NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS

MASTER THESIS

Discrete time convolution and elements of Markov Renewal theory

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NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS

Abstract

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In this thesis, we present an algebraic approach for discrete time convolutions of real and matrix valued functions. We study their properties using some well known algebraic structures such as the Rings and Groups which help for the development of thesis with concrete applications in Probability theory. Special mention is given for the convolutional inverse which plays a fundamental role in order to obtain unique solutions for renewal and Markov renewal equations.

Ultimately, the theory of convolutions is applied for the extension of usual renewal theory in which we admit the possibility that the interrarival time between two or more successive arrival times could be null. This theoretical frame can be used for application in biological systems, but in reliability theory as well, in which the thermal time is more appropriate to describe the evolution of a system. Furthermore, we use convolutional operators in order to obtain the associated results in the usual theory of Markov renewal chains.

Περίληψη

Σε αυτή τη διπλωματική παρουσιάζεται μία αλγεβρική προσέγγιση στις συνελίξεις πραγματικών και πινακικών συναρτήσεων σε διακριτό χρόνο. Εξετάζονται οι ιδιότητες τους με τη βοήθεια αλγεβρικών δομών, όπως είναι ο δακτύλιος και η ομάδα. Η χρήση αυτών των αλγεβρικών θεωριών βοηθούν στη συνέχεια της εργασίας μέσω συγκεκριμένων εφαρμογών τους στη θεωρία Πιθανοτήτων, ενώ γίνεται ιδιαίτερη μνεία στον συνελιξιακό αντίστροφο ο οποίος παίζει καταλυτικό ρόλο στη λύση γενικών μορφών ανανεωτικών και Μαρκοβιανών ανανεωτικών εξισώσεων.

Ουσιαστικά, εφαρμόζεται στην επέκταση της κλασικής ανανεωτικής θεωρίας στην οποία θα θεωρήσουμε ότι οι μηδενικές χρονικές στιγμές γεγονότων είναι επιτρεπτές, ενώ επεκτείνονται κατάλληλα έννοιες και εργαλεία που χρησιμοποιούνται στην κλασική περίπτωση με έμφαση στα οριακά θεωρήματα. Το θεωρητικό αυτό πλαίσιο θα μπορούσε να χρησιμοποιηθεί για εφαρμογές σε βιολογογικά συστήματα, αλλά και στη θεωρία αξιοπιστίας, όπου για παράδειγμα ο θερμικός χρόνος είναι πιο κατάλληλος για την περιγραφή της εξέλιξης ενός συστήματος και μπορεί να θεωρηθεί ως η πλέον κεντρικής σημασίας χρονική συνιστώσα. Επίσης, χρησιμοποιούνται συνελιξιακές πράξεις για την έκβαση αποτελεσμάτων στην κλασική θεωρία των Μαρκοβιανών ανανεωτικών αλυσίδων (μ.α.α).

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List of Abbreviations

pmf	**p**robability **m**ass **f**unction
cdf	**c**umulative **d**istribution **f**unction
r.v	**r**andom **v**ariable
rf	**r**eliability **f**unction
i.i.d	**i**ndependent and **i**dentically **d**istributed
RC	**R**enewal **C**hain
MC	**M**arkov **C**hain
MRC	**M**arkov **R**enewal **C**hain
EMC	**E**mbedded **M**arkov **C**hain
SMC	**S**emi **M**arkov **C**hain

Chapter 1

DISCRETE TIME CONVOLUTIONS

1. Introduction

In this chapter we introduce the discrete time convolution of real and matrix-valued functions. They are very helpful mathematical tools, which play an active role in several areas of Probability theory, like Renewal and Markov Renewal theory. The aim of this chapter is to study the algebraic properties of this operator and apply them in order to solve many theoretical problems in the rest of this thesis.

Our motivation originates from the work of Barbu-Limnios [4] in which they introduce discrete time convolutions for matrix–valued functions in one variable, especially for solving renewal and Markov renewal equations. A particular role in the theory is played by the left convolutional inverse of a given matrix valued function and its computation is performed recurrently. In Markov renewal theory, the convolutional inverse is used to give a form and recurrent way to compute the transition function of a semi-Markov chain, for the development of the semi-Markov reliability systems and for the desired statistical analysis in the nonparametric case [4] and [28].

The goal of this thesis is to give a unified approach to discrete time convolutions and extend their use, notably the convolutional inverse usage. Furthermore, convolutional representations will supplant the generating functions in the proofs of important results in discrete time renewal theory (for a further investigation see [4]) and make more convenient representations such as the form of the renewal function.

In this chapter, by making links with the theory of Rings, Groups and Algebras over a field, we exploit the algebraic properties of the convolution product in order to simplify significantly the development of the theory and obtain new important results. New representations of the convolutional inverse help also in this direction.

2. Discrete time convolution product of real functions

In this section we study the convolution product of a specific class of real-valued functions. We give some algebraic properties of the convolutional operator by paying special attention to the convolutional inverse of a function. The theoretical development is complemented with some applications in Probability theory.

Definition 1.1. *Let $f, g : \mathbb{N} \to \mathbb{R}$. The function $f * g : \mathbb{N} \to \mathbb{R}$, given by*

$$[f * g](k) := \sum_{l=0}^{k} f(k-l)g(l), \quad n \in \mathbb{N},$$

is said to be the discrete time convolution product of f and g.

We give now some examples, which are necessary to explain the convolution's operator usage in probability theory. An important role is played by the unitary function (the function which is identically one) and we will reserve the symbol $\mathbb{1}$ for this function.

Example 1.1. *Let X be a nonnegative integer-valued random variable and f, F its associated pmf and cdf (defined on $\mathbb{N}$). The distribution function F can be written via convolution in the form $F = \mathbb{1} * f$, since*

$$F(k) = \sum_{l=0}^{k} f(l) = \sum_{l=0}^{k} f(k-l) = \sum_{l=0}^{k} f(k-l)\mathbb{1}(l) = [f * \mathbb{1}](k) = [\mathbb{1} * f](k).$$

This shows that the unitary function $\mathbb{1}$ corresponds to the summation operator. Another simple example of compact convolutional representation is given by the reliability function of X, say it $\overline{F}$. We have

$$\overline{F}(k) = 1 - F(k) = \mathbb{1}(k) - [\mathbb{1} * f](k)$$

and consequently $\overline{F} = \mathbb{1} - \mathbb{1} * f$. A simpler representation is possible through the properties of the convolution operator that will be developed in the sequel.

Example 1.2. *Let us consider the random variable of the previous example. The expected value $\mathbb{E}(X)$ can be determined by*

$$\begin{aligned}\mathbb{E}(X) &= \sum_{n=0}^{+\infty} n\mathbb{P}(X = n) = \sum_{n=0}^{+\infty} \mathbb{P}(X > n) = \lim_{n\to\infty} \sum_{k=0}^{n} \mathbb{P}(X > k) \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \overline{F}(k) = \lim_{n\to\infty} [\mathbb{1} * \overline{F}](n).\end{aligned}$$

Consequently, the expected value of a positive and integer-valued random variable is expressed as the limit of the sequence which results from the convolution product of $\mathbb{1}$ and $\overline{F}$. In the sequel, we will also examine closer the limiting behaviour of sequences resulting from convolution products.

Example 1.3. *Let X, Y be two independent nonnegative integer-valued random variables with pmf f_X and f_Y respectively. If f_{X+Y} is the pmf of $X + Y$, then by independence we have directly that*

$$f_{X+Y}(n) = \sum_{l=0}^{n} f_X(n-l) f_Y(l) = [f_X * f_Y](n),$$

and consequently $f_{X+Y} = f_X * f_Y$. This is indeed the most famous application of convolution in probability theory.

Example 1.4. *Let X, Y be defined as in Example 1.3 and F_X, F_Y be their corresponding distribution functions. If F_{X+Y} is the distribution function of the sum $X + Y$, then*

$$\begin{aligned}F_{X+Y}(n) = \mathbb{P}(X + Y \leq n) &= \sum_{l=0}^{n} \mathbb{P}(X + Y \leq n | Y = l)\mathbb{P}(Y = l) \\ &= \sum_{l=0}^{n} \mathbb{P}(X \leq n - l | Y = l) f_Y(l) \\ &= \sum_{l=0}^{n} \mathbb{P}(X \leq n - l) f_Y(l) \\ &= \sum_{l=0}^{n} F_X(n-l) f_Y(l) \\ &= [F_X * f_Y](n).\end{aligned}$$

The above decomposition shows that $F_{X+Y} = F_X * f_Y$. Similarly, we have that $F_{X+Y} = F_Y * f_X$. Now, notice that the above decomposition could be obtained directly by using the properties of *convolution. In particular, by using the results of Examples (1.1) and (1.3) we have that*

$$F_{X+Y} = \mathbb{1} * f_{X+Y} = \mathbb{1} * (f_X * f_Y) = (\mathbb{1} * f_X) * f_Y = F_X * f_Y.$$

In the third equality we used the associativity of convolution that can be verified easily. Consequently, by using the properties of convolution we get directly that

$$F_{X+Y} = F_X * f_Y = F_Y * f_X.$$

The above examples show an interest in a systematic use of properties of the convolution operator. For this reason, we will mention some of its basic properties.

Proposition 1.1. *The discrete time convolution operator is associative, commutative, it possesses a unique identity element e_0 given by*

$$e_0(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & elsewhere, \end{cases} \tag{1.1}$$

and it is also distributive with respect to the addition of functions (componentwise addition).

Proof. The results are easy to prove and we will just show associativity. Let $f, g, h : \mathbb{N} \to \mathbb{R}$. In order to show that the discrete time convolution operator is associative we will show that for an arbitrary $k \in \mathbb{N}$ we have $[f * (g * h)](k) = [(f * g) * h](k)$. Indeed,

$$\begin{aligned}
[f * (g * h)](k) &= \sum_{l=0}^{k} f(k-l)[g * h](l) = \sum_{l=0}^{k} f(k-l) \sum_{m=0}^{l} g(l-m)h(m) \\
&= \sum_{m=0}^{k} \left(\sum_{l=m}^{k} f(k-l)g(l-m) \right) h(m) = \sum_{m=0}^{k} \left(\sum_{d=0}^{k-m} f(k-m-d)g(d) \right) h(m) \\
&= \sum_{m=0}^{k} [f * g](k-m)h(m) = [(f * g) * h](k).
\end{aligned}$$

□

It will be beneficial in some cases to identify any function $f : \mathbb{N} \to \mathbb{R}$ as an infinite dimensional vector or a formal power series. For this purpose, we will need some algebraic definitions and properties. First, let us denote by $R := \mathbb{R}^{\mathbb{N}}$ the set of all real-valued sequences. Therefore, from Proposition 1.1 we get directly that $(R, +, *)$ is a commutative ring with unity, equipped with the operations of the usual addition between sequences and of the convolution product of sequences. Obviously, e_0 is the multiplicative identity element of this ring and in the following proposition we show that R is also an integral domain (no zero divisors).

Proposition 1.2. *Let $f, g \in R$. Then, we get*

$$f * g = 0 \iff f \equiv 0 \text{ or } g \equiv 0. \tag{1.2}$$

Proof. If $f \equiv 0$ or $g \equiv 0$, then the reverse implication of (1.2) is true. Now, we will show that the direct implication is also true.

Let us assume that $f(n) \neq 0$ for some $n \in \mathbb{N}$. Then, as a consequence of the Archimedean property there exists an $n_0 \in \mathbb{N}$ such that $n_0 = \min\{l \in \mathbb{N} : f(l) \neq 0\}$. All we have to prove is that

$$[f * g](n_0 + k) = f(n_0)g(k), \quad \forall\, k \in \mathbb{N}, \tag{1.3}$$

since then by the assumption that $f * g = 0$ and the fact that $f(n_0) \neq 0$, we will get that $g \equiv 0$.

We will prove that (1.3) holds by induction. For $k = 0$, (1.3) is true, since by the definition of n_0, $f(l) = 0$ for all $l < n_0$ and $f(n_0) \neq 0$. Now, let us assume that (1.3) holds for $k = 0, 1, \ldots, k_0$ and consequently $g(0) = \ldots = g(k_0) = 0$.

Then,

$$[f * g](n_0 + k_0 + 1) = \sum_{l=0}^{n_0-1} f(l)g(n_0 + k_0 + 1 - l) + f(n_0)g(k_0 + 1) + \sum_{l=0}^{k_0} f(n_0 + k_0 + 1 - l)g(l),$$

and by using the above assumptions we get directly that (1.3) holds for $k = k_0 + 1$. Therefore, by induction we conclude that (1.3) holds for any $k \in \mathbb{N}$. This implies that R is also an integral domain. $\square$

In addition, we denote the set of power series with real coefficients $\mathbb{R}\left[[x]\right]$ equipped with the following binary operations

$$\begin{aligned} + : \ (f(0) + f(1)\,x + \ldots) \quad &+ \quad f(0) + f(1)\,x + \ldots = (f(0) + g(0)) + (f(1) + g(1))\,x + \ldots, \\ \bullet : \ (f(0) + f(1)\,x + \ldots) \quad &\bullet \quad (g(0) + g(1)\,x + \ldots) = (f(0)g(0)) + (f(0)g(1) + f(1)g(0))\,x + \ldots. \end{aligned}$$

It is well that known that $\mathbb{R}[x]$ is a commutative ring with identity element given by the constant polynomial with value one.

Let us also denote by $\mathbb{R}^{\mathbb{N}}$ the set of infinite dimensional vectors with real coordinates equipped with the following operators

$$\begin{aligned} (+): \quad (f(0), f(1), f(2), \ldots) \quad (+) \quad (g(0), g(1), g(2), \ldots) &= (f(0) + g(0), f(1) + g(1), \ldots) \\ (\bullet): \quad (f(0), f(1), f(2), \ldots) \quad (\bullet) \quad (g(0), g(1), g(2), \ldots) &= \left(\sum_{l=0}^{n} f(n-l)g(l) \right)_{n \in \mathbb{N}}. \end{aligned}$$

It is easy to notice that $\mathbb{R}^{\mathbb{N}}$ forms a commutative ring with unity. In the following proposition we show that $(R, +, *)$, $(\mathbb{R}\left[[x]\right], +, \bullet)$ and $\left(\mathbb{R}^{\mathbb{N}}, (+), (\bullet)\right)$ are algebraically identical.

Proposition 1.3. *The rings $(R, +, *)$, $(\mathbb{R}\left[[x]\right], +, \bullet)$ and $\left(\mathbb{R}^{\mathbb{N}}, (+), (\bullet)\right)$ are isomorphic.*

For the above proposition, any $f \in R$ can be identified as

$$f \cong (f(0), f(1), f(2), \ldots) \cong f(0) + f(1)\,x + f(2)\,x^2 + \ldots. \tag{1.4}$$

Since convolution of sequences corresponds to multiplication of formal power series, this property will be used for convenience in some cases. From the above representation, it is now clear that e_0 can be written as

$$e_0 \cong (1, 0, 0, \ldots) \cong 1, \tag{1.5}$$

where the equality is used here abusively, but without causing confusion, and the specific representation will depend on the context. The first equality gives a clear interpretation to our notation for the identity element as the vector which attributes 1 to the first component with index zero and the second one refers to the corresponding constant polynomial. More generally, we denote by

$$e_i \cong \underbrace{(0, \ldots, 0, 1, 0, \ldots)}_{1 \text{ in the } i\text{-th index}} \cong x^i. \tag{1.6}$$

Notation. *For simplicity, we will just suppress $\bullet$ and $(\bullet)$.*

Remark 1.1. *The function e_i clearly corresponds to the pmf of the almost surely constant random variable $X = i$. It is clear that*

$$e_i * e_j \cong x^i x^j = x^{i+j} \cong e_{i+j}. \tag{1.7}$$

We can also compute the cdf associated to e_i. In particular, if we denote by $\mathbb{1}_i$ this cdf, then

$$\mathbb{1}_i = \mathbb{1} * e_i \cong \left(\sum_{k=0}^{\infty} x^k\right) x^i = \sum_{k=0}^{\infty} x^{k+i} = \sum_{k=i}^{\infty} x^k \cong \underbrace{(0,\ldots,0,1,1,\ldots)}_{1 \text{ from the } i\text{-th index}}. \tag{1.8}$$

This intuitive notation indicates that $\mathbb{1}_i$ corresponds exactly to the sequence that starts with zeros and has 1 from the i-th index and onwards. Of course we have $\mathbb{1} = \mathbb{1}$. Similarly, we have

$$\mathbb{1}_i * e_j = \mathbb{1} * e_i * e_j = \mathbb{1} * e_{i+j} = \mathbb{1}_{i+j}. \tag{1.9}$$

The usefulness of the basis elements e_i (1.6) can be found in the simplification of the computations with convolutions. In this way, the use of formal power series can be avoided. In fact, by using the left and right member of (1.6) we can now rewrite (1.4) in the form

$$f = \sum_{k=0}^{\infty} f(k) e_k. \tag{1.10}$$

Then, we have

$$\begin{aligned} f * g &= \left(\sum_{l=0}^{\infty} f(l) e_l\right) * \left(\sum_{m=0}^{\infty} g(m) e_m\right) = \sum_{l,m=0}^{\infty} \left[f(l) g(m)\right] e_l * e_m = \sum_{l,m=0}^{\infty} f(l) g(m) e_{l+m} \\ &= \sum_{k=0}^{\infty} \left(\sum_{l,m: l+m=k} f(l) g(m)\right) e_k = \sum_{k=0}^{\infty} [f * g](k) e_k, \end{aligned} \tag{1.11}$$

as expected. Next, we define the convolutional powers (powers in the sense of convolution):

Definition 1.2. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function and $n \in \mathbb{N}$. The n-fold convolution $f^{(n)} : \mathbb{N} \to \mathbb{R}$ of the function f is defined recursively by :*

$$f^{(0)} := e_0$$

and

$$f^{(n)} := f * f^{(n-1)},\ n \geq 1.$$

From the above definition we can get directly that for all $n, k \geq 0$,

$$f^{(n)}(k) := \sum_{\substack{k_1, k_2, \ldots, k_n \geq 0 \\ k_1 + k_2 + \ldots k_n = k}} f(k_1) f(k_2) \ldots f(k_n). \tag{1.12}$$

Remark 1.2. *By rearranging the terms of (1.12) we get that for all $n, k \geq 0$,*

$$f^{(n)}(k) := \sum_{\substack{n_0, n_1, \ldots, n_k \geq 0 \\ n_0 + n_1 + \ldots n_k = n}} \binom{n}{n_0, n_1, \ldots, n_k} (f(0))^{n_0} (f(1))^{n_1} \ldots (f(k))^{n_k}.$$

For the first three terms, for $k = 0, 1, 2$, we have that for $n \geq k$,

$$\begin{aligned} f^{(n)}(0) &= (f(0))^n, \\ f^{(n)}(1) &= n (f(0))^{n-1} f(1), \\ f^{(n)}(2) &= \frac{n(n-1)}{2} (f(0))^{n-2} (f(1))^2 + n (f(0))^{n-1} f(2). \end{aligned} \tag{1.13}$$

It is easy also to see that for all $n \geq k$,

$$\begin{aligned} f^{(n)}(k) &= \sum_{l=n-k}^{n-1} \binom{n}{l} (f(0))^l \sum_{\substack{n_1,\dots,n_k \geq 0 \\ n_1+n_2+\dots n_k = n-l}} \binom{n-l}{n_1, n_2, \dots, n_k} (f(1))^{n_1} (f(2))^{n_2} \dots (f(k))^{n_k} \\ &= \sum_{l=n-k}^{n-1} \binom{n}{l} (f(0))^l f_+^{(n-l)}(k-1), \end{aligned} \tag{1.14}$$

where $f_+(k) = f(k+1)$, for all $k \geq 0$ and therefore f_+ corresponds to the translation of f one unit to the right.

Some useful properties of the convolutional powers are given below.

Proposition 1.4. *Let $f, g : \mathbb{N} \to \mathbb{R}$ and $c \in \mathbb{R}$ be a constant. Then, we get directly the following relations*

(i) $(cf) * g = f * (cg) = c(f * g)$,

(ii) $(cf)^{(n)} = c^n f^{(n)}$,

(iii) $f^{(n)} * f^{(m)} = f^{(n+m)}$,

(iv) $(f^{(n)})^{(m)} = f^{(nm)}$,

(v) $(f * g)^{(n)} = f^{(n)} * g^{(n)}$.

Proof.

(i) Using relation (1.4) we can take

$$\begin{aligned} (cf) * g &\cong (cf_0)g_0 + ((cf_1)g_0 + (cf_0)g_1)\, x + \dots \\ &= f_0(cg_0) + ((cg_1)f_0 + (cg_0)f_1)\, x + \dots \\ &= c\,(f_0 g_0 + (f_0 g_1 + f_1 g_0)\, x + \dots) \cong c(f * g) \end{aligned}$$

and consequently we obtain the desired relations.

(ii) From (i) we have

$$(c * f)^{(n)} = \underbrace{(cf) * (cf) * \cdots * (cf)}_{n-times} = c\,\underbrace{f * (cf) * \cdots * (cf)}_{n-times} = c^n \left(\underbrace{f * f * \cdots * f}_{n-times} \right) = c^n f^{(n)}.$$

$$(iii) \qquad f^{(n)} * f^{(m)} = \left(\underbrace{f * f * \cdots * f}_{n-times} \right) * \left(\underbrace{f * f * \cdots * f}_{m-times} \right) = \underbrace{f * f * \cdots * f}_{(n+m)-times} = f^{(n+m)}.$$

$$\begin{aligned} (iv)\ (f^{(n)})^{(m)} &= \underbrace{f^{(n)} * f^{(n)} \cdots * f^{(n)}}_{m-times} = \underbrace{\left(\underbrace{f * f \cdots * f}_{n-times} \right) * \cdots * \left(\underbrace{f * f \cdots * f}_{n-times} \right)}_{m-times} \\ &= \underbrace{f * f \cdots * f}_{nm-times} = f^{(nm)}. \end{aligned}$$

$$(v)\ (f*g)^{(n)} = \underbrace{(f*g)*(f*g)\cdots*(f*g)}_{n-times} = \underbrace{f*g*f*g\cdots*f*g}_{n-times}$$
$$= \left(\underbrace{f*f\cdots*f}_{n-times}\right)*\left(\underbrace{g*g\cdots*g}_{n-times}\right) = f^{(n)}*g^{(n)}.$$

□

In order to familiarize with the convolutional powers we'll use the following examples:

Example 1.5. *Consider the function of Equation 1.6. Since $e_i * e_j = e_{i+j}$, we get easily that*

$$e_i^{(n)} = \underbrace{e_i * e_i * \ldots * e_i}_{n \text{ times}} = e_{ni}.$$

In particular, we have that for all $n \in \mathbb{N}$

$$e_n = e_1^{(n)}. \tag{1.15}$$

This simple functional property is very important in the development of the theory and gives the possibility to develop a similar calculus as in the case of the generating functions to simplify the derivation of many results. This can be understood by rewriting (1.11) in the form

$$\left(\sum_{k=0}^{\infty} f(k)e_1^{(k)}\right) * \left(\sum_{k=0}^{\infty} g(k)e_1^{(k)}\right) = \sum_{k=0}^{\infty}[f*g](k)e_1^{(k)}.$$

The above functional identity justifies that the product of the generating functions of two sequences corresponds to the generating function of their convolution by using the isomorphism in Proposition 1.3.

Now, we give a proposition which gives an interesting interpretation of $\mathbb{1}^{(n)}$.

Proposition 1.5. *If $\left[n \atop k \right]$ denotes the k-combinations of n elements with repetition, then*

$$\mathbb{1}^{(n)} = \begin{bmatrix} n \\ \cdot \end{bmatrix}, \qquad \text{i.e.,} \quad \mathbb{1}^{(n)}(k) = \begin{bmatrix} n \\ k \end{bmatrix}, \text{ for all } k, n \in \mathbb{N}.$$

Additionally,

$$\mathbb{1}_i^{(n)} = \begin{bmatrix} n \\ \cdot - ni \end{bmatrix}, \qquad \text{i.e.,} \quad \mathbb{1}_i^{(n)}(k) = \begin{bmatrix} n \\ k - ni \end{bmatrix}, \text{ for all } k, n \in \mathbb{N},$$

where $\left[n \atop k \right] = 0$, for $k < 0$.

Proof. For $n = 0$, we have that the $k - combinations$ of 0 elements with repetition is one for $k = 0$ and is zero otherwise. Therefore, we have

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = e_0(k) = \mathbb{1}^{(0)}(k),$$

and assume that the above relation also holds for an arbitrary n. Then,

$$\mathbb{1}^{(n+1)}(k) = [\mathbb{1} * \mathbb{1}^{(n)}](k) = \sum_{l=0}^{k} \begin{bmatrix} n \\ l \end{bmatrix}. \tag{1.16}$$

Using Pascal's triangle and the equality $\left[{n \atop l}\right] = \binom{n+l-1}{l}$ we have

$$\begin{bmatrix} n+1 \\ l \end{bmatrix} = \begin{bmatrix} n+1 \\ l-1 \end{bmatrix} + \begin{bmatrix} n \\ l \end{bmatrix},$$

and summing over $l = 0, \ldots k$, we obtain

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \sum_{l=0}^{k} \begin{bmatrix} n \\ l \end{bmatrix}.$$

Therefore, we take

$$\mathbb{1}^{(n+1)}(k) = \begin{bmatrix} n+1 \\ k \end{bmatrix},$$

and consequently by induction we get the desired form.
From Example 1.1 and Proposition 1.4 we get

$$\mathbb{1}_i^{(n)} = (\mathbb{1} * e_i)^{(n)} = \mathbb{1}^{(n)} * e_i^{(n)} = \mathbb{1}^{(n)} * e_{ni} = \begin{bmatrix} n \\ \cdot \end{bmatrix} * e_{ni} = \begin{bmatrix} n \\ \cdot - ni \end{bmatrix}.$$

□

Example 1.6. *Let (X_n) be a sequence of i.i.d. nonnegative integer-valued random variables with common pmf and cdf f and F respectively. Set $X_0 = 0$. Also, for any $n \in \mathbb{N}$ define $S_n = \sum_{l=1}^{n} X_l$ with pmf f_n and cdf F_n. By using the properties of convolution it is straightforward to show by induction that the following relations hold :*

$$\begin{aligned} f_n &= f^{(k)} * f^{(n-k)} = f^{(n)} \qquad 0 \le k \le n, \quad n \in \mathbb{N}, \\ F_n &= F_k * f^{(n-k)} = \mathbb{1} * f^{(n)} \qquad 0 \le k \le n, \quad n \in \mathbb{N}. \end{aligned}$$

Also, recall from Example 1.1 that the associated reliability function $\overline{F}_n$ can be written in the form

$$\overline{F}_n = \mathbb{1} - F_n = \mathbb{1} - \mathbb{1} * f^{(n)} = \mathbb{1} * \left(e_0 - f^{(n)}\right).$$

Since R is a commutative ring with unity then from Proposition A.1 we get directly the Newton's binomial theorem for the discrete time convolution product of functions.

Proposition 1.6 (Binomial expressions)**.** *Let $f, g : \mathbb{N} \to \mathbb{R}$. Then, the n-fold convolution of $f + g$ is determined by*

$$(f+g)^{(n)} = \sum_{l=0}^{n} \binom{n}{l} f^{(l)} * g^{(n-l)}, \quad n \in \mathbb{N}. \tag{1.17}$$

By a direct application of the above proposition we have the following corollary which will be useful in the sequel.

Corollary 1.1. *For a function $f : \mathbb{N} \to \mathbb{R}$ we take*

$$(e_0 - f)^{(n)} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} f^{(l)}. \tag{1.18}$$

Example 1.7. *Let us consider the sequences e_0 and e_1 of relation (1.6). Then by applying Example 1.5 in Corollary 1.1 we have*

$$(e_0 - e_1)^{(n)} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e_1^{(l)} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e_l, \quad n \in \mathbb{N}$$

and each term $(e_0 - e_1)^{(n)}(k)$ can be determined by

$$(e_0 - e_1)^{(n)}(k) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e_l = (-1)^k \binom{n}{k}, \quad k \in \mathbb{N}. \tag{1.19}$$

A useful property arises when a function f satisfies $f(0) = 0$. The following result will be used in the sequel to simplify the computation of the convolutional inverse of a given function in the case that it exists.

Lemma 1.1. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function with $f(0) = 0$. Then, $f^{(n)}(k) = 0$ for all $k,\ n \in \mathbb{N}$ with $k < n$.*

Proof. By (1.10) and (1.11), we get that

$$f_1 * \ldots * f_n = \sum_{k_1,\ldots,k_n=0}^{\infty} f_1(k_1) \cdots f_n(k_n)\, e_{k_1+\ldots+k_n}. \tag{1.20}$$

Consequently,

$$f^{(n)} = \sum_{k_1,\ldots,k_n=0}^{\infty} f(k_1) \cdots f(k_n)\, e_{k_1+\ldots+k_n}. \tag{1.21}$$

Since by assumption $f(0) = 0$, if at least one of k_i is null, then the corresponding coefficient will be also null. We conclude that

$$f^{(n)} = \sum_{k_1,\ldots,k_n=1}^{\infty} f(k_1) \cdots f(k_n)\, e_{k_1+\ldots+k_n}, \tag{1.22}$$

and consequently the coefficients of e_k are null for $k < n$. □

Remark 1.3. *As an immediate consequence of the above Lemma we get that for any fixed k, the sequence $f^{(n)}(k)$ is eventually zero. Additionally, the function which results from the series $\sum_n f^{(n)}$ is well defined, since each term is actually a finite sum.*

In order to prove the next theorem, we will need a useful identity which is given in the following lemma (a well known combinatorial identity).

Lemma 1.2. *For any $k, n \in \mathbb{N}$ we have*

$$\sum_{n=l}^{l+k} \binom{n}{l} = \binom{l+k+1}{l+1}, \quad \textit{for any } k \in \mathbb{N}. \tag{1.23}$$

Proof. We prove it by induction on k.

For $k = 0$ (1.23), it is obviously true. Let us assume now that (1.23) holds also for an arbitrary $k_0 \in \mathbb{N}$. Then, for $k = k_0 + 1$ we have

$$\sum_{n=l}^{l+k_0+1} \binom{n}{l} = \sum_{n=l}^{l+k_0} \binom{l+k_0+1}{l} + \binom{n}{l} = \binom{l+k_0+1}{l+1} + \binom{l+k_0+1}{l},$$

and by using Pascal's triangle we get directly

$$\sum_{n=l}^{l+k_0+1} \binom{n}{l} = \binom{l+(k_0+1)+1}{l+1}.$$

Consequently, by induction we conclude (1.23). □

Definition 1.3. *Let $f : \mathbb{N} \to \mathbb{R}$. If there exists a function $g : \mathbb{N} \to \mathbb{R}$ such that*

$$f * g = g * f = e_0, \tag{1.24}$$

then g is called the convolutional inverse (inverse of f in the convolution sense) and it is denoted by $f^{(-1)}$.

