HELLENIC REPUBLIC
National and Kapodistrian
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DEPARTMENT OF MATHEMATICS
THEORETICAL MATHEMATICS PROGRAM
MASTER'S THESIS

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# Splitting Theorems for Semi-Riemannian Manifolds

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Athens, 2020

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# Θεωρήματα Διάσπασης για Ημι-Ρημάνιες Πολλαπλότητες

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Αθήνα, 2020

Θα ήθελα να ευχαριστήσω τον επιβλέποντα, κύριο Λάππα για την καθοδήγηση και την πρόταση του θέματος της εργασίας.

Επίσης θα ήθελα να ευχαριστήσω τον κύριο Π. Γιαννιώτη για την συνεισφορά του και για την ανεκτίμητη βοήθεια που μου προσέφερε αφιερώνοντας χρόνο να απαντήσει στις απορίες μου.

Τέλος, θέλω να ευχαριστήσω τον κύριο Α. Μελά και τον κύριο Κ. Αθανασόπουλο, για την συμβολή τους ως μέλη της επιτροπής. Ιδιεταιρα θέλω να ευχαριστήσω τον κ. Αθανασόπουλο για την εξονυχιστική μελέτη του κειμένου και τις προτάσεις του για βελτίωση.

# Abstract


Our goal is to describe the splitting theorems of Riemannian and Lorentzian geometry. In the first chapter we present a brief overview of some basic definitions and results about Riemannian and, more importantly, Lorentzian manifolds.

In 1971 Cheeger and Gromoll [5] proved that
*a complete Riemannian manifold $M$ which contains a line and has non-negative Ricci curvature is isometric to the product $\mathbb{R} \times M'$ for some Riemannian manifold $M'$.*
Similar results had been published earlier by other authors for special cases or using stronger conditions, like Toponogov's theorem for non-negative sectional curvature. A more elementary proof for the Cheeger-Gromoll theorem was given by Eschenburg and Heintze [7]. They made use of smooth support functions along with a maximum principle theorem proved by Calabi. We present this proof in the first half of the second chapter.

The theorem led Yau to postulate the following analogue for Lorentzian manifolds:
*A timelike complete Lorentz manifold $M$ which contains a line and has non-negative Ricci curvature, splits isometrically as $\mathbb{R} \times M'$ where $M'$ is a Riemannian manifold.*

Again, many weaker and different versions of the theorem were proved. At some point the Lorentzian conjecture came to the attention of Eschenburg and in 1988 he proved the splitting theorem while also assuming *global hyperbolicity* for the manifold. Soon after that, Galloway proved the theorem assuming only global hyperbolicity. In the second half of chapter 2, we prove the Lorentzian theorem as stated by Eschenburg, but we will use some ideas from the proof of Galloway.

The conjecture made by Yau was finally proved by Newmann in [15].

# Περίληψη

Σκοπός μας είναι να περιγράψουμε και να αποδείξουμε τα θεωρήματα διάσπασης για πολλαπλότητες τύπου Riemann και Lorentz. Στο πρώτο κεφάλαιο παρουσιάζουμε μια σύντομη υπενθύμιση μερικών βασικών ορισμών και αποτελεσμάτων για πολλαπλότητες τύπου Riemann και, πιο σημαντικά, τύπου Lorentz.

Το 1971 οι Cheeger και Gromoll [5] απέδειξαν ότι
*Μια πλήρης πολλαπλότητα Riemann που περιέχει ένα line και έχει μη-αρνητική Ricci καμπυλότητα είναι ισομετρική με το Καρτεσιανό γινόμενο* $\mathbb{R} \times M'$, *όπου* $M'$ *είναι μια πολλαπλότητα Riemann.*

Παρόμοια αποτελέσματα είχαν αποδειχθεί νωρίτερα από άλλους για ειδικές περιπτώσεις ή με ισχυρότερες υποθέσεις, όπως το θεώρημα του Toponogov για μη-αρνητική sectional curvature. Μια πιο στοιχειώδης απόδειξη του θεωρήματος των Cheeger και Gromoll δόθηκε από τους Eschenburg και Heintze [7]. Χρησιμοποίησαν λείες συναρτήσεις στήριξης μαζί με ένα θεώρημα αρχής μεγίστου που απέδειξε ο Calabi. Θα παρουσιάσουμε την απόδειξη αυτή στο πρώτο μισό του δεύτερου κεφαλαίου.

Το θεώρημα οδήγησε τον Yau να διατυπώσει μια ανάλογη εικασία για την περίπτωση πολλαπλοτήτων Lorentz
*Μια timelike-complete πολλαπλότητα Lorentz* $M$ *που περιέχει ένα line και έχει μη-αρνητική καμπυλότητα Ricci, διασπάται ισομετρικά σε Καρτεσιανό γινόμενο* $\mathbb{R} \times M'$ *όπου* $M'$ *είναι μια πολλαπλότητα Riemann.*

Πάλι, αρκετές ασθενέστερες εκδοχές του θεωρήματος αποδείχθηκαν. Κάποια στιγμή, η εικασία έπεσε στην προσοχή του Eschenburg και το 1988 απέδειξε ένα θεώρημα διάσπασης υποθέτοντας επι-πλέον ότι η πολλαπλότητα είναι *globally hyperbolic*. Σύντομα μετά, ο Galloway απέδειξε την διάσπαση υποθέτοντας μόνο global hyperbolicity. Στο δεύτερο μισό του δεύτερου κεφαλαίου αποδεικνύουμε το θεώρημα όπως το διατύπωσε ο Eschenburg, αλλά θα χρησιμοποιήσουμε μερικές ιδέες από την απόδειξη του Galloway.

Η εικασία που διατύπωσε ο Yau αποδείχθηκε τελικά από τον Newmann [15]

# Contents

# Chapter 1

# Introduction

In this chapter we will give some basic definitions and examples and present some elementary results. Our goal is to provide the necessary notation and results to prove the splitting theorems in the next chapter, as well as give a quick overview of semi-Riemannian geometry, focusing on Riemannian and Lorentzian manifolds.

As we shall see, most of the local theory is identical for Riemannian and and Lorentz manifolds. This is because often, the positive-definite of the Riemannian metric is not needed, so with only a non-degenerate metric, many definitions and proofs work just the same. However, there are some very important results of Riemannian geometry where the positive-definite property is essential, so these results don't hold in general in the semi-Riemannian case. At the end of the chapter we discuss some of those results, how they fail and -in some cases- what additional conditions we need to assume.

## 1.1 Definitions and Examples

### Inner Product Spaces

The objects we are interested in, are semi-Riemannian manifolds. These are $C^\infty$ manifolds equipped with an semi-Riemannian metric. We are all familiar with the natural inner product on $\mathbb{R}^n$. We remind that an inner product on a vector space $V$ is a map $g : V \times V \to \mathbb{R}$ such that:

- $g(x, y) = g(y, x)$

- $g(\lambda x + y, z) = \lambda g(x, z) + g(y, z)$
- $\forall x \; g(x, x) \geq 0$ and $g(x, x) = 0 \implies x = 0$

The first two properties mean $g$ is a symmetric bi-linear form. For the third property we say $g$ is positive-definite.

It turns out that a weaker type of inner product is of great significance in the theory of relativity, as it describes the geometry of spacetime in the same way that the canonical inner product of $\mathbb{R}^n$ describes the Euclidean geometry.

**Definition 1.** A symmetric bilinear form $g : V \times V \to \mathbb{R}$ is called a (non-degenerate) **inner product** if

$$\forall x \; g(x, u) = 0 \implies u = 0$$

It is clear that any positive-definite inner product is non-degenerate. We can also define a negative-definite inner product for $\forall x \; g(x, x) \leq 0$. A subspace of a non-degerneate inner product space can be positive or negative-definite. It can also be non-degenerate itself or it can be degenerate! This means we have to be carefull when we consider subspaces of non-degenerate inner product spaces.

From now, by inner product, we shall mean a non-degenerate, symmetric bilinear form, not necessarily positive definite- unless we mention it. We will often use the common notation $\langle \cdot, \cdot \rangle$ for $g(\cdot, \cdot)$.

Let's present some more definitions before we give some examples.

A vector $v \neq 0$ in an inner product space $V$ is called:

- **spacelike**, if $\langle v, v \rangle > 0$
- **null** or **lightlike**, if $\langle v, v \rangle = 0$
- **timelike**, if $\langle v, v \rangle < 0$

This is called the **causal character** of the vector.

With small pertubations of a timelike vector we can create an entire base of the vector space. That however gives us no information on the inner product. On the contrary the dimension of a negative-definite subspace is more interesting. The maximal dimension of a negative-definte subspace is called the **index** of the inner product

It can be proved that two inner product vector spaces of the same dimension and index are isomorphic.

Two vectors $v,\ w$ are called **orthogonal** if $\langle v, w\rangle = 0$

The **norm** of a vector $v$ is defined as $\|v\| = \sqrt{|\langle v, v\rangle|}$

A vector is called **normal** or to be of **unit length** if $\|v\| = 1$

A set of vecotrs is called **orthonormal** if they are all normal and pairwise orthogonal.

The **quadratic form** of the inner product is a map $q : V \to \mathbb{R}$ such that $q(v) = \langle v, v\rangle$

Let $(V, g)$ be an inner product space and $b_1, b_2, \ldots, b_n$ be a basis of $V$. The **matrix** of g related to that base is

$$(g)_{ij} = g(b_i, b_j) = \langle b_i, b_j\rangle$$

**Lemma 2.** The matrix of a non-degenerate inner product is symmetric and invertible.

*Proof.* The summetry is clear. Notice that, for every $j$, $g(v, b_j) = 0$ is equivalent to $\sum_i v_i g_{ij} = 0$. This is equivalent to a linear system $v^t(g) = 0$, or that $v$ lists coefficients that make the linear combination of the rows of $g$ equal zero. From that, non-degeneracy gives that the rows of $g$ are independent and, conversely, if $g$ were invertible the only solution of the system is $v = 0$ □

From the definition we have that the inner product defines the quadratic form. The following shows that the quadratic form also defines the inner product. That means the two are equivalent structures

**Lemma 3** (Polarization Identity)**.** For any two vectors $v, w$ in an inner product space, we have

$$\begin{aligned}\langle v, w\rangle &= \frac{1}{2}\big(q(v+w) - q(v) - q(w)\big)\\ &= \frac{1}{4}\big(q(v+w) - q(v-w)\big)\end{aligned}$$

*Proof.* The proof is a straightforward calculation

$$\begin{aligned}q(v+w) - q(v) - q(w) &= \langle v+w, v+w\rangle - q(v) - q(w)\\ &= q(v) + 2\langle v, w\rangle + q(w) - q(v) - q(w)\\ &= 2\langle v, w\rangle\end{aligned}$$

Similarly the other identity □

**Lemma 4.** In an inner product space $(V, g)$, there is a natural isometry $T$ between $V$ and its dual space $V^*$, given by the formula

$$v \mapsto T_v = g(v, \cdot)$$

*Proof.* It is easy to show the map $T$ is linear. Consider a basis $\{b_i\}$ of $V$ and and take some $\theta \in V^*$. Then $T_v = \theta$ is equivalent to

$$T_v(b_j) = \theta(b_j) \text{ , for all } j$$
$$\sum_i v_i g_{ij} = \theta_j \text{ , for all } j$$

But, from Lemma 2, the matrix $(g)$ is invertible and so there is a unique solution. This makes $T$ a bijection and gives a formula for $v$

$$v_i = \sum_j \theta_j g^{ij}$$

where $(g^{ij}) = (g_{ij})^{-1}$ □

This lemma gives us a way to calculate the components of a vector $v$ in a given base $\{b_i\}$, because it is usually is easier to calculate the components of the respective one-form $\langle v, \cdot \rangle$

$$v = \sum_{ij} g^{ij} \langle v, b_j \rangle b_i$$

**Proposition 5.** In an inner product space of index $\nu$, all orthonormal bases have exactly $\nu$ timelike vectors

*Proof.* Let $W$ be a maximal negative-definite subspace and $b_1, \ldots, b_m, b_{m+1}, \ldots, b_n$ be an orthonormal base with $b_1, \ldots, b_m$ timelike. The components of the inner product are then $g^{ij} = \varepsilon_i \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta and $\varepsilon_i = \langle b_i, b_i \rangle$ the "sign" of $b_i$.

From orthonormality, the vectors $b_1, \ldots, b_m$ span a negative-definite subspace of dimension $m$. Therefore $m \leq \nu = dim(W)$.

For the reverse inequality consider the linear map

$$\pi : W \to span(b_1, \ldots, b_m) : w \mapsto \sum_{i=1}^{m} \varepsilon_i \langle w, b_i \rangle b_i$$

It sufices to show that it is an injection. For this we show that its kernel is trivial. Let $\pi(w) = 0$, we have

$$\pi(w) = 0 \Leftrightarrow \forall i = 1, \ldots, m \ \langle w, b_i \rangle = 0$$

Then from orthonormality we have

$$w = \sum_{i=m+1}^{n} \varepsilon_i \langle w, b_i \rangle b_i$$

$$\Rightarrow \langle w, w \rangle = \sum_{i=m+1}^{n} \langle w, b_i \rangle^2 \geq 0$$

But $w \in W$ and therefore $\langle w, w \rangle = 0$, from which follows that $\langle w, b_i \rangle = 0$ for $i = m+1, \ldots, n$. Thus $w = 0$ □

The existence of an orthonormal basis is proved in [16]. If the index is 1, we can apply a variation of the Gram-Schmidt process. The trick is to start with a timelike vector as the normal subspace will be positive definite and Gram-Schmidt applies normally. From now on we are mostly interested in inner products of index 1 (or 0), though, for the sake of generality we will formulate some results in terms of arbitrary index.

**Lemma 6.** The timelike vectors of an inner product space of index 1 are divided in two disjoint convex sets and two timelike vectors $v, w$ are on the same component iff $\langle v, w \rangle < 0$

*Proof.* Let $u$ be a timelike vector. Consider the linear form

$$\phi_u : V \to \mathbb{R} : v \mapsto \langle v, u \rangle$$

Then $u^\perp = \phi_u^{-1}(0)$ and it is a positive-definite subspace, therefore it contains no timelike vectors. If

$$C^- = \{v | v \text{ timelike and } \phi_u(v) < 0\}$$

$$C^+ = \{v | v \text{ timelike and } \phi_u(v) > 0\}$$

then $u \in C^-$ and $-u \in C^+$. Therefore the set of timelike vectors is composed of two disjoint sets.

Take two timelike vectors $v, w$. Without loss of generality, $v \in C^-$. To show that $\phi_u(w) < 0$ iff $\langle v, w\rangle < 0$, it sufices to show that for three timelike vectors $v, u, w$, if two pairs have negative inner product, so will the third pair. Let $\langle v, w\rangle < 0$. First, for $r \in [0, 1]$ it's a straightforward calculation to see that $rv + (1 - r)w$ is timelike. If we assume that $\langle w, u\rangle \geq 0$, then for some $r_0$ we will have $\langle r_0 v + (1 - r_0)w, u\rangle = 0$, that is two linearly independent timelike vectors. This contradicts the assumption that the index is 1. $\square$

The two sets of timelike vectors of the above lemma are called the **timecones** of $V$. In a way they represent the future and the past of the observer at the start, though in a general inner product space there is no natural way to differentiate one from the other. To specify one timecone to represent the future is to **time-orient** the space.

As we see in the next example the triangle and the Cauchy-Schwarz inequalities don't hold for inner product spaces. Of course, if we restrict ourselves to a posive definite subspace, everything works as usual. Furthermore, some analogues can be proved for timelike vectors.

**Lemma 7.** For $v, w$ timelike vectors in an inner product space of index 1, we have the reverse Cauchy-Schwarz

$$|\langle v, w\rangle| \geq \|v\|\|w\|$$

Furthermore, if the vectors belong to the same timecone, we also have the reverse triangle inequality

$$\|v + w\| \geq \|v\| + \|w\|$$

*Proof.* For $\alpha = \frac{\langle v,w\rangle}{\langle v,v\rangle}$ we have $w = \alpha v + \overline{w}$, with $\overline{w} \perp v$

$$\begin{aligned}\langle w, w\rangle &= \alpha^2\langle v, v\rangle + \langle \overline{w}, \overline{w}\rangle \\ \alpha^2\langle v, v\rangle &= \langle w, w\rangle - \langle \overline{w}, \overline{w}\rangle\end{aligned}$$

Applying that to $\alpha\langle v, v\rangle = \langle v, w\rangle$

$$\begin{aligned}\langle v, w\rangle^2 &= \alpha^2\langle v, v\rangle^2 = (\alpha^2\langle v, v\rangle)\langle v, v\rangle \\ &= \big(\langle w, w\rangle - \langle \overline{w}, \overline{w}\rangle\big)\langle v, v\rangle \\ &\geq \langle w, w\rangle\langle v, v\rangle\end{aligned}$$

Where the last inequality is due to the fact that $\overline{w}$ is spacelike, threfore $\langle \overline{w}, \overline{w} \rangle \geq 0$. Since both $v$ and $w$ are timelike the last term is positive, thus taking square roots, we get the reverse C-S inequality.

