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Elliptic Systems with Variational Structure

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To my parents

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PREFACE

In this thesis we study a class of variational models from the gradient theory of phase transitions developed originally by Van der Waals and more recently by Ginzburg and Landau, Cahn and Hilliard, and others.

The thesis is structured as follows:

In Chapter 1 we give a brief description of the Calculus of Variations. In Chapter 2 we introduce the Elliptic System. In subsections 2.1-2.2 we discuss the underlying physical phenomenology and the two main mathematical tools:
(a) The Γ- limit introduced by De Giorgi;
(b) The density estimate introduced by Caffarelli and Cordoba

Next in subsection 2.3 we focus on singular potentials W. Our main result here is Lemma 2.2 (and Lemma 2.3) which implies that for such potentials the minimal symmetric solutions possess a free boundary (see Theorem 2.3). Lemma 2.2 utilizes previous results of Sperb [13] on the dead core problem that we describe briefly.

In Chapter 3 we turn our attention to the mass constraint case and to a different kind of investigation. Here we studied the preprint of G.Fusco [G.Fusco, preprint] which establishes the following result:

Let u_R be a minimizer of $J_{B_R}(v) = \int_{B_R} \frac{1}{2}|\nabla v|^2 + W(v)$ under the constraint $\frac{1}{|B_R|}\int_{B_R} v dx = \hat{m} \in \ Conv\{a_1, ..., a_N\}$, $W(a_1) = ... = W(a_N) = 0$, $W > 0$ on $\mathbb{R}^m \setminus \{a_1, ..., a_N\}$.
Then $u_R \to u$, along a sequence $R_k \to \infty$ where u is a minimal solution to $\Delta u - W_u(u) = 0$, $x \in \mathbb{R}^n$. Our main result here is the proof of Lemma 3.2 which in [G.Fusco, preprint] is stated without proof.

Finally, in the appendix our main result is a calculus inequality that implies a lower bound estimate for the energy functional J_{B_R} as defined in Chapter 3.

Εισαγωγή

Σε αυτή την διπλωματική μελετάμε μια κλάση μεταβολικών μοντέλων από την θεωρία βαθμίδας αλλαγής φάσεων που αναπτύχθηκε αρχικά από τον Van der Waals και πιο πρόσφατα από τους Ginzburg και Landau, Cahn και Hilliard, και άλλους.

Η εργασία δομείται ως εξής:

Στο Κεφάλαιο 1 δίνουμε μια σύντομη περιγραφή του Λογισμού Μεταβολών. Στο κεφάλαιο 2 εισάγουμε το Ελλειπτικό Σύστημα. Στις υποπαραγράφους 2.1-2.2 αναφέρουμε την υποκείμενη φυσική φαινομενολογία και τα δύο βασικά μαθηματικά εργαλεία:
(α) Το Γ-όριο που εισήχθη από τον De Giorgi,
(β) Την εκτίμηση πυκνότητας που εισήχθη από τους Caffarelli και Cordoba.

Έπειτα, στην υποπαράγραφο 2.3 εστιάζουμε στα ιδιόμορφα δυναμικά W. Το κύριο αποτέλεσμά μας εδώ είναι το Λήμμα 2.2 (και το Λήμμα 2.3) το οποίο μας δίνει ότι για τέτοια δυναμικά, οι συμμετρικές λύσεις που είναι και ελαχιστοποιητές έχουν ελεύθερο σύνορο (βλ. Θεώρημα 2.3). Στο Λήμμα 2.2 χρησιμοποιούνται προηγούμενα αποτελέσματα από τον Sperb [13] για το πρόβλημα "dead core" το οποίο περιγράφουμε σύντομα.

Στο κεφάλαιο 3 στρέφουμε την προσοχή μας στην περίπτωση περιορισμού της μάζας και σε ένα διαφορετικό είδος μελέτης. Εδώ μελετήσαμε κάποιες σημειώσεις του G.Fusco [G.Fusco, preprint] από τις οποίες προκύπτει το εξής αποτέλεσμα:

Έστω u_R ένας ελαχιστοποιητής του $J_{B_R}(v) = \int_{B_R} \frac{1}{2}|\nabla v|^2 + W(v)$ υπό τον περιορισμό $\frac{1}{|B_R|}\int_{B_R} v dx = \hat{m} \in Conv\{a_1, ..., a_N\}$, $W(a_1) = ... = W(a_N) = 0$, $W > 0$ στο $\mathbb{R}^m \setminus \{a_1, ..., a_N\}$. Τότε $u_R \to u$, ως προς μια υπακολουθία $R_k \to \infty$ όπου u είναι ένας ελαχιστοποιητής και είναι λύση του συστήματος $\Delta u - W_u(u) = 0$, $x \in \mathbb{R}^n$. Το αποτέλεσμά μας εδώ είναι η απόδειξη του Λήμματος 3.2 η οποία είχε παραληφθεί από τις σημειώσεις [G.Fusco, preprint] στις οποίες αναφέρεται χωρίς απόδειξη.

Τέλος, στο παράρτημα το βασικό μας αποτέλεσμα είναι μια ανισότητα λογισμού η οποία δίνει μια κάτω εκτίμηση για το συναρτησοειδές ενέργειας J_{B_R} όπως ορίστηκε στο Κεφάλαιο 3.

1 Introduction to The Calculus of Variations

Introduction

There are some different ways of thinking the idea of the Calculus of Variations. One very simple is the following: When we want to find the minimum or maximum of a real valued function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ we restrict to the points $p \in \mathbb{R}^n$ such that: $\nabla f(p) = 0$ and if (all) the eigenvalues of $(Hf)(p)$ are positive, then $f(p)$ locally minimizes f (respectively negative for maximum). Now, if we have a functional J, defined on a function space (let's say C^2 functions for example) $J : C^2(\Omega) \to \mathbb{R}$, and we want to find minimizers for example, that is : $f \in C^2(\Omega)$ such that $J(f) = \min\limits_{u \in C^2(\Omega)} J(u)$ there is a necessary condition which is a differential equation that a function must solve in order to be a minimizer (or a critical point of J). This equation (of the "first derivative" of J) is called the *Euler–Lagrange* equation. It can be a vector equation, that is a system of equations.

So, the idea is, in other words, to minimize or maximize general quantities (like functionals) and it turns out that many laws of nature can be given the form of an extremal principle together with the associated Euler–Lagrange Equation.

Another way of viewing the idea of the Calculus of Variations is identifying an important class of nonlinear problems that can be solved using relatively simple techniques from nonlinear Functional Analysis. This is the class of Variational Problems, that if Partial Differential Equations of the form: $\mathcal{F}[u] = 0$, where the nonlinear operator $\mathcal{F}[\cdot]$ is the "derivative" of an appropriate "energy" functional $I[\cdot]$. Thus we can symbolically write $\mathcal{F}[\cdot] = I'[\cdot]$ and the problem now becomes: $I'[u] = 0$. So, the class of Variational Problems is the class of PDE's that can be expressed as the Euler–Lagrange equations of an appropriate functional. An intresting question is the existence of a general criterion on whether a PDE is in this class or not.

This new formulation has the advantage that we can recognize solutions of $\mathcal{F}[u] = 0$ as being critical points of $I[\cdot]$. The point is that whereas it is usually extremely difficult to solve $\mathcal{F}[u] = 0$ directly, it may be much easier to discover minimum (or maximum, or other critical) points of the functional $I[\cdot]$. Finally, as mentioned before, many of the laws of Physics and other scientific disciplines arise directly as variational principles.

The Euler–Lagrange Equations

Let $U \subset \mathbb{R}^n$ bounded, open set with smooth boundary ∂U, and define the "energy" functional $I[\cdot]$:

$$I[w] = \int_U L(\nabla w(x), w(x), x)dx \tag{1}$$

for smooth functions $w : \overline{U} \to \mathbb{R}$ satisfying the boundary condition $w = g$ on ∂U. Where $L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$ and we call L the Lagrangian.
Notation: $L = L(p, z, x) = L(p_1, ..., p_n, z, x_1, ..., x_n)\ \ , p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in U.$

Suppose now that some particular smooth function u satisfying $u\ |_{\partial U} = g$ happens to be a minimizer of $I[\cdot]$ among all functions w satisfying the boundary condition, $I[u] = \min\limits_{w|_{\partial U}=g} I[w]$. We will demonstrate that u is then automatically a solution of a certain nonlinear partial differential equation.

For any $v \in C_c^\infty(U)$ consider the real valued function:

$$i(\tau) := I[u + \tau v]\ \ , \tau \in \mathbb{R}$$

Since u is a minimizer of $I[\cdot]$ and $u + \tau v = u = g$ on ∂U, we obsereve that $i(\cdot)$ has minimum at $\tau = 0$. Therefore : $i'(0) = 0$.

We have

$$i(\tau) = \int_U L(\nabla u + \tau \nabla v, u + \tau v, x)dx$$

Thus:

$$i'(\tau) = \int_U [\sum_{j=1}^n L_{p_j}(\nabla u + \tau \nabla v, u + \tau v, x) v_{x_j} + L_z(\nabla u + \tau \nabla v, u + \tau v, x) v]dx$$

for $\tau = 0$,

$$i'(0) = 0 \Leftrightarrow \int_U [\sum_{j=1}^{n} L_{p_j}(\nabla u, u, x) v_{x_j} + L_z(\nabla u, u, x) v] dx = 0 \tag{2}$$

Finally, since v has compact support, we can integrate by parts:

$$\int_U \frac{\partial u}{\partial x_i} v dx = \int_{\partial U} uv\hat{\nu}_j dS - \int_U \frac{\partial v}{\partial x_i} u dx \tag{3}$$

where $\overrightarrow{\nu}$ is the outword unit surface normal to ∂U and $\hat{\nu}_j$ is it's j–th component $(j = 1, ..., n)$.

So we obtain:

$$\int_U [-\sum_{j=1}^{n} (L_{p_j}(\nabla u, u, x))_{x_j} + L_z(\nabla u, u, x)] v dx = 0 \tag{4}$$

for all test functions $v \in C_c^{\infty}(U)$.

The last equation using the Fundamental Lemma of the Calculus of Variations will give us the Euler–Lagrange equations.

The Fundamental Lemma: Let $f(x)$ be a continuous, real valued function on some open set $U \subset \mathbb{R}^n$ and suppose that:

$$\int_U f(x)\eta(x) dx = 0 \tag{5}$$

for all $\eta \in C_c^{\infty}(U)$. Then we have $f(x) = 0$, $\forall\, x \in U$.

Proof.

Assume that there is a point $x_0 \in U : f(x_0) \neq 0$, without loss of generality suppose $f(x_0) > 0$. Then we can find a number $\varepsilon > 0$ and a ball $B_r(x_0) \subset\subset U$ such that $f(x) > \varepsilon$ on $B_r(x_0)$. Now define the test function

$$\eta(x) := \begin{cases} e^{-\frac{1}{r^2 - |x - x_0|^2}} & , x \in B_r(x_0) \\ 0 \quad , & x \in \mathbb{R}^n \setminus B_r(x_0) \end{cases}$$

and we arrive at the contradictory statement:

$$0 = \int_U f(x)\eta(x) dx = \int_{B_r(x_0)} f(x)\eta(x) dx > \varepsilon \int_{B_r(x_0)} \eta(x) dx > 0.$$

Thus we conclude that $f(x) = 0$, $\forall x \in U$.

$\square$

So, from (4) and the Fundamental Lemma we have:

$$-\sum_{j=1}^{n}(L_{p_j}(\nabla u, u, x))_{x_j} + L_z(\nabla u, u, x) = 0 \ , \ \ \forall x \in U. \tag{6}$$

This is the Euler–Lagrange equation associated with the energy functional $I[\cdot]$ defined by (1) (Observe that (6) is a quasilinear, second order PDE in divergence form).

Examples:

(a) Let $L(p, z, x) = \frac{1}{2}|p|^2 + W(z)$, $W : \mathbb{R} \to \mathbb{R}$, so

$$I[v] = \int_U (\frac{1}{2}|\nabla v|^2 + W(v))dx$$

and the Euler–Lagrange equation is:

$$-\Delta u + W'(u) = 0 \tag{7}$$

(b) (Minimal Surfaces) Let $L(p, z, x) = (1 + |p|^2)^{\frac{1}{2}}$, so that

$$I[w] = \int_U (1 + |\nabla w|^2)^{\frac{1}{2}} dx$$

is the area graph of the function $w : U \to \mathbb{R}$. The associated Euler–Lagrange equation is:

$$\sum_{i=1}^{n}(\frac{u_{x_i}}{(1 + |\nabla u|^2)^{\frac{1}{2}}})_{x_i} = 0 \ , \ x \in U \tag{8}$$

This partial differential equation is the minimal surface equation.
(Note: The expression: $\mathrm{div}(\frac{\nabla u}{(1 + |\nabla u|^2)^{\frac{1}{2}}}) = nH$, where H is the mean curvature of the graph of u. Thus a minimal surface has zero mean curvature.)

(c) Let

$$I[w] = \int_U (\frac{1}{2}|\nabla w|^2 + f(x)w)dx$$

with the Lagrangian: $L(p,z,x) = \dfrac{1}{2}|p|^2 + f(x)z$ and the Euler–Lagrange equation is:

$$\Delta u = f(x) \ \ , \ x \in U \tag{9}$$

the so– called Poisson equation.
(Note: If $\sigma \in C^1(\overline{G})$, then the Newtonian potential $u(x) = \int_G |x-y|^{2-n}\sigma(y)dy$ satisfies the Poisson equation with the right–hand side $f(x) = -(n-2)\omega_n\sigma(x)$, where $\omega_n = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, $\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}dx$, $t > 0$ is the surface area of the unit sphere in $\mathbb{R}^n$. Newtonian potentials play an important role as gravitational potentials and , in electrostatics, as Coulomb potentials.)