The inverse of f does not always exist. For example, for $k = 0$ we have to solve the equation

$$[f * g](0) = f(0)g(0) = 1.$$

If $f(0) = 0$, then there exists no solution. In the following proposition we give a necessary and sufficient condition for the existence and uniqueness of the convolutional inverse.

Theorem 1.1. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function. The convolutional inverse $f^{(-1)}$ exists if and only if $f(0) \neq 0$ and is given by*

$$f^{(-1)} = \frac{1}{f(0)} \sum_{n=0}^{\infty} (e_0 - f_0)^{(n)}, \tag{1.25}$$

where

$$f_0(k) = \frac{f(k)}{f(0)}, \quad k \in \mathbb{N}.$$

Additionally, each term $f^{(-1)}(k)$ in (1.26) can be represented as a finite sum and is given by

$$f^{(-1)}(k) = \frac{1}{f(0)} \sum_{n=0}^{k} (e_0 - f_0)^{(n)}(k), \tag{1.26}$$

or alternatively,

$$f^{(-1)}(k) = \frac{1}{f(0)} \sum_{n=0}^{k} (-1)^n \binom{k+1}{n+1} f_0^{(n)}(k). \tag{1.27}$$

Proof. First notice that if $f(0) = 0$, then there is no function g satisfying (1.24) (see the comments after Definition 1.1). Then, it suffices to prove that the inverse exists for $f(0) \neq 0$. For this purpose, without loss of generality we assume that $f(0) = 1$ and consequently we need to prove that

$$f^{(-1)} = \sum_{n=0}^{\infty} (e_0 - f)^{(n)}. \tag{1.28}$$

Indeed, in the general case we have $f = f(0)f_0$ and since $f_0(0) = 1$ we get easily that if the result holds for $f_0^{(-1)}$, then $f^{(-1)} = (1/f(0))f_0^{(-1)}$. This justifies that from (1.28) we get (1.26). Since convolution is commutative, in order to prove (1.28) it suffices to prove that the convolution of f with the righthand member of (1.28) is in fact e_0. Indeed, since $(e_0 - f)(0) = 0$ and by Remark 1.3 we have

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} (e_0 - f)^{(n)}\right) * f &= \left(\sum_{n=0}^{\infty} (e_0 - f)^{(n)}\right) * [e_0 - (e_0 - f)] \\
&= \sum_{n=0}^{\infty} (e_0 - f)^{(n)} - \sum_{n=1}^{\infty} (e_0 - f)^{(n)} \\
&= (e_0 - f)^{(0)} = e_0.
\end{aligned}$$

Additionally, since $(e_0 - f)(0) = 0$ by Lemma 1.1 we have that $(e_0 - f)^{(n)}(k) = 0$, for all $n > k$, and consequently (1.26) holds. Also, using Proposition 1.1 and Lemma 1.2 each $f^{(-1)}(k)$ can be written in the following form

$$
\begin{aligned}
f^{(-1)}(k) &= \sum_{n=0}^{k}(e_0 - f_0)^{(n)}(k) \stackrel{(1.6)}{=} \sum_{n=0}^{k}\sum_{l=0}^{n}(-1)^l \binom{n}{l} f_0^{(l)}(k) \\
&= \sum_{l=0}^{k}\sum_{n=l}^{k}(-1)^l \binom{n}{l} f_0^{(l)}(k) = \sum_{l=0}^{k}(-1)^l f_0^{(l)}(k) \sum_{n=l}^{k}\binom{n}{l} \\
&\stackrel{(1.23)}{=} \sum_{l=0}^{k}(-1)^l \binom{k+1}{l+1} f_0^{(l)}(k).
\end{aligned}
$$

□

Remark 1.4. *Since $f = f(0) f_u$, then by using (1.4), (1.27) can be reformulated as*

$$
f^{(-1)}(k) = \frac{1}{f(0)} \sum_{n=0}^{k} \left(-\frac{1}{f(0)}\right)^n \binom{k+1}{n+1} f^{(n)}(k). \tag{1.29}
$$

Remark 1.5. *Let $f \in \mathbb{N}$ and $n \in \mathbb{N}$. If f has a convolutional inverse then the function $\left(f^{(n)}\right)^{(-1)}$ is also defined because of Remark 1.2 and is equal to $\left(f^{(-1)}\right)^{(n)}$ for any $n \in \mathbb{N}$. We denote this sequence by $f^{(-n)}$. This observation allows us to extend the results of Proposition 1.4 on $\mathbb{Z}$.*

Example 1.8. *Let us consider the sequence of Example 1.7. We have $(e_0 - e_1)(0) = 1$. Then, using the relations (1.19) and (1.27) we take*

$$
(e_0 - e_1)^{(-1)}(k) = \sum_{n=0}^{k}(-1)^n \binom{k+1}{n+1} (-1)^k \binom{n}{k} = \sum_{n=0}^{k}(-1)^{k-n} \binom{k+1}{n+1} \binom{n}{k} = 1, \quad k \in \mathbb{N},
$$

and thus we obtain

$$
(e_0 - e_1)^{(-1)} = \mathbb{1} \quad \textit{or} \quad \mathbb{1}^{(-1)} = e_0 - e_1.
$$

*Let us now consider a nonnegative random variable X with pmf f and cdf F. From Example **??** and the above equation we get directly*

$$
f = \mathbb{1}^{(-1)} * F = (e_0 - e_1) * F,
$$

and consequently

$$
f = (e_0 - e_1) * (\mathbb{1} - \overline{F}),
$$

where $\overline{F}$ is the associated reliability function.

Remark 1.6. *As $\mathbb{1}$ corresponds to the procedure of formation of partial sums of a sequence, its inverse $e_0 - e_1$ corresponds to the procedure of formation of first order differences.*

Remark 1.7. *Set $R^* := \{f : \mathbb{N} \to \mathbb{R} |\ f(0) \neq 0\}$. From Proposition 1.1 and Theorem 1.1 we have that the pair $(R^*, *)$ forms an abelian group.*

In many cases the form of a function is complex and consequently it is difficult to compute the convolutional inverse of a function via Equation 1.26, but it is possible to use the following recurrent way,

Proposition 1.7. *Let $f \in C^*$. Its convolutional inverse can be computed as follows:*

$$f^{(-1)}(k) = \begin{cases} \dfrac{1}{f(0)} & \text{if } k = 0 \\ -\dfrac{1}{f(0)} \displaystyle\sum_{l=0}^{k-1} f(k-l) f^{(-1)}(l) & \text{otherwise.} \end{cases} \tag{1.30}$$

Proof. In order to prove this form, we'll use the equation

$$f * f^{(-1)} = e_0.$$

Since $f(0) \neq 0$ we have

$$f(0) f^{(-1)}(0) = 1 \quad \Longrightarrow f^{(-1)}(0) = \frac{1}{f(0)}.$$

Now, for $k \in \mathbb{N}^*$, we have

$$0 = e_0(k) = [f * f^{(-1)}](k) = \sum_{l=0}^{k} f(k-l) f^{(-1)}(l) = f(0) f^{(-1)}(k) + \sum_{l=0}^{k-1} f(k-l) f^{(-1)}(l),$$

and consequently we get the desired form.

□

Several functions defined usually for real or complex numbers can be extended for sequences and similar properties can be derived. We are particularly interested in the exponential function.

Definition 1.4. *Let $f : \mathbb{N} \to \mathbb{R}$. The function*

$$\exp(f) := \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}, \quad n \in \mathbb{N},$$

is called the exponential function of f.

Remark 1.8. *By Remark 1.3, it is well defined for all f such that $f(0) = 0$. For that reason the exponential function of any $f : \mathbb{N} \to \mathbb{R}$ with $f(0) = 0$, is expressed in the form*

$$\exp(f)(k) = \sum_{n=0}^{k} \frac{f^{(n)}(k)}{n!}, \quad k \in \mathbb{N}. \tag{1.31}$$

In the following proposition we give some properties of the exponential function.

Proposition 1.8. *Let $f, g : \mathbb{N} \to \mathbb{R}$, $n \in \mathbb{Z}$ and $c \in \mathbb{R}$ be a constant. The following properties hold, whenever $\exp(f)$ and $\exp(g)$ are both well defined.*

$$\begin{aligned} \exp(0) &= e_0, \\ \exp(e_0) &= e \cdot e_0, \\ \exp(c \cdot e_0) &= e^c \cdot e_0, \\ \exp(f+g) &= \exp(f) * \exp(g), \\ (\exp(f))^{(n)} &= \exp(nf), \\ (\exp(f))^{(-1)} &= \exp(-f). \end{aligned}$$

In the following proposition we study the existence and form of the exponential function.

Proposition 1.9. *Let* $f : \mathbb{N} \to \mathbb{R}$. *Then, the exponential function* $\exp(f)$ *is always well defined and is given by*

$$\exp(f)(k) = e^{f(0)} \cdot \left(\sum_{n=0}^{k} \frac{f_+^{(n)}(k-n)}{n!} \right), \quad k \in \mathbb{N}. \tag{1.32}$$

Proof. Since any f can be rewritten as $f = (f - f(0) \cdot e_0) + f(0) \cdot e_0$, i.e., as a sum of two functions for which the associated exponential function is well defined (see Remark 1.8). Then, from Proposition 1.8 we have

$$\begin{aligned} \exp(f) &= \exp\left((f - f(0) \cdot e_0) + f(0) \cdot e_0\right) = \exp\left(f - f(0) \cdot e_0\right) * \exp\left(f(0) \cdot e_0\right) \\ &= e^{f(0)} \cdot \exp\left(f - f(0) \cdot e_0\right). \end{aligned}$$

Consequently, the exponential function is well defined for any real sequence f and each term $\exp(f)(k)$ can be expressed in the following form

$$\exp(f)(k) = e^{f(0)} \cdot \left(\sum_{n=0}^{k} \frac{\left(f - f(0) \cdot e_0\right)^{(n)}(k)}{n!} \right), \quad k \in \mathbb{N}. \tag{1.33}$$

Furthermore, since $\left(f - f(0) \cdot e_0\right)(0) = 0$, we have that the following relation holds for any $k,\ n \in \mathbb{N}$ with $k \geq n$,

$$\begin{aligned} \left(f - f(0) \cdot e_0\right)^{(n)}(k) &= \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n \geq 1}} f(l_1) \cdots f(l_n) = \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n \geq 1}} f_+(l_1-1) \cdots f_+(l_n-1) \\ &= \sum_{l_1+\ldots+l_n=k-n} f_+(l_1) \cdots f_+(l_n) = f_+^{(n)}(k-n). \end{aligned}$$

Accordingly, from the previous observation and (1.33) we obtain the desired result. □

Proposition 1.10. *For any* $f, g : \mathbb{N} \to \mathbb{R}$ *we have*

$$\exp(f) = \exp(g) \Longleftrightarrow f = g.$$

Proof.

($\Longleftarrow$) It is direct by definition.

($\Longrightarrow$) For $k = 0$, we have

$$e^{f(0)} = \exp(f)(0) = \exp(g)(0) = e^{g(0)} \Longrightarrow f(0) = g(0).$$

Furthermore, since

$$e^{f(0)} f(1) = \exp(f)(1) = \exp(g)(1) = e^{g(0)} g(1),$$

we get directly that $f(1) = g(1)$ or $f_+(0) = g_+(0)$. In a similar way we take consequentially that

$$f(n) = g(n), \quad n \in \{0, 1, \ldots, k-1\}.$$

and

$$f_+(n) = g_+(n), \quad n \in \{0, 1, \ldots, k-2\}. \tag{1.34}$$

Since,

$$\begin{aligned}
\exp(f)(k) &= \exp(g)(k) \quad \Longrightarrow \\
e^{f(0)}\left(\frac{f_+^{(1)}(k-1)}{1!} + \ldots + \frac{f_+^{(k)}(0)}{k!}\right) &= e^{g(0)}\left(\frac{g_+^{(1)}(k-1)}{1!} + \ldots + \frac{g_+^{(k)}(0)}{k!}\right) \quad \Longrightarrow \\
\frac{f_+^{(1)}(k-1)}{1!} + \ldots + \frac{f_+^{(k)}(0)}{k!} &= \frac{g_+^{(1)}(k-1)}{1!} + \ldots + \frac{g_+^{(k)}(0)}{k!} \quad \overset{(1.34)}{\Longrightarrow} \\
f_+(k) &= g_+(k),
\end{aligned}$$

then we get by induction the desired result.

□

3. Convolution product of sequences of matrices

In this section we extend the results of the previous sections to the convolution product of matrix valued functions. We give its relationship with the convolutional product of real valued functions and we also provide some concrete examples in Probability theory. Also, we give the associated algebraic properties by paying special attention to the convolutional inverse of a matrix-valued function.

In order to introduce the convolution product of sequences of matrices we need some definitions. Let $E = \{1, 2, \ldots, s\}$ be a finite set. We denote by $\mathcal{M}_s := \mathbb{R}^{s\times s}$ the set of all real matrices on $E\times E$ and by $\mathcal{M}_s(\mathbb{N})$ the set of all matrix-valued functions defined on $\mathbb{N}$ with values in $\mathcal{M}_s$. For $A \in \mathcal{M}_s(\mathbb{N})$ we write $A := (A(k); k \in \mathbb{N})$, where for fixed $k \in \mathbb{N}$, the matrix $A(k) = (a_{ij}(k))_{i,j\in E}$. We also denote by I_s and 0_s the identity and the zero matrix on the set $\mathcal{M}_s$ respectively. The element A can also be interpreted as a matrix of real valued sequences, so $A = (a_{ij})_{i,j\in E}$, where $a_{ij} \in R(= \mathbb{R}^{\mathbb{N}})$. In this sense $A \in R^{s\times s}$, so it corresponds to a matrix with elements in a commutative ring. This fact will be exploited in the development of the theory.

Definition 1.5. *Let $A, B \in \mathcal{M}_s(\mathbb{N})$. The matrix-valued function $A * B \in \mathcal{M}_s(\mathbb{N})$, given by*

$$[A * B](k) := \sum_{l=0}^{k} A(k-l)B(l), \quad k \in \mathbb{N},$$

with elements

$$[A * B]_{ij}(k) := \sum_{r\in E}\sum_{l=0}^{k} \alpha_{ir}(k-l)\beta_{rj}(l), \quad i, j \in E,\ k \in \mathbb{N},$$

is said to be the discrete time convolution product of A and B.

Remark 1.9. *The Definition 1.5 emphasizes the interpretation of $A*B$ as the sequence of matrices which results from the convolution of the sequences of matrices A and B. Nevertheless, it will also be useful to interpret $A*B$ as the matrix of sequences which results from the product of the matrices of sequences A and B. In particular, $A * B = ([A * B]_{ij})_{i,j\in E}$, where*

$$[A * B]_{ij} = \sum_{r\in E} \alpha_{ir} * \beta_{rj}.$$

In this way, the convolution of two matrices in $\mathcal{M}_s(\mathbb{N})$ corresponds to the usual product of matrices in $R^{s\times s}$, where the product in R is the convolution of real valued sequences.

An important role in the rest of theory will be played by the matrix-valued function denoted by $\mathbb{I}$ with constant value I_s. In particular,

$$\mathbb{I} = (I_s, I_s, \ldots). \tag{1.35}$$

Alternatively, in view of Remark 1.9, $\mathbb{I}$ can be written as a matrix of sequences in the form

$$\mathbb{I} = \begin{pmatrix} \mathbb{1} & 0 & \ldots & 0 \\ 0 & \mathbb{1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbb{1}. \end{pmatrix}. \tag{1.36}$$

In the following examples we give some applications of the convolution product of matrix-valued functions in Probability theory.

Example 1.9. *Let J be a homogeneous Markov chain, $(X_n)_{n\geq 1}$ be a sequence of i.i.d non negative integer valued random variables and $X_0 = 0$. Let f_{ij}, F_{ij} and $\overline{F}_{ij}$ be the associated conditional sojourn time pmf, cdf and the survival function of X_n when J goes from a state i to a state j. Let us also denote by f, F and $\overline{F}$ the matrix valued functions with elements f_{ij}, F_{ij} and $\overline{F}_{ij}$ respectively. From the previous comments we get directly that any F_{ij} can be determined by*

$$F_{ij} = \mathbb{1} * f_{ij} = \sum_{r\in E} \mathbb{I}_{0_{ir}} * f_{rj}, \quad i, j \in E,$$

and by Remark 1.9 we get in matrix form

$$F = \mathbb{I} * f.$$

Therefore, each $\overline{F}_{ij}$ can be written as

$$\overline{F}_{ij} = \mathbb{1} - F_{ij}$$

and consequently

$$\overline{F} = \mathbb{II} - \mathbb{I} * f,$$

where $\mathbb{II}$ is the matrix valued function whose entries are $\mathbb{1}$.

Example 1.10. *Let us consider a sequence of i.i.d non negative integer valued random variables $(X_n)_{n\geq 1}$ and $X_0 = 0$. Let also $S_n = \sum_{l=0}^{n} X_l$, $n \in \mathbb{N}$ and $(J_n)_{n\in\mathbb{N}}$ be a homogeneous Markov chain with finite state space $E := \{1, \ldots, s\}$ i.e.*

$$\mathbb{P}(J_n = j \mid J_{0:n-1}) \overset{a.s}{=} \mathbb{P}(J_n = j \mid J_{n-1}), \quad \textit{for all } j \in E$$

Furthermore, assume that the pair (J, S) satisfies the following property

$$\mathbb{P}(J_n = j, S_n - S_{n-1} = k | J_{0:(n-2)}, \, J_{n-1} = i, \, S_{0:n-1}) = \mathbb{P}(J_n = j, S_n - S_{n-1} = k | J_n = i). \tag{1.37}$$

From this property we have that

$$\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_{0:(n-2)}, \, J_{n-1} = i, \, S_{0:n-1}) = \mathbb{P}_i(J_2 = j, X_2 = k). \tag{1.38}$$

If we denote by $P := (P(k))$ and $P_2 := (P_2(k))$ the matrix valued functions, where

$$P_{ij}(k) = \mathbb{P}(J_1 = j, \, S_1 = k \mid J_0 = i) := \mathbb{P}_i(J_1 = j, \, S_1 = k),$$

and

$$P_{2_{ij}}(k) = \mathbb{P}(J_2 = j, S_2 = k | J_0 = i) := \mathbb{P}_i(J_2 = j, S_2 = k), \quad i, j \in E, \ k, n \in \mathbb{N}.$$

Then, by using Example 1.4 and the properties (1.37), (1.38) we get directly

$$P_{2_{ij}}(k) = \sum_{r \in E} \sum_{l=0}^{k} P_{ir}(l) P_{rj}(k-l) = [P * P]_{ij}(k) \tag{1.39}$$

and consequently

$$P_2 = P * P.$$

Now, if we denote by $F_1 = (F_1(k))$ and $F_2 = (F_2(k))$ the matrix valued sequences where

$$F_{1_{ij}}(k) = \mathbb{P}_i(J_1 = j, S_1 \leq k)$$

and

$$F_{2_{ij}}(k) = \mathbb{P}_i(J_2 = j, S_2 \leq k),$$

then from Example 1.4 and relation (1.39) we conclude that

$$F_2 = \mathbb{I} * P_2 = (\mathbb{I} * P) * P = F_1 * P.$$

In the following Proposition we give some algebraic properties of the convolution product of matrix valued functions, that can be verified easily.

Proposition 1.11. *(Properties of the convolution product)[convolution algebra]*

(i) Let A,B,C $\in \mathcal{M}_s(\mathbb{N})$ be three matrix-valued functions. Then $*$ is an associative and distributive binary operation, i.e.

$$\begin{aligned} A * (B * C) &= (A * B) * C, \\ (A + B) * C &= A * C + B * C, \\ C * (A + B) &= C * A + C * B. \end{aligned}$$

(ii) The identity element for the convolution product of matrices is the matrix valued function E_0 $\in \mathcal{M}_s(\mathbb{N})$ which is given in the following form :

$$E_0(k) = \begin{cases} I_s & \text{if } k=0 \\ 0_s & \text{otherwise}. \end{cases}$$

or as a matrix of sequences in the form

$$E_0 = \begin{pmatrix} e_0 & 0 & \dots & 0 \\ 0 & e_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_0 \end{pmatrix}$$

and satisfies

$$A * E_0 = E_0 * A = A,$$

for all A $\in \mathcal{M}_s(\mathbb{N})$.

Remark 1.10. *In the general case, the equality $A * B$=$B * A$ doesn't hold true, so the operation $*$ is not commutative. This is a direct consequence of the non-commutativity of the matrix product.*

It will be useful for the rest of the theory to identify any matrix-valued function $A \in \mathcal{M}_s(\mathbb{N})$ as a formal power series with matrix coefficients or an infinite dimensional vector of matrices. In order to do this we will need some definitions and properties of the convolution product. For this purpose, from Proposition 1.11 $\mathcal{M}_s(\mathbb{N}) := R_{\mathcal{M}}$ forms a ring under the binary operations of usual addition of matrices and convolution product of matrix valued functions. Also, the function E_0 is the corresponding identity element and consequently $R_{\mathcal{M}}$ is a noncommutative ring with unity. In addition, we denote the set of all power series with $s \times s$ real matrix coefficients $\mathbb{R}^{s\times s}[[x]]$ equipped with the following binary operators

$$\begin{aligned} + :\ & (A(0) + A(1)\,x + \ldots) \quad + \quad (B(0) + B(1)\,x + \ldots) = (A(0) + B(0)) + (A(1) + B(1))\,x + \ldots, \\ \bullet :\ & (A(0) + A(1)\,x + \ldots) \quad \bullet \quad (B(0) + B(1)\,x + \ldots) = (A(0)B(0)) + (A(0)B(1) + A(1)B(0))\,x + \ldots, \end{aligned}$$

and it forms a non commutative ring with unity given by E_0.

For the latter expression we define the set of infinite dimensional vectors of matrices $(\mathbb{R}^{s\times s})^{\mathbb{N}}$ equipped with the following binary operators

$$\begin{aligned} (+) :\ & (A(0), A(1), \ldots) \quad (+) \quad (B(0), B(1), \ldots) = (A(0) + B(0), A(1) + B(1), \ldots), \\ (\bullet) :\ & (A(0), A(1), \ldots) \quad (\bullet) \quad (B(0), B(1), \ldots) = (A(0)B(0), A(0)B(1) + A(1)B(0), \ldots). \end{aligned}$$

Of course $\left((\mathbb{R}^{s\times s})^{\mathbb{N}}, (+), (\bullet)\right)$ forms a non commutative ring with unity denoted by $(I_s, 0_s, \ldots)$. It can easily be proved that $R_{\mathcal{M}}$, $\mathbb{R}^{s\times s}[[x]]$ and $\mathbb{R}^{s\times s}$ are isomorphic rings.

Proposition 1.12. *The rings $(R_{\mathcal{M}}, +, *)$, $(\mathbb{R}^{s\times s}[[x]], +, \bullet)$ and $\left((\mathbb{R}^{s\times s})^{\mathbb{N}}, (+), (\bullet)\right)$ are algebraically identical.*

As a result of the above, we can express any matrix valued function defined on $\mathbb{N}$ in the following form

$$A \cong (A(0), A(1), A(2), \ldots) \cong A(0) + A(1)\,x + A(2)\,x^2 + \ldots, \tag{1.40}$$

We can represent E_0 as

$$E_0 \cong (I_s, 0_s, 0_s, \ldots) \cong I_s. \tag{1.41}$$

As we can see E_0 is interpreted as a vector of matrices which attributes I_s to the first component with index zero and the second one refers to the corresponding constant polynomial with matrix coefficients. More generally, we denote by

$$E_i \cong \underbrace{(0_s, \ldots, 0_s, I_s, 0_s, \ldots)}_{I_s \text{ in the } i\text{-th index}} \cong I_s\,x^i. \tag{1.42}$$

Remark 1.11. *Consider an almost surely random variable $X = j$ and a Markov chain with probability transition matrix I_s. Then, we can represent E_i as the diagonal matrix in which the entries along the main diagonal are given by $\mathbb{P}(J_n = i,\ X = j \mid J_{n-1} = i) = 1$.*

$$E_i * E_j \cong \left(I_s x^i\right)\left(I_s x^j\right) = I_s x^{i+j} \cong E_{i+j}. \tag{1.43}$$

We can also compute the conditional cdf associated to E_i. More specifically, if we denote by $\mathbb{I}_i$ the sequence of matrices which starts with 0_s and is I_s after the i-th index and onwards then

$$\mathbb{I}_i = \mathbb{I} * E_i \cong \left(\sum_{l=0}^{\infty} I_s x^l\right) I_s x^i = \left(\sum_{l=0}^{\infty} I_s x^{l+i}\right) = \left(\sum_{l=i}^{\infty} I_s x^l\right) \cong \underbrace{(0_s, \ldots, 0_s, I_s, I_s, \ldots)}_{I_s \text{ from the } i\text{-th index}} \tag{1.44}$$

The obvious relation $\mathbb{I}_0 = (I_s, I_s, \ldots, I_s)$ holds. Similarly, we have

$$\mathbb{I}_i * E_j = \mathbb{I} * E_i * E_j = \mathbb{I} * E_{i+j} = \mathbb{I}_{i+j}. \tag{1.45}$$

Remark 1.12. *It is easy to notice that the basis elements E_i can be represented as a diagonal matrix of real sequences, i.e,*

$$E_i = \begin{pmatrix} e_i & 0 & \dots & 0 \\ 0 & e_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_i \end{pmatrix}, \tag{1.46}$$

where e_i are the basis elements of (1.5).

Also, in a similar way we obtain

$$\mathbb{I}_i = \begin{pmatrix} \mathbb{1}_i & 0 & \dots & 0 \\ 0 & \mathbb{1}_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{1}_i \end{pmatrix}, \tag{1.47}$$

where $\mathbb{1}_i$ are the sequences given by (1.8).

The basis elements E_i (1.42) can be used for the simplification of the computations with convolutions avoiding the formal power series with matrix coefficients. In particular, we can use (1.41) and consequently (1.40) is reformulated as

$$A = \sum_{k=0}^{\infty} A(k)\, E_k. \tag{1.48}$$

Then, since any convolutional product of E_0 with any other matrix valued function is commutative we have

$$\begin{aligned} A * B &= \left(\sum_{k=0}^{\infty} A(k)E_k\right) * \left(\sum_{l=0}^{\infty} B(l)E_l\right) = \left(\sum_{k=0}^{\infty} A(k)E_k\right) * \left(\sum_{l=0}^{\infty} E_l B(l)\right) \\ &= \sum_{k,l=0}^{\infty} A(k)\,[E_k * E_l]\, B(l) = \sum_{k,l=0}^{\infty} A(k)E_{k+l}B(l) = \sum_{k,l=0}^{\infty} [A(k)B(l)]\, E_{k+l} \\ &= \sum_{u=0}^{\infty}\left(\sum_{k,l:k+l=u} A(k)B(l)\right) E_u = \sum_{u=0}^{\infty} [A * B](u)E_u. \end{aligned} \tag{1.49}$$

Below, we introduce the convolutional powers of a matrix-valued function.

Definition 1.6. *Let $A \in \mathcal{M}_s(\mathbb{N})$ be a matrix-valued function and $n \in \mathbb{N}$. The n-fold convolution of sequences of matrices $A^{(n)}$ is the matrix-valued function defined by :*

$$\begin{aligned} A^{(0)} &:= E_0 \\ A^{(n)} &:= A * A^{(n-1)}, \quad n \geq 1 \end{aligned}$$

From the above definition we can get directly that for all $n, k0$,

$$A^{(n)}(k) := \sum_{k_1,\dots,k_n \geq 0 // k_1+k_2+\dots+k_n = k} A(k_1)A(k_2)\cdots A(k_n). \tag{1.50}$$

Proposition 1.13 (Binomial expressions)**.** *Let $A, B \in R_{\mathcal{M}}$ with $A * B = B * A$. Then, the n-fold convolution of their sum can be determined by*

$$(A+B)^{(n)} = \sum_{l=0}^{n} \binom{n}{l} \left(A^{(l)} * B^{(n-l)}\right), \quad n \in \mathbb{N}. \tag{1.51}$$

Corollary 1.2. *For a function $A \in R_{\mathcal{M}}$ we have*

$$(E_0 - A)^{(n)} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} A^{(l)}. \tag{1.52}$$

Example 1.11. *Let $A = diag\{\alpha_i\}_{i \in E} \in R_{\mathcal{M}}$. Then, the n-fold convolution powers $A^{(n)}$ are given by*

$$A^{(n)} = diag\left\{\alpha_i^{(n)}\right\}_{i \in E}. \tag{1.53}$$

The result follows easily from the fact that if $A = diag\{a_i\}_{i \in E}$ and $B = diag\{b_i\}_{i \in E}$, then $A * B = diag\{a_i * b_i\}_{i \in E}$.

Example 1.12. *Let J and $(X_n)_{n \in \mathbb{N}}$ be defined as in Example 1.10. By using the properties of convolution product of matrices it is straightforward to show by induction that the following relations hold:*

$$\begin{aligned} P_n &= P_k * P_{n-k} = P^{(n)}, \quad 0 \le k \le n, \quad n \in \mathbb{N}, \\ F_n &= F_k * P_{n-k} = \mathbb{I} * P^{(n)}, \quad 0 \le k \le n, \quad n \in \mathbb{N}. \end{aligned}$$

Example 1.13. *Consider the basis elements E_i of (1.42). Since $E_i * E_j = E_{i+j}$, we get easily that*

$$E_i^{(n)} = \underbrace{E_i * E_i * \ldots * E_i}_{n \text{ times}} = E_{ni}.$$

Also, using Example 1.53 and Remark 1.12 we get directly

$$E_i^{(n)} = diag\{e_i^{(n)}\} = diag\{e_{ni}\} = E_{ni}.$$

Example 1.14. *The n-fold convolutional powers of the matrix-valued function $\mathbb{I}_i$ is given by*

$$\mathbb{I}_i^{(n)} = \mathbb{I}_i * \cdots * \mathbb{I}_i = \begin{pmatrix} \mathbb{1}_i^{(n)} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \ldots & & \mathbb{1}_i^{(n)} \end{pmatrix},$$

where $\mathbb{1}_i^{(n)}$ is given by Example 1.5.

