13.

Obviously, it holds $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C) \subseteq S(\Omega, \partial\Omega)$. From Example 3.12, there exists a function $h \in A(R)$ (where R is the open right half plane) satisfying

$$\limsup_{\substack{z \to z_0 \\ z \in i\mathbb{R}\setminus\{z_0\}}} \left|\frac{h(z) - h(z_0)}{z - z_0}\right| = +\infty \tag{3.64}$$

for every $z_0 \in i\mathbb{R}$. In the same way, one can prove that there exists a function $\phi \in A(L)$ (where L is the open left half plane) satisfying

$$\limsup_{\substack{z \to z_0 \\ z \in i\mathbb{R}\setminus\{z_0\}}} \left|\frac{\phi(z) - \phi(z_0)}{z - z_0}\right| = +\infty \tag{3.65}$$

for every $z_0 \in i\mathbb{R}$. We consider the functions $\omega_1, \omega_2 \in A(\Omega)$ with

$$\omega_1(z) = h(e^{-i\frac{\pi}{4}}z) \tag{3.66}$$

and

$$\omega_2(z) = \phi(e^{i\frac{\pi}{4}}z) \tag{3.67}$$

for every $z \in \Omega$. We have proved that $\omega_1 \in S(\Omega, A)$ and in the same way $\omega_2 \in S(\Omega, C)$; thus, according to Theorem 3.6, the classes $S(\Omega, A)$ and $S(\Omega, C)$ are G_δ - dense in $A(\Omega)$. In addition, if f_0 is the function defined in Relation (3.4), then the restriction $(f_0 \restriction_\Omega) \in S(\Omega, B)$ and therefore, the class $S(\Omega, B)$ is also G_δ - dense in $A(\Omega)$. According to Baire's Theorem, it follows that the class $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C)$ is G_δ - dense in $A(\Omega)$. Since $S(\Omega, A) \cap S(\Omega, B) \cap S(\Omega, C) \subseteq S(\Omega, \partial\Omega)$, we obtain that it holds $S(\Omega, \partial\Omega) \neq \emptyset$ and thus, the class $S(\Omega, \partial\Omega)$ is also G_δ - dense in $A(\Omega)$.

■

Example 3.14. ([23]) We consider the open set $\Omega = \mathbb{D}\setminus[0, \frac{1}{2}]$. Then the class of functions

$$S(\Omega, \partial\Omega) = \{f \in A(\Omega) : \limsup_{\substack{z \to z_0 \\ z \in \partial\Omega\setminus\{z_0\}}} \left|\frac{f(z) - f(z_0)}{z - z_0}\right| = +\infty \text{ for every } z_0 \in \partial\Omega\} \tag{3.68}$$

is void.

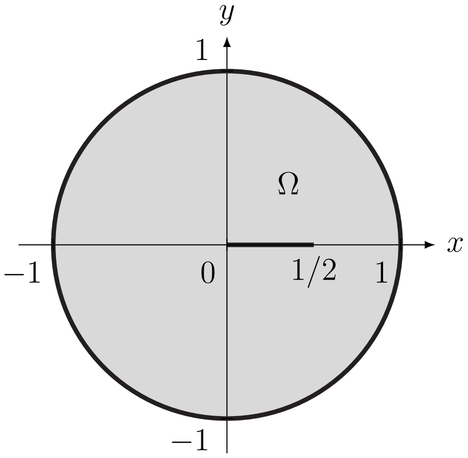

Figure 3: The bounded domain Ω of Example 3.14.

Proof. Let $f \in A(\Omega)$. Then the function f is continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}\backslash\mathbb{R}$. From a known corollary of Morera's Theorem, it follows that f is also holomorphic in $\mathbb{D}$.

Therefore, we obtain that it holds

$$\limsup_{\substack{z \to z_0 \\ z \in [0,\frac{1}{2}]\backslash\{z_0\}}} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)| < +\infty \tag{3.69}$$

for every $z_0 \in [0, \frac{1}{2}]$. Relation (3.69) clearly implies that $f \notin S(\Omega, \partial\Omega)$ and thus, we conclude that $S(\Omega, \partial\Omega) = \emptyset$.

■

Example 3.15. Let $\Omega = \{z \in \mathbb{C} : 0 < Re(z)\ \&\ 0 < Im(z) < 1\} \setminus \{z \in \mathbb{C} : 0 < Re(z)\ \&\ Im(z) = \frac{1}{2}\}$. Then it holds $S(\Omega, \partial\Omega) = \emptyset$.

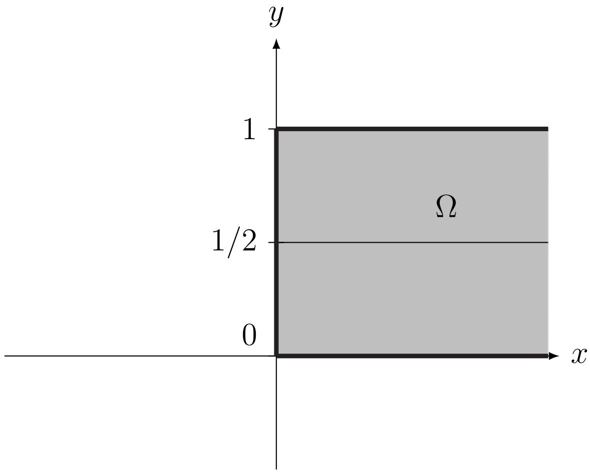

Figure 4: The unbounded domain Ω of Example 3.15.

Proof. Let $f \in A(\Omega)$. Then the function f is continuous on $\overline{\Omega}$ and holomorphic in $\Omega \setminus \{z \in \mathbb{C} : 0 < Re(z) \ \& \ Im(z) = \frac{1}{2}\}$. From a known corollary of Morera's Theorem, it follows that f is also holomorphic in Ω. The reader can fill in the necessary details as in Example 3.14.

■

4 Padé approximation

4.1 A few preliminaries

Definition 4.1 (Padé approximants). Let $\zeta \in \mathbb{C}$ be a fixed element, $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ and

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-\zeta)^n \tag{4.1}$$

be a formal power series (centered at ζ). For every $p, q \in \mathbb{N}$ we consider a function of the following form

$$[f; p/q]_\zeta(z) = \frac{A(z)}{B(z)} \tag{4.2}$$

where both functions $A(z)$ and $B(z)$ are polynomials satisfying the following conditions

(i) $$degA \leq p \text{ and } degB \leq q \tag{4.3}$$

(ii) $$B(\zeta) = 1 \tag{4.4}$$

(iii) The Taylor expansion of the function

$$[f; p/q]_\zeta(z) = \sum_{n=0}^{+\infty} b_n (z-\zeta)^n \tag{4.5}$$

at ζ satisfies

$$a_n = b_n \text{ for every } n = 0, 1, \cdots, p+q. \tag{4.6}$$

If such a rational function exists, then its irreducible form is unique, as it is well known (see for example [9]) and it is called the (p, q) - Padé approximant of f (at ζ).

If we assume that the function $A(z)/B(z)$ is irreducible, then obviously the polynomials $A(z)$ and $B(z)$ are unique. However, the polynomials $A(z)$ and $B(z)$ may not be unique in general, if we do not assume that the function $A(z)/B(z)$ is irreducible, even if it holds $degA(z) \leq p, degB(z) \leq q$ and $B(\zeta) = 1$. A necessary and sufficient condition for the uniqueness of the polynomials $A(z)$ and $B(z)$ is that $degA(z) = p$ or $degB(z) = q$ and $A(z)/B(z)$ being irreducible. This is equivalent to the non - vanishing of a particular determinant (see Relation (4.8) below). Then, it follows that the (p, q) - Padé approximant exists and it has a unique representation as $A(z)/B(z)$ with $degA(z) \leq p, degB(z) \leq q$ and $B(\zeta) = 1$.

Remark 4.2. Definition 4.1 implies that for $q = 0$ the $(p, 0)$ - Padé approximant of f exists trivially for every $p \in \mathbb{N}$, since

$$[f; p/0]_\zeta(z) = \sum_{n=0}^{p} a_n (z-\zeta)^n \tag{4.7}$$

for every $z \in \mathbb{C}$. On the other hand, for $q \geq 1$ Definition 4.1 does not necessarily imply the existence of Padé approximants. However, if a Padé approximant exists, then it is unique as a rational function. It is known ([3]) that a necessary and sufficient condition for the existence and uniqueness of the polynomials $A(z)$ and $B(z)$ in Definition 4.1 is that the following $q \times q$ Hankel determinant

$$D_{p,q}(f,\zeta) = det \begin{vmatrix} a_{p-q+1} & a_{p-q+2} & \cdots & a_p \\ a_{p-q+2} & a_{p-q+3} & \cdots & a_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q-1} \end{vmatrix} \tag{4.8}$$

is not equal to 0; or, in other words $D_{p,q}(f,\zeta) \neq 0$. In the previous determinant we set $a_k = 0$ for every $k < 0$. In addition, if $D_{p,q}(f,\zeta) \neq 0$ we also write $f \in D_{p,q}(\zeta)$. In this particular case, the (p,q) - Padé approximant of f (with center $\zeta \in \mathbb{C}$) is given by the following explicit formula

$$[f;p/q]_\zeta(z) = \frac{A(f,\zeta)(z)}{B(f,\zeta)(z)} \tag{4.9}$$

where

$$A(f,\zeta)(z) = det \begin{vmatrix} (z-\zeta)^q S_{p-q}(f,\zeta)(z) & (z-\zeta)^{q-1} S_{p-q+1}(f,\zeta)(z) & \cdots & S_p(f,\zeta)(z) \\ a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q} \end{vmatrix} \tag{4.10}$$

and

$$B(f,\zeta)(z) = det \begin{vmatrix} (z-\zeta)^q & (z-\zeta)^{q-1} & \cdots & 1 \\ a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q} \end{vmatrix} \tag{4.11}$$

where clearly

$$S_k(f,\zeta)(z) = \begin{cases} \sum_{j=0}^{k} a_j (z-\zeta)^j, & \text{if } k \geq 0 \\ 0, & \text{if } k < 0 \end{cases} \tag{4.12}$$

Relations (4.9), (4.10) and (4.11) are known as Jacobi formulas. Notice that if $f \in D_{p,q}(\zeta)$, apart from the explicit formula for the polynomials $A(f,\zeta)(z)$ and $B(f,\zeta)(z)$, these functions do not have any common zeros in $\mathbb{C}$.

It may happen $D_{p,q}(f,\zeta) = 0$ and still the Padé approximant $[f;p/q]_\zeta(z)$ may exist. In that case it holds $degA(z) < p$ and $degB(z) < q$ and there are more than one possible representations

$$[f;p/q]_\zeta(z) = \frac{\widetilde{A}(z)}{\widetilde{B}(z)} \tag{4.13}$$

with $deg\widetilde{A}(z) \leq p$ and $deg\widetilde{B}(z) \leq q$ and $\widetilde{B}(\zeta) = 1$. For instance, consider the following representation.

$$[f;p/q]_\zeta(z) = \frac{A(z)}{B(z)} = \frac{A(z)[(z-\zeta)+1]}{B(z)[(z-\zeta)+1]}. \tag{4.14}$$

Such an example comes for every rational function

$$\frac{A(z)}{B(z)} \tag{4.15}$$

with $degA(z) < p$ and $degB(z) < q$ and $B(0) = 1$ for $\zeta = 0$. Then as f we take the Taylor development at 0 of the function

$$\frac{A(z)}{B(z)} = \sum_{n=0}^{+\infty} a_n z^n. \tag{4.16}$$

An explicit example is the function

$$\frac{1}{1+z} = \sum_{n=0}^{+\infty} (-1)^n z^n \tag{4.17}$$

for $p > 0$ and $q > 1$. Then it holds

$$[f;p/q]_\zeta(z) = \frac{(1+z)}{(1+z)^2} = \frac{1}{1+z} \tag{4.18}$$

where both representations are acceptable. We will also make use of the following proposition.

Proposition 4.3. ([3]) We consider the rational function

$$f(z) = \frac{A(z)}{B(z)} \tag{4.19}$$

and for the polynomials $A(z)$ and $B(z)$ let $degA(z) = p_0$ and $degB(z) = q_0$. Also, suppose that the polynomials $A(z)$ and $B(z)$ do not have any common zero in $\mathbb{C}$. Then for every $\zeta \in \mathbb{C}$ such that $B(\zeta) \neq 0$ it holds

(i) $$f \in D_{p_0,q_0}(\zeta) \tag{4.20}$$

(ii) $$f \in D_{p,q_0}(\zeta) \text{ for every } p \geq p_0 \tag{4.21}$$

(iii) $$f \in D_{p_0,q}(\zeta) \text{ for every } q \geq q_0 \tag{4.22}$$

Moreover, for every pair $(p,q) \in \mathbb{N} \times \mathbb{N}$ with $p > p_0$ and $q > q_0$ it holds

$$f \notin D_{p,q}(\zeta). \tag{4.23}$$

In all cases above it holds $f(z) \equiv [f;p/q]_\zeta(z)$.

In some of the following results, we make use of the chordal metric χ, which is a metric defined on $\mathbb{C} \cup \{\infty\}$. The metric χ is given by the following relations

$$\chi(z,w) = \begin{cases} \dfrac{|z-w|}{\sqrt{1+|z|^2}\cdot\sqrt{1+|w|^2}}, & \text{if } z,w \in \mathbb{C} \\ \dfrac{1}{\sqrt{1+|z|^2}}, & \text{if } z \in \mathbb{C} \text{ and } w = \infty \\ 0, & \text{if } z = w \end{cases} \tag{4.24}$$

The reader could easily verify the following properties of the chordal metric

(i) $$\chi(z,w) = \chi\Big(\frac{1}{z}, \frac{1}{w}\Big) \text{ for every } z,w \in \mathbb{C} \cup \{\infty\} \tag{4.25}$$

(ii) $$\chi(z,w) \leq |z-w| \text{ for every } z,w \in \mathbb{C} \tag{4.26}$$

(iii) $$\big(\mathbb{C} \cup \{\infty\}, \chi\big) \text{ is a complete metric space.} \tag{4.27}$$

Relation (4.26) implies that if a sequence of functions $\{f_n\}_{n\in\mathbb{N}} : X \to \mathbb{C}$ converges uniformly to a function $f : X \to \mathbb{C}$ with respect to the Euclidean metric $|\cdot|$, then so does it with respect to the chordal metric χ. In addition, it is known that the metrics $|\cdot|$ and χ are uniformly equivalent on every compact subset of the complex plane.

We will also make use of the following known topological lemmas.

Lemma 4.4 (Existence of absorbing family, [29], [33], [36])**.** Let Ω be a domain in $\mathbb{C}$. Then there exists a sequence $\{K_m\}_{m\geq 1}$ of compact subsets of $\mathbb{C} \setminus \Omega$ with connected complements, such that for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement, there exists an index $m \geq 1$ satisfying $K \subseteq K_m$.

Lemma 4.5 (Existence of exhausting family, [38])**.** Let Ω be an open set in $\mathbb{C}$. Then there exists a sequence $\{L_k\}_{k\geq 1}$ of compact subsets of Ω such that

(i) $L_k \subseteq L^o_{k+1}$ for every $k \geq 1$. It follows that the sequence $\{L_k\}_{k\geq 1}$ is increasing.

(ii) For every compact set $L \subseteq \Omega$ there exists an index $k \geq 1$ such that $L \subseteq L_k$.

(iii) Every connected component of $\widetilde{\mathbb{C}} \setminus L_k$ contains at least one connected component of $\widetilde{\mathbb{C}} \setminus \Omega$ (where $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$).

Let $\Omega \subseteq \mathbb{C}$ be an open set and $\{L_k\}_{k\geq 1}$ be an exhausting family of subsets of Ω that satisfies the properties of Lemma 4.5. We consider the following class of functions

$$H(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic in } \Omega\} \tag{4.28}$$

endowed with the family of seminorms

$$\rho_k(f,g) = \sup_{z\in L_k} |f(z) - g(z)| \tag{4.29}$$

for every $k \geq 1$ and for every $f, g \in H(\Omega)$. It is known that if

$$\rho(f,g) = \sum_{k=1}^{+\infty} \frac{1}{2^k} \cdot \frac{\rho_k(f,g)}{1+\rho_k(f,g)} \tag{4.30}$$

for every $f, g \in H(\Omega)$, then ρ is a metric in $H(\Omega)$ and $\big(H(\Omega), \rho\big)$ is a complete metric space; in fact, it is a Fréchet space. Thus, Baire's theorem is at our disposal.

Generally speaking, our results concern two types of universal Padé approximants.

Definition 4.6. ([12], [14]) [Universal Padé approximants of Type I]

Let $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $p_n \to +\infty$, $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $\zeta \in \Omega$ be a fixed point. A holomorphic function $f \in H(\Omega)$ with Taylor expansion

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n \tag{4.31}$$

at $\zeta \in \Omega$ has universal Padé approximants of Type I if for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement and for every function $h \in A(K)$, there exists a subsequence $(p_{k_n})_{n\in\mathbb{N}}$ of the sequence $(p_n)_{n\in\mathbb{N}}$ satisfying the following.

(i) $f \in D_{p_{k_n}, q_{k_n}}(\zeta)$ for every $n \in \mathbb{N}$.

(ii) $\sup_{z\in K} |[f; p_{k_n}/q_{k_n}]_\zeta(z) - h(z)| \to 0$ as $n \to +\infty$.

(iii) $\sup_{z\in J} |[f; p_{k_n}/q_{k_n}]_\zeta(z) - f(z)| \to 0$ as $n \to +\infty$ for every compact set $J \subseteq \Omega$.

The set of universal Padé approximants of Type I is G_δ - dense in $H(\Omega)$, where the space $H(\Omega)$ is endowed with the topology of uniform convergence on compacta. If $q_n = 0$, then this class of functions coincides with the class of universal Taylor series.

Definition 4.7. ([35]) [Universal Padé approximants of Type II]

Let $(p_n)_{n\in\mathbb{N}}, (q_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $p_n, q_n \to +\infty$, $\Omega \subseteq \mathbb{C}$ be a domain and $\zeta \in \Omega$ be a fixed point. A holomorphic function $f \in H(\Omega)$ with Taylor expansion

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n \tag{4.32}$$

at $\zeta \in \Omega$ has universal Padé approximants of Type II if for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ and for every rational function h, there exist two subsequence $(p_{k_n})_{n\in\mathbb{N}}$ and $(q_{k_n})_{n\in\mathbb{N}}$ of the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ respectively satisfying the following.

(i) $f \in D_{p_{k_n}, q_{k_n}}(\zeta)$ for every $n \in \mathbb{N}$.

(ii) $\sup_{z\in K} \chi([f; p_{k_n}/q_{k_n}]_\zeta(z), h(z)) \to 0$ as $n \to +\infty$. The metric χ is the well known distance defined on $\mathbb{C} \cup \{\infty\}$.

(iii) $\sup_{z\in J} |[f; p_{k_n}/q_{k_n}]_\zeta(z) - f(z)| \to 0$ as $n \to +\infty$ for every compact set $J \subseteq \Omega$.

The set of universal Padé approximants of Type II is G_δ - dense in $H(\Omega)$, where the space $H(\Omega)$ is endowed with the topology of uniform convergence on compacta.

4.2 Results of approximation for Universal Padé approximants of Type I

In this section we present several results of simultaneous approximation concerning Universal Padé approximants of Type I. Our results are generic in the space of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$; that is, the space $H(\Omega)$.

Theorem 4.8. ([27]) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $L \subseteq \Omega$ be a compact set. We consider a sequence $(p_n)_{n\geq 1} \subseteq \mathbb{N}$ with $p_n \to +\infty$. Now, for every $n \geq 1$ let $q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Then, there exists a function $f \in H(\Omega)$ satisfying the following.

For every compact set $K \subseteq \mathbb{C} \backslash \Omega$ with connected complement and for every function $h \in A(K)$, there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ such that

(1) $$f \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$$

for every $\zeta \in L$, for every $n \geq 1$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2) $$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in K} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - h(z)| \to 0$$

as $n \to +\infty$.

