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Harmonic Functions on Manifolds

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Abstract

In this Master thesis we deal with harmonic functions on complete Riemannian manifolds, having as a final goal a proof of Yau's conjecture (and eventually theorem of Colding and Minicozzi) which states that the space of harmonic functions of polynomial growth of fixed degree d, $H^d(M)$, on a complete Riemannian manifold with non-negative Ricci curvature, M, is finite dimensional.

In the first chapter we give the definitions for the covariant derivatives of a function, and subsequently the Hessian and Laplacian. We also give a formula for $\triangle\big(|\nabla f|^2\big)$, where $f \in C^\infty(M)$.

In the second chapter, we first give a proof for the Laplacian comparison theorem, which bounds the Laplacian of the distance function on a complete Riemannian manifold of dimension n where the Ricci curvature satisfies $Ric \geq -(n-1)k$, where $k \geq 0$, by the Laplacian of the distance function on a space form with constant curvature $-k$. Then, we prove a gradient estimate for positive harmonic functions on such manifolds. From that result, we derive as corollaries a Liouville property, a Harnack type inequality and the finite dimensionality of the space of harmonic functions with sublinear growth of fixed degree.

In chapter 3, we give proofs of a lemma of Yau, which estimates integrals of the form $\displaystyle\int_B f^{\alpha-2}|\nabla f|$, and of the Poincaré inequality. Using these results, we give a proof for a mean value inequality.

In chapter 4, we give a proof of the theorem of Colding and Minicozzi using the ideas of Li. We also give a proof for the finite dimensionality of $H^d(M)$ in a more "relaxed" setting, where we assume a weaker mean value inequality. Then, we state two results about massive sets, where the ideas of Li find application.

Finally, in chapter 5, we state some later developments and conjectures on the subject of the dimension of $H^d(M)$.

Περίληψη

Στην παρούσα διπλωματική, ασχολούμαστε με αρμονικές συναρτήσεις σε πλήρεις πολλαπλότητες Riemann, έχοντας ως τελικό στόχο την παρουσίαση μιας απόδειξης της εικασίας του Yau (και εν τέλει θέωρημα των Colding-Minicozzi) σύμφωνα με την οποία, ο χώρος των αρμονικών συναρτήσεων πολυωνυμικής ανάπτυξης σε μια πλήρη πολλαπλότητα με μη αρνητική Ricci καμπυλότητα, είναι πεπερασμένης διάστασης.

Στην πρώτη ενότητα, δίνουμε ορισμούς για τις συναλλοίωτες παραγώγους μιας συνάρτησης, την Εσσιανή και την Λαπλασιανή. Επίσης αποδεικνύουμε έναν τύπο για την ποσότητα $\triangle\big(|\nabla f|^2\big)$, όπου $f \in C^\infty$

Στην δεύτερη ενότητα, δίνουμε μια απόδειξη για το θεώρημα σύγκρισης της Λαπλασιανής, σύμφωνα με το οποίο, η Λαπλασιανή της συνάρτησης απόστασης σε μια πλήρη πολλαπλότητα M, της οποίας η Ricci καμπυλότητα ικανοποιεί τη συνθήκη $Ric \geq -(n-1)k$, $k \geq 0$, φράσεται απ΄ την Λαπλασιανή της συνάρτησης απόστασης σε μία πολλαπλότητα σταθερής καμπυλότητας $-k$. Στη συνέχεια, αποδεικνύουμε μια εκτίμηση για την κλίση μιας θετικής αρμονικής συνάρτησης σε μια πολλαπλότητα όπως παραπάνω. Από αυτό το αποτέλεσμα, παίρνουμε ως πορίσματα την ιδιότητα Liouville, μια ανισότητα τύπου Harnack και το πεπερασμένο της διάστασης του χώρου των αρμονικών συναρτήσεων υπογραμμικής ανάπτυξης.

Στην τρίτη ενότητα, δίνουμε αποδείξεις για ένα λήμμα του Yau, το οποίο εκτιμάει ολοκληρώματα της μορφής $\displaystyle\int_B f^{\alpha-2}|\nabla f|$, και της ανισότητας Poincaré. Χρησιμοποιώντας αυτά τα αποτελέσματα, αποδεικνύουμε μια ανισότητα μέσης τιμής.

Στην τέταρτη ενότητα, δίνουμε μια απόδειξη του Li για το θεώρημα των Colding-Minicozzi, ενώ στη συνέχεια αναφέρουμε δύο αποτελέσματα για massive sets, όπου οι ιδέες του Li βρίσκουν επίσης εφαρμογή.

Τέλος, στην πέμπτη ενότητα, αναφέρουμε περαίτερω αποτελέσματα σχετικά με την διάσταση του χώρου των αρμονικών συναρτήσεων πολυωνυμικής ανάπτυξης σε πλήρεις πολλαπλότητες με μη αρνητική Ricci καμπυλότητα.

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Contents

Introduction

With the Riemannian structure on a manifold, we get the associated Levi-Civita connection and as a result, we can consider covariant derivatives. Therefore we can consider the Laplace operator and subsequently, harmonic functions. For certain types of manifolds there existed several well known results on the theory of harmonic functions. For example, on the unit disk there is the Herglotz theory (1911). On bounded domains of $\mathbb{R}^n$ we have the Martin representation of harmonic functions (1941) and on bounded symmetric domains there is the theory of Hua and Fürstenberg(1963). However, for general complete Riemannian manifolds, the first serious contributions on the theory of harmonic functions were made by the work of Yau. On such manifolds, Yau proved that for $1 < p < +\infty$, there are no harmonic functions in L^p. For $p = +\infty$ however, extra assumptions on the curvature must be made. Therefore, Yau, in 1975 ([20]), proved that complete Riemannian manifolds with non-negative Ricci curvature have a Liouville property, i.e. there are no non-constant harmonic functions. Later, he along with Cheng ([2]) localized his argument and proved a gradient estimate for harmonic functions. As corollaries, they derived the Liouville property again, a Harnack-type inequality and also the finite dimensionality of the space of harmonic functions of sublinear growth on a complete Riemannian manifold with non-negative Ricci curvature (in the same paper, Cheng also proved a similar result for harmonic maps into a Cartan-Hadamard manifold, i.e. a complete, simply connected Riemannian manifold with non-positive sectional curvature). Yau then conjectured, that the space of harmonic functions of polynomial growth of a fixed degree on such a manifold is finite dimensional. Note that in the case of $\mathbb{R}^n$, the dimension is explicitly calculated (see Chapter 5).

It was not until 1996, that Yau's conjecture was first proved. Specifically, Colding and Minicozzi proved Yau's conjecture, in the case the manifold has non-negative Ricci curvature and Euclidean volume growth ([4]). Later, they proved Yau's conjecture in the form of theorem 4.1.2 ([5]), where they also provided a sharp estimate on the dimension of the space of harmonic functions of polynomial growth. In that case, a vital role play Bishop's volume comparison theorem and the mean value inequality (theorem 3.0.3). In 1997, Li gave a proof that applied to a larger class of manifolds and that also applied to sections of vector bundles (see the beginning of chapter

4). Specifically, he assumed a volume comparison condition (definition 4.0.2) and a mean value inequality (definition 4.0.3). Note, that in the case of non-negative Ricci curvature these are satisfied (Bishop's volume comparison theorem and theorem 3.0.3). Later, Li and Wang observed that if we are not interested in getting a sharp estimate on the dimension, but only in proving the finite dimensionality of the space of harmonic functions of polynomial growth, we can relax the above conditions, by assuming the volume to have polynomial growth and a weak mean value inequality (see chapter 4).

It is worth mentioning that similar results to the ones we present here for harmonic functions, such as the gradient estimate and the Harnack inequality, hold for positive solutions of the heat equation, i.e. $\left(\triangle - \frac{\partial}{\partial t}\right) f(x,t) = 0$. (see [16])

Chapter 1

Preliminaries

Throughout this thesis let M be a *complete Riemannian* manifold of dimension n, $(n \geq 2)$. We will denote the metric tensor as $g(\cdot,\cdot)$ as well as $< \cdot,\cdot >$. For a given chart $x = (x_1, \cdots, x_n)$ and the associated frame $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$, g_{ij} will denote the metric components with respect to this chart -i.e. $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$- and g^{ij} will denote the entries of the inverse matrix of $(g_{ij})_{i,j}$. ∇ will denote the Levvi-Civita connection of M. For an $f \in C^\infty(M)$ we denote *gradf* with ∇f as well, where *gradf* is the unique vector field such that $Y(f) = \langle Y, gradf \rangle$.
The curvature R of M is $R_{XY}(Z) = \nabla_Y\big(\nabla_X Z\big) - \nabla_X\big(\nabla_Y Z\big) - \nabla_{[X,Y]}Z$. The curvature tensor is $R(X,Y,Z,W) = \langle R_{XY}(W), Z\rangle$ and with R_{ijkl} we denote its components, i.e. $R_{ijkl} = R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = \langle R_{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}}(\frac{\partial}{\partial x_l}), \frac{\partial}{\partial x_k}\rangle$
The *Ricci curvature* of M is the unique -up to sign- non-zero contraction of R, and we denote it by $Ric(\cdot,\cdot)$. If $x = (x_1, \cdots, x_n)$ is a normal coordinate chart with associated frame $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$, then the components of the Ricci curvature are $R_{ij} = \sum_{k=1}^{n} \langle R_{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}}(\frac{\partial}{\partial x_k}), \frac{\partial}{\partial x_j}\rangle$

1.1 Connection Forms

Let (U,x) be a normal coordinate chart of M, so that the frame $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$ is orthonormal at each $p \in U$. We define the 1-forms ω_{ij} by

$$\omega_{ij}(v) = \langle \nabla_v \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$$

The linearity of ∇ in the lower argument and the linearity of the metric guarantee that these are indeed 1-forms.
We call them the *connection forms* of M with respect to (U,x).

Let ω_i be the dual basis of $\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right\}$. So the ω_i's are 1-forms and $\omega_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. Cartan's first structural equation says that

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j$$

Cartan's second structural equation says that

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}$$

where $\Omega_{ij} = \sum_{l,k=1}^{n} R_{ijkl}\omega_l \wedge \omega_k$

1.2 Covariant Derivatives

Let $f \in C^\infty(M)$. Then $df = \sum_{i=1}^{n} f_i\omega_i$, where $f_i = \frac{\partial}{\partial x_i}(f)$. If we consider $\eta = df_i + \sum_{k=1}^{n} f_k\omega_{ki}$, then η is a 1-form and since the ω_i's form a basis, we must have $\eta = \sum_{j=1}^{n} \alpha^j\omega_j$ for some functions α^j.

