Notions of Galois connections for Bilattices

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Έννοιες αντιστοιχιών Galois για διπλέγματα

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ABSTRACT

Bilattices are algebraic structures, stemming from the research on knowledge representation and non-monotonic reasoning; they comprise a set equipped with two lattice orders, one modelling degree of truth and one modelling amount of information. Galois connections are very useful throughout mathematics, providing a unifying abstraction for various correspondences between ordered sets, and being in close correspondence with closure operators. We introduce notions of *Galois biconnections*, intended to be the bilattice analogue of classical Galois connections between lattices.

The first distinction we make is between bidirectional and unidirectional Galois biconnections. A *bidirectional Galois biconnection* is a (compatible) pair of Galois connections between the *truth orderings* and the *knowledge orderings* of bilattices, while a *unidirectional Galois biconnection* is actually a Galois connection equipped with extra properties that seek to capture the bilattice structure. A further distinction is between *regular Galois biconnections*, which induce order-isomorphic images of the maps, *strong Galois biconnections*, which furnish bilattice-isomorphic images.

We investigate all four species of Galois biconnections on pre-bilattices and on bilattices with negation and conflation. We examine both the survival of elegant properties of Galois connections (composability, invertibility, preservation of joins and meets, etc.) and the preservation of interesting bilattice properties (distributivity, boundedness, interlacing) for the images of the bilattices under the Galois biconnection. Finally, we discuss the naturally emerging biclosure operators on bilattices and hint at the generalisation of these concepts to sets equipped with more than two lattices.

ΣΥΝΟΨΗ

Τα διπλέγματα (bilattices) είναι αλγεβρικές δομές προερχόμενες από τα πεδία της αναπαράστασης γνώσης και της μη μονοτονικής λογικής· αποτελούνται από ένα σύνολο εφοδιασμένο με δύο πλέγματα (lattices), όπου το ένα μοντελοποιεί το βαθμό αλήθειας και το δεύτερο την ποσότητα πληροφορίας. Οι αντιστοιχίες Galois είναι πολύ χρήσιμες στα μαθηματικά, διότι αποτελούν μία ενοποιητική αφαίρεση διάφορων αντιστοιχιών μεταξύ διατεταγμένων συνόλων, καθώς και διότι σχετίζονται στενά με τους τελεστές κλειστότητας. Σε αυτή την εργασία, εισάγουμε κάποιες έννοιες δι-αντιστοιχιών Galois, που αποσκοπούν στο να αποτελέσουν το ανάλογο των αντιστοιχιών Galois για διπλέγματα.

Η πρώτη διάκριση που κάνουμε είναι ανάμεσα σε διαντιστοιχίες Galois μονής και διπλής κατεύθυνσης. Οι διαντιστοιχίες διπλής κατεύθυνσης αποτελούνται από ένα ζεύγος (συμβατών μεταξύ τους) αντιστοιχιών Galois ανάμεσα στις διατάξεις αλήθειας και πληροφορίας, ενώ οι διαντιστοιχίες μονής κατεύθυνσης είναι αντιστοιχίες Galois εφοδιασμένες με επιπλέον ιδιότητες που επιχειρούν να συλλάβουν τη διπλεγματική δομή. Μια περαιτέρω διάκριση γίνεται μεταξύ συνήθων και ισχυρών διαντιστοιχιών Galois· στις πρώτες, οι συναρτήσεις που παίρνουν μέρος έχουν ισόμορφες εικόνες ως διατάξεις, ενώ στις δεύτερες οι εικόνες είναι ισόμορφα διπλέγματα.

Εξετάζουμε τα τέσσερα είδη διαντιστοιχιών Galois που προκύπτον από τις παραπάνω διχοτομήσεις, τόσο σε διπλέγματα με τελεστές άρνησης όσο και σε διπλέγματα χωρίς τέτοιους τελεστές. Διερευνούμε την γενικευσιμότητα των κομψών ιδιοτήτων των αντιστοιχιών Galois (συνθεσιμότητα, αντιστρεψιμότητα, διατήρηση άνω και κάτω φραγμάτων κλπ), καθώς και την συμπεριφορά των εικόνων όσον αφορά ενδιαφέρουσες ιδιότητες των διπλεγμάτων. Τέλος, αναφερόμαστε στους αντίστοιχους τελεστές κλειστότητας που προκύπτουν από τις διαντιστοιχίες και κάνουμε μια νύξη του πώς οι έννοιες που παρουσιάζουμε μπορούν να γενικευτούν σε σύνολα εφοδιασμένα με περισσότερες από δύο διατάξεις.

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CONTENTS

CHAPTER 1

INTRODUCTION AND NOTATION

1.1 Introduction

Bilattices were introduced in the '80s by M. Ginsberg with a view to problems of default inference in AI [27]. Technically, *bilattices* are sets equipped with two lattice orderings aimed to model simultaneously the *validity of*, and *degree of knowledge about*, sentences from a logical language. Following their introduction, bilattices have found applications in diverse fields and have been studied from different perspectives. Within *Computer Science*, they have been used for the semantics of Logic Programming [21, 23, 1]. In *Artificial Intelligence*, they have been used to model situations in which information has a hierarchical ('prioritized') structure (*default bilattices*, [27]), as a tool for *paraconsistent reasoning* [2, 5, 4] and as a framework for *multi-valued logics* (see for instance [4, 36]). Within *Philosophical Logic*, bilattices are treated as a natural framework for generalizing Kleene's three-valued logic furnishing a tool for investigations on the '*theory of truth*' [20, 22, 24]. Finally, more recently, the *algebraic theory* of bilattices [28, 10] and their *duality theory* [32, 29, 12, 13, 14] have been thoroughly investigated.

In this thesis, we aim to contribute to the algebraic study of bilattices from an entirely different perspective: that of **Galois connections**, a simple, useful, and ubiquitous concept in algebra, logic, and, as a matter of fact, in all mathematics; a concept that '*you will discover it hidden away in almost every corner of our subject, if you keep your eyes open*' ([8, p. 38]). Technically, a Galois Connection (abbreviation GC) is a pair of opposite order-preserving (or order-reversing, depending on the tradition followed) mappings between partially ordered structures (we will only consider lattices). The orderings in the two opposite 'universes' are preserved when using the mappings to pass from the one structure to the other and, moreover, this '*back-and-forth*' process becomes stationary when iterated. Algebraists and category theorists know that there is a certain advantage in such an '*adjoint setting*': information about objects and relationships in one structure is used to obtain information in the other structure (and vice versa). The canonical example is that of classical Galois theory, where properties of permutation groups are used to study field extensions. Logicians frequently refer to the W. Lawvere '*syntax-semantics*' adjoint maps between the set of all logical 'theories' (sets of sentences of a logical language L) and the class of all structures interpreting L:

the '*semantics functor*' maps a set of sentences to its class of structures and the '*syntax functor*' maps a class of structures to the set of sentences validated in every structure; this gives rise to a sub-ordering of theories closed under logical consequence, isomorphically mapped to the corresponding class of structures (see [38] for an exposition).

Within Universal Algebra, GCs can also be seen as a simple and general method for producing closure and interior operators: the composition of the two Galois adjoints provides a closure operator and an interior operator—or two closure operators for the order reversing version [8, Sect. 2.5]. On the applications side, it is worth mentioning the field of *Formal Concept Analysis* (FCA, [26]), which is an application of GCs to Knowledge Representation. This whole field arises from G. Birkhoff's '*polarities as Galois connections*' construction which provides a GC between power sets based on a relation between the ground sets; actually, every GC between power sets can be constructed from an underlying relation between sets. Starting from such a relation between objects and their attributes, FCA is an important application of applied lattice theory and GCs to conceptual data analysis and knowledge processing (see also [16, Chapters 3 & 7]).

Clearly, GCs constitute a useful and ubiquitous algebraic concept. Here, we examine its extension in the classes of pre-bilattices and bilattices. Our aim is to develop a theory of Galois (bi)connections between pre-bilattices and bilattices with negation and conflation, focusing on non-trivial notions of such adjoint situations that retain elegant properties of the classical setting, in particular composability, invertibility, and image isomorphism. In addition, we aim to investigate the preservation of structural bilattice properties, like boundedness, interlacing, and distributivity, when passing from a pair of bilattices to a pair of isomorphic images under the Galois biconnection adjoint iteration.

The rest of the thesis is structured as follows: In Section 1.2 we introduce some general concepts and notation. In Sections 2.1, 2.2 and 2.3 we review the background on lattices, Galois connections, and bilattices respectively, establishing notation and terminology. Chapter 3 contains our results: in Section 3.1 we briefly set the stage by considering the technical desiderata, in Section 3.2 we explore Galois biconnections between pre-bilattices, and in Section 3.3 we proceed to define some additional notions of Galois biconnections for the general case of bilattices with negation and conflation. It is quite customary to see applications of bilattices equipped only with negation and thus we devote Section 3.4 to this subclass. In Section 3.5 we discuss the naturally emerging biclosure operators and in Section 3.6 we hint at the extension of these notions to sets with more than two embedded lattices. We conclude in Chapter 4 with a short discussion. In Appendix A we present some failed definitions of Galois biconnections, in hope that they will serve as a hint of the intricacies of defining such a notion on sets with multiple orders, while Appendix B contains a few remarks on a 'monstrous' (in the Lakatosian sense) yet interesting class of bilattices we came upon while working for this thesis.

1.2 Notation

Notation. Tuples (and pairs) will be denoted using parentheses, e.g. (x_1, x_2, x_3). The subset relation will be denoted with $\subseteq$, while the symbol $\subset$ will be reserved for proper subsets.

Notation. If A is a set (we will usually use capital letters for sets), the identity function on A will be denoted id_A, i.e. $\forall x \in A : \mathrm{id}_A(x) = x$. If $f\colon A \to B$ and $g\colon B \to C$ are functions, gf is their composition. If $f\colon A \to B$ and $h\colon A \to B$ are functions and R is a binary relation (usually equality or order), $f\;R\;h$ iff $\forall x \in A : f(x)\;R\;h(x)$. The image of the function $f\colon A \to B$ will be denoted by $f(A)$.

Definition 1.1. A function $f\colon A_1 \to A_2$ is called

(a) an *injection* iff $\forall x_1, x_2 \in A_1 : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$,

(b) a *surjection* iff $\forall y \in A_2 \exists x \in A_1 : f(x) = y$,

(c) a *bijection* iff it is an injection and a surjection.

If f is a bijection, its *inverse* is the (unique) function $f^{-1}\colon A_2 \to A_1$ such that $f^{-1}(y) = x$ iff $f(x) = y$.

We use some terminology form Universal Algebra (see [8, 11]) and related fields, although slightly simplified.

Definition 1.2.

- A *signature* is a family $(\sigma_i)_{i\in I}$ of (function or relation) symbols, each equipped with a non-negative integer, its *arity*.
- A *structure* $\mathcal{A}$ over a signature Σ is a non-empty set A (called the *carrier set* or *universe*) equipped with a family $(f_i)_{i\in I}$ of operations, that is functions from A^n to A, and relations, that is subsets of A^n, matching the symbols of Σ; the number of arguments each f_i can take (which for operations can be 0, in which case f_i is a *constant* of A) is called its *rank* and must be equal to the arity of the corresponding symbol (so, abusing notation, we may use the notions of rank and arity interchangeably). A *reduct* of a structure is a structure with the same universe but with some operations or relations omitted (i.e. with a smaller signature).
- An *algebra* or *algebraic structure* is a structure with no relation symbols. If $\emptyset \neq B \subseteq A$ and the image each operator f_i of A (of arity n_i) when restricted to B is a subset of B, then B forms a *subalgebra* of $\mathcal{A}$.

Notation. As above, we will use calligraphic script for (algebraic) structures. We use the notation $|\mathcal{A}|$ or—when there will be no misunderstanding—plainly A for the carrier set of $\mathcal{A}$.

Definition 1.3. Let $\mathcal{A}$, $\mathcal{B}$ be structures with a common (and with the same arity n) operator F or relation R. A function $f\colon A \to B$ is said to *preserve* or *respect* F iff $f(F(x_1,\dots,x_n)) = F(f(x_1),\dots,f(x_n))$, while F *preserves* or *respects* R iff $(x_1,\dots,x_n) \in R \Leftrightarrow (f(x_1),\dots,f(x_n)) \in R$.

Definition 1.4. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be structures with the same signatures and let $f\colon A_1 \to A_2$. f is a *homomorphism* iff it respects all operators and relations of $\mathcal{A}_1$. f is an *isomorphism* iff it is a bijection and a homomorphism.

Notation. When an operator (or relation) is shared by more than one structure, we will use the structure (or its carrier set) as a subscript of the operator for disambiguation, e.g. $\prod_{\mathcal{A}}$ or, for operators in display mode,

$$\prod_{A}.$$

Usually, the subscript will be omitted since the intended structure will be clear from context.

CHAPTER 2

LATTICES, GALOIS CONNECTIONS, BILATTICES

2.1 Orders and lattices

2.1.1 Partial orders

Definition 2.1. A binary relation $\lesssim$ on a set P is called a *preorder* iff it is

- reflexive, i.e. for all $x \in P$, $x \lesssim x$,
- transitive, i.e. for all $x, y, z \in P$, $x \lesssim y$ and $y \lesssim z$ implies $x \lesssim z$,

A binary relation $\leq$ on a set P is called a *partial order* iff it is a preorder and it is also antisymmetric, i.e. for all $x, y \in P$, $x \leq y$ and $y \leq x$ implies $x = y$.

Notation. $x < y$ stands for $x \leq y$ and $x \neq y$.

Notation. $x \parallel y$ (x is incomparable to y) iff neither $x \leq y$ nor $y \leq x$.

Definition 2.2. If $\mathcal{P}_1, \mathcal{P}_2$ are (partial) orders, $f\colon P_1 \to P_2$ is called

- *monotone* or *order-preserving* or *covariant* iff $x \leq y \Rightarrow f(x) \leq f(y)$,
- *antitone* or *order-reversing* or *contravariant* iff $x \leq y \Rightarrow f(y) \leq f(x)$.

Remark 2.3. Every partial order homomorphism h is an injection, since

$$h(x_1) = h(x_2) \Rightarrow \begin{matrix} h(x_1) \leq h(x_2) \\ h(x_2) \leq h(x_1) \end{matrix} \Rightarrow \begin{matrix} x_1 \leq x_2 \\ x_2 \leq x_1 \end{matrix} \Rightarrow x_1 = x_2.$$

Definition 2.4. If $\mathcal{P} = (P, \leq)$ is a partial order, $Q \subseteq P$, and $x \in P$, then

- x is an *upper bound* of Q iff $\forall y \in Q : y \leq x$,
- x is a *lower bound* of Q iff $\forall y \in Q : x \leq y$,
- x is the *least upper bound* or *supremum* or *join* of Q (notation $\sqcup Q$) iff x is an upper bound of Q and for all upper bounds z of Q, $x \leq z$,

- x is the *greatest lower bound* or *infimum* or *meet* of Q (notation $\sqcap Q$) iff x is a lower bound of Q and for all lower bounds z of Q, $z \leq x$,
- x is the *maximum* element of Q iff $x \in Q$ and x is an upper bound of Q,
- x is the *minimum* element of Q iff $x \in Q$ and x is a lower bound of Q,
- x is a *maximal* element of Q iff $x \in Q$ and $\nexists y \in Q : x < y$,
- x is a *minimal* element of Q iff $x \in Q$ and $\nexists y \in Q : y < z$.

Observe that if $\leq$ is a partial order (or even a preorder), then so is $\leq^{\text{op}} \stackrel{\text{def}}{=} \{ (y, x) \mid (x, y) \in \leq \}$. This gives rise to a duality: each time we establish something for $\leq$, we also establish it for $\leq^{\text{op}}$. So, statements regarding maxima and/or upper bounds, such as Remark 2.5, have dual versions regarding minima and lower bounds. Moreover, we could have defined dual notions based on each other (for example, x is maximal in Q w.r.t. $\leq$ iff x is minimal in Q w.r.t $\leq^{\text{op}}$).

Notation. Meets and joins of two elements will usually be written in infix notation; i.e. $x \sqcap y \stackrel{\text{def}}{=} \sqcap\{ x, y \}$ (same for $\sqcup$).

Observe than, when viewed as a binary operator, $\sqcap$ (and $\sqcup$) is of course commutative, associative, and idempotent, in the sense of Remark 2.11.

Remark 2.5. Let $\mathcal{P}$ be a partial order and $Q \subseteq P$. If $x \in Q$, then x is the maximum (dually minimum) element of Q iff x is the supremum (dually infimum) of Q.

Proof. We shall present the proof for upper bounds; the one for lower bounds is dual.

($\Leftarrow$) Of course, if x is the supremum of Q, it is an upper bound.

($\Rightarrow$) Let y be an upper bound of Q. Since $x \in Q$, $x \leq y$; x is also an upper bound of Q, so it is its supremum.

□

Notation. Let $x <^1 y$ (x is an *immediate predecessor* of y) iff $x < y$ and there is no z such that $x < z < y$.

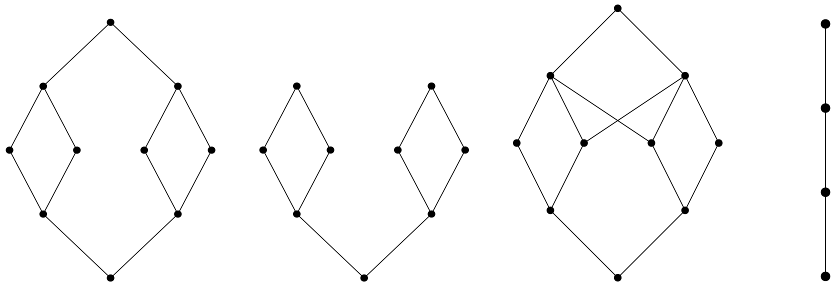
Notation (Drawing partial orders)*.* Partial orders can be drawn as follows:

(a) Each element is represented by a dot.

(b) If $x <^1 y$, the dots of x and y are connected by a line.

(c) Lines are not horizontal. The element at the lowest end of the line is smaller with respect to the order.

Some examples are drawn in Figure 2.1. Observe that reversing an order is equivalent to turning the drawing upside down.



(a) This partial order is a lattice.

(b) This partial order is not a lattice, since it has two maximal elements.

(c) This partial order is not a lattice, since a pair of elements has two minimal upper bounds.

(d) The finite total order **4**.

Figure 2.1: Some examples of partially ordered sets.

2.1.2 Lattices

Definition 2.6. A partial order $\mathcal{L} = (L, \leq)$ is called a *lattice* iff each pair of elements of L has a infimum and a supremum. A lattice $\mathcal{L} = (L, \leq)$ is called

- *bounded* iff L has maximum (notation: $\top$) and a minimum (notation: $\bot$),
- *complete* iff each (finite and infinite) subset of L has a infimum and a supremum,
- *distributive* iff $\forall x, y, z \in L : x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$.

If $\mathcal{L} = (L, \leq)$ is a lattice and $P \subseteq L$, then $\mathcal{P} = (P, \leq)$ is a *sublattice* of $\mathcal{L}$ iff $\mathcal{P}$ is a lattice and (finite) meets and joins of $\mathcal{P}$ coincide with meets and joins of $\mathcal{L}$. As expected, $\mathcal{L}^{\text{op}}$ is $(L, \leq^{\text{op}})$.

Remark 2.7. Any finite lattice is complete, since, by associativity, a finite set has a supremum (dually infimum) if each pair of its elements has a supremum (dually infimum). Moreover, any complete lattice is bounded.

Lemma 2.8. Let $\mathcal{L} = (L, \leq)$ be a lattice and $w, x, y, z \in L$.

(a) If $x \leq y$ and $w \leq z$, then $x \sqcap w \leq y \sqcap z$ and $x \sqcup w \leq y \sqcup z$.

(b) $x \leq y$ iff $x \sqcap y = x$ iff $x \sqcup y = y$.

Proof.

(a) Let $x \leq y$ and $w \leq z$. By definition, $x \sqcap w \leq x \leq y \leq y \sqcup z$ and $x \sqcap w \leq w \leq z \leq y \sqcup z$, so (by transitivity) $x \sqcap w$ is a lower bound of $\{ y, z \}$ and $y \sqcup z$ is an upper bound of $\{ x, w \}$; hence, $x \sqcap w \leq y \sqcap z$ and $x \sqcup w \leq y \sqcup z$.

(b) Now let $x \leq y$. Then, by the property we just proved and reflexivity, $x = x \sqcap x \leq y \sqcap x = x \sqcap y$; by definition, $x \sqcap y \leq x$, so the antisymmetric property gives $x = x \sqcap y$. The inverse is a direct consequence of the definition of meet. The property of $\sqcup$ is dual.

□

Example 2.9. *Total orders*, i.e. partial orders where every two elements are comparable, are (trivially) lattices, by Lemma 2.8. They are trivially distributive.

Lemma 2.10. Each finite total order is unique up to isomorphism.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be finite total orders with the same number of elements n. Let a_1 be the minimum element of A (there is one, by transitivity, finiteness, and totality), a_2 be the minimum element of $A \setminus \{ a_1 \}$, a_3 be the minimum element of $A \setminus \{ a_1, a_2 \}$ etc, so we end up with $a_1 < a_2 < a_3, < \ldots < a_n$; do the same for B to get $b_1 < b_2 < b_3 < \ldots < b_n$. Define $f\colon A \to B$ as $f(a_i) = b_i$. It is obvious that f is an isomorphism. □

Notation. Since each finite total order is unique up to isomorphism, we can denote it with a special symbol; we will use the number of its elements in bold typeface. For example, **4** is the finite total order of four elements.

Remark 2.11. There are two very similar ways to study lattices: the order-theoretic, where a lattice is viewed as a set equipped with a partial order as in the definition above, and the algebraic, where a lattice is a set equipped with two binary operators (meet and join) with a list of properties:

$$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \qquad x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \qquad \text{(associativity)}$$
$$x \sqcap y = y \sqcap x \qquad x \sqcup y = y \sqcup x \qquad \text{(commutativity)}$$
$$x = x \sqcap x \qquad x = x \sqcup x \qquad \text{(idempotency)}$$
$$x = x \sqcap (x \sqcup y) \qquad x = x \sqcup (x \sqcap y) \qquad \text{(absorption)}$$

The two ways are closely linked, since (a) for a partial order as in Definition 2.6 the operators $\sqcap$ and $\sqcup$ have the algebraic properties of a lattice and (b) the last property of Lemma 2.8 allows one to define a partial order given $\sqcap$ and $\sqcup$ (for a detailed treatment, see [16, p. 39–41]). However, the two signatures are not totally interchangeable.

