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ERGODIC AVERAGES ALONG CUBES

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Summary

We study the convergence of the ergodic averages of the integral of the product of 2^k functions and the L^2-convergence of the ergodic averages of the product of $2^k - 1$ functions, for $k = 2,\ 3$. These averages are taken along cubes whose sizes tend to $+\infty$. For each average, we show that it is sufficient to prove the convergence for special systems, the characteristic factors. From the first convergence result a combinatorial interpretation can be derived for the arithmetic structure inside a set of integers of positive upper density.

Περίληψη

Εξετάζουμε τη σύγκλιση των εργοδικών μεσων όρων του ολοκληρώματος του γινομένου 2^k το πλήθος συναρτήσεων και την L^2-σύγκλιση των εργοδικών μέσων όρων του γινομένου $2^k - 1$ το πλήθος συναρτήσεων, για $k = 2,\ 3$. Θεωρούμε τους εν λόγω μέσους όρους πάνω απο κύβους, των οποίων το μέγεθος τείνει στο άπειρο. Για κάθε ένα από τους μέσους όρους, αποδεικνύουμε ότι αρκεί να δείξουμε τη ζητούμενη σύγκλιση για κάποια συστήματα ειδικού τύπου, τους characteristic παράγοντες.

Introduction

In this dissertation we study the convergence of nonconventional ergodic averages over combinatorial cubes. The averages along cubes are concerned with demonstrating the existence of some arithmetic structure inside a set of positive upper density (see Definition 2) as mentioned in Section *Combinatorial Interpretation*, by corresponding this problem to an invertible measure preserving system (see Definition 1.2.2) and examine the behaviour of some ergodic averages, using Ergodic Theory.

Let $(X, \mathcal{X}, \mu, T)$ be an invertible measure preserving system and $A \in \mathcal{X}$. By using the method of Characteristic Factors (see Section *Characteristic Factors*) will show the following results.

Theorem 1.
The averages over $(n, m) \in [N, N'] \times [M, M'] \subseteq \mathbb{Z}^2$ *of*

$$\mu(A \cap T^n A \cap T^m A \cap T^{n+m} A)$$

converge to a limit that is equal or greater than $\mu(A)^4$ *as* $[N, N']$, $[M, M']$ *tend to* $+\infty$

Theorem 2.
The averages over $(n, m, p) \in [N, N'] \times [M, M'] \times [P, P'] \subseteq \mathbb{Z}^3$ *of*

$$\mu(A \cap T^n A \cap T^m A \cap T^{n+m} A \cap T^p A \cap T^{n+p} A \cap T^{m+p} A \cap T^{n+m+p} A)$$

converge to a limit that is equal or greater than $\mu(A)^8$ *as* $[N, N']$, $[M, M']$, $[P, P']$ *tend to* $+\infty$

We view these averages taken over the combinatorial cubes $(0, n, m, m + n)$ and $(0, n, m, n + m, p, n + p, m + p, n + m + p)$ respectively. We actually prove two stronger statements namely the convergence of averages over $n \in [N, N']$, $m \in [M, M']$ and $p \in [P, P']$ of the form

$$\int_X f_1(x) f_2(T^n x) f_3(T^m x) f_4(T^{n+m} x) \, \mathrm{d}\mu(x)$$

and

$$\int_X f_1(x) f_2(T^n x) f_3(T^m x) f_4(T^{n+m} x) f_5(T^p x) f_6(T^{n+p} x) f_7(T^{m+p} x) f_8(T^{n+m+p} x)\, \mathrm{d}\mu(x)$$

where $f_i \in L^\infty(\mu)$, $i = 1, \dots, 8$.

Furthermore, we will study the convergence in $L^2(\mu)$ of the product of 3 and 7 functions in $L^\infty(\mu)$. To be more precise, we will show the following.

Theorem 3.
The averages over $(n, m) \in [N, N'] \times [M, M'] \subseteq \mathbb{Z}^2$ *of*

$$f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

converge in $L^2(\mu)$ *when* $[N, N']$, $[M, M']$ *tend to* $+\infty$

Theorem 4.
The averages over $(n, m, p) \in [N, N'] \times [M, M'] \times [P, P'] \subseteq \mathbb{Z}^3$ *of*

$$f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x) f_4(T^p x) f_5(T^{n+p} x) f_6(T^{m+p} x) f_7(T^{n+m+p} x)$$

converge in $L^2(\mu)$ *when* $[N, N']$, $[M, M']$, $[P, P']$ *tend to* $+\infty$

Generalization of Khintchine's Theorem

Definition 1.
Let G *be a discrete abelian group and* $A \subseteq G$*. Then* A *is said to be syndetic if there exist a* $n \in \mathbb{N}$ *and* $g_1, \dots, g_n \in G$*, so that* $G = \bigcup_{i=1}^n (A + g_i)$*. In particular, if* $G = \mathbb{Z}^d$ *that means there exists an integer* $L > 0$ *such that* A *intersects every* d*-dimensional cube of size* L*.*

Khintchine proved the following result.

Theorem 5.
Let $(X, \mathcal{X}, \mu, T)$ *be an invertible measure preserving system and* $A \in \mathcal{X}$*. For every* $\varepsilon > 0$ *the set*

$$\{n \in \mathbb{Z} : \ \mu(A \cap T^n A) \geq \mu(A)^2 - \varepsilon\}$$

is syndetic.

By Theorem 1 and Theorem 2 we obtain the following generalizations of Khintchine's Theorem.

Theorem 6.
Let $(X, \mathcal{X}, \mu, T)$ *be an invertible measure presercing system and* $A \in \mathcal{X}$*. For every* $\varepsilon > 0$ *the set*

$$\{(n,m) \in \mathbb{Z}^2 : \ \mu(A \cap T^n A \cap T^m A \cap T^{n+m} A) \geq \mu(A)^4 - \varepsilon\}$$

is syndetic.

Theorem 7.
Let $(X, \mathcal{X}, \mu, T)$ *be an invertible measure preserving system and* $A \in \mathcal{X}$*. For every* $\varepsilon > 0$ *the set*

$$\{(n,m,p) \in \mathbb{Z}^3 : \ \mu(A \cap T^n A \cap T^m A \cap T^{n+m} A \cap T^p A \cap T^{n+p} A \cap T^{m+p} A \cap T^{n+m+p} A) \geq \mu(A)^8 - \varepsilon\}$$

is syndetic.

Indeed, let for example E be the subset of $\mathbb{Z}^3$ appearing in Theorem 7 and assume that E is not syndetic. Then there is a sequence of cubes $[N_{i0}, N_{i0}'] \times [M_{i0}, M_{i0}'] \times [P_{i0}, P_{i0}']$, $i \in \mathbb{N}$, in $\mathbb{Z}^3$, such that the lengths of the intervals tending to ∞ and

$$E \cap [N_{i0}, N_{i0}'] \times [M_{i0}, M_{i0}'] \times [P_{i0}, P_{i0}'] = \varnothing$$

Applying Theorem 2 gives a contradiction.

Combinatorial Interpretation

Definition 2.
The upper density, $\overline{d}$*, of a set* $B \subseteq \mathbb{Z}$ *is*

$$\overline{d}(B) := \lim_{N \to +\infty} \max_{M \in \mathbb{Z}} \frac{B \cap [M, M+N]}{N}$$

Using *Furstenberg's Correspondence Principle*, we obtain the following combinatorial statement as a corollary of Theorem 7.

Theorem 8.
Let $A \subseteq \mathbb{N}$ *with* $\overline{d}(A) > \delta > 0$*. Then the set*

$$\{(n,m,p) \in \mathbb{Z}^3 : \ \overline{d}\big(A \cap (A+n) \cap (A+m) \cap (A+n+m) \cap (A+p) \cap (A+n+p) \cap (A+m+p) \cap (A+m+n+p)\big) \ \geq \ \delta^8\}$$

is syndetic.

This theorem is closely related to other combinatorial statements, in particular Szemeredi's Theorem.

Characteristic Factors

The method of characteristic factors is used in order to prove the statements above, introduced by Furstenberg. This method consists in finding an appropriate factor (see Definition 1.4.1) of the given system, referred to as the characteristic factor, so that the limit behaviour of the averages remains unchanged when each function is replaced by its conditional expectation on this factor. Then it suffices to prove the convergence when this factor is substituted for the original system, which is facilitated when the factor has a more specific description.

In particular we will show that considering the cases of Theorem 2 and Theorem 4, the characteristic factor is approximated in some sense by another special case of systems called nilsystems (see Definition 1.11.11) and it is sufficient to prove the convergence for these systems.

Εισαγωγή

Σε αυτή την εργασία εξετάζουμε τη σύγκλιση γενικευμένων εργοδικών μέσων όρων πάνω από ¨συνδυαστικούς κύβους¨ (combinatorial cubes). Η μελέτη αυτών των μέσων όρων έγκειται στο γεγονός ότι μπορούν να περιγράψουν κάποια αριθμητική δομή σε ένα σύνολο με θετική άνω πυκνότητα (Ορισμός 2), όπως αναφέρεται στο υποκεφάλαιο *Συνδυαστική Ερμηνεία*, κάνοντας αντιστοιχία με αντιστρέψιμα συστήματα που διατηρούν το μέτρο (Ορισμός 1.2.2) και χρησιμοποιώντας εργαλεία Εργοδικής Θεωρίας.

Έστω λοιπόν $(X, \mathcal{X}, \mu, T)$ ένα αντιστρέψιμο σύστημα που διατηρεί το μέτρο και $A \in \mathcal{X}$. Θα ασχοληθούμε με τα ακόλουθα τέσσερα θεωρήματα και θα τα αποδείξουμε χρησιμοποιώντας τη αρκετά ισχυρή μέθοδο των χαρακτηριστικών παραγόντων (characteristic factors) που περιγράφεται στο υποκεφάλαιο *Μέθοδος των Χαρακτηριστικών Παραγόντων*.

Θεώρημα 1.
Οι μέσοι όροι πάνω από τα $(m,n) \in [N,N'] \times [M,M'] \subseteq \mathbb{Z}^2$ των

$$\mu(A \cap T^n A \cap T^m A \cap T^{n+m} A)$$

συγκλίνουν καθώς το μήκος των διαστημάτων $[N,N']$, $[M,M']$ πηγαίνει στο άπειρο και το όριο τους είναι μεγαλύτερο ή ίσο από το $\mu(A)^4$.

Θεώρημα 2.
Οι μέσοι όροι πάνω από τα $(m,n,p) \in [N,N'] \times [M,M'] \times [P,P'] \subseteq \mathbb{Z}^3$ των

$$\mu(A \cap T^n A \cap T^m A \cap T^{n+m} A \cap T^p \cap T^{n+p} A \cap T^{m+p} A \cap T^{n+m+p} A)$$

συγκλίνουν καθώς το μήκος των διαστημάτων $[N,N']$, $[M,M']$, $[P,P']$ πηγαίνει στο άπειρο και το όριο τους είναι μεγαλύτερο ή ίσο από το $\mu(A)^8$.

Βλέπουμε δηλαδή αυτούς του μέσους όρους πάνω από τους συνδυαστικούς κύβους $(0,n,m,m+n)$ και $(0,n,m,n+m,p,n+p,m+p,n+m+p)$ αντίστοιχα. Ιδιαίτερα, θα αποδείξουμε δύο ισχυρότερα αποτελέσματα που σχετίζονται με τη σύγκλιση μέσων όρων ολοκληρωμάτων γινομένων κάποιων συναρτήσεων. Δηλαδή θα δειξουμε τη σύγκλιση των εξής μέσων όρων πάνω απο τα $(m,n) \in [N,N'] \times [M,M'] \subseteq \mathbb{Z}^2$ και τα $(m,n,p) \in [N,N'] \times [M,M'] \times [P,P'] \subseteq \mathbb{Z}^3$ αντίστοιχα, των

$$\int\limits_X f_1(x) f_2(T^n x) f_3(T^m x) f_4(T^{n+m} x)\, \mathrm{d}\mu(x)$$

και

$$\int\limits_X f_1(x) f_2(T^n x) f_3(T^m x) f_4(T^{n+m} x) f_5(T^p x) f_6(T^{n+p} x) f_7(T^{m+p} x) f_8(T^{n+m+p} x)\, \mathrm{d}\mu(x)$$

όπου $f_i \in L^\infty(\mu)$, $i = 1, \ldots, 8$.

Επιπλέον θα εξετάσουμε τη σύγκλιση στον $L^2(\mu)$ του γινομένου 3 και του γινομένου 7 συναρτήσεων που ανήκουν στον $L^\infty(\mu)$. Ακριβέστερα θα δείξουμε τα εξής αποτελέσματα,

Θεώρημα 3.
Οι μέσοι όροι πάνω από τα $(m,n) \in [N,N'] \times [M,M'] \subseteq \mathbb{Z}^2$ των

$$f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

συγκλίνουν καθώς το μήκος των διαστημάτων $[N,N']$, $[M,M']$ πηγαίνει στο άπειρο.

Θεώρημα 4.
Οι μέσοι όροι πάνω από τα $(m,n,p) \in [N,N'] \times [M,M'] \times [P,P'] \subseteq \mathbb{Z}^3$ των

$$f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x) f_4(T^p x) f_5(T^{n+p} x) f_6(T^{m+p} x) f_7(T^{n+m+p} x)$$

συγκλίνουν καθώς το μήκος των διαστημάτων $[N,N']$, $[M,M']$, $[P,P']$ πηγαίνει στο άπειρο.

Γενίκευση του Θεωρήματος του Khintchine

Ορισμός 1.
Έστω G μια διακριτή αβελιανή ομάδα και $A \subseteq G$. Τότε το A λέγεται συνδετικό αν υπάρχει $n \in \mathbb{N}$ και $g_1, \ldots, g_n \in G$ ώστε $G = \bigcup_{i=1}^n (A + g_i)$. Στην περίπτωση όπου $G = \mathbb{Z}^d$ αυτό σημαίνει ότι υπάρχει $L > 0$ τέτοιο ώστε το A να τέμνει κάθε d-διάστατο κύβο, όγκου L.

Ο Khintchine απέδειξε το παρακάτω αποτέλεσμα.

Θεώρημα 5.
Έστω $(X, \mathcal{X}, \mu, T)$ ένα αντιστρέψιμο σύστημα που διατηρεί το μέτρο και $A \in \mathcal{X}$. Για κάθε $\varepsilon > 0$ το σύνολο

$$\{n \in \mathbb{Z} :\ \mu(A \cap T^n A) \geq \mu(A)^2 - \varepsilon\}$$

είναι συνδετικό.

Από το Θεώρημα 1 και το Θεώρημα 2 παίρνουμε, αντίστοιχα, τα εξής δύο αποτελέσματα.

Θεώρημα 6.
Έστω $(X, \mathcal{X}, \mu, T)$ ένα αντιστρέψιμο σύστημα που διατηρεί το μέτρο και $A \in \mathcal{X}$. Για κάθε $\varepsilon > 0$ το σύνολο

$$\{(n,m) \in \mathbb{Z}^2 : \ \mu(A \cap T^n A \cap T^m A \cap T^{n+m} A) \geq \mu(A)^4 - \varepsilon\}$$

είναι συνδετικό.

Θεώρημα 7.
Έστω $(X, \mathcal{X}, \mu, T)$ ένα αντιστρέψιμο σύστημα που διατηρεί το μέτρο και $A \in \mathcal{X}$. Για κάθε $\varepsilon > 0$ το σύνολο

$$\{(n,m,p) \in \mathbb{Z}^3 : \ \mu(A \cap T^n A \cap T^m A \cap T^{n+m} A \cap T^p A \cap T^{n+p} A \cap T^{m+p} A \cap T^{n+m+p} A) \geq \mu(A)^8 - \varepsilon\}$$

είναι συνδετικό.

Πράγματι, έστω E το υποσύνολο του $\mathbb{Z}^3$, όπως στο Θεώρημα 7. Υποθέτουμε ότι το E δεν είναι συνδετικό. Αυτό σημαίνει ότι υπάρχει μια ακολουθία κύβων $[N_{i0}, N'_{i0}] \times [M_{i0}, M'_{i0}] \times [P_{i0}, P'_{i0}]$, $i \in \mathbb{N}$, μέσα στο $\mathbb{Z}^3$ ώστε το μήκος των διαστημάτων να πηγαίνει στο ∞ και

$$E \cap [N_{i0}, N'_{i0}] \times [M_{i0}, M'_{i0}] \times [P_{i0}, P'_{i0}] = \varnothing$$

Εφαρμόζοντας, όμως, το Θεώρημα 2 έχουμε άτοπο.

Συνδυαστική Ερμηνεία

Ορισμός 2.
Η άνω πυκνότητα, $\overline{d}$, ενός συνόλου $B \subseteq \mathbb{Z}$ ορίζεται ως εξης

$$\overline{d}(B) := \lim_{N \to +\infty} \max_{M \in \mathbb{Z}} \frac{B \cap [M, M+N]}{N}$$

Χρησιμοποιώντας την *Αρχή Αντιστοιχίας του Furstenberg* έχουμε το παρακάτω συνδυαστικό αποτέλεσμα ως πόρισμα του Θεωρήματος 7.

Θεώρημα 8.
Έστω $A \subseteq \mathbb{N}$ με $\overline{d}(A) > \delta > 0$. Τότε το σύνολο

$$\{(n,m,p) \in \mathbb{Z}^3 : \ \overline{d}\big(A \cap (A+n) \cap (A+m) \cap (A+n+m) \cap (A+p) \cap (A+n+p) \cap (A+m+p) \cap (A+m+n+p)\big) \ \geq \ \delta^8\}$$

είναι συνδετικό.

Μέθοδος των Χαρακτηριστικών Παραγόντων

Η μέθοδος των Χαρακτηριστικών Παραγόντων, που εισήχθη από το Furstenberg, παίζει κεντρικό ρόλο στο να αποδείξουμε τα προαναφερθέντα ζητούμενα αποτελέσματα. Η μέθοδος αυτή απαρτίζεται από την εύρεση κάποιου κατάλληλου παράγοντα (Ορισμός 1.4.1) του δοθέντος αρχικού συστήματος, όπου αναφέρεται ως χαρακτηριστικος παράγοντας, ώστε η οριακή συμπεριφορά των μέσων όρων να μην επηρεάζεται αν κάθε συνάρτηση αντικατασταθεί με την conditional expecteation αυτής, ως προς αυτόν τον παράγοντα. Τότε αρκεί να αποδειχθεί η σύγκλιση των μέσων όρων, αν αντικαταστήσουμε το αρχικό σύστημα με αυτόν τον παράγοντα.

Ιδιαίτερα θα δείξουμε ότι στην περίπτωση του Θεωρήματος 2 και του Θεωρήματος 4 χρειάζεται περαιτέρω αναγωγή σε κάποια συστήματα ειδικού τύπου, τα nilsystems (Ορισμός 1.11.11) και αποδεικνύουμε ότι μας αρκεί να αποδείξουμε την εκάστοτε σύγκλιση για αυτά τα συστήματα.

Contents

Chapter 1

Preliminaries

In this chapter we establish some notation and terminology that is used in the next Chapters. Furthermore we list some probabilistic, functional and measure theoretic tools used in Ergodic Theory. Lastly we describe a special case of Ergodic systems, known as nilsystems, that play a key role in obtaining the desired results.

We omit some of the proofs of the results mentioned in this chapter as most of them are standard facts that may be found in the literature and their inclusion would increase the length of this dissertation substantially, without them being the main focus of this dissertation.

1.1 Polish Spaces, Polish Groups, Standard Borel and Lebesgue Spaces

Definition 1.1.1.
A separable completely metrizable topological space is called a **Polish space**. A topological group that is a Polish space, is called Polish group.

Proposition 1.1.2.
The product of countably many Polish spaces, endowed with the product topology, is itself a Polish space.

Definition 1.1.3.
Let $(X, \mathcal{X})$ be a measurable space, where X is a topological space and $\mathcal{X}$ is its **Borel** σ-algebra. Then $(X, \mathcal{X})$ is called standard Borel space if there is a Borel isomorphism [1]

[1] Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ two topological spaces endowed with their respective Borel σ-algebra. A

to a Polish space endowed with its Borel σ-algebra.

A standard Borel space $(X, \mathcal{X})$ equipped with a probability measure μ is called a **Lebesgue probability space** $(X, \mathcal{X}, \mu)$.

Notation. Let $(X, \mathcal{X})$ be a standard Borel space. The space of Borel probability measures on $(X, \mathcal{X})$ is denoted by $\mathcal{M}(X, \mathcal{X})$ or simply $\mathcal{M}(X)$.

Proposition 1.1.4.
Let X be a Polish space and $\mathcal{X}$ its Borel σ-algebra. Then every Borel probability measure on X is Radon. This means that $\forall \mu \in \mathcal{M}(X, \mathcal{X})$,

$$\mu(A) = \sup\{\mu(F) : \ F \subseteq A \ , \ \textit{closed}\} \ = \ \inf\{\mu(U) : \ A \subseteq U \ , \ \textit{open}\} \quad \forall A \in \mathcal{X} \ .$$

Equivalently, if $\mu \in \mathcal{M}(X, \mathcal{X})$, then for every $A \in \mathcal{X}$ and every $\epsilon > 0$, there exist $F \subseteq X$, closed and $U \subseteq X$, open with $F \subseteq A \subseteq U$ and $\mu(U \setminus F) < \epsilon$.

Theorem 1.1.5.
Let X be a Polish space that has an uncountable number of points and let $\mathcal{X}$ be its Borel σ-algebra. Then there exist a Borel isomorphism from $(X, \mathcal{X})$ to $\big([0,1], \mathcal{B}([0,1])\big)$.

Theorem 1.1.6.
Let X be a separable metric space, $\mathcal{X}$ be its Borel σ-algebra and μ be a Borel probability measure on $(X, \mathcal{X})$. Then there exists a unique closed subset C_μ of X satisfying the following,

(i) $\mu(C_\mu) = 1$

(ii) *if D is any other closed subset of X such that $\mu(D) = 1$ then $C_\mu \subseteq D$*

(iii) *C_μ is the set of all points $x \in X$ having the property that $\mu(U) > 0$ for each open subset U of X containing x.*

Theorem 1.1.7.
Let $(X, \mathcal{X})$ be a standard Borel space. Then

(i) *The σ-algebra $\mathcal{X}$ is countably generated (and thus every sub-σ-algebra of $\mathcal{X}$ is countably generated). Furthermore there exists a countable family of bounded $\mathcal{X}$-measurable functions on X that is dense in $L^p(\mu)$ for every $p \in [1, +\infty)$ and every measure $\mu \in \mathcal{M}(X, \mathcal{X})$.*
In particular $L^2(\mu)$ is separable for every $\mu \in \mathcal{M}(X, \mathcal{X})$.

function $\phi : X \to Y$ that is invertible and both ϕ, ϕ^{-1} are measurable is called a Borel isomorphism

(ii) *There exists a σ-algebra $\mathcal{M}$ on $\mathcal{M}(X,\mathcal{X})$ such that $(\mathcal{M}(X,\mathcal{X}),\mathcal{M})$ is a standard Borel space and the map $\mu \mapsto \int f \mathrm{d}\mu$ is a Borel function for every bounded Borel function f on X.*

Lemma 1.1.8.
Let H be a closed normal subgroup of the Polish group G. If H and G/H are locally compact groups, then G is locally compact. If H and G/H are compact groups, then G is compact.

1.2 Measure Preserving Systems, Ergodicity, Ergodic Theorems

Definition 1.2.1.
Let $(X,\mathcal{X},\mu)$ be a probability space and $T : X \to X$ be a measurable map. The system $(X,\mathcal{X},\mu,T)$ $\big($or $(X,\mu,T)\big)$ is called a measure preserving system if

$$\mu\big(T^{-1}A\big) = \mu(A)$$

for every $A \in \mathcal{X}$.

Notation. For a function $f : X \to X$, let $f_*\mu$ denote the measure defined by $f_*\mu(A) = \mu\big(f^{-1}A\big)$, $\forall A \in X$.

Definition 1.2.2.
A system $(X,\mathcal{X},\mu,T)$ is an invertible measure preserving system if it is a measure preserving system with T invertible and $T^{-1} : X \to X$ $\mathcal{X}$-measurable.

Definition 1.2.3.
A measure preserving system $(X,\mathcal{X},\mu,T)$ is ergodic if for every $A \in \mathcal{A}$ with $T^{-1}(A) = A$, we have $\mu(A) = 0$ or $\mu(A) = 1$.

Proposition 1.2.4.
Let $(X,\mathcal{X},\mu,T)$ be a measure preserving system. Then the following are equivalent.

(i) *The system is ergodic.*

(ii) $\mu(A) \in \{0,1\}$ *for all* $A \in \mathcal{X}$ *with* $\mu\big(A \triangle T^{-1}A\big) = 0$

(iii) *For every* $A \in \mathcal{X}$ *with* $\mu(A) > 0$, $\mu\big(\bigcup_{n\geq 1} T^{-1}(A)\big) = 1$.

Theorem 1.2.5.
Let (X, μ, T) be a measure preserving system.

(i) *If the system is ergodic then every measurable function f on X with $f \circ T = f$ μ-a.e. is equal to a constant μ-a.e.*

(ii) *If every $f \in L^\infty(\mu)$ with $f \circ T = f$ μ-a.e. is equal to a constant μ-a.e., then the system is ergodic.*

Theorem 1.2.6. ***(Von Neumann's Mean Ergodic Theorem)***
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system, $p \in [1, +\infty)$ and $f \in L^p(\mu)$. Then there exist an $\tilde{f} \in L^p(\mu)$ such that,

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \overset{L^p(\mu)}{\longrightarrow} \tilde{f}$$

and $\tilde{f} \overset{L^p(\mu)}{=} \tilde{f} \circ T$.
If the system is in addition ergodic, then

$$\tilde{f} = \int\limits_X f \, \mathrm{d}\mu \quad \mu\text{-}a.e.$$

Theorem 1.2.7. ***(Birkhoff's Pointwise Ergodic Theorem)***
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system, $f \in L^1(\mu)$. Then there exists an $\tilde{f} \in L^1(\mu)$ such that,

$$\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x) \longrightarrow \tilde{f}(x) \qquad \text{for } \mu\text{-almost every } x \in X.$$

Furthermore $\tilde{f} = \tilde{f} \circ T$, μ-a.e., $||\tilde{f}||_{L^1(\mu)} \leq ||f||_{L^1(\mu)}$ and $\int_A \tilde{f} \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu$, for every $A \in \mathcal{X}$ such that $A = T^{-1}A$.
If the system is in addition ergodic then,

$$\tilde{f} = \int\limits_X f \, \mathrm{d}\mu \quad \mu\text{-}a.e.$$

1.3 Eigenfunctions

Definition 1.3.1.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system and $p \in [1, +\infty]$. Define $U_T : L^p(\mu) \to$

$L^p(\mu)$ by

$$U_T(f) = f \circ T$$

for every $f \in L^p(\mu)$.

Remarks 1.3.2.

(i) The operator U_T is an isometry on $L^p(\mu)$, for every $p \in [1, +\infty]$.

(ii) If the system (X, μ, T) is invertible then U_T is a unitary operator on $L^2(\mu)$.

Definition 1.3.3.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. An eigenfunction of U_T (or of X) with eigenvalue $\lambda \in \mathbb{C}$, is a function $f \in L^2(\mu)$ where f is not identically zero function and $U_T f = \lambda \cdot f$, μ-a.e.

Remarks 1.3.4.

(i) $\lambda = 1$ is always an eigenvalue with eigenfunction $f = \mathbb{1}_X$ (which is an element of $L^2(\mu)$ since $\mu(X) < +\infty$).

(ii) Since every U_T is an isometry on $L^2(\mu)$, for every eigenvalue $\lambda \in \mathbb{C}$, we have $|\lambda| = 1$.

Proposition 1.3.5.
Let (X, μ, T) be a measure preserving system. Then eigenfunctions corresponding to different eigenvalues are orthogonal to each other in $L^2(\mu)$.

Corollary 1.3.6.
The eigenvalues of $U_T : L^2(\mu) \to L^2(\mu)$ are at most countably many.

Theorem 1.3.7.
Let (X, μ, T) be an ergodic measure preserving system and consider $U_T : L^2(\mu) \to L^2(\mu)$. Then

(i) *If $U_T f = \lambda f$ for some $f \in L^2(\mu)$ with $f \overset{L^2(\mu)}{\neq} 0$ then, $|\lambda| = 1$ and $|f| = c$, μ-almost everywhere, for some constant $c \in \mathbb{C} \setminus \{0\}$.*

(ii) *If $f, g \in L^2(\mu)$ are two eigenfunctions of U_T corresponding to the same eigenvalue λ, then*

$$f \overset{L^2(\mu)}{=} c \cdot g$$

for some constant $c \in \mathbb{C}$.

(iii) *The eigenvalues of* U_T *form a countable subgroup of* $\mathbb{S}^1$.

Remark 1.3.8.
Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving system. By Theorem 1.1.7 we have that $L^2(\mu)$ is separable. Combining Proposition 1.3.5 and Theorem 1.3.7 (ii) we have that each eigenspace is of dimension 1 and subspaces corresponding to different eigenvalues are orthogonal $\big($as subspaces $of L^2(\mu)\big)$. There

By Theorem 1.3.7 (i) we can consider the set of eigenfunctions of X normalized so that $|f(x)| = 1$, for μ-almost every $x \in X$. Observe that this set contains exactly one eigenfunction for each eigenvalue or equivalently, it contains exaclty one function from each eigenspace. Combining all the above we obtain that this set is countable.

1.4 Factors

Definition 1.4.1.
Let $(X, \mathcal{X}, \mu, T)$, $(Y, \mathcal{Y}, \nu, S)$ be two measure preserving systems. The system $(Y, \mathcal{Y}, \nu, S)$ is called factor of $(X, \mathcal{X}, \mu, T)$ if there exists a measurable function $\pi : X \to Y$ such that $\pi_* \mu = \nu$ and $\pi \circ T = S \circ \pi$ μ-a.e.
The function π is called factor map.

Remark 1.4.2.
If $(X, \mathcal{X}, \mu, T)$ is ergodic then $(Y, \mathcal{Y}, \nu, S)$ is also ergodic.

Definition 1.4.3.
Let $(X, \mathcal{X}, \lambda)$ be a measure space and $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{X}$ be two sub-σ-algebras then $\mathcal{A} = \mathcal{B}$ modμ if for every $A \in \mathcal{A}$, $\exists B \in \mathcal{B}$ such that $\mu(A \triangle B) = 0$ and for every $B \in \mathcal{B}$, $\exists A \in \mathcal{A}$ such that $\mu(B \triangle A) = 0$

Proposition 1.4.4.
Let $(X, \mathcal{X}, \mu, T)$ *be a measure preserving system,* $(Y, \mathcal{Y}, \nu, S)$ *be an invertible measure preserving systems and* $\pi : X \to Y$ *a factor map. Let* $\pi^{-1}(\mathcal{Y}) = \mathcal{A}$. *Then* $T^{-1}\mathcal{A} = \mathcal{A}$ *mod*μ.

Theorem 1.4.5.
Let $(X, \mathcal{X}, \mu, T)$ *be a measure preserving system and* $\mathcal{A} \subseteq \mathcal{X}$ *be a sub-*σ*-algebra such that* $T^{-1}\mathcal{A} = \mathcal{A}$, *modulo* μ. *Then there exists a measure preserving system* $(Y, \mathcal{Y}, \nu, S)$ *and a factor map* $\pi : X \to Y$ *such that* $\pi^{-1}(\mathcal{Y}) = \mathcal{A}$. *If in addition* $(X, \mathcal{X}, \mu, T)$ *is an invertible m.p.s. then* $(Y, \mathcal{Y}, \nu, S)$ *is also invertible m.p.s.*

1.5 Conditional Expectation and Conditional Measures

Proposition 1.5.1.
Let $(X, \mathcal{X}, \mu)$ be a probability space. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{X}$ and $p \in [1, +\infty)$. There exists a function $\mathbb{E}(\,\cdot \mid \mathcal{A}) : \; L^1(X, \mathcal{X}, \mu) \;\to\; L^1(X, \mathcal{A}, \mu)$ such that:

(i) *For every $f \in L^1(X, \mathcal{X}, \mu)$,*

- $\mathbb{E}(f \mid \mathcal{A})$ *is $\mathcal{A}$-measurable and*
- $\int_A f \,\mathrm{d}\mu = \int_A \mathbb{E}(f \mid \mathcal{A}) \,\mathrm{d}\mu$*, for every $A \in \mathcal{A}$.*

(ii) $\mathbb{E}(\,\cdot \mid \mathcal{A})$ *is a positive linear operator of norm* 1.

(iii) *For $f \in L^1(X, \mathcal{X}, \mu)$ and $g \in L^\infty(X, \mathcal{A}, \mu)$,*

$$\mathbb{E}(fg \mid \mathcal{A}) = g\mathbb{E}(f \mid \mathcal{A}) \quad \mu\text{-a.e.}$$

(iv) *If $\mathcal{B} \subseteq \mathcal{A}$ is a sub-σ-algebra, then if $f \in L^1(X, \mathcal{X}, \mu)$*

$$\mathbb{E}\big(\mathbb{E}(f \mid \mathcal{A}) \bigm| \mathcal{B}\big) = \mathbb{E}(f \mid \mathcal{B}) \quad \mu\text{-a.e.}$$

(v) *If $f \in L^1(X, \mathcal{A}, \mu)$ then $\mathbb{E}(f \mid \mathcal{A}) = f$, μ-a.e.*

(vi) *For any $f \in L^1(X, \mathcal{X}, \mu)$, $|\mathbb{E}(f \mid \mathcal{A})| \leq \mathbb{E}(|f| \mid \mathcal{A})$, μ-a.e.*

Remarks 1.5.2.

- The two properties in (i), characterize the conditional expectation of a given function, uniquely up to sets of measure zero.
- $\mathbb{E}(\,\cdot \mid \mathcal{A})$ can be considered as an operator from $L^2(X, \mathcal{X}, \mu)$ to $L^2(X, \mathcal{A}, \mu)$ where for each $f \in L^2(X, \mathcal{X}, \mu)$, $\mathbb{E}(f \mid \mathcal{A})$ is the projection of f on the closed subspace $L^2(X, \mathcal{A}, \mu)$ of $L^2(X, \mathcal{X}, \mu)$.
- Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{X}$. Then

  $$\mathbb{E}(f \circ T \mid T^{-1}\mathcal{A}) = \mathbb{E}(f \mid \mathcal{A}) \circ T$$

  If $\mathcal{A}$ is in addition such that $T^{-1}\mathcal{A} = \mathcal{A}$ then

  $$\mathbb{E}(f \circ T \mid \mathcal{A}) = \mathbb{E}(f \mid \mathcal{A}) \circ T$$

With the notion of conditional expectation in hand we can give the following reformulation of Theorem 1.2.6 and Theorem 1.2.7 respectively.

Notation. Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. Consider the set

$$\mathcal{I}(T) := \{A \in \mathcal{X} : \ T^{-1}(A) = A\}.$$

Then $\mathcal{I}(T)$ is a sub-σ-algebra of $\mathcal{X}$ and is referred as the invariant σ-algebra of the system $(X, \mathcal{X}, \mu, T)$.

Theorem 1.5.3.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system, $p \in [1, +\infty)$ and $f \in L^p(\mu)$. Then,

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n \overset{L^p(\mu)}{\longrightarrow} \mathbb{E}(f \mid \mathcal{I}(T))$$

. If the system is in addition ergodic, then

$$\mathbb{E}(f \mid \mathcal{I}(T)) = \int\limits_X f \, \mathrm{d}\mu \quad \mu\text{-almost everywhere}$$

.

Theorem 1.5.4.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system, $f \in L^1(\mu)$. Then,

$$\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n(x) \longrightarrow \mathbb{E}(f \mid \mathcal{I}(T))(x) \qquad \text{for } \mu\text{-almost every } x \in X \ .$$

If the system is in addition ergodic then,

$$\mathbb{E}(f \mid \mathcal{I}(T)) = \int\limits_X f \, \mathrm{d}\mu \quad \mu\text{-almost everywhere}$$

.

Theorem 1.5.5.
Let $(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ be two Lebesgue probability spaces and $\pi : X \to Y$ a measurable map. Let $\mathcal{B}$ denote the sub-σ-algebra, $\pi^{-1}(\mathcal{Y})$ of $\mathcal{X}$. Then, for every $f \in L^\infty(\mu)$ there exist a $\mathcal{Y}$-measurable function $\mathbb{E}(f \mid Y) : Y \to \mathbb{C}$, such that

$$\mathbb{E}(f \mid \mathcal{B}) = \mathbb{E}(f \mid \pi^{-1}(\mathcal{Y})) \ = \ \mathbb{E}(f \mid Y) \circ \pi \ , \qquad \mu\text{-almost everywhere.}$$

Remark 1.5.6.
Let $(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ be two Lebesgue probability spaces, $(X, \mathcal{X}, \mu, T)$, $(Y, \mathcal{Y}, \nu, S)$ and $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$, is a factor map. Assume further that $(Y, \mathcal{Y}, \nu, S)$ is an invertible measure preserving system. then

$$\mathbb{E}(f \circ T \mid Y) = \mathbb{E}(f \mid Y) \circ S \ , \qquad \nu\text{-almost everywhere,}$$

for any $f \in L^1(\mu)$.

1.6 Disintegration of a measure

Theorem 1.6.1. *(disintegration with respect to a sub-σ-algebra)*
Let $(X, \mathcal{X}, \mu)$ be a Lebesgue probability space and $\mathcal{A} \subseteq \mathcal{X}$ a sub-σ-algebra. Then there exists a $X' \in \mathcal{A}$ with $\mu(X') = 1$ and a set of Borel probability measures, $\{\mu_x^{\mathcal{A}} : \ x \in X'\}$ with the following properties

(i) *For every $x \in X'$, $\mu_x^{\mathcal{A}}$ is a Borel probability measure on X, with*

$$\mathbb{E}(f \mid \mathcal{A})(x) = \int_X f(y) \, \mathrm{d}\mu_x^{\mathcal{A}}(y)$$

for every $f \in \mathscr{L}^1(X, \mathcal{X}, \mu)$. [2]

(ii) *For $x \in X$ let $[x]_{\mathcal{A}} := \bigcap_{A \in \mathcal{A}:\, x \in A} A$ be the atom of $\mathcal{A}$ containing x. If $\mathcal{A}$ is countably generated, then*

- *The set $[x]_{\mathcal{A}}$ is an element of $\mathcal{A}$*
- *If $x, y \in X'$, with $[x]_{\mathcal{A}} = [y]_{\mathcal{A}}$, then $\mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}$.*

(iii) *The fisrt property uniquely determines the measure $\mu_x^{\mathcal{A}}$, for every $x \in X'$.*

(iv) *If $\mathcal{A}'$ is any sub-σ-algebra with $\mathcal{A} = \mathcal{A}'$ modμ, then $\mu_x^{\mathcal{A}} = \mu_x^{\mathcal{A}'}$, for μ-almost every $x \in X$*

Remark 1.6.2.
The mapping $x \mapsto \mu_x^{\mathcal{A}}$ from X to $\mathcal{M}(X, \mathcal{X})$ is $\mathcal{A}$-measurable, when $\mathcal{M}(X, \mathcal{X})$ is equipped with the σ-algebra of Theorem 1.1.7 (ii).

[2] We are forced to work with $\mathscr{L}^1$ since $\mu_x^{\mathcal{A}}$ may be singular to μ.

Theorem 1.6.3.
Let $(X, \mathcal{X}, \mu)$ be a Lebesgue probability space and $\mathcal{A} \subseteq \mathcal{X}$ a sub-σ-algebra. Suppose that there exist an $X' \in \mathcal{X}$ such that $\mu(X') = 1$ and a set $\{\nu_x : \ x \in X'\}$ of Borel probability measures with the following properties,

- $x \mapsto \nu_x$ *is measurable and for any* $f \in \mathscr{L}^\infty(\mu)$, $x \mapsto \int f \mathrm{d}\nu_x$ *is measurable.*
- *If* $[x]_\mathcal{A} = [y]_\mathcal{A}$, *for some* $x, y \in X'$ *then* $\nu_x = \nu_y$.
- $\nu([x]_\mathcal{A}) = 1$, *for any* $x \in X'$.
- *For any* $f \in \mathscr{L}^\infty(\mu)$, $\int f \mathrm{d}\mu = \int \int f \mathrm{d}\nu_x \, \mathrm{d}\mu(x)$.

Then $\nu_x = \mu_x^\mathcal{A}$ *for* μ*-almost every* $x \in X$.

Remark 1.6.4.
Theorem 1.6.1 basically says that if $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $\mathcal{A} \subseteq \mathcal{X}$ is a sub-σ-algebra, then there exist a Borel measurable map (defined almost everywhere) $x \mapsto \mu_x : \ X \to \mathcal{M}(X, \mathcal{X})$, such that for any $f \in L^\infty(\mu)$ and every $A \in \mathcal{A}$,

$$\int_A f \ \mathrm{d}\mu = \int_A \left(\int_X f \ \mathrm{d}\mu_x \right) \mathrm{d}\mu(x)$$

In the same Theorem, property (ii) can be rephrased as follows,

$$\text{for } \mu\text{-almost every } x \in X, \quad \mu_x = \mu_y, \ \mu_x\text{-almost everywhere}$$

Theorem 1.6.3 says that this decomposition is essentially unique.

Proposition 1.6.5.
Let $(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ be two Lebesgue probability spaces and $\pi : X \to Y$ a factor map. Let $\mathcal{B}$ denote the sub-σ-algebra, $\pi^{-1}(\mathcal{Y})$ of X. Then there exist a $\mathcal{Y}$-measurable map $y \mapsto \mu_y : \ Y \to \mathcal{M}(X, \mathcal{X})$, such that for every $A \in \mathcal{Y}$ and every $f \in L^\infty(\mu)$,

$$\int_{\pi^{-1}(A)} f \ \mathrm{d}\mu = \int_A \left(\int_X f \ \mathrm{d}\mu_y \right) \mathrm{d}\mu(y)$$

In addition,

$$\mathbb{E}(f \mid Y)(y) = \int_X f \ \mathrm{d}\mu_y \ , \qquad \textit{for } \nu\textit{-almost every } y \in Y$$

where $\mathbb{E}(f \mid Y)$ is defined as in Theorem 1.5.5. We call the above disintegration of the measure μ, disintegration with respect to π or disintegration over the factor Y.
It follows that $\mu_x = \mu_{\pi(x)}$, for μ-almost every $x \in X$.

Remark 1.6.6.

If $(X, \mathcal{X}, \mu, T)$ is a measure preserving system, where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{X}$, such that $T^{-1}\mathcal{A} = \mathcal{A}$, then $\mu_{T(x)} = T_*\mu_x$, for μ-almost every $x \in X$.

Let $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ be a factor map. Consider the disintegration $y \mapsto \mu_y$ of μ with respect to π. Then

$$\mu_{S(y)} = T_*\mu_y$$

for ν-almost every $y \in Y$.

Now, if $\mathcal{A} = \pi^{-1}(\mathcal{Y})$,

$$\mu_x = \mu_{\pi(x)}$$

for μ-almost every $x \in X$. That means that the disintegration with respect to $\mathcal{A}$ coincides with the disintegration with respect to π.

1.7 Ergodic Decomposition

Theorem 1.7.1.

Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system, where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space. Consider the disintegration of the measure μ over the sub-σ-algebra $\mathcal{I}(T) \subseteq \mathcal{X}$, $x \mapsto \mu_x^{\mathcal{I}(T)}$. Then for μ-almost every $x \in X$, the measure $\mu_x^{\mathcal{I}(T)}$ is T-invariant and the system $(X, \mathcal{X}, \mu_x^{\mathcal{I}(T)}, T)$ is ergodic.

Definition 1.7.2.

The disintegration over $\mathcal{I}(T)$ is called ergodic decomposition of the measure μ.

Note. (alternate presentation of the ergodic decomposition)

Let $\pi : (X, \mathcal{X}, \mu, T) \to (\Omega, \mathcal{O}, P, S)$ be the factor map associated to the invariant sub-σ-algebra $\mathcal{I}(T)$ of $\mathcal{X}$. This means that $\mathcal{I}(T) = \pi^{-1}(\mathcal{O})$, modulo μ. Then for μ-almost every $x \in X$, $\mu_x = \mu'_{\pi(x)}$ (where μ_x is as above), where the map $\omega \to \mu'_\omega$ is the disintegration of the measure μ over P. The map $\omega \to \mu'_\omega$ is also called ergodic decomposition of μ and we have that

$$\mu = \int_\Omega \mu'_\omega \, \mathrm{d}P(\omega)$$

We refer to this disintegration as alternate presentation of the ergodic decomposition.

1.8 Joinings

Definition 1.8.1.

Let (X_i, μ_i, T_i), $i = 1, 2$ be two measure preserving systems. A **joining** of these two systems is a probability measure ρ on $(X_1 \times X_2, \mathcal{X}_1 \otimes \mathcal{X}_2)$ such that $\rho(X_1 \times B) = \mu_2(B)$, for any $B \in \mathcal{X}_2$ and $\rho(A \times X_2) = \mu_1(A)$, for any $A \in \mathcal{X}_1$.

The joining ρ is ergodic if $(X_1 \times X_2, \rho, T_1 \times T_2)$ is an ergodic system.

A **self-joining** of a system, is a joining of two copies of the same system.

Remark 1.8.2.

The product measure $\mu_1 \otimes \mu_2$ is a joining. Thus the set of joinings of two systems is always nonempty

Definition 1.8.3.

When the only joining of two systems is the product measure, then the two system are said to be **disjoint**.

Definition 1.8.4.

The **diagonal self-joining** of a system (X, μ, T) is the image of the measure μ under the map $x \mapsto (x, x): \ X \to X \times X$.

Definition 1.8.5.

Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be a factor map. The **graph joining** is the image of the measure μ under the map $x \mapsto \big(x, \pi(x)\big): \ X \to X \times Y$.

Remark 1.8.6.

The graph joining is ergodic if (X, μ, T) is ergodic.

Proposition 1.8.7.

Let (X_i, μ_i, T_i), $i = 1, 2$ be two measure preserving systems and ρ be a joining of these two systems. Consider the $(x_1, x_2) \mapsto \rho_{(x_1,x_2)}^{\mathcal{I}(T_1 \times T_2)}$ be the ergodic decomposition of ρ. Then for ρ-almost every $(x_1, x_2) \in X_1 \times X_2$ the measure $\rho_{(x_1,x_2)}^{\mathcal{I}(T_1 \times T_2)}$ is a joining of μ_1 and μ_2.

Definition 1.8.8.

Let (X_i, μ_i, T_i), $i = 1, 2$ and (Y, ν, S) be three measure preserving systems and $\pi_i : X_i \to Y$, $i = 1, 2$ be two factor maps. The **relatively independent joining** of X_1 and X_2 over the common factor Y (or equivalently over the common factor $\mathcal{Y}$) is the probability measure $\mu_1 \otimes_Y \mu_2$ (or equivalently $\mu_1 \otimes_{\mathcal{Y}} \mu_2$) characterized by.

$$\int_{X_1 \times X_2} f_1(x_1)\ f_2(x_2)\ \mathrm{d}\mu_1 \otimes_Y \mu_2(x_1, x_2) = \int_Y \mathbb{E}(f_1 \mid Y)(y) \cdot \mathbb{E}(f_2 \mid Y)(y)\ \mathrm{d}\nu(y)$$

for all $f_1 \in L^\infty(\mu_1)$ and $f_2 \in L^\infty(\mu_2)$.

Definition 1.8.9.
Let (X_i, μ_i, T_i), $i = 1, 2$, and (Y_i, ν_i, S_i), $i = 1, 2$, be three measure preserving systems, $\pi_i : X_i \to Y_i$, $i = 1, 2$ be two factor maps and ρ a joining of the systems Y_1 and Y_2. The **conditional measure of** μ_1 **and** μ_2 **over** ρ is the measure $\mu_1 \otimes_\rho \mu_2$ on $X_1 \times X_2$ characterized by,

$$\int_{X_1 \times X_2} f_1(x_1)\ f_2(x_2)\ \mathrm{d}\mu_1 \otimes_\rho \mu_2(x_1, x_2) = \int_{Y_1 \times Y_2} \mathbb{E}(f_1 \mid Y_1)(y_1) \cdot \mathbb{E}(f_2 \mid Y_2)(y_2)\ \mathrm{d}\rho(y_1, y_2)$$

for all $f_1 \in L^\infty(\mu_1)$ and $f_2 \in L^\infty(\mu_2)$.

1.9 Rotations

Let Z be a compact abelian group, α a fixed element of Z, $R : Z \to Z$ the function given by $R(z) = \alpha \cdot z$ and m_Z tha Haar measure of Z. The measure preserving system (Z, m_Z, R) is called a **rotation**.

Theorem 1.9.1.
Assume that (Z, m_Z, R) is a rotation and R is the translation $z \mapsto z + \alpha$ for some fixed $\alpha \in Z$. Then (Z, m_Z, R) is ergodic if and only if $\{\alpha^n : \mathbb{Z}\}$ is dense in Z, and this holds if and only if the unique character χ of Z with $\chi(\alpha) = 1$ is the trivial character. In this case, the topological rotation (Z, R) is minimal and uniquely ergodic.
More generally, let (Z, R) be a topological rotation with R defined by a fixed element $\alpha \in Z$ and let H be the closure of $\{\alpha^n : \mathbb{Z}\}$ in Z. Then H is a subgroup of Z. Every ergodic R-invariant measure on Z is the image of m_H under the map $z \mapsto s \cdot z$ for some $s \in Z$, and every measure defined in this way is R-invariant and ergodic.

Proposition 1.9.2.
Let (Z_i, m_{Z_i}, R_i) be ergodic rotations given by $\alpha_i \in Z_i$ for $i = 1, 2$ and let H be the closure of $\{(\alpha_1, \alpha_2)^n : \mathbb{Z}\}$ in $Z_1 \times Z_2$. Then every ergodic joining λ of Z_1 and Z_2 is the image

under some translation of the Haar measure of H for some closed subgroup H of $Z_1 \times Z_2$. In particular, $(Z_1 \times Z_2, \lambda R_1 \times R_2)$ is isomorphic to an ergodic rotation.

Applying the proposition above to the graph joining, we obtain:

Corollary 1.9.3.
Every factor map $\pi : (Z_1, m_{Z_1}, R_1) \to (Z_2, m_{Z_2}, R_2)$ between ergodic rotations has the form $\pi = \beta + \phi$, where $\beta \in Z$ and ϕ is a continuous group homomorphism from Z_1 onto Z_2. Furthermore, if α_i is the element of Z_i defining the transformation on Z_i then $\phi(\alpha_1) = \alpha_2$.

Corollary 1.9.4.
Every isomorphism $\pi : (Z, m_Z, R) \to (Z, m_Z, R)$, where (Z, m_Z, R) is an ergodic rotation, is of the form $z \mapsto z + \beta$, for some $\beta \in Z$.

1.10 Group Extension and Cocycles

Definition 1.10.1.
If (X, μ, T), (Y, ν, S) are measure preserving systems and $\pi : X \to Y$ a factor map, then the system (X, μ, T) is an **extension** of Y and π can be also denoted as extension map.

Let (Y, ν, S) be a measure preserving system and K a compact group, and m_K its (normalized) Haar measure.

Definition 1.10.2.
A K-cocycle or cocycle on Y is a measurable map $\rho : Y \to K$. Let $Coc(Y, K)$ denote the set of all K-cocycles on Y.

Remark 1.10.3.
The set $Coc(Y, K)$ equipped with the operation $\rho_1 \bullet \rho_2, \quad \rho_1, \rho_2 \in Coc(Y, K)$, where $(\rho_1 \bullet \rho_2)(y) = \rho_1(y)\rho_2(y), \forall y \in Y,$ is a group.
Furthermore if K is in addition abelian then $Coc(Y, K)$ is also abelian.

Definition 1.10.4.
Let $\rho, \rho' \in Coc(Y, K)$. Let $\partial : Coc(Y, K) \to Coc(Y, K)$, defined by $\partial\rho(y) = \rho \circ S(y)\rho(y)^{-1}$.

(i) ρ is said to be a **coboundary** if there exists an $f : Y \to K$, such that $\rho = \partial f$

(ii) ρ, ρ' are said to be **cohomologous** if there exists an $f : Y \to K$, $f \in Coc(Y, K)$, such that, $\rho(y)\rho'(y)^{-1} = \partial f(y)$

(iii) ρ is said to be **quasi-coboundary** if it is cohomologous to a constant cocycle.

Lemma 1.10.5.

Let $(Y, \mathcal{Y}, \nu, S)$ be a measure preserving system, where $(Y, \mathcal{Y}, \nu)$ is a Lebesgue probability space. Let $\ell \in \mathbb{N}$. Then $Coc(Y, \mathbb{T}^\ell)$ endowed with the distance $d = d_{Coc(Y,\mathbb{T}^\ell)}$, defined by $d(\rho, \rho') = \int_Y d_{\mathbb{T}^\ell}\big(\rho(y), \rho'(y)\big)\, \mathrm{d}\nu(y)$, is a Polish group.

Furthermore then the set of coboundaries $\partial(Coc(Y, \mathbb{T}^\ell))$ is a Borel subset of $Coc(Y, \mathbb{T}^\ell)$.

Definition 1.10.6.

Let $\rho \in Coc(Y, K)$. The **isometric extension of** Y **by** K **associated to** ρ is defined to be the extension $\pi : (Y \times K, \nu \otimes m_K, S_\rho) \to (Y, \nu, S)$, where $\pi(y, k) = y$ and $S_\rho : Y \times K \to Y \times K$, defined by

$$S_\rho(y, k) = \big(S(y), \rho(y)k\big) \qquad \forall (y, k) \in Y \times K$$

The cocycle ρ is said to be ergodic, when $(Y \times K, \nu \otimes m_K, S_\rho)$ is ergodic.

Definition 1.10.7.

