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Finite type invariants for knotoids

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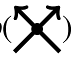
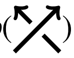
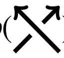
Contents

Introduction

Knots are smooth embeddings of S^1 in the interior of a 3-manifold. We usually regard equivalence classes of knots up to isotopy. Isotopy is a homotopy from one embedding of S^1 to another such that, at every time it is an embedding in the 3-manifold. A generic projection of a representative of an isotopy class of a knot into a plane is called a knot diagram. The isotopy of knots has been discretified by Reidemeister, proving that isotopy is a combination of locally planar isotopy and the three Reidemeister moves. The classification of knots up to isotopy is still an open problem, yet there are strong invariants that classify large families of knots, such as the classical polynomial invariants, Vassiliev invariants, general quantum invariants, Khovanov homology etc.

The Vassiliev invariants are extensions of knot invariants to knots which fail to be embeddings in finitely many transversal double points. The extension is possible using the relation called the Vassiliev skein relation: $\upsilon($$) = \upsilon($$) - \upsilon($$)$. A finite type invariant of type k is a knot invariant whose extension vanishes in all diagrams with $n > k$ singularities. In the original work of V. Vassiliev [26], finite type invariants correspond to the zero-dimensional classes of a special spectral sequence. In fact, the space of knots is understood as the complement of the space of mappings of S^1 in $\mathbb{R}^3$ that fail in some way to be an embedding (the so-called discriminant). Knot invariants are just locally constant functions in the space of knots and Vassiliev's formalism was trying to show that there exists such a spectral sequence that converges to the cohomology of the space of knots. These definitions and calculations have been very much simplified by the works of Bar-Natan[1], Birman & Lin[3], Stanford[23],and Polyak & Viro[22], as we are interested only in the zero-class, and as the use of the chord diagrams made a large part of the theory combinatorial, avoiding heavy technical issues of singularity theory. It is indeed a very strong family of knot invariants, as shown by the proof of the Vassiliev-Kontsevich Theorem. Nevertheless,it is not yet known whether Vassiliev invariants, or finite type invariants, can classify knots or even if they detect the unknot. Vassiliev invariants can be seen as building blocks to other known knot invariants, such as the classical Jones polynomial whose coefficients in the Taylor expansion are Vassiliev invariants.

The theory of knotoids was introduced by V. Turaev in 2011 [25]. A knotoid diagram is a generic immersion of the interval in the interior of a connected orientable surface Σ. An isotopy class of knotoid diagrams is called a knotoid. We will usually refer to spherical or planar knotoids. As Turaev showed, the theory of spherical knotoids is a faithful extension of the classical knot theory. Remarkably, and unlike classical knot theory, in the theory of knotoids, spherical knotoids that are isotopic can be in different isotopy classes of planar knotoids. Knotoids have caught the attention of the mathematical community the last years with many different results and applications to other aspects of science, such as the topological study of proteins.

In this work we introduce the theory of finite type invariants for knotoids. The main results are: a classification of knotoids with exactly one singularity, up to singular equivalence, as representatives of different homotopy classes of the annulus. We then construct the corresponding linear chord diagrams, and with that in hand we give an explicit formula for the universal invariant of type-1, and we prove that, indeed, we can generate a knotoid invariant given a real function on linear chord diagrams. We then give some examples of non-trivial v_1 invariants, such as the affine index polynomial and invariants which arise from the Taylor expansion of the Turaev extended bracket polynomial.

In Chapter 1 we give the basic definitions and results from the theory of knotoids and we give examples of some strong knotoid invariants. In

Chapter 2 we summarize the theory of Vassiliev invariants and chord diagrams. We give a proof that the chord diagrams have a rich algebraic structure and we state the Vassiliev-Kontsevich Theorem, as well as the insufficiency of this classical theory to provide a proof of the Theorem. In Chapter 3 we give the proof of the Vassiliev-Kontsevich Theorem using the Kontsevich integral and by claiming some basic facts about Knizhnik-Zamolodchikov connections. Chapters 4,5,6 are the chapters containing new results, which are joint work with S. Lambropoulou and L.H. Kauffman. In Chapter 4 we introduce the theory of singular knotoids, we give examples of Vassiliev invariants obtained by different closures and we construct the linear chord diagrams. In Chapter 5 we give some technical lemmas about a rail representation of spherical knotoids, which will be useful for proving the Classification Theorem for spherical knotoids with exactly one singularity. In Chapter 6 we prove the Classification Theorem using the results from Chapters 4 and 5, and we construct the universal Vassiliev invariant of type 1.

Acknowledgements

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Εισαγωγή

Οι κόμβοι είναι λείες εμφυτεύσεις της S^1 στο εσωτερικό μιας 3-πολλαπλότητας. Συνήθως μελετάμε κλάσεις ισοδυναμίας κόμβων ως προς την έννοια της ισοτοπίας. Ισοτοπία είναι μια ομοτοπία από μία εμφύτευση της S^1 σε μια άλλη, τέτοια ώστε σε κάθε χρονικό σημείο να είναι εμφύτευση στην 3-πολλαπλότητα. Μια τυπική προβολή ενός αντιπροσώπου μιας κλάσης ισοτοπίας ενός κόμβου σε ένα επίπεδο ονομάζεται διάγραμμα του κόμβου. Η ισοτοπία των κόμβων έχει διακριτοποιηθεί από τον Reidemeister, αποδεικνύοντας ότι μια ισοτοπία είναι ένας πεπερασμένος συνδυασμός τοπικών ισοτοπιών επιπέδου και των τριών κινήσεων Reidemeister. Η ταξινόμηση των κόμβων ως προς την έννοια της ισοτοπίας είναι ακόμα ένα ανοικτό πρόβλημα των Μαθηματικών. Όμως υπάρχουν ισχυρές αναλλοίωτες που ταξινομούν μεγάλες οικογένειες κόμβων, όπως οι κλασικές πολυωνυμικές αναλλοίωτες, οι αναλλοίωτες Vassiliev, οι κβαντικές αναλλοίωτες γενικά και η ομολογία Khovanov.

Οι αναλλοίωτες Vassiliev είναι επεκτάσεις των αναλλοίωτων κόμβων σε κόμβους που αποτυγχάνουν να είναι εμφυτεύσεις σε πεπερασμένα τω πλήθος εγκάρσια διπλά σημεία. Η επέκαση είναι δυνατή χρησιμοποιώντας τη σχέση που ονομάζεται Vassiliev skein relation $\upsilon(\ ) = \upsilon(\ ) - \upsilon(\ )$. Μια αναλλοίωτη πεπερασμένου τύπου k, είναι μια αναλλοίτη κόμβων της οποίας η επέκταση μηδενίζεται σε όλα τα διαγράμματα με $n > k$ ιδιομορφίες. Στην αρχική δουλειά του V. Vassiliev, οι αναλλοίωτες πεπερασμένου τύπου αντιστοιχούν στη μηδενικής διάστασης κλάση συνομολογίας μιας ειδικής φασματικής ακολουθίας. Ο χώρος των κόμβων γίνεται αντιληπτός ως το συμπλήρωμα του χώρου των απεικονίσεων του S^1 στον $\mathbb{R}^3$ που αποτυγχάνουν με κάποιο τρόπο να είναι εμφύτευση, και ο οποίος αποκαλείται 'διακρίνουσα'. Οι αναλλοίωτες κόμβων είναι απλά τοπικά σταθερές συναρτήσεις στο χώρο των κόμβων. Ο φρορμαλισμός αυτός προσπαθεί να δείξει ότι υπάρχει μια τέτοια φασματικής ακολουθία που να συγκλίνει στη συνομολογία του χώρου των κόμβων. Αυτοί οι ορισμοί και οι υπολογισμοί έχουν απλοποιηθεί από τη δουλειά των Bar-Natan, Birman & Lin, T. Stanford και Polyak & Viro. Αφού μας ενδιαφέρει μόνο η μηδενική κλάση, η χρήση των 'διαγραμμάτων χορδών' μετέφερε αυτή τη θεωρία στο πεδίο της συνδυαστικής θεωρίας, αποφεύγοντας μεγάλα τεχνικά προβλήματα της θεωρίας ιδιομορφιών. Είναι όντως μια πολύ δυνατή οικογένεια από αναλλοίωτες κόμβων, όπως φαίνεται από την απόδειξη του Θεωρήματος Vassiliev-Kontsevich. Φυσικά, δεν είναι ακόμα γνωστό αν οι αναλλοίωτες Vassiliev ταξινομούν τους κόμβους, ούτε καν αν αναγνωρίζουν τον τετριμμένο κόμβο. Μπορούμε να δούμε τις αναλλοίωτες Vassiliev ως βασικά κομμάτια κατασκευής άλλων αναλλοίωτων κόμβων, όπως το πολυώνυμο Jones, του οποίου οι συντελεστές στο ανάπτυγμα Taylor είναι αναλλοίωτες Vassiliev.

Η θεωρία των 'κομβοειδών' (knotoids) εισήχθη από τον V. Turaev. Ένα διάγραμμα κομβοειδούς είναι μία τυπική εμβάπτυση ενός κλειστού διαστήματος στο εσωτερικό μιας συνεκτικής προσανατολίσιμης επιφάνειας Σ. Μια κλάση ισοτοπίας από διαγράμματα κομβοειδών ονομάζεται κομβοειδές. Αξίζει να σημειωθεί ότι, αντίθετα με την κλασική θεωρία κόμβων, στη θεωρία των κομβοειδών, διαγράμματα σφαιρικών κομβοειδών που είναι ισοτοπικά μπορεί να ανήκουν σε διαφορετικές κλάσεις ισοτοπίας ως επίπεδα κομβοειδή. Τα κομβοειδή έχουν τραβήξει την προσοχή της μαθηματικής κοινότητας τα τελευταία χρόνια, με πολλά αποτελέσματα, αλλά και χρήσιμες εφαρμογές σε άλλες πτυχές της επιστήμης, όπως στην τοπολογική μελέτη των πρωτεϊνών.

Σε αυτήν την εργασία εισάγουμε τη θεωρία των αναλλοίωτων πεπερασμένου τύπου για κομβοειδή. Τα βασικά αποτελέσματα είναι μια ταξινόμηση, ως προς την ισοδυναμία ιδιομορφίας, των κομβοειδών με ακριβώς μία διασταύρωση ιδιομορφίας, εφόσον αντιστοιχούν σε αντιπροσώπους διαφορετικών κλάσεων ομοτοπίας του κύκλου. Στη συνέχεια κατασκευάζουμε τα αντίστοιχα 'γραμμικά διαγράμματα χορδών' με τα οποία μπορούμε να κατασκευάσουμε την καθολική αναλλοίωτη τύπου 1 για κομβοειδή, και αποδεικνύουμε ότι πράγματι μπορούμε να παράγουμε αναλλοίωτες κομβοειδών με δεδομένη μια πραγματική συνάρτηση στα γραμμικά διαγράμματα χορδών. Τέλος δίνουμε μερικά παραδείγματα από μη-τετριμμένες v_1 αναλλοίωτες, όπως το πολυώνυμο affine index, και αναλλοίωτες που προκύπτουν από το ανάπτυγμα Taylor του πολυωνύμου Turaev.