Indeed, consider the lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ of Figure 2.2 and let $f\colon L_1 \to L_2$ with

$$f_1(x) = \begin{cases} b_1 & \text{if } x \in \{ a_1, a_2, a_3, a_4 \} \\ b_2 & \text{if } x \in \{ a_5, a_6, a_7, a_8 \} \end{cases}$$

Since f_1 is not an injection, it is not an order-homomorphism, but it is trivial to see that for each $x_1, x_2 \in L_1$, $f_1(x_1 \sqcap x_2) = f_1(x_1) \sqcap f_1(x_2)$ and $f_1(x_1 \sqcup x_2) = f_1(x_1) \sqcup f_1(x_2)$.

However, consider $\mathcal{L}_3$ and $\mathcal{L}_4$ of Figure 2.2 and let $f_2\colon L_3 \to L_4$ with

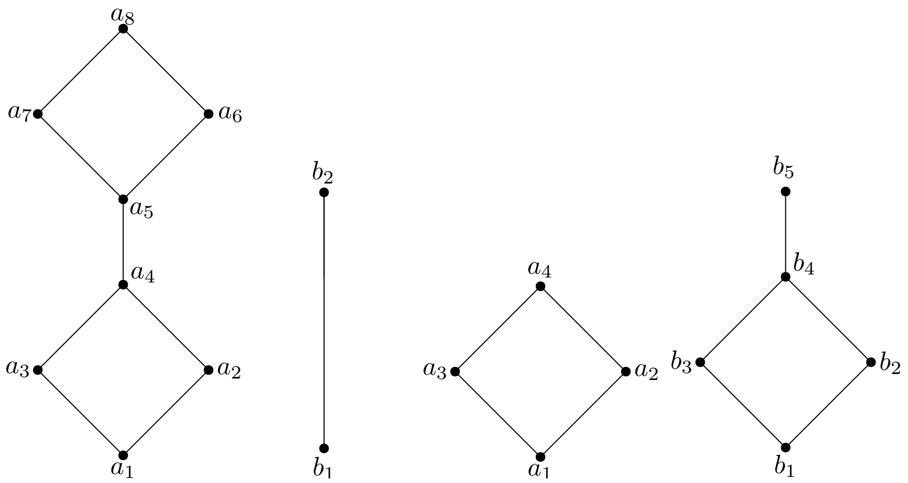
$$f_2(x) = \begin{cases} b_5 & \text{if } x = a_4 \\ b_i & \text{if } x = a_i, i \neq 4 \end{cases}$$

It is obvious that for all $x_1, x_2 \in L_3$, it holds that $f_2(x_1) \leq f_2(x_2) \Leftrightarrow x_1 \leq x_2$, but $f_2(a_3 \sqcup a_2) \neq f_2(a_3) \sqcup f_2(a_2)$.

Lemma 2.12. Consider $\mathcal{L} = (L, \leq, *)$, where $(L, \leq)$ is a lattice and $*$ is a binary operator on L such that for all $x, y \in L$, $* * x = x$ and $x \leq y \Rightarrow *y \leq *x$. Then, for any $P \subseteq L$, $* \sqcap P = \sqcup * P$ and $* \sqcup P = \sqcap * P$.

Proof. Let $x \in P$. Then $\sqcap P \leq x$, so $*x \leq * \sqcap P$; hence, $* \sqcap P$ is an upper bound of $*P$, so

$$\sqcup * P \leq * \sqcap P.$$



(a) $\mathcal{L}_1$ (b) $\mathcal{L}_2$ (c) $\mathcal{L}_3$ (d) $\mathcal{L}_4$

Figure 2.2: The lattices of Remark 2.11.

Now let $*x \in *P$. Then, $*x \leq \sqcup * P$, so $* \sqcup *P \leq * * x = x$; hence, $* \sqcup *P$ is a lower bound of P, so $* \sqcup *P \leq \sqcap P$ and

$$* \sqcap P \leq \sqcup * P.$$

Hence, $* \sqcap P = \sqcup * P$. Dually, $* \sqcup P = \sqcap * P$. □

2.2 Galois connections and closure operators

The notion of *Galois connection* is an important and ubiquitous concept in mathematics. It originated from *Galois theory* which provides a prime example of a GC between the intermediate fields of a Galois extension and the subgroups of the corresponding Galois group. The modern view of GCs is primarily attributed to G. Birkhoff's '*polarities*' [9] and O. Ore's '*Galois connexions*' [33]; the interested reader can obtain a detailed review of the origins and history of GCs in M. Erné's chapter [19] from the collective work [17] which gives a detailed treatment of the topic, its emergence in various areas of mathematics and its applications in various disciplines. For a succint technical overview, the reader is referred to [18, 31].

GCs are usually defined as correspondences between partially ordered sets; they can be even defined as a correspondence between pre-ordered sets if one is willing to sacrifice some of their properties. In this thesis, we confine ourselves to GCs between *lattices*. A GC is given by two opposite order-preserving maps, whose composition yields a closure operator and an interior operator. Another tradition prefers order-reversing maps (which give rise to two closure operators) and the literature seems to be equally divided between the two [16, p. 156]; we follow the order-preserving one.

2.2.1 Galois connections

Definition 2.13. Consider lattices $\mathcal{L}_1 = (L_1, \leq)$ and $\mathcal{L}_2 = (L_2, \leq)$ and a pair of maps $(f \colon L_1 \to L_2, g \colon L_2 \to L_1)$. The pair (f, g) is called a *Galois connection* (abbreviation GC) or *residuated pair* between $\mathcal{L}_1$ and $\mathcal{L}_2$ iff for all $x \in L_1$, $y \in L_2$,

$$f(x) \leq y \;\Leftrightarrow\; x \leq g(y).$$

Note. This kind of relationship between f and g is formulated in Category Theory as *adjunction*. For this reason, order-preserving GCs are also called *Galois adjunctions*. Hence, f is the *lower adjoint* of g and g is the *upper adjoint* of f.

It follows trivially that if ϕ is an isomorphism between $\mathcal{L}_1$ and $\mathcal{L}_2$, then (ϕ, ϕ^{-1}) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$.

The following proposition collects many important properties of GCs. One of the most important is property (a), which gives rise to yet another duality; for each property below, we state both its dual forms, so the duality will become clear by reading on. Another very important property is (j), stating that the images of the two functions (which, by property (e), comprise the stable elements of the connection) are isomorphic.

Proposition 2.14 (Properties of Galois Connections)**.** Let (f, g) be a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$.

(a) (g, f) is a GC between $\mathcal{L}_2^{\text{op}}$ and $\mathcal{L}_1^{\text{op}}$.

(b) f and g are monotone.

(c) $fgf = f$, $gfg = g$ (cancellation).

(d) $\forall x \in L_1 : x \leq gf(x)$; $\forall y \in L_2 : fg(y) \leq y$.

(e) $\forall x \in L_1 : x \in g(L_2) \Leftrightarrow x = gf(x)$; $\forall y \in L_2 : y \in f(L_1) \Leftrightarrow y = fg(y)$. For this reason, elements of $f(L_1)$ and $g(L_2)$ are referred to as the *stable elements* of L_1 and L_2 (under the GC at hand).

(f) f is an injection iff g is a surjection iff $\forall x \in L_1 : gf(x) = x$;
g is an injection iff f is a surjection iff $\forall y \in L_2 : fg(y) = y$.

(g) f and g uniquely determine one another; in fact,

$$g(y) = \bigsqcup \{\, x \mid f(x) \leq y \,\},$$
$$f(x) = \bigsqcap \{\, y \mid x \leq g(y) \,\}.$$

(h) f preserves $\sqcup$ (of finite and infinite sets);
g preserves $\sqcap$ (of finite and infinite sets).

(i) $g(L_2)$ and $f(L_1)$ (with the orders inherited from $\mathcal{L}_1$ and $\mathcal{L}_2$) are lattices, although they may not necessarily be sublattices of $\mathcal{L}_1$ and $\mathcal{L}_2$; in particular, if $P \subseteq g(L_2)$ and $Q \subseteq f(L_1)$ (and, if P or Q is infinite, provided its meet and join exist),

$$\bigsqcup_{g(L_2)} P = gf(\bigsqcup P), \qquad \bigsqcap_{g(L_2)} P = \bigsqcap P,$$
$$\bigsqcap_{f(L_1)} Q = fg(\bigsqcap Q), \qquad \bigsqcup_{f(L_1)} Q = \bigsqcup Q.$$

(j) $f(L_1) \cong g(L_2)$; in fact, (the restrictions of) f and g are inverse isomorphisms between $f(L_1)$ and $g(L_2)$.

(k) If one of $\mathcal{L}_1$, $\mathcal{L}_2$ is complete, so are both $g(L_2)$ and $f(L_1)$.

(l) If $\mathcal{L}_1$ is bounded, then

- $\top \in g(L_2)$,
- $f(L_1)$ is bounded by $f(\bot)$ and $f(\top)$.

If $\mathcal{L}_2$ is bounded, then

- $\bot \in f(L_1)$,
- $g(L_2)$ is bounded by $g(\bot)$ and $g(\top)$.

Proof.

(a) For all $y \in L_2, x \in L_1$, $g(y) \geq x \Leftrightarrow x \leq g(y) \Leftrightarrow f(x) \leq y \Leftrightarrow y \geq f(x)$.

(b) We will show monotonicity for f; monotonicity of g is shown similarly. For $x_1, x_2 \in L_1$ with $x_1 \leq x_2$, we have

$$f(x_2) \leq f(x_2) \Rightarrow x_2 \leq gf(x_2) \Rightarrow x_1 \leq gf(x_2) \Rightarrow f(x_1) \leq f(x_2).$$

(c) We will show the first property; the proof of the second is similar. For all $x \in L_1$, $gf(x) \leq gf(x) \Rightarrow fgf(x) \leq f(x)$ and $f(x) \leq f(x) \Rightarrow x \leq gf(x) \Rightarrow f(x) \leq fgf(x)$.

(d) For all $x \in L_1$, $f(x) \leq f(x) \Rightarrow x \leq gf(x)$. For all $y \in L_2$, $g(y) \leq g(y) \Rightarrow fg(y) \leq y$. These, along with properties (b) and (c) imply that gf is a closure operator and fg is an interior operator.

(e) We will show the property for $g(L_2)$; the proof for $f(L_1)$ is similar.

($\Leftarrow$) Obviously, if $x = gf(x)$, then $x \in g(L_2)$.

($\Rightarrow$) $x \in g(L_2) \Rightarrow x = g(y) = gfg(y) = gf(x)$

(f) We will show the property for the first pack of equivalences; the proof for the second is similar.

(i) $\Rightarrow$ (ii) Let $x \in L_1$. Then $f(x) = fgf(x)$ and, since f is an injection, $x = gf(x)$. Hence, $g(L_2) = L_1$.

(ii) $\Rightarrow$ (iii) Let $x \in L_1$. Since g is a surjection, $x = g(y)$, so

$$x \leq g(y) \Rightarrow f(x) \leq y \Rightarrow gf(x) \leq g(y) = x.$$

Property (d) implies $x = gf(x)$.

(iii) $\Rightarrow$ (i) For all $x_1, x_2 \in L_1$, $f(x_1) = f(x_2) \Rightarrow gf(x_1) = gf(x_2) \Rightarrow x_1 = x_2$.

(g) Let $g' \colon L_2 \to L_1$ with (f, g') a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$. Then for all $y \in L_2$,

$$g(y) = x \Rightarrow x \leq g(y) \Rightarrow f(x) \leq y \Rightarrow g(y) = x \leq g'(y),$$
$$g'(y) = x' \Rightarrow x' \leq g'(y) \Rightarrow f(x') \leq y \Rightarrow g'(y) = x' \leq g(y),$$

so $g' = g$. Similarly, for any f' such that (f', g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$, it follows that $f' = f$.

Moreover, consider some $y \in L_2$. Since $f(x) \leq y \Rightarrow x \leq g(y)$, $g(y)$ is an upper bound of $X = \{ x \in L_1 \mid f(x) \leq y \}$. Now let y' be another upper bound of X; since $fg(y) \leq y$, $g(y) \in X$, so $g(y) \leq y'$. Hence, $g(y) = \bigsqcup X$. Similarly for f.

(h) Let $P \subseteq L_1$. Since f is monotone, $f(\bigsqcup P)$ is an upper bound of $f(P)$. Let y be an upper bound of $f(P)$; then, for all $f(x) \in f(P)$, $f(x) \leq y$ (and $x \leq g(y)$), so $g(y)$ is an upper bound of P. Of course, $\bigsqcup P \leq g(y)$, so $f(\bigsqcup P) \leq y$. Hence, $f(\bigsqcup P) = \bigsqcup f(P)$. Similarly, for every $Q \subseteq L_2$, $g(\bigsqcap Q) = \bigsqcap g(Q)$.

(i) Of course, $gf(\bigsqcup P) \in g(L_2)$ is an upper bound of P (since $gf \geq \mathrm{id}_{L_1}$). Let $z \in g(L_2)$ be an upper bound of P. Since z is also an upper bound of P in L_1, $\bigsqcup P \leq z$, so $gf(\bigsqcup P) \leq gf(z) = z$. Hence, $gf(\bigsqcup P) = \bigsqcup_{g(L_2)} P$.

For $\bigsqcap_{g(L_2)} P = \bigsqcap P$, it suffices to show that $\bigsqcap P \in g(L_2)$. Indeed, for all $x \in P$, $\bigsqcap P \leq x$, so $gf(\bigsqcap P) \leq gf(x) = x$ for all $x \in P$, hence $gf(\bigsqcap P) \leq \bigsqcap P$ and, since $\bigsqcap P \leq gf(\bigsqcap P)$, we get $\bigsqcap P = gf(\bigsqcap P)$.

Dually, we obtain the equations for $f(L_1)$. Since meets and joins are well-defined for all (finite) subsets, $f(L_1)$ and $g(L_2)$ are lattices.

(j) Let $f_0 \colon g(L_2) \to f(L_1)$ be the restriction of f to $g(L_2)$ and $g_0 \colon f(L_1) \to g(L_2)$ be the restriction of g to $f(L_2)$. For each $y \in f(L_1)$, $y = fg(y)$, so f_0 is onto $f(L_1)$; similarly, g_0 is onto $g(L_2)$. Since f is monotone, so is f_0; in addition, if $fg(y_1)) \leq fg(y_2))$, then $g(y_1) = gfg(y_1)) \leq gfg(y_2)) = g(y_2)$, so f_0 is an order homomorphism. Similarly, g_0 is an order homomorphism. Partial order homomorphisms are one-to-one. Hence, both f_0 and g_0 are order (and lattice) isomorphisms. By property (d), they are inverse to each other.

(k) Without loss of generality, suppose $\mathcal{L}_1$ is complete. By property (i), $g(L_2)$ is complete; since $f(L_1)$ and $g(L_2)$ are isomorphic, $f(L_1)$ is also complete.

(l) We will prove the statement for $\mathcal{L}_1$. The one for $\mathcal{L}_2$ is proved dually.

Let $\top$ be the top element and $\bot$ be the bottom element of L_1. Since $\top \leq gf(\top)$, $\top = gf(\top)$, so $\top \in g(L_2)$. Moreover, the monotonicity of f implies that $f(L_1)$ is bounded with $f(\bot) \leq f(x) \leq f(\top)$ for all $x \in L_1$ (and of course $f(\bot), f(\top) \in f(L_1)$, so they are the infimum and supremum respectively).

□

The following easy lemma can serve as an alternative definition of GCs.

Lemma 2.15 (Characterisation of Galois connections)**.** (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$ iff

(a) f and g are monotone,

(b) $\forall x \in L_1 : x \leq gf(x)$,

(c) $\forall y \in L_2 : fg(y) \leq y$.

Proof.

($\Rightarrow$) Properties 2.14.(b) and 2.14.(d).

($\Leftarrow$) $\forall x \in L_1, y \in L_2 : f(x) \leq y \Rightarrow gf(x) \leq g(y) \Rightarrow x \leq g(y) \Rightarrow f(x) \leq fg(y) \Rightarrow f(x) \leq y$, so $f(x) \leq y \Leftrightarrow x \leq g(y)$.

□

The following results state an important property (*composability*) of GCs and a necessary and sufficient condition for the existence of a lower adjoint, given a function f for the upper adjoint.

Proposition 2.16 (Composability of Galois connections)**.** Let (f, g) be a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$ and (f', g') be a GC between $\mathcal{L}_2$ and $\mathcal{L}_3$. Then $(f'f, gg')$ is a GC between $\mathcal{L}_1$ and $\mathcal{L}_3$.

Proof. $\forall x \in L_1, z \in L_3 : x \leq gg'(z) \Leftrightarrow f(x) \leq g'(z) \Leftrightarrow f'f(x) \leq z$ □

Proposition 2.17. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices and $f \colon L_1 \to L_2$. Then, there exists $g \colon L_2 \to L_1$ such that (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$ iff

(a) f preserves $\sqcup$

(b) for every $y \in L_2$, $\bigsqcup\{\, x \mid f(x) \leq y \,\}$ exists.

Proof.

($\Rightarrow$) Properties 2.14.(h) and 2.14.(g).

($\Leftarrow$) Let $g(y) = \bigsqcup\{\, x \mid f(x) \leq_k y \,\}$; g is a function since joins are unique. Since f preserves $\sqcup$, f is monotone. g is also monotone, since $y_1 \leq y_2$ implies

$$\{\, x \mid f(x) \leq y_1 \,\} \subseteq \{\, x \mid f(x) \leq_k y_2 \,\}.$$

For $x_0 \in L_1$, $gf(x_0) = \bigsqcup\{\, x \mid f(x) \leq f(x_0) \,\}$; since $f(x_0) \leq f(x_0)$, $x_0 \leq gf(x_0)$. For $y_0 \in L_2$, $fg(y_0) = f(\bigsqcup\{\, x \mid f(x) \leq y_0 \,\})$; since f respects $\sqcup$, $fg(y_0) = \bigsqcup\{\, f(x) \mid f(x) \leq y_0 \,\}$, so, by the properties of $\vee$, $fg(y_0) \leq y_0$. The characterisation of GCs implies that (f, g) is a GC.

□

Corollary 2.18. If $\mathcal{L}_1$ is complete, all that f needs in order to be part of a GC is to preserve $\sqcup$.

Definition 2.19. If (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$, then $x_1, x_2 \in L_1$ are called *gf-equivalent* if $gf(x_1) = gf(x_2)$. Each equivalence class is called a *level* of L_1.

Levels of L_2 are defined dually, as fg-equivalence classes.

Remark 2.20. By property 2.14.(c), each level of L_1 contains an element of $g(L_2)$; by property 2.14.(e), this element is unique. So, naturally, each level will be represented by the element of $g(L_2)$ it contains.

Remark 2.21 (Drawing Galois connections)**.** An elegant way to picture GCs is given by Melton et al.:

> The subsets $g(L_2) \subseteq L_1$ and $f(L_1) \subseteq L_2$ are isomorphic skeletons with the levels attached as 'blossoms' to the 'buds' on the skeletons. The blossoms in L_1 grow downwards; in L_2 they grow upwards. The partial ordering within the levels is consistent with the ordering of the skeletons. ([31, p. 303], notation adapted)

Hence, one can draw a GC as follows: (a) the representatives of levels are drawn as filled circles, (b) the other elements are drawn as empty circles, (c) optionally, levels are drawn as closed curves, starting on their representative and containing their elements in their interior.

We do not give examples of drawing GCs here. Figures 3.3 and 3.4 can serve as such examples, if we only consider the $\leq_k$ order.

Example 2.22 ('Minimum' Galois connections). Consider the bounded lattices $\mathcal{L}_1$ and $\mathcal{L}_2$.

(a) It is easy to check that the functions $f\colon L_1 \to L_2$, $g\colon L_2 \to L_1$, where $f(x) = \bot$ for all $x \in L_1$ and $g(y) = \top$ for all $y \in L_2$, form a GC between the two lattices.

(b) Now pick $x_0 \in L_1$, $y_0 \in L_2$. The functions $f\colon L_1 \to L_2$, $g\colon L_2 \to L_1$, where

$$f(x) = \begin{cases} \bot & \text{if } x \leq x_0 \\ y_0 & \text{otherwise} \end{cases} \qquad g(y) = \begin{cases} \top & \text{if } y \geq y_0 \\ x_0 & \text{otherwise} \end{cases}$$

form a GC between the two lattices. Indeed, if $x \leq x_0$, then for all $y \in L_2$, $f(x) = \bot \leq y$ and, since in any case $x_0 \leq g(y)$, $f(x) \leq y \Leftrightarrow x \leq g(y)$ trivially; if $x > x_0$, then $f(x) \leq y \Leftrightarrow y_0 \leq y \Leftrightarrow g(y) = \top \geq x$.

(c) Pick $a_1, b_1 \in L_1$ with $a_1 \sqcap b_1 = \bot$ and $a_2 \sqcup b_2 = \top$. Then,

$$f(x) = \begin{cases} \bot & \text{if } x = \bot \\ a_2 & \text{if } \bot < x \leq a_1 \\ b_2 & \text{if } \bot < x \leq b_1 \\ \top & \text{otherwise} \end{cases} \qquad g(y) = \begin{cases} \top & \text{if } y = \top \\ a_1 & \text{if } a_2 \leq y < \top \\ b_1 & \text{if } b_2 \leq y < \top \\ \bot & \text{otherwise} \end{cases}$$

form a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$. Indeed, by definition, $x \leq gf(x)$ and $fg(y) \leq y$ follow trivially. Moreover, let $x_1 \leq x_2$ in L_1; then,

- if $f(x_2) = \top$, obviously $f(x_1) \leq f(x_2)$,
- if $f(x_2) = a_2$, then $\bot < x_2 \leq a_1$, so $\bot \leq x_1 \leq a_1$ and $f(x_1) \leq a_2$,
- if $f(x_2) = b_2$, then $\bot < x_2 \leq b_1$, so $\bot \leq x_1 \leq b_1$, hence $f(x_1) \leq b_2$,
- if $f(x_2) = \bot$, then $x_2 = \bot$, so $x_1 = \bot$ and $f(x_1) = \bot$,

so f is monotone and, dually, g is monotone. By the characterisation of GCs, f and g form a GC.

2.2.2 Residuated lattices

It is worth digressing a bit to introduce a very interesting structure which naturally gives rise to some GCs.

Definition 2.23. A *residuated lattice*[1] is an algebra $\mathcal{L} = (L, \sqcap, \sqcup, \cdot, \backslash, /)$, such that $(\mathcal{L}, \sqcap, \sqcup)$ is a lattice, $\cdot$ is associative on L, and for all $a, b, c \in L$,

$$a \cdot b \leq c \Leftrightarrow b \leq a \backslash c \Leftrightarrow a \leq c \,/\, b.$$

That is, (f_a, g_a) and (f_b, g_b), where

$$f_a(x) = a \cdot x, \qquad g_a(y) = a \backslash y, \qquad f_b(x) = x \cdot b, \qquad g_b(y) = y \,/\, b,$$

are GCs (hence the name 'residuated lattice'; remember that GCs are also called 'residuated pairs'). As a consequence, $\cdot$ is monotone with respect to both its arguments, $\backslash$ is monotone with respect to its second argument, and $/$ is monotone with respect to its first argument.