A function $f : X \to X$ is called an automorphism of $(X, \mathcal{X}, \mu, T)$ if $f_*\mu = \mu$ and $f \circ T = T \circ f$

Let $(X, \mathcal{X}, \mu, T) = (Y \times K, \mathcal{Y} \otimes \mathcal{K}, \nu \times m_K, S_\rho)$. For each $h \in K$, define $V_h : X \to X$ with $V_h(y, k) = (y, kh)$. Then clearly V_h is an automorphism of X

Definition 1.10.8.

Let $h \in K$ and $V_h : x \to X$ as above. The transformation V_h is called **vertical rotation**.

Definition 1.10.9.

We say that an extension X of the system Y is an **extension by a compact group** K if, X is isomorphic to a group extension of Y.

Now let G be a compact, metrizable group, m_G its Haar measure and H a closed subgroup of G. Then G acts on the compact space G/H by (left) translations and we write this action, $(g, z) \mapsto g \odot z : \; G \times G/H \to G/H$. Let $m_{G/H}$ be the (unique) probability measure of G/H, which is invariant under this action. Actually $m_{G/H}$ is the image of m_G under the quotient map.

Definition 1.10.10.

Let (Y, ν, S) be a measure preserving system, $\rho : Y \to G$ a G-cocycle. The **extension**

of Y **by** G/H **given by the cocycle** ρ is defined to be the extension $\pi : (Y \times G/H, \nu \otimes m_{G/H}, S_\rho) \to (Y, \nu, S)$, where $\pi(y,z) = y$ and $S_\rho : Y \times G/H \to Y \times G/H$, defined by

$$S_\rho(y,z) = (y, \rho(y) \odot z) \qquad \forall (y,z) \in Y \times G/H \ .$$

The topology of G can be defined by a metric that is invariant under left translations. Since H is a closed subgroup of G then the metric on G induces a metric on G/H that is invariant under the left action of G on G/H. When G/H is endowed with this metric then the restriction of S_ρ in each fiber of π is clearly an isometry. Thus say that the extension $\pi : Y \times G/H \to Y$ is an **isometric extension** of Y.

Without loss of generality we can assume that the subgroup H does not contain any normal subgroup of G. Under this assumption, the (left) action of G on G/H is faithful. This means that if $g \in G$ with $g \odot z = z$, $\forall z \in G/H$, then $g = e_G$.

Notation. Let $(X, \mu, T) = (Y \times G/H, \nu \otimes m_{G/H}, S_\rho)$ be an isometric extension of (Y, ν, S). Then the group G acts on X by, $(g, x) \mapsto g \star x$, where $g \star x = g \star (y,z) = (y, g \odot z)$.

Remark 1.10.11.
Clearly, if $(Y \times G/H, \nu \otimes m_K, S_\rho)$ is an isometric extension of (Y, ν, T), then $\forall g \in G$, the map $(y,z) \mapsto g \star (y,z) = (y, g \odot z)$ leaves the measure $\nu \otimes m_K$ invariant, but in general this map does not commute with S_ρ. However we have the following result.

Proposition 1.10.12.
Any factor, W of $X = Y \times G/H$ over Y has the form $Y \times G/L$, for some closed subgroup L of G containing H. In particular, the action of $g \in G$ on W induces a measure preserving transformation on this factor, written with the same notation.

Let $\pi : X \to Y$ be a factor map and assume that Y ergodic. Then the family of isometric extensions of Y, such that they are factors of X, admits a maximal element, called maximal isometric extension of Y below X.

Theorem 1.10.13.
Let (X, μ, T), (Y, ν, S) and (Z, λ, Q) be three systems, where (Y, ν, S) is in addition ergodic. Furthermore let $\pi : X \to Y$ and $p : Z \to Y$ be factor maps an W be the maximal isometric extension of Y, below X. Then the σ-algebra, $\mathcal{I}(T \times Q)$, of the $T \times Q$-invariant sets of the relatively independent joining $(X \times Z, \mu \otimes_Y \lambda, T \times Q)$, over the common factor Y, is contained in the σ-algebra $\mathcal{W} \otimes \mathcal{Z}$.

Definition 1.10.14.
Now let G be in a compact abelian group. The extension of the system (Y, ν, S) associated to a cocycle $\rho : Y \to G$ is the system $(Y \times G, \nu \otimes m_G, S_\rho)$.

Lemma 1.10.15. *(Uniqueness of the measure)*
Let (Y, ν, S) be an ergodic system, K be a compact abelian group, $\rho : Y \to K$ be an ergodic cocycle, and $(Y \times K, \nu \otimes m_K, S_\rho)$ be the extension it defines. If μ is a S_ρ-invariant measure on $Y \times K$ whose projection on Y is equal to ν, then $\mu = \nu \otimes m_K$.

Lemma 1.10.16.
Let (Y, ν, S) be an ergodic system, K be a finite dimensional torus ($K = \mathbb{T}^n$) and let $\rho : Y \to K$ be a cocycle. Then ρ is a quasi-coboundary if and only if the cocycle $\Delta\rho : Y^2 \to K$ given by $\Delta\rho(y_0, y_1) = \rho(y_0)\rho(y_1)^{-1}$ is a coboundary of the system $(Y^2, \nu \otimes \nu, T \times T)$.

1.11 Lie groups and nilsystems

Definition 1.11.1.
The commutator, $[G, G]$, of a group G is the normal subgroup spanned by the elements $[x, y] = xyx^{-1}y^{-1}$, $x, y \in G$. This means,

$$[G, G] = \langle xyx^{-1}y^{-1} : \ x, y \in G \rangle$$

Definition 1.11.2.
The center, $Z(G)$, of a group G is the normal subgroup that contains each element of G, that commutes with every other element of G. This means,

$$Z(G) = \{x \in G : \quad xg = gx, \ \forall g \in G\}$$

Definition 1.11.3.
A group G is called 2-step nilpotent when the commutator subgroup $[G, G]$ is a subset of the center, $Z(G)$, of G.

Definition 1.11.4.
A Lie group is a group G that is also a finite dimension C^∞ manifold, such that the functions

- $\gamma : G \times G \to G$, with $\gamma(g, g') = g \cdot g'$, for every $g.g' \in G$

- $\alpha : G \to G$, with $\alpha(g) = g^{-1}$, for every $g \in G$

are C^∞ maps.

Proposition 1.11.5.
If H is a closed subgroup of a Lie group G, then H is an embedded Lie group with the relative topology being the same as the group topology.

Proposition 1.11.6.
Let G be a Lie group. Then G is Hausdorff, locally compact space and locally path connected space.

Proposition 1.11.7.
Let G be a locally compact group and H a closed normal subgroup of G. If G/H and H are Lie groups then G is a Lie group.

Definition 1.11.8.
Let G be a Lie group. A discrete subgroup Λ of G is called cocompact when is closed in G and the manifold G/Λ is compact.

Definition 1.11.9.
Let G be a Lie group and $\Lambda \leq G$ a discrete, cocompact subgroup of G. If G is in addition 2-step nilpotent, the compact manifold G/Λ is called 2-step nilmanifold.

Theorem 1.11.10. ***(Malcev)***
The subgroups $[G, G]$ and $[G, G]\Lambda$ are closed subgroups of G and thus they are closed Lie subgroups of G.

Let G be a Lie group and Λ a discrete cocompact subgroup of G. The group G acts (transitively) on the manifold $X = G/\Lambda$ by

$$(g, g'\Lambda) \mapsto g \odot g'\Lambda = (gg')\Lambda : \ G \times X \to X$$

It can be proved that there exists a Borel probability measure, μ, on $X = G/\Lambda$ that is invariant under this action.

Definition 1.11.11.
Let $\alpha \in G$ and $T_\alpha = T : X \to X$ be the continuous function on X defined by $T(x) = \alpha \odot x$. Then

- The topological system $(X, T) = (G/\Lambda, T_\alpha)$ is called **topological 2-step nilsystem**.
- In addition, $T_*\mu = \mu$, thus $(X, \mu, T) = (G/\Lambda, \mu, T_\alpha)$ is a measure preserving system. This kind of systems are called **2-step nilsystems**.

Let $(X, \mu, T) = (G/\Lambda, \mu, T_\alpha)$ a 2-step nilsystem. We assume that the system is also ergodic. Let G_0 be the connected component of the identity of G. Since G is a Lie group therefore locally path connected, we have that G_0 is a clopen subgroup of G. Consider $G' = \langle G_0, \alpha \rangle$, and $\Lambda' = G' \cap \Lambda$. Then G' is an open subgroup of G and Λ' is a discrete cocompact subgroup of G'. Furthermore by ergodicity the projection of G' on X is (almost) onto X (the image of G' is an open subset of X that is T-invariant, follows that that has measure μ equal to 1). Follows that X is isomorphic to G'/Λ'.
In summary, for an ergodic 2-step nilsystem defined as above one can assume the following:

(1.1) $\quad$ *G is spanned by the identity component and the element α.*

Again let $(X, \mu, T) = (G/\Lambda, \mu, T_\alpha)$ a 2-step nilsystem. Consider Λ'' to be the largest normal subgroup of G such that $\Lambda'' \subseteq \Lambda$. Then we have that

$$G/\Lambda \cong (G/\Lambda'')/(\Lambda/\Lambda'')$$

Thus without loss of generality can G can be substituted by G/Λ'' and Λ by Λ/Λ''. Thus one can assume

(1.2) $\quad$ *The subgroup Λ does not contain any normal subgroup of G.*

In order to make the assumption that property (1.2) is satisfied, ergodicity is not necessary.

Notice that property (1.1) is satisfied for X, then after the substitutions made in order to acquire property (1.2), it is still satisfied. Furthermore the second property means that G acts faithfully on X. This means that if $g \in G$ with $g \odot x = x$, $\forall x \in X$, then $g = e_G$. This implies that $U \cap \Lambda$ is the trivial group and it follows that U is compact (thus U is a finite dimensional torus). It can be proven that U is also connected. Moreover Λ is abelian.

From here on for any ergodic 2-step nilsystem we will assume that these properties are automatically satisfied.

Let λ denote the Haar measure of U and $K = G/\Lambda U$. Let $\pi : X \to K$ be the natural (continuous and open) projection of X on the compact abelian group K. Moreover let m be the Haar measure of K and $q : G \to K$ the natural (continuous and open) projection of G on K. Define $\beta \in K$ to be the q-projection of α on K and R the rotation on K by β.

Since $U \subseteq Z(G)$, the action of U on K commutes with $T = T_\alpha$. Furthermore the action is free. This means that if $x \in X$ and $u \in U$ with $u \odot x = x$, then $u = e_U = e_G$. The quotient of X under this action is the group K.

Proposition 1.11.12. ***(Parry)***
Let (X, μ, T) be an ergodic 2-step nilsystem. Then (X, T) is uniquely ergodic and minimal 2-step topological nilsystem.

Proposition 1.11.13. ***(Parry)***
Let (X, μ, T) be an 2-step nilsystem and (K, m, R) the measure preserving system where K, m, R as above. Then

(i) *$(X.\mu, T)$ is ergodic if and only if (K, m, R) is ergodic*

(ii) *If (X, μ, T) is ergodic, the system (K, m, R) is the Kronecker factor of X, with factor map $\pi : X \to K$ where π as above.*

The previous Proposition, with an additional requirement, can be generalized in the following manner.

Proposition 1.11.14. ***(Parry, Leibman)***
Let $X = G/\Lambda$ be a manifold with Haar measure μ and $\alpha_1, \ldots, \alpha_k \in G$, $k \in \mathbb{N}$ be commuting elements of G. Assume that

$$G = \langle G_0,\ \alpha_1,\ \ldots,\ \alpha_k \rangle \tag{1.3}$$

Let $T_{\alpha_i} : X \to X$, with $T_{\alpha_i}(x) = \alpha_i \odot x$, for every $i = 1, 2, \ldots, k$ and $R_{\beta_i} : K \to K$ be the rotations on K by $\beta_i = q(\alpha_i)$, for every $i = 1, 2, \ldots, k$. Then

(i) *The joint action of $T_{\alpha_1}, \ldots, T_{\alpha_k}$ is ergodic on X if and only if the joint action of the induced transformations, $R_{\beta_1}, \ldots, R_{\beta_k}$, on K is ergodic on K.*

(ii) *If the joint action of $T_{\alpha_1}, \ldots, T_{\alpha_k}$ is ergodic on X, X is uniquely ergodic and minimal for this joint action.*

1.12 Inverse Limits

Let $(I,\leq)$ be a countable directed set. It follows that there exists an increasing sequence $(i_n)_{n\in\mathbb{N}}$ in I, such that for every $i\in I$ there exists a $n\in\mathbb{N}$ such that $i\leq i_n$.

For every $i\in I$ let (X_i,μ_i,T_i) be a measure preserving system and for every $i,j\in I$ with $i\leq j$ let $\pi_{i,j}:X_i\to X_j$ be factor maps such that

$$\pi_{i,\ell}=\pi_{i,j}\circ\pi_{j,\ell}\qquad\forall\ i\leq j\leq\ell,\quad i,j,\ell\in I$$

We say that the pair $\big((X_i,\mu_i,T_i)_{i\in I},\ (\pi_{i,j})_{i\leq j}\big)$ forms an inverse system of measure preserving systems or simply inverse system.

The inverse limit of an inverse system of measure preserving systems, is a measure preserving system (X,μ,T) endowed with the factor maps $\pi_i:X\to X_i$, $\forall i\in I$, satisfying the following properties.

(i) $\pi_i=\pi_{i,j}\circ\pi_j$, μ-almost everywhere , for all $i\leq j$, $i,j\in I$.

(ii) If (Y,ν,S) is a measure preserving system and for each $i\in I$ there exists a factor map $p_i:Y\to X_i$ such that $p_i=\pi_{i,j}\circ p_j$, $\forall i\leq j$, $i,j\in I$, then there exists a unique factor map $p:Y\to X$ such that $p_i=\pi_i\circ p$, ν-almost everywhere, for every $i\in I$.

The second property characterizes the inverse limit uniquely up to isomorphism. Thus we can assume that the inverse limit is unique and we write

$$(X,\mu,T)=\varprojlim_{i\in I}(X_i,\mu_i,T_i)\quad or\quad \varprojlim(X_i,\mu_i,T_i)$$

The inverse limit can be constructed in a more specific way. Let $X=\prod\limits_{i\in I}X_i$ and T be the diagonal transformation on X. For each $i\in I$, let $\pi_i:X\to X_i$ denote the projection to the i-th coordinate. By combining this and the properties of that inverse system we can build an invariant probability measure μ on the space X. Since I is countable, μ is a Borel probability measure on X, and thus (X,μ) is a Lebesgue space.

Now if $(X,\mu,T)=\varprojlim(X_i,\mu_i,T_i)$, then

(iii) $\mathcal{X}=\bigvee\limits_{i\in I}\pi_i^{-1}(\mathcal{X}_i)$

(iv) for every $1\leq p\leq+\infty$, the set $\bigcup\limits_{i\in I}\{f\circ\pi_i:\ f\in L^p(\mu_i)\}$ is dense in $L^p(\mu)$.

Proposition 1.12.1.

Let $\big((X_i,\mu_i,T_i)_{i\in I},\ (\pi_{i,j})_{i\leq j}\big)$ *be an inverse system and* (X,μ,T) *a measure preserving system such that* $\forall i \in I$, $\pi : X \to X_i$ *is a factor map such that the property* (i) *is satisfied. Then each of the properties* (iii) *or* (iv) *implies that* $(X,\mu,T) = \varprojlim (X_i,\mu_i,T_i)$.

Proposition 1.12.2.

Let $(i_n)_{n\in\mathbb{N}}$ *be a subset of* I *as defined in the beginning of this section. For* $n \in \mathbb{N}$, *let* $X'_n = X_{i_n}$ *and for* $m \leq n$, *let* $\pi'_{m,n} = \pi_{i_m,i_n}$. *The family of systems and factor maps form an inverse system and* $\varprojlim\limits_{n\in\mathbb{N}}(X'_n,\mu_n,T_n) = \varprojlim\limits_{i\in I}(X_i,\mu_i,T_i)$.

Remarks 1.12.3.

- This means that essentially we can reduce to the case where $I = \mathbb{N}$
- In this case property (iv) implies that for $1 \leq p \leq +\infty$ and $f \in L^p(\mu)$,

$$f \underset{L^p(\mu)}{=} \lim_{n\to+\infty} \mathbb{E}(f \mid \mathcal{X}_n)$$

Proposition 1.12.4.

The inverse limits of ergodic systems is itself an ergodic system.

1.13 Cubes

Throughout, we use 2^k-Cartesian powers of spaces for an integer $k > 0$ and need some shorthand notation. For an integer $k > 0$, let $V_k = \{0,1\}^k$. The elements of V_k are written without commas or parentheses. For $\epsilon = \epsilon_1\epsilon_2\ldots,\epsilon_k \in V_k$ and $n = (n_1,n_2,\ldots,n_k) \in \mathbb{Z}^k$ we write

$$\epsilon\cdot n = \epsilon_1 n_1 + \epsilon_2 n_2 + \ldots + \epsilon_k n_k$$

We use $\mathbf{0}$ to denote the element $00\ldots0$ of V_k and set $V_k^* = V_k \setminus \{\mathbf{0}\}$. Let X be a set. For an integer $k \geq 0$, we write $X^{[k]} = X^{2^k}$. For $k > 0$, we use the sets $V_k = \{0,1\}^k$ introduced above to index the coordinates of elements of this space, which are written $\mathbf{x} = (x_\epsilon :\ \epsilon \in V_k)$.

When f_ϵ, $\epsilon \in V_k$, are 2^k real or complex valued functions on the set X, we define a function $\bigotimes\limits_{\epsilon\in V_k} f_\epsilon$ by

$$\bigotimes_{\epsilon\in V_k} f_\epsilon(\mathbf{x}) = \prod_{\epsilon\in V_k} f_\epsilon(x_\epsilon)$$

When $g : X \to Y$ is a map, we write $g^{[k]} : X^{[k]} \to Y^{[k]}$ for the map given by $\big(g^{[k]}(\mathbf{x})\big)_\epsilon = g(x_\epsilon),\ \epsilon \in V_k$.

We often identify $X^{[k+1]}$ with $X^{[k]} \times X^{[k]}$. In this case, we write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ for a point of $X^{[k+1]}$, where $\mathbf{x}'$, $\mathbf{x}'' \in X^{[k]}$ are defined by

$$x'_\epsilon = x_{0\epsilon} \quad and \quad x''_\epsilon = x_{1\epsilon}$$

for $\epsilon \in V_k$ and 0ϵ and 1ϵ are the elements of V^{k+1} given by

$$(0\epsilon)_j = (1\epsilon)_j = \epsilon_j \quad 2 \le j \le k\ ; \quad (0\epsilon)_1 = 0,\ (1\epsilon)_1 = 1$$

The maps $\mathbf{x} \mapsto \mathbf{x}'$ and $\mathbf{x} \mapsto \mathbf{x}''$ are called the projections on the first and second side, respectively.

It is convenient to view V_k as indexing the set of vertices of the cube of dimension k, making the use of the geometric words $side$, $face$, and $edge$ for particular subsets of V_k natural. More precisely, for $0 \le \ell \le k$, consider A to be a subset of $\{1, 2, \dots, k\}$ of cardinality $k - \ell$ and let $\eta \in \{0, 1\}^A$. The subset

$$\alpha = \{\epsilon \in V_k :\ \epsilon_j = \eta_j\ ,\ \forall j \in A\}$$

of V_k is called a face of dimension ℓ of V_k, or more succinctly, an ℓ-face. Thus V_k has one face of dimension k, namely V_k itself. It has $2k$ faces of dimension $k-1$, called the **sides**, and has $k2^{k-1}$ faces of dimension 1, called **edges**. It has 2^k faces of dimension 0, each consisting in one element of V_k and called a **vertex**. We often identify the vertex $\{\epsilon\}$ with the element ϵ of V_k.

Let α be an ℓ-face of V_k. There is a natural bijection between α and V_ℓ. This bijection maps the faces of V_k included in α to the faces of V_ℓ. Moreover, for every set X, it induces a map from $X^{[k]}$ onto $X^{[\ell]}$. When α is any face, we call it a face-projection and when α is a side, we call it a a side-projection. This is a natural generalization of the projections on the first and second sides.

The symmetries of the cube V_k play an important role in the sequel. We write S_k for the group of bijections of V_k onto itself which maps every face to a face (of the same dimension, of course). This group is isomorphic to the group of the geometric cube of dimension k, meaning the group of isometries of $\mathbb{R}^k$ preserving the unit cube. It is spanned by digit permutations and reflections, which we now define.

Definition 1.13.1.
Let $\tau \in S_k$. The permutation σ on V_k given by

$$\big(\sigma(\epsilon)\big)_j = \epsilon_{\tau(j)}\ , \quad 1 \le j \le k$$

for $\epsilon \in V_k$, is called digit permutation.

Let $i \in \{1, 2, \ldots, k\}$. The permutation of V_k defined by

$$\big(\sigma(\epsilon)\big)_j = \epsilon_j \ j \neq i\ ; \quad \big(\sigma(\epsilon)\big)_i = 1 - \epsilon_i$$

for $\epsilon \in V_k$, is called reflection.

For set X the group S_k acts on $X^{[k]}$ by permutating the coordinates. More precisely, if $\sigma \in S_k$, define $\sigma_* : X^{[k]} \to X^{[k]}$ by

$$\big(\sigma_*(\mathbf{x})\big)_\epsilon = x_{\sigma(\epsilon)}$$

When σ is a digit permutation (respectively, a reflection) we also call the associated map σ_* a digit permutation (respectively, a reflection).

Chapter 2

Kronecker Factor

2.1 Ergodic decomposition of a rotation

Definition 2.1.1.

Let Z be a compact, abelian, metrizable group with additive notation, α be a fixed element of that group, $R : Z \to Z$ given by $R(z) = \alpha + z$ and m_Z the Haar measure of Z.

Then the measure preserving system (Z, m_Z, R) is called rotation. If (Z, m_Z, R) in addition is ergodic then it is called ergodic rotation.

Let (Z, m_Z, R) be an ergodic rotation.

For each $s \in Z$, set

$$Z_s = \{(z, z+s) : z \in Z\}$$

Then Z_s is a subgroup of Z^2 that is invariant under the transformation $R \times R$. The map $\theta_s : Z \to Z_s$, given by $\theta_s(z) = (z, z+s)$ is an isomorphism from the topological system (Z, R) to the topological system $(Z_s, R \times R)$. Therefore the topological system $(Z_s, R \times R)$ is uniquely ergodic and its (unique) invariant measure m_s is the image of m_Z under the map θ_s.

Proposition 2.1.2.

Let (Z, m_Z, R) be an ergodic rotation. The map $\varphi : Z \times Z \longrightarrow \mathcal{M}(Z \times Z)$ given by $\varphi(x, y) = m_{y-x}$ is an ergodic decomposition of $m_Z \otimes m_Z$

Proof. Let F be a bounded function on $X \times X$ then by the pointwise ergodic theorem

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} F\big(\, R^n(x), R^n(y) \,\big) \;=\; \mathbb{E}_{m_Z \otimes m_Z}\big(F \mid \mathcal{I}(R\times R)\big)(x,y),$$

for $m_Z \otimes m_Z$-almost every $(x, y) \in Z \times Z$.

On the other hand $\forall\ (x,y) \in Z \times Z,\ \ (x,y) \in Z_{y-x}$ and since $(Z_{y-x}, R \times R)$ is uniquely ergodic, we have that

$$\frac{1}{N} \sum_{n=1}^{N-1} F\big(\, R^n(x), R^n(y) \,\big) \underset{N\to\infty}{\longrightarrow} \int_{Z^2} F(z_1, z_2)\, \mathrm{d}m_{y-x}$$

Thus, for $m_Z \otimes m_Z$-almost every $(x, y) \in Z \times Z$, we have that

$$\mathbb{E}_{m_Z \otimes m_Z}\big(F \mid \mathcal{I}(R\times R)\big)(x,y) \;=\; \int_{Z^2} F(z_1, z_2)\, \mathrm{d}m_{y-x}$$

Equivalently,

$$\int_{Z\times Z} F(z_1,z_2)\, \mathrm{d}(m_Z \otimes m_Z)^{\mathcal{I}(R\times R)}_{(x,y)} \;=\; \int_{Z^2} F(z_1, z_2)\, \mathrm{d}m_{y-x}$$

The result follows from the fact that $Z \times Z$ has a countable base and $\mathcal{Z} \otimes \mathcal{Z}$ is its Borel σ -algebra $\big(\mathcal{B}(Z\times Z) = \mathcal{Z} \times \mathcal{Z}$ since Z is separable metric space$\big)$.

$\square$

Proposition 2.1.3.
Let $p : Z \times Z \to Z$, given by $p(x, y) = y - x$. Then

$$p^{-1}(\mathcal{Z}) = \mathcal{I}(R \times R)$$

Proof. We observe that the map p is continuous (and surjective). Thus p is $\mathcal{Z} \times \mathcal{Z}$-measurable. Let $(x, y) \in Z \times Z$,

$$p \circ (R \times R)(x, y) = p(x + \alpha, y + \alpha) = (y + \alpha) - (x + \alpha) \underset{\substack{Z\ is\\ abelian}}{=} y - x = p(x, y)$$

Thus $p^{-1}(\mathcal{Z})$ is $R \times R$-invariant. In particular $p^{-1}(\mathcal{Z}) \subseteq \mathcal{I}(R \times R)$.

On the other hand, let $\mathcal{A}$ denote the sub-σ-algebra of $\mathcal{Z} \otimes \mathcal{Z}$ that makes ϕ measurable. We consider the map $\psi : s \mapsto m_s : Z \to \mathcal{M}(Z \times Z)$ and we have that $\phi = \psi \circ p$. Thus $\mathcal{A} \subseteq p^{-1}(\mathcal{Z})$. Now from Proposition 2.1.2, $\mathcal{A} = \mathcal{I}(R \times R)$ modulo $m_Z \otimes m_Z$. Therefore $\mathcal{I}(R \times R) \subseteq p^{-1}(\mathcal{Z})$. $\square$

Remark 2.1.4.

The function ψ defined in the proof above is indeed $\mathcal{Z}$-measurable. In particular is a continuous function.

Indeed, let $f \in C(Z \times Z)$, $r \in \mathbb{R}$ and $\epsilon > 0$. We consider the subset $\mathcal{O}_{f,r,\epsilon} = \{\nu \in \mathcal{M}(Z \times Z) : \left|\int_{Z^2} f \mathrm{d}\nu - r\right| < \epsilon\}$ of $\mathcal{M}(Z \times Z)$, then

$$\psi^{-1}(\mathcal{O}_{f,r,\epsilon}) = \{s \in Z : \left|\int_{Z^2} f \ \mathrm{d}m_s - r\right| < \epsilon\} = \{s \in Z : \left|\int_Z f \circ \theta_s \ \mathrm{d}m_Z - r\right| < \epsilon\}$$

Let $s_0 \in \psi^{-1}(\mathcal{O}_{f,r,\epsilon}) \implies \left|\int_Z f \circ \theta_{s_0} \ \mathrm{d}m_Z - r\right| < \epsilon \implies \exists n_0 \in \mathbb{N}$ such that $\left|\int_Z f \circ \theta_{s_0} \ \mathrm{d}m_Z - r\right| < \epsilon - \frac{1}{n_0}$. Since f is continuous on the compact space $Z \times Z$, then is uniformly continuous. Thus there exists a $\delta = \delta(n_0) > 0$ so that, for every $\mathbf{z}, \mathbf{z}' \in Z \times Z$ with $d_{Z^2}(\mathbf{z}, \mathbf{z}') < \delta$, then $|f(\mathbf{z}) - f(\mathbf{z}')| < \frac{1}{n_0}$. Now, if $d_Z(s_0, s) < \delta$ then $d_{Z^2}(\theta_{s_0}(z), \theta_s(z)) < \delta$, for every $z \in Z \implies \left|f(\theta_{s_0}(z)) - f(\theta_s(z))\right| < \frac{1}{n_0}$, for every $z \in Z \implies \int_Z |f(\theta_{s_0}(z)) - f(\theta_s(z))| \ \mathrm{d}m_Z < \frac{1}{n_0}$.

Therefore for any $s \in Z$ that is δ-near s_0,

$$\begin{aligned}\left|\int_{Z^2} f \ \mathrm{d}m_s - r\right| &= \left|\int_Z f \circ \theta_s \ \mathrm{d}m_Z - r\right| \leq \left|\int_Z f \circ \theta_{s_0} \ \mathrm{d}m_Z - r\right| + \left|\int_Z f(\theta_{s_0}(z)) - f(\theta_s(z)) \ \mathrm{d}m_Z\right| \\ &\leq \left|\int_Z f \circ \theta_{s_0} \ \mathrm{d}m_Z - r\right| + \int_Z |f(\theta_{s_0}(z)) - f(\theta_s(z))| \ \mathrm{d}m_Z < \epsilon - \frac{1}{n_0} + \frac{1}{n_0} = \epsilon\end{aligned}$$

In other words if an $s \in Z$ is δ-near to s_0, then $s \in \psi^{-1}(\mathcal{O}_{f,r,\epsilon})$ and that completes the proof.

2.2 Existence

Theorem 2.2.1.

Let $(X, \mathcal{X}, \mu, T)$ be an invertible ergodic measure preserving system on a Borel probability space, and let $\mathcal{K}$ be the smallest -algebra with respect to which all $L^2(\mu)$ eigenfunctions of X are measurable. Then the corresponding factor $(Z, \mathcal{Z}, m_Z, R)$ of $(X, \mathcal{X}, \mu, T)$ is isomorphic to a rotation $R(z) = R_a(z) = z \cdot a$ on some compact abelian group Z .

Proof. Consider the countable set of $\{f_1, f_2, \dots, f_n, \dots\}$ of the eigenfunctions of X, such that, for every $i \in \mathbb{N}$, $|f_i| = 1$, for μ-almost everywhere (Remark 1.3.8). Let λ_i denote the corresponding eigenvalue for each f_i. Define $F : X \to \mathbb{T}^{\mathbb{N}}$ by,

$$F(x) = \big(f_1(x), f_2(x), \dots, f_n(x), \dots\big)$$

Then F is measurable. In particular $F^{-1}\big(\mathcal{B}(\mathbb{T}^{\mathbb{N}})\big) = \mathcal{K}$. Indeed for every $\prod_{i\in\mathbb{N}} B_i$, where each B_i is an open subset of $\mathbb{T}$, we have that $F^{-1}\big(\prod_{i\in\mathbb{N}} B_i\big) = \bigcap_{i\in\mathbb{N}} f_i^{-1}(B_i)$, which is an element of $\mathcal{K}$. Thus $F^{-1}\big(\mathcal{B}(\mathbb{T}^{\mathbb{N}})\big) \subseteq \mathcal{K}$. Furthermore if B is an open subset of $\mathbb{T}$, then for the set $\mathbf{B} = \mathbb{T} \times \ldots \times \mathbb{T} \times B \times \mathbb{T} \times \ldots$, where B is in the i-th position, we have that $F^{-1}(\mathbf{B}) = f_i^{-1}(B)$. This means that each of f_i is $F^{-1}\big(\mathcal{B}(\mathbb{T}^{\mathbb{N}})\big)$-measurable. By minimality of $\mathcal{K}$, follows that $\mathcal{K} \subseteq F^{-1}\big(\mathcal{B}(\mathbb{T}^{\mathbb{N}})\big)$.

Set $\alpha = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ and define $R(z) = z \cdot a$ for every $z \in \mathbb{T}^{\mathbb{N}}$. Then clearly $R \circ F = F \circ T$. Let $\mathbb{T}^{\mathbb{N}}$ be endowed with the Borel measure $\nu = F_*\mu$. Then the system $(\mathbb{T}^{\mathbb{N}}, \nu, R)$ is an ergodic system (see Remark 1.4.2). By Theorem 1.9.1 we have that the system (Z, m_Z, R), where $m_Z := \nu$ is the Haar measure of the subgroup $Z := \overline{\langle \alpha^n : n \in \mathbb{Z}\rangle}$ of $\mathbb{T}^{\mathbb{N}}$, is the ergodic factor with the required property. □

Definition 2.2.2.

The factor constructed in the Theorem above is called Kronecker factor of X.

A system that is isomporphic to its Kronecker factor is called system of order 1.

Remark 2.2.3.

Let (Z, m_Z, R) be an ergodic rotation, where $R(z) = z + \alpha$.

If $\gamma \in \widehat{Z}$ and $c \in \mathbb{T}$ then $c\gamma$ is an eigenfunction of Z, with eigenvalue $\gamma(\alpha)$. Conversely, if f is an eigenfunction of Z with eigenvalue λ, then (by density in $L^2(m_Z)$ there exists a $\gamma \in \widehat{Z}$ with $\int_Z f \cdot \overline{\gamma} \mathrm{d}m_Z \neq 0$. Now since γ is an eigenfunction of Z with eigenfunction $\gamma(\alpha)$, we have (eigenfunctions corresponding to different eigenfunctions are orthogonal in $L^2(m_Z)$) that $\lambda = \gamma(\alpha)$ and $\exists\, c \in \mathbb{T}$ such that $f = c\cdot\gamma$. Therefore all the eigenfunctions of Z are of the form $c \cdot \gamma$, for some $c \in \mathbb{T}$ and $\gamma \in \widehat{Z}$.

Since the linear subspace of $L^2(m_Z)$, consisting of characters of Z, is dense in $L^2(m_Z)$, then

$$L^2(m_Z) = \overline{\langle \gamma \in \widehat{Z}\rangle}^{||\cdot||_{L^2(m_Z)}} = \overline{\langle g \text{ is an eigenfunction of } Z\rangle}^{||\cdot||_{L^2(m_Z)}}$$

Corollary 2.2.4.

Let (X, μ, T) be an invertible ergodic system, (Z, m, R) be its Kronecker factor and $\pi : X \to Z$ the corresponding factor map. Then every eigenfunction of X is of the form $c\cdot\gamma\circ\pi$, where $c \in \mathbb{C}$ is a constant and $\gamma \in \widehat{Z}$.

Proposition 2.2.5.

Let (X, μ, T) be an ergodic system and (Z, m_Z, R) its Kronecker factor. Then:

(i) *The subspace $L^2(X, \pi^{-1}(\mathcal{Z}), \mu)$ of $L^2(\mu)$, consisting of the $\pi^{-1}(\mathcal{Z})$-measurable functions, is the closed linear space spanned by the eigenfunctions. That means*

$$L^2(X, \pi^{-1}(\mathcal{Z}), \mu) = \overline{\langle f \in L^2(\mu) : \ f \textit{ is an eigenfunction of } X \rangle}^{||\cdot||_{L^2(\mu)}}$$

(ii) *The sub-σ-algebra of $\mathcal{X}$ spanned by the eigenfunctions of X is equal to $\pi^{-1}(\mathcal{Z})$ modulo μ.*

(iii) *The system (X, μ, T) is isomorphic to an ergodic rotation if and only if its σ-algebra is spanned by its eigenfunction.*

(iv) *The factor Z is the largest factor of X that is isomorphic to a rotation.*

Proof.

(i) Let $f \in L^2(X, \pi^{-1}(\mathcal{Z}), \mu)$. Then there exists a $g \in L^2(m_Z)$ such that $f = g \circ \pi$, μ-a.e. From the Remark 2.2.3, g can be approximated in $L^2(m_Z)$ by functions of the form $\sum_{i=1}^{n} c_i \gamma_i$. Thus f is approximated in $L^2(X, \pi^{-1}(\mathcal{Z}), \mu)$, by functions of the form $\left(\sum_{i=1}^{n} c_i \gamma_i \right) \circ \pi = \sum_{i=1}^{n} c_i (\gamma_i \circ \pi)$. Observe that $\gamma_i \circ \pi$ is eigenfunction of X, with eigenvalue $\gamma(\alpha)$. Hence,

$$L^2(X, \pi^{-1}(\mathcal{Z}), \mu) = \overline{\langle f \in L^2(\mu) : \ f \text{ is an eigenfunction of } X \rangle}^{||\cdot||_{L^2(\mu)}}$$

(ii) The result is proven with same procedure applied in (i).

(iii) If the system is isomorphic to an ergodic rotation then we obtain the result by Remark 2.2.3. Now if the σ-algebra of X is spanned by its eigenfunctions, then by (i) (X, μ, T) is isomorphic to its Kronecker factor, which is an ergodic rotation.

(iv) Let $(Y, \mathcal{Y}, \nu, S)$ an ergodic rotation an $p : X \to Y$ a factor map. Then by (iii), $\mathcal{Y}$ is spanned by the eigenfunctions of Y. Thus $p^{-1}(\mathcal{Y})$ is spanned by the functions of the form $f \circ p$, where f is an eigenfunction of Y. Observe that $f \circ p$ is an eigenfunction of X. Thus $f \circ p$ is $\pi^{-1}(\mathcal{Z})$-measurable. Therefore $p^{-1}(\mathcal{Y}) \subseteq \pi^{-1}(\mathcal{Z})$.

□

2.3 Decomposition of a system via the Kronecker factor

Theorem 2.3.1.
Let (X, μ, T) be a system. Then the subspace of $L^2(\mu \otimes \mu)$, consisting of $(T \times T)$-invariant functions is the subspace spanned by functions of the form $\overline{f_1} \otimes f_2$ where each $f_i \in L^2(\mu)$, $i = 1, 2$, is an eigenfunction of U_T, such that the corresponding eigenvalues λ_1, λ_2 are equal.

Proof. Let $F \in L^2(\mu \otimes \mu)$ be an $\mathcal{I}(T \times T)$-measurable. Define $L : L^2(\mu) \to L^2(\mu)$, by setting

$$Lf(x) = \int_X F(x, y) f(y) \,\mathrm{d}\mu_2(y)$$

for any $f \in L^2(\mu)$. Then L is a Hilbert-Schmidt operator and thus is compact. Furthermore, $U_T \circ L = L \circ U_T$. Indeed

$$\begin{aligned} Lf(Tx) = U_T(Lf)(x) &= \int\limits_X F(Tx, y) f(y) \,\mathrm{d}\mu(y) \\ &\underset{T_*\mu=\mu}{=} \int\limits_X F(Tx, y) f(y) \,\mathrm{d}T_*\mu(y) = \int\limits_X F(Tx, Ty) f(Ty) \,\mathrm{d}\mu(y) \\ &\underset{F=F\circ(T\times T)}{=} \int\limits_X F(x, y) f(Ty) \,\mathrm{d}\mu(y) \;=\; L(U_T f)(x) \end{aligned}$$

By writing $F = F' + iF''$, where $F' = \frac{1}{2}\big(F(x, y) + \overline{F}(y, x)\big)$ and $F' = \frac{1}{2i}\big(F(x, y) - \overline{F}(y, x)\big)$, we have that $\overline{F'}(y, x) = F'(x, y)$ and $\overline{F''}(y, x) = F''(x, y)$. Thus, taking into account the linearity of the integral, we can assume that $\overline{F}(y, x) = F(x, y)$. With this additional assumption L is also self-adjoint.

Let $\{\lambda_n : n \in \mathbb{N}\}$ be the countable set of non-zero eigenvalues of L. For each $n \in \mathbb{N}$ we consider $V_n \leq L^2(\mu)$ to be the eigenspace corresponding to eigenvalue λ_n. We have that $dim V_n < \infty$ and since $U_T \circ L = L \circ U_T$, $U_T V_n \subseteq V_n$, for every $n \in \mathbb{N}$. Since (X, μ, T) is invertible system, then U_T is a unitary operator. It follows that $U_T \big|_{V_n}$ is a unitary operator on the finite dimensional space V_n, for any $n \in \mathbb{N}$, and thus is diagonalizable on each V_n. Hence V_n has a basis that consists of eigenfunctions of the operator U_T.

We have that $L^2(\mu \otimes \mu) = L^2(\mu) \otimes L^2(\mu)$. Hence if $\{e_n : n \in \mathbb{N}\}$ is a basis for $L^2(\mu)$ then for the $F \in L^2(\mu \otimes \mu)$,

$$F = \sum_{i,j \in \mathbb{N}} F_{ij} \cdot e_i \otimes \overline{e_j}$$

for some $F_{i,j} \in \mathbb{C}$.

Now we have that $L^2(\mu) = kerL \bigoplus \oplus_{n\in\mathbb{N}} V_n$, thus we can consider that the basis $\{e_n : n \in \mathbb{N}\}$ of $L^2(\mu)$ be the collection eigenfunctions of U_T that span V_n, for each $n \in \mathbb{N}$ plus a basis of $kerL$. We will show that $F \in \bigoplus\limits_{n\in\mathbb{N}} V_n \otimes \bigoplus\limits_{n\in\mathbb{N}} V_n$. Let $e_{j_0} \in kerL$ then

$$0 = Le_{j_0}(x) = \sum_{i,j} F_{i,j} e_i(x) \int\limits_X \overline{e_j}(y) e_{j_0}(y) \, \mathrm{d}\mu(y) = \sum_{i,j} F_{i,j} e_i(x) \langle e_{j_0}, e_j \rangle_{L^2(\mu)} = \sum_i F_{i,j_0} e_i(x)$$

Since e_i is a basis of $L^2(\mu)$ then $F_{i,j_0} = 0$ for all $i \in \mathbb{N}$. Applying the same procedure for $\overline{F}(y,x) = F(x,y)$, one can show that if $e_{i_0} \in kerL$, then $F_{i_0,j} = 0$ for all $j \in \mathbb{N}$, and we obtain the result.

Furthermore, let $e_i \circ T = \lambda_i e_i$ and $e_j \circ T = \lambda_j e_j$. We have

$$F \circ (T \times T) = \sum_{i,j\in\mathbb{N}} F_{ij} \cdot e_i \circ T \otimes \overline{e_j} \circ T = \sum_{i,j\in\mathbb{N}} F_{ij} \lambda_i \overline{\lambda_j} \cdot e_i \otimes \overline{e_j}$$

and since F is $\mathcal{I}(T \times T)$-measurable then $F \circ (T \times T) = F$. Follows that for all $i, j \in \mathbb{N}$, $F_{ij} \lambda_i \overline{\lambda_j} = F_{ij} \implies \lambda_i \overline{\lambda_j} = 1 \overset{|\lambda_j|=1}{\implies} \lambda_i = \lambda_j$. □

Corollary 2.3.2.
The sub-σ-algebra $\mathcal{I}(T \times T)$ of $\mathcal{X} \otimes \mathcal{X}$, is a subset of $\mathcal{K} \otimes \mathcal{K}$, where $\mathcal{K} = \pi^{-1}(\mathcal{Z})$. In particular, $\mathcal{K}$ is the smallest factor of X with this property.

Corollary 2.3.3.
If $F \in L^2(\mu \otimes \mu)$, is a $\mathcal{I}(T \times T)$-measurable function, then $F = \sum\limits_{i=1}^{n} \lambda_i \overline{f_i} \otimes g_i$, where f_i, g_i are eigenfunctions of X with eigenvalues $\lambda_{f_i} = \lambda_{g_i} = \lambda_i$, for every $j \in \{1, 2, \ldots, n\}$.

Definition 2.3.4.
We consider the measures m_s, $s \in Z$, on $Z \times Z$ defined in the first Section of this Chapter. For $s \in Z$ we define μ_s the conditional product square of μ with itself over m_s. That means $\mu_s = \mu \otimes_{m_s} \mu$, where for every f_1, $f_2 \in L^\infty(\mu)$ we have,

$$\begin{aligned}\int_{X\times X} f_1(x_1) f_2(x_2) \, \mathrm{d}\mu_s(x_1, x_2) &= \int_{Z\times Z} \mathbb{E}(f_1 \mid Z)(z_1) \mathbb{E}(f_2 \mid Z)(z_2) \, \mathrm{d}m_s \\ &\underset{(\theta_s)_* m_Z = m_s}{=} \int_Z \mathbb{E}(f_1 \mid Z)(z) \mathbb{E}(f_2 \mid Z)(z+s) \, \mathrm{d}m_Z(z)\end{aligned}$$

Theorem 2.3.5.
Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system, $(Z, \mathcal{Z}, m_Z, R)$ be its Kronecker factor and $\pi : X \to Z$ the associated factor map. Then

(i) *Let* $P : X \times X \to Z$, *defined by* $P(x_1, x_2) = \pi(x_1) - \pi(x_2)$. *Then*

$$\begin{aligned}\mathcal{I}(T \times T) &= P^{-1}(\mathcal{Z}) \\ &= \{A \in \mathcal{X} \otimes \mathcal{X} : \ A = \{(x_1, x_2) \in X \times X : \ P(x_1, x_2) \in B\}, \textit{for some } B \in \mathcal{Z}\} \\ &= \{A \in \mathcal{X} \otimes \mathcal{X} : \ A = \{(x_1, x_2) \in X \times X : \ \pi(x_2) - \pi(x_1) \in B\}, \textit{for some } B \in \mathcal{Z}\}\end{aligned}$$

(ii) *For* m_Z*-almost every* $s \in Z$ *the system* $(X \times X, \mu_s, T \times T)$ *is ergodic and the map* $s \mapsto \mu_s$ *is an ergodic decomposition of* $\mu \otimes \mu$.

Proof. Let $\alpha \in Z$ be the element that defining $R : Z \to Z$, $R(z) = z + \alpha$.

(i) With simple computations, clearly $P^{-1}(\mathcal{Z}) \subseteq \mathcal{I}(T \times T)$.
For the converse inclusion, we consider the multiplicative group $\widehat{Z}$. By Corollary 2.2.4 every eigenfunction of X is of the form $c \cdot (\gamma \circ \pi)$, where $c \in \mathbb{T}$ and $\gamma \in \widehat{Z}$. By Corollary 2.3.3 the σ-algebra $\mathcal{I}(T \times T)$ is spanned by functions of the form $\overline{\gamma \circ \pi} \otimes \gamma \circ \pi = \overline{\gamma} \circ \pi \otimes \gamma \circ \pi$. Observe that for every $(x_1, x_2) \in X \times X$

$$\begin{aligned}(\overline{\gamma} \circ \pi \otimes \gamma \circ \pi)(x_1, x_2) = \overline{\gamma}(\pi(x_1)) \cdot \gamma(\pi(x_2)) &\underset{\gamma \in \widehat{Z}}{=} \gamma(-\pi(x_1)) \cdot \gamma(\pi(x_2)) \\ &\underset{\substack{\gamma \in \widehat{Z}, \\ Z \text{ abelian}}}{=} \gamma(\pi(x_2) - \pi(x_1)) = \gamma(P(x_1, x_2)) = \gamma \circ P(x_1, x_2)\end{aligned}$$

In particular $\overline{\gamma} \circ \pi \otimes \gamma \circ \pi$ is $P^{-1}(\mathcal{Z})$-measurable. Hence $\mathcal{I}(T \times T) \subseteq P^{-1}(\mathcal{Z})$.

(ii) It suffices to prove that for every f_1, $f_2 \in L^\infty(\mu)$,

$$\mathbb{E}\big(f_1 \otimes f_2 \mid \mathcal{I}(T \times T)\big)(x_1, x_2) \ = \ \int_{X \times X} f_1 \otimes f_2 \, \mathrm{d}\mu_{P(x_1, x_2)} \tag{2.1}$$

By Corollary 2.3.3 $\mathcal{I}(T \times T) \subseteq \mathcal{K} \otimes \mathcal{K}$, where $\mathcal{K} = \pi^{-1}(\mathcal{Z})$. Then the left side of the above equation equals to

$$\begin{aligned}&\mathbb{E}_{\mu \otimes \mu}\big(\mathbb{E}(f_1 \mid \mathcal{K}) \otimes \mathbb{E}(f_2 \mid \mathcal{K}) \ \big| \ \mathcal{I}(T \times T)\big) \\ &\qquad \overset{(*)}{=} \ \mathbb{E}_{m_Z \otimes m_Z}\big(\mathbb{E}(f_1 \mid Z) \otimes \mathbb{E}(f_2 \mid Z) \ \big| \ \mathcal{I}(R \times R)\big) \ \circ (\pi \times \pi)\end{aligned}$$

By Proposition 2.1.2,

$$\begin{aligned}
&\mathbb{E}_{m_Z\otimes m_Z}\big(\mathbb{E}(f_1 \mid Z)\otimes\mathbb{E}(f_2 \mid Z) \bigm| \mathcal{I}(R\times R)\big) \circ (\pi\times\pi)(x_1,x_2)\\
&\qquad = \mathbb{E}_{m_Z\otimes m_Z}\big(\mathbb{E}(f_1 \mid Z)\otimes\mathbb{E}(f_2 \mid Z) \bigm| \mathcal{I}(R\times R)\big)(\pi(x_1),\pi(x_2))\\
&\qquad = \int_{Z\times Z} \mathbb{E}(f_1 \mid Z)(z_1)\mathbb{E}(f_2 \mid Z)(z_2)\,\mathrm{d}m_{\pi(x_2)-\pi(x_1)}(z_1,z_2)\\
&\qquad = \int_{Z\times Z} \mathbb{E}(f_1 \mid Z)\otimes\mathbb{E}(f_2 \mid Z)\,\mathrm{d}m_{P(x_1,x_2)}\\
&\qquad \underset{\substack{def.\\ of\mu_s}}{=} \int_{X\times X} f_1(x_1)f_2(x_2)\,\mathrm{d}\mu_{P(x_1,x_2)}(x_1,x_2)
\end{aligned}$$

$\square$

Chapter 3

Construction of the measures

$(X, \mathcal{X}, \mu, T)$ is an ergodic invertible measure preserving system, $(X, \mathcal{X}, \mu,)$ Lebesgue space.

3.1 The measure $\mu^{[1]}$

Definition 3.1.1.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. We define $\mathcal{I}^{[0]}$ to be the sub-σ-algebra of $\mathcal{X}$ that consists of the $T^{[0]} = T$-invariant subsets of X

Definition 3.1.2.
Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. We define the system ($X^{[1]}$, $\mathcal{X}^{[1]}$, $\mu^{[1]}$, $T^{[1]}$) to be the relatively independent self-joining of the system $(X, \mathcal{X}, \mu, T)$ over the factor $\mathcal{I}^{[0]}$

Remark 3.1.3.
$\mu^{[1]}$ = $\mu \otimes \mu$
Indeed, as the system $(X, \mathcal{X}, \mu, T)$ is ergodic, thus $\mathcal{I}^{[0]}$ is trivial. By definition of the relatively independent joining we have the equality

Proposition 3.1.4.
We define two sub σ-algebras of $\mathcal{X}^2$,
$\mathcal{A}_{\boldsymbol{o},1} = \{A \subseteq X^2 : \ A = B \times X (mod \mu^{[1]})$ *for some* $B \subseteq X\}$, *and*
$\mathcal{J}^{[1]} = \{A \subseteq X^2 : \ (Id \times T)^{-1}(A) = A \ (mod \mu^{1]})\}$.
Then

$$\mathcal{A}_{\boldsymbol{o},1} = \mathcal{J}^{[1]}$$

Proof. If $A \in \mathcal{A}_{\mathbf{0},1}$ then clearly A is also an element of $\mathcal{J}^{[1]}$.
Conversely, if $A \in \mathcal{J}^{[1]}$ then $(Id \times T)(A) = A$, $(mod\mu^{[1]})$, thus

$$0 = (\mu \otimes \mu)\big((Id \times T)(A) \triangle A\big) = \int\limits_{(Id\times T)(A)\triangle A} \mathrm{d}\mu \otimes \mu = \int\limits_{X} \Bigg(\int\limits_{\big((Id\times T)(A)\triangle A\big)_x} \mathrm{d}\mu(y) \Bigg) \,\mathrm{d}\mu(x)$$

$$= \int\limits_{X} \Bigg(\int\limits_{(Id\times T)(A)_x \triangle A_x} \mathrm{d}\mu(y) \Bigg) \,\mathrm{d}\mu(x)$$

Now we have that $(Id \times T)(A)_x = T(A_x)$

$$y \in (Id{\times}T)(A)_x \Leftrightarrow (x, y) \in (Id{\times}T)(A) \Leftrightarrow (x, T^{-1}(y)) \in A \Leftrightarrow T^{-1}(y) \in A_x \Leftrightarrow y \in T(A_x)$$

Thus,

$$0 = \int\limits_{X} \Bigg(\int\limits_{(Id\times T)(A)_x \triangle A_x} \mathrm{d}\mu(y) \Bigg) \,\mathrm{d}\mu(x) = \int\limits_{X} \Bigg(\int\limits_{T(A_x)\triangle A_x} \mathrm{d}\mu(y) \Bigg) \,\mathrm{d}\mu(x)$$

It follows that, for μ-almost every $x \in X$, $\mu(T(A_x) \triangle A_x) = 0$. Because (X, μ, T) is ergodic, we have that μ-almost every $x \in X$, $\mu(A_x) = 0$ *or* 1, in other words $A = B \times X$ for some $B \subseteq X$ or $A = \varnothing$ up to to a set of $\mu \otimes \mu$-measure zero. Hence $A \in \mathcal{J}^{[1]}$. □

3.2 The measure $\mu^{[k]}$

We define by induction the $T^{[k]}$-invariant measure, $\mu^{[k]}$, on $X^{[k]}$ $\forall\, k \in \mathbb{N}$.