(3) For every compact set $J \subseteq \Omega$, it holds
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in J} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. Let $\{f_i\}_{i\geq 1}$ be an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We fix a sequence $\{K_m\}_{m\geq 1}$ of compact subsets of $\mathbb{C} \setminus \Omega$ satisfying Lemma 4.4 and a sequence $\{L_k\}_{k\geq 1}$ of compact subsets of Ω satisfying Lemma 4.5.

Now, for every $i, s, n, k, m \geq 1$ and for every $j \in \{1, \cdots N(n)\}$ we consider the following sets

$$A(i,s,m,n,j) = \Big\{ f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } \zeta \in L$$
$$\text{and } \sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| < \frac{1}{s} \Big\} \tag{4.33}$$

$$A(i,s,m,n) = \Big\{ f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } \zeta \in L \text{ and for every}$$
$$j = 1, 2, \cdots, N(n) \text{ and also} \max_{j=1,\cdots,N(n)} \sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| < \frac{1}{s} \Big\}$$

$$\equiv \bigcap_{j=1}^{N(n)} A(i,s,m,n,j) \tag{4.34}$$

$$B(s,k,n,j) = \Big\{ f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } \zeta \in L$$

$$\text{and } \sup_{\zeta \in L} \sup_{z \in L_k} |[f; p_n/q_j^{(n)}]_\zeta(z) - f(z)| < \frac{1}{s} \Big\} \tag{4.35}$$

$$B(s,k,n) = \Big\{ f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } \zeta \in L \text{ and for every}$$

$$j = 1, 2, \cdots, N(n) \text{ and also } \max_{j=1,\cdots,N(n)} \sup_{\zeta \in L} \sup_{z \in L_k} |[f; p_n/q_j^{(n)}]_\zeta(z) - f(z)| < \frac{1}{s} \Big\}$$

$$\equiv \bigcap_{j=1}^{N(n)} B(s,k,n,j) \tag{4.36}$$

One can verify (mainly by using Mergelyan's theorem) that if $\mathcal{U}$ is the set of all functions satisfying the properties of Theorem 4.8, then it holds

$$\mathcal{U} = \bigcap_{i,s,k,m \geq 1} \Big(\bigcup_{n \geq 1} \big(A(i,s,m,n) \cap B(s,k,n) \big) \Big)$$

$$= \bigcap_{i,s,k,m \geq 1} \Big(\bigcup_{n \geq 1} \big([\bigcap_{j=1}^{N(n)} A(i,s,m,n,j)] \cap [\bigcap_{j=1}^{N(n)} B(s,k,n,j)] \big) \Big). \tag{4.37}$$

Since $H(\Omega)$ is a complete metric space, in order to use Baire's theorem, we will start with proving the following.

Claim 4.9. For every $i, s, n, k, m \geq 1$ and for every $j \in \{1, \cdots N(n)\}$ the sets $B(s,k,n,j)$ and $A(i,s,m,n,j)$ are open in $H(\Omega)$.

Proof of Claim 4.9 The sets $A(i,s,m,n,j)$ have been proven to be open in [29] for $q_j^{(n)} = 0$. The sets $B(s,k,n,j)$ have been proven to be open in [20] for $q_j^{(n)} \geq 1$ and in [29] for $q_j^{(n)} = 0$. We will now prove that each $A(i,s,m,n,j)$ is an open set in $H(\Omega)$ for $q_j^{(n)} \geq 1$ (see also [24]).

We fix the parameters $i, s, m, n \geq 1$ and $j \in \{1, \cdots, N(n)\}$ and we consider a function $f \in A(i,s,m,n,j)$. We want to select an $\varepsilon > 0$ such that if $g \in H(\Omega)$ with $\rho(f,g) < \varepsilon$ then it holds $g \in A(i,s,m,n,j)$. Since $f \in D_{p_n, q_j^{(n)}}(\zeta)$ for every $\zeta \in L$, the Hankel determinant $D_{p_n, q_j^{(n)}}(f, \zeta)$ is not equal to zero and that holds for every $\zeta \in L$. The previous determinant varies continuously on the parameter $\zeta \in L$ and therefore, since L is a compact set there exists a $\delta > 0$ such that

$$|D_{p_n, q_j^{(n)}}(f, \zeta)| > \frac{\delta}{2} \tag{4.38}$$

for every $\zeta \in L$. Since $\rho(f,g) < \varepsilon$ we might assume that the first $p+q+1$ Taylor coefficients of g are uniformly closed enough one by one to the corresponding Taylor coefficients of f, provided that $\varepsilon > 0$ is small enough. This follows easily by using Cauchy estimates. The function $|D_{p_n,q_j^{(n)}}(g,\zeta)|, \zeta \in L$ is continuous on L and thus we obtain

$$|D_{p_n,q_j^{(n)}}(g,\zeta)| > \frac{\delta}{2} \tag{4.39}$$

Relation (4.39) clearly implies that $g \in D_{p_n,q_j^{(n)}}$ for every ζ in the compact set L. It remains only to verify that it holds

$$\sup_{\zeta \in L} \sup_{z \in K_m} |[g; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| < \frac{1}{s}. \tag{4.40}$$

Indeed, by the triangle inequality it holds

$$\begin{aligned}\sup_{\zeta \in L} \sup_{z \in K_m} |[g; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| &\leq \sup_{\zeta \in L} \sup_{z \in K_m} |[g; p_n/q_j^{(n)}]_\zeta(z) - [f; p_n/q_j^{(n)}]_\zeta(z)| \\ &+ \sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| \end{aligned} \tag{4.41}$$

Let us denote by

$$[f; p_n/q_j^{(n)}]_\zeta(z) = \frac{A(f,\zeta)(z)}{B(f,\zeta)(z)} \tag{4.42}$$

and

$$[g; p_n/q_j^{(n)}]_\zeta(z) = \frac{A(g,\zeta)(z)}{B(g,\zeta)(z)} \tag{4.43}$$

the Padé approximants of f and g (at $\zeta \in L$) respectively. We know that the polynomials $A(f,\zeta)(z), B(f,\zeta)(z), A(g,\zeta)(z)$ and $B(g,\zeta)(z)$ are given by the Jacobi formulas and therefore, their coefficients vary continuously with respect to the parameter ζ.

Now, we have

$$\sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| < \frac{1}{s} \tag{4.44}$$

and thus, since the polynomials $A(f,\zeta)(z)$ and $B(f,\zeta)(z)$ do not have any common zeros in $\mathbb{C}$, it holds $B(f,\zeta)(z) \neq 0$ for every $\zeta \in L$ and for every $z \in K_m$. Therefore, by continuity, there exists a $\delta' > 0$ such that

$$|B(f,\zeta)(z)| > \delta' \tag{4.45}$$

for every $\zeta \in L$ and for every $z \in K_m$ (the set $L \times K_m$ is a compact one). Since the first $p+q+1$ Taylor coefficients of g are uniformly closed enough one by one to the corresponding Taylor coefficients of f for $\varepsilon > 0$ small enough, we obtain

$$|B(g,\zeta)(z)| > \frac{\delta'}{2} \tag{4.46}$$

The triangle inequality implies that

$$|[g; p_n/q_j^{(n)}]_\zeta(z) - [f; p_n/q_j^{(n)}]_\zeta(z)| = \Big|\frac{A(g,\zeta)(z)}{B(g,\zeta)(z)} - \frac{A(f,\zeta)(z)}{B(f,\zeta)(z)}\Big|$$

$$
\begin{aligned}
&\leq \Big|\frac{A(g,\zeta)(z)B(f,\zeta)(z) - B(g,\zeta)(z)A(f,\zeta)(z)}{B(f,\zeta)(z)B(g,\zeta)(z)}\Big| \\
&\leq \Big(\frac{2}{(\delta')^2}\Big)|A(g,\zeta)(z)B(f,\zeta)(z) - B(g,\zeta)(z)A(f,\zeta)(z)| \\
&\leq \Big(\frac{2}{(\delta')^2}\Big)|A(f,\zeta)(z)| \cdot |B(f,\zeta)(z) - B(g,\zeta)(z)| \\
&+ \Big(\frac{2}{(\delta')^2}\Big)|B(f,\zeta)(z)| \cdot |A(f,\zeta)(z) - A(g,\zeta)(z)|
\end{aligned} \tag{4.47}
$$

Hence, if $\varepsilon > 0$ is small enough, we obtain

$$
\sup_{\zeta\in L}\sup_{z\in K_m} |[g; p_n/q_j^{(n)}]_\zeta(z) - [f; p_n/q_j^{(n)}]_\zeta(z)| < \frac{1}{s} - \sup_{\zeta\in L}\sup_{z\in K_m} |[f; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| \tag{4.48}
$$

By combining Relations (4.44) and (4.48), we conclude that it holds

$$
\sup_{\zeta\in L}\sup_{z\in K_m} |[g; p_n/q_j^{(n)}]_\zeta(z) - f_i(z)| < \frac{1}{s} \tag{4.49}
$$

which in turn implies that $g \in A(i,s,m,n,j)$. This part of the proof is complete.

■■

Claim 4.10. For every $i, s, k, m \geq 1$ the set

$$
\begin{aligned}
\mathcal{U}(i,s,k,m) &= \bigcup_{n\geq 1} \big(A(i,s,m,n) \bigcap B(s,k,n)\big) \\
&= \bigcup_{n\geq 1} \big([\bigcap_{j=1}^{N(n)} A(i,s,m,n,j)] \bigcap [\bigcap_{j=1}^{N(n)} B(s,k,n,j)]\big)
\end{aligned} \tag{4.50}
$$

is dense in $H(\Omega)$.

Proof of Claim 4.10 We fix the parameters $i, s, k, m \geq 1$ and we want to prove that the set $\mathcal{U}(i,s,k,m)$ is dense in $H(\Omega)$. Let $g \in H(\Omega)$, $L' \subseteq \Omega$ be a compact set and $\varepsilon > 0$. Our aim is to find a function $u \in \mathcal{U}(i,s,k,m)$ such that

$$
\sup_{z\in L'} |u(z) - g(z)| < \varepsilon \tag{4.51}
$$

There is no problem if we assume that $\varepsilon < \frac{1}{s}$. According to Lemma 4.5, we are able to find an index $n_0 \geq 1$ satisfying $L \cup L' \cup L_k \subseteq L_{n_0}$. Since L_{n_0} and K_m are disjoint compact sets with connected complements, the set $L_{n_0} \cup K_m$ is also a compact one with connected complement. Consider now the following function

$$
w(z) = \begin{cases} f_i(z) & \text{if } z \in K_m \\ g(z) & \text{if } z \in L_{n_0}. \end{cases} \tag{4.52}
$$

The function w is well defined (because $L_{n_0} \cap K_m = \emptyset$) and it also holds $w \in A(L_{n_0} \cup K_m)$. We apply Mergelyan's theorem and thus we find a polynomial p such that

$$
\sup_{L_{n_0}\cup K_m} |w(z) - p(z)| < \frac{\varepsilon}{2}. \tag{4.53}
$$

Our assumption that $p_n \to +\infty$ allows us to find an index $p_{k_n} \geq 1$ such that $p_{k_n} > degp(z)$. Let $u(z) = p(z) + dz^{p_{k_n}}$, where $d \in \mathbb{C} \setminus \{0\}$ and

$$|d| < \frac{\varepsilon}{2} \cdot \frac{1}{\sup\limits_{z \in L_{n_0} \cup K_m} |z^{p_{k_n}}| + 1}. \tag{4.54}$$

It is immediate that the function u is a polynomial with $degu(z) = p_{k_n}$. We also notice that it holds

$$\sup_{z \in L_{n_0} \cup K_m} |u(z) - w(z)| < \varepsilon. \tag{4.55}$$

In order to complete the proof of Claim 4.10 we have to verify that $u \in \mathcal{U}(i, s, k, m)$.

(i) $u \in A(i, s, m, k_n, j)$ for every $j = 1, \cdots, N(k_n)$. Since u is a polynomial with $degu(z) = p_{k_n}$ we have that for every $\zeta \in L$ it holds $u \in D_{p_{k_n},0}(\zeta)$. It follows that $u \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$ for every $j = 1, 2, \cdots, N(k_n)$. We also have

$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in K_m} |[u; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - f_i(z)| = \sup_{\zeta \in L} \sup_{z \in K_m} |[u(z) - f_i(z)| < \frac{1}{s} \tag{4.56}$$

because $[u; p_{k_n}/q_j^{(k_n)}]_\zeta(z) = u(z)$ for every $\zeta \in L$ and for every $j = 1, \cdots, N(k_n)$, according to Proposition 4.3.

(ii) $u \in B(s, k, k_n, j)$ for every $j = 1, \cdots, N(k_n)$. We only have to verify that it holds

$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in L_k} |[u; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - u(z)| = 0 < \frac{1}{s} \tag{4.57}$$

which is immediate, since $[u; p_{k_n}/q_j^{(k_n)}]_\zeta(z) = u(z)$, due to the definition of the polynomial u.

Therefore

$$u \in [\bigcap_{j=1}^{N(k_n)} A(i, s, m, n, j)] \bigcap [\bigcap_{j=1}^{N(k_n)} B(s, k, n, j)] \tag{4.58}$$

or, equivalently $u \in \mathcal{U}(i, s, k, m)$. This part of the proof is complete.

■■

Since $H(\Omega)$ is a complete metric space, we apply Baire's theorem and that completes the proof.

■

We now present a consequence of Theorem 4.8.

Theorem 4.11. ([27]) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. We consider a sequence $(p_n)_{n\geq 1} \subseteq \mathbb{N}$ with $p_n \to +\infty$. Now, for every $n \geq 1$ let $q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Then there exists a function $f \in H(\Omega)$ satisfying the following.

For every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement and for every function $h \in A(K)$ there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ such that

(1) For every compact set $L \subseteq \Omega$, there exists a $n(L) \geq 1$ such that $f \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$ for every $\zeta \in L$, for every $n \geq n(L)$ (where $n(L)$ is a natural number depending on L) and for every $j \in \{1, \cdots, N(k_n)\}$.

(2)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in K} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - h(z)| \to 0$$

as $n \to +\infty$, for every compact set $L \subseteq \Omega$.

(3)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in L} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$, for every compact set $L \subseteq \Omega$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. Let $\mathcal{C}^1$ be the set of all functions satisfying Theorem 4.11. We apply Theorem 4.8 for $L = L_k$ (and that for every $k \geq 1$) and thus we obtain a G_δ - dense class of functions in $H(\Omega)$; the class $\mathcal{C}_k^1$. The reader can verify that it holds

$$\mathcal{C}^1 = \bigcap_{k \geq 1} \mathcal{C}_k^1. \tag{4.59}$$

The result follows now from Baire's theorem.

■

We now present two results similar to Theorems 4.8 and 4.11 respectively where the roles of p and q have been interchanged.

Theorem 4.12. ([27]) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $L \subseteq \Omega$ be a compact set. We consider an arbitrary sequence $(q_n)_{n \geq 1} \subseteq \mathbb{N}$ (may be bounded or unbounded). Now, for every $n \geq 1$ let $p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number, such that

$$\min_{j \in \{1,\cdots,N(n)\}} \{p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)}\} \to +\infty \tag{4.60}$$

as $n \to +\infty$. Then there exists a function $f \in H(\Omega)$ satisfying the following.

For every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement and for every function $h \in A(K)$ there exists a subsequence $(q_{k_n})_{n \geq 1}$ of the sequence $(q_n)_{n \geq 1}$ such that

(1)
$$f \in D_{p_j^{(k_n)}, q_{k_n}}(\zeta)$$

for every $\zeta \in L$, for every $n \in \mathbb{N}$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in K} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - h(z)| \to 0$$

as $n \to +\infty$.

(3) For every compact set $J \subseteq \Omega$ it holds
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta \in L} \sup_{z \in J} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - f(z)| \to 0$$
as $n \to +\infty$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. Let $\{f_i\}_{i \geq 1}$ be an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

Now, for every $i, s, n, k, m \geq 1$ and for every $j \in \{1, \cdots N(n)\}$ we consider the following sets

$$A(i,s,m,n,j) = \Big\{ f \in H(\Omega) : f \in D_{p_j^{(n)},q_n}(\zeta) \text{ for every } \zeta \in L \\ \text{and } \sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_j^{(n)}/q_n]_\zeta(z) - f_i(z)| < \frac{1}{s} \Big\} \tag{4.61}$$

$$A(i,s,m,n) = \Big\{ f \in H(\Omega) : f \in D_{p_j^{(n)},q_n}(\zeta) \text{ for every } \zeta \in L \text{ and for every} \\ j = 1,2,\cdots,N(n) \text{ and also } \max_{j=1,\cdots,N(n)} \sup_{\zeta \in L} \sup_{z \in K_m} |[f; p_j^{(n)}/q_n]_\zeta(z) - f_i(z)| < \frac{1}{s} \Big\} \\ \equiv \bigcap_{j=1}^{N(n)} A(i,s,m,n,j) \tag{4.62}$$

$$B(s,k,n,j) = \Big\{ f \in H(\Omega) : f \in D_{p_n,q_j^{(n)}}(\zeta) \text{ for every } \zeta \in L \\ \text{and } \sup_{\zeta \in L} \sup_{z \in L_k} |[f; p_j^{(n)}/q_n]_\zeta(z) - f(z)| < \frac{1}{s} \Big\} \tag{4.63}$$

$$B(s,k,n) = \Big\{ f \in H(\Omega) : f \in D_{p_j^{(n)},q_n}(\zeta) \text{ for every } \zeta \in L \text{ and for every} \\ j = 1,2,\cdots,N(n) \text{ and also } \max_{j=1,\cdots,N(n)} \sup_{\zeta \in L} \sup_{z \in L_k} |[f; p_j^{(n)}/q_n]_\zeta(z) - f(z)| < \frac{1}{s} \Big\} \\ \equiv \bigcap_{j=1}^{N(n)} B(s,k,n,j) \tag{4.64}$$

One can verify (mainly by using Mergelyan's theorem) that if $\mathcal{S}$ is the class of all functions satisfying the properties of Theorem 4.12, then it holds

$$\mathcal{S} = \bigcap_{i,s,k,m \geq 1} \Big(\bigcup_{n \geq 1} \big(A(i,s,m,n) \bigcap B(s,k,n) \big) \Big) \\ = \bigcap_{i,s,k,m \geq 1} \Big(\bigcup_{n \geq 1} \big([\bigcap_{j=1}^{N(n)} A(i,s,m,n,j)] \bigcap [\bigcap_{j=1}^{N(n)} B(s,k,n,j)] \big) \Big). \tag{4.65}$$

Since $H(\Omega)$ is a complete metric space, in order to use Baire's theorem, it suffices to prove the following.

Claim 4.13. For every $i, s, n, k, m \geq 1$ and for every $j \in \{1, \cdots N(n)\}$ the sets $B(s, k, n, j)$ and $A(i, s, m, n, j)$ are open in $H(\Omega)$.

For the proof of Claim 4.13 we refer to the proof of Claim 4.9.

Claim 4.14. For every $i, s, k, m \geq 1$ the set

$$\begin{aligned}\mathcal{S}(i,s,k,m) &= \bigcup_{n\geq 1} \big(A(i,s,m,n) \bigcap B(s,k,n)\big) \\ &= \bigcup_{n\geq 1} \big([\bigcap_{j=1}^{N(n)} A(i,s,m,n,j)] \bigcap [\bigcap_{j=1}^{N(n)} B(s,k,n,j)]\big) \end{aligned} \tag{4.66}$$

is dense in $H(\Omega)$.

Proof of Claim 4.14 We fix the parameters $i, s, k, m \geq 1$ and we want to prove that the set $\mathcal{S}(i, s, k, m)$ is a dense subset of $H(\Omega)$. Let $g \in H(\Omega)$, $L' \subseteq \Omega$ be a compact set and $\varepsilon > 0$. Our aim is to find a function $u \in \mathcal{S}(i, s, k, m)$ such that

$$\sup_{z\in L'} |u(z) - g(z)| < \varepsilon. \tag{4.67}$$

There is no problem if we assume that it holds $\varepsilon < \frac{1}{s}$. According to Lemma 4.5, we are able to find an index $n_0 \geq 1$ satisfying $L \cup L' \cup L_k \subseteq L_{n_0}$.