Now, we will consider the more general case of functionals that are defined in the smooth functions $w : \overline{U} \subset \mathbb{R}^n \to \mathbb{R}^m$, $w = (w^1, ..., w^m)$. Let $\mathbb{M}^{m\times n}$ is the space of $m \times n$ matrices, and assume the smooth Lagrangian function

$$L : \mathbb{M}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$$

is given.
Notation: $L = L(P,z,x) = L(p_1^1, ..., p_n^m, z^1, ..., z^m, x_1, ...x_n)$, for $P \in \mathbb{M}^{m\times n}$, $z \in \mathbb{R}^m$ and $x \in U$, where $P = (p_i^j)_{1\le i\le n}^{1\le j\le m}$

As previously (where m=1), we associate with L the functional

$$I[w] = \int_U L(\nabla w(x), w(x), x)dx \tag{10}$$

defined for smooth functions $w : \overline{U} \to \mathbb{R}^m$, $w = (w^1, ..., w^m)$, satisfying the boundary conditions $w = g$ on $\partial U, g : \partial U \to \mathbb{R}^m$ being given. Here $\nabla w(x) = (w_{x_i}^j)_{1\le i\le n}^{1\le j\le m}$, is the gradient matrix of w at x.

Let us now show that any smooth minimizer $u = (u^1, ..., u^m)$ of $I[\cdot]$, taken among functions equal to g on ∂U, must solve a certain system of nonlinear partial differential equations. We therefore fix $v = (v^1, ..., v^m) \in C_c^\infty(U; \mathbb{R}^m)$ and write

$$i(\tau) := I[u + \tau v]$$

so $i'(0) = 0$ and as before we deduce the equality:

$$\int_U (\sum_{j=1}^n \sum_{k=1}^m L_{p_j^k}(\nabla u, u, x)v_{x_j}^k + \sum_{k=1}^m L_{z^k}(\nabla u, u, x)v^k)dx = 0 \tag{11}$$

since this identity is valid for all choices $v^1, .., .v^m$, by taking $v^l = 0$, $\forall l = 1, ..., m$, $l \neq k$ after integrating by parts (and using the Fundamental Lemma):

$$-\sum_{j=1}^{n} (L_{p_j^k}(\nabla u, u, x))_{x_j} + L_{z^k}(\nabla u, u, x) = 0 \ , \ \ x \in U \tag{12}$$

$$\Rightarrow -\sum_{j=1}^{n} (L_{p_j^k}(\nabla u, u, x))_{x_j} + L_{z^k}(\nabla u, u, x) = 0 \ , \ \ x \in U \ , \forall \ k = 1, ..., m \tag{13}$$

This coupled, quasilinear system of PDE comprises the Euler–Lagrange equations for the functional $I[\cdot]$ defined by (10).

Isoperimetric Problems

In this subsection we shall derive necessary conditions for solutions of isoperimetric problems, that is, for local extrema $u \in C^2(\overline{U}; \mathbb{R}^m)$ of variational integrals:

$$I[u] := \int_U L(\nabla u, u, x) dx \tag{14}$$

which, besides boundary conditions on ∂U, are subject to a subsidiary condition of the kind

$$W(u) = c$$

with some constant c, where $W(u)$ is an integral of the form:

$$W(u) = \int_U G(\nabla u, u, x) dx \tag{15}$$

or

$$W(u) = \int_\Sigma G(\nabla u, u, x) d\mathcal{H}^{n-1} \tag{16}$$

respectively, where Σ is a subset of ∂U with positive Hausdorff measure measure $\mathcal{H}^{n-1}(\Sigma)$.

(For open sets Σ on smooth boundaries $\partial U, \mathcal{H}^{n-1}(\Sigma)$ can be understood as the usual surface area of Σ.)

In order to do this, we will need the idea of Lagrange multipliers. We recall what Lagrange multipliers are for ordinary functions. Let f be a smooth real valued function on $\mathbb{R}^n$ and let S be the level set of a smooth function g defined on $\mathbb{R}^n$, that is $S = \{x \in \mathbb{R}^n : g(x) = c\}$. If f restricted to S attains an extremum at some point $x_0 \in S$ where $\nabla g(x_0) \neq 0$, then there is a real number λ, called Lagrange multiplier, such that

$$\nabla f(x_0) + \lambda \nabla g(x_0) = 0 \ ,$$

i.e. the gradient of f has no tangential component along S. To give a brief geometric intuition for this, let $\{e_1, ..., e_{n-1}\}$ span the T_S (tangent space of S), we can write $\nabla f = \sum_{i=1}^{n-1} \frac{\partial f}{\partial e_i} e_i + \frac{\partial f}{\partial n} \overrightarrow{n}$ and $\min_S f(x) = f(x_0)$. At x_0 it clearly holds $\nabla_T f = \sum_{i=1}^{n-1} \frac{\partial f}{\partial e_i} e_i = 0$ and thus $\nabla f(x_0) = \frac{\partial f}{\partial n} \overrightarrow{n} \ // \ \nabla g$.

We formulate the following assumptions: Let $U \subset \mathbb{R}^n$ bounded, $u \in C^2(\overline{U}; \mathbb{R}^m)$ and let $\mathcal{U}$ be an open set in $\mathbb{R}^{n\times m} \times \mathbb{R}^m \times \mathbb{R}^n$ containing $\{(\nabla u(x), u(x), x) : x \in \overline{U}\}$. Suppose also that $L(P, z, x)$ and $G(P, z, x)$ are Lagrangians of class $C^2(\mathcal{U})$. Define W by (15) and set

$$c := W(u)$$

We finally assume that $\mathcal{J}$ is a class of mappings $v \in C^2(\overline{U}; \mathbb{R}^m)$ such that, for every $v \in \mathcal{J}$ and for every pair of functions $\phi, \psi \in C_c^\infty(U; \mathbb{R}^m)$, there exist numbers $\varepsilon_0 > 0$, $t_0 > 0$, such that $v + \varepsilon\phi + t\psi \in \mathcal{J}$ for $|\varepsilon| < \varepsilon_0$, $|t| < t_0$. We say in this case that $\mathcal{J}$ has the variation property $(\mathcal{V})$.
Remark: If a set $\mathcal{J}$ of admisible functions has the property $(\mathcal{V})$, then $\mathcal{J}$ is in a weak sense open. More precisely, any $v \in \mathcal{J}$ can be varied in all "smooth" directions ϕ with compact support in U.

Theorem: Suppose that u furnishes a weak minimum (or maximum) of the functional I in the class $\mathcal{J}_c := \mathcal{J} \cap \{v : W(v) = c\}$. Then there exist a real number λ, called the Lagrange multiplier, such that the Euler– Lagrange equations

$$-\sum_{j=1}^{n} (L_{p_j^k}(\nabla u, u, x) + \lambda G_{p_j^k}(\nabla u, u, x))_{x_j} + L_{z^k}(\nabla u, u, x) + \lambda G_{z^k}(\nabla u, u, x) = 0 \tag{17}$$

$k = 1, ..., m$, are satisfied on U.

Proof.

By assumption, we can find a function $\psi \in C_c^\infty(U;\mathbb{R}^m)$ such that $\delta W(u,\psi) = 1$. (In fact, it suffices to find $\psi \in C_c^\infty(U;\mathbb{R}^m) : \delta W(u,\psi) \neq 0$ and take $\tilde{\psi} = \dfrac{\psi}{a}$ where $a := \delta W(u,\psi)$). Define : $\delta W(u,\psi) := i'(0)$, where $i(\tau) = W(u+\tau\psi)$. With this function and an arbitrary $\phi \in C_c^\infty(U;\mathbb{R}^m)$, we define the functions

$$\Phi := I[u + \varepsilon\phi + t\psi] \quad , \quad \Psi := W(u + \varepsilon\phi + t\psi)$$

for $(\varepsilon, t) \in [-\varepsilon_0, \varepsilon_0] \times [-t_0, t_0] := Q$, and, for sufficiently small numbers $\varepsilon_0 > 0$ and $t_0 > 0$, we obtain:

$$\Phi(\varepsilon, t) \geq \Phi(0,0) \ \ (or \leq \Phi(0,0))$$

for all $(\varepsilon, t) \in Q$ with $\Psi(\varepsilon, t) = c$.. Since $\Psi_t(0,0) = \delta W(u,\psi) = 1 \neq 0$, we may apply the standard Lagrange multiplier theorem. Thus we infer the existence of a number $\lambda \in \mathbb{R}$ such that the function $\Phi(\varepsilon,t) + \lambda\Psi(\varepsilon,t)$ has $(\varepsilon,t) = (0,0)$ as a critical point. Consequently:

$$\begin{cases} \Phi_\varepsilon(0,0) + \lambda\Psi_\varepsilon(0,0) \\ \Phi_t(0,0) + \lambda\Psi_t(0,0) \end{cases}$$

or equivalently

$$\begin{cases} \delta I[u,\phi] + \lambda\delta W(u,\phi) = 0 \\ \delta I[u,\psi] + \lambda\delta W(u,\psi) = 0 \end{cases}$$

the second equation yields $\lambda = -\delta I[u,\phi]$, and we see that the value of λ is independent of the chosen variation $\phi \in C_c^\infty(U;\mathbb{R}^m)$. Hence the first relation gives that for the real valued function $h(s) = I[u+s\phi] + \lambda W(u+s\phi)$, we have $h'(0) = 0, \forall\ \phi \in C_c^\infty(U;\mathbb{R}^m)$, and thus:

$$-\sum_{j=1}^{n}(L_{p_j^k}(\nabla u, u, x) + \lambda G_{p_j^k}(\nabla u, u, x))_{x_j} + L_{z^k}(\nabla u, u, x) + \lambda G_{z^k}(\nabla u, u, x) = 0 \tag{18}$$

$k = 1, ..., m.$

□

2 The Elliptic System

2.1 Motivation and Background:

Let us briefly discuss the physical motivation of the model that we study. We are given some substance in a container, say Ω, which may exhibit two phases, which we label with " -1 " and " +1" and we would like to describe mathematically. Our approach could be that the interface formation is driven by a variational principle, that is the pattern is the outcome of the minimization of a certain energy.

For this, we may consider a "double well" function W, the potential, such that $W(\pm 1) = 0$ and $W(r) > 0$, otherwise and define the energy via:

$$E_0(u; \Omega) = \int_\Omega W(u(x))dx$$

where the function $u(x)$ represents the states of the substance at the point $x \in \Omega$.

One quickly realizes that this is minimized by any function that takes only the values ± 1 , $u(x) \in [-1, 1]$ with $u = -1$ and $u = +1$ correspoding to pure phases. In particular, the interface could be arbitrarily wild and yet the energy would not be affected.

Next thinking of surface tension energy that is related to the complexity of the interface one introduces a gradient term that penalizes the formation of interfaces and measures interface energy. This is the Van der Waals free energy functional. Surface tension is 2^{nd} order compared to the bulk energy ($\int W(u(x))$) and this explains the small parameter. Thus, we are looking at the energy:

$$E(u; \Omega) = \int_\Omega (\frac{\varepsilon^2}{2}|\nabla u(x)|^2 + W(u(x)))dx$$

where $\varepsilon > 0$ is small.

Such a gradient term indeed reduces the number interfaces of the minimizers of E, which turn out to be smooth functions interpolating between the pure phases "-1" and "+1" with level sets approaching hypersurfaces of least possible area. We explain all this below in greater detail.

Up to a scale dilatation, we may focus on the case $\varepsilon = 1$, so

$$E(u;\Omega) = \int_\Omega (\frac{1}{2}|\nabla u(x)|^2 + W(u(x)))dx$$

Note that since $E(u;\Omega) \geq 0$ and $E(\pm 1;\Omega) = 0$ we have that $u = \pm 1$ are trivial minimizers of E. We minimize E either subject to it's Dirichlet values or subject to a mass constraint, $\frac{1}{|\Omega|}\int_\Omega u(x)dx = 0$ for example (or $\frac{1}{|\Omega|}\int_\Omega u(x)dx = m \in (-1,1)$) (section 3) and thus the trivial minimizers are excluded.

We recall that if $\{u_\varepsilon\}$ is a family of global minimizers then along a subsequence $\|u_\varepsilon - u_0\|_{L^1} \to 0$ and

$$\lim_{\varepsilon\to 0} \frac{E(u_\varepsilon,\Omega)}{\varepsilon} = Per\{u_0 = 0\}$$

[24] that is the rescaled functionals Γ-converge to the perimeter functional and $\{u_0 = 0\}$ is a minimal surface. [1] We also recall the definition of Γ-convergence.

Definition 2.1 (Γ-convergence) Let X be a metric space, and for $\varepsilon > 0$ let be given $J_\varepsilon : X \to [0,+\infty]$. We say that J_ε Γ-*converge* to J on X as $\varepsilon \to 0$ if the following two conditions hold:

(LB) Lower bound inequality: for every $u \in X$ and every sequence (u_ε) s.t. $u_\varepsilon \to u$ in X there holds

$$\liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \geq J(u)$$

(UB) Upper bound inequality: for every $u \in X$ there exists (u_ε) s.t. $u_\varepsilon \to u$ in X and

$$\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = J(u)$$

[1] Γ-convergence was introduced by De Giorgi [24] and relates the *diffused interface* problem (P_ε) to the *sharp interface* problem (P_0).

Condition (LB) means that whatever sequence we choose to approximate u, the value of $J_\varepsilon(u_\varepsilon)$ is, in the limit, larger that $J(u)$; on the other hand condition (UB) implies that this bound is sharp, that is, there always exists a sequence (u_ε) which approximates u so that $J_\varepsilon(u_\varepsilon) \to J(u)$.

For studying three or more phases one naturally is lead to the vector case. $\{W = 0\} = \{a_1, ..., a_N\}$ the phases. The related functional in this case leads to minimal partitions of Ω.

Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is non–negative and for $\Omega \subset \mathbb{R}^n$ open and bounded define $J_\Omega : W^{1,2}(\Omega; \mathbb{R}^m) \to [0, +\infty]$ by

$$J_\Omega(v) = \int_\Omega (\frac{1}{2}|\nabla v|^2 + W(v))dx \tag{19}$$

In this section we will deal with bounded solutions of

$$\Delta u - W_u(u) = 0 \tag{20}$$

which are defined in an open set $\Omega \subset \mathbb{R}^n$, generally unbounded, and which are *minimal* (alternatively, *minimizers*) in the sense that they minimize, for each $U \subset \Omega$, the energy J_U ,subject to their Dirichlet values. More precisely,

Definition 2.2 (Minimality) Let $\Omega \subset \mathbb{R}^n$ open, a map $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ is called a *minimizer* or a *minimal map* if

$$J_U(u) \leq J_U(u + v) \ \ , \forall \ v \in W^{1,2}_0(U; \mathbb{R}^m) \cap L^\infty(U; \mathbb{R}^m)$$

for every open bounded Lipschitz set $U \subset \Omega$.