[1] **Addendum:** Usually, residuated lattices are required to have an identity element for the $\cdot$ operation, thus making $(L, \cdot)$ a monoid. As in [28], we don't need this, so we present a slightly less strict definition.

2.2.3 Closure operators

Closure and interior operators are encountered in algebra, topology, and logic. Here, we are interested in the former 'species' which is closely related to GCs.

Definition 2.24. Given a lattice $\mathcal{L} = (L, \leq)$, a function $\mathsf{C}\colon L \to L$ is called a *closure operator* (abbreviation CO) (resp. *interior operator*; abbreviation IO) on $\mathcal{L}$ iff for every $x \in L$,

(a) $x \leq \mathsf{C}(x)$ (extensive) (resp. $\mathsf{C}(x) \leq x$ (intensive)),

(b) $x \leq y$ implies $\mathsf{C}(x) \leq \mathsf{C}(y)$ (isotone or monotone),

(c) $\mathsf{CC}(x) = \mathsf{C}(x)$ (idempotent).

The following proposition explains the intimate relationship between GCs and COs and the next one provides a characterization of COs via the closed elements of $\mathcal{L}$ [16].

Proposition 2.25. If (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$, gf is a CO on $\mathcal{L}_1$ and fg is an IO on $\mathcal{L}_2$. Conversely, if C is a CO on $\mathcal{L}$, then $(\mathsf{C}, \mathrm{id}_L)$, where id_L is the identity function on L, is a GC between $\mathcal{L}$ and $\mathsf{C}(L)$.

Proof.

($\Rightarrow$) Properties 2.14.(b), 2.14.(c), and 2.14.(d).

($\Leftarrow$) For all $x \in L, y = c(z) \in c(L)$,

$$c(x) \leq y \Rightarrow x \leq y \Rightarrow x \leq \mathrm{id}(y) \Rightarrow c(x) \leq c(\mathrm{id}(y)) = cc(z) = c(z) = y.$$

□

Proposition 2.26. If C (resp. I) is a closure (resp. interior) operator on L, then $\mathsf{C}(x) = \bigsqcap\{\, y \in \mathsf{C}(L) \mid x \leq y \,\}$ (resp. $\mathsf{I}(x) = \bigsqcup\{\, y \in \mathsf{I}(L) \mid y \leq x \,\}$).

Proof. $c(x)$ is a lower bound of $X = \{\, y \in c(L) \mid x \leq y \,\}$, since for all $y \in X$, $x \leq y \Rightarrow c(x) \leq c(y) = y$. Moreover, $c(x) \in X$, since $x \leq c(x)$ and $c(x) \in c(L)$. Hence, $c(x) = \bigsqcap X$. □

2.2.4 Galois connections between lattices with some form of negation

In this subsection, we prove some useful facts about GCs between lattices equipped with a sort of *negation*: a unary operation satisfying certain properties. The results will be useful in Section 3.1 and Appendix A, but they may carry an independent interest.

Lemma 2.27. Consider $(L_1, \leq, *)$ and $(L_2, \leq, *)$, where

- $\mathcal{L}_1 = (L_1, \leq)$ and $\mathcal{L}_2 = (L_2, \leq)$ are lattices,
- each $*$ is a unary operator such that $x \leq y \Rightarrow *y \leq *x$.

If

- (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$,

- f, g respect $*$,

then $(L_1, \leq, *) \cong (L_2, \leq, *)$, with isomorphism f and $g = f^{-1}$.

Proof. From property 2.14.(d), for all $x \in L_1$, $x \leq gf(x)$ and $*x \leq gf(*x)$. From the defining property of $*$, we get

$$*gf(x) \leq *x \leq gf(*x)$$

and, since f and g respect $*$ and $\leq$ is a partial order, $gf(*x) = *gf(x) = *x$. Property 2.14.(e) implies that $x \in g(L_2)$. Hence, $L_1 = g(L_2)$. Similarly, $L_2 = f(L_1)$; by property 2.14.(j), $\mathcal{L}_1 \cong \mathcal{L}_2$ with isomorphism f and g its inverse. Since f and g respect $*$, $(L_1, \leq, *) \cong (L_2, \leq, *)$ with isomorphism f and $g = f^{-1}$. □

Lemma 2.28. Consider $(L_1, \sqcap, \sqcup, *)$ and $(L_2, \sqcap, \sqcup, *)$, where

- $\mathcal{L}_1 = (L_1, \sqcap, \sqcup)$ and $\mathcal{L}_2 = (L_2, \sqcap, \sqcup)$ are lattices,
- each $*$ is a unary operator such that $*(x \sqcup y) = *x \sqcap *y$ and $* * x = x$.

If

- (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$,
- $f(L_1)$, $g(L_2)$ are isomorphic subsets of $(L_1, \sqcap, \sqcup, *)$ and $(L_2, \sqcap, \sqcup, *)$,

then $g(L_2)$ is a substructure of $(L_1, \sqcap, \sqcup, *)$ (i.e. $g(L_2)$ is closed with respect to $\sqcap$, $\sqcup$, and $*$); of course, the dual holds for $f(L_1)$.

Proof. Let $x \in g(L_2)$ and $y = f(x) \in f(L_1)$. Since $x \in g(L_2)$ and (f, g) is a GC, $x = gf(x) = g(y)$. Since g is an isomorphism and $y \in f(L_1)$, $g(*y) = *g(y) = *x$, so $*x \in g(L_2)$.

Now consider $x_1, x_2 \in g(L_2)$. Property 2.14.(i) implies that $g(L_2)$ is closed with respect to $\sqcap$ and $f(L_1)$ is closed with respect to $\sqcup$. Thus, $x_1 \sqcap x_2 \in g(L_2)$; since $*x_1, *x_2 \in g(L_2)$, and g, being an isomorphism, respects $*$ for elements of $f(L_1)$, it follows that:

$$\begin{aligned} x_1 \sqcup x_2 = *(*x_1 \sqcap *x_2) &= *(*gf(x_1) \sqcap *gf(x_2)) \\ &= g(*(*f(x_1) \sqcap *f(x_2))) \in g(L_2). \end{aligned}$$

□

The following lemma provides a sufficient condition for a GC to respect a covariant negation (that is, an operator akin to conflation in the truth order of bilattices): it only has to respect it in the function images.

Lemma 2.29. Consider $(L_1, \leq, \star)$ and $(L_2, \leq, \star)$, where

- $\mathcal{L}_1 = (L_1, \leq)$ and $\mathcal{L}_2 = (L_2, \leq)$ are lattices,
- each $\star$ is a unary operator such that [2] $x \leq y \Rightarrow \star x \leq \star y$ and $\star \star x = x$.

If

- (f, g) is a GC between $\mathcal{L}_1$ and $\mathcal{L}_2$,

[2] **Addendum:** $\star \star x = x$ was missing in the previous version

- f, g respect $\star$ in $g(L_2)$ and $f(L_1)$,

then f, g respect $\star$.

Proof. By property 2.14.(g),

$$\star g(y) = \star\bigsqcup\{\, x \mid f(x) \leq y \,\} = \bigsqcup\{\, \star x \mid f(x) \leq y \,\} = \bigsqcup\{\, x \mid f(\star x) \leq y \,\},$$
$$g(\star y) = \bigsqcup\{\, x \mid f(x) \leq \star y \,\} = \bigsqcup\{\, x \mid \star f(x) \leq y \,\},$$
$$\star f(x) = \star\bigsqcap\{\, y \mid x \leq g(y) \,\} = \bigsqcap\{\, \star y \mid x \leq g(y) \,\} = \bigsqcap\{\, y \mid x \leq g(\star y) \,\},$$
$$f(\star x) = \bigsqcap\{\, y \mid \star x \leq g(y) \,\} = \bigsqcap\{\, y \mid x \leq \star g(y) \,\}.$$

For all $x \in L_1$, since $x \leq gf(x)$, it follows that $\star x \leq \star gf(x)$. By hypothesis, $\star x \leq g(\star f(x))$, so $\star f(x) \in \{\, y \mid \star x \leq g(y) \,\}$ and, since $f(\star x)$ is a lower bound of this set,

$$f(\star x) \leq \star f(x). \tag{*}$$

For all $y \in L_2$, since $fg(y) \leq y$, it follows that $\star fg(y) \leq \star y$. From inequality (*) above, we get $f(\star g(y)) \leq \star y$, so $\star g(y) \in \{\, x \mid f(x) \leq \star y \,\}$; since $g(\star y)$ is an upper bound of this set, $\star g(y) \leq g(\star y)$.

Now consider $x \in L_1$ such that $\star f(x) \leq y$. Inequality (*) implies that $f(\star x) \leq y$, so $\{\, x \mid \star f(x) \leq y \,\} \subseteq \{\, x \mid f(\star x) \leq y \,\}$; hence, $g(\star y) \leq \star g(y)$. The antisymmetric property of $\leq$ gives

$$g(\star y) = \star g(y).$$

Hence, for all $x \in L_1$, $\{\, y \mid x \leq g(\star y) \,\} = \{\, y \mid x \leq \star g(y) \,\}$, so $f(\star x) = \star f(x)$. □

2.3 Bilattices

We will review the basic definitions and facts about bilattices, mainly for establishing notation and terminology. The interested reader is referred to M. Fitting's survey [25] and O. Arieli's tutorial [3]. Bilattices are sets equipped with two lattice orders, $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$, along with some 'compatibility' requirements which 'bind' the two structures together. The connotation embodied in the orderings is that t denotes the 'degree of truth' and k denotes the 'degree of knowledge'. There is no entrenched terminology in the bilattice literature but usually the term 'bilattice' denotes a structure with a negation operation. Structures with two orders and no negation operation are called *pre-bilattices*; according to [34, p. 8], this approach is becoming standard.

2.3.1 Pre-bilattices

Definition 2.30. $\mathcal{B} = (B, \leq_{\mathrm{t}}, \leq_{\mathrm{k}})$ is called a *pre-bilattice* iff $(B, \leq_{\mathrm{t}})$ and $(B, \leq_{\mathrm{k}})$ are lattices; it is *bounded* iff both lattices are bounded; it is *complete* iff both lattices are complete.

There is no general agreement on the definition of pre-bilattices. Some authors require the underlying lattices to be bounded ([25]), others require them to be complete ([27, 4, 3]), and in many cases no restriction is given ([10, 28]).

Moreover, pre-bilattices are usually required to have at least two incomparable elements in each of their orders; we have not included this requirement in the definition of pre-bilattices, but we have done so for bilattices; when needed, we will differentiate between *trivial* and *non-trivial* pre-bilattices: non-trivial pre-bilattices have the said property, while trivial pre-bilattices may not.

$\leq_t$ is called the **truth order**; its meet and join will be denoted by $\wedge$ and $\vee$ respectively. $\leq_k$ is called the **knowledge order**; its meet and join of $\leq_k$ will be denoted by $\otimes$ (*consensus*) and $\oplus$ (*gullibility*) respectively. When they exist, the top and bottom elements of $\leq_t$ will be denoted by $\mathtt{t}$ and $\mathtt{f}$ respectively, while those of $\leq_k$ will be denoted by $\top$ and $\bot$ respectively.

Observe that the definition of a pre-bilattice is symmetric: if we swap the two lattices, we also get a bilattice. This gives rise to one more duality, so each time we have proven a statement on bilattices we can swap $\leq_t, \wedge, \vee$ with $\leq_k, \otimes, \oplus$ (respectively) and vice-versa to get another statement 'for free'. This duality can be combined with the duality of each of the two orders, so we have a 'prove one , get four' situation. We will see some examples below.

Definition 2.31. Two pre-bilattices $\mathcal{B}_1 = (B_1, \leq_t, \leq_k)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k)$ are called *isomorphic* iff there exists a bijection $\phi\colon B_1 \to B_2$ such that $x \leq_t y \Leftrightarrow \phi(x) \leq_t \phi(y)$ and $x \leq_k y \Leftrightarrow \phi(x) \leq_k \phi(y)$.

What makes a (pre)-bilattice different from a pair of lattices is the connection between the lattices that comprise it; in general, two kinds of connections have been studied: connections that bind the two lattices with algebraic properties and connections that arise from the usage of one or more involution operators.

Definition 2.32. Let $\mathcal{B} = (B, \leq_t, \leq_k)$ be a pre-bilattice. $\mathcal{B}$ is *interlaced* iff for all $w, x, y, z \in B$,

(a) $x \leq_t y$ and $w \leq_t z$ imply $x \otimes w \leq_t y \otimes z$ and $x \oplus w \leq_t y \oplus z$,

(b) $x \leq_k y$ and $w \leq_k z$ imply $x \wedge w \leq_k y \wedge z$ and $x \vee w \leq_k y \vee z$.

$\mathcal{B}$ is *distributive* iff for all $+, \cdot \in \{\wedge, \vee, \otimes, \oplus\}$ and $x, y, z \in B$,

$$x + (y \cdot z) = (x + y) \cdot (x + z).$$

Proposition 2.33. Every distributive pre-bilattice is interlaced.

Proof. Let $a_1 \leq_t a_2$ and $b_1 \leq_t b_2$. Then, by distributivity,

$$\begin{aligned} a_1 \otimes b_1 &= (a_1 \wedge a_2) \otimes (b_1 \wedge b_2) \\ &= (a_1 \otimes b_1) \wedge (a_1 \otimes b_2) \wedge (a_2 \otimes b_1) \wedge (a_2 \otimes b_2) \\ &\leq_t a_2 \otimes b_2 \end{aligned} \tag{2.1}$$

Duality within $\leq_k$ gives us $a_1 \oplus b_1 \leq_t a_2 \oplus b_2$. By bilattice duality, $a_1 \leq_k a_2$ and $b_1 \leq_k b_2$ implies $a_1 \wedge b_1 \leq_k a_2 \wedge b_2$ and $a_1 \vee b_1 \leq_k a_2 \vee b_2$. □

Proposition 2.34. In a bounded interlaced pre-bilattice,

(a) $\mathtt{t} \otimes \mathtt{f} = \bot$, $\mathtt{t} \oplus \mathtt{f} = \top$, $\top \vee \bot = \mathtt{t}$, $\top \wedge \bot = \mathtt{f}$,

(b) for each element x of the pre-bilattice

$$\bot \leq_t x \leq_t \mathtt{t} \Leftrightarrow \bot \leq_k x \leq_k \mathtt{t} \tag{2.2}$$

and dually

$$\mathtt{f} \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \bot \Leftrightarrow \bot \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{f} \tag{2.2*}$$

$$\top \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \mathtt{t} \Leftrightarrow \mathtt{t} \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \top \tag{2.2**}$$

$$\mathtt{f} \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \top \Leftrightarrow \mathtt{f} \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \top \tag{2.2***}$$

Proof.

(a) $\mathtt{t} \leq_{\mathrm{t}} \mathtt{t}$ and $\mathtt{f} \leq_{\mathrm{t}} \bot$, so interlacing gives $\mathtt{t} \otimes \mathtt{f} \leq_{\mathrm{t}} \bot$; dually (inversion of $\leq_{\mathrm{t}}$), $\bot \leq_{\mathrm{t}} \mathtt{f} \otimes \mathtt{t}$. Hence, $\mathtt{t} \otimes \mathtt{f} = \bot$. By duality within $\leq_{\mathrm{k}}$, we get $\mathtt{t} \oplus \mathtt{f} = \top$. Bilattice duality gives the two remaining equations.

(b) Let $\bot \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \mathtt{t}$. Of course, $\bot \leq_{\mathrm{k}} x$, so it is enough to prove that $x \leq_{\mathrm{k}} \mathtt{t}$. Trivially, $x \leq_{\mathrm{k}} x \oplus \mathtt{t}$ and $\bot \leq_{\mathrm{k}} \mathtt{t}$, so by interlacing $x \vee \bot \leq_{\mathrm{k}} \mathtt{t} \vee (x \oplus \mathtt{t})$. Since $\bot \leq_{\mathrm{t}} x$, it holds that $x \vee \bot = x$; since $\mathtt{t}$ is the maximum element of $\leq_{\mathrm{t}}$, $\mathtt{t} \vee (x \oplus \mathtt{t}) = \mathtt{t}$. Hence,

$$\bot \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \mathtt{t} \Rightarrow \bot \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{t}.$$

Although we could use duality to infer the opposite direction, we present the dual proof. Let $\bot \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{t}$. Of course, $x \leq_{\mathrm{t}} \mathtt{t}$, so it is enough to prove that $\bot \leq_{\mathrm{t}} x$. Trivially, $\bot \leq_{\mathrm{t}} \mathtt{t}$ and $x \wedge \bot \leq_{\mathrm{t}} x$, so by interlacing $\bot \otimes (x \wedge \bot) \leq_{\mathrm{t}} \mathtt{t} \otimes x$. Since $x \leq_{\mathrm{k}} \mathtt{t}$, it holds that $\mathtt{t} \otimes x = x$; since $\bot$ is the minimum element of $\leq_{\mathrm{k}}$, $\bot \otimes (x \wedge \bot) = \bot$. Hence,

$$\bot \leq_{\mathrm{t}} x \leq_{\mathrm{t}} \mathtt{t} \Leftarrow \bot \leq_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{t}$$

and we have established (2.2). (2.2*) is the dual version of (2.2) obtained by reversing $\leq_{\mathrm{t}}$, while (2.2**) is the dual version of (2.2) obtained by reversing $\leq_{\mathrm{k}}$; finally, (2.2***) is the dual version of (2.2) obtained by swapping the two lattices.

□

Notation. In pre-bilattices, we write $x \parallel y$ iff $x \parallel_{\mathrm{t}} y$ and $x \parallel_{\mathrm{k}} y$.

2.3.2 Bilattices

Definition 2.35. A unary operator $\neg$ on a pre-bilattice $\mathcal{B}$ is called a *negation* iff for all $x, y \in |\mathcal{B}|$,

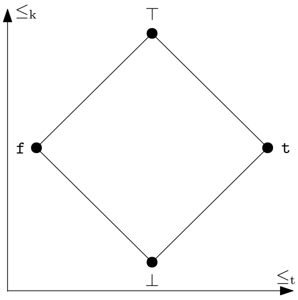
(a) $x \leq_{\mathrm{t}} y \Leftrightarrow \neg y \leq_{\mathrm{t}} \neg x$,

(b) $x \leq_{\mathrm{k}} y \Leftrightarrow \neg x \leq_{\mathrm{k}} \neg y$,

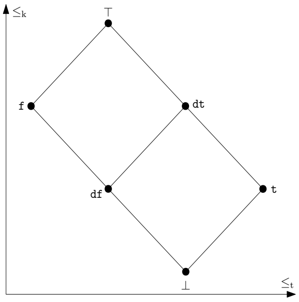
(c) $\neg\neg x = x$.

The operation of *conflation* (denoted $-$) is defined dually (swapping $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$). A non-trivial pre-bilattice with a negation (and maybe also a conflation) operator is called *bilattice*.

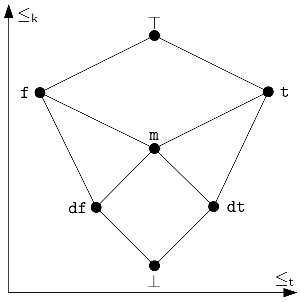
In the literature, bilattices are often defined over complete pre-bilattices; for the sake of generality, we have not made this assumption.



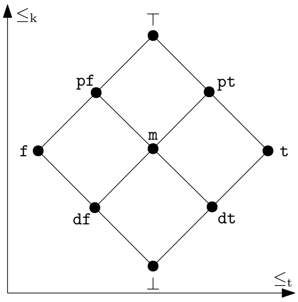
(a) $\mathcal{FOUR} \cong \mathbf{2} \odot \mathbf{2}$ is the simplest bilattice. It has both conflation and negation.



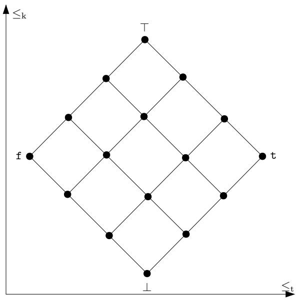
(b) The pre-bilattice $\mathcal{SIX} \cong \mathbf{2} \odot \mathbf{3}$ is distributive with neither conflation nor negation.



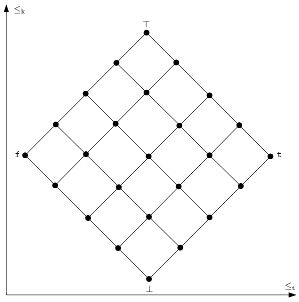
(c) $\mathcal{DEFAULT}$ is a non-interlaced bilattice. It has negation but no conflation.



(d) The distributive bilattice $\mathcal{NINE} \cong \mathbf{3} \odot \mathbf{3}$ has both conflation and negation.



(e) The distributive bilattice $\mathcal{SIXTEEN} \cong \mathbf{4} \odot \mathbf{4}$ has both conflation and negation.



(f) $\mathcal{TWENTY} - \mathcal{FIVE} \cong \mathbf{5} \odot \mathbf{5}$ is distributive and has both conflation and negation.

Figure 2.3: Some of the most common (pre-)bilattices.

Definition 2.36. In a bilattice with negation and conflation, if $\neg - = -\neg$, we say that negation and conflation *commute* and the bilattice is called *commutative*.

Definition 2.37. Two bilattices $\mathcal{B}_1 = (B_1, \leq_t, \leq_k, \neg)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k, \neg)$ are called *isomorphic* iff there exists a bijection $\phi\colon B_1 \to B_2$ such that $x \leq_t y \Leftrightarrow \phi(x) \leq_t \phi(y)$, $x \leq_k y \Leftrightarrow \phi(x) \leq_k \phi(y)$, and $\phi(\neg x) = \neg\phi(x)$.

We will use the symbol $\cong$ for all kinds of isomorphisms (bilattice, pre-bilattice). In general, the intended kind will be the one of the signature we are considering; in cases that confusion could arise, we will give a clarification.

Definition 2.38. If $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ (resp. $\mathcal{B} = (B, \leq_t, \leq_k, \neg, -)$) is a bilattice and $C \neq \emptyset$ is a subset of B, then $\mathcal{C} = (C, \leq_t, \leq_k, \neg)$ (resp. $\mathcal{C} = (C, \leq_t, \leq_k, \neg, -)$) is a *sub-bilattice* of $\mathcal{B}$ iff C is closed with respect to $\neg$ (resp. and also closed with respect to $-$) and $(C, \leq_t)$, $(C, \leq_k)$ are sublattices of $(B, \leq_t)$, $(B, \leq_k)$ respectively.

The notions of *interlaced, distributive*, and *complete* pre-bilattices extend to bilattices.

Remark 2.39. Any sub-(pre)-bilattice of an interlaced (resp. distributive) (pre)-bilattice is interlaced (resp. distributive).