For the reverse triangle inequality we need to additionaly assume that $v$ and $w$ lie on the same timecone, i.e. $\langle v, w \rangle < 0$. The reverse C-S then gives $\langle v, w \rangle \leq -\|v\|\|w\|$.

$$\begin{aligned} \|v+w\|^2 &= -\langle v, v \rangle - 2\langle v, w \rangle - \langle w, w \rangle \\ &\geq \|v\| + 2\|v\|\|w\| + \|w\| = (\|v\| + \|w\|)^2 \end{aligned}$$

Taking square roots finishes the proof. □

The above result is counterintuitive, but it probably should not be surprising, The negative nature of timelike vectors is expected to reverse things

**Example 8.** On $\mathbb{R}^n$ for any $\nu < n$ we have the inner product of index $\nu$

$$\langle x, y \rangle = -x_1 y_1 - x_2 y_2 - \cdots - x_\nu y_\nu + x_{\nu+1} y_{\nu+1} + \cdots + x_n y_n$$

Another way to write this is using the "sign" of $i$-th place

$$\varepsilon_i = \begin{cases} -1 & \text{, for } i \leq \nu \\ 1 & \text{, for } i > \nu \end{cases}$$

Then we can write

$$\langle x, y \rangle = \sum_{i=1}^{n} \varepsilon_i x_i y_i$$

We refere to this inner product space as $\mathbb{R}^n_\nu$. For the canonical base of $\mathbb{R}^n$, the matrix of the inner product is:

$$\begin{pmatrix} -I_\nu & 0 \\ 0 & I_{n-\nu} \end{pmatrix}$$

It is common to call $\mathbb{R}^4_1$ the Minkowski space for he introduced it as a mathematical model of the geometry of special relativity.

It should already be clear that orthogonality does not work in the same way as in Euclidean geometry. Null vectors are orthogonal to themselves and generally we don't have the "90 degree" angles we used to.

It is not true that a subspace of a non-degenerate vector space is non-degenerate. For example consider the subspace $\{x_1 = x_n\}$ of $\mathbb{R}^n_1$, the non-zero vector $(1, 0, ..., 0, 1)$ in orthogonal to all vectors of the subspace

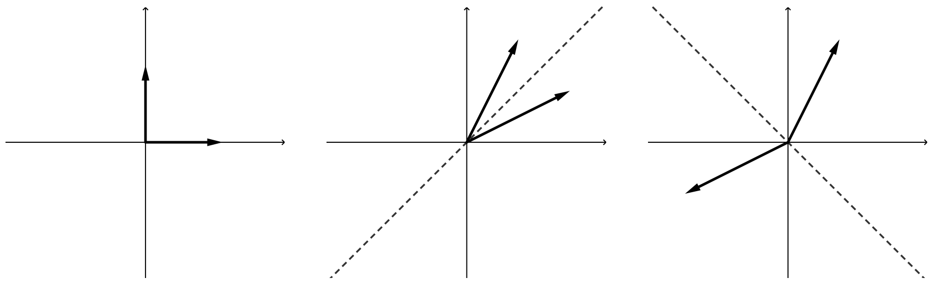


Figure 1.1: Pairs of orthogonal vectors in $\mathbb{R}^2_1$

Consider the vectors $(5,4)$ and $(1,0)$ of $\mathbb{R}^2_1$. We have

$$\|(6,4)\| = 2\sqrt{5} > 4 = \|(5,4)\| + \|(1,0)\|$$

and

$$\left|\langle(5,4),(1,0)\rangle\right| = 5 > 3 = \|(5,4)\|\|(1,0)\|$$

## Semi-Riemannian Manifolds

We assume some familiarity with the basic concepts of differential and Riemannian geometry. In Riemannian geometry we equip each tangent space of a $C^\infty$ manifold with a positive definite inner product. Now we will do the same with our more general notion of inner product.

**Definition 9.** A **semi-Riemannian** manifold is a pair $(M,g)$ of a $C^\infty$ manifold $M$ with an inner product $g_p$ for each tangent space $T_pM$. This field of inner product is called semi-Riemannian **metric** and is required to vary smoothly from point to point in the sense that if $X,Y \in \mathfrak{X}(M)$, then $g(X,Y) : M \to \mathbb{R} : p \mapsto g_p(X_p, Y_p)$ is a smooth function on $M$

**Lemma 10.** In a connected semi-Riemannian manifold, the index of $g_p$ is independent of the point $p$

*Proof.* We will show that the set of points of the same index is open and closed. It suffices to show that the index is a locally constant function.

Let $\nu$ be the index of $g_p$ for some point $p$, and let $X_1, \ldots, X_n$ be an orthonormal base with $X_1, \ldots, X_\nu$ timelike. Consider a coordinate neighbourhood $U$, in wich we have

$$X_i = \sum_j \lambda_{ij} \left.\frac{\partial}{\partial x_j}\right| p$$

Expand them to vextor fields on $U$

$$X_i = \sum_j \lambda_{ij} \frac{\partial}{\partial x_j}$$

The vector fields are linearly independent, but not orthonormal away from $p$. For $\varepsilon < \frac{1}{n}$, from the continuity of the metric we can find a neighbourhood of $p$, such that

$$\begin{aligned} \langle X_i, X_i \rangle &\leq -1 + \varepsilon, \text{ for } i = 1, \dots, \nu \\ \langle X_i, X_i \rangle &\geq 1 - \varepsilon, \text{ for } i = \nu + 1, \dots, n \\ |\langle X_i, X_j \rangle| &\leq \varepsilon, \text{ for } i \neq j \end{aligned}$$

Then for any $q$ in this neighbourhood $X_1, \dots, X_\nu$ span a negative definite subspace and $X_{\nu+1}, \dots, X_n$ span a positive definite subspace. Indeed, for $0 \neq Y = \sum_{i=1}^{\nu} \alpha_i X_i$ we'll prove $Y$ is timelike.

$$\begin{aligned} \langle Y, Y \rangle &= \sum_{i,j=1}^{\nu} \alpha_i \alpha_j \langle X_i, X_j \rangle \\ &= \sum_1^{\nu} \alpha_i^2 \langle X_i, X_i \rangle + \sum_{i \neq j} \alpha_i \alpha_j \langle X_i, X_j \rangle \\ &\leq -\sum_1^{\nu} \alpha_i^2 + \sum_{i,j=1}^{\nu} \alpha_i \alpha_j \varepsilon \\ &= -\sum_1^{\nu} \alpha_i^2 + \left( \sum_1^{\nu} \alpha_i \right)^2 \varepsilon \\ &\leq -\sum_1^{\nu} \alpha_i^2 + \left( \sum_1^{\nu} |\alpha_i| \right)^2 \varepsilon \end{aligned}$$

Treating the coefficients $\alpha_i$ as components of a vector $\alpha \in \mathbb{R}^n$, we get from a known inequality that

$$\|\alpha\|_1 \leq \sqrt{n} \|\alpha\|_2$$

Ans since $\varepsilon < \frac{1}{n}$, we have $\langle Y, Y \rangle < 0$. □

The two most important cases of Semi-Riemannian manifolds are those of index 0, these are the **Riemannian** manifolds, and those of index 1, we call them **Lorentzian** manifolds or **timespaces** because of their use in relativity

**Definition 11.** A Lorentzian manifold is called **time-orientable** if there exists a global non-zero timelike vector field $X$. Thus at each point we have the **future** timecone (resp. **past** timecone) wich contains $X$ (resp. doesn't)

A curve in $M$ is called timelike (resp. spacelike, lightlike) if the tangent velocities are timelike vectors (resp. spacelike, lightlike). Accordingly, a submanifold will be called timelike (resp. spacelike, lightlike) if all tangent vectors are timelike (resp. spacelike, lightlike). A vector or curve which is not spacelike will be called **causal**

In a time oriented Lorentz manifold we say two points are **chronologically related** $p \ll q$ (resp. **causally related** $p < q$) if there is a future pointing timelike (resp. causal) curve from $p$ to $q$. This means in a way that $p$ happened "before" $q$, and thus what happens in $p$ can cause events in $q$. This notion is non-existent in Riemann geometry and we have to keep track of how the points are causally related between them.

Using these relations we define the chronological **future** (resp. causal **future**) of a point $p$ as

$$I^+(p) = \{x \in M : p \ll x\} \text{ (resp. } J^+(p) = \{x \in M : p < x\})$$

and by revesring the relations we have the chronological past $I^-(p)$ and causal past $I^-(p)$.

A Lorentz manifold is called **chronological** (resp. **causal**), if there is no closed timelike (resp. causal) curve. It is called **strongly causal** if there exist no "almost closed" causal curves, namely no causal curves with arbitrarily close endpoints which extends arbitrarily far from them. It is clear from the definitions, that

$$\text{strongly causal} \Rightarrow \text{causal} \Rightarrow \text{chronological}$$

None of the inverses above hold though as there are counterexamples. For more details, see [16]

A Lorentz manifold is called **globally hyperbolic** if it is strongly causal and the set $J^+(p) \cap J^-(q)$ is compact for all points $p, q$.

Let $f : M \to N$ be an embedding of $M$ in a Riemannian manifold $(N, g)$. Then the **pull-back** $f^*g$ is a Riemannian metric on $M$.

$$f^*g(v, w) = g(df(v), df(w))$$

Generaly the pull-back of a non-degenerate metric is not a non-degenerate metric on $M$. The problem is the subspace $df(T_pM)$ might not be a non-degenerate subspace of $T_{f(p)}N$, so we can't benefit from $g$ being non-degenerate.

**Proposition 12.** Let $(M, \tau)$ and $(N, \sigma)$ be two semi-Riemannian manifolds of index $\mu$ and $\nu$ respectively. Consider $\pi_1$ and $\pi_2$ the projections of $M \times N$ on $M$ and $N$ respectively. Then

$$g = \pi_1^*\tau + \pi_2^*\sigma$$

Is a semi-Riemannian metric on $M \times N$ of index $\mu + \nu$

*Proof.* Symmetry and bilinearity are straightforward. We will prove the non-degeneracy. Let

$$g(v, w) = 0, \text{ for all } w \in T_{(p,q)}(M \times N)$$

Then $g(v, w) = 0$, specifically for those $w \in T_pM \oplus \{0\} \leq T_{(p,q)}(M \times N)$, and since $d\pi_2 w = 0$ for $w \in T_pM \oplus \{0\}$. We have

$$\sigma(d\pi_1(v), d\pi_1(w)) = 0$$

But these $d\pi_1(w)$ span all of $T_pM$, and therefore from non-degeneracy of $\sigma$

$$d\pi_1 v = 0$$

Likewise $d\pi_2 v = 0$, hence $v = 0$. This proves the non-degeneracy. For the index, consider two orthonormal bases of $T_pM$ and of $T_qN$. Their union creates an orthonormal base of $T_{(p,q)}(M \times N)$. Thus the index of $g$ is $\mu + \nu$ $\square$

**Example 13.** The $n$-sphere is an example of a Riemannian manifold. Its metric is derived from its embedding in $\mathbb{R}^{n+1}$

Obviously all $\mathbb{R}^n_1$ are Lorentz manifolds. Less trivial examples can be obtained if we take Cartesian products. Take $\mathbb{R}^1_1 \times S^1$ for example. In fact for any Riemann manifold $M$ the product $\mathbb{R}^1_1 \times M$ is a Lorentzian manifold

Our main goal in the next chapter is to prove a sort of inverse of Proposition 12. We will give appropriate conditions for a manifold to be writen as the product of two manifolds. These are called Splitting Theorems. We will prove the standard Riemannian Splitting Theorem, and one of many Lorentzian Splitting Theorems.

## 1.2 Some Aspects of semi-Riemannian Geometry

In the rest of this chapter we focus on some topics from the theory of Riemannian and Lorentzian manifolds. We will develop the necessary tools we need for the next chapters and highlight some interesting simmilarities and differences between Riemannian and Lorentzian Geometry.

Recall that a **connection** on a smooth manifold $M$ is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

$$\nabla_Y(\cdot) : X \mapsto \nabla_Y X \text{ is } \mathbb{R}\text{-linear and Leibniz}$$

$$\nabla_{(\cdot)} Y : X \mapsto \nabla_X Y \text{ is } \mathfrak{F}(M)\text{-linear}$$

Furthermore, we say the connection is **torsion free** if it satisfies

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

**Theorem 14.** In any semi-Riemannian manifold $M$ there exists a unique, torsion free connection that is compatible with the metric, i.e.

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

*Proof.* The theorem is proved in the same way as in Riemannian geometry. We take the Koszul formulla

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &+ \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle \end{aligned}$$

and set $F(X, Y, Z)$ to be the right hand side of the formula, then we observe that the map $Z \mapsto F(X, Y, Z)$ is a one form, therefore, from lemma 4, there exists a unique vector field $\nabla_X Y$ such that $\langle \nabla_X Y, \cdot\rangle = F(X, Y, \cdot)$ □

All the notions of Riemannian geometry based on the connection are developed in the same way for semi-Riemannian manifolds. We can use the connection to define **parallel translation** along curves, define geodesics and construct **normal neighbourhoods** using the exponential map. The **Riemann curvature** tensor is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

For a frame field $\{E_i\}$ the components of $R$ are

$$R^m_{ijk} = E^*_m(R(E_j, E_k)E_i)$$

Where $E^*_m$ is the dual frame.

The **Ricci curvature** is defined to be the contraction of the Riemann tensor

$$\begin{aligned} Ric_{ij} &= \sum_k R^k_{ikj} \\ &= \sum_k E^*_k\big(R(E_k, Ej)E_i\big) \\ &= \sum_{km} g^{km}\langle R(E_k, Ej)E_i, E_m\rangle \end{aligned}$$

For an orthonormal frame field $E_i$ the Ricci curvature tensor becomes

$$Ricc(X, Y) = \sum_i \varepsilon_i \langle R(E_i, X)Y, E_i\rangle$$

Where $\varepsilon_i = \langle E_i, E_i\rangle$ is the "sign"of $E_i$.

Next, consider a smooth hypersurface $N$ in a semi-Riemannian manifold $M$. We will give the definition of mean curvaure. First, for any two vector fields $V$, $W$ in $N$, the **second fundamental form** of $N$ is defined as

$$II(V, W) = nor(\nabla_V W)$$

where *nor* is the orthogonal projection to the normal bundle of $N$ in $M$. Let $U$ be a unit length local vector field normal to $N$. The **shape operator** of $N$ with respect to $U$ is defined by

$$\langle S(V), W\rangle = \langle II(V, W), U\rangle = \langle \nabla_V W, U\rangle = -\langle \nabla_V U, W\rangle$$

The **mean curvature** of $N$ with respect to $U$ is defined as the trace of the shape operator. which, for an orthonormal frame $\{E_i\}$ of $N$, becomes

$$H_N = trS = \sum \langle \nabla_{E_i} E_i, U\rangle = -\sum \langle \nabla_{E_i} U, E_i\rangle$$

Let $f$ be a real valued smooth function on $M$, and $X$, $Y$ be vector fields. The **Hessian** of $f$, is defined to be the two-form given by

$$H_f(X, Y) = X(Yf) - (\nabla_X Y)f$$

**Proposition 15.** Let $\nabla f$ is the gradient of a smooth function $f$ The Hessian is a symmetric bilinear form such that

$$Hess_f(X,Y) = \langle \nabla_X \nabla f, Y \rangle$$

The Hessian is also $\mathbb{R}$-linear with respect to $f$ and it depends on the values of $X$ and $Y$ only at the point where it is estimated.

