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Alexia Papalazarou

The Cahn-Hilliard Equation
Construction of a Periodic Solution

A thesis submitted in fulfillment of the requirements for the degree of
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(specialization: **Applied Mathematics**)
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Advisor: Professor Nicholas Alikakos

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I thank my supervisor for the excellent guidance,
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KX

Περίληψη

Στην διπλωματική αυτή θα ασχοληθούμε με την μελέτη της εξίσωσης Cahn-Hilliard. Πιο συγκεκριμένα, θα κατασκευάσουμε περιοδικές και μη φραγμένες λύσεις της.

Η εξίσωση Cahn-Hilliard ανήκει σε μια τάξη εξελικτικών εξισώσεων τύπου αντίδρασης-διάχυσης.

Απο φυσικής πλευράς, έχουμε να κάνουμε με ενα πρόβλημα αλλαγής φάσης, κατά το οποίο δημιουργούνται (δι)επιφάνειες. Επιπλέον, η εξίσωση Cahn-Hilliard είναι η απλούστερη εξίσωση η οποία διατηρεί την μάζα και μειώνει το συναρτησιακό της ενέργειας.

Απο γεωμετρικής πλευράς, η Cahn-Hilliard διατήρει τον όγκο που περικλείεται απο τις παραπάνω επιφάνειες και μειώνει παράλληλα την περίμετρο.

Όσον αφορά την κατασκευή των λύσεων, ενδιαφερόμαστε για την κατασκεύη τέτοιων ώστε οι επιφάνειες που τους αντιστοιχούν να μην είναι απλές σφαίρες, αλλά να είναι μη φραγμένες επιφάνειες σταθερής μέσης καμπυλότητας. Η μέθοδος κατασκευής τους είναι η εξής:

Ξεκινάμε με μια πρώτη προσέγγιση της λύσης, η οποία είναι η μονοδιάστατη ετεροκλινής.

Έπειτα, συνεχίζουμε με μια δεύτερη προσέγγιση της λύσης, στις δύο διαστάσεις. Παίρνουμε την επέκταση της προηγούμενης προσεγγιστικής λύσης κοντά σε μια επιφάνεια σταθερής μέσης καμπυλόητας. Σε αυτό το σκέλος χρησιμοποιούμε θεωρία συνήθων διαφορικών εξισώσεων και ανάλυση πεδίου φάσης.

Τέλος, επεκτείνουμε την προσεγγιστική αυτή μας λύση σε όλο τον χώρο. Ώς εκ΄ τούτου, θα χρειαστούμε θεωρία τελεστών, χρήση της μεθόδου Lyapounov-Schmidt και χρήση επιχειρημάτων σταθερού σημείου.

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1 Abstract

This thesis is devoted to the Cahn-Hilliard (CH) equation. Cahn-Hilliard, belongs in a class of evolution equations of reaction-diffusion type.

- From a physical point of view: We deal with a phase change problem where interfaces are created.
- CH equation conserves mass and reduces the Energy functional.
- At geometric level: Cahn-Hilliard conserves the volume enclosed and the reduces the perimeter.

We are interested in constructing a class of solutions which are not simple, sphere-like but instead are periodic, unbounded Constant Mean Curvature surface-like.

Method:

- First Approximation: The 1-d heteroclinic solution
- Second Approximation: Expansion of 1-d solution near a CMC surface (Key Pieces: ODE's theory, Phase plane analysis)
- Expansion to the whole space (Key Pieces: Definition of a non-linear operator through CH, the linearisation of this operator, Lyapounov-Schmidt reduction, fixed point argument)

2 The Cahn-Hilliard Equation

This section is devoted to the Cahn-Hilliard equation.

$$\begin{cases} u_t = -\Delta(\epsilon^2 \Delta u - W_u(u)) \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \frac{\partial}{\partial \nu}(\epsilon^2 \Delta u - W_u(u)) = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

where W is a double-well potential and in this thesis we will take $W(u) = \frac{1}{4}(u^2-1)^2$. This equation, is a model that describes the process of phase separation of two components of a binary alloy. Here, $\Omega \subset \mathbb{R}^d$ where $d \geqslant 1$, is a bounded domain representing the place where the isolation of the components takes place, ν as usual, denotes the outer normal on $\partial\Omega$ and the function u stands for the concentration of one of the components. Finally, ϵ is the range of intermolecular forces.

2.1 Physical Considerations

Let us consider a binary alloy (A-B). An alloy is a specific solid combination of A-B. For example, 30% A molecules - 70% B molecules = Phase I. 60% A molecules - 40% B molecules = Phase II. The state of this alloy depends not only on the temperature but also on the mean concentration of each component A,B. For a fixed mean concentration, the experiment consists in abrupt drop of temperature and then wait for equilibrium.

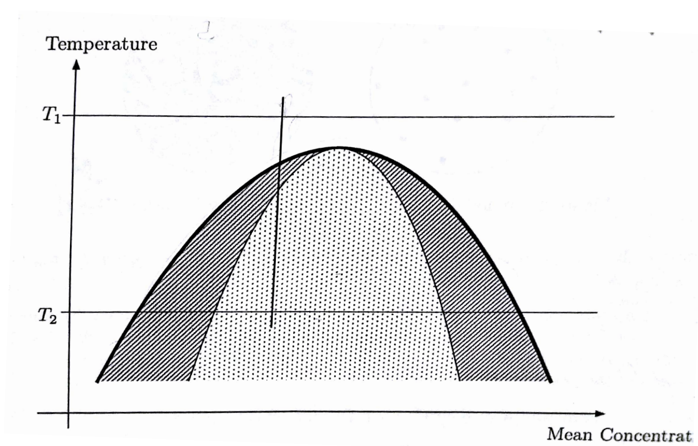

Figure 1: Phase Diagram

Explanation of the diagram

- Over the thick parabola: The coexistence of the two phases A and B.

- The stripped area: Nucleation, this is a metastable state, when the two phases separate each other and nuclei appear.

- The dotted area: Spinodal Decomposition, this is a stable state where large areas of a unique component A or B appear. The interface between phases A and B evolves. The Cahn-Hilliard equation contains big amount of information about the geometry of this interface and its evolution.

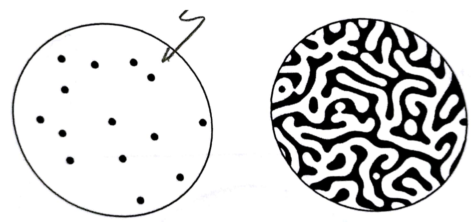

Figure 2: Nucleation and Spinodal Decomposition

We denote u the concentration of one component of the alloy, and rescaling it: $u(x) \in [-1,1]$.

We denote by u* the mean concentration:

$$u^* = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$$

Remark 1. The mean concentration u* is constant, but the local concentration evolves, and depends both in time and the location in Ω. For example, in Figure 2, the black areas, are areas where $u = 1$, and the white, areas with $u = -1$.

2.2 Notation for the Problem

The potential in its general form is shown below

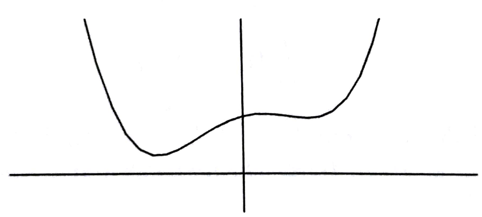

Figure 3: General Potential

Without loss of generality and thanks to conservation, it can be transformed and written in the form of W as follows.

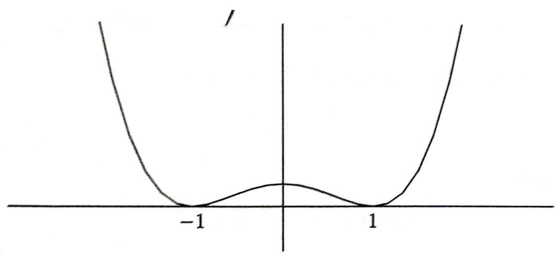

Figure 4: $W(u) = \frac{1}{4}(1-u^2)^2$

The area between the minima and the inflection points on the curve($W_{uu} > 0$), corresponds to nucleation. Between -1 and 1, the area where $W_{uu} < 0$ corresponds to the spinodal decomposition. In this thesis, we consider the double well potential W(u)=$\frac{1}{4}(1-u^2)^2$.

The function $u(x,t)$ represents the concentration of each component of the alloy at x $\in \Omega$ at time t. We will denote it by $u(x)$ or $u(t)$ depending on whether we consider space or time evolution.

2.2.1 Energy Functional

We define the free energy functional as follows

$$J_\epsilon(u) = \int_\Omega \frac{\epsilon^2}{2}|\nabla u|^2 + W(u)dx$$

This represents the free energy of the system. Thus we will try to find u which minimizes it.

The term W is a potential term, it tends to approach a system where u takes ± 1 : It is a separation term.

The gradient term is a uniformity term which will minimize the number of changes between the values ± 1 of u : It penalizes interface area and singles out a solution.

2.2.2 The Gradient Flow

Proposition 1. *Equation (1) can be derived from the gradient flow of the energy functional in $H^{-1}(\Omega)$ subject to c^* being constant.*

Proof. Let $\mathcal{H}$ be a Hilbert space. We define the function J'(u) by:

$\forall$ u,v $\in \mathcal{H}$, $\lim\limits_{t\to 0} \frac{J(u+t\nu)-J(u)}{t} := \langle J'(u), \nu \rangle_{\mathcal{H}}$

Thus, it is natural to consider the evolution of u such that:

$$\frac{\partial u}{\partial t} = -J'(u)$$

Hence

$$\frac{\partial}{\partial t}J(u) = \left\langle J'(u), \frac{\partial u}{\partial t}\right\rangle_{\mathcal{H}} = -\left|\left|\frac{\partial u}{\partial t}\right|\right|^2_{\mathcal{H}}$$

Therefore, the evolution of u with time tends to decrease $J(u)$.

The desired equation depends on the selection of the space. Here, we will choose $\mathcal{H}$ such that we get the Cahn-Hilliard equation.

Without loss of generality we can assume the total concentration to be zero. Thus, we get the space

$$\mathcal{H} = \{\Delta\Phi/\Phi \in H^1_0(\Omega), \int_\Omega \Phi = 0, \partial_n\Phi = 0\}$$

Remark 2. We have $\mathcal{H} \in H^{-1}(\Omega)$. But, for smooth functions $u \in \mathcal{H}$ the conservation of mass holds.

Let Φ be such that $\Delta\Phi = u$

$$\int_\Omega u = \int_\Omega \Delta\Phi = \int_{\partial\Omega} \partial_n\Phi = 0$$

Thus, this set is relevant according to our considerations.

Proposition 2. *For $u \in \mathcal{H}$ there is a unique $\Phi \in H^1_0(\Omega)$ such that:*

$$\begin{cases} \Delta\Phi = u \text{ on } \partial\Omega \\ \partial_n\Phi = 0 \\ \int_\Omega \Phi = 0 \end{cases}$$

Definition 1. *For Φ_1 and Φ_2 as above, we define a scalar product on $\mathcal{H}$ as:*

$$\langle u_1, u_2\rangle_{\mathcal{H}} = \langle \nabla\Phi_1, \nabla\Phi_2\rangle_{L^2}$$

Proposition 3. *For $u \in \mathcal{H}$, if $\partial_n(W_u(u) - \epsilon^2\Delta u) = 0$ on $\partial\Omega$ then:*

$$J'_\epsilon(u) = -\Delta(W_u(u) - \epsilon^2\Delta u)$$

Proof of Proposition 3. We define Φ as above for a smooth ν.

$$\begin{aligned} \lim_{t\to 0}\frac{J_\epsilon(u+t\nu) - J_\epsilon(u)}{t} &= \int_\Omega [W_u(u) - \epsilon^2\Delta u]\Delta\Phi \\ &= -\int_\Omega \nabla[W_u(u) - \epsilon^2\Delta u]\nabla\Phi \\ &= \langle -\nabla[W_u(u) - \epsilon^2\Delta u], \nabla\Phi\rangle_{L^2} \\ &= \langle -\Delta[W_u(u) - \epsilon^2\Delta u], \nu\rangle_{\mathcal{H}} \end{aligned}$$

□

With Proposition 3, the Proof of Proposition 1 is complete. □

2.3 Results About the Cahn-Hilliard Equation

The goal in this section is to present and explain some results about (1) in order to better understand its meaning and to have a general idea of what the solution will look like.