Since L_{n_0} and K_m are disjoint compact sets with connected complements, the set $L_{n_0} \cup K_m$ is also a compact one with connected complement.

Consider now the following function

$$w(z) = \begin{cases} f_i(z), & \text{if } z \in K_m \\ g(z), & \text{if } z \in L_{n_0}. \end{cases} \tag{4.68}$$

The function w is well defined (because $L_{n_0} \cap K_m = \emptyset$) and also it holds $w \in A(L_{n_0} \cup K_m)$. We apply Mergelyan's theorem and we find a polynomial p such that

$$\sup_{L_{n_0}\cup K_m} |w(z) - p(z)| < \frac{\varepsilon}{2}. \tag{4.69}$$

Since

$$\min_{j\in\{1,\cdots,N(n)\}} \{p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)}\} \to +\infty \tag{4.70}$$

as $n \to +\infty$, there exists an index $k_{n_1} \geq 1$ such that

$$\min_{j\in\{1,\cdots,N(k_{n_1})\}} \{p_1^{(k_{n_1})}, p_2^{(k_{n_1})}, \cdots, p_{N(k_{n_1})}^{(k_{n_1})}\} > degp(z). \tag{4.71}$$

Consider now the rational function

$$u(z) = \frac{p(z)}{1 + dz^{q_{k_{n_1}}}} \tag{4.72}$$

where $d \in \mathbb{C} \setminus \{0\}$ and $0 < |d|$ is small enough. We notice that it holds

$$\begin{aligned}\sup_{L_{n_0} \cup K_m} |p(z) - u(z)| &= \sup_{L_{n_0} \cup K_m} \left| p(z) - \frac{p(z)}{1 + dz^{q_{k_{n_1}}}} \right| \\ &= \sup_{L_{n_0} \cup K_m} \left| \frac{dz^{q_{k_{n_1}}} p(z)}{1 + dz^{q_{k_{n_1}}}} \right| \\ &< \frac{\varepsilon}{2} \end{aligned} \tag{4.73}$$

provided that $0 < |d|$ is small enough. It follows that

$$\sup_{L_{n_0} \cup K_m} |w(z) - u(z)| < \varepsilon. \tag{4.74}$$

In order to complete the proof we have to verify that it holds $u \in \mathcal{S}(i, s, k, m)$.

(1) $u \in D_{p_j^{(k_{n_1})}, q_{k_{n_1}}}(\zeta)$ for every $\zeta \in L_{n_0} \cup K$ and for every $j \in \{1, \cdots, N(k_{n_1})\}$, according to Proposition 4.3. In particular, this holds for every $\zeta \in L$.

(2) $[u; p_j^{(k_{n_1})}/q_{k_{n_1}}]_\zeta(z) = u(z)$ for every $\zeta \in L_{n_0} \cup K$, for every $z \in K$ and for every $j \in \{1, \cdots, N(k_{n_1})\}$, according to Proposition 4.3. In particular, this holds for every $\zeta \in L$.

Therefore

$$u \in \Big[\bigcap_{j=1}^{N(k_n)} A(i, s, m, n, j) \Big] \bigcap \Big[\bigcap_{j=1}^{N(k_n)} B(s, k, n, j) \Big] \tag{4.75}$$

or, equivalently $u \in \mathcal{S}(i, s, k, m)$. This part of the proof is complete.

■ ■

Since $H(\Omega)$ is a complete metric space, we apply Baire's theorem and that completes the proof.

■

We now present a consequence of Theorem 4.12.

Theorem 4.15. ([27]) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. We consider an arbitrary sequence $(q_n)_{n \geq 1} \subseteq \mathbb{N}$ (may be bounded or unbounded). Now, for every $n \in \mathbb{N}$ let $p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number, such that

$$\min_{j \in \{1, \cdots, N(n)\}} \{p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)}\} \to +\infty \tag{4.76}$$

as $n \to +\infty$. Then there exists a function $f \in H(\Omega)$ satisfying the following.

For every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement and for every function $h \in A(K)$, there exists a subsequence $(q_{k_n})_{n \geq 1}$ of the sequence $(q_n)_{n \geq 1}$ such that

(1) For every compact set $L \subseteq \Omega$, there exists a $n(L) \in \mathbb{N}$ such that $f \in D_{p_j^{(k_n)}, q_{k_n}}(\zeta)$ for every $n \geq n(L)$, for every $j \in \{1, \cdots, N(k_n)\}$ and for every $\zeta \in L$.

(2) $$\max_{j=1,\cdots,N(k_n)} \sup_{z\in K} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - h(z)| \to 0$$

as $n \to +\infty$.

(3) For every compact set $L \subseteq \Omega$ it holds
$$\max_{j=1,\cdots,N(k_n)} \sup_{z\in L} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. Let $\mathcal{C}^2$ be the set of all functions satisfying Theorem 4.15. We apply Theorem 4.12 for $L = L_k$ (and that for every $k \geq 1$) and thus we obtain a G_δ - dense class of functions in $H(\Omega)$; the class $\mathcal{C}_k^2$. The reader can verify that it holds

$$\mathcal{C}^2 = \bigcap_{k\geq 1} \mathcal{C}_k^2. \tag{4.77}$$

The result follows now from Baire's theorem.

■

4.3 Results of approximation for Universal Padé approximants of Type II

In this section we present several results of simultaneous approximation concerning Universal Padé approximants of Type II. Our results are generic in the space of holomorphic functions defined on an open set $\Omega \subseteq \mathbb{C}$; that is, the space $H(\Omega)$.

Theorem 4.16. ([27]) Let $\Omega \subseteq \mathbb{C}$ be an open set and $L, L' \subseteq \Omega$ two compact sets. Let also $K \subseteq \mathbb{C} \setminus \Omega$ be another compact set. We consider a sequence $(p_n)_{n\geq 1} \subseteq \mathbb{N}$ with $p_n \to +\infty$. Now, for every $n \geq 1$ let $q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Suppose also that it holds

$$\min\{q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)}\} \to +\infty. \tag{4.78}$$

Then there exists a function $f \in H(\Omega)$ such that for every rational function h, there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ satisfying the following.

(1) $$f \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$$

for every $\zeta \in L$, for every $n \in \mathbb{N}$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2) $$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in K} \chi([f; p_{k_n}/q_j^{(k_n)}]_\zeta(z), h(z)) \to 0$$

as $n \to +\infty$.

(3) $$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L}\sup_{z\in L'} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. Let $\{f_i\}_{i\geq 1}$ be an enumeration of all rational functions with the coefficients of the numerator and denominator in $\mathbb{Q} + i\mathbb{Q}$. There is also no problem to assume that for every $i \geq 1$, the numerator and the denominator do not have any common zeros in $\mathbb{C}$.

Now, for every $i, s, n \geq 1$ and for every $j \in \{1, \cdots N(n)\}$ we consider the following sets

$$A(j,n,s) = \Big\{f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ and } \sup_{\zeta\in L}\sup_{z\in L'} |[f; p_n/q_j^{(n)}]_\zeta(z) - f(z)| < \frac{1}{s}\Big\} \tag{4.79}$$

$$A(n,s) = \Big\{f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } j = 1, 2, \cdots, N(n) \text{ and } \max_{j=1,\cdots,N(n)} \sup_{\zeta\in L}\sup_{z\in L'} |[f; p_n/q_j^{(n)}]_\zeta(z) - f(z)| < \frac{1}{s}\Big\} \equiv \bigcap_{j=1}^{N(n)} A(j,n,s)$$

$$B(i,j,n,s) = \Big\{f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ and } \sup_{\zeta\in L}\sup_{z\in K} \chi([f; p_n/q_j^{(n)}]_\zeta(z), f_i(z)) < \frac{1}{s}\Big\} \tag{4.80}$$

$$B(i,n,s) = \Big\{f \in H(\Omega) : f \in D_{p_n, q_j^{(n)}}(\zeta) \text{ for every } j = 1, 2, \cdots, N(n) \text{ and } \max_{j=1,\cdots,N(n)} \sup_{\zeta\in L}\sup_{z\in K} \chi([f; p_n/q_j^{(n)}]_\zeta(z), f_i(z)) < \frac{1}{s}\Big\} \equiv \bigcap_{j=1}^{N(n)} B(i,j,n,s) \tag{4.81}$$

One can verify that if $\mathcal{U}$ is the set of all functions satisfying the properties of Theorem 4.16, then it holds

$$\mathcal{U} = \bigcap_{i,s\geq 1} \Big(\bigcup_{n\geq 1} A(n,s) \cap B(i,n,s)\Big) = \bigcap_{i,s\geq 1} \Big(\bigcup_{n\geq 1} \Big[\bigcap_{j=1}^{N(n)} A(j,n,s)\Big] \cap \Big[\bigcap_{j=1}^{N(n)} B(i,j,n,s)\Big]\Big). \tag{4.82}$$

Since $H(\Omega)$ is a complete metric space, in order to use Baire's theorem, it suffices to prove the following.

Claim 4.17. The sets $A(j, n, s)$ and $B(i, j, n, s)$ are open for every parameter.

Proof of Claim 4.17 The sets $A(j, n, s)$ and $B(i, j, n, s)$ have been proven to be open for every parameter in [35]. Thus, the sets $A(n, s)$ and $B(i, n, s)$ are also open, as a finite intersection of open sets. It follows that the class $\mathcal{U}$ is a G_δ subset of $H(\Omega)$.

■■

Claim 4.18. The set

$$\mathcal{U}(i, s) = \bigcup_{n\geq 1} \Big[\bigcap_{j=1}^{N(n)} A(j, n, s) \Big] \cap \Big[\bigcap_{j=1}^{N(n)} B(i, j, n, s) \Big] \tag{4.83}$$

is dense for every $i, s \geq 1$.

Proof of Claim 4.18 We fix the parameters $i, s \geq 1$. Let $L'' \subseteq \Omega$ be a compact set, $\phi \in H(\Omega)$ and $\varepsilon > 0$. We may assume that $\varepsilon < \frac{1}{s}$. Our aim is to find a function $g \in \mathcal{U}(i, s)$ such that

$$\sup_{z\in L''} |\phi(z) - g(z)| < \varepsilon. \tag{4.84}$$

Without loss of generality, we suppose that $L \cup L' \subseteq (L'')^o$ and also that every connected component of $\mathbb{C} \cup \{\infty\} \setminus L''$ contains a connected component of $\mathbb{C} \cup \{\infty\} \setminus \Omega$. For instance, that can be achieved by using Lemma 4.5.

Consider now the following function

$$w(z) = \begin{cases} f_i(z) & \text{if } z \in K \\ \phi(z) & \text{if } z \in L''. \end{cases} \tag{4.85}$$

The set of poles of f_i on K is finite; let μ denote the sum of the principal parts of f_i on these poles. Thus, the function $\omega - \mu$ is holomorphic in an open set containing $L'' \cup K$. We apply Runge's theorem to approximate the function $\omega - \mu$ uniformly on $L'' \cup K$ with respect to the Euclidean metric by a sequence of rational functions

$$\frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)}. \tag{4.86}$$

Hence, there exists a natural number $n_0 \geq 1$ satisfying the following

$$\sup_{z\in L''\cup K} \left| (\omega(z) - \mu(z)) - \frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)} \right| < \frac{\varepsilon}{2} \text{ for every } n \geq n_0. \tag{4.87}$$

In particular, $\widetilde{B}_n(z) \neq 0$ for every $z \in L'' \cup K$ and for every $n \geq n_0$. There is also no problem to assume that the polynomials $\widetilde{A}_n(z)$ and $\widetilde{B}_n(z)$ have no common zeros in $\mathbb{C}$. On the other hand, the sequence of functions

$$\mu(z) + \frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)} = \frac{A_n(z)}{B_n(z)} \tag{4.88}$$

defined for $n \geq n_0$, satisfies

$$\sup_{z \in K} \chi\big(f_i(z), \mu(z) + \frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)}\big) < \frac{\varepsilon}{2} \tag{4.89}$$

and

$$\sup_{z \in L''} \Big|\phi(z) - \mu(z) - \frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)}\Big| < \frac{\varepsilon}{2} \tag{4.90}$$

for every $n \geq n_0$. We notice that the polynomials $A_n(z)$ and $B_n(z)$ have no common zeros in $\mathbb{C}$, since

$$\mu(z) + \frac{\widetilde{A}_n(z)}{\widetilde{B}_n(z)} = \frac{\mu(z) + \widetilde{A}_n(z)}{\widetilde{B}_n(z)} = \frac{A_n(z)}{B_n(z)} \tag{4.91}$$

and also that it holds $B_n(z) \neq 0$ for every $z \in L''$ and for every $n \geq n_0$, because the polynomials $\widetilde{B}_n(z)$ satisfy the same property for every $n \geq n_0$.

Since $p_n \to +\infty$ and $\min\{q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)}\} \to +\infty$, there exists an index $k_{n_0} > n_0 \geq 1$ such that

$$p_{k_{n_0}} > \max\{degA_{n_0}(z), degB_{n_0}(z)\} \tag{4.92}$$

and

$$\min\{q_1^{(k_{n_0})}, q_2^{(k_{n_0})}, \cdots, q_{N(k_{n_0})}^{(k_{n_0})}\} > degB_{n_0}(z). \tag{4.93}$$

We set $t = p_{k_{n_0}} - degB_{n_0}(z)$ and we consider the function

$$\frac{A_{n_0}(z)}{B_{n_0}(z)} + dz^t = \frac{A_{n_0}(z) + dz^t B_{n_0}(z)}{B_{n_0}(z)}. \tag{4.94}$$

Now, for every $d \in \mathbb{C}$, the polynomials $A_{n_0}(z) + dz^t B_{n_0}(z)$ and $B_{n_0}(z)$ do not have common zeros in $\mathbb{C}$, because the same holds for the polynomials $A_{n_0}(z)$ and $B_{n_0}(z)$. If the parameter $d \in \mathbb{C}$ is close to zero, for instance, if

$$d \cdot \Big(\sup_{z \in L'' \cup K} |z^t|\Big) < \frac{\varepsilon}{2} \tag{4.95}$$

then, it holds

$$\sup_{z \in K} \chi\big(\frac{A_{n_0}(z)}{B_{n_0}(z)} + dz^t, \omega(z)\big) < \varepsilon \tag{4.96}$$

and

$$\sup_{z \in L''} \Big|\frac{A_{n_0}(z)}{B_{n_0}(z)} + dz^t - \omega(z)\Big| < \varepsilon. \tag{4.97}$$

Since $degB_{n_0}(z) < \min\{q_1^{(k_{n_0})}, q_2^{(k_{n_0})}, \cdots, q_{N(k_{n_0})}^{(k_{n_0})}\}$, for $d \in \mathbb{C} \setminus \{0\}$ it holds

(1) $deg(A_{n_0}(z) + dz^t B_{n_0}(z)) = p_{k_{n_0}}$.

(2) According to Proposition 4.3, it holds

$$\frac{A_{n_0}(z)}{B_{n_0}(z)} + dz^t = \frac{A_{n_0}(z) + dz^t B_{n_0}(z)}{B_{n_0}(z)} \in D_{p_{k_{n_0}}, q_j^{(k_{n_0})}}(\zeta) \tag{4.98}$$

for every $j \in \{1, 2, \cdots, N(k_{n_0})\}$ and for every $\zeta \in \mathbb{C}$ such that $B_{n_0}(\zeta) \neq 0$; in particular this holds for every $\zeta \in L$.

(3)
$$[\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t;p_{(k_{n_0})}/q_j^{(k_{n_0})}]_\zeta(z)=\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t \tag{4.99}$$

for every $j\in\{1,2,\cdots,N(k_{n_0})\}$ and for every $\zeta\in L$.

Therefore, we obtain

$$\max_{j\in\{1,2,\cdots,N(k_{n_0})\}}\sup_{\zeta\in L}\sup_{z\in K}\chi\big([\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t;p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z),f_i(z)\big)<\varepsilon \tag{4.100}$$

and

$$\max_{j\in\{1,2,\cdots,N(k_{n_0})\}}\sup_{\zeta\in L}\sup_{z\in L'}\Big|[\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t;p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z)-\big(\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t\big)\Big|=0<\frac{1}{s}<\varepsilon \tag{4.101}$$

In addition, it holds

$$\sup_{z\in L''}\Big|\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t-\phi(z)\Big|<\varepsilon. \tag{4.102}$$

Since the polynomials $A_{n_0}(z)+dz^t$ and $B_{n_0}(z)$ have no common zeros, we have that

$$\min_{z\in L'\cup K}\{|A_{n_0}(z)+dz^tB_{n_0}(z)|^2+|B_{n_0}(z)|^2\}>0. \tag{4.103}$$

One can verify that these polynomials are the ones given by the Jacobi formulas for the function

$$[\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t;p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z) \tag{4.104}$$

for any $\zeta\in\mathbb{C}$ with $B_{n_0}(\zeta)\neq 0$ and for every $j\in\{1,2,\cdots,N(k_{n_0})\}$.

Since every connected component of $\mathbb{C}\cup\{\infty\}\setminus L''$ contains a connected component of $\mathbb{C}\cup\{\infty\}\setminus\Omega$, every zero of $B_{n_0}(z)$ in $\Omega\setminus L''$ lies in the same connected componet of $\mathbb{C}\cup\{\infty\}\setminus L''$ with a point in $\mathbb{C}\cup\{\infty\}\setminus\Omega$. Therefore, we may approximate the function

$$\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t \tag{4.105}$$

by a function $g\in H(\Omega)$. The approximations is uniform on L'' with respect to the Euclidean metric. Since $L\subseteq(L'')^o$, there exists $r>0$ such that

$$\{z\in\mathbb{C}:|z-\zeta|\leq r\}\subseteq(L'')^o \text{ for all } \zeta\in L. \tag{4.106}$$

Now, Cauchy estimates allow us to show that a finite number of Taylor coefficients of g with center $\zeta\in L$ are uniformly close one by one to the corresponding coefficients of the function

$$\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t. \tag{4.107}$$

It is now easy to see that g satisfies all requirements; the only difference of g from the function $\frac{A_{n_0}(z)}{B_{n_0}(z)}+dz^t$ is that it is not true that $[g;p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z)=g(z)$, but instead, the triangle inequality implies that

$$\max_{j\in\{1,2,\cdots,N(k_{n_0})\}}\sup_{\zeta\in L}\sup_{z\in L'}|[g;p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z)-g(z)|$$

$$\leq \max_{j\in\{1,2,\cdots,N(k_{n_0})\}} \sup_{\zeta\in L} \sup_{z\in L'} \Big|[g; p_{k_{n_0}}/q_j^{(k_{n_0})}]_\zeta(z) - \big[\frac{A_{n_0}(z)}{B_{n_0}(z)} + dz^t; p_{k_{n_0}}/q_j^{(k_{n_0})}\big]_\zeta(z)\Big| +$$

$$+ \max_{j\in\{1,2,\cdots,N(k_{n_0})\}} \sup_{z\in L'} \Big|\big[\frac{A(z)}{B(z)} + dz^t; p_{k_{n_0}}/q_j^{(k_{n_0})}\big]_\zeta(z) - g(z)\Big| \tag{4.108}$$

where the last two terms are small enough because $p_{k_{n_0}}$ and $q_j^{(k_{n_0})}$ are already fixed and thus we know which set of Taylor coefficients we have to control.

This part of the proof is now complete.

■■

Since $H(\Omega)$ is a complete metric space, we apply Baire's theorem and that completes the proof.

■

The following result is a consequence of Theorem 4.16.