*Proof.* Bilinearity is obvious. For the symmetry we have

$$\begin{aligned} Hess_f(X,Y) - Hess_f(Y,X) &= \big(XY - YX\big)f - \big(\nabla_X Y - \nabla_Y X\big)f \\ &= [X,Y]f - [X,Y]f = 0 \end{aligned}$$

Next we have

$$\begin{aligned} \langle \nabla_X \nabla f, Y \rangle &= X\langle \nabla f, Y \rangle - \langle \nabla f, \nabla_X Y \rangle \\ &= X(Yf) - \big(\nabla_X Y\big)f = Hess_f(X,Y) \end{aligned}$$

Using this formula we can also easily see that $Hess_f(X,Y)$ is also $\mathfrak{F}(M)$-linear with respect to $X$ and $Y$, i.e.

$$Hess_f(gX,Y) = Hess_f(X,gY) = gHess_f(X,Y)$$

This shows $Hess_f(X,Y)$ depends on $X$ and $Y$ only on their values on the point of estimation. Finally, it is straightforward to verify $Hess_{af+g} = aHess_f + Hess_g$ for $a \in \mathbb{R}$ $\square$

**Proposition 16.** Let $\gamma$ be the geodesic with initial velocity vector $v$ at the point $p$. Then $Hess_f(v,v)$ is the second derivative of the real function $(f \circ \gamma)$ at zero

$$Hess_f(v,v) = (f \circ \gamma)''(0)$$

*Proof.* Let $X$ be a vector field defined in a neighbourhood of $p$ such that $X$ extends the velocity vector field $\gamma'$ on $\gamma$. We have

$$(f \circ \gamma)'(t) = df(\gamma'(t)) = X(f)\big(\gamma(t)\big)$$

The function $X(f)$ is defined in a neighbourhood of $p$, thus, we can differentiate it along $v = X|_p = \gamma'(0)$ and we get

$$X_p(X(f)) = (X(f) \circ \gamma)'(0) = (f \circ \gamma)''(0)$$

On the other hand

$$Hess_f(v,v) = Hess_f(X,X)|_p = X_p(X(f)) - \nabla_{X_p}X(f) = X_p(X(f))$$

Where $\nabla_{X_p}X = 0$ because $\gamma$ is a geodesic. $\square$

The **Laplacian** $\Delta f$ of a function $f$ is defined as the metric contraction of its Hessian. That means, given a frame $\{e_i\}$ it is

$$\Delta f = trHess_f = \sum g^{ij} Hess_f(e_i, e_j)$$

The following is known as the Bochner formula

**Proposition 17.** Let $f$ be a smooth function on a semi-Riemannian manifold. Then

$$\Delta\left(\frac{1}{2}g\big(\nabla f, \nabla f\big)\right) = \nabla f(\Delta f) + \|Hess_f\|^2 + Ricc(\nabla f, \nabla f)$$

*Proof.* Let $p \in M$ and take an orthonormal frame $\{E_i\}$ near $p$ which is parallel along the integral curve of $\nabla f$ that passes through $p$. We have

$$\begin{aligned} Ricc(\nabla f, \nabla f)(p) &= \sum_i \varepsilon_i \langle R(E_i, \nabla f)\nabla f, E_i\rangle|_p \\ &= \sum_i \varepsilon_i \Big\langle \nabla_{E_i}\nabla_{\nabla f}\nabla f - \nabla_{\nabla f}\nabla_{E_i}\nabla f - \nabla_{[E_i,\nabla f]}\nabla f, E_i \Big\rangle|_p \end{aligned} \tag{1}$$

Where as usual $\varepsilon_i = \langle E_i, E_i\rangle$. From construction of $\{E_i\}$, they satisfy $\nabla_{\nabla f(p)}E_i = 0$ this follows $[E_i, \nabla f](p) = \nabla_{E_i(p)}\nabla f$. This then gives

$$\begin{aligned} \sum_i \varepsilon_i \left\langle \nabla_{[E_i,\nabla f]}\nabla f, E_i\right\rangle|_p &= \sum_i \varepsilon_i \langle \nabla_{\nabla_{E_i}\nabla f}\nabla f, E_i\rangle|_p \text{ (symmetry of } Hess_f) \\ &= \sum_i \varepsilon_i \langle \nabla_{E_i}\nabla f, \nabla_{E_i}\nabla f\rangle|_p \text{ (orthonormality of } \{E_i\}) \\ &= \sum_{ij} \varepsilon_i\varepsilon_j \langle \nabla_{E_i}\nabla f, E_j\rangle^2|_p \\ &= \|Hess_f\|^2(p) \end{aligned} \tag{2}$$

In the calculation we used the symmetric nature of the Hessian and the last identity is due to the fact that for an orthonormal frame, the tensor norm of $Hess_f$ coincides with the Frobenius norm of its components.

$$
\begin{aligned}
\sum_i \varepsilon_i \langle \nabla_{\nabla f} \nabla_{E_i} \nabla f, E_i \rangle|_p &= \sum_i \varepsilon_i \big( \nabla f \langle \nabla_{E_i} \nabla f, E_i \rangle - 0 \big)(p) \\
&= \nabla f|_p \left( \sum_i \varepsilon_i \langle \nabla_{E_i} \nabla f, E_i \rangle \right) \\
&= \nabla f|_p (\Delta f) \qquad (3)
\end{aligned}
$$

Observe that for any vector field $X$

$$
\begin{aligned}
\langle \nabla_{\nabla f} \nabla f, X \rangle &= \langle \nabla_X \nabla f, \nabla f \rangle \\
&= \frac{1}{2} X \Big( g\big(\nabla f, \nabla f\big) \Big) \\
&= \frac{1}{2} \left\langle grad\Big( g\big(\nabla f, \nabla f\big) \Big), X \right\rangle
\end{aligned}
$$

Since $X$ was arbitrary, it follows that $\nabla_{\nabla f} \nabla f = \frac{1}{2} grad\Big( g\big(\nabla f, \nabla f\big) \Big)$. Applying that to the last remaining term gives

$$
\begin{aligned}
\sum_i \varepsilon_i \langle \nabla_{E_i} \nabla_{\nabla f} \nabla f, E_i \rangle|_p &= \sum_i \varepsilon_i \left\langle grad\Big( \frac{1}{2} g\big(\nabla f, \nabla f\big) \Big), E_i \right\rangle |_p \\
&= \Delta \Big( \frac{1}{2} g\big(\nabla f, \nabla f\big) \Big)(p) \qquad (4)
\end{aligned}
$$

Substituting equations (2), (3) and (4) to equation (1) gives the Bochner formula. Since $p$ was arbitrary, the formula holds for any point in the domain of $f$. □

## Some Special Aspects of Lorentz Manifolds

In this subsection we present some topics from the theory of Lorentz manifolds that either reveal essential differences from Riemannian manifolds or require special treatment. Many of these are going to be used in the Lorentzian case of the spitting theorem, but the result about the existence of Lorentz metric is here for informative reasons.

As we saw, the definition of gradient is the same for Lorentz and Riemann manifolds. We are used to thinking that the gradient points at the direction in which the function increases the fastest. That is not the case in general semi-Riemannian manifolds. If $f$ a smooth function on a Lorentz manifold with timelike gradient, then

$$\frac{\nabla f}{\|\nabla f\|}(f) = \left\langle \frac{\nabla f}{\|\nabla f\|}, \nabla f \right\rangle = -\|\nabla f\|$$

On the other hand, for any timelike unit vector $v$, by the reverse Cauchy-Schwartz inequality

$$\begin{aligned} |v(f)| &= \left|\langle v, \nabla f\rangle\right| \\ &\geq \|v\|\|\nabla f\| = \|\nabla f\| \end{aligned}$$

This means that of all the directions where $f$ decreases, it decreases the slowest at $\nabla f$ and, accordingly, of all increasing direction, $f$ increases the slowest in the direction of $-\nabla f$.

With regard to the norm of a vector, the timelike and spacelike components tend to cancel each other out. As a result, from all timelike curves in a normal neighbourhood, the timelike geodesic is the one that actually maximizes the length between two points, rather than minimize it.

Indeed, if $\alpha$ is a timelike curve in a normal neighbourhood and $U = -\nabla r$ is the unit radial vector field (where $r = \sqrt{-t^2 + x_1^2 + ... + x_{n-1}^2}$ in normal coordinates around $\alpha(0)$),

$$\alpha'(t) = -\langle \alpha'(t), U\rangle U + N$$

with $N \perp U$. For the norm of $\alpha'(t)$ this means

$$\begin{aligned} |\alpha'(t)| &= \sqrt{-\langle \alpha'(t), \alpha'(t)\rangle} = \sqrt{\langle \alpha'(t), U\rangle^2 - \langle N, N\rangle} \\ &\leq |\langle \alpha'(t), U\rangle| = -\langle \alpha'(t), U\rangle \\ &= \frac{d(r \circ \alpha)(t)}{dt} \end{aligned}$$

Where the second to last equality holds because at any $t$, $\alpha'(t)$ and $U(\alpha(t))$ have the same causality, hence $\langle \alpha'(t), U\rangle < 0$. With equality iff $N = 0$ iff $\alpha$ is radial. Then

$$L(\alpha) = \int_0^s |\alpha'(t)| dt \leq \int_0^s \frac{d(r \circ \alpha)(t)}{dt} dt = r(\alpha(s))$$

Observe that the right hand side is the length of the radial geodesic segment from $\alpha(0)$ to $\alpha(s)$.

**Lemma 18.** Let $f$ be a smooth function on a Lorentz manifold with unit length timelike gradient. Then the integral curves of $\nabla f$ are locally maximizing timelike geodesics.

*Proof.* Let $\gamma$ be an integral curve of $\nabla f$, from $p$ to $q \in I^+(p)$ in a small enough neighbourhood of $p$.

$$\begin{aligned} f(q) - f(p) = f \circ \gamma(\beta) - f \circ \gamma(\alpha) &= \int_\alpha^\beta (f \circ \gamma)' \\ &= \int_\alpha^\beta df(\gamma')_\gamma = \int_\alpha^\beta \langle \nabla f, \gamma' \rangle_\gamma \\ &= \int_\alpha^\beta -\|\nabla f\|_\gamma^2 \end{aligned}$$

Since $\|\nabla f\| = 1$, this follows $length(\gamma) = -(f(q) - f(p))$. Let $\delta : I \to M$ be another timelike curve from $p$ to $q$, defined on the interval $I$. Since $\delta$ lives in a small neighbourhood of $p$, $\delta$ must have the same causal character as $\gamma$ and $\nabla f$, which means $\langle \nabla f, \delta' \rangle \leq 0$. Thus, from the reverse Cauchy-Schwartz inequality, $\langle \nabla f, \delta' \rangle \leq -\|\nabla f\| \|\delta'\|$ which gives

$$length(\gamma) = -(f(q) - f(p)) = -\int_I \langle \nabla f, \delta' \rangle \geq \int_I \|\nabla f\| \|\delta'\| = length(\delta)$$

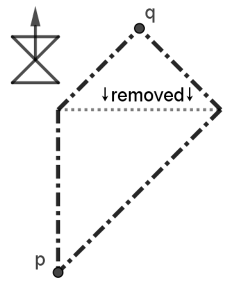
The fact that $\gamma$ maximizes the length between local timelike curves shows it is indeed a locally maximizing geodesic. $\square$

These ideas motivate the definition of the **time-separation** in a time oriented Lorentz manifold

$$d(p, q) = \begin{cases} \sup\{L(\alpha) : \alpha \text{ timelike curve from } p \text{ to } q\} & , \ p \ll q \\ 0 & , \ \text{otherwise} \end{cases}$$

The time separation is well defined, but it is not a distance. It is not symmetric, if $p \not\ll q$, then $d(p, q) = 0$, for non-chronological manifolds there are points with $d(p, p) = \infty$, and $d$ might not even be continuous

**Example 19.** Consider $\mathbb{R}^2_1$ with the interval $[(2, 2), (0, 2)]$ removed. For $p = (0, 0)$ and $q = (1, 3)$, $d$ is discontinuous at $(p, q)$. There are points




arbitrarily close to $q$ that are unaproachable by $p$ via timelike curves, thus the time separation is zero. On the other hand there are points arbitrarily close to $q$ that, by the route on the left, have time separation close to 2.

However, if the manifold is globally hyperbolic, $d$ is continuous and finite valued. This is a a consequence of the following which is proved in [16].

**Lemma 20.** For a globally hyperbolic time oriented Lorentz manifold for two causally related points, there exists a causal geodesic that realises the distance

Consider a normal coordinate neigbourhood $U$ of a point $p$. The time-separation from $p$, $d = d_p = d(\cdot, p)$ is defined in $I^-(p)$ In the intersection $U \cap I^-(p)$ we have the expression $d_p(x) = \sqrt{-\sum \varepsilon_i x_i^2}$ where as usual $\varepsilon_i = \langle \partial_i, \partial_i \rangle$. The minus sign in the coordinate expression is because $\sum \varepsilon_i x_i^2 < 0$ for $x$ in $I^-(p)$. This shows $d_p$ is smooth in $U \cap I^-(p)$. Since $\|\nabla d\| = 1$, the integral curves of $\nabla d$ are unit speed timelike geodesics.

**Proposition 21.** Let $d(x) = d(x, p)$ and $\gamma$ be an integral curve of $\nabla d$, starting from $q \in I^-(p)$ in a normal coordinate neighbourhood of $p$. If $r = d(q)$, then $\gamma(t) \underset{t \to r}{\longrightarrow} p$ and

$$\Delta d(\gamma(t)) \underset{t \to r}{\longrightarrow} +\infty$$

*Proof.* Take $v' \in T_pM$ such that $\exp(v') = q$, and let $v = \frac{v'}{\|v'\|} = \sum v_i \partial_i(p)$. Consider the radial unit speed geodesic $\delta(t) = \exp\big((r-t)v\big)$. We show that the integral curve $\gamma$ is the radial geodesic $\delta$. In coordinates $\delta(t) = \big((r-t)v_1, ..., (r-t)v_n\big)$. This gives $d(\delta(t)) = r - t$ and

$$\delta'(t) = -\sum v_i \partial_i$$

while at the same time

$$\nabla d = \sum \varepsilon_i \partial_i(d)\partial_i = \sum -\varepsilon_i^2 \frac{x_i}{d}\partial_i = -\sum \frac{x_i}{d}\partial_i$$

Then we compute the gradient along $\delta$

$$\nabla d(\delta(t)) = -\sum \frac{(r-t)v_i}{r-t}\partial_i = \delta'(t)$$

Thus $\delta$ is indeed an integral curve of $\nabla d$. The uniqueness of geodesics gives $\delta = \gamma$, and by construction $\gamma(t) \underset{t\to r}{\longrightarrow} p$ Now we move on to the computation of the Laplacian of the time separation.

$$\Delta d = \sum_{ij} g^{ij} Hess_d(\partial_i, \partial_j) = \sum_{ij} g^{ij}\Big(\partial_i\partial_j d - \sum_k \Gamma_{ij}^k \partial_k d\Big) \tag{1}$$

Calculating this along $\gamma$ while keeping in mind that $\Gamma_{ij}^k(p) = 0$ we have

$$\begin{aligned}\sum_{ijk} \Gamma_{ij}^k \partial_i d(\gamma(t)) = \sum_{ijk} -\Gamma_{ij}^k \frac{\varepsilon_i x_i}{d}(\gamma(t)) = \sum_{ijk} -\Gamma_{ij}^k(\gamma(t))\frac{\varepsilon_i(r-t)v_i}{r-t} \\ = \sum_{ijk} -\Gamma_{ij}^k(\gamma(t))\varepsilon_i v_i \underset{t\to r}{\longrightarrow} 0\end{aligned} \tag{2}$$

and

$$\sum_{ij} g^{ij}\partial_i\partial_j d = \sum_{ij}(g^{ij} - \varepsilon_i\delta_{ij})\partial_i\partial_j d + \sum_{ij}\varepsilon_i\delta_{ij}\partial_i\partial_j d$$

Since $|g^{ij} - \varepsilon_i\delta_{ij}| \le C\sum x_k^2$ for some $C > 0$ near $p$, along $\gamma$ this gives $|g^{ij} - \varepsilon_i\delta_{ij}|(\gamma(t)) \le C\sum (r-t)^2 v_k^2 = C_0(r-t)^2$ Therefore

$$\begin{aligned}\sum_{ij}(g^{ij} - \varepsilon_i\delta_{ij})\partial_i\partial_j d(\gamma(t)) &\le C_0\sum_{ij}(r-t)^2\partial_i\partial_j d(\gamma(t)) \\ &= C_0\sum_{ij}(r-t)^2\left(\frac{\varepsilon_j\delta_{ij}}{d} - \frac{\varepsilon_i x_i \varepsilon_j x_j}{d^3}\right)(\gamma(t)) \\ &= C_0\sum_{ij}(r-t)^2\left(\frac{\varepsilon_j\delta_{ij}}{r-t} - \frac{\varepsilon_i(r-t)v_i\varepsilon_j(r-t)v_j}{(r-t)^3}\right) \\ &= C_0\left(\sum_{ij}\varepsilon_j\delta_{ij} - \varepsilon_i v_i\varepsilon_j v_j\right)(r-t) \underset{t\to r}{\longrightarrow} 0\end{aligned} \tag{3}$$

Finally calculating the last term

$$\begin{aligned}\sum_{ij} \varepsilon_i \delta_{ij} \partial_i \partial_j d(\gamma(t)) &= \sum_i \varepsilon_i \left( \frac{\varepsilon_i}{d} - \frac{\varepsilon_i^2 x_i^2}{d^3} \right) (\gamma(t)) \\ &= \sum_i \left( \frac{1}{r-t} - \frac{\varepsilon_i v_i^2}{r-t} \right) \\ &= \frac{n-1}{r-t} \underset{t \to r}{\longrightarrow} +\infty \end{aligned} \tag{4}$$

Applying $(2), (3)$ and $(4)$ to $(1)$ gives $\Delta d(\gamma(t)) \underset{t \to r}{\longrightarrow} +\infty$. □

A very important theorem in Riemann geometry is the existence of a Riemannian structure in any smooth manifold. The same is not true for Lorentz metrics. The following theorem was first proved by L. Markus in [14]. More details can be found in [16].

