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Το Θεώρημα Hurewicz

Διπλωματική Εργασία για την απόκτηση
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Θεοδώρα Λεβέντη

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Περίληψη

Στην παρούσα εργασία παρουσιάζεται μια απόδειξη του θεωρήματος Hurewicz για ζεύγη χώρων (X, A). Το θεώρημα Hurewicz (σχετική περίπτωση) παρέχει έναν ισομορφισμό μεταξύ ομάδων ομολογίας και πηλίκων ομάδων ομοτοπίας για $(n-1)$-συνεκτικά ζεύγη χώρων (X, A), όπου το A είναι κατά τόξα συνεκτικό. Διατυπώθηκε από τον Witold Hurewicz το 1935, ο οποίος εισήγαγε, επίσης, την έννοια της ομοτοπίας μεταξύ χώρων, όρισε τις απόλυτες και σχετικές ομάδες ομοτοπίας διάστασης $n \geq 2$ και σχημάτισε μια μακρά ακολουθία για τις ομάδες αυτές. Η απόδειξη που επιλέξαμε να παρουσιάσουμε εδώ δεν κάνει χρήση CW προσέγγισης, αλλά ακολουθεί έναν πιο ομολογιακό τρόπο σκέψης. Οδηγό αποτέλεσαν οι αποδείξεις που περιλαμβάνονται στις πηγές [1], [2] και [3].

Η παρούσα εργασία διαρθρώνεται ως εξής:

Στο Κεφάλαιο 1 δίνονται απαραίτητες έννοιες σχετικές με τους τοπολογικούς χώρους, διάφορες τοπολογίες και τοπολογικές ιδιότητες. Παρουσιάζονται βασικές πράξεις επί χώρων, όπως κύλινδροι, κώνοι, suspensions, κύλινδροι απεικόνισης, αλλά και συγκεκριμένοι χώροι που εμφανίζονται στην αλγεβρική τοπολογία, όπως ο χώρος βρόχου, ο H-χώρος (Hopf-space), τα simplices, τα simplicial complexes και τα CW complexes. Ιδιαίτερα αναλύεται ο n-διάστατος πραγματικός προβολικός χώρος $\mathbb{R}P^n$.

Στο Κεφάλαιο 2 ασχολούμαστε με τη θεωρία ομοτοπίας. Εισάγονται οι έννοιες της ομοτοπίας μεταξύ απεικονίσεων, της παραμόρφωσης συστολής, και των ομοτοπικά ισοδύναμων χώρων. Στη συνέχεια ορίζονται η θεμελιώδης ομάδα ενός χώρου, οι ομάδες ομοτοπίας ανώτερης τάξης και οι σχετικές ομάδες ομοτοπίας ζεύγους χώρων. Κάνοντας χρήση του χώρου βρόχου, των mapping fibres και εφαρμόζοντας συναρτητές σχηματίζονται μακρές ακολουθίες ομάδων ομοτοπίας, οι οποίες αποδεικνύονται τελικά ακριβείς.

Στο Κεφάλαιο 3 ορίζεται η θεωρία ομολογίας αξιωματικά. Επιλέγεται η singular ομολογία προκειμένου να δειχθούν η ύπαρξη ομολογίας και η μοναδικότητά της ως προς ισομορφισμό, ενώ η simplicial και η cellular ομολογία εισάγονται συνοπτικά στο τέλος του κεφαλαίου. Με τη χρήση τους γίνονται ορισμένοι υπολογισμοί ομάδων, όπως του κύκλου και του προβολικού χώρου διάστασης n.

Το Κεφάλαιο 4 ξεκινά με ορισμένα στοιχεία για την simplicial προσέγγιση χώρων. Ακολούθως παρουσιάζονται κάποια σημαντικά για την αλγεβρική τοπολογία θεωρήματα και ορισμένα εξ αυτών αποδεικνύονται αναλυτικά. Πρόκειται για το θεώρημα της cellular προσέγγισης, το θεώρημα της CW προσέγγισης χώρων και το excision θεώρημα για την ομοτοπία.

Τέλος, στο Κεφάλαιο 5 περιλαμβάνεται η απόδειξη του θεωρήματος Hurewicz. Πιο συγκεκριμένα διατυπώνονται το απόλυτο και το σχετικό θεώρημα Hurewicz και αποδεικνύεται το σχετικό με μια προσέγγιση από την σκοπιά της ομολογίας. Το απόλυτο προκύπτει ως πόρισμα μέσω του σχετικού θεωρήματος.

Abstract

In this thesis we present a proof of the Hurewicz theorem. The Hurewicz theorem (relative case) provides an isomorphism between homology groups and quotients of homotopy groups of $(n-1)$-connected pairs of spaces. It was stated by Witold Hurewicz in 1935, who also introduced the notion of homotopy equivalence between spaces, defined absolute and relative homotopy groups of dimension $n \geq 2$ and formed the long sequence of these groups. Although we devote a chapter to CW approximation of spaces and the excision theorem in recognition of their significance in algebraic topology, our proof of the Hurewicz theorem does not make use of them, like many others do. Instead, it is based on a more homological approach, which follows the proofs found in [1], [2] and [3].

The thesis is structured as described below:

In Chapter 1 the necessary background about topological spaces, topologies and topological properties is introduced. Basic operations on spaces, such as cylinders, cones, suspensions and mapping cylinders, are defined. Also, particular spaces that are significant in algebraic topology are presented. More specifically, we introduce loop spaces, H-spaces, simplices, simplicial complexes and CW-complexes. There is also a special reference to the real projective n-space.

Chapter 2 contains information regarding homotopy theory. The notions of homotopy between maps, deformation retraction and homotopy equivalent spaces are presented. Their presentation is followed by the introduction of the fundamental group and higher homotopy groups of a space. The group construction for $n > 1$ is given by the fundamental group of loop spaces. After that we generalise to relative homotopy groups, or in other words homotopy groups that refer to pairs of spaces, and we form, with the help of loop spaces, of mapping fibres and with the application of functors, long sequences of these groups that turn to be exact.

In Chapter 3 homology theory is defined axiomatically. Singular homology is chosen in order to show the existence of a homology theory. Its uniqueness up to isomorphism is ensured as well. At the end simplicial and cellular homology are presented briefly and some computations of homology groups are executed.

Chapter 4 includes some major theorems that are considered basic in algebraic topology. More specifically, we shortly discuss simplicial approximation, we state and prove the cellular approximation and the CW approximation theorems in detail, while we mention without proof the excision theorem of homotopy.

In Chapter 5, finally, the Hurewicz relative theorem is shown, using a homological approach. The absolute Hurewicz theorem is just stated, since its proof follows from the relative case.

Contents

Introduction

Algebraic topology employs algebraic tools to study topological spaces. It aims at finding invariants that can lead to the classification of topological spaces up to homeomorphism or at least up to homotopy equivalence. Algebraic invariants of the kind are homotopy groups, homology groups and cohomology groups. Here we are concerned with homology and homotopy theories, as well as their similarities and their relation to one another expressed by the Hurewicz theorem.

Homology theories were historically the first to come. Their definition is axiomatic, while particular examples include simplicial homology, singular homology and cellular homology. It can be shown that all these theories give homology groups for a space that are isomorphic. Roughly speaking, homology groups classify existing n-holes in a space up to chain homotopy. One of the axioms they satisfy is the excision axiom, which facilitates their computation tremendously. Also, long exact sequences of homology groups emerge almost effortlessly, and provide an additional tool for calculations.

Moving on to homotopy theory, we start with spaces X of low dimension to see that the fundamental group $\pi_1(X)$ suffices to enable their study. The fundamental group consists of equivalence classes of basepointed maps $f : S^1 \longrightarrow X$, where homotopy of maps is the equivalence relation. Its computation can be carried out, if needed, using the van Kampen theorem or actions on covering spaces. However, $\pi_1(X)$ fails to provide helpful information when it comes to spaces of higher dimension. For example, if we take n-spheres, perhaps the simplest noncontractible spaces, we get $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) \cong 0$, $n \geq 2$. Therefore spheres S^n with $n \geq 2$ can not be distinguished via their fundamental group and finer tools are needed.

To this end, higher homotopy groups $\pi_n(X)$ are considered, since they serve as the natural higher dimensional analog of the fundamental group. They classify all continuous maps from S^n to pointed topological spaces X up to homotopy equivalence. Their study has shown that, fruitful as they might be, they are far from easily tamed. Basic machinery, like the van Kampen theorem for fundamental groups or the excision theorem and Mayer-Vietoris sequences for homology groups, is not applicable to π_n groups, if $n \geq 2$. This fact makes their computation very hard.

Let us examine n-spheres again. Based on intuition someone might conject that $\pi_m(S^n) \cong 0$, when $m > n$. After all, the same intuitive idea has quite right led us to believe that S^n possesses no $(n+1)$-holes, hence $H_m(S^n) \cong 0$ for $m > n$. But higher homotopy groups are weird. As Hopf showed in 1931 in [4], there exists a map $f : S^3 \longrightarrow S^2$ that is not homotopic to the constant map. To be more specific, f generates the 2-sphere's third homotopy group, $\pi_3(S^2) \cong \mathbb{Z}$, and this counterintuitive result crushes any hope for simplicity, when $m > n$. Computation of higher homotopy groups proves to be a challenge and investigation of the homotopy groups of spheres for $m > n$ is till this day an open research field.

Witold Hurewicz, who was the one to introduce the notion of homotopy, to define homotopy groups, absolute and relative, and to construct a long sequence of homotopy groups, which was later proved to be exact, studied the relationship between homology and homotopy groups right from their conception. He constructed the homomorphisms

$$h_n : \pi_n(X, x_0) \longrightarrow H_n(X; \mathbb{Z}),\ n \geq 1$$

$$h_n : \pi_n(X, A, x_0) \longrightarrow H_n(X, A; \mathbb{Z}),\ n \geq 2$$

which in general are neither injective nor surjective. However, they are isomorphisms if X is $(n-1)$-connected or (X, A) is $(n-1)$-connected and A simply connected, respectively. If A is just path connected in the relative case, one takes a quotient ${\pi_n}'$ of the homotopy group instead to ensure that the induced ${h_n}'$ is again an isomorphism. Hurewicz stated the homomorphisms and this 'Equivalence theorem' and said he could prove the isomorphism part, though he did not publish a proof for the relative case ([5]). Various proofs were written afterwards.

The Hurewicz theorem and its various proofs give an insight in the differences and the similarities between homology and homotopy groups. It also provides an way to calculate the first nontrivial homotopy group of a space or a pair of spaces, by executing the much easier computation of their homology group. $\pi_n(S^n) \cong \mathbb{Z}$, $n \geq 2$, comes as an easy corollary. Unfortunately its beneficial effect stops at the first non trivial homotopy group, because homotopy fails the excision axiom for larger n.

Concluding this introduction, let us ponder on a heretical why to bother with homotopy theory at all. Why should we need theorems like the Hurewicz to unlock homotopy theory's mysteries and not be content with just homology and cohomology theories? Apart from homotopy groups being an interesting structure that one would like to study just out of curiosity and for the pleasure of it, homotopy theory can also contribute to our theoretical knowledge and comprehension of spaces, help us answer classical questions regarding manifolds and maps between manifolds, deal with some extension or lifting problems, contribute to the development of new branches of algebraic topology, such as K-theory, or produce results in other fields of mathematics, such as graph theory, singularity theory and more. Additionally, it appears to have applications in physics, chemistry, biology and medical science [6].

Chapter 1

Spaces

1.1 Topological spaces

Algebraic topology revolves principally around the category of topological spaces and its subcategories. Thus we commence by defining the category of topological spaces. A brief introduction to categories in general is provided in the Appendix.

Definition 1.1.1. A **topological space** is an ordered pair (X, T), where X is a set and T is a collection of subsets of X, satisfying the following axioms:

1. $\varnothing \in T$ and $X \in T$.
2. If $A_i \in T$, $i \in I$, then $\bigcup\limits_{i \in I} A_i \in T$.
3. If $A_i \in T$, $i \in \{1, ..., n\} = S$, $n \in \mathbb{N}$, then $\bigcap\limits_{i \in S} A_i \in T$.

Elements of T are called open sets and the collection T is called a **topology** on X.

Remark. Naturally enough, a subset $K \subset X$ is closed if its complement in X is open.

Remark. Formally, a topological space is denoted with the pair (X, T). However, we will scarsely use the formal notation. We will use a simplified X instead.

Definition 1.1.2. Given a set X and two topologies T_1 and T_2 on X, we say that T_1 is coarser than T_2 or T_2 finer than T_1 when $T_1 \subset T_2$.

Example 1.1.3. On a set X $T_1 = \{\varnothing, X\}$ is the coarsest and $T_2 = P(X)$ is the finest topology.

It is generally difficult to describe all the open sets in a topology T. However, one can usually use an appropriately chosen and more easily described subcollection of spaces in T, noted with $\mathscr{B}$, which can 'produce' any open set in T as a union of sets in $\mathscr{B}$.

Definition 1.1.4. A subcollection $\mathscr{B}$ of a topology T on a topological space X is a **basis for** T if, given any open set $U \in T$ and point $x \in U$, there is an open set $B \in \mathscr{B}$ such that $x \in B \subset U$.

Example 1.1.5. The real line $(\mathbb{R}, \mathscr{U})$, where $\mathscr{U}$ is the topology produced by the basis $\mathscr{B} = \{(a, b) \subset \mathbb{R} |\ a,\ b \in \mathbb{Q}\}$, is a topological space.

Example 1.1.6. The standard topology of $\mathbb{R}^n$ is the collection of open sets in $\mathbb{R}^n$. Here a set U is open iff for every $p \in U$ there is an open ball $B(p, \varepsilon)$ with center p and radius $\varepsilon > 0$ such that $B(p, \varepsilon) \subset U$.

Definition 1.1.7. Let (X, T_X) be a topological space and A be a subset of X. A becomes a topological space (A, T_A), where T_A is the **relative topology of** A **in** X defined as $T_A = \{U \cap A |\ U \in T_X\}$.

Example 1.1.8. $I \subset \mathbb{R}$ becomes a topological space, when we endow it with the relative topology in $\mathbb{R}$.

Definition 1.1.9. Let (X, T_X), (Y, T_Y) be topological spaces. A function $f : X \longrightarrow Y$ is called **continuous** on X if the inverse image $f^{-1}(U)$ of every open subset $U \subseteq Y$ is open in X. Using only notation, we get that f is continuous, iff $f^{-1}(U) \in T_Y$ $\forall\ U \in T_X$.

Taking as a class of objects $Obj(\mathscr{T})$ all the topological spaces, as a set of morphisms between objects $Hom(X, Y)$, $X,\ Y \in Obj(\mathscr{T})$, the set of all continuous functions from X to Y and as composition rule the usual composition of functions, we form the **category** $\mathscr{T}$ **of topological spaces**.

Homeomorphisms form a subset in the set of all continuous functions in $\mathscr{T}$. They are defined as:

Definition 1.1.10. A function $f \in \mathrm{Hom}(X, Y)$ of $\mathscr{T}$ is called a **homeomorphism** if it is a bijection and its inverse f^{-1} belongs to $\mathrm{Hom}(Y, X)$. Two topological spaces X and Y are **homeomorphic** if there is a homeomorphism f between them. If so, we write $X \equiv Y$.

In topology there are several properties, called topological properties, which remain invariant under homeomorphisms. This means that, if a space possesses one topological property, then every space homeomorphic to it will possess the exact same property. Some common topological properties are separation, countability conditions, connectedness and compactness.

Below, we mention some definitions which appertain to the aforementioned topological properties.

Definition 1.1.11. A topological space X is a **Hausdorff space** if for all distinct points x, y in X there exist a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint. In that case, points x and y are called pairwise neighbourhood separable.

Definition 1.1.12. A topological space X is said to be **second countable** if it has a countable basis.

Definition 1.1.13. A space X is **connected** if it can not be written as the union of a pair of disjoint non-empty open sets. Equivalently, a space X is connected if the only sets that are simultaneously closed and open in X are $\emptyset$ and X.

Definition 1.1.14. A space X is **path-connected** if for every two points x, y in X, there is a **path** $p : I \longrightarrow X$ from x to y, i.e. a continuous map $p : [0, 1] \longrightarrow X$ with $p(0) = x$ and $p(1) = y$.

Remark. Path-connected spaces are always connected. The inverse is not always true.

Definition 1.1.15. A space X is **compact** if every open cover has a finite subcover.

Compactness proves very handy when studying spaces, since it allows us to focus on finite subcovers that are more readily handled. The following Propositions stem from these finite subcovers and are going to be needed later on.

Proposition 1.1.16. *A space X is compact if and only if any decreasing sequence of nonempty closed sets has nonempty intersection.*

Proof. We refer to Theorem 5.9 and Corollary 5.10 in [7] for the proof. □

Proposition 1.1.17. *Let (X, d) be a metric space (we refer to [7] for the definition of a metric space) that is compact and $\mathscr{U}$ be an open cover of X. Then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some $U \in \mathscr{U}$. δ is called a* ***Lebesgue number*** *of this cover.*

Proof. Let $\{U_i\}$ be a finite subcover of X, $i \in \{1, \dots, m\}$, and the result do not hold. Then $\forall\ n > 0$ there exists a set $A_n \subset X$ with $diam(A_n) < 1/n$ such that $A_n \cap (X - U_i) \neq \emptyset$ for all $i \in \{1, \dots, m\}$. If we take the closures $\bar{A}_n$, then $diam(\bar{A}_n) \leq 1/n$ and the closed sets $F_{ni} = \bar{A}_n \cap (X - U_i)$ are nonempty for all $i \in \{1, \dots, m\}$. But $\bigcap_n F_{ni} \subset \bigcap_n \bar{A}_n = \{x\}$ for all $i \in \{1, \dots, m\}$ ($\bigcap_n \bar{A}_n \neq \emptyset$ and $diam(\bigcap_n \bar{A}_n) =$

$lim_n 1/n = 0$), which means that there exists a point x that belongs in every $(X - U_i)$, thus $x \in \bigcap_i (X - U_i) \Rightarrow \exists\ x \in (X - \bigcup_i (U_i))$, which can not be, since $\{U_i\}$ is a cover of X. □

Proposition 1.1.18. *Let X, $Y \in \mathscr{T}$, $f : X \longrightarrow Y$ be a continuous function and $A \subset X$ be compact. Then the image $f(A) \subset Y$ is also compact.*

Proof. See Theorem 5.5 in [7]. □

Proposition 1.1.19. *Let $(X, T_X) \in \mathscr{T}$ be Hausdorff and A be a compact subset of X. Then A is closed in (X, T_X).*

Proof. We are going to show that $X - A$ is open. Let $x \in X - A$. Since X is Hausdorff we can find disjoint open neighbourhoods V_a and $U_{x(a)}$ for every $a \in A$. The collection $\{V_a\}_{a \in A}$ is an open cover of the compact set A. We take $\{V_{a_i}\}$, $i \in \{1, ..., m\}$, to be a finite subcover. Then $U = \bigcap_i U_{x_{a_i}} \in T_X$ and $U \cap \bigcup_i V_{a_i} = \emptyset$. But $A \subset \bigcup_i V_{a_i}$, thus U is an open neighbourhood of x such that $U \subset X - A$. □

A topological space X may possess a topological property globally. For example, X can be compact, connected or path-connected. However, the same properties may be attributed to a space X locally, occuring to sufficiently or arbitrarily small neighbourhoods of points.

Definition 1.1.20. A topological space X is called **locally compact** if $\forall x \in X$ there exists a compact neighbourhood C in X which contains x.

Remark. Every compact space is locally compact. The inverse does not hold.

Definition 1.1.21. A topological space X is called **locally Euclidean** of dimension n if every $x \in X$ has a neighbourhood U such that there exists a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair (U, ϕ) a chart, U a coordinate neighbourhood or coordinate open set, and ϕ a coordinate map or coordinate system on U.

With the general category of topological spaces and the necessary for us topological properties defined, we are now ready to present several individual types of topologies or topological spaces which will emerge frequently in our study henceforth.

Definition 1.1.22. Let (X, T_X) be a topological space and let $\sim$ be an equivalence relation on X. The **quotient space** (Y, T_Y) is defined to be the set $Y := X/_\sim = \{[x] |\ x \in X\}$ of equivalence classes of elements of X equipped with the **quotient topology** T_Y. T_Y consists of all the subsets which have an open preimage under the surjective map $p : X \longrightarrow X/_\sim$, namely $T_Y = \{U \subset Y |\ p^{-1}(U) \in T_X\}$.

Remark. This topology is the finest which makes the projection map $p : X \longrightarrow X/_\sim$ continuous.

We prove here a basic theorem for quotient spaces, that will be used further on.

Theorem 1.1.23 (Universal property of quotient spaces)**.** *Let X be a topological space, $\sim$ an equivalence relation on X and $X/_\sim$ the quotient space. Also, let $p : X \longrightarrow X/_\sim$ be the canonical projection. If $f : X \longrightarrow Z$ is a continuous map such that $a \sim b$ implies $f(a) = f(b)$ for all a and b in X, then there exists a unique continuous map $\bar{f} : X/_\sim \longrightarrow Z$ such that $f = \bar{f} \circ p$.*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & X/_\sim \arrow[dl, dashed, "\bar{f}"] \\ Y & \end{tikzcd}$$

Figure 1.1

Proof. We define a map $\bar{f} : X/_\sim \longrightarrow Y$ by the formula $\bar{f}([a]) = f(a)$, where $[a] \in X/_\sim$ is the equivalence class of $a \in X$. $\bar{f}$ is well defined, because $f(a) = f(b)$ when $[a] = [b]$. Also, $f = \bar{f} \circ p$, by the way we defined $\bar{f}$.
Now, in order to prove that $\bar{f}$ is continuous, we take U to be an open subset of Y and observe that $\bar{f}^{-1}(U) = \{[x] \in X |\ x \in f^{-1}(U)\}$. This set is open if and only if $p^{-1}(\bar{f}^{-1}(U))$ is open. The latter, however, is exactly $f^{-1}(U)$ and is open since f is continuous.
Let now h be another continuous function, with the properties described in the theorem. Then $h([a]) = h(p(a)) = f(a) = \bar{f}([a])$ for all $[a] \in X/_\sim$, which leads to $h = \bar{f}$ and $\bar{f}$ being unique. □

Definition 1.1.24. Let X be a set and $\{f_i\}_{i\in I}$ an indexed family of functions on X. A **weak topology** on X with respect to $\{f_i\}_{i\in I}$ is the coarsest topology on X that makes these functions continuous.

After topologies, we also need to define categories of spaces that we will use. In homotopy theory, where loops, spheroids and their homotopy classes are considered, one needs to choose, name and stabilise a point in each space and refer simultaneously to the space as well as the point. This point will serve as the basepoint of every loop or spheroid in this space, leading eventually to well defined structures.

Definition 1.1.25. Let (X, T_X) be a topological space and $x_0 \in X$. (X, x_0) is called a **pointed space**. Given another pointed space (Y, y_0), a map $f : (X, x_0) \longrightarrow (Y, y_0)$ is called **basepoint preserving** or **pointed map** if $f(x_0) = y_0$. Using pointed spaces as objects, basepoint preserving maps as morphisms and the usual composition of functions, we may form the category $\mathscr{T}^*$ of pointed topological spaces.

Remark. The notation $(X, *)$ will be often employed to represent a pointed space in a slightly more abstract fashion, while the point x_0 will be suppressed altogether, when confusion is improbable.

A subcategory of $\mathscr{T}$ is the category of compactly generated Hausdorff spaces, $\mathscr{CG}$. Compactly generated Hausdorff spaces, $\mathscr{CG}$, constitute a convenient category of spaces in algebraic topology, as shown in [8]. In our study, they are going to be utilised, when the homotopy groups are introduced.

Definition 1.1.26. Let (X, T) be a Hausdorff topological space. X is said to be a **compactly generated** Hausdorff space if every $K \subset X$ which intersects every compact set in a closed set is itself closed. We denote by $\mathscr{CG}$ the category of compactly generated Hausdorff spaces and their continuous functions.

Definition 1.1.27. If X is a Hausdorff space, the **associated compactly generated space** $k(X)$ is the set X with the topology defined as follows: a closed set of $k(X)$ is a set that meets each compact set of X in a closed set.

For X, Y Hausdorff spaces, $C(X, Y)$ represents the space of continuous maps $X \longrightarrow Y$ with the compact-open topology, as defined below. Furthermore, we denote with Y^X the associated compactly generated space $k(C(X, Y))$.

Definition 1.1.28. If K is a compact set in X and U is an open set in Y, let $W(K, U)$ denote the set of all functions $f \in C(X, Y)$ with $f(K) \subset U$. The family of the sets $W(K, U)$, created for all possible compact-open pairs (K, U), forms a subbasis for open sets of $C(X, Y)$. This subbasis defines the **compact-open** topology on $C(X, Y)$.

Keeping the aforementioned notation of sets of functions in mind, we may introduce the **loop space** ΩX of a pointed space $(X, *)$:

Definition 1.1.29. $\Omega X = \{\omega \in X^I | \ \omega(0) = \omega(1) = *\}$ is called the loop space of X. The loop space is an associated compactly generated space and a subspace of X^I. As a subspace, it also inherits the corresponding topology. $\Omega^n X = \Omega(\Omega^{n-1} X)$ is inductively defined and it has the constant loop at $*$ as its basepoint.

As implied by the name, the loop space consists of all the loops in X. In other words, it consinsts of all the paths f in X with $f(0) = f(1) = x_0$. This space is of great importance, a fact that will become apparent when we discuss higher homotopy groups.

If a map $f : X \longrightarrow Y$ between topological spaces is given, one can construct the map $\Omega f : \Omega X \longrightarrow \Omega Y$ between the corresponding loop spaces via the formula $\Omega f(\omega(\cdot)) = f(\omega(\cdot))$.

A very significant class of topological spaces, with applications in various fields of mathematics, is without question this of topological manifolds. These are topological spaces which locally 'resemble' an n-dimensional Euclidean space in the sense discussed below. Topological manifolds, moreover, can be equipped with additional structure. This happens, for example, in the case of differentiable manifolds, which are provided with a differentiable structure.

The following are mainly based on [9]:

Definition 1.1.30. A topological space X is called a **topological manifold** if it is a locally Euclidean, second countable, Hausdorff space. The dimension of a manifold X is n if every point has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Remark. The dimension of a manifold is well defined, because an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$ if $n \neq m$. This fact is called *invariance of dimension* and is a classical result of Brouwer, which can be proved with machinery developed in algebraic topology (see Theorem 2.26 in [10]).

Definition 1.1.31. A **smooth** or C^∞ or **differentiable manifold** is a topological manifold M together with a differentiable structure or maximal atlas. An atlas on M is a collection $\mathfrak{U} = \{(U_a, \phi_a)\}$ of pairwise C^∞-compatible charts that cover M, while a maximal atlas is the atlas containing all the charts that are compatible with $\mathfrak{U}$. Finally, two charts (U, ϕ), (V, ψ) are said to be C^∞-compatible, if the composite maps $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are differentiable on $\psi(U \cap V)$ and $\phi(U \cap V)$, respectively.

Example 1.1.32 (Unit sphere S^n)**.** The unit sphere S^n is the set of points in $\mathbb{R}^{n+1}$ with unitary distance from the origin 0. It is a smooth manifold of dimension n.
For $n = 0$, we get the 0-sphere S^0 to be a pair of points. We endow those with the discrete topology, thus producing a topological space.
For $n \geq 1$, we need a topology on our set of points. For each $i \in \{0, ..., n\}$, we define hemispheres U_i^+ and U_i^- by

$$U_i^+ = \{(x_0, ..., x_n) \in S^n | \ x_i > 0\}$$
$$U_i^- = \{(x_0, ..., x_n) \in S^n | \ x_i < 0\}$$

and maps ϕ_i^+ and ϕ_i^- by

$$\phi_i^+(x_0, ..., x_n) = \phi_i^-(x_0, ..., x_n) = (x_0, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Maps ϕ_i are homeomorphisms for all i, their inverses being

$$\phi_i^{-1}(x_0, ..., x_{i-1}, x_{i+1}, ..., x_n) = (x_0, ..., x_{i-1}, \sqrt{1 - \sum_{k \neq i} x_k^2}, x_{i+1}, ..., x_n).$$

Hemispheres $U_i^\pm$ cover S^n and it can be easily shown that $\phi_i \circ \phi_j^{-1}$ is C^∞ for all pairs of hemispheres with $U_i^\pm \cap U_j^\pm \neq \varnothing$. Thus, S^n becomes a smooth topological manifold.