Theorem 4.19. ([27]) Let $\Omega \subseteq \mathbb{C}$ be an open. We consider a sequence $(p_n)_{n\geq 1} \subseteq \mathbb{N}$ with $p_n \to +\infty$. Now, for every $n \geq 1$ let $q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Suppose that it holds

$$\min\{q_1^{(n)}, q_2^{(n)}, \cdots, q_{N(n)}^{(n)}\} \to +\infty. \tag{4.109}$$

Then there exists a function $f \in H(\Omega)$ such that for every compact set $K \subseteq \mathbb{C}\setminus\Omega$ and for every rational function h, there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ satisfying the following

(1) For every compact set $L \subseteq \Omega$, there exists a $n(L) \in \mathbb{N}$ such that $f \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$ for every $\zeta \in L$, for every $n \geq n(L)$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2) $$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in K} \chi([f; p_{k_n}/q_j^{(k_n)}]_\zeta(z), h(z)) \to 0 \text{ as } n \to +\infty$$

for every compact set $L \subseteq \Omega$.

(3) $$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in L} |[f; p_{k_n}/q_j^{(k_n)}]_\zeta(z) - f(z)| \to 0 \text{ as } n \to +\infty$$

for every compact set $L \subseteq \Omega$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. We apply Theorem 4.16 for $L = L' = L_k$ and for $K = K_m$ and that for every $k, m \geq 1$. In that way we obtain a G_δ - dense class $\mathcal{U}_{k,m}^1$ of $H(\Omega)$. One can verify (by using a diagonal argument) that if $\mathcal{U}^1$ is the class of all functions of $H(\Omega)$ satisfying the above properties, then it holds

$$\mathcal{U}^1 = \bigcup_{n,m\geq 1} \mathcal{U}_{k,m}^1. \tag{4.110}$$

The result follows once more from Baire's theorem.

■

Now we present without proof two results similar to Theorems 4.16 and 4.19 respectively where the roles of p and q have been interchanged.

Theorem 4.20. ([27]) Let $\Omega \subseteq \mathbb{C}$ be an open set and $L, L' \subseteq \Omega$ two compact sets. Let also $K \subseteq \mathbb{C} \setminus \Omega$ be another compact set. We consider a sequence $(q_n)_{n\geq 1} \subseteq \mathbb{N}$ with $q_n \to +\infty$. Now, for every $n \geq 1$ let $p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Suppose also that it holds

$$\min\{p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)}\} \to +\infty. \tag{4.111}$$

Then there exists a function $f \in H(\Omega)$ such that for every rational function h, there exists a subsequence $(q_{k_n})_{n\geq 1}$ of the sequence $(q_n)_{n\geq 1}$ satisfying the following.

(1)
$$f \in D_{p_j^{(k_n)}, q_{k_n}}(\zeta)$$

for every $\zeta \in L$, for every $n \in \mathbb{N}$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in K} \chi([f; p_j^{(k_n)}/q_{k_n}]_\zeta(z), h(z)) \to 0$$

as $n \to +\infty$.

(3)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in L'} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Theorem 4.21. ([27]) Let $\Omega \subseteq \mathbb{C}$ be an open set. We consider a sequence $(q_n)_{n\geq 1} \subseteq \mathbb{N}$ with $q_n \to +\infty$. Now, for every $n \in \mathbb{N}$ let $p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number. Suppose that it holds

$$\min\{p_1^{(n)}, p_2^{(n)}, \cdots, p_{N(n)}^{(n)}\} \to +\infty. \tag{4.112}$$

Then there exists a function $f \in H(\Omega)$ such that for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ and for every rational function h, there exists a subsequence $(q_{k_n})_{n\geq 1}$ of the sequence $(q_n)_{n\geq 1}$ satisfying the following

(1) For every compact set $L \subseteq \Omega$, there exists a $n(L) \in \mathbb{N}$ such that $f \in D_{p_j^{(k_n)}, q_{k_n}}(\zeta)$ for every $n \geq n(L)$ and for every $j \in \{1, \cdots, N(k_n)\}$.

(2)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{z\in K} \chi([f; p_j^{(k_n)}/q_{k_n}]_\zeta(z), h(z)) \to 0$$

as $n \to +\infty$ for every compact set $L \subseteq \Omega$.

(3)
$$\max_{j=1,\cdots,N(k_n)} \sup_{\zeta\in L} \sup_{\zeta\in L} |[f; p_j^{(k_n)}/q_{k_n}]_\zeta(z) - f(z)| \to 0$$

as $n \to +\infty$ for every compact set $L \subseteq \Omega$.

Moreover, the set of all functions f satisfying the above properties is G_δ - dense in $H(\Omega)$.

Proof. We apply Theorem 4.21 for $L = L' = L_k$ and for $K = K_m$ and that for every $k, m \geq 1$. In that way we obtain a G_δ - dense class $\mathcal{U}^2_{k,m}$ of $H(\Omega)$. One can verify (by using a diagonal argument) that if $\mathcal{U}^2$ is the class of all functions of $H(\Omega)$ satisfying the above properties, then it holds

$$\mathcal{U}^2 = \bigcup_{n,m \geq 1} \mathcal{U}^2_{k,m}. \tag{4.113}$$

The result follows once more from Baire's theorem.

■

4.4 Affine genericity of a class of functions

In this section we deal with the class of functions on a simply connected domain $\Omega \subseteq \mathbb{C}$ which satisfy the requirements of Theorem 4.8 for a fixed center of expansion $\zeta \in \Omega$: the class $\mathcal{A}$.

We construct a particular function f in the above class. Our construction is based on the observation that $[f;p/q]_\zeta(z) \equiv S_p(f,\zeta)(z)$ with $q \geq 1$ if and only if $a_{p+1} = a_{p+2} = \cdots = a_{p+q} = 0$, where, of course, $f(z) = \sum_{n=0}^{+\infty} a_n(z-\zeta)^n$ is the Taylor expansion of the function f, centered at $\zeta \in \Omega$.

We recall the following definition.

Definition 4.22 (Affine genericity). Let $\Omega \subseteq \mathbb{C}$ be an open set. We consider the space $H(\Omega)$ endowed with its natural topology and let $\mathcal{A} \subseteq H(\Omega)$. Let $V \leq H(\Omega)$ be a dense subspace of $H(\Omega)$ and $g \in H(\Omega)$ such that $g + V \subseteq \mathcal{A}$. Then the class $\mathcal{A}$ is called (densely) affinely generic; that is it contains the translation of a dense subspace.

Theorem 4.23. ([27]) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $\zeta \in \Omega$ be a fixed point. Let also $(p_n)_{n\geq 1} \subseteq \mathbb{N}$ be a sequence such that $p_n \to +\infty$. Now, for every $n \geq 1$, let $q_1^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$, where $N(n)$ is another natural number.

Then, there exists a function $f \in H(\Omega)$, with Taylor expansion at ζ of the form $f(z) = \sum_{n=0}^{+\infty} a_n(z-\zeta)^n$ satisfying the following.

For every compact set $K \subseteq \mathbb{C}\backslash\Omega$ with connected complement and for every function $h \in A(K)$, there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ such that

(1) $\sup\limits_{z\in K} |S_{p_{k_n}}(f,\zeta)(z) - h(z)| \to 0$, as $n \to +\infty$.

(2) $\sup\limits_{z\in J} |S_{p_{k_n}}(f,\zeta)(z) - f(z)| \to 0$, as $n \to +\infty$ for every compact set $J \subseteq \Omega$.

Furthermore, for every $n \geq 1$ it holds $a_{p_{k_n}} \neq 0$ and $a_{p_{k_n}+s} = 0$ for every $s = 1, \cdots, \max\{q_1^{(k_n)}, \cdots, q_{N(k_n)}^{(k_n)}\}$.

Proof. Let $\{f_j\}_{j\geq 1}$ be an enumeration of polynomials with coefficients of $\mathbb{Q} + i\mathbb{Q}$. Let also $\{K_m\}_{m\geq 1}$ and $\{L_k\}_{k\geq 1}$ be two fixed families of compact subsets of $\mathbb{C}$ satisfying Lemmas 4.4 and 4.5 respectively. The set $\{(K_m, f_j) : m, j \geq 1\}$ is infinite denumerable and thus we consider a function $t : \mathbb{N} \to \mathbb{N}$ such that $\{(K_m, f_j) : m, j \geq 1\} = \{(K_{m_t}, f_{j_t}) : t \geq 1\}$, where we suppose that each pair (K_{m_t}, f_{j_t}) appears infinitely many times[1].

Step 1. We consider the function

$$w_1(z) = \begin{cases} f_{j_1}(z), & \text{if } z \in K_{m_1} \\ 0, & \text{if } z \in L_1. \end{cases} \tag{4.114}$$

Notice that $w_1 \in A(K_{m_1} \cup L_1)$. We apply Mergelyan's theorem and in this way we find a polynomial h_1 such that

$$\sup_{z \in K_{m_1} \cup L_1} |w_1(z) - h_1(z)| < 1. \tag{4.115}$$

[1] In any other case, see Remark 3.21 of [27].

We consider an index $k_1 \geq 1$ such that $degh_1(z) < p_{k_1}$. Next, we select a $c_1 \in \mathbb{C} \setminus \{0\}$ such that

$$\sup_{z \in K_{m_1} \cup L_1} |w_1(z) - (h_1(z) + c_1(z-\zeta)^{p_{k_1}})| < 1. \tag{4.116}$$

Notice that such a choice is possible. We set $H_1(z) = h_1(z) + c_1(z-\zeta)^{p_{k_1}}$. Clearly, the function H_1 is a polynomial with $degH_1(z) = p_{k_1}$. Finally, we select a $t_1 \geq 1 + \max\{q_1^{(1)}, \cdots, q_{N(1)}^{(1)}\}$.

Step 2. We consider the function

$$w_2(z) = \begin{cases} \dfrac{f_{j_2}(z) - H_1(z)}{(z-\zeta)^{p_{k_1}+t_1}}, & \text{if } z \in K_{m_2} \\ 0, & \text{if } z \in L_2. \end{cases} \tag{4.117}$$

Notice that $w_2 \in A(K_{m_2} \cup L_2)$. We apply Mergelyan's theorem and in this way we find a polynomial h_2 such that

$$\sup_{z \in K_{m_2} \cup L_2} |w_2(z) - h_2(z)| < \frac{1}{2^2} \cdot \frac{1}{\max\limits_{z \in K_{m_2} \cup L_2} |z-\zeta|^{p_{k_1}+t_1} + 1}. \tag{4.118}$$

We consider an index $k_2 \geq 1$ such that $deg\big(h_2(z) \cdot (z-\zeta)^{p_{k_1}+t_1}\big) < p_{k_2}$. Next, we select a $c_2 \in \mathbb{C} \setminus \{0\}$ such that

$$\sup_{z \in K_{m_2} \cup L_2} \big|(z-\zeta)^{p_{k_1}+t_1} \cdot \big(w_2(z) - h_2(z)\big) - c_2(z-\zeta)^{p_{k_2}}\big]\big| < \frac{1}{2^2}. \tag{4.119}$$

Notice that such a choice is possible. We set $H_2(z) = (z-\zeta)^{p_{k_1}+t_1} h_2(z) + c_2(z-\zeta)^{p_{k_2}}$. Clearly, the function H_2 is a polynomial with $degH_2(z) = p_{k_2}$. Finally, we select $t_2 \geq 1 + \max\{q_1^{(2)}, \cdots, q_{N(2)}^{(2)}\}$.

Step n. So far, we have defined the polynomials $H_1, \cdots, H_{n-1}$ with $degH_i(z) = p_{k_i}$ for every $i = 1, \cdots, n-1$. We consider the function

$$w_n(z) = \begin{cases} \dfrac{f_{j_n}(z) - (H_1(z) + \sum\limits_{N=1}^{n-2} (z-\zeta)^{p_{k_N}+t_N} H_{N+1})}{(z-\zeta)^{p_{k_{n-1}}+t_{n-1}}}, & \text{if } z \in K_{m_n} \\ 0, & \text{if } z \in L_n. \end{cases} \tag{4.120}$$

Notice that $w_n \in A(K_{m_n} \cup L_n)$. We apply Mergelyan's theorem and in this way we find a polynomial h_n such that

$$\sup_{z \in K_{m_n} \cup L_n} |w_n(z) - h_n(z)| < \frac{1}{n^2} \cdot \frac{1}{\max\limits_{z \in K_{m_n} \cup L_n} |z-\zeta|^{p_{k_{n-1}}+t_{n-1}} + 1}. \tag{4.121}$$

We consider an index $k_n \geq 1$ such that

$$deg\big((z-\zeta)^{p_{k_{n-1}}+t_{n-1}} \cdot h_n(z)\big) < p_{k_n}. \tag{4.122}$$

Next, we select a $c_n \in \mathbb{C} \setminus \{0\}$ such that

$$\sup_{z\in K_{m_n}\cup L_n} \left|(z-\zeta)^{p_{k_{n-1}}+t_{n-1}} \cdot \big(w_n(z)-h_n(z)\big) - c_n(z-\zeta)^{p_{k_n}}]\right| < \frac{1}{n^2}. \tag{4.123}$$

Notice that such a choice is possible. We set $H_n(z) = (z-\zeta)^{p_{k_{n-1}}} h_n(z) + c_n(z-\zeta)^{p_{k_n}}$. Clearly, the function H_n is a polynomial with $degH_n(z) = p_{k_n}$. Finally, we select a $t_n \geq 1 + max\{q_1^{(n)}, \cdots, q_{N(n)}^{(n)}\}$.

From Weierstrass's theorem, the sequence of polynomials

$$H_1(z) + H_2(z) + \cdots + H_n(z) + \cdots \tag{4.124}$$

converges uniformly on every compact subset $J \subseteq \Omega$ to a function $f \in H(\Omega)$. We notice that from Cauchy's integral formula, for every $n \geq 1$ it holds

$$S_{p_{k_n}}(f,\zeta)(z) = H_1(z) + \sum_{N=1}^{p_{k_n}-1} H_{N+1}(z). \tag{4.125}$$

We will show that the function f meets the requirements of Theorem 4.23.

- Let $J \subseteq \Omega$ be a compact set. We consider an index $k \geq 1$ such that $J \subseteq L_k$. Then

$$\sup_{z\in L_k} |S_{p_{k_n}}(f,\zeta)(z) - f(z)| \leq \sum_{s=M(n)}^{+\infty} \frac{1}{s^2} \to 0 \tag{4.126}$$

as $n \to +\infty$, where $M(n) \geq 1$ and $M(n) \to +\infty$ as $n \to +\infty$ as well.

- Let $K \subseteq \mathbb{C} \setminus \Omega$ be a compact set with connected complement, $h \in A(K)$ and $\varepsilon > 0$. We consider an index $m \geq 1$ such that $K \subseteq K_m$. We use Mergelyan's theorem in order to find a polynomial f_j such that

$$\sup_{z\in K} |h(z) - f_j(z)| < \frac{\varepsilon}{3}. \tag{4.127}$$

Then, according to our initial hypothesis, $(K_m, f_j) = (K_{m_t}, f_{j_t})$ for infinitely many $t \geq 1$. Also, according to the construction of f, it holds

$$\sup_{z\in K_{m_t}} |f_{j_t}(z) - S_{p_{k_{j_t}}}(f,\zeta)(z)| < \frac{1}{t^2}. \tag{4.128}$$

for infinitely many $t \geq 1$. Equivalently,

$$\sup_{z\in K_m} |f_{j_t}(z) - S_{p_{k_{j_t}}}(f,\zeta)(z)| < \frac{1}{t^2}. \tag{4.129}$$

for infinitely many $t \geq 1$. For $t \to +\infty$, we select a $t_0 \geq 1$ such that

$$\sup_{z \in K_m} |f_{j_{t_0}}(z) - S_{p_{k_{j_{t_0}}}}(f,\zeta)(z)| < \frac{1}{t^2} < \frac{\varepsilon}{3}. \tag{4.130}$$

The triangle inequality yields now the result.

■

As we have already commented before, if $\mathcal{U}$ is the class of all functions satisfying Theorem 4.8 for a fixed center of expansion and f is the function constructed in Theorem 4.23, then it holds $f \in \mathcal{U}$.

Definition 4.24. We denote with $\mathcal{B} \equiv \mathcal{B}(\mathcal{A}^{(1)})$ the set of all functions satisfying Theorem 4.23.

Proposition 4.25. ([27]) Let $f \in \mathcal{B}(\mathcal{A}^{(1)})$ and p be a polynomial. Then $f + p \in \mathcal{B}(\mathcal{A}^{(1)}) \subseteq \mathcal{U}$. Thus, the class $\mathcal{U}$ contains an affine dense subspace of $H(\Omega)$. It follows that $\mathcal{U}$ is affinely generic.

Proof. This is obvious, according to the construction of the class $\mathcal{B}(\mathcal{U}^{(1)}) \subseteq \mathcal{U}$, since the condition $a_{p_{k_n}+s} = 0$ for every $s = 1, \cdots, \max\{q_1^{(k_n)}, \cdots, q_{N(k_n)}^{(k_n)}\}$ implies that $[f; p_{k_n}/q_j^{(k_n)}]_\zeta \equiv S_{p_{k_n}}(f,\zeta)$ for every $j = 1, \cdots, M(k_n)$. The previous relations combined with the fact that $a_{p_{k_n}} \neq 0$ for every $n \geq 1$ imply that $f \in D_{p_{k_n}, q_j^{(k_n)}}(\zeta)$ for every $j = 1, \cdots, M(k_n)$.

■

We will now strengthen Theorem 4.23. We consider a finite of infinite denumerable family of systems

$$\mathcal{A}^{(l)} = \left((p_n^{(l)})_{n\geq 1}, N(l,n), q_{l,i}^{(n)} \text{ for } i = 1, \cdots, N(l,n)\right), l \in I \tag{4.131}$$

where $I = \mathbb{N}$ or I is finite. As expected, for every $l \in I$ it holds

(i) $(p_n^{(l)})_{n\geq 1} \subseteq \mathbb{N}$

(ii) $p_n^{(l)} \to +\infty$ as $n \to +\infty$

(iii) $+\infty > N(l,n) \geq 1$ for every $n \geq 1$

(iv) $q_{l,i}^{(n)} \in \mathbb{N}$ for every $i = 1, \cdots, N(l,n)$

(v) $\max\{q_{l,1}^{(n)}, \cdots, q_{l,N(l,n)}^{(n)}\} \to +\infty$ as $n \to +\infty$.

Each system defines a new class of functions, namely the class $\mathcal{B}(\mathcal{A}^{(l)})$, according to Theorem 4.23.

We will now show that the class

$$\bigcap_{l \in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.132}$$

is a dense subset of $H(\Omega)$, for I a finite or an infinite denumerable set.

Theorem 4.26. ([27]) Let $I \neq \emptyset$ be a finite or infinite denumerable set. We consider a family of systems $\{\mathcal{A}^{(l)}\}_{l\in I}$ as above. Then the class

$$\bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.133}$$

is a dense subset of $H(\Omega)$ and every function

$$f \in \bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.134}$$

with Taylor expansion at ζ of the form $f(z) = \sum_{n=0}^{+\infty} a_n(z-\zeta)^n$ satisfies the following.

For every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement, for every function $h \in A(K)$ and for every $l \in I$, there exists a subsequence $(p^{(l)}_{k_n(l)})_{n\geq 1}$ of the sequence $(p^{(l)}_n)_{n\geq 1}$ such that

(1) $\sup_{z\in K} |S_{p^{(l)}_{k_n(l)}}(f,\zeta)(z) - h(z)| \to 0$, as $n \to +\infty$.

(2) $\sup_{z\in J} |S_{p^{(l)}_{k_n(l)}}(f,\zeta)(z) - f(z)| \to 0$, as $n \to +\infty$ for every compact set $J \subseteq \Omega$.

Furthermore, for every $l \in I$ and for every $n \geq 1$ it holds $a_{p^{(l)}_{k_n(l)}} \neq 0$ and $a_{p^{(l)}_{k_n(l)}+s} = 0$ for every $s = 1, \cdots, \max\{q^{(k_n(l))}_{l,1}, \cdots, q^{(k_n(l))}_{l,N(l,k_n(l))}\}$.

Proof. The proof is based on that of Theorem 4.23. We will not provide in-depth details but instead, we will give a sketch of the proof, explaining the main idea behind it. We examine each one of the two cases concerning the cardinality of the set I separately.

(1) Suppose that the set I **is finite**. In this case, without loss of the generality, we suppose that it holds $I = \{1, 2, \cdots, N\}$, for an index $N \geq 1$.