We define the *second covariant derivatives* of f by

$$f_{ij} := \alpha^j.$$

So we have

$$\sum_{j=1}^{n} f_{ij}\omega_j = df_i + \sum_{k=1}^{n} f_k\omega_{ki}$$

Lemma 1.2.1. $f_{ij} = f_{ji}$

Proof. We exterior-differentiate the equation $df = \sum_{i=1}^{n} f_i\omega_i$ and we have:

$$\begin{aligned} 0 &= \sum_i \Big[df_i \wedge \omega_i + f_i d\omega_i\Big] \\ &= \sum_k df_k \wedge \omega_k + \sum_{i,k} f_i\omega_{ik} \wedge \omega_k \text{ (Cartan's 1st equation)} \\ &= \sum_k \Big[df_k + \sum_i f_i\omega_{ik}\Big] \wedge \omega_k \\ &= \sum_{k,i} f_{ki}\omega_i \wedge \omega_k \end{aligned}$$

$\Rightarrow f_{ij} = f_{ji}$ □

Having had the second covariant derivatives defined, we consider the 1-form $df_{ij} + \sum_k \Big[f_{kj}\omega_{ki} + f_{ik}\omega_{kj}\Big]$ and we define, similarly as above, the *third covariant derivatives* f_{ijk} of f by the equation

$$\sum_k f_{ijk}\omega_k = df_{ij} + \sum_k \Big[f_{kj}\omega_{ki} + f_{ik}\omega_{kj}\Big]$$

Lemma 1.2.2. *(Ricci equation)* $\sum_{i=1}^{n}\big(f_{iji}-f_{iij}\big)=\sum_{l=1}^{n}f_l R_{lj}$

Proof. We begin by exterior-differentiating the equation $\sum_{j=1}^{n}f_{ij}\omega_j = df_i+\sum_{k=1}^{n}f_k\omega_{ki}$, which gives

$$\sum_j \Big[df_{ij}\wedge\omega_j + f_{ij}d\omega_j\Big]=\sum_j\Big[df_j\wedge\omega_{ji}+f_j d\omega_{ji}\Big]$$

Bringing everything to the right hand side, we compute (constantly renaming the indices):

$$\begin{aligned}
0 &= -\sum_j\Big[df_{ij}\wedge\omega_j+\sum_k f_{ij}\omega_{jk}\wedge\omega_k\Big]+\sum_j\Big[df_j\wedge\omega_{ji}+\sum_k f_j\omega_{jk}\wedge\omega_{ki}+f_j\Omega_{ji}\Big]\\
&= -\sum_k df_{ik}\wedge\omega_k-\sum_{j,k}f_{ij}\omega_{jk}\wedge\omega_k+\sum_k df_k\wedge\omega_{ki}+\sum_{j,k}f_j\omega_{jk}\wedge\omega_{ki}+\sum_j f_j\Omega_{ji}\\
&= -\sum_k\Big[df_{ik}+\sum_j f_{ij}\omega_{jk}\Big]\wedge\omega_k+\sum_k\Big[df_k+\sum_j f_j\omega_{jk}\Big]\wedge\omega_{ki}+\sum_j f_j\Omega_{ji}\\
&= -\sum_k\Big[df_{ik}+\sum_j f_{ij}\omega_{jk}\Big]\wedge\omega_k+\sum_{k,j}f_{kj}\omega_j\wedge\omega_{ki}+\sum_j f_j\Omega_{ji}\\
&= -\sum_k\Big[df_{ik}+\sum_j f_{ij}\omega_{jk}\Big]\wedge\omega_k-\sum_{k,j}f_{kj}\omega_{ji}\wedge\omega_k+\sum_j f_j\Omega_{ji}\\
&= -\sum_k\Big[df_{ik}+\sum_j\Big[f_{ij}\omega_{jk}+f_{jk}\omega_{ji}\Big]\Big]\wedge\omega_k+\sum_j f_j\Omega_{ji}\\
&= -\sum_{k,j}f_{ikj}\omega_j\wedge\omega_k+\frac{1}{2}\sum_{j,l,k}f_jR_{jikl}\omega_l\wedge\omega_k\\
&= -\sum_{k,j}f_{ikj}\omega_j\wedge\omega_k+\frac{1}{2}\sum_{k,j}\Big(\sum_l f_lR_{likj}\Big)\omega_j\wedge\omega_k
\end{aligned}$$

From this we get

$$\frac{1}{2}\sum_l f_lR_{likj}-f_{ikj}=\frac{1}{2}\sum_l f_lR_{lijk}-f_{ijk}\Rightarrow f_{ijk}-f_{ikj}=\frac{1}{2}\sum_l f_l\Big(R_{lijk}-R_{likj}\Big)=\sum_l f_lR_{lijk}$$

Setting $i=k$ and summing for $1\le i\le n$ we derive

$$\sum_i\big(f_{iji}-f_{iij}\big)=\sum_l f_lR_{lj}$$

□

1.3 The Hessian

The symmetric 2-tensor defined by $H_f := f_{ij}\omega_i\otimes\omega_j$ is called the *Hessian* of f.
It can be equivalently defined by $H_f\big(X,Y\big)=X(Y(f))-\big(\nabla_XY\big)(f)$ [see [8]]

Now, suppose $p \in M$ and $x \notin Cut(p)$. Let $\gamma : [0, r] \to M$ be a minimal, normal (i.e. $|\gamma'| = 1$) geodesic from p to x. Let ρ be the distance function from p. Recall, that if $\frac{\partial}{\partial r}$ is the radial, outward-pointing vector field (in a normal coordinate chart around p), then [see [17]]:

$$\nabla \rho = \frac{\partial}{\partial r}$$

Since γ is normal, $\gamma' = \frac{\partial}{\partial r} = \nabla \rho$.
Now, suppose $X \in T_xM$ such that $\langle X, \gamma' \rangle = 0$. Then, since x is not conjugate to p, there is a Jacobi field, $\widetilde{X}$, such that $\widetilde{X}(0) = 0$ and $\widetilde{X}(r) = X$ and $[\widetilde{X}, \gamma'] = 0$. Later, we will need the following:

Lemma 1.3.1. $H_\rho(X, X) = \int_0^r \Big(|\frac{D}{dt}(\widetilde{X})|^2 - \langle R(\widetilde{X}, \gamma')(\gamma'), \widetilde{X} \rangle \Big) = I_r(\widetilde{X}, \widetilde{X})$ *(the index form).*

Proof.

$$\begin{aligned} H_\rho(X, X) &= \widetilde{X}\big(\widetilde{X}(\rho)\big) - \big(\nabla_{\widetilde{X}} \widetilde{X}\big)(\rho) = \widetilde{X}\Big(\langle \widetilde{X}, \nabla \rho \rangle\Big) - \langle \nabla_{\widetilde{X}} \widetilde{X}, \nabla \rho \rangle \\ &= \widetilde{X}\Big(\langle \widetilde{X}, \gamma' \rangle\Big) - \langle \nabla_{\widetilde{X}} \widetilde{X}, \gamma' \rangle = \langle \widetilde{X}, \nabla_{\widetilde{X}} \gamma' \rangle \end{aligned}$$

The last equality is true because, since $\widetilde{X}(0) = 0$ and $\widetilde{X}(r) = X$ we have that $\langle \widetilde{X}, \gamma' \rangle(0) = \langle \widetilde{X}, \gamma' \rangle(r) = 0 \Rightarrow \langle \widetilde{X}, \gamma' \rangle \equiv 0$ [see [6]].
Also, because $[\widetilde{X}, \gamma'] = 0$, $\langle \widetilde{X}, \nabla_{\widetilde{X}} \gamma' \rangle = \langle \widetilde{X}, \nabla_{\gamma'} \widetilde{X} \rangle$. Therefore we have:

$$H_\rho(X, X) = \int_0^r \frac{d}{dt} \langle \widetilde{X}, \nabla_{\gamma'} \widetilde{X} \rangle = \int_0^r \Big(|\frac{D}{dt}(\widetilde{X})|^2 + \langle \widetilde{X}, \nabla_{\gamma'}\big(\nabla_{\gamma'} \widetilde{X}\big) \rangle \Big)$$

But since $\widetilde{X}$ is a Jacobi field, $\nabla_{\gamma'}\big(\nabla_{\gamma'} \widetilde{X}\big) + R(\widetilde{X}, \gamma')(\gamma') = 0$ Hence

$$H_\rho(X, X) = \int_0^r \Big(|\frac{D}{dt}(\widetilde{X})|^2 - \langle R(\widetilde{X}, \gamma')(\gamma'), \widetilde{X} \rangle \Big)$$

□

1.4 The Laplacian

For a $X \in \mathfrak{X}(M)$ and each $p \in M$ we can define a linear map in T_pM by $Y_p \to \big(\nabla_Y X\big)_p$. Taking the trace of that map we get the *divergence* of X, i.e.

$$divX(p) = tr\big\{Y_p \to \big(\nabla_Y X\big)_p\big\}$$

Then, for a $f \in C^\infty(M)$ we define the *Laplacian* of f by

$$\triangle f = div\big(gradf\big)$$

Locally, for a given chart $x = (x_1, \ldots, x_n)$ we can define the *Laplace - Beltrami* operator by

$$\triangle = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \big(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \big)$$

where $g = det\big((g_{ij})_{i,j}\big)$.

Alternatively, we can define the Laplacian as the trace of the Hessian, i.e. $\triangle f = \sum_{i=1}^{n} f_{ii}$.

The above different definitions are easily seen to be equivalent.

Lemma 1.4.1. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a normal coordinate chart at $p \in M$ and $f \in C^\infty(M)$. Then at p,*

$$\triangle\Big(|gradf|^2\Big) = 2\sum_{i,j} f_{ij}^2 + 2Ric(gradf, gradf) + 2\sum_i f_i (\triangle f)_i$$

Proof. Since $\boldsymbol{x}$ is a normal coordinate chart, we have that $gradf = \sum_i f_i \frac{\partial}{\partial x_i}$ and $|gradf|^2 = \sum_i f_i^2$.
So

$$\begin{aligned}
\triangle\Big(|gradf|^2\Big) &= \triangle\big(\sum_i f_i^2\big) \\
&= \sum_j \Big(\sum_i f_i^2\Big)_{jj} \\
&= \sum_j \Big(\sum_i 2 f_i f_{ij}\Big)_j \\
&= \sum_j \Big(\sum_i 2 f_i f_{ji}\Big)_j \qquad (f_{ij} = f_{ji}) \\
&= 2\sum_{i,j} f_{ij}^2 + 2\sum_i f_i \big(\sum_j f_{jij}\big) \\
&= 2\sum_{i,j} f_{ij}^2 + 2\sum_i f_i \big[\sum_j \big(f_j R_{ij} - f_{jji}\big)\big] \qquad \textit{(Ricci equation)} \\
&= 2\sum_{i,j} f_{ij}^2 + 2\sum_{i,j} f_i f_j R_{ij} + 2\sum_i f_i \big(\sum_j f_{jj}\big)_i \\
&= 2\sum_{i,j} f_{ij}^2 + 2Ric(gradf, gradf) + 2\sum_i \big(\triangle f\big)_i f_i
\end{aligned}$$

□

Definition 1.4.1. A function $f \in C^\infty(M)$ is called harmonic at $p \in M$ if $\triangle f(p) = 0$. f is called harmonic if $\triangle f = 0$ everywhere. f is called subharmonic if $\triangle f \geq 0$.

1.5 Bishop's Volume Comparison Theorem

We state a standard result from Riemannian geometry, that we will use later.

Theorem 1.5.1. (Bishop) *Suppose* $R_{ij} \geq (n-1)k$*, where* k *is a constant. Let* $p \in M$*. We denote the volume of a geodesic ball of radius* r *around* p*, by* $V_p(r)$*, and the volume of a geodesic ball of radius* r *of a space form with constant curvature* k*, by* $\bar{V}(r)$*. Then, for any* $0 < r_1 \leq r_2 < +\infty$*, we have that*

$$\frac{V_p(r_2)}{V_p(r_1)} \leq \frac{\bar{V}(r_2)}{\bar{V}(r_1)}$$

For a proof, see [12].

Chapter 2

Gradient estimate

In this chapter we will compute a bound for $\frac{|\nabla u|}{u}$, where u is a positive harmonic function, when we have a lower bound for the Ricci curvature. For this, we will need the *Laplacian Comparison Theorem* which allows us to "control" the Laplacian of the distance function. Afterwards, we will derive some important corollaries, such as a Liouville property for manifolds with non-negative Ricci curvature and a Harnack type inequality.