Proof. Suppose $\mathcal{B}$ is complete and $\mathcal{C}$ is a sub-(pre)-bilattice of $\mathcal{B}$.

- Let $a, b, c, d \in C$ with $a \leq_t c$ and $b \leq_t d$ and $\cdot \in \{\otimes, \oplus\}$; then, $a \cdot_C b = a \cdot b \leq_t c \cdot d = c \cdot_C d$; similarly, if $a \leq_k c$ and $b \leq_k d$ and $\cdot \in \{\wedge, \vee\}$, then $a \cdot_C b = a \cdot b \leq_k c \cdot d = c \cdot_C d$. Hence, $\mathcal{C}$ is interlaced.
- Let $a, b, c \in C$ and $\cdot, + \in \{\wedge, \vee, \otimes, \oplus\}$; it holds that

$$a \cdot_C (b +_C c) = a \cdot (b + c) = (a \cdot b) + (a \cdot c) = (a \cdot_C b) +_C (a \cdot_C c),$$

so $\mathcal{C}$ is distributive.

□

Note (**Addendum**). A sub-(pre)-bilattice of a complete (pre)-bilattice is not necessarily complete. Consider the lattices $\mathcal{L} = (\mathbb{N}, \leq)$ and $\mathcal{L}_\infty = (\mathbb{N} \cup \infty, \leq)$, where $\leq$ is the usual order in natural numbers with the additional property that ∞ is its top element. Using the notion of product bilattices, defined in the next section, we can construct $\mathcal{B} = \mathcal{L} \odot \mathcal{L}$ and $\mathcal{B}_\infty = \mathcal{L}_\infty \odot \mathcal{L}_\infty$. $\mathcal{B}$ is a sub-bilattice of $\mathcal{B}_\infty$, but $\mathcal{B}_\infty$ is complete and $\mathcal{B}$ is not.

Notation. If $\mathcal{B}$ is a (pre-)bilattice, $\mathcal{B}^{\mathrm{op_t}}$ is $\mathcal{B}$ where $\leq_t$ is reversed, $\mathcal{B}^{\mathrm{op_k}}$ is $\mathcal{B}$ where $\leq_k$ is reversed, and $\mathcal{B}^{\mathrm{op}}$ is $\mathcal{B}$ where both $\leq_t$ and $\leq_k$ are reversed.

In bilattices, negation corresponds to mirroring in the direction of $\leq_t$. This is why bilattices like $\mathcal{FOUR}$, $\mathcal{SIXTEEN}$, $\mathcal{TWENTY-FIVE}$, and $\mathcal{DEFAULT}$ have negation while $\mathcal{SIX}$ does not, something that becomes clear when drawing them (see Figure 2.3).

Remark 2.40. If $\mathcal{B}$ is a bilattice with negation, $\neg$ is an isomorphism between $\mathcal{B}$ and $\mathcal{B}^{\mathrm{op_t}}$.

Proof. For all $x \in B$, $x = \neg\neg x$, so $\neg$ is a surjection.
For all $x, y \in B$, $\neg x = \neg y \Rightarrow x = \neg\neg x = \neg\neg y = y$, so $\neg$ is an injection.
By definition, $x \leq_t y \Leftrightarrow \neg x \geq_t \neg y$ and $x \leq_k y \Leftrightarrow \neg x \leq_k \neg y$.
$\neg$ respects $\neg$ trivially. □

Of course, dually, conflation corresponds to mirroring in the direction of $\leq_k$. Observe $\mathcal{DEFAULT}$ (Figure 2.3c), whose symmetry grants it with a negation but not with a conflation.

2.3.3 Product and residuated bilattices

Definition 2.41. Let $\mathcal{L}_1 = (L_1, \leq)$, $\mathcal{L}_2 = (L_2, \leq)$ be lattices. Define the *product bilattice* $\mathcal{L}_1 \odot \mathcal{L}_2 = (L_1 \times L_2, \leq_t, \leq_k)$, where

$$(a_1, b_1) \leq_t (a_2, b_2) \overset{\text{def}}{\Leftrightarrow} a_1 \leq a_2 \text{ and } b_2 \leq b_1,$$
$$(a_1, b_1) \leq_k (a_2, b_2) \overset{\text{def}}{\Leftrightarrow} a_1 \leq a_2 \text{ and } b_1 \leq b_2.$$

If $\mathcal{L}_1 = \mathcal{L}_2$, we can define a negation for the product bilattice:

$$\neg(a, b) \overset{\text{def}}{=} (b, a).$$

Moreover, if $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ has a *de Morgan complement*, i.e. an operator $\bar{\cdot}$ such that $\bar{\bar{x}} = x$ and $x \leq y \Rightarrow \bar{y} \leq \bar{x}$, we can add a conflation to the product bilattice, as follows:

$$-(a, b) = (\bar{b}, \bar{a}).$$

Corollary 2.42. In a product bilattice,

$$\begin{aligned}(a_1, b_1) \wedge (a_2, b_2) &= (a_1 \sqcap a_2, b_1 \sqcup b_2)\\(a_1, b_1) \vee (a_2, b_2) &= (a_1 \sqcup a_2, b_1 \sqcap b_2)\\(a_1, b_1) \otimes (a_2, b_2) &= (a_1 \sqcap a_2, b_1 \sqcap b_2)\\(a_1, b_1) \oplus (a_2, b_2) &= (a_1 \sqcup a_2, b_1 \sqcup b_2)\end{aligned}$$

Theorem 2.43 ([7, Theorem 3.3])**.** Every bounded interlaced bilattice is isomorphic to a product bilattice.

Corollary 2.44. The number of elements of any finite interlaced bilattice is not prime.

Proposition 2.45 ([7, Proposition 3.7])**.** Every (complete, distributive) bounded interlaced bilattice with negation is isomorphic to a product of some (complete, distributive) lattice with itself.

Corollary 2.46. The number of elements of any finite interlaced bilattice with negation is a perfect square.

Intuitively, a product bilattice $\mathcal{L}_1 \odot \mathcal{L}_2$ encodes the evidence for ($\mathcal{L}_1$) and against ($\mathcal{L}_2$) a proposition. [3, p. 11–14] and [25, Sections 2.2, 3.1]) contain more information and examples on product bilattices.

When the underlying lattice is residuated (Definition 2.23), the product bilattice has some extra properties.

Definition 2.47. Given a residuated lattice $\mathcal{L} = (L, \sqcap, \sqcup, \cdot, \backslash, /)$, one can algebraically construct a *residuated bilattice*

$$\mathcal{B} = \mathcal{L} \odot \mathcal{L} = (L \times L, \wedge, \vee, \otimes, \oplus, \neg, *, \leftarrow, \rightarrow)$$

as follows:

- $(L \times L, \wedge, \vee, \otimes, \oplus, \neg)$ is the product bilattice $(L, \sqcap, \sqcup) \odot (L, \sqcap, \sqcup)$.
- For $a = (a_1, a_2), b = (b_1, b_2) \in L \times L$,

$$\begin{aligned}(a_1, a_2) \rightarrow (b_1, b_2) &= ((a_1 \setminus b_1) \sqcap (a_2 \,/\, b_2), b_2 \cdot a_1)\\a \leftarrow b &= \neg a \rightarrow \neg b\\a * b &= \neg(b \rightarrow \neg a)\end{aligned}$$

Corollary 2.48. The definitions of these operators and the construction of residuated lattices imply [28, Proposition 2.2] that for all $a, b \in L \times L$,

$$a * b \leq_{\mathrm{t}} c \Leftrightarrow b \leq_{\mathrm{t}} a \rightarrow c \Leftrightarrow a \leq_{\mathrm{t}} c \leftarrow b$$

that is, (f_a, g_a) and (f_b, g_b), where

$$f_a(x) = a * x, \qquad g_a(y) = a \rightarrow y, \qquad f_b(x) = x * b, \qquad g_b(y) = y \leftarrow b,$$

are GCs in the $\leq_{\mathrm{t}}$ order of $\mathcal{B}$.

Note. Here, we avoided stating the 'actual' definition of residuated bilattices and actually defined them as constructions on residuated lattices; in fact, these constructions are called product residuated bilattices. By [28, Theorem 3.6], any residuated bilattice is isomorphic to a product residuated bilattice, so our indirect approach is valid.

CHAPTER 3

GALOIS BICONNECTIONS AND RELATED CONCEPTS

3.1 Requirements and constraints

Having quickly reviewed the 'classical' GCs, a discussion on the desired outcome of this research is in order. Obviously, an extension of this ubiquitous construction in the bilattice setting should preferably retain many of the elegant properties mentioned in Section 2.2. The idea is that an iterated '*back-and-forth*' passage between the two bilattices should provide a pair of isomorphic images and the proposed Galois biconnections (abbreviation GB) should unconditionally enjoy composability and invertibility (duality). To obtain an interesting notion of biclosure operator (abbreviation BCO) on bilattices, the symmetry of the definition is highly desirable. In addition to that, the proposed construction should reflect the bilattice structure to the largest possible extent, clearly differentiating from a simple pair of GCs and should ideally (and perhaps, conditionally) preserve the structural properties of the bilattices involved (completeness, distributivity, interlacing) when jumping to their images under the GB. Finally, it is certainly desirable to obtain mathematically useful characterizations of GBs and an array of interesting examples; moreover, it would be nice if GBs existed between any two bilattices.

The requirement that GBs have isomorphic images needs some further scrutiny regarding the kind of their isomorphism. Should one opt for a weaker notion and require that images are just *order-isomorphic*, or is it preferable to require an *algebraic bilattice isomorphism* of the images? The first case seems to violate the requirement that GBs should take into account the structure of bilattices. The second case has serious mathematical implications. The remark below will help us to make this precise.

Remark 3.1. Let (f, g) be a pair of functions between bilattices $\mathcal{B}_1$ and $\mathcal{B}_2$ such that f is a bilattice isomorphism between $g(B_2)$ and $f(B_1)$ and g is its inverse. Then, for all $x, x_1, x_2 \in B_1, y, y_1, y_2 \in B_2, \bullet \in \{\neg, -\}, + \in \{\wedge, \vee, \oplus, \otimes\}$, it holds that

$$f(\bullet g(y)) = \bullet fg(y), \qquad f(g(y_1) + g(y_2)) = fg(y_1) + fg(y_2),$$
$$g(\bullet f(x)) = \bullet gf(x), \qquad g(f(x_1) + f(x_2)) = gf(x_1) + gf(x_2).$$

Moreover, let (f', g') be a pair of functions between bilattices $\mathcal{B}_2$ and $\mathcal{B}_3$ such that both (f', g') and $(f'f, gg')$ have the same property as (f, g). Then, for all $z, z_1, z_2 \in B_3$

$$\neg f'fgg'(z) = f'f(\neg gg'(z)) = f'(\neg fgg'(z)) \tag{3.1}$$

$$\begin{aligned} f'f(gg'(z_1) \wedge gg'(z_2)) &= f'(fgg'(z_1) \wedge fgg'(z_2)) \\ &= f'fgg'(z_1) \wedge f'fgg'(z_2) \end{aligned} \tag{3.2}$$

and similarly for $-$, $\vee$, $\otimes$, and $\oplus$.

Hence, requiring that images are *isomorphic as bilattices* implies that (3.1), (3.2), and the equations involving the other bilattice operators are necessary conditions for composability. Intuitively, it seems that these conditions are very strong and would probably fail for many interesting cases, implying in turn that GBs defined this way would *not* be unconditionally composable. An effort to make the definition strict enough to entail them may result in triviality such as an isomorphism (see Example A.2 for such a definition). Similar considerations apply to most restrictions that could be imposed on the images of the functions that form a GB.

Since bilattices contain lattices with some form of negation, the limits of what we can ask for a GC between lattices with negation apply also for bilattices. For example, if we require that GBs have bilattice-isomophic images and that they also make a GC between one of the lattices forming each bilattice, Lemma 2.28 leads to a tighter structure. This kind of restriction is more important in bilattices, since they comprise two lattices (and the corresponding negation operators) firing interesting interactions: for example, they are enough to invalidate definitions of GBs that seem natural at first glance (Appendix A).

Taking stock of the situation: these considerations lead us to distinguish **four species of GBs**. We make a distinction between *strong* and *regular* GBs, according to what the definition entails for image isomorphism: **strong GBs** have bilattice-isomorphic images, while **regular GBs** have order-isomorphic (isomorphic as pre-bilattices) images. We further distinguish between **bidirectional GBs**, defined as pairs of GCs (thus, consisting of four functions), and **unidirectional GBs**, defined as a 'classical' GC with additional properties that reflect (a part of) the bilattice structure.

3.2 Galois biconnections between pre-bilattices

3.2.1 Regular bidirectional Galois biconnections

A straightforward proposal is to define Galois biconnections as pairs of pairs of functions, the first pair being a GC with respect to $\leq_t$, the second being a GC with respect to $\leq_k$. In order to end up with a single well-defined image of the GB in each bilattice, we also add the requirement that the images of the two comprising GCs coincide.

Definition 3.2. $((f_t, g_t), (f_k, g_k))$ is called a *regular bidirectional Galois biconnection* between $\mathcal{B}_1 = (B_1, \leq_t, \leq_k)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k)$ iff

(a) (f_t, g_t) is a GC between $(B_1, \leq_t)$, $(B_2, \leq_t)$,

(b) (f_k, g_k) is a GC between $(B_1, \leq_k)$, $(B_2, \leq_k)$,

(c) $f_t(B_1) = f_k(B_1)$ and $g_t(B_2) = g_k(B_2)$

(d) $\forall x \in g_t(B_2) : f_t(x) = f_k(x)$ and $\forall y \in f_t(B_1) : g_t(y) = g_k(y)$.

A few remarks: it is straightforward that, if ϕ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$, then $((\phi, \phi^{-1}), (\phi, \phi^{-1}))$ is a regular bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$. Since the images of f_k and f_t are the same, we will usually denote both of them with $f(B_1)$. Similarly, the image of (both) g_t and g_k will usually be denoted by $g(B_2)$. The following propositions list properties of regular bidirectional GBs, generalising those of GCs.

Proposition 3.3 (Properties of regular bidirectional Galois biconnections)**.** Let $((f_t, g_t), (f_k, g_k))$ be a regular bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

(a) $((g_t, f_t), (g_k, f_k))$ is a regular bidirectional GB between $\mathcal{B}_2^{\mathrm{op}}$ and $\mathcal{B}_1^{\mathrm{op}}$.

(b) f_t, g_t are monotone with respect to $\leq_{\mathrm{t}}$;
f_k, g_k are monotone with respect to $\leq_{\mathrm{k}}$.

(c) For $i, j, l \in \{ k, t \}$, $f_j g_l f_i = f_i$, $g_j f_l g_i = g_i$.

(d) For $i \in \{ k, t \}$, $\forall x \in B_1 : g_i f_i(x) \geq_i x$ and $\forall y \in B_2 : f_i g_i(y) \leq_i y$.

(e) For $i \in \{ k, t \}$,

$$\forall x \in B_1 : x \in g(B_2) \Leftrightarrow x = g_i f_i(x)$$
$$\forall y \in B_2 : y \in f(B_1) \Leftrightarrow y = f_i g_i(y)$$

(f) For $i \in \{ k, t \}$,

f_i is an injection iff g_i is a surjection iff $\forall x \in B_1 : g_i f_i(x) = x$
g_i is an injection iff f_i is a surjection iff $\forall y \in B_2 : f_i g_i(y) = y$

(g) For $i \in \{ k, t \}$, f_i and g_i uniquely determine one another; in fact,

$$g_k(y) = \bigoplus \{ x \mid f_k(x) \leq_{\mathrm{k}} y \}, \qquad f_k(x) = \bigotimes \{ y \mid x \leq_{\mathrm{k}} g_k(y) \},$$
$$g_t(y) = \bigvee \{ x \mid f_t(x) \leq_{\mathrm{t}} y \}, \qquad f_t(x) = \bigwedge \{ y \mid x \leq_{\mathrm{t}} g_t(y) \}.$$

(h) f_k preserves $\oplus$; g_k preserves $\otimes$; f_t preserves $\vee$; g_t preserves $\wedge$.

(i) If $P \subseteq g(B_2)$ and $Q \subseteq f(B_1)$ (and, if P or Q is infinite, provided its meet, join, consensus and gullibility exist),

$$\bigoplus_{g(B_2)} P = g_k f_k \left(\bigoplus P\right), \qquad \bigotimes_{f(B_1)} Q = f_k g_k \left(\bigotimes Q\right),$$
$$\bigotimes_{g(B_2)} P = \bigotimes P, \qquad \bigoplus_{f(B_1)} Q = \bigoplus Q,$$
$$\bigvee_{g(B_2)} P = g_t f_t \left(\bigvee P\right), \qquad \bigwedge_{f(B_1)} Q = f_t g_t \left(\bigwedge Q\right),$$
$$\bigwedge_{g(B_2)} P = \bigwedge P, \qquad \bigvee_{f(B_1)} Q = \bigvee Q,$$

so $f(B_1)$ is closed with respect to $\oplus$ and $\vee$, $g(B_2)$ is closed with respect to $\otimes$ and $\wedge$, and, moreover, $f(B_1)$ and $g(B_2)$ are (maybe trivial) pre-bilattices.

(j) $f(B_1) \cong g(B_2)$ (as pre-bilattices); in fact, f_k (resp. f_t) is an isomorphism between them and g_k (resp. g_t) is its inverse.

(k) If $\mathcal{B}_1$ or $\mathcal{B}_2$ is complete, then so are both $f(B_1)$ and $g(B_2)$. Moreover, if one of the following holds:

- f_k is $\leq_t$-monotone, f_t is $\leq_k$-monotone, and $\mathcal{B}_1$ is interlaced
- g_k is $\leq_t$-monotone, g_t is $\leq_k$-monotone, and $\mathcal{B}_2$ is interlaced,

then both $f(B_1)$ and $g(B_2)$ are interlaced.

(l) If $\mathcal{B}_1$ is bounded, then

- $\mathtt{t}, \top \in g(B_2)$,
- $f(B_1)$ is bounded by $f(\top)$, $f(\mathtt{t})$, $f(\bot)$, $f(\mathtt{f})$.

If $\mathcal{B}_2$ is bounded, then

- $\mathtt{f}, \bot \in f(B_1)$,
- $g(B_2)$ is bounded by $g(\top)$, $g(\mathtt{t})$, $g(\bot)$, $g(\mathtt{f})$.

Proof. Properties (b), (d)–(i), and (l) follow directly from Proposition 2.14.

(a) Follows from the definition and the corresponding property of GCs.

(c) If $i = j = l$, the property follows from the corresponding property of GCs. Otherwise, the property follows from the definition and the case where $i = j = l$; for example, $f_k g_k f_t = f_k g_t f_t = f_t g_t f_t = f_t$.

(j) f_k is an isomorphism between $(f(B_1), \leq_k)$ and $(g(B_2), \leq_k)$ with inverse g_k, since (f_k, g_k) is a GC. As for the $\leq_t$ order, for all $x_1, x_2 \in g(B_2)$,

$$x_1 \leq_t x_2 \Leftrightarrow f_t(x_1) \leq_t f_t(x_2) \Leftrightarrow f_k(x_1) \leq_t f_k(x_2).$$

Hence, f_k is an isomorphism between $f(B_1)$ and $g(B_2)$; from property (c), g_k is its inverse. Similarly, we can show that f_t is also an isomorphism between $f(B_1)$ and $g(B_2)$ with inverse g_t.

(k) Without loss of generality, suppose that $\mathcal{B}_1$ is complete (if this happens for $\mathcal{B}_2$, we work dually). Then, by the corresponding fact for GCs, both the $\leq_k$ and the $\leq_t$ reduct of $g(B_2)$ are complete. Hence, $g(B_2)$ is complete and, since $f(B_1)$ is isomorphic to $g(B_2)$, it is complete as well.

Suppose now, without loss of generality, that f_k is monotone with respect to $\leq_t$, f_t is monotone with respect to $\leq_k$, and $\mathcal{B}_1$ is interlaced (if $\mathcal{B}_2$ is interlaced, g_k is $\leq_t$-monotone, and g_t is $\leq_k$-monotone, we work dually). Let $a, b, c, d \in g(B_2)$ with $a \leq_t c$ and $b \leq_t d$; then,

$$a \otimes_{g(B_2)} b = a \otimes b \leq_t c \otimes d = c \otimes_{g(B_2)} d$$

and, by the interlacing conditions and by monotonicity of g_t and f_k,

$$\begin{aligned} a \oplus_{g(B_2)} b &= g_k f_k(a \oplus b) = g_t f_k(a \oplus b) \\ &\leq_t g_t f_k(c \oplus d) = g_k f_k(c \oplus d) \\ &= c \oplus_{g(B_2)} d. \end{aligned}$$

Similarly, if $a \leq_k c$ and $b \leq_k d$, then $a \wedge_{g(B_2)} b \leq_k c \wedge_{g(B_2)} d$ and $a \vee_{g(B_2)} b \leq_k c \vee_{g(B_2)} d$. Hence, $g(B_2)$ is interlaced, and, since they are isomorphic, $f(B_1)$ is interlaced as well.

□

The following theorem states the *conditions* for the composability of regular bidirectional GBs. Note that *unconditional* composability can be regained if we drop the last two clauses of Definition 3.2. In this case, however, f_k and f_t (and, of course, g_t and g_k) need not have the same image. Thus, the two bilattices related with these functions would not share a skeleton, an annoying feature that raises serious doubts about the usefulness of such a construction.

Proposition 3.4 (Composability of regular bidirectional Galois biconnections)**.** Let $((f_t, g_t), (f_k, g_k))$ be a regular bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ and $((f'_t, g'_t), (f'_k, g'_k))$ be a regular bidirectional GB between $\mathcal{B}_2$ and $\mathcal{B}_3$. Then, $((f'_t f_t, g_t g'_t), (f'_k f_k, g_k g'_k))$ is a regular bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_3$ iff

$$f'_k f(B_1) = f'_t f(B_1), \qquad \forall y \in g' f' f(B_1) : g_t(y) = g_k(y),$$
$$g_k g'(B_3) = g_t g'(B_3), \qquad \forall y \in f g g'(B_3) : f'_t(y) = f'_k(y).$$

Proof.

(⇒) By definition,

$$f'_k f(B_1) = f'_k f_k(B_1) = f'_t f_t(B_1) = f'_t f(B_1)$$

and similarly for $g_k g'(B_3) = g_t g'(B_3)$.

Let $y \in g' f' f(B_1)$ and let $z = f'_k(y)$ (so, by the fact that $f'_k f(B_1) = f'_t f(B_1)$ and by property 3.3.(c), we get $z \in f'_k f(B_1)$ and $y = g'_k(z) = g'_t(z)$). Since the two regular bidirectional GBs compose, $g_k g'_k(z) = g_t g'_t(z)$, so

$$g_k(y) = g_k g'_k(z) = g_t g'_t(z) = g_t(y).$$

Similarly, f'_t and f'_k coincide on $f g g'(B_3)$.