Real and complex projective n-spaces

We begin with the real projective n-space $\mathbb{R}P^n$, $n > 0$. $\mathbb{R}P^n$ is defined to be the space of all the lines in $\mathbb{R}^{n+1}$ that pass through the origin. Each such line can be determined by a nonzero vector in $\mathbb{R}^{n+1}$, unique up to scalar multiplication.
More formally, we define in $\mathbb{R}^{n+1}\backslash\{0\}$ the equivalence relation $\sim$ where $x \sim y \Leftrightarrow x = \lambda y$, $\lambda \in \mathbb{R}\backslash\{0\}$. This relation identifies all the points of the line passing through 0 and x, so the real projective space $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1}\backslash\{0\}$ by this equivalence relation. We denote the equivalence class of a point $(x_0, x_1, ..., x_n) \neq 0$ by $[x_0 : x_1 : ... : x_n]$.
The real projective space $\mathbb{R}P^n$ is a topological space. To prove this we need to define a topology T on $\mathbb{R}P^n$, so we will find a basis of this topology. Since the equivalence relation $\sim$ is an open equivalence relation on $\mathbb{R}^{n+1}\backslash\{0\}$, i.e. the projection map $p : \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow (\mathbb{R}^{n+1}\backslash\{0\})/_\sim = \mathbb{R}P^n$ is open (see Proposition 7.14 in [9]), a basis for $\mathbb{R}P^n$ will be $\{p(B_a)\}$, where $\mathscr{B} = \{B_a\}$ is a basis for $\mathbb{R}^{n+1}\backslash\{0\}$ (see Theorem 7.9 in [9]). To be more specific, the collection of subspaces $\{B(x,r) \cap \mathbb{R}^{n+1}\backslash\{0\}\}$, $x \in \mathbb{R}^{n+1}$, $r \in \mathbb{R}$, forms a basis of open subspaces for $\mathbb{R}^{n+1}\backslash\{0\}$, thus $\{p(B(x,r) \cap \mathbb{R}^{n+1}\backslash\{0\})\}$ is a basis which defines a topology on $\mathbb{R}P^n$.
$\mathbb{R}P^n$ is also a manifold. As shown in [9], the real projective space is Hausdorff and second countable. For the locally Euclidean part and the later necessary smooth structure, we take the sets

$$U_i := \{[x_0 : x_1 : ... : x_n] \in \mathbb{R}P^n |\ x_i \neq 0\},\ i \in \{0, 1, ..., n\}$$

and the maps

$$\begin{array}{cccc} \phi_i : & U_i & \longrightarrow & \mathbb{R}^n \\ & [x_0 : x_1 : ... : x_n] & \longmapsto & (\frac{x_0}{x_i}, ..., \frac{\hat{x}_i}{x_i}, ..., \frac{x_n}{x_i}), \end{array}$$

$$\begin{array}{cccc} \phi_i^{-1} : & \mathbb{R}^n & \longrightarrow & \mathbb{R}P^n \\ & (x_0, ..., x_{i-1}, x_{i+1}, ..., x_n) & \longmapsto & [x_0 : ... : x_{i-1} : 1 : x_{i+1} : ... : x_n], \end{array}$$

where the hat sign ^ means that the particular entry is to be omitted. ϕ and ϕ^{-1} are continuous maps, one being the inverse of the other, which proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts. It can be readily proved (see [9]) that the collection $\{(U_i, \phi_i)\}_{i=0,1,...,n}$ is also a C^∞-atlas for $\mathbb{R}P^n$, which suffices to make $\mathbb{R}P^n$ a smooth manifold.

There is, however, another way to build a differentiable structure on $\mathbb{R}P^n$. The map $\pi : \mathbb{R}P^n \longrightarrow \{$1-dimensional subspaces L of $\mathbb{R}^n\}$, with $\pi(p(x)) = \pi([x]) = \{$1-dimensional subspace L of $\mathbb{R}^n$ which contains $x\}$, is a bijection. We will use this bijection in order to find charts on the real projective space. For every L we consider the following neighbourhood in $\mathbb{R}P^n$:

$$U_L = \{\text{subspaces } K \subset \mathbb{R}^n \text{ such that the projection } K \longrightarrow L \text{ is an isomorphism}\}$$

This set is in $1{-}1$ correspondence with the set $Hom(L, L^\perp)$ of morphisms from all L to their orthogonal complements $L^\perp$. For each $K \in U_L$, the projection $K \longrightarrow L^\perp$ can be

composed with the isomorphism $L \longrightarrow K$ to produce an operator $T \in Hom(L, L^{\perp})$. Reversely, for an operator $T \in Hom(L, L^{\perp})$, we define the space K to be the graph of T, i.e. $K = \{(v, Tv)|\ v \in L\} = \{v + Tv|\ v \in L\}$.
Each U_L is a chart in $\mathbb{R}P^n$ and $n+1$ of them are needed to cover the real projective space $\mathbb{R}P^n$. Using the canonical basis $\{e_0, e_1, ..., e_n\}$ of $\mathbb{R}^{n+1}$, we consider for every $0 \leq i \leq n$ the line $L_i = \{\lambda e_{n-i}|\ \lambda \in \mathbb{R}\}$ and take the neighbourhoods

$$\begin{aligned} U_{L_i} &= Hom(L_i, \oplus_{j \neq i} L_j) \\ &= \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 & a_{1i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & a_{(i-1)i} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{(i+1)i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & a_{(n+1)i} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ \middle|\ \lambda \in \mathbb{R}, a_{li} \in \mathbb{R} \right\} \\ &= \left\{ \mathbb{R} \begin{bmatrix} a_{1i} \\ \vdots \\ a_{(i-1)i} \\ 1 \\ a_{(i+1)i} \\ \vdots \\ a_{(n+1)i} \end{bmatrix} \ \middle|\ a_{li} \in \mathbb{R} \right\}. \end{aligned}$$

Maps ϕ_{L_i} take elements $\mathbb{R} \cdot (a_{1i}, \dots, a_{(i-1)i}, 1, a_{(i+1)i}, \dots, a_{(n+1)i}) \in U_{L_i}$ to vectors $(a_{1i}, \dots, a_{(i-1)i}, a_{(i+1)i}, \dots, a_{(n+1)i}) \in \mathbb{R}^n$, while their inverses, $\phi_{L_i}^{-1}$, do the opposite. These maps are obviously continuous, thus charts (U_{L_i}, ϕ_{L_i}) are formed. It is quite straigtforward to show that these charts also form a C^∞-atlas on $\mathbb{R}P^n$.

Finally, we will elaborate on the real projective space a bit more. This time we will treat $\mathbb{R}P^n$ as a homogeneous space. This way of viewing $\mathbb{R}P^n$ is not the most frequently encountered, but it is pretty interesting. Since the underlying theory needed to fully understand homogeneous spaces lies out of the scope of our study, we restrict ourselves to just stating their definition and the basic theorem that is going to be used. For a more thorough study of homogeneous spaces we refer to [11].

Definition 1.1.33. A **homogeneous space** is a smooth manifold M endowed with a transitive, smooth action of a Lie group G. Equivalently, it is a smooth manifold M of the form G/K, where G is a Lie group and K a closed subgroup of G.

Proposition 1.1.34. *Let G be a Lie group, M a homogeneous space and $p \in M$ a random point in M. The map $f : G/G_p \longrightarrow M$ with $f(gG_p) = g \cdot p$, where $g \cdot p$ represents the left action of G on M and $G_p = \{g \in G|\ g \cdot p = p\}$ is the isotropy group, is a diffeomorphism.*

We take the general linear group $GL(n+1,\mathbb{R})$, which is a Lie group (see Chapter 12 in [11]), acting on $(\mathbb{R}^{n+1}\backslash\{0\})/_\sim = \mathbb{R}P^n$, which is a manifold. So

$$\Phi : GL(n+1,\mathbb{R}) \curvearrowright (\mathbb{R}^{n+1}\backslash\{0\})/_\sim \quad | \quad (A,[v]) \longmapsto [Av]$$

$[v]$ denotes the equivalence class in the real projective space with representative $v \in \mathbb{R}^{n+1}\backslash\{0\}$, while Av denotes the usual multiplication between a matrix and a vector.
Φ is smooth: First, we observe that the action of the general linear group $GL(n{+}1,\mathbb{R})$ on $\mathbb{R}^{n+1}\backslash\{0\}$ is smooth. Now, let $(v_0, v_1, ..., v_n) \sim (u_0, u_1, ..., u_n)$. Then $(u_0, u_1, ..., u_n) = \lambda(v_0, v_1, ..., v_n)$ for some $\lambda \in \mathbb{R}\backslash\{0\}$ and

$$\Phi(A,u) = A(u_0,u_1,...,u_n) = A\lambda(v_0,v_1,...,v_n) = \lambda A(v_0,v_1,...,v_n) \sim Av = \Phi(A,v).$$

This means that the action Φ preserves the equivalence relation $\sim$, which, using Proposition 13.1.3 in [11], leads to the smoothness of Φ.
Φ is transitive: Let $[u]$, $[v] \in \mathbb{R}P^n$. We can find an $A \in GL(n{+}1,\mathbb{R})$ such that $u = Av \Rightarrow [u] = \Phi(A,[v])$.
Therefore, $\mathbb{R}P^n$ is a homogeneous space.
If we choose the point $p = [1:0:...:0] \in (\mathbb{R}^{n+1}\backslash\{0\})/_\sim$, its isotropy group is

$$[GL(n{+}1,\mathbb{R})]_p = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ 0 & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{bmatrix} \mid a_{ij} \in \mathbb{R}, i,j \in \{1,...,(n+1)\} \right\}$$

and, from 1.1.34, $\mathbb{R}P^n$ is diffeomorphic to $GL(n{+}1,\mathbb{R}) \big/ [GL(n{+}1,\mathbb{R})]_p$.

Remark. A projective space can also be defined on the complex numbers $\mathbb{C}$. This space is named the complex projective n-space $\mathbb{C}P^n$. Its points are all the complex lines that pass through the origin of $\mathbb{C}^{n+1}$. Just like $\mathbb{R}P^n$, $\mathbb{C}P^n$ can be given a topology and a differentiable structure, becoming thus a smooth manifold.

1.2 Basic operations over topological spaces

Using spaces already defined as a basis and applying various operations over them, one can create new interesting topological spaces. In this section, such operations are introduced.

Definition 1.2.1. Let $\{X_i\}$, $i \in I$, be a family of topological spaces. Their product $X := \prod\limits_i X_i$ equipped with the product topology is a topological space, known as the **product** of $\{X_i\}$. Given the projection maps $p_i : X \longrightarrow X_i$, the **product topology** on X is defined as the coarsest topology for which all the projections are continuous. More precisely, a set in X is open if it can be written as a union of sets of the form $\prod\limits_i U_i$, where U_i are open in X_i and $U_i \neq X_i$ for finitely many i.

Remark. The product space $X \times Y$ of two spaces X, Y does not necessarily inherit the topological properties of its components. For example, if $X, Y \in \mathscr{CG}$, then $X \times Y$ endowed with the product topology defined above is not always compactly generated. However, if X is locally compact and $Y \in \mathscr{CG}$, then $X \times Y \in \mathscr{CG}$. An example that usually occurs is the product $X \times I$. Proofs of this can be found in [10] and [12].

Definition 1.2.2. Let $\{X_a\}_a$ be an arbitrary collection of spaces with chosen basepoints $x_a \in X_a$. The **wedge sum** $\bigvee_a X_a$ is the quotient of the disjoint union $\bigsqcup_a X_a$ obtained by identifying all points x_a to a single point.

Remark. When it comes to representing a point in $\bigvee_a X_a$, we will treat the wedge sum as a subspace of $\prod_a X_a$. This means that a point x in X_i, will be denoted with $(x_0, x_0, ..., x, x_0, ..., x_0, ...)$.

Remark. We can also define the wedge sum of maps. Given maps $f_a : X_a \longrightarrow Y$ that agree on the basepoints, we write $\bigvee_a f_a : \bigvee_a X_a \longrightarrow Y$ to represent the function formed by applying each f_a to the respective component X_a.

Definition 1.2.3. Let X be a space and I the segment $[0, 1]$. The product $ZX = X \times I$ is called the **cylinder** over X. The quotient of ZX obtained by collapsing $X \times \{0\}$ to one point, namely $CX = X \times I/X \times \{0\}$, is called the **cone** CX of X. Finally, the quotient of ZX obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point is called the **suspension** SX of X.

Remark. Apart from spaces, maps can also be suspended. A map $f : X \longrightarrow Y$ suspends to $Sf : SX \longrightarrow SY$, the quotient map of $f \times \mathbb{1} : X \times I \longrightarrow Y \times I$.

Example 1.2.4. For $X = S^n$ we have $CX = CS^n \equiv D^{n+1}$ and $SX = SS^n \equiv S^{n+1}$. Here we must mention that D^{n+1} is the $(n+1)$-dimensional disk, i.e. the set of points

in $\mathbb{R}^{n+1}$ which are in a distance less or equal to 1 away from the origin.
To prove the first homeomorphism, we consider the map $f : ZS^n \longrightarrow D^{n+1}$ which takes $\forall\, x \in S^n$ the path $(t, x) \in ZS^n$, $t \in I$, to a path in D^{n+1} starting at $0 \in \mathbb{R}^{n+1}$ and ending at $x \in \partial D^{n+1}$ (coordinates given by the embedment of S^n in $\mathbb{R}^{n+1}$). We have $f(x, 0) = 0$, so using 1.1.23 we get a unique continuous map $\bar{f} : CX \longrightarrow D^{n+1}$ with $f = \bar{f} \circ p$. $\bar{f}$ is surjective, since f and p are. $\bar{f}$ is also injective, because f was failing injectivity on $x \in S^n \times \{0\}$ only and this has been rectified in $\bar{f}$. Finally, CS^n is compact and D^{n+1} is Hausdorff, which leads to $\bar{f}$ being a homeomorphism (Corollary A.36 in [9]).
In a similar way, using the map $f : ZS^n \longrightarrow S^{n+1}$ which takes $\forall\, x \in S^n$ the path $(t, x) \in ZS^n$ to the path in S^{n+1} starting at $(0, -1) \in \mathbb{R}^{n+2}$, passing through $(x, 0)$ in the equatorial and ending at $(0, -1)$ (coordinates given by the embedment of S^{n+1} in $\mathbb{R}^{n+2}$), we get a homeomorphism between SS^n and S^{n+1}.

We will use suspensions of spaces later on, when we will define homotopy groups. More specifically, we will use the reduced suspension ΣX of a pointed space $(X, *)$, which ensures that the basepoint remains well defined after performing the operation and allows us to define a comultiplication $f \cdot g$ between continuous maps $f,\ g \in C(\Sigma X, Y)$, $X,\ Y \in \mathscr{T}^*$.

Definition 1.2.5. The **reduced suspension** ΣX of a pointed space (X, x_0) is the space obtained by SX if we collapse $\{x_0\} \times I$ to a single point. This can be written as a quotient space in the form $\Sigma X = X \times I/(X \times \{0, 1\} \cup \{x_0\} \times I)$.

Remark. Again, maps can be suspended in this way. A map $f : X \longrightarrow Y$ suspends in a reduced way to $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ with $\Sigma f([x, t]) = [f(x), t]$.

The comultiplication $f \cdot g : \Sigma X \longrightarrow Y$, which was previously mentioned, can be formed as a composition of maps. We start by performing a 'pinching' operation c on the reduced suspension ΣX of (X, x_0), which collapses the middle copy of X to the basepoint x_0. This leads to $\Sigma X \vee \Sigma X = \Sigma X/(X \times \{1/2\})$, on which we apply $f \vee g$. To be more explicit, for $[x, t] \in \Sigma X$, where $x \in X$ and $t \in I$, the formulas are

$$c[x,t] = \begin{cases} ([x, 2t], [*]), & 0 \le t \le \dfrac{1}{2} \\ ([*], [x, 2t-1]), & \dfrac{1}{2} \le t \le 1 \end{cases}, \tag{1.1}$$

where $[x, 0] = [x, 1] = [x_0, t] := [*]$, $\forall\, t \in I$. Then

$$(f \vee g)([\ ]_1, [\ ]_2) = \begin{cases} f([x, 2t]), & [\ ]_2 = [*] \\ g([x, 2t-1]), & [\ ]_1 = [*] \end{cases}. \tag{1.2}$$

Thus

$$(f \cdot g)[x, t] := ((f \vee g) \circ c)[x, t]. \tag{1.3}$$

is the desirable comultiplication.

Definition 1.2.6. Let X, Y be topological spaces. Their **smash product** $X \wedge Y$ is defined as the quotient space $X \times Y / X \vee Y$.

Remark. The reduced suspension ΣX is actually the same as the smash product $X \wedge S^1 = X \times S^1 / X \vee S^1$. This is readily understood since both spaces can be seen as the quotient of $X \times I$ with $X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I$ identified with a point.

Proposition 1.2.7. $\Sigma S^{n-1} = S^1 \wedge S^1 \wedge ... \wedge S^1$ *(n-times).*

Proof. The result is obvious, if one uses Rem. 1.2.6 and induction on n. $\square$

Definition 1.2.8. Let X, $Y \in \mathscr{T}$ and $f : X \longrightarrow Y$. The **mapping cylinder** Mf is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by identifying each $(x, 1) \in X \times I$ with $f(x) \in Y$. We write $Mf = (X \times I) \sqcup_f Y$.

Remark. We use the notation Mf for the mapping cylinder and reserve the notation M_f for another construction, the mapping fibre, which will be defined later on.

Definition 1.2.9. Let X, $Y \in \mathscr{T}$ and $f : X \longrightarrow Y$. The **mapping cone** Cf is the quotient space of the mapping cylinder Mf obtained by collapsing $X \times \{0\}$ to a point.

1.3 Simplices and complexes

We begin this section by introducing simplicial complexes. Simplicial complexes can either be seen as objects of combinatorial topology and be given a geometric realisation later, as done in [2], or be defined as geometric objects right from the start, totally skipping the combinatorial part. The geometric point of view is the one adopted here.

The section continues with the introduction of Whitehead or CW-complexes or cell complexes. For these objects the requirement for linearity that dominates, as will be seen, simplicial complexes is aborted. They become thus structures less rigid than simplicial complexes and more appropriate for homotopy theory.

All these objects will ultimately help us examine random topological spaces by 'subdividing' them into basic, more easily treated building blocks.

Definition 1.3.1. An **n-simplex**, $n \geq 0$, is the smallest convex set in the euclidean space $\mathbb{R}^m$, $m \geq n$, that contains $n+1$ linearly independent points v_0, v_1, ..., v_n in $\mathbb{R}^m$, called **vertices** (points v_i being linearly independent is equivalent to not all of them lying in a hyperplane of dimension less than n). This simplex is denoted with $\sigma = [v_0, v_1, \ldots, v_n]$ and is given by $\sigma = \{\sum_i \lambda_i v_i \in \mathbb{R}^m | \ \sum_i \lambda_i = 1, \lambda_i \in \mathbb{R}\}$.

If we delete k of the $n+1$ vertices of $\sigma = [v_0, v_1, \ldots, v_n]$, we get a $(n-k)$-simplex τ. The new simplex is called a $(n-k)$**-face** of σ, is denoted with $\tau = [v_{i_1}, \ldots, v_{i_{n-k}}]$, and we write $\tau \prec \sigma$. Every simplex is oriented starting from the vertex with the smallest index and moving gradually through the remaining vertices by increasing indices.

Definition 1.3.2. A geometric finite **simplicial complex** K is a collection of i n_j-simplices in $\mathbb{R}^m$, $i < \infty$, $j \in \{1, ..., i\}$, $m \geq max\{n_1, ..., n_i\}$, such that the intersection of two simplices is a face of each and each face of a simplex in K is a simplex in K. We write $|K|$ for the underlying space, i.e. the union of all the simplices in K. Obviously we can have $|K| \equiv |L|$ even if $K \neq L$.

Definition 1.3.3. Let σ be a finite simplicial complex. If σ contains at least one n-simplex, but no k-simplices, $k > n$, we say that it has dimension $dim(\sigma) = n$ or that σ is n-dimensional.

We are now going to present a very useful procedure known as barycentric subdivision, which, when applied to a simplex $\sigma \in \mathbb{R}^m$, produces a simplicial complex $\mathscr{B}(\sigma)$ that consists of simplices σ_i, that are subspaces of $|\sigma|$. If we use the Euclidean metric provided by the ambient space $\mathbb{R}^m$, we can compute the mesh of $M(\mathscr{B}(\sigma))$ of $\mathscr{B}(\sigma)$. By the term mesh we mean the maximum diameter $\max\{\text{diam}(\sigma_i)| \ \sigma_i \in \mathscr{B}(\sigma)\}$ of the simplices σ_i in $\mathscr{B}(\sigma)$ expressed as a function of σ's diameter. If the initial σ was a simplicial complex instead of a simplex, a barycentric subdivision could be executed again, this time on each simplex of σ individually. The mesh would be expressed as a function of the mesh of σ, $M(\sigma)$.

The main advantage gained from this procedure is that by applying iterated barycentric subdivisions on a simplicial complex σ we can lower the mesh of the final $\mathscr{B}^k(\sigma)$ as much as we please.

Definition 1.3.4. Suppose given an n-simplex $\sigma = [v_0, v_1, ..., v_n]$. The **barycentre** of σ is the point $b(\sigma) = \frac{v_0+...+v_n}{n+1}$. This is the centre of gravity of the vertices in the usual sense.

Definition 1.3.5. Let σ be a simplicial complex consisting of the simplices σ_i. A **barycentric subdivision** of σ is a simplicial complex σ' such that

1. the vertices of σ' are the barycentres of simplices σ_i of σ and
2. the simplices of σ' are the simplices $[b(\sigma_{i_1}), ..., b(\sigma_{i_m})]$, for $i_j \prec i_{j+1}$ and $\sigma_{i_j} \neq \sigma_{i_{j+1}}$.

Proposition 1.3.6. *Every complex has a barycentric subdivision.*

Proof. See Theorem 12.16 in [12]. □

Lemma 1.3.7. *The diameter $diam(\sigma)$ of the simplex $\sigma = [v_0, ..., v_n]$ with respect to the Euclidean norm is the maximum of the $\|v_i - v_j\|$, i, $j \in \{0, ..., n\}$.*

Proof. Let x, $y \in \sigma$ and $x = \sum \lambda_j v_j$. Then $\|x - y\| = \|\sum \lambda_j(v_j - y)\| \leq \sum \lambda_j \|v_j - y\| \leq max_j \|v_j - y\|$, since $\sum \lambda_j = 1$. But now $\|v_j - y\| = \|y - v_j\| \leq max_i \|v_i - v_j\|$, which leads to $\|x - y\| \leq max_{i,j} \|v_i - v_j\|$. □

Lemma 1.3.8. *Let σ be a simplicial complex with dimension n. Then*

$$M(\mathscr{B}(\sigma)) \leq \frac{n}{n+1} M(\sigma).$$

Proof. See Proposition 12.17 in [12] or Lemma 9.4.3 in [2]. □

Proposition 1.3.9. *Let X be a space such that there exists a simplicial complex σ with $|\sigma| \equiv X$ and let $\epsilon > 0$. Then we can find a complex τ such that $|\tau| \equiv X$ and $M(\tau) < \epsilon$.*

Proof. Let $\tau = \mathscr{B}^k(\sigma)$ and $n = dim\sigma$. We have $M(\tau) \leq (\frac{n}{n+1})^k M(\sigma)$, thus we can attain the desired limit ϵ by choosing the number of iterations k to be large enough. □

Special attention is given to the n-**dimensional standard simplex** or standard n-simplex

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, t_i \geq 0 \ \forall \ i\} = [e_0, e_1, ..., e_n],$$

and its boundary, which is the union of all its faces written as

$$\partial\Delta^n = \{(t_0, \dots, t_n) \in \Delta^n |\ t_i = 0 \text{ for some } i \in \{0, 1, ..., n\}\},$$

since they are going to play a major part later, in the presentation of singular homology theory.

A very important fact regarding simplices is that a free, abelian group G_n can be constructed having n-simplices as a basis and a sum operation which represents the union of the added n-simplices.

On G_n one can define a map $\partial_n : G_n \longrightarrow G_{n-1}$, called the **boundary map**. Let $\delta_i^n : \{0, 1, \dots, n-1\} \longrightarrow \{0, 1, \dots, n\}$ be the inclusion that omits the value i and define a map $d_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ by the formula $d_i^n(\sum_{i=0}^{n-1} t_i e_i) = \sum_{i=0}^{n} t_i e_{\delta_i^n}$, which is the i-th face of Δ^n. If we take a basis element of G_n σ and identify it with a homeomorphism $\bar{\sigma} : \Delta^n \longrightarrow |\sigma|$, we can use the maps d_i^n and define $\partial_n(\bar{\sigma}) = \sum_i (-1)^i d_i^n \bar{\sigma}$. Although the choice of the homeomorphism $\bar{\sigma}$ is free, ∂_n is well defined, since we are interested only on its image $|\sigma|$. Furthermore, ∂_n is expanded linearly on G_n, thus becoming a group homomorphism.

Proposition 1.3.10. *The pair $(\Delta^n, \partial\Delta_n)$ is homeomorphic to the pair (D^n, S^{n-1}).*

Proof. See Proposition 2.3.1 in [2]. □

A powerful tool in algebraic topology are certain spaces called Whitehead complexes or CW-complexes or cell-complexes. These spaces, which were introduced by J.H.C. Whitehead in [13], are complexes possessing an additional combinatorial structure, the CW structure. Providing a space with such a structure essentially allows its breakdown into basic structural components that are easier to manipulate. The importance of this becomes apparent especially when trying to compute homology groups.

The following notation is employed hereafter:

$D^n = \{x \in \mathbb{R}^n |\ |x| \leq 1\}$ the closed n-dimensional disc,

$\partial D^n = S^{n-1} = \{x \in \mathbb{R}^n |\ |x| = 1\}$ the $(n-1)$-dimensional sphere,

$(D^n)^o = \{x \in \mathbb{R}^n |\ |x| < 1\}$ the open n-dimensional disc,

$e^n = \{x \in \mathbb{R}^n |\ |x| < 1\}$ an n-dimensional cell homeomorphic to $(D^n)^o$.

Definition 1.3.11. A **CW-complex** or **cell-complex** is a Hausdorff space X with a fixed partition $X = \bigsqcup_{n=0}^{\infty} \bigsqcup_{i \in J_n} e_i^n$ of pairwise disjoint cells e_i^n such that:

1. for every cell e_i^n there exists a map $\Phi_i^n : e_i^n \longrightarrow X$ with $\Phi_i|_{(D^n)^o} : (D^n)^o \longrightarrow e_i^n$ a homeomorphism,
2. $\varphi_i := \Phi_i|_{\partial D^n} : S^{n-1} \longrightarrow X$ mapping S^{n-1} into the union of a finite number of cells of dimension less than n,
3. a subset of X is closed if and only if it meets the closure of each cell of X in a closed set.

Each Φ_i is called a characteristic map.

Definition 1.3.12. A **subcomplex** of a CW-complex is a subspace $A \subset X$ that is the union of cells of X such that the closure of each cell in A is contained in A. A is also a CW-complex. A pair (X, A), where X is a CW-complex and A a subcomplex, will be called a **CW pair**.

Proposition 1.3.13. *A compact subspace of a CW-complex is contained in a finite subcomplex.*

Proof. See A.1 in [10]. □

In a **CW**-complex the following two properties are satisfied:

1. **C**losure-finiteness: The closure of each cell $\bar{e_i^n}$ is contained in a finite union of cells or, in other words, meets only a finite number of cells.
2. **W**eak topology: A set $A \subset X$ is open (or closed) iff the intersection $A \bigcap \bar{e_i^n}$ is open (or closed) in X^n for any cell e_i^n. Another way to describe the topology is to say that a set $A \subset X$ is open (or closed) iff $\Phi_i^{-1}(A)$ is open (or closed) in D_i^n for each characteristic map Φ_i.

A CW-complex can be constructed inductively. We start with a discrete set X^0 consisting of 0-cells. After the first n-steps, the construction continues by attaching $(n+1)$-cells e_i^{n+1} to the **n-skeleton** X^n via maps $\varphi_i : S^n \longrightarrow X^n$. This process, which can either stop for a finite n or continue indefinitely, results to a space $X = \bigcup\limits_n X^n$ with the quotient topology when $n < \infty$ (in this case it is the same as the weak topology) and the weak topology when $n = \infty$.

Example 1.3.14. I is a 1-dimensional CW-complex.
I consists of two 0-cells e_0^0, e_1^0, its ends, and a single 1-cell e^1, with boundary $\partial e^1 = \{e_0^0, e_1^0\}$. $\Phi : D^1 \longrightarrow I$ is given by $\Phi((1-t)d_0 + td_1) = t$.

Example 1.3.15. The sphere S^n is a CW-complex and can be seen as constructed by a single 0-cell e^0 and a single n-cell e^n.
The characteristic map $\Phi : D^n \longrightarrow S^n$ of the n-cell is given by the formula $\Phi(x) = (2\sqrt{1-\|\bar{x}\|^2}\ \bar{x}, 2\|\bar{x}\|^2 - 1)$, where $\bar{x}$ represents a point in D^n written as a vector in $\mathbb{R}^n$. Φ collapses $\partial D^n = S^{n-1}$ onto the 0-cell e^0, while its restriction $\Phi|_{(D^n)^o}$ is a homeomorphism between $(D^n)^o$ and $S^n - \{e^0\}$.
There is another CW-decomposition of S^n which possesses the additional advantage that S^n has each S^k, $k \leq n$, as its subcomplex. This CW-decomposition consists of two k-cells for each $k \in \{0, \ldots, n\}$ and is obtained inductively using the characteristic map $\Phi : D^n \longrightarrow D^n_\pm$, $\Phi(\bar{x}) = (\bar{x}, \pm\sqrt{1-\|\bar{x}\|^2})$, where $D^n_\pm = \{(x_i) \in S^n |\ \pm x_{n+1} \geq 0\}$.