We recall the definition of the energy functional:

$$J_\epsilon(u) = \int_\Omega \frac{\epsilon^2}{2}|\nabla u|^2 + W(u)dx$$

We will compute $J_\epsilon(u)$ for different u, to have an overview of what the minimizers can be.

2.3.1 For $u = \bar{u}$: constant

Obviously, $\bar{u}$ is a solution of (1) and we have:

$$J_\epsilon(u) = W(\bar{u})|\Omega|$$

where $|\Omega| = \int_\Omega dx$. We will study the stability of such a solution.

The linearization of (1) at $\bar{u}$:

$$\begin{cases} \frac{\partial h}{\partial t} = -\Delta[\epsilon^2 \Delta h - W_{uu}(\bar{u})h] \\ \partial_n h = \partial_n \Delta h = 0 \text{ on } \partial\Omega \\ \int_\Omega h = 0 \end{cases}$$

Assuming that h is in the form $h(t,x) = e^{\sigma t}\nu(x)$ and replacing it in the above equation we get:

$$-\Delta\nu = \mu\nu$$
$$\sigma = -\epsilon\mu^2 - W_{uu}(\bar{u})\mu$$

Since the eigenvalues μ are positive and ϵ is small, the condition for σ roughly leads to the following dichotomy:

$$\begin{cases} W_{uu}(\bar{u}) > 0 \Rightarrow \sigma < 0 \\ W_{uu}(\bar{u}) < 0 \Rightarrow \sigma \gtrsim 0 \end{cases}$$

We write $\mu = k^2$ and when $W_{uu}(\bar{u}) < 0$, the graph of $\sigma(k)$ is the following

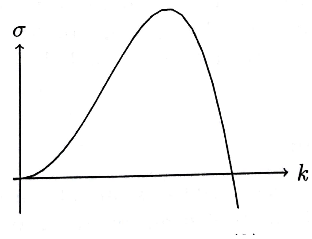

Figure 5: Dissipation $\sigma(k)$

- Thus, the case where $W_{uu}(\bar{u}) < 0$ leads to stability for big values of k.
Below we have a graph of W_u:

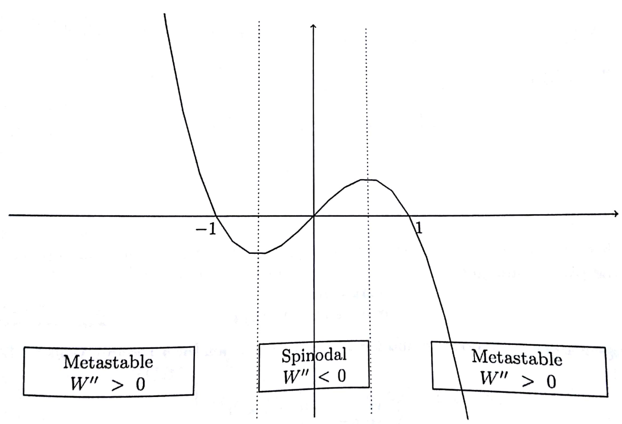

Figure 6: Graph of W'

- If $W_{uu}(\bar{u}) > 0$ then $u = \bar{u}$ is a metastable minimizer. The term "metastable" stands for the long time evolution to bubbles. This evolution is due to heteroclinic which is equal to ± 1 (stable values) at $\pm\infty$. So, the ends of the curve influence the motion of the bubble and thus the state is not stable in the usual sense.

2.3.2 $\bar{u}$ is not a global minimizer

In the definition of J_ϵ there were two influences on the system. Uniformity (the gradient term) and separation (the potential term) at ± 1. Therefore, areas will appear where u is constant and equal to ± 1, and so there will be an interface between these two areas where u will change its value from -1 to 1. The gradient term is trying to reduce the interface.

In particular, let us consider $E \in \Omega$ for which ∂E is smooth. We also consider I_ϵ a neighbourhood of ∂E of width 2ϵ. We choose u to be of the form:

$$u(x) = \frac{d(x, E \setminus I_\epsilon) - d(x, (\Omega \setminus E) \setminus I_\epsilon)}{d(x, E \setminus I_\epsilon) + d(x, (\Omega \setminus E) \setminus I_\epsilon)}$$

so that:

$$\begin{cases} -1 \leq u \leq 1 \text{ in } I_\epsilon \\ u = \pm 1 \text{ in } E \setminus I_\epsilon \\ u = \pm 1 \text{ in } (\Omega \setminus E) \setminus I_\epsilon \end{cases}$$

This is shown in the graph below:

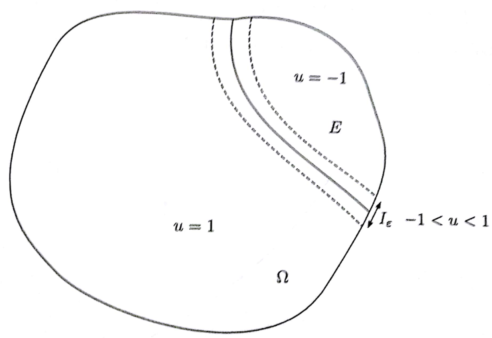

Figure 7: The I_ϵ neighbourhood of ∂E

Proposition 4. *Because of this selection of u, we have that:*

$$J_\epsilon(u) \approx CL(\partial E)\epsilon$$

where C is a constant and $L(\partial E)$ is the length of ∂E.

Proof. We denote with Σ_y the normal section of length 2ϵ to ∂E at $y \in \partial E$

$$\int_\Omega W(u(x))dx = \int_{I_\epsilon} W(u)dx \approx \int_{\partial E} \Big(\int_{\Sigma_y} W(u(x))dx \Big) dy \leq L(\partial E) L(\Sigma) \sup W \leq L(\partial E)\frac{\epsilon}{2}$$

Computation of the gradient term:
In I_ϵ we can write

$$u(x) = f(\delta(x))$$

where $f(x) = \frac{x-1}{x+1}$ and $\delta(x) = \frac{d(x, E \setminus I_\epsilon)}{d(x, (\Omega \setminus E) \setminus I_\epsilon)}$
Thus, the differential of u is:

$$\nabla u = f'(\delta) \nabla \delta$$

Proposition 5. *Close enough to a set A, the differential at x of the distance to A is the normal to A at the projection of x on A, and $|\nabla d| = 1$*

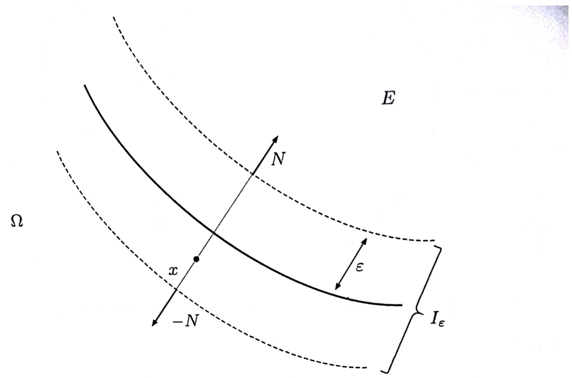

Figure 8: Notation for Computation of the Gradient Term

From Proposition 5 and because ϵ is sufficiently small (see figure) we have that:

$$\begin{aligned}\nabla_x \delta &= \frac{d(x,(\Omega \setminus E) \setminus I_\epsilon)\nabla_x[d(x,E \setminus I_\epsilon)] - d(x,E \setminus I_\epsilon)\nabla_x[d(x,(\Omega \setminus E) \setminus I_\epsilon)]}{d(x,\Omega \setminus E) \setminus I_\epsilon)^2} \\ &= \frac{[2\epsilon - d(x,E \setminus I_\epsilon)](-N) - d(x,E \setminus I_\epsilon)N}{(2\epsilon - d(x,E \setminus I_\epsilon))^2}\end{aligned}$$

For the sake of brevity we set $d = d(x, E \setminus I_\epsilon)$ and thus we have

$$|\nabla u| = \frac{2\epsilon}{(\epsilon - d)^2}\frac{2(\epsilon - d)^2}{\epsilon}$$

Finally, from the above computation of the potential and gradient terms we get:

$$J_\epsilon(u) \approx \epsilon L(\partial E).$$

□

This proof shows that minimizing the length of the interface will provide us with a better minimization of J_ϵ and since it is of order of ϵ, then $\bar{u}$ cannot be a global minimizer.

It has been shown in [1] that on a path connecting the maps $\bar{u}$ and global minimizer u_m the functional J_ϵ has a local maximum which has to be passed in order to reach the global minimizer u_m. This local maximum corresponds to a phenomenon called nucleation (u_n) which is the appearance of small regions in Ω where u=±1. As time passes, these shapes grow and merge with each other, so they become a connected area where only one component of the alloy exists. So it is natural to study the geometry of the boundary of this area: The interface. This will describe the solution of (1).

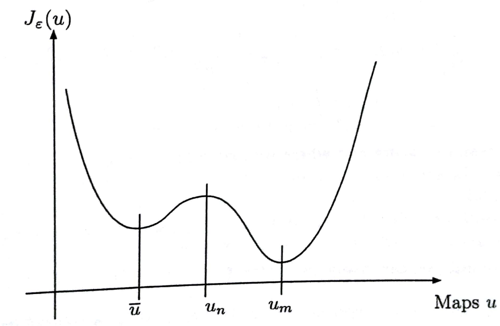

Figure 9: Nucleation and shape of the functional J_ϵ

2.3.3 Curvature of the Interface

Let's consider a minimizer of J_ϵ with a volume constraint. We will show that the interface has constant mean curvature. In order to minimize the interface with the volume constraint, we will have to introduce Lagrange multipliers.

Let X be the surface parametrized by

$$\begin{gathered} \text{X: D} \to \mathbb{R}^3 \\ (u,\nu) \longmapsto X(u,\nu) \end{gathered}$$

Let X^t be the surface parametrized by

$$\begin{gathered} X^t\text{: D} \to \mathbb{R}^3 \\ (u,\nu) \longmapsto \phi(u,\nu,t) \end{gathered}$$

where $\phi(u,\nu,t) = X(u,\nu) + th(u,\nu)N(u,\nu)$, N is the normal at X and h: X $\to \mathbb{R}$.

Let S(X) be an area of X, E, F, G be the coefficients of the first fundamental form, and e, f, g be the coefficients of the second fundamental form.

$$\begin{aligned} S(X^t) &= \int_D \sqrt{E_t G_t - F_t^2} du d\nu \\ &= \int_D \sqrt{EG - F^2 + th(u,\nu)(Eg + Ge - 2Ff)} + \mathcal{O}(t^2) \\ &= S(X) + t \int_D \frac{1}{2} \frac{1}{\sqrt{EG - F^2}} h[Eg + Ge - 2Ff] \\ &= S(X) + t \int_X hH dS \end{aligned}$$

where H is the mean curvature of the surface.

For the gradient we have:

$$\langle S', h\rangle = \int_X hHdS$$

Concerning the volume V we have:

$$\langle V', h\rangle = \int_X hdS$$

Thus, minimizing the functional S with a volume constraint leads to the existence of a Lagrange multiplier Λ:

$$\langle S', h\rangle + \Lambda\langle V', h\rangle = 0 \Rightarrow H = -\Lambda$$

Therefore, the system will evolve into a system of constant mean curvature.