Note that, under sufficient smoothness of W, from the L^∞ bound on u and elliptic regularity it follows that a minimal map $u : \Omega \to \mathbb{R}^m$ is a classical solution of (20) which is the Euler–Lagrange equation associated to J_Ω.

We assume a gradient bound on u besides the L^∞ bound required in Definition 2.2

$$|u(x) - a| < M \ \ , \ |\nabla u(x)| < M \ \ on \ \ \Omega \ , \tag{21}$$

and set

$$W_M = \max_{|u-a|\leq M} W(u)$$

We note that the gradient bound follows from the L^∞ estimate on u under sufficient regularity on W, by linear elliptic theory. The basic estimate for minimal maps is given in

Lemma 2.1 ([2]) Let $W : \mathbb{R}^m \to \mathbb{R}$ be continuous, $W \geq 0$, and assume that $\{W = 0\} \neq \emptyset$. Let u be minimal, satisfying the estimates (21). Then there is a constant $\hat{C}_0 > 0, \hat{C}_0 = \hat{C}_0(W, M)$, independent of x_0 and such that

$$B_r(x_0) \subset \Omega \Rightarrow J_{B_r(x_0)}(u) \leq \hat{C}_0 r^{n-1} \ , \ for \ \ r > 0.$$

Proof.

From (21), $g(u) := \frac{1}{2}|\nabla u|^2 + W(u)$ is bounded on Ω and it follows

$$J_{B_r(x_0)}(u) \leq C_1 r^n \leq C_1 r^{n-1} \ , \ for \ \ r \leq 1 \tag{22}$$

for some $C_1 > 0$ independent of x_0. For $r > 1$ define $v : \Omega \to \mathbb{R}^m$ by

$$v(x) = \begin{cases} a \ , & for \ \ |x - x_0| \leq r - 1 \\ (r - |x - x_0|)a + (|x - x_0| - r + 1)u(x) \ , & for \ \ |x - x_0| \in (r - 1, r] \\ u(x) \ , & for \ \ |x - x_0| > r \end{cases} \tag{23}$$

This definition and the minimality of u over balls imply

$$J_{B_r(x_0)}(u) \leq J_{B_r(x_0)}(v) = J_{B_r(x_0)\setminus B_{r-1}(x_0)}(v) \leq C_2 r^{n-1} \ , \tag{24}$$

where we have also used that (24) and (21) imply that $g(v)$ is bounded on Ω. The lemma follows from (22) and (24) with $\hat{C}_0$ =max$\{C_1, C_2\}, \hat{C}_0$ is clearly independent of x_0 and depends on u only through the bound M.

□

2.2 The Density Estimate

Now, we will give a brief idea before stating an important estimate introduced by Caffarelli and Cordoba for the scalar case [16] and extended to the vector case in [9], which complements Γ-convergence. More precisely Γ-convergence provides only L^1 convergence of u_ε to u_0 along a sequence which is very weak for controlling the level sets. The density estimate provides uniform convergence of the level sets. We begin with the sharp interface analog of the density estimate. Consider a minimal surface $\Sigma^{n-1} = \partial D$. Let $x \in \Sigma^{n-1}$, the surface Σ^{n-1} partitions the ball $B_r(x)$ into two parts, D_r and D_r^c ($D_r = D \cap B_r$ and $D_r^c = D^c \cap B_r$). Let $V(r) = \mathcal{L}^n(D_r)$,
$A(r) = \mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r)$, $\mathcal{H}^n$ the n–dimensional Hausdorff measure and S_r the spherical cup bounding D_r. Consider the following formal computation:

$$\begin{aligned} V(r) &\leq C(\mathcal{H}^{n-1}(\Sigma^{n-1} \cap B_r) + \mathcal{H}^{n-1}(S_r))^{\frac{n}{n-1}} \ \ , \textit{by the isoperimetric inequality}, \\ &\leq C(2\mathcal{H}^{n-1}(S_r))^{\frac{n}{n-1}} \ \ , \textit{by minimality since } \partial(\Sigma^{n-1} \cap B_r) = \partial S_r \ , \\ &\leq C(V'(r))^{\frac{n}{n-1}} \ \ , \textit{by the coarea formula (for instance } [1, \textit{ Appendix } C]). \end{aligned}$$

$$\Rightarrow (V(r))^{\frac{n-1}{n}} \leq \hat{C} V'(r) \tag{25}$$

From (25), it follows that

$$V(r) \geq \tilde{C} r^n \ \ , \tilde{C} = \tilde{C}(n), \ \ \forall r > 0. \tag{26}$$

The estimate (26) expresses the fact that both D and D^c have uniform positive density at each x, all the way from $r = 0$ to $r = \infty$:

$$0 < \lambda_1 \leq \frac{\mathcal{L}^n(D \cap B_r(x))}{\mathcal{L}^n(B_r(x))} \leq \lambda_2 < 1 \tag{27}$$

Our interest in (27) is at $r = \infty$, which relates to Bernstein type theorems. The estimate at $r = 0$ leads to regularity results. We recall that minimal sets of codimension 1 in $\mathbb{R}^n$ can be conveniently viewed as boundaries of minimizing partitions of open sets in $\mathbb{R}^n$. The point is that the partition P of a set U can be identified with a piecewise constant function g on U, the norm of P equals $\int_U |g(x)| dx$, and the perimeter of P, which we seek to minimize, equals $\|g\|_{BV}$, the BV norm of g, and coincides with $\mathcal{H}^{n-1}(\partial P \cap U)$. The sets of finite perimeter are those for which $\|g\|_{BV} < \infty$.

The analogy with the diffuse interface problem is via the identification

$$A(r) = \int_{B_r \cap \{|u-a| \le \lambda\}} W(u)dx \ , \quad V(r) = \mathcal{L}^n(B_r \cap \{|u-a| > \lambda\}) \ , \qquad (28)$$

where a is a phase, $W(a) = 0$, and $\lambda > 0$ is any number such that

$$d_0 = dist(a, \{W = 0\} \setminus \{a\}) \ge \lambda \qquad (29)$$

The interface corresponding to phase a is measured by the set close to a where W does not vanish, while $V(r) = \mathcal{L}^n(B_r \cap \{|u-a| > \lambda\})$ measures the volume of the set where u is close to $\{W = 0\} \setminus \{a\}$. The more singular the potential W, the less diffused the interface, and the easier the derivation of the density estimates, as it gets closer to the argument above. The basic estimate in Lemma 2.1 above is essential for localizing the (diffuse) interface, and making the specific value of $\lambda \in (0, d_0)$ irrelevant.

We consider nonnegative potentials $W \in C(\mathbb{R}^m; [0,\infty))$ with $\{W = 0\} \ne \emptyset$. Let $W(a) = 0$. We model W near a after $|u-a|^\alpha$, and thus the following hypothesis:

$$\mathbf{H} \begin{cases} 0 < \alpha < 2 : W \ \ is \ \ differentiable \ \ in \ \ a \ \ deleted \ \ neighborhood \ \ of \ \ \alpha \\ and \ \ satisfies \ \ \frac{d}{d\rho}W(a+\rho\xi) \ge \alpha C^* \rho^{\alpha-1} \ \ , \forall \rho \in (0,\rho_0] \ \ , \forall \xi \in \mathbb{R}^m : |\xi| = 1 \ , \\ for \ \ some \ \ constants \ \ \rho_0 > 0 \ , \ C^* > 0 \ \ independent \ \ of \ \ \alpha. \\ \alpha = 2 : W \ \ is \ \ C^2 \ \ in \ \ a \ \ neighborhood \ \ of \ \ a \ , \ \ and \\ c_0 \le \xi^T W_{uu}(u)\xi \le c_0' \ , \forall u : |u-a| \le q_0, \ \forall \xi : |\xi| = 1, \\ for \ \ some \ \ constants \ \ q_0 > 0, \ \ c_0' > c_0 > 0. \end{cases}$$

We note that for all $\alpha \in (0,2] \ , s \mapsto W(a + s\xi)$ is increasing near $s = 0$. In addition, for $\alpha \in [1,2] \ , W$ is also convex near $u = a$. Also note that $\mathbf{H}$ implies that a is isolated in $\{W = 0\}$, hence

$$d_0 = \min_{\xi}\{|a-z| : z \ne a, \ \ W(z) = 0\} > 0$$

Theorem 2.1 ([2]) Assume W satisfies hypothesis $\mathbf{H}$, Ω is open, $n \ge 1$, and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is minimal as in Definition 2.2. Then for any $\mu_0 > 0$ and any $\lambda \in (0, d_0)$, the condition

$$\mathcal{L}^n(B_{r_0}(x_0) \cap \{|u-a| > \lambda\}) \ge \mu_0 \qquad (30)$$

implies

$$\mathcal{L}^n(B_r(x_0) \cap \{|u-a| > \lambda\}) \geq Cr^n \ \ , \forall\, r \geq r_0\ , \tag{31}$$

as long as $B_r(x_0) \subset \Omega$, where $C = C(W, \mu_0, \lambda, r_0, M)$.

Proof.

[2,p.147-161]

Remark: It is a simple consequence of the basic estimate (Lemma 2.2) that the validity of the theorem for one of $\lambda \in (0, d_0)$, implies its validity for all $\lambda' \in (0, d_0)$. Indeed suppose

$$\mathcal{L}^n(B_{r_0'}(x_0) \cap \{|u-a| > \lambda'\}) \geq \mu_0' > 0$$

It is enough to check for $\lambda' \in (\lambda, d_0)$. Set $w_\lambda^{\lambda'} = \min\limits_{q \in [\lambda,\lambda'], \nu \in \mathbb{S}^{m-1}} W(a + q\nu) > 0$. Then,

$$w_\lambda^{\lambda'} \mathcal{L}^n(B_r(x_0) \cap \{\lambda < |u-a| \leq \lambda'\}) \leq J_{B_r(x_0)}(u) \leq C_0 r^{n-1}$$

by Lemma 2.2. Thus

$$\begin{aligned} &\mathcal{L}^n(B_r(x_0) \cap \{|u-a| > \lambda'\}) = \\ &= \mathcal{L}^n(B_{r_0'}(x_0) \cap \{|u-a| > \lambda\}) - \mathcal{L}^n(B_r(x_0) \cap \{\lambda < |u-a| \leq \lambda'\}) \\ &\geq Cr^n - \frac{C_0}{w_\lambda^{\lambda'}} r^{n-1} \ \ , \ for\ \ r \geq r_0 \end{aligned}$$

and therefore

$$\mathcal{L}^n(B_r(x_0) \cap \{|u-a| > \lambda'\}) \geq \frac{C}{2} r^n \ \ , \ \ for\ \ r \geq \overline{r} := max\{r_0, \frac{2C}{w_\lambda^{\lambda'} C_0}\}$$

This proves the claim with $C' = \frac{C}{2}$ if $\overline{r} \leq r_0'$. Otherwise we conclude by observing that

$$\mathcal{L}^n(B_{r_0'}(x_0) \cap \{|u-a| > \lambda'\}) \geq \mu_0' \geq \frac{\mu_0'}{\overline{r}^n} r^n \ \ , \ for\ \ r \in [r_0', \overline{r})$$

and by setting $C' = min\{\frac{C}{2}, \frac{\mu_0'}{\overline{r}^n}\}$. Note that here we take λ in (28),(29) strictly less than the distance from the rest of the minima of W.

2.3 Singular Potentials- The free boundary case

In this subsection we focus on the hypothesis **H** for W in the case $0 < \alpha < 2$ and also assume that there exist a minimal solution for the system (20) (and $a \in \{W = 0\}$).

We start off with a useful calculation. From the hypothesis for W we have that for $|u - a| << 1$, holds that $W_u(u) \cdot (u - a) \geq c^2|u - a|^\alpha$ and set $v(x) = |u - a|^2$.

$$\Delta v = \sum_{i=1}^{n} 2((u(x) - a)u_{x_i})_{x_i} = 2|\nabla u|^2 + 2(u(x) - a)\Delta u = \\ 2|\nabla u|^2 + 2W_u(u) \cdot (u(x) - a) \geq 2|\nabla u|^2 + 2c^2|u - a|^\alpha \tag{32}$$

Therefore,

$$\Delta v \geq 2c^2|u - a|^\alpha = 2c^2 v^{\frac{\alpha}{2}} \tag{33}$$

Let us now state some theorems from [13] that will be used for the proof of our results.

The article [13] is concerned with the problem

$$\begin{cases} \Delta u = c^2 u^p \ \ in \ \ \Omega \subset \mathbb{R}^n \\ u = 1 \ \ \ on \ \ \partial\Omega \end{cases} \tag{34}$$

with $p \in (0, 1)$. We call that a "dead core" Ω_0 develops in Ω, i.e. a region where $u \equiv 0$.

Let $X(s)$ be a solution of

$$\begin{cases} X''(s) = c^2 X^p(s) \ \ in \ \ (0, s_0) \\ X'(0) = 0 \ \ , X(s_0) = 1 \end{cases} \tag{35}$$

As a first choice of a linear problem consider the "torsion problem" , i.e.

$$\begin{cases} \Delta\psi + 1 = 0 \ \ in \ \ \Omega \\ \psi = 0 \ \ in \ \ \partial\Omega \end{cases} \tag{36}$$

One then constructs a supersolution $\overline{u}(x)$ to (34) having the same level lines as the torsion function by setting

$$\overline{u}(x) = X(s(x)) \ , \ \ x \in \Omega \tag{37}$$

where

$$s(x) = \sqrt{2(\psi_m - \psi(x))} \ , \psi_m = \max_{\Omega} \psi \tag{38}$$

In problem (35) we choose $s_0 = \sqrt{2\psi_m}$.