Then, from the above we can reformulate (1.49) as

$$A * B = \sum_{k=0}^{\infty} [A * B](k) E_1^{(k)}.$$

In the following Lemma we give a beneficial property when a matrix-valued function A satisfies $A(0) = 0_s$, which will be used in the sequel in order to simplify the computation of the convolutional inverse of a given matrix-valued function in the case that it exists.

Lemma 1.3. *Let $A \in \mathcal{M}_s(\mathbb{N})$ be a matrix-valued function with $A(0) = 0_s$. Then, $A^{(n)}(k) = 0_s$ for all $k,\ n \in \mathbb{N}$ with $k < n$.*

Proof. By (1.48) and (1.49) we get that

$$A_1 * \cdots * A_n = \sum_{k_1, \ldots k_n = 0}^{\infty} A_1(k_1) \cdots A_n(k_n) E_{k_1 + \cdots + k_n}. \tag{1.54}$$

Consequently,

$$A^{(n)} = \sum_{k_1,\dots k_n=0}^{\infty} A(k_1)\cdots A(k_n) E_{k_1+\cdots+k_n}. \tag{1.55}$$

Since by assumption $A(0) = 0_s$, if at least one of k_i is null, then the corresponding matrix coefficient will be also null. We conclude that

$$A^{(n)} = \sum_{k_1,\dots k_n=1}^{\infty} A(k_1)\cdots A(k_n) E_{k_1+\cdots+k_n}, \tag{1.56}$$

and consequently the matrix coefficient of E_k are null for $k < n$. □

Remark 1.13. *As an immediate consequence of the above lemma we get that for any fixed k, the matrix valued sequence $A^{(n)}(k)$ is eventually null. Additionally, the matrix valued function which results from the series $\sum_n A^{(n)}$ is well defined, since each term is actually a finite sum.*

Definition 1.7. *Let $A \in \mathcal{M}_s(\mathbb{N})$. If there exists a matrix valued function $B \in \mathcal{M}_s(\mathbb{N})$ such that*

$$A * B = E_0, \tag{1.57}$$

then B is called the right convolutional inverse of A (right inverse of A in the convolution sense) and it is denoted by $A_r^{(-1)}$. If there exists a matrix valued function $C \in \mathcal{M}_s(\mathbb{N})$ such that

$$C * A = E_0, \tag{1.58}$$

then C is called the left convolutional inverse of A (left inverse of A in the convolution sense) and it is denoted by $A_l^{(-1)}$. It is easy to see that if both $A_r^{(-1)}$ and $A_l^{(-1)}$ exist, then they necessarily coincide. Then, there exists a necessarily unique $B \in \mathcal{M}_s(\mathbb{N})$ such that

$$A * B = B * A = E_0, \tag{1.59}$$

it is called the convolutional inverse of A and denoted by $A^{(-1)}$.

The inverse of A does not always exist. For example, denote the matrix-valued function

$$A(k) = C^{k+1}, \quad k \in \mathbb{N},$$

where C is a nilpotent matrix of a finite index. For $k = 0$ we have to solve the equation

$$[B * A](0) = B(0)A(0) = B(0)\,C = E_0(0) = I_s.$$

Since C has been assumed to be nilpotent then it is a singular matrix and consequently there exists no solution.

Theorem 1.2. *Let $A \in \mathcal{M}_s(\mathbb{N})$ be a matrix-valued function. The convolutional inverse $A^{(-1)}$ exists if and only if $A(0)$ is non-singular and is given by the following relation*

$$A^{(-1)} = (A(0))^{-1} \left[\sum_{m=0}^{\infty} (E_0 - A_0))^{(m)}\right] = \left[\sum_{m=0}^{\infty} (E_0 - A_0^*)^{(m)}\right] (A(0))^{-1} \tag{1.60}$$

where,

$$\begin{aligned} A_0(k) &= A(k)\,(A(0))^{-1}, \quad k \in \mathbb{N}, \\ A_0^*(k) &= (A(0))^{-1}\,A(k), \quad k \in \mathbb{N}. \end{aligned}$$

Additionally, each term $A^{(-1)}(k)$ in (1.61) can be represented as a finite sum and is given by

$$A^{(-1)}(k) = (A(0))^{-1}\left[\sum_{m=0}^{k}(E_0 - A_0)^{(m)}(k)\right] = \left[\sum_{m=0}^{k}(E_0 - A_0^{*})^{(m)}(k)\right](A(0))^{-1} \quad (1.61)$$

or

$$\begin{aligned} A^{(-1)}(k) &= (A(0))^{-1}\left[\sum_{m=0}^{k}\binom{k+1}{m+1}(-1)^m A_0^{(m)}(k)\right] && (1.62)\\ &= \left[\sum_{m=0}^{k}\binom{k+1}{m+1}(-1)^m A_0^{*(m)}(k)\right](A(0))^{-1}. && (1.63)\end{aligned}$$

Proof. We prove that the middle-side of (1.61) holds. The corresponding for the right-side can be proven similarly.

First, the comments after Definition 1.7 imply that for a matrix-valued function $A \in R_{\mathcal{M}}$ where $A(0)$ forms a singular matrix, there is no $B \in R_{\mathcal{M}}$ satisfying (1.59). Then, it suffices to show that the inverse exists if $A(0)$ is non-singular. To obtain the desired form, without loss of generality we assume that $A(0) = I_s$ and consequently it suffices to prove that

$$A^{(-1)} = \sum_{m=0}^{\infty}(E_0 - A)^{(m)}, \quad (1.64)$$

because in the general case we have $A = A_0\, A(0)$ and since $A_0(0) = I_s$ we can easily get that if the result holds for $A_0^{(-1)}$, then $A^{(-1)} = (A(0))^{-1}\, A_0^{(-1)}$. In order to prove (1.64) it suffices to prove that the convolutions between A and the righthand member of (1.64) satisfy the left and middle side of (1.59). Indeed, since $E_0 - A(0) = 0_s$ then the condition of Remark (1.13) is satisfied and consequently we have

$$\begin{aligned} \left(\sum_{m=0}^{\infty}(E_0 - A)^{(m)}\right) * A &= \left(\sum_{m=0}^{\infty}(E_0 - A)^{(m)}\right) * [E_0 - (E_0 - A)]\\ &= \sum_{m=0}^{\infty}(E_0 - A)^{(m)} - \sum_{m=0}^{\infty}(E_0 - A)^{(m+1)}\\ &= (E_0 - A)^{(0)} = E_0. \end{aligned}$$

The middle member of (1.59) is proven similarly. Additionally, since $E_0 - A(0) = 0_s$ by Lemma 1.3 we have that $(E_0 - A)^{(m)}(k) = 0_s$, for all $k < m$, and consequently (1.60) holds. Also, by using Corollary 1.2, we can reform (1.60) as

$$\begin{aligned} A^{(-1)}(k) &= \sum_{m=0}^{k}\sum_{l=0}^{m}(-1)^l\binom{m}{l}A^{(l)}(k) = \sum_{l=0}^{k}\sum_{m=l}^{k}(-1)^l\binom{m}{l}A^{(l)}(k)\\ &= \sum_{l=0}^{k}(-1)^l A^{(l)}(k)\left(\sum_{m=l}^{k}\binom{m}{l}\right) = \sum_{m=0}^{k}(-1)^m\binom{k+1}{m+1}A^{(m)}(k). \end{aligned}$$

□

Remark 1.14. *We denote by $R^{*}_{\mathcal{M}}$ the set of matrix-valued functions which on 0 are non-singular matrices. From Theorem 1.60, we get that the pair $(R^{*}_{\mathcal{M}}, *)$ is a (non-abelian) Group.*

Next, we introduce the convolutional extension of the determinant and the adjugate of a given matrix-valued function in order to give an alternative representation of the convolutional inverse.

Our motivations arises from N. Limnios and G. Oprisan. ([17]) in which they wrote about the determinant in the convolutional sense. Here, we extend this idea carefully and we give some important properties of it and proving the link between the convolutional inverses for real and matrix sequences.

First, a scalar convolution between a real and a matrix sequence will be useful for the development of the theory and is given as follows

Definition 1.8. *Let $A = (\alpha_{ij}) \in R_{\mathcal{M}}$ and $f \in R$. The scalar convolution of f and A is given as*

$$f * A = (f * \alpha_{ij})_{i,j\in E} = (\alpha_{ij} * f)_{i,j\in E} = A * f.$$

Let A be a 2×2 matrix-valued function. The real valued function $det(A)$ is said to be the convolutional determinant of A given by

$$det(A) = \alpha_{11} * \alpha_{22} - \alpha_{12} * \alpha_{21}.$$

This function for a 3×3 matrix valued function A is defined in a similar way

$$det(A) = \sum_{i\in E} (-1)^{i+j} \alpha_{ij} * det\left(A_{ij}\right), \quad \text{for a fixed } j \in \{1, 2, 3\},$$

where $det(A)_{ij}$ is the convolutional determinant of a 2×2 matrix valued function which is created by excluding the i-th row and j-th column entry of A.

In the general case, the convolutional determinant of a $s \times s$ matrix-valued function A is defined recursively

$$det(A) = \sum_{i\in E} (-1)^{i+j} \alpha_{ij} * det\left(A_{ij}\right), \quad \text{for a fixed } j \in E,$$

where $det(A)_{ij}$ is the convolutional determninant of the $(s-1) \times (s-1)$ matrix sequence which is created by exclunding the i-th row and j-th column entry of A.

Remark 1.15. *It is straightforward to notice that the convolutional determinant of A on zero is the usual determinant of $A(0)$.*

In the following definition we introduce the adjugate of a matrix-valued function

Definition 1.9. *Let $A \in R_{\mathcal{M}}$. The convolutional adjugate A denoted by $adj(A$ is the sequence of matrices given by*

$$adj(A)_{ij} = (-1)^{i+j} det\left(A_{ji}\right),$$

where $det\left(A_{ji}\right)$ is the convolutional determinant of A excluding the $j-th$ row and $i-th$ column.

Some useful properties of det ar given in the following proposition.

Proposition 1.14. *For any $A, B \in R_{\mathcal{M}}$ and $f \in R$ we get*

$$\begin{aligned} det\left(E_0\right) &= e_0, && (1.65)\\ det(A * B) &= det(A) * det(B), && (1.66)\\ det\left(A^{(n)}\right) &= det(A)^{(n)}, && (1.67)\\ det(f * A) &= f^{(s)} * det(A). && (1.68) \end{aligned}$$

Furthermore, if A has two identical rows, then $det(A) \equiv 0$.

In the following Theorem we show an alternative form of the convolutional inverse for a given matrix-valued function.

Theorem 1.3. *Let $A \in R_{\mathcal{M}}$. The convolutional inverse $A^{(-1)}$ exists if and only if $A(0)$ is a nonsingular matrix and is given by*

$$A^{(-1)} = det(A)^{(-1)} * adj(A). \tag{1.69}$$

Proof. First, it's direct to show that the desired relation holds if and only if the convolutional inverse of $det(A)$ exists. From Theorem 1.60 we have that a matrix valued function has a convolutional inverse if and only if $det(A)$ is a real function which has a convolutional inverse. Indeed, for $k = 0$ we have that $A(0)$ is nonsingular iff its determinant is zero. The latter means that $det\,(A(0)) \neq 0$ and consequently $det(A)$ has a convolutional inverse.

In order to give (1.69) we need to show that

$$A * adj(A) = det(A) * E_0 = adj * A. \tag{1.70}$$

Let $i, j \in E$. If these elements are equal we have

$$det(A) = \sum_{r \in E} A_{jr} * det(A)_{jr} = \sum_{r \in E} (-1)^{r+j} A_{jr} * adj(A)_{rj}. \tag{1.71}$$

Next, assume i, j be two distinct elements of E and B be a sequence of matrices obtained by replacing row j of A with row i of A. Then, Proposition 1.14 gives

$$\begin{aligned} 0 = det(B) &= \sum_{r \in E} (-1)^{j+r} B_{jr} * det(B)_{jr} = \sum_{r \in E} (-1)^{j+r} A_{ir} * det(A)_{jr} \\ &= \sum_{r \in E} A_{ir} * adj(A)_{jr}. \end{aligned} \tag{1.72}$$

Consequently, by combining (1.71) and (1.72) we get directly

$$\sum_{r \in E} A_{ir} * adj(A)_{jr} = \begin{cases} det(A) & \text{if i=j} \\ 0 & \text{otherwise} \end{cases}$$

and thus the left-hand member of (1.70) holds.

The ride-side of (1.70) can be proven similarly. □

Remark 1.16. *Equation (1.69) gives an important relation between the forms of the convolutional inverses for real and matrix sequences.*

In the following proposition we give a recurrent way to compute the convolutional inverse of a given matrix valued function. The proof is given by Barbu and Limnios (2008) [4] in which compute the left inverse of a sequence of matrices. Furthermore, we give an alternative way to compute the convolutional inverse using the corresponding right convolutional inverse.

Proposition 1.15. *Let $A \in R^*_{\mathcal{M}}$. Its convolutional inverse is computed recursively as follows:*

$$A^{(-1)}(k) = \begin{cases} (A(0))^{-1} & \textit{if k=0} \\ (A(0))^{-1} \left[-\sum_{l=1}^{k} A^{(-1)}(k-l) A(l) \right] & \textit{otherwise}. \end{cases} \tag{1.73}$$

Furthermore, for any $k \in \mathbb{N}^*$ the convolutional inverse can be also computed by

$$A^{(-1)}(k) = \left[-\sum_{l=0}^{k-1} A(k-l) A^{(-1)}(l) \right] (A(0))^{-1}. \tag{1.74}$$

Proof. For k=0, we get

$$A^{(-1)}(0)A(0) = [A^{(-1)} * A](0) = E_0(0) = I_s$$

Since $A(0)$ is non-singular we get

$$A^{(-1)}(0) = (A(0))^{-1}.$$

For $k \neq 0$, we have $E_0(k)$= 0_s and the convolution of A and $A^{(-1)}$ can be written as

$$\left[A^{(-1)} * A\right](k) = A^{(-1)}(k)A(0) + \sum_{l=1}^{k} A^{(-1)}(k-l)A(l)$$

Since the matrix-valued function $A^{(-1)}$ is the convolutional inverse of A, we have

$$A^{(-1)}(k)A(0) + \sum_{l=1}^{k} A^{(-1)}(k-l)A(l) = 0_s$$

and we obtained the desired form. □

Definition 1.10. *Let $A \in \mathcal{M}_s(\mathbb{N})$. The matrix valued function*

$$\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^{(n)}, \quad n \in \mathbb{N},$$

is called the matrix exponential function of A.

Remark 1.17. *Let $A \in \mathcal{M}_s(\mathbb{N})$ with $A(0) = 0_s$. Then by Remark 1.13, we get directly that the associated matrix exponential function $\exp(A)$ is written as a finite sum and so is well defined.*

In the following proposition we give some elementary properties of the matrix exponential function.

Proposition 1.16. *Let $A, B \in \mathcal{M}_s(\mathbb{N})$, $n \in \mathbb{Z}$ and $C \in \mathcal{M}_s$. The following properties hold, whenever $\exp(A)$, $\exp(B)$ are both well defined.*

$$\begin{aligned}
\exp(0_s) &= E_0, \\
\exp(E_0) &= e \cdot E_0, \\
\exp(C \cdot E_0) &= e^C \cdot E_0, \\
\exp\left(A^t\right) &= \exp(A)^t, \\
\exp(A+B) &= \exp(A) * \exp(B), \quad \textit{for } A * B = B * A, \\
(\exp(A))^{(n)} &= \exp(n\,A), \\
(\exp(A))^{(-1)} &= \exp(-A).
\end{aligned}$$

Proposition 1.17. *Let $A \in \mathcal{M}_s(\mathbb{N})$. Then, the matrix exponential function $\exp(A)$ is always well defined and each term $\exp(A)(k)$ is given by*

$$\exp(A)(k) = e^{A(0)} \cdot \left(\sum_{n=0}^{k} \frac{A_+^{(n)}(k-n)}{k!}\right), \quad k \in \mathbb{N}, \tag{1.75}$$

where

$$A_+(k) = A(k+1), \quad k \in \mathbb{N}.$$

Proof. Since $A = (A - A(0) \cdot E_0) + A(0) \cdot E_0$, then any A can be expressed as a sum of two sequences of matrices for which the associated matrix exponential function is well defined (see Remark 1.17) and their convolution is commutative. Hence, from the previous proposition we have

$$\begin{aligned} \exp(A) &= \exp((A - A(0) \cdot E_0) + A(0) \cdot E_0) = \exp((A - A(0) \cdot E_0) * \exp(A(0) \cdot E_0) \\ &= e^{A(0)} \cdot (E_0 * \exp((A - A(0) \cdot E_0)) = e^{A(0)} \cdot \exp((A - A(0) \cdot E_0). \end{aligned}$$

Therefore, we obtain that the matrix exponential function is well defined for any sequence of matrices A and each term $\exp(A)(k)$ can be written as

$$\exp(A)(k) = e^{A(0)} \cdot \left(\sum_{n=0}^{k} \frac{(A - A(0) \cdot E_0)^{(n)}(k)}{n!} \right), \quad k \in \mathbb{N}. \tag{1.76}$$

Additional, since $(f - f(0) \cdot e_0)(0) = 0$, we have that the following relation holds for any $k,\ n \in \mathbb{N}$ with $k \geq n$,

$$\begin{aligned} (A - A(0) \cdot E_0)^{(n)}(k) &= \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n \geq 1}} A(l_1) \cdots A(l_n) = \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n \geq 1}} A_+(l_1 - 1) \cdots A_+(l_n - 1) \\ &= \sum_{l_1+\ldots+l_n=k-n} A_+(l_1) \cdots A_+(l_n) = A_+^{(n)}(k-n). \end{aligned}$$

Consequently, from the previous observation and (1.76) we obtain the desired result. □

4. Applications

In this section we give some applications of the convolutional inverse of a given real or matrix-valued function. We show an equality for a complex sum using the properties of the convolutional inverse in terms of real sequences.

Example 1.15. *We will show the following form*

$$\sum_{l=0}^{k} (-1)^l \binom{k+1}{l+1} \begin{bmatrix} l \\ k \end{bmatrix} = \begin{cases} 1 & \text{if } k = 0, \\ -1 & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

From Example 1.8, the unitary function $\mathbb{1}$ can be determined by

$$\mathbb{1}^{(-1)} = e_0 - e_1 \cong (1, -1, 0, \ldots).$$

This implies that

$$\mathbb{1}^{(-1)}(0) = 1, \quad \mathbb{1}^{(-1)}(1) = -1, \quad \mathbb{1}^{(-1)}(k) = 0,\ k \geq 2. \tag{1.77}$$

Furthermore, by combining Propositions 1.1 and 1.5 we get directly

$$\mathbb{1}^{(-1)}(k) = \sum_{l=0}^{k} (-1)^l \binom{k+1}{l+1} \begin{bmatrix} l \\ k \end{bmatrix}, \quad k \in \mathbb{N},$$

and consequently (1.77) gives the desired result.

Example 1.16. *Let us consider a 2×2 sequence of matrices $A \in \mathcal{M}_s(\mathbb{N})$ with*

$$A = \begin{pmatrix} f & g \\ h & w \end{pmatrix}.$$

Let us assume also that $det(A)(0) \neq 0$. Then, the convolutional inverse of A exists and is given by

$$A^{(-1)} = det(A)^{(-1)} * adj(A).$$

The convolutional determinant of A is the function

$$det(A) = f * w - g * h,$$

and the convolutional adjugate is the matrix valued function

$$adj(A) = \begin{pmatrix} w & -g \\ -h & f \end{pmatrix}.$$

Consequently, the sequence of matrices $A^{(-1)}$ is represented by

$$\begin{aligned} A^{(-1)} &= det(A)^{(-1)} * adj(A) \\ &= (f * w - g * h)^{(-1)} * \begin{pmatrix} w & -g \\ -h & f \end{pmatrix} \\ &= \begin{pmatrix} (f * w - g * h)^{(-1)} * w & -(f * w - g * h)^{(-1)} * g \\ -(f * w - g * h)^{(-1)} * h & (f * w - g * h)^{(-1)} * f \end{pmatrix}. \end{aligned}$$

From the above we get directly that any entry of $A^{(-1)}$ can be written as

$$A_{ij}^{(-1)}(k) = \left[\left(\sum_{n=0}^{\cdot} \left(-\frac{1}{det(A)(0)} \right)^n \binom{\cdot + 1}{n+1} (det(A))^{(n)} \right) * adj(A)_{ij} \right](k), \quad k \in \mathbb{N}.$$

The following example represents a famous application of convolution in the theory of distributions. More specifically, we study the distribution of a sum of i.i.d random variables which follow well known distributions. Furthermore, we obtain the generating function of each random variable by using the isomorphism between the real sequence and powerseries.

Example 1.17. (i) *Let $(X_n)_{n\geq 1}$ be an i.i.d sequence such that $X_i \sim Poisson(\lambda_i)$ with pmf*

$$f_i(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k \in \mathbb{N}.$$

Then, any f_i can be represented as

$$\begin{aligned} f_i = \sum_{k=0}^{\infty} \frac{e^{-\lambda_i}\lambda_i^k}{k!} \cdot e_1^{(k)} = e^{-\lambda_i} \cdot \sum_{k=0}^{\infty} \frac{(\lambda_i \cdot e_1)^{(k)}}{k!} &= \exp(-\lambda_i \cdot e_0) * \exp(\lambda_i \cdot e_1) \\ &= \exp(\lambda_i \cdot (e_1 - e_0)) \end{aligned}$$

Let also define the random variable $S_n = \sum_{k=0}^{n} X_k$, $n \in \mathbb{N}$ with pmf f. Then, from Example 1.6 we have

$$\begin{aligned} f = (f_1 * \ldots * f_n) &= \exp(\lambda_1 \cdot (e_1 - e_0)) * \cdots * \exp(\lambda_n \cdot (e_1 - e_0)) \\ &= \exp\left(-\sum_{j=1}^{n} \lambda_j \cdot (e_1 - e_0) \right) \end{aligned}$$

and consequently we get directly that

$$S_n \sim Poisson\left(\sum_{j=1}^{n} \lambda_j\right).$$

Furthermore, if we assume also that X_i have the same distribution (i.e $\lambda_i = \lambda$ for any i) then

$$S_n \sim Poisson\,(n \cdot \lambda)\,.$$

(ii) *Let us consider an i.i.d sequence $(X_n)_{n\geq 1}$ with $X_n \sim$ Geo(p) for any $n \in \mathbb{N}$. Then, the associated pmf f is given by*

$$f(k) = p(1-p)^k, \quad k \in \mathbb{N}.$$

Then, f can be rewritten as

$$\begin{aligned} f = \sum_{k=0}^{\infty} p(1-p)^k \cdot e^{(k)} = p \sum_{k=0}^{\infty} ((1-p)\cdot e_1)^{(k)} &= p \sum_{k=0}^{\infty} (e_0 - (e_0 - (1-p)\cdot e_1))^{(k)} \\ &= p\,(e_0 - (1-p)\cdot e_1)^{(-1)}\,. \end{aligned}$$

Then the sequence $(S_n)_{n\geq 1}$ denoted by $S_n = \sum_{l=1}^{n} X_l$ follows the Negative binomial distribution with size n and parameter p and its pmf $f^{(n)}$ is given by

$$f^{(n)} = p^n \cdot (e_0 - (1-p)\cdot e_1)^{(-n)}\,.$$

(iii) *Since any sequence f can be represented as a powereseries then from (i) and (ii) we get directly that the associated generating functions*

$$\exp\left(\lambda \cdot (e_1 - e_0)\right) \cong e^{\lambda\cdot(x-1)}, \tag{1.78}$$

$$p \cdot (e_0 - (1-p)\cdot e_1)^{(-1)} \cong \frac{p}{(1-(1-p)x)}, \tag{1.79}$$

of the Poisson and Geometric distribution respectively.

In the following table we give the pmf and generating function of some well known distributions which are obtained in a similar way as previously.

List of distributions		
Distribution of X	*pmf*	*generating function*
Poisson(λ)	$\exp\left(\lambda \cdot (e_1 - e_0)\right)$	$e^{\lambda\cdot(x-1)}$
Bernoulli(p)	$p \cdot e_0 + (1-p)\cdot e_1$	$p + (1-p)x$
Binomial(N,p)	$(p \cdot e_0 + (1-p)\cdot e_1)^{(n)}$	$(p + (1-p)x)^n$
Geom(p)	$p \cdot (e_0 - (1-p)\cdot e_1)^{(-1)}$	$\frac{p}{(1-(1-p)x)}$
Negbin(n, p)	$p^n \cdot (e_0 - (1-p)\cdot e_1)^{(-n)}$	$\left(\frac{p}{(1-(1-p)x)}\right)^n$

Chapter 2

RENEWAL CHAINS

1. Introduction

In this chapter we study renewal chains. The random system is also allowed to record multiple renewals at the same time. This implies that, the interarrival time of two successive arrival times are possible to be null. Our motivation is derived from another project in which we will use systems with this characteristic.

In classical renewal theory, the interrarival time is assumed to be strictly positive. So, in many applications, the arrival time describes single events. For example, in a system which counts the number of an engine's failures, the arrival time is presented as the time that this engine fails and the interrarival time represents the engine's lifetime between two failures.

We use the theory of convolutions in order to give new representations of important quantities and give alternative proofs. In addition, we generalize well known results of the usual renewal chain.

The renewal chain is very important for the development of the theory because we will need it to construct the theory of the Markov renewal chains. More specifically, we can make proper subsystems in which we can study as renewal chains.

In the next section we introduce the renewal chain, its delayed edition, the counting process and some other useful quantities. Furthermore, we give a unique solution for the renewal equation. The final section includes some asymptotic results in terms of renewal chains and the corresponding results in renewal theory are given as particular cases.

2. Discrete renewal theory

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d nonnegative integer-valued random variables with distribution F and probability mass function f. The random variable X_n will represent the n-th interarrival time of an arrival process $(S_n)_{n\geq 1}$ corresponding to the realization of a recurrent event. As initial conditions we take $X_0 = 0 = S_0$. Of course, the relation $S_n = \sum_{k=0}^{n} X_k$ holds for $n \in \mathbb{N}$.

Definition 2.1. *An arrival time sequence $(S_n)_{n\in\mathbb{N}}$ for which the waiting times $(X_n)_{n\geq 1}$ form an i.i.d sequence of nonnegative integer-valued random variables is called* a discrete time renewal chain *and every S_n is called a* discrete renewal time.

In the rest of theory, we refer to $(S_n)_{n\in\mathbb{N}}$ simply as a renewal chain.

In the sequel, we will see many results in which $f(0)$ plays an active role and so with the assumption that $f(0) = 0$ we get directly the corresponding results in the classical renewal theory. Also, we will accept the case that $f(0)$ is different from 1 and consequently the convolutional inverse of $(e_0 - f)$ always exists.

Remark 2.1. *If $f(0) = 0$, then $(S_n)_{n\in\mathbb{N}}$ corresponds to a usual renewal chain.*

In the following figure we give a possible sample path of a renewal chain.

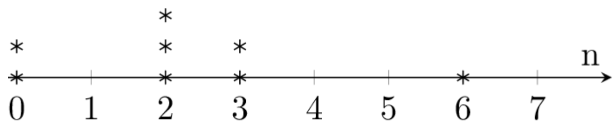

FIGURE 2.1: Renewal chain

It corresponds to a realisation, where $X_0 = 0$, $X_1 = 0$, $X_2 = 2$, $X_3 = X_4 = 0$, $X_5 = 1$, $X_6 = 0$ and $X_7 = 3$. Also, for the corresponding renewal chain we have $S_0 = S_1 = 0$, $S_2 = S_3 = S_4 = 2, S_5 = S_6 = 3$ and $S_7 = 6$.

In the development of the theory, we will need to determine an important distinction between different types of renewal chains.

Definition 2.2. *A renewal chain $(S_n)_{n\in\mathbb{N}}$ is called*

- *recurrent if*
$$\mathbb{P}(X_1 < \infty) = 1,$$
- *transient if*
$$\mathbb{P}(X_1 < \infty) \ < \ 1.$$

In the sequel, we will mainly focus on recurrent renewal chains.

Let us now denote by $\mu := \mathbb{E}(X_1)$ and $\sigma^2 := \mathbb{V}(X_1)$ if the latter is well defined ($\mu < \infty$).

Definition 2.3. *A recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$ is called*

- *positive recurrent if*
$$\mu < \infty,$$
- *null recurrent if*
$$\mu = \infty.$$

Definition 2.4. *Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent renewal chain and*

$$d := \max\Big\{l \in \mathbb{N}^* : \sum_{k=0}^{\infty} f(kl) = 1\Big\}.$$

If $d > 1$, then (S_n) is called periodic with period d or d-periodic, otherwise $(d = 1)$, it is called aperiodic.

Remark 2.2. *Since for a d-periodic renewal chain the probability mass function of X_1 is concentracted on the multiples of d, we get directly that*

$$f(k) = 0, \qquad \textit{for } k \ \not\equiv \ 0 \ (\textit{mod } d).$$

In the development of the theory a fundamental role will be played by the sequence of random variables $(Z_n)_{n\in\mathbb{N}}$, where

$$Z_n = \sum_{k=0}^{\infty} \mathbb{1}_{\{S_k = n\}}. \tag{2.1}$$

The r.v Z_n corresponds to the number of renewals that take place at time n. We also denote by u the sequence

$$u(n) = \mathbb{P}(Z_n \geq 1). \tag{2.2}$$

and we call it renewal probability. Notice that since $X_0 = 0$, we have $u(0) = 1$.