Summary of the possible evolution of the solution to the Cahn-Hilliard equation:

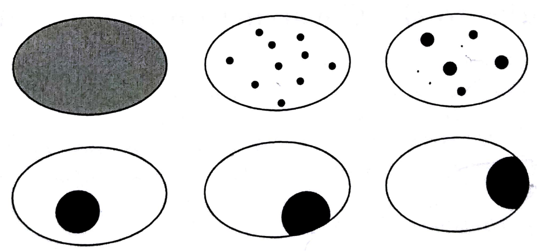

Figure 10: Evolution of a solution to Cahn-Hilliard equation-Nucleation

As we see in the figure above, we start with a uniform mixture of the two phases. As time passes by, we see nuclei appearing (phase separation), where, e.g., black correspond to u=-1 and white to u=1. Then, nuclei merge to a circle because the distribution tends to get a CMC interface with the smallest length. Subsequently, in order to minimize the length it will slowly move to $\partial\Omega$. Then it moves to the region of $\partial\Omega$ with the greatest curvature and stays there.

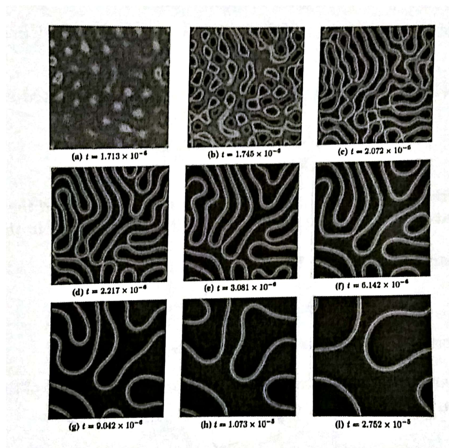

Figure 11: Evolution of a solution to Cahn-Hilliard equation-Spinodal Decomposition

As an alternative evolution of the solution, there is the spinodal decomposition. It depends on which area of the 1st figure we are in. The process is illustrated in the above figure. After the final image, the process is called "coarsening" and it has the same evolution as the one after nucleation.

3 Preliminaries

3.1 Delaunay Surfaces

Definition 2. *Delaunay surfaces are surfaces of revolution with constant mean curvature (CMC).*

Remark 3. Delaunay surfaces can be classified into two different types:

(i) Embedded Delaunay surfaces: The unduloids, D_τ: They interpolate between the cylinder and an infinite string of spheres arranged along a common axis. They are constructed as follows:

We take an ellipse and let it roll in a straight line. As it rolls the one focus creates a curve. We then rotate this curve around the horizontal axis. This surface of revolution is the unduloid.

(ii)Immersed Delaunay surfaces: The nodoids.

Here, we will deal only with the first type.

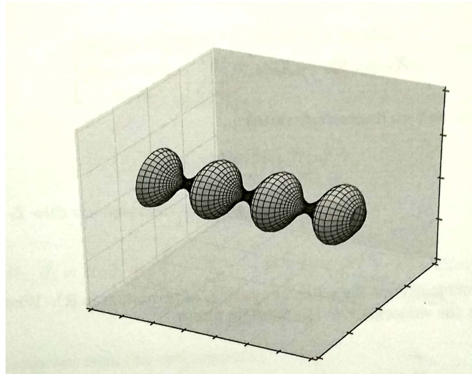

Figure 12: Delaunay Unduloid

3.1.1 Embedded Delaunay Surfaces: The Unduloids.

As we mentioned above, unduloids are surfaces of revolution, so we will use cylindrical coordinates. They are given by the parametrization:

$$x(t,\theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t)$$

where t is a linear coordinate along the axis of rotation, θ is the angular variable around it, and $\rho(t)$ solves

$$\rho_{tt} - \frac{1}{\rho}(1+\rho_t^2) + (1+\rho_t^2)^{\frac{3}{2}} = 0 \tag{2}$$

Proposition 6. *$H \equiv 1$ where H is the mean curvature $\Rightarrow$ equation (2).*

Proof. The unit normal N of D_τ at $x(t,\theta)$ is:

$$N(t,\theta) = \frac{\partial_t \times \partial_\theta}{\parallel \partial_t \times \partial_\theta \parallel} \tag{3}$$

We have that

$$\partial_t(x) \times \partial_\theta(x) = \rho(-\cos\theta, -\sin\theta, \rho_t)$$

and

$$\parallel \partial_t \times \partial_\theta \parallel = \rho\sqrt{1+\rho_t^2}$$

Thus, we replace in (3) and we get that the unit normal N is the following:

$$N(t,\theta) = \frac{1}{\sqrt{1+\rho_t^2}}(-\cos\theta, -\sin\theta, \rho_t)$$

Then the first fundamental form (in the notation of the metric tensor) is:

$$g = Edt^2 + 2Fdtd\theta + Gd\theta^2$$

where

$$\begin{aligned}
E &= X_t X_t = \rho_t^2 \cos^2\theta + \rho_t^2 \sin^2\theta + 1 = \rho_t^2 + 1 \\
F &= X_t X_\theta = 0 \\
G &= X_\theta X_\theta = \rho^2 \sin^2\theta + \rho^2 \cos^2\theta + 0 = \rho^2.
\end{aligned}$$

Therefore, $g = (1+\rho_t^2)dt^2 + \rho^2 d\theta^2$

The second fundamental form is:

$$B = Ldt^2 + 2Mdtd\theta + Nd\theta^2$$

where

$$\begin{aligned}
L &= \partial_{tt} N = \frac{\rho_{tt}}{\sqrt{1+\rho_t^2}}(-\cos^2\theta, -\sin^2\theta, 0) \\
M &= \partial_{t\theta} N = 0 \\
N &= \partial_{\theta\theta} N = \frac{\rho}{\sqrt{1+\rho_t^2}}(\cos^2\theta, \sin^2\theta, 0)
\end{aligned}$$

Thus,

$$B = -\frac{\rho_{tt}}{\sqrt{1+\rho_t^2}}dt^2 + \frac{\rho}{\sqrt{1+\rho_t^2}}d\theta^2$$

So the mean curvature is

$$H = \frac{L+N}{E} = -\rho_{tt}(1+\rho_t^2)^{-\frac{3}{2}} + \rho^{-1}(1+\rho_t^2)^{-\frac{1}{2}}$$

Thus, the condition $H \equiv 1$ yields

$$\rho_{tt} - \frac{1}{\rho}(1+\rho_t^2) + (1+\rho_t^2)^{\frac{3}{2}} = 0$$

which is the desired equation (2). □

However, here we will mostly use a new parametrization which simplifies the study of the solution of (2):

$$X_\tau(s,\theta) = (\tau e^{\sigma(s)}\cos\theta, \tau e^{\sigma(s)}\sin\theta, k(s))$$

Proposition 7. *The functions σ and k respectively satisfy:*

$$\begin{aligned}
\sigma_s^2 + \tau^2 cosh^2\sigma &= 1 \\
k_s &= \frac{\tau^2}{2}(1+e^{2\sigma})
\end{aligned}$$

3.1.2 Jacobi Fields

Definition 3. *The Jacobi operator, $\mathcal{L}$ is the linearization of $\mathcal{N}$, where $\mathcal{N}$ is the nonlinear mean curvature operator. It is given by*

$$\mathcal{L} = \Delta_{D_\tau} + |A_\tau|^2.$$

where Δ_{D_τ} is the Laplace-Beltrami operator on D_τ and $|A_\tau|^2$ is the square of the norm of the second fundamental form of D_τ .

Proposition 8. *For the Delaunay unduloid D_τ, the Jacobi operator is given by the following expression (in the (s, θ) isothermal coordinate system that we introduced earlier):*

$$\mathcal{L}_\tau = \frac{1}{\tau^2 e^{2\sigma}}(\partial_s^2 + \partial_\theta^2 + \tau^2 \cosh(2\sigma)) \tag{4}$$

Proof. We know that the following are true:

$$\Delta_{D_\tau} = -\frac{1}{\sqrt{detg}}\partial_t(\sqrt{detg}g\partial_\theta)$$

$$|A_\tau|^2 = K_1^2 + K_2^2$$

where K_1, K_2 are the principle curvatures.

But $K_1^2 = 2H^2 - K + 2H\sqrt{H^2 - K}$ and $K_2^2 = 2H^2 - K - 2H\sqrt{H^2 - K}$.

Thus, $|A_\tau|^2 = K_1^2 + K_2^2 = 4H^2 - 2K$.

From the computations in Section 2.1.1, we get the following expression for the Jacobi operator:

$$\mathcal{L}_\tau = \frac{1}{\rho\sqrt{1+\rho_t^2}}\partial_t\left(\frac{\rho}{\sqrt{1+\rho_t^2}}\partial_t\right) + \frac{1}{\rho^2}\partial_\theta^2 + \frac{\rho^2\rho_{tt}^2 + (1+\rho_t^2)^2}{\rho^2(1+\rho_t^2)^3} \tag{5}$$

The above equation becomes much simpler in the (s, θ) coordinate system.

A brief calculation shows that

$$1 + \rho_t^2 = \frac{1}{1 - \sigma_s^2}$$

and

$$\rho_{tt} = \frac{\sigma_{ss}}{\tau e^\sigma (1 - \sigma_s^2)^2}$$

Combining these two with Proposition 7, and replacing in (5) we get equation (4). □

After removing the factor $(\tau^2 e^{2\sigma})^{-1}$, it will be sufficient to study the operator

$$L_\tau w = \partial_s^2 w + \partial_\theta^2 w + \tau^2 \cosh(2\sigma) w$$

where w is a function on D_τ.

$$L_\tau w = 0 \Leftrightarrow$$
$$\partial_s^2 w + \partial_\theta^2 w + \tau^2 \cosh(2\sigma) w = 0.$$

Separation of variables: We set $w = S(s)\Theta(\theta)$ and substitute to the above equation. Then,

$$S''\Theta + S\Theta'' + \tau^2 \cosh(2\sigma(s)) S\Theta = 0$$

We devide by $S\Theta$

$$\frac{S''}{S} + \frac{\Theta''}{\Theta} + \tau^2 \cosh(2\sigma(s)) = 0 \Leftrightarrow$$
$$\frac{S''}{S} + \tau^2 \cosh(2\sigma(s)) = -\frac{\Theta''}{\Theta} = j^2.$$

Therefore,

$$\partial_s^2 + \tau^2 \cosh(2\sigma(s)) - j^2 = L_{\tau,j}, j \in \mathbb{Z}.$$

Definition 4. *A Jacobi field is a normal vector field of the form ϕN, where ϕ is a smooth map defined on the surface, and N is the unit normal such that:*

$$\mathcal{L}\phi = 0$$

Proposition 9. *The Jacobi fields on D_τ, which are elements of the kernel of L_τ, are of three types: (i) Jacobi fields arising from infinitesimal translations.*

$$\Phi_\tau^{T,e_j} = \Phi_\tau^{T,e_j}(s,\theta), j = 1, ..., d.$$

(ii) Jacobi fields arising from infinitesimal rotations.
There are $d-1$ Jacobi fields arising from from infinitesimal rotations of the axis od D_τ. We will denote them by

$$\Phi_\tau^{R,e_j}, j = 1, ..., d-1.$$

(iii) Jacobi field associated with the variation of the Delaunay parameter.

$$\Phi_\tau^D = -\partial_\tau X_\tau N_\tau.$$

Remark 4. We know that these are all Jacobi fields with at most linear growth from [14], where all the Jacobi fields of the Delaunay surface have been computed.

3.2 Fermi Coordinates

Let Σ be an embedded CMC surface in $\mathbb{R}^d$ and let H_Σ denote its mean curvature. By N we will denote its unit normal. We will assume that there exists a tubular neighbourhood $\mathcal{N}_\delta$ of Σ of width 2δ in which we can introduce a local system of coordinates (Fermi coordinates)

Definition 5. *Fermi Coordinates (y, z)*

$$
\begin{gathered}
Y : \mathcal{N} \to \Sigma \times (-\delta, \delta) \\
x \mapsto (y, z) \text{ with } x = y + zN(y).
\end{gathered}
$$

Y is a diffeomorphism whenever δ is taken sufficiently small.