Sequence \ Selection	$(p^{(1)}_n)_{n\geq 1}$	$(p^{(2)}_n)_{n\geq 1}$	$\cdots$	$(p^{(N)}_n)_{n\geq 1}$
Step 1	$\bullet$	-	$\cdots$	-
Step 2	-	$\bullet$	$\cdots$	-
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Step N	-	-	$\cdots$	$\bullet$
Step $N+1$	$\bullet$	-	$\cdots$	-
Step $N+2$	-	$\bullet$	$\cdots$	-
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Step $2N$	-	-	$\cdots$	$\bullet$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

Figure 5: A natural choice in the case where I is a finite set.

In each step of the proof we repeat the same arguments as we did in the one of Theorem 4.23, but this time, the specific argument is used for a different sequence. See Figure 5 for details. The class

$$\bigcap_{l \in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.135}$$

is proved to be a dense subset of $H(\Omega)$ in the same way. In addition, every function

$$f \in \bigcap_{l \in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.136}$$

meets the requirements of Theorem 4.26 in the same way as in Theorem 4.23.

Remark 4.27. A more general selection from the above is the following.

We fix a permutation (i.e. a 1 - 1 and onto function) $\sigma : \{1, \cdots, N\} \to \{1, \cdots, N\}$ and we repeat the same construction as above, but this time, in every "block" of N steps, starting from step $kN + 1$ and stopping at step $(k + 1)N$ (where apparently $k \geq 0$), the $kN + j$ term of the sequence is chosen from the sequence $(p_n^{(\sigma(j+1))})_{n \geq 1}$. See for instance the following table as an example.

Sequence \ Selection	$(p_n^{(1)})_{n\geq 1}$	$(p_n^{(2)})_{n\geq 1}$	$(p_n^{(3)})_{n\geq 1}$	$(p_n^{(4)})_{n\geq 1}$	$(p_n^{(5)})_{n\geq 1}$	$(p_n^{(6)})_{n\geq 1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Step $kN + 1$	-	•	-	-	-	-
Step $kN + 2$	•	-	-	-	-	-
Step $kN + 3$	-	-	•	-	-	-
Step $kN + 4$	-	-	-	-	-	•
Step $kN + 5$	-	-	-	•	-	-
Step $(k + 1)N$	-	-	-	-	•	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 6: A more general selection in the case where I is a finite set (here, $N = 6$).

(2) Suppose that the set I **is infinite denumerable**. In this case, without loss of the generality, we suppose that it holds $I = \mathbb{N}$.

In each step of the proof we repeat the same arguments as we did in the one of Theorem 4.23, but this time, the specific argument is used for a different sequence. See the following table for details.

Sequence \ Selection	$(p_n^{(1)})_{n\geq 1}$	$(p_n^{(2)})_{n\geq 1}$	$(p_n^{(3)})_{n\geq 1}$	$\cdots$	$(p_n^{(N)})_{n\geq 1}$	$(p_n^{(N+1)})_{n\geq 1}$	$\cdots$
Step 1	$\bullet$	-	-	$\cdots$	-	-	$\cdots$
Step 2	$\bullet$	$\bullet$	-	$\cdots$	-	-	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\cdots$
Step N	$\bullet$	$\bullet$	$\cdots$	$\cdots$	$\bullet$	-	$\cdots$
Step $N+1$	$\bullet$	$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\bullet$	-
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\ddots$	$\vdots$	$\cdots$	$\cdots$

Figure 7: A natural selection in the case where I is infinitely denumerable.

The class

$$\bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.137}$$

is proved to be a dense subset of $H(\Omega)$ in the same way. In addition, every function

$$f \in \bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.138}$$

meets the requirements of Theorem 4.26 in the same way as in Theorem 4.23.

■

Proposition 4.28. ([27]) Let $I \neq \emptyset$ be a finite or infinite denumerable set, $\{\mathcal{A}^{(l)}\}_{l\in I}$ a family of systems as above,

$$g \in \bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.139}$$

and p be a polynomial. Then, it holds

$$g + p \in \bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}). \tag{4.140}$$

It follows that the class

$$\bigcap_{l\in I} \mathcal{B}(\mathcal{A}^{(l)}) \tag{4.141}$$

is a dense subset of $H(\Omega)$.

Proof. The proof is the same as the one of Proposition 4.25 and therefore is omitted.
■

4.5 Algebraic genericity of a class of functions

In this section we deal with a class of functions on a simply connected domain $\Omega \subseteq \mathbb{C}$ that "almost" satisfy Theorem 4.8 for a fixed center of expansion $\zeta \in \Omega$. There are two differences.

(1) This time, we do not require the uniqueness of Padé approximants but instead, we only require their existence.

(2) We consider a slight change concerning our choice of indices, which is more general than the one we have used before.

Therefore, this new class of functions is larger than the one satisfying Theorem 4.8, mainly due to condition (1) above.

We recall the following definition.

Definition 4.29 (Algebraic genericity). Let $\Omega \subseteq \mathbb{C}$ be an open set. We consider the space $H(\Omega)$ endowed with its natural topology and let $\mathcal{A} \subseteq H(\Omega)$. Suppose that there exists a dense subspace $V \leq H(\Omega)$ such that $V \setminus \{0\} \subseteq \mathcal{A}$. Then, the class $\mathcal{A}$ is algebraically generic in $H(\Omega)$; that is, it contains a dense subspace of $H(\Omega)$ (except 0).

Theorem 4.30. ([27]) Let $(p_n)_{n\in\mathbb{N}} \geq 1$ with $p_n \to +\infty$. For every $n \geq 1$, let $N(n)$ be a natural number and $q_1^{(n)}, \cdots, q_{N(n)}^{(n)} \in \mathbb{N}$. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $\zeta \in \Omega$ be a fixed element. Then, there exists a function $f \in H(\Omega)$ with Taylor expansion at ζ of the form

$$f(z) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n \tag{4.142}$$

such that for every polynomial h and for every compact set $K \subseteq \mathbb{C} \setminus \Omega$ with connected complement there exists a subsequence $(p_{k_n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ satisfying the following.

(i) The Padé approximant $[f; p_{k_n}/q_{\sigma(k_n)}^{(k_n)}]_\zeta$ exists for every $n \geq 1$ and for every selection $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ satisfying $\sigma(k_n) \in \{1, \cdots, N(k_n)\}$ for every $n \geq 1$.

(ii)
$$\sup_{z\in K} |[f; p_{k_n}/q_{\sigma(k_n)}^{(k_n)}]_\zeta(z) - h(z)| \to 0$$

for every selection $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ satisfying $\sigma(k_n) \in \{1, \cdots, N(k_n)\}$ for every $n \geq 1$.

(iii)
$$\sup_{z\in J} |[f; p_{k_n}/q_{\sigma(k_n)}^{(k_n)}]_\zeta(z) - f(z)| \to 0$$

for every compact set $J \subseteq \Omega$ and for every selection $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ satisfying $\sigma(k_n) \in \{1, \cdots, N(k_n)\}$ for every $n \geq 1$.

We will now prove that there exists a dense subspace V of $H(\Omega)$ such that $V \setminus \{0\} \subseteq \mathcal{U}'$, where $\mathcal{U}'$ is the class of all functions $f \in H(\Omega)$ satisfying Theorem 4.30. This by definition implies that the class $\mathcal{U}'$ is algrebraically generic in $H(\Omega)$.

Theorem 4.31. ([27]) There exists a dense vector subspace V of $H(\Omega)$ such that $V \setminus \{0\} \subseteq \mathcal{U}'$.

Proof. Let $\{K_m\}_{m\geq 1}$ and $\{L_k\}_{k\geq 1}$ be two fixed families of compact subsets of $\mathbb{C}$ satisfying Lemmas 4.4 and 4.5 respectively for the specific set Ω. Let also $\{f_i\}_{i\geq 1}$ be an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

Step 1. We consider a function $g_1 \in \mathcal{B}(\mathcal{A}^{(1)})$ satisfying the following properties.

(1) $\rho(g_1, f_1) < \frac{1}{1}$.

(2) For every $m \geq 1$, there exists a subsequence $(p^{(1)}_{m,n})_{n\geq 1}$ of the sequence $(p_n)_{n\geq 1}$ such that

$$\sup_{z\in K_m} |S_{p^{(1)}_{m,n}}(g_1, \zeta)(z) - 0| \to 0 \text{ as } n \to +\infty. \tag{4.143}$$

$$\sup_{z\in J} |S_{p^{(1)}_{m,n}}(g_1, \zeta)(z) - g_1(z)| \to 0 \text{ as } n \to +\infty \tag{4.144}$$

for every compact $J \subseteq \Omega$.

So, at the end of Step 1 we have constructed infinitely many subsequences of the sequence $(p_n)_{n\geq 1}$; the sequences $(p^{(1)}_{m,n})_{n\geq 1}$ for $m \geq 1$.

Step 2. We consider the system

$$\mathcal{A}^{((1),m)} = \big((p^{(1)}_{m,n})_{n\geq 1}, N(((1), m), n), q^{(n)}_{i,((1),m)}$$
$$\text{for } i = 1, \cdots, N(((1), m), n)\big), m \geq 1. \tag{4.145}$$

According to Theorem 4.26, there exists a function

$$g_2 \in \bigcap_{m\in\mathbb{N}} \mathcal{B}(\mathcal{A}^{((1),m)}) \tag{4.146}$$

satisfying the following properties.

(1) $\rho(g_2, f_2) < \frac{1}{2}$.

(2) For every $m \geq 1$ there exists a subsequence $(p^{(2)}_{m,n})_{n\geq 1}$ of the sequence $(p^{(1)}_{m,n})_{n\geq 1}$ such that

$$\sup_{z\in K_m} |S_{p^{(2)}_{m,n}}(g_2, \zeta)(z) - 0| \to 0 \text{ as } n \to +\infty. \tag{4.147}$$

$$\sup_{z\in J} |S_{p^{(2)}_{m,n}}(g_2, \zeta)(z) - g_2(z)| \to 0 \text{ as } n \to +\infty \tag{4.148}$$

for every compact $J \subseteq \Omega$.

So, at the end of Step 2 we have constructed infinitely many subsequences of the sequence $(p_n)_{n\geq 1}$, since for every $m \geq 1$ the sequence $(p^{(2)}_{m,n})_{n\geq 1}$ is a subsequence of $(p^{(1)}_{m,n})_{n\geq 1}$.

Step N. We consider the system

$$\mathcal{A}^{((N),m)} = \big((p_{m,n}^{(N-1)})_{n\geq 1}, N(((N-1),m),n), q_{i,((N-1),m)}^{(n)}$$

$$\text{for } i = 1,\cdots, N(((N-1),m),n\big), m \geq 1. \tag{4.149}$$

According to Theorem 4.26, there exists a function

$$g_N \in \bigcap_{m\in\mathbb{N}} \mathcal{B}(\mathcal{A}^{((N),m)}) \tag{4.150}$$

satisfying the following properties.

(1) $\rho(g_N, f_N) < \frac{1}{N}$.

(2) For every $m \geq 1$ there exists a subsequence $(p_{m,n}^{(N)})_{n\geq 1}$ of the sequence $(p_{m,n}^{(N-1)})_{n\geq 1}$ such that

$$\sup_{z\in K_m} |S_{p_{m,n}^{(N)}}(g_N,\zeta)(z) - 0| \to 0 \text{ as } n \to +\infty. \tag{4.151}$$

$$\sup_{z\in J} |S_{p_{m,n}^{(N)}}(g_N,\zeta)(z) - g_N(z)| \to 0 \text{ as } n \to +\infty \tag{4.152}$$

for every compact $J \subseteq \Omega$.

So, at the end of Step N we have constructed infinitely many subsequences of the sequence $(p_n)_{n\geq 1}$, since for every $m \geq 1$ the sequence $(p_{m,n}^{(N)})_{n\geq 1}$ is a subsequence of $(p_{m,n}^{(N-1)})_{n\geq 1}$.

We consider now the linear span $< g_n : n \geq 1 > \subseteq H(\Omega)$. Let $1 \leq j_1 < \cdots < j_s$ and $a_{j_1}, \cdots, a_{j_s} \in \mathbb{C} \setminus \{0\}$. We set $g \equiv a_{j_1} g_{j_1} + \cdots + a_{j_s} g_{j_s}$. Our aim is to prove that the function g is a universal Taylor series and belong to the class $\mathcal{U}'$.

Let $K \subseteq \mathbb{C} \setminus \Omega$ be a compact set with connected complement and $h \in A(K)$. We consider an index $m \geq 1$ such that $K \subseteq K_m$. Since

$$g_{j_s} \in \bigcap_{m\in\mathbb{N}} \mathcal{B}(\mathcal{A}^{((j_s),m)}) \tag{4.153}$$

there exists a subsequence $(p_{k_n}^{(j_s)})_{n\geq 1}$ of the sequence $(g_{m,n}^{(j_s)})_{n\geq 1}$ such that

(1) $$\sup_{z\in K_m} |S_{p_{k_n}^{(j_s)}}(g_{j_s},\zeta)(z) - \frac{h(z)}{a_{j_s}}| \to 0 \text{ as } n \to +\infty.$$

(2) $$\sup_{z\in J} |S_{p_{k_n}^{(j_s)}}(g_{j_s},\zeta)(z) - g_{j_s}(z)| \to 0 \text{ as } n \to +\infty$$

for every compact set $J \subseteq \Omega$.

Since $(p_{k_n}^{(j_s)})_{n\geq 1}$ is a subsequence of the sequence $(g_{m,n}^{(j_s)})_{n\geq 1}$ we obtain that for every $t < s$ it holds

(1) $$\sup_{z\in K_m} |S_{p_{k_n}^{(j_s)}}(g_{j_t},\zeta)(z) - 0| \to 0 \text{ as } n \to +\infty.$$

(2) $$\sup_{z\in J} |S_{p_{k_n}^{(j_s)}}(g_{j_t},\zeta)(z) - g_{j_t}(z)| \to 0 \text{ as } n \to +\infty$$

for every compact set $J \subseteq \Omega$.

Thus, it follows that

(1) $$\begin{aligned} &\sup_{z\in K_m} |S_{p_{k_n}^{(j_s)}}(g,\zeta)(z) - h(z)| \\ &= \sup_{z\in K_m} |S_{p_{k_n}^{(j_s)}}(a_{j_1}g_{j_1} + \cdots + a_{j_s}g_{j_s},\zeta)(z) - h(z)| \\ &= \sup_{z\in K_m} |a_{j_1}S_{p_{k_n}^{(j_s)}}(g_{j_1},\zeta)(z) + \cdots + a_{j_s}S_{p_{k_n}^{(j_s)}}(g_{j_s},\zeta)(z) - h(z)| \\ &\leq \sup_{z\in K_m} |a_{j_1}S_{p_{k_n}^{(j_s)}}(g_{j_1},\zeta)(z)| + \cdots + \sup_{z\in K_m} |a_{j_{s-1}}S_{p_{k_n}^{(j_s)}}(g_{j_{s-1}},\zeta)(z)| + \\ &\quad + \sup_{z\in K_m} |a_{j_1}S_{p_{k_n}^{(j_s)}}(g_{j_s},\zeta)(z) - h(z)| \to 0 \end{aligned} \tag{4.154}$$

as $n \to +\infty$.

(2) $$\sup_{z\in J} |S_{p_{k_n}^{(j_s)}}(g,\zeta)(z) - g(z)| \to 0 \tag{4.155}$$

as $n \to +\infty$, for every compact set $J \subseteq \Omega$.

Finally, we notice that for every $n \geq 1$ it holds

$$[g; p_{k_n}^{(j_s)}/q_{(j_s),i}^{(k_n)}]_\zeta(z) = S_{p_{k_n}^{(j_s)}}(g,\zeta)(z) \tag{4.156}$$

for every $i = 1, \cdots N((j_s), k_n)$. That completes the proof.

■

Remark 4.32. In the above construction we may have that $a_{p_{k_n}} = 0$ for some $n \geq 1$, which would imply that $g \notin D_{p_{k_n},p_{k_n}}(\zeta)$ and also that $g \notin \mathcal{U}$. But still, we have that it holds$g \in \mathcal{U}'$.

Remark 4.33. If $q_n^{(i)} \neq 0$, then every function belonging to the class $\mathcal{U}'$ has some Taylor coefficients equal to zero. It follows that $\mathcal{U}'$ is meager in $H(\Omega)$. Since the set of universal Taylor series is G_δ - dense in $H(\Omega)$, it follows that the result of Theorem 4.31 can not be deduced from the known results about algebraic genericity of the set of universal Taylor series.

4.6 Spaceability of a class of functions

In this section we consider a simply connected domain $\Omega \subseteq \mathbb{C}$, a fixed element $\zeta \in \Omega$ and the respective class $\mathcal{U}'$ of Theorem 4.30 and we prove that the class $\mathcal{U}' \cup \{0\} \subseteq H(\Omega)$ is spaceable.

In [4] F. Bayart proved initially the spaceability of the class of Universal Taylor series in the sense of V. Nestoridis. In [7] S. Charpentier gave afterwards similar results in a more general framework, which were later improved in [30] by Q. Menet. Also, relevant results were later developed in [8] by S. Charpentier and A. Mouze.

We will need the following definitions and lemmas.

Definition 4.34 (Spaceability). Let $\Omega \subseteq \mathbb{C}$ be an open set. We consider the space $H(\Omega)$ endowed with its natural topology and let $\mathcal{A} \subseteq H(\Omega)$. Suppose that there exists a closed subspace $V \leq H(\Omega)$ of infinite dimension such that $V \setminus \{0\} \subseteq \mathcal{A}$. Then, the class $\mathcal{A}$ is spaceable in $H(\Omega)$; that is, it contains a closed subspace of infinite dimension (except 0).

Definition 4.35 (Basic sequence of a Fréchet space, [9]). Let X be a Fréchet space over a field $\mathbb{K}$ and $\{u_n\}_{n\geq 0} \subseteq X$. The sequence $\{u_n\}_{n\geq 0}$ is called a basic sequence, if it is a Schauder basis of the set $\overline{< u_n : n \geq 0 >}$; that is, if every element $x \in < u_n : n \geq 0 >$ has a unique representation in X of the following form

$$x = \sum_{n=0}^{+\infty} a_n u_n \tag{4.157}$$

for a sequence $\{a_n\}_{n\geq 0} \subseteq \mathbb{K}$.

Lemma 4.36 (Lemma 4.18 of [9], Lemme 2.2 of [30]). Let X be a Fréchet space over a field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ with a continuous norm, $\{\rho_n\}_{n\geq 0}$ be an increasing sequence of continuous norms defining its topology and $\{\varepsilon_n\}_{n\geq 0}$ be a sequence in $(0, +\infty)$ such that

$$B = \prod_{n=0}^{+\infty} (1 + \varepsilon_n) < +\infty. \tag{4.158}$$

If $\{u_n\}_{n\geq 0} \subseteq X$ is a sequence such that for every $n \geq 0$, for every $0 \leq j \leq n$ and for every $a_0, \cdots, a_{n+1} \in \mathbb{K}$ the following property holds

$$p_j(\sum_{k=0}^{n} a_k u_k) \leq (1 + \varepsilon_n) p_j(\sum_{k=0}^{n+1} a_k u_k) \tag{4.159}$$

then $\{u_n\}_{n\geq 0}$ is a basic sequence in X.

Definition 4.37 (Constant of basicity, [9]). In the previous lemma (Lemma 4.36), the infimum of the constant B satisfying Properties (4.158) and (4.159) is called constant of basicity of the sequence $\{u_n\}_{n\geq 0}$.

Lemma 4.38 (Lemma 4.10 of [9], Lemme 2.3 of [30]). Let X be a Fréchet space over a field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ with a continuous norm, $\{\rho_n\}_{n\geq 0}$ be an increasing sequence of continuous norms defining its topology and M an infinite dimensional subspace of X.