Remark 2.3. *In case that $f(0) = 0$, the sequence $(S_l)_{l\in\mathbb{N}}$ is strictly increasing and then (Z_n) is such that $Z_n \in \{0,1\}$ and $Z_0 = 1$. In this case, we also have $u(n) = \mathbb{P}(Z_n = 1)$.*

Remark 2.4. *From Remark 2.3 we get that each Z_n is almost surely zero for any n which is not a multiple of d. As a consequence of that, we get*

$$u(n) = 0, \qquad n \not\equiv 0\,(mod\; d).$$

In order to obtain easily the desired results we will need the sequence $(X_n^*)_{n\geq 1}$ which is born by the left-truncated distribution of each $(X_n)_{n\geq 1}$ at zero with $f(0) \neq 1$ i.e. $X^* \stackrel{d}{=} (X|X > 0)$. This implies that, the pmf of each X_n^*, denoted by f_*, is given by

$$f_*(k) = \mathbb{P}(X_n = k \mid X_n > 0) = \frac{f(k)}{1 - f(0)}, \quad k \in \mathbb{N}^*. \tag{2.3}$$

Furthermore, we denote by F_* and $\overline{F}_*$ the associated cdf and reliability function of X^* respectively, given by

$$\begin{aligned} F_*(k) &= [\mathbb{1} * f_*](k) = \frac{F(k)}{1 - f(0)}, \quad k \in \mathbb{N}^*, &(2.4)\\ \overline{F}_* &= \mathbb{1} - \mathbb{1} * f_* = \frac{\overline{F}}{1 - f(0)}. &(2.5)\end{aligned}$$

We assume also that $X_0^* = 0$ and $S_n^* = \sum_{l=0}^n X_l^*$. Then, the sequence $(S_n^*)_{n\in\mathbb{N}}$ forms a usual renewal chain with interrarival times with interrarival times $(X^*)_{n\in\mathbb{N}}$ and is called the associated usual renewal chain of $(S_n)_{n\in\mathbb{N}}$. Therefore, by using the sequence $(X^*)_{n\in\mathbb{N}}$, our inference is based only on the positive values of X rejecting any zero-time event.

Let us also denote by $\mu_* = \mathbb{E}(X_1^*)$ and $\sigma_*^2 := \mathbb{V}(X_1^*)$ if the latter is well defined. Then, we can obtain directly that

$$\begin{aligned} \mu_* &= \frac{\mu}{1 - f(0)},\\ \sigma_*^2 &= \frac{1}{1 - f(0)}\left(\sigma^2 - \frac{f(0)}{1 - f(0)} \cdot \mu_*^2\right).\end{aligned}$$

Furthermore, equation (2.3) can be rewritten as

$$f_* = \frac{f - f(0)\cdot e_0}{1 - f(0)},$$

and consequently we can obtain

$$f = (1 - f(0)) \cdot f_* + f(0) \cdot e_0. \tag{2.6}$$

Example 2.1. *Let us consider the sequence of r.v's of Example 1.17 (i) and $X_0 = 0$. Then the sequence $(S_n)_{n\in\mathbb{N}}$ forms a renewal chain with $f(0) \neq 1$ and consequently the associated usual renewal chain $(S_n^*)_{n\in\mathbb{N}}$ with interrarival times $(X^*)_{n\in\mathbb{N}}$ is well defined. Furthermore, the associated pmf f_* of each X_n^* can be determined by*

$$f_*(k) = \frac{e^{-\lambda}}{1 - e^{-\lambda}}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}^*.$$

Furthermore, any S_n^* follows a distribution with pmf

$$f_*^{(n)}(k) = \frac{f^{(n)}(k)}{(1 - f(0))} = \left(\frac{e^{-\lambda}}{1 - e^{-\lambda}}\right)^n \frac{(n\lambda)^k}{k!}, \quad k \in \mathbb{N}^*$$

The following proposition gives a relation between f, f_* and u via convolution.

Proposition 2.1. *Let $(S_n)_{n\in\mathbb{N}}$ be a renewal chain. If $f(0) \neq 1$, then the sequence of renewal probabilities u and the pmf f_* are linked through*

$$u = (e_0 - f_*)^{(-1)}. \tag{2.7}$$

Proof. For $n = 0$, we get $[f * u](0) = f(0) \cdot u(0) = f(0)$ since $u(0) = 1$. Then, we take

$$u(0) = 1 - f(0) + f(0) = (1 - f(0)) \cdot e_0(0) + [f * u](0).$$

Let $n \in \mathbb{N}^*$. Then, by conditioning on the value of the first arrival time S_1 we obtain

$$\begin{aligned} u(n) = \mathbb{P}(Z_n \geq 1) &= \sum_{l=0}^{n} \mathbb{P}(Z_n \geq 1 | S_1 = l)\mathbb{P}(S_1 = l) = \sum_{l=0}^{n} \mathbb{P}(Z_{n-l} \geq 1)\mathbb{P}(S_1 = l) \\ &= \sum_{l=0}^{n} u(n-l) f(l) = [f * u](n) \overset{e_0(n)=0}{=} (1 - f(0)) \cdot e_0(n) + [f * u](n) \end{aligned}$$

The above decomposition shows that u can be expressed by

$$u = (1 - f(0)) \cdot e_0 + f * u. \tag{2.8}$$

From (2.8), we get directly:

$$u * (e_0 - f) = (1 - f(0)) \cdot e_0.$$

Since the inverse of $(e_0 - f)$ exists ($f(0) < 1$), we have that

$$u = (1 - f(0)) \cdot (e_0 - f)^{(-1)} = \left(\frac{e_0 - f}{1 - f(0)}\right)^{(-1)} \overset{(2.6)}{=} (e_0 - f_*)^{(-1)}.$$

$\square$

Remark 2.5. *From Proposition 2.1 we get directly the following representations*

$$\begin{aligned} f_* &= e_0 - u^{(-1)} && (2.9) \\ F_* &= \mathbb{1} - \mathbb{1} * u^{(-1)} && (2.10) \\ \overline{F_*} &= \mathbb{1} * u^{(-1)} && (2.11) \\ \mu &= \lim_{n\to\infty} \left[\mathbb{1}^{(2)} * u^{(-1)}\right](n) && (2.12) \end{aligned}$$

Let $(L(n))_{n\in\mathbb{N}}$ be the sequence which represents the expectation of Z_n.

$$L(n) = \mathbb{E}(Z_n),\ n \in \mathbb{N}. \tag{2.13}$$

It will be beneficial for our inference to achieve an expression for $(L(n))_{n\in\mathbb{N}}$ from which we can have an easier way to compute it. For this purpose, a relation between L and u is given in the following proposition.

Proposition 2.2. *Let $(S_n)_{n\in\mathbb{N}}$ be a renewal chain. If $f(0) \neq 1$, then the sequences L and u are linked through*

$$L = \frac{u}{1 - f(0)}. \tag{2.14}$$

Furthermore, L is expressed in the convolutional form

$$L = \frac{(e_0 - f_*)^{(-1)}}{1 - f(0)}. \tag{2.15}$$

Proof. The random variable Z_0 depends on the zero-time events at time 0 and to the first jump of the chain $(S_n)_{n\in\mathbb{N}}$. Therefore, we have

$$\mathbb{P}(Z_0 = l) = \mathbb{P}(X_0 = (X_1 = \cdots = X_{l-1} = 0, X_l \geq 1) = f(0)^{l-1}(1 - f(0)),\ l \in \mathbb{N}^*. \tag{2.16}$$

Consequently, Z_0 follows the geometric distribution on $\mathbb{N}^*$ with parameter $p = 1 - f(0)$. Furthermore, we get

$$\mathbb{E}(Z_0) = \frac{1}{1 - f(0)} \overset{u(0)=1}{=} \frac{u(0)}{1 - f(0)}.$$

For $n \geq 1$, if we have the information that $Z_n \geq 1$ then this means that at least one renewal takes place at time n. This is identical to the initial condition that $Z_0 \geq 1$ and consequently we have

$$(Z_n | Z_n \geq 1) \overset{d}{=} Z_0,$$

and thus by (2.16) we have

$$\mathbb{E}(Z_n | Z_n \geq 1) = \frac{1}{1 - f(0)}.$$

Therefore, the sequence L can be determined by

$$L(n) = \mathbb{E}(Z_n) = \mathbb{E}(Z_n | Z_n \geq 1)\mathbb{P}(Z_n \geq 1) = \frac{u(n)}{1 - f(0)} \overset{(2.7)}{=} \frac{(e_0 - f_*)^{(-1)}(n)}{1 - f(0)}.$$

$\square$

Remark 2.6. *If $f(0) = 0$, then the relation (2.14) can be reformulated as follows*

$$L(n) = u(n) = \mathbb{P}(Z_n = 1),\ n \in \mathbb{N}.$$

In the following examples we give some concrete applications.

Example 2.2. *Let us consider the renewal chain of Example 2.1. Then, the associated sequences of renewal probabilities and expected number of renewals are given by*

$$u(n) = \sum_{m=0}^{n} \left(\frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^m \frac{(m\lambda)^n}{n!},$$

$$L(n) = \sum_{m=0}^{n} \left(\frac{e^{-\lambda}}{1 - e^{-\lambda}} \right)^{m+1} \frac{e^{\lambda}(m\lambda)^n}{n!},$$

respectively.

In the following definition, we introduce the notion of renewal equation:

Definition 2.5. *Let $g : \mathbb{N} \to \mathbb{R}$ be an unknown function and $b : \mathbb{N} \to \mathbb{R}$ be a known one. The equation*

$$g = b + f * g, \tag{2.17}$$

is called discrete time renewal equation.

The existence of a unique solution is given in the following proposition.

Proposition 2.3. *If $f(0) \neq 1$, then the renewal equation (2.3) has a unique solution given by*

$$g = \frac{b * u}{1 - f(0)}. \tag{2.18}$$

Proof. By using Equations (2.3) and (2.17) we get

$$g * (e_0 - f) = b \quad \Longrightarrow g = b * (e_0 - f)^{(-1)}.$$

Then, from Equation (2.7) we get the desired result. □

Remark 2.7. *If $f(0) = 0$, we have the unique solution of the usual renewal equation in discrete time.*

We define the following process

$$N(n) = \sum_{k=0}^{n} Z_k - 1 = \sum_{k=0}^{n} \sum_{l=0}^{\infty} \mathbb{1}_{\{S_l = k\}} - 1 = \sum_{l=1}^{\infty} \mathbb{1}_{\{S_l \leq n\}} = \sup\{l \in \mathbb{N} : \; S_l \leq n\}, \tag{2.19}$$

which records the number of renewals until the nth period if we exclude the one reffering to $S_0 = 0$. Since the probability that the interarrival time be zero is positive, we get that $\mathbb{P}(N(n) = k)$ for any $k \in \mathbb{N}$. Also, relation (2.19) gives us

$$\mathbb{P}\left(N(n) \geq k\right) = \mathbb{P}(S_k \leq n), \; k, n \in \mathbb{N}. \tag{2.20}$$

The following figure represents the sample path of the counting process, which corresponds to the renewal chain of Figure 2.1.

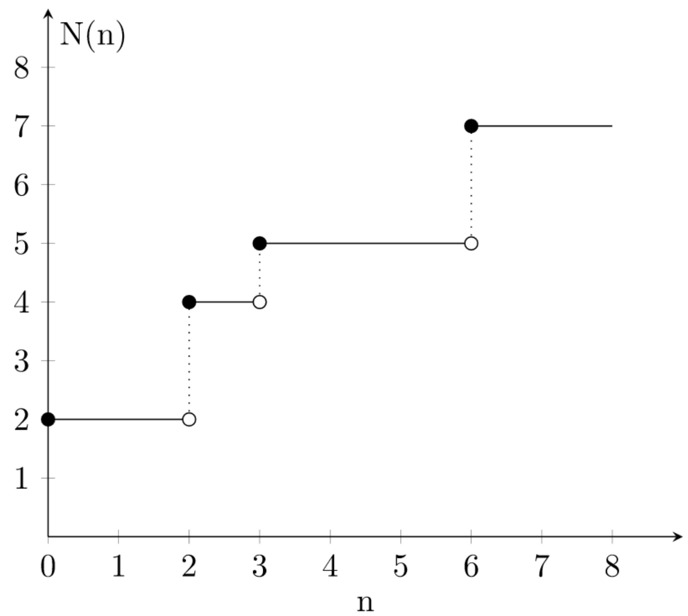

FIGURE 2.2: A sample path of counting process

We give here a typical probabilistic proof for the relation between the pmf, cdf and the mean function of $(N(n))_{n \geq 0}$.

Proposition 2.4. *For the counting process $(N(n))_{n \in \mathbb{N}}$, we have:*

(i) *The sequence of the associated cdfs* $\left(F_{N(n)}\right)_{n\in\mathbb{N}}$ *is given by*

$$F_{N(n)}(k) = 1 - F_{k+1}(n), \quad n, k \in \mathbb{N}. \tag{2.21}$$

(ii) *The sequence of the associated pmfs* $\left(f_{N(n)}\right)_{n\in\mathbb{N}}$ *is given by*

$$f_{N(n)}(k) = F_k(n) - F_{k+1}(n), \quad n, k \in \mathbb{N}. \tag{2.22}$$

(iii) *The mean function is determined by*

$$\mathbb{E}\left(N(n)\right) = \sum_{k=1}^{\infty} F_k(n), \quad n \in \mathbb{N}. \tag{2.23}$$

Proof. (i) From Equation (2.20) we get directly

$$F_{N(n)}(k) = \mathbb{P}\left(N(n) \leq k\right) = 1 - \mathbb{P}\left(N(n) \geq k+1\right) = 1 - \mathbb{P}(S_{k+1} \leq n) = 1 - F_{k+1}(n).$$

(ii) First, notice that

$$\{N(n) = k\} \quad = \quad \{N(n) \leq k\} \setminus \{N(n) \leq k-1\}, \quad n, k \in \mathbb{N}.$$

Consequently, we obtain

$$f_{N(n)}(k) = F_{N(n)}(k) - F_{N(n)}(k-1) \overset{(2.21)}{=} F_k(n) - F_{k+1}(n).$$

(iii) Since $N(n)$ is a nonnegative integer valued r.v, we directly get

$$\mathbb{E}\left(N(n)\right) = \sum_{k=0}^{\infty}\left(1 - F_{N(n)}(k)\right) \overset{(2.21)}{=} \sum_{k=0}^{\infty} F_{k+1}(n) = \sum_{k=1}^{\infty} F_k(n)),$$

or by using the definition of N we can obtain

$$\mathbb{E}\left(N(n)\right) = \mathbb{E}\left(\sum_{k=1}^{\infty} \mathbb{1}_{\{S_k \leq n\}}\right) = \sum_{k=1}^{\infty} \mathbb{E}\left(\mathbb{1}_{\{S_k \leq n\}}\right) = \sum_{k=1}^{\infty} F_k(n).$$

$\square$

Definition 2.6. *The mean function of* $N(n)$ *is called the renewal function and is denoted by*

$$M(n) := \mathbb{E}\left(N(n)\right), \; n \in \mathbb{N}. \tag{2.24}$$

Also, an other expression of this function can be achieved by the following relation

$$M = \mathbb{1} * L - \mathbb{1} = \mathbb{1} * (L - e_0) = \mathbb{1} * \left(\frac{u}{1 - f(0)} - e_0\right).$$

3. Delayed renewal chains

In this section we give a generalization of the class of renewal chains. It is referred to a delayed observation of a random system when it is happened after the beginning. In that case, we wait for the residual time of the next event and then we make a renewal chain. More precisely, this time is assumed to be the beginning of this system and we start to observe its evolution.

Another reason for which we need this model is that in the general case it's complicated to make some random systems with multiple events because we can't have an event at the 0-th period. Let us first give some definitions and notations. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables and let also X_0 be a nonegative r.v independent of $(X_n)_{n\geq 1}$, satisfying $\mathbb{P}(X_0 > 0) > 0$. We denote by $(S_n)_{n\in\mathbb{N}}$, the corresponding sequence of arrival times, that is, $S_n = \sum_{k=0}^{n} X_k$. The chain $(S'_n)_{n\in\mathbb{N}}$, where $S'_n = S_n - S_0$, is a renewal chain. Also, notice that

$$S'_n - S'_{n-1} = (S_n - S_0) - (S_{n-1} - S_0) = S_n - S_{n-1} = X_n.$$

Hence, $(X_n)_{n\geq 1}$ is the sequence of the waiting times for the renewal chain (S'_n). Furthermore, we denote by $(S^*_n)_{n\in\mathbb{N}}$ the associated usual renewal chain of $(S'_n)_{n\in\mathbb{N}}$.

Definition 2.7. *An arrival time sequence $(S_n)_{n\in\mathbb{N}}$ for which the waiting times $(X_n)_{n\geq 1}$ form an i.i.d sequence of nonegative integer-valued random variable and X_0 is independent of $(X_n)_{n\geq 1}$, is called a delayed renewal chain and every S_n is called a renewal time. The chain $(S'_n)_{n\in\mathbb{N}}$ is called the associated renewal chain which is defined after Remark 2.4.*

Remark 2.8. *A renewal chain S_n is also a delayed renewal chain, assuming that $S_0 = 0$ then we have $S'_n = S_n$. Therefore, we get directly that the renewal chain is a particular case of the delayed renewal chain.*

In the following figures we present a sample path of a delayed renewal chain with $S_0 = 1$ and its associated renewal chain respectively.

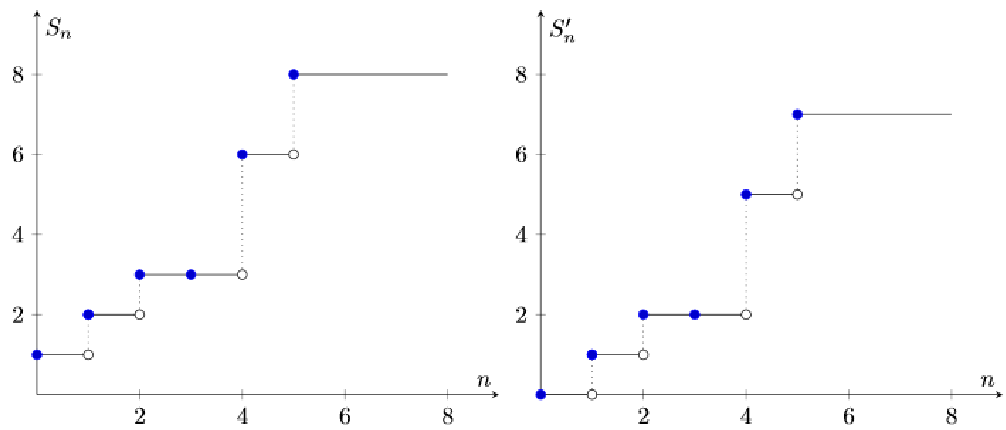

FIGURE 2.3: Delayed renewal chain.

The first figure corresponds to a realisation, where $S_0 = 1,\ S_1 = 2,\ S_2 = S_3 = 3,\ S_4 = 6,$ $S_5 = 8$ and $S_6 \geq 8$. The latter, corresponds to a realisation which results from the previous one by subtracting 1 to each S_n.

Next, we give some kinds of delayed renewal chains.

Definition 2.8. *A delayed renewal chain (S_n) is called:*

1. *Periodic with period $d > 1$ if the associated renewal chain is periodic with period d. Otherwise, if $d = 1$ both of them are called periodic.*

2. *(positive/null) recurrent if this property holds for the associated renewal chain.*

We denote by f_0, the probability mass function of X_0 and is said to be the *initial distribution* of the delayed renewal chain.

Define the following random variables

$$\begin{aligned} Z_n &= \sum_{l=0}^{\infty} \mathbb{1}_{\{S_l=n\}}, \\ Z'_n &= \sum_{l=0}^{\infty} \mathbb{1}_{\{S'_l=n\}}. \end{aligned}$$

As $v(n)$ and $u(n)$ we denote the renewal probabilities $\mathbb{P}(Z_n \geq 1)$ and $\mathbb{P}(Z'_n \geq 1)$ respectively. Since $S'_0 = 0$ we get $u(0) = 1$.

It's easy to see that $(Z'_n)_{n\in\mathbb{N}}$ is the corresponding sequence of random variables which records the number of the events for the nth period in terms of the renewal chain $(S'_n)_{n\in\mathbb{N}})$.

A useful relation between v and u is given in the following proposition:

Proposition 2.5. *If $f(0) \neq 1$, the sequences of renewal probabilities v and u satisfy*

$$v = f_0 * u. \tag{2.25}$$

Proof. Let $n \in \mathbb{N}$. By conditioning on S_0, we have

$$u(n) = \sum_{l=0}^{n} \mathbb{P}(Z_n \geq 1 \mid S_0 = l)\mathbb{P}(S_0 = l) = \sum_{l=0}^{n} \mathbb{P}(Z'_{n-l} \geq 1)\mathbb{P}(S_0 = l) = \sum_{l=0}^{n} u(n-l)f_0(l).$$

□

Denote the following sequence

$$L_D(n) = \mathbb{E}(Z^{'}_n), \; n \in \mathbb{N}. \tag{2.26}$$

This sequence is the expected number of the events at time n in terms of the delayed renewal chain. Below, we give a relation between L_D and the delayed renewal probability v.

$$L_D = \frac{v}{1-f(0)} = \frac{f_0 * u}{1-f(0)}. \tag{2.27}$$

Proposition 2.6. *If $f(0) \neq 1$, then the sequence of expected renewals L can be determined by*

$$L = \frac{f_0 * (e_0 - f_*)^{(-1)}}{1-f(0)} \tag{2.28}$$

Proof. In a similar way as in Proposition 2.14 we conclude that

$$(Z'_n | Z'_n \geq 1) \stackrel{d}{=} Y, \; n \in \mathbb{N},$$

where Y is the geometric distribution on $\mathbb{N}^*$ with parameter $1 - f(0)$. Consequently, we have

$$\mathbb{E}(Z'_n) = \mathbb{E}(Z'_n | Z'_n \geq 1)\mathbb{P}(Z'_n \geq 1) = \frac{v(n)}{1-f(0)} \stackrel{(2.15)}{=} \frac{[f_0 * u](n)}{1-f(0)}.$$

□

Remark 2.9. *From Remark 2.8, we can easily see that if $X_0 \stackrel{a.s}{=} 0$, then the sequences Z'_n and Z_n are coincise. Consequently, for the renewal probabilities u, v, and the sequences L', L coincide.*

In the following proposition we give a necessary and sufficient and condition for a positive recurrent delayed renewal chain to have constant v

Proposition 2.7. *Let $(S_n)_{n\in\mathbb{N}}$ be a positive recurrent delayed renewal chain with arrival times $(X_n)_{n\in\mathbb{N}}$ and $\mu := \mathbb{E}(X_1) \in (0, +\infty)$. The following are equivalent*

(i) *v is a constant,*

(ii) $v = \frac{\mathbb{1}}{\mu_*}$,

(iii) $f_0 = \frac{1}{\mu_*}(\mathbb{1} - \mathbb{1} * f) = \frac{\overline{F_*}}{\mu_*}$.

Proof. Since $(S_n)_{n\in\mathbb{N}}$ is assumed to be positive recurrent we get directly that $v > 0$.

$(i) \implies (ii)$ Let $c > 0$ such that $v = c \cdot \mathbb{1}$. Then, by using relation (2.5) we have:

$$f_0 = v * u^{(-1)} = c \cdot [\mathbb{1} * \mathbb{1}^{(-1)} * \overline{F_*}] \quad = \quad c \cdot \overline{F_*}. \tag{2.29}$$

Since $\lim_{n\to\infty} [\mathbb{1} * f_0](n) = 1$ and $\lim_{n\to\infty} \left[\mathbb{1} * \overline{F_*}\right](n) = \mu_*$, we obtain directly that $c = \frac{1}{\mu_*}$.

$(ii) \implies (iii)$ The result is direct by equation (2.29)

$(iii) \implies (ii)$ Since $f_0 = \frac{\overline{F_*}}{\mu_*}$, the sequence v can be written as

$$v = f_0 * u = \frac{1}{\mu_*} \cdot \left(\overline{F_*} * \overline{F_*}^{(-1)} * \mathbb{1}\right) = \left(\frac{1}{\mu_*}\right) \cdot [\mathbb{1} * e_0] = \frac{1}{\mu_*} \cdot \mathbb{1}.$$

□

Summarizing the above results, in the following definition we give a particular case of a delayed renewal chains which satisfies the conditions of Proposition 2.7.

Definition 2.9. *Let $(S_n)_{n\in\mathbb{N}}$ be a delayed renewal chain with $\mu = \mathbb{E}(X_1) < \infty$ and $v \equiv \frac{\mathbb{1}}{\mu_*}$. This chain is called a stationary renewal chain and its initial distribution defined by $f_0 = \frac{\overline{F_*}}{\mu_*}$, is called the stationary distribution of the delayed renewal chain.*

4. Asymptotic results

In this section we show some asymptotic results about (delayed or not) renewal chains. We give the asymptotic behaviour of the counting process N such as the SLLN and the central limit theorem. We include also the elementary renewal theorem which refers to the limit of the renewal function M. Furthermore, we study the convergence of the sequence of renewal probabilities and expected number of renewals.

First, we admit the following assumption The expectation of any interrarival time is finite.

Proposition 2.8. *Let (S_n) be a renewal chain. Then,*

(i) $S_n \xrightarrow[n\to\infty]{a.s} \infty$,

(ii) *if it is also recurrent we have*

$$N(n) \xrightarrow[n\to\infty]{a.s} \infty.$$

Proof. (i) Since $(X_n)_{n\in\mathbb{N}}$ is a sequence of iid random variables with $\mathbb{E}(X_1) = \mu > 0$, then by applying the SLLN we get

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s} \mu.$$

Therefore,

$$S_n = n\,\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s} \infty.$$

(ii) Note that $(N(n))_{n\in\mathbb{N}}$ is an increasing sequence and thus its limit exists (finite or infinite). Since $N(n)$ is integer-valued, if it converges to a finite number it will be an eventually constant sequence.

Let us define the following set

$$A := \left\{\lim_{n\to\infty} N(n) < \infty\right\} = \{\text{N(n) is eventually constant}\}.$$

In order to show the desired result, we will need to prove that A has probability zero. Also, this set can be rewritten in the following form

$$A = \{\text{N(n) is eventually constant}\} = \bigcup_{l=0}^{\infty}\{\text{N(n) is eventually l}\} = \bigcup_{l=0}^{\infty}\liminf_n\{N(n) = l\}.$$

Additionally, by using the properties of subadditivity and monotonicity of measures we obtain

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}\left(\bigcup_{l=0}^{\infty}\liminf_n\{N(n)=l\}\right) \le \sum_{l=0}^{\infty}\mathbb{P}\left(\liminf_n\{N(n)=l\}\right)\\ &\le \sum_{l=0}^{\infty}\liminf_n \mathbb{P}\left(\{N(n)=l\}\right) = \sum_{l=0}^{\infty}\liminf_n\left(\mathbb{P}\left(S_l \le n\right)\right) - \mathbb{P}\left(S_{l+1}\le n\right)\right).\end{aligned}$$

We assumed that S_n is recurrent. For that reason we take:

$$\mathbb{P}(S_l \le n) - \mathbb{P}(S_{l+1}\le n) \xrightarrow[n\to\infty]{} \mathbb{P}(S_l < +\infty) - \mathbb{P}(S_{l+1} < +\infty) = 0.$$

Therefore, $\mathbb{P}(A) = 0$. □

In the sequel, we give some theorems in which we describe the asymptotic behavior of $N(n)$.

Theorem 2.1 (SLLN for counting processes)**.** *If $(S_n)_{n\in\mathbb{N}}$ is a recurrent renewal chain, then*

$$\frac{N(n)}{n} \xrightarrow[n\to\infty]{a.s} \frac{1}{\mu}.$$

Proof. From the classical SLLN for the i.i.d sequence of random variables $(X_n)n \ge 1$ we have:

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{a.s} \mu.$$

Hence, the combination of the previous proposition and Theorem B.4 gives

$$\frac{S_{N(n)}}{N(n)}, \frac{S_{N(n)+1}}{N(n)+1} \xrightarrow[n\to\infty]{a.s} \mu.$$

Also, we have

$$S_{N(n)} \le n < S_{N(n)+1} \quad\Longrightarrow\quad \frac{S_{N(n)}}{N(n)} \le \frac{n}{N(n)} < \frac{N(n)+1}{N(n)}\frac{S_{N(n)+1}}{N(n)+1}.$$

As $N(n)$ converges to infinity, we have

$$\frac{N(n)+1}{N(n)} \xrightarrow[n\to\infty]{a.s} 1$$

and we get

$$\frac{N(n)}{n} \xrightarrow[n\to\infty]{a.s} \frac{1}{\mu}.$$

$\square$

Remark 2.10. *A direct consequence of the above theorem is the following result*

$$\frac{S_{N(n)}}{n} \xrightarrow[n\to\infty]{a.s} 1.$$

Theorem 2.2 (Elementary renewal theorem)**.** *For a recurrent renewal chain $(S_n)_{n\in\mathbb{N}}$, we have*

$$\frac{M(n)}{n} \xrightarrow[n\to\infty]{a.s} \frac{1}{\mu}. \tag{2.30}$$

Proof. In order to prove the desired result, we need to show that the sequence $\left(\frac{N(n)}{n}\right)_{n\geq 1}$ is uniformly integrable (Definition B.1). Let $p := 1 - f(0)$.