Example 1.3.16. The real projective space $\mathbb{R}P^n$ is a CW-complex that has a single k-cell for every $k \in \{0, 1, ..., n\}$.
Although we have elaborated on the real projective space earlier, there is yet another

way to define $\mathbb{R}P^n$ and this is the one needed here. $\mathbb{R}P^n$ can be seen as the quotient of the sphere S^n under the equivalence relation $\sim$ that identifies antipodal points.
The topology T will be defined by means of a basis. The map $p : S^n \longrightarrow S^n/\sim$ is open: Let U an open set in S^n. Since $p(U)$ is open if and only if $p^{-1}(p(U))$ is open, we only need to show that $p^{-1}(p(U))$ is open. But $p^{-1}(p(U))$ is the union of U and $V = \{p \in S^n | -p \in U\}$, the antipodal points of the points in U, where $V \equiv U$. This means that p is an open map and $\sim$ is an open equivalence relation on S^n. Therefore a basis for $\mathbb{R}P^n$ will be $\{p(B_a)\}$, where $\mathscr{B} = \{B_a\}$ is a basis for S^n (see Theorem 7.9 in [9]). Here, we take open balls $\{B(x,r)\}$, $x \in \mathbb{R}^{n+1}$, $r \in \mathbb{R}$, to be a basis for $\mathbb{R}^{n+1}$. S^n can be viewed as a subspace of $\mathbb{R}^{n+1}$, hence $\mathscr{B} = \{B(x,r) \cap S^n\}$.
If we identify in S^n all antipodal points that are not on the equatorial S^{n-1}, we get the hemisphere D^n. To get to $\mathbb{R}P^n$, we just need to identify the antipodal points of $\partial D^n = S^{n-1}$. But this actually is $\mathbb{R}P^{n-1}$. Consequently, $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell or D^n, with the quotient projection $S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$ that identifies antipodal points as the attaching map. It follows by induction on n that $\mathbb{R}P^n$ has a CW complex structure $e^0 \cup_{\alpha_1} e^1 \cup_{\alpha_2} ... \cup_{\alpha_n} e^n$ with one cell e^i in each dimension $i \leq n$.

Chapter 2

Homotopy theory

2.1 Homotopy equivalence relations

In algebraic topology spaces are studied with the help of various algebraic invariants. This allows for them to be deformed and turned into other spaces, more easily understood and analysed. The deformations applied must present some sense of continuity and either lead to homeomorphic spaces or to spaces that maintain some of the invariants intact. For example, depending on the case, a desirable tranformation can be achieved using a continuous map, a homeomorphism, or a map that will not create new or fill existing discontinuities and holes in the space. More generally we need a series of continuous maps which occur in a 'continuous flow of movement'.

Some basic deformations, which are frequently used, are presented below. Let us mention here that X, $Y \in \mathscr{T}$ and all maps between spaces are assumed continuous throughout this section.

Definition 2.1.1. Let $A \subset X$. A map $r : X \longrightarrow X$ such that $r(X) = A$ and $r|A = id_A$ is a **retraction**. More formally, a retraction is a map $r : X \longrightarrow X$ with $r^2 = r$.

Definition 2.1.2. A **homotopy** is a family of maps $h_t : X \longrightarrow Y$ such that the associated map $H : X \times I \longrightarrow Y$ given by $H(x,t) = h_t(x)$ is continuous. We say that the maps $f : X \longrightarrow Y$, $g : X \longrightarrow Y$ are **homotopic** if there exists a homotopy

H from f to g or, more explicitly, if there is a homotopy H which takes values $H(x,0) = f(x)$ and $H(x,1) = g(x)$ $\forall x \in X$. If f and g are homotopic through H, one writes $f \underset{h_t}{\sim} g$.

Proposition 2.1.3. *The relation* $\underset{h_t}{\sim}$ *is an equivalence relation.*

Proof. Let $f, g, h : X \longrightarrow Y$ be maps. Reflexivity is evident since $f \underset{f_t}{\sim} f$ by the constant homotopy $F(x,t) = f_t(x) = f(x)$ $\forall t \in I$. Symmetry holds since, if $f \underset{h_t}{\sim} g$, then $g \underset{h_{1-t}}{\sim} f$, where h_{1-t} is the inverse homotopy of h_t. For transitivity, if $f \underset{f_t}{\sim} g$ and $g \underset{g_t}{\sim} h$, then $f \underset{h_t}{\sim} h$, where

$$H(x,t) = h_t(x) = \begin{cases} f_{2t}(x), & 0 \leq t \leq \dfrac{1}{2} \\ g_{2t-1}(x), & \dfrac{1}{2} \leq t \leq 1 \end{cases} \qquad \forall x \in X.$$

The map H is continuous on $I \times I$, because it is continuous on $I \times [0, 1/2]$ and on $I \times [1/2, 1]$ and a function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately. $\square$

Definition 2.1.4. If $f : X \longrightarrow Y$ is continuous, its **homotopy class** is the equivalence class

$$[f] = \{g \in C(X,Y) \text{ such that } g \sim f\}.$$

The set of all such homotopy classes is denoted by $[X,Y]$ and it is the quotient $C(X,Y)/\!\sim$.

There are special cases of homotopies between spaces which we mention below.

Definition 2.1.5. Let $A \subset X$. A homotopy $h_t : X \longrightarrow Y$ for which $h_t|_A = h_0|_A$ holds for all $t \in I$, that is a homotopy whose restriction to A is independent of t, is called a **homotopy relative to** A, abbreviated to $\sim relA$.

Definition 2.1.6. A homotopy $h_t\ relA$ from the identity map $\mathbb{1}_X$ of X to a retraction r of X onto A is called a **deformation retraction** of X to A.

Definition 2.1.7. Let $f : X \longrightarrow Y$ be a map. Then one says that f is a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that $fg \sim \mathbb{1}_Y$ and $gf \sim \mathbb{1}_X$. If this is the case, X and Y are called **homotopy equivalent** or they are said to have the same **homotopy type** and we write $X \simeq Y$.

Remark. A deformation retraction h_t of a space X onto a subspace A is actually a homotopy equivalence. If we take the retraction $r : X \longrightarrow X$ and the inclusion $i_A : A \longrightarrow X$, we get $ri_A = \mathbb{1}_A$, while $i_A r \sim \mathbb{1}_X$ through the homotopy h_t. Thus the homotopy equivalence generalises the deformation retraction notion.

Definition 2.1.8. A space X is called **contractible** if it has the homotopy type of a point.

Remark. Saying that a space X is contractible is weaker than saying that it deformation retracts onto a point $\{x\} \in X$, since in the second case the point must remain steady throughout the homotopy h_t, while in the first is allowed to move freely.

Example 2.1.9. The cone CX of a space X deformation retracts to the point $[x, 0]$. The map $H : CX \times I \longrightarrow CX$ with $H([x,t], s) = [x, (1-s)t]$ is a homotopy $rel[x, 0]$ which takes the values $H([x,t], 0) = \mathbb{1}_{CX}[x,t]$ and $H([x,t], 1) = [x, 0] = r([x,t])$ for all $[x,t] \in CX$.

Example 2.1.10. D^n, $n \geq 2$, deformation retracts to its centre $x \in D^n$.
The homeomorphism $D^n \equiv CS^{n-1}$, which takes the centre of D^n to $[x, 0]$, along with the fact that CS^{n-1} deformation retracts to $[x, 0]$ give the result.

Example 2.1.11. The mapping cylinder Mf of a map $f : X \longrightarrow Y$ deformation retracts onto Y.
We denote the class of $(x,t) \in X \times I$ in Mf by $[x,t]$ and the class of $y \in Y$ in Mf with $[y]$. This means that $[x, 1] = [f(x)]$. Now, if we define the map $H : Mf \times I \longrightarrow Mf$ by

$$\begin{aligned} &H([x,t], s) = [x, (1-s)t + s], \quad && \text{if } x \in X, t, s \in I \\ &H([y], s) = [y], && \text{if } y \in Y, s \in I \end{aligned}$$

then H is a homotopy between $H_0 = \mathbb{1}_{Mf}$ and $H_1 = r$, where r is the retraction of Mf onto Y. H is obviously constant on Y. Therefore, we have the desired deformation retraction.

Definition 2.1.12. A map $f : X \longrightarrow Y$ is called **nullhomotopic**, if it is homotopic to the constant map.

Proposition 2.1.13. *Let X, Y be spaces and let X deformation retract to a point $x \in X$. Then any map $f : X \longrightarrow Y$ is nullhomotopic.*

Proof. Denote with $H : X \times I \longrightarrow X$ the given deformation retraction. If we compose H with f, we get $F = f \circ H : X \times I \longrightarrow Y$ with $F_0 = f \circ H_0 = f$ and $H_1 = f \circ H_1 = f \circ c_x = c_{f(x)}$ which is constant. □

Finally, we mention a proposition that stems from the important homotopy extension property (see Chapter in [2] or Chapter 0 in [10]). The **homotopy extension property**, as betrayed by the name, states that, given a map $f : X \longrightarrow Y$ and $A \subset X$, a homotopy $F : A \times I \longrightarrow Y$ with $F_0 = f|_A$ can be extended to a homotopy $H : X \times I \longrightarrow Y$ with $H_0 = f$. Not all pairs of spaces (X, A) possess this property. However, CW pairs (X, A) behave well regarding the homotopy extension property, hence the following proposition:

Proposition 2.1.14. *If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \longrightarrow X/A$ is a homotopy equivalence.*

Proof. See [10], Proposition 0.17. □

Example 2.1.15. $SS^n \simeq \Sigma S^n$.
We know that $\Sigma S^n = S^n \times I/(\{x_0\} \times I \cup S^n \times \{0,1\})$ and $SS^n = S^n \times I/(S^n \times \{0,1\})$. $SS^n \equiv S^{n+1}$ from Example 1.2.4, while Example 1.3.15 suggests that there is a CW-decomposition of the CW-complex S^{n+1} that includes every S^k with $k \geq n$ as its subcomplex. This viewpoint allows us to identify $\{x_0\} \times I \in S^n \times I$ with a closed subcomplex $\overline{e^1_\alpha}$ of S^{n+1}, which is essentially one of the two 1-cells that produce S^1. $\overline{e^1_\alpha}$ is obviously a contractible subcomplex, thus $SS^n \equiv S^{n+1} \simeq S^{n+1}/\overline{e^1_\alpha} = \Sigma S^n$ from Proposition 2.1.14.

2.2 Fundamental group

In this section, the fundamental group $\pi_1(X, x_0)$ of a pointed space (X, x_0) is introduced. As a set, $\pi_1(X, x_0)$ consists of the homotopy classes of loops based on the space's basepoint x_0. To construct a group structure on this set, though, an operation between classes of paths must be defined. This operation stems from combining consequent representatives of classes of paths. Here consequent implies that the second path begins at the same point that the first ends. Since in $\pi_1(X, x_0)$ only loops are considered, this condition is immediately satisfied and one can define a 'multiplication' like the one roughly described.

Formally, the fundamental group is defined as follows:

Definition 2.2.1. Let $X \in \mathscr{T}^*$. The **fundamental group** of X with basepoint x_0 is

$$\pi_1(X, x_0) = \{[f] |\ f \text{ is a loop based on } x_0\}$$

with binary operation $[f][g] = [f \cdot g]$, which denotes concatenation of loops. Equivalent formulas for the fundamental group are also:

$$\pi_1(X, x_0) = [(I, *), (X, x_0)] = [(S^1, *), (X, x_0)].$$

Remark. It should be mentioned here that for a space X $\pi_0(X)$ is defined to be the set of path-components of X. It is a set, not a group.

Proposition 2.2.2. *If $(X, x_0) \in \mathscr{T}^*$, then $\pi_1(X, x_0)$ is a group.*

Proof. The proof is rather easy and we refer the reader to [10]. After checking that the product operation is well-defined, one finds homotopies between $f \cdot (g \cdot h)$ and $(f \cdot g) \cdot h$ to prove associativity, amongst $c_{x_0} \cdot f$, $f \cdot c_{x_0}$ and f to ensure that the neutreal element exists and, finally, constructs the loop $\bar{f}$ from f transversed backwards to serve as the inverse of f. $\square$

Proposition 2.2.3. *π_1 is a functor (Definition A.1.7) from $\mathscr{T}^*$ to $\mathscr{G}$.*

Proof. Let $f \in Hom((X, x_0), (Y, y_0))$ and $[h] \in \pi_1(X, x_0)$. We define $\pi_1(f)([h]) = [f \circ h]$, where $f \circ h : I \longrightarrow Y$. $f \circ h$ is continuous and its image is a closed path in Y based on y_0. $\pi_1(f)$ is well defined, since $\pi_1(f)[h] = [f(h)] = [f(g)] = \pi_1(f)[h]$ if $h \sim g$ through a basepointed homotopy. Now, for $[g]$, $[h] \in \pi_1(X, x_0)$ we have $\pi_1(f)[g \cdot h] = [f \circ (g \cdot h)] = [(f \circ g) \cdot (f \circ h)] = \pi_1(f)[g]\pi_1(f)[h]$, thus $\pi_1(f)$ is a homomorphism. Finally, if $f_1 : (X, x_0) \longrightarrow (Y, y_0)$ and $f_2 : (Y, y_0) \longrightarrow (Z, z_0)$, then $\pi_1(f_2 \circ f_1)[h] = [f_2 \circ (f_1 \circ h)] = \pi_1(f_2)[f_1 \circ h] = \pi_1(f_2)\pi_1(f_1)[h]$, whilst $\pi_1(f)[c_{x_0}] = [c_{y_0}]$, which complete the proof. $\square$

Remark. We are going to use f_* instead of $\pi_1(f)$, in order to simplify notation.

Before we move on, it is essential to introduce H-spaces. The prefix H- signifies their connection with Hopf, since he was the first to study them.

Definition 2.2.4. An **H-space** (X, μ) consists of a space $X \in \mathscr{T}^*$ with basepoint x_0 and a continuous map $\mu : (X \times X, x_0 \times x_0) \longrightarrow (X, x_0)$ called multiplication. The multiplication μ must satisfy $\mu|_{X \vee X} \sim \nabla$ in $\mathscr{T}^*$, where $\nabla : X \vee X \longrightarrow X$ is a 'folding map' with $\nabla(x, x_0) = \nabla(x_0, x) = x$.

Proposition 2.2.5. *Let (X, μ) be an H-space and $Y \in \mathscr{T}^*$. Then the set of homotopy classes of functions $[(Y, *), (X, x_0)]$ has a multiplication with two-sided unit.*

Proof. We refer the reader to [12], Proposition 9.8. □

Returning to the fundamental group:

Proposition 2.2.6. *Let $(X, x_0) \in \mathscr{T}^*$ be an H-space. Then $\pi_1(X, x_0)$ is abelian.*

Proof. We define a map

$$\theta : \pi_1(X, x_0) \times \pi_1(Y, y_0) \longrightarrow \pi_1(X \times Y, (x_0, y_0))$$

through $\theta([f], [g]) = [(f, g)]$. θ is an isomorphism (see Theorem 3.7 in [14]).
For $[f]$, $[g] \in \pi_1(X, x_0)$, we have

$$\begin{aligned}
[g] &= (\mu \circ (c_{x_0}, \mathbb{1}_X))_*[g] && \text{(Definition2.2.4 gives } \mu \circ (c_{x_0}, \mathbb{1}_X) \sim \mathbb{1}_X rel\{x_0\}) \\
&= \mu_*(c_{x_0}, \mathbb{1}_X)_*[g] && (\pi_1 \text{ is a functor}) \\
&= \mu_*[c_{x_0} g, g] && \text{(Definition of induced map in 2.2.3)} \\
&= \mu_* \theta([c_{x_0} g], [g]) \\
&= \mu_* \theta([c_{x_0}], [g])
\end{aligned}$$

where both the constant map and the constant path are denoted with c_{x_0}. Similarly, using $\mu \circ (\mathbb{1}_X, c_{x_0}) \sim \mathbb{1}_X rel\{x_0\}$ we get $[f] = \mu_* \theta([f], [c_{x_0}])$.
Since $\mu_* \theta : \pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)$ is a homomorphism, we have

$$\begin{aligned}
[f][g] &= \mu_* \theta([f], [c_{x_0}]) \cdot \mu_* \theta([c_{x_0}], [g]) = \mu_* \theta(([f], [c_{x_0}]) \cdot ([c_{x_0}], [g])) \\
&= \mu_* \theta([f], [g]) = \mu_* \theta(([c_{x_0}], [g]) \cdot ([f], [c_{x_0}])) \\
&= \mu_* \theta([c_{x_0}], [g]) \cdot \mu_* \theta([f], [c_{x_0}]) \\
&= [g][f]
\end{aligned}$$

Hence, if (X, x_0) is an H-space, $\pi_1(X, x_0)$ is abelian. □

2.3 Higher homotopy groups

Definition 2.3.1. Let $(X, x_0) \in \mathscr{T}^*$ and I^n be the n-dimensional unit cube, $n \geq 1$, with boundary ∂I^n. Maps $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$ sending the cube's boundary to the basepoint are called **loops** with baseboint x_0 when $n = 1$ and n-**spheroids** with baseboint x_0 when $n > 1$.
Loops and spheroids can also be defined as maps with domain the space (S^n, s_0), where (S^n, s_0) is produced as the quotient of $(I^n, \partial I^n)$ under the relation that collapses ∂I^n onto the point s_0.

Remark. The two definitions are equivalent. Although our study will be based mainly on the second definition, we may from time to time use the first as well.

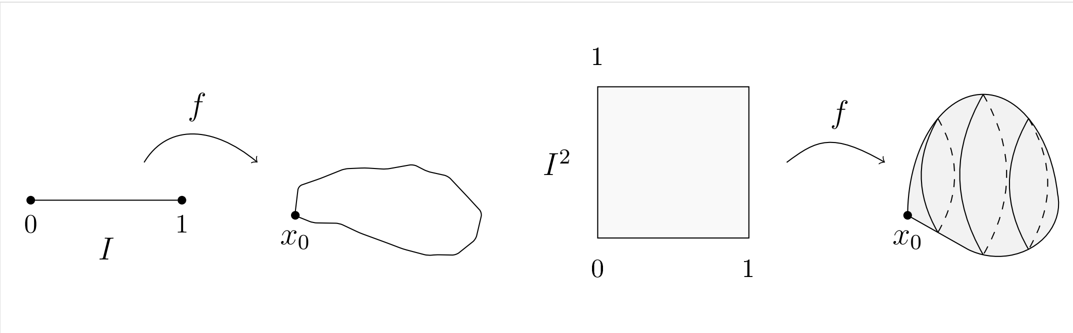


Figure 2.1: Loop and 2-spheroid

Definition 2.3.2. Let $n > 1$ and $f : (S^n, *) \longrightarrow (X, x_0)$ be spheroids. $\pi_n(X, x_0)$, which is called the **nth homotopy group** of X, is defined to be the set of homotopy classes of basepoint preserving maps f, where homotopies f_t also preserve the basepoint x_0. We write $\pi_n(X, x_0) = [(S^n, *), (X, x_0)]$. An equivalence class is denoted $[f]$ and is called the homotopy class of f. We have $[f] = [g] \Leftrightarrow f \underset{h_t}{\sim} g$.

For $n = 0$ we take I^0 to be a point and ∂I^0 to be empty, so $\pi_0(X, x_0)$ becomes the set of path-components of X, as mentioned before.

In the definition above, we have attached somehow arbitrarily the word group to the name of $\pi_n(X, x_0)$. It is still to be proved that this set of homotopy classes actually possesses a group structure.

However, before we embark on such a proof, we need to state some useful propositions and theorems. In order to start, let X, Y, $Z \in \mathscr{CG}$ (or $\mathscr{CG}^*$) and A be a closed subspace of X such that X/A is a Hausdorff space. Then we have the following results, which are stated here without proofs. The reader is referred to [8] to find them proved.

Proposition 2.3.3. $(Z,*)^{(X,*)\times(Y,*)} \equiv ((Z,*)^{(X,*)})^{(Y,*)}$, *via the homeomorphism*

$$F : ((Z,*)^{(X,*)})^{(Y,*)} \longrightarrow (Z,*)^{(X,*)\times(Y,*)}$$

where, for $f \in ((Z,*)^{(X,*)})^{(Y,*)}$, $F(f) : (X,*) \times (Y,*) \longrightarrow (Z,*)$ *is given by the formula* $F(f)(x,y) = f(y)(x)$, *with* $f(y) : (X,*) \longrightarrow (Z,*)$.

Remark. The result in Proposition 2.3.3 is called the exponential law. It also holds for X locally compact and Hausdorff and Y, $Z \in \mathscr{T}^*$. So, for $X = I$ or $X = S^1$, which are obviously both locally compact and Hausdorff, we can have Y and Z to be topological spaces with no other restrictions (see Chapter 11 in [15]).

Proposition 2.3.4. $(Y,*)^{(X/A,*)} \equiv (Y,*)^{(X,A)}$.

Theorem 2.3.5. $(Z,*)^{(Y\wedge X,*)} \equiv [(Z,*)^{(Y,*)}]^{(X,*)}$.

Theorem 2.3.6. *The homeomorphism of 2.3.5 induces a bijection*

$$\phi : [(X,*),(Z,*)^{(Y,*)}] \longrightarrow [(Y \wedge X,*),(Z,*)]$$

Notation. We employ $\equiv$ to denote a homeomorphism between spaces, $\simeq$ to denote a homotopy between spaces, $\cong$ to denote an isomorphism between groups and $\leftrightarrow$ to declare that there exists a bijection between two sets.

Proposition 2.3.7. π_n *is a functor from* $\mathscr{CG}^*$ *to* $\mathscr{G}$ *for* $n \geq 1$.

Proof. We have

$$\begin{aligned} S^n &\equiv SS^{n-1} && \text{(Example 1.2.4)} \\ &\simeq \Sigma S^{n-1} && \text{(Example 2.1.15)} \\ &\equiv S^{n-1} \wedge S^1 && (Prop.1.2.7) \end{aligned}$$

Thus, we get

$$\begin{aligned} \pi_n(X,x_0) &= [(S^n,*),(X,x_0)] \\ &\leftrightarrow [(S^{n-1}\wedge S^1,*),(X,x_0)] \\ &\leftrightarrow [(S^1,*),(X,x_0)^{(S^{n-1},*)}] && (Theorem 2.3.6) \\ &= [(S^1,*),\Omega^{n-1}(X,x_0)] \\ &= \pi_1(\Omega^{n-1}(X,x_0),*) = \pi_1((X,x_0)^{(S^{n-1},*)},*) \end{aligned}$$

which is a group, as stated in Proposition 2.2.2. This proves that $\pi_n : \mathscr{CG}^* \longrightarrow \mathscr{G}$ is a well defined function between these sets. The operation that guarantees the group structure of $\pi_n(X,x_0)$ is given by $[f] \star [g] = b^{-1}(b([f])b([g]))$, where b is the implied bijection above. Under this definition of $\star$, b becomes a homomorphism, besides being a bijection. Hence b is an isomorphism of groups.
It is shown in A.1.8 that $(\cdot)^{(Y,*)}$ is a covariant functor from $\mathscr{CG}^*$ to $\mathscr{CG}^*$. From 2.2.3 we see that π_1 is also a covariant functor from $\mathscr{CG}^* \subset \mathscr{T}^*$ to $\mathscr{G}$. The composition of two functors, when defined, is a functor as well. So $\pi_n(\cdot) = \pi_1((\cdot)^{(S^{n-1},*)})$ is a functor from $\mathscr{CG}^*$ to $\mathscr{G}$. $\square$

Proposition 2.3.8. *π_n is a functor from $\mathscr{T}^*$ to $\mathscr{G}$ for $n \geq 1$.*

Proof. The proof remains essentially the same with Proposition's 2.3.7, except for Theorem 2.3.6. In this case we use the fact that S^1 is a locally compact, Hausdorff space and the remark of Proposition 2.3.3. □

Remark. Comultiplication defined in Equation 1.3 agrees with that of Proposition 2.3.7. Although in the second case no formula was written explicitly, the multiplication $\star$ in $[(S^n, *), (X, x_0)]$ has been proved to exist. Let us recall the multiplication in π_1 defined in Proposition 2.2.2. The operation $\star$ ensures the group structure on $[(S^n, *), (X, x_0)]$. To be more specific, using the homeomorphism described in Theorem 2.3.6, we get

$$[f] \star [g] = \begin{cases} f([x, 2s]), & 0 \leq s \leq \frac{1}{2} \\ g([x, 2s-1]), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

which coincides with the comultiplication in 1.3.
This identification of multiplications would allow us to generalise the result in Proposition 2.3.7 to spaces in $\mathscr{T}^*$, even if what is said in the remark of Proposition 2.3.3 was not true. Also, let it be noted that from now on $\cdot$ will be used in place of $\star$ for this multiplication.

Remark. A more general result than this in Proposition 2.3.7 is proved in a similar way in Proposition 9.2 in [12]. It states that, if X, $Y \in \mathscr{CG}^*$, then $F(X, Y) = [(\Sigma X, *), (Y, *)]$ is a functor in two variables from $\mathscr{CG}^*$ to $\mathscr{G}$.

Proposition 2.3.9. *Let $X \in \mathscr{T}$ and (Y, y_0) be an H-space. The two multiplications in $[(SX, *), (Y, y_0)]$ are the same and they are commutative. Commutativity dictates that $f \cdot g = g \circ f$, where f, $g \in [(SX, *), (Y, y_0)]$. $\cdot$ is the multiplication defined above and $\circ$ the multiplication defined in Prop. 2.2.5.*

Proof. See the proof given in [12], Proposition 9.9. □

Theorem 2.3.10. *If $n \geq 1$, $X \in \mathscr{T}^*$, $\Omega^n X$ is an H-space.*

Proof. It is sufficient to show that ΩX is an H-space.
If we consider the constant path as a basepoint, we see that $\Omega X \in \mathscr{T}^*$. We define the multiplication $\mu : \Omega X \times \Omega X \longrightarrow \Omega X$ just like we defined composition of paths in π_1. Explicitly, the formula of μ is given by

$$\mu(\omega_1, \omega_2)(t) = \begin{cases} \omega_1(2t), & 0 \leq t \leq \frac{1}{2} \\ \omega_2(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

By Proposition 2.3.3 we deduce that $\mu : \Omega X \times \Omega X \longrightarrow \Omega X \subset X^I$ is continuous if and only if the corresponding map $\bar{\mu} : \Omega X \times \Omega X \times I \longrightarrow X$ is continuous. $\bar{\mu}|_{[0,\frac{1}{2}]}$ and $\bar{\mu}|_{[\frac{1}{2},1]}$ are continuous. They also have the same value on $t = \frac{1}{2}$. Thus $\bar{\mu}$ is continuous.
Now,

$$H([\omega, *], s)(t) = \begin{cases} \omega(\frac{2t-s}{2-s}), & 0 \leq t \leq \frac{s}{2} \\ *, & \frac{s}{2} \leq t \leq 1 \end{cases}$$

and

$$H([*,\omega],s)(t) = \begin{cases} *, & 0 \le t \le 1 - \frac{s}{2} \\ \omega(\frac{2t}{2-s}), & 1 - \frac{s}{2} \le s \le 1 \end{cases}$$

is a homotopy between $\mu|_{\Omega X \vee \Omega X}$ and ∇. Continuity is proved by Proposition 2.3.3, like before. $\square$

Proposition 2.3.11. *Let $X \in \mathscr{T}^*$. Then $\pi_n(X, x_0)$ is an abelian group, if $n \geq 2$.*

Proof.

$$\begin{aligned} \pi_n(X, x_0) &= [(S^n, *), (X, x_0)] \\ &\cong [(S^1, *), (\Omega^{n-1}(X, x_0), *)] \\ &= \pi_1(\Omega^{n-1}(X, x_0), *). \end{aligned}$$

However, $\Omega^{n-1}(X, x_0)$ is an H-space for $n - 1 \geq 1$ (Theorem 2.3.10) and the fundamental group of H-spaces is abelian (Proposition 2.2.6). $\square$

Next, we will see how a loop in $\pi_1(X, *)$ can act on a spheroid in $\pi_n(X, *)$, $n > 1$.

Proposition 2.3.12. *Let $\gamma \in \pi_1(X, x_0)$ be a loop. A map*

$$\begin{aligned} \beta : \quad \pi_1(X, x_0) &\longrightarrow Aut(\pi_n(X, x_0)) \\ [\gamma] \quad &\longmapsto \quad \beta_\gamma \quad : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0) \\ & \qquad\qquad\qquad [f] \quad \longmapsto \quad [\gamma \circ f] \end{aligned}$$

is an action called the ***action of*** π_1 ***on*** π_n.

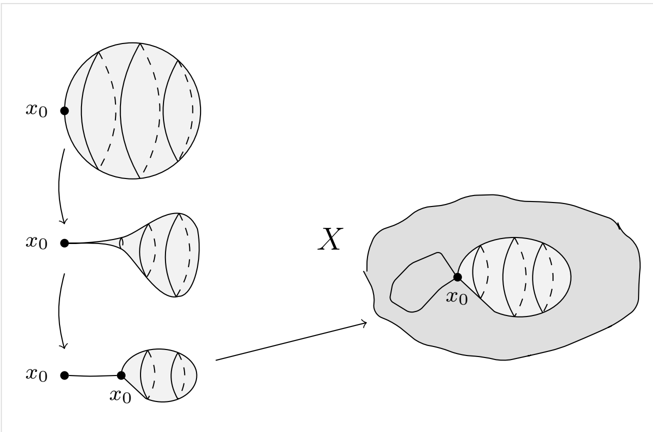


Figure 2.2: The action of the fundamental group on $\pi_n(X, x_0)$[16]

Proof. One way to prove that β is a well defined action is the following:
We first show a more general result. Namely, that $\beta_h : \pi_n(X, h(0)) \longrightarrow \pi_n(X, h(1))$, where h is a path from $h(0)$ to $h(1)$ in X, is a homomorphism with $\beta_{\bar{h}}$ its inverse. Then we prove that this β possesses the properties presented in Definition A.2.1. Figures 2.3, 2.4 and 2.5 provide a pictorial proof for the case $n = 2$. The complete proof can be found in [10]. If we replace the random path h with a loop γ, the proof works just fine and gives the desired result.