Στο πρώτο κεφάλαιο δίνουμε τους βασικούς ορισμούς και τα βασικά αποτελέσματα από τη θεωρία των κομβοειδών, και δίνουμε παραδείγματα από κάποιες ισχυρές αναλλοίωτες κομβοειδών. Στο δεύτερο κεφάλαιο παρουσιάζουμε τα βασικά αποτελέσματα των αναλλοίωτων Vassiliev και των διαγραμμάτων χορδών. Δίνουμε απόδειξη ότι τα διαγράμματα χορδών έχουν πλούσια αλγεβρική δομή και διατυπώνουμε το Θεώρημα Vassiliev-Kontsevich, το οποίο αποδεικνύουμε στο Κεφάλαιο 3 εισάγοντας την Θεωρία των συνοχών Khnizhnik-Zamolodchikov και τη χρήση του ολοκληρώματος Kontsevich. Τα κεφάλαια 4,5,6 περιέχουν νέα αποτελέσματα, τα οποία είναι κοινή δουλειά με τη Σ. Λαμπροπούλου και τον L.H. Kauffman. Στο Κεφάλοιο 4 εισάγουμε τη Θεωρία των ιδιομορφικών κομβοειδών, δίνουμε παραδείγματα από αναλλοίωτες Vassiliev που παίρνουμε από διαφορετικά κλεισίματα των κομβοειδών, και κατασκευάζουμε τα γραμμικά διαγράμματα χορδών. Στο Κεφάλαιο 5 δίνουμε κάποια τεχνικά λήμματα για μια ἀναπαράσταση σε ράγες' των σφαιρικών κομβοειδών, τα οποία θα είναι χρήσιμα για να αποδείξουμε το Θεώρημα Ταξινόμησης των σφαιρικών κομβοειδών με ακριβώς ένα σημείο ιδιομορφίας. Στο κεφάλαιο 6 αποδεικνύουμε το Θεώρημα Ταξινόμησης χρησιμοποιώντας τα κεφάλαια 4,5, και κατασκευάζουμε την καθολική αναλλοίωτη τύπου ένα.

Chapter 1

Knotoids

1.1 Definition

Definition 1.1.1. Let Σ be a path connected, oriented 2-manifold without boundary, smoothly embedded in $\mathbb{R}^3$. A *knotoid diagram* is an immersion γ of the interval in Σ, whose singularities are only finitely many transversal double points, which are endowed with over- or undercrossing data. These double points are called crossings. $\gamma(0)$ is called the *leg* and $\gamma(1)$ is called the *head* of the knotoid diagram, while both comprise the *endpoints* of the knotoid. We will actually call a knotoid diagram its image in Σ rather than the immersion itself. Similarly, a *multi-knotoid diagram* is an immersion of a disjoint union of the interval and some copies of the circle, and an isotopy class of a multi-knotoid diagram is a *multi-knotoid*.

In classical knot theory, ambient isotopy is generated combinatorially by three local moves on diagrams, the *Reidemeister moves*, see Fig. 1.1, together with planar or disc isotopy. Note that in the knotoid theory an isotopy may displace the endpoints, but may not pull a strand adjacent to an endpoint over or under a transversal strand. This justifies the notion of the *forbidden moves* Ω_+, Ω_-, illustrated in Fig. 1.2.

Definition 1.1.2. Two knotoid diagrams K_1, K_2 in a surface Σ are called *isotopic* if they differ by disc isotopies of Σ and the Reidemeister moves away from the endpoints. An equivalence class of a knotoid diagram (up to isotopy) is called a *knotoid*.

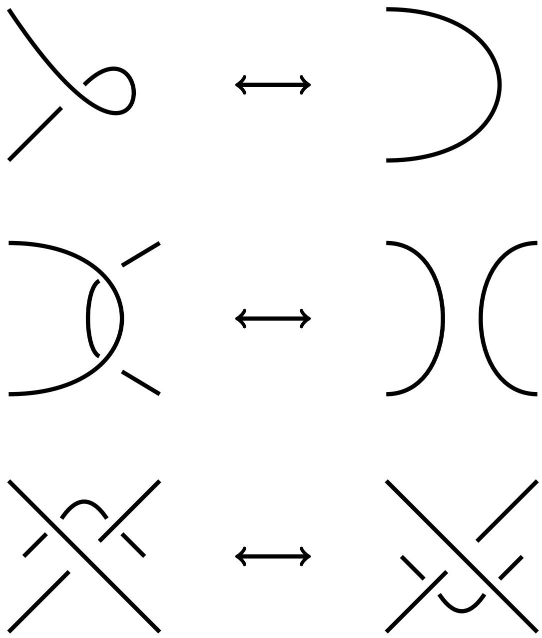


Figure 1.1: The Reidemeister moves I, II and III

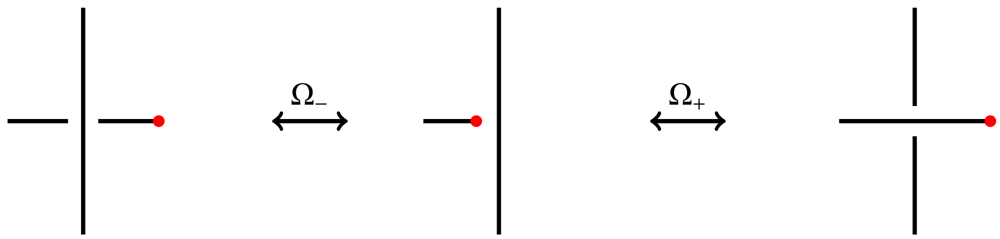


Figure 1.2: The forbidden moves for knotoids

The height (or complexity) of a knotoid K is the minimum number of crossings over all diagrams of K that are created when joining the leg and the head of K by a simple unknotted arc. The knotoids of zero height are the *knot-type knotoids* and their isotopy classes correspond bijectively to the classical knot types. The knotoids of height greater than zero are called *proper knotoids*.

Definition 1.1.3. Let $\mathcal{M}$ be a category of mathematical structures (e.g.Laurent polynomials, abelian groups, commutative rings). An invariant of knotoids is as usual a mapping $I : \mathit{Knotoids} \to \mathcal{M}$ such that equivalent knotoids map to equivalent structures in $\mathcal{M}$.

Knotoids in $\mathbb{R}^2$ and S^2 are called planar and spherical knotoids respectively and are denoted $K(\mathbb{R}^2)$ and $K(S^2)$. Writing $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ we get the well-defined surjective map $\imath : K(\mathbb{R}^2) \to K(S^2)$. On the other hand, any knotoid in S^2 can be represented by a knotoid diagram in $\mathbb{R}^2$ by pushing a representative diagram in S^2 away from the point at infinity. The knotoid represented by an embedding $[0, 1] \hookrightarrow \Sigma$ is called trivial in Σ. There exist examples of non-trivial planar knotoids which are trivial in $K(S^2)$. The simplest such example is the unifoil. See Fig 1.3

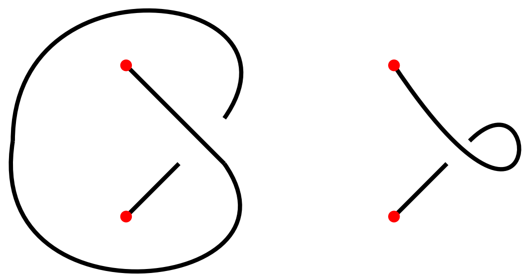


Figure 1.3: Two non-equivalent planar knotoids which are equivalent as spherical

In [25] and [8] several invariants for knotoids have been defined, mainly for spherical ones. One of them is the bracket polynomial.

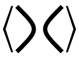
In a knotoid diagram we can take an arc joining the leg and the head without passing through any other existing crossing but only creating new ones. Such an arc will be called a *shortcut* and there are various ways to declare the new crossings to obtain a knot diagram in an invariant manner. We will later deal with the different type of closures for singular knotoids. So we will postpone for the moment the proofs concerning the under, over or virtual closure of a knotoid.

If K is a knotoid diagram with n crossings and for the moment ignore the over/under extra data, then we get a graph in S^2 with n 4-valent vertices and 2 1-valent vertices which correspond to the endpoints.This graph is called the underlying graph of the knotoid diagram K, and straightforwardly from the Euler's formula it divides S^2 into $n + 1$ regions, which we will call regions of the knotoid. The same hold identically for $\mathbb{R}^2$ instead of S^2.

1.2 The Kauffman bracket polynomial and the Jones polynomial

We first introduce the Kauffman bracket polynomial in an inductive way, exactly as for classical knots.

Definition 1.2.1. Let K be a knotoid or a multi-knotoid diagram. Let $< K >$ be the element of the ring $Z[A, B, d]$ defined by means of the rules:

1. $\langle$  $\rangle = 1$ meaning that the Kauffman bracket of the trivial knotoid is trivial but since the trivial knotoid is knot type of course also $\langle$  $\rangle = 1$
2. $\langle L \sqcup$  $\rangle = d\langle L\rangle$
3. $\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$

We must determine A, B, d in order to have knotoid isotopy invariance or invariance under Reidemeister moves. Rule (1) says that $< K >$ takes the value 1 on a single unknotted linear or circular diagram. Rule (2) says that $< K >$ is multiplied by d when the knotoid diagram $\tilde{K}$ is a disjoint union of K with a circular or linear (unknoted) component. This component, if circular, can surround other parts of the diagram. Rule (3) applies to diagrams that differ locally at a neighbourhood of a crossing. Applying rule (3) several times we expand the formulas until we reach diagrams consisting of disjoint unions of circles (or generally Jordan curves), and one unknotted arc embedded in the surface (i.e. the trivial knotoid). Rule (2) then implies that the value of <> on a disjoint collection of $n-1$ circles and one arc is d^{n-1}. And of course rule (3) implies the formula

$$\langle \times \rangle = B \langle \asymp \rangle + A \langle)(\rangle$$

Definition 1.2.2. We define a *state* on a knotoid diagram $K \subset \Sigma$, when Σ is an orientable surface, a mapping from the set of crossings of K to the set $\{-1, 1\}$.