From the proof of Proposition 2.14 we take

$$\mathbb{E}\left(Z_n^2 | Z_n \geq 1\right) = \frac{1+f(0)}{(1-f(0))^2} = \frac{2-p}{p^2}$$

and consequently

$$\mathbb{E}\left(Z^2\right) = \frac{2-p}{p^2}\cdot u.$$

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{l=0}^{n} Z_l\right)^2\right) &= \sum_{l=0}^{n}\mathbb{E}(Z_l^2) + \sum_{k\neq m}\mathbb{E}(Z_k Z_m) \leq \frac{2-p}{p^2}\sum_{l=0}^{n} u(l) + \frac{1}{2}\sum_{k\neq m}\mathbb{E}(Z_k^2 + Z_m^2) \\
&= \frac{2-p}{p^2}\left(\sum_{l=0}^{n} u(l) + \frac{1}{2}\sum_{k\neq m}\{u(k)+u(m)\}\right) \\
&\leq \frac{(2-p)}{p^2}\left(n+1+\binom{n+1}{2}\right) = \frac{(2-p)}{p^2}\left(n+1+\frac{n(n+1)}{2}\right).
\end{aligned}$$

The combination of the previous inequality and the relation $N(n) = \sum_{l=0}^{n} Z_l - 1 < \sum_{l=0}^{n} Z_l$ gives us

$$\mathbb{E}\left(N(n)^2\right) < \mathbb{E}\left(\left(\sum_{l=0}^{n} Z_l\right)^2\right) \leq \frac{(2-p)}{p^2}\left(n+1+\frac{n(n+1)}{2}\right)$$

and consequently the following relation holds

$$\mathbb{E}\left(\left(\frac{N(n)}{n}\right)^2\right) < \frac{(2-p)}{p^2}\left(\frac{n+1}{n^2} + \frac{n(n+1)}{2n^2}\right) \leq 3\left(\frac{2-p}{p^2}\right)$$

and is independent of n. Hence, we have

$$\sup_{n\geq 1}\mathbb{E}\left(\left(\frac{N(n)}{n}\right)^2\right) \leq 3\left(\frac{2-p}{p^2}\right) < \infty.$$

From Proposition B.1 we have that the sequence $\left(\frac{N(n)}{n}\right)_{n\geq 1}$ is uniformly integrable. Combining this with the SLLN for counting processes and Theorem B.3, we obtain:

$$\lim_{n\to\infty}\frac{M(n)}{n}=\frac{1}{\mu}.$$

□

Remark 2.11. *if $f(0)=0$, then we get that $\frac{N(n)}{n}\leq 1$ and from the bounded convergence theorem we get directly the desired convergence.*

Theorem 2.3. *Let $(S_n)_{n\in\mathbb{N}}$ be a positive recurrent renewal chain with $\mu=\mathbb{E}(X_1)$ and $0<\mathbb{V}(X_1)=\sigma^2<\infty$. Then, N is described asymptotically by*

$$\sqrt{\frac{n\mu^3}{\sigma^2}}\left(\frac{N(n)}{n}-\frac{1}{\mu}\right)\xrightarrow[n\to\infty]{d}\mathcal{N}(0,1).$$

Proof. Let $x\in\mathbb{R}$. In order to show the desired result we need to prove the following limiting relation:

$$\mathbb{P}\left(\sqrt{\frac{n\mu^3}{\sigma^2}}\left(\frac{N(n)}{n}-\frac{1}{\mu}\right)\leq x\right)\xrightarrow{n\to\infty}\Phi(x),$$

where Φ is the distribution function of the standard Normal distribution. Then,

$$\begin{aligned}\mathbb{P}\left(\sqrt{\frac{n\mu^3}{\sigma^2}}\left(\frac{N(n)}{n}-\frac{1}{\mu}\right)\leq x\right) &= \mathbb{P}\left(N(n)\leq\frac{n}{\mu}+x\sqrt{\frac{n\sigma^2}{\mu^3}}\right)\\ &= \mathbb{P}\left(N(n)\leq\left[\frac{n}{\mu}+x\sqrt{\frac{n\sigma^2}{\mu^3}}\right]\right),\end{aligned}$$

where $[x]$ is the integer part of the real number x. Set $\xi_n=\left[\frac{n}{\mu}+x\sqrt{\frac{n\sigma^2}{\mu^3}}\right]$. For large n we have the following approximations

$$\begin{aligned}\xi_n &\sim \frac{n}{\mu}+x\sqrt{\frac{n\sigma^2}{\mu^3}},\\ (n-\mu\xi_n) &\sim -x\sqrt{\frac{n\sigma^2}{\mu}},\\ \sigma\sqrt{\xi_n} &\sim \sigma\sqrt{\frac{n}{\mu}}.\end{aligned}$$

Thus, as $n\to\infty$ we conclude that

$$\frac{n-\xi_n\mu}{\sigma\sqrt{\xi_n}}\sim -x.$$

Using the CLT for the sequence $(X_n)_{n\geq 1}$ and the symmetry of the Normal distribution we have

$$\begin{aligned}\mathbb{P}\left(\sqrt{\frac{n\mu^3}{\sigma^2}}\left(\frac{N(n)}{n}-\frac{1}{\mu}\right)\leq x\right) &= \mathbb{P}\left(N(n)\leq\xi_n\right)=\mathbb{P}(S_{\xi_n}\geq n)\\ &= \mathbb{P}\left(\frac{S_{\xi_n}-\xi_n\mu}{\sigma\sqrt{\xi_n}}\geq\frac{n-\xi_n\mu}{\sigma\sqrt{\xi_n}}\right)\sim 1-\Phi(-x)=\Phi(x).\end{aligned}$$

□

Remark 2.12. *The CLT for counting processes gives us an alternative way to show the corresponding WLLN.*

Theorem 2.4 (renewal theorem)**.** *Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent renewal chain, then:*

(i) *if it is aperiodic we have:*

$$u(n) \xrightarrow[n\to\infty]{} \frac{1}{\mu_*}.$$

(ii) *if it is periodic of period $d > 1$ we have:*

$$u(dn) \xrightarrow[n\to\infty]{} \frac{d}{\mu_*},$$

and

$$u(n) = 0, \quad \forall n \in \{l \in \mathbb{N} |\, l \not\equiv 0 \,(mod\ d)\}.$$

(i) *The first proof is based on the coupling technique. It is a generalization of the proof in [15]. The difference is identified in the possibility of a zero-time event. The latter doesn't admit some properties which exist in a system with time jumps such as the positive reucrrence of a Markov chain. So this problem is solved using different conditions and properties.*

Coupling tecnhique. Let $(R_n)_{n\in\mathbb{N}}$ be the stationary b-renewal chain associated to the b-renewal chain $(S_n)_{n\in\mathbb{N}}$, with interarrival times $(Y_n)_{n\in\mathbb{N}}$ for which we have:

$$X_n \stackrel{d}{=} Y_n, \; n \geq 1.$$

Also, we take

$$\mathbb{P}(R_0 = n) = \frac{\mathbb{P}(X_1^* > n)}{\mu_*}, \quad n \in \mathbb{N}.$$

Define the chain $(U_n)_{n\in\mathbb{N}}$ which is expressed by

$$\begin{aligned} U_0 &= X_0 - Y_0 = -Y_0 \\ U_n &= S_n - R_n = U_{n-1} + (X_n - Y_n), \; n \geq 1. \end{aligned}$$

Obviously, the chain $(U_n - U_0)_{n\in\mathbb{N}}$ forms a Markov Chain.

As N we describe the first time when the chains $(S_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ have the same renewals, i.e.

$$N = \inf\{l \in \mathbb{N} : U_l = U_0\}.$$

As $\mathbb{E}(X_1 - Y_1) = 0$, we obtain that the Markov Chain $(U_n - U_0)_{n\in\mathbb{N}}$ implies the condition of Theorem C.2 and consequently is recurrent. This satisfies that

$$\mathbb{P}(N < \infty) = 1.$$

For any $n \in \mathbb{N}$, define the following sets

$$A_n^- = \bigcup_{l=0}^{N-1} \{S_l = n\}, \quad A_n^+ = \cup_{l=N}^{\infty} \{S_l = n\}, \quad A_n = \bigcup_{l=0}^{\infty} \{S_l = n\},$$

$$B_n^- = \bigcup_{l=0}^{N-1} \{R_l = n\}, \quad B_n^+ = \cup_{l=N}^{\infty} \{R_l = n\}, \quad B_n = \bigcup_{l=0}^{\infty} \{R_l = n\}.$$

Consequently, we get for n $\geq$N that the rv's S_n and R_n have the same distribution. Hence, the events $A_n^+ \ and \ B_n^+$ have equal probabilities.

Therefore,

$$\begin{aligned}
u(n) &= \mathbb{P}(Z_n \geq 1) = \mathbb{P}(A_n) = \mathbb{P}(A_n^+ \cup A_n^-) = \mathbb{P}(A_n^+) + \mathbb{P}(A_n^-) - \mathbb{P}(A_n^+ \cap A_n^-) \\
&= \mathbb{P}(B_n^+) + \mathbb{P}(A_n^-) - \mathbb{P}(A_n^+ \cap A_n^-) = \mathbb{P}(B_n \setminus B_n^-) + \mathbb{P}(A_n^-) - \mathbb{P}(A_n^+ \cap A_n^-) \\
&= \mathbb{P}(B_n) - \mathbb{P}(B_n \setminus B_n^+) + \mathbb{P}(A_n^-) - \mathbb{P}(A_n^+ \cap A_n^-).
\end{aligned}$$

Since $(R_n)_{n\in\mathbb{N}}$ is a stationary b-renewal chain then from Definition 2.9 we infer that

$$\mathbb{P}(B_n) = v(n) = \frac{1}{\mu_*}.$$

As a result of $\mathbb{P}(N < \infty) = 1$ and the recurrence of $(S_n)_{n\in\mathbb{N}}$ we have:

$$\mathbb{P}(A_n^-) \leq \mathbb{P}(S_N \geq n) \xrightarrow[n\to\infty]{} 0$$

and consequently

$$\mathbb{P}(A_n^+ \cap A_n^-) \leq \mathbb{P}(A_n^-) \xrightarrow[n\to\infty]{} 0.$$

In a similar way we obtain

$$\mathbb{P}(B_n \setminus B_n^+) \leq \mathbb{P}(B_n^-) \xrightarrow[n\to\infty]{} 0.$$

and thus, we get

$$\lim_{n\to\infty} u(n) = \frac{1}{\mu_*}.$$

□

The following is based on the relation $\overline{F}_* * u = \mathbb{1}$ and an analogous proof is given by [12].

Convolutional . From $\overline{F}_* * u = \mathbb{1}$ we have

$$\left(\sum_{k=0}^{\infty} \overline{F}_*(k) e_1^{(k)}\right) \cdot \left(\sum_{k=0}^{\infty} u(k) e_1^{(k)}\right) = \sum_{k=0}^{\infty} \mathbb{1}(k) e_1^{(k)},$$

and if we use the isomorphism in Proposition 1.3 we will get that

$$\left(\sum_{k=0}^{\infty} \overline{F}_*(k) x^k\right) \cdot \left(\sum_{k=0}^{\infty} u(k) x^k\right) = \sum_{k=0}^{\infty} x^k.$$

Therefore, for any $x \in (-1, 1)$ we can easily obtain that

$$\begin{aligned}
\left(\sum_{k=0}^{\infty} \overline{F}_*(k) x^k\right) \cdot \left(\sum_{k=0}^{\infty} u(k) x^k\right) &= \frac{1}{1-x} \Longrightarrow \\
(1-x) \cdot \left(\sum_{k=0}^{\infty} u(k) x^k\right) &= \frac{1}{\sum_{k=0}^{\infty} \overline{F}_*(k) x^k}. \qquad (2.31)
\end{aligned}$$

The left-hand member of (2.31) can be rewritten as

$$\begin{aligned}(1-x)\cdot\left(\sum_{k=0}^{\infty}u(k)x^k\right) &= \sum_{k=0}^{\infty}u(k)x^k-\sum_{k=0}^{\infty}u(k)x^{k+1}\\ &= u(0)+\sum_{k=1}^{\infty}u(k)x^k-\sum_{k=1}^{\infty}u(k-1)x^k\\ &= 1+\sum_{k=1}^{\infty}\left(u(k)-u(k-1)\right)x^k.\end{aligned}$$

Since $\sum_{k=0}^{\infty}\overline{F}_*(k)=\mu<\infty$, then from Abel's theorem (Theorem D.1) we have

$$\lim_{x\to 1^-}\sum_{k=0}^{\infty}\overline{F}_*(k)x^k=\sum_{k=0}^{\infty}\overline{F}_*(k)=\mu,$$

and consequently from the above we conclude that

$$\begin{aligned}1+\sum_{k=1}^{\infty}\left(u(k)-u(k-1)\right) &= \frac{1}{\mu} \quad\Longrightarrow\\ 1+\lim_{n\to\infty}[u(n)-u(0)] &= \frac{1}{\mu} \quad\overset{u(0)=1}{\Longrightarrow}\\ \lim_{n\to\infty}u(n) &= \frac{1}{\mu}.\end{aligned}$$

□

(ii) *Proof.* Define the sequence $(X'_n)_{n\in\mathbb{N}}$ as

$$X'_n=\frac{X_n}{d},\ n\in\mathbb{N}.$$

Since $X_n\in d\cdot\mathbb{N}$ then $(X'_n)_{n\in\mathbb{N}}$ forms a sequence of i.i.d random variables defined on $\mathbb{N}$ with probability mass function f_d and expected values μ_d given by

$$f_d(k)=\mathbb{P}(X'_1=k)=\mathbb{P}(X_1=dk)=f(dk),\ k\in\mathbb{N} \tag{2.32}$$

and

$$\mu_d=\mathbb{E}(X'_1)=\frac{\mathbb{E}(X_1)}{d}=\frac{\mu}{d}.$$

Also, define

$$S'_n=\sum_{l=0}^{n}X'_l$$

and obviously we have

$$S'_n=\frac{S_n}{d}.$$

The sequence $(S'_n)_{n\in\mathbb{N}}$ forms an aperiodic renewal chain with waiting times $(X'_n)_{n\geq 1}$ and mean time $\mu_d=\frac{\mu}{d}$. From (2.32), we have that the corresponding sequence of renewal probabilities u_d is given by $u_d(n)=u(dn)$. Thus, from (i) we obtain

$$u(dn)=u_d(n)\xrightarrow[n\to\infty]{}\frac{1-f(0)}{\mu_d}=\frac{d}{\frac{\mu}{1-f(0)}}=\frac{d}{\mu_*}.$$

□

The difference between the proof with convolutions and this in [12] is that we had already proven that the genereting function of the reliability function has no zero values on $(-1,1)$, since $\overline{F}_* * u = \mathbb{1}$. The latter arises from the fact that we used convolutional techniques such as the existence of the convolutional inverse. In [12] the proof starts from (2.8) using generating functions without convolutional properties and representations. Ultimately, these proofs are crossed in the end with different starting points.

Remark 2.13. *An alternative proof of the elementary renewal theorem is born via renewal theorem. Indeed, since u converges to μ_* then its average converges also to μ_*. Consequently,*

$$\frac{M(n)}{n} = \frac{1}{1-f(0)} \cdot \frac{[\mathbb{1} * u](n)}{n} - \frac{1}{n} \xrightarrow[n\to\infty]{} \frac{1}{1-f(0)} \cdot \frac{1}{\mu_*} = \frac{1}{\mu}.$$

In the aperiodic case we obtain similarly that

$$\frac{M(dn)}{n} \xrightarrow[n\to\infty]{} \frac{d}{\mu}.$$

The following is a direct consequence of the renewal theorem.

Corollary 2.1. *Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent renewal chain, then:*

(i) *if it is aperiodic, we have:*

$$L(n) \xrightarrow[n\to\infty]{} \frac{1}{\mu}$$

(ii) *if it is aperiodic of period $d > 1$ we have*

$$L(dn) \xrightarrow[n\to\infty]{} \frac{d}{\mu}.$$

and

$$L(n) = 0, \quad \textit{for all } n \in \{l \in \mathbb{N} |\, l \not\equiv 0 \ (mod\ d)\}.$$

Remark 2.14. *If $f(0)$=0, then we take the usual renewal theorem.*

As a result of the renewal theorem, we have the following theorem.

Theorem 2.5 (Key renewal theorem)**.** *Let $(S_n)_{n\in\mathbb{N}}$ be a recurrent renewal chain and a real sequence $(f_0(n))_{n\in\mathbb{N}}$ with $\sum_{n=0}^{\infty}|f_0(n)| < \infty$ then:*

(i) *if it is aperiodic we have:*

$$\lim_{n\to\infty}[f_0 * u](n) \xrightarrow[n\to\infty]{} \frac{1}{\mu_*}\sum_{n=0}^{\infty} f_0(n) \tag{2.33}$$

(ii) *if it is periodic of period $d > 1$, then for any positive integer $l \in$ [0,d-1], we have:*

$$[f_0 * u](l + nd) \xrightarrow[n\to\infty]{} \frac{d}{\mu_*}. \tag{2.34}$$

Proof. (i) From the renewal theorem we get

$$\lim_{n\to\infty} u(n-l) = \frac{1}{\mu_*}.$$

Since $u(n) \leq 1$, we get

$$|f_0(l)u(n-l)| \leq |f_0(l)|, \ \forall\, l = 0, \ldots, n.$$

Then

$$\sum_{l=0}^{\infty} |f_0(l)u(n-l)| \leq \sum_{l=0}^{\infty} |f_0(l)| < \infty.$$

Thus, the dominated convergence theorem for sequences (Theorem D.2) gives us:

$$\lim_{n\to\infty} [f_0 * u](n) = \sum_{l=0}^{\infty} \lim_{n\to\infty} f_0(l)u(n-l) = \frac{1}{\mu_*} \sum_{l=0}^{\infty} f_0(l).$$

(ii) For any positive integer n we have

$$[f_0 * u](l+nd) = \sum_{k=0}^{l+nd} f_0(k)u(l+nd-k).$$

Since S_n is periodic of period $d > 1$, we have

$$u(l+nd-k) > 0 \implies l \equiv k \ (mod\, d).$$

Therefore,

$$[f_0 * u](l+nd) = \sum_{k=0}^{n} f_0(l+d(n-k))u(kd)$$

and in a similar way as (i) we conclude the desired result.

□

Now, we can give a corresponding renewal theorem for a delayed recurrent renewal chain.

Theorem 2.6. *[Renewal theorem for delayed renewal chains]*
Let $(S_n)_{n\in\mathbb{N}}$ be a delayed recurrent renewal chain with initial distribution f_0. Then

1. *In the case of aperiodicity*

$$v(n) \xrightarrow[n\to\infty]{} \frac{1}{\mu_*} \sum_{n=0}^{\infty} f_0(n) \tag{2.35}$$

2. *In the case of periodicity of period $d > 1$, we have*

$$v(l+nd) \xrightarrow[n\to\infty]{} \frac{d}{\mu_*} \sum_{n=0}^{\infty} f_0(l+nd)$$

for any integer $l \in [0, d-1)$.

Proof. From Proposition 2.5 we have:

$$v(n) = [f_0 * u](n) = \sum_{l=0}^{n} f_0(l)u(n-l), \ n \in \mathbb{N}.$$

Consequently, both of these results are considered as applications of the key renewal theorem. □

The following sequence is a consequence by the combination of Proposition 2.14 and Theorem 2.6. Let $(S_n)_{n\in\mathbb{N}}$ be a delayed recurrent renewal chain with initial distribution f_0. Then,

1. in the aperiodic case we have

$$L_D(n) \xrightarrow[n\to\infty]{} \frac{1}{\mu}\sum_{n=0}^{\infty} f_0(n) \tag{2.36}$$

2. in the periodic case of period d > 1, we have

$$L_D(l+nd) \xrightarrow[n\to\infty]{} \frac{d}{\mu}\sum_{n=0}^{\infty} f_0(l+nd) \tag{2.37}$$

for any integer l $\in [0, d-1)$.

5. Examples

In this section we give some applications. First, We give a usual parametric renewal chain and its asymptotic results. Furthermore, we show some useful results for two important processes. The following example summarizes the main results of this chapter for a renewal chain with Poisson interrarival times and studies the convergence of the sequence of renewal probabilities for different rates.

Example 2.3. *Let $(X_{n\geq 1})$ be a sequence of i.i.d random variables which follow the Poisson distribution with rate $\lambda \in (0, \infty)$, pmf*

$$f(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k \in \mathbb{N},$$

expectation and variance

$$\mathbb{E}(X_n) = \mathbb{V}(X_n) = \lambda,\ n \in \mathbb{N}^*.$$

So the function L is expressed as

$$L(n) = \frac{u(n)}{1-f(0)} = \frac{u(n)}{1-e^{-\lambda}},\ n \in \mathbb{N}.$$

From the SLLN for counting processes and the elementary renewal theorem we have

$$\frac{N(n)}{n} \xrightarrow[n\to\infty]{} \frac{1}{\lambda},\ \frac{M(n)}{n} \xrightarrow[n\to\infty]{} \frac{1}{\lambda}$$

respectively.

Furthermore, Theorem 2.3 gives us

$$\sqrt{n}\left(\frac{N(n)}{n} - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{1}{\lambda^2}\right).$$

From renewal theorem, we have

$$u(n) \xrightarrow[n\to\infty]{} \frac{1-e^{-\lambda}}{\lambda}$$

and consequently

$$L(n) \xrightarrow[n\to\infty]{} \frac{1}{\lambda}.$$

In the following figure we summarize values of renewal probability and function L until time 20 for different rates

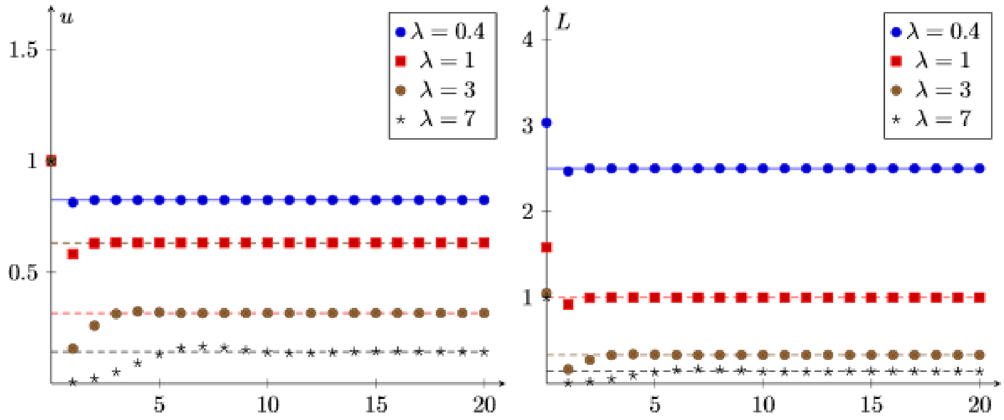

FIGURE 2.4: Plots of u and L until time 20.

From the above figure we can easily see that the corresponding rate of convergence for the sequences u and L decreases as the rate λ increases.

Next, we give the corresponding backward and residual times for a renewal chain and we study their asymptotic properties which will be used in the sequel.

Example 2.4. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d nonnegative integer valued r.v's with $X_0 = 0$. Also, set*

$$U_n = n - S_{N(n)}, \quad R_n = S_{N(n)+1} - n.$$

For an arbitrary $x \in \mathbb{N}$, denote the following sequences

$$p_x(n) = \mathbb{P}(U_n \geq x), \quad g_x(n) = \mathbb{P}(R_n \geq x), \quad x \in \mathbb{N}.$$

The function g_x can be rewritten as

$$g_x(n) = \mathbb{P}(R_n > x) = \sum_{l=0}^{\infty} \mathbb{P}(R_n > x | S_1 = l) f(l),$$

where

$$\mathbb{P}(R_n > x | S_1 = l) = \begin{cases} 0 & \text{if } l - n < x, \ l > n \\ 1 & \text{if } l - n \geq x, \ l > n \\ g(n-l) & \text{otherwise}. \end{cases}$$

and we have

$$g_x(n) = \sum_{l=n+x}^{\infty} f(l) + \sum_{l=0}^{n} g_x(n-l) f(l) = \overline{F}(x+n+1) + [g_x * f](n) := \overline{F_x}(n) + [g_x * f](n).$$

This relation forms a renewal equation and thus, from Proposition 2.3 we have

$$g_x(n) = [u * \overline{F_x}](n).$$

Also, notice that

$$\sum_{l=0}^{\infty} \overline{F_x}(l) = \sum_{l=x}^{\infty} \overline{F}(l) \leq \sum_{l=0}^{\infty} \overline{F}(l) = \mu < \infty.$$

Then, the condition of key renewal theorem is satisfied and consequently we have

$$g_x(n) = \frac{1}{1-f(0)}[u * \overline{F_x}](n) \xrightarrow[n\to\infty]{} \frac{\sum_{l=x+1}^{\infty} \overline{F}(l)}{\mu}$$

or

$$1 - g_x(n) \xrightarrow[n\to\infty]{} \frac{\sum_{l=0}^{x} \overline{F}(l)}{\mu}.$$

Therefore, the sequence of r.v's $(R_n)_{n\in\mathbb{N}}$ *converges in law to a random variable with distribution function*

$$g_\infty(x) = \frac{\sum_{l=0}^{x} \overline{F}(l)}{\mu}, \quad x \in \mathbb{N}.$$

Furthermore, the sequence $(U_n)_{n\in\mathbb{N}}$ *is studied under the following relation*

$$\begin{aligned} \{U_n > x\} &= \{n - S_{N(n)} > x\} = \{S_{N(n)} < n - x\} = \{N(n) \leq N(n-x)\} \\ &= \{N(n) < N(n-x) + 1\} = \{n < S_{N(n-x)+1}\} = \{n - x < S_{N(n-x)+1} - x\} \\ &= \{x < S_{N(n-x)+1} - (n-x)\} = \{R_{n-x} > x\}. \end{aligned}$$

Hence, the limiting distribution of U_n *is*

$$1 - \lim_{n\to\infty} p_x(n) = 1 - \lim_{n\to\infty} g_x(n-x) = 1 - \lim_{m\to\infty} g_x(m) \xrightarrow[m\to\infty]{} \frac{\sum_{l=0}^{x} \overline{F}(l)}{\mu}.$$

Chapter 3

MARKOV RENEWAL THEORY

1. Introduction

In this chapter we introduce the discrete-time Markov renewal theory and we study its probabilistic properties. We use elements of convolutional theory in order to achieve more convenient representations for well known sequences and quantities. Furthermore, we give some alternative proofs for well known theorems in order to avoid classic techniques and methods of renewal theory such as the Markov renewal theorem. Some famous results of Markov renewal theory are given in order to make a well organized chapter.

The structure of this chapter is as follows: In the next section we introduce the notion of discrete-time Markov renewal and semi-Markov chains and obtain the unique solution for the discrete-time Markov renewal equation. In section 3, we give the associated asymptotic results for this class of stochastic processes and in the last section we present some concrete examples.

2. Markov Renewal theory

Here, we give the theory of Markov renewal chains. We introduce the semi-Markov kernel, the Markov -Renewal chain and we give the unique solution for the Markov renewal equation. Also, we study some kinds of Markov renewal chains and their probabilistic properties.

In order to introduce the Markov renewal chains we will need some definitions. Consider a random system with finite state space $E = \{1, \ldots, s\}$. We assume that the system is evolving with jumps in time and a chain (J_n) records the successively visited states. The interarrival jump times are recorded by the sequence of strictly positive integer-valued random variables $(X_n)_{n\geq 1} \in \mathbb{N}$ and $X_0 = 0$. Denote by S_n the sequence of partial sums of $(X_n)_{n\in\mathbb{N}}$, i.e. $S_n = \sum_{k=0}^{n} X_k$.

Below, we introduce the notion of semi-Markov kernel.

Definition 3.1. *A matrix valued-function $q = (q_{ij}(k)) \in \mathcal{M}_s(\mathbb{N})$ is said to be a discrete time semi-Markov kernel, if it satisfies the following conditions:*

(i) $q_{ij}(k) \geq 0$, $i, j \in E$, $k \in \mathbb{N}$,

(ii) $q_{ij}(0) = 0$, i, $j \in E$,

(iii) $\sum_{j\in E} \sum_{k=0}^{\infty} q_{ij}(k) = 1$, $i \in E$.

In the development of the theory we assume that $q_{ii}(k) = 0$ for any state i because we will study systems in which the direct transitions are possible only between different states.

Definition 3.2. *The chain $(J, S) := (J_n, S_n)_{n\in\mathbb{N}}$ is said to be a Markov-renewal chain (MRC) if for all $n \in \mathbb{N}$, $i, j \in E$ and $k \in \mathbb{N}$ it satisfies almost surely*

$$\mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_{0:n}, S_{0:n}) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_n).$$

Furthermore, if the above equation is independent of n, then (J, S) is said to be homogeneous and

$$q_{ij}(k) := \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n = k | J_n = i).$$

Remark 3.1. *The process (J, S) is a homogeneous Markov chain and its transition matrix is determined by the semi-Markov kernel.*

Remark 3.2. *We can also define $p_{ij} = \sum_{k=0}^{\infty} q_{ij}(k)$. Then, the matrix $p = (p_{ij})_{i,j\in E}$ is a Markov transition matrix.*

Proposition 3.1. *Let (J, S) a MRC and q the associated semi-Markov kernel. Then, the processes (J, S), (J, X) and J are Markov chains with transition probabilities*

$$\begin{aligned} \mathbb{P}(J_{n+1} = j, S_{n+1} = s_n + k | J_n = i, S_n = s_n) &= q_{ij}(k) && (3.1)\\ \mathbb{P}(J_{n+1} = j, X_{n+1} = k | J_n = i, X_n = k') &= q_{ij}(k) && (3.2)\\ \mathbb{P}(J_{n+1} = j | J_n = i) &= p_{ij}. && (3.3) \end{aligned}$$

The process J is a Markov chain, called the embedded Markov chain associated to the MRC (J, S). Its associated transition matrix is given by p (Remark 3.2).

Example 3.1. *Let us consider a Markov renewal chain (J, S) with associated semi-Markov kernel*

$$q = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix},$$

where f and g are two probability functions with support on $\mathbb{N}^*$. Then, J is a Markov chain and its transition probabilities are given by the identity matrix I_2.

Assumption 3.1. *The embedded Markov chain J is irreducible and aperiodic and its stationary distribution is given by the stochastic vector π^*.*

Remark 3.3. *For the embedded Markov chain J we denote by μ^*_{ll} and μ^*_{jl}, with $l \neq j$ the mean recurrence time in state l and the passage time in state l, starting from a state j, respectively. Furthermore, if Assumption 3.1 holds, then by classical results on Markov chain theory, we have*

$$\mu^*_{jj} = \frac{1}{\pi^*_j}.$$

In the development of the theory a fundamental role will be played by the sequence of matrices $E_0 - q$ and in the following proposition we examine the existence and form of its convolutional inverse.

Proposition 3.2. *The convolutional inverse of the matrix-valued function $E_0 - q$ exists and is given by*

$$(E_0 - q)^{(-1)} = \sum_{n\geq 0} q^{(n)}. \tag{3.4}$$

Proof. Since $q(0)$ is assumed to be the null matrix then $[E_0 - q](0) = I_s$ and consequently the convolutional inverse of the sequence of matrices $E_0 - q$ exists because of Theorem 1.2. Furthermore, from Equation (1.60) we get directly that

$$(E_0 - q)^{(-1)} = \sum_{n\geq 0} (E_0 - (E_0 - q))^{(l)} = \sum_{n\geq 0} q^{(n)}.$$

□

Example 3.2. *A Markov chain with state space $E = \{1, \ldots, s\}$ and transition matrix $p = (p_{ij})_{i,j\in E}$ is a special case of a Markov renewal chain with semi Markov kernel*

$$q_{ij}(k) = \begin{cases} p_{ij}p_{ii}^{k-1}, & \text{if } k \in \mathbb{N}^* \text{ and } i \neq j \\ 0, & otherwise. \end{cases}$$

In the following proposition we give a useful form for the n-th step transition probabilities $\big(\mathbb{P}_i(J_n = j, S_n = k)\big)_{i,j\in E,\ n\in\mathbb{N}}$.