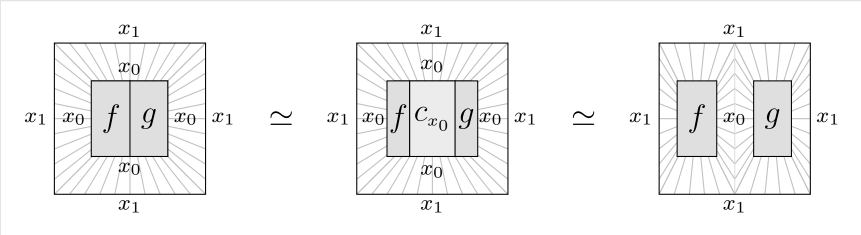
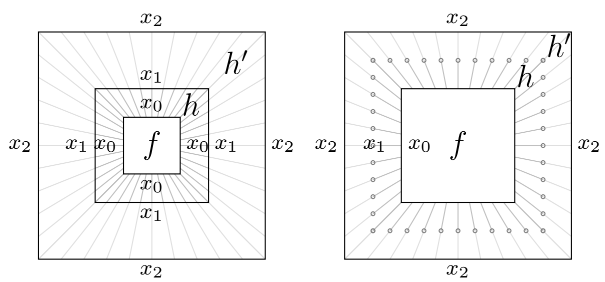


Figure 2.3: Homotopy proving that β_h is a homomorphism.

There is another interesting way to prove the Proposition. This employs the notions of a fundamental groupoid $\Pi(X)$ and transport functors. In this case the intuitive sense of a path acting on a spheroid remains unaltered, but the properties of an action come as a special case of the properties of a transport functor. We refrain ourselves from presenting this particular proof, since we have not even touched upon the prerequisites and just refer anyone who feels interested in it to [2]. $\square$



Domain of $h'\diamond(h\diamond f)$ Domain of $(h' \cdot h)\diamond f$

Figure 2.4: $\beta_{h'}(\beta_h[f]) = \beta_{h'\cdot h}[f]$

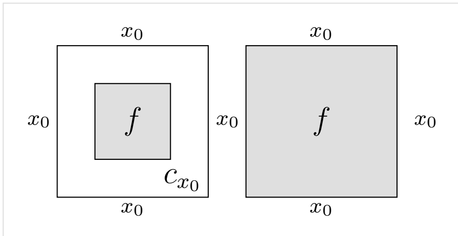


Figure 2.5: $c_{x_0}\diamond f \sim f$

2.4 Relative homotopy groups

Definition 2.4.1. Let $(X, A) \in \mathscr{T}^{2*}$ (Definition in A.1.3) and $x_0 \in A$ a basepoint. Let also I^n be the n-dimensional unit cube, $n \geq 2$, ∂I^n its boundary, I^{n-1} the face of I^n with the last coordinate equal to 0 and $J^{n-1} = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\} \subset \partial I^n \subset I^{n-1} \times I$. The **relative homotopy group** of (X, A, x_0), $\pi_n(X, A, x_0)$, is defined as the set $[(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$. Equivalently, $\pi_n(X, A, x_0)$ can be defined as the set $[(D^n, S^{n-1}, s_0), (X, A, x_0)]$, due to the existing homeomorphism between $(I^n, \partial I^n, J^{n-1})$ and (D^n, S^{n-1}, s_0).

Remark. The definition $\pi_n(X, A, x_0) = [(D^n, S^{n-1}, *), (X, A, *)]$ will be preferred throughout our study.

In the definition above, we misused the term group, so our immediate task is to find out if and when $\pi_n(X, A, x_0)$ is indeed a group. The main idea is to try to construct a bijection between the set $\pi_{n+1}(X, A, x_0) = [(D^{n+1}, S^n, *), (X, A, *)]$, $n > 0$, and the group $[S^n, M_{i_A}]$, where M_{i_A} is the mapping fibre of the inclusion $i_A : A \hookrightarrow X$ defined below. If we do so, the group operation in $\pi_n(M_{i_A}) = [S^n, M_{i_A}]$ will be transferred via the bijection to an operation in $\pi_{n+1}(X, A, *)$ and thus make it a group.

Definition 2.4.2. Let (X, x_0), $(Y, y_0) \in \mathscr{T}^*$ and $f : (X, x_0) \longrightarrow (Y, y_0)$ be a pointed map. The **mapping fibre** of f is the pointed space

$$M_f = \{(x, \omega) \in X \times Y^I |\ \omega(0) = y_0 \text{ and } \omega(1) = f(x)\}.$$

M_f has the induced topology and its basepoint is (x_0, ω_0), where ω_0 is the constant path at y_0.

Remark. According to the definition, the mapping fibre of M_{i_A} is

$$M_{i_A} = \{(a, \omega) \in A \times X^I |\ \omega(0) = x_0 \text{ and } \omega(1) = i_A(a) = a \in A\}.$$

Proposition 2.4.3. *Let $(X, A, x_0) \in \mathscr{T}^{2*}$ and $i_A : A \hookrightarrow X$ be the inclusion. Then there is a bijection $\theta : [S^n, M_{i_A}]_* \longrightarrow [(D^{n+1}, S^n, v_0), (X, A, x_0)]_*$.*

Proof. In order to define $\theta : [S^n, M_{i_A}] \longrightarrow [(D^{n+1}, S^n, v_0), (X, A, x_0)]$, we initially need to be able to allocate a map $\overline{f} : (D^{n+1}, S^n, v_0) \longrightarrow (X, A, x_0)$ to every map $f : S^n \longrightarrow M_{i_A}$.
Let $f : S^n \longrightarrow M_{i_A}$ with $f(v) = (a_v, \omega_v) \in M_{i_A}$, $v \in S^n$, be a basepoint preserving map. If we take the projection $q : M_{i_A} \longrightarrow X^I$ with $q(a, \omega) = \omega$ and compose it with f, we form the map $q \circ f : S^n \longrightarrow X^I$, which is continuous being the composition of continuous functions. From the remark in Proposition 2.3.3, we get that the map $\overline{q \circ f} : S^n \times I \longrightarrow X$ with $\overline{q \circ f}(v, t) = \omega_v(t)$ is continuous. Also, $\overline{q \circ f}(v, 0) = \omega_v(0) = x_0$, $\overline{q \circ f}(v, 1) = \omega_v(1) = a_v \in A$ and $\overline{q \circ f}(v_0, t) = \omega_0(t) = x_0$.

Finally, from the universal property of quotient spaces 1.1.23, there is a continuous, basepointed map

$$f_p : CS^n = \frac{S^n \times I}{S^n \times \{0\}} \longrightarrow X$$

with $f_p([v,t]) = \omega_v(t)$, which finally gives the map $\bar{f} : D^{n+1} \longrightarrow X$ as $\bar{f}(tv) = (f_p \circ \phi^{-1})(tv) = \omega_v(t)$, where ϕ is the homeomorphism from CS^n to D^{n+1} (1.2.4).
We define

$$\begin{aligned} \theta : [S^n, M_{i_A}] &\longrightarrow [(D^{n+1}, S^n, v_0), (X, A, x_0)] \\ [f] &\longmapsto [\bar{f}] \end{aligned}$$

For $[f] = [f']$, there is a homotopy $F : S^n \times I \longrightarrow M_{i_A}$ with $F(v,s) = F_s(v)$, $F_0(v) = f(v)$, $F_1(v) = f'(v)$, $F_s(v_0) = (a_0, \omega_0)$. Now, based on what was expounded previously, for each $F_s : S^n \longrightarrow M_{i_A}$ the continuous map $\overline{F}_s : (D^{n+1}, S^n, v_0) \longrightarrow (X, A, x_0)$ can be formed. Thus we get another homotopy, namely $G : (D^{n+1}, S^n, v_0) \times I \longrightarrow (X, A, x_0)$ with $G_s = \overline{F}_s$ and $G_s(v_0) = \omega_0(1) = x_0$ for all $s \in I$, which leads eventually to $[\bar{f}] = [\bar{f}']$ and the fact that θ is well defined.
We will now construct the inverse of θ. Let $h : (D^{n+1}, S^n, v_0) \longrightarrow (X, A, x_0)$ with $h(0) = x_0$. If $h(0) \neq x_0$, we compose h with the homotopy $T_s(tv) = tv + (1-t)s(v_0)$, $T_0 = \mathbb{1}_{D^{n+1}}$ to take a new homotopy between h and $h'(tv) = h(tv + (1-t)(v_0))$. Obviously $[h'] = [h]$ and $h'(0) = h(v_0) = x_0$. Next, we define a path ω_v for each $v \in S^n$ via the formula $\omega_v(t) = h(tv) = h(\phi([v,t]))$. Since $\omega_v(0) = h(0) = x_0$ and $\omega_v(1) = h(v) \in A$, we have $(\omega_v(1), \omega_v) \in M_{i_A}$. We define $\theta^{-1}([h])(v) = (h(\phi([v,1]), h(\phi([v,\cdot])),$ for all $v \in S^n$. This map is well defined and a continuous pointed map. Finally, we check that

$$\begin{aligned} (\theta \circ \theta^{-1})([h]) &= \theta[(\omega_{(\cdot)}(1), h \circ \phi)] = [h \circ \phi \circ \phi^{-1}] = [h] \\ (\theta^{-1} \circ \theta)([f]) &= \theta^{-1}[\bar{f}] = [(\omega_{(\cdot)}(1), \omega_{(\cdot)})] = [(a_{(\cdot)}, \omega_{(\cdot)})] = [f] \end{aligned}$$

and conclude that θ is a bijection. □

Proposition 2.4.4. *$\pi_n(X, A, x_0)$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$.*

Proof. In Proposition 2.4.3, a bijection $[(D^n, S^{n-1}, *), (X, A, *)] \overset{\theta}{\longleftrightarrow} [S^{n-1}, M_{i_A}]$ was constructed. Since $\pi_n(X, A, *) = [(D^n, S^{n-1}, *), (X, A, *)]$ and $\pi_{n-1}(M_{i_A}) = [S^{n-1}, M_{i_A}]$, the group structure which $\pi_{n-1}(M_{i_A})$ possesses when $n \geq 2$ is transferred through the bijection θ to $\pi_n(X, A, *)$, same way it did via b in 2.3.7. In addition, we have proved (Prop. 2.3.11) that $\pi_{n-1}(M_{i_A})$ is an abelian group when $n \geq 3$ and this property is transferred through θ to $\pi_n(X, A, *)$ for the same n. □

Proposition 2.4.5. *Adjusting appropriately the ideas in Proposition 2.3.12, we can define an action*

$$\begin{aligned} \beta : \quad \pi_1(A, x_0) &\longrightarrow Aut(\pi_n(X, A, x_0)) \\ [\gamma] &\longmapsto \beta_\gamma \quad : \pi_n(X, A, x_0) \longrightarrow \pi_n(X.A, x_0) \\ & \qquad\qquad [f] \longmapsto [\gamma \diamond f] \end{aligned}$$

for $n \geq 2$ in the relative case.

Proposition 2.4.6 (Compression criterion). *A map $f : (D^n, S^{n-1}, *) \longrightarrow (X, A, *)$ represents zero in $\pi_n(X, A, *)$ if and only if it is homotopic $relS^{n-1}$ to a map with image contained in A.*

Proof. $\Rightarrow$ Let $[f] \in \pi_n(X, A, x_0)$ with $[f] = 0$. Then $\exists$ homotopy $F : D^n \times I \longrightarrow X$ such that $F(x, 0) = f$, $F(x, 1) = c_{x_0}$, $F(S^{n-1}, t) \subset A$ for $t \in I$. We form a new homotopy $H : D^n \times I \longrightarrow X$ via the formula $H(s, t) = F(p_t(s))$ where each p_t is a radial projection of $D^n \times \{0\}$ to $\{D^n \times t\} \cup \{S^{n-1} \times [0, t]\}$, for example the radial projection from the point $(\bar{0}, -1) \in D^n \times \mathbb{R}$. For an $s \in \partial D^n = S^{n-1}$ we have $H(s, t) = F(s, 0)$, $\forall\ t \in I$. If s is any point in D^n, $H(s, 0) = F(s, 0) = f(s)$ and $H(s, 1) = F(p_1(s))$. But $p_1(s)$ belongs either to $D^n \times 1$ or to $S^{n-1} \times [0, 1]$ and this translates to either $H(s, 1) = c_{x_0}$ or $H(s, 1) \subset A$. Thus, $[f] \sim_{relS^{n-1}} [F \circ p_1]\ relS^{n-1}$ with $F \circ p_1(D^n) \subset A$.
$\Leftarrow$ Let $[f] \sim [g]\ relS^{n-1}$ where the image of g is in A. We have $[g] \sim 0$ in $\pi_n(X, A, x_0)$, because D^n deformation retracts to a point via a homotopy d_t from id_{D^n} to c_{x_0} and using d_t we can construct the homotopy $g_t = g \circ d_t$ with $g_t(\partial D^n) \subset g(D^n) \subset A$, $\forall\ t \in I$. So, $[f] \sim 0$ in $\pi_n(X, A, x_0)$. □

2.5 Exact sequences of homotopy groups

Exact sequences are an extremely useful tool in algebraic topology. In homotopy theory, however, they do not arise very often. A long exact sequence of homotopy groups can be meticulously constructed using the Puppe sequence and the functor $[S^0, \cdot]$. The Puppe sequence, in its turn, can be constructed in two ways: the first employs mapping fibres, leading to the exact Puppe sequence, while the second uses quotient spaces and leads to the coexact Puppe sequence.

Here, [14] serves as our main reference for the construction of the exact Puppe sequence and we refer the reader to [3] for the coexact Puppe sequence.

We recall the definition of the mapping fibre M_f given in 2.4.2 and we observe that there exist an inclusion $j : \Omega Y \longrightarrow M_f$ and a projection $q : M_f \longrightarrow X$, which are formed as $j(\omega) = (x_0, \omega)$ and $q(x, \omega) = x$ for all $x \in X$, $\omega \in \Omega Y$. Composing these maps, along with f and Ωf, we produce the sequence:

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} M_f \xrightarrow{q} X \xrightarrow{f} Y \tag{2.1}$$

As we can see in A.1.8, Ω is a covariant functor from $\mathscr{T}^*$ to $\mathscr{T}^*$. If we apply this functor on the previous sequence, we get

$$\Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega j} \Omega M_f \xrightarrow{\Omega q} \Omega X \xrightarrow{\Omega f} \Omega Y \tag{2.2}$$

Sequence 2.2 overlaps with sequence 2.1. We can splice these two together to form a longer sequence of spaces and, by iterating this construction, we conclude in the long sequence

$$\ldots \xrightarrow{\Omega^2 q} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega j} \Omega M_f \xrightarrow{\Omega q} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} M_f \xrightarrow{q} X \xrightarrow{f} Y \tag{2.3}$$

This is the **Puppe sequence** related to the map f.

Remark. For $f = i_A : A \hookrightarrow X$ in particular, one gets the Puppe sequence:

$$\ldots \xrightarrow{\Omega^2 q} \Omega^2 A \xrightarrow{\Omega^2 i_A} \Omega^2 X \xrightarrow{\Omega j} \Omega M_{i_A} \xrightarrow{\Omega q} \Omega X \xrightarrow{\Omega i_A} \Omega Y \xrightarrow{j} M_{i_A} \xrightarrow{q} X \xrightarrow{i_A} Y \tag{2.4}$$

Here we state the two theorems that lead eventually to the long exact sequence of homotopy groups. Theorem 2.5.1, specifically, calls for significant preliminary work, so its proof is postponed until later in this section.

Theorem 2.5.1 (Puppe sequence)**.** *If X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, then the Puppe sequence related to f (2.3) is exact in $h\mathscr{T}^*$.*

Theorem 2.5.2 (Homotopy sequence of a pair)**.** *If $(X, A, *) \in \mathscr{T}^{2*}$, then there is an exact sequence of homotopy groups*

$$\ldots \pi_{n+1}(A) \xrightarrow{\Omega^n i_{A*}} \pi_{n+1}(X) \xrightarrow{\Omega^{n-1} j_*} \pi_{n+1}(X, A) \xrightarrow{\Omega^{n-1} q_*} \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \ldots$$
$$\ldots \pi_1(A) \longrightarrow \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)$$

Proof. According to Theorem 2.5.1, the Puppe sequence 2.4 is exact in $h\mathscr{T}^*$. From the definition of exactness in A.2.5 this is equivalent to saying that, if $Z \in \mathscr{T}^*$, applying the functor $[Z, \cdot]$ on the sequence in 2.4 results to an exact sequence in $\mathscr{S}ets^*$. So, if we apply $[S^0, \cdot]$, we get the long exact sequence of pointed sets:

$$\begin{gathered}\ldots \pi_{n+1}(A) \xrightarrow{\Omega^n i_{A*}} \pi_{n+1}(X) \xrightarrow{\Omega^{n-1} j_*} [S^0, \Omega^n M_{i_A}] \xrightarrow{\Omega^{n-1} q_*} \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \ldots \\ \ldots \longrightarrow \pi_1(A) \longrightarrow \pi_1(X) \longrightarrow [S^0, M_{i_A}] \longrightarrow \pi_0(A) \longrightarrow \pi_0(X)\end{gathered}$$

The long exact sequence of homotopy groups is formed finally by making use of the bijection constructed in Proposition 2.4.3 and the fact that every map between groups in this sequence is a group homomorphism (in a more pedantic approach the functor $[S^1, \cdot]$ is initially applied on 2.4 to ensure group morphisms, then $[S^0, \cdot]$ to produce the tail of the sequence and finally these sequences are joined appropriately). $\square$

Lemma 2.5.3. *Let X, $Y \in \mathscr{T}^*$, $f : X \longrightarrow Y$ be a pointed map and $r : M_f \longrightarrow Y$ be a map defined by $r(x, \omega) = \omega(1)$. f is nullhomotopic rel$\{x_0\}$ if and only if there exists a pointed map ϕ which makes the diagram in 2.6 commute.*

Proof. $\Rightarrow$ Since f is nullhomotopic rel$\{x_0\}$, there is homotopy $F : X \times I \longrightarrow Y$ with $F(x, 0) = f(x_0) = y_0$, $F(x, 1) = f(x)$ $\forall$ $x \in X$ and $F(x_0, \cdot) = \omega_0$. We define $\phi : X \longrightarrow M_f$ by $\phi(x) = (x, F(x, \cdot))$. ϕ is a pointed map, because $\phi(x_0) = (x_0, F(x_0, \cdot)) = (x_0, \omega_0)$. Furthermore, $r(\phi(x)) = r(x, F(x, \cdot)) = F(x, 1) = f(x)$ for all $x \in X$, which proves that the diagram commutes.

$$\begin{tikzcd} & M_f \arrow[dr, "r"] & \\ X \arrow[ur, "\phi"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

Figure 2.6

$\Leftarrow$ Conversely, assume that such a ϕ exists. Then $\phi(x) = (a(x), \omega_x) \in M_f$, with $a : X \longrightarrow X$ and ω_x a path in Y. From the commutativity of the diagram we get that $r(\phi(x)) = \omega_x(1) = f(x)$ for all $x \in X$. A homotopy $F : X \times I \longrightarrow Y$ with $F(x, t) = \omega_x(t)$ can be defined. Since ϕ is a pointed map, we calculate $\phi(x_0) = (a(x_0), \omega_{x_0}) = (x_0, \omega_0)$, which means that ω_{x_0} is the constant path in Y, $F(x_0, t) = \omega_0(t) = y_0$ for all $t \in I$ and, subsequently, the homotopy preserves the basepoint. Also $F(x, 0) = \omega_x(0) = y_0$ and $F(x, 1) = \omega_x(1) = f(x)$, for all $x \in X$, which concludes that $\omega_0 \sim f$ rel$\{x_0\}$. $\square$

Lemma 2.5.4. *If X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, the sequence*

$$M_f \xrightarrow{q} X \xrightarrow{f} Y$$

is exact in $h\mathscr{T}^*$.

Proof. We will work in the category $h\mathscr{T}^*$. If $Z \in h\mathscr{T}^*$, we need to prove that the sequence $[Z, M_f] \xrightarrow{q_*} [Z, X] \xrightarrow{f_*} [Z, Y]$ is exact in $\mathscr{S}ets^*$.
img$q_* \subset$ kerf_*:
Let $h \in [Z, M_f]$. We want to show that $q_*([h])$ belongs to kerf_* or, in other words, that $f \circ q \circ h$ is nullhomotopic.

Obviously, if we proved instead that $f \circ q$ is nullhomotopic, it would suffice. From 2.5.3 this is equivalent to proving that there is a map ϕ_1 which makes the diagram in Fig.2.7 commute.
Define ϕ_1 by $\phi_1(x, \omega) = ((x, \omega), \omega) \in M_{f\circ q}$. Then $r_1(\phi_1(x, \omega)) = r_1((x, \omega), \omega) = \omega(1) = f(q(x, \omega))$ for all $(x, \omega) \in M_f$.
$\ker f_* \subset \operatorname{im} q_*$:
Let $[h] \in [Z, X]$ and $f \circ h$ be nullhomotopic via a homotopy F with $F_0 = \omega_0$ and $F_1 = f \circ h$. Then the diagram in Fig.2.8 commutes for $\phi_2 : Z \longrightarrow M_{f\circ h}$ with $\phi_2(z) = (z, F(z, \cdot))$. Indeed, $r_2(\phi_2(z)) = r_2(z, F(z, \cdot)) = F(z, 1) = f(h(z))$ for all $z \in Z$.
Thus, we conclude that $\operatorname{im} q_* = \ker f_*$ and our sequence is exact in $h\mathscr{T}^*$. $\square$

$$\begin{tikzcd} & M_{f\circ q} \arrow[dr, "r_1"] & \\ M_f \arrow[ur, "\phi_1"] \arrow[rr, "f \circ q"] & & Y \end{tikzcd}$$

Figure 2.7

$$\begin{tikzcd} & M_{f\circ h} \arrow[dr, "r_2"] & \\ Z \arrow[ur, "\phi_2"] \arrow[rr, "f \circ h"] & & Y \end{tikzcd}$$

Figure 2.8

Corollary. *If X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, the sequence*

$$\dots M_{\bar{q}} \xrightarrow{\bar{\bar{q}}} M_q \xrightarrow{\bar{q}} M_f \xrightarrow{q} X \xrightarrow{f} Y$$

is exact in $h\mathscr{T}^*$.
Here we have $M_q = \{((x, \omega), \gamma) \in M_f \times X^I |\ \gamma(0) = x_0,\ \gamma(1) = q(x, \omega) = x\}$, $\bar{q} : M_q \longrightarrow M_f$ with $\bar{q}((x, \omega), \gamma) = (x, \omega)$, $M_{\bar{q}} = \{(((x, \omega), \gamma), \xi) \in M_q \times M_f^I |\ \xi(0) = (x_0, \omega_0),\ \xi(1) = (x, \omega)\}$, $\bar{\bar{q}} : M_{\bar{q}} \longrightarrow M_q$ with $\bar{\bar{q}}(((x, \omega), \gamma), \xi)$.

Lemma's 2.5.4 corollary along with the next proposition will be used to prove that, if X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, the sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} M_f \xrightarrow{q} X \xrightarrow{f} Y$$

is exact in $h\mathscr{T}^*$.

Proposition 2.5.5. *If X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, then the following diagram is commutative in $h\mathscr{T}^*$.*

$$\begin{tikzcd} \Omega X \arrow[r, "\Omega f"] \arrow[d, "r_1"] & \Omega Y \arrow[r, "j"] \arrow[d, "r_2"] & M_f \arrow[r, "q"] \arrow[d, "\mathbb{1}"] & X \arrow[r, "f"] \arrow[d, "\mathbb{1}"] & Y \arrow[d, "\mathbb{1}"] \\ M_{\bar{q}} \arrow[r, "\bar{\bar{q}}"] & M_q \arrow[r, "\bar{q}"] & M_f \arrow[r, "q"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Figure 2.9

Proof. Let us first define the undefined maps in the diagram. We have

$r_2 : \Omega Y \longrightarrow M_q$ with $r_2(\omega) = ((x_0, \omega), \gamma_0)$
$r_1 : \Omega X \longrightarrow M_{\bar{q}}$ with $r_1(\gamma) = (((x_0, \omega_0), \gamma), \xi_0)$

Counting from left to right, the third and fourth squares obviously commute. In the second square we have $\mathbb{1}(j(\omega)) = \mathbb{1}(x_0, \omega) = (x_0, \omega)$ and $\bar{q}(r_2(\omega)) = \bar{q}((x_0, \omega), \gamma_0) = (x_0, \omega)$ for all $\omega \in \Omega Y$. So it commutes in $\mathscr{T}^*$ and, consequently, in $h\mathscr{T}^*$ too.
Now, we examine the first square of the diagram. Going one way, we get $r_1(\Omega f(\gamma)) = r_1(f \circ \gamma) = ((x_0, f \circ \gamma), \gamma_0)$, while going the other way $\bar{\bar{q}}(r_2(\gamma)) = \bar{\bar{q}}(((x_0, \omega_0), \gamma), \xi_0) = ((x_0, \omega_0), \gamma)$. If we find a pointed homotopy between these functions, then the square commutes. First, we define for each $t \in I$ and each $\gamma \in X^I$ a path in X with

$$\beta_{\gamma,t}(s) = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \gamma(t), & t \leq s \leq 1 \end{cases}$$

where $s \in I$. These paths coincide with γ up until s reaches t. After that, meaning from $s = t$ till $s = 1$, they remain constant on their end point $\gamma(t)$.
Let $F : X^I \times I \longrightarrow M_q$ with $F(\gamma, t) = ((x_0, f \circ \beta_{\gamma,t}), \beta_{\gamma,1-t})$. F is continuous and gives $F(\gamma, 0) = ((x_0, f \circ \beta_{\gamma,0}), \beta_{\gamma,1}) = ((x_0, f \circ \gamma_0), \gamma) = ((x_0, \omega_0), \gamma) = \bar{\bar{q}}(r_2(\gamma))$, $F(\gamma, 1) = ((x_0, f \circ \beta_{\gamma,1}), \beta_{\gamma,0}) = ((x_0, f \circ \gamma), \gamma_0) = r_1(\Omega f(\gamma))$ and $F(\gamma_0, t) = ((x_0, f \circ \beta_{\gamma_0,t}), \beta_{\gamma_0,1-t}) = ((x_0, \omega_0), \gamma_0)$. Therefore the first square also commutes in $h\mathscr{T}^*$. □

Corollary. *If X, $Y \in \mathscr{T}^*$ and $f : X \longrightarrow Y$ is a pointed map, then the sequence*

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{j} M_f \xrightarrow{q} X \xrightarrow{f} Y$$

is exact in $h\mathscr{T}^*$.

Proof. The diagram in Figure 2.9 commutes, the vertical maps are equivalences in $h\mathscr{T}^*$ (see Definition A.1.13) and the bottom row is exact. Apply the functor $[Z, \cdot]$ to this diagram and use the fact that $(r_1)_*$ and $(r_2)_*$ are isomorphisms, which is derived by the aforementioned equivalences. The top row is exact in $\mathscr{S}ets^*$, consequently the top row in the original diagram is exact in $h\mathscr{T}^*$. □

Lemma 2.5.6. *If $X \xrightarrow{f} Y \xrightarrow{g} W$ is an exact sequence in $h\mathscr{T}^*$, then so is the looped sequence $\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\Omega g} \Omega W$.*

$$\begin{array}{ccccc} [\Sigma Z, X] & \longrightarrow & [\Sigma Z, Y] & \longrightarrow & [\Sigma Z, W] \\ \downarrow & & \downarrow & & \downarrow \\ [Z, \Omega X] & \longrightarrow & [Z, \Omega Y] & \longrightarrow & [Z, \Omega W] \end{array}$$

Figure 2.10

Proof. From the adjointness of (Σ, Ω) we get the commutative diagram in Fig. 2.10, where the vertical maps are pointed bijections. The top row is exact in $\mathscr{S}ets^*$, by hypothesis, and so it follows that the bottom row is exact as well for all $Z \in \mathscr{T}$. □

Proof of Theorem 2.5.1 The result is given if we splice together the exact sequence of Lemma's 2.5.4 Corollary with the exact sequence of Lemma 2.5.6 and akcnowledge that $\Omega^n X = \Omega(\Omega^{n-1} X)$. $\square$

Therefore, it has been proved that for every pair $(X, A, *) \in \mathscr{T}^{2*}$, there is an exact sequence of homotopy groups

$$\begin{gathered} \ldots \longrightarrow \pi_{n+1}(A) \xrightarrow{\Omega^n i_{A*}} \pi_{n+1}(X) \xrightarrow{\Omega^{n-1} j_*} \pi_{n+1}(X, A) \xrightarrow{\Omega^{n-1} q_*} \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \ldots \\ \ldots \longrightarrow \pi_1(A) \longrightarrow \pi_1(X) \longrightarrow \pi_1(X, A) \longrightarrow \pi_0(A) \longrightarrow \pi_0(X) \end{gathered}$$

Chapter 3

Homology

3.1 Axiomatic definition of homology

The axiomatic foundation of homology was given by Eilenberg and Steenrod in [17]. In their book, which was published in 1952, they stated the seven axioms for a homology theory. Here, we have grouped some of the initial axioms together or replaced them with notions equivalent to them, and we have added the additivity axiom, which was firstly introduced by Milnor, in 1962 ([18]). Our presentation follows at points the presentations of Spanier, in [3], and Hatcher, in [10].