If G is the underlying planar graph for K, then a state of G is a choice of splitting marker for every vertex of G. We can choose between the A-splitting, which is $\asymp$, and the B-splitting, which is $)($. We call the underlying planar graph for a diagram K the universe for K. This terminology distinguishes the underlying planar graph from the link projection and from other graphs that can arise. Thus we speak of the states of a universe.

Since splitting all the vertices of a state results in a configuration of disjoint circles and an embedded segment, we see that the states are in one-to-one correspondence with final configurations in the expansion of the bracket. Accordingly, we define for a diagram K and a state S the bracket $\langle K|S \rangle$, given by the formula $\langle K|S \rangle = A^m B^l$, where m is the number of state markers that correspond to the A-splitting and l is the number of state markers that correspond to the B-splitting. The total contribution of a given state to the polynomial $\langle K \rangle$ is then given by the formula $\langle K|S \rangle \, d^{|S|-1}$, where $|S|$ is the number of the connected components of the 1-manifold formed when all the splittings are done. Then we have:

Proposition 1.2.3. $\langle K \rangle$ is uniquely determined from the rules (1), (2), (3) and is given by the formula

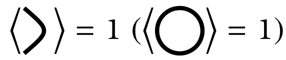
$$\langle K \rangle = \sum_{S:state} \langle K|S \rangle \, d^{|S|-1}$$

Lemma 1.2.4.

 $= AB \langle)(\rangle + (ABd + A^2 + B^2) \langle \asymp \rangle$

where the three diagrams represent the same projection except in the area indicated. And hence  $= \langle)(\rangle$ iff $AB = 1$ and $d = -A^2 - A^{-2}$.

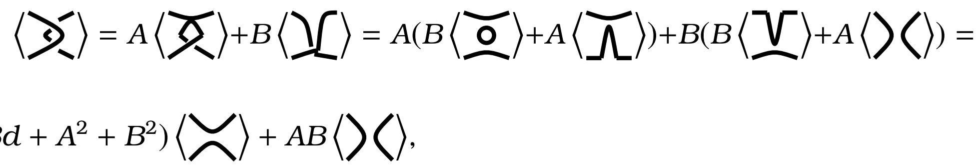
The rules of the Kauffman bracket are now deformed to the following

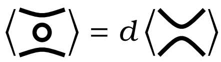
1. 
2. $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$
3. 

and the formula for the bracket becomes

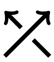
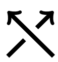
$$\langle K \rangle = \sum_{S:state} \langle K|S \rangle \, (-A^2 - A^{-2})^{|S|-1} = \sum_{S:state} A^{\sigma_s}(-A^2 - A^{-2})^{|S|-1}$$

where $\sigma_s \in \mathbb{Z}$ is the sum of the values ± 1 of the states over all crossings of K.

Proof. 

where the first and the second equality follow by rule (3) and then we use rule (2) to deduce that , and again rule (3). □

One can easily show that the bracket polynomial ($\langle K \rangle \in \mathbb{Z}[A^{\pm 1}]$) is a Laurent polynomial which is regular isotopy invariant (i.e. invariant under Reidemeister II and III moves). Indeed, it is only left to check the Reidemeister III move, which is a straightforward calculation. So $\langle K \rangle$ is a regular isotopy invariant for knotoids and an isotopy invariant when consider up to multiplication of integer powers of $-A^3$.

Furthermore, $\langle K \rangle$ is multiplied by $-A^{\pm 3}$ under the Reidemeister I moves. We call writhe $wr(K) = n_+ - n_-$ where n_+ is the number of positive crossings  and n_- is the number of positive crossings . The writhe is clearly a regular isotopy invariant and changes by ± 1 under Reidemeister I moves.

From the above we have:

Proposition 1.2.5. $\langle K \rangle$ is a regular isotopy invariant for knotoids. Moreover,

$$f_K = (-A^3)^{-wr(K)} \sum_{s:state} A^{\sigma_s}(-A^2 - A^{-2})^{|s|-1}$$

is an isotopy invariant for knotoids, which is called the *normalized bracket polynomial* for knotoids (and which is the analogue of the *Jones polynomial* by an appropriate change of variable).

1.3 The Turaev extended bracket polynomial

The *Turaev extended bracket polynomial* is a generalisation of the Kauffman bracket polynomial for knotoids, introduced by V. Turaev. In the original paper [25] it is denoted by $\langle\langle K\rangle\rangle_o$, we will denote it by T_K. It is a two-variable Laurent polynomial $T_K \in \mathbb{Z}[A^{\pm1}, u^{\pm1}]$. T_K is defined as follows: Let K be a knotoid diagram in S^2. Pick a shortcut a for K, meaning a generic arc in S^2 which joins the leg with the head of the knotoid, creating the least number of extra crossings. Then, take a state s and the smoothed 1-manifold K_s described in the bracket polynomial. Let k_s be the segment component of K_s. Note that k_s coincides with K in a small neighborhood of the endpoints of K. In particular, both k_s and a have the same endpoints as K. We orient K, k_s and a, from the leg of K to the head of K. Let $k_s \cdot a$ be the number of times k_s crosses a from right to left minus the number of times k_s crosses a from left to right (the common endpoints are not counted). Similarly, let $K \cdot a$ be the algebraic number of intersections of K with a. We then define

$$T_K(A, u) := (-A^3)^{-wr(K)} u^{-K\cdot a} \sum_{s:state} A^{\sigma_s} u^{k_s\cdot a} (-A^2 - A^{-2})^{|s|-1}.$$

Clearly $T(A, u)$ is an isotopy invariant for knotoids.

1.4 The affine index polynomial

The affine index polynomial was defined for virtual knots and links by L.H. Kauffman [13] and then for knotoids, virtual or classical, by N. Gügümcü and L.H. Kauffman [8]. It is based on an integer labeling assigned to flat knotoid diagrams (i.e. diagrams with the information 'under' or 'over' on classical crossings omitted) in the following way. A flat knotoid diagram, classical or virtual, is associated with a graph where the flat classical crossings and the endpoints are regarded as the vertices of the graph. An arc of an oriented flat knotoid diagram is an edge of the graph it represents, that extends from one vertex to the next vertex.

We start labeling the edges as exemplified in Fig. 1.4. At each flat crossing, the labels of the arcs change by one. If the incoming arc labeled by $a \in \mathbb{Z}$ crosses another arc which goes to the right then the next arc is labeled by $a + 1$; if the incoming arc $b \in \mathbb{Z}$ crosses another arc which goes to the left then it is labeled by $b - 1$. There is no change of labels at virtual crossings. Note that it is convenient to label the first arc from the leg to the first crossing by 0.

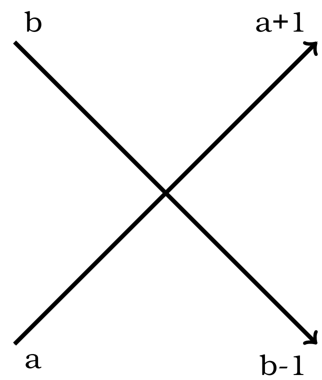


Figure 1.4: Integer labeling

Let c be a classical crossing of the knotoid diagram K. We denote $w_+(c), w_-(c)$ the following integers deriving from the labels at the flat crossing corresponding to c:

$$w_+(c) := b - (a + 1)$$

$$w_-(c) := a - (b - 1)$$

where a and b are the labels for the left and the right incoming arcs, respectively. The numbers $w_+(c)$ and $w_-(c)$ are called positive and negative weights of c, respectively. Define, now, the *weight* of c to be:

$$w_K(c) := w_{sgn(c)}(c)$$

Definition 1.4.1. The *affine index polynomial* of K is defined as

$$P_K(t) := \sum_{c \in Cr(K)} sgn(c)(t^{w_K(c)} - 1)$$

where $Cr(K)$ is the set of all classical crossings of K.

The affine index polynomial is an isotopy invariant and has some very interesting properties, as shown in [8], including that its higher degree is smaller than or equal to the height of the knotoid. Furthermore it is very easy to calculate by hand, as in the following example which will be very useful in what follows.

Example 1.4.2. We will calculate the affine index polynomial of the knotoid below:

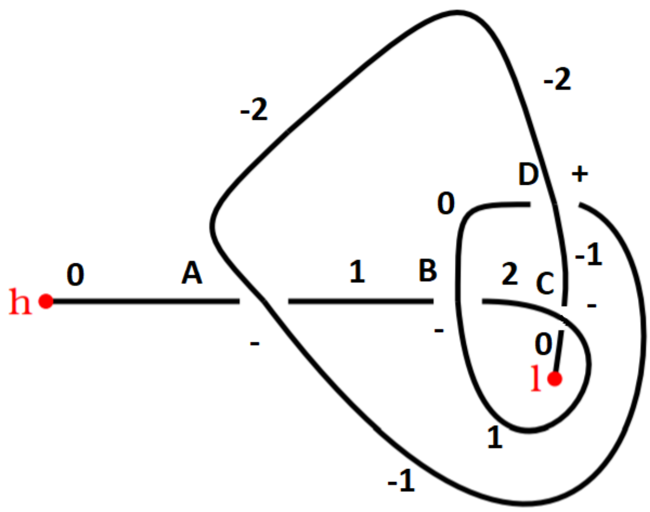


Figure 1.5: Labeling the knotoid

	$w_+ = b - (a+1)$	$w_- = a - (b-1)$
A	2	-2
B	1	-1
C	-2	2
D	-1	1

Table 1.1: Weights for the affine index polynomial

With the labeling of Fig. 1.5 and of Table 1.1 we have all we need for calculating the polynomial. Namely:

$$P_K = \sum_{c \in cr(K)} sgn(c)(t^{\omega_k(c)} - 1) = -(t^{-2} - 1) - (t^{-1} - 1) - (t^2 - 1) + (t^{-1} - 1) =$$

$$= -(t^2 + t^{-2} - 2).$$

This calculation will be used afterwards, but for now note that we have a simple proof that K, as illustrated in Fig. 1.5, has height 2.

Chapter 2

Finite type invariants for knots and chord diagrams

In the original work of V. Vassiliev [26], finite type invariants correspond to the zero-dimensional classes of a special spectral sequence. In the theory, the space of knots is the complement of the so-called discriminant. Knot invariants are just locally constant functions in the space of knots and Vassiliev's formalism was trying to show that there exists such a spectral sequence that converges to the cohomology of the space of knots. The sequel works of Bar-Natan[1], Birman & Lin[3], Stanford[23, 23] and Polyak & Viro[22] simplified greatly Vassiliev's theory, making it combinatorial, especially by the use of the chord diagrams. In this work we are mainly interested in this combinatorial approach.