Proposition 3.3. *Let (J,S) be a Markov renewal chain. The following relation holds for any $i,j \in E$ and $k,n \in \mathbb{N}$.*

$$\mathbb{P}(J_n = j, S_n = k | J_0 = i) = q_{ij}^{(n)}(k). \tag{3.5}$$

Proof. We show it by induction. For $n = 0$ we have that the quantity $\mathbb{P}(J_n = j, S_n = k|J_0 = i)$ is zero for $i \neq j$, otherwise is one and thus, from Definition 1.2 we can directly observe that

$$\mathbb{P}(J_0 = j, S_0 = k|J_0 = i) = q_{ij}^{(0)}(k).$$

Furthermore, assume that for an arbitrary $n \in \mathbb{N}$ the following relation holds

$$\mathbb{P}(J_n = j, S_n = k|J_0 = i) = q_{ij}^{(n)}(k), \quad k \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}_i(J_{n+1} = j, S_{n+1} = k) &= \sum_{r\in E}\sum_{l=0}^{k} \mathbb{P}_i(J_{n+1} = j, S_{n+1} = k|S_n = l, J_n = r)\mathbb{P}_i(S_n = l, J_n = r) \\
&= \sum_{l=0}^{k} \mathbb{P}_i(J_{n+1} = j, S_{n+1} - S_n = k - l|S_n = l, J_n = r)q_{ir}^{(n)}(l) \\
&= \sum_{l=0}^{k} q_{rj}(k-l)q_{ir}^{(n)}(l) \\
&= q_{ij}^{(n+1)}(k)
\end{aligned}$$

and by induction we conclude that the desired form holds. □

Remark 3.4. *From Proposition 3.3 we get directly that the n-step transition kernel $q^{(n)}$ represents the n-step transition probability from (i,k') to (j,k) for the homogeneous Markov Chain (J,S) since*

$$p^{(n)}\big((i,k'),(j,k)\big) = q_{ij}^{(n)}(k-k').$$

Consequently, the n-step transition probabilities of the embedded Markov chain J are determined by

$$p_{ij}^{(n)} = \sum_{k=n}^{\infty} q_{ij}^{(n)}(k).$$

Example 3.3. *Let us consider the MRC which is defined in Example 3.1. We compute the inverse of the sequence of matrices $E_0 - q$ using the following ways*

1. *Using Theorem 1.2*
2. *Using Theorem 1.3.*

1. The n-fold convolutions of q are given by

$$q^{(n)} = \begin{cases} \begin{pmatrix} 0 & f^{(\frac{n+1}{2})} * g^{(\frac{n-1}{2})} \\ f^{(\frac{n-1}{2})} * g^{(\frac{n+1}{2})} & 0 \end{pmatrix} & \text{if } n \text{ is odd,} \\[2ex] \begin{pmatrix} f^{(\frac{n}{2})} * g^{(\frac{n}{2})} & 0 \\ 0 & f^{(\frac{n}{2})} * g^{(\frac{n}{2})} \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

Then, from (3.4) we have

$$(E_0-q)^{(-1)}(k) = \sum_{\substack{n-0 \\ 2\not| n}}^{k} \begin{pmatrix} 0 & f^{(\frac{n+1}{2})} * g^{(\frac{n-1}{2})} \\ f^{(\frac{n-1}{2})} * g^{(\frac{n+1}{2})} & 0 \end{pmatrix}(k) + \sum_{\substack{n-0 \\ 2|n}}^{k} \begin{pmatrix} f^{(\frac{n}{2})} * g^{(\frac{n}{2})} & 0 \\ 0 & f^{(\frac{n}{2})} * g^{(\frac{n}{2})} \end{pmatrix}(k).$$

2. The convolutional determinant of $E_0 - q$ and its inverse are the real sequences

$$\begin{aligned} det(E_0 - q) &= det\begin{pmatrix} e_0 & -f \\ -g & e_0 \end{pmatrix} = e_0 - f * g, \\ det(E_0 - q)^{(-1)} &= (e_0 - f * g)^{(-1)} = \sum_{l=0}^{\cdot} f^{(l)} * g^{(l)}. \end{aligned}$$

The associated adjugate matrix function is given by

$$adj(E_0 - q) = adj\begin{pmatrix} e_0 & -f \\ -g & e_0 \end{pmatrix} = \begin{pmatrix} e_0 & f \\ g & e_0 \end{pmatrix}.$$

Consequently, the inverse $(E_0 - q)^{(-1)}$ can be also expressed by

$$(E_0 - q)^{(-1)} = (e_0 - f * g)^{(-1)} * \begin{pmatrix} e_0 & f \\ g & e_0 \end{pmatrix}.$$

Therefore, any associated element is given as follows

$$\begin{aligned} (E_0 - q)_{11}^{(-1)} &= (e_0 - f * g)^{(-1)}, & (E_0 - q)_{12}^{(-1)} &= (e_0 - f * q)^{(-1)} * f, \\ (E_0 - q)_{21}^{(-1)} &= (e_0 - f * g)^{(-1)} * g, & (E_0 - q)_{22}^{(-1)} &= (e_0 - f * q)^{(-1)}. \end{aligned}$$

Remark 3.5. *Since $q(0)$ is assumed to be null then from Remark (1.3) we take $q^{(n)}(k)$ is also null for any $n, k \in \mathbb{N}$ with $n > k$.*

The previous result can be also obtained through the following reasoning: Since in k-time units we cannot have more than k jumps (almost surely) we directly obtain that $q^{(n)}(k) = 0_E$, $n, k \in \mathbb{N}$ with $n > k$.

Some other quantities of interest related to a specific state of a MRC are given in the following definition:

Definition 3.3. *For all $i, j \in E$ and $k \in \mathbb{N}$, we introduce:*

(i) *The cumulated semi-Markov kernel $Q = (Q(k)) \in \mathcal{M}_E(\mathbb{N})$,*

$$Q = \mathbb{I} * q,$$

where $\mathbb{I} \equiv I_s$,

(ii) *the conditional probability mass function of X_{n+1}, $n \in \mathbb{N}$, given by*

$$f_{ij}(k) := \mathbb{P}(X_{n+1} = k \mid J_n = i, J_{n=1} = j),$$

(iii) *the conditional distribution function of X_{n+1}, given by*

$$F_{ij} = \mathbb{1} * f_{ij},$$

(iv) *the sojourn time probability mass function*

$$h_i(k) = \mathbb{P}(X_{n+1} = k | J_n = i),$$

(v) *the sojourn time cumulative distribution function*

$$H_i = \mathbb{1} * h_i,$$

(vi) *the survival function of state* i

$$\overline{H}_i = \mathbb{1} - H_i = \mathbb{1} - \mathbb{1} * h_i,$$

(vii) *the mean sojourn time of state* i

$$m_i = \lim_{k\to\infty} \left[\mathbb{1} * \overline{H_i}\right](k).$$

It's easy to notice that the conditional probability mass function satisfy the following relation

$$f_{ij}(k) = \begin{cases} \frac{q_{ij}(k)}{p_{ij}}, & \text{if } p_{ij} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We can also introduce two kinds of semi-Markov kernels for which $f_{ij}(k)$ is independent from i or j. This case is satisfied by a sequence of matrices which satisfy the form $q_{ij}(k) = p_{ij}f_i(k)$ or $q_{ij}(k) = p_{ij}f_j(k)$. The first form corresponds to a Markov renewal chain in which the sojourn times depend only on the present visited state and the latter on the next. In the development of the theory, we use the general form of f.

Example 3.4. *Let (J_n, S_n) be a Markov renewal chain, $q_{ij}(k) = p_{ij}f_{ij}(k)$ and $h_i(k)$ be the associated semi-Markov kernel and sojourn time probability mass function. The pair (J_n, S_{n+1}) forms also a Markov renewal chain with semi-Markov kernel $q^*_{ij}(k) = p_{ij}h_j(k)$. Furthermore, the pair (J_{n+1}, S_n) is a Markov renewal chain and the form of the semi-Markov kernel is given by $\tilde{q}_{ij}(k) = p_{ij}h_i(k)$.*

Example 3.5. *Let us consider again the MRC of Example 3.1. For the associated conditional distributions we have*

$$\begin{aligned} f_{12} = f, \quad f_{21} = g, \quad h_1 = f, \quad h_2 = g, \\ Q_{12} = F, \quad Q_{21} = G, \quad H_1 = F, \quad H_2 = G, \\ \overline{H_1} = \overline{F}, \quad \overline{H_2} = \overline{G} \end{aligned},$$

where

$$\begin{aligned} F &= \mathbb{1} * f, & \overline{F} &= \mathbb{1} - F, \\ G &= \mathbb{1} * g, & \overline{G} &= \mathbb{1} - G. \end{aligned}$$

Then, the mean sojourn times of states 1 and 2 are computed by

$$m_1 = \lim_{n\to\infty} \left[\mathbb{1} * \overline{F}\right](n), \quad m_2 = \lim_{n\to\infty} \left[\mathbb{1} * \overline{G}\right](n).$$

Define the array of random variables $(R_{nj})_{n\in\mathbb{N}, j\in E}$ which is described by

$$R_{nj} = \sum_{l=0}^{n} \mathbb{1}_{\{J_l=j, S_l=n\}}, \ n \in \mathbb{N}, \ j \in E. \tag{3.6}$$

Since S is strictly increasing, clearly $R_{nj} \in \{0, 1\}$, and it can be interpreted as an indicator variable which records if a renewal of type j has occured at time n. The corresponding probabilities starting from state i, are denoted by $u_{ij}(n)$, that is,

$$u_{ij}(n) = \mathbb{P}(R_{nj} = 1 | J_0 = i) := \mathbb{P}_i(R_{nj} = 1), n \in \mathbb{N},\ i, j \in E, \tag{3.7}$$

and we refer it as the sequence of Markov renewal probabilities. Notice that since $X_0 = 0$, we have $u_{ii}(0) = 1,\ i \in E$.

The following proposition gives a relationship between u and q.

Proposition 3.4. *The sequence of matrices u and q are linked through*

$$u = (E_0 - q)^{(-1)}. \tag{3.8}$$

Proof. From the monotonicity of $(S_n)_{n\in\mathbb{N}}$ and the definition of R, the latter is expressed by the following union of disjoint sets

$$\{R_{nj} = 1\} = \bigcup_{l=0}^{n} \{J_l = j, S_l = n\}, \quad i, j \in E,\ n \in \mathbb{N}.$$

Thence, the sequence u is given by

$$u_{ij}(n) = \sum_{l=0}^{n} \mathbb{P}_i \left(J_l = j,\ S_l = n\right) = \sum_{l=0}^{n} q_{ij}^{(l)}(n),$$

and the desired result is obtained by Proposition 3.4. □

From the above proposition, we can directly obtain that

$$u = E_0 + q * u.$$

This representation forms a class of equations known as Markov renewal equations.

Definition 3.4. *Let $L \in \mathcal{M}_E(\mathbb{N})$ be a known matrix valued function and $G \in \mathcal{M}_E(\mathbb{N})$ be an unknown. The equation*

$$G = L + q * G \tag{3.9}$$

is called discrete time Markov renewal equation.

The existence of the unique solution is given in the following proposition.

Proposition 3.5. *The solution of the discrete time Markov renewal equation exists, is unique and is given by*

$$G = u * L. \tag{3.10}$$

Proof. From Equation (3.9) we have

$$(E_0 - q) * G = L \quad \Longrightarrow G = (E_0 - q)^{(-1)} * L \overset{(3.8)}{=} u * L.$$

□

Definition 3.5. *Let (J, S) be a Markov renewal chain and $\tilde{N}_j(k)$ be the number of visits of the MC J to state j until time $N(k)$, up to time $k \in \mathbb{N}$ given by*

$$\tilde{N}_j(k) := \sum_{n=0}^{N(k)} \mathbb{1}_{\{J_n = j\}} = \sum_{n=0}^{k} \mathbb{1}_{\{J_n = j, S_n \leq k\}} = \sum_{n=0}^{k} \sum_{l=0}^{k} \mathbb{1}_{\{J_n = j, S_n = l\}}. \tag{3.11}$$

The matrix-valued function $U = (U(k), k \in \mathbb{N}) \in \mathcal{M}_s$, where

$$U_{ij}(k) = \mathbb{E}_i \left(\tilde{N}_j(k)\right), \quad i, j \in E,\ k \in \mathbb{N}, \tag{3.12}$$

is called Markov renewal function.

Proposition 3.6. *The matrix valued functions U and u are linked through*

$$U = \mathbb{I} * u. \tag{3.13}$$

Proof. From the linearity of expectation we get directly that

$$U_{ij}(k) \quad = \quad \mathbb{E}_i\left(\sum_{n=0}^{k}\sum_{l=0}^{k}\mathbb{1}_{\{J_n=j,S_n=l\}}\right) = \sum_{n=0}^{k}\sum_{l=0}^{k}\mathbb{E}_i\left(\mathbb{1}_{\{J_n=j,S_n=l\}}\right) = \sum_{n=0}^{k}u_{ij}(n). \tag{3.14}$$

□

Remark 3.6. *Since*

$$u = E_0 + q * u,$$

then from equation (3.13) we get directly that

$$U = u * \mathbb{I} = \mathbb{I} * E_0 + q * u * \mathbb{I} = \mathbb{I} + q * U,$$

and consequently U is the unique solution of the Markov renewal equation

$$U = \mathbb{I} + q * U.$$

Next, we give the notion of a semi-Markov chain and its probabilistic characteristics.

Definition 3.6. *Let (J, S) be a Markov renewal chain and*

$$N(k) := \sup\{n \in \mathbb{N} : S_n \leq k\},$$

the counting process of the number of jumps in $[0, k]$. The chain $Z = (Z_k)_{k\in\mathbb{N}}$

$$Z_k = J_{N(k)}, \quad k \in \mathbb{N},$$

is said to be a semi-Markov chain associated to the MRC (J, S).

We denote by α=$(\alpha_i)_{i\in E}$ the initial distribution of the SMC Z. This is, α_i = $\mathbb{P}(Z_0 = i)$ = $\mathbb{P}(J_0 = i)$, $i \in E$.

Definition 3.7. *The transition function of the semi-Markov chain Z denoted by P is the matrix valued function given by*

$$P_{ij}(k) = \mathbb{P}(Z_k = j|Z_0 = i), \quad i, j \in E,\ k \in \mathbb{N}.$$

In the following proposition we give an exact form of the transition function P making links with the Markov renewal equations.

Proposition 3.7. *The transition function P of a semi-Markov chain Z is given by*

$$P = u * \overline{H}, \tag{3.15}$$

where

$$\overline{H} = \left(diag\{\overline{H_j}\}\right)_{j\in E}.$$

Proof. Let $i, j \in E$ be two arbitrary elements and $k \in \mathbb{N}$. Any $P_{ij}(k)$ can be written as

$$P_{ij}(k) = \mathbb{P}(Z_k = j|Z_0 = i) = \mathbb{P}(Z_k = j, X_1 > k|J_0 = i) + \mathbb{P}(Z_k = j, X_1 \leq k|J_0 = i). \tag{3.16}$$

We have also

$$\mathbb{P}(Z_k = j, X_1 > k | J_0 = i) = \mathbb{1}_{\{i=j\}}\mathbb{P}(X_1 > k | J_0 = i) = \mathbb{1}_{\{i=j\}}\overline{H_i}(k) \tag{3.17}$$

and

$$\begin{aligned}
\mathbb{P}(Z_k = j, X_1 \le k | J_0 = i) &= \sum_{r\in E}\sum_{l=0}^{k} \mathbb{P}(Z_k = j, X_1 = l, J_1 = r | J_0 = i) \\
&= \sum_{r\in E}\sum_{l=0}^{k} \mathbb{P}(Z_k = j | X_1 = l, J_1 = r, J_0 = i)\mathbb{P}(X_1 = l, J_1 = r | J_0 = i) \\
&= \sum_{r\in E}\sum_{l=0}^{k} \mathbb{P}(Z_{k-l} = j | J_0 = r) q_{ir}(l) \\
&= \sum_{r\in E}\sum_{l=0}^{k} q_{ir}(l) P_{rj}(k-l) = \sum_{r\in E} \left[q_{ir} * P_{rj}\right](k). \qquad (3.18)
\end{aligned}$$

Therefore, by adding (3.17) and (3.18) relation (3.16) becomes

$$P_{ij} = \mathbb{1}_{\{i=j\}}\overline{H_i} + \sum_{r\in E} q_{ir} * P_{rj}.$$

This implies that the sequence of matrices P is represented by the Markov renewal equation

$$P = \overline{H} + q * P$$

Consequently, from Proposition 3.5 we conclude that

$$P = u * \overline{H}.$$

□

Example 3.6. *Let us consider the two-state Markov renewal chain of Example 3.1. The matrix-valued functions u and $\overline{H}$ are given in Examples 3.3 and 3.5 respectively. Then, Z is a two state semi-Markov chain with transition function*

$$\begin{aligned}
P &= u * \overline{H} = (e_0 - f * g)^{(-1)} * \begin{pmatrix} e_0 & f \\ g & e_0 \end{pmatrix} * \begin{pmatrix} \overline{F} & 0 \\ 0 & \overline{G} \end{pmatrix} \\
&= (e_0 - f * g)^{(-1)} * \begin{pmatrix} \overline{F} & f * \overline{G} \\ g * \overline{F} & \overline{G} \end{pmatrix}.
\end{aligned}$$

Then, each function $P_{ij}(k)$ is given by

$$P_{11}(k) = \left[(e_0 - f * g)^{(-1)} * \overline{F}\right](k), \qquad P_{12}(k) = \left[(e_0 - f * g)^{(-1)} * f * \overline{G}\right](k),$$
$$P_{21}(k) = \left[(e_0 - f * g)^{(-1)} * g * \overline{F}\right](k), \qquad P_{22}(k) = \left[(e_0 - f * g)^{(-1)} * \overline{G}\right](k).$$

Definition 3.8. *For a Semi-Markov chain $(Z_k)_{k\in\mathbb{N}}$, the limit distribution $\pi = (\pi_i)_{i\in E}$ is defined, when it exists, by $\pi_j = \lim_{k\to\infty} P_{ij}(k)$, for every $i, j \in E$.*

Also we can express u in terms of renewal chains embedded in the MRC. To do this, we need the definitions of successive passage times in a certain state and their distributions.

Definition 3.9. *Let (J, S) be a Markov renewal chain. We denote by $\left(S_n^j\right)_{n\in\mathbb{N}}$ the sequence of succesive passage times in a fixed state $j \in E$, given by*

$$S_0^j = S_m,\ with\, m = \inf\{l \in \mathbb{N} | J_l = j\}$$

$$S_n^j = S_m,\ with\, m = \inf\{l \in \mathbb{N} | J_l = j, S_l > S_{n-1}^j\}$$

Definition 3.10. *For any states $i, j \in E$ we consider:*

1. *The probability mass function of the recurrence time in state j*

$$g_{jj}(k) := \mathbb{P}_j\left(S_1^j = k\right), \tag{3.19}$$

2. *the cumulative distribution function of the recurrence time in state j,*

$$G_{jj} := \mathbb{1} * g_{jj}, \tag{3.20}$$

3. *the survival function of the recurrence time in state j,*

$$\overline{G_{jj}} = \mathbb{1} - G_{jj},$$

4. *the mean recurrence of state j for the SMC Z :*

$$\mu_{jj} := \mathbb{E}_j(S_1^j) = \lim_{n\to\infty}[\mathbb{1} * \overline{G_{jj}}](n),$$

5. *the probability mass function of the first hitting time of state j, starting from state i :*

$$g_{ij}(k) := \mathbb{P}_i\left(S_0^j = k\right), \tag{3.21}$$

6. *the cumulative distribution function of the first hitting time of state j, starting from state i :*

$$G_{ij} := \mathbb{1} * g_{ij}, \tag{3.22}$$

7. *the survival function of the first hitting time of state j, starting from state i :*

$$\overline{G_{ij}} := \mathbb{1} - G_{ij}, \tag{3.23}$$

8. *the mean first passage time from state i to state j for the semi-Markov chain Z :*

$$\mu_{ij} := \mathbb{E}_i(S_0^j) = \lim_{n\to\infty}[\mathbb{1} * \overline{G_{ij}}](n).$$

From the above definitions, we obtain easily that $\left(S_n^j\right)$, with $S_0^j = 0$ is a usual renewal chain which describes the visits of Z in state j and the function g_{jj} is the associated pmf of the interrarival times. If we assume that $J_0 = i$ then $S_0^j > 0$ and consequently S_n^j forms a delayed renewal chain and the sequence $\left(S_n^j - S_0^j\right)_{n\in\mathbb{N}}$ is the associated usual renewal chain, with initial distribution g_{ij} and pmf g_{jj}. From this observation, we give a concrete result which gives an expression for u in terms of Renewal chains.

For this purpose, define the matrix valued functions $g = (g_{ij})_{i,j\in E}$, $\check{u} = diag\{u_{jj}\}_{j\in E}$. The latter has a convolutional inverse since $u(0) = I_s$.

Proposition 3.8. *The matrix valued functions u and g satisfy the following relation*

$$g = (u - E_0) * \check{u}^{(-1)}. \tag{3.24}$$

Proof. Note that u_{jj} is a sequence of renewal probabilities of the usual renewal chain which records renewals on j with probability mass function g_{jj}. Therefore, from Proposition 2.1 we have

$$u_{jj} = e_0 + g_{jj} * u_{jj}.$$

Now, let us consider two distinct states $i, j \in E$. Then, we have a delayed renewal chain which records the visits on j when the system starts from i with $f_0 = g_{ij}$. In this case, u_{ij} and u_{jj} are the sequences of renewal probabilities for the delayed renewal chain and the simple renewal chain respectively. Therefore, from Proposition 2.5 we obtain

$$u_{ij} = g_{ij} * u_{jj}.$$

Consequently, from the above we have the following matrix form

$$u = E_0 + g * \check{u}. \tag{3.25}$$

From (3.25) we obtain that

$$g * \check{u} = u - E_0$$

and we get directly that (3.24) holds. □

From the above proposition and Example 1.35 we get directly the following corollary for the matrix valued sequences $G = (G_{ij})_{i,j\in E}$, $\overline{G} = (\overline{G_{ij}})_{i,j\in E}$ and the matrix $\mu = (\mu_{ij})_{i,j\in E}$.

Corollary 3.1. *The sequences of matrices G, $\overline{G}$ and the matrix μ are given by*

$$G = (U - \mathbb{I}) * \check{u}^{(-1)}, \tag{3.26}$$
$$\overline{G} = \mathbb{II} - (U - \mathbb{I}) * \check{u}^{(-1)}, \tag{3.27}$$
$$\mu = \lim_{n\to\infty} \left[\mathbb{I} * \left(\mathbb{II} - (U - \mathbb{I}) * \check{u}^{(-1)}\right)\right](n). \tag{3.28}$$

Remark 3.7. *We derive from relation (3.25) that q can be determined by*

$$q = E_0 - (E_0 + g * \check{u})^{(-1)}.$$

Remark 3.8. *The representations in Proposition 3.8 and Corollary 3.1 can be written differently. Then, for the recurrence time in a state j we get*

$$g_{jj} = e_0 - u_{jj}^{(-1)}, \qquad G_{jj} = \mathbb{1} - \mathbb{1} * u_{jj}^{(-1)}, \tag{3.29}$$
$$\overline{G}_{jj} = \mathbb{1} * u_{jj}^{(-1)}, \qquad \mu_{jj} = \lim_{n\to\infty} \left[\mathbb{1}^{(2)} * u_{jj}^{(-1)}\right](n). \tag{3.30}$$

Furthermore, for the passage time in a state j, starting from a different state i, we have

$$g_{ij} = u_{ij} * u_{jj}^{(-1)}, \qquad G_{ij} = U_{ij} * u_{jj}^{(-1)}, \tag{3.31}$$
$$\overline{G}_{ij} = \mathbb{1} - U_{ij} * u_{jj}^{(-1)}, \qquad \mu_{ij} = \lim_{n\to\infty} \left[\mathbb{1}^{(2)} - \mathbb{1} * U_{ij} * u_{jj}^{(-1)}\right](n). \tag{3.32}$$

Example 3.7. *Let us consider the MRC of Example 3.1. Then, the quantities of Definition 3.10 are expressed by*

$$g_{11} = g_{22} = f * g, \quad g_{12} = f, \quad g_{21} = g,$$
$$G_{11} = G_{22} = F * g = f * G, \quad G_{12} = F, \quad G_{21} = G,$$
$$\overline{G_{11}} = \mathbb{1} - F * g, \quad \overline{G_{12}} = \overline{F}, \quad \overline{G_{21}} = \overline{G}.$$

In addition, the associated mean passage and recurrence times are determined by

$$\mu_{11} = \mu_{22} = m_1 + m_2, \quad \mu_{12} = m_1, \quad \mu_{21} = m_2.$$

Definition 3.11. *Let $(Z_k)_{k\in\mathbb{N}}$ be a semi-Markov chain with state space E and $(J_n, S_n)_{n\in\mathbb{N}}$ the associated Markov renewal chain.*

1. *If $G_{ij}(\infty) \cdot G_{ji}(\infty) > 0$, we say that i and j communicate and we denote this by $i \leftrightarrow j$.*
2. *The SMC is said to be irreducible if all the states communicate with each other.*
3. *A state i is said to be recurrent if $G_{ii}(\infty) = 1$ and transient if $G_{ii}(\infty) < 1$.*
4. *A recurrent state i is positive recurrent if $\mu_{ii} < \infty$ and null recurrent $\mu_{ii} = \infty$.*
5. *A subset of states C is said to be a final set if $\sum_{i\in C} P_{ij}(k) = 0$ for any $j \in E\backslash C$ and any $k \in \mathbb{N}$.*
6. *The SMC (MRC) is said to be ergodic if it is irreducible and positive recurrent.*
7. *Let d be an element of $\mathbb{N}$ with $d > 1$. A state $i \in E$ is said to be d-periodic (aperiodic) if the distribution $g_{ii}(\circ)$ is d-periodic (aperiodic).*
8. *An irreducible SMC is d-periodic, if all states are d-periodic. Otherwise, it is called aperiodic*

In order to show that in an irreducible SMC either all states are periodic or none we will need some definitions. The latter and the proof of this result were given by Çinlar [9]. Let (J, S) be a Markov renewal chain defined over $(\Omega, \mathcal{M})$, where Ω is a set and $\mathcal{M}$ the corresponding σ-Algebra.

Consider the sequence of σ-Algebras

$$\mathcal{H}_n \subset \mathcal{M}.$$

and assume each (J_n, S_n) be $\mathcal{H}_n$-measurable. Define the process

$$N_n^{(A)} = \inf\{k > N_{n-1}^{(A)} : J_k \in A\},\ n \geq 1,\ N_0^{(A)} := 0,$$

where A is a proper nonempty subset of set E. Namely, any N^A counts the successive entrance times in the set A. Its easy to notice that each N^A is an $\mathcal{H}_n$ stopping time and consequently the sequence

$$\hat{\mathcal{H}}_n = \{\Lambda \in \mathcal{M} : \Lambda \cap \{N_n = m\} \in \mathcal{H}_n\ \forall m \in \mathbb{N}\}$$

is a filtration. Also, define the $\hat{\mathcal{H}}_n$-measurable chain $(\hat{J}_n, \hat{S}_n)_{n\in\mathbb{N}}$ by

$$\bar{J}_n = J_{N_n^{(A)}},\ \bar{S}_n = S_{N_n^{(A)}},\ n \in \mathbb{N}.$$

with state space $A \cup \mathbb{N}$.

Lemma 3.1. *The process $(\hat{J}, \hat{S})$ is a Markov renewal chain. Furthermore, if $\hat{u}$ is the corresponding Markov renewal probability, then $\hat{u}_{ij} = u_{ij}$ for any $i, j \in A$.*

Proof. Since N^A is a stopping time then by applying the strong Markov property to the Markov chain (J, S) we have that $(\hat{J}, \hat{S})$ forms a Markov renewal chain.

In addition, we have

$$\bigcup_{l=0}^{\infty}\{J_l = j, S_l = n\} = \bigcup_{l=0}^{\infty}\{\hat{J}_l = j, \hat{S}_l = n\}, \quad j \in A.$$

Hence, from (3.7) we obtain

$$u_{ij} = \hat{u}_{ij}, \quad i, j \in A.$$

□

Proposition 3.9.

1. *Two communicating states of a MRC are either both periodic or both aperiodic. In the first case they have the same period.*

2. *If the embedded Markov chain J is irreducible and the MRC is d-periodic then $q_{ij}(k)$ has the support $\{\alpha_{ij} + rd; r \in \mathbb{N}\}$, where α_{ij} are nonnegative constants depending on states i and j.*

Proof. (i) Let us consider the Markov renewal chain of Lemma 3.1 and set $A = \{i, j\}$. The corresponding semi-Markov kernel $\hat{q}$ is of form, say

$$\hat{q} = \begin{pmatrix} C & V \\ B & L \end{pmatrix},$$

where V, B are accordingly chosen to imply the link about i and j.

Also, by Lemma 3.1 we obtain that the Markov renewal probabilities $u_{ij}, u_{ii}, u_{ji}, u_{jj}$ are the same whether they are computed from (J, S) or $(\hat{J}, \hat{S})$. As a result of Proposition 3.8 we have that the same holds true for $g_{ij}, g_{ii}, g_{ji}, g_{jj}$ since they are completely determined by the sequences u. Also, any g can be determined in terms of $(\hat{J}, \hat{S})$ by

$$g_{ii} = C + V * B + V * L * B + V * L * L * B + \cdots , \tag{3.33}$$

$$g_{jj} = L + B * V + B * C * V + B * C * C * V + \cdots , \tag{3.34}$$

$$g_{ij} = V + C * V + C * C * V + C * C * C * V + \cdots . \tag{3.35}$$

Let i be periodic with period d. Then, the function g_{ii} is positive over $\{0, d, 2d, \dots\}$ and from Equation (3.33) we have that the same holds for $C, V * B, V * L * B, \cdots$. Consequently, the sequences $C, V * B$ and L are $d - periodics$ and thus from Equation (3.34) we infer that the sequence g_{jj} is also periodic of period d. Hence, j is $d - periodic$.

(ii) Consider x, y be two elements of $\mathbb{N}$ for which the sequence V is positive and set c be a point such that $B(c) > 0$. Then, for $z_1 = x+c$ and $z_2 = y+c$, we have $[B*V](z_1), [B*V](z_2) > 0$. Since i, j are assumed to be $d - periodics$ we have that the support of $B * V$ is a subset of $d\mathbb{N}$ and we have $z_1, z_2 \in d\mathbb{N}$. Therefore, the element $|x - y| = |z_1 - z_2|$ is a member of $d\mathbb{N}$. By Equation (3.35) we infer that the support of V is the set $\alpha_{ij} + d\mathbb{N}$ for some $\alpha_{ij} \in \mathbb{N}$. Similarly, g_{ij} is supported by the set $\alpha_{ji} + d\mathbb{N}$ for some $\alpha_{ji} \in \mathbb{N}$.