Then, for every $\varepsilon > 0$, for every $u_0, \cdots, u_n \in X$, there exists a $u_{n+1} \in M$ such that $\rho_1(u_{n+1}) = 1$ and also, for every $0 \leq j \leq n$ and for every $a_0, \cdots, a_{n+1} \in \mathbb{K}$ it holds

$$p_j(\sum_{k=0}^{n} a_k u_k) \leq (1+\varepsilon_n) p_j(\sum_{k=0}^{n+1} a_k u_k) \tag{4.160}$$

Definition 4.39 (Equivalent sequences of a Fréchet space, Definition 4.20 of [9]). Two basic sequences $\{g_n\}_{n\geq 0}$ and $\{f_n\}_{n\geq 0}$ of a Fréchet space X are equivalent if for any sequence $\{a_n\}_{n\geq 0} \subseteq \mathbb{K}$ it holds

$$\Big[\text{the sequence } \sum_{k=0}^{+\infty} a_k g_k \text{ converges in} X\Big] \iff \Big[\text{the sequence } \sum_{k=0}^{+\infty} a_k f_k \text{ converges in} X\Big]. \tag{4.161}$$

Lemma 4.40 (Lemma 4.21 of [9], Lemme 2.5 of [30]). Let X be a Fréchet space over a field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ with a continuous norm, $\{\rho_n\}_{n\geq 0}$ be an increasing sequence of continuous norms defining its topology. If $\{u_n\}_{n\geq 0}$ is a basic sequence in X such that for every $k \geq 0$ it holds $p_1(u_k) = 1$ and for every $n \geq 0$, the sequence $\{u_k\}_{k\geq n}$ is basic in $(X, \rho_n) \equiv X_n$ with constant of basicity less than B, then every sequence $\{f_n\}_{n\geq 0} \subseteq X$ satisfying

$$\sum_{n=0}^{+\infty} 2B\rho_n(u_n - f_n) < 1 \tag{4.162}$$

is basic in X. Moreover, the sequences $\{u_n\}_{n\geq 0}$ and $\{f_n\}_{n\geq 0}$ are equivalent in X.

Lemma 4.41 (Similar to Lemma 4.4 of [9]; slightly extended, [9]). Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $a \in \mathbb{C}\backslash\Omega, \zeta \in \Omega$ be two fixed elements. Let also $K \subseteq \mathbb{C}\backslash\Omega\cup\{a\}$ and $L \subseteq \Omega$ two compact sets with connected complements and $h \in A(K)$. Then, for every $\varepsilon > 0$, for every denumerable set $\emptyset \neq W \subseteq \mathbb{C}$ and for every $p \in \mathbb{N}^*$, there exists a polynomial

$$P(z) = \sum_{k=p}^{q} a_k z^k \tag{4.163}$$

satisfying the following properties

(1) $P(a) \notin W \cup \{\zeta\}$

(2) $||P - h||_K < \varepsilon$

(3) $||P||_L < \varepsilon$.

Proof. Let $\varepsilon > 0$, $\emptyset \neq W \subseteq \mathbb{C}$ a denumerable set and $p \in \mathbb{N}^*$. We consider the function $f : K \cup L \to \mathbb{C}$ defined as follows

$$f(z) = \begin{cases} 0, & \text{if } z \in L \\ \dfrac{h(z)}{(z-\zeta)^p}, & \text{if } z \in K. \end{cases} \tag{4.164}$$

Then, it holds that $f \in A(K \cup L)$, where the set $K \cup L$ is a compact one with connected complement. We apply Mergelyan's theorem and thus we find a polynomial $Q(z)$ such that

$$||Q - f||_{K\cup L} < \frac{\varepsilon}{2} \cdot \frac{1}{1 + ||(z-\zeta)^p||_{K\cup L}}. \tag{4.165}$$

Consider now any $c \in \mathbb{C}$. We set

$$P(z) = (z-\zeta)^p(Q(z)+c) = (z-\zeta)^p Q(z) + c(z-\zeta)^p. \tag{4.166}$$

The polynomial $P(z)$ satisfies the following properties.

- $$\begin{aligned} ||P||_L &= ||(z-\zeta)^p Q(z) + c(z-\zeta)^p||_L \\ &\leq ||(z-\zeta)^p||_L \cdot \big(||Q(z)+c||_L\big) \\ &\leq ||(z-\zeta)^p||_L \cdot \big(||Q||_L + |c|\big) \\ &< ||(z-\zeta)^p||_L \cdot \Big(\frac{\varepsilon}{2} \cdot \frac{1}{1+||(z-\zeta)^p||_{K\cup L}} + |c|\Big) \\ &< \frac{\varepsilon}{2} + |c| \cdot ||(z-\zeta)^p||_{K\cup L}. \end{aligned} \tag{4.167}$$

  Thus, Property (3) is verified, provided that $|c|$ is small enough.

- $$\begin{aligned} ||P-h||_K &= ||(z-\zeta)^p Q(z) + c(z-\zeta)^p - h(z)||_K \\ &\leq ||(z-\zeta)^p||_L \cdot \Big(||Q(z) + c - \frac{h(z)}{(z-\zeta)^p}||_K\Big) \\ &\leq ||(z-\zeta)^p||_K \cdot \Big(||Q - \frac{h(z)}{(z-\zeta)^p}||_K + |c|\Big) \\ &< ||(z-\zeta)^p||_K \cdot \Big(\frac{\varepsilon}{2} \cdot \frac{1}{1+||(z-\zeta)^p||_{K\cup L}} + |c|\Big) \\ &< \frac{\varepsilon}{2} + |c| \cdot ||(z-\zeta)^p||_{K\cup L}. \end{aligned} \tag{4.168}$$

  Thus, Property (2) is also verified, provided that $|c|$ is small enough.

In particular, if

$$|c| < \frac{\varepsilon}{2} \cdot \frac{1}{1+||(z-\zeta)^p||_{K\cup L}} \tag{4.169}$$

it holds $||P||_L < \varepsilon$ and at the same time $||P-h||_K < \varepsilon$. In addition, in order to verify Property (1), it suffices to select

$$c \notin \Big\{\frac{w}{(a-\zeta)^p} - Q(a) : w \in W \cup \{\zeta\}\Big\}. \tag{4.170}$$

and that completes the proof.

■

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $\zeta \in \Omega$ be a fixed element. If $\mathcal{M} = <(z-\zeta)^k : k \geq 0>$, then $\mathcal{M}$ is an infinite dimensional subspace of $H(\Omega)$. The set $\mathcal{M}$ will be fixed from now on.

We now present the main result of this section. Its proof is s close adaptation to that of Theorem 4.17 in [9].

Theorem 4.42. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and $\zeta \in \Omega$ be a fixed element. Then, the class $\mathcal{U}' \cup \{0\}$ is spaceable.

Proof. In order to prove Theorem 4.42, we fix the following.

(1) $\{L_n\}_{n\geq 0}$ a family of compact subsets of Ω satisfying Lemma 4.5 with the extra property $L_k' \cap \Omega \neq \emptyset$ for every $k \geq 0$. Notice that such a choice is possible.

(2) $\{K_m\}_{m\geq 0}$ a family of compact subsets of $\mathbb{C} \setminus \Omega$ satisfying Lemma 4.4.

(3) $\{P_n\}_{n\geq 0}$ an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

(4) Two functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$ such that for every pair $(a,b) \in \mathbb{N} \times \mathbb{N}$, there exist infinitely many $n \in \mathbb{N}$ such that $(\phi(n), \psi(n)) = (a,b)$.

We denote with $\preceq$ the lexicographical order on $\mathbb{N} \times \mathbb{N}$. Thus, by definition, for every $(a,b), (a',b') \in \mathbb{N} \times \mathbb{N}$ it holds

$$(a,b) \preceq (a',b') \Leftrightarrow [a < a'] \text{ or } [a = a' \text{ and } b \leq b']. \tag{4.171}$$

Suppose that $(N_n)_{n\geq 0} \subseteq (0, +\infty)$ is a sequence decreasing to 0 "fast enough"[2]. We will build by induction three sequences of polynomials; namely the sequences $\{u_k\}_{k\geq 0}$, $\{g_{n,k}\}_{n\geq k\geq 0}$ and $\{f_{n,k}\}_{n\geq k\geq 0}$ satisfying the following properties for every $n \geq k \geq 0$.

(0) $\{u_k\}_{k\geq 0}$ is a basic sequence in $H(\Omega)$ (according to Definition 4.35).

Remark 4.43. We notice that Lemmas 4.36 and 4.38 imply the existence of a basic sequence in $H(\Omega)$. Thus, Relation (0) has meaning.

(1) $$||P_{\Phi(n)} - g_{n,k}||_{K_{\Psi(n)}} \leq N_n. \tag{4.172}$$

(2) $$||f_{n,k}||_{K_{\Psi(n+1)}} \leq N_n. \tag{4.173}$$

(3) $$||f_{n+1,k} - f_{n,k}||_{L_{n+1}} \leq N_n. \tag{4.174}$$

(4) $$||f_{n,k} - g_{n,k}||_{L_n} \leq N_n. \tag{4.175}$$

(5) $$||f_{k,k} - u_k||_{L_k} \leq N_k. \tag{4.176}$$

[2]This point will be clarified further on.

(6) For every $j \geq k$ we set $g_{j+1,k} = f_{j,k} + P(g_{j+1,k}, f_{j,k})$, where $P(g_{j+1,k}, f_{j,k})$ is a polynomial with

$$val(P(g_{j+1,k}, f_{j,k})) \geq \min\{p_n : p_n \geq \max_{(n',k') \preceq (j,k)} \{deg(f_{n',k'})\} + \max_{(n',k') \preceq (j,k)} \{q_1^{(n')}, \cdots, q_{N(n')}^{(n')}\} + 2\}. \tag{4.177}$$

We remind that if $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial, then we denote with $val(p)$ the following

$$val(p) = \min\{n \in \mathbb{N} : a_n \neq 0\}. \tag{4.178}$$

In addition, for every $k \in \mathbb{N}$ we set $g_{k,k} = u_k + P(g_{k,k}, u_k)$, where $P(g_{k,k}, u_k)$ is a polynomial with

$$\mathrm{val}(P(g_{k,k}, u_k)) \geq \min\{p_n : p_n \geq deg(u_k) + \max\{q_1^{(k)}, \cdots, q_{N(k)}^{(k)}\} + 2\}. \tag{4.179}$$

(7) For every $j \geq k$ we set $f_{j,k} = g_{j,k} + R(f_{j,k}, g_{j,k})$, where $R(f_{j,k}, g_{j,k})$ is a polynomial with

$$\mathrm{val}(R(f_{j,k}, g_{j,k})) \geq \min\{p_n : p_n \geq \max_{(n',k') \preceq (j,k)} \{deg(g_{n',k'})\} + \max_{(n',k') \preceq (j,k)} \{q_1^{(n')}, \cdots, q_{N(n')}^{(n')}\} + 2\}. \tag{4.180}$$

(8)
$$||u_k||_{L_0} = 1 \text{ for every } k \in \mathbb{N}. \tag{4.181}$$

In the next figure (Figure 7) we present the first steps of our construction.

Of course, we have to explain why such a construction is possible. This is mainly done by using Lemma 4.38 and Lemma 4.41. See Figure 7 for further details.

Lemma 4.38 is used to move from the top of one column to the top of the column on the immediate right, in order to build the basic sequence $\{u_k\}_{k\geq 0}$ with the desired properties, while Lemma 4.41 is used in two cases.

(i) In each single column, in order to move from one block to the block immediately below.

(ii) At the top of each column, in order to build the functions $g_{k,k}$ and $f_{k,k}$.

However, suppose that such a construction is possible for the time being. Next, for every $k \in \mathbb{N}$ we set

$$f_k = \sum_{n=k}^{+\infty} (f_{n+1,k} - f_{n,k}) + f_{k,k} = \lim_{N\to+\infty} f_{N+1,k}. \tag{4.182}$$

$$
\begin{array}{cccc}
\text{Column } 1 & \text{Column } 2 & \text{Column } 3 & \text{Column } 4 \\[1em]
\boxed{u_0 \to g_{0,0} \to f_{0,0}} & & & \\[1em]
\text{Lemma} \downarrow 4.41 & \searrow \text{Lemma } 4.38 & & \\[1em]
\boxed{g_{1,0} \to f_{1,0}} & \boxed{u_1 \to g_{1,1} \to f_{1,1}} & & \\[1em]
\text{Lemma} \downarrow 4.41 & \text{Lemma} \downarrow 4.41 & \searrow \text{Lemma } 4.38 & \\[1em]
\boxed{g_{2,0} \to f_{2,0}} & \boxed{g_{2,1} \to f_{2,1}} & \boxed{u_2 \to g_{2,2} \to f_{2,2}} & \\[1em]
\text{Lemma} \downarrow 4.41 & \text{Lemma} \downarrow 4.41 & \text{Lemma} \downarrow 4.41 & \searrow \text{Lemma } 4.38 \\[1em]
\boxed{g_{3,0} \to f_{3,0}} & \boxed{g_{3,1} \to f_{3,1}} & \boxed{g_{3,2} \to f_{3,2}} & \boxed{u_3 \to g_{3,3} \to f_{3,3}}
\end{array}
$$

Figure 8: The very first steps in the construction of the sequences $\{u_k\}_{k\geq 0}$, $\{g_{n,k}\}_{n\geq k\geq 0}$ and $\{f_{n,k}\}_{n\geq k\geq 0}$.

Relation (3) implies that for every $k \in \mathbb{N}$ it holds $f_k \in H(\Omega)$. In order to explain this, we fix a $k \in \mathbb{N}$ and we notice that for every $N \geq k$ it holds

$$\sum_{n=k}^{N}(f_{n+1,k} - f_{n,k}) + f_{k,k} = f_{N+1,k}. \tag{4.183}$$

We fix any $j \in \mathbb{N}$ and we want to show that the sequence

$$S_N = \sum_{n=k}^{N}(f_{n+1,k} - f_{n,k}) + f_{k,k} = f_{N+1,k} \tag{4.184}$$

(defined for $N \geq k$) converges uniformly on L_j. We select an index $N_0 \in \mathbb{N}$ such that $N_0 > j, k$. Thus, for every $N > N_0$ we obtain

$$
\begin{aligned}
S_N &= \sum_{n=k}^{N}(f_{n+1,k} - f_{n,k}) + f_{k,k} \\
&= \sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k} + \sum_{n=N_0+1}^{N}(f_{n+1,k} - f_{n,k}) \\
&= f_{N+1,k}.
\end{aligned} \tag{4.185}
$$

The first term

$$\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k} \tag{4.186}$$

is fixed, while the second one

$$\sum_{n=N_0+1}^{N} (f_{n+1,k} - f_{n,k}) \tag{4.187}$$

changes, depending on $N \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
||S_N||_{L_j} &= ||\sum_{n=k}^{N}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} \\
&\le ||\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} + ||\sum_{n=N_0+1}^{N}(f_{n+1,k} - f_{n,k})||_{L_j} \\
&\le ||\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} + \sum_{n=N_0+1}^{N}||(f_{n+1,k} - f_{n,k})||_{L_j}
\end{aligned}$$

$$\begin{aligned}
&\le ||\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} + \sum_{n=N_0+1}^{N}||(f_{n+1,k} - f_{n,k})||_{L_{n+1}} \\
&\le ||\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} + \sum_{n=N_0+1}^{N} N_n \\
&\le ||\sum_{n=k}^{N_0}(f_{n+1,k} - f_{n,k}) + f_{k,k}||_{L_j} + \sum_{n=0}^{+\infty} N_n \\
&< +\infty.
\end{aligned}$$

Thus, according to Weierstrass's theorem, the sequence $\{S_N\}_{N\ge k}$ converges uniformly on every compact set $L \subseteq \Omega$ to the function $f_k \in H(\Omega)$.

By combining Relation (5) and Lemma 4.40 it follows that $\{f_k\}_{k\ge 0}$ is a basic sequence of $H(\Omega)$ equivalent to the sequence $\{u_k\}_{k\ge 0}$. Indeed, if B is the constant of basicity for the sequence $\{u_k\}_{k\ge 0}$ then, according to Lemma 4.40, it suffices to prove that

$$\sum_{n=0}^{+\infty} 2B\rho_n(u_n - f_n) < 1 \Leftrightarrow \sum_{n=0}^{+\infty} ||u_n - f_n||_{L_n} < \frac{1}{2B}. \tag{4.188}$$

Since for every $k \in \mathbb{N}$ it holds

$$f_k - u_k = \sum_{n\ge k}^{+\infty}(f_{n+1,k} - f_{n,k}) + f_{k,k} - u_k \tag{4.189}$$

we obtain the following

$$\begin{aligned} ||f_k - u_k||_{L_k} &\leq \sum_{n\geq k}^{+\infty} ||f_{n+1,k} - f_{n,k}||_{L_k} + ||f_{k,k} - u_k||_{L_k} \\ &\leq \sum_{n\geq k}^{+\infty} ||f_{n+1,k} - f_{n,k}||_{L_k} + N_k \\ &\leq \sum_{n\geq k}^{+\infty} ||f_{n+1,k} - f_{n,k}||_{L_{n+1}} + N_k \\ &\leq \sum_{n\geq k}^{+\infty} N_n + N_k. \end{aligned} \tag{4.190}$$

Thus, the desired inequality (Relation (4.188)) holds[3] provided that

$$\sum_{k=0}^{+\infty}(\sum_{n\geq k}^{+\infty} N_n + N_k) < \frac{1}{2B}. \tag{4.191}$$

Next, we consider the following set

$$\mathcal{F} = \overline{< f_k : k \geq 0 >}. \tag{4.192}$$

Since the sequences $\{f_k\}_{k\geq 0}$ and $\{u_k\}_{k\geq 0}$ are equivalent, Relation (0) implies that $\{f_k\}_{k\geq 0}$ is linearly independent and thus $\mathcal{F}$ is an infinite dimensional subspace of $H(\Omega)$. Indeed, let $j_1 < \cdots < j_N \in \mathbb{N}$, $a_{j_1}, \cdots, a_{j_N} \in \mathbb{C} \setminus \{0\}$ and suppose that $a_{j_1} f_{j_1} + \cdots + a_{j_N} f_{j_N} = 0$. The previous relation implies that the element $0 \in \mathcal{F}$ has (at least) two distinct representations as an infinite linear combination of the functions $\{f_k\}_{k\geq 0}$, which contradicts the basicity of $\{f_k\}_{k\geq 0}$.

Our aim now is to show that for every $f \in \mathcal{F} \setminus \{0\}$ it holds $f \in \mathcal{U}'$. Let

$$f = \sum_{k=0}^{+\infty} a_k f_k \in \mathcal{F} \setminus \{0\} \tag{4.193}$$

and $j > k \geq 0$. We set

$$p_{n(j,k)} = \min\{n \in \mathbb{N} : p_n \geq g_{j,k}\}. \tag{4.194}$$

According to our construction, the $(p_{n(j,k)}, q_s^{(n(j,k))})$ - Padé approximant of f (centered at $\zeta \in \Omega$) will be the sums of all blocks of the form $g_{j',k'}$ or $f_{j',k'}$ (up the coefficients a_k) appearing in the f_k's with $(j',k') \preceq (j,k)$ and that holds for every $s = 1, \cdots, N(n(j,k))$. This sum is a polynomial with degree $\leq p_{n(j,k)}$ by definition and since the valuation of any other blocks $g_{j',k'}$ or $f_{j',k'}$ with $(j,k) \prec (j',k')$ is strictly bigger than $p_{n(j,k)} + q_s^{(n(j,k))} + 1$, the Taylor expansion of this sum (centered at $\zeta \in \Omega$) will

[3]Property 1 $(p1)$ for the sequence $(N_n)_{n\geq 0}$. Here, we can also consider that it holds $\sum_{k=0}^{+\infty} N_k < +\infty$ and also that the sequence $(N_n)_{n\geq 0}$ is strictly decreasing.

coincide with that of the function f up to the index $p_{n(j,k)} + q_s^{(n(j,k))} + 1$ and that holds for every $s = 1, \cdots, N(n(j,k))$.