Definition 3.2.1.
$\mathcal{I}^{[k]}$ is the sub-σ-algebra of $\mathcal{X}^{[k]}$ that consists of all $T^{[k]}$-invariant sets

Definition 3.2.2.
We define the system ($X^{[k+1]}$, $\mathcal{X}^{[k+1]}$, $\mu^{[k+1]}$, $T^{[k+1]}$) to be the relatively independent self-joining of the system ($X^{[k]}$, $\mathcal{X}^{[k]}$, $\mu^{[k]}$, $T^{[k]}$) over the factor $\mathcal{I}^{[k]}$
This means that if f_ϵ, $\epsilon \in V_{k+1}$ are $2^{[k]}$ bounded functions on X then,

$$\int_{X^{[k+1]}} \bigotimes_{\epsilon\in V_{k+1}} f_\epsilon \;\mathrm{d}\mu^{[k+1]} = \int_{X^{[k]}} \mathbb{E}(\bigotimes_{\eta\in V_k} f_{0\eta} \mid \mathcal{I}^{[k]}) \cdot \mathbb{E}(\bigotimes_{\eta\in V_k} f_{1\eta} \mid \mathcal{I}^{[k]}) \;\mathrm{d}\mu^{[k]}$$

Lemma 3.2.3.
Let $k, \ell \geqslant 0$ be integers and $\mu^{[k]} = \int_{\Omega_k} \mu_\omega^{[k]} \, \mathrm{d}P_k(\omega)$ be an ergodic decomposition of the measure $\mu^{[k]}$. Let $(\mu_\omega^{[k]})^{[\ell]}$ be the measure on $(X^{[k]})^{[\ell]} = X[k+\ell]$ built from the ergodic system $(X^{[k]}, \mu_\omega^{[k]}, T^{[k]})$ in the same way that $\mu^{[k]}$ was built from (X, μ, T). Then

$$\mu^{[k+\ell]} = \int_{\Omega_k} (\mu_\omega^{[k]})^{[\ell]} \, \mathrm{d}P_k(\omega)$$

Proof. For $\ell = 1$, the equation holds by the definition of $\mu^{[k+1]}$ as the relatively independent selfjoinig of $\mu^{[k]}$ over $\mathcal{I}^{[k]}$

We will show by induction tha it holds for every $\ell \geqslant 1$.

Assume that it holds for some $\ell \geqslant 1$. Let $\mathcal{J}_\omega$ denote the invariant σ-algebra of the system $\big((X^{[k]})^{[\ell]}, (\mu_\omega^{[k]})^{[\ell]}, (T^{[k]})^{[\ell]}\big) = (X^{[k+\ell]}, \mu_\omega^{[k+\ell]}, T^{[k+\ell]})$

Let f, g be two bounded functions on $X^{[k+\ell]}$.By applying the Pointwise Ergodic theorem for the two sytems $(X^{[k+\ell]}, \mu^{[k+\ell]}, T^{[k+\ell]})$, $(X^{[k+\ell]}, \mu_\omega^{[k+\ell]}, T^{[k+\ell]})$,

(1) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} f \circ (T^{[k+\ell]})^n(x) \; = \; \mathbb{E}\big(\, f \mid \mathcal{I}^{[k+\ell]} \, \big)(x),$ for $\mu^{[k+\ell]}$-almost every $x \in X^{[k+\ell]}$

and

(2) $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} f \circ (T^{[k+\ell]})^n(x) \; = \; \mathbb{E}\big(\, f \mid \mathcal{J}_\omega \, \big)(x),$ for $(\mu_\omega^{[k]})^{[\ell]}$-almost every $x \in X^{[k+\ell]}$

We define $N = \{x \in X^{[k+\ell]}:$ (1) does not hold $\}$. Then $\mu^{[k+\ell]}(N) = 0$ thus, by induction, $\int_{\Omega_k} (\mu_\omega^{[k]})^{[\ell]}(N) \, \mathrm{d}P_k(\omega) = 0$. That is $(\mu_\omega^{[k]})^{[\ell]}(N) = 0,$ for P_k-almost every $\omega \in \Omega_k$

In particular $\exists \; \Omega' \subset \Omega_k$, such that $P_k(\Omega') = 1$, and $\forall \; \omega \in \Omega'$, equation (1) holds for $(\mu_\omega^{[k]})^{[\ell]}$-almost everywhere.

In sum, if $\omega \in \Omega'$ then

$\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} f \circ (T^{[k+\ell]})^n(x) \; = \; \mathbb{E}\big(\, f \mid \mathcal{I}^{[k+\ell]} \, \big)(x),$ for $(\mu_\omega^{[k]})^{[\ell]}$-almost every $x \in X^{[k+\ell]}$

and

$\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=0}^{N-1} f \circ (T^{[k+\ell]})^n(x) \; = \; \mathbb{E}\big(\, f \mid \mathcal{J}_\omega \, \big)(x),$ for $(\mu_\omega^{[k]})^{[\ell]}$-almost every $x \in X^{[k+\ell]}$

Equivelantly, for P_k-almost every $\omega \in \Omega$, $\mathbb{E}\big(\, f \mid \mathcal{I}^{[k+\ell]}\big) = \mathbb{E}\big(\, f \mid \mathcal{J}_\omega\big)$, $(\mu_\omega^{[k]})^{[\ell]}$-almost every $x \in X^{[k+\ell]}$

The same holds for g. Hence,

$$
\begin{aligned}
\int\limits_{X^{[k+\ell+1]}} f \otimes g \,\mathrm{d}\mu^{[k+\ell+1]} &= \int\limits_{X^{[k+\ell]}} \mathbb{E}(\ f \mid \mathcal{I}^{[k+\ell]}\ ) \cdot \mathbb{E}(\ g \mid \mathcal{I}^{[k+\ell]}\ ) \,\mathrm{d}\mu^{[k+\ell]}(x) \\
&= \int\limits_{\Omega_k} \Big(\int\limits_{X^{[k+\ell]}} \mathbb{E}(\ f \mid \mathcal{I}^{[k+\ell]}\ ) \cdot \mathbb{E}(\ g \mid \mathcal{I}^{[k+\ell]}\ ) \,\mathrm{d}(\mu_\omega^{[k]})^{[\ell]}(x) \Big) \,\mathrm{d}P_k(\omega) \\
&= \int\limits_{\Omega_k} \Big(\int\limits_{X^{[k+\ell]}} \mathbb{E}(\ f \mid \mathcal{J}_\omega\ ) \cdot \mathbb{E}(\ g \mid \mathcal{J}_\omega\ ) \,\mathrm{d}(\mu_\omega^{[k]})^{[\ell]}(x) \Big) \,\mathrm{d}P_k(\omega) \\
&= \int\limits_{\Omega_k} \Big(\int\limits_{X^{[k+\ell]}} f \cdot g \,\mathrm{d}(\mu_\omega^{[k]})^{[\ell+1]}(x) \Big) \,\mathrm{d}P_k(\omega)
\end{aligned}
$$

That is

$$
\mu^{[k+\ell+1]} = \int\limits_{\Omega_k} (\mu_\omega^{[k]})^{[\ell+1]} \mathrm{d}P_k(\omega)
$$

$\square$

Lemma 3.2.4.
Let $p : (X, \mu, T) \to (Y, \nu, S)$ be a factor map and let $k \geq 0$ be an integer. Then the map $p^{[k]} : (X^{[k]}, \mu^{[k]}, T^{[k]}) \to (Y^{[k]}, \nu^{[k]}, S^{[k]})$ is a factor map.

Proof. Clearly $p^{[k]} \circ T^{[k]} = S^{[k]} \circ p^{[k]}$. We are left with showing that the image of $\mu^{[k]}$ under $p^{[k]}$ is $\nu^{[k]}$. For $k = 0$ is obvious and we assume that it holds for some $k \geq 0$. Let f_ϵ, $\epsilon \in V_k$ be sounded functions on Y. Since $p^{[k]}$ is a factor map, we have that

$$
\mathbb{E}\big((\bigotimes_{\epsilon \in V_k} f_\epsilon) \circ p^{[k]} \mid \mathcal{I}^{[k]}(X) \big) = \mathbb{E}\big(\bigotimes_{\epsilon \in V_k} f_\epsilon \mid \mathcal{I}^{[k]}(Y) \big) \circ p^{[k]}
$$

By the definitions of the measures $\mu^{[k+1]}$, $\nu^{[k+1]}$ the statement follows for $k + 1$. $\square$

3.3 The measure $\mu^{[2]}$

Already stated above, $\mu^{[2]}$ is the ralatively independent self-joining of $\mu^{[1]} = \mu \otimes \mu$ over $\mathcal{I}^{[1]}$. In particular,

$$
\int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \,\mathrm{d}\mu^{[2]} = \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{I}^{[1]}) \,\mathrm{d}(\mu \otimes \mu)
$$

Proposition 3.3.1.
The measure $\mu^{[2]}$ is $T^{[2]}$ invariant.

Proof. Let $f_{00}, f_{01}, f_{10}, f_{11} \in L^\infty(\mu)$ then,

$$
\begin{aligned}
&\int_{X^4} f_{00} \circ T \otimes f_{01} \circ T \otimes f_{10} \circ T \otimes f_{11} \circ T \, \mathrm{d}\mu^{[2]} \\
&= \int_{X^2} \mathbb{E}(f_{00} \circ T \otimes f_{01} \circ T \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \circ T \otimes f_{11} \circ T \mid \mathcal{I}^{[1]}) \, \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^2} \mathbb{E}((f_{00} \otimes f_{01}) \circ (T \times T) \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}((f_{10} \otimes f_{11}) \circ (T \times T) \mid \mathcal{I}^{[1]}) \, \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{I}^{[1]}) \ \circ \ T \times T \, \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{I}^{[1]}) \, \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \, \mathrm{d}\mu^{[2]}
\end{aligned}
$$

The third equality holds by the definition of $\mathcal{I}^{[1]}$. □

Proposition 3.3.2.
The measure $\mu^{[2]}$ is relatively independent with respect to $\mathcal{K}^{[2]}$
(and the Kronecker factor is minimal with this property)
This means that

$$\int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \, \mathrm{d}\mu^{[2]} = \int_{X^4} \mathbb{E}(f_{00} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \otimes \mathbb{E}(f_{10} \mid \mathcal{K}) \otimes \mathbb{E}(f_{11} \mid \mathcal{K}) \, \mathrm{d}\mu^{[2]}$$

Proof. Let f,g be bounded functions on X and $\Gamma \in \mathcal{K} \otimes \mathcal{K}$
For every $x, y \in X$ we denote $\Gamma_x, \Gamma_y X$ the sets $\{y \in X | \ (x,y) \in \Gamma\}, \ \{x \in X | \ (x,y) \in \Gamma\}$,

respectively. Then for μ-almost every x, $\Gamma_x \in \mathcal{K}$ and for μ-almost every y, $\Gamma_y \in \mathcal{K}$

$$\begin{aligned}
\int\limits_{\Gamma} \mathbb{E}(f \mid \mathcal{K}) \otimes \mathbb{E}(g \mid \mathcal{K})\ (x,y)\ \mathrm{d}(\mu \otimes \mu)(x,y) &= \int\limits_{X} \mathbb{E}(f \mid \mathcal{K})(x) \Big(\int\limits_{\Gamma_x} \mathbb{E}(g \mid \mathcal{K})(y)\ \mathrm{d}\mu(y)\ \Big)\ \mathrm{d}\mu(x) \\
&= \int_X \mathbb{E}(f \mid \mathcal{K})(x) \Big(\int\limits_{\Gamma_x} g(y)\ \mathrm{d}\mu(y)\ \Big)\ \mathrm{d}\mu(x) \\
&= \int\limits_{\Gamma} \mathbb{E}(f \mid \mathcal{K}) \otimes g\ (x,y)\ \mathrm{d}(\mu \otimes \mu)(x,y) \\
&= \cdots \\
&= \int\limits_{\Gamma} f \otimes g\ (x,y)\ \mathrm{d}(\mu \otimes \mu)(x,y) \\
&= \int\limits_{\Gamma} \mathbb{E}(f \otimes g \mid \mathcal{K} \otimes \mathcal{K})(x,y)\ \mathrm{d}(\mu \otimes \mu)(x,y)
\end{aligned}$$

In particular, $\mathbb{E}(f \mid \mathcal{K}) \otimes \mathbb{E}(g \mid \mathcal{K}) = \mathbb{E}(f \otimes g \mid \mathcal{K} \otimes \mathcal{K})$ $(\mu \otimes \mu)$-a.e.

Now let $f_{00},\ f_{01},\ f_{10},\ f_{11}$ be four bounded functions on X.

$$\begin{aligned}
&\int_{X^4} \mathbb{E}(f_{00} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \otimes \mathbb{E}(f_{10} \mid \mathcal{K}) \otimes \mathbb{E}(f_{11} \mid \mathcal{K})\ \mathrm{d}\mu^{[2]} \\
&= \int_{X^2} \mathbb{E}\big(\mathbb{E}(f_{00} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \mid \mathcal{I}^{[1]}\big) \cdot \mathbb{E}\big(\mathbb{E}(f_{10} \mid \mathcal{K}) \otimes \mathbb{E}(f_{11} \mid \mathcal{K}) \mid \mathcal{I}^{[1]}\big)\ \mathrm{d}\mu^{[1]} \\
&= \int\limits_{X^2} \mathbb{E}\big(\mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{K} \otimes \mathcal{K}) \mid \mathcal{I}^{[1]}\big) \cdot \mathbb{E}\big(\mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{K} \otimes \mathcal{K}) \mid \mathcal{I}^{[1]}\big)\ \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{I}^{[1]})\ \mathrm{d}(\mu \otimes \mu) \\
&= \int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}\ \mathrm{d}\mu^{[2]}
\end{aligned}$$

The third equality holds by the previous proposition. □

Remark 3.3.3.

Let Z, as mentioned above, denote the Kronecker factor of X. Define $Z_4 = \{(z, sz, tz, tsz) : z, s, t \in K\}$ Then Z_4 is a closed subgroup of Z^4.

Proof. Let $(\mathbf{z}_n)_{n \in \mathbb{N}}$ a sequence in Z_4 and $\mathbf{z} \in Z^4$ such that $\mathbf{z}_n \to \mathbf{z}$. Each $\mathbf{z}_n$ is of the form $(z_n, s_n z_n, t_n z_n, t_n s_n z_n)$ and $\mathbf{z} = (z_{00}, z_{01}, z_{10}, z_{11})$.

Hence $z_n \to z_{00}$ and $s_n z_n \to z_{01}$ and $t_n z_n \to z_{10}$ and $s_n t_n z_n \to z_{11}$
$\Rightarrow$ $s_n = (s_n z_n)(z_n^{-1}) \to z_{01} z_{00}^{-1}$ and $t_n = (t_n z_n)(z_n^{-1}) \to z_{10} z_{00}^{-1}$
By setting $z = z_{00}$, $s = z_{01} z_{00}^{-1}$ and $t = z_{10} z_{00}^{-1}$, follows $\mathbf{z} = (z, sz, tz, tsz)$. Therefore Z_4 is closed in Z^4 □

Proposition 3.3.4.
Let m_4 be the Haar measure of Z_4, $\pi_4 = \pi \times \pi \times \pi \times \pi : X^4 \to Z^4$ and g_ϵ, $\epsilon \in V_2$, be four bounded functions on Z. Then,

$$\int_{X^{[2]}} \bigotimes_{\epsilon \in V_2} g_\epsilon \circ \pi_4 \, \mathrm{d}\mu^{[2]} = \int_{Z_4} \bigotimes_{\epsilon \in V_2} g_\epsilon \, \mathrm{d}m_4$$

Proof. Let $\phi : Z^3 \to Z_4$ given by $\phi(z, s, t) = (z, z + s, z + t, z + s + t)$. Then ϕ is a bijection, continuous and open function. In particular, ϕ is homeomorphism.
Let $m^3 = m \otimes m \otimes m$, then $\phi_* m^3$ is a Borel probability measure on Z_4 that is invariant under any rotation, thus $\phi_* m^3 = m_4$. Now let $g_{00}, g_{01}, g_{10}, g_{11}$ be four bounded functions on Z, then

$$\begin{aligned} \int_{Z^4} \bigotimes_{\epsilon \in V_2} g_\epsilon \, \mathrm{d}m_4 &= \int_{Z^3} \bigotimes_{\epsilon \in V_2} g_\epsilon \circ \phi \, \mathrm{d}m^3 \\ &= \int_{Z^3} g_{00}(z) \cdot g_{01}(z + s) \cdot g_{10}(z + t) \cdot g_{11}(z + s + t) \, \mathrm{d}m(z)\mathrm{d}m(s)\mathrm{d}m(t) \end{aligned}$$

With the additional assumption that g_ϵ, $\epsilon \in V_2$ are characters of Z, $g_\epsilon \circ \pi$, $\epsilon \in V_2$ are eigenfunctions of X, thus $\mathcal{K}$-measurable
Now, remember that $s \mapsto \mu_s$ is an ergodic decomposition of $\mu \otimes \mu$, where

$$\begin{aligned} \int_{X \times X} f_1(x_1) \cdot f_2(x_2) \, \mathrm{d}\mu_s(x_1, x_2) &= \int_{Z \times Z} \mathbb{E}(f_1 \mid Z)(z_1) \cdot \mathbb{E}(f_2 \mid Z)(z_2) \, \mathrm{d}m_s(z_1, z_2) \\ &= \int_Z \mathbb{E}(f_1 \mid Z)(z) \cdot \mathbb{E}(f_2 \mid Z)(z + s) \, \mathrm{d}m(z) \end{aligned}$$

Then, by Lemma 3.2.3 ,

$$\mu^{[2]} = \int_Z \mu_s \otimes \mu_s \, \mathrm{d}m(s)$$

Thus,

$$\begin{aligned}
&\int_{X^4} \bigotimes_{\epsilon\in V_2} g_\epsilon \circ \pi_4 \,\mathrm{d}\mu^{[2]} \\
&= \int_Z \Big(\int_Z \mathbb{E}(g_{00} \mid Z)(z) \cdot \mathbb{E}(g_{01} \mid Z)(z+s)\,\mathrm{d}m(z)\Big)\Big(\int_Z \mathbb{E}(g_{10} \mid Z)(z') \cdot \mathbb{E}(g_{11} \mid Z)(z'+s)\,\mathrm{d}m(z')\Big)\,\mathrm{d}m(s) \\
&= \int_Z \Big(\int_Z g_{00}(z) \cdot g_{01}(z+s)\,\mathrm{d}m(z)\Big)\Big(\int_Z g_{10}(z') \cdot g_{11}(z'+s)\,\mathrm{d}m(z')\Big)\,\mathrm{d}m(s) \\
&= \int_{Z^3} g_{00}(z) \cdot g_{01}(z+s) \cdot g_{10}(z') \cdot g_{11}(z'+s)\,\mathrm{d}m(z)\mathrm{d}m(z')\mathrm{d}m(s) \\
&= \int_{Z^3} g_{00}(z) \cdot g_{01}(z+s) \cdot g_{10}(z+t) \cdot g_{11}(z+t+s)\,\mathrm{d}m(z)\mathrm{d}m(t)\mathrm{d}m(s) \\
&= \int_{Z^3} \bigotimes_{\epsilon\in V_2} g_\epsilon \circ \phi \,\mathrm{d}m^3 \;=\; \int_{Z^4} \bigotimes_{\epsilon\in V_2} g_\epsilon \,\mathrm{d}m_4
\end{aligned}$$

The second equality holds beacause $g_\epsilon \circ \pi$ is $\mathcal{K}$-measurable, thus $\mathbb{E}(g_\epsilon \circ \pi \mid \mathcal{K}) = g_\epsilon \circ \pi$. Furthermore $\mathbb{E}(g_\epsilon \circ \pi \mid \mathcal{K}) = \mathbb{E}(g_\epsilon \circ \pi \mid Z) \circ \pi$. Hence $g_\epsilon \circ \pi = \mathbb{E}(g_\epsilon \circ \pi \mid Z) \circ \pi$, μ-a.e. $\Rightarrow g_\epsilon = \mathbb{E}(g_\epsilon \circ \pi \mid Z)$, $\pi_* \mu = m$-a.e.
The general case follows from density. □

From the two propositions above we deduce that:

Corollary 3.3.5.
The measure m_4 is the image of $\mu^{[2]}$ under the $\pi_4 = \pi\times\pi\times\pi\times\pi :\ X^4 \to Z^4$. Furthermore, if f_ϵ, $\epsilon \in V_2$ are four bounded functions on X, then

$$\int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \,\mathrm{d}\mu_4 = \int_{Z^4} \mathbb{E}(f_{00} \mid Z) \otimes \mathbb{E}(f_{01} \mid Z) \otimes \mathbb{E}(f_{10} \mid Z) \otimes \mathbb{E}(f_{11} \mid Z)\,\mathrm{d}m_4$$

Proposition 3.3.6.
By identifying Z^4 with $Z\times Z^3$, we have that the projection of m_4 on Z^3 is $m^3 = m\otimes m\otimes m$.

Proof. Let $\psi :\ Z^4 \to Z^3$ given by $\psi(z_{00}, z_{01}, z_{10}, z_{11}) = (z_{01}, z_{10}, z_{11})$ then $\psi_* m_4$ is a Borel probability measure on Z^3 and if f a bounded function on Z^3 then,

$$\begin{aligned}\int_{Z^3} f((z_{01}, z_{10}, z_{11}) &+ (z'_{01}, z'_{10}, z'_{11}))\, \mathrm{d}\psi_* m_4 \\ &= \int_{Z^4} f \circ \psi\, ((z_{00}, z_{01}, z_{10}, z_{11}) + (z'_{00}, z'_{01}, z'_{10}, z'_{11}))\, \mathrm{d}m_4 \\ &= \int_{Z^4} f \circ \psi\, ((z_{00}, z_{01}, z_{10}, z_{11}))\, \mathrm{d}m_4 \\ &= \int_{Z^3} f(z_{01}, z_{10}, z_{11})\, \mathrm{d}\psi_* m_4\end{aligned}$$

Thus $\psi_* m_4$ is the Haar measure on Z^3 In particular, $\psi_* m_4 = m \otimes m \otimes m$ □

Since μ_4 is relatively independent with respect $\mathcal{K}^4$ we have the following,

Corollary 3.3.7.
By identifying X^4 with $X \times X^3$, we have that the projection of $\mu^{[2]}$ on X^3 is $\mu^3 = \mu \otimes \mu \otimes \mu$.

Proof. Let $\Psi : X^4 \to X^3$ given by $\Psi(x_{00}, x_{01}, x_{10}, x_{11}) = (x_{01}, x_{10}, x_{11})$, $f\epsilon \in L^\infty(\mu)$, $\epsilon \in V_2^*$, and $f_{00} = \mathbb{1}_X$, then $f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} = (f_{01} \otimes f_{10} \otimes f_{11}) \circ \Psi$, $\mu^{[2]}$-a.e. Now we have that,

$$\begin{aligned}\int_{X^3} \bigotimes_{\epsilon \in V_2^*} f_\epsilon \, \mathrm{d}\Psi_* \mu^{[2]} &= \int_{X^4} \bigotimes_{\epsilon \in V_2^*} f_\epsilon \circ \Psi \, \mathrm{d}\mu^{[2]} = \int_{X^4} \bigotimes_{\epsilon \in V_2} f_\epsilon \, \mathrm{d}\mu^{[2]} \\ &= \int_{Z^4} \bigotimes_{\epsilon \in V_2} \mathbb{E}(f_\epsilon \mid Z)\, \mathrm{d}m_4 = \int_{Z^4} \bigotimes_{\epsilon \in V_2^*} \mathbb{E}(f_\epsilon \mid Z) \circ \psi \, \mathrm{d}m_4 \\ &= \int_{Z^3} \bigotimes_{\epsilon \in V_2^*} \mathbb{E}(f_\epsilon \mid Z)\, \mathrm{d}\psi_* m_4 = \int_{Z^3} \bigotimes_{\epsilon \in V_2^*} \mathbb{E}(f_\epsilon \mid Z)\, \mathrm{d}m^3 \\ &= \prod_{\epsilon \in V_2^*} \int_Z \mathbb{E}(f_\epsilon \mid K)\, \mathrm{d}m = \prod_{\epsilon \in V_2^*} \int_X \mathbb{E}(f_\epsilon \mid \mathcal{K})\, \mathrm{d}\mu = \prod_{\epsilon \in V_2^*} \int_X f_\epsilon \, \mathrm{d}\mu = \int_{X^3} \bigotimes_{\epsilon \in V_2^*} f_\epsilon \, \mathrm{d}\mu^3\end{aligned}$$

The fourth equality holds for the same reason the second equality holds and because $\mathbb{E}(\mathbb{1}_X \mid Z) = \mathbb{1}_Z$, and the sixth equality holds from Proposition 3.3.6 □

Note.
Let $\varphi : K^3 \to K$ be defined by $\varphi(z_{01}, z_{10}, z_{11}) = z_{01} z_{10} z_{11}^{-1}$. Then for each $(z_{00} z_{01}, z_{10}, z_{11}) \in K_4$, $z_{00} = (z_{01}, z_{10}, z_{11})$ and this relation holds for m_4-almost every $(z_{00}, z_{01}, z_{10}, z_{11}) \in$

K^4. Since m_4 is the projection of $\mu^{[2]}$ on K^4 we have that for any $\mathcal{K}$ function f on X there exists a bounded function F on X^3, measurable with respect to $\mathcal{K}^3 = \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}$ such that:

$$f(x_{00}) = F(x_{01}, x_{10}, x_{11}), \tag{3.1}$$

for $\mu^{[2]}$-almost every $\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^4$.
Indeed, if f is $\mathcal{K}$-measurable then $\exists\ h \in L^\infty(m)$ such that $h \circ \pi = f$. Furthermore $h(z) = h(z_{00}) = h \circ \varphi(z_{01}, z_{10}, z_{11})$, m_4-a.e. . Setting $H :\ Z^3 \to \mathbb{R}$, with $H = h \circ \varphi$, we have that $H(z_{01}, z_{10}, z_{11}) = h(z_{00})$, for m_4-almost every $(z_{00}, z_{01}, z_{10}, z_{11}) \in Z^4$ Then the requested function is defined by $F = H \circ \pi_3$, because,

$$\int_{X^4} |F(x_{01}, x_{10}, x_{11}) - f(x_{00})|\ \mathrm{d}\mu^{[2]} = \int_{Z^4} |H(z_{01}, z_{10}, z_{11}) - h(z_{00})|\ \mathrm{d}m_4 = 0$$

The equality holds by Corollary 3.3.5. Therefore $f(x_{00}) = F(x_{01}, x_{10}, x_{11})$, $\mu^{[2]}$-a.e.

On the other hand, if f a bounded function on X and F is a bounded function on X^3 satisfying equation (3.1) , then f is $\mathcal{K}$-measurable.
Indeed, we have

$$\begin{aligned} ||f||_{L^2(\mu)} &= \int_{X^4} f(x_{00}) \cdot F(x_{01}, x_{10}, x_{11})\ \mathrm{d}\mu^{[2]}(\mathbf{x}) \\ &= \int_{X^4} \mathbb{E}(f) \mid \mathcal{K})(x_{00}) \cdot F(x_{01}, x_{10}, x_{11})\ \mathrm{d}\mu^{[2]}(\mathbf{x}) \quad = \int_X \mathbb{E}(f \mid \mathcal{K})(x) \cdot f(x)\ \mathrm{d}\mu(x) \end{aligned} \tag{3.2}$$

For the second equality:
We have that $F(x_{01}, x_{10}, x_{11})$ can be approached in $L^1(\mu \otimes \mu \otimes \mu)$ by finite sums of the form $\sum\limits_i F_i(x_{00})G_i(x_{10})H_i(x_{11})$. This last argument can be easily shown for characteristic function and the fact than finite unions of rectangles of $\mathcal{K}$-measurable sets is an *algebra* generating $\mathcal{K}^3$. Now the finite sums of the form $f(x_{00}) \cdot \sum\limits_i F_i(x_{00})G_i(x_{10})H_i(x_{11})$ approach $f(x_{00}) \cdot F(x_{01}, x_{10}, x_{11})$ in $L^1(\mu^{[2]})$. Indeed if a $\sum\limits_i F_i(x_{01})G_i(x_{10})H_i(x_{11})$ is

$\varepsilon / \|F\|_{L^\infty(\mu^3)}$-near $F(x_{01}, x_{10}, x_{11})$ in $L^1(\mu^3)$ then,

$$\begin{aligned}
&\int_{X^4} |f(x_{00}) \cdot \sum_i F_i(x_{00}) G_i(x_{10}) H_i(x_{11}) - f(x_{00}) \cdot F(x_{01}, x_{10}, x_{11})| \, \mathrm{d}\mu^{[2]}(\mathbf{x}) \\
&= \int_{X^4} |f(x_{00})| \cdot |F(x_{01}, x_{10}, x_{11}) - \sum_i F_i(x_{01}) G_i(x_{10}) H_i(x_{11})| \, \mathrm{d}\mu^{[2]}(\mathbf{x}) \\
&= \int_{X^3} |F(x_{01}, x_{10}, x_{11})| \cdot |F(x_{01}, x_{10}, x_{11}) - \sum_i F_i(x_{01}) G_i(x_{10}) H_i(x_{11})| \, \mathrm{d}\mu^3(\tilde{\mathbf{x}}) \; < \; \varepsilon
\end{aligned}$$

where $\mathbf{x} = (x_{00}, \tilde{\mathbf{x}}) \in X \times X^3$. The last equality holds because f,F satisfy equation (3.1). Now,

$$\begin{aligned}
\int_{X^4} f \otimes \sum_i F_i G_i H_i \, \mathrm{d}\mu^{[2]} &= \sum_i \int_{X^4} f \otimes F_i \otimes G_i \otimes H_i \, \mathrm{d}\mu^{[2]} \\
&= \sum_i \int_{X^4} \mathbb{E}(f \mid \mathcal{K}) \otimes F_i \otimes G_i \otimes H_i \, \mathrm{d}\mu^{[2]} = \int_{X^4} \mathbb{E}(f \mid \mathcal{K}) \otimes \sum_i F_i G_i H_i \, \mathrm{d}\mu^{[2]}
\end{aligned}$$

and the last equality holds by Proposition 3.3.2.
But $\int\limits_{X^4} \mathbb{E}(f \mid \mathcal{K}) \otimes \sum\limits_i F_i G_i H_i \, \mathrm{d}\mu^{[2]}$, $\big($in the same manner as $\int\limits_{X^4} f \otimes \sum\limits_i F_i G_i H_i \, \mathrm{d}\mu^{[2]}$, approaches in $L^1(\mu^{[2]})$, $\int\limits_{X^4} f(x_{00}) F(x_{01}, x_{10}, x_{11}) \, \mathrm{d}\mu^{[2]} = ||f||_{L^2(\mu)}\big)$ approaches $\int\limits_{X^4} \mathbb{E}(f) \mid \mathcal{K})(x_{00}) \cdot F(x_{01}, x_{10}, x_{11}) \, \mathrm{d}\mu^{[2]}(\mathbf{x}) = \int\limits_X \mathbb{E}(f \mid \mathcal{K})(x) \cdot f(x) \, \mathrm{d}\mu(x)$. By uniqueness of $|| \cdot ||_{L^1(\mu^{[2]})}$-limit we have that the relation (3.2) is proven.
It is easy now to deduce that f is $\mathcal{K}$-measurable, since by relation (3.2):
$\langle\, f - \mathbb{E}(f \mid \mathcal{K}), f \rangle = 0 \;\Rightarrow\; f = \mathbb{E}(f \mid \mathcal{K})$, μ-a.e.

By summarizing the previous results we obtain the following.

Proposition 3.3.8.
Let A be a subset of X. Then

$$A \in \mathcal{K} \quad \iff \quad \exists\, B \subseteq X^3 \text{ such that } A \times X^3 = X \times B \; (mod \mu^{[2]})$$

Note.
The rotations $R_{4,1} = \mathbf{Id} \times R \times \mathbf{Id} \times R$ and $R_{4,2} = \mathbf{Id} \times \mathbf{Id} \times R \times R$ of Z^4 clearly leave Z_4 invariant, thus leave the measure m_4 invariant. Since $\mu^{[2]}$ is relatively independent over m_4, it follows that,

Proposition 3.3.9.
The measure $\mu^{[2]}$ is invariant under the transformations

$$T_{4,1} = \boldsymbol{Id} \times T \times \boldsymbol{Id} \times T \text{ and } T_{4,2} = \boldsymbol{Id} \times \boldsymbol{Id} \times T \times T$$

Proof. Let $f_{00}, f_{01}, f_{10}, f_{11} \in L^\infty(\mu)$

$$\begin{aligned}
\int_{X^4} \bigotimes_{\epsilon \in V_2} f_\epsilon \circ T_{4,1} \, \mathrm{d}\mu^{[2]} &= \int_{X^4} \mathrm{E}(f_{00} \mid \mathcal{K}) \otimes \mathrm{E}(f_{01} \circ T \mid \mathcal{K}) \otimes \mathrm{E}(f_{10} \mid \mathcal{K}) \otimes \mathrm{E}(f_{11} \circ T \mid \mathcal{K}) \, \mathrm{d}\mu^{[2]} \\
&= \int_{Z^4} \mathrm{E}(f_{00} \mid Z) \otimes \mathrm{E}(f_{01} \circ T \mid Z) \otimes \mathrm{E}(f_{10} \mid Z) \otimes \mathrm{E}(f_{11} \circ T \mid Z) \, \mathrm{d}m_4 \\
&= \int_{Z^4} \mathrm{E}(f_{00} \mid Z) \otimes \mathrm{E}(f_{01} \mid Z) \circ R \otimes \mathrm{E}(f_{10} \mid Z) \otimes \mathrm{E}(f_{11} \mid Z) \circ R \, \mathrm{d}m_4 \\
&= \int_{Z^4} \mathrm{E}(f_{00} \mid Z) \otimes \mathrm{E}(f_{01} \mid Z) \otimes \mathrm{E}(f_{10} \mid Z) \otimes \mathrm{E}(f_{11} \mid Z) \, \mathrm{d}m_4 \\
&= \int_{X^4} \bigotimes_{\epsilon \in V_2} f_\epsilon \, \mathrm{d}\mu^{[2]}
\end{aligned}$$

The first equality holds by Proposition 3.3.2, the second equality holds by Proposition 3.3.4 and the fourth holds by the Note above.
Now for the four equality:
Let $f \in L^\infty(\mu)$ and $A \in \mathcal{Z}$, then

$$\begin{aligned}
\int_A \mathbb{E}(f \circ T \mid Z) \, \mathrm{d}m &= \int_{\pi^{-1}(A)} \mathbb{E}(f \circ T \mid \mathcal{K}) \, \mathrm{d}\mu = \int_{\pi^{-1}(A)} f \circ T \, \mathrm{d}\mu = \int_{T^{-1}(\pi^{-1}(A))} f \, \mathrm{d}\mu \\
&= \int_{\pi^{-1}(R^{-1}(A))} f \, \mathrm{d}\mu = \int_{\pi^{-1}(R^{-1}(A))} \mathbb{E}(f \mid \mathcal{K}) \, \mathrm{d}\mu = \int_{R^{-1}(A)} \mathbb{E}(f \mid Z) \, \mathrm{d}m = \int_A \mathbb{E}(f \mid Z) \circ R \, \mathrm{d}m
\end{aligned}$$

Thus $\mathbb{E}(f \circ T \mid Z) = \mathbb{E}(f \mid Z) \circ R$ for m-a.e. and by the fact that

$$\begin{aligned}
&\int_{Z^4} \mathrm{E}(f_{00} \mid Z) \otimes \mathrm{E}(f_{01} \circ T \mid Z) \otimes \mathrm{E}(f_{10} \mid Z) \otimes \mathrm{E}(f_{11} \circ T \mid Z) \, \mathrm{d}m_4 \\
&= \int_{Z^3} \mathrm{E}(f_{00} \mid Z)(z) \cdot \mathrm{E}(f_{01} \circ T \mid Z)(sz) \cdot \mathrm{E}(f_{10} \mid Z)(tz) \cdot \mathrm{E}(f_{11} \circ T \mid Z)(stz) \, \mathrm{d}m^3(z, s, t)
\end{aligned}$$

it is easy to deduce that the requested equality indeed holds.
The proof is exactly the same for $T_{4,2}$. $\square$

Remark 3.3.10.

For $T_{4,2}$ there is a more direct proof:

$$\int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \circ T \otimes f_{11} \circ T \, \mathrm{d}\mu^{[2]}$$

$$= \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}((f_{10} \otimes f_{11}) \circ (T \times T) \mid \mathcal{I}^{[1]}) \, \mathrm{d}(\mu \otimes \mu)$$

$$= \int_{X^2} \mathbb{E}(f_{00} \otimes f_{01} \mid \mathcal{I}^{[1]}) \cdot \mathbb{E}(f_{10} \otimes f_{11} \mid \mathcal{I}^{[1]}) \, \mathrm{d}(\mu \otimes \mu)$$

$$= \int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \, \mathrm{d}\mu^{[2]}$$

Proposition 3.3.11.

We define two sub σ-algebras of $\mathcal{X}^4$,
$\mathcal{A}_{\mathbf{0},2} = \{A \subseteq X^4 : \ A = B \times X^3 (mod \mu^{[2]})$ *for some* $B \subseteq X\}$, *and*
$\mathcal{J}^{[2]} = \{A \subseteq X^4 : \ T_{4,1}^{-1}(A) = A \ (mod \mu^{[2]})$ *and* $T_{4,2}^{-1}(A) = A \ (mod \mu^{[2]})\}$.
Then

$$\mathcal{A}_{\mathbf{0},2} = \mathcal{J}^{[2]} \qquad mod(\mu^{[2]})$$

Proof. If $A \in \mathcal{A}_{\mathbf{0},2}$ then clearly A is an element of $\mathcal{J}^{[2]}$.

Conversely, if $A \in \mathcal{J}^{[2]}$ then $T_{4,1}^{-1}(A) = A$ and $T_{4,2}^{-1}(A) = A$. Let F be a bounded $\mathcal{J}^{[2]}$-measurable function on X^4. Since $(X^4, \mu^{[2]}, T^{[2]})$ is a self-joining of $(X^2, \mu \otimes \mu, T; [1])$, over $\mathcal{I}^{[1]}$, we have that the function $F(\mathbf{x}) = F(\mathbf{x}', \mathbf{x}'')$ on $X^4 = X^2 \times X^2$ can be approximated in $L^2(\mu^{[2]})$ by finite sums of the form $\sum_i F_i(\mathbf{x}')G_i(\mathbf{x}'')$, where F_i, G_i are bounded functions on X^2. Since $F \circ T_{4,2} = F$ and $T_{4,2} = \mathbf{Id}_{X^2} \times T^{[1]}$, by passing to ergodic averages we can assume that each G_i is $T^{[1]}$-invariant. Indeed, let $\varepsilon > 0$ and consider a finite sum $\sum_i F_i(x')G_i(x'')$, such that $||F - \sum_i F_i(x')G_i(x'')||_{L^2(\mu^{[2]})} < \varepsilon/2$. Since F and $\mu^{[2]}$ are $T^{4,2}$-invariant, we have that for every $j \in \mathbb{N}$, the sum

$$\big(\sum_i F_i G_i\big) \circ T_{4,2}(\mathbf{x}', \mathbf{x}'') = \sum_i F_i(\mathbf{x}') G_i\big((T^{[1]})^j(\mathbf{x}'')\big)$$

is ε-near F in $L^2(\mu^{[2]})$. By Von Neumann's Mean Ergodic Theorem (Theorem 1.2.6) we have that

$$A_n = \frac{1}{n}\sum_{j=0}^{n-1}\left(\sum_i F_i(\mathbf{x}') G_i\big((T^{[1]})^j \mathbf{x}''\big)\right) \to \sum_i F_i(\mathbf{x}')\widetilde{G}_i(\mathbf{x}'')$$

in $L^2(\mu^{[2]})$, as n tends to $+\infty$, where each $\widetilde{G}_i$ is T^2-invariant. Now we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}
||A_n - F||_{L^2(\mu^{[2]})} &= \left\| \frac{1}{n} \sum_{j=0}^{n-1} \left(\sum_i F_i(\mathbf{x}') G_i\big((T^{[1]})^j(\mathbf{x}'')\big) \right) - F \right\|_{L^2(\mu^{[2]})} \\
&\leq \frac{1}{n} \sum_{j=0}^{n-1} \left\| \sum_i F_i(\mathbf{x}') G_i\big((T^{[1]})^j(\mathbf{x}'')\big) - F \right\|_{L^2(\mu^{[2]})} \\
&< \frac{1}{n} \sum_{j=0}^{n-1} \frac{\varepsilon}{2} = \varepsilon/2
\end{aligned}$$

Choose, $n_0 \in \mathbb{N}$ so that A_{n_0} is $\varepsilon/2$-near to $\sum\limits_i F_i \otimes \widetilde{G}_i$, in $L^2(\mu^{[2]})$. Then,

$$\left\| \sum_i F_i \otimes \widetilde{G}_i - F \right\|_{L^2(\mu^{[2]})} \leq \left\| \sum_i F_i \otimes \widetilde{G}_i - A_{n_0} \right\|_{L^2(\mu^{[2]})} + \|A_{n_0} - F\|_{L^2(\mu^{[2]})} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence F can be approximated in $L^2(\mu^{[2]})$ by finite sums of the form $\sum\limits_i F_i(\mathbf{x}')G_i(\mathbf{x}'')$, where F_i, G_i are bounded functions and in addition each G_i is $T^{[1]}$-invariant.

Now we will show that by the construction of the measure $\mu^{[2]}$ we have that for each i, $G_i(\mathbf{x}') = G_i(\mathbf{x}'')$, for $\mu^{[2]}$-almost every $\mathbf{x} = (\mathbf{x}', \mathbf{x}'') \in X^4 = X^2 \times X^2$. Since each G_i is $T^{[1]}$-invariant, $G_i(\mathbf{y}) = \mathbb{E}(G_i \mid \mathcal{I}^{[2]})(\mathbf{y}) = \int\limits_{X^2} G_i \, \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}$, for $\mu \times \mu$-almost every $\mathbf{y} \in X^2$.

$$\begin{aligned}
&\int\limits_{X^4} |G_i(\mathbf{x}') - G_i(\mathbf{x}'')| \, \mathrm{d}\mu^{[2]} \\
&= \int\limits_{X^2} \left(\int\limits_{X^2 \times X^2} |G_i(\mathbf{x}') - G_i(\mathbf{x}'')| \, \mathrm{d}(\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}) \otimes (\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}})(\mathbf{x}', \mathbf{x}'') \right) \mathrm{d}(\mu \otimes \mu)(\mathbf{y}) \\
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} \Big(\int\limits_{X^2} (G_i(\mathbf{x}') - G_i(\mathbf{x}''))\mathbb{1}_{[G_i(\mathbf{x}') > G_i(\mathbf{x}'')]} \, \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}') \\
&\qquad + \int\limits_{X^2} (G_i(\mathbf{x}'') - G_i(\mathbf{x}'))\mathbb{1}_{[G_i(\mathbf{x}') < G_i(\mathbf{x}'')]} \, \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}') \Big) \quad \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \Bigg) \, \mathrm{d}(\mu \otimes \mu)(\mathbf{y})
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} \quad \Big(\int\limits_{X^2} G_i(\mathbf{x}')(\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{x}') \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}') \\
&\qquad\qquad - \int_{X^2} G_i(\mathbf{x}'')(\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{x}')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}') \\
&\qquad\qquad + \int\limits_{X^2} G_i(\mathbf{x}'')(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{x}')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}') \\
&\qquad\qquad - \int\limits_{X^2} G_i(\mathbf{x}')(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{x}')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}')\ \Big) \quad \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'')\Bigg)\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} \big(G_i(\mathbf{y})((\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{y})) - G_i(\mathbf{x}'')((\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{y})) \\
&\qquad + G_i(\mathbf{x}'')(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{y}) - G_i(\mathbf{y})(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{y})\ \big)\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'')\Bigg)\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} G_i(\mathbf{y})((\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') - \int\limits_{X^2} G_i(\mathbf{x}'')((\mathbb{1}_{(G_i(\mathbf{x}''),+\infty)} \circ G_i)(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \\
&\qquad + \int\limits_{X^2} G_i(\mathbf{x}'')(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \\
&\qquad - \int\limits_{X^2} G_i(\mathbf{y})(\mathbb{1}_{\left(-\infty,G_i(\mathbf{x}'')\right)} \circ G_i)(\mathbf{y})\ \big)\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'')\Bigg)\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})>G_i(\mathbf{x}'')]}(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') - \int\limits_{X^2} G_i(\mathbf{x}'')\mathbb{1}_{[G_i(\mathbf{y})>G_i(\mathbf{x}'')]}(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \\
&\qquad + \int\limits_{X^2} G_i(\mathbf{x}'')\mathbb{1}_{[G_i(\mathbf{y})<G_i(\mathbf{x}'')]}(\mathbf{y})\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') - \int\limits_{X^2} G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})<G_i(\mathbf{x}'')]}(\mathbf{y})\ \big)\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'')\Bigg)\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= \int\limits_{X^2} \Bigg(\int\limits_{X^2} G_i(\mathbf{y})(\mathbb{1}_{\left(-\infty,G_i(\mathbf{y})\right)} \circ G_i)(\mathbf{x}'')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') - \int\limits_{X^2} G_i(\mathbf{x}'')(\mathbb{1}_{\left(-\infty,G_i(\mathbf{y})\right)} \circ G_i)(\mathbf{x}'')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \\
&\qquad + \int\limits_{X^2} G_i(\mathbf{x}'')(\mathbb{1}_{(G_i(\mathbf{y},+\infty)} \circ G_i)(\mathbf{x}'')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'') \\
&\qquad - \int\limits_{X^2} G_i(\mathbf{y})(\mathbb{1}_{(G_i(\mathbf{y},+\infty)} \circ G_i)(\mathbf{x}'')\ \mathrm{d}\mu_{\mathbf{y}}^{\mathcal{I}^{[1]}}(\mathbf{x}'')\Bigg)\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= \int\limits_{X^2} G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})>G_i(\mathbf{y})]}\ -\ G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})>G_i(\mathbf{y})]} \\
&\qquad\qquad\qquad +\ G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})<G_i(\mathbf{y})]}\ -\ G_i(\mathbf{y})\mathbb{1}_{[G_i(\mathbf{y})<G_i(\mathbf{y})]}\ \mathrm{d}(\mu\otimes\mu)(\mathbf{y}) \\
&= 0.
\end{aligned}
$$

Thus each sum $\sum_i F_i(\mathbf{x}')G_i(\mathbf{x}'') = \sum_i F_i(\mathbf{x}')G_i(\mathbf{x}')$, for $\mu^{[2]}$-almost every $(\mathbf{x}', \mathbf{x}'') \in X^4$. Passing to the limit, we have that there exists a function H on X^2 such that $F(\mathbf{x}', \mathbf{x}'') = H(\mathbf{x}')$, for $\mu^{[2]}$-almost every $(\mathbf{x}', \mathbf{x}'') \in X^4$. Since F is $T_{4,1}$-invariant then H is $\mathbf{Id} \times T$ invariant and thus by Proposition 3.1.4 is $\mathcal{A}_{\mathbf{0},1}$-measurable. Follows that F is $\mathcal{A}_{\mathbf{0},2}$-measurable. □

Corollary 3.3.12.
The measure $\mu^{[2]}$ is ergodic under the joint action spanned by $T^{[2]}$, $T_{4,1}$, $T_{4,2}$.

Proof. Let $A \subseteq X^4$ be an invariant set under all $T^{[2]}$, $T_{4,1}$, $T_{4,2}$. From Proposition 3.3.11 we have that there exists a set $B \subseteq X$ such that $A = B \times X^3$ Now, $T^{[2]}(A) = A$, and thus $T(B) = B$. By ergodicity, $\mu(B) = 1$ or 0, it follows that $\mu^{[2]}(A) = 1$ or $\mu^{[2]}(A) = 0$. □

Proposition 3.3.13.
The measure $\mu^{[2]}$ is invariant under the group of isometries of the unit Euclidean square, acting on X^4 by permutation of the coordinates

Proof. Since the group of isometries of the unit Euclidean square is spanned by digit permutations and reflections, it suffices to prove it for any reflection and any digit permutation,

- reflection:
  There are only two reflection σ_1^2, σ_2^2 defined by
  $\sigma_1^2(\boldsymbol{\epsilon}) = \sigma_1^2(\epsilon_1\epsilon_2) = (1-\epsilon_1)\epsilon_2$, for each $\boldsymbol{\epsilon} \in V_2$ and
  $\sigma_2^2(\boldsymbol{\epsilon}) = \sigma_2^2(\epsilon_1\epsilon_2) = \epsilon_1(1-\epsilon_2)$, for each $\boldsymbol{\epsilon} \in V_2$
  and they act on X^4 by
  $(\sigma_1^2)_*((x_{00}, x_{01}, x_{10}, x_{11})) = (x_{10}, x_{11}, x_{00}, x_{01})$ for each $\mathbf{x} \in X^4$ and
  $(\sigma_1^2)_*((x_{00}, x_{01}, x_{10}, x_{11})) = (x_{01}, x_{00}, x_{11}, x_{10})$ for each $\mathbf{x} \in X^4$.

Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_2$

$$\int\limits_{X^4} (f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}) \circ (\sigma_1^2)_* \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_{X^4} f_{00}(x_{10}) \cdot f_{01}(x_{11}) \cdot f_{10}(x_{00}) \cdot f_{11}(x_{01}) \, \mathrm{d}\mu^{[2]}(\mathbf{x})$$
$$= \int\limits_{Z^4} \mathbb{E}(f_{00} \mid Z)(z_{10}) \cdot \mathbb{E}(f_{01} \mid Z)(z_{11}) \cdot \mathbb{E}(f_{10} \mid Z)(z_{00}) \cdot \mathbb{E}(f_{11} \mid Z)(z_{01}) \, m_4(\mathbf{z})$$
$$= \int\limits_{Z^3} \mathbb{E}(f_{00} \mid Z)(tz) \cdot \mathbb{E}(f_{01} \mid Z)(stz) \cdot \mathbb{E}(f_{10} \mid Z)(z) \cdot \mathbb{E}(f_{11} \mid Z)(sz) \, m(z)m(s)m(t) =$$

Now by setting $z' = tz$ and $t^{-1} = t'$ we have that the Haar measure, m, of Z is invariant under this transformations. Thus

$$= \int\limits_{Z^3} \mathbb{E}(f_{00} \mid Z)(z') \cdot \mathbb{E}(f_{01} \mid Z)(sz') \cdot \mathbb{E}(f_{10} \mid Z)(t'z') \cdot \mathbb{E}(f_{11} \mid Z)(st'z') \, m(z')m(s)m(t')$$
$$= \int\limits_{X^4} (f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}) \, \mathrm{d}\mu^{[2]}$$

The same procedure can be applied for $(\sigma_2^2)_*$.

- digit permutation:

  The only one nontrivial digit permutation of V_2, $\tau \in S_2$. In other words $\tau = (12)$ and it defies σ on V_2, with $\sigma(\boldsymbol{\epsilon}) = \sigma(\epsilon_1\epsilon_2) = \epsilon_2\epsilon_1$. Now σ acts on X^4 by $\sigma_*(x_{00}, x_{01}, x_{10}, x_{11}) = (x_{00}, x_{10}, x_{01}, x_{11}$

  Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_2$

$$
\begin{aligned}
&\int_{X^4} (f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}) \circ (\sigma_*) \, \mathrm{d}\mu^{[2]} \\
&= \int_{X^4} f_{00}(x_{00}) \cdot f_{01}(x_{10}) \cdot f_{10}(x_{01}) \cdot f_{11}(x_{11}) \, \mathrm{d}\mu^{[2]}(\mathbf{x}) \\
&= \int_{Z^4} \mathbb{E}(f_{00} \mid Z)(z_{00}) \cdot \mathbb{E}(f_{01} \mid Z)(z_{10}) \cdot \mathbb{E}(f_{10} \mid Z)(z_{01}) \cdot \mathbb{E}(f_{11} \mid Z)(z_{11}) \, \mathrm{d}m_4(\mathbf{z}) \\
&= \int_{Z^3} \mathbb{E}(f_{00} \mid Z)(z) \cdot \mathbb{E}(f_{01} \mid Z)(tz) \cdot \mathbb{E}(f_{10} \mid Z)(sz) \cdot \mathbb{E}(f_{11} \mid Z)(stz) \, \mathrm{d}m(z)\mathrm{d}m(s)\mathrm{d}m(t) \\
&= \int_{Z^3} \mathbb{E}(f_{00} \mid Z)(z) \cdot \mathbb{E}(f_{01} \mid Z)(s'z) \cdot \mathbb{E}(f_{10} \mid Z)(t'z) \cdot \mathbb{E}(f_{11} \mid Z)(s't'z) \, \mathrm{d}m(z)\mathrm{d}m(s')\mathrm{d}m(t') \\
&= \int_{X^4} (f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}) \, \mathrm{d}\mu^{[2]}
\end{aligned}
$$

□

Lemma 3.3.14.
Let $f \in L^\infty(\mu)$. Then

$$\int_{X^{[2]}} f \otimes f \otimes f \otimes f \mathrm{d}\mu^{[2]} \ \geq \ \left(\int_X f \, \mathrm{d}\mu \right)^4$$

Proof. Applying the Cauchy-Schwarz inequality we have,

$$
\begin{aligned}
&\int_{X^{[2]}} f \otimes f \otimes f \otimes f \mathrm{d}\mu^{[2]} = \int_{X^{[1]}} \mathbb{E}(f \otimes f \mid \mathcal{I}^{[1]})^2 \, \mathrm{d}\mu \otimes \mu \geq \left(\int_{X^{[1]}} \mathbb{E}(f \otimes f \mid \mathcal{I}^{[1]}) \, \mathrm{d}\mu \otimes \mu \right)^2 \\
&= \left(\int_{X^{[1]}} f \otimes f \, \mathrm{d}\mu \otimes \mu \right)^2 = \left(\left(\int_X f \, \mathrm{d}\mu \right)^2 \right)^2 = \left(\int_X f \, \mathrm{d}\mu \right)^4
\end{aligned}
$$

□

By applying the previous Lemma for the function $f = \mathbb{1}_A$ we have that

Corollary 3.3.15.
For any subset A of X , $\mu^{[2]}(A \times A \times A \times A) \ \geq \ \mu(A)^4$

Definition 3.3.16.
The sub-σ-algebra of $\mathcal{X}^3$ that contains all the subsets of X^3 that satisfy the property in Proposition 3.3.8 is denoted as $\mathcal{J}_3$.

Lemma 3.3.17.

(i) *$\mathcal{J}_3$ is T^3-invariant.*

(ii) *$\mathcal{J}_3$ is the σ-algebra that contains the subset of X^3 that are invariant by $T_{3,1}$ and $T_{3,2}$.*

Proof.

(i) Clearly this statement holds since $\mathcal{K}$ is T-invariant.