Then, by conditioning on the first step we obtain

$$g_{ij} = q_{ij} + \sum_{r \neq j} q_{rj} * g_{rj}, \quad i, j \in E.$$

This relation implies that each $(q_{ij})_{i,j \in E}$ is positive over $\alpha_{ij} + d\mathbb{N}$. Obviously, for $i = j$ we take $\alpha_{ij} = 0$.

□

Corollary 3.2. *Let (J, S) be an irreducible Markov renewal chain. Then either any state is aperiodic or else is periodic with period $d > 1$.*

In the following proposition we give a useful link between the mean recurrence times and sojourn times. This relation will be useful for the sequel and is achieved with the acceptance of assumption 3.1.

Proposition 3.10. *Let (J, S) be an aperiodic an ergodic Markov renewal chain. The mean recurrence time of an arbitrary state $j \in E$ is decribed by*

$$\mu_{jj} = \frac{\overline{m}}{\pi_j^*}, \tag{3.36}$$

where

$$\overline{m} = \sum_{i \in E} m_i \, \pi_i^*.$$

Proof. For $i, j \in E$ we have

$$\begin{aligned}
\mu_{ij} = \mathbb{E}_i(S_0^j) &= \sum_{r \in E} \mathbb{E}_i(S_0^j | J_1 = r) p_{ir} = p_{ij} m_i + \sum_{r \neq j} \mathbb{E}_i(S_0^j | J_1 = r) p_{ir} \\
&= p_{ij} m_i + \sum_{r \neq j} \left(\mathbb{E}_r(S_0^j) + \mathbb{E}(S_1) \right) p_{ir} = p_{ij} m_i + \sum_{r \neq j} p_{ir} \mu_{rj} + (1 - p_{ij}) m_i \\
&= m_i + \sum_{r \neq j} p_{ir} \mu_{rj}
\end{aligned}$$

and consequently by using the properties of the stationary distribution of the Markov chain J we get

$$\begin{aligned}
\sum_{i \in E} \mu_{ij} \pi_i^* &= \sum_{i \in E} \pi_i^* m_i + \sum_{i \in E} \pi_i^* \sum_{k \neq j} p_{ik} \mu_{kj} \\
&= \sum_{i \in E} \pi_i^* m_i + \sum_{k \neq j} \left(\sum_{i \in E} \pi_i^* p_{ik} \right) \\
&= \sum_{i \in E} \pi_i^* m_i + \sum_{k \in E} \pi^*(k) \mu_{kj} - \pi^*(j) \mu_{jj}.
\end{aligned}$$

Hence, from the above equality we get the desired form. □

Example 3.8. *For the embedded Markov chain J of the Example 3.1 the corresponding stationary distribution is given by the vector $(\frac{1}{2}, \frac{1}{2})$ and so if we apply it in (3.36) we get directly the mean reccurence times which are given in Example 3.7.*

3. Asymptotic Results for Markov renewal chains

In this section we study some asymptotic properties of Markov renewal chains. First, we examinate closely the asymptotic behavior of the process $(S_n)_{n \in \mathbb{N}}$, of the number of visits to a certain state, and the number of transitions between two states. We give also the renewal theorem and the key renewal theorem for Markov renewal chains. Furthermore, we show the SLLN and CLT for Markov renewal chains.

In the sequel, we will suppose that the following Assumptions hold.

Assumption 3.2. *The semi-Markov chain is irreducible.*

Assumption 3.3. *The mean sojourn time in any state is finite*

$$m_i := \mathbb{E}_i(S_1) = \lim_{n \to \infty} \left[\mathbb{1} * \overline{H_i} \right](n) < \infty.$$

Now, we define the processes which count the number of visits to a certain state and the number of direct transition between two states, respectively.

Definition 3.12. *For all $i, j \in E$, define :*

(i) $N_i(k) := \sum_{n=0}^{N(k)-1} 1_{\{J_n=i\}} = \sum_{n=0}^{k} 1_{\{J_n=i,S_{n+1}\leq k\}}$,

(ii) $N_{ij}(k) := \sum_{n=0}^{N(k)} 1_{\{J_{n-1}=i,J_n=j\}} = \sum_{n=1}^{k} 1_{\{J_{n-1}=i,J_n=j,S_{n+1}\leq k\}}$.

Proposition 3.11. *Let (J, S) be an aperiodic Markov renewal chain. Then, we have*

$$\lim_{k\to\infty} \frac{N_i(k)}{N(k)} \overset{a.s}{=} \pi_i^* \tag{3.37}$$

$$\lim_{k\to\infty} \frac{N_{ij}(k)}{N(k)} \overset{a.s}{=} \pi_i^* p_{ij} \tag{3.38}$$

Proof. From the ergodic theorem we get:

$$\frac{1}{n}\sum_{l=0}^{n-1} \mathbb{1}_{\{J_l=i\}} \xrightarrow[n\to\infty]{a.s} \pi_i^*$$

and

$$\frac{1}{n}\sum_{l=1}^{n} \mathbb{1}_{\{J_{l-1},J_l=i\}} \xrightarrow[n\to\infty]{a.s} \pi_i^* p_{ij}.$$

Since $N(k) \xrightarrow[k\to\infty]{a.s} \infty$, then by Theorem B.4 we have

$$\frac{N_i(k)}{N(k)} = \frac{1}{N(k)} \sum_{l=0}^{N(k)-1} \mathbb{1}_{\{J_l=i\}} \xrightarrow[k\to\infty]{a.s} \pi_i^*$$

and

$$\frac{N_{ij}(k)}{N(k)} = \frac{1}{N(k)} \sum_{l=1}^{N(k)} \mathbb{1}_{\{J_l=i\}} \xrightarrow[k\to\infty]{a.s} \pi_i^* p_{ij}.$$

□

Proposition 3.12. *For an aperiodic Markov renewal chain we have*

$$\lim_{k\to\infty} \frac{N_i(k)}{k} \overset{a.s}{=} \frac{1}{\mu_{ii}}, \quad i \in E. \tag{3.39}$$

Proof. Consider an arbitrary state $i \in E$. Let $J_0 \overset{a.s}{=} i$ $(S_0^i \overset{a.s}{=} 0)$. Since $(S_n^i)_{n\in\mathbb{N}}$ is a renewal chain on the event $\{J = i\}$, then it's easy to observe that the process $\tilde{N}_i(k) - 1$ is its counting process. Then, the application of the SLLN for counting processes gives

$$\lim_{k\to\infty} \frac{\tilde{N}_i(k)}{k} \overset{a.s}{=} \frac{1}{\mu_{ii}}, \quad j \in E$$

and from Remark **??**, we conclude the desired result.

The case in which J_0 is not necessarily equal to i $\big(i.e.\ \mathbb{P}(J_0 = i) < 1\ and\ \mathbb{P}(S_0^i > 0) > 0\big)$ is studied in a similar way through the delayed renewal chain $(S_n^i)_{n\in\mathbb{N}}$ and its associated renewal chain $(S_n^i - S_0^i)_{n\in\mathbb{N}}$.

□

The following results are corollaries from the last two propositions

Corollary 3.3. *Let (J, S) be an aperiodic Markov renewal chain. Then, we have*

$$\lim_{k\to\infty} \frac{N_{ij}(k)}{k} \overset{a.s}{=} \frac{p_{ij}}{\mu_{ii}}, \tag{3.40}$$

$$\lim_{k\to\infty} \frac{N(k)}{k} \overset{a.s}{=} \frac{1}{\pi_i^* \mu_{ii}}. \tag{3.41}$$

Theorem 3.1 (Markov renewal theorem). *Let $(J_n, S_n)_{n\in\mathbb{N}}$ be a recurrent Markov renewal chain, then:*

(i) *if it is aperiodic we have:*

$$\lim_{n\to\infty} u_{ij}(n) = \frac{1}{\mu_{jj}}.$$

(ii) *if it is periodic of period d>1 we have:*

$$\lim_{n\to\infty} u_{ij}(n) = \frac{d}{\mu_{jj}},$$

and

$$u_{ij}(n) = 0, \quad \forall\, n \in \{l \in \mathbb{N} |\, l \not\equiv 0 \; mod\; d\}\,.$$

Proof. (i) [Renewal theory] First, we assume that $i = j$. In this case, the chain S_n^j forms a renewal chain on the event $\{J = j\}$ with probability mass function g_{jj} and sequence of renewal probability $u_{jj}(n)$. Hence, by applying the renewal theorem for an aperiodic recurrent renewal chain we have the desired result.

Now, consider i and j be distinct and then the chain S_n^j is a delayed renewal chain on the event $\{J = j\}$ and $S_n^j - S_0^j$ is the associated renewal chain. Furthermore, the sequence g_{ij} is its initial distribution and $u_{ij}(n)$ is the corresponding renewal probability. Therefore, the application of the renewal theorem for a delayed recurrent renewal theorem gives

$$\lim_{n\to\infty} u_{ij}(n) \overset{a.s}{=} \frac{1}{\mu_{jj}} \sum_{l=0}^{\infty} g_{ij}(n) = \frac{1}{\mu_{jj}}.$$

(i) [Convolutional] From $\overline{G}_{jj} * u_{jj} = \mathbb{1}$ we have

$$\left(\sum_{k=0}^{\infty} \overline{G}_{jj}(k) e_1^{(k)}\right) \cdot \left(\sum_{k=0}^{\infty} u_{jj}(k) e_1^{(k)}\right) = \sum_{k=0}^{\infty} \mathbb{1}(k) e_1^{(k)},$$

and if we use the isomorphism in Proposition 1.3 we will get that

$$\left(\sum_{k=0}^{\infty} \overline{G}_{jj}(k) x^k\right) \cdot \left(\sum_{k=0}^{\infty} u_{jj}(k) x^k\right) = \sum_{k=0}^{\infty} x^k.$$

Therefore, for any $x \in (-1, 1)$ we can easily obtain that

$$\begin{aligned} \left(\sum_{k=0}^{\infty} \overline{G}_{jj}(k) x^k\right) \cdot \left(\sum_{k=0}^{\infty} u_{jj}(k) x^k\right) &= \frac{1}{1-x} \Longrightarrow \\ (1-x) \cdot \left(\sum_{k=0}^{\infty} u_{jj}(k) x^k\right) &= \frac{1}{\sum_{k=0}^{\infty} \overline{G}_{jj}(k) x^k}. \end{aligned} \tag{3.42}$$

The left-hand member of (2.31) can be rewritten as

$$\begin{aligned}(1-x)\cdot\left(\sum_{k=0}^{\infty}u_{jj}(k)x^k\right) &= \sum_{k=0}^{\infty}u_{jj}(k)x^k-\sum_{k=0}^{\infty}u_{jj}(k)x^{k+1}\\ &= u_{jj}(0)+\sum_{k=1}^{\infty}u_{jj}(k)x^k-\sum_{k=1}^{\infty}u_{jj}(k-1)x^k\\ &= 1+\sum_{k=1}^{\infty}\left(u_{jj}(k)-u_{jj}(k-1)\right)x^k.\end{aligned}$$

Since $\sum_{k=0}^{\infty}\overline{G}_{jj}(k)=\mu_{jj}<\infty$, then from Abel's theorem (Theorem D.1) we have

$$\lim_{x\to 1^-}\sum_{k=0}^{\infty}\overline{G}_{jj}(k)x^k=\sum_{k=0}^{\infty}\overline{G}_{jj}(k)=\mu_{jj},$$

and consequently from the above we conclude that

$$\begin{aligned}1+\sum_{k=1}^{\infty}\left(u_{jj}(k)-u_{jj}(k-1)\right)=\frac{1}{\mu_{jj}} \quad &\Longrightarrow \quad 1+\lim_{n\to\infty}\left[u_{jj}(n)-u_{jj}(0)\right]=\frac{1}{\mu_{jj}}\\ &\overset{u_{jj}(0)=1}{\Longrightarrow} \quad \lim_{n\to\infty}u_{jj}(n)=\frac{1}{\mu_{jj}}.\end{aligned}$$

Since $g_{ij}*u_{jj}=u_{ij}$, we get directly from Theorem D.2 that

$$\lim_{n\to\infty}u_{ij}(n)\overset{a.s}{=}\frac{1}{\mu_{jj}}\sum_{l=0}^{\infty}g_{ij}(n)=\frac{1}{\mu_{jj}}.$$

(ii) Define the sequence $(X'_n)_{n\in\mathbb{N}}$ as

$$X'_n=\frac{X_n}{d},\ n\in\mathbb{N},$$

and set

$$S'_n=\sum_{l=0}^{n}X'_n.$$

Then the pair $(J_n,S'_n)_{n\in\mathbb{N}}$ is also a Markov renewal chain with sequence of Markov renewal probabilities u_d which satisfies that $u_d(n)=u(dn)$ for any $n\in\mathbb{N}$. Furthermore, for the associated mean recurrence time we obtain that $\mu_d=\frac{\mu}{d}$. Therefore from (i) we have

$$u_{jjd}(n)\xrightarrow[n\to\infty]{}\frac{1}{\mu_{jjd}}=\frac{d}{\mu_{jj}}.$$

For the sequence u_{ij} with $i\neq j$ the result is obtained similarly.

□

In the previous theorem we give again the proof which is presented in Theorem 2.4. We did it because we want to make the theory of Markov renewal chains less dependent from the techniques of Renewal theory.

Since $U_{ij}=\mathbb{1}*u_{ij}$ then the following result is a direct consequence of the Markov renewal theorem

Corollary 3.4 (Elementary Markov renewal theorem)**.** *Let $(J_n,S_n)_{n\in\mathbb{N}}$ be a recurrent Markov renewal chain, then:*

(i) *if it is aperiodic we have:*

$$\lim_{n\to\infty}\frac{U_{ij}(n)}{n}=\frac{1}{\mu_{jj}}.$$

(ii) *if it is periodic of period d>1 we have:*

$$\lim_{n\to\infty}\frac{U_{ij}(dn)}{n}=\frac{d}{\mu_{jj}},$$

and

$$U_{ij}(n)=0,\quad \forall\, n\in\{l\in\mathbb{N}|\, l\not\equiv 0\, mod\, d\}\,.$$

Theorem 3.2 (Key Markov renewal theorem)**.** *Let (J,S) be an periodic Markov renewal chain. Also, consider a sequence $t_j(l)$ with $\sum_{l=0}^{\infty}|t_j(l)|<\infty$. For any state $i,j\in E$ we have*

$$\lim_{n\to\infty}[u_{ij}*t_j](n)\overset{a.s}{=}\frac{\sum_{n=0}^{\infty}t_j(n)}{\mu_{jj}}. \tag{3.43}$$

Proof. First, we have

$$[u_{ij}*t_j](n)=\sum_{l=0}^{n}u_{ij}(n-l)t_j(l),\quad n\in\mathbb{N}.$$

Since $u_{ij}(n-l)$ is a bounded sequence which converges to $\frac{1}{\mu_{JJ}}$ for any $l\le n$, $|t_j(l)u_{ij}(n-l)|<|t_j(l)|$ and $\sum_{l=0}^{\infty}|t_j(l)|<\infty$ the conditions of Proposition D.2 are satisfied. For that reason we get directly the desired result.

□

AN important application of the key Markov renewal theorem is to obtain the limit distribution of a semi-Markov chain which satisfies Assumptions 3.2 and 3.3.

Proposition 3.13. *Let Z be an aperiodic Markov renewal chain. Then its limit distribution is determined by*

$$\pi_j=\frac{m_j}{\mu_{jj}}. \tag{3.44}$$

Proof. From Proposition 3.7 we have

$$P_{ij}=u_{ij}*\overline{H}_j.$$

Furthermore, from Assumption 3.3 we get directly

$$\lim_{k\to\infty}\left[\mathbb{1}*\overline{H_j}\right](k)=m_j<\infty.$$

Therefore, we can apply the key Markov renewal theorem and thus we have

$$\lim_{k\to\infty}P_{ij}(k)=\frac{1}{\mu_{jj}}\cdot\lim_{k\to\infty}\left[\mathbb{1}*\overline{H_i}\right](k)=\frac{m_j}{\mu_{jj}}.$$

□

Remark 3.9. *By using Proposition 3.10, relation (3.44) can be refrormulated as*

$$\pi_j=\frac{\pi_j^* m_j}{\sum_{i\in E}\pi_i^* m_i}.$$

Furthermore, we can compute μ_{jj} via Corollary 3.1. This implies that, it is not necessary to find the stationary distribution of the embedded Markov chain J and we represent the limit distribution of Z using convolutional forms, i.e

$$\pi_j = \lim_{n\to\infty} \frac{\left[\mathbb{1} * \overline{H}_i\right](n)}{\left[\mathbb{1}^{(2)} * u_{jj}^{(-1)}\right](n)}.$$

Example 3.9. *Let us consider the MRC of Example 3.1 again. Then, the limit distribution of the Semi-Markov chain Z is given by*

$$\pi_1 = \frac{m_1}{m1 + m_2}, \quad \pi_2 = \frac{m_2}{m_1 + m_2}.$$

Next, we give the functional SLLN and CLT for Markov renewal chains. The corresponding results in the continuous-time Markov renewal chains are given in [20]. In order to present these results we will need some definitions.

Define the function $f : E \times E \times \mathbb{N}$ and the functional

$$W_f(k) = \sum_{n=1}^{N(k)} f(J_{n-1}, J_n, X_n), \quad k \in \mathbb{N}. \tag{3.45}$$

It's easy to check that $(J_n, J_{n+1}, X_{n+1},)_{n\in\mathbb{N}}$ is a Markov chain and its transition probabilities are given by:

$$p\left((r,s,x),(i,j,k)\right) = \mathbb{1}_{\{s=i\}} p_{ij} q_{ij}(k), \quad r,s,i,j \in E,\ x,k \in \mathbb{N}. \tag{3.46}$$

The n-th step transition probabilities can be determined by

$$\begin{aligned} p^{(n)}\left((r,s,x),(i,j,k)\right) &:= \mathbb{P}(J_n = i, J_{n+1} = j, X_{n+1} = k \mid J_0 = r, J_1 = s, X_1 = x) \\ &= \mathbb{P}(J_{n+1} = j, X_{n+1} = k \mid J_n = i)\mathbb{P}(J_n = i \mid J_0 = r, J_1 = s, X_1 = x) \\ &= q_{ij}(k)\mathbb{P}(J_{n-1} = i \mid J_1 = s) := q_{ij}(k) p_{si}^{(n-1)}. \end{aligned}$$

So its stationary distribution, when it exists, is given by

$$\pi(i,j,k) = \pi_i^* q_{ij}(k),\ i,j \in E,\ k \in \mathbb{N}. \tag{3.47}$$

Also, set the series

$$A_{ij} = \sum_{x=0}^{\infty} f(i,j,x) q_{ij}(x), \quad A_i = \sum_{j=1}^{s} A_{ij},$$

$$B_{ij} = \sum_{x=0}^{\infty} f^2(i,j,x) q_{ij}(x), \quad B_i = \sum_{j=1}^{s} B_{ij}.$$

if they converge. Define the quantities:

$$\pi_f = \sum_{j=1}^{s} A_j \pi^*(j), \quad \pi(f) = \mu_{ii} \frac{\pi_f}{\pi_i^*}$$

$$\sigma_i^2 = \sum_{i\in E} B_i \pi_i^* - \left(\sum_{i\in E} A_i \pi_i^*\right)^2 + 2\sum_{r\in E}\sum_{l\neq i}\sum_{r\neq i} A_{rl} A_k \frac{\mu_{li}^* + \mu_{ik}^* - \mu_{lk}^*}{\mu_{rr}\mu_{kk}^*},$$

, where

$$\mu_{ij}^* = \mathbb{E}\left(\inf\{l \in \mathbb{N} : J_l = j\} \mid J_0 = i\right),$$

is the mean passage time from state i to state j.

$$\sigma_f^2 = \frac{\sigma_i^2}{\pi_i^* \mu_{ii}}. \tag{3.48}$$

Theorem 3.3. *For an aperiodic Markov renewal chain (J, S) we have*

$$\lim_{n\to\infty} \frac{W_f(n)}{n} \overset{a.s}{=} \pi(f). \tag{3.49}$$

Proof. From the functional ergodic theorem we get

$$\begin{aligned}
\frac{\sum_{l=1}^{n} f(J_{l-1}, J_l, X_l)}{n} \xrightarrow[n\to\infty]{a.s} \sum_{i,j\in E}\sum_{x=0}^{\infty} \pi_i^* q_{ij}(x) f(i,j,x) &= \sum_{i\in E} \pi_i^* \sum_{j\in E}\sum_{x=0}^{\infty} q_{ij}(x) f(i,j,x) \\
&= \sum_{i\in E} \pi_i^* A_i \\
&= \pi_f.
\end{aligned}$$

Then, by applying the Gut's theorem for the process $W_f(n) = \sum_{l=1}^{N(n)} f(J_{l-1}, J_l, X_l)$ we obtain

$$\frac{W_f(n)}{n} = \frac{N(n)}{n}\frac{W_f(n)}{N(n)} \xrightarrow[n\to\infty]{a.s} \frac{\pi_f}{\pi_i^* \mu_{ii}} = \pi(f).$$

□

The following theorem is the functional CLT for Markov renewal chains and its proof is deferred to [20] and is based on convolutional forms.

Theorem 3.4. *Let (J, S) be an aperiodic Markov renewal chain. Then, the asymptotic distribution of W_f is given by*

$$\sqrt{n}\left[\frac{W_f(n)}{n} - \pi(f)\right] \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \sigma_f^2\right),$$

where σ_f^2 is given in (3.48).

4. Examples

In this section we apply our results in three concrete examples. First, we study the asymptotic properties such as asymptotic consistency of an important process. Second, we construct a semi-Markov model and analyze its characteristics which are applied in a numerical result by using simulated data.

Example 3.10. *Here, we give the asymptotic properties of the process $\frac{N_{ijx}(k)}{N_i(k)}$ where $N_{ijx}(k)$ counts the number of the transitions from state i to state j when it takes x-time units, until time k denoted by*

$$N_{ijx}(k) = \sum_{l=1}^{N(k)} \mathbb{1}_{\{J_{l-1}=i, X_l=x, J_l=j\}}.$$

First, let us consider the function

$$g(r,s,y) = \mathbb{1}_{\{r=i,s=j,y=x\}} - q_{ij}(x)\mathbb{1}_{\{r=i\}} := f(r,s,x) - q_{ij}(x)\mathbb{1}_{\{r=i\}}.$$

In order to find the desired asymptotic behaviour of this process we have to compute the associated quantities A_r, A_{rs}, B_r and B_{rs} for all $r, s \in E$.

$$\begin{aligned} A_{rs} &= \sum_{l=0}^{\infty} g(r,s,l)q_{rs}(l) = \sum_{l=0}^{\infty} f(r,s,l)q_{rs}(l) - \sum_{l=0}^{\infty} \mathbb{1}_{\{r=i\}} q_{ij}(x) q_{rs}(l) \\ &= \mathbb{1}_{\{r=i,s=j\}} q_{ij}(x,y) - \mathbb{1}_{\{r=i\}} q_{ij}(x) \sum_{l,v=0}^{\infty} q_{is}(l) = \mathbb{1}_{\{r=i,s=j\}} q_{ij}(x) - \mathbb{1}_{\{r=i\}} q_{ij}(x) p_{is} \\ &= \mathbb{1}_{\{r=i\}} q_{ij}(x) \left(\mathbb{1}_{\{s=j\}} - p_{is} \right) \end{aligned}$$

and consequently

$$A_r = \sum_{s \in E} A_{rs} = \mathbb{1}_{\{r=i\}} q_{ij}(x,y) \sum_{s \in E} \left(\mathbb{1}_{\{s=j\}} - p_{is} \right) = 0.$$

$$\begin{aligned} B_{rs} &= \sum_{l=0}^{\infty} g^2(r,s,l) q_{ij}(l) \\ &= \sum_{l=0}^{\infty} f(r,s,l) q_{rs}(l) + \sum_{l,v=0}^{\infty} \mathbb{1}_{\{r=i\}} q_{ij}^2(x) q_{rs}(l) - 2 \sum_{l,v=0}^{\infty} f(r,s,l) q_{ij}(x) q_{rs}(l) \\ &= \mathbb{1}_{\{r=i,s=j\}} q_{ij}(x) + q_{ij}^2(x) p_{is} - \mathbb{1}_{\{r=i,s=j\}} q_{ij}^2(x) \\ &= \mathbb{1}_{\{r=i\}} q_{ij}(x) \left(\mathbb{1}_{\{s=j\}} + q_{ij}(x) p_{is} - 2 q_{ij}(x) \mathbb{1}_{\{s=j\}} \right), \end{aligned}$$

and

$$\begin{aligned} B_r &= \sum_{s \in E} B_{rs} \\ &= \mathbb{1}_{\{r=i\}} q_{ij}(x) \sum_{s \in E} \left(\mathbb{1}_{\{s=j\}} + q_{ij}(x) p_{is} - 2 q_{ij}(x) \mathbb{1}_{\{s=j\}} \right) \\ &= \mathbb{1}_{\{r=i\}} q_{ij}(x) \left(1 + q_{ij}(x) - 2 q_{ij}(x) \right) \\ &= \mathbb{1}_{\{r=i\}} q_{ij}(x) \left(1 - q_{ij}(x) \right). \end{aligned}$$

Therefore, summing up the above results we get

$$\sigma_g^2 = \sum_{r \in E} B_r \pi^*(r) = \pi_i^* q_{ij}(x) \left(1 - q_{ij}(x) \right), \quad \pi_g = 0$$

and

$$\sigma^2(g) = \frac{\sigma_g^2}{\mu_{ii} \pi_i^*} = \frac{q_{ij}(x) \left(1 - q_{ij}(x) \right)}{\mu_{ii}}.$$

Hence, by Theorems 3.3 and 3.4 we can obtain

$$\lim_{k \to \infty} \frac{W_g(k)}{k} = \lim_{k \to \infty} \left(\frac{W_f(k)}{k} - \frac{S_{N(k)}}{k} \pi(f) \right) = 0$$

and

$$\frac{W_g(k)}{\sqrt{k}} \xrightarrow[k \to \infty]{d} \mathcal{N}\left(0, \sigma^2(g)\right)$$

respectively.

Therefore, some elements about the asymptotic behaviour of $\frac{N_{ijx}(k)}{N_i(k)}$ are given as follows

$$\begin{aligned}\lim_{k\to\infty}\frac{N_{ijx}(k)}{N_i(k)} &= \lim_{k\to\infty}\frac{k}{N_i(k)}\left(\frac{W_g(k)}{k}+\frac{N_i(k)}{k}q_{ij}(x)\right)\\ &= \mu_{ii}\frac{q_{ij}(x)}{\mu_{ii}}\\ &= q_{ij}(x).\end{aligned}$$

and
$$\frac{1}{\sqrt{k}}\left(\frac{N_{ijx}(k)}{N_i(k)}-q_{ij}(x)\right)=\frac{W_g(k)}{\sqrt{k}}\xrightarrow[k\to\infty]{d}\mathcal{N}\left(0,\sigma^2(g)\right).$$

Example 3.11. *Let us assume a Markov renewal chain (J,S), where the embedded Markov chain J with state space $\{1,2,3\}$ has the following transition matrix*

$$P=\begin{pmatrix}0 & 1 & 0\\ 0.7 & 0 & 0.3\\ 1 & 0 & 0\end{pmatrix},$$

with sojourn times given by

$$\begin{aligned}(X_1|J_0=1,\ J_1=2) &\sim Geom\left(\frac{1}{2}\right)+1\\ (X_1|J_1=2,\ J_2=3) &\sim DWeibull\left(\frac{1}{3},2\right)\\ (X_1|J_1=2,\ J_2=1) &\sim Dweibull\left(\frac{1}{2},3\right)+1\\ (X_1|J_1=3,\ J_2=1) &\sim Geom\left(\frac{1}{3}\right)+1.\end{aligned}$$

Then, the associated pmfs are given by

$$\begin{aligned}f_{12}(k) &= \frac{1}{2^k}\cdot\mathbb{1}_{\{k>0\}},\\ f_{23}(k) &= \left(\frac{1}{3^{(k-1)^2}}+\frac{1}{3^{k^2}}\right)\cdot\mathbb{1}_{\{k>0\}},\\ f_{21}(k) &= \left(\frac{1}{2^{(k-1)^3}}+\frac{1}{2^{k^3}}\right)\cdot\mathbb{1}_{\{k>0\}},\\ f_{31}(k) &= \frac{1}{3}\cdot\left(\frac{2}{3}\right)^{k-1}\cdot\mathbb{1}_{\{k>0\}},\quad k\in\mathbb{N}.\end{aligned}$$

Then the semi-Markov kernel is the matrix valued function q given by

$$q=\begin{pmatrix}0 & f_{12} & 0\\ 0.7\cdot f_{21} & 0 & 0.3\cdot f_{23}\\ f_{31} & 0 & 0\end{pmatrix},$$

and the sequence E_0-q is expressed by

$$E_0-q=\begin{pmatrix}e_0 & -f_{12} & 0\\ -0.7\cdot f_{21} & e_0 & -0.3\cdot f_{23}\\ -f_{31} & 0 & e_0\end{pmatrix},$$

We can compute the convolutional inverse of E_0-q using 3 different ways

1. *Using Proposition 1.15*
2. *Using Theorem 1.2*
3. *Using Theorem 1.3.*

Here, we show the third way for mathematical reasons. The first is used only for numerical results. Thence, the convolutional determinant of $E_0 - q$ is the real sequence

$$\begin{aligned} det\,(E_0 - q) &= det\begin{pmatrix} e_0 & -f_{12} & 0 \\ -0.7\cdot f_{21} & e_0 & -0.3\cdot f_{23} \\ -f_{31} & 0 & e_0 \end{pmatrix} \\ &= e_0 + f_{12} * det\begin{pmatrix} -0.7\cdot f_{21} & -0.3\cdot f_{23} \\ -f_{31} & e_0 \end{pmatrix} \\ &= e_0 - 0.3\cdot(f_{12} * f_{21}) - 0.7\cdot(f_{12} * f_{23} * f_{31}), \end{aligned}$$

and consequently its convolutional inverse is

$$det\,(E_0 - q)^{(-1)} = \sum_{l\geq 0} [0.3\cdot(f_{12} * f_{21}) + 0.7\cdot(f_{12} * f_{23} * f_{31})]^{(l)} := w.$$

Furthermore, the corresponding adjugate matrix function is the sequence

$$\begin{aligned} adj(E_0 - q) &= adj\begin{pmatrix} e_0 & -f_{12} & 0 \\ -0.7\cdot f_{21} & e_0 & -0.3\cdot f_{23} \\ -f_{31} & 0 & e_0 \end{pmatrix} \\ &= \begin{pmatrix} e_0 & f_{12} & 0.7\cdot(f_{12} * f_{23}) \\ 0.7\cdot f_{21} + 0.3\cdot(f_{23} * f_{31}) & e_0 & -0.3\cdot f_{23} \\ f_{31} & f_{12} * f_{31} & e_0 - 0.3\cdot(f_{12} * f_{21}) \end{pmatrix}, \end{aligned}$$

and hence u is the matrix valued function given by

$$u = w\begin{pmatrix} e_0 & f_{12} & 0.7\cdot(f_{12} * f_{23}) \\ 0.7\cdot f_{21} + 0.3\cdot(f_{23} * f_{31}) & e_0 & -0.3\cdot f_{23} \\ f_{31} & f_{12} * f_{31} & e_0 - 0.3\cdot(f_{12} * f_{21}). \end{pmatrix}$$

In order to find the transition function of the SMC Z we will need the sequence $\overline{H}$. The latter is given by the elements

$$\begin{aligned} \overline{H_1}(k) &= \frac{1}{2^k}, \\ \overline{H_2}(k) &= 0.3\cdot\frac{1}{3^{k^2}} + 0.7\cdot\frac{1}{2^{k^3}}, \\ \overline{H_3}(k) &= \frac{2}{3}\cdot\frac{1}{3^k}, \quad k\in\mathbb{N}. \end{aligned}$$

Therefore, the transition function P is expressed by

$$\begin{aligned} P &= u * \overline{H} \\ &= w * \begin{pmatrix} e_0 & f_{12} & 0.7\cdot(f_{12} * f_{23}) \\ 0.7\cdot f_{21} + 0.3\cdot(f_{23} * f_{31}) & e_0 & -0.3\cdot f_{23} \\ f_{31} & f_{12} * f_{31} & e_0 - 0.3\cdot(f_{12} * f_{21}) \end{pmatrix} * diag\left\{\overline{H_i}\right\} \\ &= \begin{pmatrix} w * \overline{H_1} & w * f_{12} * \overline{H_2} & 0.7\cdot(w * f_{12} * f_{23} * \overline{H_3}) \\ w * (0.7\cdot f_{21} + 0.3\cdot(f_{23} * f_{31})) * \overline{H_1} & w * \overline{H_2} & -0.3\cdot\left(w * f_{23} * \overline{H_3}\right) \\ w * f_{31} * \overline{H_1} & w * f_{12} * f_{31} * \overline{H_2} & w * (e_0 - 0.3\cdot(f_{12} * f_{21})) * \overline{H_3} \end{pmatrix}. \end{aligned}$$

In the following figure we give the evolution of the transition matrix P until the 40-th period.