(⇐) Since GCs always compose, it suffices to demonstrate that properties (c) and (d) of the definition hold.

First, since $f'_k f(B_1) = f'_t f(B_1)$ (resp. $g_k g'(B_3) = g_t g'(B_3)$), we have

$$f'_k f_k(B_1) = f'_k f(B_1) = f'_t f(B_1) = f'_t f_t(B_1),$$

(and similarly $g_k g'_k(B_3) = g_t g'_t(B_3)$) so it makes sense to use $f' f(B_1)$ (resp. $g g'(B_3)$) for both.

Let $z \in f' f(B_1)$ and $y = g'_t(z)$ (so $y = g'_k(z)$). By hypothesis,

$$g_t g'_t(z) = g_t(y) = g_k(y) = g_k g'_k(z).$$

Similarly, for all $x \in g g'(B_3)$, $f'_k f_k(x) = f'_t f_t(x)$.

□

An alternative characterization of the GBs we introduced in this section, follows.

Proposition 3.5 (Characterisation of regular bidirectional Galois biconnections)**.** $((f_t, g_t), (f_k, g_k))$ is a regular bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff

(a) g_t and f_t are $\leq_t$-monotone; g_k and f_k are $\leq_k$-monotone,

(b) For all $x \in B_1, y \in B_2$,

$$x \leq_t g_t f_t(x), \qquad f_t g_t(y) \leq_t y,$$
$$x \leq_k g_k f_k(x), \qquad f_k g_k(y) \leq_k y$$

(c) $f_t g_k f_k = f_k$, $f_k g_t f_t = f_t$, $g_k f_t g_t = g_t$, $g_t f_k g_k = g_k$.

Proof. By the characterisation of GCs, it suffices to show that the last two properties of the definition are equivalent to the last property of the characterisation.

($\Rightarrow$) Property 3.3.(c).

($\Leftarrow$) Let $y = f_t(x) \in f_t(B_1)$; since $f_t = f_k g_t f_t$, $y \in f_k(B_1)$. Similarly, $f_t g_k f_k = f_k$ implies $f_k(B_1) \subseteq f_t(B_1)$, so $f_k(B_1) = f_t(B_1) = f(B_1)$. Moreover, for each $y = f_t(x) \in f(B_1)$,

$$g_t(y) = g_t f_t(x) = (g_k f_t g_t)(f_t(x)) = g_k(f_t g_t f_t(x)) = g_k f_t(x) = g_k(y),$$

so g_t and g_k coincide on $f(B_1)$. The rest of the proof works dually.

□

Notation (Drawing bidirectional Galois biconnections). We can draw bidirectional GBs in a manner similar to the one for GCs. Since f_k and f_t (and, dually, g_k and g_t) share a common image, we keep drawing each of their elements with a filled circle and each other element with an empty circle. The only difference is that each element may have two different closures, one for $\leq_k$ and one for $\leq_t$; we draw the resulting levels as follows: levels for $\leq_k$ are drawn with solid coloring and levels for $\leq_t$ are drawn with striped coloring, as in Figure 3.1.

Example 3.6. Consider $\mathcal{NINE}$ and let $\mathcal{B}$ be the pre-bilattice that is induced from the subset $\{\top, \bot, \mathtt{t}, \mathtt{f}, \mathtt{df}\}$ of $|\mathcal{NINE}|$. Figure 3.2 features a regular bidirectional GB between $\mathcal{NINE}$ and $\mathcal{B}$. Observe that the set of stable elements is not an interlaced pre-bilattice, even though $\mathcal{NINE}$ is; this differentiates regular bidirectional GBs from the other kinds of GBs we present in this thesis. Moreover, neither f_k is monotone with respect to $\leq_t$ (since $\mathtt{df} \leq_t \mathtt{m}$ in $\mathcal{NINE}$ but $\mathtt{df}$ is not $\leq_t$-related to $\top$ in the other pre-bilattice) nor is f_t monotone with respect to $\leq_k$ (again, observe $\mathtt{df}$ and $\mathtt{m}$).

Example 3.7 ('Minimum' regular bidirectional Galois biconnections). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bounded bilattices. Then, $((f_t, g_t), (f_k, g_k))$, where

$$f_t(x) = \begin{cases} \mathtt{f} & \text{if } x \leq_t \top \\ \bot & \text{otherwise} \end{cases} \qquad g_t(y) = \begin{cases} \mathtt{t} & \text{if } \bot \leq_t y \\ \top & \text{otherwise} \end{cases}$$

$$f_k(x) = \begin{cases} \bot & \text{if } x \leq_k \mathtt{t} \\ \mathtt{f} & \text{otherwise} \end{cases} \qquad g_k(y) = \begin{cases} \top & \text{if } \mathtt{f} \leq_k y \\ \mathtt{t} & \text{otherwise} \end{cases}$$

is a regular bidirectional GB between the two bilattices. Indeed, by Example 2.22, (f_t, g_t) is a GC between the $\leq_t$ orders and (f_k, g_k) is a GC between the $\leq_k$ orders; moreover, $g_t f_t$ and $g_k f_k$ (resp. $f_t g_t$ and $f_k g_k$) have the same stable elements, $\top$ and $\mathtt{t}$ (resp. $\mathtt{f}$ and $\bot$), and f_t, f_k (resp. g_t, g_k) are equal on them. Observe that, in this case, the stable elements form trivial pre-bilattices.

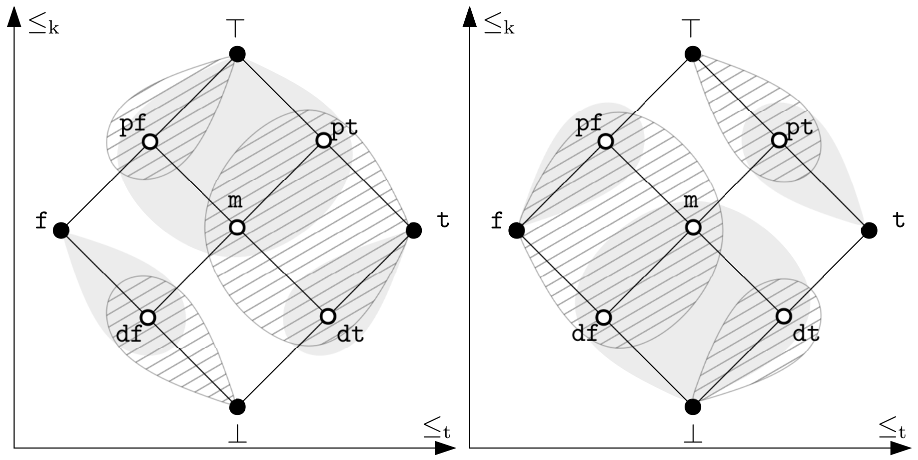


Figure 3.1: Drawing a regular bidirectional GB between $\mathcal{NINE}$ and itself, with $\top$, $\bot$, t, and f as stable elements.

3.2.2 Regular unidirectional Galois biconnections

Inside the preceding section, a GB has been defined as a pair of GCs. Another possibility is to define a GB between two bilattices as a GC between some projections of the bilattices at hand, equipped with some additional properties. The current section along with Section 3.3.2 discuss this species of GBs. So, we proceed to define a GB (f, g) as a GC with respect to $\leq_k$, whose functions are both $\leq_t$-monotone. Of course, this definition has a dual construction requesting that the pair (f, g) makes a GC with respect to $\leq_t$ and both its functions are $\leq_k$-monotone; we will show that the two dual definitions do not coincide. Since there are two kinds of regular unidirectional GBs we could distinguish them by referring to the one presented here as **k-directional** and to its dual as **t-directional**.

Definition 3.8. (f, g) is called a *regular k-directional Galois biconnection* between $\mathcal{B}_1 = (B_1, \leq_t, \leq_k)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k)$ iff

(a) (f, g) is a GC between $(B_1, \leq_k)$, $(B_2, \leq_k)$

(b) Both f and g are monotone with respect to $\leq_t$

As expected, an isomorphism ϕ between $\mathcal{B}_1$ and $\mathcal{B}_2$ gives rise to a regular unidirectional GB (ϕ, ϕ^{-1}) between $\mathcal{B}_1$ and $\mathcal{B}_2$. We can collect a list of properties of regular k-directional GBs.

Proposition 3.9 (Properties of regular k-directional Galois biconnections)**.** Let (f, g) be a regular k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

(a) (g, f) is a regular k-directional GB between $\mathcal{B}_2^{\text{op}}$ and $\mathcal{B}_1^{\text{op}}$.

(b) f and g are monotone (with respect to both $\leq_k$ and $\leq_t$).

(c) $fgf = f$, $gfg = g$.

(d) gf is a CO on $(B_1, \leq_k)$; fg is an IO on $(B_2, \leq_k)$.

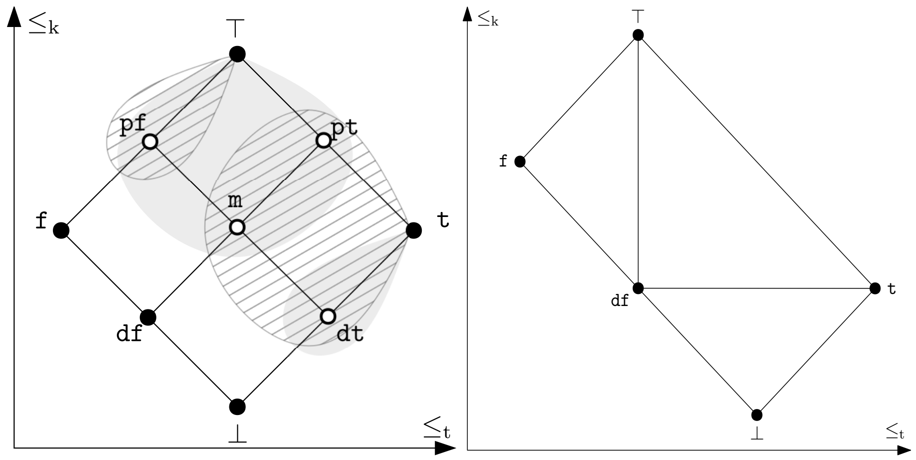


Figure 3.2: A regular bidirectional GB between $\mathcal{NINE}$ and a pre-bilattice it contains.

(e) $\forall x \in B_1 : x \in g(B_2) \Leftrightarrow x = gf(x)$
$\forall y \in B_2 : y \in f(B_1) \Leftrightarrow y = fg(y)$

(f) f is an injection iff g is a surjection iff $\forall x \in B_1 : gf(x) = x$;
g is an injection iff f is a surjection iff $\forall y \in B_2 : fg(y) = y$.

(g) f and g uniquely determine one another; in fact,

$$g(y) = \bigoplus \{ x \mid f(x) \leq_k y \}, \qquad f(x) = \bigotimes \{ y \mid x \leq_k g(y) \}.$$

(h) f preserves $\oplus$; g preserves $\otimes$.

(i) $g(B_2)$ and $f(B_1)$ are pre-bilattices; in particular, if $P \subseteq g(B_2)$ and $Q \subseteq f(B_1)$ (and, if P or Q is infinite, provided its meet, join, consensus and gullibility exist),

$$\bigoplus_{g(B_2)} P = gf\left(\bigoplus P\right), \qquad \bigotimes_{f(B_1)} Q = fg\left(\bigotimes Q\right),$$
$$\bigotimes_{g(B_2)} P = \bigotimes P, \qquad \bigoplus_{f(B_1)} Q = \bigoplus Q,$$
$$\bigvee_{g(B_2)} P = gf\left(\bigvee P\right), \qquad \bigwedge_{f(B_1)} Q = fg\left(\bigwedge Q\right),$$
$$\bigwedge_{g(B_2)} P = gf\left(\bigwedge P\right), \qquad \bigvee_{f(B_1)} Q = fg\left(\bigvee Q\right),$$

so $f(B_1)$ is closed with respect to $\oplus$ and $g(B_2)$ is closed with respect to $\otimes$.

(j) $f(B_1) \cong g(B_2)$ (as pre-bilattices); in fact, (the restrictions of) f and g are inverse isomorphisms between $f(B_1)$ and $g(B_2)$.

(k) If $\mathcal{B}_1$ or $\mathcal{B}_2$ is complete (resp. interlaced), then so are both $f(B_1)$ and $g(B_2)$.

(l) If $\mathcal{B}_1$ is bounded, then

- $\top \in g(B_2)$,
- $f(B_1)$ is bounded by $f(\top)$, $f(\mathtt{t})$, $f(\bot)$, $f(\mathtt{f})$.

If $\mathcal{B}_2$ is bounded, then

- $\bot \in f(B_1)$,
- $g(B_2)$ is bounded by $g(\top)$, $g(\mathtt{t})$, $g(\bot)$, $g(\mathtt{f})$.

Proof. Properties (c)–(h) follow directly from the fact that (f, g) is a GC between the $\leq_{\mathrm{k}}$ orders of $\mathcal{B}_1$ and $\mathcal{B}_2$.

(a)–(b) Follow from the corresponding property for GCs and from the definition of regular k-directional GBs.

(i) The equations for $\oplus$ and $\otimes$ follow from the corresponding property of GCs. We will only extract the remaining equations for $g(B_2)$; the ones for $f(B_1)$ can be extracted dually.

Let $P \subseteq g(B_2)$. Since f, g are $\leq_{\mathrm{t}}$-monotone and for all $x = gf(x) \in P$, $\wedge P \leq_{\mathrm{t}} x$ (resp. $x \leq_{\mathrm{t}} \vee P$), it follows that $gf(\wedge P) \leq_{\mathrm{t}} x$ (resp. $x \leq_{\mathrm{t}} gf(\vee P)$), so $gf(\wedge P)$ (resp. $gf(\vee P)$) is a lower (resp. upper) bound of P. Now let $x' = gf(x')$ be an arbitrary lower (resp. upper) bound of P in $g(B_2)$; then, of course $x' \leq_{\mathrm{t}} \wedge P$ (resp. $\vee P \leq_{\mathrm{t}} x'$), so—again by $\leq_{\mathrm{t}}$-monotonicity—, we get $x' \leq_{\mathrm{t}} gf(\wedge P)$ (resp. $gf(\vee P) \leq_{\mathrm{t}} x'$).

(j) Since (f, g) is a GC between the $\leq_{\mathrm{k}}$ orders (so the result holds for $\leq_{\mathrm{k}}$), it suffices to show that f is an isomorphism between the $\leq_{\mathrm{t}}$ orders of $f(B_1)$, $g(B_2)$ with inverse g.

Let $x_1, x_2 \in g(B_2)$ with $x_1 \leq_{\mathrm{t}} x_2$. Since f, g are monotone and x_1, x_2 are stable elements, $x_1 \leq_{\mathrm{t}} x_2 \Rightarrow f(x_1) \leq_{\mathrm{t}} f(x_2) \Rightarrow x_1 = gf(x_1) \leq_{\mathrm{t}} gf(x_2) = x_2$, so f is a $\leq_{\mathrm{t}}$-isomorphism and g is its inverse.

(k) Similar to 3.3.(k).

(l) Follows from the corresponding property of GCs and property (b).

□

As a ‘reward’ for the simplification of GBs, we obtain unconditional composability.

Proposition 3.10 (Composability of regular unidirectional Galois biconnections)**.** Let (f, g) be a regular unidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ and (f', g') be a regular unidirectional GB between $\mathcal{B}_2$ and $\mathcal{B}_3$. Then $(f'f, gg')$ is a regular unidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_3$.

Proof. Immediate from the composability of GCs and the fact that composing monotone functions yields a monotone function. □

Proposition 3.11 (Characterisation of regular k-directional Galois biconnections)**.** (f, g) is a regular k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff

(a) $\forall x \in B_1 : x \leq_{\mathrm{k}} gf(x), \forall y \in B_2 : fg(y) \leq_{\mathrm{k}} y$,

(b) f and g are monotone with respect to both $\leq_{\mathrm{t}}$ and $\leq_{\mathrm{k}}$.

Proof. Immediate from the definition and the corresponding property of GCs. □

As mentioned above, Definition 3.8 has a dual. It is easy to demonstrate that the two definitions do not always coincide, by providing a pair of functions that respect the one but not the other. Observe that for k-directional GBs `t` and `f` may or may not be in $g(B_2)$ and $f(B_1)$ respectively; this helps to differentiate this definition from its dual, in which `t` and `f` have to be in $g(B_2)$ and $f(B_1)$ respectively, but there is no constraint for $\top$ and $\bot$.

Example 3.12. Consider $\mathcal{TWENTY-FIVE}$ and $\mathcal{SIXTEEN}$. The two bilattices have $\mathcal{NINE}$ embedded in them in various ways; many of these ways can give rise to regular unidirectional GBs. Figure 3.3 represents two such GBs between the two bilattices, which only differ on the definition of the left adjoint function. Observe that `t` (and `f`) of $\mathcal{TWENTY-FIVE}$ does not belong to its set of stable elements, so the dual definition does not hold for this pair of functions. Notice that since `t` is not a stable element for $\mathcal{TWENTY-FIVE}$, the GBs of this example cannot be modelled by bidirectional or diagonal GBs. Notice also that the second GB of this example can neither be modelled by strong unidirectional GBs.

Remark 3.13 (Addendum[1]). Let $\mathcal{L} = (L, \sqcap, \sqcup, \cdot, \backslash, /)$ be a residuated lattice and $\mathcal{B} = \mathcal{L} \odot \mathcal{L} = (L \times L, \wedge, \vee, \otimes, \oplus, \neg, *, \leftarrow, \rightarrow)$. Recall that (f_a, g_a) and (f_b, g_b), where

$$f_a(x) = a * x, \qquad g_a(y) = a \rightarrow y, \qquad f_b(x) = x * b, \qquad g_b(y) = y \leftarrow b,$$

are GCs in the $\leq_\mathrm{t}$ order of $\mathcal{B}$. Can they be regular t-directional GBs?

(a) It would suffice for f_a, g_a, f_b, g_b to respect $\leq_\mathrm{k}$. Focusing on g_a, this means that if $(a_1, a_2) \leq_\mathrm{k} (a'_1, a'_2)$ and $(b_1, b_2) \leq_\mathrm{k} (b'_1, b'_2)$, then the following holds:

$$((a_1 \backslash b_1) \sqcap (a_2 \,/\, b_2), b_2 \cdot a_1) \leq_\mathrm{k} ((a'_1 \backslash b_1) \sqcap (a'_2 \,/\, b_2), b_2 \cdot a'_1)$$

By definition of product bilattices, this is equivalent to

$$(a_1 \backslash b_1) \sqcap (a_2 \,/\, b_2) \leq (a'_1 \backslash b_1) \sqcap (a'_2 \,/\, b_2) \qquad b_2 \cdot a_1 \leq b_2 \cdot a'_1$$

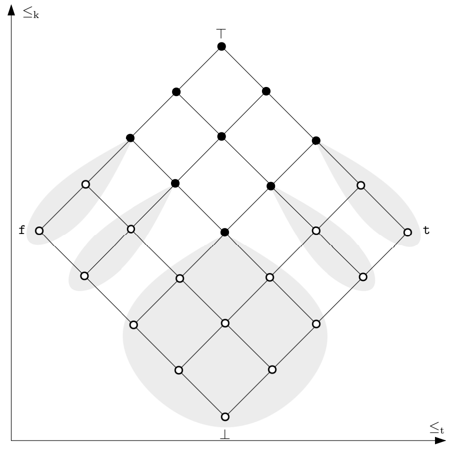
The second property is always true, given that $\cdot$ is monotone in residuated lattices. The first property implies

$$(a_1 \backslash b_1) \sqcap (a_2 \,/\, b_2) \leq a'_1 \backslash b_1.$$

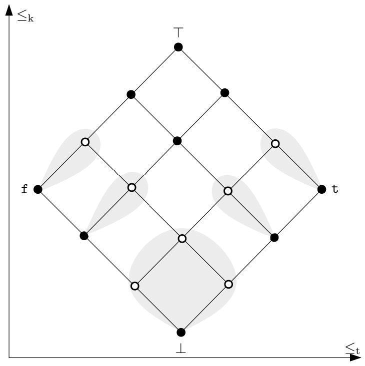
Hence, (f_a, g_a) is a regular t-directional GBs only if for all $a, a', b, c \in L$, if $a \leq a'$ and $c = x \,/\, y$ for some $x, y \in L$, then $(a \backslash b) \sqcap c \leq a' \backslash b$. Dually, (f_b, g_b) is a regular t-directional GBs ony if for all $a, b, b', c \in L$, if $b \leq b'$ and $c = x \backslash y$ for some $x, y \in L$, then $(a \,/\, b) \sqcap c \leq a \,/\, b'$.

(b) If $\backslash$ is monotone with respect to its first argument and $/$ is monotone with respect to its second argument, then L is bounded and for all $x, y \in L$, $x/y = x\backslash y = \top_L$ and $x \cdot y = \bot_L$ (hence (f_a, g_a) and (f_b, g_b) are —trivially— t-directional GBs, as in Example 3.14 below).

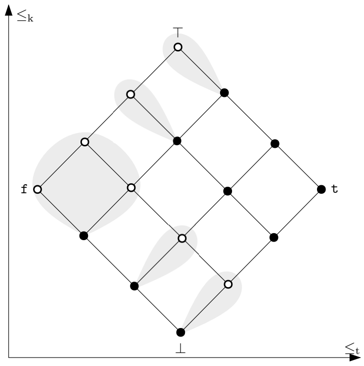
[1]This was originally stated as a proposition, which included only point (b) phrased quite optimistically



(a) $\mathcal{TWENTY} - \mathcal{FIVE}$ as the left bilattice in both GBs.



(b) $\mathcal{SIXTEEN}$ as the right bilattice in (f_1, g).



(c) $\mathcal{SIXTEEN}$ as the right bilattice in (f_2, g).

Figure 3.3: Representation of two regular k-directional GBs, (f_1, g) and (f_2, g), demonstrating two of the ways that $\mathcal{NINE}$ can be seen as a common core between $\mathcal{TWENTY} - \mathcal{FIVE}$ and $\mathcal{SIXTEEN}$.

Proof. First, we can show that under these assumptions $\backslash$ is constant with respect to its first argument. Indeed, let $a_1 \leq a_2$; then $a_1 \backslash c \leq a_2 \backslash c$, hence, by the definition of residuated lattices, $a_2 \leq c \,/\, (a_1 \backslash c)$. By our assumptions, we get $a_1 \leq c \,/\, (a_2 \backslash c)$, which is turned back into $a_2 \backslash c \leq a_1 \backslash c$ by the properties of residuated lattices. Hence, $a_1 \leq a_2$ implies $a_1 \backslash c = a_2 \backslash c$. Now let $x, y \in L$; since $x \leq x \sqcup y$ and $y \leq x \sqcup y$, we have $x \backslash z = x \sqcup y \backslash z = y \backslash z$ for all $z \in L$.