(ii) We identify X^4, with $X \times X^3$, by $x = (x_{00}, \tilde{\mathbf{x}})$, where $x_{00} \in X$ and $\tilde{\mathbf{x}} = (x_{01}, x_{10}, x_{11}) \in X^3$. Let X^3 be endowed with the measure $\mu^3 = \mu \otimes \mu \otimes \mu$, the projection of $\mu^{[2]}$ on X^3 and with the transformations $T_{3,1}$, $T_{3,2}$ induced on X^3 by the transformations $T_{4,1}$, $T_{4,2}$ respectively.

Let $B \subseteq X^3$. If B is $\mathcal{J}_3$-measurable. Then $X \times B$ is invariant by all $T_{4,1}$, $T_{4,2}$ and by Proposition 3.3.11 there exists an $A \subseteq X$ such that $X \times B = A \times X^3 \; mod \mu^{[2]}$. Conversely, if there exists an $A \subseteq X$ such that $X \times B = A \times X^3 \; mod \mu^{[2]}$, then clearly B is is $\mathcal{J}_3$-measurable.

□

Lemma 3.3.18.
Let f and g be two bounded functions on X and X^3 respectively, then

$$\int_{X^4} f(x_{00}) g(\tilde{\boldsymbol{x}}) \mathrm{d}\mu^{[2]}(\boldsymbol{x}) = \int_{X^4} \mathbb{E}(f \mid \mathcal{K})(x_{00}) \mathbb{E}(g \mid \mathcal{J}_3)(\tilde{\boldsymbol{x}}) \, \mathrm{d}\mu^{[2]}(\boldsymbol{x})$$

In other words $(X^{[2]}, \mu^{[2]})$ is the relatively independent joining of (X, μ) and (X^3, μ^3) over $\mathcal{K}$ when identified with $\mathcal{J}_3$.

Proof. Let f be a bounded function on X and g a bounded function on X^3, Since $\mu^{[2]}$ is $T_{4,1}$, $T_{4,2}$-invariant, then for every n_1, $n_2 \in \mathbb{N}$, we have,

$$\int_{X^4} f(x_{00}) g(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[2]}(\mathbf{x}) = \int_{X^4} f(x_{00}) g(T_{3,1}^{n_1} T_{3,2}^{n_2}(\tilde{\mathbf{x}})) \, \mathrm{d}\mu^{[2]}(\mathbf{x})$$

Thus we have

$$\int\limits_{X^4} f(x_{00})g(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[2]}(\mathbf{x}) = \frac{1}{n_1n_2}\sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2}\int\limits_{X^4} f(x_{00})g(T_{3,1}^{\alpha}T_{3,2}^{\beta}(\tilde{\mathbf{x}}))\,\mathrm{d}\mu^{[2]}(\mathbf{x})$$

From the L^1-ergodic theorem,

$$\frac{1}{n_1n_2}\sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2}\int\limits_{X^4} f(x_{00})g(T_{3,1}^{\alpha}T_{3,2}^{\beta}(\tilde{\mathbf{x}}))\,\mathrm{d}\mu^{[2]}(\mathbf{x}) \xrightarrow[n_1,n_2\to\infty]{L^1(\mu^{[2]})} \int\limits_{X^4} f(x_{00})\mathbb{E}(g\mid\mathcal{J}_3)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[2]}(\mathbf{x})$$

Thus

$$\int\limits_{X^4} f(x_{00})g(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[2]}(\mathbf{x}) = \int\limits_{X^4} f(x_{00})\mathbb{E}(g\mid\mathcal{J}_3)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[2]}(\mathbf{x}) = \int\limits_{X^4} \mathbb{E}(f\mid\mathcal{K})(x_{00})\mathbb{E}(g\mid\mathcal{J}_3)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[2]}(\mathbf{x})$$

□

Definition 3.3.19.

Let $f \in L^\infty(\mu)$. We define

$$|||f|||_{[2]} \;=\; \left(\int_{X^4} f\otimes f\otimes f\otimes f\mathrm{d}\mu^{[2]}\right)^{1/4}$$

Lemma 3.3.20.

(i) *Let f_ϵ, $\epsilon \in V_2$ be four bounded functions on X, then*

$$\left|\int_{X^4}\bigotimes_{\epsilon\in V_2} f_\epsilon\,\mathrm{d}\mu^{[2]}\right| \le \prod_{\epsilon\in V_2} |||\, f_\epsilon\, |||_{[2]}$$

(ii) $|||\cdot|||_{[2]}$ *is a semi-norm on $L^\infty(\mu)$*

Proof.

(i) Let f_ϵ, $\epsilon\in V_2$ be four bounded functions on X.

$$\begin{aligned}
&\left(\int\limits_{X^4}\bigotimes_{\epsilon\in V_2} f_\epsilon \mathrm{d}\mu^{[2]}\right)^2\\
&\le ||\mathbb{E}(f_{00}\otimes f_{01}\otimes f_{00}\otimes f_{01}\ |\mathcal{I}^{[1]})||^2_{L^2(\mu\otimes\mu)}\cdot||\mathbb{E}(f_{10}\otimes f_{11}\otimes f_{10}\otimes f_{11}\ |\mathcal{I}^{[1]})||^2_{L^2(\mu\otimes\mu)}\\
&= \left(\int\limits_{X^4} f_{00}\otimes f_{01}\otimes f_{00}\otimes f_{01}\mathrm{d}\mu^{[2]}\right)\cdot\left(\int\limits_{X^4} f_{10}\otimes f_{11}\otimes f_{10}\otimes f_{11}\mathrm{d}\mu^{[2]}\right)
\end{aligned}$$

By using the σ_*, defined in the Proposition 3.3.13, we have that

$$\left(\int\limits_{X^4} f_{00}\otimes f_{01}\otimes f_{00}\otimes f_{01}\mathrm{d}\mu^{[2]}\right)\cdot\left(\int\limits_{X^4} f_{10}\otimes f_{11}\otimes f_{10}\otimes f_{11}\mathrm{d}\mu^{[2]}\right)$$
$$=\left(\int\limits_{X^4} f_{00}\otimes f_{00}\otimes f_{01}\otimes f_{01}\mathrm{d}\mu^{[2]}\right)\cdot\left(\int\limits_{X^4} f_{10}\otimes f_{10}\otimes f_{11}\otimes f_{11}\mathrm{d}\mu^{[2]}\right)$$

Applying the same procedure on each of the last two integrals, we obtain the result.

(ii) Let f be a bounded function on X.

$$|||f|||_{[2]}^4=\left(\int_{X^4} f\otimes f\otimes f\otimes f\mathrm{d}\mu^{[2]}\right)=\int_{X^2}\left(\mathbb{E}(f\otimes f\mid\mathcal{I}^{[1]})\right)^2\,\mathrm{d}(\mu\otimes\mu)$$
$$\geqslant\left(\int_{X^2}\mathbb{E}(f\otimes f\mid\mathcal{I}^{[1]})\,\mathrm{d}(\mu\otimes\mu)\right)^2=\left(\int_{X^2} f\otimes f\,\mathrm{d}(\mu\otimes\mu)\right)^2$$
$$=\left(\int_{X^2} f(x)\cdot f(y)\,\mathrm{d}(\mu\otimes\mu)(x,y)\right)^2=\left[\left(\int_X f\,\mathrm{d}\mu\right)^2\right]^2=\left(\int_X f\,\mathrm{d}\mu\right)^4\geqslant 0$$

It is trivial to prove that $|||c\cdot f|||_{[2]}=|c|\cdot|||f|||_{[2]}$, $\forall c\in\mathbb{C}$
The remaining property that needs to be proven is subadditivity. Let $f,g\in L^\infty(\mu)$.

$$|||f+g|||_{[2]}^4$$
$$=\binom{4}{0}\int\limits_{X^4} f^{\{4\}}\,\mathrm{d}\mu^{[2]}+\binom{4}{1}\int\limits_{X^4} f^{\{3\}}g^{\{1\}}\,\mathrm{d}\mu^{[2]}$$
$$+\binom{4}{2}\int\limits_{X^4} f^{\{2\}}g^{\{2\}}\,\mathrm{d}\mu^{[2]}+\binom{4}{3}\int\limits_{X^4} f^{\{1\}}g^{\{3\}}\,\mathrm{d}\mu^{[2]}+\binom{4}{4}\int\limits_{X^4} g^{\{4\}}\,\mathrm{d}\mu^{[2]}$$

where the notation $f^{\{n\}}$ (respectively $g^{\{n\}}$) implies the number that f (respectively g) appears in the integral regardless of the position. By **(i)** we have that,

$$|||f+g|||_{[2]}^4$$
$$\leq\binom{4}{0}|||f|||_{[2]}^4+\binom{4}{1}|||f_{[2]}|||^3|||g|||_{[2]}+\binom{4}{2}|||f|||_{[2]}^2|||g|||_{[2]}^2+\binom{4}{3}|||f|||_{[2]}|||g|||_{[2]}^3+\binom{4}{4}|||g|||_{[2]}^4$$
$$=\left(|||f|||_{[2]}+|||g|||_{[2]}\right)^4$$

□

Proposition 3.3.21.
Let $f \in L^\infty(\mu)$, then

$$|||f|||_{[2]} = 0 \iff \mathbb{E}(f \mid \mathcal{K}) = 0$$

Proof. Assume that $\mathbb{E}(f \mid \mathcal{K}) = 0$. By Lemma 3.3.18 applied for $g(\tilde{\mathbf{x}}) = \prod\limits_{\epsilon \in V_2^*} f(x_\epsilon)$, we have that $|||f|||_{[2]} = 0$

Conversely, let $|||f|||_{[2]} = 0$. Then by Lemma 3.3.20, if $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_2^*$.

$$\int\limits_{X^4} f(x_{00}) \cdot \prod_{\epsilon \in V_2^*} f_\epsilon(x_\epsilon)\ \mathrm{d}\mu^{[2]} = 0$$

By density we have that $\int\limits_{X^4} f(x_{00}) g(\tilde{\mathbf{x}})\ \mathrm{d}\mu^{[2]} = 0$, function on X^3. In particular, this holds for every $\mathcal{J}_3$-measurable function on X^3 and thus f is orthogonal in $L^2(\mu)$ to every $\mathcal{K}$-measurable function on X. Thus $\mathbb{E}(f \mid \mathcal{K}) = 0$. □

Proposition 3.3.22.
Let $f \in L^2(\mu)$, then
f is $\mathcal{K}$-measurable if and only if f is orthogonal in $L^2(\mu)$ to every $g \in L^\infty(\mu)$ with $|||g|||_{[2]} = 0$

Proof. Let $f \in L^2(\mu)$, $\mathcal{K}$-measurable and $g \in L^\infty(\mu)$ with $|||g|||_{[2]} = 0$. Then there exists a function F in X^3 that is $\mathcal{K}^3$-measurable such that, for $\mu^{[2]}$-almost every $\mathbf{x} \in X^4$

$$f(x_{00}) = F(x_{01}, x_{10}, x_{11})$$

As in **Note** on page 11, we have that

$$\left| \int\limits_X f \cdot g\ \mathrm{d}\mu \right| = \left| \int\limits_{X^4} F(x_{01}, x_{10}, x_{11}) \cdot g(x_{00})\ \mathrm{d}\mu^{[2]}(\mathbf{x}) \right| \backsim \left| \sum_i \int_{X^4} g \otimes F_i \otimes G_i \otimes H_i\ \mathrm{d}\mu^{[2]}(\mathbf{x}) \right|$$
$$\leq \sum_i |||g|||_{[2]} \cdot |||F_i|||_{[2]} \cdot |||G_i|||_{[2]} \cdot |||H_i|||_{[2]} = 0$$

Conversely, let $f \in L^2(\mu)$ such that $\forall g \in L^\infty(\mu)$ with $|||g|||_{[2]} = 0$,

$$\int_X f \cdot g \mathrm{d}\mu$$

We can write $f = f' + f''$, where f' is $\mathcal{K}$-measurable and $\mathbb{E}(f'' \mid \mathcal{K}) = 0$. Then,

$$\|f\|_{L^2(\mu)} = \int_X f \cdot f\ \mathrm{d}\mu = \int_X f \cdot (f' + f'')\ \mathrm{d}\mu = \int_X f \cdot f'\ \mathrm{d}\mu + \int_X f \cdot f''\ \mathrm{d}\mu$$

Now, $\mathbb{E}(f'' \mid \mathcal{K}) \iff |||f'''|||_{[2]} = 0$ thus,

$$\int_X f \cdot f'' \, \mathrm{d}\mu \;=\; 0$$

In other words

$$\int_X f \cdot f \, \mathrm{d}\mu = \int_X f \cdot \mathbb{E}(f \mid \mathcal{K}) \, \mathrm{d}\mu$$

Hence f is $\mathcal{K}$-measurable □

Proposition 3.3.23.

Let (X, μ, T) and (Y, ν, S) be two systems a,d $p : X \to Y$ be a factor map. Let $\mathcal{K}(X)$, $\mathcal{K}(Y)$ be the Kronecker factor of X, Y respectively. Then $p^{-1}\big(\mathcal{K}(Y)\big) = \mathcal{K}(X) \cap p^{-1}(\mathcal{Y})$.

Proof. Let $p^3 = p \times p \times p : (X^3, \mu^3, T^3) \to (Y^3, \nu^3, S^3)$ be the natural map. Then p^3 is a factor map. Let f be a bounded function on X that is $p^{-1}\big(\mathcal{K}(Y)\big)$-measurable. Then $f = g \circ p$ for some bounded function on Y that is $\mathcal{K}(Y)$-measurable. By Proposition 3.3.8 there exist a $\mathcal{K}^3(Y)$-measurable function G on Y^3 such that $g(y_{00}) = G(y_{01}, y_{10}, y_{11})$, for $\nu^{[2]}$-almost every $\mathbf{y} = (y_{00}, y_{01}, y_{10}, y_{11}) \in Y^4$. Thus $g \circ p(x_{00}) = G \circ p^3(x_{01}, x_{10}, x_{11})$, for $\mu^{[2]}$-almost every $(x_{00}, x_{01}, x_{10}, x_{11}) \in X^4$. Again by Proposition 3.3.8 we have that $f = g \circ p$ is $\mathcal{K}(X)$-measurable. Thus $p^{-1}\big(\mathcal{K}(Y)\big) \subseteq \mathcal{K}(X) \cap p^{-1}(\mathcal{Y})$.

Now let f be a bounded function on X that is $\mathcal{K}(X) \cap p^{-1}(\mathcal{Y})$-measurable. Then $f = g \circ p$ for some bounded function on Y. Write $g = g' + g''$, where g' is $\mathcal{K}(Y)$-measurable and $\mathbb{E}(g'' \mid \mathcal{K}(Y)) = 0$. By the first part of this proof, we have that $g' \circ p$ is $\mathcal{K}(X)$-measurable. Since $\mathbb{E}(g'' \mid \mathcal{K}(Y)) = 0$, we have that $|||g''|||_{Y^4} = 0$, and thus $|||g'' \circ p|||_{X^4} = 0$ thus $\mathbb{E}(g'' \circ p \mid \mathcal{K}(X)) = 0$. Since $f = g' \circ p + g'' \circ p$ is $\mathcal{K}(X)$-measurable, we have that $g'' \circ p = 0$. Thus $g'' = 0$ and g is $\mathcal{K}(Y)$-measurable. □

Remark 3.3.24.

This means that the Kronecker factor of any ergodic system is a system of order 1. Furthermore the Kronecker factor is the largest system of order 1 under any ergodic system.

Note.

The sub-σ-algebra $\mathcal{K}^4 \subseteq \mathcal{X}^4$ is $T^{[2]}$-invariant, hence the conditional expectation with respect to $\mathcal{I}^{[2]}$ commutes with the conditional expectation with respect to $\mathcal{K}^4$

Lemma 3.3.25.

Let $f_\epsilon \in L^\infty(\mu)$ for $\epsilon \in V_2$.

(i) *If f_γ is $\mathcal{K}$-measurable for some $\gamma \in V_2$, then* $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big)$ *is $\mathcal{K}^4$-measurable.*

(ii) *If* $\mathbb{E}(f_\delta \mid \mathcal{K}) = 0$ *for some $\delta \in V_2$, then* $\mathbb{E}\big(\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}) \mid \mathcal{K}^4\big) = 0.$

(iii) *If the conditions in both (i) and (ii) are satisfied, then* $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) = 0$

Proof. (i) By Proposition 3.3.13 it suffices to prove it for $\gamma = 00$
Since f_{00} is $\mathcal{K}$-measurable , $\exists\, F$ on X^3, that is $\mathcal{K}^3$-measurable, such that for $\mu^{[2]}$-almost every $\mathbf{x} \in X^4$, $f(x_{00}) = F(x_{01}, x_{10}, x_{11})$. Thus we have that,
$\prod_{\epsilon\in V_2} f(x_\epsilon) = F(x_{01}, x_{10}, x_{11}) \cdot \prod_{\epsilon\in V_2^*} f_\epsilon(x_\epsilon)$ for $\mu^{[2]}$-almost every $\mathbf{x} \in X^4$

By Corollary 2.3.2 we also obtain that the T^3-invariant sets of $(X^3, \mu\otimes\mu\otimes\mu)$ are $\mathcal{K}^2$-measurable

By corollary 3.3.7 we have that the 3 dimensional marginal of $\mu^{[2]}$ is $\mu\otimes\mu\otimes\mu$. Thus $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big)$ is $\mathcal{K}^4$-measurable. Indeed, let $A \in \mathcal{I}^{[2]}$

$$\begin{aligned}
\int_A \mathbb{E}\big(\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}) \mid \mathcal{K}^4\big)\, \mathrm{d}\mu^{[2]} &= \int_A \mathbb{E}\big(\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4) \mid \mathcal{I}^{[2]}\big)\, \mathrm{d}\mu^{[2]} \\
= \int_A \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big)\, \mathrm{d}\mu^{[2]} &= \int_A \mathbb{1}_X \otimes \mathbb{E}\big(F \cdot \bigotimes_{\epsilon\in V_2^*} f_\epsilon \mid \mathcal{K}^3\big)\mathrm{d}\mu^{[2]} \\
= \int_A \mathbb{1}_X \otimes F \cdot \bigotimes_{\epsilon\in V_2^*} f_\epsilon\, \mathrm{d}\mu^{[2]} = \int_A \bigotimes_{\epsilon\in V_2} f_\epsilon\, \mathrm{d}\mu^{[2]} &= \int_A \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big)\, \mathrm{d}\mu^{[2]}
\end{aligned}$$

The first equality holds by the previous **Note** and the third by the fact that, $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big) = \mathbb{E}\big(\mathbb{1}_X \otimes F \cdot \bigotimes_{\epsilon\in V_2^*} f_\epsilon \mid \mathcal{K}^4\big) = \mathbb{1}_X \otimes \mathbb{E}\big(F \cdot \bigotimes_{\epsilon\in V_2^*} f_\epsilon \mid \mathcal{K}^3\big)$

(ii) By Proposition 3.3.2 we have that, $\bigotimes_{\epsilon\in V_2} \mathbb{E}(f_\epsilon \mid \mathcal{K}) = \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big)$
Indeed, if $A \in \mathcal{K}^4$, then

$$\int_A \bigotimes_{\epsilon\in V_2} \mathbb{E}(f_\epsilon \mid \mathcal{K})\, \mathrm{d}\mu^{[2]} = \int_A \bigotimes_{\epsilon\in V_2} f_\epsilon\, \mathrm{d}\mu^{[2]} = \int_A \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big)\, \mathrm{d}\mu^{[2]}$$

Hence

$$\mathbb{E}(f_\delta \mid \mathcal{K}) = 0 \implies \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big) = 0$$
$$\implies \mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big) \bigg| \mathcal{I}^{[2]}\bigg) = 0 \implies \mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) \bigg| \mathcal{K}^4\bigg) = 0$$

(iii) If one f_γ is $\mathcal{K}$-measurable then from (i)

$$\mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big) \bigg| \mathcal{I}^{[2]}\bigg) = \mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big)$$

Furthermore if $\mathbb{E}(f_\delta \mid \mathcal{K}) = 0$ then from (ii)

$$\mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) \bigg| \mathcal{K}^4\bigg) = 0$$

and as mentioned before

$$\mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{K}^4\big) \bigg| \mathcal{I}^{[2]}\bigg) = \mathbb{E}\bigg(\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) \bigg| \mathcal{K}^4\bigg)$$

It follows that

$$\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) = 0$$

□

3.4 The measure $\mu^{[3]}$

As stated above, the measure $\mu^{[3]}$ is the relatively independent joining over $\mathcal{I}^{[2]}$. That means

$$\int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon \mathrm{d}\mu^{[3]} = \int_{X^4} \mathbb{E}\big(\bigotimes_{\substack{\epsilon\in V_3\\ \epsilon_1=0}} f_\epsilon \mid \mathcal{I}^{[2]}\big) \cdot \mathbb{E}\big(\bigotimes_{\substack{\epsilon\in V_3\\ \epsilon_1=1}} f_\epsilon \mid \mathcal{I}^{[2]}\big) \,\mathrm{d}\mu^{[2]}$$
$$= \int_{X^4} \mathbb{E}\big(\bigotimes_{\eta\in V_2} f_{0\eta} \mid \mathcal{I}^{[2]}\big) \cdot \mathbb{E}\big(\bigotimes_{\eta\in V_2} f_{1\eta} \mid \mathcal{I}^{[2]}\big) \,\mathrm{d}\mu^{[2]}$$

for $f_{000},\ f_{001},\ f_{010},\ f_{011},\ f_{100},\ f_{101},\ f_{110},\ f_{111} \in L^\infty(\mu)$.

Proposition 3.4.1.
The measure $\mu^{[3]}$ is $T^{[3]} = T\times T\times T\times T\times T\times T\times T\times T$ -invariant.

Proof. The proof is exactly the same as Proposition 3.3.1. $\square$

Proposition 3.4.2.
The measure $\mu^{[3]}$ is invariant under the tranformations

$$T_{8,1} = T_{4,1} \times T_{4,1}\ ,\ T_{8,2} = T_{4,2} \times T_{4,2} \text{ and } T_{8,3} = \textbf{Id} \times \textbf{Id} \times \textbf{Id} \times \textbf{Id} \times T \times T \times T \times T$$

where $T_{4,1}, T_{4,2}$ are as in Proposition 3.3.9

Proof. Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3$ be 8 functions on X

$$\begin{aligned}
\int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon \circ T_{8,1}\, \mathrm{d}\mu^{[3]} &= \int_{X^4} \mathbb{E}(\bigotimes_{\eta\in V_2} f_{0\eta} \circ T_{4,1} \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(\bigotimes_{\eta\in V_2} f_{1\eta} \circ T_{4,1} \mid \mathcal{I}^{[2]})\, \mathrm{d}\mu^{[2]} \\
&= \int_{X^4} \mathbb{E}(\bigotimes_{\eta\in V_2} f_{0\eta} \mid \mathcal{I}^{[2]}) \circ T_{4,1} \cdot \mathbb{E}(\bigotimes_{\eta\in V_2} f_{1\eta} \mid \mathcal{I}^{[2]}) \circ T_{4,1}\, \mathrm{d}\mu^{[2]} \\
&= \int_{X^4} \Big(\mathbb{E}(\bigotimes_{\eta\in V_2} f_{0\eta} \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(\bigotimes_{\eta\in V_2} f_{1\eta} \mid \mathcal{I}^{[2]})\Big) \circ T_{4,1}\, \mathrm{d}\mu^{[2]} \\
&= \int_{X^4} \mathbb{E}(\bigotimes_{\eta\in V_2} f_{0\eta} \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(\bigotimes_{\eta\in V_2} f_{1\eta} \mid \mathcal{I}^{[2]})\, \mathrm{d}\mu^{[2]} \;=\; \int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon\, \mathrm{d}\mu^{[3]}
\end{aligned}$$

The second equality holds because $T_{4,1} \circ T^{[2]} = T^{[2]} \circ T_{4,1}$ and the fifth holds by Proposition 3.3.9
The remaining two cases can be proved in the exact same manner. $\square$

Proposition 3.4.3.
The measure $\mu^{[3]}$ is invariant under the group of isometries of the unit Euclidean cube, acting on X^8 by permutation of the coordinates.

Note. We will use the transformations $(\sigma_1^2)_*$, $(\sigma_2^2)_*$, $(\sigma)_*$ defined in Proposition 3.3.13, acting on X^4 by
$(\sigma_1^2)_*((x_{00}, x_{01}, x_{10}, x_{11})) = (x_{10}, x_{11}, x_{00}, x_{01})$ for each $\mathbf{x} \in X^4$,
$(\sigma_2^2)_*((x_{00}, x_{01}, x_{10}, x_{11})) = (x_{01}, x_{00}, x_{11}, x_{10})$ for each $\mathbf{x} \in X^4$ and
$\sigma_*(x_{00}, x_{01}, x_{10}, x_{11}) = (x_{00}, x_{10}, x_{01}, x_{11})$ for each $\mathbf{x} \in X^4$

Proof. Since the group of isometries of the unit Euclidean cube is spanned by digit permutations and reflections, it suffices to prove it for any reflection and any digit permutation,

- reflections:

  There are 3 reflections σ_1^3, σ_2^3, σ_3^3 where

$$
\begin{aligned}
&(\sigma_1^3)_*(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}) = \\
&\qquad\qquad = (x_{100}, x_{101}, x_{110}, x_{111}, x_{000}, x_{001}, x_{010}, x_{011}) \\
&(\sigma_2^3)_*(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}) = \\
&\qquad\qquad = (x_{010}, x_{011}, x_{000}, x_{001}, x_{110}, x_{111}, x_{100}, x_{101}) \\
&(\sigma_3^3)_*(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}) = \\
&\qquad\qquad = (x_{001}, x_{000}, x_{011}, x_{010}, x_{101}, x_{100}, x_{111}, x_{110})
\end{aligned}
$$

Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3$.

$$
\begin{aligned}
&\int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \circ (\sigma_1^3)_* \; \mu^{[3]} \\
&= \int_{X^8} f_{000}(x_{100}) f_{001}(x_{101}) f_{010}(x_{110}) f_{011}(x_{111}) f_{100}(x_{000}) f_{101}(x_{001}) f_{110}(x_{010}) f_{111}(x_{011}) \; \mathrm{d}\mu^{[3]} \\
&= \int_{X^4} \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111}) \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011}) \mid \mathcal{I}^{[2]}) \; \mathrm{d}\mu^{[2]} \\
&= \int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \; \mathrm{d}\mu^{[3]}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \circ (\sigma_2^3)_* \; \mathrm{d}\mu^{[3]} \\
&= \int_{X^8} f_{000}(x_{010}) f_{001}(x_{011}) f_{010}(x_{000}) f_{011}(x_{001}) f_{100}(x_{110}) f_{101}(x_{111}) f_{110}(x_{100}) f_{111}(x_{101}) \; \mathrm{d}\mu^{[3]} \\
&= \int_{X^4} \mathbb{E}(f_{010} \otimes f_{011} \otimes f_{000} \otimes f_{001}) \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{110} \otimes f_{111} \otimes f_{100} \otimes f_{101}) \mid \mathcal{I}^{[2]}) \; \mathrm{d}\mu^{[2]} \\
&= \int_{X^4} \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011}) \mid \mathcal{I}^{[2]}) \circ (\sigma_1^2)_* \cdot \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111}) \mid \mathcal{I}^{[2]}) \circ (\sigma_1^2)_* \; \mathrm{d}\mu^{[2]}
\end{aligned}
$$

$$= \int_{X^4} \Big(\mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011}) \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111}) \mid \mathcal{I}^{[2]}) \Big) \circ (\sigma_1^2)_* \, \mathrm{d}\mu^{[2]}$$

$$= \int_{X^4} \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011}) \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111}) \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[2]}$$

$$= \int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]}$$

For the third equality, let $A \in \mathcal{I}^{[2]}$, then

$$\int_A \mathbb{E}(f_{010} \otimes f_{011} \otimes f_{000} \otimes f_{001}) \mid \mathcal{I}^{[2]}) = \int_A f_{010}(y_{00}) \cdot f_{011}(y_{01}) \cdot f_{000}(y_{10}) \cdot f_{001}(y_{11}) \, \mathrm{d}\mu^{[2]}$$

$$= \int_A \big(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \big) \circ (\sigma_1^2)_* \, \mathrm{d}\mu^{[2]}$$

$$= \int_{\left((\sigma_1^2)_*\right)^{-1}(A)} f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \, \mathrm{d}\mu^{[2]} \qquad (\mathbf{1})$$

Now it is easy to deduce that $(\sigma_1^2)_* \circ T^{[2]} = T^{[2]} \circ (\sigma_1^2)_*$. Thus $\left((\sigma_1^2)_*\right)^{-1}(A) \in \mathcal{I}^{[2]}$ and by that we have

$$(\mathbf{1}) = \int_{\left((\sigma_1^2)_*\right)^{-1}(A)} \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[2]}$$

$$= \int_A \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \circ (\sigma_1^2)_* \, \mathrm{d}\mu^{[2]}$$

Thus $\mathbb{E}(f_{010} \otimes f_{011} \otimes f_{000} \otimes f_{001}) \mid \mathcal{I}^{[2]}) = \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \circ (\sigma_1^2)_*$, $\mu^{[2]}$-a.e. and correspondingly $\mathbb{E}(f_{110} \otimes f_{111} \otimes f_{100} \otimes f_{101}) \mid \mathcal{I}^{[2]}) = \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111}) \mid \mathcal{I}^{[2]}) \circ (\sigma_1^2)_*$, $\mu^{[2]}$-a.e., and the relation follows.
The fifth equality holds by Proposition 3.3.13
The case for $(\sigma_3^3)_*$ is proved in the exact same way as $(\sigma_2^3)_*$, using now the transformation $(\sigma_2^2)_*$

- There are five nontrivial elements in S_3, $\tau_i \in S_3$, $i \in \{1, 2, 3, 4, 5\}$, where $\tau_1 = (12)$, $\tau_2 = (13)$, $\tau_3 = (23)$, $\tau_4 = (123)$, $\tau_5 = (132)$. They defy five nontrivial digit permutation on V_2,

$\sigma_1(\epsilon_1\epsilon_2\epsilon_3)=\epsilon_2\epsilon_1\epsilon_3$ $\sigma_2(\epsilon_1\epsilon_2\epsilon_3)=\epsilon_3\epsilon_2\epsilon_1$ $\sigma_3(\epsilon_1\epsilon_2\epsilon_3)=\epsilon_1\epsilon_3\epsilon_2$
$\sigma_4(\epsilon_1\epsilon_2\epsilon_3)=\epsilon_3\epsilon_1\epsilon_2$ $\sigma_5(\epsilon_1\epsilon_2\epsilon_3)=\epsilon_2\epsilon_3\epsilon_1$
and those transformations act on X^8 by

$$(\sigma_1)_*(x_{000},x_{001},x_{010},x_{011},x_{100},x_{101},x_{110},x_{111}) \\ =(x_{000},x_{001},x_{100},x_{101},x_{010},x_{011},x_{110},x_{111})$$

$$(\sigma_2)_*(x_{000},x_{001},x_{010},x_{011},x_{100},x_{101},x_{110},x_{111}) \\ =(x_{000},x_{100},x_{010},x_{110},x_{001},x_{101},x_{011},x_{111})$$

$$(\sigma_3)_*(x_{000},x_{001},x_{010},x_{011},x_{100},x_{101},x_{110},x_{111}) \\ =(x_{000},x_{010},x_{001},x_{011},x_{100},x_{110},x_{101},x_{111})$$

$$(\sigma_4)_*(x_{000},x_{001},x_{010},x_{011},x_{100},x_{101},x_{110},x_{111}) \\ =(x_{000},x_{010},x_{100},x_{110},x_{001},x_{011},x_{101},x_{111})$$

$$(\sigma_5)_*(x_{000},x_{001},x_{010},x_{011},x_{100},x_{101},x_{110},x_{111}) \\ =(x_{000},x_{100},x_{001},x_{101},x_{010},x_{110},x_{011},x_{111})$$

We have that $\tau_2=\tau_1\circ\tau_3\circ\tau_1$, $\tau_4=\tau_3\circ\tau_1$ and $\tau_5=\tau_1\circ\tau_3$
$\Rightarrow \sigma_2=\sigma_1\circ\sigma_3\circ\sigma_1$, $\sigma_4=\sigma_3\circ\sigma_1$ and $\sigma_5=\sigma_1\circ\sigma_3$
$\Rightarrow\ (\sigma_2)_*=(\sigma_1)_*\circ(\sigma_3)_*\circ(\sigma_1)_*$, $(\sigma_4)_*=(\sigma_3)_*\circ(\sigma_1)_*$ and $(\sigma_5)_*=(\sigma_1)_*\circ(\sigma_3)_*$,
thus it suffices to show it just for $(\sigma_1)_*$ and $(\sigma_3)_*$

Let $f_\epsilon\in L^\infty(\mu)$, $\epsilon\in V_3$.

$$\int_{X^8}\bigotimes_{\epsilon\in V_3} f_\epsilon\circ(\sigma_3)_*\,\mathrm{d}\mu^{[3]}$$
$$=\int_{X^8} f_{000}(x_{000})f_{001}(x_{010})f_{010}(x_{001})f_{011}(x_{011})f_{100}(x_{100})f_{101}(x_{110})f_{110}(x_{101})f_{111}(x_{111})\,\mathrm{d}\mu^{[3]}$$
$$=\int_{X^4}\mathbb{E}(f_{000}\otimes f_{010}\otimes f_{001}\otimes f_{011}\mid\mathcal{I}^{[2]})\cdot\mathbb{E}(f_{000}\otimes f_{110}\otimes f_{101}\otimes f_{111}\mid\mathcal{I}^{[2]})\,\mathrm{d}\mu^{[2]}$$
$$=\int_{X^4}\mathbb{E}(f_{000}\otimes f_{001}\otimes f_{010}\otimes f_{011}\mid\mathcal{I}^{[2]})\circ(\sigma)_*\cdot\mathbb{E}(f_{000}\otimes f_{101}\otimes f_{110}\otimes f_{111}\mid\mathcal{I}^{[2]})\circ\sigma_*\,\mathrm{d}\mu^{[2]}$$

$$= \int\limits_{X^4} \Big(\mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{000} \otimes f_{101} \otimes f_{110} \otimes f_{111} \mid \mathcal{I}^{[2]}) \Big) \circ \sigma_* \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_{X^4} \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \cdot \mathbb{E}(f_{000} \otimes f_{101} \otimes f_{110} \otimes f_{111} \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]}$$

For the third equality, let $B \in \mathcal{I}^{[2]}$, then

$$\int\limits_B \mathbb{E}(f_{000} \otimes f_{010} \otimes f_{001} \otimes f_{011} \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_B f_{000}(y_{00}) \cdot f_{010}(y_{01}) \cdot f_{001}(y_{10}) \cdot f_{011}(y_{11}) \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_B \big(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011}\big) \circ \sigma_* \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_{(\sigma_*)^{-1}(B)} f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \, \mathrm{d}\mu^{[2]} \qquad (\mathbf{2})$$

Now it is easy to deduce the $\sigma_* \circ T^{[2]} = T^{[2]} \circ \sigma_*$. Thus $(\sigma_*)^{-1}(B) \in \mathcal{I}^{[2]}$ and by that we have,

$$(\mathbf{2}) = \int\limits_{(\sigma_*)^{-1}(B)} \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_B \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \circ (\sigma_*) \, \mathrm{d}\mu^{[2]}$$

Thus $\mathbb{E}(f_{000} \otimes f_{010} \otimes f_{001} \otimes f_{011} \mid \mathcal{I}^{[2]}) = \mathbb{E}(f_{000} \otimes f_{001} \otimes f_{010} \otimes f_{011} \mid \mathcal{I}^{[2]}) \circ (\sigma_*)$, $\mu^{[2]}$-a.e. and correspondingly $\mathbb{E}(f_{100} \otimes f_{110} \otimes f_{101} \otimes f_{111} \mid \mathcal{I}^{[2]}) = \mathbb{E}(f_{100} \otimes f_{101} \otimes f_{110} \otimes f_{111} \mid \mathcal{I}^{[2]}) \circ (\sigma_*)$, $\mu^{[2]}$-a.e. and the equality holds.
The fifth equality holds by Proposition 3.3.13

Now for $(\sigma_1)_*$
We define the system $(Y, \mathcal{Y}, \nu, S) = (X \times X, \mathcal{X} \otimes \mathcal{X}, \mu \otimes \mu, T \times T)$. Then the measure $\nu^{[2]}$, built from the system $(Y, \mathcal{Y}, \nu, S)$ in the same way that $\mu^{[2]}$ was built from $(X, \mathcal{X}, \mu, T)$, is equal to $\mu^{[3]}$

Indeed let $\nu = \mu \otimes \mu = \int\limits_{\Omega} \mu_\omega \, \mathrm{d}P(\omega)$ be the ergodic decomposition of $\nu = \mu \otimes \mu$. Then by Lemma 3.2.3, $\nu^{[2]} = \int\limits_{\Omega} \mu_\omega^{[2]} \, \mathrm{d}P(\omega)$ where $\mu_\omega^{[2]}$ is constructed in the same way that $\mu^{[2]}$ was built from $(X, \mathcal{X}, \mu, T)$. From the same Lemma, we have that $\mu^{[3]} = \int\limits_{\Omega} \mu_\omega^{[2]} \, \mathrm{d}P(\omega)$. Thus $\nu^{[2]} = \mu^{[3]}$.

Now we identify X^8 with Y^4 as follows

$$(x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111})$$
$$= \Big(\underbrace{(x_{(00)0}, x_{(00)1})}_{y_{00}}, \underbrace{(x_{(01)0}, x_{(01)1})}_{y_{01}}, \underbrace{(x_{(10)0}, x_{(10)1})}_{y_{10}}, \underbrace{(x_{(11)0}, x_{(11)1})}_{y_{11}}\Big)$$

Then $(\sigma_3)_*$ acts on X^8 as σ_* acts on Y^4. By Proposition 3.3.13 we have that $(\sigma_*)_*\nu = \nu$. Hence $((\sigma_3)_*)_*\mu^{[3]} = \mu^{[3]}$

□

Corollary 3.4.4.
The image of $\mu^{[3]}$ under any side projection $X^{[3]} \to X^{[2]}$ is $\mu^{[2]}$.

Lemma 3.4.5.
Let $f \in L^\infty(\mu)$. Then

$$\int_{X^{[3]}} f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \, \mathrm{d}\mu^{[3]} \;\geq\; \left(\int_X f \, \mathrm{d}\mu\right)^8$$

Proof. By applying the Cauchy-Schwarz inequality and Lemma 3.3.14 we have that,

$$\int_{X^{[3]}} f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \mathrm{d}\mu^{[3]} = \int_{X^{[2]}} \mathbb{E}(f \otimes f \otimes f \otimes f \mid \mathcal{I}^{[2]})^2 \, \mathrm{d}\mu^{[3]}$$
$$\geq \left(\int_{X^{[2]}} \mathbb{E}(f \otimes f \otimes f \otimes f \mid \mathcal{I}^{[2]}) \, \mathrm{d}\mu^{[3]}\right)^2 = \left(\int_{X^{[2]}} f \otimes f \otimes f \otimes f \, \mathrm{d}\mu^{[3]}\right)^2$$
$$\geq \left(\left(\int_X f \, \mathrm{d}\mu\right)^4\right)^2 = \left(\int_X f \, \mathrm{d}\mu\right)^8$$

□

Applying the previous Lemma for the function $\mathbb{1}_A$, we have that,

Corollary 3.4.6.
For any subset A of X,

$$\mu^{[3]}(A \times A \times A \times A \times A \times A \times A \times A) \;\geq\; \mu(A)^8$$

Chapter 4

Conze-Lesigne Factor

4.1 Construction of the factor

Proposition 4.1.1.
We define $\mathcal{A}_{\boldsymbol{0},3} = \{B \subseteq X^8 : \ B = A \times X^7 \ (mod\mu^{[3]})\}$, *and*

$$\mathcal{J}^{[3]} = \{A \subseteq X^8 : \ T_{8,1}^{-1}(A) = A \ (mod\mu^{[3]}), \ T_{8,2}^{-1}(A) = A \ \ (mod\mu^{[3]}) \text{ and } T_{8,3}^{-1}(A) = A \ (mod\mu^{[3]})\}$$

Then

$$\mathcal{A}_{\boldsymbol{0},3} = \mathcal{J}^{[3]} \qquad (mod\mu^{[3]})$$

We omit the proof of this Proposition as the procedure is exaclty the same as in Proposition 3.3.11.

Corollary 4.1.2.
The measure $\mu^{[3]}$ is ergodic under the joint action spanned by $T^{[3]}$, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$.

Proof. Let $B \subseteq X^8$ be an invariant set under all $T^{[3]}$,$T_{8,1}$, $T_{8,2}$, $T_{8,3}$. From Proposition 4.1.1 we have that there exists a set $B \subseteq X$ such that $A = B \times X^7$
Now, $(T^{[3]})^{-1}(B) = 0 \Rightarrow T^{-1}(A) = A$. By ergodicity, $\mu(A) = 1$ or 0, it follows that $\mu^{[3]}(B) = 1$ or $\mu^{[3]}(B) = 0$. □

We establish some more notation. We identify X^8, with $X \times X^7$, by $\mathbf{x} = (x_{000}, \tilde{\mathbf{x}})$, where $x_{000} \in X$ and $\tilde{\mathbf{x}} = (x_\epsilon : \ \epsilon \in V_3^*) \in X^7$. Let X^7 be endowed with the measure μ_7, the projection of $\mu^{[3]}$ on X^7 and with the transformations $T_{7,1}$, $T_{7,2}$, $T_{7,3}$ induced on

X^7 by the transformations $T_{8,1}$, $T_{8,2}$, $T_{8,3}$, respectively. We define $\mathcal{J}_7$ be the σ-algebra of subsets of X^7, invariant under the transformations $T_{7,1}$, $T_{7,2}$, $T_{7,3}$.

Let $B \subseteq X^7$. If B is $\mathcal{J}_7$-measurable. Then $X \times B$ is invariant by all $T_{8,1}$, $T_{8,2}$, $T_{8,3}$ and by Proposition 4.1.1 there exists an $A \subseteq X$ such that $X \times B = A \times X^7$ $(mod\mu^{[3]}$. Conversely, if there exists an $A \subseteq X$ such that $X \times B = A \times X^7$ $(mod\mu^{[3]})$, then clearly B is is $\mathcal{J}_7$-measurable. So we have,

Lemma 4.1.3.
Let X^7 be endowed with the measure μ_7. A set $B \subseteq X^7$ is $\mathcal{J}_7$-measurable if and only if there exists a set $A \subseteq X$ so that

$$X \times B = A \times X^7 \quad (mod\mu^{[3]})$$

Equivalently,

$$B \in \mathcal{J}_7 \quad \Leftrightarrow \quad \mathbb{1}_A(x_{000}) = \mathbb{1}_B(\tilde{\mathbf{x}}), \text{ for } \mu^{[3]}\text{-almost every } \mathbf{x} = (x_{000}, \tilde{\mathbf{x}}) \in X^8 \tag{4.1}$$

Definition 4.1.4.
The Conze-Lesigne σ-algebra, $\mathcal{CL}$, on X contains the sets $A \subseteq X$ so that there exists a set $B \subseteq X^7$ such that $A \times X^7 = B \times X$ mod$\mu^{[3]}$
Equivalently, is the σ-algebra of subsets of X such that the relation (4.1) is satisfied.
A system is called Conze-Lesigne system or system of order 2 when it is isomorphic to its Conze-Lesigne factor.

Remarks 4.1.5.

- By Lemma 4.1.3 and the definition 4.1.4 of the Conze-Lesigne σ-algebra we can identify the σ-algebras $\mathcal{CL} \subseteq \mathcal{X}$ and $\mathcal{J}_7 \subseteq \mathcal{X}^7$ by identifying a subset B of X^7 belonging to $\mathcal{J}_7$ in with the corresponding subset A of X, in $\mathcal{CL}$
- By Lemma 4.1.3, if f is a bounded function on X then, f is $\mathcal{CL}$-measurable if and only if there exists a $\mathcal{J}_7$-measurable function F on X^7 with $f(x_{000}) = F(\tilde{\mathbf{x}})$ for $\mu^{[3]}$-almost every $\mathbf{x} = (x_{000}, \tilde{\mathbf{x}}) \in X^8$

Proposition 4.1.6.
The σ-algebra $\mathcal{CL}$ is invariant under T (thus is a factor of X).

Proof. From the previous Remark (4.1.5) it suffices to prove that $T^7(\mathcal{J}_7) = \mathcal{J}_7$. By the fact that $T \circ T_{7,i} = T_{7,i}$, $\forall i = 1, 2, 3$ we have $T^7(\mathcal{J}_7) = \mathcal{J}_7$. as desired. □

Lemma 4.1.7.
Let f and g be two bounded functions on X and X^7 respectively, then

$$\int_{X^8} f(x_{000})g(\tilde{\boldsymbol{x}})\,\mathrm{d}\mu^{[3]}(\boldsymbol{x}) = \int_{X^8} \mathbb{E}(f \mid \mathcal{CL})(x_{000})\mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\boldsymbol{x}})\,\mathrm{d}\mu^{[3]}(\boldsymbol{x})$$

In other words $(X^{[3]}, \mu^{[3]})$ is the relatively independent joining of (X, μ) and (X^7, μ_7) over $\mathcal{CL}$ when identified with $\mathcal{J}_7$

Proof. Since $\mu^{[3]}$ is $T_{8,1}$ and $T_{8,2}$ and $T_{8,3}$ invariant (Proposition 3.4.2). then for every n_1, n_2, $n_3 \in \mathbb{N}$, we have

$$\int_{X^8} f(x_{000})g(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} f(x_{000})g(T_{7,1}^{n_1}T_{7,2}^{n_2}T_{7,3}^{n_3}(\tilde{\mathbf{x}}))\,\mathrm{d}\mu^{[3]}(\mathbf{x})$$

$\big((f\cdot g)\circ T_{8,i}(x_{000}, \tilde{\mathbf{x}}) = f(x_{000})\cdot(g\circ T_{7,i})(\tilde{\mathbf{x}})\ ,\quad \forall i = 1,2,3$ and the order is not important since $T_{8,i}\circ T_{8,j} = T_{8,j}\circ T_{8,i}\ ,\quad \forall i,j = 1,2,3\ \big)$
Thus we have,

$$\int_{X^8} f(x_{000})g(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x}) = \frac{1}{n_1n_2n_3}\sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2}\sum_{\gamma=0}^{n_3}\int_{X^8} f(x_{000})g(T_{7,1}^{\alpha}T_{7,2}^{\beta}T_{7,3}^{\gamma}(\tilde{\mathbf{x}}))\,\mathrm{d}\mu^{[3]}(\mathbf{x})$$

From Von Neumann's Mean Ergodic Theorem (Theorem 1.5.3),

$$\frac{1}{n_1n_2n_3}\sum_{\alpha=0}^{n_1}\sum_{\beta=0}^{n_2}\sum_{\gamma=0}^{n_3}\int_{X^8} f(x_{000})g(T_{7,1}^{\alpha}T_{7,2}^{\beta}T_{7,3}^{\gamma}(\tilde{\mathbf{x}}))\,\mathrm{d}\mu^{[3]}(\mathbf{x}) \xrightarrow[n_1,n_2,n_3\to\infty]{L^1(\mu^{[3]})} \int_{X^8} f(x_{000})\mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x})$$

Hence,

$$\int_{X^8} f(x_{000})g(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} f(x_{000})\mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} \mathbb{E}(f \mid \mathcal{CL})(x_{000})\mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\mathbf{x}})\,\mathrm{d}\mu^{[3]}(\mathbf{x})$$

□

Lemma 4.1.8.
Let $f \in L^\infty(\mu)$. The following are equivalent.

(i) $\mathbb{E}(f \mid \mathcal{CL}) = 0$

(ii) $\int_{X^8} f(x_{000})g(\tilde{\boldsymbol{x}})\,\mathrm{d}\mu^{[3]} = 0$ *for every bounded function g on X^7*

Proof. Let $f \in L^\infty(\mu)$ and $g \in L^\infty(\mu_7)$.
If such that $\mathbb{E}(f \mid \mathcal{CL}) = 0$, by Lemma 4.1.7 we have that

$$\int_{X^8} f(x_{000}) g(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} \mathbb{E}(f \mid \mathcal{CL})(x_{000}) \mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x})$$

Hence,

$$\int_{X^8} f(x_{000}) g(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} \mathbb{E}(f \mid \mathcal{CL})(x_{000}) \mathbb{E}(g \mid \mathcal{J}_7)(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = 0$$

Conversely, let $f \in L^\infty(\mu)$ such that $\int_{X^8} f(x_{000}) g(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]} = 0$ for every bounded function g on X^7. By Lemma 4.1.7 it suffices to prove it with the additional property that f is $\mathcal{CL}$-measurable. Then there exists a $\mathcal{J}_7$-measurable function F on X^7 such that $f(x_{000}) = F(\tilde{\mathbf{x}}) \quad$ for $\mu^{[3]}$-almost every $\mathbf{x} = (x_{000}, \tilde{\mathbf{x}}) \in X^8$. By hypothesis we have

$$0 = \int_{X^8} f(x_{000}) F(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = \|f\|^2_{L^2(\mu)}$$

□

Proposition 4.1.9.
The measure $\mu^{[3]}$ is relatively independent with respect to its projection on $\mathcal{CL}^8$, meaning that,

$$\int_{X^8} \bigotimes_{\in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]} = \int_{X^8} \bigotimes_{\in V_3} \mathbb{E}(f_\epsilon \mid \mathcal{CL}) \, \mathrm{d}\mu^{[3]}$$

Proof. It suffices to prove that if there exists $\eta \in V_3$ such that $\mathbb{E}(f_\eta \mid \mathcal{CL}) = 0$, then $\int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]} = 0.$
Indeed the statement above is enough, by writing the arbitrary f_ϵ, $\epsilon \in V_3$, as the direct sum of f'_ϵ and f''_ϵ, where $f'_\epsilon = \mathbb{E}(f_\epsilon \mid \mathcal{CL})$ and f''_ϵ the complementary part, on $L^2(\mu)$. That is f_ϵ is $\mathcal{CL}$-measurable and $\mathbb{E}(f''_\epsilon \mid \mathcal{CL}) = 0$. Expanding the the $\int_{X^8} \bigotimes_\epsilon (f'_\epsilon + f''_\epsilon) \, \mathrm{d}\mu^{[3]}$, we get the sum of 16 integrals where $\int_{X^8} \bigotimes_\epsilon f'_\epsilon \, \mathrm{d}\mu^{[3]}$ is the only integral which does not contain any function with conditional expectation with respect to $\mathcal{CL}$ zero.
Thus if that condition holds, then the desired relation holds.
Now let $\eta \in V_3$ such that $\mathbb{E}(f_\eta \mid \mathcal{CL}) = 0$. By Lemma 4.1.8, that is equivalent with, $\int_{X^8} f_\eta(x_{000}) g(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]} = 0$ for every bounded function g on X^7. If $\eta = 000$ then we choose

g to be $\bigotimes_{\epsilon\in V_3,\ \epsilon\neq\eta} f_\epsilon$ and we have $\int_{X^8} f_\eta(x_{000})\big(\bigotimes_{\epsilon\in V_3,\ \epsilon\neq\eta} f_\epsilon\big)(\tilde{\mathbf{x}})\, \mathrm{d}\mu^{[3]} = 0$ as desired. If $\eta \neq \mathbf{0}$ then $\eta = \eta_1\eta_2\eta_3 \in V_3$ where at least one of the η_i, $i \in \{1,2,3\}$ is not zero. Then by using the necessary of the reflections, σ_i^3, $i \in \{1,2,3\}$ so f_η goes to the right position and setting g to be the proper product of the rest of f_ϵ so $f \otimes g = f_\eta \bigotimes \big(\bigotimes_{\epsilon\in V_3,\ \epsilon\neq\eta} f_\epsilon\big)$ when composed with the reflections used above, equals $\bigotimes_{\epsilon\in V_3} f_\epsilon$ and by Proposition 3.3.13 we obtain the desired result. □

Definition 4.1.10.
Let $f \in L^\infty(\mu)$. We define

$$|||f|||_{[3]} = \left(\int_{X^8} f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \otimes f \mathrm{d}\mu^{[3]}\right)^{1/8}$$

Lemma 4.1.11.

(i) *Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3$, then*

$$\left|\int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon \,\mathrm{d}\mu^{[3]}\right| \leq \prod_{\epsilon\in V_3} |||f_\epsilon|||_{[3]}$$

(ii) $|||\cdot|||_{[3]}$ *is a seminorm on* $L^\infty(\mu)$.

Proof. The procedure is exactly the same as in the proof of Lemma 3.3.20. □

Similar to Kronecker factor, the Conze-Lesigne algebra has good behaviour with respect to factor. More precisely, if $q: X \to Y$ is a factor map, then $q^{-1}(\mathcal{CL}(Y)) = \mathcal{CL}(X) \cap q^{-1}(\mathcal{Y})$, where $\mathcal{CL}(X)$, and $\mathcal{CL}(Y)$ are the Conze-Lesigne algebras of X and Y respectively.