**Theorem 22.** A smooth manifold admits a Lorentz metric iff it is either non-compact or the Euler characteristic vanishes $\chi(M) = 0$

*Scetch of proof.* It is a known result that non-compact manifolds admit nowhere vanishing vector fields (see theorem 4.8 of [11]). For compact manifolds the existence of such vector field is equivalent to $\chi(M) = 0$ (see [12]). In any case we have a non-vanishing vector field $X$ Lets consider an auxiliary Riemannian metric $g$ on $M$. Without loss of generality we assume that $|X| = 1$, then we can take the $(0, 2)$ form $g - 2\theta \otimes \theta$, where $\theta = g(\cdot, X)$. This can be proved to be a Lorentz metric.

For the inverse, if $M$ is orientable, there is a timelike vector field. Hence if $M$ is compact, it must have $\chi(M) = 0$.

If is not time orientable, we can create a two sheeted covering $\tilde{M}$, by taking $\tilde{M}$ to be the set of timecones for each point of $M$. This $\tilde{M}$ can be equipped with a Lorentz metric and is time orientable. Therefore, there is a non-vanishing vector field on $\tilde{M}$. The proof is completed by observing that $\tilde{M}$ is compact iff $M$ is compact and $\chi(\tilde{M}) = 2\chi(M)$ □

# Chapter 2

# The Splitting Theorems

## 2.1 The Riemannian Case

Here we will prove the splitting theorem for Riemannian manifolds, which states that under certain curvature condidions, we can 'split' the Riemmanian manifold in a Cartesian product of $\mathbb{R}^m$, for some $m \leq n$, and some Riemannian manifold $M'$ which contains no lines. This is done by induction removing one line at a time. Several versions of this result were proved in special cases or with stronger conditions. The result as presented here was first proved by Cheeger and Gromoll [5], but their proof uses some heavy machinery and does not provide much insight in the Lorentz case. We follow the proof given by Eschenburg and Heintze [7].

### Busemann Functions

**Definition 23.** A **line** (resp. **ray**) is a geodesic curve $\gamma : \mathbb{R} \to M$ (resp. $\gamma : [0, \infty) \to M$ which preserves length

$$d(\gamma(r), \gamma(r')) = |r - r'|$$

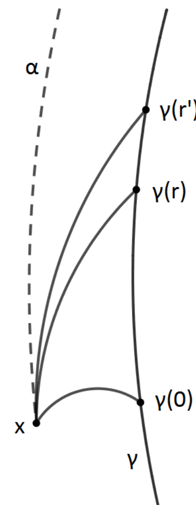
where $d$ denotes the Riemannian distance.

Consider a ray $\gamma$ starting from $p$ and for each $n \in \mathbb{N}$ let $v_n$ be the initial unit velocity of a minimal geodesic from $p$ to $\gamma(n)$. From the compactness of the unit sphere of $T_pM$, there is a subsequence which converges to some $v$

**Definition 24.** A ray with initial velocity such a $v$ is called an **asymptote** of $\gamma$

Let $\gamma : [0, \infty) \to M$ be a ray in a Riemannian manifold $M$. For $r \geq 0$ we define the functions

$$b_r(x) = r - d(x, \gamma(r))$$




**Lemma 25.** The directed family $\{b_r\}_{r \geq 0}$ is pointwise non-decreasing and bounded. The pointwise limit, $b$, is Lipschitz continuous

*Proof.* For $r \leq r'$ we have from the triangle inequality

$$\begin{aligned} b_r(x) - b_{r'}(x) &= r - r' - d(x, \gamma(r)) + d(x, \gamma(r')) \\ &\leq r - r' + d(\gamma(r), \gamma(r')) = 0 \end{aligned}$$

Again from the triangle inequality we have

$$b_r(x) = d(\gamma(0), \gamma(r)) - d(x, \gamma(r)) \leq d(x, \gamma(0))$$

Thus the functions $b_r$ converge pointwise to some function $b$. The Lipschitz continuity of $b$ again follows from the triangle inequality.

$$\begin{aligned} |b(x) - b(y)| &= \lim_{r \to \infty} |b_r(x) - b_r(y)| \\ &= \lim_{r \to \infty} |r - r - d(x, \gamma(r)) + d(y, \gamma(r))| \\ &\leq \lim_{r \to \infty} |d(x, y)| = d(x, y) \end{aligned}$$

$\square$

The function $b = \lim b_r$ is called the **Busemann** function associated to the ray $\gamma$ and the functions $b_r$ are called the **pre-Busemann** functions. The Busemann function need not behave nicely, but it has a simple formula on an asymptote of $\gamma$

**Lemma 26.** Let $\alpha$ be an asymptote of $\gamma$. Then

$$b(\alpha(t)) = t + b(\alpha(0))$$

*Proof.* Let $p = \alpha(0)$, $v$ be the initial velocity of $\alpha$, and $v_n$ the initial velocities of the minimal geodesics from $p$ to $\gamma(n)$ with $v_n \to v$

Let $\varepsilon > 0$ and fix some $t$, there is $n$ such that

- $|b(p) - \big(n - d(p, \gamma(n))\big)| < \varepsilon$
- $|d\big(exp(tv), \gamma(n)\big) - d\big(exp(tv_n), \gamma(n)\big)| < \varepsilon$
- $|b(exp(tv)) - \big(n - d(exp(tv), \gamma(n))\big)| < \varepsilon$

We have

$$\begin{aligned} |b(p) + t - b(\alpha(t))| &\leq \varepsilon + |n - d(p, \gamma(n)) + t - b(\alpha(t))| \\ &= |n - d\big(exp(tv_n), \gamma(n)\big) - b(\alpha(t))| + \varepsilon \\ &\leq |d\big(exp(tv), \gamma(n)\big) - d\big(exp(tv_n), \gamma(n)\big)| + \\ &\qquad |n - d\big(exp(tv), \gamma(n)\big) - b(\alpha(t))| + \varepsilon \\ &\leq 3\varepsilon \end{aligned}$$

Since $\varepsilon$ was arbitrary, $b(\alpha(t)) = t + b(\alpha(0))$ □

Now consider the case where $\gamma$ is a line. It consists of two rays, $\gamma^+$ and $\gamma^-$, and the corresponding Busemann functions are $b^+$ and $b^-$

**Lemma 27.** For a line $\gamma$ it is $b^+ + b^- \leq 0$ and $b^+ + b^- = 0$ on $\gamma$

*Proof.*

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{r\to\infty} \big(2r - d(x, \gamma^+(r)) - d(x, \gamma^-(r))\big) \\ &= \lim_{r\to\infty} \big(2r - d(x, \gamma(r)) - d(x, \gamma(-r))\big) \\ &\leq \lim_{r\to\infty} \big(2r - d(\gamma(r), \gamma(-r))\big) \\ &= 0 \end{aligned}$$

Where the inequality becomes equality when $x$ lies on the line, for big enough $r \geq 0$ □

A problem with the Busemann functions is that they might not be differentiable. This makes working with them challenging. It will instead prove usefull to use support functions. These have good differential behaviour and are still closelly related to the original function.

Consider an asymptote of the ray $\gamma$ with initial velocity $v$, starting at the point $p$. We define the functions

$$b_{p,r}(x) = b(p) + r - d(x, exp(rv))$$

**Lemma 28.** The functions $\{b_{p,r}\}$ are support functions of $b$, at the point $p$

*Proof.*

$$b_{p,r}(p) = b(p) + r - d(p, exp(rv)) = d(p)$$

$$\begin{aligned} b(x) + d(x, exp(rv)) &= \lim_{t\to\infty} \big(t - d(x, \gamma(t)) + d(x, exp(rv))\big) \\ &\geq \lim_{t\to\infty} \big(t - d(exp(rv), \gamma(t))\big) \\ &= b\big(exp(rv)\big) = b(p) + r \end{aligned}$$

Thus $b(x) \geq b_{p,r}(x)$ □

## Preliminaries

**Lemma 29.** Let $f$ be a smooth function on a Riemannian manifold with $\|\nabla f\| = 1$. Then the integral curves of $\nabla f$ are minimizing geodesics

The proof is similar to the analogous result (lemma 18) for Lorentz manifolds.

*Proof.* Let $\gamma : (\alpha, \beta) \to M$ be an integral curve of $\nabla f$, from $p$ to $q$.

$$\begin{aligned} f(q) - f(p) = f \circ \gamma(\beta) - f \circ \gamma(\alpha) &= \int_\alpha^\beta (f \circ \gamma)' \\ &= \int_\alpha^\beta df(\gamma')_\gamma = \int_\alpha^\beta \langle \nabla f, \gamma' \rangle|_\gamma \\ &= \int_\alpha^\beta \|\nabla f\|^2|_\gamma \end{aligned}$$

Since $\|\nabla f\| = 1$, this follows $length(\gamma) = f(q) - f(p)$. Let $\delta : I \to M$ be another curve from $p$ to $q$, defined on the interval $I$. From the Cauchy-Schwartz inequality $\langle \nabla f, \delta' \rangle \leq \|\nabla f\| \|\delta'\|$ which gives

$$length(\gamma) = f(q) - f(p) = \int_I \langle \nabla f, \delta' \rangle \leq \int_I \|\nabla f\| \|\delta'\| = length(\delta)$$

The fact that $\gamma$ minimizes the length shows it is indeed a minimizing geodesic. □

**Lemma 30.** Let $f$ be a smooth function with $\|\nabla f\|$ constant, on a Riemannian manifold. Then

$$\begin{aligned} -Ricc(\nabla f, \nabla f) &= \nabla f(\Delta f) + \|Hess_f\|^2 \\ &\geq \nabla f(\Delta f) + \frac{(\Delta f)^2}{n-1} \end{aligned}$$

*Proof.* The equality follows immediately from the Bochner formula (proposition 17). Just observe that $(\Delta f \circ \gamma)' = \gamma'(\Delta f)$ and $\Delta\left(\frac{1}{2}\|\nabla f\|^2\right) = 0$.

Notice that since the gradient of $f$ has constant norm, it is parallel to itself. We take an orthonormal frame $\{E_i\}$, such that $E_n = \frac{\nabla f}{\|\nabla f\|}$. Since $\|\nabla f\|$ is constant, $E_n$ is parallel along itself and since $Hess_f$ is symmetric, the non-zero entries of the Hessian matrix lie in an $(n-1) \times (n-1)$ submatrix. Treating $Hess_f$ as an $(n-1)^2$ vector, we can use the canonical inner product of $\mathbb{R}^{(n-1)^2}$ to write the trace of $Hess_f$ as

$$tr(Hess_f) = \langle Hess_f, I_{n-1} \rangle$$

Then from the Cauchy-Schwartz inequality we get

$$\Delta f = tr(Hess_f) \leq \|Hess_f\| \|I_{n-1}\| = \sqrt{n-1} \|Hess_f\|$$

Applying that to the equation gives the conclusion. □

The following result is analogous to proposition 21. The proof of this however is somewhat simpler.

**Proposition 31.** Let $p$ a point in $M$, and $d(x) = d(p, x) = \|exp_p - 1(x)\|$ the Riemannian distance. $d$ is smooth in a small neighbourhood around $p$ (except at $p$) and

$$\Delta d(x) \xrightarrow[x \to p]{} +\infty$$

*Proof.* Take a normal coordinate neighbourhood centered around $p$. In coordinates $d(x) = \sqrt{\sum x_i^2}$, which is smooth in a neighbourhood around $p$, excluding $p$, and gives $\partial_i d = \frac{x_i}{d}$. Also, because we use normal coordinates, $\Gamma_{ij}^k(p) = 0$ and there is some $C > 0$ such that

$$|g^{ij} - \delta_{ij}| \leq Cd^2$$

The coordinate expression of the Laplacian is

$$\Delta d = \sum_{ij} g^{ij} \left( \partial_i \partial_j d - \sum_k \Gamma_{ij}^k \partial_k d \right)$$

First we calculate the Christoffel symbol term

$$\begin{aligned} \left| \sum_{ij} g^{ij} \Gamma_{ij}^k \partial_k d \right| &= \left| \sum_{ij} g^{ij} \Gamma_{ij}^k \frac{x_i}{d} \right| \\ &\leq \sum_{ij} \left| g^{ij} \Gamma_{ij}^k \right| \xrightarrow[x \to p]{} 0 \end{aligned}$$

The other term we split in two, as in the Lorentzian proof

$$\sum_{ij} g^{ij} \partial_i \partial_j d = \sum_{ij} (g^{ij} - \delta_{ij}) \partial_i \partial_j d + \sum_{ij} \delta_{ij} \partial_i \partial_j d$$

For the first of those therms we get

$$\begin{aligned} \left| \sum_{ij} (g^{ij} - \delta_{ij}) \partial_i \partial_j d \right| &\leq C \sum_{ij} d^2 \left| \frac{\delta_{ij}}{d} - \frac{x_j x_i}{d^3} \right| \\ &= C \sum_{ij} \left| \delta_{ij} d - \frac{x_i x_j}{d} \right| \\ &\leq C \sum_{ij} |\delta_{ij} d - x_i| \xrightarrow[x \to p]{} 0 \end{aligned}$$

Finally, the last term becomes

$$\begin{aligned} \sum_{ij} \delta_{ij} \partial_i \partial_j d &= \sum_i \left( \frac{1}{d} - \frac{x_i^2}{d^3} \right) \\ &= \frac{n-1}{d} \xrightarrow[x \to p]{} +\infty \end{aligned}$$

$\square$

**Corollary 32.** For a complete Riemannian manifold of non-negative Ricci curvature, if $d_p$ is the Riemannian distance from $p$, then

$$\Delta d_p \leq \frac{n-1}{d_p}$$

*Proof.* Let $c$ be an integral curve of $\nabla d_p$ with $c(0) = p$. We take $\phi = \Delta d_p \circ c$. Then from the above proposition 31, $\lim_{t \to 0} \phi(t) = \infty$, and from lemma 30, $\frac{\phi'}{\phi^2} + \frac{1}{n-1} \leq 0$. We integrate $\left(\frac{1}{\phi}\right)'$

$$\int_0^r -\frac{\phi(t)'}{\phi^2(t)} dt \geq \int_0^r \frac{1}{n-1} dt$$
$$\int_0^r \left(\frac{1}{\phi(t)}\right)' dt \geq \frac{r}{n-1}$$
$$\frac{1}{\phi(r)} - \frac{1}{\phi(0)} \geq \frac{r}{n-1}$$

As we mentioned $\phi(0) = \infty$, therefore $\phi(r) \leq \frac{n-1}{r}$ □

**Lemma 33.** Let $\alpha$ be a line on a complete Riemannian manifold with non-negative Ricci curvature, then for any $t \in \mathbb{R}$

$$\lim_{r \to \infty} Hess_{b_r^\pm}(\alpha(t)) = 0$$

*Proof.* Let $t \in \mathbb{R}$, $v \in T_{\alpha(t)}M$ and $H_r^\pm(t) = Hess_{b_r^\pm}(\alpha(t))$. From the polarization property, the Hessian is equivalent to $h_r^\pm(t)(v) = Hess_r^\pm(t)(v, v)$ and proposition 16 says $h_r^\pm(v) = (b_r^\pm \circ \gamma_v)''(0)$, where $\gamma_v$ is the geodesic with initial velocity $v$. From lemma 25 $(b_r^\pm)_r$ is non-decreasing with $r$, which combined with lemma 27 implies that for any $r$, $b_r^+ \leq -b_r^-$, with equality along $\alpha$. It follows that the second derivatives of $(b_r^\pm \circ \gamma_v)_r$ at $t$ are non-decreasing with $r$ and that $(b_r^+ \circ \gamma_v)''(t) \leq -(b_r^- \circ \gamma_v)''(t)$. Therefore, as $r \to \infty$ we can define

$$h^\pm(v) = \lim h_r^\pm(v)$$

As a limit of quadratic forms, $h^\pm$ is itself a quadratic form, which defines a bilinear form $H^\pm$. This bilinear form is also the limit of $H_r^\pm$ as $r \to \infty$.

We have $h^+ \leq -h^-$ which implies $trH^+ \leq -trH^-$. On the other hand,

$$trH_r^\pm(t) = \Delta b_r^\pm(\alpha(t)) = -\Delta d_{\alpha(t)}(\alpha(r)) \geq -\frac{n-1}{d_{\alpha(t)}(\alpha(r))} \xrightarrow[r \to \infty]{} 0$$

This gives $trH^{\pm} \geq 0$, which in combination with $trH^+ \leq -trH^-$, gives $trH^{\pm} = 0$. Specifically, $(trH^+ + trH^-) = \sum (h^+ + h^-)(E_i) = 0$ for some orthonormal frame $\{E_i\}$. Since $h^+ + h^- \leq 0$ it follows that $(h^+ + h^-)(E_i) = 0$ for all $i$. This implies that $H^+ + H^- = 0$

If $H^+(t) \neq 0$, there is a vector field $V$ around $\alpha(t)$ such that $h^+(t)(V) \neq 0$ and, for big enough $r$, $h_r^+(t)(V) \neq 0$. From the continuity of $h_r^+$ with respect to $t$, we can find an interval $I$ around $t$, such that $\|H_r^+\|^2 \geq |h_r^+(V)|^2 / \|V\|^4 > \varepsilon$. The monotonicity with respect to $r$ secures this holds for all larger $r$. Lemma 30 gives $\frac{d}{dt}(trH_r^+) \leq -\|H_r^+\|^2 < -\varepsilon$. But acording to the above $trH_r^+ \xrightarrow[r \to \infty]{} 0$. This contradiction completes the proof. □

Before we go on with the splitting theorem itself, we will prove a version of the maximum principle due to Calabi [3]

**Lemma 34.** (The Hopf-Calabi Maximum Principle) Let $M$ be a connected Riemannian manifold, $f \in C^0(M)$ such that $\forall\, p \in M\ \forall \varepsilon > 0\ \exists f_{p,\varepsilon} \in C^2(M)$ support function with $\Delta f_{p,\varepsilon} \geq -\varepsilon$. Then $f$ attains no maximum or it is constant.