It remains to enumerate all those blocks of $g_{j',k'}$ or $f_{j',k'}$ appearing in the f_k's with $(j',k') \preceq (j,k)$. The block with the bigger degree is $g_{j,k}$ by definition. The blocks that have been built before $g_{j,k}$ are the $f_{j-1,k'}$'s with $0 \leq k' \leq j-1$ and the $f_{j,k'}$'s with $0 \leq k' \leq k-1$. Next, we observe that the block $f_{j-1,k}$ is a part of $g_{j,k}$, according to the construction, while every $f_{j-1,k'}$ is a part of $f_{j,k'}$ for $0 \leq k' \leq k-1$. Moreover, any other blocks $g_{j',k'}$ or $f_{j',k'}$ with $j' \leq j-2$ are a part of $f_{j-1,k'}$. So, by avoiding to count more than once the same blocks, we obtain

$$[f; p_{n(j,k)}/q_s^{(n(j,k))}]_\zeta = S_{p_{n(j,k)}}(f) = a_k g_{j,k} + \sum_{k'=0}^{k-1} a'_k f_{j,k'} + \sum_{k'=k+1}^{j-1} a'_k f_{j-1,k'} \qquad (4.195)$$

and that holds for every $s = 1, \cdots, N(n(j,k))$.

Next, we observe that the sequence $(a_k)_{k\geq 0} \subseteq \mathbb{N}$ is bounded by some constant M. In order to explain this, we notice that since the sequences $\{f_k\}_{k\geq 0}$ and $\{u_k\}_{k\geq 0}$ are equivalent, the series

$$\sum_{k=0}^{+\infty} a_k u_k \qquad (4.196)$$

converges and according to Relation (9) the following holds

$$|a_k| = ||a_k u_k||_{L_0} \leq 2B||\sum_{k=0}^{+\infty} a_k u_k||_{L_0}. \qquad (4.197)$$

We set $M = 2B$. In order to complete the proof, we have to show that the function f has the desired universal approximation properties.

(•) Let $K \subseteq \mathbb{C} \setminus \Omega$ be a compact set with connected complement and $h \in A(K)$. Let also $k_0 = \min\{k \in \mathbb{N} : a_k \neq 0\}$. We select an index $r \in \mathbb{N}$ such that $K \subseteq K_r$ and we also consider a polynomial P_l. By the initial hypothesis for the functions ϕ and ψ, there exists a sequence $(v_j)_{j\geq 0} \subseteq \mathbb{N}$ such that $(v_j)_{j\geq 0}$ is strictly increasing and also $(\phi(v_j), \psi(v_j)) = (l, r)$ for every $j \geq 0$, while $v_j > k_0$ for every $j \geq 0$.

We set $p_{n(v_j,k_0)} = \min\{p_n : p_n \geq deg(g_{v_j,k_0})\}$. From Properties (1) and (2) and the previous, for every $s = 1, \cdots, N(n(v_j,k_0))$ we obtain the following

$$\begin{aligned} &||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - a_{k_0} P_l||_{K_r} \\ &= ||a_{k_0} g_{v_j,k_0} + \sum_{k'=k_0+1}^{v_j-1} a_{k'} f_{v_j-1,k'} - a_{k_0} P_{\phi(v_j)}||_{K_{\psi(v_j)}} \\ &\leq \sum_{k'=k_0+1}^{v_j-1} |a_{k'}| \cdot ||f_{v_j-1,k'}||_{K_{\psi(v_j)}} + |a_{k_0}| \cdot ||g_{v_j,k_0} - P_{\phi(v_j)}||_{K_{\psi(v_j)}} \\ &\leq M \sum_{k'=k_0+1}^{v_j-1} N_{v_j-1} + M N_{v_j} \end{aligned}$$

$$\leq M(v_j - k_0)N_{v_j-1}. \tag{4.198}$$

By choosing $(N_n)_{n\geq 0}$ decreasing to 0 fast enough[4], we obtain the desired result.

(•) Let $n \in \mathbb{N}$. For $j \in \mathbb{N}$ large enough and for every $s = 1, \cdots, N(n(v_j, k_0))$, by using Relations (3) and (4), one can verify that it holds

$$||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - f||_{L_n}$$

$$= ||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - \sum_{k'=k_0}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$= ||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - \sum_{k'=k_0}^{v_j-1} a_{k'} f_{k'} - \sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq ||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - \sum_{k'=k_0}^{v_j-1} a_{k'} f_{k'}||_{L_n} + ||\sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq ||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - \sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'} - f_{n,k'}) - \sum_{k'=k_0}^{v_j-1} a_{k'} f_{v_j-1,k'}||_{L_n}$$

$$+||\sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq ||[f; p_{n(v_j,k_0)}/q_s^{(n(v_j,k_0))}]_\zeta - (\sum_{k'=k_0+1}^{v_j-1} a_{k'} f_{v_j-1,k'} + a_{k_0} f_{v_j-1,k_0})||_{L_n}$$

$$+||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'} - f_{n,k'})||_{L_n} + ||\sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq ||a_{k_0} g_{v_j,k_0} - a_{k_0} f_{v_j-1,k_0}||_{L_n} + ||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'} - f_{n,k'})||_{L_n}$$

$$+||\sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq ||a_{k_0} g_{v_j,k_0} - a_{k_0} f_{v_j,k_0} + a_{k_0} f_{v_j,k_0} - a_{k_0} f_{v_j-1,k_0}||_{L_n}$$

$$+||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'} - f_{n,k'})||_{L_n} + ||\sum_{k'=v_j}^{+\infty} a_{k'} f_{k'}||_{L_n}$$

$$\leq |a_{k_0}| \cdot ||g_{u_j,k_0} - f_{u_j,k_0}||_{L_n} + |a_{k_0}| \cdot ||f_{u_j,k_0} - f_{u_j-1,k_0}||_{L_n}$$

[4]Property 2 ($p2$) for the sequence $(N_n)_{n\geq 0}$. Here it suffices to demand that it holds $\lim_{n\to+\infty} nN_{n-1} = 0$.

$$+||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'}-f_{n,k'})||_{L_n}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}$$
$$\leq M(||g_{v_j,k_0}-f_{v_j,k_0}||_{L_{v_j}}+||f_{v_j,k_0}-f_{v_j-1,k_0}||_{L_{v_j}})$$
$$+||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'}-f_{n,k'})||_{L_n}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}$$
$$\leq 2MN_{v_j}+||\sum_{k'=k_0}^{v_j-1} a_{k'}(\sum_{n\geq v_j-1}^{+\infty} f_{n+1,k'}-f_{n,k'})||_{L_n}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}$$
$$\leq 2MN_{v_j}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}+\sum_{k'=k_0}^{v_j-1}|a_{k'}|(\sum_{n\geq v_j-1}^{+\infty}||f_{n+1,k'}-f_{n,k'}||_{L_n})$$
$$\leq 2MN_{v_j}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}+\sum_{k'=k_0}^{v_j-1}|a_{k'}|(\sum_{n\geq v_j-1}^{+\infty}N_n)$$
$$\leq 2MN_{v_j}+||\sum_{k'=v_j}^{+\infty} a_{k'}f_{k'}||_{L_n}+M|v_j-k_0|\sum_{n\geq v_j-1}^{+\infty}N_n. \tag{4.199}$$

Thus, by choosing $(N_n)_{n\geq 0}$ decreasing to 0 fast enough[5], we obtain the result.

This completes the proof.

■

Remark 4.44. It suffices to set $N_n = \frac{1}{2^n}$ for every $n \in \mathbb{N}$, since the sequence $(N_n)_{n\geq 0}$ satisfies all properties $(p1)$, $(p2)$ and (p_3).

[5]Property 3 $(p3)$ for the sequence $(N_n)_{n\geq 0}$. Here it suffices to demand that it holds $\lim\limits_{k\to+\infty} k(\sum\limits_{n\geq k-1}^{+\infty} N_n)=0$.

5 Universal Laurent series

In this section we deal with compact subsets F of $(\mathbb{C} \cup \{\infty\}, \chi)$ which satisfy the following topological condition ([34]).

Condition 5.1. Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus F$ be a domain. Suppose that it holds $\infty \in F$; therefore $\Omega \subseteq \mathbb{C}$. We assume that the following conditions hold.

(i) Among the connected components of F, there exists a distinct sequence $F_1, F_2, \cdots$ such that every F_j is a compact subset of $\mathbb{C}$.

(ii)
$$F = cl_\chi(\bigcup_{j=1}^{+\infty} F_j) \tag{5.1}$$

where χ is the chordal distance on $\mathbb{C} \cup \{\infty\}$.

(iii) For every $\ell \geq 1$, there exists a $\delta_\ell > 0$ such that

$$\chi(F_\ell, F_j) \equiv \inf\{\chi(z, w) : z \in F_\ell \;\&\; w \in F_j\} > \delta_\ell \tag{5.2}$$

for every $j \geq 1$ with $j \neq \ell$.

(iv) For every $j \geq 1$ it holds ${F_j}^o \neq \emptyset$.

Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be a domain such that its complement $F = \mathbb{C} \cup \{\infty\} \setminus \Omega$ satisfies Condition (5.1). We assume that $\infty \in F$; thus $\Omega \subseteq \mathbb{C}$. Moreover, for every $\ell \geq 1$ we select a $c_\ell \in F_j^o$ and we set $\Gamma = cl_\chi\{c_\ell : \ell \geq 1\}$. Obviously, the set Γ is a compact subset of $(\mathbb{C} \cup \{\infty\}, \chi)$ but this does not necessarily imply that $\infty \in \Gamma$.

We assume that there exists a sequence of compact subsets of $\overline{\Omega}$, namely the sequence $\{L_n\}_{n\geq 1}$, satisfying the following properties.

(0) $\{L_n\}_{n\geq 1}$ is increasing; that is $L_n \subseteq L_{n+1}$ for every $n \geq 1$.

(1) $\overline{L_n \cap \Omega} = L_n$ for every $n \geq 1$.

(2) Each connected component of $\mathbb{C} \cup \{\infty\} \setminus L_n$ contains a connected component of $\mathbb{C} \cup \{\infty\} \setminus \overline{\Omega}$.

(3) For every compact set $J \subseteq \Omega$, there exists an index $n \geq 1$ such that $J \subseteq L_n$.

(4) Every connected component of $\mathbb{C} \cup \{\infty\} \setminus L_n$ contains a $c_\ell \in F$.

Under the above assumptions, we define the space $T^\infty(\Omega)$ as the space of all (analytic) functions $f \in H(\Omega)$ such that for every derivative $f^{(\ell)}$ of f $(\ell \geq 0)$ and for every $n \geq 1$, the function $f^{(\ell)}_{|(L_n \cap \Omega)}$ is uniformly continuous on $L_n \cap \Omega$ and therefore it extends continuously on $\overline{L_n \cap \Omega} = L_n$ (see also [40]; there the space $T^\infty(\Omega)$ has been defined without property (4) above).

The space $T^\infty(\Omega) \equiv T^\infty(\Omega, \{L_n\}_{n\geq 1})$ is endowed with the topology induced by the seminorms

$$\sup_{z \in L_n} |f^{(\ell)}(z)| \tag{5.3}$$

(for every $\ell \geq 0$ and for every $n \geq 1$). It is known that in this way $T^{\infty}(\Omega)$ becomes a Fréchet space. We also consider the set $Y^{\infty}(\Omega)$ to be the closure of the set of all rational functions with poles off $\bigcup_{n=1}^{+\infty} L_n$ with respect to the topology of $T^{\infty}(\Omega)$. Thus, $Y^{\infty}(\Omega)$ is a closed subset of a complete metric space and therefore is a complete metric space itself.

We also assume that there exists a sequence $\{K_m\}_{m\geq 1}$ of compact subsets of $\mathbb{C}\setminus(\Omega\cup\Gamma) \subseteq \mathbb{C}$ such that every K_m has connected complement and also it holds $L_n \cap K_m = \emptyset$ for every $n, m \geq 1$.

One could naturally ask whether the sequences $\{L_n\}_{n\geq 1}$ and $\{K_m\}_{m\geq 1}$ satisfying the requested properties may exist or not. For this purpose, we present the following examples.

Example 5.2. For every $\ell \in \mathbb{N}$ we set $c_\ell = \ell$ and let $F_\ell = \overline{B(c_\ell, \frac{1}{3})}$ be the Euclidean disc with center $c_\ell = \ell$ and radius $\frac{1}{3}$. We set $F = cl_\chi(\bigcup_{\ell=0}^{+\infty} F_\ell) = \left(\bigcup_{\ell=0}^{+\infty} \overline{B(c_\ell, \frac{1}{3})}\right) \cup \{\infty\}$ and $\Omega = \mathbb{C} \cup \{\infty\} \setminus F$. Also, let $\Gamma = cl_\chi\{c_\ell : \ell \geq 0\} = \mathbb{N} \cup \{\infty\}$. It is obvious that $\infty \in F$ and therefore $\infty \notin \Omega$. It is easy to check that the set F satisfies Condition (5.1).

For each $n \in \mathbb{N}^*$ we set $L_n = \overline{\Omega} \cap \overline{B(0,n)}$. Again, it is easy to see that the family $\{L_n\}_{n\geq 1}$ meets the requirements to define the space $T^{\infty}(\Omega)$.

Let $K \subseteq \mathbb{C}\setminus(\Omega\cup\Gamma)$ be a compact set with connected complement such that $K \cap L_n = \emptyset$ for every $n \geq 1$. By definition of the sequence $\{L_n\}_{n\geq 1}$ it holds that $\overline{\Omega} = \bigcup_{n=1}^{+\infty} L_n$. The previous relation implies that $K \cap \overline{\Omega} = \emptyset \Rightarrow K \subseteq \mathbb{C} \setminus (\overline{\Omega} \cup \Gamma)$. In addition, since K is a compact set, there exists a $n \in \mathbb{N}$ such that $K \subseteq \overline{B(0,n)}$.

It follows that $K \subseteq K(n, s, t)$, where

$$K(n,s,t) = \overline{B(0,n)} \cap \left\{ z \in \mathbb{C} : d(z, \overline{\Omega}) \geq \frac{1}{s} \right\} \cap \left\{ z \in \mathbb{C} \cup \{\infty\} : \chi(z, \Gamma) \geq \frac{1}{t} \right\} \tag{5.4}$$

for some $t, s \in \mathbb{N}^*$. Now, it is easy to check that every $K(n,s,t)$ is a compact subset of $\mathbb{C}$ and also that it holds

$$\begin{aligned} \mathbb{C} \cup \{\infty\} \setminus K(n,s,t) = {} & (\{z \in \mathbb{C} : |z| > n\} \cup \{\infty\}) \\ & \cup (\{z \in \mathbb{C} : d(z, \overline{\Omega}) < \frac{1}{s}\} \cup \{\infty\}) \\ & \cup \{z \in \mathbb{C} \cup \{\infty\} : \chi(z, \Gamma) < \frac{1}{t}\}. \end{aligned} \tag{5.5}$$

It is easy to see that

$$n + \frac{1}{3} \in (\{z \in \mathbb{C} : |z| > n\} \cup \{\infty\}) \cap (\{z \in \mathbb{C} : d(z, \overline{\Omega}) < \frac{1}{s}\} \cup \{\infty\}) \tag{5.6}$$

and also that

$$n + 1 \in (\{z \in \mathbb{C} : |z| > n\} \cup \{\infty\}) \cap \{z \in \mathbb{C} \cup \{\infty\} : \chi(z, \Gamma) < \frac{1}{t}\}; \tag{5.7}$$

thus, the union $\mathbb{C} \cup \{\infty\} \setminus K(n,s,t)$ is also a connected set. In other words, each $K(n,s,t)$ has connected complement.

We also notice that $K(n,s,t) \subseteq \mathbb{C} \setminus (\overline{\Omega} \cup \Gamma) \subseteq \mathbb{C} \setminus (\Omega \cup \Gamma)$. This implies that $K(n,s,t) \cap L_m = \emptyset$ for every parameter n, s, t and m. An enumeration $\{K_m\}_{m\geq 1}$ of the sets $K(n,s,t)$ yields the result.

Remark 5.3. In Example 5.2 we have that $\partial\Omega \subseteq \bigcup_{n=1}^{+\infty} L_n$; in particular, it holds $\partial\Omega \cap L_n \neq \emptyset$ for every $n \geq 1$. In addition, $K_m \cap \partial\Omega = \emptyset$ for every $m \geq 1$. It can be easily seen that in this particular case it holds $T^\infty(\Omega) = A^\infty(\Omega)$ and $Y^\infty(\Omega) = X^\infty(\Omega)$, where $X^\infty(\Omega)$ denotes the closure in $A^\infty(\Omega)$ of the set of rational functions with poles off $\overline{\Omega}$ (see [37] for the relevant definitions). Moreover, in this example it also holds $X^\infty(\Omega) = A^\infty(\Omega)$ (see [37]).

Lemma 5.4. (Lemma 2.2 from [11].) Let G be a domain in $\mathbb{C}$, with $G \neq \mathbb{C}$. We assume that $(\mathbb{C} \cup \{\infty\} \setminus G)$ has a finite number of components $A_0, A_1, \cdots, A_k$, for some $k \geq 0$ and we fix $\infty \in A_0, a_1 \in A_1, \cdots, a_k \in A_k$. Then there exists a sequence of compact sets $\Lambda_m \subseteq \mathbb{C} \setminus (\{a_1, \cdots, a_k\} \cup G)$ with connected complement such that for every compact set $K \subseteq \mathbb{C} \setminus (\{a_1, \cdots, a_k\} \cup G)$ with connected complement, there exists an index $m \geq 1$ such that $K \subseteq \Lambda_m$.

Example 5.5. Consider the same $F, F_\ell, c_\ell, \Gamma$ and Ω as in Example 5.2. For every $n \geq 0$, we apply Lemma 5.4 for the domain $\Omega_n = \Omega \cap B(0, n + \frac{1}{2})$. Each $\mathbb{C} \cup \{\infty\} \setminus \Omega_n$ has precisely $n + 2$ connected components; the sets F_ℓ (for $0 \leq \ell \leq n$) and the set $\mathbb{C} \cup \{\infty\} \setminus B(0, n + \frac{1}{2})$. Since $c_\ell \in F_\ell$ for every $\ell \geq 0$ and $\infty \in \mathbb{C} \cup \{\infty\} \setminus B(0, n + \frac{1}{2})$, it follows from Lemma 5.4 that there exists a sequence of compact sets $\{\Lambda_{n,m}\}_{m\geq 1}$ with connected complement, such that $\Lambda_{n,m} \subseteq \mathbb{C} \setminus (\{c_\ell : 0 \leq \ell \leq n\} \cup \Omega_n)$ for every $m \geq 1$ and also for every compact set $K \subseteq \mathbb{C} \setminus (\{c_\ell : 0 \leq \ell \leq n\} \cup \Omega_n)$ with connected complement, there exists an index $m \geq 1$ such that $K \subseteq \Lambda_{n,m}$.

At this point, we notice the following: let $K \subseteq \mathbb{C} \setminus (\{c_\ell : 0 \leq \ell \leq n\} \cup \Omega_n)$ be a compact set with connected complement. Then we can split K in two disjoint pieces $K = K_1 \cup K_2$, where K_1 and K_2 are compact sets with connected complements such that $K_1 \subseteq (\bigcup_{\ell=0}^n F_\ell) \setminus \{c_\ell : 0 \leq \ell \leq n\}$ and $K_2 \subseteq \mathbb{C} \cup \{\infty\} \setminus B(0, n + \frac{1}{2})$. We leave the proof of this claim to the reader.

We split each $\Lambda_{n,m}$ separately in two disjoint compact pieces with connected complements as $\Lambda_{n,m} = \Lambda_{n,m}^{(1)} \cup \Lambda_{n,m}^{(2)}$, where $\Lambda_{n,m}^{(1)} \subseteq (\bigcup_{\ell=0}^n F_\ell) \setminus \{c_\ell : 0 \leq \ell \leq n\}$ for every $m \geq 1$ and $\Lambda_{n,m}^{(2)} \subseteq \mathbb{C} \cup \{\infty\} \setminus B(0, n + \frac{1}{2})$ for every $m \geq 1$. An enumeration of the set $\{\Lambda_{n,m}^{(1)} : n, m \geq 1\}$ gives us the family $\{K_m\}_{m\geq 1}$.