Proposition 4.1.12.
If $q: X \to Y$ is a factor map, then $q^{-1}\big(\mathcal{CL}(Y)\big) = \mathcal{CL}(X) \cap q^{-1}\big(\mathcal{Y}\big)$, where $\mathcal{CL}(X)$, and $\mathcal{CL}(Y)$ are the Conze-Lesigne algebras of X and Y respectively

Proof. Let $q^7 : X^7 \to Y^7$ be the natural map. By Lemma 3.2.4 the map q^7 is a factor map. Let f be a bounded function on X that is $q^{-1}\big(\mathcal{CL}(Y)\big)$. Then $f = g \circ q$ for some bounded function g on Y, that is $\mathcal{CL}(Y)$-measurable. Thus there exists a bounded $\mathcal{J}_7$-measurable function G on Y^7, such that $g(y_{000}) = G(\tilde{\mathbf{y}})$, for $\nu^{[3]}$-almost every $\mathbf{y} = (y_{000}, \tilde{\mathbf{y}}) \in Y^8$. Thus $g \circ q(x_{000}) = G \circ q^7(\tilde{\mathbf{x}})$, for $\mu^{[3]}$-almost every $\mathbf{x} = (x_{000}, \tilde{\mathbf{x}}) \in X^8$. In other words f is $\mathcal{CL}(X)$-measurable. Thus $q^{-1}\big(\mathcal{CL}(Y)\big) \subseteq \mathcal{CL}(X) \cap q^{-1}\big(\mathcal{Y}\big)$

Let f be a bounded function on X that is $\mathcal{CL}(X)\cap q^{-1}\big(\mathcal{Y}\big)$-measurable. Then $f = g\circ q$ for some bounde function g on Y. Write $g = g' + g''$, where g' is $\mathcal{CL}(Y)$-measurable and $\mathbb{E}\big(g'' \mid \mathcal{CL}(Y)\big) = 0$. By the first part of this proof, we have that $g' \circ q$ is $\mathcal{CL}(X)$ measurable. Now, since $\mathbb{E}\big(g'' \mid \mathcal{CL}(Y)\big) = 0$, $|||g''|||_{Y^8} = 0$ and thus $|||g'' \circ q|||_{X^8} = 0$, follows that $\mathbb{E}\big(g'' \circ q \mid \mathcal{CL}(X)\big) = 0$. Since $f = g' \circ q + g'' \circ q$ is $\mathcal{CL}(X)$-measurable, we have that $q'' \circ q = 0$. Hence $g'' = 0$ and thus g is $\mathcal{CL}(Y)$-measurable. □

Remark 4.1.13.

This means that the Conze-Lesigne factor of any ergodic system is a Conze-Lesigne system. Furthermore the Conze-Lesigne factor is the largest Comze-Lesigne system under any ergodic system.

Proposition 4.1.14.

Let $f \in L^\infty(\mu)$ *then,*

$$\mathbb{E}(f \mid \mathcal{CL}) = 0 \quad \Leftrightarrow \quad |||f|||_{[3]} = 0$$

Proof. If $\mathbb{E}(f \mid \mathcal{CL}) = 0$, by Lemma 4.1.8 $\int\limits_{X^8} f(x_{000})g(\tilde{\mathbf{x}})\ \mathrm{d}\mu^{[3]} = 0$ for every bounded function g on X^7.By setting $g(\tilde{\mathbf{x}}) = \prod\limits_{\epsilon\in V_3^*} f(x_\epsilon)$ we have that

$$|||f|||_{[3]}^8 = \int\limits_{X^8} f\otimes f\otimes\cdots\otimes f\ \mathrm{d}\mu^{[3]} = 0$$

Conversely, let $|||f|||_{[3]} = 0$, then by Lemma 4.1.11, if $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3^*$

$$\int\limits_{X^8} f \otimes \bigotimes_{\epsilon\in V_3^*} f_\epsilon\ \mathrm{d}\mu^{[3]} = 0$$

By density we have that $\int\limits_{X^8} f(x_{000})g(\tilde{\mathbf{x}})\ \mathrm{d}\mu^{[3]} = 0$ for every bounded function g on X^7 and by Lemma 4.1.8, $\mathbb{E}(f \mid \mathcal{CL}) = 0$. □

Remark 4.1.15.

By the definition of the measure $\mu^{[3]}$ and of the seminorm $|||\cdot|||_{[3]}$

$$|||f|||_{[3]} = 0 \quad \Leftrightarrow \quad E(f\otimes f\otimes f\otimes f \mid \mathcal{I}^4) = 0\ ,\ \mu^{[2]} - a.e.$$

Lemma 4.1.16.

Let $f \in L^\infty(\mu)$*. Then* $|||f|||_{[2]} \le |||f|||_{[3]}$.

Proof. By Cauchy-Schwartz inequality,

$$|||f|||_{[3]}^8 = \|\mathbb{E}(\bigotimes_{\eta\in V_2} f \mid \mathcal{I}^{[2]})\|^2_{L^2(\mu^{[2]})} \geq \Big(\int_{X^4} \bigotimes_{\eta\in V_2} f \, \mathrm{d}\mu^{[2]}\Big)^2 = |||f|||_{[2]}^8$$

□

Corollary 4.1.17.
For the factors $\mathcal{K}, \mathcal{CL}$ the following holds $\mathcal{K} \subseteq \mathcal{CL}$.

Proof. It suffices to prove that for every bounded function f on X with, $\mathbb{E}(f \mid \mathcal{CL}) = 0$, then $\mathbb{E}(f \mid \mathcal{K}) = 0$. Indeed, assume that the statement holds and let g is a $\mathcal{K}$-measurable function. Write $g = g' + g''$, where g' is $\mathcal{CL}$-measurable and $\mathbb{E}(g'' \mid \mathcal{CL}) = 0$. By the statement, follows that $\mathbb{E}(g'' \mid \mathcal{K}) = 0$. Thus $g = g'$.
Now we have,

$$\mathbb{E}(f \mid \mathcal{CL}) = 0 \;\Leftrightarrow\; |||f|||_{[3]} = 0 \;\Rightarrow\; |||f|||_{[2]} = 0 \;\Leftrightarrow\; \mathbb{E}(f \mid \mathcal{K}) = 0$$

□

Proposition 4.1.18.
$\mathcal{CL}$ is the <u>smallest</u> sub-σ-algebra of $\mathcal{X}$ so that $\mathcal{I}^{[2]} \subseteq \mathcal{CL} \otimes \mathcal{CL} \otimes \mathcal{CL} \otimes \mathcal{CL} = \mathcal{CL}^4$.

Proof. To prove that $\mathcal{I}^{[2]} \subseteq \mathcal{CL}^4$ it suffices to show that if $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_2$ and $\exists$ $\eta \in V_2$ so that $\mathbb{E}(f_\eta \mid \mathcal{CL}) = 0$, then $\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}) = 0$

$$\int_{X^4} |\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]})|^2 \, \mathrm{d}\mu^{[2]} = \int_{X^8} \bigotimes_{\epsilon\in V_2} f_\epsilon(x_{0\epsilon}) \bigotimes_{\epsilon\in V_2} f_\epsilon(x_{1\epsilon}) \, \mathrm{d}\mu^{[3]} \leq \prod_\epsilon |||f_\epsilon|||_{[3]} = 0$$

Hence $\mathbb{E}(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}) = 0$ as desired.

Now let $\mathcal{W}$ be a factor of X such that $\mathcal{I}^{[2]} \subseteq \mathcal{W}^4$ and let $f \in L^\infty(\mu)$, so that $\mathbb{E}(f \mid \mathcal{W}) = 0$. By the projection $(x_{00}, x_{01}, x_{10}, x_{11}) \mapsto (x_{00}, x_{01}) : X^4 \to X^2$, we have that $\mathcal{I}^{[1]} \subseteq \mathcal{W}^2$. By minimality, $\mathcal{K} \subseteq \mathcal{W}$. Now since $\mathcal{K} \subseteq \mathcal{W}$ and $\int_{X^4} \bigotimes_{\epsilon\in V_2} f_\epsilon \, \mathrm{d}\mu^{[2]} = \int_{X^4} \bigotimes_{\epsilon\in V_2} \mathbb{E}(f_\epsilon \mid \mathcal{K}) \, \mathrm{d}\mu^{[2]}$, we obtain that,

$$\int_{X^4} \bigotimes_{\epsilon\in V_2} f_\epsilon \, \mathrm{d}\mu^{[2]} = \int_{X^4} \bigotimes_{\epsilon\in V_2} \mathbb{E}(f_\epsilon \mid \mathcal{W}) \, \mathrm{d}\mu^{[2]}$$

Thus $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{W}^4\big) = \bigotimes_{\epsilon\in V_2} \mathbb{E}(f_\epsilon \mid \mathcal{W})$, $\mu^{[2]}$-almost everywhere. Applying this equality for $f_\epsilon = f$, for all $\epsilon \in V_2$, we have that $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{W}^4\big) = 0$ and by hypothesis follows that $\mathbb{E}\big(\bigotimes_{\epsilon\in V_2} f_\epsilon \mid \mathcal{I}^{[2]}\big) = 0$. Equivalently, $|||f|||_{[3]} = 0$ and thus $\mathbb{E}(f \mid \mathcal{CL}) = 0$. Therefore $\mathcal{CL} \subseteq \mathcal{W}$. □

4.2 Group Extensions

Lemma 4.2.1.
Let (X, μ, T) be a Conze-Lesigne system and (Z, m, R) its Kronecker factor. Then (X, μ, T) is an isometric extension of (Z, m, R).

Proof. Let $(W, \mathcal{W}, \lambda, S)$ denote the maximal isometric extension of Z below X. We will show that $W = X$. Since W is a factor of X, it suffices to show that for every $f \in L^\infty(\mu)$, with $\mathbb{E}(f\,|\mathcal{W}) = 0$, then $f = 0$, μ-a.e.
Let $f \in L^\infty(\mu)$, with $\mathbb{E}(f\,|\mathcal{W}) = 0$. Define $F(\mathbf{x}) = F(x_{00}, x_{01}, x_{10}, x_11) = \prod_{\epsilon\in V_2} f(x_\epsilon)$, $\forall \mathbf{x} \in X^4$. Considering $(X^{[2]}, \mu^{[2]}, T^{[2]})$ as the relatively independent joining of (X, μ, T) and (X^3, μ^3) over the common factor $\mathcal{Z} \equiv \mathcal{J}_3$, then by Theorem 1.10.13, $\mathcal{I}^{[2]} \subseteq \mathcal{W}\otimes\mathcal{X}^3$. We have that

$$\mathbb{E}(F \mid \mathcal{W}\otimes\mathcal{X}^3)(\mathbf{x}) = \mathbb{E}(f \mid \mathcal{W})(x_{00}) \prod_{\epsilon\in V_2^*} f(x_\epsilon) = 0$$

for $\mu^{[3]}$-almost every $\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^4$. It suffices to show that for every $A \times B \in \mathcal{W}\otimes\mathcal{X}^3$,

$$\int_{A\times B} \mathbb{E}(F \mid \mathcal{W}\otimes\mathcal{X}^3)(\mathbf{x})\, \mathrm{d}\mu^{[2]} = \int_{A\times B} \mathbb{E}(f \mid \mathcal{W})(x_{00}) \prod_{\epsilon\in V_2^*} f(x_\epsilon)\, \mathrm{d}\mu^{[2]}$$

which clearly holds. Now since $\mathbb{E}(F \mid \mathcal{W}\otimes\mathcal{X}^3) = 0$ and $\mathcal{I}^{[2]} \subseteq \mathcal{W}\otimes\mathcal{X}^3$, follows that $\mathbb{E}(F \mid \mathcal{I}^{[2]}) = 0$. This means $|||f|||_{[3]} = 0$. Equivalently, $\mathbb{E}(f \mid \mathcal{CL}) = 0$. Since (X, μ, T) is a Conze-Lesigne system, this means that $f = 0$. □

Let (Y, ν, S) be system, $(X, \mu, T) = (Y \times G/H, m \otimes m_{G/H}, R_\rho)$ be an isometric extension of Y and α an edge of V_2. Let $g \in G$, then $V_g : X \to X$, with $V_g(y, q) = (y, g\cdot q)$. Difine $g^{(\alpha)} \in G^4$, where

$$(g^{(\alpha)})_\epsilon = \begin{cases} \mathrm{g}, & if\ \epsilon \in \alpha \\ \mathrm{e}_G, & otherwise \end{cases}$$

Now define, $V_g^{(\alpha)} : X^4 \to X^4$ with $V_g^{(\alpha)} = (\tilde{y}, \tilde{q}) = (\tilde{y},\ g^{(\alpha)} \cdot \tilde{q})$, where

$$(g^{(\alpha)} \cdot \tilde{q})_\epsilon = \begin{cases} \mathrm{g}_\epsilon \cdot q_\epsilon, & if\ \epsilon \in \alpha \\ \mathrm{q}_\epsilon, & otherwise \end{cases}$$

Lemma 4.2.2.
Let (Y, ν, S) be system, $(X, \mu, T) = (Z \times G/H, m \otimes m_{G/H}, S_\rho)$ be an isometric extension of Y and α an edge of V_2. Then every $A \in \mathcal{I}^{[2]}(T)$ is $V_g^{(\alpha)}$-invariant.

Proof. First, let $g \in G$, $\eta \in V_1$ and $\mu \otimes \mu = \int\limits_\Omega \mu_\omega\ \mathrm{d}P(\omega)$ the ergodic decomposition of $\mu \otimes \mu$. Then by Lemma 3.2.3, $\mu^{[2]} = \int\limits_\Omega \mu_\omega \otimes \mu_\omega\ \mathrm{d}P(\omega)$. Define $J_4^\omega = \{B \in \mathcal{X}^4 : \mu_\omega \otimes \mu_\omega \big(B \triangle (T^{[2]})^{-1} B\big) = 0\}$, the invariant σ-algebra of the system $(X^{[2]}, \mu_\omega \otimes \mu_\omega, T^{[2]})$.

Let $A \in \mathcal{I}^{[2]}$. Then $\mu^{[2]}\big(A \triangle (T^{[2]})^{-1} A\big) = 0$. Follows that $\mu_\omega \otimes \mu_\omega \big(A \triangle (T^{[2]})^{-1} A\big) = 0$, for P-almost every $\omega \in \Omega$. This means that $A \in J_4^\omega$ for P-almost every $\omega \in \Omega$. In other words $\mathcal{I}^{[2]} \subseteq J_4^\omega$, dor P-almost every $\omega \in \Omega$.

Since every μ_ω is ergodic on X^2, the system $(X^{[2]}, \mu_\omega \otimes \mu_\omega, T^{[2]})$ is relatively independent joining of (X^2, μ_ω, T^2) over the trivial factor. Furthermore (X^2, μ_ω, T^2) is isometric extension of (Y^2, ν_ω, S^2) by $G^2/H^2 \simeq \big(G/H\big)^2$.

By Lemma $A.5$, for every $(g, g') \in G^2$ the map,

$$\big((x_{(0)0}, x_{(0)1}),\ (x_{(1)0}, x_{(1)1})\big) \mapsto \big((g, g') \cdot (x_{(0)0}, x_{(0)1}),\ (g, g') \cdot (x_{(1)0}, x_{(1)1})\big) = \big((g \cdot x_{(0)0}, g' \cdot x_{(0)1}),\ (g \cdot x_{(1)0}, g' \cdot x_{(1)1})\big)$$

leaves each set of $\mathcal{I}(X^4, \mu_\omega \otimes \mu_\omega, T^4) = J_4^\omega$, invariant. Follows that $V_g^{(\eta)} \times V_g^{(\eta)}$ leaves every set of $\mathcal{J}_4^\omega$, modulo $\mu_\omega \otimes \mu_\omega$. Follows that $V_g^{(\eta)} \times V_g^{(\eta)}$ leaves every set of $\mathcal{I}^{[2]}$, modulo $\mu_\omega \otimes \mu_\omega$, for P-almost every $\omega \in \Omega$. Thus $V_g^{(\eta)} \times V_g^{(\eta)}$ leaves every set of $\mathcal{I}^{[2]}$, modulo $\mu^{[2]}$.

Now we have that $V_g^{(\eta)} \times V_g^{(\eta)} = V_g^{(\alpha)}$, where α is an edge of V_2. Thus $V_g^{(\alpha)}$ leaves every set of $\mathcal{I}^{[2]}$, modulo $\mu^{[2]}$, for an edge α of V_2. Since with digit permutations and reflections we can obtain from every edge of V_2, any other edge of V_2 and $\mu^{[2]}$ is invariant under this transformations, we have that, for every edge β of V_2, V_g^β leaves every set of $\mathcal{I}^{[2]}$, modulo $\mu^{[2]}$. $\square$

Proposition 4.2.3.
Let (X, μ, T) be a Conze-Lesigne system and (Z, m, R) its Kronecker factor. Then (X, μ, T) is an extension of (Z, m, R) by a compact abelian group U.

Proof. By Lemma 4.2.1 we have that (X, μ, T) is an isometric extension of (Z, m, R). This means there exist a compact, metrizable group G, a closed subgroup of G, H and a cocycle $\rho : Z \to G$, susch that

$$(X, \mu, T) = (Z \times G/H, m \otimes m_{G/H}, R_\rho)$$

where $R_\rho(z, y) = (R(z), \rho(z) \cdot y)$, for every $z \in Z$ and every $y \in G/H$. We will show that G is abelian, thus G/H is a compact abelian group as desired.

Let $\epsilon \in V_2$ and $g, g' \in G$. Let α, β be two faces of V_2 such that $\alpha \cap \beta = \{\epsilon\}$. In particular α, β can be chosen to be edges of V_2. Then by Lemma 4.2.2 for every $A \in \mathcal{I}^{[2]}$ is $V_g^{(\alpha)}$, $V_{g'}^{(\alpha)}$ invariant. By some relatively simple computations we have that $u = [g^{(\alpha)}, g'^{(\beta)}] = [g, g']^{\alpha \cap \beta} = [g, g']^{(\{\epsilon\})}$, thus $V_g^{(\{\epsilon\})}$, leaves every set of $\mathcal{I}^{[2]}$, invariant. Since every generator of $[G, G]$ can be obtained this way, we have that for every $u \in [G, G]$, $V_u^{(\{\epsilon\})}$ leaves each set of $\mathcal{I}^{[2]}$ invariant and thus we clearly have the same result for every $u \in \overline{[G, G]}$. Since this holds for every $\epsilon \in V_2$ we obtain that $\mathcal{I}^{[2]}$ is contained to $\mathcal{W}^{[2]}$, where W is the quotient of $X = Y \times G/H$ under the action of $\overline{[G, G]}$.

By Remark 4.1.13 and Proposition 4.1.18 and since X is a Conze-Lesigne system, we have that $W = X$. Thus the action of $\overline{[G, G]}$ on X is trivial and sice the action of G on X is faithful, we have that $\overline{[G, G]} = \{e_G\}$. In particular $[G, G] = \{e_G\}$. Thus G is abelian. $\square$

Definition 4.2.4.
Let (Y, ν, S) be a system and G a compact abelian group.

(i) For a function $F : Y^{[k]} \to G$, define $\partial^{[k]} F : Y^{[k]} \to K$, with $\partial^{[k]} F(\mathbf{y}) = F \circ T^{[k]}(\mathbf{y}) F(y)^{-1}$, for every $\mathbf{y} \in Y^{[k]}$.

(ii) For a cocycle $\rho : Y \to K$, define $\Delta^{[k]} \rho : Y^{[k]} \to K$, with $\Delta^{[k]} \rho(\mathbf{y}) = \sum\limits_{\epsilon \in V_k} \rho(y_\epsilon)^{(-1)^{|\epsilon|}}$, for every $\mathbf{y} \in Y^{[k]}$, where $|\epsilon| = \epsilon_1 + \ldots + \epsilon_k$.

(iii) A cocyle $\rho : Y \to K$ is said to be a **cocycle of type k**, if there exist an $F : Y^{[k]} \to K$, such that $\Delta^{[k]} \rho = \partial^{[k]} F$, $\nu^{[k]}$-a.e. . In other words $\Delta^{[k]} \rho$ is a coboundary of $Y^{[k]}$.

Lemma 4.2.5.
Let (Y, ν, S) be an ergodic system. K a compact abelian group and $\rho : Y \to K$ a cocycle. Then ρ is a coboundary if and only if for every $\chi \in \widehat{K}$, $\chi \circ \rho : Y \to \mathbb{T}$ is a coboundary.

Proof. If ρ is o coboundary then clearly for for every $\chi \in \widehat{K}$, $\chi \circ \rho : Y \to \mathbb{T}$ is a coboundary.

Conversely, assume that $\rho : Y \to K$ is a cocycle and for every $\chi \in \widehat{K}$, $\chi \circ \rho : Y \to \mathbb{T}$ is a coboundary. Thus for each character χ there exists a $f_\chi : Y \to \mathbb{T}$ such that,

$$\chi\big(\rho(y)\big) = f\big(S(y)\big) \cdot \overline{f}(y) \tag{4.2}$$

for ν-almost every $y \in Y$. Now, if $\chi, \chi' \in \widehat{K}$, the function $f_{\chi\chi'}\overline{f}_\chi\overline{f}_{\chi'}$ is S-invariant and, by ergodicity, is a constant almost everywhere. Thus there exists a constant $c(\chi, \chi') \in \mathbb{T}$, such that,

$$f_{\chi\chi'}(y) = c(\chi, \chi') f_\chi(y) f_{\chi'}(y) \tag{4.3}$$

for ν-almost every $y \in Y$.

Since K is compact, $\widehat{K}$ is countable. Thus there exists a $Y_1 \in \mathcal{Y}$, with $S(Y_1) \subseteq Y_1$, $\nu(Y_1) = 1$ and relations (4.2) and (4.3) hold for every $y \in Y_1$. Picking a $y_1 \in Y_1$ we replace every f_χ by $\overline{f_\chi(y_1)} f_\chi$. Then equality (4.2) still holds for every $y \in Y_1$ and for every $\chi, \chi' \in \widehat{K}$, relations (4.3) becomes,

$$f_{\chi\chi'}(y) = f_\chi(y) f_{\chi'}(y)$$

for every $y \in Y_1$.
Hence there exists a function $F : Y_1 \to K$ such that for every $y \in Y_1$, $f_\chi(y) = \chi\big(F(y)\big)$, for every $\chi \in \widehat{K}$. Indeed, $\forall y \in Y_1$ we consider $\Phi_y : \widehat{K} \to \mathbb{T}$, with $\Phi(y)(\chi) = f_\chi(y)$, for every $\chi \in \widehat{K}$. Then $\Phi \in \widehat{\widehat{K}} \cong K$ and thus considering each Φ_y as an element of K, $\chi(\Phi_y) = \Phi_y(\chi)$. By setting $F : Y_1 \to K$, with $F(y) = \Phi_y$, we have that $\chi\big(F(y)\big) = \chi(\Phi_y) = \Phi_y(\chi) = f_\chi(y)$.
Extending F on Y, by (4.2), we have that for ν-almost every $y \in Y$, $\chi\big(F(Sy) \cdot F(y)^{-1}\big) = \chi\big(\rho(y)\big)$, and this holds for every $\chi \in \widehat{K}$. Thus $F(S(y)) \cdot F(y)^{-1} = \rho(y)$, for ν-almost every $y \in Y$. Thus ρ is a coboundary, as desired. $\square$

Proposition 4.2.6.
Let (Y, ν, S) be an ergodic system, K a compact abelian group, $\rho \in Coc(Y, K)$ an ergodic cocycle. Le (X, μ, T) be the extension of Y, by K, associated to ρ and $\pi : X \to Y$ the corresponding factor map. Then, if X is a type 2 system the ρ is a cocycle of type 2.

Proof. We have $X = Y \times K$ and $X^4 = Y^4 \times K^4$. Let $\chi : K \to \mathbb{T}$ be a character of K. Define $\psi : Y \to \mathbb{T}$, by $\psi(y,k) = \chi(k)$ $\Phi : X^4 \to \mathbb{T}$, by

$$\begin{aligned}\Psi(\mathbf{y},\mathbf{k}) = \prod_{\epsilon\in V_2} \chi(k_\epsilon)^{(-1)^{|\epsilon|}} &= \chi(k_{00})\,\chi(k_{01})^{-1}\,\chi(k_{10})^{-1}\,\chi(k_{11}) \\ &= \chi(k_{00})\,\overline{\chi(k_{01})}\,\overline{\chi(k_{10})}\,\chi(k_{11})\end{aligned}$$

Then for every $(\mathbf{y},\mathbf{k}) \in Y^4 \times K^4$, $\Psi \circ T^4(\mathbf{y},\mathbf{k}) = \Psi(\mathbf{y},\mathbf{k}) \cdot \chi\big(\Delta^{[2]}\rho(\mathbf{y})\big)$. In other words

$$\Psi \circ T^4 = \Psi \cdot \chi\big(\Delta^{[2]}\rho\big) \circ \pi^4 \tag{4.4}$$

Now by definition of ψ and because X is a Conze-Lesigne system we have that $0 \neq \psi = \mathbb{E}(\psi \mid \mathcal{CL})$. Thus $|||\psi|||_{[3]} \neq 0$. By definition of the simenorm and $\mu^{[3]}$,

$$||\mathbb{E}(\Psi \mid \mathcal{I}^{[2]})||_{L^2(\mu^{[2]})} = |||\psi|||_{[3]}^8 \neq 0$$

Let $J : L^2(\nu^{[2]}) \to L^2(\mu^{[2]})$ be the linear map defined by $J(f) = \Psi \cdot f \circ \pi^4$. Since $\mu^{[2]} = \nu^{[2]} \otimes m_K^4$, J is an isometry and its range is an closed subspace of $L^2(\mu^{[2]})$. By relation (4.4) for an $f \in L^2(\nu^{[2]})$ we have,

$$\begin{aligned} J(f)\circ T^4 &= (\Psi\circ T^4)\cdot(f\circ\pi^4\circ T^4) = (\Psi\circ T^4)\cdot(f\circ S^4\circ\pi^4) \\ &= \Psi\cdot\chi\big(\Delta^{[2]}\rho\big)\circ\pi^4\cdot(f\circ S^4)\circ\pi^4 = \Psi\cdot\Big(\chi\big(\Delta^{[2]}\rho\big)\cdot(f\circ S^4)\Big)\circ\pi^4 \\ &= J\Big(\chi\big(\Delta^{[2]}\rho\big)\cdot(f\circ S^4)\Big)\end{aligned}$$

Therefore the range of J is T^4-invariant. By Theorem 1.5.3 and since $J\big(L^2(\nu^{[2]})\big)$ is a closed and T^4-invariant subspace of $L^2(\mu^{[2]})$, we have that the range of J is also invariant under taking conditional expectation with respect to $\mathcal{I}^{[2]}(X)$. Since the function $\Psi = J(1)$, we have that $\mathbb{E}\big(\Psi \mid \mathcal{I}^{[2]}(X)\big)$ is also contained in $J\big(L^2(\nu^{[2]})\big)$. This means that there exists a non identically zero function $f \in L^2(\nu^{[2]})$ with $J(f) = \mathbb{E}\big(\Psi \mid \mathcal{I}^{[2]}(X)\big)$. Because $J(f)\circ T^4 = J\Big(\chi\big(\Delta^{[2]}\rho\big)\cdot(f\circ S^4)\Big)$ and J is an isometry we have that for $\nu^{[2]}$-almost everywhere,

$$\big(\chi(\Delta^{[2]}\rho)\big)\cdot f\circ S^4 = f \tag{4.5}$$

In particular, since $|\big(\chi(\Delta^{[2]}\rho)\big)\cdot f\circ S^4| = |f\circ S^4| = |f|\circ S^4$, $|f|$ is S^4-invariant. Replacing f by $\mathbb{1}_{\{\mathbf{y}\in Y^4:\ f(\mathbf{y})\neq 0\}}\frac{f}{|f|}$, we have that (4.5) still holds and thus we can assume that $|f| = 0$

or 1, $\nu^{[2]}$-almost everywhere. Let $A = \{\mathbf{y} \in Y^4 : |f(\mathbf{y})| = 1\}$. Then A is S^4-invariant and $\nu^{[2]}(A) > 0$.

Now we have that $(Y^{[2]}, \nu^{[2]})$ is ergodic under the joint action of S^4, $S_{4,1}$, $S_{4,2}$. Since A is S^4-invariant and has a positive measure, by Proposition 1.2.4, there exist a countable family of T^4-invariant subsets, A_n of A and a family of transformations on Y^4, Q_n, where each Q_n is of the form $T_{4,1}^{i_n} \circ T_{4,2}^{j_n}$, where $i_n, j_n, \in \mathbb{N} \cup \{0\}$, such that the sets $B_n = Q_n^{-1}(A_n)$, forms a partition of Y^4 into S^4 invariant subsets.

Note that each Q_n commutes with S^4. By the construction of B_n and relation (4.5) we have that on each B_n, $f \circ Q_n$ takes values on $\mathbb{T}$ and

$$\big(\chi(\Delta^{[2]}\rho \circ Q_n)\big) \cdot f \circ Q_n \circ S^4 = f \circ Q_n \;\Leftrightarrow\; (f \circ Q_n \circ S^4) \cdot \overline{f \circ Q_n} = \overline{\chi(\Delta^{[2]}\rho \circ Q_n)}$$

Now since $Q_n = T_{4,1}^{i_n} \circ T_{4,2}^{j_n}$, for each n, for every $\epsilon \in V_2$ there exists an integer $m = m(n, \epsilon)$, such that

$$\big(Q_n(\mathbf{y})\big)_\epsilon = S^{m(n,\epsilon)}(y_\epsilon) \qquad \forall\, \mathbf{y} \in Y^4$$

Thus for every $n \in \mathbb{N}$,

$$\begin{aligned}\Delta^{[2]}\rho \circ Q_n(\mathbf{y}) \cdot \Delta^{[2]}\rho(\mathbf{y})^{-1} &= \sum_{\epsilon \in V_2} \Big(\rho\big(((Q_n(\mathbf{y}))_\epsilon\big)\rho(y_\epsilon)^{-1}\Big) \\ &= \sum_{\epsilon \in V_2} \Big(\rho\big(S^{m(n,\epsilon)}(y_\epsilon)\big)\rho(y_\epsilon)^{-1})\Big)\end{aligned}$$

For every $n \in \mathbb{N}$, define $F_n : Y^4 \to K$, by

$$F_n(\mathbf{y}) = \sum_{\substack{\epsilon \in V_2 \\ m(n,\epsilon) \neq 0}} \sum_{i=0}^{m(n,\epsilon)} \rho\big(S^{i-1}(y_\epsilon)\big) \qquad \forall\, \mathbf{y} \in Y^4$$

Then

$$\Delta^{[2]}\rho \circ Q_n(\mathbf{y}) \cdot \Delta^{[2]}\rho(\mathbf{y})^{-1} = \partial F(\mathbf{y}) = F \circ S^4(\mathbf{y}) \cdot F(\mathbf{y})^{-1} \qquad \forall\, \mathbf{y} \in Y^4 \tag{4.6}$$

In other words the map

$$\mathbf{y} \mapsto \Delta^{[2]}\rho \circ Q_n(\mathbf{y}) \cdot \Delta^{[2]}\rho(\mathbf{y})^{-1} \;:\; Y^4 \to K$$

is a coboundary of the system $(Y^4, \nu^{[2]}, S^4)$.

By setting, for each n, $\phi_n = \overline{\chi} \circ F_n : Y^4 \to \mathbb{T}$ and by equality (4.6), we have,

$$\overline{\chi}\big(\Delta^{[2]}\rho \circ Q_n\big) \cdot \chi\big(\Delta^{[2]}\rho\big) = \phi_n \circ S^4 \cdot \overline{\phi_n} \tag{4.7}$$

For each n, define the function $\psi_n = \overline{f} \circ Q_n \cdot \phi_n : B_n \to \mathbb{T}$. By (4.6) and (4.7) we have that $\chi\big(\Delta^{[2]}\rho\big) = \psi_n \circ S^4 \cdot \overline{\psi}$, on B_n. Since the sets B_n is S^4-invariant and form a partition of Y^4, we can define the function $G : Y^4 \to \mathbb{T}$ by $G = \sum\limits_{n \in \mathbb{N}} \mathbb{1}_{B_n}\psi_n$ and it satisfies the property

$$G \circ S^4 \cdot \overline{G} = \chi\big(\Delta^{[2]}\rho\big)$$

Therefore for every $\chi \in \widehat{K}$, the cocycle $\chi\big(\Delta^{[2]}\rho\big)$ is a coboundary of Y^4. By Lemma 4.2.5 , $\Delta^{[2]}\rho$ is a coboundary of Y^4, an hence ρ is o cocycle of type 2. □

By summarizing the previous results we have the following Theorem.

Theorem 4.2.7.
Let (X, μ, T) be a Conze-Lesigne system and let (K, m, R) be its Kronecker factor. Then X is an extension of K by a compact abelian group U and the cocycle $\rho : K \to U$ that defines this extension, is a cocycle of type 2.

Lemma 4.2.8.
Let (Z, m, R) be an ergodic rotation, U be a finite dimensional torus and $\rho : Z \to U$ a cocycle of type 2. Then for every $s \in Z$, there exist $f : Z \to U$ and $c \in U$ such that

$$\rho(sx)\rho(x)^{-1} = f(Rx)f(x)^{-1}c$$

*This equation is called **Conze-Lesigne Equation**.*

Proof. Let $s \in Z$.
The cocycle, σ, defined by $x \mapsto \rho(sx)\rho(x)$ is a cocyle of type 1. Indeed, since ρ is a cocycle of type 2 thre exists an $F : Z^4 \to U$, such that

$$\Delta^{[2]}(\mathbf{x}) = \rho(x_{00})\rho(x_{01})^{-1}\rho(x_{10})^{-1}\rho(x_{11}) = F \circ T^4(\mathbf{x})F(\mathbf{x})^{-1}$$

for every $\mathbf{x} \in Z^4$ Now

$$\begin{aligned}\Delta^{[1]}\sigma(\mathbf{x}) = \sigma(x_0)\sigma(x_1) &= \rho(sx_0)\rho(x_0)^{-1}\rho(sx_1)^{-1}\rho(x_1) \\ &= F \circ T^4(sx_0, x_0, sx_1, sx_0)F(sx_0, x_0, sx_1, sx_0)^{-1}\end{aligned}$$

for every $\mathbf{x} \in Z^2$. Define $G : Z^2 \to U$, by setting $G(x_0, x_1) = F(sx_0, x_0, sx_1, sx_1)$. Then we have that,

$$\Delta^{[1]}\sigma(\mathbf{x}) = G \circ T^2(\mathbf{x})G(\mathbf{x})^{-1} \qquad \forall\, \mathbf{x} \in Z^2$$

In other words, the cocyle $\sigma : Z \to U$, is a cocycle of type 1. By Lemma 1.10.16 and since U is a torus, ρ is a quasi-coboundary. This means that there exist a function $f : Z \to U$ and a constant $c \in U$, sucht that. for every $x \in Z$,

$$\sigma(x) = f(Rx)f(x)^{-1}c \quad \Longleftrightarrow \quad \rho(sx)\rho(x)^{-1} = f(Rx)f(x)^{-1}c$$

□

Lemma 4.2.9.
Let (Y, ν, S) and K a compact abelian group. A cocycle cohomologous to a cocycle of type k is itself a cocycle of type k

Proof. For this proof we will use the additive notation, due to convenience.
Let ρ, $\rho' \in Coc(Y, K)$ two cocyles, such that ρ is cohomologous to ρ'. This means that there exists an $f : Y \to K$ such that

$$\rho(y) - \rho(y) = f \circ S(y) - f(y)$$

Assume that ρ is a type k cocycle. This means that there exists an $F : Y^{[k]} \to K$ such that

$$\Delta^{[k]}\rho(\mathbf{y}) = F \circ S^{[k]}(\mathbf{y}) - F(\mathbf{y})$$

Since K is abelian we have that

$$\begin{aligned}
\Delta^{[k]}\rho'(\mathbf{y}) &= \Delta^{[k]}\rho(\mathbf{y}) - \Delta^{[k]}(\rho - \rho')(\mathbf{y}) \\
&= F \circ S^{[k]}(\mathbf{y}) - F(\mathbf{y}) - \sum_{\epsilon \in V_k} \left(|-1|^{|\epsilon|} f \circ S(y_\epsilon) - |-1|^{|\epsilon|} f(y_\epsilon)\right) \\
&= \left(F \circ S^{[k]}(\mathbf{y}) + \sum_{\epsilon \in V_k} |-1|^{|\epsilon|} f \circ S(y_\epsilon)\right) - \left(F(\mathbf{y}) + \sum_{\epsilon \in V_k} |-1|^{|\epsilon|} f(y_\epsilon)\right)
\end{aligned}$$

Define $G : Y^{[k]} \to K$, by $G(\mathbf{y}) = F(\mathbf{y}) + \sum_{\epsilon \in V_k} |-1|^{|\epsilon|} f(y_\epsilon)$, for every $\mathbf{y} \in Y^{[k]}$. Then $\Delta^{[k]}\rho' = \partial G$. Thus ρ' is also a cocycle of type k. □

Remark 4.2.10.
From the previous Lemma we have that if a cocycle is a quasi-coboundary, equivalently is cohomologous to a constant, then is a cocycle of type k, for every $k \in \mathbb{N}$.

Lemma 4.2.11.
Let (Z, m, R), where $R(x) = \alpha x$, be an ergodic rotation, and $\rho : Z \to \mathbb{T}$ a cocycle of type 2. Assume that there exist a $n \in \mathbb{N}$ such that the cocycle $\rho_n : Z \to \mathbb{T}$, defined by $\rho_n(z) = \rho(z)^n$, is a quasi-coboundary of Z. Then ρ is also a quasi-coboundary.

Proof. Let s, f, c be such that they satisfy the Conze-Lesigne Equation. This means that

$$\rho(sx)\rho(x)^{-1} = f(Rx)f(x)^{-1}c \Leftrightarrow \rho(sx)\overline{\rho(x)} = f(Rx)\overline{f(x)}c$$

Since ρ_n is a quasi-coboundary the cocycle, there exist an $g : Y \to \mathbb{T}$ and a constant $c' \in \mathbb{T}$, such that

$$\rho_n(x) = g \circ R(x)g(x)^{-1}c'$$

Define $\sigma_s : Z \to \mathbb{T}$ by $\sigma_s(z) = \rho_n(sz)\rho_n(z)^{-1} = \rho_n(sz)\overline{\rho_n(z)}$. Then

$$\begin{aligned}\sigma_s(x) = \rho_n(sz)\rho_n(z)^{-1} &= g \circ R(sx)g(sx)^{-1}c' \; g \circ R(x)^{-1}g(x)c'^{-1} \\ &= g \circ R(sx)g(sx)^{-1} \;\; g \circ R(x)^{-1}g(x)\end{aligned}$$

By setting $G : Z \to \mathbb{T}$, to be $G(x) = g(sx)g(x)^{-1}$, we have that $\sigma_s(x) = \partial G(x) = G(Rx)G(x)^{-1} = G(Rx)\overline{G(x)}$. In other words σ_s is a coboundary. Substituting in the Conze-Lesigne Equation,

$$\begin{aligned}\sigma_s(x) = \rho_n(sz)\rho_n(z)^{-1} = f_n(Rx)\overline{f_n(x)}c^n \quad &\Longrightarrow \quad G(Rx)\overline{G(x)} = f_n(Rx)\overline{f_n(x)}c^n \\ &\Longrightarrow \quad G(Rx)\overline{f_n(Rx)} = G(x)\overline{f_n(x)}c^n\end{aligned}$$

where $f_n(x) = f(x)^n$. In other words c^n is an eigenvalue of Z, with eigenfunction $F = G \cdot \overline{f}$.
Therefore, for all s, f, c, satisfying the Conze-Lesigne Equation, we have that $c \in \Gamma$, where

$$\Gamma = \{c \in \mathbb{T} : \; c^n \text{ is an eigenvalue of Z}\}$$

Since Z is compact, Γ is countable. Furthermore Γ is a subgroup of $\mathbb{T}$.

If $s \in Z$, consider $V_s : Z \to Z$, with $V_s(x) = sx$. Since Z is abelian, for all $s \in Z$, $R \circ V_s = V_s \circ R$. With this notation, $\sigma_s = \rho \circ V_s \, \overline{\rho}$

Define

$$Z_0 = \{s \in Z : \; \text{the cocyle } \sigma_s \text{ is a coboundary}\}$$

Let $s, s' \in Z_0$. This means that for the cocycles σ_s and $\sigma_{s'}$, there exist $h_s, h_{s'} : Z \to \mathbb{T}$, such that $\sigma_s = \partial h_s$ and $\sigma_{s'} = \partial h_{s'}$. Then for the cocycle $\sigma_{ss'} : x \mapsto \rho(ss'x)\overline{\rho(x)}$ we have,

$$\begin{aligned}\sigma_{ss'} = \rho \circ V_{ss'}\overline{\rho} = \big(\rho \circ V_s\, \overline{\rho}\big) \circ V_{s'} \cdot \big(\rho \circ V_{s'}\, \overline{\rho}\big) &= \big(h_s \circ R \circ V_{s'}\, \overline{h_s} \circ V_{s'}\big) \cdot \big(h_{s'} \circ R\, \overline{h_{s'}}\big) \\ &= \big(h_s \circ V_{s'}\, h_{s'}\big) \circ R \cdot \overline{\big(h_s \circ V_{s'}\, h_{s'}\big)}\end{aligned}$$

Thus $\sigma_{ss'}$ is also a coboundary. Equivalently $ss' \in Z_0$. Now for the cocycle $\sigma_{s^{-1}} : x \mapsto \rho(s^{-1}x)\overline{\rho(x)}$ we have,

$$\sigma_{s^{-1}} = \rho \circ V_{s^{-1}}\, \overline{\rho} = \big(\rho\, \overline{\rho \circ V_s}\big) \circ V_{s^{-1}} = \big(h\, \overline{h \circ R}\big) \circ V_{s^{-1}} = \big(\overline{h \circ V_{s^{-1}}}\big) \circ R \cdot \overline{(\overline{h \circ V_{s^{-1}}})}$$

Thus $\sigma_{s^{-1}}$ is also a coboundary. Equivalently $s^{-1} \in Z_0$. Lastly, $e_{\mathbb{T}}$ clearly is an element of Z_0. Follows that Z_0 is a subgroup of Z.

Define $\Phi : Z \to Coc(Z, \mathbb{T})$, by $\Phi(s) = \sigma_s = \rho \circ V_s\, \overline{\rho}$. Then we have that Φ is measurable $\big($see Lemma 4.2.11.1 below$\big)$ and clearly $\Phi^{-1}\big(\partial(Coc(Z, \mathbb{T}))\big) = Z_0$. Thus by Lemma 1.10.5, Z_0 is a Borel subset of Z. Define $\Psi : Z \to \Gamma$, by $\Psi(z) = c_z$, where c_z is the constant arising from the Conze-Lesigne Equation. Then Ψ is an group epimorphism and $ker\Psi = Z_0$. Hence $Z/Z_0 \simeq \Gamma$, and Γ is countable. Thus $m(Z_0) > 0$. (Follows that Z_0 is an open subgroup of Z.)

Furthermore α clearly is an element of Z_0. Follows that $Z_0 = R(Z_0)$, modulo m. Since Z is an ergodic rotation, and $m(Z_0) > 0$, we have that $Z_0 = Z$, modulo m. Thus for m-almost every $s \in Z$, σ_s is a coboundary. In other words the cocycle $(x_0, x_1) \mapsto \rho(x_1)\overline{\rho(x_0)}$ is a coboundary of the system $Z \times Z$. By Lemma 1.10.16, ρ is a quasi-coboundary. $\square$

Lemma 4.2.11.1.

For $\zeta \in Z$ let $V_\zeta : Z \to Z$ denote the translation on X by ζ, defined by $V_\zeta(z) = \zeta \cdot z$ and $f : Z \to \mathbb{T}$ be a continuous function. The function $\Phi : Z \to Coc(Z, \mathbb{T})$ defined by $\Phi(z) := (f \circ V_z)\overline{f}$, is continuous. Indeed, if

$$\begin{aligned}d_{Coc(Z,\mathbb{T})}\big(\Phi(z), \Phi(z')\big) &= \int_Z d_{\mathbb{T}}\Big(\big(f \circ V_z(\zeta)\big)\overline{f}(\zeta), \big(f \circ V_{z'}(\zeta)\big)\overline{f}(\zeta)\Big)\, \mathrm{d}\lambda_Z(\zeta) \\ &= \int_Z d_{\mathbb{T}}\big(f \circ V_z(\zeta), f \circ V_{z'}(\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta) = \int_Z d_{\mathbb{T}}\big(f(z\zeta), f(z'\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta).\end{aligned}$$

The second equality holds since $d_{\mathbb{T}}$ is invariant under translations. Now since f is considered continuous we have that if $z' \to z$ then for every $\zeta \in Z$ we have that

$d_{\mathbb{T}}\big(f(z\zeta), f(z'\zeta)\big) \to 0$. Now since $d_{\mathbb{T}}$ is bounded, applying Lebesgue's Dominated Convergence Theorem we have that if $(z_n)_{n\in\mathbb{N}}$ is a sequence in Z such that $z_n \to z$, for some $z \in Z$, then

$$\int_Z d_{\mathbb{T}}\big(f(z\zeta), f(z_n\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta) \underset{n\to\infty}{\longrightarrow} 0.$$

To summarize, we proved that if $f : Z \to \mathbb{T}$ is a continuous function then Φ is Borel measurable (in particular continuous).

For the general case, let ρ be an element of $Coc(Z,\mathbb{T})$. By Lemma A.1 there exists a sequence of continuous functions $f_n : Z \to \mathbb{T}$, $n \in \mathbb{N}$, such that $dCoc(Z,\mathbb{T})(r, f_n) \to 0$, as $n \to +\infty$. Consider the functions $\Phi, \Phi_n : Z \to Coc(Z,\mathbb{T})$ defined by $\Phi := (\rho \circ V_z)\overline{\rho}$ and $\Phi_n := (f \circ V_z)\overline{f}$, for every $z \in Z$ and every $n \in \mathbb{N}$. Then we have,

$$\begin{aligned}
d_{Coc(Z,\mathbb{T})}\big(\Phi(z),\Phi(z')\big) &= \int_Z d_{\mathbb{T}}\Big(\big(\rho \circ V_z(\zeta)\big)\overline{\rho}(\zeta), \big(f_n \circ V_z(\zeta)\big)\overline{f_n}(\zeta)\Big)\, \mathrm{d}\lambda_Z(\zeta)\\
&\leq \int_Z d_{\mathbb{T}}\Big(\big(\rho \circ V_z(\zeta)\big)\overline{\rho}(\zeta), \big(f_n \circ V_z(\zeta)\big)\overline{\rho}(\zeta)\Big)\, \mathrm{d}\lambda_Z(\zeta)\\
&\quad + \int_Z d_{\mathbb{T}}\Big(\big(f_n \circ V_z(\zeta)\big)\overline{\rho}(\zeta), \big(f_n \circ V_z(\zeta)\big)\overline{f_n}(\zeta)\Big)\, \mathrm{d}\lambda_Z(\zeta)\\
&= \int_Z d_{\mathbb{T}}\Big(\big(\rho \circ V_z(\zeta)\big), \big(f_n \circ V_z(\zeta)\big)\Big)\, \mathrm{d}\lambda_Z(\zeta) \;+\; \int_Z d_{\mathbb{T}}\big(\overline{\rho}(\zeta), \overline{f_n}(\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta)\\
&= \int_Z d_{\mathbb{T}}\big(\rho(\zeta), f_n(\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta) \;+\; \int_Z d_{\mathbb{T}}\big(\rho(\zeta), f_n(\zeta)\big)\, \mathrm{d}\lambda_Z(\zeta)\\
&= 2d_{Coc(Z,\mathbb{T})}(\rho, f_n) \to 0.
\end{aligned}$$

The second equality holds since $d_{\mathbb{T}}$ is invariant under translations and the third equality holds since the Haar measure of Z, λ_Z, is invariant under the translation V_z and $d_{\mathbb{T}}(\overline{z}, \overline{w}) = d_{\mathbb{T}}(z, w)$. It follows that Φ is the pointwise limit of Φ_n, and since every Φ_n is Borel measurable, Φ is also Borel measurable.

Corollary 4.2.12.

Let (Z, m, R), where $R(x) = \alpha x$, be an ergodic rotation, and $\rho : Z \to \mathbb{T}$ a cocycle of type 2. Assume that there exist a $n \in \mathbb{N}$ such that the cocycle $\rho_n : Z \to \mathbb{T}$, defined by $\rho_n(z) = \rho(z)^n$, is a quasi-coboundary of Z. Then ρ is a cocyle of type 1.

Proof. It is a direct result combining the previous Lemma and the Remark above. □

Proposition 4.2.13.

Let (Z, m, R) be an ergodic rotation, U be a compact abelian group, $\rho \in Coc(Z, U)$ an *ergodic cocycle and* $\pi : (Y, \nu, S) \to (Z, m, R)$ *be the extension of* Z *by* U *defined by* ρ. *Then, if* ρ *is a cocycle of type* 1 *the* (Y, ν, S) *is a cocycle of order* 1.

Proof. (Additive Notation)
Let $u \in U$ and $V_u : X \to X$, eith $V_u(z, g) = (z, u + g)$. Since ρ is a cocycle of type 1, there exists an function $F : Z^{[1]} \to U$ such that $\Delta^{[1]}\rho = \partial F = F \circ R^{[1]} - F$. Define $\Phi : Y^{[1]} \to U$, by $\Phi(\mathbf{z}, \mathbf{u}) = F(\mathbf{z}) - \sum_{\epsilon \in V_1} (-1)^{|\epsilon|} u_\epsilon$. Then,

$$\begin{aligned} \Phi \circ S^{[1]}(\mathbf{z}, \mathbf{u}) \; = \; \Phi \circ R^{[1]}_\rho \; &= \; F \circ R^{[1]}(\mathbf{z}) - \Delta^{[1]}\rho(\mathbf{z}) - \sum_{\epsilon \in V_1} (-1)^{|\epsilon|} u_\epsilon \\ &= \; F(\mathbf{z}) - \sum_{\epsilon \in V_1} (-1)^{|\epsilon|} u_\epsilon \; = \; \Phi(\mathbf{z}, \mathbf{u}) \end{aligned}$$

Consider $\mathcal{Z}(Y)$ to be the Kronecker factor of Y and $\pi_2 : Y \to \mathcal{Z}(Y)$ the corresponding factor map. Then, since Z is a factor of order 1 of Y, then Z is a factor of $\mathcal{Z}(Y)$ and let π_3 denote the corresponding factor map. By Lemma $A.4$, $\mathcal{Z}(Y)$ is an extension of Z by the compact abelian group K/L where $L = \{u \in U : \; \pi_2 \circ V_u = \pi_2\}$.

Let $u \in L$. Consider the function $V_u^{(\{\tilde{0}\})} : Y^{[1]} \to Y^{[2]}$, where, $V_u(z_0, \; z_1, \; g_0, \; g_1) = (z_0, \; z_1, \; u + g_0, \; g_1)$. By Lemma 4.2.2 we have that $V_u^{(\{\tilde{0}\})}$ leaves each set of $\mathcal{I}(R^{[1]})$ invariant. Since Φ is $R^{[1]}$-invariant, we have that $\Phi \circ V_u^{(\{\tilde{0}\})} = \Phi$, $\mu^{[1]}$- almost everywhere. This means that

$$F(\mathbf{z}) - \sum_{\epsilon \in V_1} (-1)^{|\epsilon|} u_\epsilon = F(\mathbf{z}) - g_0 + u - \sum_{\epsilon \in V_1^*} (-1)^{|\epsilon|} u_\epsilon \; \Rightarrow \; g_0 = g_0 + u \; \Rightarrow \; u = 0$$

Thus $L = \{0\}$. In other words $Y = \mathcal{Z}(Y)$. Thus Y is a system of oreder 1. □

With a similar procedure we abtain the following.

Lemma 4.2.14.
Let (Y, ν, S) *be an ergodic system of order* 2, U *be a compact abelian group,* $\rho \in Coc(Z, U)$ *an ergodic cocycle and* $\pi : (X, \mu, T) \to (Y, \nu, S)$ *be the extension of* Y *by* U *defined by* ρ. *Then, if* ρ *is a cocycle of type* 2 *the* (Y, ν, S) *is a cocycle of order* 2.

Theorem 4.2.15.
Let (X, μ, T) *be a Conze-Lesigne system and* (K, m, R) *its Kronecker factor. Then* X *is an extension of* K *by a compact connected abelian group* U *and the cocycle* $\rho : K \to U$ *that defines the extension is a cocycle of type* 2.

Proof. By Theorem 4.2.7 we are left with showing that U is connected.
Assume that U is not connected. Then U admits an open subgroup U_0 and an integer $n > 1$, such that $U/U_0 \simeq \mathbb{Z}_n$. Let $q : U \to U/U_0$ denote the natural (continuous and open) group epimorphism.

Define the cocycle $\sigma : Z \to U/U_0$ by $\sigma(z) = \rho(z)U_0$. In other words $\sigma = q \circ \rho$. The cocycle σ is also of type 2. Consider the group isomorphisms $\phi : U/U_0 \to \mathbb{Z}_n$ and $\psi : \mathbb{Z}_n \to \{e^{2\pi \frac{k}{n} i} : k \in \mathbb{Z}\}$. Define the (non-ergodic) cocycle $\tau : Z \to \mathbb{T}$, with $\tau := \psi \circ \phi \circ \sigma$. Then $\tau_n \equiv 1$ and thus, by the Corollary above, τ is of type 1. Follows that σ is of type 1.

Now let $\tilde{\pi} : (\tilde{X}, \tilde{\mu}, \tilde{T}) \to (Z, m, R)$ be the extension of Z by the compact abelian gorup U/U_0, defined by the cocycle σ. Since σ is of type 1, by Theorem 4.2.13 is a system of order 1. By maximality of the Kronecker factor in X, we have that $Z = \tilde{X}$. Thus $n = 1$ which is a contradiction. □

4.3 Conze-Lesigne systems and 2-step nilsystems

Lemma 4.3.1.
Let (Y, ν, S) be an ergodic system, K a compact abelian group and $\rho \in Coc(Y, K)$. Then ρ is <u>not</u> ergodic if and only if there exists a character $\chi \in \widehat{K}$, where χ is not identically equal to 1, such that $\chi \circ \rho$ is a coboundary of $\mathbb{T}$.