Theorem 2.2 ([13]) Assume that the mean curvature of $\partial\Omega$ is nonnegative everywhere. Then

$$\begin{gathered} \overline{u}(x) = X(s(x)) \ \ is \ \ a \ \ supersolution, \ \ i.e. \\ \Delta\overline{u} \leq c^2 \overline{u}^p \ \ in \ \ \Omega \\ \overline{u} = 1 \ \ on \ \ \partial\Omega \end{gathered} \tag{39}$$

One of the corollaries of this Theorem is the information on the location and the size of the "dead core" Ω_0, which may be stated as

Corollary 2.1 ([13]) The dead core Ω_0 contains the set

$$\{x \in \Omega | \psi(x) \geq d(p,c)[\sqrt{2\psi_m} - \frac{1}{2} d(p,c)]\} \ ,$$

where $d(p,c) := \dfrac{\sqrt{2(p+1)}}{(1-p)c}$.

Next, utilizing the results above we will prove the following lemmas.

Lemma 2.2 Let $\Omega = B_R(x_0) \subset \mathbb{R}^n$ and $v \in C^2(\Omega; \mathbb{R}_+)$ satisfy the following assumptions:

$$\begin{gathered} \Delta v(x) \geq c^2 v^{\frac{\alpha}{2}}(x) \ \ , \ x \in \Omega \\ v(x) \leq \delta \ \ , \ x \in \partial\Omega \end{gathered} \tag{40}$$

$\alpha \in (0,2) \Leftrightarrow \frac{\alpha}{2} = p \in (0,1)$.
Then if $y_0 \in \Omega$ such that $dist(y_0, \partial\Omega) > R_0 \Rightarrow v(y_0) = 0$.

where $R_0 := \begin{cases} \sqrt{n}d(p,c) & , R \geq \sqrt{n}d(p,c) \\ 2R - \sqrt{n}d(p,c) & , \frac{1}{2}\sqrt{n}d(p,c) < R < \sqrt{n}d(p,c) \end{cases}$.

Proof.

From the maximum principle we have that $v(x) \leq \delta$ in Ω
Define $\hat{v} := \dfrac{v}{\delta}$ and $\hat{c} := \dfrac{c}{\delta^{\frac{1-p}{2}}}$, then we have:

$$\begin{cases} \Delta\hat{v}(x) \geq \hat{c}^2\hat{v}^{\frac{\alpha}{2}}(x) \ , \ x \in \Omega. \\ \hat{v}(x) \leq 1 \ , \ x \in \partial\Omega \end{cases}$$

For $\Omega = B_R(x_0)$ we have that

$$\psi(x) = \frac{R^2}{2n} - \frac{1}{2n}|x - x_0|^2 \ , \ \psi_m = \frac{R^2}{2n} \tag{41}$$

is a solution to the problem:

$$\begin{cases} \Delta\psi(x) + 1 = 0, \ x \in \Omega \\ \psi(x) = 0 \ , \ x \in \partial\Omega \end{cases} \tag{42}$$

Also, we have that if:

$$\begin{cases} \Delta u \leq c^2u^p, \ x \in \Omega \\ \Delta v \geq c^2v^p, \ x \in \Omega \\ v \leq u \ , \ x \in \partial\Omega \end{cases} \tag{43}$$

then $v \leq u$, in Ω. So since $u, v \geq 0$, if $u(x_1) = 0 \Rightarrow v(x_1) = 0$.
Such $\overline{u}$ is defined in [13] via ψ in Theorem 2.2 (supersolution with $u = 1 \geq \hat{v}$ on the boundary). Then by Corollary 2.1 the dead core of $\overline{u}$ contains the set $\{x \in \Omega | \psi(x) \geq C_0 := d(p,c)[\frac{R}{\sqrt{n}} - \frac{1}{2}d(p,c)]\}$, that is if
$y_0 \in \{\psi(x) \geq C_0\} \Rightarrow \overline{u}(y_0) = 0$ and thus $\hat{v}(y_0) = v(y_0) = 0$.
Since ψ has the form (41) we can see that

$$\{x \in \Omega | \psi(x) \geq C_0\} = \{dist(x, \partial\Omega) \geq R_0\}$$

as follows:

$$\psi(x) \geq C_0 \Leftrightarrow \frac{R^2}{2n} - \frac{1}{2n}|x-x_0|^2 \geq C_0 \Leftrightarrow \sqrt{R^2 - 2nC_0} \geq |x - x_0|$$

$$\Leftrightarrow R - |x-x_0| \geq R - \sqrt{R^2 - 2nC_0} = R - \sqrt{R^2 - 2\sqrt{n}d(p,c)R + n(d(p,c))^2} =$$
$$= R - |R - \sqrt{n}d(p,c)| = R_0$$

and notice that: $dist(x, \partial\Omega) = dist(x, \partial B_R(x_0)) = R - dist(x, x_0)$

$\square$

<u>Notes:</u> (1) $\hat{c}$ depends on δ and tends to infinity as δ tends to zero.
(2) $d(p,\hat{c})$ tends to zero as δ tends to zero, and so does C_0.
(3) $d(p,\hat{c})$ tends to a finite limit ($= \frac{\sqrt{2}}{\hat{c}}$) as p tends to zero. Hence the estimate appears uniform all the way down to $p = 0$.

Remark: If we take $\tilde{\Omega}$ open set, such that $B_R(x_0) \subset \tilde{\Omega}$ and

$$\begin{cases} \Delta\tilde{\psi}(x) + 1 = 0, \ x \in \tilde{\Omega} \\ \tilde{\psi}(x) = 0 \ \ , \ x \in \partial\tilde{\Omega} \end{cases}$$

then, we have: $\psi \leq \tilde{\psi} \Rightarrow \{\psi(x) \geq C_0\} \subset \{\tilde{\psi}(x) \geq C_0\} \Rightarrow \{x \in B_R(x_0) : dist(\partial B_R(x_0), x) \geq R_0\} \subset \{\tilde{\psi}(x) \geq C_0\}$.
Thus, the above theorem holds for more general open sets (open sets that contain a ball $B_R(x_0)$).

Lemma 2.3 Let D open, convex $\subset \mathbb{R}^n$ and for some $d_0 > 0$,
$\Omega := \{x \in D : dist(x, \partial D) \geq d_0\}$ and let $v \in C^2(D; \mathbb{R}_+)$ satisfying:

$$\begin{aligned} \Delta v(x) \geq c^2 v^{\frac{\alpha}{2}}(x) \ \ , \ x \in \Omega \\ v(x) \leq \delta \ \ , \ x \in \Omega \end{aligned} \tag{44}$$

$\alpha \in (0,2) \Leftrightarrow \frac{\alpha}{2} = p \in (0,1)$.
Then if $x_0 \in D$ such that $dist(x_0, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)c} \Rightarrow v(x_0) = 0$.

Proof.

We have that:

$$\{x \in D : dist(x, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)c}\} = \{x \in \Omega : dist(x, \partial\Omega) \geq 2\frac{\sqrt{2n(p+1)}}{(1-p)c}\}$$

and Ω is convex (parallel sets have at the same side of supporting planes).
Let $x_0 \in D$ such that $dist(x_0, \partial D) \geq d_0 + 2\frac{\sqrt{2n(p+1)}}{(1-p)c}$. Since $dist(\partial D, \partial\Omega) = d_0 \Rightarrow dist(x_0, \partial\Omega) \geq 2\frac{\sqrt{2n(p+1)}}{(1-p)c}$ and since Ω is convex there exist a ball $B_R(x_0) \subset \Omega$ for $R = 2\frac{\sqrt{2n(p+1)}}{(1-p)c} = 2\sqrt{n}d(p,c) > R_0 = \sqrt{n}d(p,c)$, $d(p,c)$ as defined above.
Therefore we can apply Lemma 2.2 in the ball $B_R(x_0)$ and we have that $v(x) = 0 \, , \forall x \in B_{R_0}(x_0) = \{x \in B_R(x_0) : dist(\partial B_R(x_0), x) \geq R_0\} \Rightarrow v(x_0) = 0$.

$\square$

Next, we are going state our main Theorem that can be proved with one of the main ingredients being the Lemma 2.3. Before that, we give some notation and definitions.

A *Coxeter group*, or more simply a *reflection group* G, is a finite subgroup of the orthogonal group $O(\mathbb{R}^n)$, generated by a set of reflections. The notation $|G|$ stands for the order of G, that is, the number of elements of G. For the rest of this subsection we assume that the same reflection group G acts both on the domain space $\mathbb{R}^n$ or $B_R \subset \mathbb{R}^n$ and on the target space $\mathbb{R}^m$, and take $n = m$ and thus we consider maps $u : B_R \subset \mathbb{R}^n \to \mathbb{R}^n$. If G is a reflection group acting on $\mathbb{R}^n$, a *reflection* $\gamma \in G$ is a map $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\gamma x = x - 2(x \cdot \eta_\gamma)\eta_\gamma \;\; , x \in \mathbb{R}^n$$

for some unit vector $\eta_\gamma \in \mathbb{S}^{n-1}$ which, aside from its orientation, is uniquely determined by γ. The hyperplane

$$\pi_\gamma = \{x \in \mathbb{R}^n : x \cdot \eta_\gamma = 0\},$$

is the set of the points that are fixed by γ. The open half space $S_\gamma^+ = \{x \in \mathbb{R}^n : x \cdot \eta_\gamma = 0\}$ depends on the orientation of η_γ. We let $\Gamma \subset G$ denote set of all reflections in G. Every finite subgroup of $O(\mathbb{R}^n)$ has a *fundamental region*, that is, a subset $F \subset \mathbb{R}^n$ with the properties:

1.F is open and convex,
2.$F \cap gF = \emptyset$, for $I \neq g \ \in G$, where I is the identity,
3.$\mathbb{R}^n = \bigcup\{g\overline{F} : g \in G\}$.

The set $\cup_{\gamma\in\Gamma}\pi_\gamma$ divides $\mathbb{R}^n \setminus \bigcup_{\gamma\in\Gamma}\pi_\gamma$ in exactly $|G|$ congruent conical regions. Each one of these regions can be identified with the fundamental region F for the action of G on $\mathbb{R}^n$. We assume the orientation of η_γ is such that $F \subset S_\gamma^+$ and we have

$$F = \bigcap_{\gamma\in\Gamma} S_\gamma^+$$

Given $a \in \mathbb{R}^n$, the *stabilizer* of a, denoted by $G_a \subset G$, is the subgroup of the elements $g \in G$ that fix a:

$$G_a = \{g \in G : ga = a\}.$$

The Hypothesis of the Theorem

H1 (N nondegenerate global minima) The potential W is differentiable in a deleted neighborhood of α and satisfies $W(a_i) = 0$, for $i = 1, ..., N$ and $W > 0$ on $\mathbb{R}^n\setminus\{a_1, ..., a_N\}$. Furthermore, there holds $\frac{d}{d\rho}W(a+\rho\xi) \geq \alpha C^*\rho^{\alpha-1}$, $\forall\, \rho \in (0, \rho_0]$, $\forall\, \xi \in \mathbb{R}^m : \ |\xi| = 1$, for some constants $\rho_0 > 0, C^*$ independent of α.
H2 (Symmetry) The potential W is invariant under a finite reflection group G acting on $\mathbb{R}^n$, that is,

$$W(gu) = W(u) \ \ , \forall\, g \in G \ \ \textit{and} \ \ u \in \mathbb{R}^n.$$

Moreover, there exists $M > 0$ such that $W(su) \geq W(u)$, for $s \geq 1$ and $|u| = M$. We seek equivariant solutions of system (20), that is, solutions satisfying

$$u(gx) = gu(x) \ \ , \forall\, g \in G \ \ \textit{and} \ \ x \in \mathbb{R}^n.$$

H3 (Location of global minima) Let $F \subset \mathbb{R}^n$ be a fundamental region of G. We assume that $\overline{F}$ (the closure of F) contains a single global minimum of W, say a_1, and let G_{a_1} be the stabilizer of a_1. Setting $D := Int(\bigcup_{g \in G_{a_1}} g\overline{F})$, a_1, is also the unique global minimum of W in the region D.

Notice that, by the invariance of W, Hypothesis **H3** implies that the number of minima of W is

$$N = \frac{|G|}{|G_{a_1}|}$$

Theorem 2.3 ([A,G,Z (in preparation)]) Under Hypotheses **H1-H3**, there exists an equivariant classical solution to system (20) and a $d_0 = d_0(M)$, such that
1. $u(x) = a_1$, for $x \in D := Int(\bigcup_{g \in G_{a_1}} g\overline{F})$ with $dist(x, \partial D) \geq d_0$
2. $u(\overline{F}) \subset \overline{F}$ and $u(D) \subset D$.

Remarks:
1) The free boundary is defined to be the boundary of the set $\{x \in D : |u - a_1| > 0\}$.
2) The case where $\alpha = 2$ in **(H)**, it holds a similar result in [2, p.186] where instead of 1., there is an asyptotic estimate of the form $|u - a_1| \leq Ke^{-kdist(x,\partial D)}$.

Outline of the Proof

The proof proceeds in several steps. We begin by minimizing

$$J_{B_R}(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 + W(u))dx$$

over balls B_R centered at the origin, and then taking the limit

$$u(x) = \lim_{R \to \infty} u_R(x) \quad u \in W_E^{1,2}(B_R; \mathbb{R}^n)$$

along subsequences of minimizers u_R, we denote by $W_E^{1,2}(B_R; \mathbb{R}^n) \subset W^{1,2}(B_R; \mathbb{R}^n)$ the subspace of equivariant maps (as defined in **H2**). Minimizing over compact sets is forced by the fact that the action evaluated over $\mathbb{R}^n$ is finite only for trivial, constant maps ([2, Theorem 3.5, p.100]). Minimizing in the equivariant class does not affect the Euler– Lagrange equation (by classical facts) and relatively easily renders the estimate $J_{B_r}(u_R) \leq Cr^{n-1}$, $0 < r < R-1$. This estimate implies the existence of a nontrivial solution $u(x)$ in the equivariant class under only Hypotheses **H1** and **H2**, and very mild regularity assumptions on W, and also very mild nondegeneracy hypotheses on a_i. To obtain information on the behavior of the solution, we introduce the notion of positivity, $u(\overline{F}) \subset \overline{F}$, as a constraint in the minimization process. This, in principle, could affect the Euler–Lagrange equation. It is shown that the associated gradient flow with Neumann condition on B_R preserves positivity, and since it reduces J_{B_R}, we conclude that positivity is a removable constraint.