2.1 Vassiliev invariants of finite type

Definition 2.1.1. A *singular knot* is a mapping of S^1 in $\mathbb{R}^3$ that fails to be an embedding only in finitely many points where we have only transversal self-intersections, the *singular crossings*. If $f : S^1 \to \mathbb{R}^3$ is an almost everywhere smooth mapping, then a simple transversal double point $p \in im(f)$ is a point in which $f^{-1}(p) = \{t_1, t_2\}$ such that $f'(t_1), f'(t_2)$ are linearly independent.

Definition 2.1.2. Two singular knots are said to be *rigid vertex isotopic* if any two diagrams of theirs differ by a fine sequence of disc isotopies, the Reidemeister moves for classical knots, and the rigid vertex isotopy moves involving singular crossings, some variations of which are illustrated in Figs. 1.1 and 2.1.

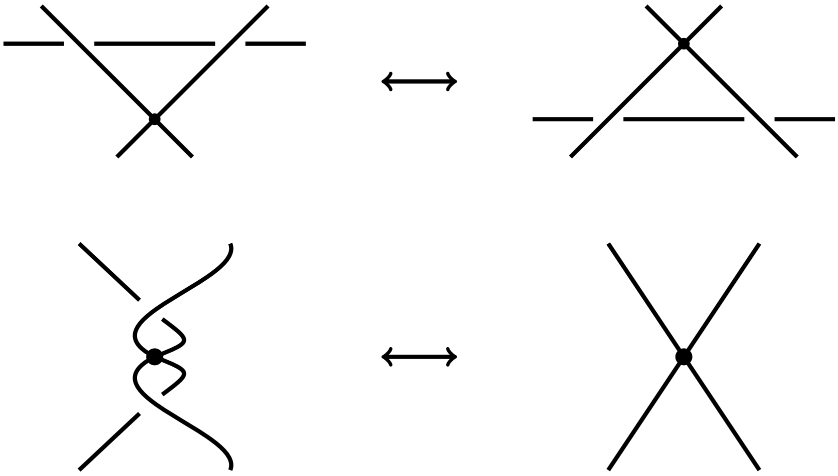
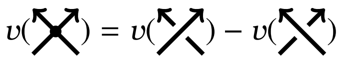


Figure 2.1: Rigid vertex isotopy moves for singular crossings

Any knot invariant v can be extended to an invariant of singular knots using the *Vassiliev skein relation*[26]:



(2.1.1)

The small diagrams in the relation denote diagrams that are identical except for the regions in which they differ only as indicated in the small diagrams. Using this relation successively, one can extend v to singular knots with an arbitrary number of singular crossings. Such an invariant is said to be a *Vassiliev invariant.*

On the other hand, given a singular knot K, there are many choices when resolving a sequence of singular crossings of K; in fact the complete resolution yields the alternating sum

$$\sum_{\varepsilon_1=\pm1,\ldots,\varepsilon_n=\pm1} (-1)^{|\varepsilon|} v(K_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n})$$

where $|\varepsilon|$ is the number of -1's in the sequence of ε_i and $K_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}$ is the knot obtained by a positive/negative resolution of each singular crossing of K.

Definition 2.1.3. A knot invariant v is said to be of *finite type* $\leqslant k$ if there exists a $k \in \mathbb{N}$ such that v vanishes on every singular knot with more than k singular crossings. We say that v *is of type* k if it is of type $\leqslant k$ and not of type $\leqslant k-1$.

Symbolically, the set of all $\mathbb{C}$-valued Vassiliev invariants of finite type $\leqslant k$ is V_K and it has a natural grading

$$V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots \subset V := \bigcup_{k=0}^{\infty} V_k$$

It follows from the definition that each V_k and V are complex vector spaces of finite and infinite dimension respectively. We could, if needed, replace $\mathbb{C}$ by any commutative ring R; thus each V_k is an R-module and so is its graded completion.

Definition 2.1.4. The *closure of* V, denoted $\bar{V}$, is defined as:

$v \in \bar{V} \iff \forall K_1, K_2$ with $v(K_1) \neq v(K_2)$ there exists an $n \in \mathbb{N}$ and a Vassiliev invariant of finite type $\leqslant n$, v_n, such that $v_n(K_1) \neq v_n(K_2)$.

2.2 Chord diagrams

Definition 2.2.1. A *chord diagram* of order n is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. The set of all chord diagrams of order n will be denoted by A_n. This means that only the relative combinatorial positions of the ends of the chords are important. Their precise geometrical locations on the circle are irrelevant.

By a chord diagram of a singular knot K we mean a circle parametrising K with the two preimages of each singular crossing connected by a chord. Furthermore, a *top row diagram* is a knot with precisely n singular crossings.

Lemma 2.2.2. The value of a Vassiliev invariant, v, of order $\leq n$ on a top row diagram depends only on the chord diagram and not on the graph embedding in the space.

Proof. Let K_1, K_2 be singular knots with n singularities, which have diffeomorphic chord diagrams. Then there is a bijection between the chords of the two chord diagrams and hence between the singularities. Place K_1, K_2 in $\mathbb{R}^3$ so that the corresponding singularities coincide together with both branches of the knot in a neighbourhood of each singular crossing. Now, K_1, K_2 are two embeddings of the same abstract graph, and we know that we can obtain one from the other by a finite sequence of crossing switches. Now we can deform K_1 into K_2 in such a way that some small neighbourhoods of the singular crossings do not move (rigid-vertex isotopy). By a general position argument, we may assume that the only new singularities created in the process of this deformation, that is, of the crossing switches, are a finite number of transversal double points, all at distinct values of the deformation parameter. By the Vassiliev skein relation, in each of these switching events the value of v does not change, and this implies that $v(K_1) = v(K_2)$. □

Remark 2.2.3. The technique of (classical) crossing switches can be applied to any finite type invariant of order k when applied to a top row diagram. Namely, one can use Eq. 2.1.1 for switching any real crossing at will. As a result, the value of a finite type invariant of order k depends only on the nodal structure (the underlying 4-valent graph) and not on the embedding of the graph in space.

2.3 1-term and 4-term relations

Definition 2.3.1. The *symbol* of a Vassiliev invariant of finite type $\leqslant n$ is the restriction of v to the set of singular knots with precisely n singular crossings, considered as a function on the set of chord diagrams.

It would be a question whether the symbol of an invariant is an arbitrary function on chord diagrams. We shall explore this question by following the story of the Vassiliev-Kontsevich theorem. In fact, it satisfies certain relations. The so-called one-term relation (1T) is very easy to see. Namely, the symbol of an invariant v always vanishes on a chord diagram with an isolated chord (not intersecting other chords). This follows from the fact that we can choose a singular knot representing such a chord diagram as having a small kink parameterized by the arc between the endpoints of the isolated chord. Then the Vassiliev skein relation 2.1.1 and the Reidemeister I move gives what is illustrated in fig. 2.2. In the following relations, for reasons of simplicity, when summing diagrams we mean the evaluations of the Vassiliev invariant v on the relevant diagrams.

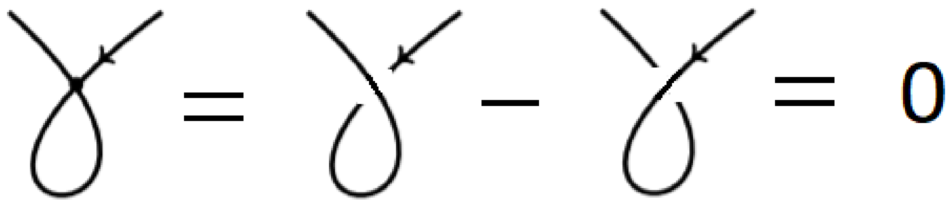


Figure 2.2: One-term relation

The second topological property of v is the so-called four-term relation (4T), see Fig. 2.3.

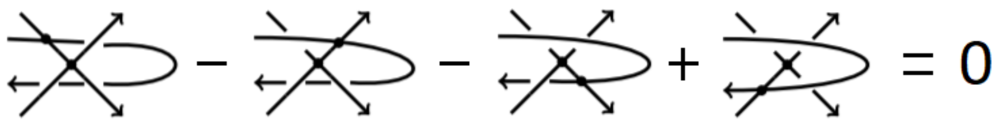


Figure 2.3: Four-term relation

The proof is topological and pure skein theoretic, as it uses consecutively the Vassiliev skein relation to get the result, see Fig 2.4.

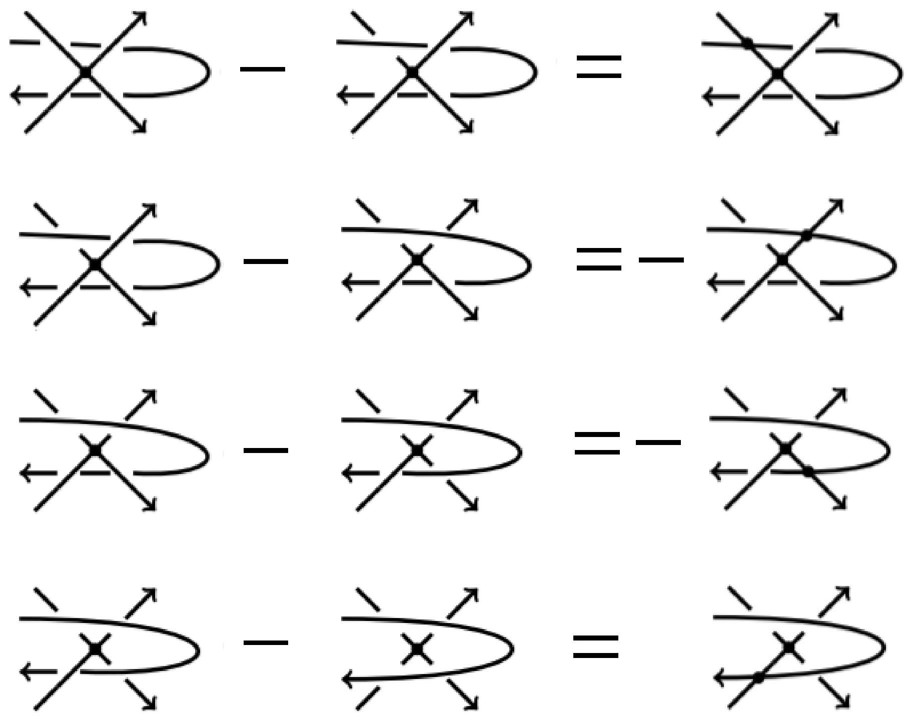


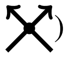
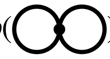
Figure 2.4: Proof of the four-term relation

Summing over these four relations by parts, the left hand side vanishes and hence we derive the desired formula.

So the four-term relation holds for all singular knots with more than two singularities, as well as for the corresponding chord diagrams.