If $\backslash$ is constant with respect to its first argument, it follows that it is constant and L is bounded. Let $a, b \in L$. Of course, $a \cdot b \leq a \cdot b$, hence, $b \leq a \backslash (a \cdot b)$; since $\backslash$ is constant with respect to its first argument, for all $x \in L$, we have $b \leq x \backslash (a \cdot b)$ which is equivalent to $x \leq (a \cdot b) \,/\, b$; this implies that $(a \cdot b) \,/\, b = \top_L$. Since $/$ is constant with respect to its first argument and b is arbitrary, it follows that $x \,/\, y = \top_L$ for all $x, y \in L$. Dually, $x \backslash y = \top_L$.

Since $/$ and $\cdot$ as well as $\backslash$ and $\cdot$ form GCs and their images must be equal in size,

$\cdot$ is constant with respect to both its arguments. By the properties of GCs, the single element of its image can only be $\bot_L$.

Hence, for all $a, b \in B$, $a \to b = (\top_L, \bot_L) = \mathtt{t}$, $a \leftarrow b = \neg a \to \neg b = \mathtt{t}$, and $a * b = \neg(b \to \neg a) = \neg\mathtt{t} = \mathtt{f}$. □

(c) If L is bounded and (f_a, g_a) and (f_b, g_b) are regular t-directional GBs, then for all $a, b \in L$, $a * b = \mathtt{f}$ and $a \leftarrow b = a \to b = \mathtt{t}$.

Proof. Since L is bounded, $x / \bot_L = \top_L$ for all $x \in L$. Indeed, $\bot_L \le \top_L \setminus x$, which, by the definition of residuated lattices, becomes $\top_L \le x / \bot_L$; hence $\top_L = x / \bot_L$.

Since (f_a, g_a) is a regular t-directional GB, it follows by point (a) that for all $a, a', b, c \in L$, if $a \le a'$ and $c = x / y$ for some $x, y \in L$, then $(a \setminus b) \sqcap c \le a' \setminus b$; by using $\top_L$ as c, we get that $\setminus$ is monotone with respect to its first argument. Dually, we can get that $/$ is monotone with respect to its second argument. We can then apply point (b). □

(d) If L is commutative (i.e. for all $a, b \in L$, $a \cdot b = b \cdot a$) and (f_a, g_a) and (f_b, g_b) are regular t-directional GBs, then for all $a, b \in L$, $a * b = \mathtt{f}$ and $a \leftarrow b = a \to b = \mathtt{t}$.

Proof. Since L is commutative, for all $a, b \in L$, $a \setminus b = b / a$. Indeed, by the properties of residuated lattices, we have $a \setminus b \le a \setminus b$, hence $a \cdot (a \setminus b) \le b$, hence $(a \setminus b) \cdot a \le b$, hence $a \setminus b \le b / a$; similarly, $b / a \le b / a$ implies $b / a \le a \setminus b$.

Since (f_a, g_a) is a regular t-directional GB, it follows by point (a) that for all $a, a', b, c \in L$, if $a \le a'$ and $c = x / y$ for some $x, y \in L$, then $(a \setminus b) \sqcap c \le a' \setminus b$; by using b / a as c, we get that $\setminus$ is monotone with respect to its first argument. Dually, we can get that $/$ is monotone with respect to its second argument. We can then apply point (b). □

Example 3.14 ('Minimum' regular k-directional Galois biconnections). Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bounded bilattices. Then, (f, g), where $f(x) = \bot$ and $g(y) = \top$ is a regular k-directional GB, since both functions are trivially $\le_\mathrm{t}$-monotone and, by Example 2.22, they form a GC between the $\le_\mathrm{k}$ orders.

3.2.3 Diagonal Galois biconnections

A special case of regular GBs is interesting in its own right and deserves a closer examination.

Definition 3.15. (f, g) is called a *diagonal Galois biconnection* between $\mathcal{B}_1 = (B_1, \le_\mathrm{t}, \le_\mathrm{k})$, $\mathcal{B}_2 = (B_2, \le_\mathrm{t}, \le_\mathrm{k})$ iff (f, g) is a GC between $(B_1, \le_\mathrm{t})$, $(B_2, \le_\mathrm{t})$ and between $(B_1, \le_\mathrm{k})$, $(B_2, \le_\mathrm{k})$.

Pre-bilattice isomorphisms are diagonal GBs. Observe also that diagonal GBs are simultaneously regular unidirectional GBs (t-directional and k-directional) and regular bidirectional GBs. The following propositions list properties of diagonal GBs.

Proposition 3.16 (Properties of diagonal Galois biconnections)**.** Let (f, g) be a diagonal GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

(a) (g, f) is a diagonal GB between $\mathcal{B}_2^{\mathrm{op}}$ and $\mathcal{B}_1^{\mathrm{op}}$.

(b) f and g are monotone (with respect to both $\leq_{\mathrm{k}}$ and $\leq_{\mathrm{t}}$).

(c) $fgf = f, gfg = g$.

(d) For $i \in \{ t, k \}, \forall x \in B_1 : x \leq_i gf(x)$ and $\forall y \in B_2 : fg(y) \leq_i y$.

(e) $\forall x \in B_1 : x \in g(B_2) \Leftrightarrow x = gf(x), \forall y \in B_2 : y \in f(B_1) \Leftrightarrow y = fg(y)$.

(f) f is an injection iff g is a surjection iff $gf = \mathrm{id}_{B_1}$;
g is an injection iff f is a surjection iff $fg = \mathrm{id}_{B_2}$.

(g) f and g uniquely determine one another; in fact,

$$g(y) = \bigvee \{ x \mid f(x) \leq_{\mathrm{t}} y \} = \bigoplus \{ x \mid f(x) \leq_{\mathrm{k}} y \} ,$$

$$f(x) = \bigwedge \{ y \mid x \leq_{\mathrm{t}} g(y) \} = \bigotimes \{ y \mid x \leq_{\mathrm{k}} g(y) \} .$$

(h) f preserves $\vee$ and $\oplus$; g preserves $\wedge$ and $\otimes$.

(i) $g(B_2)$ and $f(B_1)$ are pre-bilattices (although they may not necessarily be sub-pre-bilattices of $\mathcal{B}_1$ and $\mathcal{B}_2$); in particular, if $P \subseteq g(B_2)$ and $Q \subseteq f(B_1)$,

$$\bigoplus_{g(B_2)} P = gf\left(\bigoplus P\right), \qquad \bigotimes_{f(B_1)} Q = fg\left(\bigotimes Q\right),$$

$$\bigotimes_{g(B_2)} P = \bigotimes P, \qquad \bigoplus_{f(B_1)} Q = \bigoplus Q,$$

$$\bigvee_{g(B_2)} P = gf\left(\bigvee P\right), \qquad \bigwedge_{f(B_1)} Q = fg\left(\bigwedge Q\right),$$

$$\bigwedge_{g(B_2)} P = \bigwedge P, \qquad \bigvee_{f(B_1)} Q = \bigvee Q,$$

so $f(B_1)$ is closed with respect to $\wedge$ and $\otimes$ and $g(B_2)$ is closed with respect to $\vee$ and $\oplus$.

(j) $f(B_1) \cong g(B_2)$ (as pre-bilattices); in fact, (the restrictions of) f and g are inverse isomorphisms between $f(B_1)$ and $g(B_2)$.

(k) If one of $\mathcal{B}_1, \mathcal{B}_2$ is complete (resp. interlaced), so are both $f(B_1)$ and $g(B_2)$.

(l) All bimaximal elements of $\mathcal{B}_1$ (that is, all x for which no x' exists such that $x \leq_{\mathrm{t}} x'$ and $x \leq_{\mathrm{k}} x'$) are in $g(B_2)$. All biminimal elements of $\mathcal{B}_2$ (that is, all y for which no y' exists such that $y' \leq_{\mathrm{t}} y$ and $y' \leq_{\mathrm{k}} y$) are in $f(B_1)$.

Proof. Properties (a)–(i) follow directly from the corresponding ones for GCs. Properties (j) and (k) follow from the fact that diagonal GBs are regular unidirectional GBs. Property (l) follows from properties (d) and (e). □

Composability and characterisation are immediate from the corresponding properties of GCs.

Proposition 3.17 (Composability of diagonal Galois biconnections)**.** Let (f, g) be a diagonal GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ and (f', g') be a diagonal GB between $\mathcal{B}_2$ and $\mathcal{B}_3$. Then $(f'f, gg')$ is a diagonal GB between $\mathcal{B}_1$ and $\mathcal{B}_3$.

Proposition 3.18 (Characterisation of diagonal Galois biconnections)**.** (f, g) is a diagonal GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff for all $x \in B_1$ and $y \in B_2$, $x \leq_{\mathrm{k}} gf(x)$, $x \leq_{\mathrm{t}} gf(x)$, $fg(y) \leq_{\mathrm{k}} y$, and $fg(y) \leq_{\mathrm{t}} y$.

Proposition 3.19. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be pre-bilattices and $f\colon B_1 \to B_2$. Then, there exists $g\colon B_2 \to B_1$ such that (f, g) is a diagonal GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff

(a) f preserves $\vee$ and $\oplus$,

(b) for every $y \in B_2$, $\bigvee\{\, x \mid f(x) \leq_{\mathrm{t}} y \,\}$ and $\bigoplus\{\, x \mid f(x) \leq_{\mathrm{k}} y \,\}$ exist,

(c) for every $y \in B_2$, $\bigvee\{\, x \mid f(x) \leq_{\mathrm{t}} y \,\} = \bigoplus\{\, x \mid f(x) \leq_{\mathrm{k}} y \,\}$.

The dual of the above holds if we have g and search for f.

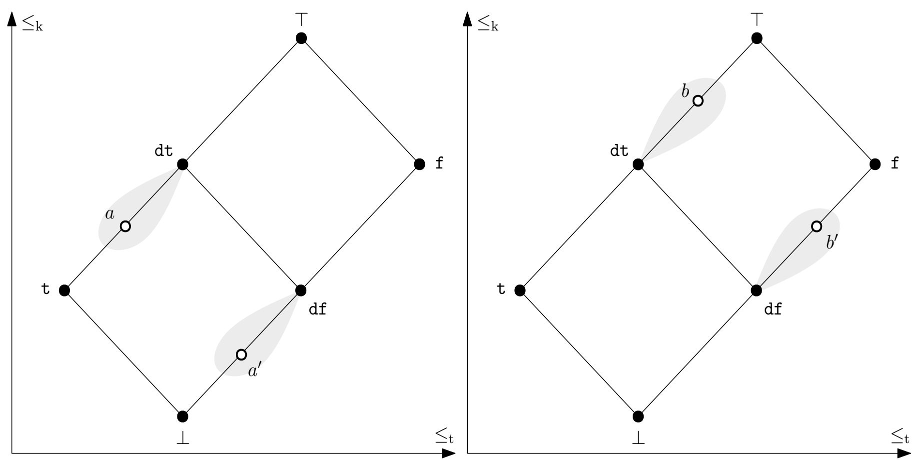
Proof.

($\Rightarrow$) Follows from proposition 3.16.

($\Leftarrow$) The first two conditions imply that there exist g and g' such that (f, g) is a GC between $(B_1, \leq_{\mathrm{t}})$ and $(B_2, \leq_{\mathrm{t}})$ and (f, g') is a GC between $(B_1, \leq_{\mathrm{k}})$ and $(B_2, \leq_{\mathrm{k}})$. By the third condition, $g = g'$.

□

Example 3.20. Consider the bilattices $\mathcal{B}_1$ and $\mathcal{B}_2$ of Figure 3.4, which are refinements of $\mathcal{SIX}^{\mathrm{op_t}}$. Define $f\colon B_1 \to B_2$ as follows: $\forall x \in \{\, \top, \bot, \mathtt{t}, \mathtt{f}, \mathtt{df}, \mathtt{dt} \,\} : f(x) = x$ and $f(a) = \mathtt{dt}$, $f(a') = \mathtt{df}$. By Proposition 3.19, (f, g) is a diagonal GB, with g defined as follows: $\forall x \in \{\, \top, \bot, \mathtt{t}, \mathtt{f}, \mathtt{df}, \mathtt{dt} \,\} : g(x) = x$ and $g(b) = \mathtt{dt}$, $g(b') = \mathtt{df}$.



(a) $\mathcal{B}_1$ (b) $\mathcal{B}_2$

Figure 3.4: The GB of example 3.20.

Not every pair of bilattices can be related through a diagonal GB. Consider $\mathcal{SIXTEEN}$ and $\mathcal{TWENTY-FIVE}$; by property 3.16.(l), all biminimal (or bimaximal, if we consider $\mathcal{TWENTY-FIVE}$ as a the left bilattice) elements of the bilattice $\mathcal{TWENTY-FIVE}$ must be stable. There are five such elements, forming a chain which is simultaneously increasing with respect to $\leq_k$ and decreasing with respect to $\leq_t$; no such chain can be found in $\mathcal{SIXTEEN}$.

3.3 Galois biconnections between bilattices with negation and conflation

The results of Section 3.2 naturally carry through to bilattices with *negation* and *conflation*: they just 'disregard' these extra operators. We have hinted earlier at the difficulties of incorporating a unary operation of 'negation' into GCs and we experiment here with variants of GBs that take into account the extended signatures of bilattices with *negation* and *conflation*. Diagonal GBs cannot be strengthened in this manner as they collapse to isomorphisms.

3.3.1 Strong bidirectional Galois biconnections

The following definition augments the one of Section 3.2.1 with a few more properties to ensure that the images of the functions are isomorphic bilattices.

Definition 3.21. $((f_t, g_t), (f_k, g_k))$ is called a *strong bidirectional Galois biconnection* between $\mathcal{B}_1 = (B_1, \leq_t, \leq_k, \neg, -)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k, \neg, -)$ iff

(a) $((f_t, g_t), (f_k, g_k))$ is a regular bidirectional GB,

(b) f_t and g_t respect $-$,

(c) f_k and g_k respect $\neg$.

Corollary 3.22. If ϕ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$, then $((\phi, \phi^{-1}), (\phi, \phi^{-1}))$ is a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proposition 3.23 (Properties of strong bidirectional Galois biconnections)**.** Let $((f_t, g_t), (f_k, g_k))$ be a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$. In addition to the properties of Proposition 3.3, the following hold:

(a) $((g_t, f_t), (g_k, f_k))$ is a strong bidirectional GB between $\mathcal{B}_2^{\text{op}}$ and $\mathcal{B}_1^{\text{op}}$.

(b) $\forall x \in B_1 : x \in g(B_2) \Leftrightarrow \neg x \in g(B_2) \Leftrightarrow -x \in g(B_2)$,
$\forall y \in B_2 : y \in f(B_1) \Leftrightarrow \neg y \in f(B_1) \Leftrightarrow -y \in f(B_1)$.

(c) For all $x \notin g(B_2)$, $f_t(x) \neq f_k(x)$ or $f_t(\neg x) \neq f_k(\neg x)$;
for all $y \notin f(B_1)$, $g_t(y) \neq g_k(y)$ or $g_t(\neg y) \neq g_k(\neg y)$.

(d) $g(B_2)$ and $f(B_1)$ with the orders inherited from $\mathcal{B}_1$ and $\mathcal{B}_2$ are bilattices; in particular, if $P \subseteq g(B_2)$ and $Q \subseteq f(B_1)$ (P and Q finite),

$$\bigoplus_{g(B_2)} P = \bigoplus P, \qquad \bigotimes_{f(B_1)} Q = \bigotimes Q,$$

$$\bigotimes_{g(B_2)} P = \bigotimes P, \qquad \bigoplus_{f(B_1)} Q = \bigoplus Q,$$

$$\bigvee_{g(B_2)} P = \bigvee P, \qquad \bigwedge_{f(B_1)} Q = \bigwedge Q,$$

$$\bigwedge_{g(B_2)} P = \bigwedge P, \qquad \bigvee_{f(B_1)} Q = \bigvee Q,$$

so $g(B_2)$ and $f(B_1)$ with the orders inherited from $\mathcal{B}_1$ and $\mathcal{B}_2$ are sub-bilattices of $\mathcal{B}_1$ and $\mathcal{B}_2$.

(e) $f(B_1) \cong g(B_2)$ (as bilattices); in fact, f_k (resp. f_t) is an isomorphism between them and g_k (resp. g_t) is its inverse.

(f) If $\mathcal{B}_1$ or $\mathcal{B}_2$ is complete (resp. interlaced, distributive), then so are both $f(B_1)$ and $g(B_2)$.

(g) If $\mathcal{B}_1$ is bounded, then $\mathtt{t}, \mathtt{f}, \top, \bot \in g(B_2)$;
if $\mathcal{B}_2$ is bounded, then $\mathtt{t}, \mathtt{f}, \top, \bot \in f(B_1)$.

Proof.

(a) Follows from the definition and the corresponding property of regular bidirectional GBs.

(b) Let $x = g_k(y) = g_t(y') \in g(B_2)$. Since g_k preserves $\neg$, we have that $\neg x = g_k(\neg y) \in g(B_2)$ and since g_t preserves $-$, $-x = g_t(-y') \in g(B_2)$. Now let $\neg x \in g(B_2)$, so $\neg\neg x = x \in g(B_2)$. Similarly, $-x \in g(B_2)$ implies $--x = x \in g(B_2)$. The proof for $f(B_1)$ is dual.

(c) Let $x \notin g(B_2)$. It suffices to show that $f_t(x) = f_k(x)$ implies $f_t(\neg x) \neq f_k(\neg x)$. Indeed, by properties (b) and 3.3.(d), $\neg x <_t g_t f_t(\neg x)$ and $x <_t g_t f_t(x)$, that is $\neg g_t f_t(x) <_t g_t f_t(\neg x)$. Since $f_t(x) = f_k(x)$, we get

$$\neg g_t f_t(x) = \neg g_k f_t(x) = \neg g_k f_k(x) = g_k f_k(\neg x) = g_t f_k(\neg x),$$

so $g_t f_k(\neg x) <_t g_t f_t(\neg x)$, which of course implies $f_k(\neg x) \neq f_t(\neg x)$.

(d) We shall prove the property only for $g(B_2)$; the proof for $f(B_1)$ works dually. By property (b), $g(B_2)$ is closed with respect to negation. The equation for $\wedge$ follows by the corresponding property of regular bidirectional GBs. Let $P \subseteq g(B_2)$. Observe that $\bigwedge \neg P = \bigwedge_{g(B_2)} \neg P \in g(B_2)$ and, hence, $g_k f_k(\bigwedge \neg P) = \bigwedge \neg P$. By the corresponding property of regular bidirectional GBs,

$$\bigvee_{g(B_2)} P = g_t f_t \left(\bigvee P\right) = g_t f_t \left(\neg \bigwedge \neg P\right)$$
$$= g_k f_k \left(\neg \bigwedge \neg P\right) = \neg g_k f_k \left(\bigwedge \neg P\right) = \bigvee P.$$

Similarly, $g(B_2)$ is closed with respect to conflation, consensus, and gullibility.

(e) Of course, f_k and f_t are pre-bilattice isomorphisms. f_k respects $\neg$ by definition. As for $-$, for all $x \in g(B_2)$, since $-x$ is also in $g(B_2)$, $f_k(-x) = f_t(-x) = -f_t(x) = -f_k(x)$. Hence, f_k is an isomorphism between $f(B_1)$ and $g(B_2)$; from property 3.3.(c), g_k is its inverse. Similarly, we can show that f_t is also an isomorphism between $f(B_1)$ and $g(B_2)$ with inverse g_t.

(f) Let $\mathcal{B}_1$ be interlaced (resp. distributive); if so happens for $\mathcal{B}_2$, we work dually. By Remark 2.39 $g(B_2)$ is also interlaced (resp. distributive) and, by isomorphism, so is $f(B_1)$. Completeness follows from property (d).

(g) Let $\mathcal{B}_1$ be bounded (the proof for the case that $\mathcal{B}_2$ is bounded is similar). By the corresponding property for regular bidirectional GBs, $\mathtt{t}$ and $\top$ are in $g(B_2)$. Since $g(B_2)$ is closed with respect to negation and consensus, $\bot$ and $\mathtt{f}$ are also in $g(B_2)$.

□

Proposition 3.24 (Composability of strong bidirectional Galois biconnections)**.** Let $((f_t, g_t), (f_k, g_k))$ be a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ and $((f'_t, g'_t), (f'_k, g'_k))$ be a strong bidirectional GB between $\mathcal{B}_2$ and $\mathcal{B}_3$. Then, $((f'_t f_t, g_t g'_t), (f'_k f_k, g_k g'_k))$ is a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_3$ iff

$$f'_k f(B_1) = f'_t f(B_1), \qquad g_k g'(B_3) = g_t g'(B_3),$$
$$g' f' f(B_1) \subseteq f(B_1), \qquad f g g'(B_3) \subseteq g'(B_3).$$

Proof. Obviously, $f'_t f_t$ and $g_t g'_t$ respect $-$ and $f'_k f_k$ and $g_k g'_k$ respect $\neg$. Since strong bidirectional GBs are also regular bidirectional GBs, it suffices to show that g_t and g_k coincide on $g' f' f(B_1)$ iff $g' f' f(B_1) \subseteq f(B_1)$ and that f'_t and f'_k coincide on $f g g'(B_3)$ iff $f g g'(B_3) \subseteq g'(B_3)$. We shall prove the first equivalence; the proof for the second is similar.

Let $y \in g' f' f(B_1)$. Of course, $\neg y \in g' f' f(B_1)$.

($\Rightarrow$) Since $g_t(y) = g_k(y)$ and $g_t(\neg y) = g_k(\neg y)$, property 3.23.(c), implies $y \in f(B_1)$.

($\Leftarrow$) Since $y \in f(B_1)$, the definition of strong bidirectional GBs implies $g_k(y) = g_t(y)$.

□

Proposition 3.25 (Characterisation of strong bidirectional Galois biconnections)**.** $((f_t, g_t), (f_k, g_k))$ is a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff

(a) g_t and f_t are $\leq_\mathrm{t}$-monotone; g_k and f_k are $\leq_\mathrm{k}$-monotone,

(b) For all $x \in B_1, y \in B_2$,

$$x \leq_\mathrm{t} g_t f_t(x), \qquad f_t g_t(y) \leq_\mathrm{t} y,$$
$$x \leq_\mathrm{k} g_k f_k(x), \qquad f_k g_k(y) \leq_\mathrm{k} y$$

(c) $f_t g_k f_k = f_k$, $f_k g_t f_t = f_t$, $g_k f_t g_t = g_t$, $g_t f_k g_k = g_k$,

(d) g_t and f_t respect $-$; g_k and f_k respect $\neg$.