By Hypothesis **H3**, there is a unique minimum a_1 of W in $\overline{F}$. Thus, the aforementioned estimate $J_{B_r}(u_R) \leq Cr^{n-1}$, with $r \in (0, R-1)$ (which also holds under the positivity constraint), implies easily that $\mathcal{L}^n(A_{\overline{q}} \cap B_r) \leq Kr^{n-1}$, where $A_{\overline{q}} = \{x \in F : |u(x) - a_1| \geq \overline{q}\}$, $\overline{q}$, and arbitrary otherwise. Using this estimate and the density estimate, Theorem 3.1, which holds for W with the **H1** assumption jointly with [Alikakos,Zarnescu (in preparation)], the solution in most of D is close to a_1 and more precisely that, given $\delta_0 > 0$, there is $\hat{d}_0 > 0$, depending on δ_0, such that

$$|u(x) - a_1| \leq \delta_0 \quad \forall x \in D, \;\; dist(x, \partial D) \geq \hat{d}_0 \tag{45}$$

A sketch of the proof of the inequality above is the following: By contradiction, suppose that $\nexists$ such $\hat{d}_0$. Hence $\exists\ x_k$, $dist(x_k, \partial D) \to +\infty$ and with the property $|u(x_k) - a_1| \geq \delta_0 > 0$. By positivity, $u|_D$ is bounded away from $\{W = 0\} \setminus \{a_1\}$. Therefore, from the contradiction hypothesis and the density estimate we have $|\{x \in B(x_k, R_k) : |u(x) - a_1| \geq \frac{\delta_0}{2}\}| \geq CR_k^n$ and thus $\int_{B(x_k,R_k)} W(u) \geq C(\delta_0)R_k^n$ and we have contradiction by the basic estimate. Finally, the inequality above (45) , (33) , and Lemma 2.3 imply that $u(x) = a_1$ for $x \in D\ : dist(x, \partial D) \geq d_0$.

3 The mass constraint case

3.1 Introduction

We assume:

(H1) $W : \mathbb{R}^m \to \mathbb{R}$ is nonnegative and satisfies

$0 = W(a_j) < W(u)$, for $\quad j = 1, ..., N \quad$ and $u \in \mathbb{R}^m \setminus \{a_1, ..., a_N\}$ for some $N \geqslant 1$ and some $a_1, ..., a_N \in \mathbb{R}^m$, $a_i \neq a_j$ for $i \neq j$. Moreover W is C^2 in a neighborhood of a_j ,where a_j , $j = 1, ..., N$ are nondegenerate zero of W and $c_0 \leq \xi^T W_{uu}(u)\xi \leq c_0'$,$\forall u : |u - a| \leq q_0$, $\forall \xi : |\xi| = 1$,
for some constants $q_0 > 0$, $c_0' > c_0 > 0$ and

$$\text{span}(\{a_1, ..., a_N\}) = \mathbb{R}^m.$$

We also assume that there is $M > 0$ such that

$$W(su) \geq W(u)\ , \ \text{ for } s \geq 1 \text{ and } |u| = M.$$

<u>Notation:</u> We use $|\cdot|$ for both the Lebesgue measure and the Euclidean or the matrix norm.

For later reference we note that from (H1) we have

Lemma 3.1 There exist positive constants q, d, c_1, C_1 such that

$$u \in \mathbb{R}^m \setminus \cup_j B_q(a_j) \Rightarrow W(u) \geq d\ ,$$

$$u \in B_q(a_j) \Rightarrow \begin{cases} \frac{1}{2}c_1|u - a_j|^2 \leq W(u) \\ |W_u(u)| \leq C_1|u - a_j|. \end{cases}$$

Proof.
The first inequality hold from the continuity of W and the fact that u is

bounded and the other inequalities turn out from the Taylor expansion of W, W_{u_i} near a_j.

$$W(u) = W(a_j) + W_u(a_j) \cdot (u - a_j) + \frac{1}{2}\langle W_{uu}(a_j) \cdot (u - a_j), u - a_j\rangle + o(|u - a_j|^2)$$
$$\Rightarrow W(u) = \frac{1}{2}\langle W_{uu}(a_j) \cdot (u - a_j), u - a_j\rangle + o(|u - a_j|^2) \geq \frac{1}{2}c_1|u - a_j|^2$$

and also,

$$W_u(u) = W_u(a_j) + W_{uu}(a_j) \cdot (u - a_j) + o(|u - a_j|)$$
$$\Rightarrow |W_u(u)| \leq |W_{uu}(a_j)||u - a_j| + o(|u - a_j|) = |u - a_j|(|W_{uu}(a_j)| + \frac{o(|u - a_j|)}{|u - a_j|})$$
$$\leq C_1|u - a_j|$$

□

For each $R > 1$ we let $u_R : B_R \subset \mathbb{R}^n \to \mathbb{R}^m$ a minimizer of the problem:

$$J_{B_R}(u_R) = \min_{\mathcal{A}^R} J_{B_R}(v)\ ,\ \ J_{B_R}(v) = \int_{B_R} (\frac{1}{2}|\nabla v|^2 + W(v))dx \qquad (46)$$
$$\mathcal{A}^R := \{v \in W^{1,2}(B_R; \mathbb{R}^m) : \frac{1}{|B_R|}\int_{B_R} vdx = \hat{m}\} \qquad (47)$$

where $\hat{m} \in \mathrm{Conv}(\{a_1, ..., a_N\})$ is a suitable vector and $\hat{m} \neq a_j\ , j = 1, ..., N$ and also, without loss of generality suppose that $|u_R| \leq M$ (we can use the fact that (H1) implies that we can produce a minimizer u_R which in addition satisfies the estimate $|u_R(x)| \leq M$ as proved in [2, p.188]).

Proposition 3.1 The minimizer u_R is a smooth solution of the system

$$\Delta u = W_u(u) - \frac{1}{|B_R|}\int_{B_R} W_u(u)dx\ ,\ \ x \in B_R \qquad (48)$$

Proof.

We have that the Euler–Lagrange equations of J_{B_R} subject to the constraint $\frac{1}{|B_R|}\int_{B_R} udx = \hat{m}$,is the following PDE system:

$$\Delta u - W_u(u) + \lambda = 0$$

where λ is the Lagrange multiplier ($u = (u_1, ..., u_m)$).

Integrating the system over B_R we have that

$$\lambda = -\frac{1}{|B_R|}[\int_{B_R} (\Delta u - W_u(u))dx] = -\frac{1}{|B_R|}(\int_{B_R} (\Delta u_1 - W_{u_1}(u))dx, ..., \int_{B_R} (\Delta u_m - W_{u_m}(u))dx)$$

If u is a minimizer of J_{B_R} subject to the constraint $\frac{1}{|B_R|}\int_{B_R} udx = \hat{m}$, we have that $\forall \phi \in C^\infty(B_R; R^m) : \int_{B_R} \phi(x)dx = 0$ $(= (0, ..., 0))$, it holds:

$$\Rightarrow \frac{d}{d\delta}|_{\delta=0} J(u + \delta\phi) = 0 \Leftrightarrow \int_{B_R} (\nabla u \cdot \nabla\phi + W_u(u)\phi)dx = 0 \qquad (49)$$

where $\nabla u = (\frac{\partial u_i}{\partial x_j})^{1\leq i\leq m}_{1\leq j\leq n}$ and $\nabla u \cdot \nabla \phi = \sum_{1\leq j\leq n} \sum_{1\leq i\leq m} \frac{\partial u_i}{\partial x_j}\frac{\partial \phi_i}{\partial x_j}$

So from (49) and $\int_U v\Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} v dS - \int_U \nabla u \cdot \nabla v dx$ we have that:

$$\int_{\partial B_R} \frac{\partial u}{\partial \nu}\phi - \int_{B_R} \Delta u\phi + \int_{B_R} W_u(u)\phi = 0 \ \ ,\forall \phi \in C^\infty(B_R; R^m) : \int_{B_R} \phi(x)dx = 0$$

$$\Rightarrow \int_{\partial B_R} \frac{\partial u}{\partial \nu}\phi + \lambda \int_{B_R} \phi = 0 \Rightarrow \int_{\partial B_R} \frac{\partial u}{\partial \nu}\phi = 0 \ \ ,\forall \phi \in C^\infty(B_R; R^m) : \int_{B_R} \phi(x)dx = 0$$

and thus: $\frac{\partial u}{\partial \nu}|_{\partial B_R} = 0$ and $\int_{B_R} \Delta u dx = \int_{\partial B_R} \frac{\partial u}{\partial \nu} = 0$.

Therefore, $\lambda = \frac{1}{|B_R|}\int_{B_R} W_u(u)dx$

$\square$

Elliptic theory implies that u_R for some $K > 0$ and $a \in (0, 1)$ independent of $R > 1$, it results:

$$\|u_R\|_{C^{2,a}(B_R;\mathbb{R}^m)} < K \qquad (50)$$

To see this, we state the Theorem 4.8 (p.62) in [4].

Theorem 3.1 Let $u \in C^2(\Omega)$, $f \in C^a(\Omega)$ satisfy: $\Delta u = f$ in an open set Ω of $\mathbb{R}^n$. Then:

$$|u|^*_{2,a;\Omega} \leq C(|u|_{0;\Omega} + |f|^{(2)}_{0,a;\Omega})$$

where $C = C(n, a)$(independent of Ω).

We note that $|u|^*_{k;\Omega}$ and $|u|^*_{k,a;\Omega}$ are norms on the subspaces $C^k(\Omega)$ and $C^{k,a}(\Omega)$ respectively for which they are finite. The classical norms of $C^k(\Omega), C^{k,a}(\Omega)$ are:

$$\|u\|_{C^k(\Omega)} := |u|_{k;\Omega} := \sum_{j=0}^{k} [u]_{j,0;\Omega} = \sum_{j=0}^{k} |\nabla^j u|_{0;\Omega}$$

$$[u]_{k,0;\Omega} := |\nabla^k u|_{0;\Omega} = \sup_{|b|=k} \sup_{\Omega} |\nabla^b u|$$

$$\|u\|_{C^{k,a}(\Omega)} := |u|_{k,a;\Omega} := |u|_{k;\Omega} + [u]_{k,a;\Omega} = |u|_{k;\Omega} + [\nabla^k u]_{a;\Omega}$$

$$[f]_{a;\Omega} := \sup_{x,y \in \Omega\ , x \neq y} \frac{|f(x) - f(y)|}{|x - y|^a}$$

If Ω is bounded and $d = diam\Omega$, then these norms are related. More precisely, we have that the norms $|u|^*_{k,a;\Omega}$, $\|u\|_{C^{k,a}(\Omega)}$ are equivalent:

$$|u|^*_{k,a;\Omega} \leq max(1, d^{k+a}) \|u\|_{C^{k,a}(\Omega)}$$

and if $\tilde{\Omega} \subset \Omega$ and $\sigma = dist(\tilde{\Omega}, \partial\Omega)$, then

$$min(1, \sigma^{k+a}) \|u\|_{C^{k,a}(\tilde{\Omega})} \leq |u|^*_{k,a;\Omega}$$

So, if we set $\Omega = B_{2R}$ and $\tilde{\Omega} = B_R$ we have that: $\|u\|_{C^{k,a}(B_R)} \leq |u|^*_{k,a;B_{2R}} < K$ from the Theorem above supposing that u and f are bounded.

Now we can apply this to the system (48) and we have the estimate (50).

Notes: (i) The above equation $\Delta u = f$ is the Poisson equation and $f = f(x)$ while in the PDE system (48) we have $f = f(u) = W_u(u) - \frac{1}{|B_R|} \int_{B_R} W_u(u) dx$ but we can set $g(x) = W_u(u(x)) - c$, $c = \frac{1}{|B_R|} \int_{B_R} W_u(u) dx$ since $u = u(x)$ and apply the Theorem.
(ii) Note that the Lagrange multiplier in the system (48) is a vector while in the Introduction we proved that it's a real number. This is due to the

constraint, in this problem we have a vector constraint.

We will also need the Ascoli-Arzela theorem:

Theorem 3.2 Let (X, ρ) be a compact metric space. Then each bounded and equicontinuous sequence in C(X) has a subsequence that converges uniformly.

The estimate (50) and the Ascoli-Arzela theorem implies (via diagonal argument) the existence of a subsequence $\{u_{R_k}\}_{k=0}^{\infty}$ that converges in $C^2_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ to a map $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$.

The main result is the following

Theorem 3.3 [G.Fusco,preprint] The map u defined above solves

$$\Delta u - W_u(u) = 0 \ , \quad u : \mathbb{R}^n \to \mathbb{R}^m \ . \tag{51}$$

and is minimal.

We recall that $u \in W^{1,2}_{loc}(\mathbb{R}^n;\mathbb{R}^m) \cap L^{\infty}(\mathbb{R}^n;\mathbb{R}^m)$, is *minimal* if

$$J_{\Omega}(u) \leq J_{\Omega}(u+v) \text{ , for } v \in W^{1,2}_0(\Omega;\mathbb{R}^m)$$

for every open bounded Lipschitz set $\Omega \subset \mathbb{R}^n$.

Theorem 3.3 is the main result in Fusco [G.Fusco, preprint]. Its proof proceeds in a series of lemmas the first of which is the, so called, "Basic Estimate", Lemma 3.2 below, which appears without proof in [G.Fusco, preprint]. Our main task in section 3 is to provide a proof. The analog of Lemma 3.2 in the case without constraint is well known (see page 23, Lemma 2.1). In the mass constraint case the proof is more demanding since the constructed energy comparison map has to satisfy the constraint.