Proof. Let $\pi : (Y \times K, \nu \otimes m, S_\rho) \to (Y, \nu, S)$ is the extension of Y by K defined by ρ. Let $\chi \in \widehat{K}$, where χ is not identically equal to 1 such that $\chi \circ \rho$ is a coboundary of $\mathbb{T}$. This means there exists a function $f : Y \to \mathbb{T}$ such that $\chi \circ \rho(y) = f \circ S(y) - \overline{f}(y)$. Define $F : Y \times K \to \mathbb{T}$ by $F(y, k) = f(y)\overline{\chi}(k)$. Then

$$
\begin{aligned}
F \circ S_\rho(y, k) &= F\big(S(y), \rho(y)k\big) \overset{\chi\ chracter}{=} f\big(S(y)\big)\ \overline{\chi}\big(\rho(y)\big)\ \overline{\chi}(k) \\
&= f(y)\ \overline{\chi}(k) = F(y, k)
\end{aligned}
$$

Thus F is S_ρ-invariant and is not equal to a constant. Indeed if $F(y, k) = c$, for all (y, k) then $F(y, k_1) = c$ for all y. Follows that $f(y) = c\chi(k_1)$ for all y. Follows that $\chi(y) = f \circ S(y)\overline{f}(y) = c\overline{c}\ \chi(k_1)\overline{\chi}(k_1) = 1$, for all y and that contradicts with the choice of χ. Thus ρ is not ergodic.

Conversely, since ρ is note ergodic, there exists an $f : Y \times K \to \mathbb{C}$, S_ρ-invariant that is not everywhere equal to a constant. By relation $A.1$ there exist a vertical Fourier coefficient $\hat{f}_\chi$, where $\chi \in \widehat{K}$ and χ is not identically equal to 1. Thus $\hat{f}_\chi$ is a vertical character with frequency χ. This means that $\hat{f}_\chi(y,k) = \phi(y)\chi(k)$, for some $\phi : Y \to \mathbb{C}$ that is not everywhere equal to 0.

Since $V_h \circ S_\rho = S_\rho \circ V_h$, for all $h \in K$ we have that $\hat{f}_\chi \circ S_\rho = \hat{f}_\chi$. Follows that $\phi(y) = \phi\big(S(y)\big)\ \chi\big(\rho(y)\big)$. Follows that $|\phi|$ is S-invariant an since (Y,ν,S) is ergodic, $|\phi| = c$, almost everywhere for some constant $c \in \mathbb{C} \setminus \{0\}$.

By defining $\psi : Y \to \mathbb{T}$, with $\psi = \frac{\overline{\phi}}{c}$, we have that $\psi \circ S\ \overline{\psi} = \chi \circ \rho$. This means that $\chi \circ \rho$ is a coboundary. □

Lemma 4.3.2. *Let (Y,ν,S) be an ergodic system, K a compact connected abelian group and $\rho : Y \to K$ a cocycle. Then there exists an $c \in K$ such that the cocycle ρ is ergodic.*

Proof. Let $\chi \in \widehat{K}$, where χ is not identically equal to 1. Define the sets

$$A_\chi = \{c \in K :\ \chi \circ (\rho \cdot c)\ is\ a\ coboundary\}$$
$$B_\chi = \{c \in K :\ \chi(c)\ is\ an\ eigenfunction\ of\ (Y,\nu,S)\}$$

Clearly B_χ is a subgroup of K and $ker(\chi)$ is a subgroup of B_χ. If $\chi(a) = \chi(b) = \lambda$, where λ is an eigenvalue of Y, then $a\ ker(\chi) = b\ ker(\chi)$. Since Y admits countably many eigenvalues, there are c_i, $i \in \mathbb{N}$ such that

$$B_\chi = \bigsqcup_{\substack{c\in K:\\ \chi(c)\\ eigenvalue}} c\ ker(\chi) = \bigsqcup_{i\in\mathbb{N}} c_i\ ker(\chi)$$

Thus that $ker(\chi)$ has has a countable index in B_χ. Furthermore, since $ker(\chi)$ is closed on K and $B_\chi = \bigsqcup_{i\in\mathbb{N}} c_i\ ker(\chi)$, B_χ is a Borel subset of K.

Now since χ is a non-trivial character of K and K is connected, $\chi(K)$ is a non-trivial connected subgroup of $\mathbb{T}$. Thus $\chi(K) \simeq \mathbb{T}$. By the First Isomorphism Theorem $K/ker(\chi) \simeq \mathbb{T}$. In particular $ker(\chi)$ has has a uncountable index in K.

Applying Lagrange Theorem we obtain that B_χ has also an uncountable index in K. In particular is infinite. Thus, if m_K is the Haar measure of K then $m_K(B_\chi) = 0$.

Observe that A_χ is not empty, is a coset of B_χ. Indeed, let $c, c' \in A\chi$, meaning that there exist function $f_c, f_{c'} : Y \to \mathbb{T}$ $\big($clearly $f_c, f_{c'} \in L^2(\nu)\big)$ such that $\chi \circ (\rho \cdot c) = \partial f_c$ and $\chi \circ (\rho \cdot c') = \partial f_{c'}$. This means that for ν-almost every $y \in Y$,

$$\chi\big(\rho(x)\big)\chi(c) = f_c\big(S(y)\big) \cdot f_c(y)^{-1}(y) = f_c\big(S(y)\big) \cdot \overline{f_c}(y)$$
$$\chi\big(\rho(x)\big)\chi(c') = f_{c'}\big(S(y)\big) \cdot f_{c'}(y)^{-1}(y) = f_{c'}\big(S(y)\big) \cdot \overline{f_{c'}}(y)$$

Thus we have,

$$\chi(cc'^{-1}) = \chi(c)\overline{\chi}(c') = f_c\big(S(y)\big)\cdot\overline{f_c}(y) \ \cdot\ \overline{f_{c'}\big(S(y)\big)}\cdot f_{c'}(y) = \big(f_c\cdot\overline{f_{c'}}\big)\circ S(y)\cdot\overline{\big(f_c\cdot f_{c'}\big)}(y)$$

Thus $cc'^{-1}\in B_\chi$. Follows that A_χ is a subset of a coset of B_χ $\big($ in particular $A_\chi\subseteq c\,B_\chi$, where $c\in A_\chi$ $\big)$.

Conversely, since the union of all cosets of B_χ covers K, then there exists a $c\in K$ such that $A_\chi\cap cB_\chi\neq\emptyset$. Since $cB_\chi=\beta B_\chi$ for every $\beta\in cB_\chi$, we can assume that $c\in A_\chi$. Let $c'\in B_\chi$. Then $cc'^{-1}\in B_\chi$. Thus we have, that there exist $f,g:Y\to\mathbb{T}$, such that,

$$\begin{aligned}\chi\circ\rho\,\chi(c) \ &= \ f\circ S\cdot\overline{f}\\ g\circ S \ = \ \chi(cc'^{-1})g \ &= \ \chi(c)\overline{\chi}(c')g\end{aligned}$$

Follows that,

$$\begin{aligned}\chi\circ(\rho\cdot c') = \chi\circ\rho\cdot\chi(c') = \chi\circ\rho\,\chi(c)\;g\,\overline{g}\circ S &= \chi\circ(\rho\cdot c)\;g\,\overline{g}\circ S\\ &= f\circ S\,\overline{f}\;g\,\overline{g}\circ S = (f\overline{g})\circ S\;\overline{(f\overline{g})}\end{aligned}$$

This means that $c'\in A_\chi$. Therefore $cB_\chi\subseteq A_\chi$. Thus $m_K(A_\chi)=0$.

Thus the countable union, $\bigcup_{\substack{\chi\in\widehat{K}\\ \chi\neq 1}} A_\chi$ cannot be equal to K. This means that there exists a $c\in K$, such that $\chi\circ(\rho+c)$ is not e coboundary. By Lemma 4.3.1, $\rho+c$ is ergodic. $\square$

Proposition 4.3.3.

Assume that (X,μ,T) and (Y,ν,S) are ergodic systems and that X is of order 2 for some integer. Assume that $\pi:X\to Y$ is a factor map and $\rho:Y\to U$ is a cocycle. Then ρ is of type 2 on Y if and only if $\rho\circ\pi$ is if type 2 on X.

Proof. Let ρ be of type 2. Then $\Delta^{[2]}\rho=\partial f$ for some $f:Y\to U$. Follows that

$$\big(\Delta^{[2]}\rho\big)\pi = (\partial f)\circ\pi \Rightarrow \Delta^{[2]}(\rho\circ\pi) = \partial(f\circ\pi)$$

. Thus $\rho\circ\pi$ is of type 2.

Conversely, assume that $\rho\circ\pi$ is of type 2. By Lemma 4.2.5 it suffices to show that $\chi\circ\rho$ for every $\chi\in\widehat{U}$. Since $\chi\circ\rho\circ\pi$, we can assume that $U=\mathbb{T}$. By Lemma 4.3.1 the set $\{c\in\mathbb{T}:\ \rho\cdot c \text{ is not ergodic}\}$ is a coset of the countable (since Y admits countably many eigenvalues) subgroup $\{c\in\mathbb{T}:\ c^n \text{ is an eigenvalue of } Y \text{ for some } n\in\mathbb{N}\}$. Thus there exists a $c \in \mathbb{T}$ so that the cocyle $\rho \cdot c$ is ergodic. By substituting $\rho \cdot c$ with ρ, we can assume that ρ is ergodic.

By Lemma 4.2.14 the extension $\tilde{X}$, of X by K defined by $\rho \circ \pi$ is a system of order 2 since $\rho \circ \pi$ is of type 2. Furthermore the extension $\tilde{Y}$, of Y by K defined by ρ is a factor of $\tilde{X}$ thus it is also a system of order 2. By Proposition 4.2.6, ρ is a cocycle of type 2. $\square$

Theorem 4.3.4.
Let (Z, m, R) be an ergodic roatation, U a finite dimensional torus and $\rho : Z \to U$ a cocycle of type 2. Then there exist a closed subgroup Z_0 of Z so that Z/Z_0 is a compact abelian Lie group and a cocycle $\rho' : Z/Z_0 \to U$ of type 2 so that ρ is chomologous to $\rho' \circ \pi$, where $\pi : Z \to Z/Z_0$ is the natural projection.

Proof. By the Conze-Lesigne Equation, we have that for every $s \in Z$ the map $z \mapsto \rho(sz)$ is a quasi-coboundary. The compact abelian group Z acts on itself with automorphisms by $S_s : \ z \mapsto s$. In other words for every $s \in Z$ $\rho \circ S_s - \rho$ is a quasi-coboundary. By Lemma $A.7$ there exist a closed subgroup Z_0 of Z and a cocycle $\sigma : Z \to U$, such that $Z/Z_0 \simeq \mathbb{T}^m \times G$, where G is a finite abelian group and σ is cohomologous to ρ and for every $s \in Z_0$, $\sigma \circ S_s = \sigma$.

Since for every $s \in Z_0$, $\sigma \circ S_s = \sigma$ we can define $\rho' : Z/Z_0 \to U$ by $\rho'(z + Z_0) = \sigma(z)$. Now, by Proposition 4.3.3 we have that ρ' is of type 2 iff $\rho' \circ \pi$ is of type 2. Observe that $\rho' \circ \pi = \sigma$ and σ is cohomologous to ρ which is of type 2. Thus ρ' is indeed of type two. $\square$

Definition 4.3.5.
Let (X, μ, T) be a system of order 2. Then X is called **toral** if its Kronecker factor Z ia a compact abelian Lie group and X ia an extension of Z by a finite dimensional torus.

Lemma 4.3.6.
Let G be a compact abelian group and H a subgroup of G. Define th set $Ann_G(H) = \left\{\chi \in \widehat{G} \;\middle|\; \forall h \in H : \ \chi(h) = 1_{\mathbb{C}}\right\}$. Then $Ann_G(H)$ is a closed subgroup of $\widehat{G}$ and $\widehat{H} \simeq \widehat{G}/Ann_G(H)$

Corollary 4.3.7.
Let G be a compact abelian group and H a subgroup of G. Then each character of H is of the form $\chi|_H$, where $\chi \in \widehat{G}$.

Lemma 4.3.8.
Let U be a compact abelian group. Then U is the inverse limit of a sequence of sompact abelian Lie group. If in addtion U is connected then U is the inverse limits of finite-dimensional tori.

Proof. As stated in the beginning, U is considered to be metrizable.
Since Z is compact abelian then Z is separable and let $A = \{z_1, z_2, \dots\}$ be a countable dense subset of Z. By Potryagin's Theorem $\widehat{\widehat{Z}} = \widehat{(\widehat{Z})} \simeq Z$. Furthermore the dual group of a compact group, separates its points. Thus Z separates the points of $\widehat{Z}$. Equivalently A separates the points of $\widehat{Z}$ and since A is s countable set follows that $\widehat{Z}$ is also countable. Now let $\widehat{Z} = \{\chi_n :\ n \in \mathbb{N}\}$. Define the subgroups of $\widehat{Z}$, $\Lambda_i = \langle \chi_1, \chi_2, \dots, \chi_i \rangle$, $i \in \mathbb{N}$. Then $\{\Lambda_i\}_{i \in \mathbb{N}}$ is an increasing sequence of finitely generated subgroups of $\widehat{Z}$ and $Z = \bigcup_{i \in \mathbb{N}} \Lambda_i$.

Let $Z_i = \widehat{\Lambda_i}$ be the dual group of Λ_i. Let $z \in \widehat{\widehat{Z}} = Z$, a character of $\widehat{Z}$. Then the restriction of z on each Λ_i, $z|_{\Lambda_i} : \Lambda_i \to \mathbb{T}$, is clearly an element of $\widehat{\Lambda_i} = Z_i$. Define for each $i \in \mathbb{N}$, $p_i : Z \to Z_i$, with $p_i(z) = z|_{\Lambda_i}$. Then p_i is a continuous group epimorphism:

- p_i is clearly a group homomorphism
- π is continuous
  Let $z_m \to z$, where $z_m, z \in \widehat{\widehat{Z}} = Z$, $m \in \mathbb{N}$. This means that $z_m(\chi)$ converges to $z(\chi)$, for every $\chi \in \widehat{Z}$. In particular this holds for every $\chi \in \Lambda_i$ and thus $z_m|_{\Lambda_i} \to z|_{\Lambda_i}$. Follows that $p_i(z_m) \to p(z)$
- by the corollary above p_i is onto

Let $i \le j$, $i, j \in \mathbb{N}$ and $p_{i,j} : Z_j \to Z_i$, defined by $p_{i,j}(z) = z|_{\Lambda_i}$. Then $p_{i,j}$ is a continuous group epimorphism. Furthermore, clearly $p_{i,j} \circ p_j = p_i$ and $p_{i,k} = p_{i,j} \circ o_{j,k}$, $\forall i \le j \le k \in \mathbb{N}$. Thus $Z = \varprojlim_i Z_j$ $\big($ in the algebraic sense where the homomorphisms are in continuous and thus measurable $\big)$.

Since each Λ_i is finitely generated then Λ_i is the direct product of a free abelian group and a finite group. This means that $\Lambda_i = \mathbb{Z}^{n_i} \times H_i$, where $H_i = \{g_1, \dots, g_{k_i}\}$, for some $n_i,\ k_i \in \mathbb{N}$. Now for G_1, G_2 we have that $\widehat{G_1 \times G_2} = \widehat{G_1} \times \widehat{G_2}$, $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{H} \simeq H$, for any finite group H. Thus $\widehat{\Lambda_i} = \mathbb{T}^{n_i} \times H_i = \bigsqcup_{j=1}^{k_i} \mathbb{T}^{n_i} \times \{g_j\}$, which is a compact abelian Lie group.

Let, now assume that Z is in addition connected. Then we have that $\widehat{Z}$ is torsion free. Indeed, if $\chi \in \widehat{Z}$ and $n \in \mathbb{N}$ such that $\chi^n \equiv 1$. Then $\chi(Z)$ is a closed subgroup of the discrete group $\mathbb{Z}_n \leq \mathbb{T}$. Since χ is continuous and Z is connected and $\chi(Z)$ a discrete set, then $\chi \equiv c$ for some constant $c \in \mathbb{T}$. Since χ is a character, follows that $c = 1$. Thus $\chi \equiv 1$. Thus $\widehat{Z}$ does not contain any non trivial element with finite order. In other words $\widehat{Z}$ is torsion free. Thus each $\Lambda_i \leq \widehat{Z}$ is torsion free. This means that $H_i = \{1\}$, for each $i \in \mathbb{N}$. Thus isomorphic to $\mathbb{T}^{n_i}$. □

Remark 4.3.9.

With the exact same procedure one can show that if (Z, m_Z, R) is an ergodic rotation and $\alpha \in Z$ defines R, then $(Z, m_Z, R) = \varprojlim_i (Z_i, m_{Z_i}, R_i)$, where Z_i is as above, m_{Z_i} is its Haar measure and R_i is the rotation on Z_i defined by $\alpha_i = p_i(\alpha)$.

Proposition 4.3.10.

Every system of order 2 is the inverse limit of a sequence of toral system, of order two.

Proof. Let (X, μ, T) be a system of order 2. By Theorem 4.2.15, X is an extension of its Kronecker factor, Z, by a compact connected abelian group, given by a cocycle $\rho : Z \to U$ of type 2. By Lemma 4.3.8, U is an inverse limit of tori. Let $p_n : U \to \mathbb{T}$ be the continuous group homomorphisms defined in the proof above. Setting $A_n = kerp_n$ we have that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed subgroups, with $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$, such that $U_n = U/A_n$ is a finite dimensional torus.

Define $\pi_n : Z \to U_n$, by $\xi_n(z) = \rho(z)U_n$. In other words $\xi_n = p_n \circ \rho$. Consider (X_n, μ_n, T_n) to be the extension of Z by the compact connected abelian gorup $U_n = \mathbb{T}^{m(n)}$, associated with the cocycle ξ_n. For $n, m \in \mathbb{N}$, with $n < m$, define $\pi_{n,m} : X_m \to X_n$, by $\pi_{n,m}(z, u_m) = \big(z, p_{n,m}(u_m)\big)$ and $\pi_n : X \to X_n$, by $\pi_n(z, u) = \big(z, p_n(u)\big)$ Then $\big((X_n, \mu_n, T_n)_{n\in\mathbb{N}},\ (\pi_{n,m})_{n,m\in\mathbb{N}}, n \leq m\big)$ is an inverse system and $(X, \mu, T) = \varprojlim_n (X_n, \mu_n, T_n)$.

By Lemma 4.3.4 for each n there exists a subgroup K_n of Z, such that Z/K_n is a compact abelian Lie group and a cocycle $\xi'_n : Z/K_n \to U_n$, such that ξ_n is chomologous to ξ'_n. In addition we can modify the groups K_n such that $\{K_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subgroups and $\bigcap_n K_n = \{1\}$. Now for each n, let (Y_n, ν_n, S_n) be the extension of the compact abelian Lie group Z/K_n by the torus U_n, associated with the cocycle ξ'_n. Then $(X, \mu, T) = \varprojlim_n (Y_n, \nu_n, S_n)$ and each (Y_n, ν_n, S_n) is a toral system. □

Throughout the remaining section, we assume that (X, μ, T) is a system of order 2, (Z, m, R) its Kronecker factor, where $R(z) = \alpha z$, U a finite dimensional torus, $\mathbb{T}^n$, and $\rho : Z \to U$ a cocycle, such that

$$(X, \mu, T) = (Z \times U, m \otimes m_U, R_\rho)$$

Definition 4.3.11.

- Define $\mathscr{G}$ to be the group of the measure preserving transformation $S_{u,f} : Z \times U \to Z \times U$ given by
$$S_{u,f}(z, u) = \big(s \cdot z,\ u \cdot f(z)\big)$$
where $s \in Z$, $f \in Coc(Z, U)$ satisfy the Conze-Lesigne Equation for some constant $c \in U$.

- A map $f : Z \to U$ is **affine** if there exist a continuous group homomorphism $\phi : Z \to U$ and a constant $c \in U$, such that
$$f \ = \ \phi \cdot c$$

- Define $\mathcal{A}(Z, U) = \{f : Z \to U |\ f \text{ is affine }\}$.

Remarks 4.3.12.

(i) Remember if $s,\ s' \in Z$ and $f_s,\ f_{s'}$ are the corresponding functions arrising from the Conze-Lesigne Equation, then the function defined by
$$f_{ss'}(z) = f_{s'}(sz) f_s(z)$$
satisfies the Conze-Lesigne Equation for ss' and thus $S_{s',f_{s'}} \circ S_{s,f_s} = S_{ss',f_{ss'}} \in \mathscr{G}$. Therefore $\mathscr{G}$ is indeed a group equipped with the operation of composition

(ii) T is an element of $\mathscr{G}$ since $T = R_\rho = S_{\alpha,\rho}$ and (a, ρ) satisfies the Conze-Lesigne Equation with constant $c = e_U$.

(iii) $\mathcal{A}(Z, U)$ is a closed subgroup of the (Polish group) $Coc(Z, U)$.

(iv) $\mathcal{A}(Z, U) \simeq U \times \Gamma$, where Γ is the discrete group, consisting of the continuous group homomorphisms on Z to the torus U. In other words $\Gamma = \widehat{Z}^n \times U$, where $U = \mathbb{T}^n$.

Proposition 4.3.13.

For every toral system of order 2, $\mathscr{G}$ *is* 2*-step nilpotent.*

Proof. For each $w \in Z$ let V_w denote the topological isomorphism on Z defined by

$$V_w(z) = x \cdot w$$

Let S_{s,f_s}, $S_{s',f'_s} \in \mathscr{G}$, where (s, f_s, c_s) and $(s', f_{s'}, c_{s'})$ satisfy the Conze-Lesigne Equation

(i) the inverse element of S_{s,f_s} in $\mathscr{G}$, is denoted as S^{-1}_{s,f_s} and is equal to $S_{s^{-1},\overline{f_s}\circ V_{s^{-1}}}$, where $(s^{-1}, \overline{f_s} \circ V_{s^{-1},\overline{c_s}})$ satisfy the Conze-Lesigne Equation.

(ii) the functions $f_{ss'} = (f_s \circ V_{s'}) \cdot f_{s'}$ and $g_{s's} = (f_{s'} \circ V_s) \cdot f_s$ satisfy the Conze Lesigne equation for ss', $c_{ss'} = c_s c_{s'}$ and $s's$, $c_{s's} = c_{s'} c_s$, respectively. Since Z, $U = \mathbb{T}^n$ are abelian groups we have that $ss' = s's$ and $c_{ss'} = c_{s's}$.

By (i) and some simple computations we have that

$$S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s}(z,u) = (z, f_{ss'}(z) \cdot \overline{g_{s's}}(z) \cdot u)$$

Let $S_{s'',f_{s''}} \in \mathscr{G}$. Then

$$S_{s'',f_{s''}} \circ \big(S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s}\big)(z,u) \;=\; (s''z,\; f_{s''}(z) \cdot f_{ss'}(z) \cdot \overline{g_{s's}}(z) \cdot u)$$

and

$$\big(S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s}\big) \circ S_{s'',f_{s''}}(z,u) \;=\; (s''z,\; f_{s''}(z) \cdot f_{ss'}(s''z) \cdot \overline{g_{s's}}(s''z) \cdot u)$$

Now by (ii), $ss' = s's$ and $c_{ss'} = c_{s's}$ and thus by the Conze-Lesigne Equation we have that

$$f_{ss'}(z) \cdot \overline{g_{s's}}(z) \;=\; f_{ss'}(\alpha z) \cdot \overline{g_{s's}}(\alpha z) \;=\; (f_{ss'} \cdot \overline{g_{s's}}) \circ R(z)$$

By ergodicity of the system (Z, m, R) we have that $f_{ss'} \cdot \overline{g_{s's}} = \beta$, m-almost everywhere for some constant $\beta \in U$. Hence for m-almost every $s'' \in Z$

$$f_{ss'}(z) \cdot \overline{g_{s's}}(z) \;=\; f_{ss'}(s''z) \cdot \overline{g_{s's}}(s''z)$$

for m-almost every $z \in Z$. This means that

$$S_{s'',f_{s''}} \circ \big(S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s}\big) \;=\; \big(S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s}\big) \circ S_{s'',f_{s''}}$$

$\mu = m \otimes m_U$, almost-everywhere. Thus $S^{-1}_{s,f_s} \circ S^{-1}_{s',f_{s'}} \circ S_{s,f_s} \circ S_{s',f'_s} \in Z(\mathscr{G})$ and since $[\mathscr{G}, \mathscr{G}]$ is spanned by elements of this form we have that $[\mathscr{G}, \mathscr{G}] \subseteq Z(\mathscr{G})$. Therefore $\mathscr{G}$ is 2-step nilpotent. $\square$

Let $\mathscr{G}$ be endowed with the topology of convergence in probability. The map $p : \mathscr{G} \to Z : S_{s,f} \mapsto s$ is a continuous group homomorphism. In particular by Conze-Lesigne Equation p is an epimorphism. The kernel of this homomorphism is the group of transformations of the kind $S_{1,f}$, where (again by Conze-Lesigne Equation) $f(tz)f(z)^{-1}$ is constant. A map $f \in Coc(Z, U)$ satisfies this condition if and only if it is affine. Indeed if a function $f = (f_1, \ldots, f_n) : Z \to U = \mathbb{T}^n$, satisfies this property, then $f \circ R = c \cdot f$ for some constant. Follows that each f_i is an eigenfunction of Z and thus $f = c_i \cdot \gamma_i$, where c_i is a constant and $\gamma_i \in \widehat{Z}$. Thus $f = c \cdot \phi$, where $c = (c_1, \ldots, c_n)$ is a constant and $\phi = (\gamma_1, \ldots, \gamma_n)$. The function ϕ is clearly a continuous group homomorphism. The converse clearly holds.

The map $f \mapsto S_{1,f}$ is then an algebraic and topological embedding of $\mathcal{A}(Z, U)$ in $\mathscr{G}$ with range $ker(p)$. In the sequel we identify $\mathcal{A}(Z, U)$ with $ker(p)$.

By Lemma 1.1.8, $\mathscr{G}$ is locally compact, since $\mathcal{A}(Z, U) = \Gamma \times U$ (as mentioned in Remarks 4.3.12) which is locally compact and $\mathscr{G}/\mathcal{A}(Z, U) \simeq Z$ which is compact and thus locally compact.

Lemma 4.3.14.

$\mathscr{G}$ acts transitively on X by automorphisms.

This means that for every x_1, $x_2 \in X$, there exists a $g = S_{s,f} \in \mathscr{G}$, so that $g \bullet x_1 := S_{s,f}(x_1) = x_2$.

Proof. Let x_1, $x_2 \in X$. Identifying $X = Z \times U$, we have that $x_1 = (z_1, u_1)$ and $x_2 = (z_2, u_2)$. There exist a $s_0 \in Z$ such that $z_1 = s_0 z_2$. Consider $S_{s_0,f_0} \in \mathscr{G}$. Then $S_{s_0,f_0}(z_1, u_1) = (s_0 z_1, f_0(z_1)u_1) = (z_2, f_0(z_1)u_1)$. Now there exists a $c \in U$ so that $cf_0(z_1)u_1 = u_2$. Consider $\phi : Z \to U$ be a homomorphism so that $\phi(z_2) = c$. Furthermore we have that ϕ satisfies the Conze-Lesigne Equation for $s = 1$ and constant $\phi(\alpha)$. Thus $S_{1,\phi}$ is an element of $\mathscr{G}$ and $S_{1,\phi} \circ S_{s_0,f_0}(z_1, u_1) = S_{1,\phi}(z_2, f_0(z_1)u_1) = (z_2, \phi(z_2)f_0(z_1)u_1) = (z_2, cf_0(z_1)u_1)$. Thus for $S_{s,f} = S_{1,\phi} \circ S_{s_0,f_0}$ we have the statement. □

Proposition 4.3.15.

Every toral system of order 2 is isomorphic to a 2-step nilsystem.

Proof. Let (X, μ, T) be a toral system of order 2. That is its Kronecker factor is a compact abelian Lie group and is an extension by $U = \mathbb{T}$, defined by some cocycle $\rho : Z \to U$. The kernel of the map p, $\mathcal{A}(Z, U)$ is the direct product of the torus U and the discrete group Γ. Thus $\mathcal{A}(Z,U)$ is a Lie group. Since $\mathscr{G}/\mathcal{A}(Z,U) \simeq Z$ and $\mathcal{A}(Z,U)$ are Lie groups, follows that $\mathscr{G}$ is a Lie group (Proposition 1.11.7). Furthermore by Proposition 4.3.13 $\mathscr{G}$ is 2-step nilpotent.

Define the stabilizer of $(e_Z, e_U) = (1,1)$, under the action of $\mathscr{G}$, $C = Stab_{\mathscr{G}}\big((1,1)\big) = \{S \in \mathscr{G} :\ S \bullet (1,1) = (1,1)\} = \{S \in \mathscr{G} :\ S(1,1) = (1,1)\}$. Since $\mathscr{G} \simeq Z \times \mathcal{A}(Z,U)$, $\mathscr{G}$ consist of transformations associated to (s,f) where $s \in Z$ and $f \in \mathcal{A}(Z,U)$. Thus C consists of transformations associated to $(1,f)$, where $1 \in Z$ and $f \in \mathcal{A}(Z,U)$. This means $C = \{(1,f) :\ f \in \mathcal{A}(Z,U)\}$. Now, since $\mathcal{A}(Z,U)$ is the direct product of a torus and a discrete group, then C is also a discrete group.

Now since $\mathscr{G}$ acts transitively on X, the onto map $\mathscr{G} \to X :\ \ S_{s,f} \mapsto S_{s,f}(1,1)$ induces a bijection ϕ from the manifold $\mathscr{G}/C$ onto X. For every $S_{s,f} = g \in \mathscr{G}$. define $\xi_g : \mathscr{G}/C \to \mathscr{G}/C$, by setting $\xi_g = \phi^{-1} \circ g \circ \phi$. Then ξ_g is the translation on $\mathscr{G}/C$, by $g \in \mathscr{G}$. Furthermore since the measure μ is invariant under any $g \in \mathscr{G}$, then the image of μ under ϕ^{-1} is an invariant measure on $\mathscr{G}/C$ that is invariant under the action of $\mathscr{G}$ on $\mathscr{G}/C$. Thus $(\phi^{-1})_*\mu$ is the Haar measure of $\mathscr{G}/C$. Lastly define $\widetilde{T}$ on $\mathscr{G}/C$ by $\widetilde{T} = \phi^{-1} \circ T$, we have that $\phi : (X,\mu,T) \to (\mathscr{G}/C, m_{\mathscr{G}/C}, \widetilde{T})$ is an isomorphism and $(\mathscr{G}/C, m_{\mathscr{G}/C}, \widetilde{T})$ is a 2-step nilsystem. □

Summarizing the previous results we have the following.

Theorem 4.3.16.
Every Conze-Lesigne system is the inverse limit of a sequence of 2*-step nilsystems.*

Chapter 5

Some first results

5.1 The product of 3 terms and the integral with 4 terms

We now prove the Bergelson's two dimensional generalization of Khintchine's Theorem. We consider sequences of averages of the form

$$\frac{1}{(N_1-M_1)(N_2-M_2)}\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2} f_{01}(T^{n_1}x)f_{10}(T^{n_2}x)f_{11}(T^{n_1+n_2}x) \tag{5.1}$$

where f_{01},$f_{01}, f_{01} \in L^{\infty}(X)$ for some invertible ergodic probability measure preserving system $(X,\mathcal{B},\mu,T)$

Proposition 5.1.1.

When N_1-M_1 and N_2-M_2 tend to $+\infty$ the average 5.1 *converges to* 0 *in* $L^2(\mu)$ *whenever there exists an* $\eta\in V_2^*$ *with* $\mathbb{E}(f_\eta \mid \mathcal{K})=0$.

In other words, Kronecker factor is characteristic for the convergence in $L^2(\mu)$ *for average* 5.1

Proof. Let $k,\ell\in\mathbb{Z}$. Because μ is $T^{n_1+n_2}$-invariant, we have,

$$\int\limits_X f_{01}(T^{n_1+k}x)\cdot f_{10}(T^{n_2+\ell}x)\cdot f_{11}(T^{n_1+n_2+k+\ell}x)\cdot f_{01}(T^{n_1}x)\cdot f_{10}(T^{n_2}x)\cdot f_{11}(T^{n_1+n_2}x)\,\mathrm{d}\mu(x)$$

$$=\int\limits_X \big(f_{01}\circ T^k\cdot f_{01}\big)(T^{-n_2}x)\cdot\big(f_{10}\circ T^{\ell}\cdot f_{10}\big)(T^{-n_1}x)\cdot\big(f_{11}\circ T^{k+\ell}\cdot f_{11}\big)(x)\,\mathrm{d}\mu(x)$$

Taking average over n_1 and n_2, we have as N_1-M_1 and N_2-M_2 tend to $+\infty$, by ergodicity, this converges to

$$\gamma_{k,\ell} = \int_X f_{01} \circ T^k \cdot f_{01} \, \mathrm{d}\mu \int_X f_{10} \circ T^\ell \cdot f_{10} \, \mathrm{d}\mu \int_X f_{11} \circ T^{k+\ell} \cdot f_{11} \, \mathrm{d}\mu$$

Now, if $\mathbb{E}(f_\eta \mid \mathcal{K}) = 0$ for some $\eta \in V_2^*$, then $\frac{1}{\Gamma \cdot \Delta} \sum_{k=0}^{\Gamma-1} \sum_{\ell=0}^{\Delta-1} |\gamma_{k,\ell}| \to 0$, as $\Gamma, \Delta \to \infty$. Indeed, if for example $\eta = 01$ (the other cases are similar) then, by ergodicity $\frac{1}{\Delta} \int_X f_{01} \circ T^k \cdot f_{01} \, \mathrm{d}\mu \to \left(\int_X f_{01} \, \mathrm{d}\mu \right)^2$, as $\Delta \to \infty$. By Lemma 3.3.14

$$\begin{aligned} \left(\int_X f_{01} \, \mathrm{d}\mu \right)^4 &\leq \int_{X^4} f_{01} \otimes f_{01} \otimes f_{01} \otimes f_{01} \, \mathrm{d}\mu^{[2]} \\ &= \int_{X^4} \mathbb{E}(f_{01} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \otimes \mathbb{E}(f_{01} \mid \mathcal{K}) \, \mathrm{d}\mu^{[2]} \end{aligned}$$

and thus, if $\mathbb{E}(f_{01} \mid \mathcal{K}) = 0$ then the last integral is also 0. Hence

$$\lim_{\Gamma, \Delta \to \infty} \frac{1}{\Gamma \cdot \Delta} \sum_{k=0}^{\Gamma-1} \sum_{\ell=0}^{\Delta-1} |\gamma_{k,\ell}| = 0$$

In particular

$$\lim_{L \to \infty} \frac{1}{L^2} \sum_{k=0}^{L-1} \sum_{\ell=0}^{L-1} |\gamma_{k,\ell}| = 0$$

The result follows from the two dimensional Van der Corput Lemma (Section $A.4$). □

Proposition 5.1.2.
The average 5.1 *converges in* $L^2(\mu)$ *to*

$$\int_{K \times K} \mathbb{E}\big(f_{01} \mid Z\big)\big(\pi(x)\big) \, \mathbb{E}\big(f_{10} \mid Z\big)\big(s\pi(x)\big) \, \mathbb{E}\big(f_{11} \mid Z\big)\big(st\pi(x)\big) \, \mathrm{d}m(s)\mathrm{d}m(t)$$

Proof. By Proposition 5.1.1, it suffices to prove it when all of f_ϵ, $\epsilon \in V_2^*$ are $\mathcal{K}$-measurable. We consider $g_\epsilon = \mathbb{E}(f_\epsilon \mid Z)$, $\epsilon \in V_2^*$ on Z and $f_{00} = \mathbb{1}_X$, thus $g_{00} = \mathbb{E}(f_{00} \mid Z) = \mathbb{1}_Z$. By Von Neumann's Ergodic Theorem (Theorem 1.5.3) and because $(X, \mathcal{B}, \mu, T)$ is ergodic, the average 5.1 converges in $L^2(\mu)$ to

$$\int_X \mathbb{E}(f_{01} \mid \mathcal{K}) \, \mathrm{d}\mu \cdot \int_X \mathbb{E}(f_{10} \mid \mathcal{K}) \, \mathrm{d}\mu \cdot \int_X \mathbb{E}(f_{11} \mid \mathcal{K}) \, \mathrm{d}\mu$$

Now we have,

$$\int_X \mathbb{E}(f_{01} \mid \mathcal{K})\ \mathrm{d}\mu \cdot \int_X \mathbb{E}(f_{10} \mid \mathcal{K})\ \mathrm{d}\mu \cdot \int_X \mathbb{E}(f_{11} \mid \mathcal{K})\ \mathrm{d}\mu = \int_X f_{01}\ \mathrm{d}\mu \cdot \int_X f_{10}\ \mathrm{d}\mu \cdot \int_X f_{11}\ \mathrm{d}\mu$$
$$= \int_{X^3} f_{01} \otimes f_{10} \otimes f_{11}\ \mathrm{d}\mu \otimes \mu \otimes \mu = \int_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11}\ \mathrm{d}\mu^{[2]}$$
$$= \int_{Z^3} g_{00}(z)\ g_{01}(sz)\ g_{10}(tz)\ g_{11}(stz)\ \mathrm{d}m(z)\ \mathrm{d}m(s)\ \mathrm{d}m(t)$$
$$= \int_{Z^3} \mathbb{1}_Z(z)\ g_{01}(sz)\ g_{10}(tz)\ g_{11}(stz)\ \mathrm{d}m(z)\ \mathrm{d}m(s)\ \mathrm{d}m(t)$$
$$= \int_{Z^3} g_{01}(sz)\ g_{10}(tz)\ g_{11}(stz)\ \mathrm{d}m(z)\ \mathrm{d}m(s)\ \mathrm{d}m(t)$$
$$= \int_{Z^3} g_{01}(z')\ g_{10}(t'z')\ g_{11}(st'z')\ \mathrm{d}m(z')\ \mathrm{d}m(s)\ \mathrm{d}m(t')$$
$$= \int_X \int_{Z^2} g_{01}(\pi(x))\ g_{10}(p\pi(x))\ g_{11}(pq\pi(x))\ \mathrm{d}m(p)\ \mathrm{d}m(q)\ \mathrm{d}\mu(x)$$

First equality holds because every f_ϵ, $\epsilon \in V_2^*$ is $\mathcal{K}$-measurable. Second equality holds by Proposition 3.3.2 and Proposition 3.3.4. Seventh equality by setting $z' = sz$ and $t' = ts^{-1}$ and because the Haar measure m is invariant under those translations. Finally, eighth because $\pi :\ X \to Z$ is factor map, thus for every $z \in Z$ there exists $x \in X$ such that $z = \pi(x)$ and $m = (\pi)_*\mu$.
Therefore, indeed, the average 5.1 converges in $L^2(\mu)$ to

$$\int_{K\times K} \mathbb{E}\big(f_{01} \mid Z\big)\big(\pi(x)\big)\ \mathbb{E}\big(f_{10} \mid Z\big)\big(s\pi(x)\big)\ \mathbb{E}\big(f_{11} \mid Z\big)\big(st\pi(x)\big)\ \mathrm{d}m(s)\mathrm{d}m(t)$$

□

Theorem 5.1.3.
Let f_{00},f_{01},f_{10},$f_{11} \in L^\infty(\mu)$, $n_1 \in [M_1, N_1]$ *and* $n_2 \in [M_2, N_2]$, *then*

$$\frac{1}{(N_1 - M_1)(N_2 - M_2)} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \int_X f_{00}(x) f_{01}(T^{n_1}x) f_{10}(T^{n_2}x) f_{11}(T^{n_1+n_2}x)\ \mathrm{d}\mu(x)$$
$$\longrightarrow \int_{X^4} \bigotimes_{\epsilon\in V_2} f_\epsilon\ \mathrm{d}\mu^{[2]}$$

as $N_1 - M_1$ *and* $N_1 - M_1$ *tend to* $+\infty$.

Proof. By Proposition 5.1.2 we have,

$$\frac{1}{n_1 n_2} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \int\limits_X f_{00}(x) f_{01}(T^{n_1}x) f_{10}(T^{n_2}x) f_{11}(T^{n_1+n_2}x) \, \mathrm{d}\mu(x)$$
$$\longrightarrow \int\limits_X \Big(f_{00}(x) \cdot \int\limits_{K\times K} \mathbb{E}\big(f_{01} \mid Z\big)\big(\pi(x)\big) \, \mathbb{E}\big(f_{10} \mid Z\big)\big(s\pi(x)\big) \, \mathbb{E}\big(f_{11} \mid Z\big)\big(st\pi(x)\big) \, \mathrm{d}m(s)\mathrm{d}m(t)\Big) \, \mathrm{d}\mu(x)$$

By Proposition 5.1.1 we can consider the case where f_ϵ, $\epsilon \in V_2$ are $\mathcal{K}$-measurable. Now, with this additional assumption we have,

$$\int\limits_{X^4} f_{00} \otimes f_{01} \otimes f_{10} \otimes f_{11} \, \mathrm{d}\mu^{[2]}$$
$$= \int\limits_{Z^3} \mathbb{E}(f_{00} \mid Z)(z) \, \mathbb{E}(f_{01} \mid Z)(sz) \, \mathbb{E}(f_{10} \mid Z)(tz) \, \mathbb{E}(f_{11} \mid Z)(stz) \, \mathrm{d}m(z)\mathrm{d}m(s)\mathrm{d}m(t)$$
$$= \int\limits_X \int\limits_{Z^2} \mathbb{E}(f_{00} \mid Z)(\pi(x)) \, \mathbb{E}(f_{01} \mid Z)(s\pi(x)) \, \mathbb{E}(f_{10} \mid Z)(t\pi(x)) \, \mathbb{E}(f_{11} \mid Z)(st\pi(x)) \, \mathrm{d}m(s)\mathrm{d}m(t) \, \mathrm{d}\mu(x)$$
$$= \int\limits_X \mathbb{E}(f_{00} \mid Z)(\pi(x)) \int\limits_{Z^2} \mathbb{E}(f_{01} \mid Z)(\pi(x)) \, \mathbb{E}(f_{10} \mid Z)(s\pi(x)) \, \mathbb{E}(f_{11} \mid Z)(st\pi(x)) \, \mathrm{d}m(s)\mathrm{d}m(t) \, \mathrm{d}\mu(x)$$
$$= \int\limits_X \mathbb{E}(f_{00} \mid \mathcal{K})(x) \int\limits_{Z^2} \mathbb{E}(f_{01} \mid Z)(\pi(x)) \, \mathbb{E}(f_{10} \mid Z)(s\pi(x)) \, \mathbb{E}(f_{11} \mid Z)(st\pi(x)) \, \mathrm{d}m(s)\mathrm{d}m(t) \, \mathrm{d}\mu(x)$$
$$= \int\limits_X \Big(f_{00}(x) \cdot \int\limits_{K\times K} \mathbb{E}\big(f_{01} \mid Z\big)\big(\pi(x)\big) \, \mathbb{E}\big(f_{10} \mid Z\big)\big(s\pi(x)\big) \, \mathbb{E}\big(f_{11} \mid Z\big)\big(st\pi(x)\big) \, \mathrm{d}m(s)\mathrm{d}m(t)\Big) \, \mathrm{d}\mu(x)$$

□

Theorem 5.1.4. ***(Bergelson)***
Let A be a subset of X with $\mu(A) > 0$. Then for any $\epsilon > 0$ the set

$$\{(n,m) \in \mathbb{Z} \ : \quad \mu\big(A \cap T^n A \cap T^m A \cap T^{n+m} A)\big) \ \geq \ \mu(A)^4 - \epsilon\}$$

is syndetic.

Proof. By setting for every $\epsilon \in V_2$ in Theorem 5.1.3, $f_\epsilon = \mathbb{1}_A$ and by Corollary 3.3.15 we deduce the requested.

□

5.2 The measure μ_7

Recall that μ_7 is the projection of $\mu^{[3]}$ on X^7 and $\mathcal{J}_7$ is the σ-algebra on X^7 consisting of sets that are invariant under $T_{7,1}$, $T_{7,2}$ and $T_{7,3}$, as defined in Chapter 4

Lemma 5.2.1.

The measure μ_7 is relatively independent with respect to $\mathcal{K}^7$.

That means that if f_ϵ, $\epsilon \in V_3^*$ are seven bounded functions on X, then,

$$\int_{X^7} \bigotimes_{\epsilon \in V_3^*} f_\epsilon \, \mathrm{d}\mu_7 = \int_{X^7} \bigotimes_{\epsilon \in V_3^*} \mathbb{E}(f_\epsilon \mid \mathcal{K}) \, \mathrm{d}\mu_7$$

Proof. Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3^*$ and assume that $\exists\, \eta \in V_3^*$ with $\mathbb{E}(f_\eta \mid \mathcal{K}) = 0$. We will show that $\int_{X^7} \bigotimes_{\epsilon \in V_3^*} f_\epsilon \, \mathrm{d}\mu_7 = 0$

Define $f_{000} = 1$. By definition of μ_7,

$$\int_{X^7} \bigotimes_{\epsilon \in V_3^*} f_\epsilon \, \mathrm{d}\mu_7 = \int_{X^7} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]} = \int_{X^4} \mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) \, \mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 1}} f_\epsilon \mid \mathcal{I}^4) \, \mathrm{d}\mu^{[2]}$$

Firstly assume that $\eta_1 = 0$. Since f_{000} is $\mathcal{K}$-measurable and $\mathbb{E}(f_\eta \mid \mathcal{K}) = 0$ by Lemma 4.1.14 and Lemma 4.1.11, $\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) = 0$, thus the integral above equals to zero.

Now assume that $\eta_1 = 1$. Since f_{000} is $\mathcal{K}$-measurable by Lemma 3.3.25, $\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4)$ is $\mathcal{K}^4$-measurable. Since $\mathbb{E}(f_\eta \mid \mathcal{K}) = 0$, by the same Lemma $\mathbb{E}\big(\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 1}} f_\epsilon \mid \mathcal{I}^4) \mid \mathcal{K}^4\big) = 0$, thus the integral above is also zero. To be more precise

$$\int_{X^4} \mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) \, \mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 1}} f_\epsilon \mid \mathcal{I}^4) \, \mathrm{d}\mu^{[2]} = \int_{X^4} \mathbb{E}\big(\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) \mid \mathcal{K}^4\big) \, \mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 1}} f_\epsilon \mid \mathcal{I}^4) \, \mathrm{d}\mu^{[2]}$$

$$= \int_{X^4} \mathbb{E}\big(\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) \mid \mathcal{K}^4\big) \, \mathbb{E}\big(\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 1}} f_\epsilon \mid \mathcal{I}^4) \mid \mathcal{K}^4\big) \, \mathrm{d}\mu^{[2]}$$

$$= \int_{X^4} \mathbb{E}\big(\mathbb{E}(\bigotimes_{\substack{\epsilon \in V_3 \\ \epsilon_1 = 0}} f_\epsilon \mid \mathcal{I}^4) \mid \mathcal{K}^4\big) \cdot 0 \, \mathrm{d}\mu^{[2]} \; = \; 0$$

□

Corollary 5.2.2.

Let X^7 be endowed with the measure μ_7. Then $\mathcal{J}^7 \subseteq \mathcal{CL}^7$.

Proof. $\mathcal{CL}$ is T-invariant, thus $\mathcal{CL}^7$ is invariant under $T_{7,1}$, $T_{7,2}$ and $T_{7,3}$. So we have that the conditional expectations on $\mathcal{CL}^7$ and on $\mathcal{J}_7$ commute. Therefore it suffices to show that if $f \in L^\infty(\mu_7)$ with $\mathbb{E}(f \mid \mathcal{CL}) = 0$, then $\mathbb{E}(f \mid \mathcal{J}_7) = 0$. Equivalently, it suffices to show that the unique $f \in L^\infty(\mu_7)$, that is also $\mathcal{J}_7$-measurable, with $\mathbb{E}(f \mid \mathcal{CL}) = 0$, is the zero function.
Let f be a function as above. By Lemma 4.1.3 there exists a $\mathcal{CL}$-measurable function $g \in L^\infty(\mu)$ such that

$$g(x_{000}) = f(\tilde{\mathbf{x}}) \qquad \text{for } \mu^{[3]}\text{-almost every } \mathbf{x} = (x_{000}, \tilde{\mathbf{x}}) \in X^8$$

Now,

$$\int_{X^7} f^2 \, \mathrm{d}\mu_7 = \int_{X^8} g(x_{000}) f(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = \int_{X^8} \mathbb{E}(g \mid \mathcal{CL})(x_{000}) \mathbb{E}(f \mid \mathcal{CL}^7)(\tilde{\mathbf{x}}) \, \mathrm{d}\mu^{[3]}(\mathbf{x}) = 0$$

The first equality holds by Proposition 4.1.9 and the last by hypothesis.
Hence $f = 0$ as desired. □

Corollary 5.2.3.
Let X^7 be endowed with the measure μ_7. Let $f_\epsilon \in L^\infty(\mu)$, $\epsilon \in V_3^*$ and assume that $\exists\, \eta \in V_3^*$ such that $\mathbb{E}(f_\eta \mid \mathcal{CL}) = 0$. Then $\mathbb{E}(\bigotimes_{\epsilon \in V_3^*} f_\epsilon \mid \mathcal{J}_7) = 0$.

Proof. Let g_ϵ, $\epsilon \in V_3^*$ be seven bounded functions on X, that are also measurable with respect to $\mathcal{CL}$. By Lemma 5.2.1,

$$\int_{X^7} \bigotimes_{\epsilon \in V_3^*} f_\epsilon \cdot \bigotimes_{\epsilon \in V_3^*} g_\epsilon \mathrm{d}\mu_7 = \int_{X^7} \mathbb{E}(f_\epsilon g_\epsilon \mid \mathcal{CL}) \, \mathrm{d}\mu_7$$

Now, since g_η is $\mathcal{CL}$-measurable, $\mathbb{E}(f_\eta g_\eta \mid \mathcal{CL}) = g_\epsilon \mathbb{E}(f_\eta \mid \mathcal{CL}) = 0$, by hypothesis. Thus $\mathbb{E}(f_\eta g_\eta \mid \mathcal{K}) = 0$, thus $\int_{X^7} \mathbb{E}(f_\epsilon g_\epsilon \mid \mathcal{CL}) \, \mathrm{d}\mu_7 = 0$. To summarize, we proved that $\bigotimes_{\epsilon \in V_3^*} f_\epsilon$ is orthogonal to every $\bigotimes_{\epsilon \in V_3^*} g_\epsilon$ where g_ϵ are $\mathcal{CL}$-measurable, in $L^2(\mu_7)$. By density $\bigotimes_{\epsilon \in V_3^*} f_\epsilon$ is orthogonal, in $L^2(\mu_7)$, to every $\mathcal{CL}^7$-measurable function, thus $\mathbb{E}(\bigotimes_{\epsilon \in V_3^*} f_\epsilon \mid \mathcal{CL}^7) = 0$.
By Corollary 5.2.2 follows that, $\mathbb{E}(\bigotimes_{\epsilon \in V_3^*} f_\epsilon \mid \mathcal{J}_7) = 0$ □

5.3 Reduction to nilsystems

Given seven bounded functions f_ϵ, $\epsilon \in V_3^*$ we consider averages over $n_1 \in [N_1, M_1]$, $n_2 \in [N_2, M_2]$, $n_3 \in [N_3, M_3]$, of the form,

$$\int_X \prod_{\epsilon \in V_3^*} f_\epsilon \circ T^{n\cdot\epsilon} \, \mathrm{d}\mu \tag{5.2}$$

and take limit when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$ tend to $+\infty$ We will show that for the average of 5.2, the Kronecker factor is characteristic. In other words the average converges to 0 whenever $\mathbb{E}(f_\epsilon \mid \mathcal{K}) = 0$ for at least one $\epsilon \in V_3^*$.

Lemma 5.3.1.
Let $f_\eta \in L^\infty(\mu)$, $\eta \in V_2$. Then the limsup, as $N_1 - M_1 \to +\infty$, $N_2 - M_2 \to +\infty$, of

$$\frac{1}{(N_1-M_1)(N_2-M_2)} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \int_X \Big| \frac{1}{N_3-M_3} \sum_{n_3=M_3}^{N_3} \prod_{\eta \in V_2} f_\eta \circ T^{n_1\eta_1+n_2+\eta_2-n_3} \Big|^2 \, \mathrm{d}\mu$$

is less than or equal to

$$\int_{X^4} \Big| \mathbb{E}\big(\otimes_{\eta\in V_2} f_\eta \mid \mathcal{I}^{[2]} \big) \Big|^2 \, \mathrm{d}\mu^{[2]}$$

Proof. Without loss of generality, we can assume that $||f_\eta||_\infty \leq 1$, for each $\eta \in V_2$. Fix an integer $L > 0$. We consider $x_{n_3} = \prod_{\eta \in V_2} f_\eta^{n\cdot\eta - n_3}$, $n = (n_1, n_2) \in \mathbb{Z}^2$ and $n_3 \in \mathbb{Z}$.By van der Corput Lemma (Section $A.4$) for each $n = (n_1, n_2)$,

$$\begin{aligned}
&\int_X \Big| \frac{1}{N_3-M_3} \sum_{n_3=M_3}^{N_3} \prod_{\eta\in V_2} f_\eta \circ T^{n_1\eta_1+n_2+\eta_2-n_3} \Big|^2 \, \mathrm{d}\mu = \left\| \frac{1}{N_3-M_3} \sum_{n_3=M_3}^{N_3} x_{n_3} \right\|_{L^2(\mu)}^2 \\
&\leq \frac{4L}{N_3-M_3} + \sum_{\ell=-L}^{L} \frac{L-|\ell|}{L^2} \frac{1}{N_3-M_3} \sum_{n=M_3}^{N_3} \langle x_{n_3}, x_{n_3+\ell} \rangle \\
&= \frac{4L}{N_3-M_3} + \sum_{\ell=-L}^{L} \frac{L-|\ell|}{L^2} \frac{1}{N_3-M_3} \sum_{n=M_3}^{N_3} \int_X \prod_{\eta\in V_2} f_\eta \circ T^{n\cdot\eta-n_3} \cdot \prod_{\eta\in V_2} f_\eta \circ T^{n\cdot\eta-n_3+\ell} \, \mathrm{d}\mu \\
&= \frac{4L}{N_3-M_3} + \sum_{\ell=-L}^{L} \frac{L-|\ell|}{L^2} \frac{1}{N_3-M_3} \sum_{n=M_3}^{N_3} \int_X \prod_{\eta\in V_2} (f_\eta \circ T^{n\cdot\eta-n_3} \cdot f_\eta \circ T^{n\cdot\eta-n_3+\ell}) \, \mathrm{d}\mu
\end{aligned}$$

$$= \frac{4L}{N_3 - M_3} + \sum_{\ell=-L}^{L} \frac{L - |\ell|}{L^2} \frac{1}{N_3 - M_3} \sum_{n=M_3}^{N_3} \int_X \prod_{\eta \in V_2} (f_\eta \circ T^{n \cdot \eta} \cdot f_\eta \circ T^{n \cdot \eta + \ell}) \, \mathrm{d}\mu$$

$$= \frac{4L}{N_3 - M_3} + \sum_{\ell=-L}^{L} \frac{L - |\ell|}{L^2} \frac{1}{N_3 - M_3} (N_3 - M_3) \int_X \prod_{\eta \in V_2} (f_\eta \circ T^{n \cdot \eta} \cdot f_\eta \circ T^{n \cdot \eta + \ell}) \, \mathrm{d}\mu$$

$$= \frac{4L}{N_3 - M_3} + \sum_{\ell=-L}^{L} \frac{L - |\ell|}{L^2} \int_X \prod_{\eta \in V_2} (f_\eta \circ T^{n \cdot \eta} \cdot f_\eta \circ T^{n \cdot \eta + \ell}) \, \mathrm{d}\mu$$

Thus the limsup of the average stated above is bounded by

$$\sum_{\ell=-L}^{L} \frac{L - |\ell|}{L^2} \limsup_{\substack{N_1 - M_1 \to \infty \\ N_2 - M_2 \to \infty}} \frac{1}{(N_1 - M_1)(N_2 - M_2)} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \int_X \prod_{\eta \in V_2} (f_\eta \cdot f_\eta \circ T^\ell) \, \mathrm{d}\mu$$

By Theorem 5.1.3 the limsup above is

$$\sum_{\ell=-L}^{L} \frac{L - |\ell|}{L^2} \int_{X^4} \bigotimes_{\eta \in V_2} (f_\eta \cdot f_\eta \circ T^\ell) \, \mathrm{d}\mu^{[2]}$$

By taking the limits as $L \to \infty$ we obtain the bound. $\square$

Lemma 5.3.2.