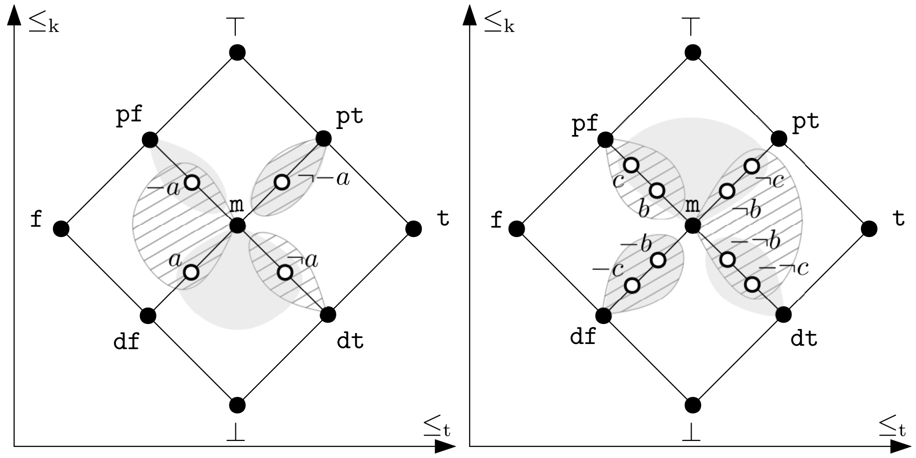
Proof. Immediate by the characterisation of regular bidirectional GBs and the definition of strong bidirectional GBs. □

Example 3.26. Let $\mathcal{B}$ be such that $B^{\mathrm{i}} = \{\, x \in B \mid x \parallel_{\mathrm{t}} \neg x \text{ or } x \parallel_{\mathrm{k}} -x \,\} \neq \emptyset$ (for such a bilattice, see Appendix B). Then $((f_t, \mathrm{id}_{B\setminus B^{\mathrm{i}}}), (f_k, \mathrm{id}_{B\setminus B^{\mathrm{i}}}))$ where

$$f_t(x) = \begin{cases} x \vee -x & \text{if } x \in B^{\mathrm{i}} \\ x & \text{otherwise} \end{cases} \qquad f_k(x) = \begin{cases} x \oplus \neg x & \text{if } x \in B^{\mathrm{i}} \\ x & \text{otherwise} \end{cases}$$

is a strong bidirectional GB between B and $B \setminus B^{\mathrm{i}}$.

Example 3.27. Consider the bilattices $\mathcal{B}_1$ and $\mathcal{B}_2$ of Figure 3.5. Then $((f_t, g_t), (f_k,$



(a) $\mathcal{B}_1$ (b) $\mathcal{B}_2$

Figure 3.5: The GB of Example 3.27.

$g_k))$, where

- $f_k(a) = \mathtt{m}$, $f_k(\neg a) = \mathtt{m}$, $f_t(a) = \mathtt{m}$, $f_t(\neg a) = \mathtt{dt}$, $f_k(-a) = \mathtt{pf}$, $f_k(\neg -a) = \mathtt{pt}$, $f_t(-a) = \mathtt{m}$, $f_t(\neg -a) = \mathtt{pt}$,
- f_k and f_t map every other element of $\mathcal{B}_1$ to the element of $\mathcal{B}_2$ with the same name,
- g_t, g_k are defined as in property 3.3.(g),

is a strong bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Strong bidirectional GBs are stronger than regular bidirectional GBs, but, of, course, this strength comes at the price of rigidity: the sub-bilattice structure of the sets of stable elements heavily limits the range of objects falling under the definition. The following two examples bear witness to this fact.

Example 3.28 ('Minimum' strong bidirectional Galois biconnections). In general, there may not be a strong bidirectional GB between every pair of bilattices. For example, the bilattice $\mathcal{B}$ of Figure 3.6 has only two sub-bilattices: $\{\, \top, \bot, \top \wedge \bot, \top \vee \bot \,\}$ and itself. Since its stable elements must include $\mathtt{t}$, it follows that all of its elements must be stable; of course, not every other bilattice may contain a sub-bilattice isomorphic to $\mathcal{B}$.

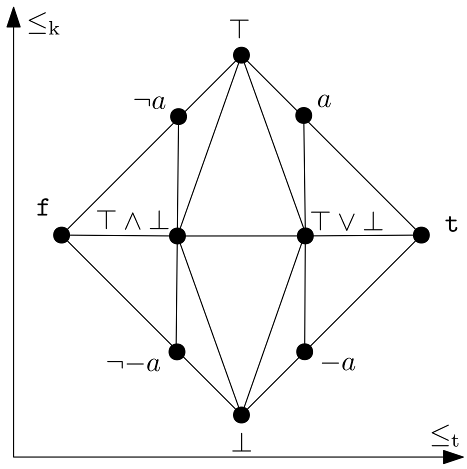


Figure 3.6: If this bilattice takes part in a strong bidirectional or k-directional GB, all its elements must be stable.

However, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bounded interlaced bilattices. Then, $((f_t, g_t), (f_k, g_k))$, where

$$f_t(x) = \begin{cases} \texttt{f} & \text{if } x = \texttt{f} \\ \top & \text{if } \texttt{f} <_t x \leq_t \top \\ \bot & \text{if } \texttt{f} <_t x \leq_t \bot \\ \texttt{t} & \text{otherwise} \end{cases} \qquad g_t(y) = \begin{cases} \texttt{t} & \text{if } y = \texttt{t} \\ \top & \text{if } \top \leq_t y <_t \texttt{t} \\ \bot & \text{if } \bot \leq_t y <_t \texttt{t} \\ \texttt{f} & \text{otherwise} \end{cases}$$

$$f_k(x) = \begin{cases} \bot & \text{if } x = \bot \\ \texttt{f} & \text{if } \bot <_k x \leq_k \texttt{f} \\ \texttt{t} & \text{if } \bot <_k x \leq_k \texttt{t} \\ \top & \text{otherwise} \end{cases} \qquad g_k(y) = \begin{cases} \top & \text{if } y = \top \\ \texttt{f} & \text{if } \texttt{f} \leq_k y <_k \top \\ \texttt{t} & \text{if } \texttt{t} \leq_k y <_k \top \\ \bot & \text{otherwise} \end{cases}$$

is a strong bidirectional GB, since (f_k, g_k) and (f_t, g_t) are GCs by Example 2.22, $f_k(B_1) = f_t(B_1) = \{\texttt{t}, \texttt{f}, \top, \bot\}$, $g_k(B_2) = g_t(B_2) = \{\texttt{t}, \texttt{f}, \top, \bot\}$, f_t and f_k are equal on $\{\texttt{t}, \texttt{f}, \top, \bot\}$, g_t and g_k are equal on $\{\texttt{t}, \texttt{f}, \top, \bot\}$, and it is easy to see that f_t, g_t preserve conflation and f_k, g_k preserve negation.

Example 3.29. Consider $\mathcal{TWENTY} - \mathcal{FIVE}$. Contrary to the situation in Example 3.12, there is only one subset X of it that is isomorphic to $\mathcal{NINE}$ and can serve as the set of stable elements of a strong bidirectional GB, the one depicted in Figure 3.7. Indeed, the four extreme elements must be stable by property 3.23.(g), the only element that can play the part of the 'middle' element of $\mathcal{NINE}$ is the 'middle' element of $\mathcal{TWENTY} - \mathcal{FIVE}$, and closure with respect to $\wedge$ and $\vee$ forces the selection of the rest. We can thus build a 'canonical' strong bidirectional GB between $\mathcal{NINE}$ and $\mathcal{TWENTY} - \mathcal{FIVE}$, by taking the isomorphism between $\mathcal{NINE}$ and X as both $f_k\colon |\mathcal{NINE}| \to X$ and $f_t\colon |\mathcal{NINE}| \to X$ and using Proposition 2.17 to find g_k and g_t, resulting in the levels depicted in Figure 3.7.

Observe that f_k and g_k fall also under the definition of strong k-directional GBs below, since they are $\leq_t$-monotone and they respect negation and conflation in X.

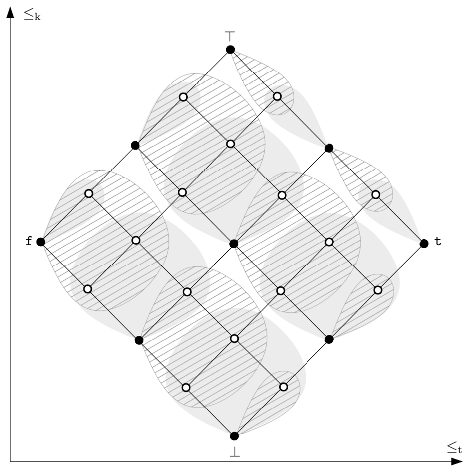


Figure 3.7: $\mathcal{TWENTY-FIVE}$ as the right bilattice in a strong bidirectional GB with a set of stable elements isomorphic to $\mathcal{NINE}$.

3.3.2 Strong unidirectional Galois biconnections

The definition below is a stronger version of the one in Section 3.2.2; in order to achieve bilattice isomorphism, it contains properties for the preservation of negation and conflation in the images. Just like its regular version, this definition has a dual[2].

Definition 3.30. (f, g) is called a *strong k-directional Galois biconnection* between $\mathcal{B}_1 = (B_1, \leq_t, \leq_k, \neg, -)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k, \neg, -)$ iff

(a) (f, g) is a regular k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$,

(b) $g(\neg f(x)) = \neg g(f(x))$ and $f(\neg g(y)) = \neg f(g(y))$,

(c) $g(-f(x)) = -g(f(x))$ and $f(-g(y)) = -f(g(y))$.

Corollary 3.31. If ϕ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$, then (ϕ, ϕ^{-1}) is a strong unidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proposition 3.32 (Properties of strong k-directional Galois biconnections)**.** Let (f, g) be a strong k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$. In addition to the properties of Proposition 3.9, the following hold:

(a) (g, f) is a strong k-directional GB between $\mathcal{B}_2^{op}$ and $\mathcal{B}_1^{op}$.

(b) $\forall x \in B_1 : x \in g(B_2) \Leftrightarrow \neg x \in g(B_2) \Leftrightarrow -x \in g(B_2)$
$\forall y \in B_2 : y \in f(B_1) \Leftrightarrow \neg y \in f(B_1) \Leftrightarrow -y \in f(B_1)$.

(c) f and g respect negation.

(d) $g(B_2)$ and $f(B_1)$ with the orders inherited from $\mathcal{B}_1$ and $\mathcal{B}_2$ are bilattices; in particular, if $P \subseteq g(B_2)$ and $Q \subseteq f(B_1)$ (P and Q finite),

$$\bigoplus_{g(B_2)} P = \bigoplus P, \qquad \bigotimes_{f(B_1)} Q = \bigotimes Q,$$

[2]**Addendum:** Although the two dual definitions probably do not coincide, we do not demonstrate it.

so $f(B_1)$ and $g(B_2)$ are closed with respect to (both) $\otimes$ and $\oplus$.

(e) $f(B_1) \cong g(B_2)$ (as bilattices); in fact, (the restrictions of) f and g are inverse isomorphisms between $f(B_1)$ and $g(B_2)$.

(f) If $\mathcal{B}_1$ is bounded, then $\top, \bot \in g(B_2)$; moreover if $\mathcal{B}_1$ is also interlaced, then $\mathtt{t}, \mathtt{f} \in g(B_2)$.
If $\mathcal{B}_2$ is bounded, then $\top, \bot \in f(B_1)$; moreover, if $\mathcal{B}_2$ is also interlaced, then $\mathtt{t}, \mathtt{f} \in f(B_1)$.

Proof.

(a) Follows from the definition and the corresponding property of GCs.

(b) Similar to 3.23.(b).

(c) Lemma 2.29.

(d) Let $P \subseteq g(B_2)$. We know that, since (f, g) is a GC, $\bigoplus_{g(B_2)} P = gf(\bigoplus P)$. But property (b) implies that $-P \subseteq g(B_2)$; hence, $\bigotimes -P \in g(B_2)$ and $\bigoplus P = -\bigotimes -P \in g(B_2)$. But this in turn implies that $gf(\bigoplus P) = \bigoplus P$ and we are done; this implies that $g(B_2)$ is closed with respect to $\oplus$. The equation for $f(B_1)$ can be proved dually.

Since strong k-directional GBs are also regular k-directional GBs, $g(B_2)$ and $f(B_1)$ are pre-bilattices. Property (b) implies they are also closed with respect to $\neg$ and $-$, so they are bilattices.

(e) $(f(B_1), \leq_\mathrm{k})$ and $(g(B_2), \leq_\mathrm{k})$ are isomorphic as lattices, since (f, g) is a GC. Points (b) and (c) of the definition mean exactly that the restrictions of f and g respect negation and conflation. It now suffices to show that $\forall x_1, x_2 \in g(B_2) : f(x_1) \leq_\mathrm{t} f(x_2) \Leftrightarrow x_1 \leq_\mathrm{t} x_2$ and $\forall y_1, y_2 \in f(B_1) : g(y_1) \leq_\mathrm{t} g(y_2) \Leftrightarrow y_1 \leq_\mathrm{t} y_2$; these follow directly from monotonicity and property 3.9.(b).

(f) Let $\mathcal{B}_1$ be bounded (the proof for the case that $\mathcal{B}_2$ is bounded is similar). By the corresponding property for regular k-directional GBs, $\top$ is in $g(B_2)$. Since $g(B_2)$ is closed with respect to conflation, $\bot$ is also in $g(B_2)$.

Now let $\mathcal{B}_1$ be bounded and interlaced (the proof for $\mathcal{B}_2$ is dual). Then, $\mathtt{f} = \top \wedge \bot$, so $\mathtt{f} \leq_\mathrm{t} \top$ and $\mathtt{f} \leq_\mathrm{t} \bot$. By $\leq_\mathrm{t}$-monotonicity, $gf(\mathtt{f}) \leq_\mathrm{t} gf(\top) = \top$ and $gf(\mathtt{f}) \leq_\mathrm{t} gf(\bot) = \bot$, so $gf(\mathtt{f}) \leq_\mathrm{t} \top \wedge \bot = f$ and, of course, $gf(\mathtt{f}) = \mathtt{f}$. Dually, $gf(\mathtt{t}) = \mathtt{t}$.

□

Proposition 3.33 (Characterisation of strong k-directional Galois biconnections)**.** (f, g) is a strong k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ iff

(a) f and g are $\leq_\mathrm{t}$-monotone and $\leq_\mathrm{k}$-monotone,

(b) $\forall x \in B_1 : x \leq_\mathrm{k} gf(x)$ and $\forall y \in B_2 : fg(y) \leq_\mathrm{k} y$,

(c) f and g respect $\neg$,

(d) $g(-f(x)) = -g(f(x))$ and $f(-g(y)) = -f(g(y))$.

Proof.

($\Rightarrow$) Follows from the definition, Proposition 3.32, and Lemma 2.15.

($\Leftarrow$) Since f and g respect $\neg$, we immediately get $g(\neg f(x)) = \neg g(f(x))$ and $f(\neg g(y)) = \neg f(g(y))$. Then, Lemma 2.15 is enough to show that (f, g) is a strong k-directional GB.

□

Proposition 3.34 (Composability of strong k-directional Galois biconnections)**.** Let (f, g) be a strong k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$ and (f', g') be a strong k-directional GB between $\mathcal{B}_2$ and $\mathcal{B}_3$. Then, $(f'f, gg')$ is a strong k-directional GB between $\mathcal{B}_1$ and $\mathcal{B}_3$ iff $fgg'(B_3) \subseteq g'(B_3)$ and $g'f'f(B_1) \subseteq f(B_1)$.

Proof. From the corresponding property of GCs, $(f'f, gg')$ is a GC between the $\leq_k$-reducts of $\mathcal{B}_1$ and $\mathcal{B}_3$. Since all of f, g, f', and g' preserve $\neg$, $\wedge$, and $\vee$, so do $f'f$ and gg'. So, it only remains to prove that

$$gg'(-f'f(x)) = -gg'f'f(x) \text{ and } f'f(-gg'(z)) = -f'fgg'(z)$$

iff

$$fgg'(B_3) \subseteq g'(B_3) \text{ and } g'f'f'(B_1) \subseteq f(B_1).$$

($\Rightarrow$) Let $y_0 = fgg'(z)$. Then,

$$\begin{aligned} g'f'(-y_0) &= g'f'(-fgg'(z)) = g'f'f(-gg'(z)) \\ &= g'(-f'fgg'(z)) = -g'f'fgg'(z) \\ &= -g'f'(y_0). \end{aligned}$$

Since $\forall y \in B_2 : y \leq_k g'f'y$ (and $-y \in B_2$), we have $-g'f'y \leq_k -y \leq g'f'(-y)$, so $g'f'(-y_0) = -y_0$, which implies that $-y_0$ (and, in turn, y_0) is in $g'(B_3)$. Hence, $fgg'(B_3) \subseteq g'(B_3)$. The other case can be proved dually.

($\Leftarrow$) By the definition of strong k-directional GBs, if $fgg'(B_3) \subseteq g'(B_3)$, then

$$\begin{aligned} f'f(-gg'(z)) &= f'(-fgg'(z)) = f'(-g'(z')) \\ &= -f'g'(z') = -f'fgg'(z). \end{aligned}$$

The other case can be proved dually.

□

Example 3.35 ('Minimum' strong k-directional Galois biconnections). As in the bidirectional case, given two arbitrary bilattices, there may not exist a strong k-directional GB between them. For example, take the bilattice $\mathcal{B}$ of Figure 3.6. Its stable elements have to include $\top, \bot, \top \wedge \bot$, and $\top \vee \bot$, since the closure of $\top \vee \bot$ must be greater than or equal to $\top \vee \bot$ in both directions (and similarly for $\top \wedge \bot$). Hence, the only options for the closure of $-a$ are itself, $\mathtt{t}$ and $\top \vee \bot$; however, since $-a \leq_k \mathtt{t}$ and $\top \vee \bot \not\leq_k \mathtt{t}$, $-a$ or $\mathtt{t}$ must also be a stable element. In both cases, all the elements of $\mathcal{B}$ must be stable (if $-a$ is stable, then $\mathtt{t}$ must be stable for the same reason as $\top \vee \bot$; if $\mathtt{t}$ is stable, then $\otimes$-closure implies that $-a$ is stable; closing with respect to $\neg$ and $-$ completes the argument). Since a strong k-directional GB between $\mathcal{B}$ and some other bilattice $\mathcal{B}'$ would induce an isomorphism between $\mathcal{B}$ and (a subset of) $\mathcal{B}'$, it is obvious that this may not always be possible.

Again, the situation is different if we restrict ourselves to interlaced bilattices. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bounded interlaced bilattices, then (f, g), where

$$f(x) = \begin{cases} \bot & \text{if } x = \bot \\ \mathtt{f} & \text{if } \bot <_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{f} \\ \mathtt{t} & \text{if } \bot <_{\mathrm{k}} x \leq_{\mathrm{k}} \mathtt{t} \\ \top & \text{otherwise} \end{cases} \qquad g(y) = \begin{cases} \top & \text{if } y = \top \\ \mathtt{f} & \text{if } \mathtt{f} \leq_{\mathrm{k}} y <_{\mathrm{k}} \top \\ \mathtt{t} & \text{if } \mathtt{t} \leq_{\mathrm{k}} y <_{\mathrm{k}} \top \\ \bot & \text{otherwise} \end{cases}$$

is a strong k-directional GB between them. Indeed, by referring to Example 2.22 and the definitions of f and g, the only non-trivial property is $\leq_{\mathrm{t}}$-monotonicity. Let $x_1 \leq x_2$ in $\mathcal{B}_1$. We prove monotonicity by case analysis on the image of x_1.

- If $f(x_1) = \mathtt{f}$, there is nothing to prove.
- If $f(x_1) = \bot$, then $x_1 = \bot$, so $\bot \leq_{\mathrm{t}} x_2$ and, since $\mathcal{B}_1$ is interlaced, $x_2 \leq_{\mathrm{t}} \mathtt{t}$, hence $\bot \leq_{\mathrm{t}} f(x_2) \leq_{\mathrm{t}} \mathtt{t}$, meaning that $f(x_2) \in \{\mathtt{t}, \bot\}$.
- If $f(x_1) = \mathtt{t}$, then $\bot <_{\mathrm{k}} x_1 \leq_{\mathrm{k}} \mathtt{t}$, so, by (2.2) and $x_1 \leq_{\mathrm{t}} x_2$, we get $\bot <_{\mathrm{k}} x_2 \leq_{\mathrm{k}} \mathtt{t}$, meaning $f(x_2) = \mathtt{t}$.
- If $f(x_1) = \top$, then, by the definition of f, $\bot \leq_{\mathrm{k}} x_1 \leq_{\mathrm{k}} \mathtt{f}$ does not hold. Hence, (2.2*) implies that $\mathtt{f} \leq_{\mathrm{t}} x_1 \leq_{\mathrm{t}} \bot$ does not hold, so neither does $\mathtt{f} \leq_{\mathrm{t}} x_2 \leq_{\mathrm{t}} \bot$, meaning (again by (2.2*)) that $f(x_2) \in \{\mathtt{t}, \top\}$.

3.4 Galois biconnections between bilattices with negation

In this section, we consider how the definitions presented in Sections 3.2 and 3.3 behave when we limit ourselves to bilattices with negation and no conflation (of course, the case of bilattices with conflation and no negation is dual).

Since the GBs of Section 3.2 do not take negation into account, their properties as demonstrated also hold in this case.

As for strong bidirectional GBs, an inspection of the proofs clearly shows that they retain (almost) all their other properties, except of course those properties that directly mention conflation. Given a strong bidirectional GB (f, g) between bilattices $\mathcal{B}_1$ and $\mathcal{B}_2$, while $f(B_1)$ and $g(B_2)$ are still bilattices, they may fail to be sub-bilattices of $\mathcal{B}_1$ and $\mathcal{B}_2$: in particular, in property 3.23.(d), we can only have that $\bigoplus_{g(B_2)} P = g_k f_k(\bigoplus P)$ and $\bigotimes_{f(B_1)} Q = f_k g_k(\bigotimes Q)$, so $f(B_1)$ is not necessarily closed with respect to $\otimes$ and $g(B_2)$ is not necessarily closed with respect to $\oplus$; the same holds for property 3.3.(i) of strong k-directional GBs. Moreover, in the absence of conflation, their required stable elements do not include $\bot$.

Given their asymmetry, strong k-directional GBs are affected a little more deeply by the removal of conflation. Since the condition on their composability has to do with conflation, in the case where there is none (or we disregard it), they have unconditional composability. In addition, in the absence of conflation, their only required stable elements are those of regular unidirectional GBs and the functions of Example 3.14 form a strong k-directional GB between any pair of bilattices.

Of course, the situation for bilattices with conflation and no negation is dual.

3.5 Biclosure operators

Just like GCs give rise to COs, each notion of GB gives rise to a corresponding notion of BCO.