Remark 5. In the sequel we will use the inverse of Y:

$$
\begin{gathered}
Y^{-1} : \Sigma \times (-\delta, \delta) \to \mathcal{N}_\delta \\
(y, z) \mapsto x
\end{gathered}
$$

Definition 6. *Pullback of a function in Fermi coordinates*
Given a function $w : \mathcal{N}_\delta \to \mathbb{R}^d$ we define its pullback Y^*w to $\Sigma \times (-\delta, \delta)$ as:

$$
Y^*w(y, z) = w \circ Y^{-1}(y, z)
$$

Definition 7. *Shifted Fermi coordinates (y, t)*
Let $h : \Sigma \to \mathbb{R}$ be a given smooth function such that the map Y_h be a diffeomorphism.

$$
\begin{gathered}
Y_h : \mathcal{N}_\delta \to \Sigma \times (-\delta, \delta) \\
x \mapsto (y, t) \text{ with } x = y + (t + h(y))N(y).
\end{gathered}
$$

Definition 8. *Pullback of a function in shifted Fermi coordinates.*
Given a function $w : \mathcal{N}_\delta \to \mathbb{R}^d$ we define its pullback Y_h^*w as:

$$
Y_h^*w(y, t) = w \circ Y_h^{-1}(y, t)
$$

Definition 9. *Stretched shifted Fermi coordinates $(\bar{t}, \bar{y})$.*
Let $h : \Sigma \to \mathbb{R}$ be a given smooth function such that the map $Y_{\epsilon,h}$ be a diffeomorphism.

$$
\begin{gathered}
Y_{\epsilon,h} : \mathcal{N}_\delta \to \Sigma \times (-\tfrac{\delta}{\epsilon}, \tfrac{\delta}{\epsilon}) \\
x \mapsto (\bar{t}, \bar{y}) \text{ where } \bar{t} = \tfrac{t}{\epsilon} \text{ and } \bar{y} = y.
\end{gathered}
$$

Definition 10. *Pullback of a function in stretched shifted Fermi coordinates.*
Given a function $w : \mathcal{N}_\delta \to \mathbb{R}^d$ we define its pullback $Y_{\epsilon,h}^*$ as:

$$
Y_{\epsilon,h}^*w(\bar{y}, \bar{t}) = w \circ Y_{\epsilon,h}^{-1}(\bar{y}, \bar{t})
$$

Proposition 10. *Expressions for the Laplacian in Fermi, shifted Fermi and stretched shifted Fermi coordinates:*
(i) Laplacian in Fermi coordinates.

$$
\Delta = \Delta_\Sigma + \partial_z^2 - (H_\Sigma + z|A_\Sigma|^2)\partial_z + z\mathbb{B}_{\Sigma,z} + z^2\mathbb{Q}_{\Sigma,z}
$$

where $\Sigma_z = \Sigma + zN$ = the original CMC surface shifted in the direction of the normal by z, $z\mathbb{B}_{\Sigma,z} = \Delta_{\Sigma_z} - \Delta_\Sigma$, the operator $\mathbb{B}_{\Sigma,z}$ is a second order differential operator and

$$
\mathbb{Q}_\Sigma(y, z) = z^2 \sum_{j=1}^{d-1} \mathbb{K}_j^3 + z^3 \sum_{j=1}^{d-1} \mathbb{K}_j^4 + \dots
$$

$\mathbb{K}_j$ *denote the principal curvatures of* Σ.

(ii) Laplacian in shifted Fermi coordinates.

$$\Delta = \Delta_\Sigma + (1+|\nabla_\Sigma h|^2)\partial_t^2 - (H_\Sigma + \Delta_\Sigma h + (t+h)|A_\Sigma|^2)\partial_t + (t+h)\mathbb{B}_{\Sigma,t+h} + (t+h)^2\mathbb{Q}_{\Sigma,t+h}$$

where $|A_\Sigma|^2$ *is the square of the norm of the second fundamental form of* Σ.

(iii) Laplacian in stretched shifted Fermi coordinates.

$$\Delta = \Delta_\Sigma + \epsilon^{-2}(1+|\nabla_\Sigma h|^2)\partial_{\bar{t}}^2 - \epsilon^{-1}(H_\Sigma + \Delta_\Sigma h + (\epsilon\bar{t}+h)|A_\Sigma|^2)\partial_{\bar{t}} + (\epsilon\bar{t}+h)\mathbb{B}_{\Sigma,\epsilon\bar{t}+h} + (\epsilon\bar{t}+h)^2\mathbb{Q}_{\Sigma,\epsilon\bar{t}+h}$$

4 A Periodic Solution

We considered the problem (1) on an open set Ω, and we saw that is interesting to find stationary solutions (which are minimizers for J_ϵ) to this equation. We have found that these solutions are constant and equal to ± 1 in subsets of Ω.

Stationary solutions of (1) satisfy the Euler-Lagrange equation

$$\begin{cases} \epsilon^2 \Delta u - W_u(u) = \delta_\epsilon \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \\ \frac{1}{|\Omega|}\int_\Omega u dx = m \text{ on } \partial\Omega \end{cases} \tag{6}$$

where δ_ϵ is a Lagrange multiplier.

Scaling variables $x \mapsto \frac{x}{\epsilon}$ and letting ϵ tend to 0 leads us to the following problem:

$$\Delta u - W_u(u) = \delta_\epsilon \tag{7}$$

Since ϵ is assumed to be a small parameter, we can consider this problem on the whole space $\mathbb{R}^d$. Thus, from now on, we will consider the problems on $\mathbb{R}^d$.

Dilating of the independent variable by a large factor ϵ^{-1} >0

$$x \mapsto \epsilon^{-1} x$$

we obtain an equivalent form of (7):

$$\epsilon \Delta u - \frac{1}{\epsilon} W_u(u) = l_\epsilon \tag{8}$$

where, $l_\epsilon = \frac{\delta_\epsilon}{\epsilon}$.

Remark 6. We do this rescaling having in mind the Γ-convergence. In particular, the Modica-Mortola theorem asserts that:

$$\tfrac{1}{\epsilon} J_\epsilon \to \int \sqrt{W(u)} Per_\Omega(A) \text{ as } \epsilon \to 0.$$

Moreover, with this convergence, for a sequence of minimizers of $\frac{1}{\epsilon}J_\epsilon$, the limit is a minimizer of the limit operator and thus we have a solution to our problem.

Theorem 1. *For each CMC surface D_τ and for sufficiently small ϵ the equation (8)*

$$\epsilon \Delta u - \frac{1}{\epsilon} W_u(u) = l_\epsilon$$

has a solution u_ϵ in $\mathbb{R}^3$, which is rotationally symmetric and periodic in the direction of the axis of D_τ.
As $\epsilon \to 0$ we have:

$$l_\epsilon = C + \mathcal{O}(\epsilon)$$

where C is a constant, and also, u_ϵ satisfies uniformly over compacts

$$u_\epsilon \to 1 \ \textit{as} \ \epsilon \to 0 \ \textit{in} \ \Omega_\tau^+$$
$$u_\epsilon \to -1 \ \textit{as} \ \epsilon \to 0 \ \textit{in} \ \Omega_\tau^-$$

where $\Omega_\tau^\pm$ *are the two disjoint components in which the* D_τ *divides the space.*

4.1 The Solution in One Dimension

The homogeneous equation (7) in one dimension is:

$$u'' - W_u(u) = 0 \tag{9}$$

Proposition 11. *The equation (9) has a unique odd increasing solution* Θ *such that:*

$$\lim_{t \to +\infty} \Theta(t) = 1$$
$$\lim_{t \to -\infty} \Theta(t) = -1$$

This solution is called the heteroclinic solution.

Proof. We rewrite our equation in the form of a system:

$$\begin{cases} u_1' = u_2 \\ u_2' = W_u(u_1) \end{cases}$$

By definition, we have that the level curves are $E(u_1, u_2) = \frac{1}{2}u_2^2 - W(u_1)$

In order to draw the phase portrait we think of a ball starting on the left maximum of the curve $-W$. The ball will reach the maximum on the right and then it will return back from where it starded. So we get the following:

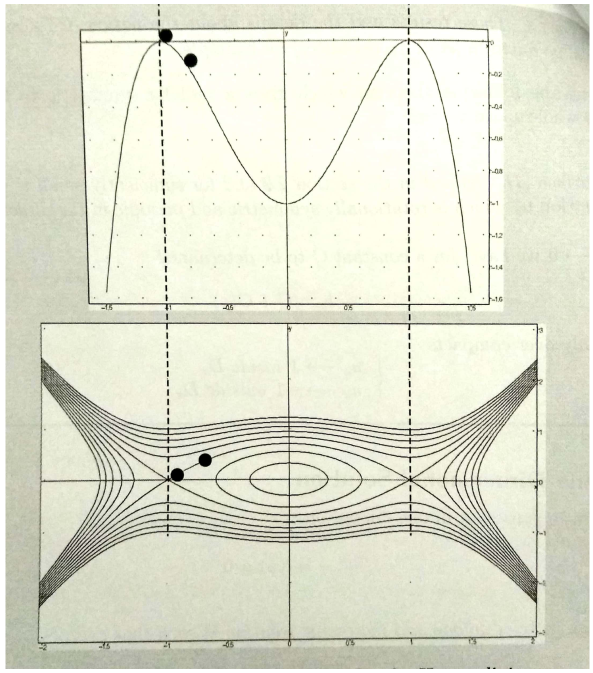

Figure 13: The curve $-W$ and the phase plane for the heteroclinic solution

□

4.2 The Solution Near D_τ

In this subsection we will focus on solving equation (8). Having in mind subsection 2.2 we will express the solution of (8),w, in stretched and shifted Fermi coordinates $(y, \bar{t})$ in the following way

$$Y^*_{\epsilon,h} w(y, \bar{t}) = U(\bar{t}) + \epsilon^2 \psi_0(y, \bar{t}) \tag{10}$$

for some functions U and ψ_0 which will be determined. Moreover, we assume that $h : D_\tau \to \mathbb{R}$ is

$$h = \epsilon^2 h_0$$

where h_0 is a constant and we will choose it later.
Now, the error in equation (8) in the tubular neighbourhood $\mathcal{N}_\delta$ of D_τ is of the form:

$$N_\epsilon(w) - l_\epsilon = \epsilon \Delta w + \frac{1}{\epsilon}(1 - w^2)w - l_\epsilon$$

where, $N_\epsilon(w) := \epsilon \Delta w - \frac{1}{\epsilon} W'(w)$

Thus,

$$\epsilon \Delta w + \frac{1}{\epsilon}(1-w^2)w - l_\epsilon = \epsilon^{-1}\partial_{tt}U - H_{D_\tau}\partial_t U - \epsilon^{-1}W_U(U) - l_\epsilon \quad (11)$$

$$+\epsilon\partial_{tt}\psi_0 - \epsilon^2 H_{D_\tau}\partial_t\psi_0 - \epsilon W_{UU}(U)\psi_0 - \epsilon(t+\epsilon h_0)|A_{D_\tau}|^2\partial_t U \quad (12)$$

$$-\frac{1}{\epsilon}W_w(w) + \frac{1}{\epsilon}W_U(U) + \epsilon W_{UU}(U)\psi_0$$

$$+\epsilon^3\Delta_{D_\tau}\psi_0 - \epsilon^3(t+\epsilon h_0)|A_{D_\tau}|^2\partial_t\psi_0 + \epsilon(\epsilon t + h)\mathbb{B}_{D_\tau,\epsilon t+h}(w) + \epsilon(\epsilon t+h)^2\mathbb{Q}_{D_\tau,\epsilon t+h}(w)$$

We will simplify this equation. We will consider separately the different orders in ϵ.
We first have the equation (11):

$$U'' - \epsilon H_{D_\tau}U' - W_U(U) = \epsilon l_\epsilon \quad (13)$$

Then we have the equation (12):

$$\partial_{\bar{t}\bar{t}}\psi_0 - \epsilon H_{D_\tau}\partial_{\bar{t}}\psi_0 + (1-3U^2)\psi_0 = (\bar{t}+\epsilon h_0)|A_{D_\tau}|^2\partial_{\bar{t}}U \quad (14)$$

Remark 7. The other terms in the error are of order ϵ^3. In order to see this we have to notice that the 3rd line has a Taylor expansion form, and the for the last line $\mathbb{B}(U) = 0$ since U only depends on t.