*Proof.* We prove the set of points where $f$ attains maximum is open and closed, therefore $f$ is either constant or has no maximum. As the inverse image of a singleton, the set is closed.

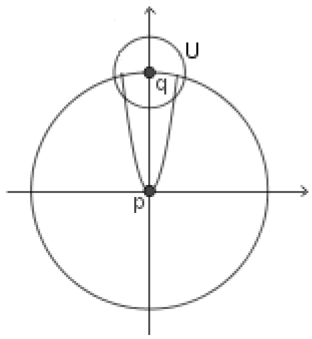
Let $f$ attain local maximum at a point $p$ where $f$ is not constant in any neighbourhood of $p$. Choose a neighbourhood $U$ of $p$, diffeomorphic to an open ball $B$ of $\mathbb{R}^n$. Let $K = \{x \in \partial B | f(x) = f(p)\}$

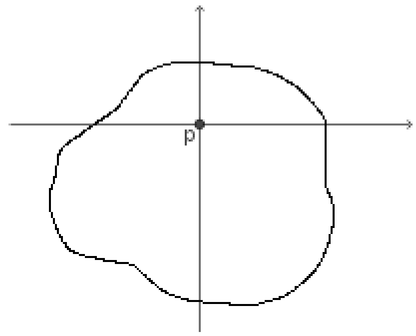
We claim there is a real function $\phi$ defined in a neighbourhood of $\overline{B}$ such that

$$\begin{aligned} &\phi(p) = 0 \\ &\phi|_K < 0 \\ &\nabla\phi \neq 0 \text{ at } \overline{B} \end{aligned}$$

Indeed, let $q \in \partial B \backslash K$, Since $K$ is a closed set, there is an open neighbourhood $U$ of $q$ which does not intersect it. For an appropriate coordinate system $(x_1, ..., x_n)$, we can have $x_1(p) = 0$ and the pre-image $x_1^{-1}(\mathbb{R}^+) \cap \partial B \subset U$. Thus $\phi = x_1$ is an appropriate function.

Now we define $h = e^{\alpha\phi} - 1$, with $\alpha > -\frac{\Delta\phi}{\|\nabla\phi\|^2}$. Notice that since $\overline{B}$ is compact, both $\Delta\phi$ and $\|\nabla\phi\|$ are bounded on $\overline{B}$. This means the number $\alpha$






is well defined. This $h$ satisfies

$$
\begin{aligned}
h(p) &= 0 \\
h|_K &< 0 \\
\Delta h &> 0 \text{ on } B
\end{aligned}
$$

The first two are clear. Lets consider an orthonormal frame $\{E_i\}$. Observe that $E_i(e^{\alpha\phi}) = d(e^{\alpha\phi})(E_i) = \alpha E_i(\phi)e^{\alpha\phi}$ Now we calculate the Laplacian

$$
\begin{aligned}
\Delta(e^{\alpha\phi} - 1) = \Delta(e^{\alpha\phi}) &= \sum_i \big(E_i E_i(e^{\alpha\phi}) - (\nabla_{E_i} E_i)(e^{\alpha\phi})\big) \\
&= \big(\alpha\Delta\phi + \alpha^2\|\nabla\phi\|^2\big)e^{\alpha\phi}
\end{aligned}
$$

Which is positive since we chose $\alpha > -\frac{\Delta\phi}{\|\nabla\phi\|^2}$. $h$ might have a positive maximum at $\partial B$ but since $\partial B$ is compact, for a small enough $\eta > 0$ we get $(f + \eta h)(x) < f(p)$ for any $x \in \partial B$. At the same time $(f + \eta h)(p) = f(p)$, therefore $(f + \eta h)$ attains maximum at some point $z$ in the interior of $B$. But $\Delta(f_{z,\varepsilon} + \eta h) \geq \eta\Delta h - \varepsilon$, which is positive for small enough $\varepsilon$. This contradicts the fact that the Hessian must be negative-definite at a local maximum. Thus $f$ is maximised in a neighbourhood of $p$. $\square$

## The Splitting Theorem

**Theorem 35.** Let $(M, g)$ be a complete connected Riemannian manifold with non-negative Ricci curvature, and $\gamma : \mathbb{R} \to M$ be a line. If $b^{\pm}$ are the Busemann functions associated with the rays $\gamma^{\pm}$, then $M$ is isometric with $\mathbb{R} \times (b^+)^{-1}(0)$

*Proof.* Let $p$ be a point of $M$. Consider two aymptotes of $\gamma^+$ and $\gamma^-$ staring at $p$ with initial velocities $v^+$ and $v^-$. First we prove that the Busemann functions $b^\pm$ are differentiable on $p$, with $\|\nabla b^\pm\| = 1$. We take the support functions $b^\pm_{p,r}$ of $b^\pm$ with respect to those initial velocities. We have that $(b^+_{p,r} + b^-_{p,r})$ is a support function of $b^+ + b^-$. From lemma 32 we get

$$\begin{aligned}\Delta(b^+_{p,r} + b^-_{p,r}) &= -\Delta\Big(d_p(\exp(rv^+)) + d_p(\exp(rv^-))\Big) \\ &\geq -\frac{2(n-1)}{r}\end{aligned}$$

By lemma 27 the function $(b^+ + b^-)$ attains maximum on the points of $\gamma$. From the maximum principle $b^+ + b^- = 0$ everywhere. We have $b^+_{p,r} \leq b^+ = -b^- \leq -b^-_{p,r}$ with equality at p, therefore $b^+, b^-$ are differentiable at $p$ with $\nabla b^+(p) = \nabla b^+_{p,r}(p) = -\nabla b^-_{p,r}(p) = -\nabla b^-(p)$, and since $p$ was arbitrary, $b^\pm$ are differentiable in $M$.

The gradient of the support functions is

$$\nabla b^\pm_{p,r}|_p = \nabla(b^+(p) + r - d(p, \exp(rv^\pm)))|_p = -\nabla\rho_p(\exp(rv^\pm))|_p = -v^\pm.$$

Therefore, $v^+ = -v^-$, which means there is only one asymptote per orientation of $\gamma$ and they fit together to make a line passing from $p$. From the above we can also see that the gradient of the Busemann function at $p$ has $\|\nabla b^\pm\| = 1$ since $v^\pm$ is initial velocity for asymptote.

Next we will prove the Busemann functions are "linear", meaning they have vanishing Hessian. If $\beta^\pm_r$ are the pre-Busemann functions of the new line, then $\beta^\pm_r = b^\pm_{p,r} - b^\pm(p)$. Hence from lemma 33

$$\lim_{r\to\infty} Hess(b^\pm_{p,r}(p)) = \lim_{r\to\infty} Hess(\beta^\pm_r + b^\pm(p)) = 0$$

Consider a geodesic $c$ of $M$ and take $b^\pm \circ c : I \to \mathbb{R}$. The above means there are support functions for $b^\pm \circ c$ with arbitrarily small second derivative. From the maximum principle for one dimention we get that $b^\pm \circ c$ is either constant or has no maximum. The same holds for $(b^\pm \circ c - l)$ with $l : I \to \mathbb{R}$ any affine function. This can be shown to be true only when $b^\pm \circ c$ is convex. And since $b^+ = -b^-$, $b^\pm \circ c$ are both convex and concave. In othe words $b^\pm \circ c$ is affine. Therefore, for the initial velocity $v = c'(0)$ of $c$, $Hess_{b^\pm}(v, v) = (b^\pm \circ c)''(0) = 0$. Since the geodesic was arbitrary, this last identity holds for any tangent vector. From polarization it follows that $Hess_{b^\pm} = 0$. This in turn means the Lie derivative of the metric along $\nabla b^\pm$ vanishes $\mathfrak{L}_{\nabla b} g = 0$. Hence the flow $\Phi$ of $b$ acts by isometries. The integral curves of $\nabla b^\pm$ are lines composed of two asymptotes as we saw above. This means they are defined on all of $\mathbb{R}$, and also $b(\Phi(t, x)) = t + b(x)$. Using these we take a map $F : \mathbb{R} \times b^{-1}(0) \to M$ which is a diffeomorphism and an isometry along level sets and the integral curves. $\square$

## 2.2 The Lorentzian Case

The interest and success of the Riemannian splitting theorem led S.T. Yau to formulate an analogue for the Lorentzian Geometry in 1982. Specifically, he speculated that a Lorentzian manifold $M$ with non-negative Ricci curvature that is timelike geodesically complete and has a timelike line, splits isometrically as a cartesian product $\mathbb{R}^1_1 \times M'$, where $M'$ is a Riemannian manifold. Various versions of splitting were proved for Lorentzian manifolds using stronger conditions.

In 1988 J. Eschenburg proved a splitting theorem for space-times which were timelike geodesically complete and globally hyperbolic. In 1989 G. Galloway proved the thorem without the timelike completeness assumption. In 1990 R. Newman finally proved the original conjecture by Yau

Here we study the theorem as proved by Eschenburg, but in proving it we will use techniques utilized by Galloway in his proof. Like in the Riemannian proof, we make heavy use of support functions

### Lorentz Busemann Functions

In this section $(M, g)$ denotes a time oriented, globally hyperbolic, timelike geodesically complete Lorentz manifold. We will require some objects analogous to the Riemannian case. A timelike ray (line) is a timelike geodesic defined on $[0, \infty)$ (resp. on $\mathbb{R}$) that respects the Lorentz distance. In defining the Busemann function with respect to a timelike ray $\gamma$, in order for $d(x, \gamma(r))$ to make sense $x$ must lie in the past of $\gamma$ and since we also want to have $d(\gamma(0), x)$ we also require that $x$ lies in the future of $\gamma(0)$. thus we define for $x \in I^+(\gamma(0)) \cap I^-(\gamma) = I(\gamma)$

$$b(x) = \lim_{r\to\infty} \big(r - d(x, \gamma(r))\big)$$

**Lemma 36.** $b(p) \geq d(p,q) + b(q)$

*Proof.* From the revesre triagle inequality, the pre-Busemann functions $b_r(x) = r - d(x, \gamma(r))$ are non-increasing and bounded from bellow by $d(\gamma(0), x)$. Thus the Busemann function $b$ is indeed well defined. Also by the reverse triangle inequality, for $p \ll q \ll \gamma(r)$ we have

$$\begin{aligned} b_r(q) &= r - d(q, \gamma(r)) \\ &\geq r - d(p, \gamma(r)) + d(p,q) \\ &= b_r(p) + d(p,q) \end{aligned}$$

, which, taking limits becomes

$$b(q) \geq d(p,q) + b(p)$$

□

This is the Lorentzian equivalent of the Riemannian Busemann function being Lipschitz continuous, so we will call this property "reverse Lipschitz" property, even though this has nothing to do with continuity at all.

Asympotes from $p$ can also be defined as limits of maximal timelike geodesic segments from $p$ to $\gamma(r_n)$ with $r_n \to \infty$, though as the limit of timelike curves need not be timelike, asymptotes to $\gamma$ might be timelike or lightlike. As in the Riemannian case, the Busemann function has a nice behaviour on asymptotes

**Lemma 37.** If $\alpha$ is an asymptote to $\gamma$ and $b$ is its Busemann function, then

$$b(\alpha(t)) = t + b(\alpha(0))$$

*Proof.* The inequality $b(\alpha(t)) \geq t + b(\alpha(0))$, follows immediately from the "reverse Lipschitz" property above. For the reverse inequality, take maximal segments $\alpha_s : [0, r_s] \to M$ from $\alpha(0)$ to $\gamma(s)$ with $\alpha_s \to \alpha$ as $s \to \infty$. Notice that $r_s \to \infty$. This means that if $u > 0$ then for all big enough $s > 0$, $\alpha_s(u)$ is defined with $\alpha_s(u) \to \alpha(u)$

Fix some $t > 0$. We will prove the inequality for $\alpha(t)$. Take $u > t$. Then $\alpha_s(u) \to \alpha(u) \in I^+(\alpha(t))$ which is open. Therefore, for all big enough $s$,

$\alpha_s(u) \in I^+(\alpha(t))$ and

$$\begin{aligned}
b_s(\alpha(t)) &= s - d(\alpha(t), \gamma(s)) \\
&\leq s - d(\alpha(t), \alpha_s(u)) - d(\alpha_s(u), \gamma(s)) \\
&= s - d(\alpha_s(u), \alpha_s(r_s)) - d(\alpha(t), \alpha_s(u)) \\
&= s - r_s + u - d(\alpha(t), \alpha_s(u)) \\
&= s - d(\alpha(0), \alpha_s(r_s)) + u - d(\alpha(t), \alpha_s(u)) \\
&= s - d(\alpha(0), \gamma(s)) + u - d(\alpha(t), \alpha_s(u)) \\
&= b_s(\alpha(0)) + u - d(\alpha(t), \alpha_s(u))
\end{aligned}$$

letting $s \to \infty$, this becomes

$$b(\alpha(t)) \leq b(\alpha(0)) + u - d(\alpha(t), \alpha(u)) = b(\alpha(0)) + t$$

□

The proof of Lemma 30 can be transfered to Lorentz manifolds. This gives us the following

**Lemma 38.** Let $f$ be a smooth function on a Lorentz manifold with $\|\nabla f\| =$ constant. Then

$$\begin{aligned}
-Ricc(\nabla f, \nabla f) &= \nabla f(\Delta f) + \|Hess_f\|^2 \\
&\geq \nabla(\Delta f) + \frac{(\Delta f)^2}{n-1}
\end{aligned}$$

As in the Riemannian case, the identity is immediate from the Bochner formula (proposition 17) and the inequality follows from a Caushy-Schwartz inequality

**Lemma 39.** Let $p$ be a point in a Lorentz manifold with non-negative Ricci curvature and $d = d_p$ be the time separation from $p$, defined in $I^-(p)$ and smooth in a small enough neighbourhood of $p$. Then

$$\Delta d_p \geq -\frac{n-1}{d_p}$$

*Proof.* Let $c$ be an integral curve of $\nabla d$ starting at $c(0) = q$. Then if $r = d(q)$, $c(t)$ tends to $p$ as $t$ tends to $r$. We take $\phi = \Delta d_p \circ c$. Then from proposition

21, $\lim_{t\to r} \phi(t) = \infty$, and from the above lemma 38, $\frac{\phi'}{\phi^2} + \frac{1}{n-1} \le 0$. We integrate $\left(\frac{1}{\phi}\right)'$

$$\begin{aligned}\int_0^r -\frac{\phi(t)'}{\phi^2(t)}dt &\ge \int_0^r \frac{1}{n-1}dt\\ \int_0^r \left(\frac{1}{\phi(t)}\right)' dt &\ge \frac{r}{n-1}\\ \frac{1}{\phi(r)} - \frac{1}{\phi(0)} &\ge \frac{r}{n-1}\end{aligned}$$

It follows that $\phi(0) \ge -\frac{n-1}{r}$ □

**Lemma 40.** Assume $Ricc(v, v) \ge 0$ for all timelike vectors $v$. If the Busemann function is smooth in an open $U \subset I(\gamma)$ and has unit length, past directed timelike gradient, then $\Delta b \le 0$ (super-harmonic)

*Proof.* Assume $\Delta b(p) > 0$ for some point $p \in U$ and consider the level surface $b^{-1}(c)$ with $c = b(p)$. For some $r_0$ we have $p \in I^-(\gamma(r_0))$, we define $U_0 = U \cap I^-(\gamma(r_0))$ and $\Sigma = \{b = c\} \cap U_0$. We calculate the mean curvature $H_\Sigma$ of $\Sigma$ with respect to the future pointing timelike normal to $\Sigma$. The fact that $b$ has unit length past timelike gradient implies

$$\begin{aligned}H_\Sigma &= \sum \frac{\langle \nabla_{E_i} \nabla b, E_i\rangle}{|\nabla b|}\\ &= \sum \langle \nabla_{E_i} \nabla b, E_i\rangle - \frac{1}{2}\nabla b \langle \nabla b, \nabla b\rangle\\ &= \Delta b\end{aligned}$$

This gives $H = H_\Sigma(p) = \Delta b(p) > 0$. We choose a $q \in I^+(p) \cap U_0$ close enough to $p$ that $H_\Sigma \ge H/2$. We can slightly deform $\Sigma$ around $p$ so that the new surface $\Sigma'$ satisfies

- $A = \Sigma' \setminus \Sigma \subset I^-(q)$
- $A \cap I^-(p) \ne \emptyset$
- $H_{\Sigma'}(x) \ge H/3$ for all $x \in A$

By the reverse "Lipchitz" inequality, there is a big enough $r \geq 0$ such that $b_r < c$ for some points in $A$. Thus, $b_r|_{\Sigma'}$ attains a minimum $c' < c$ at some point $z \in A$, hence $\Sigma' \subset \{b_r \geq c'\}$.