Homology theory is defined on a suitable category of pairs and maps called admissible category. The definition of this category given by Eilenberg and Steenrod can be found in [17]. In our study, it suffices to mention that $\mathscr{T}^2$ is an admissible category of pairs and maps.

Let $\mathscr{A}$ be an admissible category and (X, A), $(Y, B) \in \mathscr{A}^2$. A **homology theory** (h, ∂) consists of a functor h and a natural transformation ∂. Specifically, h is a covariant functor from the category $\mathscr{A}^2$ to the category $\mathscr{G}_{ab}$ of graded abelian groups and homomorphisms of degree 0 (Definition A.2.7). That is $h(X, A) = \{h_n(X, A)\}$. On the other hand, the natural transformation ∂, which is called the **boundary operator**, is a map of degree -1 from the functor h on (X, A) to the functor h on $(A, \varnothing)$. That is $\partial(X, A) : \{\partial_n(X, A) : h_n(X, A) \longrightarrow h_{n-1}(A)\}$. Naturality of ∂ means that diagram in Fig.3.1 is commutative for every $f : (X, A) \longrightarrow (Y, B)$.

$$\begin{array}{ccc} h_n(X,A) & \xrightarrow{\partial_n} & h_{n-1}(A) \\ \Big\downarrow{\scriptstyle h_n(f)} & & \Big\downarrow{\scriptstyle h_n(f|_A)} \\ h_n(Y,B) & \xrightarrow{\partial_n} & h_{n-1}(B) \end{array}$$

Figure 3.1

For the pair (h, ∂) to be a homology theory, the following axioms must be satisfied:

1. **Homotopy Axiom** If f_0, $f_1 : (X, A) \longrightarrow (Y, B)$ are homotopic, then $h(f_0) = h(f_1) : h(X, A) \longrightarrow h(Y, B)$.
   For simplicity, we will employ f_{*n} instead of $h_n(f)$ when referring to the induced maps.

2. **Exactness Axiom** For any pair $(X, A) \in \mathscr{A}^2$ with inclusion maps $i_A : A \longrightarrow X$ and $j_X : X \longrightarrow (X, A)$ there is an exact sequence

   $$\ldots \overset{\partial_{n+1}(X,A)}{\longrightarrow} h_n(A) \overset{h_n(i_A)}{\longrightarrow} h_n(X) \overset{h_n(j_X)}{\longrightarrow} h_n(X, A) \overset{\partial_n(X,A)}{\longrightarrow} h_{n-1}(A) \overset{h_{n-1}(i_A)}{\longrightarrow} \ldots$$

   This sequence is called the homology sequence of (X, A).

3. **Excision Axiom** For any pair $(X, A) \in \mathscr{A}^2$, if U is an open subset of X such that $\bar{U} \subset int(A)$, then the excision map $j : (X - U, A - U) \longrightarrow (X, A)$ induces an isomorphism
   $$h(j) : h(X - U, A - U) \longrightarrow h(X, A).$$

4. **Additivity Axiom** If X is the disjoint union of open subsets X_a with inclusion maps $i_a : X_a \longrightarrow X$, $\{X_a\} \subset \mathscr{A}$, then the homomorphisms

   $$h_n(i_a) = i_{a*} : h_n(X_a) \longrightarrow h_n(X)$$

   must provide an injective representation of $h_n(X)$ as a direct sum. In other words, the direct sum map $\oplus_a i_{a*} : \oplus_a h_n(X_a) \longrightarrow h_n(X)$ is an isomorphism.

5. **Dimension Axiom** If $P \in \mathscr{A}$ consists of a single point, then $h_n(P) = 0$ for all $n \neq 0$.

Our target is to prove that a homology theory like the one axiomatically described above really exists. For that reason we are going to introduce and study singular homology. It will be proved that singular homology is indeed such a homology theory. Moreover, as it was shown by Milnor in [18], if we fix $h_0(P) = G$ for some abelian group G, the singular homology with coefficient group G (see below) becomes the only homology theory that satisfies the axioms of homology.

3.2 Singular homology

A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$, where Δ^n is the standard n-simplex defined in Section 1.3. We denote with $C_n(X)$ the free abelian group with basis the set of singular n-simplices in X. The elements of $C_n(X)$, called **singular n-chains**, are finite formal sums $\sum_i n_i\sigma_i$ for coefficients $n_i \in \mathbb{Z}$, $\sigma_i : \Delta^n \longrightarrow X$.

Let now $\{e_0, e_1, ..., e_n\}$ be the standard unit vectors in $\mathbb{R}^{n+1}$. We define the map $\delta_i^n : \{0, 1, \ldots, n-1\} \longrightarrow \{0, 1, \ldots, n\}$ as the inclusion that omits the value i and $d_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ by the formula $d_i^n(\sum_{i=0}^{n-1} t_ie_i) = \sum_{i=0}^{n} t_ie_{\delta_i^n}$, which is the i-th face of Δ^n. Using the maps d_i^n, a boundary map $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ can be defined for each n via $\partial_n(\sigma) = \sum_i(-1)^i\sigma d_i^n$. Maps ∂_n lead to the following sequence:

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(X) \longrightarrow 0 \tag{3.1}$$

A sequence like this is called a **chain complex** whenever $\partial^2 = 0$.

Lemma 3.2.1. *The composition* $C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$ *is zero.*

Proof. It is easy to check that $\delta_i^n\delta_j^{n-1} = \delta_j^n\delta_{i-1}^{n-1}$ for $j < i$. Hence

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \sum_j (-1)^j \left(\sum_i (-1)^i\sigma d_i^n\right) d_j^{n-1} = \\
&= \sum_{j<i}(-1)^i(-1)^j\sigma d_i^n d_j^{n-1} + \sum_{j\geq i}(-1)^i(-1)^j\sigma d_i^n d_j^{n-1} = \\
&= \sum_{j<i}(-1)^i(-1)^j\sigma d_j^n d_{i-1}^{n-1} + \sum_{j>i}(-1)^i(-1)^{j-1}\sigma d_i^n d_{j-1}^{n-1} = 0
\end{aligned}$$

since the second sum becomes the negative of the first after switching i and j in it. □

Since $\partial_{n-1}\partial_n = 0$, 3.1 becomes a chain complex, $Im\partial_{n+1} \subset ker\partial_n$ for all n, and the n-th **singular homology** of X can be defined via the formula $H_n(X) := \dfrac{ker\partial_n}{Im\partial_{n+1}}$. To complete the terminology, let us say that elements of $ker\partial$ are called **cycles**, while elements in $Im\partial$ are called **boundaries**.

If we use the augmented sequence below instead of 3.1, we compute the **reduced homology groups** $\widetilde{H}_n(X)$.

$$\ldots \longrightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{3.2}$$

Here $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ takes a chain $\sum_i n_i\sigma_i$ to the integer $\sum_i n_i$. As one may immediately notice, $H_n(X) = \widetilde{H}_n(X)$ when $n \geq 1$ and $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Deviating slightly from our course, we mention here that if we had built singular n-chains with coefficients $n_i \in G$, $G \in \mathscr{G}_{ab}$, instead of $n_i \in \mathbb{Z}$, we would have created the abelian group $C(X; G)$. This would have led, in a similar way as above, to the definition of the so called **singular homology with coefficients in** G, denoted with $H_n(X; G)$. Having clarified this, the remainder of the section is dedicated to proving the existence and uniqueness of a homology theory which satisfies the axioms and has $H_0(P) = \mathbb{Z}$, or, to state it differently, has coefficient group $\mathbb{Z}$.

The homology that qualifies for the task of fulfilling all the axioms of homology is the **relative singular homology**. Let us define it for $(X, A) \in \mathscr{A}^2$:

Let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. This means that chains with image in A are trivial in $C_n(X, A)$. These chains are called **relative cycles**. Since the boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a well defined quotient boundary map $\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$. Elements in $Im\partial$ are called **relative boundaries**.
We form a sequence

$$\ldots \longrightarrow C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \longrightarrow \ldots,$$

exactly like the one in the absolute case, for which the relation $\partial^2 = \partial_n \partial_{n+1} = 0$ holds. This means that the sequence is a chain complex for the relative case with $Im\partial_{n+1} \subset ker\partial_n$, which eventually allows us to define the relative homology groups $H_n(X, A) = \dfrac{ker\partial_n}{Im\partial_{n+1}}$. Elements of $H_n(X, A)$ are represented by relative cycles $a \in ker\partial_n$.

Remark. Singular homology groups can be derived from relative singular homology groups, if we employ the pair $(X, \emptyset) \in \mathscr{A}^2$ in place of just $X \in \mathscr{A}$.

For a map $f : X \longrightarrow Y$, an induced homomorphism $f_\sharp : C_n(X) \longrightarrow C_n(Y)$ is defined by composing each singular n-simplex $\sigma : \Delta^n \longrightarrow X$ with f to get a singular n-simplex $f_\sharp(\sigma) = f\sigma : \Delta^n \longrightarrow Y$. $f_\sharp$ is then extended linearly. A simple calculation shows that each $f_\sharp$ satisfies $\partial_n f_\sharp = f_\sharp \partial$ for all n, therefore the diagram in Fig.3.2 commutes. This construction can be modified appropriately to incorporate

$$\begin{tikzcd}
\ldots \arrow[r] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[r, "\partial_{n-1}"] \arrow[d, "f_\sharp"] & C_{n-2}(X) \arrow[r] \arrow[d, "f_\sharp"] & \ldots \\
\ldots \arrow[r] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r, "\partial_{n-1}"] & C_{n-2}(Y) \arrow[r] & \ldots
\end{tikzcd}$$

Figure 3.2

the case of a map $f : (X, A) \longrightarrow (Y, B)$, leading to an $f_\sharp$ which ultimately can pass to quotients. More specifically, it gives $f_\sharp : C_n(X, A) \longrightarrow C_n(Y, B)$ with $f_\sharp([\sigma]) = [f\sigma]$, which is a well defined map.

The fact that maps $f_\sharp$ satisfy $\partial_n f_\sharp = f_\sharp \partial$ for all n is also expressed by saying that maps $f_\sharp$ define a **chain map** from the singular chain complex of (X, A) to that of (Y, B). The same relation also implies that $f_\sharp$ take cycles to cycles and boundaries to boundaries. f finally leads to an induced homomorphism $H_n(f) = f_* : H_n(X, A) \longrightarrow H_n(Y, B)$. This result is summarised in the next Proposition:

Proposition 3.2.2. *A chain map between chain complexes induces homomorphisms between the relative homology groups of the two complexes.*

It can be verified using the associativity of compositions $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$ that $H_n(fg) = (fg)_* = H_n(f)H_n(g) = f_*g_*$ for $(X, A) \xrightarrow{g} (Y, B) \xrightarrow{f} (Z, C)$. It also holds that $H_n(\mathbb{1}) = \mathbb{1}_* = \mathbb{1}$, where $\mathbb{1}$ denotes the identity map of a space or a group. Therefore, H_n is a covariant functor from $\mathscr{A}$ to $\mathscr{G}_{ab}$.

Proposition 3.2.3. *Let $X = \{x_0\}$. Then $H_n(X) = 0$ for $n \neq 0$ and $H_0(X) = \mathbb{Z}$ or, in other words, singular homology satisfies the Dimension Axiom and has $\mathbb{Z}$ as its group of coefficients.*

Proof. There is a unique singular n-simplex σ_n for each n. $\partial\sigma_n = \sum_{i=0}^{n}(-1)^i \sigma_n d_i^n = \sum_{i=0}^{n}(-1)^i \sigma_{n-1}$, hence ∂ is either 0 for n odd or an isomorphism for n even. We have the chain complex

$$\ldots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

The conclusion is readily drawn from this chain complex. □

Proposition 3.2.4. *Let (X, A), $(Y, B) \in \mathscr{A}^2$ and f, $g \in Hom((X, A), (Y, B))$. If f and g are homotopic, then $f_* = g_* : H_n(X, A) \longrightarrow H_n(Y, B)$. Thus, singular relative homology satisfies the Homotopy Axiom.*

Proof. In the prism $\Delta^n \times I$ we denote with $[v_0, v_1, \ldots, v_n]$ the n-simplex $\Delta^n \times \{0\}$ and with $[w_0, w_1, \ldots, w_n]$ the n-simplex $\Delta^n \times \{1\}$. Then $\Delta^n \times I$ consists of all the $(n + 1)$-simplices of the form $[v_0, \ldots, v_i, w_i, \ldots, w_n]$, where $i \in \{0, 1, \ldots, n\}$. Given a homotopy $F : X \times I \longrightarrow Y$ from f to g and a singular simplex $\sigma : \Delta^n \longrightarrow X$, the composition $G = F \circ (\sigma \times \mathbb{1}) : \Delta^n \times I \longrightarrow X \times I \longrightarrow Y$ can be formed. Through this we define the prism operators $P : C_n(X) \longrightarrow C_{n+1}(Y)$ with:

$$P(\sigma) = \sum_i (-1)^i (F \circ (\sigma \times \mathbb{1}))|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}.$$

Obviously, P take $C_n(A)$ to $C_{n+1}(B)$, hence they induce relative prism operators $P : C_n(X, A) \longrightarrow C_{n+1}(Y, B)$ with $P(\sigma) = \sum_i (-1)^i (F \circ (\sigma \times \mathbb{1}))|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$, with σ denoting the equivalence class in $C_n(X)/C_n(A)$ from here on.
If we compute $\partial P(\sigma)$ and $P\partial(\sigma)$ and then take their sum, we get $\partial P(\sigma) + P\partial(\sigma) = g\sigma - f\sigma = g_\sharp(\sigma) - f_\sharp(\sigma)$. Now, if $a \in C_n(X, A)$ is a relative cycle, then we have $g_\sharp(a) - f_\sharp(a) = \partial P(a) + P\partial(a) = \partial P(a) + P(0) = \partial P(a) \in Im\partial_{n+1}$, hence $g_*(\sigma) - f_*(\sigma) = 0$ in $H_n(X, A)$. If the equation $\partial P + P\partial = g_\sharp - f_\sharp$ holds for all $\sigma \in C_n(X, A)$, we say that $f_\sharp$ and $g_\sharp$ are chain homotopic on relative chain groups. In this case they induce equal homomorphisms f_* and g_* on relative homology groups, as it has been shown. □

Remark. If two maps are chain homotopic in relative homotopy groups, we get equality of the induced maps. Chain homotopy is not as strict a requirement as regular homotopy. In a way, it is easier to find maps f and g with $f_* = g_*$ in homology, since no basepoint is required to be fixed and $f_\#$ and $g_\#$ are allowed to differ by boundaries.

Proposition 3.2.5. *There is a long exact sequence of homology groups:*

$$\dots \xrightarrow{\bar{\partial}_{n+1}} H_n(A) \xrightarrow{i_{A*}} H_n(X) \xrightarrow{j_{X*}} H_n(X,A) \xrightarrow{\bar{\partial}_n} H_{n-1}(A) \xrightarrow{i_{A*}} \dots$$

which proves that singular relative homology satisfies the Exactness Axiom.

Proof. We present here the proof in a descriptive manner and refer to [10] for its detailed and complete version.
In the long sequence of homology groups i_{A*} and j_{X*} are induced by the inclusions $i_A : (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $j_X : (X, \emptyset) \hookrightarrow (X, A)$, respectively. Therefore, the only map left to define is the boundary map $\partial : H_n(X,A) \longrightarrow H_{n-1}(A)$. In order to do so we employ the following diagram, which is commutative and has short exact sequences as columns and chain complexes as rows. ∂ is built using the relevant

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"] & C_{n-1}(A) \arrow[r] \arrow[d, "i"] & \dots \\
\dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "j"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "j"] & C_{n-1}(X) \arrow[r] \arrow[d, "j"] & \dots \\
\dots \arrow[r] & C_{n+1}(X,A) \arrow[r, "\partial"] \arrow[d] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 3.3

part of this diagram as a 'stair' from $C_n(X,A)$ to $C_{n-1}(A)$ (up-right-up). A relative cycle $c \in C_n(X,A)$ moves through the epimorphism j up to an element $b \in C_n(X)$ with $j(b) = c$, then goes to $\partial b \in C_{n-1}(X)$ and from there up again to an element $a \in C_{n-1}(A)$ such that $i(a) = \partial b$.
The commutativity of the diagram, the exactness of the columns and the properties of the rows ensure that this map is well defined. Now, one has just to check if the produced sequence of homology groups is exact, which is routine and analytically addressed in [10]. □

Remark. Generalising the concept of a pair (X,A) and its exact sequence, we form triples of spaces (X,A,B), $B \subset A \subset X$, and the sequence:

$$\ldots \longrightarrow H_n(A,B) \longrightarrow H_n(X,B) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A,B) \longrightarrow \ldots,$$

which is proved to be exact. For the proof a diagram similar to the one in Figure 3.3 is used, where the columns are the short exact sequences

$$0 \longrightarrow C_n(A,B) \longrightarrow C_n(X,B) \longrightarrow C_n(X,A) \longrightarrow 0.$$

Remark. Obviously, the same proof works for the construction of a long exact sequence of reduced homology groups of a pair (X,A), $A \neq \emptyset$,

$$\ldots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X,A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \ldots,$$

if we add the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mathbb{1}} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

in dimension -1. $H_n(X,A) = \widetilde{H}_n(X,A)$ for all n, when $A \neq \emptyset$.

Example 3.2.6. $H_n(X,x_0) \cong \widetilde{H}_n(X)$
The long exact sequence for the reduced homology is

$$\ldots \longrightarrow \widetilde{H}_n(x_0) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X,x_0) \longrightarrow \widetilde{H}_{n-1}(x_0) \longrightarrow \ldots$$

But $\widetilde{H}_n(x_0) \cong 0$ for all n, thus $H_n(X,x_0) = \widetilde{H}_n(X,x_0) \cong \widetilde{H}_n(X)$.

Theorem 3.2.7. *Let $(X,A) \in \mathscr{A}^2$ and $U \subset X$ open such that $\bar{U} \subset int(A)$. The excision map $j : (X-U, A-U) \longrightarrow (X,A)$ induces isomorphisms $j_* : H_n(X-U, A-U) \longrightarrow H_n(X,A)$ for all n. In other words, singular relative homology satisfies the Excision Axiom.*
Equivalently, for subspaces A, $B \subset X$, whose interiors cover X, the inclusion $i : (B, A \cup B) \hookrightarrow (X,A)$ induces isomorphisms $i_* : H_n(B, A \cap B) \longrightarrow H_n(X,A)$ for all n.

Proof. The complete proof is rather lengthy and technical. We refer the reader to [10] for it. Here we confine ourselves to discussing its general idea and some of its basic steps.
We start by considering a collection $\mathscr{U} = \{U_i\}$ of subspaces of X, whose interiors cover X, and denoting with $C_n^{\mathscr{U}}(X)$ the subgroup of $C_n(X)$ generated by singular maps σ with image in some $U_i \in \mathscr{U}$. Observe that $\partial(C_n^{\mathscr{U}}(X)) \subset C_{n-1}^{\mathscr{U}}(X)$, so $H_n^{\mathscr{U}}(X)$ can be formed, as usual. For this cover $\mathscr{U}$ and $\sigma \in C_n(X)$, $\{\sigma^{-1}(U_i)\}$ is a cover of open sets of the compact metric space $\Delta^n \subset \mathbb{R}^n$. We name $m(\sigma)$ a fixed Lebesgue number for the cover $\{\sigma^{-1}(U_i)\}$ (see Proposition 1.1.17). Based on it we will try to 'break down' the domain Δ^n and subsequently each map σ to components fitting inside some U_i.
Having said that, the main idea behind the proof is to construct a map $\rho : C_n(X) \longrightarrow C_n^{\mathscr{U}}(X)$ which decomposes each generator σ to a sum of such 'small' component maps $\sigma_i \in C_n^{\mathscr{U}}(X)$ and makes $i\rho$ and ρi chain homotopic to the identity. Then $i_* : H_n^{\mathscr{U}}(X) \longrightarrow H_n(X)$ will become an isomorphism. Applying this result for the cover

$\mathscr{U} = \{A, B\}$ and the inclusion i between quotients $C_n^{\mathscr{U}}(X)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$ gives $H_n^{\mathscr{U}}(X, A) \cong H_n(X, A)$, which, combined with the isomorphism $C_n(B)/C_n(A \cap B) \cong C_n^{\mathscr{U}}(X)/C_n(A)$, will ultimately lead to the desired isomorphism $H_n(B, A \cap B) \cong H_n(X, A)$.
In order to achieve what has been just described, we need to decompose every σ's domain Δ^n into simplices with diameters smaller than $m(\sigma)$. Iterated barycentric subdivision of Δ^n is employed to this end (see 1.3.5, 1.3.6 and 1.3.9). This procedure is extended to maps and produces an operator $S : C_n(X) \longrightarrow C_n(X)$, which subdivides generators σ. S leads to a new operator $D : C_n(X) \longrightarrow C_{n+1}(X)$, for which $\partial D + D\partial = \mathbb{1} - i\rho$ holds. Also $\rho i = \mathbb{1}$, since $m(\sigma) = 0$ if $\sigma \in C_n^{\mathscr{U}}(X)$. $\square$

Proposition 3.2.8. *Let $X \in \mathscr{T}$ and $\{X_\alpha\}$ be its path components. Then the inclusions $i_\alpha : X_\alpha \longrightarrow X$ induce an isomorphism $\oplus_\alpha i_{\alpha *} : \oplus_\alpha H_n(X_\alpha) \longrightarrow H_n(X)$. In other words, singular homology satisfies the Additivity Axiom.*

Proof. A singular simplex has always a path connected image. Thus $C_n(X)$ can be written as the direct sum of its subgroups $C_n(X_\alpha)$. This decomposition remains unchanged even after applying the boundary maps ∂_n, so $C_n(X_\alpha)$ is taken to $C_{n-1}(X_\alpha)$. $ker\partial_n$ and $Im\partial_{n+1}$ split similarly into direct sums. Therefore, the homology groups also split, giving finally the isomorphism $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$. $\square$

We have proved that singular homology H_n, together with its boundary map, constitute a homology theory, indeed. Next, some useful Propositions and examples will be mentioned, right before we prove the uniqueness of H_n.

Proposition 3.2.9. *A pair $(X, A) \in \mathscr{T}^2$ is called a good pair if A is a nonempty closed subspace of X and a deformation retraction of some neighbourhood U in X. Let (X, A) be a good pair and $q : (X, A) \longrightarrow (X/A, A/A)$ the quotient map. Then q induces isomorphisms $q_* : H_n(X, A) \longrightarrow H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$ for all n.*

Proof. In the commutative diagram in Figure 3.4 i_{1*} are isomorphisms, because $H_n(U, A) = 0$ and $H_n(U/A, A/A) = 0$ for all n in the exact sequences of the triple (X, U, A) and $(X/A, U/A, A/A)$, respectively. i_{2*} are isomorphisms too, a conclusion reached through excision. Finally, the right-hand vertical map q_* is an isomorphism,

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "i_{1*}"] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "q_*"] & H_n(X-A, U-A) \arrow[l, "i_{2*}"'] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r, "i_{1*}"] & H_n(X/A, U/A) & H_n(X/A - A/A, U/A - A/A) \arrow[l, "i_{2*}"']
\end{tikzcd}$$

Figure 3.4

because q restricts to a homeomorphism on the complement of A. The commutativity of the diagram leads to the desired result. $\square$

Example 3.2.10.

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z} \text{ for } i = n \\ 0 \text{ otherwise} \end{cases}$$

From the long exact sequence of the pair $(D^n, \partial D^n)$, where $\partial D^n \equiv S^{n-1}$ and D^n is contractible, we get $H_i(D^n, \partial D^n) \cong H_{i-1}(S^{n-1})$. Proposition 3.2.9 also gives $H_i(D^n, \partial D^n) \cong H_i(D^n/\partial D^n) \cong \widetilde{H}_i(S^n)$. Thus $\widetilde{H}_i(S^n) \cong H_{i-1}(S^{n-1})$. Induction and the homology groups of S^1 calculated in Example 3.3.3 produce the result.

Example 3.2.11. The identity map $id_n : \Delta^n \longrightarrow \Delta^n$ is a relative cycle generating $H_n(\Delta^n, \partial\Delta^n) \cong H_n(D^n, \partial D^n) \cong \mathbb{Z}$ (see Example 3.2.10).
It is obvious that $[id_n]$ is a relative cycle, so we need only to prove that it generates $H_n(\Delta^n, \partial\Delta^n)$. We will use induction on n. For $n = 0$ the result is trivial. Let it also hold till $n-1$. We denote the union of all but one of the $(n-1)$-dimensional faces of Δ^n with K. More specifically, we choose to work with the inclusion $d_0^n : \Delta^{n-1} \longrightarrow \Delta^n$ and excise the face $[d_0^n]$ in every case. The following maps are isomorphisms

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, K) \xleftarrow{(d_0^n)_*} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

∂ is the boundary map in the long exact sequence of the triple $(K, \partial\Delta^n, \Delta^n)$

$$\longrightarrow H_n(\partial\Delta^n, K) \longrightarrow H_n(\Delta^n, K) \longrightarrow H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, K) \longrightarrow$$

and an isomorphism, because Δ^n deformation retracts to K. $(d_0^n)_*$ is an isomorphism as well, since $\Delta^{n-1}/\partial\Delta^{n-1} \equiv \partial\Delta^n/K$ and these pairs are good (Proposition 3.2.9). According to the definition of the boundary map we have $\partial[id_n] = \sum_j (-1)^n [d_j^n]$. However, $[d_j^n] = 0$ for $j \neq 0$ in the relative group $H_n(\partial\Delta^n, K)$, so we get $\partial[id_n] = (-1)^n[d_0^n]$. In conclusion, the composite isomorphism $\partial^{-1}(d_0^n)_* : H_n(\Delta^n, \partial\Delta^n) \longrightarrow H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$ sends the generator $[id_{n-1}]$ to $[id_n]$.

Proposition 3.2.12. *If X is a CW-complex, the inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_i(X^n) \longrightarrow H_i(X)$ for all $i < n$.*

Proof. Since (X^{n+1}, X^n) is a good pair and $X^{n+1}/X^n \equiv \bigvee_\alpha S_\alpha^{n+1}$, we deduce that $H_i(X^{n+1}, X^n) = 0$ for $i \neq n+1$ and $H_{n+1}(X^{n+1}, X^n) \cong H_{n+1}(\bigvee_\alpha S_\alpha^{n+1})$. Generally, for a wedge sum $\bigvee_\alpha X_\alpha$ at basepoints $x_\alpha \in X_\alpha$ such that the pairs (X_α, x_α) are good, we get $\widetilde{H}_{n+1}(\coprod_\alpha X_\alpha/\coprod_\alpha x_\alpha) \cong H_{n+1}(\coprod_\alpha X_\alpha, \coprod_\alpha x_\alpha) \cong \oplus_\alpha H_{n+1}(X_\alpha, x_\alpha) \cong \oplus_\alpha \widetilde{H}_{n+1}(X_\alpha)$ from the excision axiom, the additivity axiom and Example 3.2.6. Knowing that $H_{n+1}(S^{n+1}) \cong \mathbb{Z}$, we conclude that $H_{n+1}(X^{n+1}, X^n) \cong \oplus_\alpha \mathbb{Z}$.
Let us consider the long exact sequence of the pair (X^{n+1}, X^n)

$$.. \longrightarrow H_{i+1}(X^{n+1}, X^n) \longrightarrow H_i(X^n) \longrightarrow H_i(X^{n+1}) \longrightarrow H_i(X^{n+1}, X^n) \longrightarrow .. \qquad (3.3)$$

If $i \neq n+1$, $i \neq n$, then $H_i(X^{n+1}) \cong H_i(X^n)$, and, if we take the long exact sequences of the pairs (X^{n+k}, X^{n+k-1}) the one after the other, for $k \geq 1$, we conclude that $H_i(X^n) \cong H_i(X^{n+k})$, when $i < n$. This finishes the case of a finite dimensional CW-complex.
Let X have infinite dimension. A singular chain $\sum_k n_k \sigma_k \in C_i(X)$ has compact image in the CW-complex X (Proposition 1.1.18), hence meets only finitely many cells of X. Take $m \in \mathbb{Z}$ such that $\sum_k n_k \sigma_k(\Delta^i) \subset X^m$. For a i-cycle c and with the finite dimensional case already proved, we know that there exists an n such that $c \in H_i(X^n)$ for all $i < n$. Therefore, $i_* : H_i(X^n) \longrightarrow H_i(X)$ is surjective. For this n,

$i_* : H_i(X^n) \longrightarrow H_i(X)$ is also injective. If we take representatives c_1, $c_2 \in H_i(X^n)$ such that $i_*[c_1] = i_*[c_2]$, then these are homologous, so they differ by a boundary of a chain b in $C_{i+1}(X)$, $i < n$. Since b has compact image in the CW-complex X, we get an m, $n \leq m$, such that $b(\Delta^{i+1}) \subset X^m$ and this leads to $[c_1] = [c_2]$ in $H_i(X^m)$. But then $H_i(X^m) \cong H_i(X^n)$ for all $i < n$ from the finite dimensional case, which completes the proof. $\square$

We are going to complete our presentation of singular homology by calling up the following Theorem, which was stated and proved by Milnor in 1962, regarding the uniqueness of H_n:

Theorem 3.2.13. *Let (h, ∂) be an additive homology theory on the category $\mathscr{W}$ (Example A.1.3, 6) with coefficient group G. Then for each $(X, A) \in \mathscr{W}$ there is a natural isomorphism between $h_n(X, A)$ and the nth singular homology group of (X, A) with coefficients in G, $H_n((X, A); G)$.*

Proof. It can be found in [18]. $\square$

Therefore, if one chooses a particular coefficient group, there is exactly one homology theory which conforms with all the axioms. Furthemore, this homology theory is isomorphic to the singular homology theory. In other words, any homology theory in compliance with the aforementioned requirements coincides with the singular relative homology theory.