Then, finding solutions to (13) and (14) will give us a w for which the error in equation (8) is of order ϵ^3.

We have to determine U and ψ_0 in order to find w. Also we need to choose h_0 and to find a suitable lagrange multiplier l_ϵ.

4.2.1 Construction of a solution U for equation (13)

$$U'' - cU' - W_U(U) + \lambda = 0, \quad (15)$$

where $W(U) = \frac{1}{4}(U^2-1)^2$ and $W_U(U) = U^3 - U$.

We rewrite our equation as it follows

$$\begin{cases} U_\lambda'' - (W_U(U_\lambda) - \lambda) = cU_\lambda' \\ \\ \lim_{\eta\to\infty} U(\eta) = \alpha_\lambda, \lim_{\eta\to\infty} U(\eta) = \beta_\lambda \end{cases} \quad (16)$$

where, $U = \alpha_\lambda > -1$, $U = \beta_\lambda > 1$ and $W_U(\alpha_\lambda) = W_U(\beta_\lambda) = \lambda$.

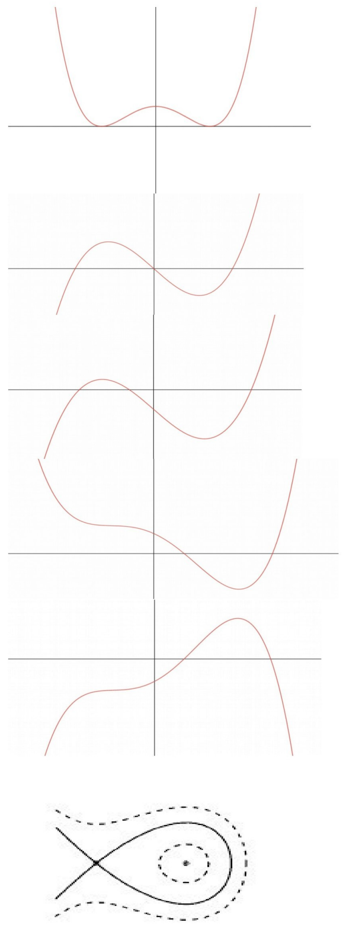

Figure 14: Diagrams(W,W_u,$W_u - \lambda$,$W_u - \lambda u$,$-W_u + \lambda u$)-Phase portrait for $c = 0$

Theorem 2. *For* $0 < \lambda < \lambda_0 := \frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)$, $\exists c = c(\lambda)$*,* $U = U_\lambda(\eta)$ *that is a solution of (15).*

Moreover,
(i) $c(\lambda) \to 0$ *as* $\lambda \to 0^+$
(ii) $c'(0) \neq 0$

Proof. We write equation (16) in system form:

$$\begin{cases} x' = y \\ \\ y' = cy + W_U(x) - \lambda \end{cases} \tag{17}$$

As we see, for the equilibrium points we have that y=0 and $W_U(x) = \lambda$.They also are independent of c. So let $x = \alpha_\lambda, \beta_\lambda \, and \, \gamma_\lambda$ be the values such as $W_U(\alpha_\lambda) = W_U(\beta_\lambda) = W_U(\gamma_\lambda) = \lambda$.
So now we take the linearization of the system (17) at the equilibrium point $(\alpha_\lambda, 0)$:

$$\begin{cases} x' = y \\ \\ y' = cy + W_{UU}(\alpha_\lambda)x \end{cases} \tag{18}$$

The corresponding eigenvalues are:

$$\mu^2 - \mu c - W_{UU}(\alpha_\lambda) = 0 \Rightarrow \mu^{\pm} = \frac{c \pm \sqrt{c^2 + 4W_{UU}(\alpha_\lambda)}}{2}$$

So the points $(\alpha_\lambda, 0)$ and $(\beta_\lambda, 0)$ are both saddles.
The corresponding eigenvectors are:

$$r^{\pm} = \begin{pmatrix} 1 \\ \mu^{\pm} \end{pmatrix}$$

The existence of the stable and unstable manifolds of the equilibrium point is obtained by the following proposition.

Proposition 12. *Let the system (17) with $\mu^- < 0 < \mu^+$. Then there exists $\delta > 0$ such that in the region $V = \{(x,y)/|x|, |y| < \delta \}$ the W^s, W^u are non-empty sets.*

Combining the above information, eigenvectors are tangent to the manifolds at the equilibrium points.
Therefore, each manifold has the same slope as the corresponding eigenvector.

Step 1: Existence of the desired solution

We begin with a shooting argument. We fix $\lambda \in (0, \lambda_0)$ and a parameter c.

Case (i): $0 < c \ll 1$.

Let $U(\eta; \lambda, c)$ a solution of (16).
The level curves are given from the expression:

$$E_\lambda(x,y) = \frac{1}{2}y^2 - (W(x) - \lambda) = \mu$$

where μ is a constant
We take

$$\frac{d}{d\eta}E_\lambda(U, U') = U'U'' - (W_U(U) - \lambda)U' = U'[U'' - (W_U(U) - \lambda)] = cU'^2 > 0$$

So the solution moves through increasing values of the level curves.

Because of continuous dependence on the parameter c, the solution for $c << 1$ will nearly follow the trajectory of the homoclinic solution. In particular, we get that there exists T >0 such that

(U,U') hit the U-axis at a point $(\xi(c), 0)$ with $\xi(c)$ <1 <β_λ.

Case (ii): $c \gg 1$.

Lemma 1. *For $c \gg 1$ the solution $(U(\eta;\lambda,c), U'(\eta;\lambda,c))$ with $(U(0;\lambda,c), U'(0;\lambda,c))$ on the unstable manifold of (16) at $(\alpha_\lambda, 0)$ intersects the unstable manifold of the corresponding system for c=0, from the $(\beta_\lambda, 0)$.*

Proof of Lemma 1. We take

$$\frac{dy}{dx} = \frac{cy + W_U(x) - \lambda}{y} = c + \frac{W_U(x) - \lambda}{y} \geqslant c, 0 \leqslant x \leqslant \gamma_\lambda.$$

The term $W_U(x) - \lambda > 0$ as we see in Figure 14.
Therefore,

$$\int_0^{\gamma_\lambda} \frac{dy}{dx} \geqslant c\gamma_\lambda \Rightarrow$$
$$y(\gamma_\lambda) - y(0) \geqslant c\gamma_\lambda, y(0) > 0.$$

Consequently, $y(\gamma_\lambda) \geqslant c\gamma_\lambda$

So, for c $\gg 1$, our solution intersects the desired unstable manifold before $x = \gamma_\lambda$. $\square$

Remark 8.
Since $\frac{d}{d\eta}E_\lambda(U, U')$ >0, the solution $(U(\eta;\lambda,c), U'(\eta;\lambda,c))$ cannot hit the unstable manifold of the system for c=0 again.
Also, $x' = y$ >0. So, the solution for $c \gg 1$ does not intersect the x-axis at the interval $[0, \beta_\lambda]$.

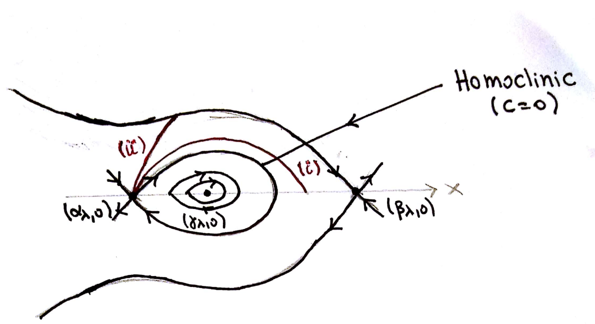

Figure 15: Phase Portrait for $c << 1$ and $c >> 1$ in comparison with the one for $c = 0$

Now, let Σ = $\{c$>0 / The solution of (16) intersects the x-axis in finite time at the interval $(\kappa, \beta_\lambda)\}$, where k is the point where the homoclinic orbit intersects x-axis.

Proposition 13.

1. $\Sigma \neq \emptyset$
2. Σ: *Bounded*
3. $c^* \notin \Sigma$

Proof of Proposition 13.
1. True because of case (a).

2. True because of case (b) and the remark. Therefore, there exists c*:= sup Σ.

3. Let $c^* \in \Sigma$.
Hence, $\exists$ (x,y) solution of (16) such that for finite time τ

$$(x(\tau), y(\tau)) = (\xi, 0)$$

and

$$y'(\tau) = W_U(x(\tau)) - \lambda < 0$$

where k<x<β_λ.
Consequently it intersects transversally the x-axis and due to continuous dependence on the parameter c, $\exists\epsilon$ >0 such that c*+$\epsilon \in \Sigma$. This is a contradiction, since c* is the supremum of the set Σ.
Therefore, the proof is complete. $\square$

Now, for n $\in \mathbb{N}$ we define a sequence $c_n \in \Sigma$ such that $c_n \to$ c* as $n \to +\infty$.
Also, we define the sequence of times in which the solution intersects the x-axis, $\{T_n\}$, $n \to +\infty$.

Proposition 14.

1. $T_n \to +\infty$
2. $(x(T_n; c*), y(T_n; c*)) \to (\beta_\lambda, 0)$ *as* $T_n \to +\infty$.

Proof of Proposition 14.
1. Let $T_n \to \tau$, as $n \to +\infty$. Hence, $(T_n, c_n) \to (\tau, c*)$ and due to continuity of the solution we get: $x(T_n, c_n) \to x(\tau, c*)$ and $y(T_n, c_n) \to y(\tau, c*)$.
Since τ is a finite time of the intersection of the solution with x-axis we have that: x(τ,c*) <β_λ and y(τ,c*)=0.
But this means that c* $\in \Sigma$, and that is a contradiction.

2. From the Poincaré - Bendixson Theorem, we know that in infinite time, in the closed and bounded domain R, where R is the domain that consists of the $x > 0$ and $y > 0$ axis and the level curve $\frac{1}{2}y^2 + W(x) = W(\beta_\lambda)$, then our trajectory will either be a limit cycle, or it will approach a limit cycle spirally, or it will end up on an equilibrium point.

But due to the direction of the vector field the two first options are rejected. So as $T_n \to +\infty$ our solution goes to the only equilibrium point in the first quadrant, $(\beta_\lambda, 0)$. $\square$

Step 2: Proof of properties (i) and (ii) of the Theorem 2.

(i) We multiply equation (16) with U'_λ and we get

$$U_\lambda" U'_\lambda - W_U(U_\lambda)U'_\lambda = c(U'_\lambda)^2 - \lambda U'_\lambda.$$

We now take the derivative of the energy functional and integrate on $\mathbb{R}$:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d}{d\eta}(\frac{1}{2}(U'_\lambda)^2 - W_U(U_\lambda))d\eta = c\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta - \lambda(\beta_\lambda - \alpha_\lambda) \Rightarrow \\ W(\beta_\lambda) - W(\alpha_\lambda) = \lambda(\beta_\lambda - \alpha_\lambda) - c\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta \Rightarrow \\ W(\beta_\lambda) - \lambda\beta_\lambda = -c(\lambda)\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta \end{aligned} \tag{19}$$

For $\lambda \to 0^+$ we have that

$$\begin{aligned}&W(\beta_\lambda) = W(\beta_0) = 0\\ &\lambda(\beta_\lambda) = 0\\ &\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta \to \int_{-\infty}^{+\infty}(U'_0)^2 d\eta \neq 0\end{aligned}$$

Hence, from (19) we get that $c(\lambda) \to 0$.