Let $\eta_r : I \to M$ be a maximal geodesic from $z$ to $\gamma(r)$ with $y_r$ being the halfway point. Notice that $d(z, \gamma(r)) = r - c'$, therefore $\eta : [0, r - c'] \to M$ and $y_r = \eta\left(\frac{r-c'}{2}\right)$. We define the function

$$\beta_r(x) = r - \left(\frac{r - c'}{2} + d(x, y_r)\right)$$

Observe that $\beta_r(z) = c'$, $\beta_r$ is smooth around $z$ and $\beta_r \geq b_r$ by the reverse triangle inequality. The above Lorentzian inequality for the Laplacian gives

$$\Delta\beta_r(z) \leq \frac{2(n-1)}{r - c'} \xrightarrow[r\to\infty]{} 0$$

However, $\Sigma'$ can be the level set set of an appropriate future increasing smooth function $f \leq b_r$. Then $f(z) = b_r(z) = \beta_r(z) = c'$ and we have $f \leq b_r \leq \beta_r$, which implies $\Delta f(z) \leq \Delta b_r(z) \leq \Delta\beta_r(z)$. But the last inequality gives $H_{\Sigma_r} \geq H_{\Sigma'} \geq H/3 > 0$ This is a contradiction. Therefore $\Delta b \leq 0$ $\square$

## Nice Neighbourhoods

The proof of the Lorentzian splitting theorem follows a different approach than the Riemannian one. The local splitting will be proved using small enough neighbourhoods of the line $\gamma$ called "**nice neighbourhoods**". In this section we introduce nice neighbourhoods and show they exist around each point of the line.

**Definition 41.** A nice neighbourhood of a ray $\gamma$ is an open set $U \subset I(\gamma)$, for which there exists $K > 0$ and $T > 0$ such that for any $q \in U$, $r > T$ any unit speed maximal geodesic from $q$ to $\gamma(r)$ with initial velocity $v$, satisfies

$$g_R(v, v) \leq K$$

where $g_R$ is some auxiliary Riemannian metric

**Proposition 42.** Nice neighbourhoods of $\gamma$ exist

The proof is by construction. Consider the interval $I = [-T, T]$ and a tubular neighbourhood of $\gamma|_I$, diffeomorphic to $I \times B(R) \subset \mathbb{R}^n$, where $B(R)$

is the ball of radius $R > 0$ in $\mathbb{R}^n$. We take an orthonormal frame $\{e_i\}$ parallel along $\gamma$, such that $e_1(\gamma(t)) = \gamma'(t)$. Then we define the Fermi coordinates using the exponential map. If $(x_1, x_2..., x_n) \in I \times B(R)$

$$\Phi(x_1, x_2..., x_n) = exp\left(\sum_{i\neq 1} x_i e_i(x_1)\right)$$

This $\Phi$ is the normal exponential map of $\gamma|_I$, with $\Phi(\gamma(t), 0) = \gamma(t)$. Since $D\Phi$ is a linear isomorphism at each point of $\gamma(t)$ and from the compactness of $I = [-T, T]$, there exists a small enough $R > 0$ such that $\Phi : I \times B(R) \to U(T, R) = \Phi(I \times B(R))$ is a smooth diffeomorphism. Then $\Phi^{-1} = \chi = (x_1, x_2, ..., x_n)$ is the Fermi coordinates with $\partial_i = e_i$ on $\gamma$

In a way simmilar to the normal coordinates at a point, it can be proved that, along $\gamma$, we have $g_{ij} = \varepsilon_i \delta_{ij}$ (where $\varepsilon_i$ the sign of $g(e_i, e_i)$), and $\Gamma^k_{ij} = 0$, the last one implies that $dg_{ij} = 0$ at $\gamma$ and thus

$$|g_{ij} - \varepsilon_i \delta_{ij}| \leq c|x|^2$$

where $|x|^2 = \sum x_i^2$. But for each $x \in U$, we may assume that $x_1(x) = 0$ (shift the starting point of $\gamma$, if necessary), this means

$$|g_{ij} - \varepsilon_i \delta_{ij}| \leq c \sum_{i\neq 1} x_i^2$$

This essentially means that the given metric $g$ is close to a flat Minkowski metric $g_0$ associated to the Fermi coordinates $g \approx \sum \varepsilon_i dx_i^2 = -dx_1^2 + dx_2^2 + ... + dx_n^2 = g_0$. More precisely, for some $\varepsilon > 0$ and small enough $R > 0$, we have

$$\|g - g_0\| \leq C\varepsilon$$

At this point we introduce the flat Riemannian metric $g_1$ associated to the Fermi coordinates, as we shall need it later. It is defined as $g_1 = \sum dx_i^2 = dx_1^2 + dx_2^2 + ... + dx_n^2$. By $\nabla^0$ and $\nabla^1$ we denote the respective Levi-Civita connections of $g_0$ and $g_1$. Observe that the Christoffel symbols for both of them are all zero, because the coordinate components of both metrics are constant. This means $\nabla^0 = \nabla^1$.

Even though a connection is not a tensor, the difference of connections is, as the non-linear factors from the Leibniz properties cancel each other out. This allows us to take the tensor norm of the deifference of two connections.

We measure the difference $\nabla - \nabla^0$ with respect to the tensor norm that extends the Riemann metric $g_1$. The components of $(\nabla - \nabla^0)$ are

$$\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$$

Where $\Gamma_{ij}^k$ and $\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols of $\nabla$ and $\nabla^0$ respectively. But as we mentioned $\tilde{\Gamma}_{ij}^k = 0$, therefore

$$\begin{aligned}\|\nabla - \nabla^0\|_{g_1}^2 &= \sum (g_1)_{k_1 k_2} (g_1)^{i_1 i_2} (g_1)^{j_1 j_2} \Gamma_{i_1 j_1}^{k_1} \Gamma_{i_2 j_2}^{k_2} \\ &= \sum (\Gamma_{ij}^k)^2\end{aligned}$$

we recall that $\Gamma_{ij}^k = 0$ on $\gamma$. From the continuity of $\Gamma_{ij}^k$ and the compactness of $I$, there is a small enough $R$ that

$$\|\nabla - \nabla^0\|_1 \leq \varepsilon, \text{ in } U(R,T)$$

The next step to prove the existence of nice neighbourhoods is an aproximation result:

**Lemma 43.** For small enough $R > 0$, there is a $C > 0$, such that for all $s > 2T$

$$|b_s - x_1| \leq C\|\overline{x}\|^2, \text{ where } \|\overline{x}\|^2 = \sum_{i \neq 1} x_i^2$$

*Proof of lemma.* Take $x \in U(T,R)$ and consider the functions

$$f(x) = 2T - d(x, \gamma(2T)) \text{ and } e(x) = -2T + d(\gamma(-2T), x)$$

These functions are smooth for $x$ outside of the cut-locus of $\gamma(2T)$ (or of $\gamma(-2T)$). No point of the cut locus lies on $\gamma$, therefore, from compactness of $I$, for a small enough $R > 0$ both $f$ and $e$ are smooth in $U$. Also notice that along $\gamma$, $\nabla f|_\gamma = -\nabla d_{\gamma(2T)} = \gamma'$ and thus $Df$ vanishes on the normal space of $\gamma$. The same can be proved for $e$. Furthermore $f(\gamma(t)) = t = e(\gamma(t))$. That implies

$$|f - x_1| \leq c_1\|\overline{x}\|^2 \text{ and } |e - x_1| \leq c_2\|\overline{x}\|^2$$

Finally, observe that $f \geq b_s \geq e$, this implies that for any $s \geq 2T$

$$|b_s - x_1| \leq \left\{ \begin{array}{lll} b_s - x_1 \leq & f - x_1 \leq & c_1\|\overline{x}\|^2 \\ x_1 - b_s \leq & x_1 - e \leq & c_2\|\overline{x}\|^2 \end{array} \right.$$

Thus indeed $|b_s - x_1| \leq C\|\overline{x}\|^2$ for some $C > 0$.

(lemma) $\square$

Continuing, we will show that $U(T, R)$ is a nice neighbourhood if $R$ is small enough. Take $U(T, R_0)$ as in the previous lemma 43. Let $R_1 \leq R_0$ and $R_1 \leq T$ such that $\mu = CR_1 \leq R_1/10$. In $U_1 = U(T, R_1)$ we have

$$|b_s - x_1| \leq \mu$$

We define $U = U(T/2, R_1/2)$, take $q \in U$ and $s > 2T$ and let $a$ be a maximal unit speed geodesic from $q$ to $\gamma(s)$. While $a$ is in $U_1$ we have the coordinate expression $a(t) = (a_1(t), \overline{a}(t))$ and since $||a'(t)|| = 1$ and $\|g - g_0\| \leq \varepsilon$ we have

$$\begin{aligned}
|g(a'(t), a'(t)) - g_0(a'(t), a'(t))| &\leq \varepsilon \\
-1 + a_0'(t)^2 - \|\overline{a}'(t)\|_0^2 &\geq -\varepsilon \\
a_0'(t) &\geq 1 - \varepsilon \\
a_0'(t) &\geq \frac{1}{2},
\end{aligned}$$

where $\varepsilon$ is small enough. If $\|g - g_0\| < 1$, then the causal character of any vector coincides for the two metrics. This means $a$ is also a timelike curve with respect to the metric $g_0$ and since the starting point is $a(0) = q \in U(T/2, R_1/2)$, $a$ cannot escape from $U_1$ as long as

$$a_0(\tau) - a_0(0) < R_1/2.$$

Recall that $R_1 < T$. The function

$$t(\tau) = a_0(\tau) - a_0(0)$$

has positive derivative as $a_0'(\tau) > 1/2$, i.e. $t$ is strictly increasing. Therefore, for some $t_0 \in (4\mu, R_1/2)$ there is a unique $\tau_0$ such that

$$t(\tau_0) = t_0 = a_0(\tau_0) - a_0(0)$$

This implies that $a|_{[0,\tau_0]} \subset U_1$. From the definitions of $b_s$ and $a$ we have

$$b_s(a(\tau)) - b_s(a(0)) = \tau$$

Also, applying lemma 43 in $U_1$

$$\begin{aligned}|b_s(a(\tau_0)) - a_0(\tau_0)| &\leq \mu \\ |b_s(a(0)) - a_0(0)| &\leq \mu\end{aligned}$$

Combining the above we get

$$\begin{aligned}|b_s(a(\tau_0)) - b_s(a(0)) - a_0(\tau_0) + a_0(0)| &< 2\mu \\ |\tau_0 - t_0| &< 2\mu \\ \tau_0 - t_0 &> -2\mu \\ \tau_0 &> t_0 - 2\mu \\ \tau_0 &> \frac{t_0}{2} \\ \frac{t_0}{\tau_0} &< 2 \\ \frac{a_0(\tau_0) - a_0(0)}{\tau_0} &< 2\end{aligned}$$

This means the average speed of $a_0$ is less than 2 in $[0, \tau_0]$. By the intermediate value theorem, there is a $\tau_1 \in (0, \tau_0)$ such that

$$a_0'(\tau_1) < 2$$

and since $a_0'(\tau) > \|\overline{a}'(\tau)\|_0$ we have

$$\|a'(\tau_1)\|_1 = a_0'(\tau_1)^2 + \|\overline{a}'(\tau_1)\|_0^2 < 4$$

As we mentioned we have $\|\nabla - \nabla^0\| < \varepsilon$. Consider the function $f(\tau) = \|a'(\tau)\|_1$. Whose derivative satisfies

$$\begin{aligned}|f'(\tau)| &= \left|\frac{d}{d\tau}\sqrt{\langle a'(\tau), a'(\tau)\rangle_1}\right| \\ &= \frac{1}{2\|a'(\tau)\|_1}\left|\frac{d}{d\tau}\langle a'(\tau), a'(\tau)\rangle_1\right| \\ &= \frac{1}{2\|a'(\tau)\|_1}\left|2\langle \nabla^1_{a'(\tau)} a'(\tau), a'(\tau)\rangle_1\right| \\ &\leq \frac{1}{\|a'(\tau)\|_1}\|a'(\tau)\|_1 \|\nabla^1_{a'(\tau)} a'(\tau)\|_1 \\ &= \|(\nabla^1 - \nabla)_{a'(\tau)} a'(\tau) + \nabla_{a'(\tau)} a'(\tau)\|_1 \\ &\leq \varepsilon \|a'(\tau)\|_1^2 \\ &= \varepsilon f(\tau)^2\end{aligned}$$

It follows $\left(\frac{1}{f}\right)'(\tau) = \frac{f'(\tau)}{f(\tau)^2} \leq \varepsilon$. Integrating we get

$$\int_0^{\tau_1} \left(\frac{1}{f}\right)'(\tau) d\tau \leq \int_0^{\tau_1} \varepsilon d\tau$$

$$\frac{1}{f(\tau_1)} - \frac{1}{f(0)} \leq \varepsilon\tau_1$$

$$\frac{1}{f(0)} \geq \frac{1}{f(\tau_1)} - \varepsilon\tau_1 \geq \frac{1}{4}\varepsilon\tau_0$$

Notice that from $|\tau_0 - t_0| \leq 2\mu$ we get $\tau_0 \leq t_0 + 2\mu \leq R_1/2 + R_1/5 < R_1$. Therefore, if $R_1$ is small enough that $\frac{1}{4} - \varepsilon R_1 > \frac{1}{10}$, then we have

$$\|a'(0)\|_1 \leq \frac{1}{\frac{1}{4} - \varepsilon R_1} \leq 10$$

And that proves the existence of nice neighbourhoods for $\gamma$ □

The first important consequence of the existence of nice neighbourhoods is that, although the Minkowski unit ball on $T_qM$ is unbounded, the initial velocities of maximal segments from the neighbourhood to $\gamma(s)$, for any $s$, live in a compact subset. This allows us to have limits of maximal segments and form timelike asymptotes like in the Riemannian case.

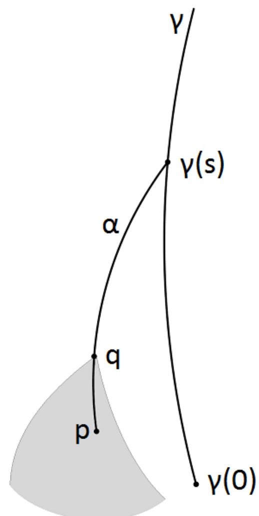
Also we notice that even though the construction was done for a nice neighbourhood around $\gamma(0)$, a shift of the starting point can give us a nice neighbourhood at any point of $\gamma$.

## Preliminaries

In this section we give the necessary results concerning the behaviour of the Busemann function and the structure of the manifold in a nice neighbourhood.

**Lemma 44.** The Busemann function, in a nice neighbourhood $U$, is Lipschitz continuous with respect to the auxiliary Riemannian metric

*Sketch of proof* We show that the pre-Busemann functions $b_s$ are all Lipschitz continuous with respect to $g_1$ and with the same constant. The function $d_s(x) = d(x, \gamma(s))$ is defined on $I^-(\gamma(s))$ and continuous but not smooth everywhere. Take $p \in I^-(\gamma(s))$ and let $a$ be a maximal unit speed geodesic from $p$ to $\gamma(s)$. Consider a point $q$ on $a$ close enough to $p$ so that $p$ is not in




the cut locus of $q$. Then $d(\cdot, q)$ is smooth in a neighbourhood of $p$. Take the function

$$f_p(x) = d(x, q) + d(q, \gamma(s))$$

It is smooth near $p$ because both $p$.

We have $f_p(p) = d_s(p)$ and by the reverse triangle inequality, $d_s \geq f_p$. This means $f_p$ is a lower support function of $d_s$ at $p$ with $\nabla f_p(p) = a'(0)$. For this support function, for any $v \in T_pM$ we get

$$\begin{aligned} |Df_p(v)| &= |g(a'(o), v)| \\ &\leq \|g\|_1 \|a'(0)\|_1 \|v\|_1 \\ &\leq GM\|v\|_1 \end{aligned}$$

where $M > 0$ is from the definition of a nice neighbourhood and $G$ is an upper bound for $\|g\|_1$ in $U$. This means that $\|Df_p\|_1 \leq GM$, and since $p$ was arbitrary, we the same constant $GM$ for any point in $I^-(\gamma(s))$. It can be proved that under these conditions $b_s$ is Lipschitz continuous with constant $GM$. And since the constant does not depend of $s$ either we have $b$ Lipschitz continuous as well □

For details see [8]. In summary, nice neighbourhoods have the following properties

1. for any $t$, $\gamma(t)$ is contained in a nice neighbourhood

2. Asymptotes from points in a nice neighbourhood are timelike
3. $\{b_r\}$ converges locally uniformly to $b$, therefore $b$ is continuous on a nice neighbourhood

Before the next lemma, we need a definition

**Definition 45.** Let $A$ be a chronological set in a Lorentzian manifold. The **edge** of $A$, denoted by $edge\{A\}$, is the set of points $p$ in $\overline{A}$ such that for any neighbourhood $U$ of $p$ there is a timelike curve from $U \cap I^-(p)$ to $U \cap I^+(p)$ which does not intersect $A$.