3.2 Basic Lemmas

Lemma 3.2 There is a constant $C_0 > 0$ independent of $R > 1$ such that

$$J_{B_R}(u_R) \leq C_0 R^{n-1} \text{ , for } R > 1.$$

There are some facts that will be used for the proof of the Lemma 3.2.

• Let $\varphi \in C_c^\infty(\mathbb{R}^n)$: $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, $supp\varphi \subseteq \overline{B}_1$ ($\Rightarrow \varphi(x) = 0$, $|x| > 1$).
Set $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$, for $\varepsilon > 0$. Then $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$.

• Given $f \in L^1(\mathbb{R}^n)$,we define :

$$f_\varepsilon(x) = (f * \varphi_\varepsilon)(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)dy = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)dy \tag{52}$$

<u>Fact (1):</u>

$$\int_{\mathbb{R}^n} f_\varepsilon(x)dx = \int_{\mathbb{R}^n} f(x)dx \tag{53}$$

Proof.

$$\int_{\mathbb{R}^n} f_\varepsilon(x)dx = \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)dy)dx = \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x-y)\varphi_\varepsilon(y)dx)dy$$

$$= \int_{\mathbb{R}^n} \varphi_\varepsilon(y)(\int_{\mathbb{R}^n} f(x-y)dx)dy = \int_{\mathbb{R}^n} \varphi_\varepsilon(y)(\int_{\mathbb{R}^n} f(z)dz)dy$$

$$= (\int_{\mathbb{R}^n} f(x)dx)(\int_{\mathbb{R}^n} \varphi_\varepsilon(y)dy) = \int_{\mathbb{R}^n} f(x)dx$$

(the second equality follows from Fubini's theorem).

<u>Fact (2):</u> For every $f \in L^1(\mathbb{R}^n)$ and $\varphi \in C^k(\mathbb{R}^n)$, if φ^m bounded for $m = 1, ..., k$, then: $f * \varphi_\varepsilon \in C^k(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_i}(f * \varphi_\varepsilon)(x) = \int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial x_i}\varphi(x-y)dy \text{ , } i = 1, ..., n \text{ .} \tag{54}$$

Proof.

$$\frac{\partial}{\partial x_i}(f * \varphi_\varepsilon)(x) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^n} f(y)\varphi(x-y)dy = \int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial x_i}\varphi(x-y)dy$$

where the last equality follows from the dominated convergence theorem and the fact that: $|f(y)\frac{\partial \varphi}{\partial x_i}(x-y)| \le |f(y)| \sup(|\frac{\partial \varphi}{\partial x_i}(x-y)|)$ and $|f(y)| \sup(|\frac{\partial \varphi}{\partial x_i}(x-y)|) \in L^1(\mathbb{R}^n)$

Fact (3):

$$|\frac{\partial f_\varepsilon}{\partial x_i}| \le \frac{c}{\varepsilon} \, , f \in L^1 \cap L^\infty \tag{55}$$

Proof.

By Fact (2) $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial f_\varepsilon}{\partial x_i} = \int_{\mathbb{R}^n} f(y) \frac{\partial}{\partial x_i} \varphi_\varepsilon(x-y)dy = \int_{\mathbb{R}^n} f(y) \frac{\partial}{\partial x_i}(\frac{1}{\varepsilon^n}\varphi(\frac{x-y}{\varepsilon}))dy =$$

$$= \int_{\mathbb{R}^n} f(y) \frac{1}{\varepsilon^n} \frac{\partial \varphi}{\partial u_i}(\frac{x-y}{\varepsilon}))\frac{1}{\varepsilon}dy = -\frac{1}{\varepsilon} \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial u_i}(u) f(x-\varepsilon u)du$$

$$\Rightarrow |\frac{\partial f_\varepsilon}{\partial x_i}| \le \frac{1}{\varepsilon} \int_{\mathbb{R}^n} |\frac{\partial \varphi}{\partial u_i}(u)| \, |f(x-\varepsilon u)|du \le \|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\frac{\partial \varphi}{\partial u_i}(u)|du$$

$$\Rightarrow |\frac{\partial f_\varepsilon}{\partial x_i}| \le \frac{c}{\varepsilon}$$

Proof. (of Lemma 3.2.)
Given $\hat{m} \in \mathbb{R}^n$, $\hat{m} \in \text{Conv}(a_1, ..., a_N) \Rightarrow m = \sum_{i=1}^N t_i a_i$, $\sum_{i=1}^N t_i = 1$

We define:

$$\hat{v}(y) = \begin{cases} a_1 \, , B_{r_1} \\ a_2 \, , B_{r_2} \setminus B_{r_1} \\ \vdots \\ a_N \, , B_1 \setminus B_{r_{N-1}} \\ 0 \quad , \mathbb{R}^n \setminus B_1 \end{cases} \tag{56}$$

we choose $r_1, ..., r_{N-1}$ such that:

$$\frac{1}{|B_1|}\int_{B_1}\hat{v}(y)dy = \hat{m} \Leftrightarrow a_1\frac{|B_{r_1}|}{|B_1|} + a_2(\frac{|B_{r_2}| - |B_{r_1}|}{|B_1|}) + ... + a_N(\frac{|B_1| - |B_{r_{N-1}}|}{|B_1|}) = \sum_{i=1}^{N} t_i a_i$$

$$\Leftrightarrow \begin{cases} |B_{r_1}| = t_1|B_1| \\ |B_{r_2}| - |B_{r_1}| = t_2|B_1| \\ \vdots \\ |B_1| - |B_{r_{N-1}}| = t_N|B_1| \end{cases} \quad \Leftrightarrow \begin{cases} r_1 = t_1^{\frac{1}{n}} \\ r_2 = (t_1 + t_2)^{\frac{1}{n}} \\ \vdots \\ r_{N-1} = (t_1 + t_2 + ... + t_{N-1})^{\frac{1}{n}} = (1 - t_N)^{\frac{1}{n}} \end{cases}$$

So,

$$\int_{\mathbb{R}^n}\hat{v}(y)dy = \int_{B_1}\hat{v}(y)dy = \hat{m}|B_1| \tag{57}$$

• Define:

$$\hat{v}_\varepsilon(y) = (\hat{v} * \varphi_\varepsilon)(y) \tag{58}$$

Then, by Fact (1): $\int_{\mathbb{R}^n}\hat{v}_\varepsilon(y)dy = \int_{\mathbb{R}^n}\hat{v}(y)dy = \hat{m}|B_1|$ and $\hat{v} \in L^1 \cap L^\infty$

and by Fact (3) $\Rightarrow |\frac{\partial \hat{v}_\varepsilon}{\partial y_i}(y)| \leq \frac{c}{\varepsilon}$

We notice that $\hat{v}_\varepsilon(y) = \hat{v}(y)$ outside the $2\varepsilon-$ annuli centered about $|x| = r_1$:

Let $y : |y| \leq r_1 - \varepsilon$, then:

$$\hat{v}_\varepsilon(y) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\hat{v}(z)\varphi(\frac{y-z}{\varepsilon})dz = \frac{1}{\varepsilon^n}\int_{|y-z|\leq\varepsilon}\hat{v}(z)\varphi(\frac{y-z}{\varepsilon})dz \tag{59}$$

$$\Rightarrow \hat{v}_\varepsilon(y) = \int_{|y-z|\leq\varepsilon}\hat{v}(z)\varphi_\varepsilon(y-z)dz \tag{60}$$

$|y| \leq r_1 - \varepsilon \Rightarrow |z| \leq |y| + \varepsilon \leq r_1 \Rightarrow \hat{v}(z) = a_1$

So, from (60) we have:

$$\hat{v}_\varepsilon(y) = a_1\int_{|y-z|\leq\varepsilon}\varphi_\varepsilon(y-z)dz = a_1\int_{\mathbb{R}^n}\varphi_\varepsilon(y-z)dz = a_1 \tag{61}$$

Similarly, $\hat{v}_\varepsilon(y) = a_{i+1}$,in $r_i + \varepsilon \leq |y| \leq r_{i+1} - \varepsilon$, $i = 1, ..., N-2$ and $\hat{v}_\varepsilon(y) = a_N$, in $r_{N-1} + \varepsilon \leq |y| \leq 1 - \varepsilon$ and $\hat{v}_\varepsilon(y) = 0$, in $\mathbb{R}^n \setminus B_{1+\varepsilon}$.

Now if we define:

$$\widetilde{v}_{R+1}(x) = \hat{v}_\varepsilon(\frac{x}{R}) = \hat{v}_\varepsilon(y) \ , \ x = Ry = \frac{y}{\varepsilon} \tag{62}$$

Then:

$$\frac{1}{B_{R+1}} \int_{B_{R+1}} \widetilde{v}_{R+1}(x)dx = \frac{1}{B_{R+1}} \int_{B_{R+1}} \hat{v}_\varepsilon(\frac{x}{R})dx = \frac{R^n}{|B_{R+1}|} \int_{B_{1+\frac{1}{R}}} \hat{v}_\varepsilon(y)dy$$

$$= \frac{R^n}{(R+1)^n |B_1|} \int_{B_{1+\varepsilon}} \hat{v}_\varepsilon(y)dy = \frac{R^n}{(R+1)^n |B_1|} \int_{\mathbb{R}^n} \hat{v}_\varepsilon(y)dy = \frac{R^n}{(R+1)^n} m$$

we also have:

$$|\frac{\partial \widetilde{v}_{R+1}}{\partial x_i}(x)| \leq |\frac{\partial \hat{v}_\varepsilon}{\partial y_i}| \, |\frac{\partial y_i}{\partial x_i}| \leq \frac{c}{\varepsilon}\varepsilon = c \Rightarrow |\nabla \widetilde{v}_{R+1}| \leq c \tag{63}$$

Thus, if u_{R+1} minimizes $J_{B_{R+1}}(u_{R+1})$ with:

$$\frac{1}{|B_{R+1}|} \int_{B_{R+1}} u_{R+1}(x)dx = \frac{R^n}{(R+1)^n} \hat{m} \tag{64}$$

Then:

$$\begin{aligned} J_{B_{R+1}}(u_{R+1}) \leq J_{B_{R+1}}(\widetilde{v}_{R+1}) = \sum_{i=1}^{N-1} \int_{Rr_i - 1 \leq |x| \leq Rr_i + 1} (\frac{1}{2}|\nabla \widetilde{v}_{R+1}|^2 + W(\widetilde{v}_{R+1}))dx \\ + \int_{R-1 \leq |x| \leq R+1} (\frac{1}{2}|\nabla \widetilde{v}_{R+1}|^2 + W(\widetilde{v}_{R+1}))dx \end{aligned} \tag{65}$$

and we have that the first $N-1$ terms of the sum in (65) is zero since: $Rr_i - 1 \leq |x| \leq Rr_i + 1 \Leftrightarrow r_i - \varepsilon \leq |y| \leq r_i + \varepsilon$. Therefore from (61) and the similar relations $\widetilde{v}_{R+1} = a_i$, $i \in \{1, ..., N\} \Rightarrow |\nabla \widetilde{v}_{R+1}|^2 + W(\widetilde{v}_{R+1}) = 0$, in $Rr_i - 1 \leq |x| \leq Rr_i + 1$. Finally, from (65) ,the fact that W is bounded and that the volume:
$|\{R - 1 \leq |x| \leq R + 1\}| \leq c_1 R^{n-1}$, we have:

$$J_{B_{R+1}}(\widetilde{v}_{R+1}) \leq C(R+1)^{n-1} \Rightarrow J_{B_{R+1}}(u_{R+1}) \leq C(R+1)^{n-1} \tag{66}$$

So, we proved the estimate for functions that satisfy:

$$\frac{1}{|B_R|}\int_{B_R} u(x)dx = \frac{(R-1)^n}{R^n}\hat{m} \tag{67}$$

Finally, let $J_{B_R}(u) = \int_{B_R}(\frac{1}{2}|\nabla u|^2 + W(u))dx$, $u \in \mathcal{A}^R$, $\hat{m} \in$Conv$(\{a_1, ..., a_N\})$ and $\{W = 0\} = \{a_1, ..., a_N\}$.

We define $\tilde{W}(u) := W(u\frac{(R-1)^n}{R^n})$, $\tilde{m} := \frac{R^n}{(R-1)^n}\hat{m}$ and $\tilde{J}_{B_R}(v) := \int_{B_R}(\frac{1}{2}|\nabla v|^2 + \tilde{W}(v))dx$, $v \in \{u \in W^{1,2}(B_R;\mathbb{R}^m) : \frac{1}{|B_R|}\int_{B_R} udx = \tilde{m}\}$ and thus $\tilde{m} \in$Conv$(\{\tilde{a}_1, ..., \tilde{a}_N\})$, where $\tilde{a}_j := \frac{R^n}{(R-1)^n}a_j$ and since we have the estimate for functionals minimized subject to (67), we use $\tilde{v}_R$ as constructed above for the functional $\tilde{J}_{B_R}$. If we define $\tilde{u}_R := \tilde{v}_R\frac{(R-1)^n}{R^n}$, then $\tilde{u}_R \in \mathcal{A}^R$ and $\tilde{u}_R = a_i \Leftrightarrow \tilde{v}_R = \tilde{a}_i$ which gives the estimate:
$J_{B_R}(\tilde{u}_R) \leq CR^{n-1}$.

$\square$

Lemma 3.3 Given $\delta > 0$ we have

$$|\{x \in B_R : W(u_R(x)) < \delta\}| \geq |B_R|(1 - \frac{C_0}{\omega\delta R}) \tag{68}$$

where $\omega = |B_1|$.

Basic Estimate implies that the set where $dist(u(x), \{W = 0\}) \geq \delta$ is small.

Proof.