(ii) From (19) we have $c(0) = 0$. We take the derivative of (19)

$$W_U(\beta_\lambda)\frac{d\beta_\lambda}{d\lambda} - \Big(\beta_\lambda + \lambda\frac{d\beta_\lambda}{d\lambda}\Big) = -c'(\lambda)\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta - c(\lambda)\frac{d}{d\lambda}\int_{-\infty}^{+\infty}(U'_\lambda)^2 d\eta \tag{20}$$

We substitute with $\lambda = 0$ in the equation (20):

$$\begin{aligned}c'(0) = \frac{1}{\int_{-\infty}^{+\infty}(U'_0)^2 d\eta} \neq 0 \Leftrightarrow \\ \lim_{\lambda\to 0}\frac{c(\lambda)}{\lambda} \neq 0.\end{aligned}$$

$\square$

Remark 9. The terms $\epsilon, \epsilon l_\epsilon$ of the equation from [9] can be replaced by the terms c, λ respectively.

Proof. We write $c(\lambda)$ for $\lambda = \epsilon l_\epsilon$ as follows

$$c(\lambda) = \frac{c(\lambda)}{\lambda}\lambda = \frac{c(\epsilon l_\epsilon)}{\epsilon l_\epsilon} = \Big(\frac{c(\epsilon l_\epsilon) l_\epsilon}{\epsilon l_\epsilon}\Big)\epsilon$$

We will prove that $\frac{c(\epsilon l_\epsilon l_\epsilon)}{\epsilon l_\epsilon} = 1$

For l_ϵ we know that $l_\epsilon \to l_0$. So it is in our own convenience how it will be defined.

Thus we set

$$\frac{1}{l_\epsilon} = \frac{c(\epsilon l_\epsilon)}{\epsilon l_\epsilon}$$

$$\lim_{\lambda\to 0}\frac{1}{l_\epsilon} = \frac{1}{l_0} = c'(0) \neq 0$$

Hence our choice is correct and therefore we proved that $c(\lambda) = 1\epsilon = \epsilon$.

□

4.2.2 Construction of a solution ψ_0 for equation (14)

Let us write, for convenience, the equation (14) as follows:

$$\phi'' - \epsilon H_\Sigma \phi' + (1 - 3U^2)\phi = g(y, \bar{t}) \tag{21}$$

We have that $\phi_1 = U'$ is a solution of the homogeneous equation of (21). We can easily confirm that, by multiplying equation (13) with U' and then substitute U' in the equation.

Proposition 15. *The fundamental set of the ODE (21) is spanned by the functions*

$$U' = \mathcal{O}((cosh\bar{t})^{\eta^\pm}) \text{ and } V(\bar{t}) = \mathcal{O}((cosh\bar{t})^{\nu^\pm})$$

where

$$\eta^\pm = \tfrac{1}{2}\Big(\epsilon H_\Sigma - \sqrt{-4k(\pm\infty) + \epsilon^2 H_\Sigma^2}\Big),\ \nu^\pm = \tfrac{1}{2}\Big(\epsilon H_\Sigma + \sqrt{-4k(\pm\infty) + \epsilon^2 H_\Sigma^2}\Big)$$

Proof. We write equation (21) in the form of a system

$$\begin{cases} \phi_1' = \phi_2 \\ \phi_2' = -(1 - 3U^2)\phi_1 + \epsilon H_\Sigma \phi_2 \end{cases}$$

The characteristic polynomial is

$$\eta^2 - \epsilon H_\Sigma \eta + (1 - 3U^2) = 0 \Leftrightarrow$$

$$\eta, \nu = \frac{1}{2}\Big(\epsilon H_\Sigma \pm \sqrt{\epsilon^2 H_\Sigma^2 - 4(1 - 3U^2)}\Big)$$

where $1 - 3U^2 = k$ and $k(\pm\infty) = 1 - 3(\pm 1 + \sigma_\epsilon^\pm)^2$ since $U(\pm\infty) = \pm 1 + \sigma_\epsilon^\pm$.

As $t \to \pm\infty$ we have a homogeneous ODE with constant coefficients, so we find two linearly independent solutions:

$$\phi_1 = c_1 e^{\eta^{\pm} t} = c_1 (e^t)^{\eta^{\pm}}$$

$$\phi_2 = c_2 e^{\nu^{\pm} t} = c_2 (e^t)^{\nu^{\pm}}$$

But, because $t \to \pm\infty$ we substitute e^t with $cosht$ for brevity.

$\square$

By Liouville's formula we have that the Wronskian is:

$$W(t) = W(0) \int_0^t trAds$$

where A is the matrix of the system of the equation (21).
Thus, if we assume that $W(0) = 1$

$$W(t) = e^{\epsilon H_\Sigma t}$$

But it is known that:

$$\phi = \mathcal{G}(g)$$

where

$$\mathcal{G}(\nu)(t,y) = -U'(t) \int_0^t V(s) e^{-\epsilon H_\Sigma s} \nu(s,y) ds + V(t) \int_{-\infty}^t U'(s) e^{-\epsilon H_\Sigma s} \nu(s,y) ds$$

If $g(y,t)$ satisfies the orthogonality condition

$$\int_{\mathbb{R}} g(t,y) U'(t) e^{-\epsilon H_\Sigma t} dt = 0$$

the function $\mathcal{G}(g)$ is exponentially decaying whenever g is exponentially decaying.

Lets assume for instance that

$$|g(y,t)|(\cosh t)^\mu \leqslant C$$

where $\mu \in (\eta + \epsilon H_\Sigma, -\eta]$ and $\eta = max \eta^+, \eta^- < 0$

Then we have

$$|\phi(y,t)|(\cosh t)^\mu \leqslant C$$

Now we will determine h_0 in order to determine next the function ψ_0. To do this, we will choose h_0 such that it satisfies the orthogonally condition mentioned above:

$$\int_{\mathbb{R}} (t + \epsilon h_0)(U'(t))^2 e^{-\epsilon H_\Sigma t} dt = 0$$

Hence,

$$h_0 = \frac{\int_{\mathbb{R}} t (U'(t))^2 e^{-\epsilon H_\Sigma t} dt}{\epsilon \int_{\mathbb{R}} (U'(t))^2 e^{-\epsilon H_\Sigma t} dt}$$

With this choice we define

$$\psi_0(y,t) = \mathcal{G}((t + \epsilon h_0) U') |A_\Sigma|^2.$$

4.3 The Solution in the Whole Space

In the previous subsection we built a special solution near the Delaunay unduloid. Here, we will expand this solution in $\mathbb{R}^d$ for $d > 3$.

4.3.1 The Lyapunov-Schmidt Reduction Method

This method is a way to solve equations of the form

$$Ax = N(x)$$

where A is a linear operator and N a nonlinear. They are both continuous.
In particular, $A : X \to Z$, $N : X \to Z$ where, X, Z are Banach spaces.

Lemma 2. *Let π_{KerA} and π_{ImA}be the continuous projections on the corresponding subspaces. Then there is a continuous linear operator*

$$K : ImA \to Im(I - \pi_{KerA})$$

such that

$$\begin{cases} KA = I - \pi_{KerA} \ on \ X \\ AK = I \ on \ ImA \end{cases}$$

Proposition 16. *The equation $Ax = N(x)$ is equivalent to the following system*

$$\begin{cases} x = y + z \ where \ y \in KerA \ and \ z \in Im(I - \pi KerA) \\ z - K\pi_{ImA}N(y + z) = 0 \\ (I - \pi_{ImA}N(y + z) = 0 \end{cases}$$

The idea of the Lyapunov-Schmidt reduction is to manage to invert the linear operator A, then solve the first equation for a given y using a fixed point argument (this gives a certain $z^*(y)$), and finally solve the following

$$(I - \pi_{ImA})N(y + z^*(y)) = 0$$

Now, let us recall the definition of the nonlinear operator N_ϵ in section 4.2 by:

$$N_\epsilon = \epsilon\Delta u - \frac{1}{\epsilon}W'(u)$$

Equation (8) now takes the form:

$$N_\epsilon(u) = l_\epsilon \tag{22}$$

Remark 10. Since we are interested in finding a solution periodic in the direction of the x_d axis with the minimal period equal to that of D_τ we define the manifold $\mathring{D}_\tau$ by identifying the set $D_\tau \cap \{x_d = 0\}$ with the set $D_\tau \cap \{x_d = 2T_\tau\}$.

We want to linearise N_ϵ around the local solution $w = U + \epsilon^2 \psi_0$. For that, we will first extend w to the whole space and define the map $\mathbb{H}$:

Definition 11.

$$\mathbb{H} = \begin{cases} 1 + \sigma_\epsilon & \text{if } x \in \mathring{D}_\tau^+ \\ -1 + \sigma_\epsilon & \text{if } x \in \mathring{D}_\tau^- \end{cases}$$

where $\mathring{D}_\tau^+$ and $\mathring{D}_\tau^-$ denote the interior and the exterior of $\mathring{D}_\tau$ respectively.

We will also need a cutoff function χ such that:

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \leqslant \frac{1}{2} \\ 0 & \text{if } |s| \geqslant 1 \end{cases}$$

Definition 12. *Let χ^* the cutoff function supported in the tubular neighbourhood of $\mathring{D}_\tau$ such that*

$$Y^*_{\epsilon,h} \chi^*(t) = \chi\left(\frac{\epsilon t}{\delta}\right)$$

We can now define a global approximate solution w^* by:

$$w^*(x) = w(x)\chi^*(x) + \mathbb{H}(x)(1 - \chi(x)) \tag{23}$$

Now, we linearise N_ϵ near w^* and the solution we are searching for has the form:

$$u = w^* + \phi$$

where ϕ is a small, in a way to be specified, function. That is why it seems natural to linearise N_ϵ.

4.3.2 The Linear Operator L_ϵ

In this section we will study the linear operator since we will have to invert it according to Lyapunov-Schmidt reduction.

Definition 13. *The linearisation of N_ϵ near w^* is:*

$$L_\epsilon \phi := \epsilon \Delta \phi - \frac{1}{\epsilon} W''(w^*)\phi$$

Now, we can rewrite equation (8), with $u = w^* + \phi$:

$$L_\epsilon \phi = -N_\epsilon(w^*) - Q_\epsilon(\phi) + l_\epsilon \tag{24}$$

where, $L_\epsilon \phi = DN_\epsilon(w^*)\phi$ and $Q_\epsilon(\phi) = N(w^* + \phi) - N_\epsilon(w^*) - L_\epsilon \phi$.

Definition 14. *The expression of L_ϵ in Fermi coordinates is:*

$$\mathbb{L}_\epsilon := \epsilon\Delta_{\mathring{D}_\tau} - (H_{\mathring{D}_\tau} + \epsilon(\bar{t} + \epsilon h_0)\chi\Big(\frac{\epsilon\bar{t}}{\delta}\Big)|A_{\mathring{D}_\tau}|^2)\partial_{\bar{t}} + \epsilon^{-1}\partial_{\bar{t}}{}^2 - \epsilon^{-1}W''(w^*)$$

Remark 11.

$$Y^*_{\epsilon,h}L_\epsilon \approx \mathbb{L}_\epsilon$$

Proof of Remark 11. The expression for the Laplacian is:

$$\Delta = \Delta_{\mathring{D}_\tau} + \epsilon^{-2}\partial_{tt} - \epsilon^{-1}(H_{\mathring{D}_\tau} + (\epsilon t + h)|A_{\mathring{D}_\tau}|^2)\partial_t + (\epsilon t + h)\mathbb{B}_{\mathring{D}_\tau,\epsilon t+h} + (\epsilon t + h)^2\mathbb{Q}_{\mathring{D}_\tau,\epsilon t+h}$$

Thus,

$$\mathbb{L}_\epsilon - Y^*_{\epsilon,h}L_\epsilon = \epsilon(t + \epsilon h_0)|A_{\mathring{D}_\tau}|^2\Big[1 - \chi\Big(\frac{\epsilon t}{\delta}\Big)\Big]\partial_t + \epsilon^2(t + \epsilon h_0)\mathbb{B} + \epsilon^3(t + \epsilon h_0)\mathbb{Q}$$

From Remark 7 at page 23, we get the desired result. □

Now, we will focus on solving the following linear equation:

$$L_\epsilon\phi = g(x) \Rightarrow, \text{ in } \mathbb{R}^{d-1} \times S_{2T_\tau} \tag{25}$$
$$\epsilon\Delta\phi - \frac{1}{\epsilon}W''(w^*)\phi = g(x)$$

We want to find a solution with a reasonably bounded norm. That happens only if the right hand side of (25) satisfies some extra condition, or equivalently, we need to introduce Lagrange multipliers. Thus, we will solve the following:

$$\epsilon\Delta\phi - \frac{1}{\epsilon}W''(w^*)\phi = g(x) + \chi^* \sum_{j=1}^{d} c_j Z^{T,e_j}_{\tau,\epsilon} \tag{26}$$

where,
c_j are the Lagrange multipliers,
$Y^*_{\epsilon,h}\chi^*(\bar{t}) = \chi\Big(\frac{\epsilon\bar{t}}{\delta}\Big)$,
$Y^*_{\epsilon,h}Z^{T,e_j}_{\tau,\epsilon}(y,\bar{t}) = V(y,\bar{t})\Phi^{T,e_j}_\tau(y)$
$V = \partial_{\bar{t}}w = \partial_{\bar{t}}(U + \epsilon^2\psi_0)$ and
$e_j =$ the coordinate axis.