**Lemma 46.** Let $U$ be a nice neighbourhood of $\gamma$. The level sets of the Busemann function are partial Cauchy surfaces i.e. $\Sigma_c = \{b = c\} \cap U$ is closed and edgeless in $U$ and acausal

*Proof.* Since $b$ is continuous in $U$, $\Sigma_c$ is closed in $U$. From the "reverse Lipschitz" property of the Busemann function, it is impossible for two chronologically related points to live on the same level set. Thus $\Sigma_c$ is chronological.

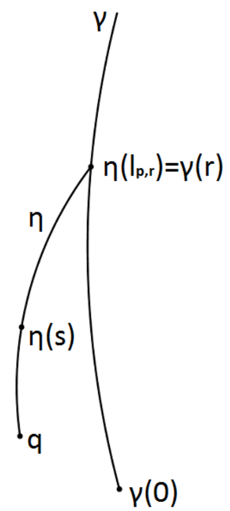
From the continuity it follows that it is edgeless i.e. the edge is the empty set. Indeed, if $p \in I^-(U)$ and $q \in I^+(U)$ and $\beta$ a timelike curve in $U$ from $p$ to $q$, remember that the reverse triangle inequality gives $b(y) \geq b(x)+d(x,y)$, for $x \ll y$. This gives $b(p) < c$ and $b(q) > c$. Applying the intermidiate value theorem to $b \circ \beta$, we see that $\beta$ meets $\Sigma_c$

Suppose that $\Sigma_c$ is not acausal, and let $p$, $q \in \Sigma_c$ be connected by a null geodesic $\alpha$ from $p$ to $q$. If $\eta_n$ are maximal geodesic segments from $q$ to $\gamma(r_n)$ (with $r_n \to \infty$), such that $\eta_n$ converge to an asymptote $\eta$, then by cutting the corners of $\alpha \cup \eta$ and $\alpha \cup \eta_n$, we have that, for big enough $n \in \mathbb{N}$

$$b_{r_n}(q) - b_{r_n}(p) = d(p, \gamma(r_n)) - d(q, \gamma(r_n)) > \varepsilon$$

But this contradicts the fact that $p$, $q \in \Sigma_c \subset \{b = c\}$ $\square$

The last preliminary result is a convexity lemma. To prove it we make use of a technical result. Suppose $q \in I^-(\gamma(r))$ for some $r$, then $\eta = \eta_{q,r} : [0, l_{q,r}] \to M$ is a maximal geodesic segment from $q$ to $\gamma(r)$ and for $0 \leq s \leq l_{q,r}$ we define $d^s_{q,r}(x) = d(x, \eta_{q,r}(s))$




**Lemma 47.** There is a small enough nice neighbourhood $U$ of $\gamma|_{[\alpha,r_0]}$ for which, given any compact, spacelike hypersurface $S$ with boundary in $U$, there is $C > 0$, $\tau > 0$ and $r_1 > r_0$ such that $\forall q \in S$, $r > r_1$, $\tau < s < l_{q,r}$

$$Hess_{d^s_{q,r}}(w,w) \geq -C$$

for all $w \in T_qM$ with $\langle w, w\rangle \leq 1$

*Sketch of proof.* Let $X = \tau\gamma'(t)$ for $\alpha < t < r_0$ and let $V$ be a neighbourhood of $X$ in $TU$. If $v \in V$ we define $\rho_v(x) = d(x, \exp(v))$. For appropriate choice of $U$, $V$, $\tau$ we have that the map $(v, w) \mapsto Hess_{\rho_v}(w, w)$ is smooth for $v \in V$, $w \in TU$. Therefore, there is $C > 0$ such that

$$Hess_{\rho_v}(w,w) \geq -C$$

for $g_R(w, w) \leq 1$, where $g_R$ is an auxiliary Riemannian metric. Then by continuity, for some $r_1 > r_0$ we have $\forall r > r_1$ it is

$$Hess_{d^s_{q,r}}(w,w) \geq -C$$

for $g_R(w, w) \leq 1$. Finally, restricting ourselves to an spacelike hypersurface, by compactness of $S$, the auxiliary metric is uniformly equivalent to the given Lorentz metric on $S$ $\square$

For details on the proof, see the Appendix in [9]. We now present and prove the convexity lemma, which is a maximum principle type result

**Lemma 48.** Assume $Ricc \geq 0$ for all timelike vectors. Let $\Sigma$ be a connected, smooth, spacelike hypersurface in a nice neighbourhood $U$ as in the technical lemma above. Assume $H_\Sigma \geq 0$ with respect to the future pointing unit normal. Then if $b$ achieves a minimum on $\Sigma$, it is constant.

*Proof.* Let $U \subset I^-(\gamma(r_0))$ a nice neighbourhood as in the technical lemma and $\Sigma \subset U$. Assume $b$ achieves a minimum at $q \in \Sigma$ but $b$ is not constant and let $\alpha = b(q)$. Then there is a coordinate ball centered at $q$ with

$$\partial^0 B = \{x \in \partial B : b(x) = \alpha\} \neq \emptyset$$

There is a smooth function $h : \Sigma \to \mathbb{R}$ with

- $h(q) = 0$
- $\Delta_\Sigma h \leq -D$ for some $D > 0$, where $\Delta_\Sigma$ is the induced Laplacian on $\Sigma$
- $h > 0$ on $\partial^0 B$
- $|\nabla_\Sigma h| \leq 1$ where $\nabla_\Sigma$ is the gradient along $\Sigma$

The construction of such a function is just like in the proof of the Hopf-Calabi maximum principle (lemma 34).

For a small enough $\varepsilon > 0$, consider the function $f_\varepsilon = b + \varepsilon h$. It has $f_\varepsilon(q) = \alpha$ and $f_\varepsilon > \alpha$ on $\partial^0 B$. Then, for big enough $r > 0$, the function $f_{r,\varepsilon} = b_r + \varepsilon h$ also has $f_{r,\varepsilon}(q) = \alpha$ and $f_{r,\varepsilon} > \alpha$ on $\partial^0 B$. Therefore, $f_{r,\varepsilon}$ minimizes at some point $p \in B$.

From the technical lemma, for $\overline{B}$ we have $\eta_r : [0, l] \to M$ maximal geodesic from $p$ to $\gamma(r)$. The reverse triangle inequality gives $l = d(p, \gamma(r)) \geq d(p, \gamma(r_0)) + d(\gamma(r_0), \gamma(r)) \geq r - r_0$. Thus we can assume that $r$ is big enough that $l/2 > \tau$, where $\tau$ is as in the technical lemma.

Let $y_r = \eta_r(l/2)$ and define $\beta_r = r - \big(l/2 + d(x, y_r)\big)$. From the reverse triangle inequality, $\beta_r$ is an upper support function of $b_r$ at $p$. Thus the function $\phi_{r,\varepsilon} = \beta_r + \varepsilon h$ is an upper support of $f_{r,\varepsilon}$ at $p$. As a consequence of that $\phi_{r,\varepsilon}$ minimizes at $p$, which means that $Hess_{\phi_{r,\varepsilon}}$ is positive definite at $p$ and thus $\Delta_\Sigma \phi_{r,\varepsilon}(p) \geq 0$

The contradiction arises by calculating the Laplacian of $\phi_{r,\varepsilon}$ and showing that for small enough $\varepsilon > 0$ and big enough $r > 0$ it becomes negative. We have

$$\Delta_\Sigma \phi_{r,\varepsilon} = \Delta_\Sigma \beta_r + \varepsilon \Delta_\Sigma h \tag{2.1}$$

Calculating $\Delta_\Sigma \beta_r = \sum \langle \nabla_{E_i} \nabla_\Sigma \beta_r, E_i \rangle$ for an orthonormal frame $\{E_i\}$ of $\Sigma$ gives

$$\Delta_\Sigma \beta_r = \Delta \beta_r + Hess_{\beta_r}(N, N) - H_\Sigma \langle \nabla \beta_r, N \rangle \tag{2.2}$$

From the Lorentzian inequality for the Laplacian we get

$$\Delta \beta_r(p) \leq \frac{n-1}{d_{y_r}(p)} = \frac{2(n-1)}{l} \leq \frac{2(n-1)}{r - r_0} \tag{2.3}$$

Observe that $\nabla \beta_r = -\nabla d_{y_r}$. This combined with the fact that $\eta_r$ is a maximal geodesic from $p$ to $\gamma(r)$ gives $\nabla \beta_r = -\eta_r'(0)$, in particular, past directed timelike. Using the assumption for the mean curvature $H_\Sigma \geq 0$ we have

$$H_\Sigma \langle \nabla \beta_r, N \rangle|_p \geq 0 \tag{2.4}$$

Since $\phi_{r,\varepsilon}$ minimizes at $p$, $\nabla_\Sigma \phi_{r,\varepsilon}(p) = 0$, which implies $\nabla_\Sigma \beta_r(p) + \varepsilon \nabla_\Sigma h(p) = 0$. By projecting $\nabla \beta_r$ to $T_p\Sigma$, we get the expression

$$\nabla_\Sigma \beta_r(p) = \nabla \beta_r(p) + \langle \nabla \beta_r, N \rangle|_p N_p$$

The plus sign is due to the timelike nature of $N_p$. The above gives an expression for $N_p$, combined with $\nabla \beta_(p) = -\eta_r'(0)$:

$$N_p = \frac{1}{\langle \eta_r'(0), N_p \rangle} \Big( \eta_r'(0) + \varepsilon \nabla_\Sigma h(p) \Big)$$

We use this expression to calculate the Hessian of $\beta_r$ at $p$

$$\begin{aligned} Hess_{\beta_r}(N, N) = \frac{1}{\langle \eta_r'(0), N_p \rangle} \Big( & Hess_{\beta_r}\big(\eta_r'(0), \eta_r'(0)\big) \\ & + \varepsilon Hess_{\beta_r}\big(\eta_r'(0), \nabla_\Sigma h(p)\big) \\ & + \varepsilon^2 Hess_{\beta_r}\big(\nabla_\Sigma h(p), \nabla_\Sigma h(p)\big) \Big) \end{aligned}$$

$\beta_r$ has unit length gradient, therefore its integral curves are geodesics. This toghether with the fact that $\nabla \beta_r = -\eta_r'(0)$ implies $Hess_{\beta_r}\big(\eta_r'(0), \eta_r'(0)\big) = Hess_{\beta_r}\big(\eta_r'(0), \nabla_\Sigma h(p)\big) = 0$. Thus

$$Hess_{\beta_r}(N, N) = \frac{\varepsilon^2}{\langle \eta_r'(0), N_p \rangle} Hess_{\beta_r}\big(\nabla_\Sigma h(p), \nabla_\Sigma h(p)\big)$$

By the reverse Cauchy-Schwartz inequality, $\langle \eta_r'(0), N_p \rangle \geq 1$. Also, since $Hess_{\beta_r} = -Hess_{d_{yr}}$ and $|\nabla_\Sigma h| \leq 1$, from the technical lemma we have

$$Hess_{\beta_r}(N, N) \leq \varepsilon^2 C \tag{2.5}$$

Substituting the inequalities 2.3, 2.4 and 2.5 to equation 2.2

$$\Delta_\Sigma \beta_r(p) \leq \frac{2(n-1)}{r - r_0} + \varepsilon^2 C$$

By its construction, $h$ satisfies $\Delta|_\Sigma h \leq -D$. Substituting to equation 2.1 we get

$$\Delta_\Sigma \phi_{r,\varepsilon}(p) \leq \frac{2(n-1)}{r - r_0} + \varepsilon^2 C - \varepsilon D$$

Notice that the dominant factor of the right hand side is $-\varepsilon D$. This means for $r$ big enough and $\varepsilon$ small enough, $\Delta_\Sigma \phi_{r,\varepsilon}(p)$ is negative, which is our contradiction.

We have proved that $b$ is constant in a neighbourhood of the minimizing point $q$. A connectivity argument finishes the proof $\square$

An immediate consequence of the convexity lemma is the following

**Corollary 49.** Let $\Sigma$ be a smooth, maximal ($H_\Sigma = 0$), spacelike hypersurface in a small enough nice neighbourhood $U$. If $edge(\Sigma) \subset \{b \geq c\}$, then $\Sigma \subset \{b \geq c\}$.

## Local Splitting

**Theorem 50.** Let $M$ be a time-oriented, globally hyperbolic, timelike-complete Lorentz manifold which contains a timelike line $\gamma$ and satisfies $Ricc(v, v) \geq 0$ for all timelike vectors. There is a neighbourhood of $\gamma$ in which the Busemann function is smooth, the integral curves of its gradient form asymptotes to $\gamma$, and the flow of its gradient acts by isometries.

The proof is divided in 4 steps. In the process we construct a map $E$ using the normal exponential map of a well behaved smooth spacelike hypersurface. This map will give the splitting of the nice neighbourhood. It turns out the hypersurface is a subset of the level set of $b$, and the map defined is the restriction of the flow of $\nabla b$

So far the Busemann function $b$ need not be smooth, so its level sets might have bad behaviour. However a result by Bartnik [1] can guarantee the existence of some smooth spacelike hypersurface for appropriate boundary data

**Step 1.** Existence of a spacelike hypersurface.
Let $U$ be a nice neigbourhood of $\gamma$ and consider the level sets $S^+ = \{b^+ = 0\} \cap U$ and $S^- = \{b^- = 0\} \cap U$. By lemma 2.2, $S^+$ is causal and edgeless in $U$. This implies that it is an imbeded topological submanifold (see [16]). Let $W$ be a small coordinate neighbourhood of $\gamma(0)$ in $S^+$.

Bartnik's result (theorem 4.1 in [1]) implies that there exists a smooth maximal surface $\Sigma$ such that $\Sigma$ is achronal in $U$, $\overline{\Sigma}$ is compact, $edge(\Sigma) = edge(W)$ and $\Sigma$ meets $\gamma$. We note that the acausality of $S^+$ ensures the smoothness of $\Sigma$.

From the reverse triangle inequality $b^+ + b^- \geq 0$. Since $S^+ \subset \{b^+ = 0\}$ we have $edge(W) = edge(\Sigma) \subset \{b^- \geq 0\}$. Applying corollary 49 for $b^+$ and $b^-$ on $\Sigma$ we get $\Sigma \subset \{b^+ \geq 0\} \cap \{b^- \geq 0\}$. This implies that $\Sigma$ must meet $\gamma$ at $\gamma(0)$ since $b^+ + b^- = 0$ on $\gamma$ and $b^+(\gamma(t)) = t$ But his means $b^+$ and $b^-$ minimize at $\gamma(0) \in \Sigma$. From the convexity lemma follows that $b^+ = b^- = 0$ on $\Sigma$, so $\Sigma \subset \{b^\pm = 0\}$. We haven't proven that $b^+$ or $b^-$ is smooth yet.

**Step 2.** For each point of $\Sigma$ there is a unique asympote to $\gamma^+$ and a unique asympotoe to $\gamma^-$. Furthermore, these asympotes fit toghether to lines orthogonal to $\Sigma$.
Let $\alpha^+$ and $\alpha^-$ be asymptotes of $\gamma^+$ and $\gamma^-$ respectively, both starting at $q$. By the formula for the Busemann functions on asymtpotes we have

$$b^+(\alpha^+(t)) = b^+(\alpha^+(0)) + t = t$$

$$b^-(\alpha^-(t)) = t$$

From the "reverse Lipschitz" property of the Busemann function

$$b^+(\alpha^-(t)) \leq b^+(\alpha^-(0)) - d(\alpha^-(t), \alpha^-(0)) = -t$$

$$b^-(\alpha^+(t)) \leq b^-(\alpha^+(0)) - d(\alpha^+(0), \alpha^+(t)) = -t$$

Also, from the fact that $b^+ + b^- \geq 0$ follows

$$b^+(\alpha^+(t)) + b^-(\alpha^+(t)) \geq 0 \iff b^-(\alpha^+(t)) \geq -t$$

$$b^+(\alpha^-(t)) + b^-(\alpha^-(t)) \geq 0 \iff b^+(\alpha^-(t)) \geq -t$$

Combining the above, we see that $b^-(\alpha^+(t)) = -t$ and $b^+(\alpha^-(t)) = -t$.

We denote by $\alpha$ the (possibly) broken geodesic

$$\alpha(t) = \begin{cases} \alpha^+(t), & t \geq 0 \\ \alpha^-(-t), & t \leq 0 \end{cases}$$

The above can be written as $b^\pm(\alpha(t)) = t$.