2.1 The Laplacian Comparison Theorem

In order to prove this chapter's main result we will need the following:

Theorem 2.1.1. *Suppose $R_{ij} \geq -(n-1)k$, where $k \geq 0$ is a constant. Let $p \in M$ and $\gamma : [0, b] \to M$ a minimal, normal geodesic in M with $\gamma(0) = p$. Let N be a space form of curvature $-k$, $\widetilde{p} \in N$ and $\widetilde{\gamma} : [0, b] \to N$ a minimal, normal geodesic in N, with $\widetilde{\gamma}(0) = \widetilde{p}$. Moreover, let ρ be the distance function from p, in M, and $\widetilde{\rho}$ the distance function from $\widetilde{p}$, in N. Then*

$$\triangle_M \rho\big(\gamma(t)\big) \leq \triangle_N \widetilde{\rho}\big(\widetilde{\gamma}(t)\big)$$

Proof. We fix a $t \in [0, b]$. Let $e_1, \cdots, e_n$ be an orthonormal, parallel frame along γ, with $e_1 = \gamma'$. For $i \geq 2$, let $X_i(s)$ be a normal Jacobi field along γ such that $X_i(0) = 0$ and $X_i(t) = e_i(t)$. Then by Lemma 1.3, $H_\rho\big(e_i(t), e_i(t)\big) = I_t(X_i, X_i)$. Also, observe that, since γ is a geodesic, $H_\rho(e_1, e_1) = H_\rho(\gamma', \gamma') = 0$.
Therefore,

$$\triangle_M \rho\big(\gamma(t)\big) = \sum_{i=2}^{n} H_\rho\big(e_i(t), e_i(t)\big) = \sum_{i=2}^{n} I_t(X_i, X_i)$$

Analogously, we define, in N, $\widetilde{e_1}, \cdots, \widetilde{e_n}$ and $\widetilde{X_i}(s)$, $i = 2, \cdots, n$. Since N has constant curvature $-k$, $\widetilde{X_i}(s) = \frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)} \widetilde{e_i}(s)$ [see [6]].

Now, we define along γ, for $i \geq 2$,the vector fields:

$$X_i'(s) = \frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)} e_i(s)$$

Each X_i' has the same boundary values as X_i and the X_is are Jacobi fields, hence by the *Index Lemma* [see [6]]:

$$I_t(X_i, X_i) \leq I_t(X_i', X_i')$$

Therefore, it suffices to show that $\sum_{i\geq 2} I_t(X_i', X_i') \leq \sum_{i \geq 2} I_t(\widetilde{X_i}, \widetilde{X_i}) = \triangle_N \widetilde{\rho}\big(\widetilde{\gamma}(t)\big)$. Indeed, we have

$$\begin{aligned}
\sum I_t(X_i', X_i') &= \sum \int_0^t \Big(|\frac{D}{ds} X_i'|^2 - \big\langle R(X_i', \gamma')(\gamma'), X_i' \big\rangle \Big) ds \\
&= \sum \int_0^t \Big(|\frac{D}{ds} X_i'|^2 - [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 \big\langle R(e_i, \gamma')(\gamma'), e_i \big\rangle \Big) ds \\
&= \sum \int_0^t \Big([\frac{\frac{d}{ds}(sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 - [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 K(\gamma', e_i) \Big) ds \\
&= \int_0^t \Big([n-1] [\frac{\frac{d}{ds}(sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 - [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 Ric(\gamma', \gamma') \Big) ds \\
&\leq \int_0^t \Big([n-1] [\frac{\frac{d}{ds}(sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 + [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 (n-1)k \Big) ds \\
&= \sum \int_0^t \Big(|\frac{\widetilde{D}}{ds} \widetilde{X}_i|^2 - [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 \big\langle \widetilde{R}(\widetilde{X}_i, \widetilde{\gamma}')(\widetilde{\gamma}'), \widetilde{X}_i \big\rangle \Big) ds \\
&= \sum I_t(\widetilde{X}_i, \widetilde{X}_i)
\end{aligned}$$

□

In the same setting as in the above proof, we want to calculate $\triangle_N \widetilde{\rho}$. So, we have:

$$\begin{aligned}
\triangle_N \widetilde{\rho} = \sum I_t(\widetilde{X}_i, \widetilde{X}_i) &= \int_0^t \Big([n-1] [\frac{\frac{d}{ds}(sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 + [\frac{sinh(\sqrt{k}s)}{sinh(\sqrt{k}t)}]^2 (n-1)k \Big) ds \\
&= (n-1) \int_0^t \Big(\frac{k \cdot cosh^2(\sqrt{k}s)}{sinh^2(\sqrt{k}t)} + k \cdot \frac{sinh^2(\sqrt{k}s)}{sinh^2(\sqrt{k}t)} \Big) ds \\
&= (n-1)k \int_0^t \frac{cosh^2(\sqrt{k}s) + sinh^2(\sqrt{k}s)}{sinh^2(\sqrt{k}t)} \\
&= \frac{(n-1)k}{sinh^2(\sqrt{k}s)} \int_0^t cosh(2\sqrt{k}s) ds = \frac{(n-1)k}{sinh^2(\sqrt{k}s)} \int_0^t \frac{d}{ds} (\frac{sinh(2\sqrt{k}s)}{2\sqrt{k}}) ds \\
&= \frac{(n-1)\sqrt{k}}{2} \cdot \frac{sinh(2\sqrt{k}t)}{sinh^2(\sqrt{k}t)} = \frac{(n-1)\sqrt{k}}{2} \cdot \frac{2 sinh(\sqrt{k}t) cosh(\sqrt{k}t)}{sinh^2(\sqrt{k}t)} \\
&= (n-1)\sqrt{k} \quad coth(\sqrt{k}t)
\end{aligned}$$

It is easy to check that $\sqrt{k}coth(\sqrt{k}t) \leq \dfrac{1+\sqrt{k}t}{t}$. But t is the distance from p of $\gamma(t)$, i.e. $t = \rho\big(\gamma(t)\big)$. Therefore, we have the following:

Corollary 2.1.1.1. *If $R_{ij} \geq -(n-1)k$, then, at any point where ρ is smooth,*

$$\triangle\rho \leq \frac{n-1}{\rho}(1+\sqrt{k}\rho)$$

2.2 The Gradient Estimate

Theorem 2.2.1. *Suppose $R_{ij} \geq -(n-1)k$, where $k \geq 0$ is a constant. Let u be a positive, harmonic function on M, and $B_\alpha(x)$ be a geodesic ball at $x \in M$. Then on $B_{\frac{\alpha}{2}}(x)$ we have:*

$$\frac{|\nabla u|}{u} \leq C_n\left(\frac{1+\alpha\sqrt{k}}{\alpha}\right)$$

where C_n depends only on n.

Proof. By Lemma 1.4.1, we have $\frac{1}{2}\triangle\big(|\nabla u|^2\big) = \sum\limits_{i,j} u_{ij}^2 + \sum\limits_i u_i(\triangle u)_i + Ric(\nabla u, \nabla u) = \sum\limits_{i,j} u_{ij}^2 + Ric(\nabla u, \nabla u)$. ($u$ harmonic)
Now, from the hypothesis on the Ricci curvature:

$$\frac{1}{2}\triangle\big(|\nabla u|^2\big) \geq \sum_{i,j} u_{ij}^2 - (n-1)k|\nabla u|^2 \tag{2.1}$$

Let $p \in B_\alpha(x)$ such that $\nabla u(p) \neq 0$, otherwise the inequality of the theorem holds trivially. Then we may choose normal coordinates at p, $x = (x_1, \cdots, x_n)$ such that $\frac{\partial}{\partial x_1} = \frac{1}{|\nabla u|}\nabla u$ and $\big\langle \frac{\partial}{\partial x_i}, \nabla u\big\rangle = 0$ at p, for $i = 2, \cdots, n$. Then at p, $u_1 = \frac{\partial}{\partial x_1}(u) = \frac{1}{|\nabla u|}\nabla u(u) = du\big(\frac{1}{|\nabla u|}\nabla u\big) = \big\langle \frac{1}{|\nabla u|}\nabla u, \nabla u\big\rangle = |\nabla u|$ and $u_i = 0$ for $i \neq 1$. Indeed, $u_i = \frac{\partial}{\partial x_i}(u) = \big\langle \frac{\partial}{\partial x_i}, \nabla u\big\rangle = 0$.
Now, at p,

$$\frac{\partial}{\partial x_j}\big(|\nabla u|\big) = \frac{\partial}{\partial x_j}\Big(\sqrt{\sum u_i^2}\Big) = \frac{\sum u_i u_{ij}}{|\nabla u|} = u_{1j}$$

Hence,

$$|\nabla\big(|\nabla u|\big)|^2 = \sum_j u_{1j}^2$$

Recalling the formula $\triangle(f\cdot g) = g\cdot\triangle(f) + f\cdot\triangle(g) + 2\langle\nabla f, \nabla g\rangle$, we have:

$$\begin{aligned}\triangle\big(|\nabla u|^2\big) &= \triangle\big(|\nabla u|\cdot|\nabla u|\big)\\ &= 2|\nabla u|\cdot\triangle\big(|\nabla u|\big) + 2\big\langle\nabla\big(|\nabla u|\big), \nabla\big(|\nabla u|\big)\big\rangle\\ &= 2|\nabla u|\cdot\triangle\big(|\nabla u|\big) + 2|\nabla\big(|\nabla u|\big)|^2\\ &= 2|\nabla u|\cdot\triangle\big(|\nabla u|\big) + 2\sum_j u_{1j}^2\end{aligned}$$

Combining with (2.1) we get:

$$
\begin{aligned}
|\nabla u| \cdot \triangle\big(|\nabla u|\big) + \sum_j u_{1j}^2 &\geq \sum_{i,j} u_{ij}^2 - (n-1)k|\nabla u|^2 \\
\Rightarrow |\nabla u| \cdot \triangle\big(|\nabla u|\big) + (n-1)k|\nabla u|^2 &\geq \sum_{i,j} u_{ij}^2 - \sum_j u_{1j}^2 \\
&= \sum_{i\neq 1,j} u_{ij}^2 \\
&\geq \sum_{i\neq 1} u_{i1}^2 + \sum_{i\neq 1} u_{ii}^2
\end{aligned} \tag{2.2}
$$

Now, in general, we have $\sum_{\nu=1}^N \alpha_\nu^2 \geq \frac{1}{N}\Big(\sum_{\nu=1}^N \alpha_\nu\Big)^2$. Indeed, $\Big(\sum_{\nu=1}^N \alpha_\nu\Big)^2 = \sum_\nu \alpha_\nu^2 + 2\sum_{\kappa<\lambda}\big(\alpha_\kappa\alpha_\lambda\big) \leq \sum_\nu \alpha_\nu^2 + \sum_{\kappa<\lambda}\big(\alpha_\kappa^2 + \alpha_\lambda^2\big) = N\cdot\sum_{\nu=1}^N \alpha_\nu^2$ Moreover, since u is harmonic, $\sum_i u_{ii} = 0 \Rightarrow u_{11}^2 = \Big(\sum_{i\neq 1} u_{ii}\Big)^2$ Hence, we get

$$
\sum_{i\neq 1} u_{i1}^2 + \sum_{i\neq 1} u_{ii}^2 \geq \frac{1}{n-1}\sum_{i\neq 1} u_{i1}^2 + \frac{1}{n-1}\Big(\sum_{i\neq 1} u_{ii}\Big)^2 = \frac{1}{n-1}\sum_{i\neq 1} u_{i1}^2 + \frac{1}{n-1}u_{11}^2 = \frac{1}{n-1}\sum_i u_{i1}^2
$$

But, by a previous calculation:

$$
|\nabla\big(|\nabla u|\big)|^2 = \sum_i u_{1i}^2 = \sum_i u_{i1}^2
$$

Combing this with (2.2), we have

$$
|\nabla u| \cdot \triangle\big(|\nabla u|\big) + (n-1)k|\nabla u|^2 \geq \frac{1}{n-1}|\nabla\big(|\nabla u|\big)|^2 \tag{2.3}
$$

Since we can do the same calculations for any $p \in B_\alpha(x)$ such that $\nabla u(p) \neq 0$, (2.3) holds for every such p.