The idea is to solve (26) by gluing a solution defined near $\mathring{D}_\tau$ and another one defined away from it. To describe this construction we need some preparation.

First, the solution we are searching for, has the form:

$$\phi = \chi^* \bar{\phi} \circ Y_{\epsilon,h} + \psi$$

where $\bar{\phi}$ describes the solution near $\mathring{D}_\tau$ and ψ far from it. To avoid complicated notions we will omit the composition with $Y_{\epsilon,h}$.

We also, introduce the function $q(x)$ as follows:

$$q(x) = \begin{cases} -W''(1+\sigma_\epsilon^+) & \text{if } dist(x, \mathring{D}_\tau) > \delta/2 \\ -W''(-1+\sigma_\epsilon^-) & \text{if } dist(x, \mathring{D}_\tau) < -\delta/2. \end{cases}$$

Finally, we need another cutoff function $\bar{\chi}$ such that:

$$\bar{\chi}\chi^* = \chi^*$$

We can now split our equation into:

$$\begin{cases} \mathbb{L}_\epsilon \bar{\phi} = \bar{\chi}\Big(g + \chi^* \sum_{j=1}^d c_j Z_{\tau,\epsilon}^{T,e_j} + (\mathbb{L}_\epsilon - L_\epsilon)\bar{\phi} - [\chi^*, L_\epsilon]\bar{\phi} + \epsilon^{-1}(q + W''(w^*))\psi\Big) \\ \epsilon\Delta\psi + \epsilon^{-1}[-(1-\chi^*)W''(w^*) + \chi^* q]\psi = (1-\chi^*)\Big(g + \chi^* \sum_{j=1}^d c_j Z_{\tau,\epsilon}^{T,e_j} - [\chi^*, L_\epsilon]\bar{\phi}\Big) \end{cases} \tag{27}$$

A solution to (27) is a solution of (26). Indeed, multiplying the first equation by χ^* , adding the two equations and using the fact that $\bar{\chi}\chi^* = \chi^*$ gives us the solution to our problem. We also notice that the first equation is written in Fermi coordinates on $\mathring{D}_\tau \times \mathbb{R}$ and the second one is defined on $\mathbb{R}^{d-1} \times S_{2T_\tau}$.

We have to invert these linear operators and find estimates of the solution in order to apply a fixed pointed argument later.

Let us introduce a functional space.

Definition 15.

$$\mathcal{X} := \Big\{\phi \in L^2(\mathring{D}_\tau \times \mathbb{R}) / \int_{\mathbb{R}} \phi(y,\bar{t})V(y,\bar{t})d\bar{t} = 0\Big\}$$

We decompose:

$$\phi = \phi^{||} + \phi^{\perp}$$

where

$$\begin{cases} Y_{\epsilon,h}^* \phi^{||} \in \mathcal{X} \\ Y_{\epsilon,h}^* \phi^{\perp} \in \mathcal{X}^{\perp} \end{cases}$$

are the orthogonal projections.

Remark 12. Some explanations about space $\mathcal{X}$:
First, we define:

$$K := \Big\{ Z \in L^2 | \exists \phi, Y^*_{\epsilon,h} Z = \phi(y) V(y,\bar{t}) \Big\}$$

Thus,

$$\begin{aligned} K^\perp &= \Big\{ \phi \in L^2 | \forall \varphi, \int \int_{\mathring{D}_\tau \times \mathbb{R}} \varphi(y) V(y,\bar{t}) \phi(y,\bar{t}) dy d\bar{t} \Big\} \\ &= \Big\{ \phi \in L^2 | \forall \varphi, \int_{\mathring{D}_\tau} \varphi \Big[\int_{\mathbb{R}} V(y,\bar{t}) \phi(y,\bar{t}) \Big] dy \Big\} \\ &= \mathcal{X} \end{aligned}$$

Thus, every function which is orthogonal to $\mathcal{X}$ can be written as φV.
Next, we define:

$$\mathcal{Y} := \Big\{ \phi(y,\bar{t}) = V(y,\bar{t}) Z(y) | \forall j, \int_{\mathring{D}_\tau} Z(y) \Phi^{T,e_j}_\tau (y) dy = 0 \Big\}$$

Using the expression of the Laplacian in Fermi coordinates and the fact that we constructed a good approximation of the solution, we can easily compute:

$$Y^*_{\epsilon,h}(L_\epsilon Z) = \mathcal{O}(\epsilon)$$

From the above, it follows that:

$$\mathcal{X}^\perp \cap (\ker L_\epsilon)^\perp \subset \mathcal{Y}$$

Definition 16 (Weighted Holder Spaces)**.** *The weighted Holder norms on $\mathring{D}_\tau \times \mathbb{R}$:*

$$\begin{aligned} \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} &= \sup_{\bar{t}\in\mathbb{R}} (cosh\bar{t})^\mu \|u\|_{\mathcal{C}^{0,\alpha}(\mathring{D}_\tau \times (\bar{t}-1,\bar{t}+1))} \\ \|u\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} + \|\nabla_{\mathring{D}_\tau \times \mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} \\ \|u\|_{\mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} &= \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} + \|\nabla_{\mathring{D}_\tau \times \mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} + \|\nabla^2_{\mathring{D}_\tau \times \mathbb{R}} u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} \end{aligned} \tag{28}$$

where, $\nabla_{\mathring{D}_\tau \times \mathbb{R}}$ and $\nabla^2_{\mathring{D}_\tau \times \mathbb{R}}$ denote the gradient and the Hessian on the manifold $\mathring{D}_{\tau\times\mathbb{R}}$ respectively.
We also define the norm:

$$\|u\|_{\mathcal{E}^{l,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} = \sum_{0 \leqslant k+m \leqslant l} \epsilon^m \|\partial^k_{\bar{t}} D^m_{\mathring{D}_\tau} u\|_{\mathcal{C}^{0,\alpha}_\mu}$$

Remark 13. With this definition we have for $l = 0$:

$$\|u\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})} = \|u\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau \times \mathbb{R})}$$

Thus, spaces $\mathcal{C}$, $\mathcal{E}$ are basically the same for $l = 0$.

Lemma 3. *Let us consider the following equation:*

$$\mathbb{L}_\epsilon \phi = g(y, \bar{t})$$

With the above notation for the projections on $\mathcal{X}$ and its orthogonal complement, we have the following estimates for ϕ:

$$\left\{ \begin{array}{l} \|\phi^{||}\|_{\mathcal{E}^{l,\alpha}_\mu} \leqslant C\epsilon^{1-\alpha}(\|g^{||}_{\mathcal{E}^{0,\alpha}_\mu} + \epsilon^2 \|g^\perp\|_{\mathcal{E}^{0,\alpha}_\mu}) \\ \|\phi^\perp\|_{\mathcal{C}^{l,\alpha}_\mu} \leqslant C\epsilon^{-1}\|g^\perp\|_{\mathcal{C}^{0,\alpha}_\mu} \end{array} \right. \tag{29}$$

Lemma 4 (a priori estimate for (27b))**.**

$$\|\psi\|_{\mathcal{C}^{l,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \leqslant C\epsilon^{1-l-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + e^{-c\epsilon^{-1/4}}(\sum_{j=1}^{d}|c_j| + \|\bar{\phi}\|_{\mathcal{C}^{1,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})})$$

Lemma 5 (a priori estimate for (27a))**.** *If we set*

$$\mathfrak{g} = \bar{\chi}\Big\{g + \chi^*\sum_{j=1}^{d} c_j Z^{T,e_j}_{\tau,\epsilon} + (\mathbb{L}_\epsilon - L_\epsilon)\bar{\phi} - [\chi^*, L_\epsilon]\bar{\phi} + \epsilon^{-1}[q + W''(w^*)]\psi\Big\}$$

then using Lemma 3 (29a) we have:

$$\|\bar{\phi}^{||}\|_{\mathcal{E}^{l,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} \leqslant C\epsilon^{1-\alpha}(\|\mathfrak{g}^{||}\|_{\mathcal{E}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} + \epsilon^2\|\mathfrak{g}^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})})$$

Now, using Lemma 3 (29b) we get:

$$\|\bar{\phi}^\perp\|_{\mathcal{C}^{l,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} \leqslant C\epsilon^{-1}\|\mathfrak{g}^\perp\|_{\mathcal{C}^{0,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})}$$

These two Lemmas lead us in solving the system (27) and therefore the equation (26). This is summed up in the following proposition:

Proposition 17. *For each sufficiently small ϵ there is a solution ϕ to equation (26) of the form $\phi = \chi^*\bar{\phi}\circ Y_{\epsilon,h} + \psi$*
where

$$(\bar{\phi}^{||}, \bar{\phi}^\perp, \psi) \in \mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R}) \times \mathcal{C}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})$$

such that the following estimates hold:

$$
\begin{cases}
\|\bar{\phi}^{||}\|_{\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} \leqslant C\epsilon^{1-\alpha}\Big\{\|(\bar{\chi}g)^{||}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \epsilon^{-1}\delta(\epsilon)\|(\bar{\chi}g)^{\perp}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \epsilon^{-3-\alpha}\delta(\epsilon)\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S
\\
\|\bar{\phi}^{\perp}\|_{\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} \leqslant C\epsilon^{-1}\Big\{\|(\bar{\chi}g)^{\perp}\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \epsilon^{1-\alpha}\|(\bar{\chi}g)^{||}\|_{\mathcal{E}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})} + \epsilon^{-2-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})}\Big\}
\\
\|\psi\|_{\mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} \leqslant C\epsilon^{-1-\alpha}\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})} + \mathcal{O}(e^{-c\epsilon^{-1/8}})\|\bar{\chi}g\|_{\mathcal{C}^{0,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R})}
\end{cases}
\tag{30}
$$

Lemma 6. *Proposition 17 is a result of the following:*
Using a priori estimates (30) we can solve the system (27) by a standard fixed point argument.

To do this we replace the functions $\bar{\phi}^{||}$, $\bar{\phi}^{\perp}$, ψ on the right hand of the system by known functions $\bar{\Phi}^{||}$, $\bar{\Phi}^{\perp}$, Ψ which satisfy estimates of the same type as (30) but with constants bigger than those appearing in (30).

Then we have a map

$$
(\bar{\Phi}^{||}, \bar{\Phi}^{\perp}, \Psi) \longmapsto (\bar{\phi}^{||}, \bar{\phi}^{\perp}, \psi)
$$

from a certain ball in the space $\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R}) \times \mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R}) \times \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau})$ into itself.

This and the Lipschitz character of this map (from the way we derived a priori estimates), allows for an application of the Banach fixed point theorem.