From Lemma 3.2 it follows in particular

$$|\{x \in B_R : W(u_R(x)) \geq \delta\}| \leq \int_{\{x\in B_R : W(u_R(x))\geq\delta\}} W(u_R(x))dx \leq$$

$$\leq \int_{B_R} W(u_R(x))dx \leq C_0R^{n-1}$$

$$\Rightarrow |\{x \in B_R : W(u_R(x)) < \delta\}| \geq |B_R| - \frac{C_0}{\omega\delta R}|B_R|$$

$\square$

Lemma 3.4 There exist a constant $C_2 > 0$ such that, given $\delta \in (0, d)$, it results

$$\frac{1}{|B_R|} | \int_{B_R} W_u(u_R(x))dx | \leq C_2(\sqrt{\delta} + \frac{1}{\delta R}) \tag{69}$$

and therefore, for $\delta = \frac{1}{R^{\frac{2}{3}}}$

$$\Rightarrow \frac{1}{|B_R|} | \int_{B_R} W_u(u_R(x))dx | \leq \frac{2C_2}{R^{\frac{1}{3}}} \tag{70}$$

This lemma establishes that the Lagrange multiplier tends to zero, hence the limiting equation is $\Delta u = W_u(u)$.

Proof.

Assume that $\delta \in (0, d)$ then $W(u_R(x)) < \delta$ and Lemma 3.1 imply the existence of $a \in \{a_1, ..., a_N\}$ such that $|u_R(x) - a| < q$ and therefore Lemma 3.1 implies
$|u_R(x) - a| < \sqrt{\frac{2\delta}{c_1}}$ and therefore $|W_u(u_R(x))| < C_1 \sqrt{\frac{2\delta}{c_1}}$.
From this and Lemma 3.4 we have

$$| \int_{B_R} W_u(u_R(x))dx \ | \leq | \int_{W(u_R(x)) \geq \delta} W_u(u_R(x))dx \ | + | \int_{W(u_R(x)) < \delta} W_u(u_R(x))dx \ |$$

$$\leq |B_R| \frac{C_0 C_W}{\omega \delta R} + |B_R| C_1 \sqrt{\frac{2\delta}{c_1}} \leq |B_R| (C_1 \sqrt{\frac{2\delta}{c_1}} + \frac{C_0 C_W}{\omega \delta R})$$

where $C_W = \max\limits_{u \leq M} |W_u(u)|$. It follows

$$\frac{1}{|B_R|} \ | \int_{B_R} W_u(u_R(x))dx \ | \leq \ C_1 \sqrt{\frac{2\delta}{c_1}} + \frac{C_0 C_W}{\omega \delta R} \leq C_2(\delta + \frac{1}{\delta R})$$

that concludes the proof provided we set $\delta = \frac{1}{R^{\frac{2}{3}}}$ and $C_2 = \max\{C_1 \sqrt{\frac{2}{c_1}}, \frac{C_0 C_W}{\omega}\}$.

$\square$

Lemma 3.5 Let $A \subset\subset B_R$ an open set with Lipschitz boundary. Assume that for some $x_0 \in \overline{A}$ and $\delta \in (0,d)$(d as in Lemma 3.1) it results $W(u_R(x_0)) \geq \delta$.
Then

$$J_A(u_R) \geq \eta \frac{\delta}{2} |B_{\frac{\delta}{2C_W K}(x_0)}| = C_3 \delta^{n+1} \tag{71}$$

where $\eta \in (0,1)$ is a constant that depends only on n and the Lipschitz constant of ∂A.

Proof.

We have

$$|W(u_R(x)) - W(u_R(x_0))| \leq \; |\int_0^1 W_u(u_R(x_0) + s(u_R(x) - u_R(x_0)) \cdot (u_R(x) - u_R(x_0)) ds|$$

$$\leq C_W \int_0^1 |u_R(x) - u_R(x_0)| ds \leq C_W K |x - x_0|$$

the last inequality holds from (50).Thus,

$$|W(u_R(x_0))| - |W(u_R(x))| \leq |W(u_R(x)) - W(u_R(x_0))| \leq \; C_W K |x - x_0|$$

So , it follows

$$|x - x_0| \leq \frac{\delta}{2C_W K} \Rightarrow |W(u_R(x))| \geq \frac{\delta}{2}$$

and therefore

$$\frac{\delta}{2} |B_{\frac{\delta}{2C_W K}(x_0)} \cap A| \leq \int_A W(u_R(x)) dx \leq J_A(u_R)$$

which conclude the proof since the assumption that A has a Lipschitz boundary implies the existence of a number $\eta \in (0,1)$ (that depends only on n and the Lipschitz constant) such that

$$\eta |B_{\frac{\delta}{2C_W K}(x_0)}| \leq |B_{\frac{\delta}{2C_W K}(x_0)} \cap A|$$

By [18] the desired inequality is true for Sobolev extension domains. So for domains D for which there exists a continuous linear operator $E : W^{1,p}(D) \to W^{1,p}(\mathbb{R}^n)$ such that $(Eu)_{\rceil D} = u$.
It is a classical result that a Lipschitz domain is such an extension domain, e.g. Theorem 1.4.3.1 in [19].

$\square$

Lemma 3.6 Given $\delta \in (0, d)$ and $l > 0$ there is $R_{\delta,l}$ such that $R \leq R_{\delta,l}$ implies the existence of a cube $Q_l \subset B_R$ of side l such that $W(u_R(x)) < \delta$, for $x \in Q_l$. More precisely the number N_δ^R of such cubes satisfies the bound

$$N_\delta^R \geq \frac{\omega R^n}{2^n l^n}(1 - \frac{2^n l^n C_0}{\omega C_3 \delta^{n+1} R}). \tag{72}$$

Proof.

For each $z = (z_1, ..., z_n) \in \mathbb{Z}^n$ let $Q_l(z) \subset \mathbb{R}^n$ the cube $Q_l(z) = \{x : z_j < \frac{x_j}{l} < z_j + 1,\ 1 \leq j \leq n\}$. Let N^R the number of the $z \in \mathbb{Z}^n$ such that $Q_l \subset B_R$ and let N_δ^R the number of the $z \in \mathbb{Z}^n$ such that

$$W(u_R(x)) < \delta \text{ , for } x \in Q_l(z).$$

From Lemma 3.5 and Lemma 3.6 we have

$$N^R - N_\delta^R \leq \frac{C_0 R^{n-1}}{C_3 \delta^{n+1}}.$$

This and the obvious inequality

$$\frac{|B_{R-\sqrt{n}l}|}{l^n} \leq N^R$$

imply that, provided $R > R_{\delta,l} = \frac{2^n l^n C_0}{\omega C_3 \delta^{n+1}}$, we have $(\frac{R}{2} > \sqrt{n}l \Leftrightarrow \frac{R}{2} < R - \sqrt{n}l \Rightarrow |B_{\frac{R}{2}}| \leq |B_{R-\sqrt{n}l}|)$

$$N^R - N^R_\delta \le \frac{C_0 R^{n-1}}{C_3 \delta^{n+1}} \le \frac{|B_{\frac{R}{2}}|}{l^n} \le N^R$$

and therefore

$$\frac{|B_{\frac{R}{2}}|}{l^n} - N^R \le \frac{C_0 R^{n-1}}{C_3 \delta^{n+1}} \Rightarrow N^R_\delta \ge \tfrac{\omega R^n}{2^n l^n}(1 - \tfrac{2^n l^n C_0}{\omega C_3 \delta^{n+1} R}).$$

□

3.3 The comparison function

Assume $Q_l \subset B_R$ as in Lemma 3.6 and let x_0 be the center of Q_l and set $r = \frac{l}{2}$. Given a constant $\mu > 0$ and constants $a_1, ..., a_N$ let $\tilde{v}_R : B_R \to \mathbb{R}^m$ be defined by

$$\tilde{v}_R = \begin{cases} u_R + \mu \sum_j \alpha_j a_j (1 - \frac{|x-x_0|}{r}), & x \in B_r(x_0), \\ u_R \ , \ x \in B_R \setminus B_r(x_0) \end{cases} \tag{73}$$

Then we have

$$\int_{B_r(x_0)} (\tilde{v}_R - u_R) dx = \mu \sum_j \alpha_j a_j \frac{\gamma}{n(n+1)} r^n = \mu \sum_j \alpha_j a_j \frac{\gamma}{\omega n(n+1)} |B_r(x_0)| \tag{74}$$

where γ is the measure of $\mathbb{S}^{n-1}$.

Lemma 3.7 For fixed $\alpha_1, ..., \alpha_N$ and provided $\mu \le \sqrt{\delta}$, it results

$$|\int_{B_r(x_0)} (W(\tilde{v}_R) - W(u_R)) dx| \le C_4 \sqrt{\delta} \mu |B_r(x_0)| \tag{75}$$

Proof.

Set $w_R = \tilde{v}_R - u_R$. Since $|w_R| = O(\mu)$ and $W(u_R) < \delta$ implies $|u_R - a| = O(\sqrt{\delta})$, for $x \in B_r(x_0)$ we have $|W_u(u_R + s w_R)| = O(\sqrt{\delta})$ and therefore

$$|W(\tilde{v}_R) - W(u_R)| \le \int_0^1 |W_u(u_R + s w_R)| \cdot w_R| ds \le C_4 \sqrt{\delta} \mu$$

$\square$

Lemma 3.8 Assume that $R \geq \max\{\delta^{-1}, R_{\delta,l}\}$, then it results

$$|\int_{B_r(x_0)}(|\nabla\tilde{v}_R|^2 - |\nabla u_R|^2)dx| \leq \frac{\mu}{r}|B_r(x_0)|\,|\sum_j \alpha_j a_j|(\frac{\mu}{r}|\sum_j \alpha_j a_j| + 2C_5\delta^{\frac{1}{4}}) \tag{76}$$

Proof.

From (73) we have $|\nabla w_R| \leq \frac{\mu}{r}|\sum_j \alpha_j a_j|$ and from the definition of $w_R, \tilde{v}_R$,

$|\nabla\tilde{v}_R|^2 - |\nabla u_R|^2 = |\nabla w_R|^2 + 2\nabla w_R \nabla u_R$, therefore

$$\begin{aligned} |\int_{B_r(x_0)}(|\nabla\tilde{v}_R|^2 - |\nabla u_R|^2)dx| \leq \int_{B_r(x_0)}|\nabla w_R|(|\nabla w_R| + 2|\nabla u_R|)dx \\ \leq (\frac{\mu}{r}|\sum_j \alpha_j a_j|)^2|B_r(x_0)| + 2\frac{\mu}{r}|\sum_j \alpha_j a_j||B_r(x_0)|^{\frac{1}{2}}(\int_{B_r(x_0)}|\nabla u_R|^2dx)^{\frac{1}{2}} \end{aligned} \tag{77}$$

from the Cauchy–Schwarz inequality.

On the other hand, by multiplying with $u_R - a$ in (48) and using the integration by parts:

$$\int_\Omega v\Delta u = \int_{\partial\Omega} v\frac{\partial u}{\partial\nu}dS - \int_\Omega \nabla v\nabla u dx \tag{78}$$

we have

$$\int_{B_r(x_0)}\Delta u_R\cdot(u_R - a)dx = \int_{\partial B_r(x_0)}(u_R - a)\frac{\partial u}{\partial\nu}dS - \int_{B_r(x_0)}|\nabla u_R|^2dx$$

and

$$\int_{B_r(x_0)}\Delta u_R\cdot(u_R - a)dx = \int_{B_r(x_0)}W_u(u_R)\cdot(u_R - a)dx - \int_{B_r(x_0)}(\frac{1}{|B_R|}\int_{B_R}W_u(u_R)dx)\cdot(u_R - a)dx$$

$$\begin{aligned} \Rightarrow \int_{B_r(x_0)}|\nabla u_R|^2dx = -\int_{B_r(x_0)}W_u(u_R)\cdot(u_R - a)dx + \\ + \int_{B_r(x_0)}(\frac{1}{|B_R|}\int_{B_R}W_u(u_R)dx)\cdot(u_R - a)dx + \int_{\partial B_r(x_0)}(u_R - a)\frac{\partial u}{\partial\nu}dS \end{aligned}$$

$$\Rightarrow \int_{B_r(x_0)} |\nabla u_R|^2 dx \leq \int_{B_r(x_0)} |W_u(u_R)| \cdot |u_R - a| dx + \\ +(\frac{1}{|B_R|}\int_{B_R} |W_u(u_R)| dx) \int_{B_r(x_0)} |u_R - a| dx + \int_{\partial B_r(x_0)} |u_R - a| |\frac{\partial u}{\partial \nu}| dS \tag{79}$$

Recalling that in Q_l we have $|W_u(u_R)| \leq C_1\sqrt{\frac{2\delta}{c_1}}$ and $|u-a| < \sqrt{\frac{2\delta}{c_1}}$, from (79) we have

$$\int_{B_r(x_0)} |\nabla u_R|^2 dx \leq C_1 \frac{2\delta}{c_1}|B_r(x_0)| + (C_1\sqrt{\frac{2\delta}{c_1}} + \frac{C_0 C_W}{\omega \delta R})\sqrt{\frac{2\delta}{c_1}}|B_r(x_0)| + \sqrt{\frac{2\delta}{c_1}} K\gamma r^{n-1}$$

the second term in the inequality cames from the estimate (in the proof of Lemma 3.4)

$$\frac{1}{|B_R|} \mid \int_{B_R} W_u(u_R(x)) dx \mid \leq \; C_1\sqrt{\frac{2\delta}{c_1}} + \frac{C_0 C_W}{\omega \delta R}$$

So,

$$\int_{B_r(x_0)} |\nabla u_R|^2 dx \leq \sqrt{\frac{2\delta}{c_1}} |B_r(x_0)| (2C_1\sqrt{\frac{2\delta}{c_1}} + \frac{C_0 C_W}{\omega \delta R} + \frac{K\gamma}{\omega r})$$

From this and $\delta R \geq 1$ it follows

$$(\int_{B_r(x_0)} |\nabla u_R|^2 dx)^{\frac{1}{2}} \leq C_5 \delta^{\frac{1}{4}} |B_r(x_0)|^{\frac{1}{2}}$$

Therefore, (77) implies

$$\begin{aligned} &| \int_{B_r(x_0)} (|\nabla \tilde{v}_R|^2 - |\nabla u_R|^2) dx | \leq \\ &\leq (\frac{\mu}{r} |\sum_j \alpha_j a_j|)^2 |B_r(x_0)| + 2C_5 \frac{\mu \delta^{\frac{1}{4}}}{r} |\sum_j \alpha_j a_j| |B_r(x_0)| \\ &= \frac{\mu}{r} |B_r(x_0)| \; |\sum_j \alpha_j a_j| (\frac{\mu}{r} |\sum_j \alpha_j a_j| + 2C_5 \delta^{\frac{1}{4}}) \end{aligned} \tag{80}$$

□

Lemma 3.9 Assume there is a ball $B_\rho(x_1) \subset B_R$, a map $v \in W^{1,2}(B_\rho(x_1);\mathbb{R}^m)$ and $E > 0$ such that

$$J_{B_\rho(x_1)}(u) - J_{B_\rho(x_1)}(v) = E \; ,$$
$$v(x) = u(x) \; , \;\; x \in \partial B_\rho(x_1)$$

Then there exists a map $\hat{v}_R \in W^{1,2}(B_{2\rho}(x_1);\mathbb{R}^m)$ such that

$$\lim_{R\to+\infty}(J_{B_{2\rho}(x_1)}(u_R) - J_{B_{2\rho}(x_1)}(\hat{v}_R)) = E \; ,$$
$$\hat{v}_R(x) = u_R(x) \; , \; x \in \partial B_{2\rho}(x_1) \; .$$

Proof.