Next, we consider the function $\phi = \frac{|\nabla u|}{u}$ and we will find a lower bound for $\triangle\phi$. Firstly, we have

$$
\nabla\phi = \frac{\nabla\big(|\nabla u|\big)}{u} - \frac{|\nabla u|\cdot\nabla u}{u^2}
$$

Also,

$$
\begin{aligned}
\triangle\big(|\nabla u|\big) &= \triangle\big(\phi\cdot u\big) \\
&= u\cdot\triangle\phi + \phi\cdot\triangle u + 2\big\langle\nabla\phi, \nabla u\big\rangle \\
&= u\cdot\triangle\phi + 2\big\langle\nabla\phi, \nabla u\big\rangle
\end{aligned}
$$

$$
\Rightarrow \triangle\phi = \frac{\triangle\big(|\nabla u|\big)}{u} - 2\frac{\big\langle\nabla\phi, \nabla u\big\rangle}{u} = \frac{|\nabla u|\cdot\triangle\big(|\nabla u|\big)}{|\nabla u|\cdot u} - 2\frac{\big\langle\nabla\phi, \nabla u\big\rangle}{u}
$$

So, for any p such that $\nabla u(p) \neq 0$, using (2.3), we get:

$$
\begin{aligned}
\triangle\phi &\geq \frac{1}{|\nabla u|\cdot u}\left(\frac{1}{n-1}|\nabla\big(|\nabla u|\big)|^2 - (n-1)k|\nabla u|^2\right) - 2\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} \\
&= \frac{1}{(n-1)u\cdot|\nabla u|}|\nabla\big(|\nabla u|\big)|^2 - (n-1)\frac{|\nabla u|}{u}k - 2\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} \\
&= \frac{1}{(n-1)u\cdot|\nabla u|}|\nabla\big(|\nabla u|\big)|^2 - (n-1)\phi\cdot k - 2\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u}
\end{aligned}
\tag{2.4}
$$

However, using our calculation of $\nabla\phi$ above, we have:

$$
\begin{aligned}
2\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} &= \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{2}{n-1}\cdot\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} \\
&= \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{2}{n-1}\cdot\frac{\big\langle\frac{\nabla(|\nabla u|)}{u},\nabla u\big\rangle}{u} + \frac{2}{n-1}\cdot\frac{\big\langle-\frac{|\nabla u|}{u^2}\cdot\nabla u,\nabla u\big\rangle}{u} \\
\text{(C-S)} &\leq \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{2}{n-1}\cdot\frac{|\nabla\big(|\nabla u|\big)|\cdot|\nabla u|}{u^2} - \frac{2}{n-1}\cdot\frac{|\nabla u|^3}{u^3} \\
&= \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{2}{n-1}\cdot\frac{|\nabla\big(|\nabla u|\big)|\cdot|\nabla u|}{u^2} - \frac{2}{n-1}\cdot\phi^3
\end{aligned}
$$

In addition,

$$
\frac{2}{n-1}\cdot\frac{|\nabla\big(|\nabla u|\big)|\cdot|\nabla u|}{u^2} = \frac{2}{n-1}\cdot\frac{|\nabla\big(|\nabla u|\big)|}{|\nabla u|^{1/2}u^{1/2}}\cdot\frac{|\nabla u|^{3/2}}{u^{3/2}} \leq \frac{1}{n-1}\left(\frac{|\nabla\big(|\nabla u|\big)|^2}{|\nabla u|u} + \phi^3\right)
$$

Taking these and (2.4) into account, we derive:

$$
\begin{aligned}
\triangle\phi &\geq \frac{1}{n-1}\cdot\frac{1}{u|\nabla u|}\cdot|\nabla\big(|\nabla u|\big)|^2 - (n-1)\phi\cdot k - 2\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} \\
&\geq \frac{1}{n-1}\cdot\frac{1}{u|\nabla u|}\cdot|\nabla\big(|\nabla u|\big)|^2 - (n-1)\phi\cdot k - \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{2}{n-1}\phi^3 - \\
&\quad - \frac{1}{n-1}\cdot\frac{|\nabla\big(|\nabla u|\big)|^2}{|\nabla u|u} - \frac{1}{n-1}\phi^3 \\
&= -(n-1)\phi\cdot k - \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{u} + \frac{1}{n-1}\phi^3
\end{aligned}
\tag{2.5}
$$

To get our estimate, we consider the function $F = (\alpha^2-\rho^2)\phi = (\alpha^2-\rho^2)\frac{|\nabla u|}{u}$, on $B_\alpha(x)$
Then, $F \geq 0$ and $F = 0$ on $\partial B_\alpha(x)$. Hence, if $\nabla u \not\equiv 0$ on $B_\alpha(x)$, F attains its maximum, at some $x_0 \in B_\alpha(x)$. We consider two cases:

Case 1: x_0 is not a cut-point of x

In that case ρ is smooth near $x_0 \Rightarrow F$ is smooth near x_0, and therefore:

$$\nabla F(x_0) = 0$$

and

$$\triangle F(x_0) \le 0$$

The former gives:

$$-\nabla\rho^2 \cdot \phi + (\alpha^2 - \rho^2)\nabla\phi = 0 \Rightarrow \frac{\nabla\rho^2}{\alpha^2-\rho^2} = \frac{\nabla\phi}{\phi} \tag{2.6}$$

The Laplacian of F is,

$$\begin{aligned}\triangle F &= -\triangle\rho^2 \cdot \phi + (\alpha^2-\rho^2)\triangle\phi + 2\big\langle \nabla(\alpha^2-\rho^2), \nabla\phi\big\rangle \\ &= -\triangle\rho^2 \cdot \phi + (\alpha^2-\rho^2)\triangle\phi - 2\big\langle \nabla\rho^2, \nabla\phi\big\rangle\end{aligned}$$

Hence, from $\triangle F(x_0) \le 0$, we have:

$$\begin{aligned}&-\triangle\rho^2 \cdot \phi + (\alpha^2-\rho^2)\triangle\phi - 2\big\langle \nabla\rho^2, \nabla\phi\big\rangle \le 0 \\ &\Rightarrow -\frac{\triangle\rho^2}{\alpha^2-\rho^2} + \frac{\triangle\phi}{\phi} - 2\frac{\big\langle \nabla\rho^2, \nabla\phi\big\rangle}{(\alpha^2-\rho^2)\phi} \le 0 \\ &\Rightarrow -\frac{\triangle\rho^2}{\alpha^2-\rho^2} + \frac{\triangle\phi}{\phi} - 2\frac{|\nabla\rho^2|^2}{(\alpha^2-\rho^2)^2} \le 0 \text{ (using (2.6))}\end{aligned}$$

But, we have that $|\nabla\rho^2| = 2\rho|\nabla\rho| = 2\rho$ and $\triangle\rho^2 = 2\rho\triangle\rho + 2|\nabla\rho|^2 = 2\rho\triangle\rho + 2$. Now, using Corollary 2.1.1.1:

$$\begin{aligned}&\triangle\rho^2 = 2\rho\triangle\rho + 2 \le 2(n-1)(1+\sqrt{k}\rho) + 2 \\ &\le 2(n-1)(1+\sqrt{k}\rho) + 2(1+\sqrt{k}\rho) \le C(1+\sqrt{k}\rho)\end{aligned}$$

where C depends only on n. So we have that

$$\begin{aligned}0 &\ge -\frac{\triangle\rho^2}{\alpha^2-\rho^2} + \frac{\triangle\phi}{\phi} - 2\frac{|\nabla\rho^2|^2}{(\alpha^2-\rho^2)^2} \\ &\ge -\frac{C(1+\sqrt{k}\rho)}{\alpha^2-\rho^2} + \frac{\triangle\phi}{\phi} - 2\frac{4\rho^2}{(\alpha^2-\rho^2)^2} \\ \text{(using (2.5))} &\ge -\frac{C(1+\sqrt{k}\rho)}{\alpha^2-\rho^2} - (n-1)k - \big(2-\frac{2}{n-1}\big)\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{\phi\cdot u} + \frac{1}{n-1}\phi^2 - \frac{8\rho^2}{(\alpha^2-\rho^2)^2}\end{aligned} \tag{2.7}$$

Now, since $\dfrac{\nabla\phi}{\phi} = \dfrac{\nabla\rho^2}{\alpha^2-\rho^2} = \dfrac{2\rho\nabla\rho}{\alpha^2-\rho^2}$, we get

$$\begin{aligned}\frac{\big\langle\nabla\phi,\nabla u\big\rangle}{\phi\cdot u} &= \frac{2\rho\big\langle\nabla\rho,\nabla u\big\rangle}{(\alpha^2-\rho^2)u} \\ \text{(C-S)} &\le \frac{2\rho}{\alpha^2-\rho^2}\frac{|\nabla u|}{u} = \frac{2\rho}{\alpha^2-\rho^2}\phi\end{aligned}$$

So, in (2.7), we multiply by $(\alpha^2-\rho^2)^2$, and we get:

$$\begin{aligned} 0 &\geq \frac{1}{n-1}F^2 - \big(2-\frac{2}{n-1}\big)2\rho F - C(1+\sqrt{k}\rho)(\alpha^2-\rho^2) - (n-1)k(\alpha^2-\rho^2)^2 - 8\rho^2 \\ &\geq \frac{1}{n-1}F^2 - 2C_1\rho F - C(1+\sqrt{k}\rho)\alpha^2 - 8\alpha^2 - (n-1)k\alpha^4 \\ &\geq \frac{1}{n-1}F^2 - 2C_1\alpha F - C_2(1+\sqrt{k}\alpha)^2\alpha^2 \end{aligned} \tag{2.8}$$

The last line can be justified as follows:

$$8\alpha^2 \leq 8(1+\sqrt{k}\alpha)\alpha^2 \leq 8(1+\sqrt{k}\alpha)^2\alpha^2$$

and

$$(n-1)k\alpha^4 = (n-1)(\sqrt{k}\alpha)^2\alpha^2 \leq (n-1)(1+\sqrt{k}\alpha)^2\alpha^2$$

Also, the constants C_1, C_2 depend only on n.

If we consider $P(s) = \frac{1}{n-1}s^2 - 2C_1\alpha s - C_2(1+\sqrt{k}\alpha)^2\alpha^2$, (2.8) says that $P(F) \leq 0$, therefore F must lie between the roots of P. In particular,

$$\begin{aligned} &F \leq \frac{2C_1\alpha + \sqrt{4C_1^2\alpha^2 + 4\frac{C_2}{n-1}(1+\sqrt{k}\alpha)^2\alpha^2}}{2\frac{1}{n-1}} \\ \Rightarrow &F \leq C_n'(1+\sqrt{k}\alpha)\alpha \end{aligned}$$

Moreover, on $B_{\alpha/2}(x)$, $\rho \leq \frac{\alpha}{2} \Rightarrow \alpha^2 - \rho^2 \geq \frac{3}{4}\alpha^2$. As a result, on $B_{\alpha/2}(x)$, we have:

$$\begin{aligned} (\alpha^2-\rho^2)\frac{|\nabla u|}{u} &\leq C_n'(1+\sqrt{k}\alpha)\alpha \\ \Rightarrow \frac{|\nabla u|}{u} &\leq \frac{4}{3}\frac{1}{\alpha^2}C_n'(1+\sqrt{k}\alpha)\alpha \\ \Rightarrow \frac{|\nabla u|}{u} &\leq C_n\Big(\frac{1+\sqrt{k}\alpha}{\alpha}\Big) \end{aligned}$$

where C_n depends only on n.

Case 2: x_0 is a cut-point of x

We consider a minimizing, normal geodesic $\gamma : [0,r] \to M$ such that $\gamma(0) = x$ and $\gamma(r) = x_0$. For $\epsilon > 0$ small, we consider $y = \gamma(\epsilon)$. We note the following fact:

Since γ is minimizing, there cannot be a conjugate point to y somewhere in γ, for in that case, there would have to be a cut-point of y somewhere in between, and γ would stop to minimize the distance after that, a contradiction.