Remark 2.3.2. Every knot invariant of type 1 is trivial.

Indeed, let υ be an invariant of type 1 and let K be a singular knot with exactly 1 singular crossing (υ vanishes on knots with more singularities). The singular crossing divides the knot into two disjoint closed curves. After appropriate classical crossing switches, using the Vassiliev skein relation, the two closed curves can turn into simple unknotted closed curves with no self-crossings or shared crossings between them. Now, these crossing switches cost nothing, because each use of Eq. 2.1.1 will add a singular crossing to the one of the two resulting diagrams and, by definition, υ will vanish on this diagram. Hence, we derive the relation:

$v(K :=$  $) = v($  $)$

which, again, using Eq. 2.1.1, is clearly equal to zero, by the Reidemeister I move.

Proposition 2.3.3. There exists an injection $\bar{a}_n : V_n \setminus V_{n-1} \to \mathcal{R}A_n$ where $\mathcal{R}A_n$, the dual space of A_n modulo (1T) and (4T).

Proof. Let $\{e_1, e_2, \ldots, e_k\}$ be a set of representatives of non-equivalent chord diagrams of order n. Then every symbol of type n depends only on the values on $\{e_1, e_2, \ldots, e_k\}$. Thus, for every $v_n \in V_n \setminus V_{n-1}$ we have the values $\{v_n(e_1), v_n(e_2), \ldots, v_n(e_k)\}$ which define a linear function in $\mathbb{R}^k$. So, we have the $1-1$ operator $T : V_n/V_{n-1} \to (\mathbb{R}^k)^*$ which is not an epimorphism. One can see that easily, since, for example, every non-trivial element of V_1/V_0 cannot correspond to a linear function which does not vanish in the unique chord diagram with one chord.

Let, now, U be a vector space with basis $(e_1, \ldots, e_k)$, U^* its dual space and $F \subseteq U^*$ such that $\forall f \in F$

$$\sum_{i=1}^{k} \lambda_{1i} f(e_i) = 0, \ldots, \sum_{i=1}^{k} \lambda_{pi} f(e_i) = 0,$$

for λ_{ji} fixed. Let, further, $U_1 \subseteq U$ subspace generated by

$$w_1 = \sum_{i=1}^{k} \lambda_{1i} e_i, \ldots, w_p = \sum_{i=1}^{k} \lambda_{pi} e_i$$

Then every $f \in F$ vanishes in U_1, thus F is can be thought of as a subset of $(U/U_1)^*$.

We return to $\mathbb{R}^k$ with basis $(e_1, \ldots, e_k)$ that corresponds to chord diagrams of order n. In this basis we want to express the equations that come from (1T) and (4T), such that:

- $v(e_s) = 0$, for some indices s.
- $v(e_a) - v(e_b) + v(e_c) - v(e_d) = 0$, for appropriate indices.

Then the symbol of a Vassiliev invariant of type $\leqslant n$ can be thought of as a linear function in $\mathbb{R}^k/U_1$, where U_1 is spanned by the expressions

$$e_s, e_a - e_b + e_c - e_d.$$

On the other hand, the space A_n modulo (1T) and (4T) could be presented by $\{e_1, \ldots, e_k\}$ such that

- $e_s = 0$
- $e_a - e_b + e_c - e_d = 0$

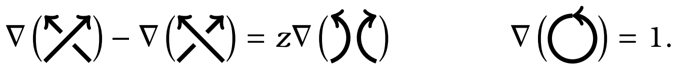
Hence, we have shown that there exists a monomorphism of V_n/V_{n-1} in $\mathcal{R}A_n$.

$\square$

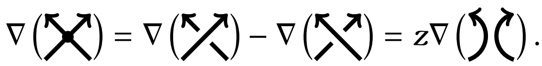
2.4 Examples of Vassiliev invariants of knots

2.4.1 The Conway polynomial

The Conway polynomial may be defined by the skein relation and its initial condition on the unknot:



Its coefficients are Vassiliev invariants. Indeed, comparing the Conway skein relation with the Vassiliev skein relation (2.1.1) we conclude that the Conway polynomial of a knot with a singular crossing is divisible by z:



Consequently, the Conway polynomial of a knot with $k > n$ singular crossings is divisible by z^k. Thus, its coefficient $c_n(K)$ at z^n vanishes on singular knots with $> n$ singular crossings. Therefore, it is a Vassiliev invariant of order $\leqslant n$.

2.4.2 Finite type invariants and the Jones polynomial

Recall the definition of the normalized bracket for a knot K:

$$f_K(A) := (-A^{-3})^{-w(K)} \langle K \rangle (A)$$

where $w(K) = n_+ - n_-$ the writhe of K for $n_\pm$ the number of positive/negative crossings of K.

Making the substitution $t^{1/4} = A^{-1}$ we obtain the classical Jones polynomial. The skein relation for the Jones polynomial together with an initial condition for the unknot is

$$t^{-1}J(\;) - tJ(\;) = (t^{1/2} - t^{-1/2})J(\;) \;; \qquad J(\;) = 1.$$

Making a further substitution $t = e^x$ and then taking the Taylor expansion into a formal power series in x, we can represent the Jones polynomial of a knot K as a power series:

$$J(K) = \sum_{n=0}^{\infty} j_n(K) x^n.$$

We claim that the coefficient $j_n(K)$ is a Vassiliev invariant of order $\leqslant n$. Indeed, substituting $t = e^x$ into the skein relation gives:

$$(1 - x + \ldots) \cdot J(\;) - (1 + x + \ldots) \cdot J(\;) = (x + \frac{x^2}{4} + \ldots) \cdot J(\;)$$

From which we obtain:

$$J(\;) = J(\;) - J(\;) = x\big(j_0(\;) + j_0(\;) + j_0(\;)\big) + \ldots$$

This means that the value of the Jones polynomial on a knot with a single singular crossing is divisible by x. Therefore, the Jones polynomial of a singular knot with $k > n$ singular crossings is divisible by x^k, and thus its nth coefficient vanishes on such a singular knot.

2.4.3 The HOMFLYPT polynomial

The skein relation and the initial condition for the HOMFLYPT polynomial is

$$aP(\;) - a^{-1}P(\;) = zP(\;) \;; \qquad P(\;) = 1 .$$

Making a substitution $a = e^x$ and taking the Taylor expansion in x, we represent $P(K)$ as a Laurent polynomial in z and a power series in x, $P(K) = \sum p_{k,l}(K)\, x^k z^l$. Similarly to the case of the Jones polynomial, one can show that $p_{k,l}(K)$ is a Vassiliev invariant of order $\leqslant k + l$.

Remark 2.4.1. Despite the fact that the coefficients of the Conway, the Jones and the HOMFLYPT polynomials are finite type invariants, the degrees of these polynomials are not of finite type. Nevertheless, they belong to the closure $\bar{\mathcal{V}}$.

2.5 Lie algebras

Definition 2.5.1. A *Lie algebra* g over $\mathbb{C}$ is a vector space equipped with a bilinear operation $(x, y) \to [x, y]$, the so-called Lie bracket, which also satisfies the identities

$$[x, y] = -[y, x]$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

We will call g abelian if $[x, y] = 0 \quad \forall x, y \in g$. Obviously, any vector space V endowed with the trivial bracket is an abelian Lie algebra.

We usually consider the bilinear operation of Lie bracket as a product, and hence one can speak about rings, homomorphisms, ideals etc. An ideal in a Lie algebra g is a vector subspace I so that $[a, x] \in I \quad \forall a \in g, x \in I$.

Definition 2.5.2. A Lie algebra is called *simple* if it is not abelian and does not contain any proper ideals. It is called *semi-simple* if it is isomorphic to a direct sum of simple Lie algebras.

For $x \in g$ we write ad_x for the linear map $ad_x : g \to g$ defined by $ad_x(y) = [x, y]$. The *Killing form* on a Lie algebra g is defined by

$$< x, y >^K := Tr(ad_x ad_y)$$

Note that this bilinear form is non-degenerate iff g is semi-simple.

Definition 2.5.3. A bilinear form $<,>: g \otimes g \to \mathbb{C}$ is ad-invariant iff

$$< ad_z(x), y > + < x, ad_z(y) >= 0 \quad \forall x, y, z \in g$$

A Lie algebra equipped with an ad-invariant metric is said to be *metrized.*

If (e_i) is a basis of the vector space (Lie algebra) g we have

$$[e_i, e_j] = \sum_{k=1}^{d} c_{ijk} e_k$$

and the coefficients c_{ijk} are called the *structure constants* of g with respect to the given basis. Note that, in a metrized Lie algebra if (e_i) is an orthonormal basis with respect to the ad-invariant metric, the structure constants are anti-symmetric with respect to permutations of the indices, which is obvious by the definition of ad-invariance.

Given an associative algebra A one can think of it as a Lie algebra equipped with the commutator $[a, b] = ab - ba$. The converse is not true since not every Lie algebra can be thought to be in this context, but in fact every Lie algebra g is a subspace of an associative algebra closed under the commutator.

Definition 2.5.4. A *representation* of a Lie algebra g in a vector space V is a Lie algebra homomorphism $\rho : g \to gl(V)$, where $gl(V)$ is the Lie algebra of linear operators of V.

This means that ρ maps g to $gl(V)$ so that,

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

Equivalently, one could say that ρ is a g-action on V and V is a g-module. The invariants of the action are the elements of V that lie in the kernel of ρ for all $x \in g$. Symbolically V^g is the space of all invariants in V.

Definition 2.5.5. The *Universal Enveloping Algebra* of g, which is denoted by $U(g)$, is the quotient of the tensor algebra $T(g)$ by the two-sided ideal generated by all expressions of the form:

$$x \otimes y - y \otimes x - [x, y]$$

One could understand that what we did here was to force the commutator of two elements of $g \subset T(g)$ to be equal to their Lie bracket in g.

2.6 Quantum knot invariants

Let g be a semisimple Lie algebra and V a finite dimensional representation of g. One can view V as a representation of the universal enveloping algebra $U(g)$. This representation can be also deformed with parameter q to a representation of the quantum group $U_q(g)$. The vector space V remains the same, but the action now depends on q. For a generic value of q all irreducible representations of $U_q(g)$ can be obtained in this way. Using representation theory we can go through the technicalities also for non-generic values of q and obtain compact results on 3-manifold invariants, but the generic values are enough for our purposes.