Let $w_R : B_{2\rho}(x_1) \setminus B_\rho(x_1) \to \mathbb{R}^m$ be the map

$$w_R(x) = (2 - \frac{|x-x_1|}{\rho})(u(x) - u_R(x)) \; , \;\; x \in B_{2\rho}(x_1) \setminus B_\rho(x_1)$$

and define $\hat{v}_R$ by setting

$$\hat{v}_R = \begin{cases} v \; , \;\; x \in B_\rho(x_1) \\ u_R + w_R \; , \;\; x \in B_{2\rho}(x_1) \setminus B_\rho(x_1) \end{cases}$$

We have

$$J_{B_{2\rho}(x_1)}(u_R) - J_{B_{2\rho}(x_1)}(\hat{v}_R) = J_{B_\rho(x_1)}(u_R) - J_{B_\rho(x_1)}(v) +$$

$$+ J_{B_{2\rho}(x_1)\setminus B_\rho(x_1)}(u_R) - J_{B_{2\rho}(x_1)\setminus B_\rho(x_1)}(u_R + w_R) =$$

$$= E + J_{B_\rho(x_1)}(u_R) - J_{B_\rho(x_1)}(u) + J_{B_{2\rho}(x_1)\setminus B_\rho(x_1)}(u_R) - J_{B_{2\rho}(x_1)\setminus B_\rho(x_1)}(u_R + w_R)$$

Since u_R converges to u in $C^1(\overline{B_{2\rho}(x_1)}, \mathbb{R}^m)$ we have that w_R also converges to zero in $C^1(\overline{B_{2\rho}(x_1)}, \mathbb{R}^m)$.
This and the above inequality conclude the proof.

$\square$

3.4 The proof of Theorem 3.3

We are now in the position of completing the proof of Theorem 3.3.

Lemma 3.6 implies that, fixed δ and l, by taling $R > 0$ sufficiently large, we can assume

$$B_r(x_0) \cap B_{2\rho}(x_1) = \emptyset$$

where $B_r(x_0) \subset Q_l$ is used in (73) and $B_{2\rho}(x_1)$ is the ball in the definition of $\hat{v}_R$. It follows that by setting

$$v_R = \begin{cases} \tilde{v}_R \ , & x \in B_r(x_0) \ , \\ \hat{v}_R \ , & x \in B_{2\rho}(x_1) \ , \\ u_R \ , & x \in B_R \setminus (B_r(x_0) \cup B_{2\rho}(x_1)) \end{cases}$$

we have a well defined map $v_R \in W^{1,2}(B_R; \mathbb{R}^m)$. Next we show that we can choose $\mu, \alpha_1, ..., \alpha_N, r$ and δ in the definition of $\tilde{v}_R$ in such a way that

$$\int_{B_R} (u_R - v_R) dx = 0$$
$$J_{B_R}(u_R) - J_{B_R}(v_R) > 0$$

in contradiction with the minimality of u_R.

Note that, since $u_R \to u$ and $w_R \to 0$, from

$$\int_{B_{2\rho}(x_1)} (u_R - \hat{v}_R) dx = \int_{B_\rho(x_1)} (u - v) dx + \int_{B_\rho(x_1)} (u_R - u) dx - \int_{B_{2\rho}(x_1) \setminus B_{rho}(x_1)} w_R dx$$

it follows that there is $V_0 > 0$ independent of $R > 0$ such that

$$1 + \left| \int_{B_{2\rho}(x_1)} (u_R - \hat{v}_R) dx \right| \leq V_0 \tag{81}$$

Set

$$\mu = \frac{V_0 \omega n(n+1)}{\gamma |B_r(x_0)|} = \frac{C_6}{|B_r(x_0)|}$$

Then the definition of v_R it follows that the condition $\int_{B_R}(u_R - v_R)dx = 0$ is equivalent to

$$\int_{B_r(x_0)}(u_R - \tilde{v}_R)dx + \int_{B_{2\rho}(x_1)}(u_R - \hat{v}_R)dx = 0$$

and from (74) is equivalent to

$$-\int_{B_{2\rho}(x_1)}(u_R - \hat{v}_R)dx = \int_{B_r(x_0)}(u_R - \tilde{v}_R)dx = -\mu \sum_j \alpha_j a_j \frac{\gamma}{\omega n(n+1)}|B_r(x_0)| =$$
$$= -V_0 \sum_j \alpha_j a_j$$

This equation determines $\sum_j \alpha_j a_j$ and shows that $|\sum_j \alpha_j a_j| \leq 1$ (from (74)).

Next we consider the energy. We have

$$J_{B_R}(u_R) - J_{B_R}(v_R) = J_{B_{2\rho(x_1)}}(u_R) - J_{B_{2\rho(x_1)}}(\hat{v}_R) + J_{B_r(x_0)}(u_R) - J_{B_r(x_0)}(\tilde{v}_R)$$

From Lemma 3.9 for $R > 0$ sufficiently large, it results

$$J_{B_{2\rho(x_1)}}(u_R) - J_{B_{2\rho(x_1)}}(\hat{v}_R) \geq \frac{E}{2}. \tag{82}$$

On the other hand from Lemma 3.7 and Lemma 3.8 and the expression of μ above (and the fact that $|\sum_j \alpha_j a_j| \leq 1$) it follows

$$|J_{B_r(x_0)}(u_R) - J_{B_r(x_0)}(\tilde{v}_R)| \leq C_4 C_6 \sqrt{\delta} + \frac{C_6}{2r}\left(\frac{C_6}{r|B_r(x_0)|} + 2C_5 \delta^{\frac{1}{4}}\right).$$

This expression shows that we can fix $\delta > 0$ small enough and $r > 0$ large enough to obtain $|J_{B_r(x_0)}(u_R) - J_{B_r(x_0)}(\tilde{v}_R)| \leq \frac{E}{4}$ which combined with (82) gives $J_{B_R}(u_R) - J_{B_R}(v_R) > 0$ and contradicts the minimality of u_R, so that concludes the proof of Theorem 3.3.

□

Remark. In the proof of Theorem 3.3 we only showed that u satisfies the definition of being minimal on balls. Actually this implies that u is minimal on each bounded set. Indeed, if $\Omega \subset \mathbb{R}^n$ is a bounded set and $J_\Omega(u) - J_\Omega(v) > 0$ for some v and coincides with u on $\partial\Omega$, we can choose $\rho > 0$ and x_1 such that $\Omega \subset B_\rho(x_1)$ and define $v^* = v$ on Ω and $v^* = u$ on $B_\rho(x_1) \setminus \Omega$. Then we have $J_{B_\rho(x_1)}(u) - J_{B_\rho(x_1)}(v^*) = J_\Omega(u) - J_\Omega(v) > 0$ and can procced as in the proof of Theorem 3.3. Finally, the fact that u satisfies (51) is obvious from Lemma 3.4.

4 Appendix: A lower bound Estimate

We will prove a gradient estimate and as a result we will have a lower bound for the functional J_{B_R} defined in (46). But first, we will state a known result in the case without mass constraint.

Theorem 4.1 Assume $W \geq 0$ and let u be a $W^{1,2}_{loc}(\mathbb{R}^n;\mathbb{R}^m) \cap L^{\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^m)$ solution to (20). Then, we have

$$\frac{d}{dr}(r^{-(n-2)}J_{B_r}(u)) \geq 0 \ , \ for \ \ r > 0, \tag{83}$$

where

$$J_{B_r}(u) = \int_{B_r} (\frac{1}{2}|\nabla u|^2 + W(u))dx \ , \tag{84}$$

with $x_0 \in \mathbb{R}^n$ arbitrary and $B_r := B_r(x_0)$ the r–ball in $\mathbb{R}^n$ centered at x_0.

Proof.

[2 ,p.89-91]

<u>Notes:</u> (1) An immediate consequence of (83) is the lower bound $J_{B_r}(u) \geq cr^{n-2}$ for nonconstant solutions.
(2) In the proof of this theorem the main tool is the stress– energy tensor, which is an algebraic fact implying several useful identities like the monotonicity formula (83).

Next, we will prove a similar lower bound estimate for the mass constaint case in a different way, without using the stress–energy tensor, which does not seem applicable due to the lagrange multiplier in the equation (48). We state a well–known theorem that we need for the proof.

Theorem 4.2 Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that, for any $u \in W^{1,p}_0(U)$

$$\|u\|_{L^{p^*}(U)} \leq C\|\nabla u\|_{L^p(U)} \ , \tag{85}$$

where $p^* = \frac{np}{n-p}$.

Proof.

[5 , p.155 -157]

Proposition 4.3 Let $u \in \mathcal{A}^R := \{v \in W^{1,p}(B_R;\mathbb{R}^m) : \frac{1}{|B_R|}\int_{B_R} v dx = \hat{m}\}$ and $1 \leq p < n$. If $u \in W_0^{1,p}(B_R)$, then

$$\int_{B_R} |\nabla u(x)|^p dx \geq CR^{n-p} \tag{86}$$

where C is a constant that depends on $\hat{m}, p$ and $n > 2$.

Proof.

Let $u \in \mathcal{A}^R$,

$$\frac{1}{|B_R|}\int_{B_R} u(x)dx = \hat{m} \Rightarrow \int_{B_R} u(x)dx = \hat{m}|B_R| \Rightarrow \int_{B_R} |u(x)|dx \geq |\hat{m}|\,|B_R| \tag{87}$$

$$\Rightarrow |\hat{m}||B_R| \leq \int_{B_R} |u(x)|dx = \int_{B_R\cap\{|u|>1\}} |u(x)|dx + \int_{B_R\cap\{|u|\leq 1\}} |u(x)|dx \tag{88}$$

From Theorem 4.2, we have

$$\int_{B_R} |u(x)|^{p^*} dx + |B_R \cap \{|u| \leq 1\}| \leq C(\int_{B_R} |\nabla u|^p)^{\frac{p^*}{p}} + |B_R \cap \{|u| \leq 1\}| \tag{89}$$

$$\Rightarrow C(\int_{B_R} |\nabla u|^p)^{\frac{p^*}{p}} \geq |\hat{m}|\,|B_R| - |B_R \cap \{|u| \leq 1\}| \tag{90}$$

(i) If $|\hat{m}| > 1$: Then,

$$C(\int_{B_R} |\nabla u(x)|^p dx)^{\frac{p^*}{p}} \geq |\hat{m}|\,|B_R| - |B_R| = (|\hat{m}| - 1)|B_R| \tag{91}$$

$$\Rightarrow (\int_{B_R} |\nabla u(x)|^p dx)^{\frac{p^*}{p}} = (\int_{B_R} |\nabla u(x)|^p dx)^{\frac{n}{n-p}} \geq \frac{|\hat{m}| - 1}{C}|B_1|R^n = C_1 R^n \tag{92}$$

$$\Rightarrow \int_{B_R} |\nabla u(x)|^p dx \geq \widehat{C} R^{n-p} \tag{93}$$

where $\widehat{C} = (\frac{|\hat{m}|-1}{C}|B_1|)^{\frac{n-p}{n}}$ and C depends only on p, n.
(ii) If $|\hat{m}| \leq 1$:

$$\frac{1}{|B_R|}\int_{B_R} u(x)dx = \hat{m} \Rightarrow \frac{1}{|B_R|}\int_{B_R} (\lambda u(x))dx = \lambda\hat{m} \tag{94}$$

we choose λ such that $|\lambda\hat{m}| > 1$ (for example $\lambda = \frac{2}{|\hat{m}|}$)
So from (i)

$$\Rightarrow \int_{B_R} |\nabla(\lambda u(x))|^p dx \geq \widehat{C} R^{n-p} \Rightarrow \int_{B_R} |\nabla u(x)|^p dx \geq \widehat{C}_1 R^{n-p} \tag{95}$$

where $\widehat{C}_1 = \frac{1}{\lambda^p}(\frac{\lambda|\hat{m}|-1}{C}|B_1|)^{\frac{n-p}{n}}$.

□

So, for $p = 2$, this results the following lower bound estimate

Corollary 4.4 Let $u \in \mathcal{A}^R := \{v \in W^{1,2}(B_R; \mathbb{R}^m) : \frac{1}{|B_R|}\int_{B_R} v dx = \hat{m}\}$. If $u \in W_0^{1,2}(B_R)$, and $W \geq 0$, then

$$J_{B_R}(u) := \int_{B_R} (\frac{1}{2}|\nabla u(x)|^2 + W(u(x)))dx \geq CR^{n-2} \ , \ (n > 2) \tag{96}$$

Remark: Proposition 4.3 holds for more general domains with a similar proof. In other words, if $u \in \{v \in W_0^{1,p}(U; \mathbb{R}^m) : \frac{1}{|U|}\int_U v dx = \hat{m}\}$, then

$$\int_U |\nabla u|^p \geq \tilde{C}|U|^{\frac{n-p}{n}}$$

where $\tilde{C} = \tilde{C}(n, p, \hat{m})$.

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