As a result we can find a neighborhood of γ such that it contains no conjugate to y points. Next, we consider the distance function from y, $\bar{\rho}$. From the triangle inequality we have

$$\begin{aligned} &d(z,x) \leq d(z,y) + d(y,x) = d(z,x) + \epsilon \\ \Rightarrow &\rho \leq \bar{\rho} + \epsilon \end{aligned} \tag{2.9}$$

Additionally, since γ is minimizing: $\rho(x_0) = \bar{\rho}(x_0) + \epsilon$. Finally, we consider the function $\bar{F} = \big(\alpha^2 - (\bar{\rho}+\epsilon)^2\big)\frac{|\nabla u|}{u}$. By (9):

$$\bar{F} \leq F \qquad \text{and} \qquad \bar{F}(x_0) = F(x_0)$$

Therefore, $\bar{F}$ attains its maximum at x_0. Since $\bar{\rho}$ is smooth near x_0, we can do the same work as before and get the same estimate for $\bar{F}(x_0) = F(x_0)$. Passing to the limit as $\epsilon \to 0$ completes the proof of the theorem □

Corollary 2.2.1.1. *On a non-compact, complete Riemannian manifold M, such that $R_{ij} \geq 0$, there do not exist positive harmonic functions, other than the constants.*

(For a compact manifold, that is true, even without any curvature assumption)

Proof. In Theorem 2.2.1 we can take $k = 0$. Then, if u is a harmonic function we have:

$$\frac{|\nabla u|}{u} \leq C\frac{1}{\alpha}$$

Letting $\alpha \to +\infty$, gives $\nabla u \equiv 0$ which implies that u is constant. □

Corollary 2.2.1.2. (Harnack inequality) *Suppose $R_{ij} \geq -(n-1)k$. Let u be a harmonic function on the geodesic ball B_α. Then, there exists a constant $C(n,\alpha,k)$ such that*

$$\sup_{B_{\alpha/2}} u \leq C(n,\alpha,k) \cdot \inf_{B_{\alpha/2}} u$$

Proof. By Theorem 2.2.1, $\frac{|\nabla u|}{u} \leq C(n,\alpha,k)$. Let $x_1, x_2 \in B_{\alpha/2}$ such that $u(x_1) = \sup\limits_{B_{\alpha/2}} u$ and $u(x_2) = \inf\limits_{B_{\alpha/2}} u$.

Let γ be a minimizing, normal geodesic that connects x_1 and x_2. Then, $l(\gamma) = d(x_1,x_2) \leq \alpha$. So, we have

$$\int_\gamma \frac{|\nabla u|}{u} ds \leq C(n,\alpha,k)\int_\gamma ds \leq C(n,\alpha,k)\cdot\alpha$$

Next, we note that $\frac{d}{ds}u\big(\gamma(s)\big) = \gamma'(s)\big(u\big) = \big\langle \nabla u, \gamma' \big\rangle \Rightarrow |\frac{d}{ds}u\big(\gamma(s)\big)| \leq |\nabla u|$, by the Cauchy-Schwarz inequality and since $|\gamma'| = 1$. Finally, we compute

$$log\Big(\frac{u(x_1)}{u(x_2)}\Big) = \left|\int_\gamma \frac{d}{ds} log\Big[u\big(\gamma(s)\big)\Big] ds\right| \leq \int_\gamma \frac{1}{u}|\nabla u| ds \leq C(n,\alpha,k)\cdot\alpha$$

As a result

$$u(x_1) \leq e^{C(n,\alpha,k)\cdot\alpha}\cdot u(x_2)$$

which proves the Corollary. □

Corollary 2.2.1.3. (Yau-Cheng) *Suppose* $R_{ij} \geq 0$. *Let* u *be a harmonic function on* M *and let* $B_\alpha(p)$ *be a geodesic ball around a point* $p \in M$. *We set* $i(\alpha) = \inf_{B_\alpha} u(x)$. *Then, there exists a constant* C, *depending only on* n, *such that*

$$|\nabla u|(x) \leq C \frac{u(x) - i(\alpha)}{\alpha}$$

for every $x \in B_{\alpha/2}$.
In particular there does not exist a non-constant harmonic function, satisfying the growth estimate

$$\liminf_{x \to \infty} \frac{u(x)}{\rho(x)} \geq 0$$

where ρ *is the distance function from* p.

Proof. We observe that $u - i(\alpha)$ is a positive harmonic function. We apply theorem 2.2.1 for $u - i(\alpha)$, with $k = 0$. Then, we have

$$\frac{|\nabla u|}{u - i(\alpha)} \leq C \frac{1}{\alpha}$$
$$\Rightarrow |\nabla u| \leq C \frac{1}{\alpha} \big(u - i(\alpha)\big)$$

for every $x \in B_{\alpha/2}$.
Now, by the maximum principle, $i(\alpha) = u(y)$, where $y \in \partial B_\alpha(p)$. Hence $\frac{i(\alpha)}{\alpha} = \frac{u(y)}{\rho(y)}$. As a result, if we assume the growth estimate on u,

$$\lim_{\alpha \to \infty} \frac{i(\alpha)}{\alpha} \geq 0$$

Fix an $x \in M$. By our estimate, $|\nabla u|(x) \leq C \frac{u(x) - i(\alpha)}{\alpha}$. Hence, if we let $\alpha \to \infty$, we get $|\nabla u| \leq 0 \Rightarrow \nabla u = 0$. x was arbitrary, so $\nabla u \equiv 0$ and u is constant. $\square$

Chapter 3

Mean Value Inequality

In this chapter we prove a mean value inequality, which will be used later in the proof of the theorem of Colding and Minicozzi. We begin with two lemmas.

Lemma 3.0.1. (Yau) *Let $p \in M$, $r_4 > 0$ and $B_{r_4}(p)$ be a geodesic ball around p, such that $B_{r_4}(p) \cap \partial M = \emptyset$. Suppose f is a non-negative, subharmonic function on $B_{r_4}(p)$. Then, for every $\alpha > 1$ and $0 \leq r_1 < r_2 < r_3 < r_4$, if we denote $B_i = B_{r_i}(p)$, we have the following estimates:*

$$\int_{B_3 \setminus B_2} f^{\alpha-2} |\nabla f|^2 \leq \frac{4}{(\alpha-1)^2} \left[\frac{1}{(r_2 - r_1)^2} \int_{B_2 \setminus B_1} f^\alpha + \frac{1}{(r_4 - r_3)^2} \int_{B_4 \setminus B_3} f^\alpha \right]$$

and

$$\int_{B_3} f^{\alpha-2} |\nabla f|^2 \leq \frac{4}{(\alpha-1)^2} \frac{1}{(r_4 - r_3)^2} \int_{B_4 \setminus B_3} f^\alpha$$

Proof. We define the function

$$\phi(\rho(x)) = \begin{cases} 0 & \text{for } \rho \leq r_1 \\ \frac{\rho - r_1}{r_2 - r_1} & \text{for } r_1 \leq \rho \leq r_2 \\ 1 & \text{for } r_2 \leq \rho \leq r_3 \\ \frac{r_4 - \rho}{r_4 - r_3} & \text{for } r_3 \leq \rho \leq r_4 \\ 0 & \text{for } r_4 \leq \rho \end{cases}$$

Since $\phi \equiv 0$, on ∂B_4, we have that

$$\int_{B_4} \phi^2 f^{\alpha-1} \triangle f + \int_{B_4} \langle \nabla(\phi^2 f^{\alpha-1}), \nabla f \rangle = 0$$

Therefore

$$\int_{B_4} \phi^2 f^{\alpha-1} \triangle f = -\int_{B_4} \langle \nabla(\phi^2 f^{\alpha-1}), \nabla f \rangle = -\int_{B_4} (\alpha-1) \phi^2 f^{\alpha-2} |\nabla f|^2 - \int_{B_4} 2\phi f^{\alpha-1} \langle \nabla \phi, \nabla f \rangle$$

Observe that $\int_{B_4} \phi^2 f^{\alpha-1} \triangle f \geq 0$, hence

$$\int_{B_4} (\alpha-1)\phi^2 f^{\alpha-2}|\nabla f|^2 \leq -\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle \tag{3.1}$$

But the left-hand side is non-negative, so the previous inequality implies that

$$-\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle = \left|\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle\right|$$

.

Now, we compute

$$\begin{aligned}\left|\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla f, \nabla\phi\rangle\right| &\leq 2\int_{B_4} \left[\phi f^{\alpha/2-1}|\nabla f|\right]\cdot\left[f^{\alpha/2}|\nabla\phi|\right] \\ &= 2\int_{B_4} \left[\frac{\sqrt{\alpha-1}}{\sqrt{2}}\phi f^{\alpha/2-1}|\nabla f|\right]\cdot\left[\frac{\sqrt{2}}{\sqrt{\alpha-1}} f^{\alpha/2}|\nabla\phi|\right] \\ &\leq \frac{\alpha-1}{2}\int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 + \frac{2}{\alpha-1}\int_{B_4} f^\alpha|\nabla\phi|^2\end{aligned}$$

Combining this with (3.1) we get

$$\begin{aligned}(\alpha-1)\int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 &\leq -\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle = \left|\int_{B_4} 2\phi f^{\alpha-1}\langle\nabla\phi, \nabla f\rangle\right| \\ \Rightarrow (\alpha-1)\int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 &\leq \frac{\alpha-1}{2}\int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 + \frac{2}{\alpha-1}\int_{B_4} f^\alpha|\nabla\phi|^2 \\ \Rightarrow \frac{\alpha-1}{2}\int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 &\leq \frac{2}{\alpha-1}\int_{B_4} f^\alpha|\nabla\phi|^2 \\ \Rightarrow \int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2 &\leq \frac{4}{(\alpha-1)^2}\int_{B_4} f^\alpha|\nabla\phi|^2\end{aligned}$$

But $\int_{B_3\setminus B_2} f^{\alpha-2}|\nabla f|^2 \leq \int_{B_4} \phi^2 f^{\alpha-2}|\nabla f|^2$ and $\int_{B_4} f^\alpha|\nabla\phi|^2 = \int_{B_2\setminus B_1} \frac{1}{(r_2-r_1)^2} f^\alpha + \int_{B_4\setminus B_3} \frac{1}{(r_4-r_3)^2} f^\alpha$, and the first estimate is proved. The second estimate is proved in an analogous manner by considering

$$\phi = \begin{cases} 1 & \text{for } \rho \leq r_3 \\ \frac{r_4-\rho}{r_4-r_3} & \text{for } r_3 \leq \rho \leq r_4 \\ 0 & \text{for } r_4 \leq \rho \end{cases}$$

□

Lemma 3.0.2. (Poincaré Inequality) *Suppose $p \in M$ and $B_{2r}(p)$ is a geodesic ball around p such that $B_{2r}(p) \cap \partial M = \emptyset$. Moreover, $R_{ij} \geq -(n-1)k$, $k \geq 0$, and $\alpha \geq 1$. Then there exist constants $C_1(\alpha), C_2(\alpha, n)$ such that for any function g, supported on $B_r(p)$:*

$$\int_{B_r(p)} |\nabla g|^\alpha \geq C_1 \cdot \frac{1}{r^\alpha} \cdot e^{-C_2(1+r\sqrt{k})} \int_{B_r(p)} |g|^\alpha$$

Proof. Let $q \in \partial B_{2r}(p)$. If $x \in B_r(p)$, then $d(x,p) \leq d(x,p) + d(p,q) < 3r$, while $d(p,q) \leq d(x,p) + d(x,q) \Rightarrow d(x,q) \geq 2r - d(p,x) > r$. In other words, $B_r(p) \subseteq B_{3r}(q) \backslash B_r(q)$. Let ρ denote the distance function from q, $\rho(x) = d(x,q)$. Then, by lemma 2.1.1.1, $\triangle \rho \leq (n-1)(\sqrt{k} + \frac{1}{\rho})$. Now, for $\lambda > n-2$ (and recalling that $\sum \rho_i^2 = |\nabla \rho|^2 = 1$), we have

$$\begin{aligned}
\triangle \rho^{-\lambda} &= -\lambda \rho^{-\lambda-1} \triangle \rho + \lambda(\lambda+1)\rho^{-\lambda-2} \\
&\geq -\lambda \rho^{-\lambda-1}(n-1)(\sqrt{k} + \frac{1}{\rho}) + \lambda(\lambda+1)\rho^{-\lambda-2} \\
&= \lambda \rho^{-\lambda-1}\big[-\sqrt{k}(n-1) - (n-1)\rho^{-1} + (\lambda+1)\rho^{-1}\big] \\
&= \lambda \rho^{-\lambda-1}\big[(\lambda+2-n)\rho^{-1} - (n-1)\sqrt{k}\big] \\
&\geq \lambda \rho^{-\lambda-1}\big[(\lambda+2-n)\frac{1}{3r} - (n-1)\sqrt{k}\big]
\end{aligned}$$

on $B_r(p)$.