For $s < 0 < t$, the length of the curve $\alpha|_{(s,t)}$ is

$$length(\alpha|_{(s,t)}) = t - s = b^+(t) - b^+(s) \geq d(\alpha(s), \alpha(t))$$

where the last inequality is obtained from the "reverse Lipschitz" property. This shows the length of $\alpha|_{(s,t)}$ is maximal and thus it is an unbroken geodesic.

For the orthogonality, we show that $b^+$ is smooth at the point $q$. For some $r > 0$, we consider the functions defined near $q$ given by the formulas

$$f_r(x) = b^+(q) + r - d(x, \alpha(r))$$
$$g_r(x) = b^+(q) + r - d(\alpha(-r), x)$$

These are upper and lower support function respectively of $b^+$ at $q = \alpha(0)$ We verify this for $f_r$

$$f_r(q) = b^+(q) + r - d(\alpha(0), \alpha(r)) = b^+(q)$$

A consequence from the proof of lemma 37 is that the asymptotes of $\gamma$ live inside of $I(\gamma)$. This means that for big enough $s$, we have $\alpha(r) \ll \gamma(s)$ and so, we can use the reverse triangle inequality

$$\begin{aligned} f_r(x) - b^+(x) &= \lim_s \Big( b^+(q) + r - d(x, \alpha(r)) - s + d(x, \gamma(s)) \Big) \\ &\geq b^+(q) + r - \lim_s \Big( s - d(\alpha(r), \gamma(s)) \Big) \\ &= b^+(q) + r - b^+(\alpha(r)) = 0 \end{aligned}$$

This shows $f_r$ is an upper support function. Similarly, $g_r$ is a lower support function of $b^+$ at $q$. Notice that for appropriate choice of $r$, both $f_r$ and $g_r$ are smooth at $q$ and as we showed $g_r \leq b^+ \leq f_r$ with equality at $q$. It follows that $b^+$ is smooth at the point $q$ with $\nabla b^+(q) = \nabla f_r(q) = -\nabla d_{\alpha(r)}(q) = -\alpha'(0)$

Finally, since $\Sigma \subset \{b^+ = 0\}$, and the gradient of $b^+$ must be orthogonal to the level set, the line $\alpha$ is also orthogonal to $\Sigma$.

We now introduce the map that will give the local splitting. Take $B \subset \Sigma$ be a small ball of radius $R$ centered at $\gamma(0)$. Let $N$ be the unit length future pointing vector field normal to $B$. As we saw, $N$ consists of the initial velocities of the asymptote lines starting at $B$. Set $U = \mathbb{R}^1_1 \times B$ and define the map $E : U \to M$

$$E(t,p) = exp(tN_p)$$

Notice that by timelike completeness, the map $E$ is indeed defined on the entire $U$.

**Step 3.** The map $E$ is a diffeomorphism

We know the map $E$ is smooth, we need to show it is injective and non-singlular.

First, we assume $E$ is not injective, this means $E(t,p) = E(s,q)$ for some points $p$, $q$ in $B$ and $t, s \in \mathbb{R}$. This is equivalent to $\alpha(t) = \beta(s)$, where $\alpha$ and $\beta$ are asymptotes starting at $p$ and $q$ respectively. In the previous step, we proved that $b^+$ is smooth at $p = \alpha(0)$. In fact we proved it for any line that satisfies the formula $b^+(\alpha(t)) = b^+(\alpha(0)) + t$. By shifting the initial point of the line, the formula still holds for the new reparametrized line. It follows that $b^+$ is smooth at $\alpha(t)$ with $\alpha'(t) = \nabla b^+(\alpha(t))$. And also that $b^+$ is smooth at $\beta(s)$ with $\beta'(s) = \nabla b^+(\beta(s)) = \nabla b^+(\alpha(t)) = \alpha'(t)$. From the uniqueness of geodesics, $\alpha = \beta$, thus $(t,p) = (s,q)$.

It is known that $E$ has non-singular points iff $B$ has no focal points (see [16] pr. 10.30). Let $\alpha(t_0)$ be the first focal point along the future pointing normal $\alpha$. Then there is a neighbourhood $V$ of $\alpha|[0,t_0]$ which contains no focal points of $B$. Thus $E$ is a diffeomorphism to $V$ and $b^+ = pr_2 \circ E^{-1}$ on $V$. Where the last comes from the behaviour of $b^+$ on asymptotes. This implies $b^+$ is smooth at $V$, with unit length past-directed gradient. Denote by $\Sigma_t = \{b^+ = t\} \cap V$, then this follows $\Delta b^+|_{\Sigma_t} = H_{\Sigma_t}$. From the superharmonicity of the Lorentz Busemann function (lemma 40), $H_{\Sigma_t} = \Delta b^+ \leq 0$.

On the other hand, the mean curvature of $\Sigma_t$ is the trace of the shape operator $S_t$, and the fact that $\alpha(t_0)$ is a focal point of $B$ and of $\Sigma_t$, implies that $\frac{1}{t-t_0}$ is an eigenvalue of $S_t$. Therefore, as $t \to t_0$, $trS_t \to \infty$ This gives a contradiction and shows there are no future focal points. A similar argument for $b^-$ shows there are no past focal points.

**Step 4.** The map $E : U \to M$ is an isometry

First, notice that $b^\pm = \pm pr_2 \circ E^{-1}$ on $E(U)$, which implies $b^+$, and $b^-$ are smooth on $E(U)$, with $b^+$ having past directed unit length gradient and $b^-$ having future pointing unit length gradient. We can apply the superhamonicity lemma 40 for $b^+$ directly, and, after reversing the time orientation, we can apply it to $b^-$ as well. This gives $\Delta b^{\pm} \leq 0$. But, we have $b^+ + b^- = 0$ on $E(U)$, thus $\Delta b^+ = \Delta b^- = 0$.

Since $b$ has unit length gradient, applying the Bochner formula (proposition 17) to $b$ we get

$$\nabla b^+(\Delta b^+) + |Hessb^+|^2 + Ricc(\nabla b^+, \nabla b^+) = 0$$

Combining $\Delta b^+ = 0$ with the hypothesis $Ricc \geq 0$ for timelike vectors, gives $|Hessb^+|^2 = 0$ which implies that $\nabla b^+$ is parallel and $Hessb^+ = 0$. For gradient fields this is equivalent to the gradient of $b^+$ being a Killing vector field, i.e. $\mathfrak{L}_{\nabla b^+} g = 0$. This is implies that the flow of $\nabla b^+$ acts by isometries on $E(U)$. We only need to notice that $E$ is the flow of $\nabla b^+$ restricted to $\mathbb{R} \times B$

## Global Splitting

**Theorem 51.** Let $M$ be a time-oriented, globally hyperbolic, timelike-complete Lorentz manifold which contains a timelike line $\gamma$ and satisfies $Ricc(v, v) \geq 0$ for all timelike vectors. There is a spacelike hypersurface $M'$ of $M$ such that $M$ is isometric to $\mathbb{R}^1_1 \times M'$

A **flat strip** is an isometric immersion $f : \mathbb{R}^1_1 \times [0, \alpha] \to M$. As a consequence, for all $s \in [0, \alpha]$ the curve $t \mapsto f(t, s)$ is a timelike line in $M$. Two timelike lines $\gamma$ and $\gamma'$ are called **strictly parallel**, if they bound a flat strip and **parallel** if there is a finite sequence of lines $\{\gamma_n\}_{n=1}^N$ such that $\gamma_1 = \gamma$, $\gamma_N = \gamma'$ and for any $1 \leq n < N$ $\gamma_n$ and $\gamma_{n+1}$ are strictly parallel.

**Step 1.** If $\gamma_1$ and $\gamma_2$ are strictly parallel, then $I(\gamma_1) = I(\gamma_2)$ and their Busemann functions agree $b_1 = b_2$

Let $f : \mathbb{R}^1_1 \times [0, \alpha] \to M$ be a flat strip with $\gamma_1$ and $\gamma_2$ as its boundaries. We will utilize the well behaved geometry of the Cartesian product $\mathbb{R}^1_1 \times [0, \alpha]$.

Take some $s \in \mathbb{R}$. We identify $\gamma_1(s)$ as $(s, 0) \in \mathbb{R}^1_1 \times [0, \alpha]$. It is clear we can find future and past pointing timelike curves from $(s, 0)$ to the line $\mathbb{R}^1_1 \times \{\alpha\}$. Therefore $\gamma_1 \subset I(\gamma_2)$. Similarly $\gamma_2 \subset I(\gamma_1)$. Thus $I(\gamma_1) = I(\gamma_2)$.

We will show that $b_1 \leq b_2$ and $b_2 \leq b_1$. To do that, we show that for any $r \in \mathbb{R}$ the pre-Busemann function satisfies $b_{1,r} \geq b_2$ and vice versa. But before that we need to show that for any $t \in \mathbb{R}$ $b_1(\gamma_2(t)) = t$ and vice versa.

Again we identify $\mathbb{R}^1_1 \times [0, \alpha]$ with its image and use it's nice geometry.

$$\begin{aligned}
b_1(\gamma_2(t)) &= \lim_{r\to\infty}\Big(r - d\big(\gamma_2(t), \gamma_1(r)\big)\Big) \\
&= \lim_{r\to\infty}\Big(r - d\big((t,\alpha),(r,0)\big)\Big) \\
&= \lim_{r\to\infty}\Big(r - |(r-t, -\alpha)|\Big) \\
&= \lim_{r\to\infty}\Big(r - \sqrt{(r-t)^2 - \alpha^2}\Big) \\
&= t + \lim_{r\to\infty}\Big((r-t) - \sqrt{(r-t)^2 - \alpha^2}\Big) = t
\end{aligned}$$

Similarly we show $b_2(\gamma_1(t) = t$. Now, for any $r > 0$ we have

$$\begin{aligned}
b_{1,r}(x) - b_2(x) &= \lim_{s\to\infty}\Big(r - d\big(x, \gamma_1(r)\big) - s + d\big(x, \gamma_2(s)\big)\Big) \\
&\geq \lim_{s\to\infty}\Big(r - s + d\big(\gamma_1(r), \gamma_2(s)\big)\Big) \\
&= r - b_2(\gamma_1(r)) = 0
\end{aligned}$$

Thus $b_{1,r} \geq b_2$ and similarly we show $b_{2,r} \geq b_1$. Letting for $r \to \infty$, we get $b_1 \geq b_2$ and $b_2 \geq b_1$. Therefore $b_1 = b_2$.

**Step 2.** Let $c : [0,1] \to M$ be a geodesic starting at $\gamma(0)$ with $c \neq \gamma$. Then, there is a flat strip containing both $\gamma$ and $c$.

From the local splitting, there is an isometry $E : \mathbb{R}^1_1 \times B \to M$. The geodesic $c$ might not lie in $B$, but is decomposed, near $\gamma$, as $c = (c_1, c_2)$ with $c_1$ and $c_2$ geodesics in $\mathbb{R}^1_1$ and $B$ respectively. For $s \in \mathbb{R}$, $c_1(s) = ks$ for some $k \in \mathbb{R}$ and is $m$ is the speed of $c_2$, the reparametrization $s \mapsto c_2\left(\frac{s}{m}\right)$ is a unit speed geodesic ($m \neq 0$ since $c \neq \gamma$). Restricting the isometry $E$ on the image of $c_2$ gives a flat strip

$$f(t,s) = E\left(t, c_2\left(\frac{s}{m}\right)\right)$$

defined for $t \in \mathbb{R}$ and $s$ in some interval $[0, \varepsilon]$. With this flat strip $\gamma(t) = f(t, 0)$ and $c(s) = f(ks, ms)$.

The problem is that this flat strip is not defined beyond a nice neighbourhood of $\gamma$. We extend the flat strip. Take $X$ to be the parallel translation of $\gamma'(0)$ along $c$. Notice that in the nice neighbourhood, X is a restriction of $\nabla b$ which is parallel. Now define $\gamma_s$ to be the geodesic with initial velocity

$X(s)$. Timelike completeness ensures each $\gamma_s$ are defined on all of $\mathbb{R}$. In the nice neighbourhood we have

$$\gamma_s(t) = f(t + ks, ms)$$

This allows us to describe $f$ in terms of $\gamma_s$'s

$$f(t, s) = \gamma_{s/m}\left(t - \frac{k}{m}s\right)$$

Take $A = \{s \in [0, 1] : f|_{\mathbb{R}\times[0,s]}$ is an isometry$\}$ and let $\alpha = \sup A$. As we saw, $\alpha > 0$. This means

$$f : \mathbb{R} \times [0, \alpha) \to M$$

is a flat strip. The extension to $\alpha$ is smooth, since for any $t \in \mathbb{R}$

$$f(t, s) = \gamma_{s/m}\left(t - \frac{k}{m}s\right) \longrightarrow \gamma_{\alpha/m}\left(t - \frac{k}{m}\alpha\right) = f(t, \alpha)$$

and it is an isometry because the geodesic $\gamma_{\alpha/m}$ is of unit speed.

This shows that $f|_{\mathbb{R}\times[0,\alpha]}$ is indeed a flat strip and from the local splitting for the line $\gamma_{\alpha/m}$, the flat strip can be extended beyond $\alpha$. Therefore the flat strip can be defined on the entire $\mathbb{R} \times [0, 1]$, which means, it contains both $\gamma$ and $c$.

**Step 3.** The split
Consider the set $\mathcal{P}_\gamma \subset M$ of points that lie on some parallel line of $\gamma$ and let $p \in \mathcal{P}_\gamma$. This means there are some consecutive strictly parallel lines with the first one passing through $p$ and the last one being $\gamma$. Utilizing the respective flat strips, we can construct future-pointing and past-pointing piece-wise smooth timelike curves from $p$ to $\gamma$. Thus $p \in I(\gamma)$ and since $p$ was arbitrary, $\mathcal{P}_\gamma \subset I(\gamma)$.

Taking this argument one step further, we can connect any $p \in \mathcal{P}_\gamma$ to $\gamma(0)$ by a piece-wise smooth curve in $\mathcal{P}_\gamma$. This makes $\mathcal{P}_\gamma$ connected. from step 2, we see that $\mathcal{P}_\gamma$ is open. These imply that $\mathcal{P}_\gamma = M$

This means the Busemann function $b$ is defined in all of $M$. Also, for any point $p \in M$, by induction on step 1, b agrees with the Busemann function of the parallel line $\gamma_p$ through $p$, which is smooth near $p$. Furthermore, the gradient of $b_p$ is parallel near $p$ and $\gamma_p$ is an integral curve of $b_p$.

It follows that $b$ is smooth everywhere and the parallel lines are the integral curves of $\nabla b$, which is parallel. This gives the desired isometry

$$j : \mathbb{R} \times b^{-1}(0) \to M : (t, p) \mapsto \exp(-t\nabla b(p))$$

# Bibliography


[1] Bartnik, R. Regularity of variational maximal surfaces. *Acta Mathematica 161* (1988), 145–181.

[2] Brickell, and Clark. *Differentiable manifolds*. Van Nostrand Reinhold, 1970.

[3] Calabi, E. An extension of E. Hopf's maximum principle with an application to geometry. *Duke Math. J. 25* (1965), 285–303.

[4] Cheeger, and Ebin. *Comparison theorems in Riemannian geometry*. American Mathematical Society, 2008.

[5] Cheeger, and Gromoll. The splitting theorem for manifolds of nonnegative Ricci curvature. *Journal of Differential Geometry 6*, 1 (1971), 119–128.

[6] do Carmo, M. *Riemannian geometry*. Birkhäuser, 1992.

[7] Eschenburg, and Heintze. An elementary proof of the Cheeger-Gromoll splitting theorem. *Annals of Global Analysis and Geometry 2*, 2 (1984), 141–151.

[8] Eschenburg, J.-H. The splitting theorem for space-times with strong energy condition. *J. Differential Geometry 27*, 3 (1988), 477–491.

[9] Galloway, G. The Lorentzian splitting theorem without the completeness assumption. *J. Differential Geometry 29*, 2 (1989), 373–387.

[10] Haesen, S., and Verstraelen, L., Eds. *Topics in modern differential geometry*. Atlantis Press, 2017.

[11] Hirsch, M. On imbedding differentiable manifolds in Euclidean space. *Annals of Mathematics 73*, 3 (1961), 556–571.

[12] Hirsch, M. *Differential topology*, vol. 33 of *GTM*. Springer-Verlag, 1976.

[13] Kobayashi, and Nomizu. *Foundations of differential geometry*, vol. 1. Interscience publishers, 1963.

[14] Markus, L. Line field elements and Lorentz structures on differentiable manifolds. *Annals of Mathematics 62*, 3 (1955), 411–417.

[15] Newman, R. A proof of the splitting conjecture of S.T. Yau. *Journal of Differential Geometry 31*, 1 (1990), 163–184.

[16] O'Neil, B. *Semi Riemannian geometry.* Academic Press, 1983.

[17] Petersen, P. *Riemannian geometry*, 2 ed. Springer, 2006.

[18] Steenrod, N. *The topology of fibre bundles.* Princeton University Press, 1951.