3.3 Computations of homological groups

Singular homology proves extremely handy for the theoretical study of homology theory. However, it does not offer similar advantages when it comes to computing homology groups of spaces. There are other, more suitable homology theories that can do the trick. In this section, we will briefly introduce two such homologies: simplicial and cellular. We will restrict ourselves to very limited information, since we would rather make some calculations than be thorough in presenting these theories.

3.3.1 Simplicial homology

Definition 3.3.1. Let $X \in \mathscr{T}$ and Δ^n be the standard n-simplex. A **Δ-complex** stucture is a collection of maps $\sigma_\alpha : \Delta^{n(\alpha)} \longrightarrow X$ such that:

1. the restriction $\sigma_\alpha|_{(\Delta^{n(\alpha)})^o}$ is injective and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{(\Delta^{n(\alpha)})^o}$;
2. each restriction of σ_α to a face in $\partial\Delta^n \equiv \Delta^{n-1}$ of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \longrightarrow X$;
3. a set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices $e^n_\alpha := \sigma((\Delta^{n(\alpha)})^o)$ of X. Elements of $\Delta_n(X)$ are written as finite sums $\sum_\alpha n_\alpha e^n_\alpha$ with $n_\alpha \in \mathbb{Z}$. Equivalently, we can write $\sum_\alpha n_\alpha \sigma_\alpha$ where σ_α is the characteristic map of e^n_α. A boundary map $\partial : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is defined by the formula $\partial(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha d^n_i$, just like it did in singular homology. The proof of 3.2.1 suffices to show that $\partial^2 = 0$, hence the groups $\Delta_*(X)$ form a chain complex.

$$\ldots \longrightarrow \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} \Delta_0(X) \longrightarrow 0 \tag{3.4}$$

The n-th **simplicial homology** of X can be defined via the formula $H^\Delta_n(X) := \dfrac{ker\partial_n}{Im\partial_{n+1}}$ using the chain complex in 3.4.

Theorem 3.3.2. *The homomorphisms $H^\Delta_*(X,A) \longrightarrow H_*(X,A)$ are isomorphisms for all Δ-complex pairs (X,A) (X is a Δ-complex and $A \subset X$ is also a Δ-complex).*

Proof. See Theorem 2.27 in [10]. □

Example 3.3.3.

$$H_n(S^1) \cong \begin{cases} \mathbb{Z}, & \text{for } n = 0,\ 1 \\ 0, & \text{for } n \geq 2 \end{cases}$$

We equip S^1 with a Δ-complex and see it as a space with vertex s_0 and an edge e^1. $\Delta_0(S^1) \cong \mathbb{Z}$, $\Delta_1(S^1) \cong \mathbb{Z}$ and $\Delta_k(S^1) \cong 0$ for all $k > 1$. Now for the boundary map $\partial_1 : \Delta_1(S^1) \longrightarrow \Delta_0(S^1)$ we have $\partial_1(e) = s_0 - s_0 = 0$, thus $\partial_1 = 0$. These observations lead us to $ker\partial_1 \cong \mathbb{Z}$, $ker\partial_0 \cong \mathbb{Z}$, $Im\partial_2 \cong 0$, $Im\partial_1 \cong 0$, which eventually produce the groups $H_n^{\Delta}(S^1) \cong H_n(S^1)$.

3.3.2 Cellular homology

Proposition 3.3.4. *Let X be a CW-complex. The sequence*

$$\ldots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \ldots \tag{3.5}$$

is a chain complex called the ***cellular chain complex*** *of X. The homology groups of this chain map are called the* ***cellular homology groups*** *of X and we denote the n-th cellular homology group with $H_n^{CW}(X)$.*

Proof. We recall Proposition 3.2.12 and intersperse the sequence 3.5 with portions of the long exact sequences of the form 3.3 to formulate the following diagram. d_{n+1}

$$\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) \cong H_n(X) \arrow[ur] & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\ldots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \ldots \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & 0 \arrow[ur] & & &
\end{tikzcd}$$

Figure 3.5

and d_n are defined to be $j_n \circ \partial_{n+1}$ and $j_{n-1} \circ \partial_n$, respectively. The composition $d_n \circ d_{n+1} = j_n \circ \partial_{n+1} \circ j_{n-1} \circ \partial_n = 0$, because $\partial_{n+1} \circ j_{n-1} = 0$ being the composition of successive maps in the chain complex

$$0 \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^n) \cong H_{n-1}(X) \to 0$$

$\square$

Theorem 3.3.5. $H_*^{CW}(X) \cong H_*(X)$.

Proof. $H_n(X) \cong H_n(X^n)/Im\partial_{n+1}$ as can be extracted from Fig.3.5. Since j_n is injective $Im\partial_{n+1}$ is mapped isomorphically onto $Im(j_n\partial_{n+1}) = Imd_{n+1}$ and $H_n(X^n)$ isomorphically onto $Imj_n = ker\partial_n$. The injectivity of j_{n-1} gives $ker\partial_n = kerd_n$. Thus j_n induces the isomorphism $H_n(X^n)/Im\partial_{n+1} \cong kerd_n/Imd_{n+1}$. □

If we want to calculate some homology groups, we need a more concrete way to estimate maps $d_n = j_n\partial_{n+1}$. The appropriate formula of d_n uses the notion of the degree of a map $f : S^n \longrightarrow S^n$, $n > 0$, which will be introduced in a bit, and is built via the commutative diagram in Fig. 3.6. Neither the details of the formula's construction nor the properties and the theory relevant to the degree map are presented here. The reader can find them in [10].

Definition 3.3.6. Let $f : S^n \longrightarrow S^n$, $n > 0$. The induced map $f_* : H_n(S^n) \longrightarrow H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself. Therefore it can be expresses as $f_*(\alpha) = d\alpha$, where α is the generator of $H_n(S^n)$ and $d \in \mathbb{Z}$. d depends only on f, it is called the **degree of** f and has the notation $d = deg\ f$.

Remark. Although $deg\ f$ has several properties, we are going to need only two of them, namely:

1. $deg\ id = 1$ and $deg\ f = \pm 1$, if f is a homotopy equivalence.
2. If f is the antipodal map, i.e. the map that takes each point of S^n to its antipodal point, $deg\ f = (-1)^{n+1}$.

Definition 3.3.7. Let $f : S^n \longrightarrow S^n$, $n > 0$, be a map such that there exists a point $y \in S^n$ whose preimage $f^{-1}(y) = \{x_1, ..., x_m\}$ consists of finitely many points. If we take $U_{x_1}, ..., U_{x_m}$ to be disjoint neighbourhoods of the points x_i, with $f(U_{x_i}) \subset V_y$, V_y neighbourhood of y, then the map $f_* : H_n(U_{x_i}, U_{x_i} - \{x_i\}) \longrightarrow H_n(V_y, V_y - \{y\})$ can be expressed as $f_*(\alpha) = d_i\alpha$, $d_i \in \mathbb{Z}$. d_i is called the **local degree of** f **at** x_i and is written as $deg\ f|_{x_i}$.

Remark. The local degree $deg\ f|_{x_i}$ is well defined, because exactness and excision axioms provide the following isomorphisms for the contractible subspaces U_{x_i} and $S^n - \{x_i\}$:

$$\begin{aligned} H_n(U_{x_i}, U_{x_i} - \{x_i\}) &\cong H_{n-1}(U_{x_i} - \{x_i\}) \\ &\cong H_n(S^n - \{x_i\}, U_{x_i} - \{x_i\}) \\ &\cong H_n(S^n, U_{x_i}) \cong H_n(S^n) \end{aligned}$$

Similarly, $H_n(V_y, V_y - \{y\}) \cong H_n(S^n)$.

Proposition 3.3.8. *$deg\ f = \sum_i deg\ f|_{x_i}$, where the aforementioned notation is adopted.*

Proof. See Proposition 2.30 in [10]. □

The cellular boundary formula of $d_n : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$ is $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$, where $d_{\alpha\beta}$ is the degree of the map $f : S^{n-1}_\alpha \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{q} S^{n-1}_\beta$. ϕ_α is the attaching map of the cell e^n_α, while q is the quotient map that collapses $X^{n-1} - e^{n-1}_\beta$ to a point.

During the proof of Proposition 3.2.12 we saw that $H_n(X^n, X^{n-1}) \cong \oplus_\alpha \mathbb{Z}$, where the index α counts the n-cells of X. For this reason it is enough to define the formula of d_n on the basis elements e^n_α. Also, in spite of the fact that e^{n-1}_β might be infinite, the summation has a finite number of summands, because ϕ_α has a compact image in the CW-complex X^{n-1} (Proposition 1.1.18).

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Phi_{\alpha *}"] & \widetilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "\Delta_{\alpha\beta *} = q_{\beta *} q_* \phi_{\alpha *}"] \arrow[d, "\phi_{\alpha *}"] & \widetilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}$$

Figure 3.6

Example 3.3.9.

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0,\ k = n \text{ odd} \\ \mathbb{Z}_2, & \text{if } k \text{ odd},\ 0 < k < n \\ 0, & \text{otherwise} \end{cases}$$

In Example 1.3.16 we saw that $\mathbb{R}P^n$ has a CW structure with a single k-cell for each $k \leq n$. The attaching map of the k-cell e^k is the quotient projection $\phi : S^{k-1} \longrightarrow \mathbb{R}P^{k-1}$ that identifies antipodal points of S^{k-1} and 'glues' them to $\mathbb{R}P^{k-1}$. Consider the composition $f : S^{k-1} \xrightarrow{\phi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$ and a point $y \in S^{k-1}$. The preimage $f^{-1}(y)$ consists of two points x_1, x_2, one in each hemisphere of the domain S^{k-1}. The restrictions f_1 and f_2 of f on neighbourhoods of U_{x_1} and U_{x_2}, respectively, are homeomorphisms. To be more exact one is homotopic to the identity, while the other is homotopic to the antipodal map. Now, $d_k = deg\ f = deg\ f_1 + deg\ f_2 = 1 + (-1)^{k-1+1}$. Concequently, d_k will be 0 or a multiplication by 2, depending on whether k is odd or even. This result leads to the following cellular chain complexes for $\mathbb{R}P^n$:

$$H_{n+1}([\mathbb{R}P^n]^{n+1}, \mathbb{R}P^n) \longrightarrow H_n(\mathbb{R}P^n, [\mathbb{R}P^n]^{n-1}) \longrightarrow .. \longrightarrow H_1([\mathbb{R}P^n]^1, [\mathbb{R}P^n]^0) \rightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \ldots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0, \text{ if } n \text{ is even}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \ldots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0, \text{ if } n \text{ is odd}$$

which give the values of $H^{CW}_k(\mathbb{R}P^n) \cong H_k(\mathbb{R}P^n)$ which were stated at the beginning.

Remark. With the notation $[\mathbb{R}P^n]^k$ we mean the k-skeleton of $\mathbb{R}P^n$.

Chapter 4

Theorems preceding Hurewicz'

Theorems that are often employed to prove the relative Hurewicz theorem are presented in this Chapter. Although the way we chose to prove the Theorem does not use them, we seized the opportunity to include them due to their significance in algebraic topology in general.

4.1 Elements of simplicial approximation

Definition 4.1.1. Let K be a simplicial complex and $f : |K| \longrightarrow \mathbb{R}^n$. f is said to be **linear** if for each simplex $[v_0, ..., v_l]$ of K and for each $x \in [v_0, ..., v_l]$, we have $f(x) = f(\sum_{i=0}^{l} t_i v_i) = \sum_{i=0}^{l} t_i f(v_i)$. This means that a linear map is completely determined by its value on the vertices of the domain.

Definition 4.1.2. A set $X \subset \mathbb{R}^m$ is said to have linear dimension $\leq k$, $k \leq m$, if there exist affine k-planes A_1, ..., A_l with $X \subset \bigcup_{i=0}^{l} A_i$. For the empty set $\emptyset$ we define $lindim(\emptyset) = -1$.

Proposition 4.1.3. *Let X be a set in $\mathbb{R}^m$ with $lindim(X) < m$. Then X is nowhere dense.*

Proof. See Proposition 12.5 in [12]. □

Proposition 4.1.4. *Let K be a complex, $f : |K| \longrightarrow \mathbb{R}^m$ be a linear map and $X \subset K$. Then $lindim(f(X)) \leq lindim(X)$.*

Proof. See Proposition 12.6 in [12]. □

Lemma 4.1.5. *Let X be a CW-complex, $f : I^n \longrightarrow X \cup e^k$, $n < k$. Then there is an open set $U \subset I^n$ and a homotopy $h_t : \overline{U} \longrightarrow e^k$ $rel \partial U$ such that:*

1. *$h_0 = f|_{\overline{U}}$;*
2. *there is a complex $N \subset U$ such that $h_1|_N$ is linear;*
3. *$h_1^{-1}(e^k(1/2)) \subset N^o$, where $e^k(1/2) = \Phi(\bar{B}(k, 1/2))$, Φ is the characteristic map and $D^k = \bar{B}(k, 1)$.*

Proof. See Lemma 13.4 in [12]. □

Corollary (4.1.5)**.** *$\pi_n(X \cup e^k, X, *) = 0$ for $n < k$.*

Proof. Let $f : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X \cup e^k, X, *)$ represent a homotopy element. We apply Lemma 4.1.5 for f. For the open set U given by the Lemma, $f(\overline{U}) \subset e^k$ and $f(\partial I^n) \subset X$. Thus $\overline{U} \cap \partial I^n = \emptyset$ and h_t can be extended to a homotopy $H_t : I^n \longrightarrow X \cup e^k$ with $H_t|_{\partial I^n} = f|_{\partial I^n}$ for all t. More specifically, we have

$$H_t(u) = \begin{cases} h_t(u), & u \in \overline{U} \\ f(u), & u \in I^n - U \end{cases}$$

$H_0 = f$. Let us choose a point $p \in e^k(1/2)$ such that $p \notin H_1(N)$. This we can do, because $H_1(N)$ has linear dimension $\leq n < k = lindim(e^k)$ (Proposition 4.1.4), hence it is nowhere dense (Proposition 4.1.3). $H_1^{-1}(p) \subset H_1^{-1}(e^n(1/2)) \subset N$, so $H_1^{-1}(p) = \emptyset$. This leads us to the conclusion that H_1 is in the image of $i_* : \pi_n(X \cup e^k - \{p\}, X, *) \longrightarrow \pi_n(X \cup e^k - \{p\}, X, *)$ induced by inclusion. But $X \cup (e^k - \{p\})$ deformation retracts to X. Thus $\pi_n(X \cup e^k - \{p\}, X, *) \cong \pi_n(X, X, *) = 0$. □

4.2 Cellular approximation

The cellular approximation theorem states that any map between CW-complexes is homotopic to one that sends cells to cells of the same or lower dimension. It ensures in a way that a homotopically equivalent version of maps between CW complexes can be found, which is more easily contained and manipulated.
[12], [2] and [19] prove the Cellular Approximation Theorem in the general context of relative CW-complexes and subspaces of CW-complexes. To this end, they use the notions of colimits or direct limits. Since we are only interested in CW-pairs and subcomplexes of CW-complexes in our study, we choose to present the less general version found in [10] and [20].

Definition 4.2.1. Let X and Y be CW-complexes. A map $f : X \longrightarrow Y$ is called **cellular** if $f(X^n) \subset Y^n$.

Remark. The exact same definition also holds for a map $f : (X, A) \longrightarrow (Y, B)$ between CW-pairs.

Theorem 4.2.2. *Let (X, A) be a CW pair, $(Y, B) \in \mathscr{T}^{2*}$ with $B \neq \emptyset$, $f : (X, A) \longrightarrow (Y, B)$ and S be a set of integers. Suppose that, if $e^k \subset X - A$, then $k \in S$ and $\pi_k(Y, B, *) = 0$ for any choice of the base point $*$. Then there is a map $g : X \longrightarrow B$ with $g \sim f$ rel$A$.*

Proof. See Lemma 4.6 in [10]. □

Theorem 4.2.3 (Cellular approximation theorem)**.** *Every map $f : X \longrightarrow Y$ between CW-complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on A.*

Proof. The proof will be by induction. Assume that f has already been made cellular on X^{n-1} and take e^n to be an n-cell of X (or $X - A$). The closure $\overline{e^n}$ is compact in X, so its image under f is also compact in Y. Proposition 1.3.13 ensures that $S = \{j \in I | \ f(\overline{e^n}) \cap e_j \neq \emptyset\}$ is a finite set, where I is a set of indices. Let $e_j = e^k \in S$ be the cell with the maximal dimension k. If $n \leq k$, then f is already cellular on e^k and we need to do nothing. If, however, $k > n$, we take the composition $g = f \circ \Phi$ of the characteristic map $\Phi : I^n \equiv D^n \longrightarrow X^{n-1} \cup e^n$ with the given map $f : X^{n-1} \cup e^n \longrightarrow Y^k$ and apply Lemma 4.1.5. From now on arguments are very similar to the arguments in the proof of Corollary 4.1.5. Using the Lemma we obtain an open subset $U \subset I^n$ and a homotopy h_t with the properties described there. Since $g(\overline{U}) \subset e^k$ and $g(\partial I^n) \subset (Y^k - e^k)$, we take $\overline{U} \cap \partial I^n = \emptyset$. Hence h_t can be extended to a homotopy $H_t : I^n \longrightarrow (Y^k - e^k) \cup e^k$ with $H_t|_{\partial I^n} = g|_{\partial I^n}$ for all t. More specifically, we have

$$H_t(u) = \begin{cases} h_t(u), & u \in \overline{U} \\ g(u) = f \circ \Phi(u), & u \in I^n - U \end{cases}$$

Obviously, $H_0 = g$, while H_1 is linear on $N \subset U$. Let us choose a point $p \in e^k(1/2)$ such that $p \notin H_1(N)$, just like we did in the corollary. Using now what was found for H_t, we can define the homotopy

$$G_t(x) = \begin{cases} h_t \circ \phi^{-1}(x), & x \in \phi(\overline{U}) \\ f(x), & x \in X^{n-1} - \phi(U) \end{cases}$$

where $G_t|_{X^{n-1}} = f|_{X^{n-1}}$ for all t and there exists $p \notin G_1(\phi(N))$. In other words, our initial $f|_{X^{n-1} \cup e^n}$ is homotopic $relX^{n-1}$ to a map whose image misses one point on $e^k \subset Y^k$. If we compose the homotopy G_t with a deformation retraction of $Y^k - \{p\}$ to $Y^k - e^k$ we can deform the map $f|_{X^{n-1} \cup e^n}$ $relX^{n-1}$ to a map whose image misses the whole cell e^k.

Using finitely many repetitions of this procedure we find a homotopic map whose image $f(e^n)$ misses all cells e_j, $j \in S$, with dimension greater than n. If we do this for all n-cells in X (or $X - A$), we obtain a homotopy of $f|_{X^n}$ $relX^{n-1}$ (or $f|_{X^n}$ $relX^{n-1} \cup A^n$) to a cellular map. The induction step is completed with the application of the homotopy extension property, which extends this homotopy to one defined on all X.

If X has infinite dimension, we let n go to ∞ and the resulting infinite string of homotopies becomes a single, coherent one with the nth homotopy being performed during the t-interval $[1-1/2^n, 1-1/2^{n+1}]$. The continuity of this homotopy is ensured by the weak homotopy axiom of CW complexes. □

Example 4.2.4. $\pi_n(S^k) = 0$ for $n < k$.

S^n is a CW-complex with a single 0-cell and a single n-cell (1.3.15). If we take $f : (S^n, *) \longrightarrow (S^k, *)$ to represent an element in the homotopy group, we get that $[f] = [g]$ where $g(S^n) \subset [S^k]^n$. However, since $n < k$ and with given the CW-decomposition of S^k, $[S^k]^n = \{*\}$ and $g = 0$, which means that f is homotopic to the constant map.

4.3 CW approximation

Many general statements in algebraic topology can be proved using CW approximations of spaces. Their study is easier since, in their case, any problem can be tackled using a cell-by-cell approach. Conclusions are more readily reached for individual cells, so the process of decomposing a CW complex to its building blocks and reconstructing it from its elements can efficiently lead to the desired results. Of course, these CW complexes must be equivalent to the studied spaces in a way that will be defined shortly.

Definition 4.3.1. A map $f : X \longrightarrow Y$ is called a **weak homotopy equivalence** if $f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0))$ is an isomorphism for all $n \geq 0$ and all $x_0 \in X$. Generalising the definition to pairs, a map $f : (X, X_0) \longrightarrow (Y, Y_0)$ is called a weak homotopy equivalence if the associated maps $f : X \longrightarrow Y$ and $f|_{X_0} : X_0 \longrightarrow Y_0$ are weak homotopy equivalences.

Definition 4.3.2. Let $X \in \mathscr{T}^*$. A **CW approximation** or resolution of X is a pair (K, f) where K is a CW-complex and $f : K \longrightarrow X$ is a weak homotopy equivalence. A CW approximation of a pair (X, X_0), now, is a CW pair (K, K_0) and a weak homotopy equivalence $f : (K, K_0) \longrightarrow (X, X_0)$.

It will be shown that CW approximations of spaces and pairs always exist and any two of them are homotopically equivalent. However, before proving their existence, let us prove that CW approximations behave well with respect to homology. We follow here the proof presented in Section 9.5 of [2], altering slightly the notation.

Let $(X, A, *) \in \mathscr{T}^*$ and $[\Delta^k]^n$ the n-skeleton of the standard simplicial complex Δ^k. Let also $C_k^{(n,A)}(X) \subset C_k(X)$ for $n \geq 0$ be the subgroup generated by singular simplices $\sigma : \Delta^k \longrightarrow X$ with the property $\sigma([\Delta^k]^n) \subset A$. The groups $\{C_k^{(n,A)}(X) | \, k \geq 0\}$ form the Eilenberg subcomplex $C_*^{(n,A)}(X)$ of $C_*(X)$.

Proposition 4.3.3. *Let (X, A) be n-connected. Then the inclusion of the Eilenberg subcomplex $i : C_*^{(n,A)}(X) \longrightarrow C_*(X)$ is a chain equivalence.*

Proof. We refer the reader to [2].
First of all, it is pretty straigtforward to check that the Eilenberg subcomplex does produce a chain complex ($\partial^2 = 0$) and thus homology groups. In the proof, that was referred to above, a map $P : C_k(X) \longrightarrow X^{\Delta^k \times I}$ is constructed using induction on k, the n-connectedness of (X, A) and the homotopy extension property (2.1.14). For $\sigma \in C_k(X)$, $P(\sigma) : \Delta^k \times I \longrightarrow X$ is actually a homotopy between $P(\sigma)_0 = \sigma$ and $P(\sigma)_1$ which satisfies $P(\sigma)_1([\Delta^k]^n) \subset A$ and $P(\sigma) \circ (d_i^k \times id) = P(\sigma \circ d_i^k)$. This P allows us to define the chain map $\rho : C_k(X) \longrightarrow C_k^{(n,A)}(X)$ with $\rho(\sigma) = P(\sigma)_1$, for which $\rho \circ i = id$ holds by construction. The operator $s : C_k(X) \longrightarrow C_{k+1}(X)$ with $s(\sigma) =$

$P(\sigma)_{\#}(h(i_k))$ is also formed, where $i_k \in C_k(\Delta^k)$ and $h : C_k(\Delta^k) \longrightarrow C_{k+1}(\Delta^k \times I)$ is given in 9.3.3 in [2]. Computations result to the relation $\partial s + s\partial = i \circ \rho - id$, which, if combined with the previous equality $\rho \circ i = id$, shows that i is a chain equivalence. □

Now, for $k \leq n$ $[\Delta^k]^n = \Delta^k$. Consequently, $C_k^{(n,A)}(X) = C_k(A)$ and the exact homology sequence of (X, A) gives $H_k(A) \cong H_k(X)$ for (X, A) n-connected.

Theorem 4.3.4. *A weak homotopy equivalence $f : X \longrightarrow Y$ induces isomorphisms $f_* : H_n(X) \longrightarrow H_n(Y)$ for all n.*

Proof. The mapping cylinder Mf deformation retracts to Y (see Example 2.1.11), thus Mf is homotopy equivalent to Y and homotopy equivalent spaces have isomorphic homotopy and homology groups. Let us denote with r the retraction from Mf to Y and point out here that $r \circ i_X = r|_X = f$.
Taking the long exact sequence of homotopy of (Mf, X)

$$\ldots \longrightarrow \pi_i(X) \longrightarrow \pi_i(Mf) \longrightarrow \pi_i(Mf, X) \longrightarrow \pi_{i-1}(X) \longrightarrow \ldots$$

the weak homotopy equivalence guarantees that $H_n(Mf, X) = 0$ for all n or equivalently (Mf, X) is n-connected for all n. From Proposition 4.3.3 we conclude that $f_* : H_n(X) \xrightarrow{i_{X*}} H_n(Mf) \xrightarrow{r_*} H_n(Y)$ is an isomorphism for all n. □

CW approximations can simplify proofs regarding homotopy and homology groups of spaces by reducing them to 'equivalent' CW-complexes. However, both their existence and their uniqueness up to homotopy equivalence are yet to be proved. Existence will be proven by construction, while a theorem known as Whitehead's theorem will be employed for the uniqueness part.

Theorem 4.3.5. *Every space X has a CW approximation (K, f). If X is path connected, K can be chosen to have a single 0-cell with all other cells attached by basepoint preserving maps.*

Proof. Let X be path connected. If not, the described construction still works, but we need to perform small alterations, which are pointed out below.
The construction of a CW approximation $f : K \longrightarrow X$ of X is inductive on $n \geq 1$. Generally, the induction step for $n > 1$ begins with a given CW-complex L, a map $f : L \longrightarrow X$ and the fixed basepoints $p \in L$, $x_0 \in X$ with $f(p) = x_0$. During the n-th step, $n \in \mathbb{Z}$, we attach n-cells to L to form a new CW-complex $M = L \cup_j e_j^n$ and a map $\bar{f} : M \longrightarrow X$ extending the previous f, in a way that ensures that the induced map $\pi_i(f) : \pi_i(M, p) \longrightarrow \pi_i(X, f(p))$ is injective for $i = n-1$ and surjective for $i = n$. This way, we eventually build isomorphisms $\pi_i(f)$ for all i.
Let $n = 1$. We choose a basepoint $x_0 \in X$, form $L^0 = \{p\}$ and create $f : L^0 \longrightarrow X$ with $f(p) = x_0$. Obviously, $\pi_0(f)$ is an isomorphism. If X was not path connected, then L^0 would include a separate point p_i for each path component of X and everything would be repeated for each basepoint/path component. Now, we need to

make $\pi_1(f)$ surjective. We choose maps $f_\beta : (S^1, s_0) \longrightarrow (X, f(p))$, which are representatives of generators of $\pi_1(X, f(p))$. For each f_β we attach a 1-cell e^1_β to L^0 via $\phi(\partial S^1) = c_p$ the constant map at p. This results to $M = L^0 \cup_\beta e^1_\beta$. f is extended over M using the f_βs on each S^1_β. Surjectivity is dictated by construction.
Let the inductive step be true for $n-1$.
Entering the n-th step, we aim at making $\pi_{n-1}(\bar{f})$ an isomorphism and $\pi_n(\bar{f})$ surjective, where $\bar{f}$ will be an extension of $f : L^{n-1} \longrightarrow X$ on a new CW complex M. Since we already have the surjectivity of $\pi_{n-1}(f)$ from induction and any new n-cell will not influence that, $\pi_{n-1}(\bar{f})$ will be surjective whatever the extension and we can focus on injectivity. We choose representatives of generators of $\ker\pi_{n-1}(f)$ $\phi_\alpha : (S^{n-1}, s_0) \longrightarrow (L, p)$. From Theorem 4.2.3, each representative ϕ_α can be chosen to be cellular. Viewing S^{n-1} as a CW-complex with its usual structure of a single 0-cell and a single $(n-1)$-cell presented in Example 1.3.15, we use ϕ_αs as characteristic maps and glue n-cells e^n_α to L with them. Let $L^n = L \cup_\alpha e^n_\alpha$. The map $f : L \longrightarrow X$ extends to a map $\bar{f} : L^n \longrightarrow X$, because $f\phi_\alpha$ is nullhomotopic in L^n (see Lemma 4.3.6 below). In order to make $\pi_n(\bar{f})$ surjective, we do what we did for $\pi_1(f)$, using n-cells instead of 1-cells. As a result, $M = L \cup_\alpha e^n_\alpha \cup_\beta e^n_\beta$ is formed, along with the extension $\bar{f} : M \longrightarrow X$.
Again, $\pi_n(\bar{f})$ is surjective thanks to its construction, while the injectivity of $\pi_{n-1}(\bar{f})$ stems from our freedom to choose cellular representatives $h \in ker\pi_{n-1}(\bar{f})$. Since $h(S^{n-1}) \subset M^{n-1} = L$, we get $h \in ker\pi_{n-1}(f) \Rightarrow h = \sum_i n_i(\phi_\alpha)_i \sim 0$ in M. □

Lemma 4.3.6. *Suppose given maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathscr{T}$. Then, if $f \circ g$ is nullhomotopic, there exists $h : Y \cup_f CX \longrightarrow Z$ with $h|_Y = g$.*

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[d, "g"] & Y \cup_f CX \arrow[dl, "h"] \\
& Z &
\end{tikzcd}$$

Proof. See Proposition 14.15 in [12]. □

Theorem 4.3.7. *Let $(X, X_0) \in \mathscr{T}^2$. There exists a CW pair (K, K_0) and a weak equivalence $f : (X, X_0) \longrightarrow (K, K_0)$ or in other words (X, X_0) has a CW approximation (K, K_0).*

Proof. The technique used to prove Theorem 4.3.5 can be applied to the pair of spaces (X, X_0) as well. We first construct a CW approximation (K_0, f_0) of X_0. Then, we start the inductive step for the construction of the desired CW approximation considering given the CW-complex K_0 and the map $i_{X_0} \circ f_0 : K_0 \longrightarrow X$ and attaching cells to K_0 to create a weak homotopy equivalence (K, f) extending f_0. Finally, using the Five-Lemma (see A.2.9), we conclude that the map $f : (K, K_0) \longrightarrow (X, X_0)$ induces isomorphisms on relative as well as absolute homotopy groups. □

Remark. If (X, X_0) is n-connected for some $n \geq 0$, then (K, K_0) is n-connected too, since the weak equivalence f induces isomorphisms on all homotopy groups.