We take $\lambda = n - 1 + 3(n-1)r\sqrt{k}$. Then

$$\triangle \rho^{-\lambda} \geq \lambda \rho^{-\lambda-1}\frac{1}{3r} \geq \lambda(3r)^{-\lambda-2}$$

on $B_r(p)$.

Suppose f is a non-negative function, supported on $B_r(p)$. Multiplying the above inequality with f and integrating over $B_r(p)$, we get:

$$\lambda(3r)^{-\lambda-2}\int_{B_r(p)} f \leq \int_{B_r(p)} f \triangle \rho^{-\lambda} \tag{3.2}$$

Since f is supported on $B_r(p)$, it is zero on $\partial B_r(p)$. Therefore

$$\int_{B_r(p)} f \triangle \rho^{-\lambda} = -\int_{B_r(p)} \big\langle \nabla f, \nabla \rho^{-\lambda} \big\rangle$$

But the left hand side of (3.2) is non-negative, hence

$$\begin{aligned}
&-\int_{B_r(p)} \big\langle \nabla f, \nabla \rho^{-\lambda} \big\rangle = \Big| \int_{B_r(p)} \big\langle \nabla f, \nabla \rho^{-\lambda} \big\rangle \Big| \\
&= \Big| \int_{B_r(p)} \lambda \rho^{-\lambda-1} \big\langle \nabla f, \nabla \rho \big\rangle \Big| \leq \lambda \int_{B_r(p)} \rho^{-\lambda-1} |\nabla f| \\
&\leq \lambda r^{-\lambda-1} \int_{B_r(p)} |\nabla f|
\end{aligned}$$

Combining with (3.2) we get

$$\int_{B_r(p)} |\nabla f| \geq \frac{1}{3^{\lambda+2}} \cdot \frac{1}{r} \int_{B_r(p)} f \tag{3.3}$$

Now, $\frac{1}{3^{\lambda+2}} = \frac{1}{3^{n-1+3(n-1)r\sqrt{k}+2}} = exp\big[-log3(n-1+3(n-1)r\sqrt{k}+2)\big] \geq$
$exp\big[-log3(3(n-1)+3(n-1)r\sqrt{k}+2)\big] = exp\big[-2log3\big] \cdot exp\big[-3log3(n-1)(1+r\sqrt{k})\big]$. So, we

can rewrite (3.3) as

$$\int_{B_r(p)} |\nabla f| \geq C_1 \cdot e^{-C_2(1+r\sqrt{k})} \cdot \frac{1}{r} \int_{B_r(p)} f \tag{3.4}$$

Finally, for $\alpha = 1$, we apply (3.4) for $f = |g|$. For $\alpha > 1$, by Hölder's inequality

$$\begin{aligned}
&\alpha\Big(\int_{B_r(p)} |\nabla g|^\alpha\Big)^{\frac{1}{\alpha}} \cdot \Big(\int_{B_r(p)} |g|^\alpha\Big)^{\frac{\alpha-1}{\alpha}} \\
\geq &\alpha \int_{B_r(p)} |g|^{\alpha-1}|\nabla g| = \int_{B_r(p)} |\nabla g^\alpha| \\
\geq &C_1 \cdot \frac{1}{r} \cdot e^{-C_2(1+r\sqrt{k})} \int_{B_r(p)} |g|^\alpha
\end{aligned}$$

where for the last inequality we have applied (3.4) for $f = |g|^\alpha$. The desired inequality follows. □

Theorem 3.0.3. (Mean Value Inequality) *Let $p \in M$ and $B_{4r}(p)$ be a geodesic ball around p such that $B_{4r}(p) \cap \partial M = \emptyset$. We denote the volume of a geodesic ball of radius r around p, by $V_p(r)$. Suppose $R_{ij} \geq -(n-1)k$, where $k \geq 0$, and suppose that f is a non-negative, subharmonic function on $B_{4r}(p)$. Then, there exist constants C_1, C_2, with C_2 depending only on n, such that*

$$\sup_{B_r(p)} f^2 \leq C_1\big(1 + e^{C_2 r\sqrt{k}}\big)\frac{1}{V_p(4r)} \int_{B_{4r}(p)} f^2$$

Proof. We begin by considering the function h on $B_{2r}(p)$, which solves the problem

$$\triangle h = 0$$

$$h = f \text{ on } \partial B_{2r}(p)$$

Since $f \geq 0$, by the maximum principle $h \geq 0$ on $B_{2r}(p)$. Moreover, $f - h$ is subharmonic on $B_{2r}(p)$ and $f - h \equiv 0$ on $\partial B_{2r}(p)$, so by the maximum principle again $f \leq h$ on $B_r(p)$. By the Harnack inequality

$$\sup_{B_r(p)} h \leq \inf_{B_r(p)} h \cdot e^{C(1+r\sqrt{k})}$$

As a result

$$\sup_{B_r(p)} f^2 \leq \sup_{B_r(p)} h^2 \leq \inf_{B_r(p)} h^2 \cdot e^{2C(1+r\sqrt{k})} \leq e^{2C(1+r\sqrt{k})}\frac{1}{V_p(r)} \int_{B_r(p)} h^2 \tag{3.5}$$

Next, we estimate $\int_{B_r(p)} h^2$:

$$\begin{aligned}
\int_{B_r(p)} h^2 &= \int_{B_r(p)} (h-f+f)^2 = \int_{B_r(p)} (h-f)^2 + \int_{B_r(p)} 2f(h-f) + \int_{B_r(p)} f^2 \\
&\leq 2\int_{B_r(p)} (h-f)^2 + 2\int_{B_r(p)} f^2 \\
&\leq 2\int_{B_{2r}(p)} (h-f)^2 + 2\int_{B_{4r}(p)} f^2
\end{aligned} \tag{3.6}$$

Now, $h - f \equiv 0$ on $\partial B_{2r}(p)$. By the Poincare inequality (lemma 3.0.2)

$$\int_{B_{2r}(p)} (h-f)^2 \leq C_1' r^2 e^{C_2(1+r\sqrt{k})} \int_{B_{2r}(p)} |\nabla(h-f)|^2 \tag{3.7}$$

But $\int_{B_{2r}(p)} |\nabla(h-f)|^2 \leq 2\int_{B_{2r}(p)} |\nabla h|^2 + 2\int_{B_{2r}(p)} |\nabla f|^2$. However, since h is harmonic and $h = f$ on $\partial B_{2r}(p)$, by Dirichlet's principle,

$$\int_{B_{2r}(p)} |\nabla h|^2 \leq \int_{B_{2r}(p)} |\nabla f|^2$$

As a result

$$\int_{B_{2r}(p)} |\nabla(h-f)|^2 \leq 4\int_{B_{2r}(p)} |\nabla f|^2 \tag{3.8}$$

Using lemma 3.0.1 (with $\alpha = 2, r_3 = 2r, r_4 = 4r$) we get

$$\int_{B_{2r}(p)} |\nabla f|^2 \leq \frac{1}{r^2}\int_{B_{4r}(p)} f^2 \tag{3.9}$$

Combining (3.5),(3.6),(3.7),(3.8) and (3.9), we get

$$\begin{aligned}
\sup_{B_r(p)} f^2 &\leq e^{2C(1+\sqrt{k}r)} \frac{1}{V_p(r)} \left[4C_1' e^{C_2(1+r\sqrt{k})} \int_{B_{4r}(p)} f^2 + 2\int_{B_{4r}(p)} f^2 \right] \\
&\leq \widetilde{C_1}\big(1 + e^{C_2(1+r\sqrt{k})}\big) \frac{1}{V_p(r)} \int_{B_{4r}(p)} f^2 \\
&\leq C_1\big(1 + e^{C_2(1+r\sqrt{k})}\big) \frac{1}{V_p(4r)} \int_{B_{4r}(p)} f^2
\end{aligned}$$

where the last inequality is justified by the volume comparison theorem. □

Chapter 4

Polynomial Growth Harmonic Functions

Definition 4.0.1. For $d \geq 1$, we define $H^d(M)$ to be the vector space of harmonic functions of M which are of polynomial growth of order at most d:

$$H^d(M) = \Big\{ f | \quad \triangle f = 0 \qquad \& \qquad f = O(\rho^d) \Big\}$$

where ρ is the distance function from a fixed point $p \in M$.
We denote the dimension of $H^d(M)$ by $h^d(M)$.

Shortly after Colding and Minicozzi proved Yau's conjecture in the form of theorem 4.1.2, Li gave a proof that applies to a larger class of manifolds, i.e. those that satisfy a volume comparison estimate and a mean value inequality (see difinitions below).

Definition 4.0.2. We say that a manifold satisfies a *Volume Comparison Condition* $(\mathcal{V}_\mu)$ for a $\mu > 1$, if for any $0 < r_1 \leq r_2 < +\infty$ and $p \in M$, we have that:

$$V_p(r_2) \leq C_{\mathcal{V}} V_p(r_1) \Big(\frac{r_2}{r_1} \Big)^\mu$$

for some constant $C_{\mathcal{V}} > 0$.

Definition 4.0.3. We say that a manifold satisfies a *Mean Value Inequality* $(\mathcal{M})$ if for any $r > 0$ and $p \in M$ and any non-negative subharmonic function f on M we have that:

$$f^2(p) \leq C_{\mathcal{M}} \frac{1}{V_p(r)} \int_{B_r(p)} f^2(y) dy$$

for some constant $C_{\mathcal{M}} > 0$.

Also, Li's argument can be applied to sections of vector bundles. More specifically, Li proved the following theorem:

Theorem 4.0.1. (Li) *Suppose M satisfies $(\mathcal{V}_\mu)$ and $(\mathcal{M})$. Let E be a rank-m vector bundle over M and $S_d(M,E) \leq \Gamma(E)$ a vector subspace of sections of E, such that for each $u \in S_d(M,E)$ we have that*

- $\triangle |u| \geq 0$
- $|u| = O(\rho^d)$

where ρ is the distance function from a fixed point. Then the dimension of $S_d(M,E)$ is finite.