Definition 3.36. $(\mathsf{C}_t, \mathsf{C}_k)$ is called a *regular bidirectional biclosure operator* on a bilattice $\mathcal{B}$ iff

(a) C_t is a CO on the $\leq_{\mathrm{t}}$ reduct of $\mathcal{B}$,

(b) C_k is a CO on the $\leq_{\mathrm{k}}$ reduct of $\mathcal{B}$,

(c) for all $x \in B$, $\mathsf{C}_t(x) = x \Leftrightarrow \mathsf{C}_k(x) = x$.

Definition 3.37. $(\mathsf{C}_t, \mathsf{C}_k)$ is called a *strong bidirectional biclosure operator* on a bilattice $\mathcal{B}$ iff

(a) $(\mathsf{C}_t, \mathsf{C}_k)$ is a regular bidirectional BCO,

(b) C_t respects $-$ and C_k respects $\neg$.

Corollary 3.38. Let $(\mathsf{C}_t, \mathsf{C}_k)$ be a regular or strong bidirectional BCO on $\mathcal{B}$. Then, $\mathsf{C}_t(B) = \mathsf{C}_k(B)$.

Proof. $x \in \mathsf{C}_t(B) \Leftrightarrow \mathsf{C}_t(x) = x \Leftrightarrow \mathsf{C}_k(x) = x \Leftrightarrow x \in \mathsf{C}_k(B)$ □

Notation. Given the corollary above, if $(\mathsf{C}_t, \mathsf{C}_k)$ is a regular or strong bidirectional BCO, we will use $\mathsf{C}(B)$ to denote both $\mathsf{C}_t(B)$ and $\mathsf{C}_k(B)$.

Definition 3.39. C is called a *regular k-directional biclosure operator* on a bilattice $\mathcal{B}$ iff

(a) C is a CO on the $\leq_{\mathrm{k}}$ reduct of $\mathcal{B}$,

(b) C is monotone with respect to $\leq_{\mathrm{t}}$.

Regular t-directional BCOs are defined dually.

Definition 3.40. C is called a *strong k-directional biclosure operator* on a bilattice $\mathcal{B}$ iff

(a) C is a regular k-directional BCO,

(b) C respects $\neg$,

(c) $\mathsf{C}(B)$ is closed with respect to $-$.

Strong t-directional BCOs are defined dually.

Definition 3.41. C is a *diagonal biclosure operator* on a bilattice $\mathcal{B}$ iff it is a CO for both orders of $\mathcal{B}$.

Definition 3.42. I will be called a *bi-interior* operator on a bilattice $\mathcal{B}$ iff I is a BCO on $\mathcal{B}^{\mathrm{op}}$. Of course, each definition of BCO gives rise to a corresponding BIO.

In the manner of GCs and COs/IOs, each kind of GB is closely related to the corresponding kind of BCO/BIO. The correspondence between diagonal GBs and diagonal BCOs is a direct consequence of the correspondence between GCs and COs. The rest are proved in Propositions 3.43 and 3.44.

Proposition 3.43. If $((f_t, g_t), (f_k, g_k))$ is a regular (resp. strong) bidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$, then $(g_t f_t, g_k f_k)$ is a regular (resp. strong) bidirectional BCO on $\mathcal{B}_1$ and $(f_t g_t, f_k g_k)$ is a regular (resp. strong) bidirectional BIO on $\mathcal{B}_2$. Conversely, if $(\mathsf{C}_t, \mathsf{C}_k)$ is a regular (resp. strong) bidirectional BCO on $\mathcal{B}$, then $((\mathsf{C}_t, \mathrm{id}_B), (\mathsf{C}_k, \mathrm{id}_B))$ is a regular (resp. strong) bidirectional GB between $\mathcal{B}$ and $\mathsf{C}(B)$.

Proof. We will only prove the parts concerning biclosures. The parts for bi-interiors are similar.

For regular bidirectional GBs:

($\Rightarrow$) From the corresponding property for GCs and COs, it follows that $g_t f_t$ is a CO on the $\leq_\mathrm{t}$ reduct of $\mathcal{B}_1$, $g_k f_k$ is a CO on the $\leq_\mathrm{k}$ reduct of $\mathcal{B}_1$. Moreover,

$$g_t f_t(x) = x \Leftrightarrow x \in g_t(B_2) \Leftrightarrow x \in g_k(B_2) \Leftrightarrow g_k f_k(x) = x,$$

so $(g_t f_t, g_k f_k)$ is a regular bidirectional BCO on $\mathcal{B}_1$.

($\Leftarrow$) From the corresponding property for GCs and COs, it follows that $(\mathsf{C}_t, \mathrm{id}_B)$ is a GC between the $\leq_\mathrm{t}$ reducts of $\mathcal{B}$ and $\mathsf{C}_t(B)$ and $(\mathsf{C}_k, \mathrm{id}_B)$ is a GC between the $\leq_\mathrm{k}$ reducts of $\mathcal{B}$ and $\mathsf{C}_k(B)$. The last two properties of the definition of regular bidirectional GBs follow from the fact that $\mathsf{C}_t(B) = \mathsf{C}_k(B)$ and the idempotency of COs.

For strong bidirectional GBs:

($\Rightarrow$) Since strong bidirectional GBs are also regular bidirectional GBs, it suffices to show that $g_t f_t$ respects $-$ and $g_k f_k$ respects $\neg$. These properties follow from the definition of strong bidirectional GBs.

($\Leftarrow$) Since strong bidirectional BCOs are also regular bidirectional BCOs, it suffices for C_k to respect $\neg$ and C_t to respect $-$, which they do by definition.

□

Proposition 3.44. If (f, g) is a regular (resp. strong) unidirectional GB between $\mathcal{B}_1$ and $\mathcal{B}_2$, then gf is a regular (resp. strong) unidirectional BCO on $\mathcal{B}_1$ and fg is a regular (resp. strong) unidirectional BCO on $\mathcal{B}_2$. Conversely, if C is a regular (resp. strong) unidirectional BCO on $\mathcal{B}$, then $(\mathsf{C}, \mathrm{id}_B)$ is a regular (resp. strong) unidirectional GB between $\mathcal{B}$ and $\mathsf{C}(B)$.

Proof. We will only prove the parts concerning biclosures. The parts for bi-interiors are similar.

For regular k-directional GBs (t-directional work dually):

($\Rightarrow$) From the corresponding property for GCs and COs, it follows that gf is a CO on the $\leq_\mathrm{k}$ reduct of $\mathcal{B}_1$. Moreover, since f and g are $\leq_\mathrm{t}$-monotone, so is gf, so it is a regular k-directional BCO.

($\Leftarrow$) Let C be a regular k-directional BCO. By the corresponding property for COs, (C, id_B) is a CO between the $\leq_\mathrm{k}$ reducts of $\mathcal{B}$ and $\mathsf{C}(B)$. Moreover, since C respects $\leq_\mathrm{t}$ by definition (and id_B respects it trivially), $(\mathsf{C}, \mathrm{id}_B)$ is a regular k-directional GB.

For strong unidirectional GBs, the situation is similar to the one of strong bidirectional GBs. □

3.6 Galois n-connections

Our definitions can be easily generalised to sets with more than two embedded lattices (such as those studied in [35, 37]). We shall only present the generalised definitions, without any properties; however, we expect that their properties will be analogous to those of the corresponding biconnections.

Note. In the literature, sets with more than two embedded lattices are called either n-lattices or multilattices. The term multilattice has also been used for structures comprising of one partial order with some generalised notions of suprema and infima [15]; for this reason, we will use the term n-lattice.

Definition 3.45. A *n-lattice*, is a set equipped with n (distinct) partial orders, such that each of its partial orders is a lattice. Each of the lattices $\leq_i$ comprising a n-lattice may have an associated negation, i.e. an operator $\neg_i$ such that $\neg_i\neg_i x = x$, $x \leq_i y \Rightarrow \neg_i y \leq_i \neg_i x$, and, for all other lattices $\leq_j$ of $\mathcal{N}$, $x \leq_j y \Rightarrow \neg_i x \leq_j \neg_i y$.

Definition 3.46. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are n-lattices, a *regular n-directional Galois n-connection* is formed by n pairs of functions (f_i, g_i), such that

(a) Each (f_i, g_i) is a GC between the $\leq_i$ orders of $\mathcal{N}_1$ and $\mathcal{N}_2$,

(b) for all f_i, $f_i(N_1)$ is the same set $(f(N_1))$ and all g_i restricted to $f(N_1)$ are equal; for all g_i, $g_i(N_2)$ is the same set $(g(N_2))$ and all f_i restricted to $g(N_2)$ are equal.

Definition 3.47. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are n-lattices, a *strong n-directional Galois n-connection* is a regular n-directional Galois n-connection such that each f_i and g_i respect the negation operators of all lattices except for $\leq_i$.

Definition 3.48. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are n-lattices, a *regular i-directional Galois n-connection* is a pair of functions (f, g) such that (f, g) is a GC between the $\leq_i$-th orders of $\mathcal{N}_1$ and $\mathcal{N}_2$ and f, g are monotone with respect to all other lattices.

Definition 3.49. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are n-lattices, a *strong i-directional Galois n-connection* is a pair of functions (f, g), such that (f, g) is a regular i-directional Galois n-connection and

(a) f, g respect the negation operators of all lattices except for $\leq_i$,

(b) for all $x \in N_1$, $y \in N_2$, $g(\neg_i f(x)) = \neg_i g f(x)$ and $f(\neg_i g(y)) = \neg_i f g(y)$.

In the manner of biconnections and BCO, it is natural to expect that each kind of Galois n-connection will give rise to an n-closure operator and vice versa.

CHAPTER 4

CONCLUSION

The Galois connection is an ‘old’, useful and widely used construction: it connects two structures in an ‘*adjoint situation*’ which allows us to use information about one structure to gain information about the other one, and vice versa. Technically speaking,

> it has to be admitted that deep theorems rarely are immediate consequences of the general theory of Galois connections, but usually require some extra tools and ideas stemming from the specific theory under consideration. But the general framework, supporting the intuition and suggesting the appropriate concepts, is established by the discovery of the **‘right’ underlying Galois connection**, followed by a good characterization of the ‘Galois-closed’ (or ‘Galois-open’) elements or sets. And there is no doubt that many proofs become shorter, more elegant and more transparent in the language of Galois connections. ([17, Preface, p. viii], emphasis added).

To name one of its applications for Computer Science, the Galois connection has been used to provide concept formation methods out of objects/attributes relations (*Formal Concept Analysis*, [26]). On the other hand, bilattices is a recent (mid '80s) outcome of considerations in logic-based Knowledge Representation [27] and have been recently investigated from different perspectives.

In this thesis, we have contributed to the algebraic investigations on bilattices, by introducing Galois biconnections, i.e. generalisations of Galois connections to bilattices. The properties we aimed (or wished) for were: (a) isomorphic images, i.e. when two bilattices are related via a Galois biconnection, a common skeleton is revealed between them, (b) inversibility, giving rise to duality such as the one of Galois connections, (c) unconditional composability, thus giving rise to a category, (d) symmetry, i.e. lack of a distinct dual definition, (e) preservation of bilattice properties: completeness, interlacing, distributivity, (f) existence of a biconnection between any arbitrary pair of bilattices, (g) wide range of examples.

Due to the existence of more than one order (made even more complicated in the presence of negation), the conflicting nature of some of the properties, and the power of Galois connections, we did not manage to come up with a single definition equipped with all the desired properties. However, we produced four definitions, each with its own merits. Table 4.1 compares our definitions regarding (some of) the above properties; the only properties shared by all of them (and, for this reason, not included in the table) are isomorphic images (however, there is a difference in the kind of isomorphism obtained) and inversibility. Regular biconnections are more widely applicable, since they do not take negation and conflation into account; on the other hand, strong biconnections, when they exist, reveal a greater similarity between bilattices. Bidirectional biconnections are symmetric, thus giving rise to more elegant closure operators (which, moreover, simultaneously provide a $\leq_t$-closure and a $\leq_k$-closure for each element), but they do not always compose; on the other hand, definitions of unidirectional biconnections come in distinct dual pairs, but their regular versions have unconditional composability. Finally, diagonal Galois biconnections are a special case of both unidirectional and bidirectional biconnections.

We hope that our results will help understanding the rich structure of bilattices. Moreover, we hope that fresh ideas may emerge on the exploitation of bilattices in Knowledge Representation, much like the way(s) Galois connections have led to the ontology-theoretic construction of concept lattices in AI.

Biconnection	Image isomorphism	Composability	Symmetry	Image closed with respect to		Required stable elements		Inherited properties	Always exists
				left	right	left	right		
regular bidirectional, Definition 3.2	(trivial) pre-bilattice	conditional	full	$\otimes, \wedge$	$\oplus, \vee$	$\top$, $\mathtt{t}$	$\bot$, $\mathtt{f}$	completeness	yes
strong bidirectional, Definition 3.21	bilattice	conditional	full	**in general** $\otimes, \wedge$ **with conflation** also $\oplus, -$ **with negation** also $\vee, \neg$	**in general** $\oplus, \vee$ **with conflation** also $\otimes, -$ **with negation** also $\wedge, \neg$	**in general** $\top$, $\mathtt{t}$ **with conflation** also $\bot$ **with negation** also $\mathtt{f}$	**in general** also $\bot$, $\mathtt{f}$ **with conflation** also $\top$ **with negation** also $\mathtt{t}$	**in general** completeness, interlacing **with conflation and negation** distributivity	no (yes)*
regular k-directional, Definition 3.8	(trivial) pre-bilattice	unconditional	has dual	$\otimes$	$\oplus$	$\top$	$\bot$	completeness, interlacing	yes
regular t-directional (dual to above)	(trivial) pre-bilattice	unconditional	has dual	$\wedge$	$\vee$	$\mathtt{t}$	$\mathtt{f}$	completeness, interlacing	yes
strong k-directional, Definition 3.30	bilattice	**no conflation** unconditional **with conflation** conditional	has dual	**no conflation** $\otimes, \neg$ **with conflation** $\oplus, \otimes, \neg, -$	**no conflation** $\oplus, \neg$ **with conflation** $\oplus, \otimes, \neg, -$	**no conflation** $\top$ **with conflation** $\top, \bot, (\mathtt{t}, \mathtt{f})$*	**no conflation** $\bot$ **with conflation** $\top, \bot, (\mathtt{t}, \mathtt{f})$*	completeness, interlacing	**with confl.** no (yes)* **no confl.** yes
strong t-directional (dual to above)	bilattice	**no negation** unconditional **with negation** conditional	has dual	**no negation** $\wedge, -$ **with negation** $\wedge, \vee, \neg, -$	**no negation** $\vee, -$ **with negation** $\wedge, \vee, \neg, -$	**no negation** $\mathtt{t}$ **with negation** $\mathtt{t}, \mathtt{f}, (\top, \bot)$*	**no negation** $\mathtt{f}$ **with negation** $\mathtt{t}, \mathtt{f}, (\top, \bot)$*	completeness, interlacing	**with neg.** no (yes)* **no neg.** yes
diagonal, Definition 3.15 (only for pre-bilattices)	(trivial) pre-bilattice	unconditional	full	$\wedge, \otimes$	$\vee, \oplus$	all bimaximal	all biminimal	completeness, interlacing	no

Table 4.1: This table lists properties of GBs and their images between a given pair of bounded (pre-)bilattices $(\mathcal{B}_1, \mathcal{B}_2)$. In the table, 'left' refers to the stable elements of $\mathcal{B}_1$ and 'right' to the stable elements of $\mathcal{B}_2$ under the biconnection. Conditions in bold (for example **no conflation**) refer to the signatures of $\mathcal{B}_1$ and $\mathcal{B}_2$. Entries marked with * are valid only when the (pre-)bilattices are interlaced.

APPENDIX A

SOME TOO STRONG DEFINITIONS FOR GALOIS BICONNECTIONS

Example A.1. Let $\mathcal{B}_1 = (B_1, \leq_t, \leq_k, \neg)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k, \neg)$ be bilattices and let $f\colon B_1 \to B_2$, $g\colon B_2 \to B_1$ such that

(a) (f, g) is a GC between $(B_1, \leq_t)$, $(B_2, \leq_t)$ and between $(B_1, \leq_k)$, $(B_2, \leq_k)$,

(b) $g(\neg f(x)) = \neg g(f(x))$ and $f(\neg g(y)) = \neg f(g(y))$.

Then, f and g are isomorphisms.

Indeed, by Lemma 2.29, f and g respect negation. Since (f, g) is a GC for the $\leq_t$ orders, by Lemma 2.27, $(B_1, \leq_t, \neg) \cong (B_2, \leq_t, \neg)$ with f an isomorphism between them and $g = f^{-1}$. Since (f, g) is a GC for the $\leq_k$ orders and $f(B_1) = B_2$, $g(B_2) = B_1$, by property 2.14.(j), $(B_1, \leq_k) \cong (B_2, \leq_k)$ with f an isomorphism between them and $g = f^{-1}$. Hence, $\mathcal{B}_1 \cong \mathcal{B}_2$ with f an isomorphism between them and $g = f^{-1}$.

Example A.2. Let $\mathcal{B}_1 = (B_1, \leq_t, \leq_k, \neg)$, $\mathcal{B}_2 = (B_2, \leq_t, \leq_k, \neg)$ be bilattices and let $f\colon B_1 \to B_2$, $g\colon B_2 \to B_1$ such that

(a) (f, g) is a Galois connection between $(B_1, \leq_t)$, $(B_2, \leq_t)$,

(b) f and g are order homomorphisms with respect to $\leq_t$,

(c) $g(\neg f(x)) = \neg g(f(x))$ and $f(\neg g(y)) = \neg f(g(y))$.

Then, f and g are isomorphisms.

Indeed, Definition 3.30 holds for (f, g), so properties 3.32.(e) and 3.9.(f) also hold for (f, g). Every partial order homomorphism is an injection, so f and g are injections; it follows (by property 3.9.(f)) that both f and g are surjections. Hence, $B_2 = f(B_1)$ and $B_1 = g(B_2)$, which implies (by property 3.32.(e)) $\mathcal{B}_1 \cong \mathcal{B}_2$ with f an isomorphism between them and $g = f^{-1}$.

APPENDIX B

AN ASIDE: PRETTY BILATTICES

During our research on Galois biconnections, we came upon some monstrous (in a Lakatosian sense [30] of the term 'monstrous') bilattices which, to our knowledge, have not been studied. We chose the name '*non-pretty*' for them; below, we state some questions regarding their relation to other classes of bilattices (precise, commutative, interlaced).

Definition B.1. A bilattice $\mathcal{B}$ is called *pretty* iff $B^{\mathrm{i}} \stackrel{\mathrm{def}}{=} \{\, x \in B \,|\, x \parallel_{\mathrm{t}} \neg x \text{ or } x \parallel_{\mathrm{k}} -x \,\} = \emptyset$.

Remark B.2. $x \parallel_{\mathrm{t}} \neg x \Rightarrow x \parallel \neg x$ and, dually, $x \parallel_{\mathrm{k}} -x \Rightarrow x \parallel -x$.

Proof. Let $x \parallel_{\mathrm{t}} \neg x$. If $x \leq_{\mathrm{k}} \neg x$, then $\neg x \leq_{\mathrm{k}} \neg\neg x = x$, so $\neg x = x$, which is absurd; similarly $\neg x \not\leq_{\mathrm{k}} x$. □

Example B.3. A non-pretty bilattice is shown in Figure B.1; notice that, in this bilattice, negation and conflation commute.

Remember that in Example 3.26 we have shown that we can always find a strong bidirectional BCO in a non-pretty bilattice $\mathcal{B}$, such as the closed elements are exactly those of $B \setminus B^{\mathrm{i}}$.

Definition B.4 ([6, Definition 5.(b)])**.** Let $x <^1_{\mathrm{t}} y$ (resp. $x <^1_{\mathrm{k}} y$) mean that x is an immediate predecessor of y with respect to $\leq_{\mathrm{t}}$ (resp. $\leq_{\mathrm{k}}$). A pre-bilattice is called *precise* iff (a) $x <^1_{\mathrm{t}} y$ implies $x <^1_{\mathrm{k}} y$ or $y <^1_{\mathrm{k}} x$ and (b) $x <^1_{\mathrm{k}} y$ implies $x <^1_{\mathrm{t}} y$ or $y <^1_{\mathrm{t}} x$.

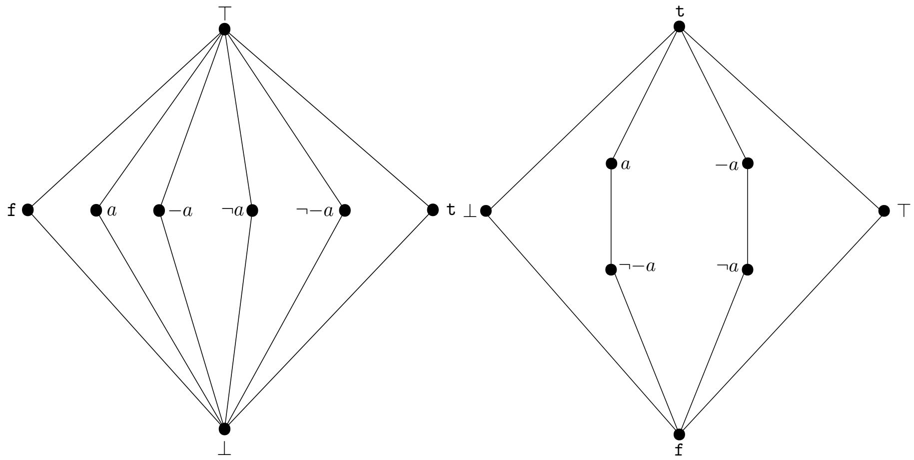
Example B.5. A non-precise bilattice is $\mathcal{FIVE}$, shown in Figure B.2.

Proposition B.6 ([6, Theorem 3])**.** Every interlaced pre-bilattice is precise.

Conjecture. Every precise bilattice is pretty.

Remark B.7. The opposite does not hold. See, for example, $\mathcal{FIVE}$ (Figure B.2); it is pretty but not precise.

Question. Are all pretty bilattices commutative?



(a) The knowledge order. (b) The truth order.

Figure B.1: A non-pretty bilattice: observe that $a \parallel \neg a$ and $a \parallel -a$.

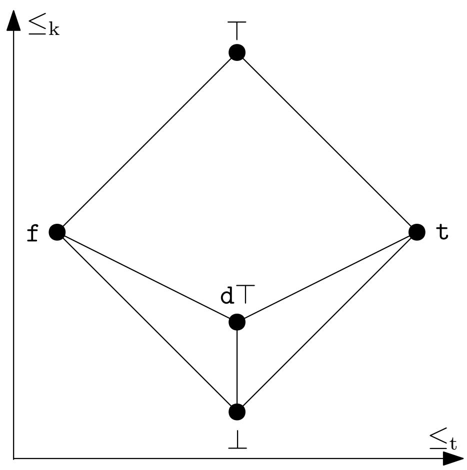


Figure B.2: $\mathcal{FIVE}$

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