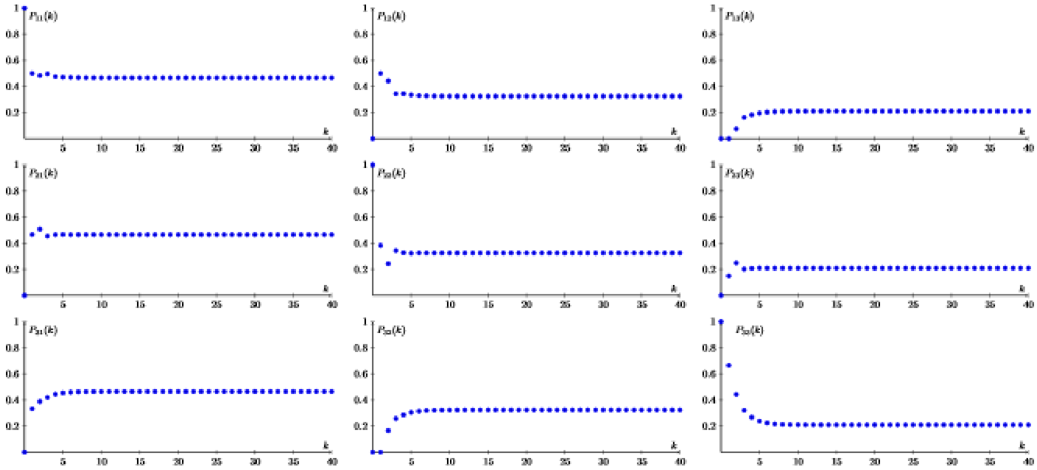

FIGURE 3.1: Transition probabilities of Z over time

From Figure 3.1 we observe that each of the transition probabilities converge, after time 10, to some values. This implies that, from this observation we get that Z has a limit distribution. From the statistical program R we have that the limit distribution of Z is

$$\pi = (0.4658548, 0.3245105, 0.2096347). \tag{3.50}$$

We can also show it using (3.44). In order to do this, we need to compute some useful quantities as the mean sojourn time, and the limiting behavior of u. For the latter, from the statistical program R, we have that the function u for a large n will be eventually

$$u = \begin{pmatrix} 0.2329274 & 0.2329274 & 0.06987823 \\ 0.2329274 & 0.2329274 & 0.06987823 \\ 0.2329274 & 0.2329274 & 0.06987823 \end{pmatrix},$$

and consequently the mean recurrence time of any state is

$$\mu_{11} = 0.2329274, \quad \mu_{22} = 0.232974, \quad \mu_{33} = 0.6987823$$

Furthermore, the mean sojourn times are

$$m_1 = \mathbb{E}[X_1|J_0 = 1, \ J_1 = 2] = 2, \tag{3.51}$$
$$m_2 = 0.7 \cdot \mathbb{E}[X_1|J_0 = 2, \ J_1 = 1] + 0.3 \cdot \mathbb{E}[X_1|J_0 = 2, \ J_1 = 3] = 1.393183, \tag{3.52}$$
$$m_3 = \mathbb{E}[X_1|J_0 = 3, \ J_1 = 1] = 3, \tag{3.53}$$

where the second quantity is computed numerically. By summarizing the above and apply them in (3.44), we get directly that the limiting distribution of Z is the stochastic vector π with coordinates

$$\pi_1 = 0.4658548, \quad ,\pi_2 = 0.3245105, \quad \pi_3 = 0.2096347 \tag{3.54}$$

The stationary distribution of J is given by the stochastic vector

$$\pi^* = (\pi_1^*, \pi_2^*, \pi_3^*) = (0.4347826, \ 0.4347826, \ 0.1304348).$$

Then, we find the mean recurrence times using Proposition 3.36 then we get again the limiting distrubution which is presented above.

In the following plot, we add in Figure 3.1 the limiting distrubution of Z for which any coordinate is represented by a red line. We observe that the transition probabilities of Z and the limiting distrubution coincide, after time 10.

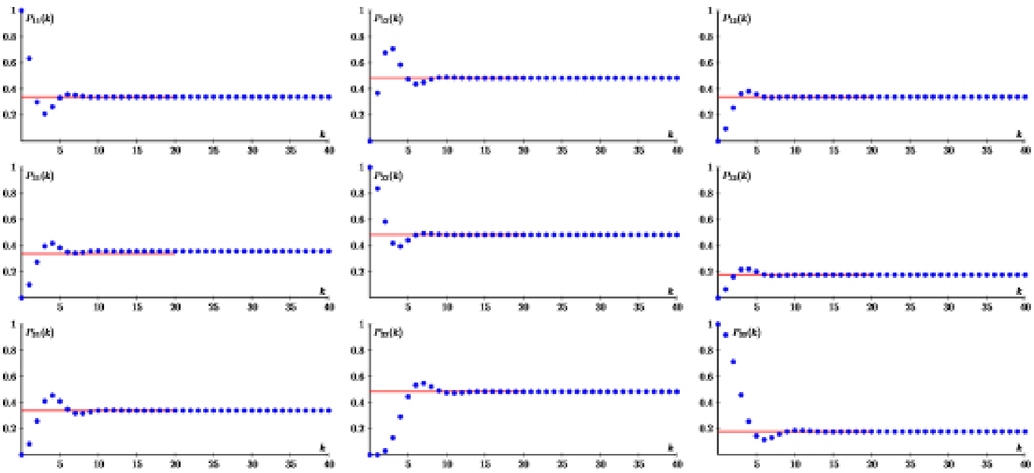

FIGURE 3.2: Transition probabilities of Z over time vs. limiting distrubution

In the following plot we give 6 independent observations of the desired semi-Markov chain for $n = 30$ periods via simulated data. This simulation is obtained by the following steps

1. *Set $X_0 = 0$ and draw a $J_0 \sim \alpha$..*
2. *Draw a $J_m \sim P_{J_{m-1}}.$, $m \geq 1$.*
3. *Draw a $X_m \sim f_{J_{m-1}J_m}(\cdot)$, $m \geq 1$.*
4. *Set $S_m = S_{m-1} + X_m$.*
5. *If $S_m \geq n$, then end.*
6. *Else, set $m \leftarrow m + 1$ and go to step 2.*

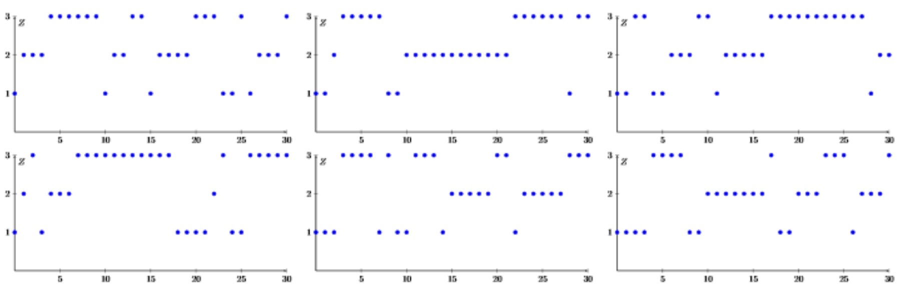

FIGURE 3.3: 6 independent observations of a Semi-Markov chain until time 30

Appendix A

Algebra

1. Group theory

In this section we give the necessary theory of theory of groups in order to apply it in the theory of Rings. In order to do this, first we need the definition of semi-group. For the sequel, we denote by G a non-empty set equipped with a binary operation $*$.

Definition A.1. *A pair $(G, *)$ is said to be a semi-Group if the operation $*$ is associative, i.e*

$$a * (b * c) = (a * b) * c, \quad \forall\, a, b, c \in G.$$

Definition A.2. *A set G equipped with a binary operation $*$ is said to be a group if*

(i) *$(G, *)$ is a semi-Group.*

(ii) *Exists a special element e for which we have,*

$$\forall\, a \in G : \quad a * e = e * a = a,$$

which is called the idenity element of G.

(iii) *For any $a \in G$, there is an element $b \in G$ such that*

$$a * b = b * a = a.$$

The element b is said to be the inverse element of a.

Definition A.3. *A group $(G, *)$ is said to be an abelian group if $*$ is also commutative, i.e.*

$$a * b = b * a, \quad \forall\, a, b \in G.$$

Example A.1. *Consider the set of integers $\mathbb{Z}$ equipped with the operation of usual multiplication. The pair $(\mathbb{Z}, *)$ forms a semi-group but it doesn't satisfy the conditions of Definition A.2.*

Example A.2. *The set $\mathbb{Z}$ equipped with the operation of usual addition is an abelian group. The identity element is represented by the zero number. The inverse element of any integer a is expressed by the number $-a$.*

Example A.3. *Let us consider s be a finite natural number and the set $G = \{A \in \mathbb{R}^{s\times s} | \, det(A) \neq 0\}$. G is a group under multiplication of matrices.*

2. Ring Theory

Definition A.4. *A nonempty set R is called ring if the binary operations addition $+$ and multiplication $*$ are defined in R and*

$a)$ *$(R, +)$ is an abelian group.*

b) $(R, *)$ *is a semi group.*

c) *Both left and right distributive laws hold in it, i.e.* $\forall\, a, b, c \in R$

$$\begin{aligned} a*(b+c) &= a*b+a*c \qquad (A.1)\\ (b+c)*a &= b*a+c*a. \qquad (A.2)\end{aligned}$$

Definition A.5. *Let* R *be a ring.* R *is said to be a commutative ring if for any* $a, b \in R$ *we have*

$$a*b = b*a.$$

Definition A.6. *Let* R *be a ring in which for any* $a \in R$ *exists an element* $e \in R$ *such that*

$$e*a = a*e = a.$$

Then, R *is said to be a ring with unity and* e *is called the identity element of* R*.*

Example A.4. *Let us consider the set of integers* $\mathbb{Z}$*. Then,* $\mathbb{Z}$ *is a commutative ring with unity.*

Example A.5. *The set of all even numbers* $2\mathbb{Z}$=$\{0, \pm 2, \pm 4, \ldots\}$ *is a commutative ring without unity, equipped with the binary operations of the usual addition and multiplication.*

Definition A.7. *Let* R *be a commutative ring. Then a nonzero element* $a \in R$ *is called zero advisor if there is a nonzero element* $b \in R$ *such that* $a*b=0$*.*

Definition A.8. *If a commutative ring* R *is said to be an integral domain if it has no zero advisors i.e., for any* $a, b \in R$ *we have*

$$a*b=0 \Longrightarrow\ a=0 \text{ or } b=0.$$

Proposition A.1. *Let* R *be a ring with unity. Then, for any* $\alpha,\ \beta \in R$ *with* $\alpha*\beta = \beta*\alpha$ *we have*

$$(\alpha+\beta)^n = \sum_{l=0}^{n}\binom{n}{l}\alpha^{n-l}*\beta^l. \qquad (A.3)$$

Proof. Obviously, (A.3) holds for $n=0$. Now, let us assume that for an arbritrary $n \in \mathbb{N}$ (A.3) is true. Then, we have

$$\begin{aligned}(\alpha+\beta)^{(n+1)} &= (\alpha+\beta)*(\alpha+\beta)^{(n)} = \alpha*(\alpha+\beta)^{(n)}+\beta*(\alpha+\beta)^{(n)}\\ &= \sum_{l=0}^{n}\binom{n}{l}\alpha^{(l+1)}*\beta^{(n-l)}+\sum_{l=0}^{n}\binom{n}{l}\alpha^{(l)}*\beta^{(n-l+1)}\\ &= \alpha^{(n+1)}+\beta^{(n+1)}+\sum_{l=0}^{n-1}\binom{n}{l}\alpha^{(l+1)}*\beta^{(n-l)}+\sum_{l=1}^{n}\binom{n}{l}\alpha^{(l)}*\beta^{(n-l+1)}\\ &= \alpha^{(n+1)}+\beta^{(n+1)}+\sum_{l=1}^{n}\binom{n}{l-1}\alpha^{(l)}*\beta^{(n-l+1)}+\sum_{l=1}^{n}\binom{n}{l}\alpha^{(l)}*\beta^{(n-l+1)}\\ &= \alpha^{(n+1)}+\beta^{(n+1)}+\sum_{l=1}^{n}\left(\binom{n}{l-1}+\binom{n}{l}\right)\alpha^{(l)}*\beta^{(n+1-l)}.\end{aligned}$$

Therefore, by using Pascal's triangle we take

$$(f+g)^{(n+1)} = \sum_{l=0}^{n+1}\binom{n+1}{l}f^{(l)}*g^{(n+1-l)},$$

and consequently by the principle of mathematical induction we get the desired form. $\square$

Definition A.9. *Let $(R, +, *)$ and $(R', (+), (*))$ be two rings. A mapping $\phi : R \to R'$ is said to be ring homomorphism if for any $a, b \in R$*

(i) $\phi(a + b) = \phi(a)(+)\phi(b)$,

(ii) $\phi(a * b) = \phi(a)(*)\phi(b)$.

Definition A.10. *A ring homomorphism which is an one to one and onto function is said to be an isomorphism. If $\phi : R \to R'$ is such an isomporphism, we call the rings R and R' isomorphic and we write $R \cong R'$.*

Example A.6. *Let us consider the function $\phi : \mathbb{C} \to \mathcal{M}_2$ given by*

$$\phi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

is an isomorphism. Then we can identify any complex number as a concrete 2×2 matrix.

Appendix B

Probability theory

We give some basic elements of Probability theory which we used in this project. All r.v's and r..v's are defined on an abritrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. Elements of Measure theory

Theorem B.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_n, A \in \mathcal{F}$. Then,*

(i) *if $A_n \uparrow A$ then $\mathbb{P}(A_n) \uparrow \mathbb{P}(A)$.*

(ii) *if $A_n \downarrow A$ then $\mathbb{P}(A_n) \downarrow \mathbb{P}(A)$.*

Theorem B.2 (Dominated convergence theorem)**.**
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\lim_{n\to\infty} X_n \overset{a.s}{=} X$ and there is a r.v Y with $\mathbb{E}(Y) < \infty$ such that $|X_n| < Y$ for all n. Then

$$\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X).$$

Definition B.1. *A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is said to be uniformly integrable if*

$$\lim_{\alpha\to\infty} \sup_{n\geq 0} \mathbb{E}\big(X_n \mathbb{1}_{\{X_n \geq \alpha\}}\big) = 0.$$

Proposition B.1. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Assume that for some $\delta > 0$, we have*

$$\sup_{n\geq 0} \mathbb{E}\big(|X_n|^{1+\delta}\big) < \infty.$$

Then, any X_n is uniformly integrable.

Theorem B.3. *Let $(X_n)_{n\in\mathbb{N}}$ be an uniformly integrable sequence of random variables. Suppose that $\lim_{n\to\infty} X_n = X$. Then,*

(i) $\mathbb{E}(X) < \infty$.

(ii) $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

(iii) $\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0$.

2. Stochastic Convergence

Theorem B.4 (Gut's theorem)**.**
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $(N_n)_{n\in\mathbb{N}}$ a positive integer-valued stochastic process. Assume that

$$\lim_{n\to\infty} X_n \overset{a.s}{=} X \quad \& \quad \lim_{n\to\infty} N_n \overset{a.s}{=} \infty.$$

Then,

$$\lim_{n\to\infty} X_{N_n} \overset{a.s}{=} X.$$

Theorem B.5 (Anscombe's theorem)**.**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $(N_n)_{n\in\mathbb{N}}$ a positive integer-valued stochastic process. Suppose that

$$\frac{1}{\sqrt{n}}\sum_{l=0}^{n} X_l \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2) \quad \& \quad \frac{N_n}{n} \xrightarrow[n\to\infty]{P} \theta \in (0,\infty).$$

Then,

$$\frac{1}{\sqrt{N_n}}\sum_{l=0}^{N_n} X_l \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2).$$

Theorem B.6 (Continuous mapping theorem)**.**

Let X_n be a sequence of r..v's in $\mathbb{R}^k$, $k \geq 1$, and let $f : \mathbb{R}^k \to \mathbb{R}^m$ be a function which is continuous on a set $C \subset \mathbb{R}^k$ such that $\mathbb{P}\big(X \in C\big) = 1$. Then, the following limits hold

$$If\ X_n \xrightarrow[n\to\infty]{a.s/p/d} X,\ then\ f(X_n) \xrightarrow[n\to\infty]{a.s/p/d} f(X).$$

Theorem B.7 (Slutsky's theorem)**.**

Let X_n, Y_n be sequences of random elements. Also, suppose that X_n converges in law to a random variable X and Y_n converges in probability to a constant c. Then,

(i) $X_n + Y_n \xrightarrow[n\to\infty]{d} X + c$

(ii) $X_n Y_n \xrightarrow[n\to\infty]{d} cX$

(iii) $\dfrac{X_n}{Y_n} \xrightarrow[n\to\infty]{d} \dfrac{X}{c}$*, for* $c, Y_n \neq 0$.

Appendix C

Markov chains

1. Introduction

Definition C.1. *A sequence* $\boldsymbol{X} = \{X_n\}_{n\in\mathbb{N}}$ *of random variables is said to be a discrete time Markov chain, if it satisfies the Markov property, that is, for any* $n \in \mathbb{N}$, $i, j \in E$ *and* $i_{0:n-1} \in E^n$, *such that* $\mathbb{P}\big(X_{0:n} = (i_{0:n-1}, i)\big) > 0$, *we have*

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{0:n1} = i_{0:n1}) = P(X_{n+1} = j | X_n = i). \tag{C.1}$$

If, additionally, these probabilities do not depend on n, *then* $\boldsymbol{X}$ *is called (time-)homogeneous.*

- The conditional probabilities given by C.1, for a time homogeneous Markov Chain $\mathbf{X}$, are denoted by p_{ij}, and we say that p_{ij} is the one-step transition probability from state i to state j. The matrix $P = (p_{ij})_{i,j\in E}$ called the transition matrix of $\mathbf{X}$.
- The vector $\alpha = (\alpha_i)_{i\in E}$ with $\alpha_i = \mathbb{P}(X_0 = i)$ is called the initial probability of $\mathbf{X}$.

Proposition C.1. *For any* $n \in \mathbb{N}$, $i, j \in E$, *and* $B_k \subset E$, $k = 0, 1, \ldots, n-1$ *such that* $\mathbb{P}(X_n = j, X_{n-1} \in B_{n-1}, \ldots, X_0 \in B_0) > 0$, *we have*

$$\mathbb{P}(X_n = j | X_{n-1} = i, \ldots, X_0 \in B_0) = \mathbb{P}(X_n = j | X_{n-1} = i). \tag{C.2}$$

in order to study much better the behavior of a Markov chain we will need the following definition

Definition C.2. *Let* $n \in \mathbb{N}$ *and* $i, j \in E$ *we denote by*

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i).$$

We refer to the conditional probability $p_{ij}^{(n)}$ *as the* n*-step transition probability from* i *to (state)* j. *It corresponds to the probability that* $\boldsymbol{X}$ *will visit* j, *starting from* i, *after* n*-steps (transitions). For a xed* $n \in \mathbb{N}$, *these probabilities form the* n*-step transition matrix of the MC* $\boldsymbol{X}$, *that is,*

$$P^{(n)} = (p_{ij}^{(n)})_{i,j\in E}.$$

where $P^{(1)} = P$ *and* $P^{(0)} = I_E$.

2. Strong Markov property

Definition C.3. *A random variable* T, *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$, *with values in* $\mathbb{N} \cap \{\infty\}$, *is called a stopping time with respect to the sequence* $(X_n)_{n\in\mathbb{N}}$ *if the occurrence of the event* $\{T = n\}$ *is determined by the past of the chain up to time* n, $(X_k; k \leq n)$

Definition C.4. *The Markov chain* $(X_n)_{n\in\mathbb{N}}$ *is said to have the strong Markov property if, for any stopping time* T, *for any integer* $n \in \mathbb{N}$ *and state* $j \in E$ *we have*

$$\mathbb{P}(X_{n+T} = j | X_k, k \leq T) \stackrel{a.s}{=} \mathbb{P}_{X_T}(X_n = j).$$

Proposition C.2. *Any Markov chain has the strong Markov property.*

3. Stationarity and states classification

Definition C.5. *A Markov chain **X** with initial distribution α is said to be stationary if its marginal distribution is invariant in time, that is, for any $A \subset E$ and $n \in \mathbb{N}$*

$$\mathbb{P}(X_n \in A) = \alpha(A).$$

Definition C.6. *A probability vector $\pi \in \mathbb{R}^s$ is said to be a stationary vector of **X** if*

$$\pi_j = \sum_{i \in E} p_{ij} \pi_i, \quad j \in E.$$

This property can be written equivalently in matrix form as follows:

$$\pi = \pi P.$$

A rst obvious property of the stationary distribution is given in the following property.

Proposition C.3. *Let **X** be a Markov chain which admits π as a stationary vector. If the initial vector $\alpha = \pi$, then for any $A \subset E$ and $n \in \mathbb{N}$:*

$$\mathbb{P}(X_n \in A) = \pi(A).$$

*or, in other words, **X** is a stationary Markov chain with stationary distribution $\pi(\circ)$.*

Definition C.7. *Let $i \in E$ and define*

$$T_i = \inf\{n \in \mathbb{N} : X_n = i\}$$

The above random variable is said to be the rst return time on i. If $X_0 = i$ with probability 1, then it is called rst recurrence time.

With the help of the above denition the states are classied as follows:

Definition C.8. *A state $i \in E$ is said to be*

(i) *recurrent, if* $\mathbb{P}(T_i < \infty | X_0 = i) = 1$

(ii) *transient, if* $\mathbb{P}(T_i < \infty | X_0 = i) < 1$.

Definition C.9. *A recurrent state $i \in E$ is said to be*

(i) *Positive recurrent if* $\mu^*_{ii} \mathbb{E}_i(T_i) < \infty$.

(ii) *Null recurrent if* $\mu^*_{ii} \mathbb{E}_i(T_i) = \infty$.

Also, two useful processes are defined as follows

Definition C.10. *Let $i, j \in E$ and define*

- *The time spent by the chain in state i, during the time interval* $[0, n-1]$

$$N_i(n) = \sum_{l=0}^{n-1} \mathbb{1}_{\{X_l = i\}}$$

- *The number of direct transitions from i to j, until n-th period.*

$$N_{ij}(n) = \sum_{l=1}^{n} \mathbb{1}_{\{X_{l-1}=i, X_l=j\}}$$

Definition C.11. (i) *we say that the state $i \in E$ leads to a state $j \in E$ if there exists $n \in \mathbb{N}$, such that $p_{ij}^{(n)} > 0$. We denote this by $i \to j$.*

(ii) *If both states lead to each other ($i \to j$ and $j \to i$), then we say that i and j communicate, and we denote this by $i \longleftrightarrow j$.*

Theorem C.1. *A state $i \in E$ is recurrent if and only if*

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty.$$

and for a transient state j, we have that for all $\in E$,

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty.$$

Proposition C.4. (i) *The relation of communication given in Denition C.11, denes an equivalence relation on the state space E.*

(ii) *Recurrence and transience are properties of the communication (equivalence) class, or, in other words, if we qualify a state of a class as recurrent or transient, then this is automatically transferred to all the states of the same class.*

Definition C.12. *A Markov chain $\boldsymbol{X}$ is said to be irreducible if all of its states communicate. It is said to be recurrent or transient if all of its states are recurrent or transient respectively.*

For finite Markov chains the following result holds.

Proposition C.5. *An irreducible Markov chain $\boldsymbol{X}$ (with finite state space) is necessarily positive recurrent.*

Definition C.13. *A recurrent and aperiodic state $i \in E$ is called ergodic.*

Definition C.14. *The Markov chain $\boldsymbol{X}$ is said to be periodic, aperiodic or ergodic, if all its states are periodic, aperiodic or ergodic respectively.*

By the above denitions and Proposition C.5 we have:

Corollary C.1. *An irreducible and aperiodic Markov chain X (with nite state space) is necessarily ergodic.*

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d integer-valued r.v's. Also, set $S_n = \sum_{l=0}^{n} X_l$ for any $n \in \mathbb{N}$. It's obvious to conclude that $(S_n)_{n\in\mathbb{N}}$ forms a Markov chain with state space the set $\mathbb{Z}$. In the following theorem we give a sufficient condition for its recurrence.

Theorem C.2. *Suppose that the Markov chain $(S_n)_{n\in\mathbb{N}}$ is irreducible. Also, assume that*

$$\mathbb{E}(|X_k|) < \infty \quad \& \quad \mathbb{E}(X_k) = 0, \quad k \in \mathbb{N}.$$

Then the Markov chain $(S_n)_{n\in\mathbb{N}}$ is recurrent.

4. Asymptotic results

A useful characterization of the stationary distributions of a nite Markov chain **X** is given via the limit distributions of **X** .

Definition C.15. *Let **X** be a MC with transition matrix P and $i \in E$. If*

$$p_{ij}^{(n)} = P_{ij}^{n} \xrightarrow[n\to\infty]{a.s} \pi_j^{(i)}, \quad i \in E.$$

where $\pi^{(i)}$ is a probability vector, then $\pi^{(i)}$ is called limit vector of the chain and the associated distribution $\pi^{(i)}(\circ)$ is called limit distribution of the chain.

Note that in general a MC can have many limit distributions. The following result holds.

Proposition C.6. *If a probability vector π is a limit vector of **X**, then it is stationary for the MC X.*

Proposition C.7. *Let i and j be two recurrent states Then,*

$$\begin{aligned} \lim_{n\to\infty} \frac{N_i(n)}{n} &= \frac{1}{\mu_{ii}^*} \\ \lim_{n\to\infty} \frac{N_{ij}(n)}{n} &= \frac{p_{ij}}{\mu_{ii}^*} \end{aligned}$$

Theorem C.3 (Law of large numbers for MCs - **Birkhoff**)**.**
If X is an irreducible MC with stationary vector π, and if $f : E \to \mathbb{R}$, then

$$\frac{1}{n}\sum_{l=1}^{n} f(X_l) \xrightarrow[n\to\infty]{a.s} \mu_f = \mathbb{E}_\pi(X_0)$$

where in the last expression X_0 is assumed to follow the stationary distribution $\pi(\circ)$.

Theorem C.4 (CLT for Markov chains)**.**
*Let **X** be an irreducible MC with stationary distribution π and $f : E \to \mathbb{R}$. Then, independently of the initial distribution*

$$\frac{1}{\sqrt{n}}\Big(\sum_{l=1}^{n} f(X_l) - n\mu_f\Big) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma_f^2),$$

where

$$\begin{aligned} \mu_f &= \mathbb{E}_\pi(X_0), \\ \sigma_f^2 &= \mathbb{V}_\pi(X_0) + 2\sum_{l=0}^{\infty} Cov_\pi\big(f(X_0), f(X_l)\big). \end{aligned}$$

Appendix D

Real sequences

Theorem D.1 (Abel's Theorem)**.** *Let $G(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ be a powerseries with real coefficients α_k with radius of convergence 1. Suppose that the series $\sum_{k=0}^{\infty} \alpha_k$ converges. Then $G(x)$ is continuous from the left at $x = 1$, i.e.*

$$\lim_{x \to 1^-} G(x) = \sum_{k=0}^{\infty} \alpha_k.$$

Theorem D.2 (Domitated convergence theorem for sequences)**.**
Let $x(n,l)$ be a double real sequence for which $\lim\limits_{n\to\infty} x(n,l)$ exists for any $l \in \mathbb{N}$ and that $|x(n,l)| \leq y(l)$ with $\sum\limits_{l=0}^{\infty} y(l) < \infty$. Then,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{l=0}^{\infty} x(n,l) &= \sum_{l=0}^{\infty} \lim_{n\to\infty} x(n,l) \\
\lim_{n\to\infty} \sum_{l=0}^{n} x(n,l) &= \sum_{l=0}^{\infty} \lim_{n\to\infty} x(n,l)
\end{aligned}$$

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