The Kronecker factor is characteristic for the average of the integral 5.2*. In other words the average converges to* 0 *whenever* $\mathbb{E}(f_\epsilon \mid \mathcal{K}) = 0$ *for at least one* $\epsilon \in V_3^*$*.*

Proof. Firstly, lets assume that $\mathbb{E}(f_\epsilon \mid \mathcal{K}) = 0$ for some $\epsilon \in \{001,\ 010,\ 011\}$. Set $g_\eta = f_{0\eta_1\eta_2}$, for $\eta \in V_2^*$ and $g_{00} = 1$. Then g_{00} is $\mathcal{K}$-measurable and $\mathbb{E}(g_\eta \mid \mathcal{K}) = 0$ for some $\eta \in V_2^*$. By Lemma 3.3.25, $\mathbb{E}(\bigotimes_{\eta \in V_2} g_\eta \mid \mathcal{I}^{[2]}) = 0$. By Lemma 5.3.1,

$$\frac{1}{(N_1 - M_1)(N_2 - M_2)} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \int_X \Big| \frac{1}{N_3 - M_3} \sum_{n_3=M_3}^{N_3} \prod_{\eta \in V_2} f_\eta \circ T^{n_1\eta_1 + n_2 + \eta_2 - n_3} \Big|^2 \, \mathrm{d}\mu \to 0$$

as $N_1 - M_1 \to +\infty$, $N_2 - M_2 \to +\infty$. For $n = (n_3, n_1, n_2)$ we can rewrite the above average of the integral 5.2, as the average over $n_1 \in [M_1, N_1]$ and $n_2 \in [M_2, N_2]$ of

$$\int_X \left(\frac{1}{N_3 - M_3} \sum_{n_3=M_3}^{N_3} \prod_{\eta \in V_2} g_\eta \circ T^{n_1\eta_1 + n_2\eta_2 - n_3} \right) \left(\prod_{\epsilon \in V_3, \epsilon_0 = 1} f_\epsilon \circ T^{n_1\epsilon_1 + n_2\epsilon_2} \right) \mathrm{d}\mu$$

Applying Cauchy-Schwartz to this average and using the limit above, we have that this average converges to 0. The same process can be applied for $\epsilon \in \{100,\ 101,\ 111\}$.

Finally we consider the case where $\mathbb{E}(f_{111} \mid \mathcal{K}) = 0$. By the preceding steps we can assume that each of the other functions is $\mathcal{K}$-measurable. Set $h_\eta = f_{1\eta_1\eta_2}$ for $\eta \in V_2^*$. Then (as mentioned before) $h_{00},\ h_{01},\ h_{10}$ are $\mathcal{K}$-measurable and $\mathbb{E}(h_{11} \mid \mathcal{K}) = 0$. By Lemma 3.3.25, $\mathbb{E}(\bigotimes_{\eta\in V_2} h_\eta \mid \mathcal{I}^{[2]}) = 0$. We conclude as above, by using again Lemma 5.3.1 and the Cauchy-Schwartz inequality. □

For the following part we omit some of the proofs, as they are similar to those of Chapter 3. Define Z_7 to be the closed subgroup of Z^7 where

$$Z_7 = \{(za, zb, zab, zc, zac, zbc, zabc) \ : \quad z, a, b, c \in Z\}$$

and let m_7 be the Haar measure of Z_7. Then the measure m_7 is the projection of μ_7 on Z^7 and if g_ϵ, $\epsilon \in V_3^*$ are seven bounded functions on Z

$$\int_{Z^7} \bigotimes_{\epsilon\in V_3^*} g_\epsilon \ \mathrm{d}m_7 = \int_{Z^4} \prod_{\epsilon\in V_3^*} g_\epsilon(zs_1^{\epsilon_1} s_2^{\epsilon_2} s_3^{\epsilon_3}) \ \mathrm{d}m(s_1)\mathrm{d}m(s_2)\mathrm{d}m(s_3)\mathrm{d}m(z)$$

Proposition 5.3.3.
Let $f_\epsilon \in L^\infty(\mu)$. The average of integral 5.2 *converges to*

$$\int_{X^7} \bigotimes_{\epsilon\in V_3^*} f_\epsilon \ \mathrm{d}\mu_7$$

Proof. By Lemma 5.2.1 and $m_7 = (\pi_7)_*\mu_7$

$$\int_{X^7} \bigotimes_{\epsilon\in V_3^*} f_\epsilon \ \mathrm{d}\mu_7 = \int_{Z^7} \bigotimes_{\epsilon\in V_3^*} \mathbb{E}(f_\epsilon \mid Z) \ \mathrm{d}m_7 = \int_{Z_7} \bigotimes_{\epsilon\in V_3^*} \mathbb{E}(f_\epsilon \mid Z) \ \mathrm{d}m_7$$

By Lemma 5.3.2 it suffices to prove the result when f_ϵ are all $\mathcal{K}$-measurable. $\big($ Thus from now on we can continue with the assumption that $(X, \mu, T) = (Z, m, R)$ and $(X^7, \mu_7, T^7) = (Z_7, m_7, R^7)\ \big)$.
By density it suffices to prove the result for functions of the form $\bigotimes_{\epsilon\in V_3^*} g_\epsilon$ where each of the functions g_ϵ is a character of Z, in particular is continuous. Now since (Z_7, R^7) is uniquely ergodic (by the uniqueness of the Haar measure), the map $Z_7 \to \mathbb{R}$, defined by

$$(z_\epsilon)_{\epsilon\in V_3^*} \mapsto \frac{1}{N_1 - M_1} \frac{1}{N_2 - M_2} \frac{1}{N_3 - M_3} \sum_{n_1=M_1}^{N_1} \sum_{n_2=M_2}^{N_2} \sum_{n_3=M_3}^{N_3} \prod_{\epsilon\in V_3^*} g_\epsilon \circ R^{\epsilon_1 n_1 + \epsilon_2 n_2 + \epsilon_3 n_3}(z_\epsilon)$$

converge uniformly to the constant

$$\int_{Z_7} \bigotimes_{\epsilon\in V_3^*} g_\epsilon \ \mathrm{d}\mu_7$$

Thus this average converges uniformly to the same constant on the diagonal subset of Z_7 , $\{(z,z,z,z,z,z,z)\ :\ \ z \in Z\}$. In other words the average

$$\frac{1}{N_1-M_1}\frac{1}{N_2-M_2}\frac{1}{N_3-M_3}\sum_{n_1=M_1}^{N_1}\sum_{n_2=M_2}^{N_2}\sum_{n_3=M_3}^{N_3}\prod_{\epsilon\in V_3^*} g_\epsilon \circ R^{\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3}(z)$$

converges uniformly to the same constant. Taking the integral , we get that the average

$$\frac{1}{N_1-M_1}\frac{1}{N_2-M_2}\frac{1}{N_3-M_3}\sum_{n_1=M_1}^{N_1}\sum_{n_2=M_2}^{N_2}\sum_{n_3=M_3}^{N_3}\int_X\prod_{\epsilon\in V_3^*} g_\epsilon \circ R^{\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3}(z)\ \mathrm{d}m$$

converges to

$$\int_{Z_7}\bigotimes_{\epsilon\in V_3^*} g_\epsilon\ \mathrm{d}\mu_7$$

□

We now study averages over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$ and $n_3 \in [M_3, N_3]$ of

$$\prod_{\epsilon\in V_3^*} f_\epsilon\big(T^{\epsilon_1 n_1+\epsilon_2 n_2+\epsilon_3 n_3} x_\epsilon\big) \tag{5.3}$$

in $L^2(\mu)$ as $N_1 - M_1$, $N_2 - M_2$ and $N_3 - M_3$ tend to $+\infty$

Lemma 5.3.4.
The factor $\mathcal{CL}$ of X is characteristic for the convergence in $L^2(\mu)$ of the average of the product 5.3. *This means that this average converges to* 0 *if there exists at least one $\epsilon \in V_3^*$ such that $\mathbb{E}(f_\epsilon \mid \mathcal{CL}) = 0$.*

Proof. For $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$ set

$$u_n = \prod_{\epsilon\in V_3^*} f_\epsilon \circ T^{n\cdot\epsilon} = \prod_{\epsilon\in V_3^*} f_\epsilon \circ T^{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}$$

Now, for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ we have

$$\langle u_{k+n}, u_n\rangle = \int_X u_{k+n}u_n\ \mathrm{d}\mu = \int_X (f_\epsilon \circ T^{k\cdot\epsilon}\cdot f_\epsilon)\circ T^{n\cdot\epsilon}\ \mathrm{d}\mu$$

By Proposition 5.3.3, for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, the averages over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$ and $n_3 \in [M_3, N_3]$ as $N_1 - M_1$, $N_2 - M_2$ and $N_3 - M_3$ tend to $+\infty$ of

$$\int_X u_{k+n} u_n \, \mathrm{d}\mu$$

converge to

$$\gamma_k = \int_{X^7} \prod_{\epsilon \in V_3^*} f_\epsilon(T^{k\cdot\epsilon} x_\epsilon) f_\epsilon(x_\epsilon) \, \mathrm{d}\mu_7 = \int_{X^7} F(T_{7,1}^{k_3} T_{7,2}^{k_2} T_{7,3}^{k_1} \tilde{\mathbf{x}}) F(\tilde{\mathbf{x}}) \, \mathrm{d}\mu_7$$

where $F(\tilde{\mathbf{x}}) = \prod_{\epsilon \in V_3^*} f_\epsilon(x_\epsilon)$ for $\tilde{\mathbf{x}} = (x_\epsilon)_{\epsilon \in V_3^*}$. Thus the average of γ_k over $k_1, k_2, k_3 \in [0, K]$ converges to

$$\int_{X^7} \mathbb{E}(F \mid \mathcal{CL}) F \, \mathrm{d}\mu_7 = \int_{X^7} \mathbb{E}^2(F \mid \mathcal{CL}) \, \mathrm{d}\mu_7 = \|\mathbb{E}(F \mid \mathcal{CL})\|^2_{L^2(\mu_7)}$$

By Lemma 5.2.3 if there exists at least one $\epsilon \in V_3^*$ such that $\mathbb{E}(f_\epsilon \mid \mathcal{CL}) = 0$ the average of γ_k converges to 0. By van der Corput Lemma $(A.4)$ the result follows. □

The product of 7 terms

In order to prove Theorem4 it suffices to prove it when all the functions are measurable with respect to $\mathcal{CL}$ since, if f_ϵ, $\epsilon \in V_3^*$, are seven bounded functions on X, by Lemma 5.3.4, the difference between the average of the product (5.3) and the same average with $\mathbb{E}(f_\epsilon \mid \mathcal{CL})$ substituted for each f_ϵ, converges to 0 in $L^2(\mu)$. Therefore, we can restrict to the case that the system itself is a $\mathcal{CL}$-system, meaning that the system is equal to its Conze-Lesigne algebra.

Furthermore, a $\mathcal{CL}$-system is an inverse limit of 2-step nilsystems. By density, it suffices to prove Theorem4 for such systems.

The integral with 8 terms

Theorem 5.3.5.
Let f_ϵ, $\epsilon \in V_3$ be eight functions on the ergodic system (X, μ, T). Then the average over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$, $n_3 \in [M_3, N_3]$ of

$$\int_X \prod_{\epsilon \in V_3} f_\epsilon\big(T^{n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3} x\big) \, \mathrm{d}\mu(x) = \int_X f_{000}(x) \prod_{\epsilon \in V_3^*} f_\epsilon\big(T^{n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3} x\big) \, \mathrm{d}\mu(x) \tag{5.4}$$

converges to

$$\int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]}$$

when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$ *tend to* $+\infty$.

We will take as a fact that the theorem holds for 2-step nilsystems, which is proven in the next chapter (Theorem 6.2.6).
By Lemma 5.3.4 the difference between the average 5.4 and the same average with each of the function f_ϵ is replaced by $\mathbb{E}(f_\epsilon \mid \mathcal{CL})$. By Proposition 4.1.9,

$$\int_{X^8} \bigotimes_{\epsilon \in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]} = \int_{X^8} \bigotimes_{\epsilon \in V_3} \mathbb{E}(f_\epsilon \mid \mathcal{CL}) \, \mathrm{d}\mu^{[3]}$$

Thus it suffices to prove it when (X, μ, T) is a Conze-Lesigne system. In this case by Theorem 4.3.16, X is the inverse limit of a sequence of 2-step nilsystems. So it suffices to prove it for 2-step nilsystems and the result follows from density.

Combining Corollary 3.4.6 and the Theorem above, we gain Theorem 2.

Chapter 6

Convergence for a nilsystem

We are left with showing that Theorem 5.3.5 holds for ergodic 2-step nilsystems. The proof uses the precise description of measures $\mu^{[2]}$ and $\mu^{[3]}$ in the particular case of a 2-step nilsystem.

Throughout this chapter the system (X, μ, T) is an ergodic 2-step nilsystem. We assume that the hypotheses H_1 and H_2 are satisfied.

6.1 The manifold X_4

Define

$$G_4 = \{\mathbf{g} = (g_{00}, g_{01}, g_{10}, g_{11}) \in G^4 \ : \quad g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} \in [G, G]\},$$
$$\Lambda_4 = \{\boldsymbol{\lambda} = (\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}) \in \Lambda^4 \ : \quad \lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11} = 1\} \ = \ \Lambda^4 \cap G_4$$

Proposition 6.1.1.
G_4 is closed subgroup of G^4, thus is a closed Lie subgroup of G^4. In addition G_4 is 2-step nilpotent.

Proof.

- $G_4 \leq G^4$
  Let $\mathbf{g}$, $\mathbf{h} \in G_4 \ \Rightarrow \ g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} = u_1, \ h_{00}h_{01}^{-1}h_{10}^{-1}h_{11} = u_2 \in [G, G]$

We will show $\mathbf{g} \cdot \mathbf{h} \in G_4$:

$$
\begin{aligned}
&g_{00}h_{00}h_{01}^{-1}g_{01}^{-1}h_{10}^{-1}g_{10}^{-1}g_{11}h_{11} = g_{00}(u_2h_{11}^{-1}h_{10})g_{01}^{-1}h_{10}^{-1}(g_{01}g_{00}^{-1}u_1)h_{11} \\
&= g_{00}u_2h_{11}^{-1}\underbrace{(h_{10})g_{01}^{-1}h_{10}^{-1}g_{01})}_{=u_3\in[G,G]}g_{00}^{-1}u_1h_{11} = g_{00}u_2h_{11}^{-1}u_3g_{00}^{-1}u_1h_{11} \\
&= u_2\underbrace{g_{00}h_{11}^{-1}g_{00}^{-1}h_{11}}_{\in[G,G]}u_3u_1 \quad \in [G,G] \trianglelefteq G
\end{aligned}
$$

The last equality holds because $u_1, u_2, u_3 \in [G,G] \subseteq Z(G)$.
Now we will show $\mathbf{g}^{-1} \in G_4$. We have that $\mathbf{g}^{-1} = (g_{00}^{-1}, g_{01}^{-1}, g_{10}^{-1}, g_{11}^{-1})$ and $g_{00}^{-1}g_{01} = g_{10}^{-1}g_{11}u_1^{-1}$. Now,

$$
g_{00}^{-1}g_{01}g_{10}g_{11}^{-1} = g_{10}^{-1}g_{11}u^{-1}g_{10}g_{11}^{-1} = \underbrace{g_{10}^{-1}g_{11}g_{10}g_{11}^{-1}}_{\in[G,G]}u^{-1} \quad \in [G,G] \trianglelefteq G
$$

- G_4 is closed in G^4

  Let $(\mathbf{g}_n)_{n\in\mathbb{N}}$ a sequence in G_4 and $\mathbf{g} \in G^4$ such that $\mathbf{g}_n \to \mathbf{g}$. Equivalently, $g_\epsilon^n \to g_\epsilon$, for any $\epsilon \in V_2$. By the continuity of the maps $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2 : \ G \times G \to G$ and $\alpha \mapsto \alpha^{-1} : \ G \to G,$ we have that,

  $$
  g_{00}^n(g_{00}^n)^{-1}(g_{00}^n)^{-1}g_{00}^n \to g_{00}g_{01}^{-1}g_{10}^{-1}g_{11}
  $$

  Now $g_{00}^n(g_{00}^n)^{-1}(g_{00}^n)^{-1}g_{00}^n \in [G,G]$, $\forall n \in \mathbb{N}$ and by Theorem 1.11.10, $[G,G]$ is closed in G. Hence $g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} \in [G,G] \ \Rightarrow \mathbf{g} \in G_4$

- G_4 is 2-step nilpotent

  That means $[G_4, G_4] \subseteq Z(G_4)$. In other words, if $\mathbf{g} \cdot \mathbf{h} \in G_4$ then $\mathbf{g}\mathbf{h}\mathbf{g}^{-1}\mathbf{h}^{-1} \iff \alpha\mathbf{g}\mathbf{h}\mathbf{g}^{-1}\mathbf{h}^{-1} = \mathbf{g}\mathbf{h}\mathbf{g}^{-1}\mathbf{h}^{-1}\alpha,$ for every $\alpha \in G_4 \iff \alpha_\epsilon g_\epsilon h_\epsilon g_\epsilon^{-1}h_\epsilon^{-1} = g_\epsilon h_\epsilon g_\epsilon^{-1}h_\epsilon^{-1}\alpha_\epsilon$, $\forall \epsilon \in V_2,\ \forall \alpha = (\alpha_\epsilon)_{\epsilon\in V_2} \in G_4$ and the last statement is true, since $g_\epsilon h_\epsilon g_\epsilon^{-1}h_\epsilon^{-1} \in [G,G]$ and $[G,G] \subseteq Z(G)$

□

Proposition 6.1.2.

The commutator, $[G_4, G_4]$, of G_4 is equal to $[G,G]^4$.

Proof. For showing $[G_4, G_4] \subseteq [G,G]^4$ it suffices to show it for the generators of $[G_4, G_4]$. Let $\mathbf{g}, \mathbf{h} \in G_4$. Then $\mathbf{g}\mathbf{h}\mathbf{g}^{-1}\mathbf{h}^{-1} = (g_{00}h_{00}g_{00}^{-1}h_{00}^{-1}, \ldots)$ which is clearly element of $[G,G]^4$.

Conversely, let $u \in [G,G]$ then $(1,1,1,u),\ (1,1,u,1),\ (1,u,1,1),\ (u,1,1,1) \in G^4$. In particular they are in $[G_4, G_4]$. Now if $(u_{00}, u_{01}, u_{10}, u_{11}) \in [G,G]^4$ then $(u_{00}, u_{01}, u_{10}, u_{11}) = (u_{00}, 1, 1, 1)(1, u_{01}, 1, 1)(1, 1, u_{10}, 1)(1, 1, 1, u_{11}) \quad \in [G_4, G_4] \trianglelefteq G_4$ □

Proposition 6.1.3.
Λ_4 is discrete, cocompact subgroup of G_4, with $\Lambda_4 \cap [G,G]^4 = \{1\}$.

Proof.

- $\Lambda_4 \cap [G,G]^4 = \{1\}$
  Since $\Lambda \cap [G,G] = \{1\}$, then $\Lambda^4 \cap [G,G]^4 = \{1\}$.
  Thus $\Lambda_4 \cap [G,G]^4 = \Lambda^4 \cap G_4 \cap [G,G]^4 = \{1\}$.

- $\Lambda_4 \leq G_4$
  $\Lambda_4 = \Lambda^4 \cap G_4 \leq G^4 \ \Rightarrow \Lambda_4 \subseteq G_4$.
  Λ^4, G_4 are subgroups of G^4, thus $\Lambda_4 = \Lambda^4 \cap G_4 \leq G^4$.
  Hence $\Lambda_4 \leq G_4$.

- Λ_4 is closed in G_4
  Since Λ is closed in G then, Λ^4 is closed in G^4. Thus $\Lambda_4 = \Lambda^4 \cap G_4$ is closed in G_4

- Λ_4 is discrete subgroup of G_4

- G_4/Λ^4 is compact
  By Lemma $A.8$ it suffices to prove that $G_4\Lambda_4$ is a closed subset of G^4. Define the continuous, onto maps $\phi : G^4 \to G$, with $\phi(g_{00}, g_{01}, g_{10}, g_{11}) = g_{00}g_{01}^{-1}g_{10}^{-1}g_{11}$ and $\pi : G \to G/[G,G]$ to be the natural projection. Now define $h : G^4 \to G/[G,G]$ with $h(g_{00}, g_{01}, g_{10}, g_{11}) = g_{00}g_{01}^{-1}g_{10}^{-1}g_{11}[G,G]$, which is as well onto and continuous. Since G is 2-step nilpotent, $G/[G,G]$ is an abelian group. Thus h is a group homomorphism:

$$
\begin{aligned}
&h\big((g_{00}, g_{01}, g_{10}, g_{11})(g'_{00}, g'_{01}, g'_{10}, g'_{11})\big) \ = \ g_{00}g'_{00}(g'_{01})^{-1}g_{01}^{-1}(g'_{10})^{-1}g_{10}^{-1}g_{11}g'_{11}[G,G] \\
&= (g_{00}[G,G])(g'_{00}[G,G])((g'_{01})^{-1}[G,G])(g_{01}^{-1}[G,G])((g'_{10})^{-1}[G,G])(g_{10}^{-1}[G,G])(g_{11}[G,G])(g'_{11}[G,G]) \\
&= (g_{00}[G,G])(g_{01}^{-1}[G,G])(g_{10}^{-1}[G,G])(g_{11}[G,G])(g'_{00}[G,G])((g'_{01})^{-1}[G,G])((g'_{10})^{-1}[G,G])(g'_{11}[G,G]) \\
&= (g_{00}g_{01}^{-1}g_{10}^{-1}g_{11}[G,G])\ (g'_{00}{g'_{01}}^{-1}{g'_{10}}^{-1}g'_{11}[G,G]) \\
&= h\big((g_{00}, g_{01}, g_{10}, g_{11})\big)\ h\big((g'_{00}, g'_{01}, g'_{10}, g'_{11})\big)
\end{aligned}
$$

Therefore if $(g_{00}, g_{01}, g_{10}, g_{11}) \in G_4$ and $(\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}) \in \Lambda^4$, we have that,

$$\begin{aligned} &g_{00}\lambda_{00}\lambda_{01}^{-1}g_{01}^{-1}\lambda_{10}^{-1}g_{10}^{-1}g_{11}\lambda_{11}[G,G] = h\big((g_{00}, g_{01}, g_{10}, g_{11})(\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11})\big) \\ &= h(g_{00}, g_{01}, g_{10}, g_{11})\ h(\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}) = \ (g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})(\lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11})[G,G] \end{aligned}$$

Hence there exists a $u \in [G,G]$ so that

$$g_{00}\lambda_{00}\lambda_{01}^{-1}g_{01}^{-1}\lambda_{10}^{-1}g_{10}^{-1}g_{11}\lambda_{11} = (g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})(\lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11})u$$

and since $[G,G] \subseteq Z(G)$,

$$(g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})(\lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11})u \ = \ u(g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})(\lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11})$$

and the equality above becomes,

$$g_{00}\lambda_{00}\lambda_{01}^{-1}g_{01}^{-1}\lambda_{10}^{-1}g_{10}^{-1}g_{11}\lambda_{11} \ = \ u(g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})(\lambda_{00}\lambda_{01}^{-1}\lambda_{10}^{-1}\lambda_{11}) \quad \in [G,G]\Lambda$$

In other words, $\phi(G_4\Lambda^4) \subseteq [G,G]\Lambda$, and since ϕ is onto it follows that

$$G_4\Lambda^4 \subseteq \phi^{-1}\big(\phi(G_4\Lambda^4)\big) \subseteq \phi^{-1}([G,G]\Lambda)$$

Conversely, let $\phi(g_{00}, g_{01}, g_{10}, g_{11}) \in [G,G]\Lambda$. Then there exists a $v \in [G,G]$ and a $\lambda \in \Lambda$, such that $g_{00}g_{01}^{-1}g_{10}^{-1}g_{11} = v\lambda$. Equivalently, $g_{00}g_{01}^{-1}g_{10}^{-1}g_{11}\lambda^{-1} = v \in [G,G]$. Thus, by the definition of G_4, $(g_{00}, g_{01}, g_{10}, g_{11}\lambda^{-1}) \in G_4$. Follows that,

$$(g_{00}, g_{01}, g_{10}, g_{11}) = (g_{00}, g_{01}, g_{10}, g_{11}\lambda^{-1})(1_G, 1_G, 1_G, \lambda) \quad \in G_4\Lambda^4$$

This means that $\phi^{-1}([G,G]\Lambda) \subseteq G_4\Lambda_4$.
By summarizing, we have that $\phi^{-1}([G,G]\Lambda) = G_4\Lambda^4$. By Theorem 1.11.10 we have that $[G,G]\Lambda$ is a closed subgroup of G and since ϕ is continuous, it follows that $G_4\Lambda^4$ is a closed subset of G^4 and that completes the proof.

□

There is a natural embedding of the manifold $G_4/\Lambda_4 \hookrightarrow X^4 = G^4/\Lambda^4$ and we can identify this manifold with its image in X^4

(6.1) $$X_4 = \{\mathbf{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^4 \ : \quad q(x_{00})q(x_{01})^{-1}q(x_{10})^{-1}q(x_{11}) = 1\}$$

where $q : G/\Lambda \to G/\Lambda[G,G]$ is the natural projection of G/Λ on $G/\Lambda[G,G]$.

Lemma 6.1.4.
The measure $\mu^{[2]}$ is the Haar measure of $X_4 = G_4/\Lambda_4$

Proof. Let $\mu'^{[2]}$ be the Haar measure of X_4. The transformations $T_4 = T^{[2]}$, $T_{4,1}$, $T_{4,2}$ of X^4 are the translations by the elements $\alpha_4 = (\alpha, \alpha, \alpha, \alpha)$, $\alpha_{4,1} = (1, \alpha, 1, \alpha)$ and $\alpha_{4,2} = (1, 1, \alpha, \alpha)$, of G^4, respectively. In addition α_4, $\alpha_{4,1}$, $\alpha_{4,2}$ are elements of G_4. Then these transformation are translation on $X_4 \simeq G_4/\Lambda_4$. In particular they leave the measure $\mu'^{[2]}$ invariant.
We will show that $\mu'^{[2]}$ is ergodic under the action of $T_4 = T^{[2]}$, $T_{4,1}$, and $T_{4,2}$. We will use the Proposition 1.11.14. We observe that the hypothesis (1.3) of this Proposition is satisfied for $X_4 \simeq G_4/\Lambda_4$, since $G = \langle G_0, \alpha\rangle$ $\big($Property (1.1) $\big)$.
Let $q : G \to K$, where $K = G/\Lambda[G, G]$ $\big($ K is compact and by Proposition 1.11.13 is the Kronecker factor of $X = G/\Lambda$ $\big)$ and q is the natural projection (q is continuous, open, group epimorphism). Define $q_4 : G_4 \to K^4$ with

$$q_4(g_{00}, g_{01}, g_{10}, g_{11}) = (q(g_{00}), q(g_{01}), q(g_{10}), q(g_{11}))$$

Then q_4 is a continuous group homomorphism. In addition

$$\begin{aligned} q_4\big(G_4\big) &= \{(k_{00}, k_{01}, k_{10}, k_{11}) \in K^4 : \; k_{00}k_{01}^{-1}k_{10}^{-1}k_{11} = 1\} \\ &= \big\{(z, sz, tz, stz) : \; z, t, s \in K\big\} = K_4 \end{aligned}$$

and $kerq_4 = (\Lambda^4[G, G]^4) \cap G_4 = (\Lambda^4[G_4, G_4]) \cap G_4 = \Lambda_4[G_4, G_4] = \Lambda_4[G, G]^4$. Thus, by first Group isomorphism theorem, $K_4 \simeq G_4/\Lambda_4[G, G]^4$. Under this identification the transformations R_4, $R_{4,1}$, $R_{4,2}$, induced by T_4, $T_{4,1}$, $T_{4,2}$ through q_4, on K_4 are the rotations by $\beta_4 = (\beta, \beta, \beta, \beta)$, $\beta_{4,1} = (1, \beta, 1, \beta)$ and $\beta_{4,2} = (1, 1, \beta, \beta)$, respectively, where $\beta = q(\alpha)$.
Now (K, m, R) (where $R(k) = \beta \cdot k$) is ergodic since, (X, μ, T) is ergodic. Equivalently (Theorem 1.9.1) the subgroup of K generated by the element β, $\langle\beta\rangle$, is dense in K. It follows that the subgroup, $\langle\beta_4, \beta_{4,1}, \beta_{4,2}\rangle$, of K_4 is dense in K_4 (since for any element of K_4, $(z, sz, tz, stz) = (z, z, z, z)(1, s, 1, s)(1, 1, t, t)$ and β, $\langle\beta\rangle$, is dense in K). Equivalently the joint action of the rotations R_4, $R_{4,1}$, $R_{4,2}$ is ergodic on K_4. By Proposition 1.11.14 the measure $\mu'[2]$ is ergodic on X_4 under the joint action of T_4, $T_{4,1}$, $T_{4,2}$. By Proposition 1.11.12 the measure $\mu'^{[2]}$ is the unique measure on X_4 that is invariant under the T_4, $T_{4,1}$, $T_{4,2}$.

Now the measure $\mu^{[2]}$ is also invariant under those transformations on X_4. By Proposition 3.3.4 and 6.1, $\mu^{[2]}\big(X_4\big) = \mu^{[2]}\big(q_4^{-1}(K_4)\big) = m_4\big(K_4\big) = 1$. In other word $\mu^{[2]}$ is concentrated on X_4. It follows $\mu^{[2]} = \mu'^{[2]}$. □

6.2 The manifold X_8

Define

$$H = \{(h, hu, hv, huv) \ : \quad h \in G,\ u, v \in [G,G]\}$$

and

$$G_8 = \{(g_{00}, g_{01}, g_{10}, g_{11}, h_{00}g_{00}, h_{01}g_{01}, h_{10}g_{10}, h_{11}g_{11}) \ : \\ (g_{00}, g_{01}, g_{10}, g_{11}) \in G_4,\ (h_{00}, h_{01}, h_{10}, h_{11}) \in H\}$$

Proposition 6.2.1.
H is a normal subgroup of G_4. Furthermore H is closed in G_4, thus is a closed Lie subgroup of G_4

Proof.

- $H \leq G_4$
  Let $\alpha_1 = (h_1, h_1u, h_1v_1, h_1u_1v_1)$, $\alpha_2 = (h_2, h_2u_2, h_2v_2, h_2u_2v_2) \in H$. Now since $u_1, v_1, u_2, v_2 \in [G,G] \subseteq Z(G)$, then $gu_i = u_ig$ and $gv_i = v_ig$, $\forall g \in G$ and $\forall i \in \{1,2\}$. Thus,

$$\begin{aligned}\alpha_1\alpha_2^{-1} &= (h_1h_2^{-1}, h_1u_1u_2^{-1}h_2^{-1}, h_1v_1v_2^{-1}h_2^{-1}, h_1u_1v_1v_2^{-1}u_2^{-1}h_2^{-1}) \\ &= ((h_1h_2^{-1}, h_1h_2^{-1}u_1u_2^{-1}, h_1h_2^{-1}v_1v_2^{-1}, h_1h_2^{-1}u_1v_1v_2^{-1}u_2^{-1}) \quad \in H\end{aligned}$$

- H is closed in G_4
  Let $\big((h_n, h_nu_n, h_nv_n, h_nu_nv_n)\big)_{n\in\mathbb{N}}$ a sequence in H and $(g_{00}, g_{01}, g_{10}, g_{11}) \in G_4$ such that $(h_n, h_nu_n, h_nv_n, h_nu_nv_n) \to (g_{00}, g_{01}, g_{10}, g_{11})$. This means $h_n \to g_{00}$ and $h_nu_n \to g_{01}$ and $h_nv_n \to g_{10}$ and $h_nu_nv_n \to g_{11}$.
  Now we have $[G,G] \ni u_n = (h_nu_n)h_n^{-1} \to g_{01}g_{00}^{-1}$ and since $[G,G]$ is closed in G, $g_{01}g_{00}^{-1} \in [G,G]$. Then $g_{01} = \big(\underbrace{g_{01}g_{00}^{-1}}_{=u\in[G,G]}\big)\, g_{00} = uh \underset{u\in[G,G]}{=} hu$. In the same manner $g_{10} = hv$, where $h = g_{00}$ and $v = g_{10}g_{00}^{-1} = \lim\limits_{n\to\infty} v_n$. Finally for g_{11}, by continuity, $g_{11} = \lim\limits_{n\to\infty} h_nu_nv_n = huv$. To summarize, $\big(g_{00}, g_{01}, g_{10}, g_{11}\big) = (h, hu, hv, huv)$ for some $h \in G$ and $u,\ v \in [G,G]$, thus it is an element of H.

- $H \trianglelefteq G_4$

  It suffices to prove that $\mathbf{g}H\mathbf{g}^{-1} \subseteq H$ for every $\mathbf{g} \in G_4$. Then $\forall \mathbf{g} \in G_4$ we will have that $\mathbf{g}H\mathbf{g}^{-1} \subseteq H$ and $\mathbf{g}^{-1}H\mathbf{g} \subseteq H$ and thus $\mathbf{g}H \subseteq H\mathbf{g}$ and $H\mathbf{g} \subseteq \mathbf{g}H$. In particular we wil have that $H\mathbf{g} = \mathbf{g}H$, $\forall \mathbf{g} \in G_4$.

  Let $\mathbf{g} = (g_{00},\ g_{01},\ g_{10},\ g_{11}) \in G_4$, $u, v \in [G, G]$ and $h \in G$. We will show that

$$(g_{00},\ g_{01},\ g_{10},\ g_{11}) \cdot (h,\ hu,\ hv,\ huv) \cdot (g_{00},\ g_{01},\ g_{10},\ g_{11})^{-1} \ \in H$$

  Now since $u, v \in [G, G] \subseteq Z(G)$,

$$\begin{aligned}
&(g_{00},\ g_{01},\ g_{10},\ g_{11}) \cdot (h,\ hu,\ hv,\ huv) \cdot (g_{00},\ g_{01},\ g_{10},\ g_{11})^{-1}\\
&= (g_{00}hg_{00}^{-1},\ g_{01}hug_{01}^{-1},\ g_{10}hvg_{10}^{-1},\ g_{11}huvg_{11}^{-1})\\
&= (g_{00}hg_{00}^{-1},\ g_{01}hg_{01}^{-1}u,\ g_{10}hg_{10}^{-1}v,\ g_{11}hg_{11}^{-1}uv)
\end{aligned}$$

  By setting $h' = g_{00}hg_{00}^{-1}$, $u' = g_{00}h^{-1}g_{00}^{-1}g_{01}hg_{01}^{-1}u$ and $v' = g_{00}h^{-1}g_{00}^{-1}g_{10}hg_{10}^{-1}v$ we have that $h'u' = g_{01}hg_{01}^{-1}u$ and $h'v' = g_{10}hg_{10}^{-1}v$. Furthermore we have,

$$\begin{aligned}
u' &= g_{00}h^{-1}g_{00}^{-1}g_{01}hg_{01}^{-1}u \ = \ (g_{01}g_{01}^{-1})g_{00}h^{-1}g_{00}^{-1}g_{01}hg_{01}^{-1}u\\
&= g_{01}\underbrace{(g_{01}^{-1}g_{00}h^{-1}g_{00}^{-1}g_{01}h)}_{\in[G,G]\subseteq Z(G)}g_{01}^{-1}u \ = \ (g_{01}^{-1}g_{00}h^{-1}g_{00}^{-1}g_{01}h)u \quad \in [G, G]
\end{aligned}$$

  In the same manner we have that $v' \in [G, G]$.

  We are left with showing that

$$h'u'v' = g_{11}hg_{11}^{-1}uv$$

  We have,

$$\begin{aligned}
h'u'v' &= g_{01}hg_{01}^{-1}u \cdot g_{00}h^{-1}g_{00}^{-1} \cdot g_{10}hg_{10}^{-1}v \ = \ g_{01}hg_{01}^{-1} \cdot g_{00}h^{-1}g_{00}^{-1} \cdot g_{10}hg_{10}^{-1} \cdot uv\\
&= g_{00}g_{00}^{-1} \cdot g_{01}hg_{01}^{-1} \cdot g_{00}h^{-1}g_{00}^{-1} \cdot g_{10}hg_{10}^{-1} \cdot uv\\
&= g_{00} \cdot \underbrace{(g_{00}^{-1}g_{01}hg_{01}^{-1}g_{00}h^{-1})}_{\in[G,G]\subseteq Z(G)} \cdot g_{00}^{-1}g_{10}hg_{10}^{-1} \cdot uv\\
&= g_{00}g_{00}^{-1}g_{10} \cdot (g_{00}^{-1}g_{01}hg_{01}^{-1}g_{00}h^{-1})h \cdot g_{10}^{-1} \cdot uv\\
&= g_{10} \cdot (g_{00}^{-1}g_{01}hg_{01}^{-1}g_{00}) \cdot g_{10}^{-1} \cdot uv\\
&= g_{10}g_{00}^{-1}g_{10}g_{00}^{-1}g_{01} \cdot h \cdot g_{01}^{-1}g_{00}g_{10}^{-1} \cdot uv
\end{aligned}$$

Now by the definition of G_4,

$$\begin{aligned} g_{11}hg_{11}^{-1}uv &= g_{10}g_{01}g_{00}^{-1}\underbrace{(g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})}_{\in[G,G]\subseteq Z(G)}h(g_{00}g_{01}^{-1}g_{10}^{-1}g_{11})^{-1}g_{00}g_{01}^{-1}g_{10}^{-1}uv \\ &= g_{10}g_{01}g_{00}^{-1}hg_{00}^{-1}g_{01}^{-1}g_{10}^{-1}uv \end{aligned}$$

Since $g_{01}^{-1}g_{00}g_{01}g_{00}^{-1} \in [G,G] \subseteq Z(G)$, we have

$$\begin{aligned} hg_{01}^{-1}g_{00}g_{01}g_{00}^{-1} = g_{01}^{-1}g_{00}g_{01}g_{00}^{-1}h \iff g_{00}^{-1}g_{01}hg_{01}^{-1}g_{00} &= g_{01}g_{00}^{-1}hg_{00}g_{01}^{-1} \\ \iff g_{10}g_{00}^{-1}g_{01}hg_{01}^{-1}g_{00}g_{10}^{-1} &= g_{10}g_{01}g_{00}^{-1}hg_{00}g_{01}^{-1}g_{10}^{-1} \iff h'u'v' = g_{11}hg_{11}^{-1}uv \end{aligned}$$

Summarizing, we have that $(g_{00},\ g_{01},\ g_{10},\ g_{11})\cdot(h,\ hu,\ hv,\ huv)\cdot(g_{00},\ g_{01},\ g_{10},\ g_{11})^{-1} = (h',\ h'u',\ h'v',\ hu'v')$, where $h' \in G$ and $u', v' \in [G,G]$. In other words $(g_{00},\ g_{01},\ g_{10},\ g_{11})\cdot(h,\ hu,\ hv,\ huv)\cdot(g_{00},\ g_{01},\ g_{10},\ g_{11})^{-1} \in H$ and that completes the proof.

$\square$

Proposition 6.2.2.
G_8 is a closed subgroup of G^8, thus is a closed Lie subgroup of G^8.

Proof.

- $G_8 \leq G^8$
  Let $z, z' \in G_8$, where $z = (g_{00}, g_{01}, g_{10}, g_{11}, h_{00}g_{00}, h_{00}g_{00}, h_{01}g_{01}, h_{10}g_{10}, h_{11}g_{11})$ and $z' = (g'_{00}, g'_{01}, g'_{10}, g'_{11}, h'_{00}g'_{00}, h'_{00}g'_{00}, h'_{01}g'_{01}, h'_{10}g'_{10}, h'_{11}g'_{11})$. Then
  $zz'^{-1} =$

  $$\begin{aligned} &(g_{00}{g'_{00}}^{-1}, g_{01}{g'_{01}}^{-1}, g_{10}{g'_{10}}^{-1}, g_{11}{g'_{11}}^{-1}, h_{00}g_{00}{g'_{00}}^{-1}{h'_{00}}^{-1}, \\ &\qquad h_{01}g_{01}{g'_{01}}^{-1}{h'_{01}}^{-1}, h_{10}g_{10}{g'_{10}}^{-1}{h'_{10}}^{-1}, h_{11}g_{11}{g'_{11}}^{-1}{h'_{11}}^{-1}) \end{aligned}$$

  Now

  $$(g_{00}{g'_{00}}^{-1}, g_{01}{g'_{01}}^{-1}, g_{10}{g'_{10}}^{-1}, g_{11}{g'_{11}}^{-1}) = \big(g_\epsilon\big)_{\epsilon\in V_2}\big({g'_\epsilon}^{-1}\big)_{\epsilon\in V_2} = \big(g_\epsilon\big)_{\epsilon\in V_2}\big(g'_\epsilon\big)_{\epsilon\in V_2}^{-1} \in G_4$$

  , since $\big(g_\epsilon\big)_{\epsilon\in V_2}$, $\big(g_\epsilon\big)_{\epsilon\in V_2}$ are elements of the group G_4.
  Furthermore,

  $$\begin{aligned} \big(h_\epsilon g_\epsilon {g'_\epsilon}^{-1}{h'_\epsilon}^{-1}\big)_{\epsilon\in V_2} &= (h_\epsilon)_{\epsilon\in V_2}(g_\epsilon)_{\epsilon\in V_2}(g'^{-1})_{\epsilon\in V_2}(h'^{-1})_{\epsilon\in V_2} \\ &= (h_\epsilon)_{\epsilon\in V_2}(g_\epsilon)_{\epsilon\in V_2}(g')^{-1}_{\epsilon\in V_2}(h')^{-1}_{\epsilon\in V_2} \underset{notation}{=} (h_\epsilon)_\epsilon(g_\epsilon)_\epsilon(g')^{-1}_\epsilon(h')^{-1}_\epsilon \end{aligned}$$

Since $H \trianglelefteq G_4$, there exists an $(h''_\epsilon)_\epsilon \in H$, such that

$$(g_\epsilon)_\epsilon(g')_\epsilon^{-1}(h')_\epsilon^{-1} = (h''_\epsilon)_\epsilon(g_\epsilon)_\epsilon(g')_\epsilon^{-1}$$

Thus $(h_\epsilon)_\epsilon(g_\epsilon)_\epsilon(g')_\epsilon^{-1}(h')_\epsilon^{-1} = (h_\epsilon)_\epsilon(h''_\epsilon)_\epsilon(g_\epsilon)_\epsilon(g')_\epsilon^{-1}$. By setting $(H_\epsilon)_\epsilon = (h_\epsilon)_\epsilon(h''_\epsilon)_\epsilon \in H$, we have

$$zz'^{-1} = \big((g_\epsilon)_\epsilon(g')_\epsilon^{-1},\ (H_\epsilon)_\epsilon\big(g_\epsilon)_\epsilon(g')_\epsilon^{-1}\big)$$

where $(g_\epsilon)_\epsilon(g')_\epsilon^{-1} \in G_4$ and $(H_\epsilon)_\epsilon \in H$.

- G_8 is closed in G^8

  Let $(g_n)_{n\in\mathbb{N}}$ a sequence in G_8 and $g \in G^8$ such that $g_n \to g \iff$

  $$(g^n_{00}, g^n_{01}, g^n_{10}, g^n_{11}, h^n_{00}g^n_{00}, h^n_{00}g^n_{00}, h^n_{01}g^n_{01}, h^n_{10}g^n_{10}, h^n_{11}g^n_{11}) \to (g_\epsilon)_{\epsilon\in V_3}$$

  Since $(g^n_{00}, g^n_{01}, g^n_{10}, g^n_{11}) \in G_4$, $\forall n \in \mathbb{N}$ and G_4 closed subgroup of G^4, then $g_{000}, g_{001}, g_{010}, g_{011}) \in G_4$. Now,

  $$h^n_{00}g^n_{00} \to g_{100}, \quad h^n_{01}g^n_{01} \to g_{101}, \quad h^n_{10}g^n_{10} \to g_{110}, \quad h^n_{11}g^n_{11} \to g_{111}$$

  By continuity, we have that $h^n_{00} = (h^n_{00}g^n_{000})(g^n_{000})^{-1} \to g_{100}g^{-1}_{000} \quad \in G$. In the same manner each of the h^n_{01}, h^n_{10} and h^n_{11} converges to some element of G. Hence $(h^n_{00}, h^n_{01}, h^n_{10}, h^n_{11})$ converges in G_4, to some element of G^4. Since $(h^n_{00}, h^n_{01}, h^n_{10}, h^n_{11}) \in H$, $\forall n \in \mathbb{N}$ and H is closed in G_4,

  $$(h^n_{00}, h^n_{01}, h^n_{10}, h^n_{11}) \to (h_{00}, h_{01}, h_{10}, h_{11}) \quad \in H$$

  It follows that (again from continuity) that

  $$g_{100} = h_{00}g_{000}, \quad g_{101} = h_{01}g_{001}, \quad g_{110} = h_{10}g_{010}, \quad g_{111} = h_{11}g_{011}$$

  Hence, to summarize,

  $$(g_\epsilon)_{\epsilon\in V_3} = (g_{000}, g_{001}, g_{010}, g_{011}, h_{00}g_{000}, h_{01}g_{001}, h_{10}g_{010}, h_{11}g_{011})$$

  where $(g_{000}, g_{001}, g_{010}, g_{011}) \in G_4$ and $(h_{00}, h_{01}, h_{10}, h_{11}) \quad \in H$

$\square$

Proposition 6.2.3.

The group G_8 is 2-step nilpotent.

Proof. $G_8 \leq G_4 \times HG_4 \underset{H \trianglelefteq G_4}{=} G_4 \times G_4$. Since G_4 is 2-step nilpotent then, $G_4 \times G_4$ is 2-step nilpotent. Now every subgroup of a k-step nilpotent group is k'-step nilpotent group, with $k' \leq k$. Thus G_8 is either 1-step nilpotent (in other words abelian) or 2-step nilpotent). Since G_8 is clearly non abelian group, we obtain the result. □

Proposition 6.2.4.

$$\begin{aligned} G_8 = \{\boldsymbol{g} = (g_\epsilon)_{\epsilon \in V_3} \ : \ & g_{000}g_{001}^{-1}g_{011}g_{010}^{-1}g_{100}g_{111}^{-1}g_{101}g_{100}^{-1} = 1 \ ; \ (g_{000}, g_{001}, g_{010}, g_{011}) \in G_4; \\ & (g_{000}, g_{001}, g_{100}.g_{101} \in G_4) \ ; \ (g_{000}, g_{010}, g_{100}, g_{110}) \in G_4\} \end{aligned}$$

Define

$$\begin{aligned} \Lambda_8 &= \Lambda^8 \cap G_8 \\ U_8 &= [G, G]^8 \cap G_8 \\ &= \{\mathbf{u} = (u_\epsilon)_{\epsilon \in V_3} \ : \quad u_{000}u_{001}^{-1}u_{011}u_{010}^{-1}u_{110}u_{111}u_{101}u_{100}^{-1} = 1\} \end{aligned}$$

As in the preceding section:

Λ_8 is discrete cocompact subgroup of G_8. There is a natural embedding of G_8/Λ_8 in $X^8 = (G/\Lambda)^8 = G^8/\Lambda^8$. We identify the manifold G_8/Λ_8 with its image X_8 in X^8.

The commutator, $[G_8, G_8]$, of G_8 is equal to U_8 and $U_8 \cap \Lambda_8 = \{1\}$

Lemma 6.2.5.
The measure $\mu^{[3]}$ is the Haar measure of the manifold $X_8 = G_8/\Lambda_8$.

Proof. We proceed as in Lemma 6.1.4.
Let $\mu'^{[3]}$ be the Haar measure of the manifold X_8. The transformations $T_8 = T^{[3]}$, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$ on X^8 are thr translations by the four elements of G^8,

$$\begin{aligned} \alpha_8 &= (\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha), \quad \alpha_{8,1} = (1, \alpha, 1, \alpha, 1, \alpha, 1) \\ \alpha_{8,2} &= (1, 1, \alpha, \alpha, 1, 1, \alpha, \alpha), \quad \alpha_{8,3} = (1, 1, 1, 1, \alpha, \alpha, \alpha, \alpha) \end{aligned}$$

In particular those four element belong to G_8 and thus the corresponding transformations are translations on the manifold $X_8 = G_8/\Lambda_8$

We will show that $\mu'^{[3]}$ is ergodic on X_8, for the joint action of $T_8 = T^{[3]}$, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$. Observe that the hypothesis (1.3) of Proposition 1.11.14 is satisfied, since the Property (1.1) is satisfied. We define $q_8 : G_8 \to K^8$ with

$$q_8\big((g_\epsilon)_{\epsilon \in V_3}\big) = \big(q(g_\epsilon)\big)_{\epsilon \in V_3}$$

Then q_8 is continuous group homomorphism. Furthermore, by the definition of $H \trianglelefteq G_4$, $K = G/\Lambda$ and $q_8 = (q_4 \times q_4)|_{G_8}$

$$\begin{aligned} q_8(G_8) &= \{(k_\epsilon)_{\epsilon \in V_3} \in G^8 \ : \ (k_\epsilon)_{\epsilon \in V_3} = (k_{000}, k_{001}, k_{010}, k_{011}, ck_{000}, ck_{001}, ck_{010}, ck_{011}), \\ &\qquad\qquad \text{where } (k_{000}, k_{001}, k_{010}, k_{011}) \in K_4 \text{ and } c \in K\} \\ &= \{(z, za, zab, zab, zc, zac, zbc, zabc) \ : \ z, a, b, c \in K\} \\ &= K_8 \end{aligned}$$

Clearly K_8 is a closed subgroup of K^8 and its kernel is $\Lambda^8[G,G]^8 \cap G_8 = \Lambda_8 U_8$. Thus we can identify $G_8/\Lambda_8 U_8 = K_8$. Under this identification the transformations induced by $T_8 = T^{[3]}$, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$, on K_8 are rotations by

$$\begin{aligned} \beta_8 = q_8(\alpha_8) = (\beta, \beta, \beta, \beta, \beta, \beta, \beta, \beta), \quad \beta_{8,1} = q_8(\alpha_{8,1}) = (1, \beta, 1, \beta, 1, \beta, 1) \\ \beta_{8,2} = q_8(\alpha_{8,2}) = (1, 1, \beta, \beta, 1, 1, \beta, \beta), \quad \beta_{8,3} = q_8(\alpha_{8,3}) = (1, 1, 1, 1, \beta, \beta, \beta, \beta) \end{aligned}$$

where $\beta = q_(\alpha)$.

As in the proof of Lemma 6.1.4, it can be shown that the joint action of these rotations on K_8, is ergodic on K_8. By Proposition 1.11.14 the joint action of $T_8 = T^{[3]}$, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$ is ergodic on X_8. By Proposition 1.11.14 $\mu'^{[3]}$ is the unique measure on X_8 that is invariant under these transformations. The measure $\mu^{[3]}$ is also invariant under these transformations, and we will show that is concentrated on X_8.

By Lemma $A.8$, $H\Lambda_4$ is a closed subgroup of G_4. Let Y denote the compact space $G_4/H\Lambda_4$. Define $\xi : X_4 \to Y$, be the natural projection of $X_4 = G_4/\Lambda_4$ on $Y = G_4/H\Lambda_4$. Since $\alpha_4 = (\alpha, \alpha, \alpha, \alpha)$ belongs to H, clearly the map ξ is invariant under T_4.