Remark 2.6.1. An important property of quantum groups is that every representation gives rise to a linear transformation $R : V \otimes V \to V \otimes V$ which is a solution of the *quantum Yang-Baxter equation*:

$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R) \tag{2.6.1}$$

The general construction of the matrix corresponding to the linear transformation R from a representation of a semi-simple Lie algebra g is the following. Consider a knot diagram in the plane and take a generic horizontal line. To each intersection point of the line with the diagram assign either the representation space V or its dual V^*, depending on whether the orientation of the knot at this intersection is directed upwards or downwards. See left-hand illustration of Fig. 2.5. Then take the tensor product of all such spaces over the whole horizontal line. If the knot diagram does not intersect the line, then the corresponding vector space is the ground field $\mathbb{C}$. A portion of a knot diagram between two such horizontal lines represents a tangle T. We assume that this tangle is framed by the blackboard framing. Consecutive tangles can be viewed to be multiplied by concatenation.

To T we associate a linear transformation $\partial^{fr}(T)$ from the vector space corresponding to the bottom of T to the vector space corresponding to the top of T. See right-hand illustration of Fig. 2.5. The following three properties hold for the linear transformation $\partial^{fr}(T)$:

- $\mathcal{J}^{fr}(T)$ is an invariant of the isotopy class of the framed tangle T.

- $\mathcal{J}^{fr}(T_1 \cdot T_2) = \mathcal{J}^{fr}(T_1) \circ \mathcal{J}^{fr}(T_2)$

- $\mathcal{J}^{fr}(T_1 \otimes T_2) = \mathcal{J}^{fr}(T_1) \otimes \mathcal{J}^{fr}(T_2)$

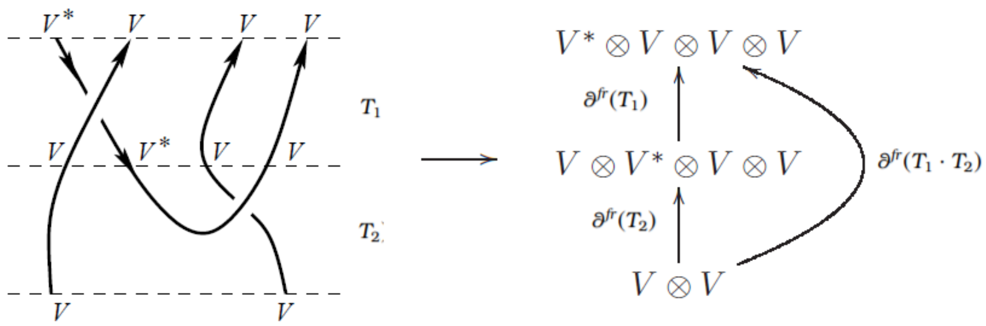


Figure 2.5: Multiplicity of quantum invariants of tangles

Because of the multiplicativity property, it is enough to define the invariant only for elementary tangles such as a crossing, a minimum and a maximum point (a cup and a cap). So, given a quantum group $U_q(g)$ and a finite-dimensional representation, V, one can associate certain linear transformations with elementary tangles in a consistent way. Of course, for a trivial tangle consisting of a single string connecting top to bottom, the corresponding linear operator should be the identity transformation. The R-matrix appears here as the linear transformation corresponding to a positive crossing, while R^{-1} corresponds to a negative crossing. So, the validity of the Reidemeister II move is straightforward. We are ready to check the validity of the Reidemeister III move which is the most complicated one. Fig. 2.6 shows clearly that it is the same as the quantum Yang-Baxter equation.

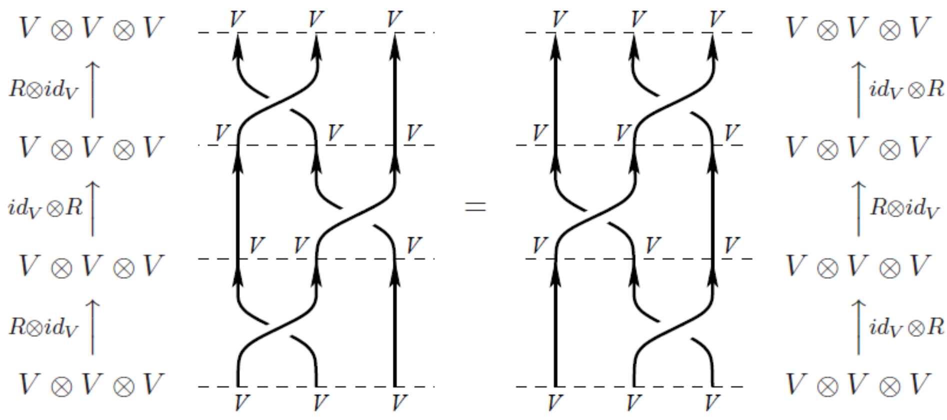


Figure 2.6: Reidemeister III move for quantum invariants

This process is valid for all quantum groups. Since we are not interested to go in deep with the quantum invariants, we will conclude the chapter with the construction for $g = sl_2$ and its standard 2-dimensional representation.

$$R : \begin{cases} e_1 \otimes e_1 \mapsto q^{\frac{1}{4}} e_1 \otimes e_1 \\ e_1 \otimes e_2 \mapsto q^{-\frac{1}{4}} e_2 \otimes e_1 \\ e_2 \otimes e_1 \mapsto q^{-\frac{1}{4}} e_1 \otimes e_2 + (-q^{\frac{3}{4}} + q^{-\frac{1}{4}}) e_2 \otimes e_1 \\ e_2 \otimes e_2 \mapsto q^{-\frac{1}{4}} e_2 \otimes e_2 \end{cases}$$

One could see in a typical way that the quantum invariant corresponding to sl_2 is a substitution of the Jones polynomial. Generally, the known polynomial invariants have quantum invariant descriptions. This fact is crucial for studying categorifications, homology theories for knots and links and, of course, Topological Quantum Field Theory.

J. Birman and X.-S. Lin in [3] proved that all quantum invariants produce Vassiliev invariants in the way similar to the previous examples. Namely, making a substitution $q = e^x$, one can show that the coefficient of x^n in the Taylor expansion of a quantum invariant is a finite type invariant of type $\leqslant n$. Thus all quantum invariants belong to the closure $\bar{\mathcal{V}}$.

2.7 The graded algebra of circular (knot) chord diagrams

Recall that a chord diagram encodes the order of singular crossings along a singular knot. A Vassiliev invariant of order k gives rise to a function on chord diagrams with k chords. The conditions that a function on chord diagrams should satisfy in order to come from a Vassiliev invariant are the so-called one-term and four-term relations (as adapted to chord diagrams). Indeed, for the chord diagrams we can reinterpret the (4T) relation to what Fig. 2.7 illustrates, using the injection $\bar{a}_n$.

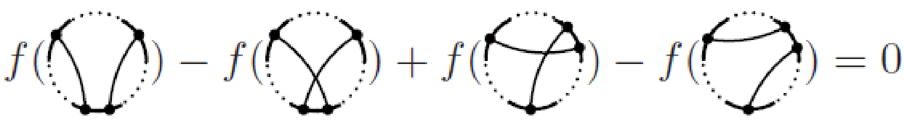


Figure 2.7: Four-term relation on chord diagrams

The vector space spanned by chord diagrams, A_n, modulo the (4T) relations is denoted by $\mathcal{A}_n^{fr}$, and, as we will see, it has the structure of a Hopf algebra.

Definition 2.7.1. We call a function $f : A_n \to R$ which satisfies the (4T) relations a *(framed) weight system*. Here R is a commutative ring, but we usually think of $R = \mathbb{C}$. We will call a weight system that also satisfies the (1T) relations an *unframed weight system*.

Our goal is to define an algebra structure on $\mathcal{A}^{fr} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n^{fr}$.

Definition 2.7.2. We define the *product* of two chord diagrams D_1, D_2 to be their connected sum along the two circles, in a way that the cutting and gluing take place in regions of the circles that do not involve any endpoint of any chord.

The product of chord diagrams is extended linearly to a map $\mu : \mathcal{A}_n^{fr} \otimes \mathcal{A}_m^{fr} \to \mathcal{A}_{n+m}^{fr}$.

Lemma 2.7.3. The product of chord diagrams is well-defined modulo the (4T) relations.

Proof. It is enough to prove that if one of the two diagrams, say D_2, is turned inside the product diagram by as much as it is required for the cutting to happen in the region that was next to it without moving D_1, then the result is the same modulo (4T) relations.

Let n be the order of D_1 and a chord of D_2 with endpoints a, b adjacent to D_1. What we shall prove is that the chord diagram is stable when moving a across an endpoint of a chord while fixing b. In this process we obtain $2n+1$ chord diagrams, say $P_0, P_1, \dots, P_{2n}$. We must prove that P_0 is equivalent to P_{2n} modulo (4T) relations. But summing over all (4T) relations we get every $P_k, k \in \{1, \dots, 2n-1\}$ an even number of times, with opposite signs, and only once P_0 and the last $-P_{2n}$. Hence $P_0 - P_{2n} = 0$ which was the desired result. □

Definition 2.7.4. We define the *co-product* in the algebra $\mathcal{A}_n^{fr}$ as

$$\delta : \mathcal{A}_n^{fr} \to \bigotimes_{k+l=n} \mathcal{A}_k^{fr} \otimes \mathcal{A}_l^{fr}$$

with value on a chord diagram $D \in \mathcal{A}_n^{fr}$

$$\delta(D) = \sum_{J \subseteq [D]} D_J \otimes D_{\bar{J}}$$

where the summation is taken over all subsets J of $[D]$, the set of chords of D. Here D_J is the diagram consisting of the chords that belong to J and $\bar{J} = [D] \setminus J$ is the complementary subset of chords.

Lemma 2.7.5. *The co-product in $\mathcal{A}_n^{fr}$ is well-defined modulo the (4T) relations.*

Proof. Let $D_1 - D_2 + D_3 - D_4 = 0$ be a (4T) relation. We must show that the sum $\delta(D_1) - \delta(D_2) + \delta(D_3) - \delta(D_4)$ can be written as a combination of (4T) relations. Recall that a specific (4T) relation is determined by the choice of a moving chord m and a fixed chord a. Now, take a chord diagram and the splitting $A \cup B$ of the set of chords in the diagrams D_i, the same for each i, and denote by A_i, B_i the resulting chord diagrams giving the contributions $A_i \otimes B_i$ to $\delta(D_i), i = \{1, 2, 3, 4\}$. Suppose, without loss of generality, that the moving chord m belongs to the subset A. Then $B_1 = B_2 = B_3 = B_4$ and

$A_1 \otimes B_1 - A_2 \otimes B_2 + A_3 \otimes B_3 - A_4 \otimes B_4 = (A_1 - A_2 + A_3 - A_4) \otimes B_1$. If the fixed chord a also belongs to A, then $A_1 - A_2 + A_3 - A_4$ is a (4T) relation. If a belongs to B then it is $A_1 = A_2$ and $A_3 = A_4$. □

In Fig. 2.8 we compute an example of a co-product.