4.1 The Theorem of Colding-Minicozzi

We will adapt Li's proof, to prove directly the theorem of Colding-Minicozzi. The proof is based on the following lemma:

Lemma 4.1.1. (Li) *Let K be a finite dimensional vector space (of dimension k) of functions of M such that each $u \in K$ has polynomial growth of order at most d, i.e.*

$$|u| \leq C\rho^d$$

Moreover, assume that M has polynomial volume growth at most of order μ, i.e.

$$V_p(r) \leq Cr^\mu$$

for $p \in M$. For $\lambda > 0$ we consider the inner products

$$A_\lambda(u,v) = \int_{B_\lambda(p)} u \cdot v$$

Then, for any $\beta > 1, \delta > 0$ and $r_0 > 0$, there exists $r > r_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $A_{\beta r}$, then

$$\sum_{i=1}^k \int_{B_r(p)} |u_i|^2 \geq k\beta^{-(2d+\mu+\delta)}$$

Proof. Let $r, r' > 0$. We denote the trace and the determinant of A_r with respect to $A_{r'}$ with $tr_{r'}A_r$ and $det_{r'}A_r$, respectively. We assume that the lemma is false. Then, for all $r > r_0$,

$$\sum_{i=1}^k \int_{B_r(p)} |u_i|^2 = tr_{\beta r}A_r < k\beta^{-(2d+\mu+\delta)}$$

Since each A_r is positive definite, by the geometric-arithmetic mean inequality applied on the eigenvalues

$$\left(det_{\beta r} A_r\right)^{\frac{1}{k}} \leq \frac{1}{k} tr_{\beta r} A_r$$

As a result,

$$det_{\beta r} A_r \leq \beta^{-k(2d+\mu+\delta)}$$

for each $r > r_0$. Setting $r = \beta^i r$, for $0 \leq i \leq j-1$ and iterating the above inequality j times, we get

$$det_{\beta^j r} A_r \leq \beta^{-jk(2d+\mu+\delta)} \tag{4.1}$$

However, for a fixed A_r-orthonormal basis $\{u_i\}_{i=1}^k$,

$$\int_{B_{r_1}(p)} |u_1|^2 \leq CV_p(r_1) r_1^2 d \leq C^2 r_1^{2d+\mu}$$

for each $r_1 > r$, and each $1 \leq i \leq k$. Then, since $A_{\beta^j r}$ is positive definite, by Hadamard's inequality:

$$det_r A_{\beta^j r} \leq C^{2k} \beta^{jk(2d+\mu)} r^{k(2d+\mu)}$$

and this contradicts (4.1) as $j \to +\infty$. Therefore, the lemma is true. □

Theorem 4.1.2. (Colding-Minicozzi) *Suppose M has non-negative Ricci curvature. Then, the dimension of $H^d(M)$ is finite. Moreover, there exists a constant C, depending only on n, such that*

$$h^d(M) \leq C \cdot d^{n-1}$$

Proof. (Li) First of all, observe that, since $R_{ij} \geq 0$, by Bishop's volume comparison theorem, $(\mathcal{V}_\mu)$ holds with $C_\mathcal{V} = 1$ and $\mu = n$, and also $(\mathcal{M})$ holds by theorem 3.0.3 and $C_\mathcal{M}$ depends only on n.

Let K be a finite dimensional subspace of $H^d(M)$, with $dimK = k$, and suppose $\{u_i\}_{i=1}^k$ is a basis of K. We claim that for any $p \in M, r > 0$ and $0 < \epsilon < \frac{1}{2}$, we have the estimate

$$\sum_{i=1}^k \int_{B_r(p)} |u_i|^2 \leq C_1 \epsilon^{-(n-1)} \sup_{u \in \mathcal{T}} \int_{B_{[1+\epsilon]r}(p)} |u|^2 \tag{4.2}$$

where C_1 is a constant depending only on n and
$\mathcal{T} = \{u = a_1 u_1 + \cdots + a_k u_k \quad | \quad (a_1, \cdots, a_k) \in \mathbb{R}^k \text{unit vector}\}$.
Indeed, for any $x \in B_r(p)$, there exists a subspace K_x of K, defined by
$K_x = \{f \in K \quad | \quad f(x) = 0\}$.
We observe that K_x has co-dimension at most 1, since otherwise there would exist two linearly independent function $f, g \in K$ such that $f(x), g(x) \neq 0$. But then, the function $f(x) \cdot g - g(x) \cdot f$ is a linear combination of f and g that belongs in K_x, a contradiction. Therefore, after an orthonormal

change of basis, we may assume that $u_i \in K_x$, for $2 \leq i \leq k$ and then $\sum_{i=1}^{k} |u_i|^2(x) = |u_1|^2(x)$. Now, u_1 is harmonic, so (denoting with $\rho(x)$ the distance from p to x, i.e. $\rho(x) = d(p,x)$) by the mean value inequality $(\mathcal{M})$ for radius $[1+\epsilon]r - \rho(x)$ and noting that $B_{[1+\epsilon]r-\rho(x)}(x) \subseteq B_{[1+\epsilon]r}(p)$ we have

$$
\begin{aligned}
\sum_{i=1}^{k} |u_i|^2(x) &= |u_1|^2(x) \\
&\leq C_{\mathcal{M}} \frac{1}{V_p([1+\epsilon] - \rho(x))} \int_{B_{[1+\epsilon]r-\rho(x)}(x)} |u_1|^2 \\
&\leq C_{\mathcal{M}} \frac{1}{V_p([1+\epsilon] - \rho(x))} \int_{B_{[1+\epsilon]r}(p)} |u_1|^2 \\
&\leq C_{\mathcal{M}} \frac{1}{V_p([1+\epsilon] - \rho(x))} \sup_{u \in \mathcal{T}} \int_{B_{[1+\epsilon]r}(p)} |u|^2
\end{aligned}
\tag{4.3}
$$

Now, since $[1+\epsilon]r - \rho(x) < [1+\epsilon]r < 2r$, by the volume comparison theorem, we have

$$V_x(2r) \leq V_x([1+\epsilon]r - \rho(x)) \Big(\frac{2r}{[1+\epsilon]r - \rho(x)} \Big)^n$$

By a simple triangle inequality we observe that $B_r(p) \subset B_{2r}(x)$. Therefore we have

$$V_x([1+\epsilon]r - \rho(x)) \geq \Big(\frac{[1+\epsilon]r - \rho(x)}{2r} \Big)^n V_x(2r) \geq \Big(\frac{[1+\epsilon]r - \rho(x)}{2r} \Big)^n V_p(r)$$

$$\Rightarrow \frac{1}{V_x([1+\epsilon]r - \rho(x))} \leq \Big(\frac{2r}{[1+\epsilon]r - \rho(x)} \Big)^n \frac{1}{V_p(r)}$$

Using this and (4.3) we get

$$\sum_{i=1}^{k} |u_i|^2(x) \leq \frac{C_{\mathcal{M}} 2^n}{V_p(r)} \cdot \big([1+\epsilon] - \frac{\rho(x)}{r} \big)^{-n} \cdot \sup_{u \in \mathcal{T}} \int_{B_{[1+\epsilon]r}(p)} |u|^2$$

Integrating over $B_r(p)$ we get

$$\sum_{i=1}^{k} \int_{B_r(p)} |u_i|^2(x) \leq \frac{C_{\mathcal{M}} 2^n}{V_p(r)} \cdot \sup_{u \in \mathcal{T}} \int_{B_{[1+\epsilon]r}(p)} |u|^2 \cdot \int_{B_r(p)} \big([1+\epsilon] - \frac{\rho(x)}{r} \big)^{-n} dx \tag{4.4}$$

To continue, we consider the function

$$f(\rho) = \big([1+\epsilon] - \frac{\rho}{r} \big)^{-n}$$

Then

$$f'(\rho) = \frac{n}{r} \big([1+\epsilon] - \frac{\rho}{r} \big)^{-n-1} \geq 0$$

for $\rho < r$. Using polar coordinates gives

$$\int_{B_r(p)} \big([1+\epsilon] - \frac{\rho(x)}{r} \big)^{-n} dx = \int_0^r A_p(t) f(t) dt$$

where $A_p(t)$ is the area of the sphere of radius t, around p.
Now, integrating by parts, we have

$$\int_0^r A_p(t)f(t)dt = \big[f(t)V_p(t)\big]_0^r - \int_0^r f'(t)V_p(t)dt \tag{4.5}$$

However, since $f' \geq 0$, by the volume comparison theorem,

$$\begin{aligned}\int_0^r f'(t)V_p(t)dt &\geq r^{-n}V_p(r)\int_0^r f'(t)t^n dt\\ &= r^{-n}V_p(r)\Big(\big[f(t)t^n\big]_0^r - n\int_0^r f(t)t^{n-1}dt\Big)\\ &= f(r)V_p(r) - nr^{-n}V_p(r)\int_0^r f(t)t^{n-1}dt\end{aligned}$$

Combining with (4.5) we get

$$\begin{aligned}\int_0^r f(t)A_p(t)dt &\leq nr^{-n}V_p(r)\int_0^r f(t)t^{n-1}dt\\ &\leq nr^{-n}V_p(r)r^{n-1}\int_0^r f(t)dt\\ &= nr^{-1}V_p(r)\int_0^r f(t)dt\\ &= \frac{n}{r}V_p(r)\int_0^r \big([1+\epsilon] - \frac{t}{r}\big)^{-n}dt\\ &= \frac{n}{r}V_p(r)\Big[\frac{-r}{-n+1}\big([1+\epsilon] - \frac{t}{r}\big)^{-n+1}\Big]_0^r\\ &= \frac{n}{r}V_p(r)\frac{r}{n-1}\big(\epsilon^{1-n} - (1+\epsilon)^{1-n}\big)\\ &\leq \frac{n}{n-1}V_p(r)\epsilon^{1-n}\end{aligned} \tag{4.6}$$

From (4.4) and (4.6) our claim is proved.

On the other hand, note that condition $(\mathcal{V}_\mu)$ implies that the volume growth of M is at most of order r^μ. Therefore, if we take $\beta = 1+\epsilon$ and a $\delta > 0$ in lemma 4.1.1, we have that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K, with respect to the inner product $A_{\beta r}$, then there exists $r > 0$ such that

$$\sum_{i=1}^k \int_{B_r(p)} |u_i|^2 \geq k \cdot \beta^{-(2d+n+\delta)} \tag{4.7}$$

However, if $\{u_i\}_{i=1}^k$ is an $A_{\beta r}$-orthonormal basis, then

$$\int_{B_{[1+\epsilon]r}(p)} |u|^2 = 1$$

for any $u \in \mathcal{T}$.

Hence, combining (4.2) and (4.7) we get

$$k \cdot \beta^{-(2d+n+\delta)} \leq C_1 \cdot \epsilon^{-(n-1)}$$

Taking $\delta \to 0$:

$$k \leq C_1 \cdot \epsilon^{-(n-1)} \cdot (1+\epsilon)^{2d+n}$$

Taking $\epsilon = \dfrac{1}{2d}$, we get

$$k \leq C_2 \cdot d^{n-1} \cdot \Big(1 + \frac{1}{2d}\Big)^{2d+n}$$

The proof of the theorem will be complete if we show that $\Big(1 + \dfrac{1}{2d}\Big)^{2d+n}$ is bounded by a constant depending only on n. To that end, we consider the function

$$H(x) = \big(1 + \frac{1}{x}\big)^{x+n}$$

for $x \geq 2$. Then

$$\begin{aligned} H'(x) &= \big(1 + \frac{1}{x}\big)^{x+n} \cdot \Big(log(1 + \frac{1}{x}) + [x+n]\frac{1}{1+\frac{1}{x}}(-\frac{1}{x^2})\Big) \\ &= \big(1 + \frac{1}{x}\big)^{x+n}\Big(log(1 + \frac{1}{x}) - \frac{x+n}{x^2+x}\Big) \end{aligned}$$

We set $h(x) = log(1 + \dfrac{1}{x}) - \dfrac{x+n}{x^2+x}$ and compute:

$$\begin{aligned} h'(x) &= -\frac{x}{x+1} \cdot \frac{1}{x^2} - \frac{x^2 + x - (x+n)(2x+1)}{(x^2+x)^2} \\ &= \frac{-x^2 - x}{(x^2+x)^2} - \frac{-x^2 - 2nx - n}{(x^2+x)^2} \\ &= \frac{(2n-1)x + n}{(x^2+x)^2} > 0 \end{aligned}$$

Therefore h is increasing, and since $\lim_{x \to \infty} h = 0$, we have that $h < 0$. As a result, $H' < 0$ and H is decreasing, hence

$$H(x) \leq H(2) = C(n)$$

□

Next, we derive finite dimensionality for $H^d(M)$ in a more "relaxed" setting.