On the other hand, for every $n \geq 1$, we set $L_n = \{z \in \Omega : |z| \leq n \text{ and } d(z, \mathbb{C}\setminus\Omega) \geq \frac{1}{n}\}$. Again, it is easy to check that the sequence $\{L_n\}_{n\geq 1}$ satisfies properties $(0) - (4)$. In addition, for every $n, m \geq 1$ it holds $L_n \cap K_m = \emptyset$, since for every $s, t \geq 1$ it holds $\Lambda_{s,t}^{(1)} \subseteq F$, while $L_n \subseteq \Omega$.

Remark 5.6. We notice that in Exampe 5.5 it holds $L_n \cap \partial\Omega = \emptyset$ for every $n \geq 1$ and also that $\partial\Omega \subseteq \bigcup_{m=1}^{+\infty} K_m$. In this case $Y^\infty(\Omega) = H(\Omega)$, where $H(\Omega)$ is the space of all holomorphic functions in Ω.

Example 5.7. Consider the same $F, F_\ell, c_\ell, \Gamma$ and Ω as in Example 5.2. We consider an exhausting family $\{L_{n,1}\}_{n\geq 1}$ of compact sets of Ω (see [38]). By definition, the family $\{L_{n,1}\}_{n\geq 1}$ satisfies the following properties.

(i')
$$L_{n,1}^o \subseteq L_{n+1,1}$$

for every $n \geq 1$.

(ii') If $J \subseteq \Omega$ is a compact set, then there exists an index $n \geq 1$ such that $J \subseteq L_{n,1}$.

(iii') Every connected component of $\mathbb{C} \cup \{\infty\} \setminus L_{n,1}$ contains a connected component of $\mathbb{C} \cup \{\infty\} \setminus \Omega$.

Next, for every $n, m \geq 1$ we consider the following sets

$$L_{n,2} = \{z \in \overline{\Omega} : |z| \leq n \text{ and } Imz \geq 0\} \tag{5.8}$$

$$K_m = \left\{z \in \mathbb{C} \setminus \Omega : |z| \leq m \text{ and } \; Imz \leq -\frac{1}{m}\right\} \tag{5.9}$$

We set $L_n = L_{n,1} \cup L_{n,2}$ for every $n \geq 1$. The families $\{L_n\}_{n \geq 1}$ and $\{K_m\}_{m \geq 1}$ consist of compact sets and also it is easy to verify that for every $n, m \geq 1$ it holds $L_n \cap K_m = \emptyset$. Now, for every $m \geq 1$ we have that it holds $\mathbb{C} \setminus K_m = \Omega \cup (\mathbb{C} \setminus \overline{B(0,m)}) \cup \{z \in \mathbb{C} : Imz > -\frac{1}{m}\}$, where $(m+1)i \in \Omega \cap (\mathbb{C} \setminus \overline{B(0,m)})$ and $\frac{2}{3}i \in \Omega \cap (\mathbb{C} \setminus \{z \in \mathbb{C} : Imz > -\frac{1}{m}\})$; therefore the set $\mathbb{C} \setminus K_m$ is connected (or equivalently, K_m has connected complement).

In order to complete this example we have to show that the family $\{L_n\}_{n \geq 1}$ satisfies properties $(0) - (4)$. Properties $(0), (1)$ & (3) are almost immediate. For Properties (2) and (4) we work as it follows.

For every $n \geq 1$ it holds $\mathbb{C} \cup \{\infty\} \setminus L_n = (\mathbb{C} \cup \{\infty\} \setminus L_{n,1}) \cap (\mathbb{C} \cup \{\infty\} \setminus L_{n,2}) = (\mathbb{C} \cup \{\infty\} \setminus L_{n,1}) \cap [(\mathbb{C} \cup \{\infty\} \setminus \overline{\Omega}) \cup (\{z \in \mathbb{C} : Imz < 0\} \cup \{\infty\}) \cup (\mathbb{C} \cup \{\infty\} \setminus \overline{B(0,n)})]$, where the set $\mathbb{C} \cup \{\infty\} \setminus L_{n,2}$ is connected. Thus, every connected component B of $\mathbb{C} \cup \{\infty\} \setminus L_n$ is of the form $A \cap \mathbb{C} \cup \{\infty\} \setminus L_{n,2}$, where A is a connected component of $\mathbb{C} \cup \{\infty\} \setminus L_{n,1}$. It follows that B contains an entire F_ℓ^o and therefore a c_ℓ and we are done.

Remark 5.8. In Example 5.7 we have that $\partial\Omega \subseteq \bigcup_{n,m \geq 1}(L_n \cup K_m)$ and also that for every $n, m \geq 1$ it holds $\partial\Omega \cap L_n \neq \emptyset$ and $\partial\Omega \cap K_m \neq \emptyset$.

Now, we return to the general case. Let $f \in Y^\infty(\Omega)$ be a function and $\ell \geq 1$. We consider a closed polygonal curve $\gamma_\ell \subseteq \Omega \cap \mathbb{C}$ such that $Ind(\gamma_\ell, c_\ell) = -1$ and $Ind(\gamma_j, c_j) = 0$ for every $j \neq \ell$. This can be done due to assumption (iii) of Condition (5.1). For every $\ell \geq 1$, we consider the function

$$f_\ell(z) = \frac{1}{2\pi i} \int_{\gamma_\ell} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{5.10}$$

which is well defined, extends holomorhically in $\mathbb{C} \cup \{\infty\} \setminus F_\ell$ and also satisfies $f_\ell(\infty) = 0$. In addition, each such function f_ℓ has a Laurent expansion in $\mathbb{C} \cup \{\infty\} \setminus F_\ell$ (centered at $c_\ell \in F$) of the following form

$$f_\ell(z) = \sum_{m=1}^{+\infty} a_m(f_\ell) \frac{1}{(z - c_\ell)^m}. \tag{5.11}$$

Consider $\{e_n\}_{n \geq 0}$ an enumeration of the set $\{a_m(f_\ell)\frac{1}{(z-c_\ell)^m} : \ell \geq 1 \text{ and } m \geq 0\}$. Then, one may consider the operators $T_n : Y^\infty(\Omega) \to H_0(\Omega \setminus \Gamma_n)$, where Γ_n is a finite subset of $\{c_\ell : \ell \geq 1\}$, to be the sum $T_n = e_0 + \cdots + e_n$ and that for every $n \geq 0$. We recall that $H_0(U)$ for an open set U denotes the set of holomorphic functions $f : U \to \mathbb{C}$ such that

$$\lim_{\substack{z\to\infty \\ z\in U}} f(z) = 0. \tag{5.12}$$

In our case $U = \Omega \setminus \Gamma_n$ contains a neighbourhood of ∞ in $\mathbb{C}$. Then, the family $\{T_n\}_{n\geq 0}$ satisfies the following conditions.

(5) Each T_n is a continuous function of f in $Y^\infty(\Omega)$.

(6) For every rational function $g \in Y^\infty(\Omega)$ with poles in $\{c_\ell : \ell = 1, 2, \cdots\}$ there exists an index $k_0 \in \mathbb{N}$ such that $T_k(g) = g$ for every $k \geq k_0$.

(7) For every $\lambda \geq 0$, it also holds that $T_n^{(\lambda)} : Y^\infty(\Omega) \to H_0(\Omega \setminus \Gamma)$, where $T_n^{(\lambda)}$ denotes the λth derivative of T_n.

A particular case is when

$$T_n(f) = \sum_{\ell=0}^{n} S_n(f_\ell), \tag{5.13}$$

where $S_n(f_\ell)$ is the nth partial sum of the series

$$f_\ell(z) = \sum_{m=1}^{+\infty} a_m(f_\ell)\frac{1}{(z-c_\ell)^m}. \tag{5.14}$$

Our results are valid for general operators $\{T_n\}_{n\geq 0}$ satisfying properties (5) - (7) above.

We will also use the following lemma (see [34], [15] and [39]).

Lemma 5.9. Let $K \subseteq \mathbb{C}\cup\{\infty\}$ be a compact set and $A \subseteq \mathbb{C}\cup\{\infty\}$ be a set intersecting every connected component of $\mathbb{C}\cup\{\infty\} \setminus K$. Let also $K \subseteq V \subseteq \mathbb{C}\cup\{\infty\}$ be an open set. Then, there exists an open set $K \subseteq W \subseteq V \subseteq \mathbb{C}\cup\{\infty\}$ such that every connected component of $\mathbb{C}\cup\{\infty\} \setminus W$ intersects A.

5.1 A generic result of Laurent approximation

We consider an open set $\Omega \subseteq \mathbb{C}$ and F its complement in $\mathbb{C}\cup\{\infty\}$ that satisfies Condition 5.1. Thus, it holds $\infty \in F$. Moreover, for every $\ell \geq 1$ we select a $c_\ell \in F_j^o$ and we set $\Gamma = cl_\chi\{c_\ell : \ell \geq 1\}$.

We assume that there exists an increasing sequence of compact subsets of $\overline{\Omega}$, namely the sequence $\{L_n\}_{n\geq 1}$, satisfying properties (0) - (4). We fix such a sequence $\{L_n\}_{n\geq 1}$. Therefore, the spaces $T^\infty(\Omega)$ and $Y^\infty(\Omega)$ can be defined as in the preliminaries section.

We also assume that there exists a sequence $\{K_m\}_{m\geq 1}$ of compact subsets of $\mathbb{C}\setminus(\Omega\cup\Gamma) \subseteq \mathbb{C}$ such that every K_m has connected complement and also it holds $L_n \cap K_m = \emptyset$ for every $n, m \geq 1$. We also fix such a sequence $\{K_m\}_{m\geq 1}$.

Definition 5.10 (Definition of the class $\mathcal{L}$)**.** We define a class of functions $\mathcal{L} \subseteq Y^\infty(\Omega)$ as follows: $f \in \mathcal{L} \subseteq Y^\infty(\Omega)$ if and only if for every compact set $K = K_m$ for some $m = 1, 2, \cdots$ and for every polynomial p, there exists a sequence $(\lambda_n)_{n\geq 1} \subseteq \mathbb{N}$ so that the following hold.

$$\sup_{z\in K} |T^{(\ell)}_{\lambda_n}(f)(z) - p^{(\ell)}(z)| \to 0 \text{ as } n \to +\infty \tag{$\mathcal{L}$.1}$$

for every $\ell \in \mathbb{N}$.

$$\sup_{z\in L} |T^{(\ell)}_{\lambda_n}(f)(z) - f^{(\ell)}(z)| \to 0 \text{ as } n \to +\infty \tag{$\mathcal{L}$.2}$$

for every $\ell \geq 0$ and for every compact set $L = L_\tau \subseteq \Omega$, for some $\tau = 1, 2, \cdots$.

We now present the main result of this section.

Theorem 5.11. The class $\mathcal{L}$ is a G_δ - dense subset of $Y^\infty(\Omega)$ and therefore $\mathcal{L} \neq \emptyset$.

Proof. We consider $\{f_j\}_{j\geq 1}$ an enumeration of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

Next, for every parameter τ, m, j, k, s and N, we consider the following sets

$$E(K_m, f_j, k, s, N) = \{f \in Y^\infty(\Omega) : \sup_{z\in K_m} |T^{(\ell)}_k(f)(z) - f^{(\ell)}_j(z)| < \frac{1}{s} \text{ for every } \ell = 0, \cdots, N\}. \tag{5.15}$$

$$F(L_\tau, k, s, N) = \{f \in Y^\infty(\Omega) : \sup_{z\in L_\tau} |T^{(\ell)}_k(f)(z) - f^{(\ell)}(z)| < \frac{1}{s} \text{ for every } \ell = 0, \cdots, N\}. \tag{5.16}$$

Then, one should verify that the following relation holds.

$$\mathcal{L} = \bigcap_{\tau=1}^{+\infty} \bigcap_{m=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcap_{j=1}^{+\infty} \bigcap_{N=0}^{+\infty} \big(\bigcup_{k=0}^{+\infty} E(K_m, f_j, k, s, N) \bigcap F(L_\tau, k, s, N)\big). \tag{5.17}$$

In order to apply Baire's theorem, we have to prove the following.

Claim 5.12. For every parameter, the set

$$A(\tau, m, s, j, N) \equiv \bigcup_{k=0}^{+\infty} (E(K_m, f_j, k, s, N) \bigcap F(L_\tau, k, s, N)) \tag{5.18}$$

is dense in $Y^\infty(\Omega)$.

Proof of Claim 5.12 We fix the parameters $\tau, m, s, j \geq 1$ and $N \geq 0$ and we want to prove that the set $A(\tau, m, s, j, N)$ is dense in $Y^\infty(\Omega)$.

Let $f \in Y^\infty(\Omega)$ and V_f be an open basic neighbourhood of f in $Y^\infty(\Omega)$. We may assume that

$$V_f = \{g \in T^\infty(\Omega) : \sup_{z\in L_{n_1}} |f^{(\ell)}(z) - g^{(\ell)}(z)| < \varepsilon \text{ for every } \ell = 0, \cdots, M\} \cap Y^\infty(\Omega), \tag{5.19}$$

where $M \geq N$ and $L_\tau \subseteq L_{n_1}$. Our aim is to find a function $g \in V_f \cap A(\tau, m, s, j, N)$. We notice the following.

(8) Every connected component of $\mathbb{C} \cup \{\infty\} \setminus K_m \cup L_{n_1}$ contains also a $c_\ell \in F$.

(9) $K_m \cap L_{n_1} = \emptyset$.

From our initial assumptions about the sequence $\{L_n\}_{n\geq 1}$ we have that every connected component of $\mathbb{C} \cup \{\infty\} \setminus L_{n_1}$ contains a c_ℓ.

Let V be a component of $\mathbb{C} \cup \{\infty\} \setminus L_{n_1}$ (which is an open set). Then, the set $V \cap K_m$ has connected complement and also the set $V \setminus K_m$ is connected. In order to explain property (8), if $\mathbb{C} \cup \{\infty\} \setminus L_{n_1} = \cup_{i\in I} A_i$ is the disjoint union of its connected components, then every connected component of $\mathbb{C} \cup \{\infty\} \setminus (K_m \cup L_{n_1})$ is exactly of the form $(\mathbb{C} \cup \{\infty\} \setminus K_m) \cap A_i = A_i \setminus K_m$, where $\mathbb{C} \cup \{\infty\} \setminus K_m$ is connected, $\Gamma \subseteq \mathbb{C} \cup \{\infty\} \setminus K_m$ and therefore, according to property (7), for every single $i \in I$ we are able to select a $c_\ell \in (\mathbb{C} \cup \{\infty\} \setminus K_m) \cap A_i$. Property (9) is immediate.

We consider the function $H : K_m \cup L_{n_1} \to \mathbb{C}$ as follows.

$$H(z) = \begin{cases} f_j(z), & \text{if } z \in K_m \\ f(z), & \text{if } z \in L_{n_1}. \end{cases} \tag{5.20}$$

According to Lemma 5.9, it is possible to find an open neighbourhood $S_1 \subseteq \mathbb{C}\cup\{\infty\}$ of $K_m \cup L_{n_1}$, such that every bounded connected component of $\mathbb{C} \cup \{\infty\} \setminus S_1$ contains a $c_{\ell'} \in F$. This can be done by setting $A = \{c_\ell : \ell \geq 1\} \cup \{\infty\}$, $V = \mathbb{C} \cup \{\infty\}$ and $K = K_m \cup L_{n_1}$. We apply Runge's theorem in order to approximate the function H uniformly on each compact subset of S_1 with rational functions with poles only in $\{c_\ell : \ell \geq 1\} \cup \{\infty\}$. Since S_1 is open, Weierstrass' theorem implies that the previous approximation is valid for every finite set of derivatives. Thus, it is possible to find a rational function g with poles only in $\{c_\ell : \ell \geq 1\} \cup \{\infty\}$ and an index $k_0 \in \mathbb{N}$ such that $T_k(g) = g$ for every $k \geq k_0$ and $g \in V_f \cap A(\tau, m, s, j, N)$. Obviously, $g \in Y^\infty(\Omega)$. ■■

Claim 5.13. For every parameter, the sets $E(K_m, f_j, k, s, N)$ and $F(L_\tau, k, s, N)$ are open subsets of $Y^\infty(\Omega)$.

Proof of Claim 5.13 Let $\{g_r\}_{r\geq 1} \in Y^\infty(\Omega) \setminus F(L_\tau, k, s, N)$ and $g \in Y^\infty(\Omega)$ such that $g_r \to g$ as $r \to +\infty$ in $Y^\infty(\Omega)$. It follows that there exists an $\ell_0 \in \{0, \cdots, N\}$ such that

$$\begin{aligned} \frac{1}{s} &\leq \sup_{z\in L_\tau} |T_k^{(\ell_0)}(g_r)(z) - g_r^{(\ell_0)}(z)| \\ &\leq \sup_{z\in L_\tau} |T_k^{(\ell_0)}(g_r)(z) - T_k^{(\ell_0)}(g)(z)| \\ &+ \sup_{z\in L_\tau} |T_k^{(\ell_0)}(g)(z) - g^{(\ell_0)}(z)| \\ &+ \sup_{z\in L_\tau} |g^{(\ell_0)}(z) - g_r^{(\ell_0)}(z)|. \end{aligned} \tag{5.21}$$

Since every $T_k^{(\ell_0)}$ is a continuous function, by taking limits in Relation (5.21) as $r \to +\infty$ it follows that.

$$\frac{1}{s} \leq \sup_{z\in L_\tau} |T_k^{(\ell_0)}(g)(z) - g^{(\ell_0)}(z)| \tag{5.22}$$

and thus, the set $Y^{\infty}(\Omega) \setminus F(L_n, k, s, N)$ is a closed one in $Y^{\infty}(\Omega)$. The proof that every $E(K_m, f_j, k, s, N)$ is also an open subset of $Y^{\infty}(\Omega)$ is similar and therefore is omitted.

■■

We apply Baire's theorem and that completes the proof.

■

Remark 5.14. In the class $\mathcal{L}$ the approximation $(\mathcal{L}.1)$ is more generally valid for every set K satisfying $K \subseteq K_m$ for some $m \geq 1$; in particular for every compact set K with connected complement such that $K \subseteq K_m$ for some $m \geq 1$.

Indeed, let K be such a compact set and $f \in \mathcal{L}$. We consider the following class of functions.

$$\mathcal{L}(K) = \bigcap_{\tau=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcap_{j=1}^{+\infty} \bigcap_{N=0}^{+\infty} \Big(\bigcup_{k=0}^{+\infty} E(f_j, K, k, s, N) \bigcap F(L_\tau, k, s, N) \Big) \tag{5.23}$$

where for every parameter τ, j, k, s and N, we consider the following sets

$$E(f_j, K, k, s, N) = \{ f \in Y^{\infty}(\Omega) : \sup_{z \in K} |T_k^{(\ell)}(f)(z) - f_j^{(\ell)}(z)| < \frac{1}{s} \text{ for every } \ell = 0, \cdots, N \}. \tag{5.24}$$

$$F(L_\tau, k, s, N) = \{ f \in Y^{\infty}(\Omega) : \sup_{z \in L_\tau} |T_k^{(\ell)}(f)(z) - f^{(\ell)}(z)| < \frac{1}{s} \text{ for every } \ell = 0, \cdots, N \}. \tag{5.25}$$

It is almost immediate that $\mathcal{L} \subseteq \mathcal{L}(K)$ because if $K \subseteq K_m$ it follows that

$$E(f_j, K_m, k, s, N) \subseteq E(f_j, K, k, s, N). \tag{5.26}$$

The last relation yields the result.

Remark 5.15. Theorem 5.11 in the case of Example 5.2 gives a generic result in the space $A^{\infty}(\Omega)$ of holomorphic functions in Ω whose all derivatives extend continuously on $\overline{\Omega}$. In that example the universal approximation in not requested at any point of the boundary. In the case of Example 5.5, Theorem 5.11 gives a generic result in $H(\Omega)$ (see [34]). Finally, in the case of Example 5.7, Theorem 5.11 gives a generic result where the universal approximation is valid on a part of the boundary and on another disjoint part of the boundary the universal function is smooth.

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