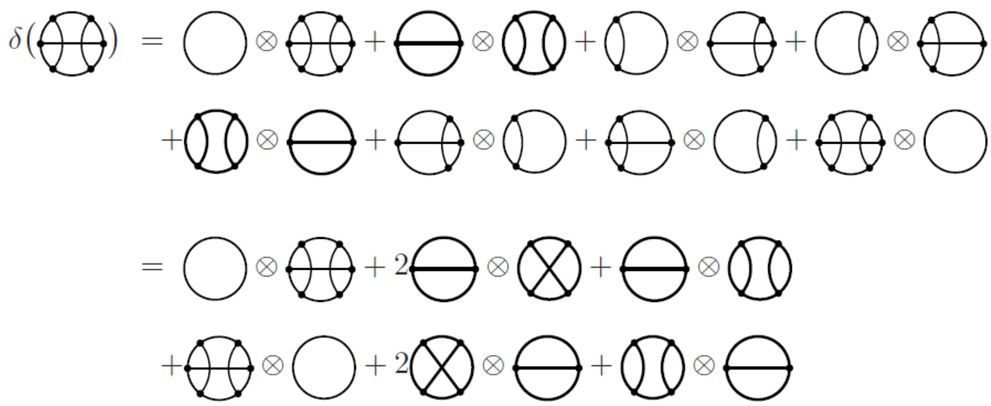


Figure 2.8: Example of co-product

The *unit*, ι, and the *co-unit*, ε, in the algebra $\mathcal{A}^{fr}$ are:

$$\iota : R \to \mathcal{A}^{fr} \quad , \quad \iota(x) = x\bigcirc$$

$$\varepsilon : \mathcal{A}^{fr} \to R \quad , \quad \varepsilon(x\bigcirc + \dots) = x$$

Remark 2.7.6. Note that we could end up with the same construction of Hopf algebra structure if in the chord diagrams we allowed two types of trivalent vertices:

- Internal vertices in which three segments, which start from the circle, meet. These vertices are oriented by specifying one of the two possible cyclic orderings of the arcs emanating from such a vertex.
- External vertices in which a chord ends on the circle.

Then we would take the set of all such new diagrams, symb. D^t, and consider it modulo the so-called STU relation, as illustrated in Fig. 2.9, which is a purely algebraic construction. Indeed: take a (matrix) Lie algebra with generators T_a. Then, a natural equation expresses the closure of the Lie algebra under commutators, namely: $T^aT^b - T^bT^a = f_{abc}T^c$, which is diagrammatically equivalent to the STU relation. Denote $A^t = D^t/\{STU\}$, which is isomorphic to the algebra of chord diagrams that we defined earlier, as the STU relation implies the (4T) relation easily. Hence, another approach to the theory could be purely algebraic and, so, one would pay attention to the properties obtained by the Lie-algebra structures. This approach is used for proving the flatness of the KZ-connection.

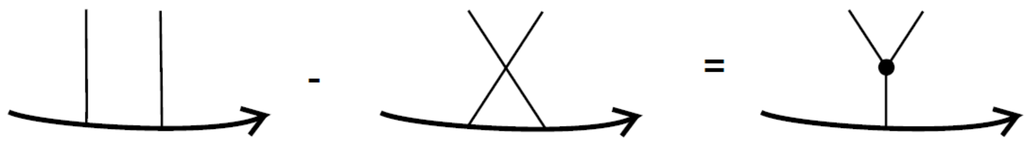


Figure 2.9: The STU relation

2.8 The Vassiliev-Kontsevich Theorem

Recall that the map $\bar{a}_n : \mathcal{V}_n/\mathcal{V}_{n-1} \longrightarrow RA_n$ is injective. For $R = \mathbb{C}$ the map $\bar{a}_n$ identifies $\mathcal{V}_n/\mathcal{V}_{n-1}$ with the subspace of unframed weight systems $W_n \subset RA_n$. In other words, the space of unframed weight systems is isomorphic to the graded space associated with the filtered space of Vassiliev invariants:

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} \mathcal{W}_n \cong \bigoplus_{n=0}^{\infty} \mathcal{V}_n/\mathcal{V}_{n-1}$$

Theorem 2.8.1. The theorem consists of two parts:

1. (V. Vassiliev) The symbol of every Vassiliev invariant is an unframed weight system.

2. (M. Kontsevich) Every unframed weight system is the symbol of a certain Vassiliev invariant.

With the technology we developed so far we can prove for the moment the first and easy part of the Vassiliev-Kontsevich theorem and wait to develop some analytic and geometric tools for proving the converse.

Proof. The idea is to prove that given a function $f \in RA_n$ coming from the symbol of an invariant v, it satisfies the (1T) and (4T) relations.

Let K be a singular knot whose chord diagram contains an isolated chord. The double point p that corresponds to the preimage of the isolated chord divides the knot into two disjoint parts K_1, K_2. The fact that the chord is isolated means that K_1 and K_2 do not have common double points. There may, however, be crossings involving branches from both parts, which can be ruled out by crossing switches using the Vassiliev skein relation. So we can untangle K_1 from K_2 and obtain a singular knot K' with the same chord diagram as K and with the property that the two parts lie on either side of some plane in $\mathbb{R}^3$ that passes through the singular crossing p. Thus, easily by the Vassiliev skein relation: $v(K) = v(K') = v(K'_+) - v(K'_-) = 0$

The (4T) relation, as already said, is purely topological property, so the proof that any function $f \in RA_n$ coming from the symbol of an invariant v satisfies the (4T) relation is illustrated in Figs. 2.3, 2.4.

□

Chapter 3

The Kontsevich integral

The space $\mathcal{A}_n$ of unframed chord diagrams of order n is the quotient of $\mathcal{A}_n^{fr}$ by the subspace spanned by all diagrams with an isolated chord, i.e. an (1T) relation. Let $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$. Then, roughly speaking, an idea of the proof of the Kontsevich theorem is to construct an element $Z(K) \in \mathcal{A}$ of the algebra of chord diagrams for every knot K. Having a weight system w we can apply it to $Z(K)$ and prove that $w(Z(K))$ is a Vassiliev invariant whose symbol is w. However, when we try to realize this idea, several complications occur. The first one is that $Z(K)$ is going to be an element of the graded completion $\widehat{\mathcal{A}}$ of the algebra $\mathcal{A}$, or in other words, it is going to be an infinite sum of elements of $\mathcal{A}_n$ for all values of n, like a formal power series. The second one is that $Z(K)$ is not quite an invariant of knots. We will have to correct it before applying the weight system to it.

3.1 The construction

Let $z \in \mathbb{C}$ and $t \in \mathbb{R}$ be coordinates (z, t) in $\mathbb{R}^3$. The planes t = const are thought of being horizontal. We define the Kontsevich integral for strict Morse knots, i.e. knots with the property that the coordinate t restricted to the knot has only non-degenerate critical points with distinct critical values.

Definition 3.1.1. The *Kontsevich integral* $Z(K)$ of a strict Morse knot K is given by the following formula

$$Z(K) := \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int\limits_{\substack{t_{\min}<t_1<\cdots<t_m<t_{\max} \\ t_j \text{ are noncritical}}} \sum_{P=\{(z_j,z_j')\}} (-1)^{\downarrow_P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \,,$$

where

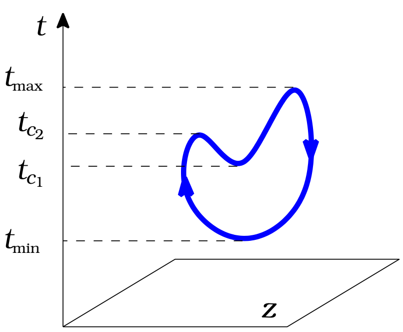
- the numbers $t_{\min}$ and $t_{\max}$ are the minimum and the maximum of the function t on K.
- the integration domain is the set of all points of the m-dimensional simplex $t_{\min} < t_1 < \cdots < t_m < t_{\max}$ none of whose coordinates t_i is a critical value of t; the m-simplex is divided by the critical values into several connected components.
- the number of summands in the integrand is constant in each connected component of the integration domain, but can be different for different components. in each plane $\{t = t_j\} \subset \mathbb{R}^3$ choose an unordered pair of distinct points (z_j, t_j) and (z_j', t_j) on K, so that $z_j(t_j)$ and $z_j'(t_j)$ are continuous functions; we denote by $P = \{(z_j, z_j')\}$ the set of such pairs for $j = 1, \ldots, m$ and call it a *pairing*; the integrand is the sum over all choices of the pairing P.
- for a pairing P, the symbol '$\downarrow_P$' denotes the number of points (z_j, t_j) or (z_j', t_j) in P where the coordinate t decreases as one goes along K;
- for a pairing P, consider the knot K as an oriented circle and connect the points (z_j, t_j) and (z_j', t_j) by a chord; we obtain a chord diagram with m chords (thus, intuitively, one can think of a pairing as a way of inscribing a chord diagram into a knot in such a way that all chords are horizontal and are placed on different levels).
- over each connected component, z_j and z_j' are smooth functions in t_j; by $\bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$ we mean the pullback of this form to the integration

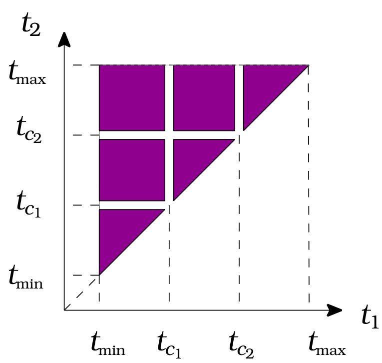
domain of the variables $t_1, \ldots, t_m$; the integration domain is considered with the orientation of the space $\mathbb{R}^m$ defined by the natural order of the coordinates $t_1, \ldots, t_m$.

- by convention, the term in the Kontsevich integral corresponding to $m = 0$ is the (only) chord diagram of order 0 taken with coefficient one, which is the unit of the algebra $\widehat{\mathcal{A}}$.