Proposition 4.3.8. *Let $f : (K, K_0) \longrightarrow (X, X_0)$ and $g : (L, L_0) \longrightarrow (Y, Y_0)$ be CW approximations and suppose given a map $h : (X, X_0) \longrightarrow (Y, Y_0)$. Then there is a map $\psi : (K, K_0) \longrightarrow (L, L_0)$ that is unique in $h\mathscr{T}^{2*}$ and makes the diagram in Fig.*

$$\begin{tikzcd}
(L, L_0) \arrow[r, "g"] & (Y, Y_0) \\
(K, K_0) \arrow[u, "\psi"] \arrow[r, "f"] & (X, X_0) \arrow[u, "h"']
\end{tikzcd}$$

Figure 4.1

4.1 commute in the same category.

Proof. Analytically, the proof can be found in Chapter 16 of [12]. Here we give a general idea of its structure.
Lemma 4.3.9 and the homotopy extension property in 2.1.14 are consecutively used. First one finds a map $\psi_0 : K_0 \longrightarrow L_0$ with $g \circ \psi_0 \sim h \circ f$. This homotopy is extended to create $H_1 : K \times I \longrightarrow Y$ and through this and Lemma 4.3.9 the desired map $\psi : (K, K_0) \longrightarrow (L, L_0)$ is found. For the uniqueness part two such maps are assumed given, ψ_1, ψ_2, which satisfy $g \circ \psi_1 \sim h \circ f \sim g \circ \psi_2$. We start with a homotopy between $\psi_1|_{L_0} \sim \psi_2|_{L_0}$ that arises from applying Lemma 4.3.9 on the right diagram and conclude to a homotopy $\psi_1 \sim \psi_2$ after using repeatedly the lemma and the homotopy extension property. $\square$

Lemma 4.3.9. *Let X, $Y \in \mathscr{T}^*$ and $f : (X, *) \longrightarrow (Y, *)$. f is a weak homotopy equivalence if and only if given any CW pair (K, K_0) and maps $k_0 : K_0 \longrightarrow X$, $k : K \longrightarrow Y$ with $f \circ k_0 = k|_{K_0}$ there is a map $g : K \longrightarrow X$ with $g|_{K_0} = k_0$ and $f \circ g \sim k\ relK_0$, as shown in Fig. 4.2.*

Proof. For the if part, let the property be satisfied and $[k] \in \pi_n(Y, *)$. From the diagram in Fig. 4.3 we get that there exists a map $[g_1] \in \pi_n(X, *)$ such that $[fg_1] = [k]$, which means that f_* is onto. Now, let $[\alpha] \in \pi_n(X, *)$ such that $[f\alpha] = 0$ and β

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
K_0 \arrow[u, "k_0"] \arrow[r] & K \arrow[u, "k"'] \arrow[ul, "g"]
\end{tikzcd}$$

Figure 4.2

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
* \arrow[u, "x"] \arrow[r] & K \arrow[u, "k"'] \arrow[ul, "g_1"]
\end{tikzcd}$$

Figure 4.3

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
S^n \arrow[u, "\alpha"] \arrow[r] & D^{n+1} \arrow[u, "\overline{\beta}"'] \arrow[ul, "g_2"]
\end{tikzcd}$$

Figure 4.4

be a homotopy from $f\alpha$ to the constant map. For the induced map $\overline{\beta} : D^{n+1} \longrightarrow Y$ and the diagram in Fig.4.4 there exists a map g_2 with $g_2|_{S^n} = \alpha$. From g_2 we get a homotopy from α to the constant map, so f_* is also injective. For the only if part, we form the diagram in Fig.4.5 and define $\overline{F} : K \times 0 \cup K_0 \times I \longrightarrow Mf$ by $\overline{F}(l, 0) = j(k(l))$ and $\overline{F}(l, t) = (k_0(l), t)$ for $l \in K_0$. We may extend $\overline{F}$ to the $F : K \times I \longrightarrow Mf$ using the homotopy extension property. Let now $\gamma : K \longrightarrow Mf$ be given by $\gamma(l) = F(l, 1)$. We have $\gamma(K_0) \subset X \times \{1\}$. If we use Theorem 4.2.2, we

$$\begin{tikzcd}
& Mf \arrow[d, shift left, "\pi"] \\
X \arrow[ur, "i"] \arrow[r, "f"] & Y \arrow[u, shift left, "j"] \\
K_0 \arrow[u, "k_0"] \arrow[r] & K \arrow[u, "k"'] \arrow[ul, "g_3"']
\end{tikzcd}$$

Figure 4.5

can construct a map $g_3 : K \longrightarrow X \times \{1\}$ with $g_3 \sim \gamma \ rel K_0$. Now $g_3|_{K_0} = k_0$ and $fg_3 = \pi g_3 \sim \pi\gamma \sim k \ rel K_0$, where the last homotopy is $G = \pi F$. □

Theorem 4.3.10 (Whitehead's Theorem)**.** *Let X, Y be connected CW complexes. If a map $f : X \longrightarrow Y$ between them induces isomorphisms $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case X is a subcomplex of Y and f is the inclusion $X \hookrightarrow Y$, X is a deformation retract of Y.*

Proof. See [13]. □

Theorem 4.3.11. *Let $(X, X_0) \in \mathscr{T}^{2*}$. (X, X_0) has a CW approximation (K, K_0) that is unique up to homotopy equivalence.*

Proof. Let $((K, K_0), f)$ and $((L, L_0), g)$ be CW approximations of the pair (X, X_0). Proposition 4.3.8 provides a map $\psi : (K, K_0) \longrightarrow (L, L_0)$ with $g \circ \psi \sim f$. Thus ψ induces isomorphisms in all homotopy groups. Moreover, using Whitehead's theorem we get that ψ is a homotopy equivalence from (K, K_0) to (L, L_0). □

4.4 Excision in homotopy

One last theorem that we will mention here without its proof is the one referred to as excision in homotopy. Generally, the excision axiom which was presented in Chapter 3 fails in homotopy and this is rather important, since it essentially distinguishes homotopy from homology. If it didn't fail, homotopy theory would become one more example of homology theory. However, some sort of the excision property holds for a specific range of dimensions if we pose additional hypotheses on the homotopy groups in question.

Theorem 4.4.1 (Excision in homotopy). *Let $X \in \mathscr{T}$ which is decomposed as the union of subspaces A and B. Let also $C = A \cap B \neq \emptyset$, (A, C) be m-connected and (B, C) be n-connected, m, $n \geq 1$. Then the map $i_* : \pi_i(A, C) \longrightarrow \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Proof. For the proof see Sections 6.4 to 6.9 in [2]. □

Chapter 5

The Hurewicz theorem

5.1 Absolute and relative Hurewicz theorems

The Hurewicz theorem in its elementary, absolute form states that the first nonzero homotopy and homology groups of a simply connected space happen simultaneously and are isomorphic. For its general and relative form, which refers to all spaces, not just simply connected, quotients of homotopy groups are taken.
The central idea behind the general proof is that there exists a homomorphism h between the long exact sequences of homology and quotients of homotopy groups that is proved to be an isomorphism. In the relative case, elements in both homotopy and homology groups of a space X can be represented by maps from Δ^n to X, since Δ^n is homeomorphic to D^n. However, in homotopy groups constant attention must be paid to the chosen basepoint x_0 of X, keeping maps and homotopies pointed, while in homology groups a certain freedom of movement is allowed, which leads to homologous maps that can differ by a boundary. In order to amend this discrepancy and attain the same level of 'freedom' such boundaries must become trivial. Also n-spheroids that differ by a loop must be considered equivalent.

The proof of the general Hurewicz theorem that will be presented here is inductive, starting with the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ and increasing n's value from there. A specific map h is constructed and then it is checked and proved to be an isomorphism.

Presentations of the theorem can be found in a plethora of textbooks. To name a few, we mention [10], [2], [15], [3], [19]. The proof we present here for the case $n = 1$ has been influenced by [10] and [15], while we follow the more homological approach of [2] and [3] for $n > 1$, which does not use CW approximation and the excision of homotopy, like others do.

Theorem 5.1.1 (Hurewicz - $n = 1$)**.** *Let X be path connected. Then $\pi_1(X, x_0)^{ab} \cong H_1(X)$, where $G^{ab} = G/[G, G]$.*

Proof. Let $h : \pi_1(X, x_0) \longrightarrow H_1(X)$ which takes the homotopy class of a loop $[f]^\#$ to the homology class $[fa]$ of the same loop seen as an 1-cycle. $a : \Delta^1 \longrightarrow I$ is the homeomorphism that takes $(1-t)d_0^1 + td_1^1$ to t, while the superscript $^\#$ is being employed to distinguish homotopy elements from homology elements. In what follows we will show that h is a well defined isomorphism.

h is well-defined: Let f_1, f_2 be representatives of a class $[f]^\#$, such that $f_1 \neq f_2$. If we consider the homotopy $F : I \times I \longrightarrow X$ from f_1 to f_2 and draw one of $I \times I$'s diagonals, we can view F as a pair of singular 2-simplices, namely $\sigma_1 = F|_{t_1}$ and $\sigma_2 = F|_{t_2}$, as can be seen in the Figure 5.1. Now, $\partial(\sigma_1 - \sigma_2) = \sum_i (-1)^i \sigma_1 d_i^2 - \sum_i (-1)^i \sigma_2 d_i^2 = f_1 a + c_1 a - f_2 a - c_2 a$, since the two restrictions of F cancel on the

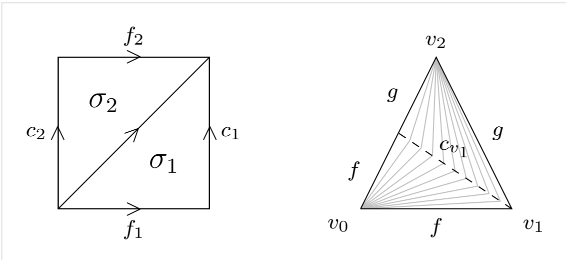


Figure 5.1

diagonal. The constant maps $c_1 a$, $c_2 a \in C_0(X)$ can be written as the boundary of the constant map $c : \Delta^2 \longrightarrow X$, therefore they are homologous to zero. This leads us to $f_1 a - f_2 a = \partial(\sigma_1 - \sigma_2) - c_1 a + c_2 a \in Im\partial^2$ and $[f_1 a] = [f_2 a]$.

h is a homomorphism: For $[f]^\#$, $[g]^\# \in \pi_1(X, x_0)$, we define a singular map $\sigma : \Delta^2 \longrightarrow X$, as shown in Figure 5.1, whose restriction on each ray under c_{v_1} identifies with fa, while over c_{v_1} the restriction identifies with ga. For this σ, we have $\partial\sigma = \sum_{i=0}^{3} (-1)^i \sigma d_i^2 = ga - (f \cdot g)a + fa \Rightarrow fa + ga - fa \cdot ga \in Im\partial^2$, thus $h([f]^\#[g]^\#) = [fa] + [ga] = h([f]^\#) + h([g]^\#)$.

h is a surjection: Take the chain $c_1 = \sum_i n_i \sigma_i$ to be a representative of a class $[c]$ in $H_1(X)$. We will transform c_1 until we reach such a homologous 1-cycle c' that $c'a^{-1}$ is a loop based on x_0. First, we 'explode' the sum of c_1, relabeling σ_is, so all its coefficients become equal to ± 1. Then we substitute each σ_j with coefficient -1 in the new sum with its inverse path $\overline{\sigma}_j$. This way we get a homologous c_2 with $n_i = 1$ for all i. Since $\partial c_2 = 0 \Rightarrow \sum_j \sigma_j(d_0^1 - d_1^1) = 0$ and every $\sigma_j d_k^1$ is a basis element in the free abelian group $C_0(X)$, we conclude that they must be mutually eliminated two by two. This means that we can combine the relevant paths $\sigma_j d_k^1 a^{-1}$ till we are left with loops $\tau_j a^{-1}$ each one based on a point x_j in X, whose summation now forms $[c_3] = [c]$. We choose a path γ_j from x_0 to x_j for each j in the path-connected space

X and form the composite paths $\gamma_j \cdot \tau_j a^{-1} \cdot \overline{\gamma}_j$ with $h[\tau_j a^{-1}]^\# = h[\gamma_j \cdot \tau_j a^{-1} \cdot \overline{\gamma}_j]^\#$. Therefore, we have managed to transform c_1 to its homologous $c_4 = \sum_j \gamma_j a \cdot \tau_j \cdot \overline{\gamma}_j a$, for which the loop $c_4 a^{-1}$ based on x_0 gives $h([c_4 a^{-1}]^\#) = [c]$.

kerh $= [\pi_1, \pi_1]$: $[\pi_1, \pi_1] \subset kerh$, since h is a homomorphism and $H_1(X)$ abelian. For the inverse inclusion, take an element $[f]^\# \in kerh$. We want to show that f is homotopic to a map in $[\pi_1, \pi_1]$. From the hypothesis $\exists\ \sigma_i \in C_2(X)$ such that $\partial(\sum_i n_i \sigma_i) = fa$. We 'explode' this sum, relabeling σ_js, in order to get the chain $\sum_j m_j \sigma_j$ with $m_j = \pm 1$. This gives the equation $fa = \partial(\sum_j m_j \sigma_j) = \sum_j m_j \partial(\sigma_j) = \sum_j m_j \sum_k (-1)^k \sigma_j d_k^2 = \sum_j m_j(\sigma_j d_0^2 - \sigma_j d_1^2 + \sigma_j d_2^2)$ in the free abelian group $C_1(X)$. But fa and $\sigma_j d_k^2$ are basis elements of $C_1(X)$. This means that all but one of the $\sigma_j d_k^2$s must cancel two by two. The only one eventually left will be the one equal to f, namely $f = m_p \sigma_p d_k^2$ for some p and $k \in \{0, 1, 2\}$.
In the path connected space X we choose paths γ_{j0}, γ_{j1} and γ_{j2} from x_0 to $\sigma_j d_0^2 d_0^1$, $\sigma_j d_1^2 d_1^1$ and $\sigma_j d_2^2 d_0^1$, respectively. If any of the ends happens to be x_0, we choose the constant path on x_0 as γ, and the same path is chosen for all coinciding ends. Applying composition of paths, we get for each path $\sigma_j d_k^2 a^{-1}$, $k \in \{0, 1, 2\}$, a class $L_{j0} = [\gamma_{j0} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j1}]^\#$, $L_{j1} = [\gamma_{j2} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j1}]^\#$ or $L_{j2} = [\gamma_{j2} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j0}]^\#$ in π_1. According to what we have defined so far $[f]^\# = L_{pk} = [\gamma_{j\cdot} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j\cdot}]^\# = [\sigma_j d_k^2 a^{-1}]^\#$, since γ paths are constant in this case. Lemma 5.1.2 below allows the substitution $\langle L_{pk} \rangle = \langle f \rangle = \prod_j (\langle L_{j0} \rangle \cdot \langle L_{j1} \rangle^{-1} \cdot \langle L_{j2} \rangle)^{m_j}$, where the brackets $\langle\ \rangle$ are now used to denote elements in the quotient multiplicative abelian group $\pi_1/[\pi_1, \pi_1]$. If we examine representatives of the cosets in $\pi_1/[\pi_1, \pi_1]$, we get

$$\begin{aligned} L_{j0} \cdot L_{j1}^{-1} \cdot L_{j2} &= [\gamma_{j0} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j1} \cdot (\gamma_{j2} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j1})^{-1} \cdot \gamma_{j2} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j0}]^\# \\ &= [\gamma_{j0} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j1} \cdot \gamma_{j1} \cdot (\sigma_j d_k^2 a^{-1})^{-1} \cdot \overline{\gamma}_{j2})^{-1} \cdot \gamma_{j2} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j0}]^\# \\ &= [\gamma_{j0} \cdot \sigma_j d_k^2 a^{-1} \cdot (\sigma_j d_k^2 a^{-1})^{-1} \cdot \sigma_j d_k^2 a^{-1} \cdot \overline{\gamma}_{j0}]^\# = [c_{x_0}]^\#, \end{aligned}$$

because $\sigma_j d_k^2 a^{-1} \cdot (\sigma_j d_k^2 a^{-1})^{-1} \cdot \sigma_j d_k^2 a^{-1} \sim (\sigma|_{\partial\Delta^2}) a^{-1}$ is nullhomotopic. Since $\langle c_{x_0} \rangle = \langle 1 \rangle$, we conclude that $\langle f \rangle = \langle 1 \rangle$, hence $[f]^\# \in [\pi_1, \pi_1]$. □

Lemma 5.1.2. *Let F be a free abelian group with basis B, $\{x_0, \ldots, x_k\}$ be a subset of B (repetitions of elements are allowed) and assume that*

$$m_0 x_0 = m_1 x_1 + \cdots + m_k x_k,$$

where $m_i \in \mathbb{Z}$. Now, take an abelian group $(G, +')$ and a set $\{y_0, \ldots, y_k\}$ of elements of G for which $y_i = y_j$ whenever $x_i = x_j$. Then we can substitute x_i with y_i in the previous equation, i.e. $m_0 y_0 = m_1 y_1 +' \cdots +' m_k y_k$ holds.

Proof. The proof can be found in [15]. □

Remark. A different way to formulate the homomorphism h is by using the equivalent definition $[S^1, X] = \pi_1(X, *)$ for the fundamental group. If we do so and choose as z_1 the generator of $H_1(S^1)$ corresponding to the singular simplex $\Phi : \Delta^1 \longrightarrow I/\partial I = S^1$, we get $h([f]^\#) = f_*[z_1]$, which equates with the previous definition.

We move now to the case $n \geq 2$. For $(X, A, *) \in \mathscr{T}^{2*}$ we define maps h, called

Hurewicz homomorphisms,

$$\begin{aligned} h_{(X,A,*)} &= h : \pi_n(X, A, *) \longrightarrow H_n(X, A), & n \geq 2, \\ h_{(X,A)} &= h : \pi_n(X, *) \longrightarrow H_n(X) & n \geq 1. \end{aligned}$$

We are going to provide a specific formula for these maps and prove that they are natural homomorphisms that make the diagrams in Figure 5.2 commute.

$$\begin{tikzcd} \pi_n(X,*) \arrow[r] \arrow[d, "h"] & \pi_n(X,A,*) \arrow[r, "\partial"] \arrow[d, "h"] & \pi_{n-1}(A,*) \arrow[d, "h"] \\ H_n(X) \arrow[r] & H_n(X,A) \arrow[r, "\partial"] & H_{n-1}(A) \end{tikzcd}$$

Figure 5.2

Let us consider the definitions for homotopy groups $[(S^n, *), (X, *)] = \pi_n(X, *)$ and $[(D^n, S^{n-1}, *), (X, A, *)] = \pi_n(X, A, *)$. We also choose generators $z_n \in H_n(S^n)$ and $\overline{z}_n \in H_n(D^n, S^{n-1})$ such that $\partial \overline{z}_n = z_{n-1}$ and $q_*(\overline{z}_n) = z_n$, where $q : D^n \longrightarrow D^n/S^{n-1} = S^n$ is the quotient map. If we start our selection by fixing the $z_1 = \Phi$ that was picked in a remark earlier, then the other generators are uniquely determined through induction and the previous relations.

Using the notation that has just been introduced:

Definition 5.1.3. The **Hurewicz map** is defined via the formulas $h_{(X,A,*)}([f]^\#) = f_*[\overline{z}_n]$, $n \geq 2$, and $h_{(X,*)}([f]^\#) = f_*[z_n]$, $n \geq 1$.

The naturality of h and their compatibility with exact sequences (commutativity of the diagram in Figure 5.2) arise directly from the definitions. The fact that h is a homomorphism, however, requires a bit more work.

Proposition 5.1.4. *The Hurewicz map $h_{(X,A,*)}$, $n \geq 2$, is a homomorphism of groups.*

Proof. h is well defined due to the homotopy axiom (Section 3) of the homology theory h_*. Our target is to show that $(f + g)_* = f_* + g_*$ for every pair of maps $f,\ g : (D^n, \partial D^n, *) \longrightarrow (X, A, *)$, because then $h([f + g]^\#) = h([f]^\#) + h([g]^\#)$. Although we use the $+$ symbol between maps here, $f + g$ can be identified with the comultiplication $\cdot$ defined in Equation 1.2. Moreover, for $n = 2$ $+$ does not imply commutativity, but we prefer it for the sake of uniformity.
We employ the map $c : D^n \longrightarrow D^n \vee D^n$, which performs a 'pinch' in the middle of D^n by collapsing the equatorial D^{n-1} to a point, the quotient maps $q_1, q_2 : D^n \vee D^n \longrightarrow D^n$, where:

$$q_1(x_1, x_2) = \begin{cases} x_1, & \text{if } x_2 = * \\ *, & \text{if } x_1 = * \end{cases} \qquad q_2(x_1, x_2) = \begin{cases} *, & \text{if } x_2 = * \\ x_2, & \text{if } x_1 = * \end{cases} \tag{5.1}$$

and the map $f \vee g : D^n \vee D^n \longrightarrow X$, $f, g : D^n \longrightarrow X$, $f(\partial D^n)$ and $g(\partial D^n) \subset A$ with

$$(f \vee g)(x_1, x_2) = \begin{cases} f(x_1), & \text{if } x_1 \neq *, \ x_2 = * \\ g(x_2), & \text{if } x_1 = *, \ x_2 \neq * \ . \\ f(*) = g(*) = *, & \text{if } x_1 = x_2 = * \end{cases} \tag{5.2}$$

Their induced maps produce the diagram in Figure 5.3.
It has been shown, while proving Proposition 3.2.12, that, if X is the wedge sum of pointed spaces X_α, i.e. $X = \bigvee_\alpha X_\alpha$, then the inclusions $i_\alpha : X_\alpha \hookrightarrow X$ induce an

$$\begin{tikzcd} H_n(D^n, \partial D^n) \arrow[r, "c_*"] \arrow[dr, "{[a] \longmapsto ([a],[a])}"] & H_n(D^n \vee D^n, \partial D^n \vee \partial D^n) \arrow[r, "(f \vee g)_*"] \arrow[d, "q_{1*} \oplus q_{2*}"] & H_n(X, A) \\ & H_n(D^n, \partial D^n) \oplus H_n(D^n, \partial D^n) & \end{tikzcd}$$

Figure 5.3

isomorphism $i_* = \oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{H}_n(X_\alpha, x_\alpha) \longrightarrow \widetilde{H}_n(X)$. Applying this result on the space $S^n \vee S^n$, along with the isomorphism $H_n(D^n, \partial D^n) \cong \widetilde{H}_n(S^n)$, we conclude that $q_{1*} \oplus q_{2*}$ is the inverse of $i_{1*} \oplus i_{2*}$, hence an isomorphism.
We denote the diagonal map $[a] \longmapsto ([a], [a])$ with the letter D. The left triangle in the diagram commutes and $(f \vee g)_*(i_{1*} + i_{2*})$ sends $([a], [0])$ to $f_*([a])$ and $([0], [a])$ to $g_*([a])$, since the composite maps give $(f \vee g)i_1 = f$ and $(f \vee g)i_2 = g$. This means that $([a], [a])$ is being sent to $f_*([a]) + g_*([a])$ or equivalently $(f \vee g)_*(i_{1*} + i_{2*})D([a]) = f_*([a]) + g_*([a]) = (f \vee g)_* c_*([a])$. But $(f \vee g)_* c_*([a]) = (f + g)_*([a])$, which proves the desired. $\square$

Basepoints of X are undeniably significant when examining homotopic elements in π_n, while completely absent in H_n. Since our goal is to find an isomorphism between a homology group and a group that has spheroids as elements, we need to treat spheroids that are homotopic via a homotopy which does not respect basepoints as equivalent, because their images under h are equal. If we are to have a chance to get an isomorphism, we need $ker h$ to be trivial. Since it is not, we need to factor this kernel out.
We consider the actions

$$\beta_X : \pi_1(X, *) \times \pi_n(X, *) \longrightarrow \pi_n(X, *),\ n \geq 1$$
$$\beta_A : \pi_1(A, *) \times \pi_n(X, A, *) \longrightarrow \pi_n(X, A, *),\ n \geq 2$$

When $n = 1$, $\beta_{\{\cdot\}}([\gamma], [f]^\#) = [\gamma \cdot f \cdot \overline{\gamma}]^\#$, while, for $n \geq 2$, the actions are those defined in Propositions 2.3.12 and 2.4.5. Now $h(\beta_{\{\cdot\}}([\gamma], [f]^\#) - [f]^\#) = 0$, since $\beta_{\{\cdot\}}([\gamma], [f]^\#)$ is homotopic to $[f]^\#$ through a homotopy that does not respect basepoints. Concequently, if we name $N \trianglelefteq \pi_n$ the normal subgroup generated by all elements of the form $\beta_{\{\cdot\}}([\gamma], [f]^\#) - \beta_{\{\cdot\}}([c_*], [f]^\#) = [\gamma][f]^\# - [f]^\#$ and $\pi_n' = \pi_n / N$, then $N \subset ker h$ and h induce the homomorphisms

$$h' : {\pi_n}'(X, x_0) \longrightarrow H_n(X),\ n \geq 1$$
$$h' : {\pi_n}'(X, A, x_0) \longrightarrow H_n(X, A),\ n \geq 2.$$

Finally, for $n = 1$ it is easy to see that $N = [\pi_1(X, x_0), \pi_1(X, x_0)]$ and ${\pi_1}'$ coincides with its definition as an abelianisation given earlier.

Lemma 5.1.5. *Let $(X, A, *) \in \mathscr{T}^{2*}$. Then ${\pi_n}'(X, A, *)$ is abelian for $n \geq 2$.*

Proof. For $n \geq 3$ the result is readily obtained by Proposition 2.4.4.
Let $n = 2$ and $[f]^\#, [g]^\# \in \pi_2(X, A, *)$. Recall the boundary map $\partial : \pi_2(X, A, *) \longrightarrow \pi_1(A, *)$ and observe that ∂f is a loop in A that can act on g. Since $[\partial f][g]^\# = [g]^\#$, ${\pi_n}'(X, A, *)$ would prove to be abelian, if we managed to show the validity of the equation $[f]^\#[g]^\#[\overline{f}]^\# = [\partial f][g]^\#$.
The homotopy that provides us with the result is depicted in Figure 5.4. □

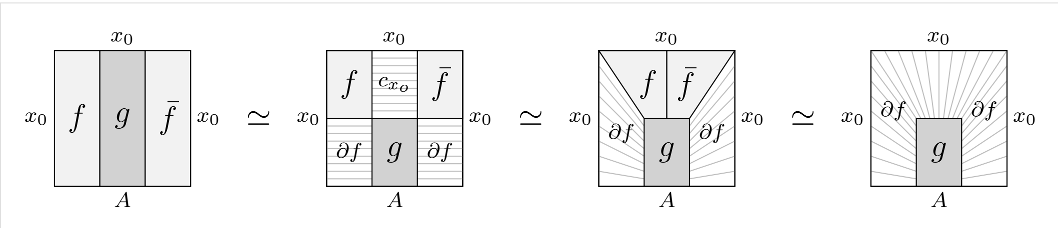


Figure 5.4: Homotopy proving that ${\pi_n}'(X, A, *)$ is abelian.

We mention the absolute form of the Hurewicz theorem without a proof, because it can be seen as a special case of the relative form.