Definition 4.1.1. We say that M satisfies a *Weak Mean Value Inequality* ($\mathcal{WM}$) if there exist constants $C_{\mathcal{WM}} > 0$ and $b > 1$, such that for any non-negative subharmonic function f on M

$$f(x) \leq C_{\mathcal{WM}} \frac{1}{V_x(r)} \int_{B_{br}(x)} f(y)dy$$

for all $x \in M$ and $r > 0$.

Theorem 4.1.3. *Suppose M satisfies $(\mathcal{WM})$ and the volume growth of M satisfies $V_p(r) = O(r^\mu)$ for some $p \in M$. Then $H^d(M)$ has finite dimension and $h^d(M) \leq C_{\mathcal{WM}}(2b+1)^{2d+\mu}$*

Proof. Let $K \leq H^d(M)$ be a finite dimensional subspace of $H^d(M)$, with $dimK = k$. We set $\beta = 2b+1$, where b is the constant in $(\mathcal{WM})$. By lemma 4.1.1, for any $\delta > 0$, there exists $r > 0$ such that if $\{u_i\}_{i=1}^k$ is an $A_{\beta r}$-orthonormal basis of K, then

$$\sum_{i=1}^{k} \int_{B_r(p)} u_i^2 \geq k \cdot \beta^{-(2d+\mu+\delta)} \tag{4.8}$$

On the other hand, since $\sum_{i=1}^{k} u_i^2$ is subharmonic, by the maximum principle there exists a point $q \in \partial B_r(p)$ such that

$$\sum_{i=1}^{k} u_i^2(x) \leq \sum_{i=1}^{k} u_i^2(q)$$

for any $x \in B_r(p)$.

As in the proof of theorem 4.1.2, we can assume (after an orthonormal change of basis, if necessary) that $u_i(q) = 0$, for $2 \leq i \leq k$. We note that
$B_r(p) \subset B_{2r}(p) \subset B_{(2b+1)r}(p)$ and then, by $(\mathcal{WM})$ on u_1^2 we have

$$\begin{aligned} V_p(r)u_1^2(q) &\leq V_q(2r)u_1^2(q) \\ &\leq C_{\mathcal{WM}} \int_{B_{2r}(q)} u_i^2 \\ &\leq C_{\mathcal{WM}} \int_{B_{(2b+1)r}(p)} u_i^2 \\ &= C_{\mathcal{WM}} \end{aligned}$$

As a result

$$\begin{aligned} \sum_{i=1}^{k} \int_{B_r(p)} u_i^2 &\leq V_p(r) \sum_{i=1}^{k} u_i^2(q) \\ &\leq V_p(r)u_1^2(q) \\ &\leq C_{\mathcal{WM}} \end{aligned} \tag{4.9}$$

By (4.8) and (4.9),

$$k \cdot \beta^{-(2d+\mu+\delta)} \leq C_{\mathcal{WM}}$$

Taking $\delta \to 0$,

$$\begin{aligned} &k \cdot \beta^{-(2d+\mu)} \leq C_{\mathcal{WM}} \\ \Rightarrow &k \leq (2b+1)^{2d+\mu} C_{\mathcal{WM}} \end{aligned}$$

As a result

$$h^d(M) < C_{\mathcal{WM}}(2b+1)^{2d+\mu} < +\infty$$

$\square$

4.2 Two Results about Massive Sets

Li's ideas in the proof of theorem 4.1.2 (in particular lemma 4.1.1), can be used to prove similar-looking results about massive sets. The notion of a *massive set* (more specifically that of a *0-massive set*) was first introduced by Grigor'yan in [7]. Later, Li and Wang generalized this idea and considered *d-massive sets*, in [15]. We begin with the difinition.

Definition 4.2.1. Let $d \geq 0$ be any real number. Let $\Omega \subset M$ be a subset of M such that it admits a non-negative, subharmonic function f defined on Ω, such that

$$f = 0 \qquad \text{on} \quad \partial\Omega$$

and

$$|f(x)| \leq C \cdot \rho^d$$

for each $x \in \Omega$ and for some constant $C > 0$. We then call Ω a *d-massive* set (The function f is called a *potential* function of Ω). We also denote the maximum number of disjoint d-massive sets admissible on M by $m^d(M)$.

Li and Wang proved the following three theorems:

Theorem 4.2.1. (Li-Wang) *Suppose M satisfies $(\mathcal{V}_\mu)$ and $(\mathcal{M})$. Let $d \geq 1$. Then, there exists a constant $C > 0$, depending only on μ, such that*

$$m^d(M) \leq C \cdot C_{\mathcal{M}} \cdot d^{\mu-1}$$

The proof of the theorem above is essentially the same with that of theorem 4.1.2. Instead of considering an orthonormal basis of a finite dimensional subspace, we consider a set of potential functions corresponding to disjoint d-massive sets, with normalized L^2-norm. Note that these are automatically orthogonal, since they are supported on disjoint sets. Also, we do not need to consider a change of basis and the subspaces K_x, because for each $x \in M$, there exists at most one potential function not vanishing at x. The rest of the argument is almost the same.

Similarly, they proved the next theorem, which mirrors theorem 4.1.3:

Theorem 4.2.2. Li-Wang *Suppose M satisfies $(\mathcal{WM})$ and the volume growth of M satisfies:*

$$V_p(r) = O(r^\mu)$$

for some point $p \in M$*.Then*

$$m^d(M) \leq C_{\mathcal{WM}} \cdot (2b+1)^{2d+\mu}$$

In the next theorem, they provided a sharp estimate of $m^d(M)$ on $\mathbb{R}^2$:

Theorem 4.2.3. *On* $\mathbb{R}^2$*,*

$$m^d(\mathbb{R}^2) \leq 2d$$

for all $d \geq 0$*.*

For a proof, see [15].

In general, it is not known whether there exists a direct relation between m^d and h^d. However for $d = 0$ we have the following theorem (see [7])

Theorem 4.2.4. (Grigor'yan) *The maximum number of disjoint 0-massive sets admissible on* M *is given by the dimension of the space of bounded harmonic functions on* M*, i.e.*

$$m^0(M) = h^0(M)$$

Chapter 5

Further Results about $h^d(M)$

In this final chapter, we present some further results about the dimension of $H^d(M)$, as well as some conjectures on the subject.

First of all, for $d \in \mathbb{Z}^+$, the case $M = \mathbb{R}^n$ has been explicitly calculated, and it is

$$h^d(\mathbb{R}^n) = \binom{n+d-1}{d} + \binom{n+d-2}{d-1}$$

see for example [12].

Along with his conjecture about whether the space of harmonic functions with polynomial growth is finite dimensional on a manifold M with non-negative Ricci curvature, Yau also raised the question if the following relation holds:

$$dim H^d(M) \leq dim H^d(\mathbb{R}^n)$$

In this direction, Li and Tam considered the case $d = 1$ and proved the following in [13]

Theorem 5.0.1. (Li-Tam) *Suppose M has non-negative Ricci curvature and that the volume growth of M satisfies*

$$V_p(r) = O(r^k)$$

for a constant $k > 0$. Then

$$dim H^1(M) \leq dim H^1(\mathbb{R}^k) = k + 1$$

Moreover, as it turns out, such a constant k must exist and $1 \leq k \leq n$. As a result

Corollary 5.0.1.1. *If M has non-negative Ricci curvature, then*

$$dim H^1(M) \leq n + 1$$

This, led Li and Tam to consider two questions:

Question 1: *Suppose M has non-negative Ricci curvature and satisfies the volume growth condition*

$$V_p(r) = O(r^k)$$

for some constant $k > 0$. Is it true then that

$$dimH^d(M) \leq dimH^d(\mathbb{R}^k) = \binom{k+d-1}{d} + \binom{k+d-2}{d-1}?$$

In the case $n = 2$ the question above was affirmatively answered by Kasue in [9], and independently by Li and Tam in [14].

Question 2: *What can be said about the manifolds on which equality is achieved in corollary 5.1.1?*

Li proved in [10] the following

Theorem 5.0.2. (Li) *Suppose M is complete Kähler manifold with non-negative Ricci curvature. If*

$$dimH^1(M) = 2m + 1$$

where $m = dim_{\mathbb{C}} M$, then M must be isometrically biholomorphic to $\mathbb{C}^m$.

Later, Cheeger, Colding and Minicozzi gave a proof in [1] that did not require the Kähler assumption. They proved a splitting type theorem for the tangent cone at infinity that had the following corollary:

Theorem 5.0.3. (Cheeger-Colding-Minicozzi) *Suppose M has non-negative Ricci curvature. If*

$$dimH^1(M) = n + 1$$

then M must be isometric to $\mathbb{R}^n$.

Finally, Wang in [19], estimated $dimH^1(M)$ when M has non-negative Ricci curvature outside a compact set and has finite 1st Betti number. Recall, that the *n-th Betti number* of a topological space is the rank of the n-th homology group.

Theorem 5.0.4. (Wang) *Suppose M has non-negative Ricci curvature outside a geodesic ball $B_r(p)$, for $p \in M$ and $r > 0$. Moreover suppose that the 1st Betti number of M is finite and that the Ricci curvature in $B_r(p)$ satisfies*

$$R_{ij} \geq -k$$

for some constant $k > 0$. Then there exists a constant C depending on n, r and k such that

$$dimH^1(M) \leq C$$

Bibliography

[1] J. Cheeger, T. Colding and W. Minicozzi, *Linear growth harmonic functions on complete manifolds with non-negative Ricci curvature*, Geom. Func. Anal. **5** (1995), 948-954

[2] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **27** (1975), 333-354

[3] T. Colding and W. Minicozzi, *On function theory on spaces with a lower Ricci curvature bound*, Math. Res. Lett. **3** (1996), 241-246

[4] T. Colding and W. Minicozzi, *Harmonic functions with polynomial growth*, J. Diff. Geom. **46** (1997), 1-77

[5] T. Colding and W. Minicozzi, *Harmonic functions on manifolds*, Ann. Math. **146** (1997), 725-747

[6] M. P. doCarmo, *Riemannian Geometry*, Birkhäuser, 1992

[7] A. Grigor'yan, *On the dimension of spaces of harmonic functions*, Math. Notes **48** (1990), 1114-1118

[8] J. Jost, *Riemannian Geometry and Geometric Analysis*, 4th ed., Universitext, Springer, 2005

[9] A. Kasue, *Harmonic functions with growth conditions on manifold of asymptotically non-negative curvature II*, Recent Topics in Differential and Analystical Geometry, Adv. Stud. Pure Math., vol. 18, North-Holland, 1989

[10] P. Li, *Harmonic functions of linear growth on Kähler manifolds with non-negative Ricci curvature*, Math. Research Letters **2** (1995), 79-94

[11] P. Li, *Curvature and function theory on Riemannian manifolds*, Surv. Diff. Geom. VII (2000), 375-432

[12] P. Li, *Geometric Analysis*, Cambridge Studies in Advanced Mathematics 134, Cambridge, 2012

[13] P. Li and L. F. Tam, *Linear growth harmonic functions on a complete manifold*, J. Diff. Geom. **29** (1989), 421-425

[14] P. Li and L. F. Tam, *Complete surfaces with finite total curvature*, J. Diff. Geom. **33** (1991), 139-168

[15] P. Li and J. P. Wang, *Convex hull properties of harmonic maps*, J. Diff. Geom. **48** (1998), 497-530

[16] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operators*, Acta Math. **156** (1986), 153-201

[17] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, 1983

[18] R. Schoen and S. T. Yau, *Lecture on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. I, International Press, 1994

[19] J. Wang, *Linear growth harmonic functions on complete manifolds*, Comm. Anal. Geom. **4** (1995), 683-698

[20] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **27** (1975), 201-228