Now measure $\mu^{[3]}$ is the relatively independent self-joining of $\mu^{[2]}$ over $\mathcal{I}^{[2]}$. By Lemma 6.1.4 the measure $\mu^{[2]}$ is concentrated on X_4 and thus, $\mu_8(X_4 \times X_4) \underset{\substack{\textit{property of} \\ \textit{joining}}}{=} \mu^{[2]}(X_4) \cdot \mu^{[2]}(X_4) = 1$. In addition ξ is T_4-invariant, hence (as in the proof of Proposition 3.3.11) $\xi(x') = \xi(x'')$, for $\mu^{[3]}$-almost every $\mathbf{x} = (x', x'') \in X^4 \times X^4$. In other words, $\mu^{[3]}$ is concentrated on the set

$$\Xi = \{\mathbf{x} = (x', x'') \in X_4 \times X_4 \ : \quad \xi(x') = \xi(x'')\} \ .$$

We are left with showing that Ξ is actually X_8. Let $\Lambda_8 g \in X_8 = G_8/\Lambda_8$, where $g = \big((g_\eta)_{\eta \in V_2}, (h_\eta g_\eta)_{\eta \in V_2}\big)$. Since $\Lambda_8 = \Lambda^8 \cap G_8$,

$$\Lambda_8 g = \Lambda^8 g = (\Lambda^4 \times \Lambda^4)\big((g_\eta)_{\eta \in V_2}, (h_\eta g_\eta)_{\eta \in V_2}\big) = \big(\Lambda^4 (g_\eta)_{\eta \in V_2}, \Lambda^4 (h_\eta g_\eta)_{\eta \in V_2}\big)$$

Now, by the definition of G_8 and because, $H \trianglelefteq G_4$, $(g_\eta)_{\eta\in V_2} \in G_4$, and $(h_\eta g_\eta)_{\eta\in V_2} \in HG_4 \leq G_4$. Since $\Lambda_4 = \Lambda^4 \cap G_4$,

$$\big(\Lambda_4(g_\eta)_{\eta\in V_2}, \Lambda_4(h_\eta g_\eta)_{\eta\in V_2}\big) = \big(\Lambda^4(g_\eta)_{\eta\in V_2}, \Lambda^4(h_\eta g_\eta)_{\eta\in V_2}\big) = (x', x'') \ .$$

So, $\Lambda_8 g = (x', x'') \in \Xi$.

Conversely, let $(x', x'') \in \Xi$ then $(x', x'') = \big(\Lambda_4(g'_\eta)_{\eta\in V_2}, \Lambda_4(g''_\eta)_{\eta\in V_2}\big)$, where $(g'_\eta)_{\eta\in V_2}$, $(g''_\eta)_{\eta\in V_2} \in G_4$, with $\xi(x') = \xi(x'')$. This means

$$\Lambda_4 H(g'_\eta)_{\eta\in V_2} = \Lambda_4 H(g''_\eta)_{\eta\in V_2} \implies (g''_\eta)_{\eta\in V_2} = (\lambda_\eta)_{\eta\in V_2}(h_\eta)_{\eta\in V_2}(g'_\eta)_{\eta\in V_2}$$

where, $(\lambda_\eta)_{\eta\in V_2} \in \Lambda_4$ and $(h_\eta)_{\eta\in V_2} \in H$.

Thus (as above),

$$\begin{aligned}
(x', x'') = \big(\Lambda_4(g'_\eta)_\eta \ , \ \Lambda_4(\lambda_\eta)_\eta(h_\eta)_\eta(g'_\eta)_\eta\big) \ &= \ \big(\Lambda_4(g'_\eta)_\eta \ , \ \Lambda_4(h_\eta)_\eta(g'_\eta)_\eta\big) \\
= \ \big(\Lambda^4(g'_\eta)_\eta \ , \ \Lambda^4(h_\eta)_\eta(g'_\eta)_\eta\big) \ &= \ (\Lambda^4 \times \Lambda^4)\big((g'_\eta)_\eta \ , \ (h_\eta)_\eta(g'_\eta)_\eta\big) \\
= \ \Lambda^8\big((g'_\eta)_\eta \ , \ (h_\eta)_\eta(g'_\eta)_\eta\big) & \\
= \ \Lambda_8\big((g'_\eta)_\eta \ , \ (h_\eta)_\eta(g'_\eta)_\eta\big) \quad \in X_8 &
\end{aligned}$$

□

We are now ready to proceed with the proof of Theorem 5.3.5 in the case where the system (X, μ, T) is an ergodic 2-step nilsystem:

Theorem 6.2.6.

Let (X, μ, T) be a 2-step nilsystem and f_ϵ, $\epsilon \in V_3$ be eight bounded functions on X. Then the average over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$, $n_3 \in [M_3, N_3]$ of

$$\int_X \prod_{\epsilon\in V_3} f_\epsilon\big(T^{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}x\big)\ \mathrm{d}\mu(x) = \int_X f_{000}(x) \prod_{\epsilon\in V_3^*} f_\epsilon\big(T^{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}x\big)\ \mathrm{d}\mu(x) \tag{6.2}$$

converges to

$$\int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon \ \mathrm{d}\mu^{[3]}$$

when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$ tend to $+\infty$.

Proof. The measure $\mu^{[3]}$ is ergodic under the joint action of T_8, $T_{8,1}$, $T_{8,2}$, $T_{8,3}$ (Corollary 4.1.2). By Proposition 1.11.14 X_8 is uniquely ergodic for this action. Observe that $\prod_{\epsilon\in V_3} f_\epsilon \circ T^{p+n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}(x_\epsilon) = \bigotimes_{\epsilon\in V_3} f_\epsilon \circ \big((T^{[3]})^p \circ T_{8,1}^{n_1} \circ T_{8,2}^{n_2} \circ T_{8,3}^{n_3}\big)(\mathbf{x})$, for every

$\mathbf{x} = (x_\epsilon : \ \epsilon \in V_3) \in X^{[3]}$. Thus if f_ϵ, $\epsilon \in V_3$ are eight continuous functions on X, the averages

$$\frac{1}{N_1 - M_1}\frac{1}{N_2 - M_2}\frac{1}{N_3 - M_3}\sum_{n_1=M_1}^{N_1}\sum_{n_2=M_2}^{N_2}\sum_{n_3=M_3}^{N_3}\frac{1}{B-A}\sum_{p=A}^{B}\prod_{\epsilon\in V_3} f_\epsilon \circ T^{p+n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}(x_\epsilon)$$

converges uniformly to the constant, when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$, $B - A$ tend to $+\infty$

$$\int_{X^8} \bigotimes_{\epsilon\in V_3} f_\epsilon \, \mathrm{d}\mu^{[3]}$$

Since the diagonal $\{(x, x, x, x, x, x, x, x) : \ x \in X\}$ is a subset of X_8, the average

$$\frac{1}{N_1 - M_1}\frac{1}{N_2 - M_2}\frac{1}{N_3 - M_3}\sum_{n_1=M_1}^{N_1}\sum_{n_2=M_2}^{N_2}\sum_{n_3=M_3}^{N_3}\frac{1}{B-A}\sum_{p=A}^{B}\prod_{\epsilon\in V_3} f_\epsilon \circ T^{p+n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}(x)$$

converges uniformly to the same constant, when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$, $B - A$ tend to $+\infty$. Taking the integral and since $T_*\mu = \mu$, we obtain the result for continuous functions. The general case holds by density. □

Now we will prove the Theorem 4, for nilsytems:

Theorem 6.2.7.
Let (X, μ, T) be a 2-step nilsystem and f_ϵ, $\epsilon \in V_3^*$ be seven bounded functions on X. Then the average over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$, $n_3 \in [M_3, N_3]$ of

$$\prod_{\epsilon\in V_3^*} f_\epsilon\big(T^{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}x\big) \tag{6.3}$$

converges in $L^2(\mu)$ to the function

$$x \mapsto \mathbb{E}\Big(\bigotimes_{\epsilon\in V_3^*} f_\epsilon \mid \mathcal{J}_7\Big)(x)$$

where we have identified $\mathcal{J}_7$ with $\mathcal{CL}$, when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$ tend to $+\infty$.

Proof. Let f_ϵ, $\epsilon \in V_3^*$ be seven continuous functions on X. By Theorem $A.9$, assuming $f_{000} = \mathbb{1}_X$, we can deduce that the averages

$$\frac{1}{N_1 - M_1}\frac{1}{N_2 - M_2}\frac{1}{N_3 - M_3}\sum_{n_1=M_1}^{N_1}\sum_{n_2=M_2}^{N_2}\sum_{n_3=M_3}^{N_3}\prod_{\epsilon\in V_3^*} f_\epsilon \circ T^{n_1\epsilon_1+n_2\epsilon_2+n_3\epsilon_3}(x_\epsilon)$$

converge for every $\mathbf{x} = (x_\epsilon : \epsilon \in V_3)$. In particular they converge for every diagonal point $(x, x, x, x, x, x, x, x) \in X_8$. Therefore the averages of (6.3) converge for every $x \in X$. The $L^2(\mu)$-convergence follows by Proposition 1.11.14.

Assume $F(x)$ is the limit. By Theorem 6.2.6, for every bounded function g on X,

$$\int_X g(x)F(x)\,\mathrm{d}\mu(x) = \int_{X^8} g(x_{000}) \cdot \prod_{\epsilon \in V_3^*} f_\epsilon(x_\epsilon)\,\mathrm{d}\mu^{[3]}(\mathbf{x})$$

By Lemma 4.1.7, the last integral is equal to

$$\int_{X^8} g(x) \cdot \mathbb{E}\Big(\bigotimes_{\epsilon \in V_3^*} f_\epsilon \mid \mathcal{J}_7\Big)(x)\,\mathrm{d}\mu(x)$$

when the σ-algebra $\mathcal{J}_7$ is identified with the σ-algebra $\mathcal{CL}$. Follows that $F(x) = \mathbb{E}\big(\bigotimes_{\epsilon \in V_3^*} f_\epsilon \mid \mathcal{J}_7\big)(x))$, for μ-almost every $x \in X$. The convergence in The general case holds by density. $\square$

Appendix A

Some more notation

In this Appendix we establish some additional notation and results that, regardless their importance, are being used less frequently in this dissertation.

A.1 More about Group Extensions and Cocycles

Lemma A.1.
Let X be a compact metric space, $\mathcal{X}$ its Borel σ-algebra and μ a Borel probability space on $(X, \mathcal{X})$. Then $\big(C(X, \mathbb{T}^\ell), d_{Coc(X,\mathbb{T}^\ell)}\big)$ is dense in $\big(Coc(X, \mathbb{T}^\ell), d_{Coc(X,\mathbb{T}^\ell)}\big)$, where $d_{Coc(X,\mathbb{T}^\ell)}$ is a metric on $Coc(X, \mathbb{T}^\ell)$, defined by $d_{Coc(X,\mathbb{T}^\ell)}(\rho, \rho') = \int_X d_{\mathbb{T}^\ell}\big(\rho(x), \rho'(x)\big)\, \mathrm{d}\mu(x)$.

Proof. We prove this statement when $\ell = 1$. The general case is proven in a similar manner.
Let $f : X \to [0, 1)$ be an element of $Coc(X, \mathbb{T})$. For all $n \in \mathbb{N}$ define, $s_n(x) := k/n$, where $x \in X$ with $f(x) \in [k/n, (k + 1)/n)$ and $A_{k,n} := f^{-1}\big([k/n, (k + 1)/n)\big)$, $\forall k \in \{0, 1, \ldots, n-1\}$. Notice that for every $n \in \mathbb{N}$, $A_{k,n}$, $k \in \{0, 1, \ldots, n-1\}$ is pairwise disjoint and $\bigcup_{k=0}^{n-1} A_{n,k} = X$. Then for every $n \in \mathbb{N}$,

$$s_n \sum_{k=0}^{n-1} \frac{k}{n} \mathbb{1}_{A_{k,n}}$$

Furthermore $0 \leq s_n \leq f$, $\forall n \in \mathbb{N}$ and $s_n \to f$, since $|s_n(x) - f(x)| < 1/n$, $\forall x \in X$. In particular,

$$\sup_{x \in X} |s_n(x) - f(x)| \leq \frac{1}{n}$$

Since μ is a Radon measure on $(X,\mathcal{X})$ $\big($Proposition 1.1.4$\big)$, for every $n\in\mathbb{N}$ and for every $k\in\{0,1,\dots,n-1\}$, there exists a compact set $B_{k,n}\subseteq A_{k,n}$ such that $\mu(A_{k,n}\setminus B_{k,n})<1/n^2$.

For every $n\in\mathbb{N}$ the set $\bigcup_{k=0}^{n-1}B_{k,n}$ is a closed set, thus the restriction of s_n in this set extents to a continuous function $f_n:X\to[0,1)$. Indeed, he restriction of s_n in each $B_{k,n}$ is a constant and thus continuous. Moreover $B_{k,n}$, $k\in\{0,1,\dots,n-1\}$ are compact and pairwise disjoint the distance between each and every pair of them is positive. Hence the restriction of s_n on $\bigcup_{k=0}^{n-1}B_{k,n}$ is a continuous function. By Tietze Extension Theorem $\big($see [5]$\big)$ the function s_n extents to a continuous function $f_n:X\to[0,1)$. In particular,

$$\max_{x\in X}f_n(x)=\max_{x\in\bigcup_{k=0}^{n-1}B_{k,n}}s_n(x)\qquad\text{and}\qquad\min_{x\in X}f_n(x)=\min_{x\in\bigcup_{k=0}^{n-1}B_{k,n}}s_n(x)$$

Notice that for every $x\in X$, $|f_n(x)-f(x)|=|s_n(x)-f(x)|\le 1/n$ and for every $x\in X$, $|f_n(x)-f(x)|\le 1$. Moreover $d_{\mathbb{T}}\big(f_n(x),f(x)\big)\le|f_n(x)-f(x)|$, for every $x\in X$. We have,

$$\begin{aligned}
d_{Coc(X,\mathbb{T})}(f_n,f) &= \int_Z d_{\mathbb{T}}\big(f_n(x),f(x)\big)\,\mathrm{d}\mu(x)\le\int_Z|f_n(x)-f(x)|\,\mathrm{d}\mu(x)\\
&=\int_{\bigcup_{k=0}^{n-1}B_{k,n}}|f_n(x)-f(x)|\,\mathrm{d}\mu(x)+\int_{X\setminus\bigcup_{k=0}^{n-1}B_{k,n}}|f_n(x)-f(x)|\,\mathrm{d}\mu(x)\\
&\le\frac{1}{n}\mu(X)+\mu\Big(X\setminus\bigcup_{k=0}^{n-1}B_{k,n}\Big)\;=\;\frac{1}{n}+\mu\Big(\bigcup_{k=0}^{n-1}A_{k,n}\setminus\bigcup_{k=0}^{n-1}B_{k,n}\Big)\\
&\le\frac{1}{n}+n\frac{1}{n^2}\;=\;2/n
\end{aligned}$$

In other words $d_{Coc(X,\mathbb{T})}(f_n,f)\to 0$. □

Proposition A.2.
Let (Y,ν,T) be an ergodic system, K a compact abelian group and $\rho:Y\to K$ a cocycle. Then there exists a closed subgroup H of K and an ergodic cocycle $\sigma:Y\to H$ such that ρ is cohomologous to σ when considered as a K-valued cocycle.

Proof. Define $\Lambda=\{\chi\in\widehat{K}:\ \chi\circ\rho\text{ is a coboundary}\}$. Then Λ is a subgroup of $\widehat{K}$. Define $H=\Lambda^{\perp}=\{g\in K:\ \chi(g)=1,\ \forall\chi\in\Lambda\}$. Then $\Lambda=H^{\perp}=\{\chi\in\widehat{K}:\ \chi(g)=1,\ \forall g\in H\}$. The dual group of K/H is naturally identified with Λ.

Let $\rho':Y\to K/H$ defined by $\rho'(y)=\rho(y)H$. Then for every $\chi\in\widehat{K/H}=\Lambda$, $\chi\circ\rho'$ is a coboundary. By Lemma 4.2.5, ρ' is a coboundary. This means there exist a function

$f' : Y \to K/H$, such that $\rho' = f' \circ \rho' - f' = \partial f'$. Lift f' to a function $f : Y \to K$. Then $\rho - (f' \circ \rho' - f') = \sigma$, where σ takes values on H. By this equation follows that ρ is cohomologous to σ.

Now assume that σ is not ergodic. By Lemma 4.3.1 there exists a nontrivial $\chi \in \widehat{H}$ such that $\chi \circ \sigma$ is a coboundary. By extending the character χ of H to a character χ' of K, we have that $\chi \circ \rho$ is a coboundary. Thus χ is an element of Λ and hence has e trivial restriction on H, which contradicts with the hypothesis. $\square$

By this proposition we obtain the following.

Lemma A.3.
Let (X, μ, T), (X', μ', T'), (Y, ν, S), (Y', ν', S') be measure preserving systems and let $\pi : X \to Y$, $\pi' : X' \to Y'$, $r : X \to X'$ and $p : Y \to Y'$ be factor maps such that $p \circ \pi = \pi' \circ r$. Assume that X is ergodic and X' is an ectension of Y' by a compact abelian group K associated with a cocyle $\rho' : Y' \to K'$. Furthermore assume that $\mathcal{X} = r^{-1}(\mathcal{X}') \bigvee \pi^{-1}(\mathcal{Y})$. Then X is an extension of Y by a compact ablian group H.

Proof. Reference [14, Part 1, Chaprer 5, Section 3 : Section 3, Subsection 3.6 : The Mackey group of a cocycle, Lemma 18, page 72] $\square$

Lemma A.4.
Let $\pi : (X, \mu, T) \to (Y, \nu, S)$ be an ergodic extension by a compact abelian group K and let $p : W \to Y$ be an intermediate extension, meaning that there exists a factor map $q : X \to W$ such that $p \circ q = \pi$. Then W is the quotient of X under the action of some closed subgroup H of K and is an extension of Y by the group K/H.

Proof. Reference [14, Part 1, Chaprer 5, Section 3 : Section 3, Subsection 3.6 : The Mackey group of a cocycle, Lemma 19, page 73] $\square$

Lemma A.5.
Let (X, μ, T) be an ergodic isometric extension of (Y, ν, S), where $X = Y \times G/H$, $\mu = \nu \times m_{G/H}$ and $T = S_\rho$ for a cocyle $\rho : Y \to G$. Then for every $g \in G$ the transformation

$$(x_0, x_1) \mapsto (g \cdot x_0,\ g \cdot x_1) :\ X^2 \to X^2$$

leaves each set of $\mathcal{I}^{[1]}$ invariant.

Proof. Let Z be the Kronecker factor of X. and let W be the factor associated to the σ-algebra $\mathcal{Y} \vee \mathcal{Z}$. Then W is a factor of X. Thus there exists o closed subgroup K of G such that $W = Y \times G/K$. By Lemma 3.3.23 follows that X and W share the same Kronecker factor. Thus $\mathcal{I}^{[2]}$ is contained in $\mathcal{W} \otimes \mathcal{W}$. Thus, without loss of generality, we can assume that $\mathcal{X} = \mathcal{Y} \vee \mathcal{Z}$.

Applying Lemma $A.3$ with Z instead of X' and the trivial system instead of Y', we have that X is an extension of Y by a compact abelian group L. Comparing the two representations of X, we obtain that H is normal subgroup of G and G/H is isomorphic to L. In particular G/H is abelian.

Let $g \in G$ and $p : X \to Z$ the factor map. The transformation $V_g : X \to X$, with $V_g(y, q) = (y, g \cdot q)$ commutes with $T = S_\rho$. Thus by Corollary 1.9.4 there exists a R_g on Z which is a rotation defined by a $\beta \in Z$, such that $R_g \circ p = p \circ V_g$. Therefore, for $x_1, x_2 \in X$, $p(V_g x_2) \cdot p(V_g x_1)^{-1} = p(x_2) \cdot p(x_1)^{-1}$. By Theorem 2.3.5 follows that every $A \in \mathcal{I}^{[2]}$ is $V_g \times V_g$ invariant. □

Lemma A.6.

Let (X, μ, T) be an ergodic system, U a compact abelian group and let $(u, x) \mapsto u \times x$ be a free action of U on X by automorphisms. Let $\rho \in Coc(X, U)$ be a cocycle so that $\rho \circ V_u - \rho$ is a coboundary for every $u \in U$. Then there exists an open subgroup U_0 of U and a cocycle $\rho' \in Coc(X, U)$, cohomologous to ρ, with $\rho' \circ V_u = \rho'$ for every $u \in U_0$.

Proof. Reference [12, Lemma C.9, APPENDIX] □

Lemma A.7.

Let (X, μ, T) be an ergodic system, U a compact abelian group and let $(u, x) \mapsto u \times x$ be a free action of U on X by automorphisms. Let $\rho \in Coc(X, U)$ be a cocycle so that $\rho \circ V_u - \rho$ is a quasi-coboundary for every $u \in U$. Then there exists a closed subgroup U_1 of U so that U/U_1 is a torus and there exists a cocycle ρ', cohomologous to ρ, with $\rho' \circ V_u = \rho'$ for every $u \in U_0$.

Proof. Reference [12, Lemma C.10, APPENDIX] □

A.2 More about nilmanifolds

Lemma A.8.

Let $X = G/\Gamma$ be a nilmanifold. For a subgroup H of G the following are equivalent

(i) *The subset $H \cdot e_X$ of X is closed in X.*

(ii) *the subset $H\Gamma$ of G is closed in G.*

(iii) *The subgroup $\Gamma \cap H$ is cocompact in H meaning that $H/(H \cap \Gamma)$ is a compact.*

(iv) *There exists a compact subset K of H such that $K(H \cap \Gamma) = H$.*

Theorem A.9.
Let $X_8 = G_8/\Lambda_8$ be a manifold and $\alpha_{8,1}$, $\alpha_{8,2}$, $\alpha_{8,3}$ the three commuting elements of X_8 that define $T_{8,1}$, $T_{8,1}$, $T_{8,3}$ respectively. Then for every continuous function f on X, the average over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$ and $n_3 \in [M_3, N_3]$ of

$$f\big(T_{8,1}^{n_1}T_{8,2}^{n_2}T_{8,3}^{n_3}(x)\big)$$

converges for all $x \in X$, when $N_1 - M_1$, $N_2 - M_2$, $N_3 - M_3$ tend to $+\infty$.

Proof. (Sketch of the proof)
For every $x \in X$ define the set,

$$X_{8,x} := \{\mathbf{x} = (x_\epsilon : \epsilon \in V_3) \in X_8 : \ x_{000} = x\}$$

Clearly these sets form a closed partision of X_8.

Now define,

$$G_8' := \{\mathbf{g} = (x_\epsilon : \epsilon \in V_3) \in G_8 : \ g_{000} = 1_G\}$$

Clearly G_8' is a closed subgroup of G_8 and thus it is a closed Lie subgroup of G_8. Moreover G_8' is 2-step nilpotent, since G_8 is 2-step nilpotent and G_8' is not abelian. We observe that for every $x \in X$, (left) translations by elements of this group leaves each $X_{8,x}$ invariant and acts transitively on each of this spaces. We can now give to $X_{8,x}$ the structure of a nilmanifold by identifying it with the manifold G_8'/Λ_x, where $\Lambda_x := \Lambda_8 \cap G_8'$ is a discrete cocompact subgroup of G_8'.

Let $\mu_{8,x}$ denote the Haar measure of $X_{8,x}$. This measure is invariant under the action of G_8' on $X_{8,x}$. In particular it is invariant under the transformations $T_{8,1}$, $T_{8,1}$, $T_{8,3}$.

We have that

$$\mu^{[3]} = \int\limits_X \mu_{8,x} \, \mathrm{d}\mu(x)$$

since the right hand side of this equation is a measure on X_8 that is invariant under the action of G_8

We give a second interpretation of this formula. Let $\pi_{000} : X^8 \to X$ be the first projection. The family of measures $(\mu_{8,x} : \ x \in X)$ is the conditional probability measures given the σ-algebra $\pi_{000}^{-1}(\mathcal{X}) \equiv \mathcal{A}_{\mathbf{0},3}$ which coincides with the σ-algebra $\mathcal{J}^{[3]}$ modulo $\mu^{[3]}$. Thus the equation above can be viewed as the ergodic disintegration of $\mu^{[3]}$ for the action spanned by the joint action of $T_{8,1}$, $T_{8,1}$, $T_{8,3}$.

It follows that for μ-almost every $x \in X$, the measure $\mu_{8,x}$ is ergodic for this action. By Proposition 1.11.14 applied to the nilmanifold $X_{8,x}$, we have that for μ-almost every $x \in X$, $X_{8,x}$ is uniquely ergodic for the action spanned by the three transformations $T_{8,1}$, $T_{8,1}$, $T_{8,3}$. Therefore, for any continuous function F on $X_{8,x}$, for μ-almost every $x \in X$, the average over $n_1 \in [M_1, N_1]$, $n_2 \in [M_2, N_2]$ and $n_3 \in [M_3, N_3]$ of

$$F\big(T_{8,1}^{n_1} T_{8,2}^{n_2} T_{8,3}^{n_3}(x)\big)$$

converges to $\int\limits_{X_{8,x}} F \, \mathrm{d}\mu_{8,x}$. The result follows. $\square$

A.3 Vertical characters and vertical Fourier transforms

Let (Y, ν, S) be a system, K be a compact abelian group and $\rho : Y \to K$ a cocycle. Let $(X, \mu, T) = (Y \times K.\nu \otimes m_K, S_\rho)$ be the extension defined by ρ.

If $\chi \in \widehat{K}$, a vertical character of X with frequency χ is a function $F \in L^2(\mu)$ such that

$$\widehat{F}_\chi(V_h x) = \chi(h)\widehat{F}_\chi(x)$$

for every $h \in K$. Since $X = Y \times K$, this means that $F(y, g) = f(y)\chi(g)$ for some $f \in L^2(\nu)$.

The vertical characters with frequency χ form a closed subspace L_χ of $L^2(\mu)$. This subspace is invariant under multiplication by functions of the form $\phi \circ \pi$, where $\phi \in L^\infty(\nu)$ and is also S_ρ-invariant. For example, for the trivial character 1 of K, the subspace L_1 is the set of functions of the form $(y, g) \mapsto f(y)$ with $f \in L^2(\nu)$. For distinct characters, the associated spaces L_χ for $\chi \in \widehat{K}$ are pairwise orthogonal subspaces of $L^2(\mu)$.

Now, for a function $F \in L^2(\mu)$ and $\chi \in \widehat{K}$ define the function $\widehat{F}_\chi$ on X by,

$$\widehat{F}_\chi(x) = \int\limits_K F(V_h x)\overline{\chi}(h) \, \mathrm{d}m_K(h)$$

and we call $\widehat{F}_\chi$ the vertical Fourier coefficient of frequency χ of F. Identifying X with $Y \times K$, this formula becomes

$$\widehat{F}_\chi(y,g) = f_\chi(y)\chi(g) \quad \text{where } f_\chi = \int_K F(y,h)\overline{\chi}(h)\, \mathrm{d}m_K(h)$$

Clearly, every vertical Fourier coefficient $\widehat{F}_\chi$ is a vertical character with frequency χ.

Via the identification of X with $Y \times K$, if $F \in L^2(\mu)$ then for μ-almost every $y \in Y$, the function f_y on K given by $f_y(g) = f(y,g)$ belongs to $L^2(m_K)$ and thus is the sum in this space of its Fourier Transform. It follows that

$$F = \sum_{\chi \in \widehat{K}} \widehat{F}_\chi \tag{A.1}$$

where the convergence holds in $L^2(\mu)$ and the series is called the vertical Fourier series of F. It follows from that $L^2(\mu)$ is the orthogonal sum of the spaces $L_\chi, \chi \in \widehat{K}$.

A.4 Van der Corput Lemma

Proposition A.10.
Assume that $\{x_n\}$ is a sequence in a Hilbert space with norm $||x_n|| \leq 1$ for all $n \in Z$. Let L,M and N be integers with $L > 0$ and $N > M$. Then

$$\left\| \frac{1}{N-M} \sum_{n=M}^{N-1} x_n \right\|^2 \leq \frac{4L}{N-M} + \sum_{\ell=-L}^{L} \frac{L-|\ell|}{L^2} \frac{1}{N-M} \sum_{n=M}^{N} \langle x_n, x_{n+\ell} \rangle$$

Proposition A.11.
Assume that $\{x_n\}$ is a sequence in a Hilbert space with norm $||x_n|| \leq 1$ for all $n \in Z$. Then

$$\limsup_{N\to+\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\|^2 \leq \limsup_{H\to+\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to+\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_n, x_{n+h} \rangle \right|$$

Proposition A.12.
Assume that $\{u_{n_1,n_2,n_3} : n_1, n_2, n_3 \in \mathbb{Z}\}$ be a bounded triple sequence of vectors on a Hilbert space. If the limit of

$$\frac{1}{K^3} \sum_{k_1,k_2,k_3=0}^{K-1} \left| \lim_{N-M\to+\infty} \frac{1}{N-M} \sum_{n_1,n_2,n_3=M}^{N} \langle u_{n_1,n_2,n_3},\ u_{n_1+k_1,n_2+k_2,n_3+k_3} \rangle \right|$$

for $K \to +\infty$*, is equal to* 0*, then*

$$\lim_{N-M \to +\infty} \left\| \frac{1}{N-M} \sum_{n_1,n_2,n_3=M}^{N} u_{n_1,n_2,n_3} \right\| = 0$$

Appendix B

Proofs

In this Appendix we give proofs of or, mainly, references for the results mentioned in Chapter 1.

B.1 Section 1.1

Proof of Proposition 1.1.2. A topological space is called topologically complete if it is completely metrizable. The Cartesian product of a countable family of topologically complete topological spaces, endowed with the product topology, is topologically complete [5, Theorem 2.5(4) in Ch. XIV]. Also, the product of countably many separable Hausdorff topological spaces, endowed with the product topology again, is, obviously, separable [5, Theorem 7.2(3) in Ch. VIII]. Another reference for this fact is [15, Proposition 3.3 (iii)]. □

Proof of Proposition 1.1.4. References for this fact are, for example, [2, Theorem 1.1], [15, Theorem 17.10], [22, Theorem 1.2, Chapter II]. □

Proof of Theorem 1.1.5. This is the Isomorphism Theorem for standard Borel spaces [15, Theorem 15.6] or [22, Theorem 2.12, Chapter I]. □

Proof of Theorem 1.1.7. (i) That $\mathcal{X}$ is countably generated is obvious: the open balls with centers in a countable dense subset of X and rational radii obviously generate $\mathcal{X}$. To show the other assertion, first note that, in case X uncountable, it is enough to only consider the standard Borel space $\big([0,1], \mathcal{B}([0,1])\big)$, by Theorem 1.1.5. For if $\{f_n \colon n \in \mathbb{N}\}$ is a countable family of bounded Borel measurable functions on $[0,1]$

which is dense in any $L^p\big([0,1],\mathcal{B}([0,1]),\mu\big)$, $p \in [1,+\infty)$, for any Borel probability measure μ on $\big([0,1],\mathcal{B}([0,1])\big)$, and if $f\colon X \to [0,1]$ is the Borel isomorphism of Theorem 1.1.5, then $\{f_n \circ f\colon n \in \mathbb{N}\}$ is a countable family of bounded Borel functions on X which is dense in any $L^p(X,\mathcal{X},\mu)$, for any $p \in [1,+\infty)$ and any Borel probability measure μ on $(X,\mathcal{X})$. Now in the case of the interval $[0,1]$, the continuous functions are dense in $L^p\big([0,1],\mathcal{B}([0,1]),\mu\big)$ for any $p \in [1,+\infty)$, for any finite Borel measure μ [8, Proposition 7.9], and on the other hand, $C\big([0,1]\big)$ is separable because $[0,1]$ is a compact metric space [3, Theorem 6.6, Chapter V]. It follows that any countable dense subset $\{f_n\colon n \in \mathbb{N}\}$ of $C\big([0,1]\big)$ is dense in $L^p\big([0,1],\mathcal{B}([0,1]),\mu\big)$ for any $p \in [1,+\infty)$, for any finite Borel measure μ. Finally, if X is countable, (finite) linear combinations of the characteristic functions of the singletons with coefficients in a countable dense subset of $\mathbb{C}$, say with coefficients in $\mathbb{Q}+i\mathbb{Q}$, form a countable set of bounded Borel functions on X which is dense in any $L^p\big([0,1],\mathcal{B}([0,1]),\mu\big)$, $p \in [1,+\infty)$, for any finite Borel measure μ on X.

Alternatively one may argue directly as follows, without appealing to the Isomorphism Theorem 1.1.5. If X is a Polish space, it may be embedded as a dense Borel subset of a compact metrizable space. In fact X is homeomorphic to a G_δ subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ with the product topology [15, Theorem 4.14]. If $h\colon X \to [0,1]^{\mathbb{N}}$ is the homeomorphism of X onto its image in $[0,1]^{\mathbb{N}}$ and $d(x,y) := \rho(h(x),h(y))$, $x,y \in X$, where ρ is the metric on the Hilbert cube $[0,1]^{\mathbb{N}}$, then the completion $\big(\hat{X},\hat{d}\big)$ of the metric space (X,d) is a compact metric space. Indeed $\big(\hat{X},\hat{d}\big)$ is complete and totally bounded, because for every $\varepsilon > 0$, on can choose $\delta \in (0,\varepsilon)$, and cover the closure $\overline{h(X)}$ of the image $h(X)$ of X in $[0,1]^{\mathbb{N}}$ with a finite number of open δ-balls with centers $h(x_1),\dots,h(x_n)$ in $h(X)$, by compactness of $\overline{h(X)}$; then the open balls with centers $x_1,\dots,x_n$ and radii ε cover $\hat{X}$. It follows that $C\big(\hat{X}\big)$ is separable and has therefore a countable dense subset $\big\{\hat{f}_n\colon n \in \mathbb{N}\big\}$ [3, Theorem 6.6, Chapter V]. Then, the restrictions $f_n := \hat{f}_n|_X$ of the $\hat{f}_n$ to X, $n \in \mathbb{N}$, form a countable set of bounded Borel functions on X which is dense in $L^p\big(X,\mathcal{X},\mu\big)$ for any $p \in [1,+\infty)$, for any finite Borel measure μ on X. Indeed, if μ is a Borel probability measure in X, then $\hat{\mu}(B) := \mu(B \cap X)$, for B a Borel subset of $\hat{X}$, defines a Borel measure on $\hat{X}$, because X is G_δ and hence Borel in $\hat{X}$, so $B \cap X$ is a Borel subset of X whenever B is a Borel subset of $\hat{X}$. For any $p \in [1,+\infty)$, $C\big(\hat{X}\big)$ is dense in $L^p\big(\hat{X},\hat{\mathcal{X}},\hat{\mu}\big)$ [8, Proposition 7.9], where $\hat{\mathcal{X}}$ is the Borel σ-algebra of $\big(\hat{X},\hat{d}\big)$, and it follows that already $\big\{\hat{f}_n\colon n \in \mathbb{N}\big\}$ is dense in $L^p\big(\hat{X},\hat{\mathcal{X}},\hat{\mu}\big)$. Then, if $g \in L^p(X,\mathcal{X},\mu)$, for

some $p \in [1, +\infty)$, extend g to a function $\hat{g}$ on $\hat{X}$, say by setting $\hat{g}(x) = 0$ for $x \in \hat{X} \smallsetminus X$; then $\hat{g}$ is Borel-measurable, $\int |\hat{g}|^p \,\mathrm{d}\hat{\mu} = \int |g|^p \mathrm{d}\mu < +\infty$ and hence $\hat{g} \in L^p\big(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}\big)$. For any $\varepsilon > 0$ there exists then an $n \in \mathbb{N}$ such that $\big(\int \big|\hat{g} - \hat{f}_n\big|^p \mathrm{d}\hat{\mu}\big)^{1/p} < \varepsilon$, and because $\hat{\mu}\big(\hat{X} \smallsetminus X\big) = 0$, one has that $\varepsilon > \big(\int \big|\hat{g} - \hat{f}_n\big|^p \mathrm{d}\hat{\mu}\big)^{1/p} = \big(\int |g - f_n|^p \,\mathrm{d}\mu\big)^{1/p}$. This shows that $\{f_n \colon n \in \mathbb{N}\}$ is dense in $L^p(X, \mathcal{X}, \mu)$.
(ii) When X is a Polish space, the space of Borel probability measures $\mathrm{M}(X, \mathcal{X})$ on X, where $\mathcal{X}$ is the Borel σ-algebra of X, is completely metrizable and separable, i.e., Polish [15, Theorem 17.23]. Therefore, with $\mathcal{M}$ the Borel σ-algebra of $\mathrm{M}(X, \mathcal{X})$, $(\mathrm{M}(X, \mathcal{X}), \mathcal{M})$ is a standard Borel space. Furthermore, $\mathcal{M}$ is in fact generated by the maps $\mu \mapsto \int f \,\mathrm{d}\mu$ as f varies over bounded real Borel functions on X [15, Theorem 17.24], which by definition means that $\mathcal{M}$ is the smallest σ-algebra with respect to which all these maps are measurable. $\square$

Proof of Lemma 1.1.8. For this fact see [1, Chapter 1] or [19, CHAPTER II, Section 2.2 : The class of locally compact groups]

$\square$

B.2 Section 1.2

Proof of Proposition 1.2.4. References for these facts are, for example, [6, Theorem 2.14] or [25, Theorem 1.5]. $\square$

Proof of Theorem 1.2.5. References for these facts are, [6, Theorem 2.14] again or [25, Theorem 1.6 and Remark (1) following it]. $\square$

Proof of Theorem 1.2.6. A reference for this facts is [25, Corollary 1.14.1]. See also [6, Theorem 2.21 and Corollary 2.22]. $\square$

Proof of Theorem 1.2.7. [6, Theorem 2.30] again or [25, Theorem 1.14]. $\square$

B.3 Section 1.3

Proof of Proposition 1.3.5. Reference [25, Theorem 3.1]. $\square$

Proof of Corollary 1.3.6. When $(X, \mathcal{X})$ is a standard Borel space, $L^2(X, \mathcal{X}, \mu)$ is separable, by Theorem 1.1.7. Hence any orthogonal set is countable, and by Proposition 1.3.5, there can only be countably many distinct eigenvalues. $\square$

Proof of Theorem 1.3.7. Reference [25, Theorem 3.1]; see also [6, Lemma 6.9]. □

B.4 Section 1.4

Proof of Proposition 1.4.4. Let $E := \{x \in X : \pi \circ T(x) = S \circ \pi(x)\}$. Let $A \in \mathcal{A}$. Then $A = \pi^{-1}(B)$ for some $B \in \mathcal{Y}$. Since $(Y, \mathcal{Y}, \nu, S)$ is invertible, there exists $B' \in \mathcal{Y}$ such that $B = S^{-1}(B')$. Let $A' := \pi^{-1}(B')$. Then $T^{-1}(A') \mathbin{\triangle} A \subseteq E^{\mathrm{c}}$, because when $x \in E$, then

$$x \in T^{-1}(A') \Leftrightarrow T(x) \in A' \Leftrightarrow \pi \circ T(x) \in B' \Leftrightarrow S \circ \pi(x) \in B' \Leftrightarrow \pi(x) \in B \Leftrightarrow x \in A.$$

Hence $\mu\big(T^{-1}(A') \mathbin{\triangle} A\big) = 0$. Conversely, given $T^{-1}(A) \in T^{-1}\mathcal{A}$, i.e., given $A \in \mathcal{A}$, there exists $B \in \mathcal{Y}$ such that $\pi^{-1}(B) = A$; since S is measurable, $S^{-1}(B) \in \mathcal{Y}$, whence $A' := \pi^{-1}\big(S^{-1}(B)\big) \in \mathcal{A}$, and again $A' \mathbin{\triangle} T^{-1}(A) \subseteq E^{\mathrm{c}}$. □

Proof of Theorem 1.4.5. A reference for this is [6, Theorem 6.5]. The setting there is slightly different than ours in that a Borel probability space in that reference is defined to be a dense Borel subset of a compact metric space $\overline{X}$ endowed with the restriction of the Borel σ-algebra $\mathcal{B}\big(\overline{X}\big)$ to X and a probability measure defined on this restriction. As explained in the proof of Theorem 1.1.7, when X is a Polish space, it may be embedded as a dense G_δ and hence Borel subset of a compact metrizable space. Thus a Polish space endowed with its Borel σ algebra and a Borel probability measure is a Borel probability space in the sense of [6]. Finally, if X is merely Borel isomorphic to a Polish space X' and $h\colon X \to X'$ is a Borel isomorphism, i.e., h is invertible and both h and h^{-1} are measurable with respect to the Borel σ-algebras on X and X', and if $\mathcal{A}$ is a sub-σ-algebra of the Borel σ-algebra $\mathcal{X}$ of X satisfying $\mathcal{A} = T^{-1}\mathcal{A} \bmod \mu$, then $h\mathcal{A}$ is a sub-σ-algebra of the Borel σ-algebra $\mathcal{X}'$ of X' satisfying $(T')^{-1}h\mathcal{A} = h\mathcal{A} \bmod \mu'$, where $T' := h \circ T \circ h^{-1}$ and $\mu' := h_*\mu$. Note that $h\mathcal{X} = \mathcal{X}'$. If $(Y, \mathcal{Y}, \nu, S)$ is a factor of $(X', \mathcal{X}', \mu', T')$ with $(Y, \mathcal{Y})$ standard Borel and $\mathcal{A} = \pi^{-1}(\mathcal{Y})$, where $\pi\colon X' \to Y$ is a factor map, then $(Y, \mathcal{Y}, \nu, S)$ is also a factor of the original system $(X, \mathcal{X}, \mu, T)$ with factor map $\pi \circ h$ and satisfies $(\pi \circ h)^{-1}(\mathcal{Y}) = \mathcal{A}$. □

B.5 Section 1.5

Proof of Proposition 1.5.1. These are standard results in Probability theory, see, e.g., Section 34 of [2]. Alternatively, see [6, Theorem 5.1] for the Proposition as stated here. □

Proof of Remarks 1.5.2.

- Reference [6, Theorem 5.1]
- Reference [6, page 125]
- Note that $\mathbb{E}(f \mid \mathcal{A}) \circ T$ is $T^{-1}(\mathcal{A})$-measurable because $\mathbb{E}(f \mid \mathcal{A})$ is $\mathcal{A}$-measurable. Let $A \in \mathcal{A}$

$$\int\limits_{T^{-1}(A)} \mathbb{E}(f \circ T \mid T^{-1}\mathcal{A}) \, \mathrm{d}\mu = \int\limits_{T^{-1}(A)} f \circ T \, \mathrm{d}\mu$$
$$= \int\limits_{A} f \, \mathrm{d}\mu = \int\limits_{A} \mathbb{E}(f \mid \mathcal{A}) \, \mathrm{d}\mu = \int\limits_{T^{-1}(A)} \mathbb{E}(f \mid \mathcal{A}) \circ T \, \mathrm{d}\mu$$

□

Proof of Theorem 1.5.5. Reference [6, Lemma 5.25]. This is standard again, see e.g., [4, Theorem 4.2.8].

Alternatively, the statement is proven when X, Y and Z are dense Borel subsets of some compact spaces $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ respectively. In particular we have the result when X, Y and Z are compact spaces. Via Theorem 1.1.5 we obtain the same result when X, Y and Z are Polish spaces. □

Proof of Remark 1.5.6. When $(Y, \mathcal{Y}, \nu, S)$ is invertible, $S^{-1}\mathcal{Y} \subseteq \mathcal{Y}$, because S is measurable, and also $S\mathcal{Y} \subseteq \mathcal{Y}$, because S^{-1} is measurable, whence $\mathcal{Y} \subseteq S^{-1}\mathcal{Y}$, because $A = S^{-1}(S(A)) \in S^{-1}(S\mathcal{Y}) \subseteq S^{-1}\mathcal{Y}$ for all $A \in \mathcal{Y}$. It follows that $S^{-1}\mathcal{Y} = \mathcal{Y}$, and then that

$$T^{-1}(\pi^{-1}\mathcal{Y}) = (\pi \circ T)^{-1}\mathcal{Y} = (S \circ \pi)^{-1}\mathcal{Y} = \pi^{-1}(S^{-1}\mathcal{Y}) = \pi^{-1}\mathcal{Y}.$$

Let $f \in L^1(X, \mathcal{X}, \mu)$. One then has that, for μ-almost all $x \in X$,

$$\mathbb{E}(f \circ T \mid Y)(\pi(x)) = \mathbb{E}(f \circ T \mid \pi^{-1}\mathcal{Y})(x) = \mathbb{E}(f \circ T \mid T^{-1}(\pi^{-1}\mathcal{Y}))(x)$$
$$= \mathbb{E}(f \mid \pi^{-1}\mathcal{Y})(T(x)) = \mathbb{E}(f \mid Y)(\pi(T(x))) = \mathbb{E}(f \mid Y)(S(\pi(x))) = \mathbb{E}(f \mid Y) \circ S(\pi(x)).$$

Let $A := \{y \in Y \colon \mathbb{E}(f \circ T \mid Y)(y) \neq \mathbb{E}(f \mid Y) \circ S(y)\}$. Then

$$\nu(A) = \mu\big(\pi^{-1}(A)\big) = \mu\big(\{x \in X \colon \mathbb{E}(f \circ T \mid Y)(\pi(x)) \neq \mathbb{E}(f \mid Y) \circ S(\pi(x))\}\big) = 0. \quad \square$$

B.6 Section 1.6

Proof of Theorem 1.6.1. Reference [6, Theorem 5.14]. The statement is proven when X is a dense Borel subset of some compact spaces $\tilde{X}$. In particular we have the result when X is a compact space. Via Theorem 1.1.5 we obtain the same result when X is Polish space.
Another reference [14, Part 1, Chapter 2, Section 2:Probability Spaces, Subsection 2.5:Disintegration of a measure, page 19] □

Proof of Theorem 1.6.3. Reference [6, Theorem 5.19] □

Proof of Proposition 1.6.5. Reference [14, Part 1, Chapter 2, Section 2:Probability Spaces, Subsection 2.5:Disintegration of a measure, page 19] □

Proof of Remark 1.6.6. Reference [14, Part 1, Chapter 3, Section 2:Ergodic Theory, Subsection 2.8:Disintegration of a measure, page 34] □

B.7 Section 1.7

Proof of Theorem 1.7.1. Reference [6, Theorem 6.2]. The statement is proven when X is a dense Borel subset of some compact spaces $\tilde{X}$. In particular we have the result when X is a compact space. Via Theorem 1.1.5 we obtain the same result when X is Polish space. □

B.8 Section 1.8

References for this Section are, for example, [6], [11] and [14]

B.9 Section 1.9

Proof for Theorem 1.9.1. A reference for this facts is [14, Part 1, Chapter 4, Section 1:Topological and measurable rotations, Subsection 1.2:Measurable rotations, Proposition 5, page 47]. See also [6, Theorem 4.14] □

Proof for Proposition 1.9.2. Reference [14, Part 1, Chapter 4, Section 1:Topological and measurable rotations, Subsection 1.2:Joinings and factors of rotations, Proposition 7, page 49] □

Proof for Corollary 1.9.3. Reference [14, Part 1, Chapter 4, Section 1:Topological and measurable rotations, Subsection 1.2:Joinings and factors of rotations, Corollary 8, page 49] □

Proof for Corollary 1.9.4. Reference [14, Part 1, Chapter 4, Section 1:Topological and measurable rotations, Subsection 1.2:Joinings and factors of rotations, Corollary 9, page 49] □

B.10 Section 1.10

Proof for Lemma 1.10.5. d is clearly a metric. By Theorem 1.1.5 there exists a Borel isomorphism $\phi : (Y, \mathcal{Y}) \to ([0,1], \mathcal{B}([0,1]))$. Define the Borel probability measure on $([0,1], \mathcal{B}([0,1]))$, $\lambda := \phi_*\nu$. We have that $\phi : (Y, \mathcal{Y}, \nu) \to ([0,1], \mathcal{B}([0,1]), \lambda)$ is an isomorphism between these two Lebesgue probability spaces. This isomorphism induces a isometric isomorphism between $Coc(Y, \mathbb{T}^\ell)$ and $Coc([0,1], \mathbb{T}^\ell)$. In particular by defining $\Phi : \big(Coc(Y, \mathbb{T}^\ell), d_{Coc(Y,\mathbb{T}^\ell)}\big) \to \big(Coc([0,1], \mathbb{T}^\ell), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$, where $\Phi(\rho) = \rho \circ \phi^{-1}$ one can easily check that is Φ is indeed an isometric isomorphism. Hence it sufficient to show that $\big(Coc([0,1], \mathbb{T}^\ell), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$ is a Polish space.

Firstly, by Lemma A.1 $\big(C([0,1]), \mathbb{T}^\ell), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$ is a dense subset of $(Coc([0,1], \mathbb{T}^\ell)$. Furthermore $\big(C([0,1]), \mathbb{T}^\ell), d_{sup}\big)$ where d_{sup} is the metric on $C([0,1]$ induced by $\|\cdot\|_\infty$, is separable and $\forall \rho, \rho' \in C([0,1])$, $d_{Coc([0,1],\mathbb{T}^\ell)}(\rho, \rho') \leq d_{sup}(\rho, \rho')$. It follows that $\big(C([0,1]), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$ is separable. Hence $\big(Coc([0,1], \mathbb{T}^\ell), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$ is separable.

Now let $(\rho_n)_{n\in\mathbb{N}}$ be a $d_{Coc([0,1],\mathbb{T}^\ell)}$-Cauchy sequence in $\big(Coc([0,1], \mathbb{T}^\ell), d_{Coc([0,1],\mathbb{T}^\ell)}\big)$. This means that for every $\epsilon > 0$ there exists a $n_0 = n_0(\epsilon)$ such that $\forall n, m \geq n_0$, $d_{Coc([0,1],\mathbb{T}^n)}(\rho_n, \rho_m) < \epsilon \iff \int_{[0,1]} d_{\mathbb{T}^\ell}(\rho_n(x), \rho_m(x))\, \mathrm{d}\lambda(x) < \epsilon \implies \mathrm{d}_{\mathbb{T}^\ell}(\rho_n(x), \rho_m(x))$ for λ-almost every $x \in [0,1]$. In particular there exists a $X' \in \mathcal{B}([0,1])$ such that $\lambda(X') = 1$ and for every $x \in X'$ and every $\epsilon > 0$ there exists a $n_0 = n_0(\epsilon)$ such that $\forall n, m \geq n_0$, $d_{\mathbb{T}^\ell}(\rho_n(x), \rho_m(x)) < \epsilon$. This means that $\forall x \in [0,1]$ the sequence $\big(\rho_n(x)\big)$ is $d_{\mathbb{T}^\ell}$-Cauchy in $\mathbb{T}^\ell$. Therefore there exists an element $t = t(x) \in \mathbb{T}^\ell$. such that $\rho_n(x) \xrightarrow{d_{\mathbb{T}^\ell}} t(x)$. Define ($\lambda$-a.e.) $\rho : [0,1] \to \mathbb{T}^\ell$ by $\rho(x) = t(x)$. Then ρ is measurable, since is it is (λ-a.e.) the pointwise limit of ρ_n. In particular $\rho \in Coc([0,1], \mathbb{T}^\ell)$. We are left with showing that ρ is the $d_{Coc([0,1],\mathbb{T}^\ell)}$-limit of ρ_n. Let $\epsilon > 0$ and $x\ X'$. Then,

(B.1) there exists a $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that, $\forall n, m \geq n_0,\ d_{\mathbb{T}^\ell}(\rho_n(x), \rho_m(x)) < \epsilon/4.$

(B.2) there exists a $n_1 = n_1(\epsilon, x) \in \mathbb{N}$ such that, $\forall n \geq n_1,\ d_{\mathbb{T}^\ell}(\rho_n(x), \rho(x)) < \epsilon/4$.

Let $N = \max\{n_0, n_1\}$ and $m \geq N$. Then $\forall n \geq (\epsilon)$ we have that,

$$d_{\mathbb{T}^\ell}(\rho_n(x), \rho(x)) \leq d_{\mathbb{T}^\ell}(\rho_n(x), \rho_m(x)) + d_{\mathbb{T}^\ell}(\rho_m(x), \rho(x)) \overset{\substack{(B.1),\\(B.2)}}{<} \epsilon/4 + \epsilon/4 = \epsilon/2$$

. It follows that for all $n \geq n_0(\epsilon)$,

$$d_{Coc([0,1],\mathbb{T}^\ell)}(\rho_n, \rho) = \int_{[0,1]} d_{\mathbb{T}^\ell}(\rho_n(x), \rho(x))\ \mathrm{d}\lambda(x) = \int_{X'} d_{\mathbb{T}^\ell}(\rho_n(x), \rho(x))\ \mathrm{d}\lambda(x) \leq \epsilon/2 < \epsilon$$

In other words ρ is the $d_{Coc([0,1],\mathbb{T}^\ell)}$-limit of ρ_n.

For the second part of this Lemma see [14, Part 1, Chapter 5, Section 3:Cocycles and coboundaries, Subsection 3.3:Measurability properties, Lemma 10, page 65] □

Proof for Proposition 1.10.12. Reference [10]. They show that if $X = Y \times G$ is an ergodic extension of Y and W is an intermediate extension of Y , then X is an extension of W by a closed subgroup H of G. From the proof, it follows that W is an isometric extension of Y , and it is of the form $W = Y \times G/H$. The same result holds more generally when X is an isometric extension of Y . □

Proof of Theorem 1.10.13. Reference [11, Theorem 9.21] □

Proof of Lemma 1.10.15. Reference of this fact is [14, Part 1, Chapter 5, Section 2:Extensions by a compact abelian group, Subsection 2.2:Uniqueness of the measure, Lemma 4, page 61] □

Proof of Lemma 1.10.16. Reference [14, Part 1, Chapter 5, Section 3:Cocycles and coboundaries, Subsection 3.4:Cocycles on a Cartesian square, Lemma 13, page 67] □

B.11 Section 1.11

Proof of Proposition 1.11.5. Reference [16, Theorem 20.10] □

Proof of Theorem 1.11.10. Reference [18] □

Proof of Proposition 1.11.12. Reference [20] □

Proof of Proposition 1.11.13. Reference [20] □

Proof of Proposition 1.11.14. Reference of this fact are [20] or [17] □

B.12 Section 1.12

References for this Section are, for example [11] and [14].

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