Theorem 5.1.6 (Hurewicz - Absolute form)**.** *Let $X \in \mathscr{T}^*$ be $(n-1)$-connected $(n \geq 1)$. Then $h' : {\pi_n}'(X, *) \longrightarrow H_n(X)$ is an isomorphism.*

Theorem 5.1.7 (Hurewicz - Relative form)**.** *If (X, A) is a $(k-1)$-connected pair of path connected spaces, $k \geq 2$ and $A \neq \varnothing$, then $h' : {\pi_k}'(X, A, x_0) \longrightarrow H_k(X, A)$ is an isomorphism and $H_i(X, A) = 0$ for $i < k$.*

Proof. The proof is by induction on k. The absolute case for $k = 1$ has already been proved in Theorem 5.1.1. First we use the inductive step described below to prove the Theorem for $k = 2$. Then, assuming the absolute theorem for $1 \leq k \leq n-1$, we prove the relative theorem for $k = n$ following the same inductive step and the absolute case is simultaneously true for $k \geq 2$, if we set $A = \{*\}$.
We extend here the definition of the Eilenberg subcomplexes introduced in 4.3.3 to cover for the relative, basepointed case. Let $C_k^{(n-1,A,*)}(X, A)$ be the quotient of the abelian group generated by maps $\sigma \in C_k(X)$ such that $\sigma([\Delta^k]^{n-1}) \subset A$, $\sigma([\Delta^k]^0) = \{*\}$, modulo $C_k^{(0,\{*\})}(A)$. $\partial\partial = 0$ still holds and the chain complex that is created produces the homology group $H_k^{(n-1,A,*)}(X, A)$. The inclusion $i : C_*^{(n-1,A,*)}(X, A) \hookrightarrow C_*(X, A)$ induces isomorphisms $H_*^{(n-1,A,*)}(X, A) \cong H_*(X, A)$ for an $(n-1)$-connected pair (X, A) with path connected A, just like it did in Proposition 4.3.3.

Elements $[f]^{\#}$ in $\pi_n(X,A,*)$ are homotopy classes of maps from $(D^n,\partial D^n,*)$ to $(X,A,*)$, while elements $[f]$ in $H_n^{(n-1,A,*)}(X,A)$ are equivalence classes of n-chains of maps from $(\Delta^n,\partial\Delta^n,\Delta^0)$ to $(X,A,*)$. Since the n-dimensional disk is homeomorphic to the n-simplex and $[(D^n,\partial D^n,*),(X,A,*)]\cong[(\Delta^n,\partial\Delta^n,v_0),(X,A,*)]$, v_0 vertex of Δ^n, we can look at ${\pi_n}'(X,A,*)$ as consisting of homotopy classes in the group $[(\Delta^n,\partial\Delta^n,v_0),(X,A,*)]$ and perform a minor alteration to the Hurewicz homomorphism so as to match our change in view.
Let $D^n \overset{\alpha}{\equiv} \Delta^n$, where α is the homeomorphism that sends the generator $[\bar{z}_n]$ of $H_n(D^n,\partial D^n)$ to the generator $[id_n]$ of $H_n(\Delta^n,\partial\Delta^n)$ (Example 3.2.11). We adjust slightly the Hurewicz homomorphism h and now it takes the homotopy class $[f]^{\#}\in\pi_n(X,A,*)$ to $f_*(\alpha_*(\bar{z}_n))=f_*(id_n)$ in $H_n(X,A)\cong H_n^{(n-1,A,*)}(X,A)$.
In what follows we are going to gradually build an inverse homomorphism ψ of the adjusted Hurewicz homomorphism h'. Let $\psi_1: C_n^{(n-1,A,*)}(X)\longrightarrow {\pi_n}'(X,A,*)$ which assigns to a singular simplex $\sigma:(\Delta^n,\partial\Delta^n,\Delta^0)\longrightarrow(X,A,*)$ the corresponding element $\sigma^{\#'}$ in ${\pi_n}'(X,A,*)=[(\Delta^n,\partial\Delta^n,v_0),(X,A,*)$ (the superscript $'$ is used to dinstinguish elements in ${\pi_n}'$). ψ_1 is well defined and a homomorphism, since ${\pi_n}'$ is abelian (see Lemma 5.1.5). If $\sigma(\Delta^n)\subset A$, then by the compression criterion (2.4.6) the corresponding homotopy class is zero, i.e. $C_n(A)\subset ker\psi_1$. Therefore, we can extend the homomorphism ψ_1 to the homomorphism $\psi_2: C_n^{(n-1,A,*)}(X,A)\longrightarrow{\pi_n}'(X,A,*)$. Each $\sigma\in C_n^{(n-1,A,*)}(X,A)$ is a relative cycle, because $\partial\sigma$ belongs to $C_{n-1}^{(n-1,A,*)}(X,A)$ which leads to $\partial\sigma(\Delta^{n-1})\subset A\Rightarrow\partial\sigma\in C_{n-1}(A)$. This leads us to $C_n^{(n-1,A,*)}(X,A)=ker\partial_n$. We have yet to show that $\psi_2\circ\partial: C_{n+1}^{(n-1,A,*)}(X,A)\longrightarrow{\pi_n}'(X,A,*)$ is trivial, because then $\psi_2(Im\partial_{n+1})=0\Rightarrow Im\partial_{n+1}\subset ker\psi_2$ and we will be able to form a well defined homomorphism

$$\psi: H_n^{(n-1,A,*)}(X,A)\longrightarrow{\pi_n}'(X,A,*),$$

with $\psi([c]+Im\partial_{n+1})=\psi(p([c]))=\psi_2([c])=[c]^{\#'}$, where $p: ker\partial_n\longrightarrow ker\partial_n/Im\partial_{n+1}$ is the quotient projection.
Take $\tau:(\Delta^{n+1},[\Delta^{n+1}]^{n-1},[\Delta^{n+1}]^0)\longrightarrow(X,A,*)$ such that $[\tau]\in C_{n+1}^{(n-1,A,*)}(X,A)$ is a basis element. $\psi(\partial([\tau]))=\sum_{i=0}^{n+1}(-1)^i[\tau d_i^{n+1}]^{\#'}$ and, if we define elements $[b_n]^{\#}\in\pi_n(\partial\Delta^{n+1},[\Delta^{n+1}]^{n-1},v_0)$ via

$$[b_2]^{\#}=([v_1v_0][d_0^3])[d_2^3][d_1^3]^{-1}[d_3^3]^{-1},$$

$$[b_n]^{\#}=[v_1v_0][d_0^{n+1}]+\sum_{i=1}^{n+1}(-1)^i[d_i^{n+1}],\ n\geq 3,$$

where $[v_1v_0]$ denotes the affine path class in Δ^{n+1} from v_1 to v_0, then

$$\psi(\partial([\tau]))=\tau_*^{\#'}[b_n]^{\#'}=\tau_*^{\#'}(j_*^{\#'}[b_n]^{\#'})$$

in ${\pi_n}'(X,A,*)$, since $[v_1v_0][d_0^{n+1}]^{\#'}=[d_0^{n+1}]^{\#'}$, $n\geq 2$.
Elements $[b_2]^{\#}$ and $[b_n]^{\#}$ are closely related to the homological boundary operator. Their main difference from ∂ is that in their case provisions have been made in order for the face maps to be transported to the base point v_0. Also, in the formula above we write $j_*^{\#}$ for the map induced from the inclusion $j:\partial\Delta^{n+1}\longrightarrow\Delta^{n+1}$ and we insert $j_*^{\#}$ freely in the calculation, since τ gets restricted on Δ^{n+1}'s faces when $\psi\circ\partial$

is applied on it.
The $(n-1)$-skeleton of Δ^{n+1} is $(n-2)$-connected. This can be proved by induction on n: For $n=2$, $[\Delta^3]^1$ is 0-connected. Let $[\Delta^n]^{n-2}$ be $(n-3)$-connected. Then, combining the homeomorphism $D^{n+1} \equiv \Delta^{n+1}$ with the fact $H_{n-2}(D^{n+1}) \cong 0$, the isomorphism $H_{n-2}(X) \cong H_{n-2}(X^{n+i})$ for $i \geq -1$ from Proposition 3.2.12 and the theorem's induction hypothesis applied to the absolute case, we conclude that $[\Delta^{n+1}]^{n-1}$ is $(n-2)$-connected.

The induction assumption also gives $\pi_{n-1}'([\Delta^{n+1}]^{n-1}, v_0) \overset{h'}{\cong} H_{n-1}([\Delta^{n+1}]^{n-1})$ for the $(n-2)$-connected skeleton $[\Delta^{n+1}]^{n-1}$. Commutativity of the diagram in Figure 5.5 gives $\partial_n h_n' = h_{n-1}'\partial_n^{\#}$ and it is easy to see that $\partial_n h_n'[b_n]^{\#} = \partial_n(b_{n*}(id_n)) = 0$, since

$$\begin{tikzcd}
\pi_n'(\partial\Delta^{n+1}, [\Delta^{n+1}]^{n-1}, v_0) \arrow[r, "\partial_n^{\#}"] \arrow[d, "h_n'"] & \pi_{n-1}'([\Delta^{n+1}]^{n-1}, v_0) \arrow[d, "h_{n-1}'"] \\
H_n(\partial\Delta^{n+1}, [\Delta^{n+1}]^{n-1}) \arrow[r, "\partial_n"] & H_{n-1}([\Delta^{n+1}]^{n-1})
\end{tikzcd}$$

Figure 5.5

it is a boundary. This leads to $h_{n-1}'\partial_n^{\#}[b_n]^{\#} = 0$ and finally results to $\partial_n^{\#}[b_n]^{\#} = 0$, because h_{n-1}' is an isomorphism.
We split the boundary map $\partial_n^{\#}$ into

$$\partial_n^{\#} : \pi_n'([\Delta^{n+1}]^n, [\Delta^{n+1}]^{n-1}, v_0) \xrightarrow{j_*^{\#}} \pi_n'(\Delta^{n+1}, [\Delta^{n+1}]^{n-1}, v_0) \xrightarrow{\partial'} \pi_{n-1}'([\Delta^{n+1}]^{n-1}, v_0)$$

and, since Δ^{n+1} is contractible, ∂' is an isomorphism, as one can see from the long exact homotopy sequence of the pair $(\Delta^{n+1}, [\Delta^{n+1}]^{n-1})$. Thus, $j_*^{\#}[b_n]^{\#} = 0 \Rightarrow \psi \circ \partial = 0$, which means that the inverse map ψ can be extended and this finishes the inductive step for the relative Hurewicz theorem. $\square$

Corollary. $\pi_n(S^n) \cong \mathbb{Z}$, $n \geq 2$.

Appendix

A.1 Categories and functors

The following are based on [12] and [15].

Definition A.1.1. A **category** $\mathscr{C}$ consists of:

1. A class of objects $\mathrm{Obj}(\mathscr{C})$.
2. A set of morphisms $\mathrm{Hom}(X, Y)$ for every ordered pair of objects (X, Y) such that it is pairwise disjoint and for each $A \in \mathrm{Obj}(\mathscr{C})$ it includes an 'identity' morphism $\mathbb{1}_A \in \mathrm{Hom}(A, A)$.
3. A function for the composition of morphisms $\circ$: $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$ for every ordered triple of objects (X, Y, Z), which is associative when defined and satisfies $f \circ \mathbb{1}_A = f$ and $\mathbb{1}_A \circ g = g$, for all $f \in \mathrm{Hom}(X, A)$, $g \in \mathrm{Hom}(A, X)$, X, $A \in \mathrm{Obj}(\mathscr{C})$.

Definition A.1.2. Let $\mathscr{A}$ and $\mathscr{C}$ be categories with $\mathrm{Obj}\mathscr{C} \subset \mathrm{Obj}\mathscr{A}$. If A, $B \in \mathrm{Obj}\mathscr{C}$, let us denote the two possible Hom sets by $\mathrm{Hom}_{\mathscr{C}}(A, B)$ and $\mathrm{Hom}_{\mathscr{A}}(A, B)$. Then $\mathscr{C}$ is a **subcategory** of $\mathscr{A}$ if $\mathrm{Hom}_{\mathscr{C}}(A, B) \subset \mathrm{Hom}_{\mathscr{A}}(A, B)$ for all objects A, B in $\mathscr{C}$ and if composition in $\mathscr{C}$ is the same as composition in $\mathscr{A}$. A subcategory $\mathscr{C}$ that inherits all morphisms between its objects from $\mathscr{A}$, namely $\mathrm{Hom}_{\mathscr{A}}(A, B) =$

$\mathrm{Hom}_{\mathscr{A}}(A,B)$, is called a **full subcategory**.

Example A.1.3.

1. The category of all topological spaces $\mathscr{T}$ has topological spaces X as objects, continuous functions between spaces as morphisms and the usual composition of maps as composition.
2. The category of all pointed topological spaces $\mathscr{T}^*$ has pointed spaces (X, x_0) as objects, basepoint preserving continuous functions between spaces as morphisms and the usual composition of maps as composition.
3. The category of all pointed sets $\mathscr{S}ets^*$ has pointed sets (X, x_0) as objects, basepoint preserving functions between sets as morphisms and the usual composition of maps as composition.
4. The category of all algebraic groups $\mathscr{G}$ has groups as objects, morphisms between groups as morphisms and the usual composition of morphisms as composition.
5. The category of ordered pairs of topological spaces $\mathscr{T}^2$ has ordered pairs (X, A), $A \subset X$, as objects, functions $f : (X, A) \longrightarrow (Y, B)$ with $f(A) \subset B$ as morphisms and the usual composition as composition. The category of pointed ordered pairs, $\mathscr{T}^{2*}$, can be defined combining the definitions of $\mathscr{T}^2$ and $\mathscr{T}^*$.
6. The category $\mathscr{W}$ of topological spaces that have the homotopy type of a CW complex has spaces $X \in \mathscr{T}$ with $X \simeq K$, K a CW complex, as objects and all morphisms between these spaces as morphisms. It is a full subcategory of $\mathscr{T}$.

Definition A.1.4. A **congruence** on a category $\mathscr{C}$ is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)}\mathrm{Hom}(A, B)$ of all morphisms in $\mathscr{C}$, such that $f \in\mathrm{Hom}(A, B)$ and $f \sim f'$ implies $f' \in\mathrm{Hom}(A, B)$ and, if $f \sim f'$, $g \sim g'$ and $g\circ f$ exists, then $g\circ f \sim g'\circ f'$.

Theorem A.1.5. *Let $\mathscr{C}$ be a category with congruence $\sim$ and let $[f]$ denote the equivalence class of a morphism f. Define $\mathscr{C}'$ as follows:*

$$\begin{aligned} Obj\mathscr{C}' &= Obj\mathscr{C} \\ Hom_{\mathscr{C}'}(A,B) &= \{[f] |\ f \in Hom(A,B)\} \\ [g]\circ[f] &= [g\circ f]. \end{aligned}$$

Then $\mathscr{C}'$ is a category.

Proof. See Theorem 0.4 in [15]. □

Remark. The category $\mathscr{C}'$ is called a **quotient category** of $\mathscr{C}'$. We usually denote $Hom_{\mathscr{C}'}(A,B)$ by $[A,B]$.

Example A.1.6. A very common category in algebraic topology is the homotopy category $h\mathscr{T}$. It is constructed as a quotient category of $\mathscr{T}$, where the congruence relation $\sim$ is the homotopy relation between morphisms in $\mathscr{T}$. Thus $h\mathscr{T}$ has topological spaces as objects and equivalence classes of continuous functions as morphisms. In a similar way, we get $h\mathscr{T}^*$ from the category of pointed topological spaces $\mathscr{T}^*$ and $h\mathscr{T}^2$ from the category of ordered pairs of topological spaces $\mathscr{T}^2$. The homotopy relations are realised again through morphisms in the respective categories.

Definition A.1.7. If $\mathscr{A}$ and $\mathscr{C}$ are categories, a **covariant functor** $F : \mathscr{A} \longrightarrow \mathscr{C}$ is a function that assigns to each object $A \in \mathrm{Obj}\mathscr{A}$ an object $FA \in \mathrm{Obj}\mathscr{C}$ and to each morphism $f \in \mathrm{Hom}(A, A')$ in $\mathscr{A}$ a morphism $Ff \in \mathrm{Hom}(FA, FA')$ in $\mathscr{C}$, in such a way that:

1. if f, g are morphisms in $\mathscr{A}$ for which $g \circ f$ is defined, then $F(g \circ f) = Fg \circ Ff$ and
2. $F\mathbb{1}_A = \mathbb{1}_{FA}$ for every $A \in \mathrm{Obj}\mathscr{A}$.

Example A.1.8.

1. Fixing (X, A), $[(X, A), (Y, B)]$ is a covariant functor from $\mathscr{T}^2$ or $h\mathscr{T}^2$ to the category of sets and functions.
2. Fixing $(X, *) \in \mathscr{CG}^*$, $(\cdot)^{(X,*)}$ is a covariant functor from $\mathscr{CG}^*$ to $\mathscr{CG}^*$.

   Proof. Let $(Y, *) \in \mathscr{CG}^*$. Then $(Y, *)^{(X,*)}$ is the compactly generated topological space produced by all the pointed continuous functions from $(X, *)$ to $(Y, *)$. Thus $(\cdot)^{(X,*)}$ takes objects from the category $\mathscr{CG}^*$ to $\mathrm{Obj}(\mathscr{CG}^*)$.
   For $f_1 \in \mathrm{Hom}((Y, *), (Z, *))$, $(\cdot)^{(X,*)}f_1 : (Y, *)^{(X,*)} \longrightarrow (Z, *)^{(X,*)}$ is given by $(\cdot)^{(X,*)}f_1\ (g) = f_1 \circ g$, $g \in (Y, *)^{(X,*)}$, which is a pointed continuous map. In other words $f_1 \circ g$ belongs to $C((X, *), (Z, *))$, which proves that $(\cdot)^{(X,*)}f_1 \in \mathscr{CG}^*$.
   For $f_1 \in \mathrm{Hom}((Y, *), (Z, *))$ and $f_2 \in \mathrm{Hom}((Z, *), (W, *))$ we have $(\cdot)^{(X,*)}(f_1 \circ f_2)\ (g) = (f_1 \circ f_2) \circ g = f_1(f_2 \circ g) = (\cdot)^{(X,*)}(f_1) \circ (\cdot)^{(X,*)}(f_2)\ (g)$, $g \in (Y, *)^{(X,*)}$ and $(\cdot)^{(X,*)}id\ (g) = id \circ g = g$. □

3. Σ is a functor from $h\mathscr{T}^* \longrightarrow h\mathscr{T}^*$.

   Proof. $h\mathscr{T}^*$ is a quotient category, so it suffices to prove that Σ is a functor from $\mathscr{T}^*$ to $\mathscr{T}^*$ and $\Sigma f \sim \Sigma g$ for $f \sim g$.
   Let $X \in \mathscr{T}^*$. Then the reduced suspension ΣX also belongs to $\mathscr{T}^*$, since it is a quotient space of $X \times I$ that that preserves a well defined basepoint. For $f \in \mathrm{Hom}((X, x_0), (Y, f(x_0)))$, $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ is given by $\Sigma f([x, t]) = [f(x), t]$. From the quotient topology and the commutative diagram in A.1 $(\Sigma f)^{-1}(U)$ is open for all $U \subset \Sigma Y$ open neighbourhoods of $f(x_0)$, which leads ultimately to Σf being continuous. Moreover, $\Sigma f[x_0, t] = [f(x_0), t]$, where $[x_0, t]$ is the basepoint of ΣX and $[f(x_0), t]$ is the basepoint of ΣY. Thus $\Sigma f \in \mathrm{Hom}((\Sigma X, [x_0, t]), (\Sigma Y, [f(x_0), t]))$.
   Taking $f \in \mathrm{Hom}((X, x_0), (Y, f(x_0)))$, $g \in \mathrm{Hom}((Y, f(x_0)), (Z, g(f(x_0))))$, we

have $\Sigma(g\circ f)([x,t]) = [(g\circ f)(x),t] = [g(f(x)),t] = \Sigma g([f(x),t]) = \Sigma g(\Sigma f([x,t]))$ $= (\Sigma g \circ \Sigma f)([x,t])$. Finally, $\Sigma id([x,t]) = [x,t]$ and Σ has been proved to be a functor from $\mathscr{T}^*$ to $\mathscr{T}^*$.
Now, let $f \sim g$. Taking $f_{\Sigma X}, g_{\Sigma X} : (ZX, (X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)) \longrightarrow (ZY, (Y \times \{0\} \cup Y \times \{1\} \cup \{f(x_0)\} \times I))$ with $f_{\Sigma X}(x,t) = (f(x),t)$ we have $f_{ZX} \sim g_{ZX}$ via a homotopy $F : (ZX \times I \longrightarrow ZY)$. From A.2 we can finally form the maps $\Sigma f, \Sigma g : (\Sigma X, [x_0,t]) \longrightarrow (\Sigma Y, [f(x_0),t])$, which are homotopic via the homotopy $\bar{F}$. □

4. Ω is a functor from $h\mathscr{T}^* \longrightarrow h\mathscr{T}^*$. This is actually Theorem 11.8 in [14].

$$\begin{tikzcd}
(X,x_0) \arrow[r, "f"] \arrow[d, "\Sigma"] & (Y,f(x_0)) \arrow[d, "\Sigma"] \\
(\Sigma X,[x_0,t]) \arrow[r, "\Sigma f"] & (\Sigma Y,[f(x_0),t])
\end{tikzcd}$$

Figure A.1

$$\begin{tikzcd}
ZX \times I \arrow[r, "F"] \arrow[d, "p \times \mathbb{1}"] & Y \arrow[d, "q"] \\
ZX/_{(X\times\{0,1\}\cup\{x_0\}\times I)} \times I \arrow[r, "\bar{F}"] & Y/_{(Y\times\{0,1\}\cup\{f(x_0)\}\times I))}
\end{tikzcd}$$

Figure A.2

Definition A.1.9. If $\mathscr{A}$ and $\mathscr{C}$ are categories, a **contravariant functor** $F : \mathscr{A} \longrightarrow \mathscr{C}$ is a function that assigns to each object $A \in \mathrm{Obj}\mathscr{A}$ an object $FA \in \mathrm{Obj}\mathscr{C}$ and to each morphism $f \in \mathrm{Hom}(A, A')$ in $\mathscr{A}$ a morphism $Ff \in \mathrm{Hom}(FA', FA)$ in $\mathscr{C}$, in such a way that:

1. if f, g are morphisms in $\mathscr{A}$ for which $g \circ f$ is defined, then $F(g \circ f) = Ff \circ Fg$ and
2. $F\mathbb{1}_A = \mathbb{1}_{FA}$ for every $A \in \mathrm{Obj}\mathscr{A}$.

Example A.1.10.

1. Fixing (Y,B), $[(X,A),(Y,B)]$ is a contravariant functor from $\mathscr{T}^2$ or $h\mathscr{T}^2$ to the category of sets and functions.
2. Fixing $(Y,*) \in \mathscr{CG}^*$, $(Y,*)^{(\cdot)}$ is a contravariant functor from $\mathscr{CG}^*$ to $\mathscr{CG}^*$. This is proved similarly to Example A.1.8, 2.

Definition A.1.11. Let $F : \mathscr{A} \longrightarrow \mathscr{C}$ and $G : \mathscr{C} \longrightarrow \mathscr{A}$ be functors. The ordered pair (F, G) is an **adjoint pair** of functors if, for each object A in $\mathscr{A}$ and each object C in $\mathscr{C}$, there is a bijection

$$\tau = \tau_{AC} : Hom(FA, C) \longrightarrow Hom(A, GC),$$

which is natural in each variable.

Naturality means that diagrams in Fig.A.3 and Fig.A.4 commute for all $f \in Hom(A', A)$ in $\mathscr{A}$ and $g \in Hom(C, C')$ in $\mathscr{C}$.

$$\begin{tikzcd} Hom(FA, C) \arrow[r, "Ff \circ (\cdot)"] \arrow[d, "\tau"] & Hom(FA', C) \arrow[d, "\tau"] \\ Hom(A, GC) \arrow[r, "f \circ (\cdot)"] & Hom(A', GC) \end{tikzcd}$$

Figure A.3

$$\begin{tikzcd} Hom(FA, C) \arrow[r, "g \circ (\cdot)"] \arrow[d, "\tau"] & Hom(FA, C') \arrow[d, "\tau"] \\ Hom(A, GC) \arrow[r, "Gg \circ (\cdot)"] & Hom(A, GC') \end{tikzcd}$$

Figure A.4

Example A.1.12. (Σ, Ω) are adjoint functors from $h\mathscr{T}^*$ to $h\mathscr{T}^*$.

Proof. See the proof for Theorem 11.2 in [15]. □

Remark. Using the previous result inductively we can conclude that (Σ^n, Ω^n) are adjoint functors.

Definition A.1.13. An **equivalence** in a category $\mathscr{A}$ is a morphism $f : A \longrightarrow B$ for which there exists a morphism $g : B \longrightarrow A$ with $f \circ g = \mathbb{1}_B$ and $g \circ f = \mathbb{1}_A$.

Remark. For example, a homotopy equivalence between pointed topological spaces is an equivalence in the category $h\mathscr{T}^*$.

A.2 Basic notions in algebra

The following are based on [14], [15] and [21].

Definition A.2.1. Let X be a set and G a group. Then the function

$$\begin{aligned} \tau: \quad G \times X &\longrightarrow \quad X \\ (g,x) \quad &\longmapsto \quad g \cdot x \end{aligned}$$

is called the **action of G on X**, if the following are valid:

1. $1 \cdot x = x$, for all $x \in X$
2. $g \cdot (h \cdot x) = (gh) \cdot x$, for all $g, h \in G$ and $x \in X$.

Definition A.2.2. Let R be a ring. A **left R-module** is an abelian group M on which R acts linearly; that is, there is a map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r,m) &\longmapsto rm \end{aligned}$$

for $r \in R$, $m \in M$, for which

1. $(r+s)m = rm + sm$
2. $r(m+n) = rm + rn$
3. $(rs)m = r(sm)$
4. $1m = m$.

Exact sequences of objects and morphisms can be defined in any category with kernels and cokernels.

Definition A.2.3. In $\mathscr{G}$, a sequence $... \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_i} A_i \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_{i-2}} ...$ of groups and group morphisms is called an **exact sequence** if $ker f_i = Im f_{i+1}$, $\forall i \in \mathbb{N}$. The sequence may be either finite or infinite.

Definition A.2.4. In $\mathscr{S}ets^*$ a sequence

$$... \xrightarrow{f_{i+1}} (A_{i+1}, *) \xrightarrow{f_i} (A_i, *) \xrightarrow{f_{i-1}} (A_{i-1}, *) \xrightarrow{f_{i-2}} ...$$

of pointed sets and pointed sets morphisms is called **exact in $\mathscr{S}ets^*$** if $ker f_i = Im f_{i+1}$, $\forall i \in \mathbb{N}$, where $ker f_i = f_i^{-1}(*)$.

Definition A.2.5. In $h\mathscr{T}^*$ a sequence

$$\ldots \xrightarrow{f_{i+1}} (A_{i+1},*) \xrightarrow{f_i} (A_i,*) \xrightarrow{f_{i-1}} (A_{i-1},*) \xrightarrow{f_{i-2}} \ldots$$

of pointed topological spaces and pointed maps is called **exact in** $h\mathscr{T}^*$ if the induced sequence

$$\ldots \xrightarrow{\overline{f_{i+1}}} [(X,*),(A_{i+1},*)] \xrightarrow{\overline{f_i}} [(X,*),(A_i,*)] \xrightarrow{\overline{f_{i-1}}} [(X,*),(A_{i-1},*)] \xrightarrow{\overline{f_{i-2}}} \ldots$$

is exact in $\mathscr{S}ets^*$ for every $X \in \mathscr{T}^*$.

Definition A.2.6. An exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called a **short exact sequence**.

Definition A.2.7. A **graded group** $C = \{C_q\}$ consists of a collection of abelian groups C_q indexed by integers q. Elements of C_q are said to have **degree q**. For C and D graded groups, a **homomorphism of degree** d, $\tau : C \longrightarrow D$, consists of a collection of morphisms $\tau = \{\tau_q : C_q \longrightarrow D_{q+d}\}$.

Definition A.2.8. In homological algebra, a **chain complex (A,d)** is a sequence of abelian groups $\{A_i\}$, $i \in \mathbb{Z}$, connected by homomorphisms $d_n : A_n \longrightarrow A_{n-1}$, such that the composition of any two consecutive of them is the zero map, i.e. $d_n \circ d_{n-1} = 0$, for all n. These maps are called boundary operators and a chain complex usually is written as:

$$\ldots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} \ldots$$

Lemma A.2.9. *Let A, B, C, D, E, A', B', C', D', $E' \in \mathscr{G}^{ab}$ be in a commutative as the one depicted in Figure A.5. If the two rows are exact and α, β, δ and ϵ are isomorphisms, then γ is an isomorphism too. The result is known as the Five-Lemma.*

Proof. It suffices to show that γ is surjective if β and δ are surjective and ϵ is injective,

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r, "k"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

Figure A.5

and γ is injective if β and δ are injective and α is surjective. The proof is based on a technique called diagram chasing. It is simple yet rather tedious, thus we prove here the first statement and refer the reader to Chapter 2 in [10] for the second.
Let $c' \in C'$. Since δ is surjective, the second row exact and ϵ injective, we have $k'(c') = \delta(d)$ for some $d \in D$, $\epsilon(l(d)) = l'(\delta(d)) = l'(k'(c')) = 0$ and $l(d) = 0$.

Hence $d = k(c)$ for some $c \in C$ by exactness of the upper row. Now commutativity gives $k'(c' - \gamma(c)) = \delta(d) - \delta(k(c)) = 0$. Therefore $c' - \gamma(c) = j'(b')$ for some $b' \in B'$ by exactness and, since β is surjective and $b' = \beta(b)$ for some $b \in B$, we get $\gamma(c + j(b)) = \gamma(c) + \gamma(j(b)) = \gamma(c)j'(\beta(b)) = \gamma(c) + j'(b') = c'$ which shows that γ is surjective. $\square$

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