4.3.3 Proof of Theorem 1

Now we can finish solving the nonlinear problem (24):

$$
L_\epsilon\phi = l_\epsilon - N_\epsilon(w^*) - Q_\epsilon(\phi)
$$

As we saw above, we need to modify this equation by introducing Lagrange multipliers. Thus we will consider:

$$
L_\epsilon\phi = l_\epsilon - N_\epsilon(w^*) - Q_\epsilon(\phi) + \chi^* \sum_{j=1}^{d} c_j Z^{T,e_j}_{\tau,\epsilon}. \tag{31}
$$

Proposition 18. *There is a solution to the nonlinear equation (31), where for a certain constant K and suitable $\bar{\alpha}$, m we have:*

$$
\begin{cases}
\bar{\phi}^{||} \in \mathcal{X}\cap\mathcal{E}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R}) \quad \textit{and } \|\bar{\phi}^{||}\|_{\mathcal{E}^{2,\alpha}_{\mu}} \leqslant K\epsilon^{4-\bar{\alpha}} \\
\bar{\phi}^{\perp} \in \mathcal{Y}\cap\mathcal{C}^{2,\alpha}_{\mu}(\mathring{D}_\tau\times\mathbb{R}) \quad \textit{and } \|\bar{\phi}^{\perp}\|_{\mathcal{C}^{2,\alpha}_{\mu}} \leqslant K\epsilon^{2} \\
\psi \in \mathcal{C}^{2,\alpha}(\mathbb{R}^{d-1}\times S_{2T_\tau}) \quad \textit{and } \|\psi\|_{\mathcal{C}^{2,\alpha}} \leqslant Ke^{-m\epsilon^{-1}}.
\end{cases}
$$

Proof. Let us consider functions $\tilde{\phi}, \tilde{\psi}$ such that the inequalities in the proposition hold.

Lemma 7. *We can solve the equation for $\tilde{\phi}$, $\tilde{\psi}$ using the previous section about the linear operator, and get a solution ϕ, ψ such that the inequalities hold again with a smaller constant $\tilde{K}$.*

Thus we have a nonlinear map

$$(\tilde{\phi}^{||}, \tilde{\phi}^{\perp}, \tilde{\psi}) \mapsto (\bar{\phi}^{||}, \bar{\phi}^{\perp}, \psi)$$

between small balls in the corresponding spaces.

This map is a contraction, and it is straightforward using the quadratic nature of the nonlinear function $Q(\phi)$.

We can now use a Banach fixed point argument to conclude. $\square$

Proof of Lemma 7. Let us write:

$$\hat{\phi} := \chi^* \tilde{\phi} \circ Y_{\epsilon,h} + \tilde{\psi}.$$

We also define:

$$\begin{cases} A_1 := \chi^*[l_\epsilon - N_\epsilon(w^*)] \\ A_2 := N(w^*) - \chi^* N(w) - (1 - \chi^*) N(\mathbb{H}) \end{cases}$$

Thus, we have:

$$l_\epsilon - N_\epsilon(w^*) = \chi^*[l_\epsilon - N_\epsilon(w^*)] + N(w^*) - \chi^* N(w) - (1 - \chi^*) N(\mathbb{H}) = A_1 + A_2$$

The above holds because of the definition of w^* and N_ϵ and the expression of Laplace operator in stretched shifted Fermi coordinates.

Next, it is easy to obtain the following estimates for A_1 and A_2:

$$\begin{cases} \|Y^*_{\epsilon,h} \bar{\chi} A_2\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant C_0 e^{-c_\mu \epsilon^{-1/3}} \\ \|(1 - \chi^*) A_2\|_{\mathcal{C}^{0,\alpha}} \leqslant C_0 e^{-\theta \epsilon^{-1}} \\ \|(Y^*_{\epsilon,h} \bar{\chi} A_1)^{||}\|_{\mathcal{E}^{0,\alpha}_\mu} \leqslant C_0 \epsilon^3 \\ \|Y^*_{\epsilon,h} \bar{\chi} A_1)^{\perp}\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant C_0 \epsilon^3 \\ \|(1 - \chi^*) A_1\|_{\mathcal{C}^{0,\alpha}} \leqslant C_0 e^{-\theta \epsilon^{-1}} \end{cases}$$

where C_0, c_μ, θ are positive constants.
In this way we estimated the size of the error of the approximation $l_\epsilon - N_\epsilon(w^*)$.

Using these five inequalities and Proposition 17 we get the desired results. Indeed, lets prove that for the first inequality of Proposition 18:

$$\begin{aligned}
\|\bar{\phi}^{||}\|_{\mathcal{E}^{2,\alpha}_\mu(\mathring{D}_\tau\times\mathbb{R})} &\leqslant C\epsilon^{1-\alpha}\Big\{\|(\bar{\chi}g)^{||}\|_{\mathcal{E}^{0,\alpha}_\mu} + \epsilon^{-1}\delta(\epsilon)\|(\bar{\chi}g)^{\perp}\|_{\mathcal{C}^{0,\alpha}_\mu} + \epsilon^{-3-\alpha}\delta(\epsilon)\|(1-\chi^*)g\|_{\mathcal{C}^{0,\alpha}}\Big\} \\
&\leqslant C\epsilon^{1-\alpha}\Big\{\|(\bar{\chi}A_1)^{||}\| + \|(\bar{\chi}A_2)^{||}\| + \|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| \\
&+ \epsilon^{-1}\delta(\epsilon)\|(\bar{\chi}A_1)^{||}\| + \epsilon^{-1}\delta(\epsilon)\|\bar{\chi}A_1)^{\perp}\| + \epsilon^{-1}\delta(\epsilon)\|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| \\
&+ \epsilon^{-3-\alpha}\delta(\epsilon)\big[\|(1-\chi^*)A_1\| + \|(1-\chi^*)A_2\| + \|(1-\chi^*)Q_\epsilon(\hat{\phi})\|\big]\Big\} \\
&\leqslant C\epsilon^{1-\alpha}\Big[C_0\epsilon^3 + C_0e^{-c_\mu\epsilon^{-1/3}} + \|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| + \epsilon^{-1}\delta(\epsilon)C_0\epsilon^3 + \epsilon^{-1}\delta(\epsilon)C_0e^{-c_\mu\epsilon^{-1/3}} \\
&+ \epsilon^{-1}\delta(\epsilon)\|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| + 2\epsilon^{-3-\alpha}\delta(\epsilon)C_0e^{-\theta\epsilon^{-1}} + \epsilon^{-3-\alpha}\delta(\epsilon)\|(1-\chi^*)Q_\epsilon(\hat{\phi})\|\Big] \\
&\leqslant \tilde{C}[\epsilon^{4-\tilde{\alpha}} + \mathcal{Q}_\epsilon]
\end{aligned}$$

In the last inequality we have used the definition of δ and α. Indeed, the lowest orders in ϵ are:

- The first term in $\epsilon^{4-\alpha}$ thus we choose $\tilde{\alpha} > \alpha$.
- The fourth term in fifth row in $\delta(\epsilon)\epsilon^{3-\alpha}$ thus we choose $\tilde{\alpha} > \alpha + 1/3$.

$\mathcal{Q}_\epsilon$ is composed of terms in $Q_\epsilon(\hat{\phi})$:

$$\mathcal{Q}_\epsilon := \epsilon^{1-\alpha}\big(\|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| + \epsilon^{-1}\delta(\epsilon)\|\bar{\chi}Q_\epsilon(\hat{\phi})^{||}\| + \epsilon^{-3-\alpha}\delta(\epsilon)\|(1-\chi^*)Q_\epsilon(\hat{\phi})\|\big)$$

and thus for a certain λ:

$$\mathcal{Q}_\epsilon \leqslant \lambda\|\hat{\phi}\|^2(\epsilon^{1-\alpha} + \epsilon^{-\alpha}\delta(\epsilon) + \epsilon^{-2-2\alpha}\delta(\epsilon)) := \lambda_\epsilon\|\hat{\phi}\|^2$$

Then, using the hypothesis for $\hat{\phi}$ we have:

$$\|\bar{\phi}^{||}\|_{\mathcal{E}^{2,\alpha}_\mu} \leqslant \tilde{C}(1 + \lambda_\epsilon K^2\epsilon^{4-\tilde{\alpha}})\epsilon^{4-\tilde{\alpha}}$$

Thus, we fix $K = 4\tilde{C}$ and for a sufficiently small ϵ we get:

$$\|\bar{\phi}^{||}\|_{\mathcal{E}^{2,\alpha}_\mu} \leqslant 2\tilde{C}\epsilon^{4-\tilde{\alpha}} := \tilde{K}\epsilon^{4-\tilde{\alpha}} < K\epsilon^{4-\tilde{\alpha}}$$

□

Finally, we have a solution of the equation

$$\epsilon\Delta u - \frac{1}{\epsilon}W_u(u) = l_\epsilon + \chi^*\sum_{j=1}^{d} c_j Z^{T,e_j}_{\tau,\epsilon} \tag{32}$$

which is satisfying the assertions of the Theorem 1 by construction:

$$\begin{aligned}
u &= w^* + \phi \\
&= w^* + \chi^*\bar{\phi}\circ Y_{\epsilon,h} + \psi \\
&= w\chi^* + \mathbb{H}(1-\chi^*) + \phi
\end{aligned}$$

and we recall that:

$$Y^*_{\epsilon,h}w(y,\bar{t}) = \Theta(y,\bar{t}) + \epsilon\varphi + \epsilon^2\psi_0$$

The convergences of Theorem 1 are obvious with the above equality and the requirement for l_ϵ has been proven before.

The last step is to show that in fact $c_j = 0$. For that, we will need the following Lemma:

Lemma 8 (Balancing formula)**.** *Let $X = \sum \alpha_j \partial_{x_j}$ be the infinitesimal generator of translations or rotations in $\mathbb{R}^d$. For any $C^2(\mathbb{R}^d)$ function, it holds:*

$$div\Big[\Big(\frac{\epsilon}{2}|\nabla u|^2 - \frac{1}{\epsilon}W(u)\Big)X(u) - \epsilon X(u)\nabla u\Big] = -[\epsilon\Delta u + \frac{1}{\epsilon}W'(u)]X(u) \tag{33}$$

Proposition 19. $c_j = 0$

Proof. We will take $X_j = \partial_{x_j}$ for some $1 \leqslant j \leqslant d$ and integrate formula (33) over the cylinder $\mathcal{C}_R = B_R \times S_{2T_\tau}$. Using (32) and Green's Theorem we get:

$$\int_{\partial\mathcal{C}_R}\Big(\frac{\epsilon}{2}|\nabla u|^2 - \frac{1}{\epsilon}W(u) + l_\epsilon u\Big)n_j dS - \int_{\partial\mathcal{C}_R}\partial_{x_j}u\partial_n u dS = -\int_{\mathcal{C}_R}\Big(\sum_{j'=1}^{d}\chi^* c_{j'} Z^{T,e_{j'}}_{\tau,\epsilon}\Big)\partial_{x_j}u.$$

The first integral is 0 on the top and the bottom of the cylinder $\mathcal{C}_R$ and on the other hand, using the asymptotic behaviour of the solution we get finally:

$$\lim_{R\to\infty} I_R = 0$$

where I_R, II_R, III_R denote the first, second and third term respectively.
For the second integral, we have that the integrals on the top and the bottom of $\mathcal{C}_R$ are cancelled because u is periodic. Then, from exponential decay of the derivatives of u we get:

$$\lim_{R\to\infty} II_R = 0$$

Finally, we note that

$$\partial_{x_j}u \approx Z^{T,e_j}_{\tau,\epsilon}$$

hence,

$$III_R = c_j\int_{\mathring{D}_\tau\times\mathbb{R}}|Z^{T,e_j}_{\tau,\epsilon}|^2 + \circ(1)\sum_{j'=1}^{d}c_{j'}$$

from which we get immediately that $c_j = 0$, $j = 1, ..., d$. □

This Proposition ends the construction of our solution as stated in Theorem 1.

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