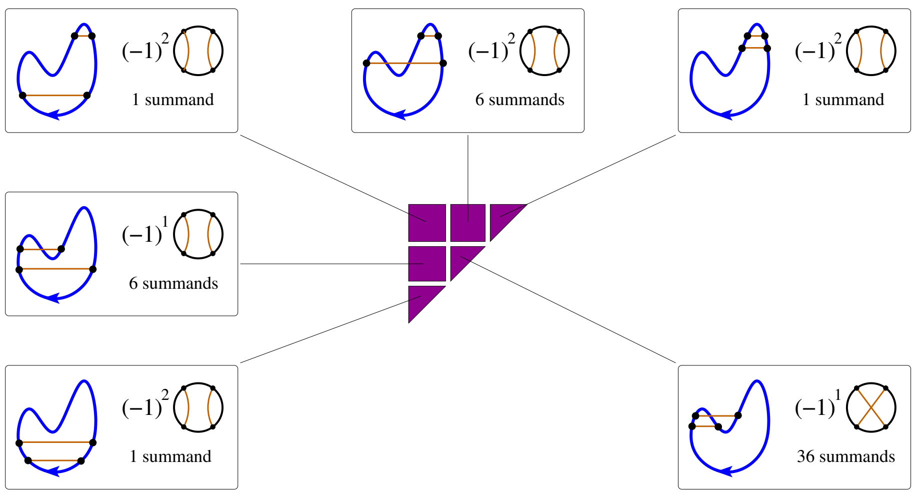
Example 3.1.2. *(see [5])*

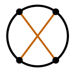
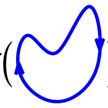
We exemplify the integration domain for a strict Morse knot with two local maxima and two local minima for the case of $m = 2$ chords.

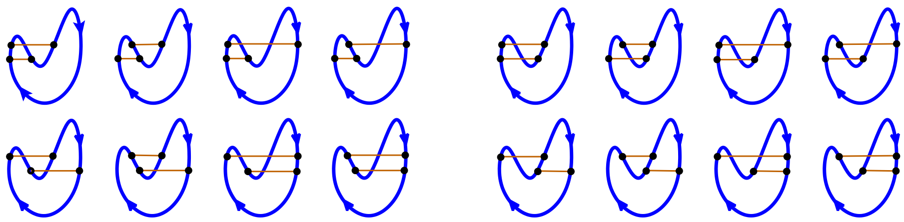


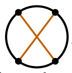


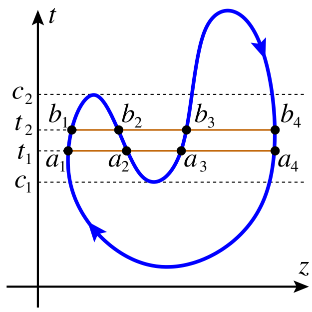
For each connected component of the integration domain, the number of summands corresponding to different choices of the pairing, a typical pairing P, and the corresponding chord diagram $(-1)^{\downarrow_P} D_P$ are shown in the picture.



Now let us calculate the coefficient of the chord diagram  in $Z($$)$. Out of the 51 pairings, the following 16 contribute to the coefficient:



All of them appear on the middle triangular component, $t_{c_1} < t_1 < t_2 < t_{c_1}$ of the integration domain. To handle the integral which appears as the coefficient at , we denote the z-coordinates of the four points in the pairings on the level $\{t = t_1\}$ by a_1, a_2, a_3, a_4. Correspondingly, we denote the z-coordinates of the four points in the pairings on the level $\{t = t_2\}$ by b_1, b_2, b_3, b_4:



Then each of the four possible pairings $z_1 - z_1'$ on the level $\{t = t_1\}$ will look like $a_{jk} := a_k - a_j$ for $(jk) \in A := \{(12), (13), (24), (34)\}$. Similarly, each of the four possible pairings $z_2 - z_2'$ on the level $\{t = t_2\}$ will look like $b_{lm} := b_m - b_l$ for $(lm) \in B := \{(13), (23), (14), (24)\}$. The integral we are interested in now can be written as

$$\frac{1}{(2\pi i)^2} \int\limits_{\Delta} \sum_{(jk)\in A} \sum_{(lm)\in B} (-1)^{j+k+l+m} d \ln a_{jk} \wedge d \ln b_{lm}$$

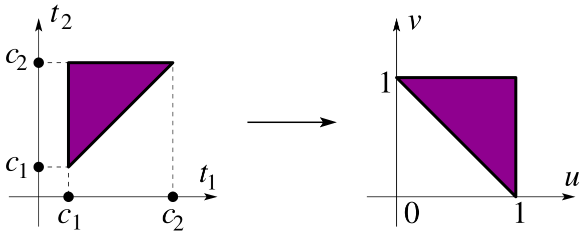
$$= -\frac{1}{4\pi^2} \int\limits_{\Delta} \Bigg(\sum_{(jk)\in A} (-1)^{j+k+1} d \ln a_{jk} \Bigg) \wedge \Bigg(\sum_{(lm)\in B} (-1)^{l+m-1} d \ln b_{lm} \Bigg)$$

$$= -\frac{1}{4\pi^2} \int\limits_{\Delta} d \ln \frac{a_{12} a_{34}}{a_{13} a_{24}} \wedge d \ln \frac{b_{14} b_{23}}{b_{13} b_{24}} \ .$$

The change of variables

$$u := \frac{a_{12}a_{34}}{a_{13}a_{24}}, \qquad v := \frac{b_{14}b_{23}}{b_{13}b_{24}}$$

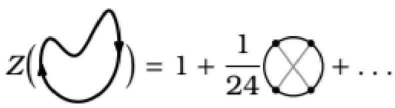
transforms the component Δ of the integration domain into the standard triangle Δ'

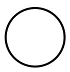


Since it changes the orientation of the triangle (has a negative Jacobian), our integral becomes

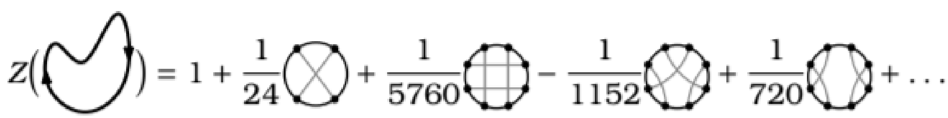
$$\begin{aligned}\frac{1}{4\pi^2}\int_{\Delta'} d\ln u \wedge d\ln v &= \frac{1}{4\pi^2}\int_0^1\left(\int_{1-u}^1 d\ln v\right)\frac{du}{u}\\ &= -\frac{1}{4\pi^2}\int_0^1 \ln(1-u)\frac{du}{u} = \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\int_0^1 \frac{u^k}{k}\frac{du}{u}\\ &= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\zeta(2)}{4\pi^2} = \frac{1}{24}\,.\end{aligned}$$

Therefore,



where the free term 1 stands for the unit in the algebra $\mathcal{A}$ of chord diagrams, $1 =$  $\in \mathcal{A}$.

The following terms of this integral are of degree 4:



3.2 The universal Vassiliev invariant

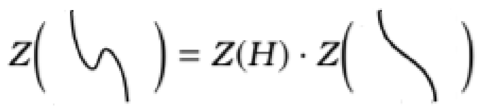
Here we deal with the second complication in the proof of the Kontsevich theorem. The Kontsevich integral possesses several basic properties:

- $Z(K)$ converges for any strict Morse knot K.
- It is invariant under the deformations of the knot in the class of Morse knots with the same number of critical points.
- It behaves in a predictable way under the deformations that add a pair of new critical points to a Morse knot.

We will only sketch the proof of these properties here referring the reader to [1], [5]. The last property may be specified as follows. Let H stand for the unknot from Example 3.1.2:

$H :=$  .

If a knot isotopy creates one extra local minimum and one local maximum, the Kontsevich integral is multiplied by $Z(H)$:



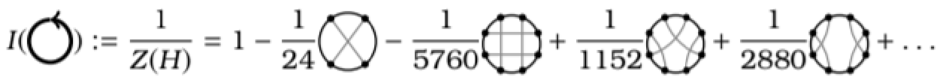
where the multiplication is understood in sense of the formal infinite series from the graded completion of the algebra of chord diagrams $\widehat{\mathcal{A}}$.

Definition 3.2.1. With the help of this formula, we can define the *universal Vassiliev invariant* by either

$$I(K) := \frac{Z(K)}{Z(H)^{c/2}} \qquad \text{or} \qquad I'(K) := \frac{Z(K)}{Z(H)^{c/2-1}},$$

where c denotes the number of critical points of K in an arbitrary strict Morse representation, and the quotient means division in the algebra $\widehat{\mathcal{A}}$ following the law: $(1+a)^{-1} = 1 - a + a^2 - a^3 + \ldots$.

Both $I(K)$ and $I'(K)$ are invariants of topological knots with values in $\widehat{\mathcal{A}}$. The version $I'(K)$ has the advantage of being multiplicative with respect to the connected sum of knots; in particular, it vanishes (more precisely, takes the value 1) on the unknot. However, the version $I(K)$ is also used as it has a direct relationship with the quantum invariants. In particular,



Proposition 3.2.2. The Kontsevich integral is multiplicative for any Morse knot K.

$$Z(K_1 \cdot K_2) = Z(K_1) \cdot Z(K_2)$$

Actually the whole discussion of the Kontsevich integral applies in any morse tangle just by replacing the algebra $\hat{A}$ of chord diagrams by the graded completion of the vector space of tangle chord diagrams on the skeleton of T,and take t_{min} and t_{max} to correspond to the bottom and the top of T. So what we mean with the dot product is a multiplication of tangles. So we rewrite the proposition as $Z(T_1 \cdot T_2) = Z(T_1) \cdot Z(T_2)$ for any strict Morse tangles T_1, T_2

Proof. Let t_{min} and t_{max} correspond to the bottom and the top of $T_1 \cdot T_2$, respectively, and let t_{mid} be the level of the top of T_2 (or the bottom of T_1, which is the same). In the expression for the Kontsevich integral of the tangle $T_1 \cdot T_2$ let us remove from the domain of integration all points with at least one coordinate t equal to t_{mid}. This set is of codimension one, so the value of integral remains unchanged. On the other hand, the connected components of the new domain of integration are precisely all products of the connected components for T_1 and T_2, and the integrand for $T_1 \cdot T_2$ is the exterior product of the integrands for T_1 and T_2. From which we take the desired result using the Fubini theorem. ◻

3.3 Convergence of the Kontsevich integral

Proposition 3.3.1. For any strict Morse tangle T the Kontsevich integral $Z(T)$ converges.

Proof

A ”short chord” is a chord whose endpoints are adjacent, that is, one of the arcs that it bounds contains no endpoints of other chords. In particular, short chords are isolated. The linear span of all diagrams with a short chord and all four-term relation contains all diagrams with an isolated chord.

The integration domain may have many connected components, in the boundary of whom the integrand may have singularities. This happens near a critical point of a tangle when a pairing includes a "short chord" i.e. its ends are on the branches of the knot that come together at a critical point. In this case the integrand could have a problem as the denominator $z_j - z_j'$ may explode.

First of all, from the multiplicity of the Kontsevich integral for tangles we have that the Kontsevich integral of a product converges iff the integral of the factors do so. With this in hand and the fact that we can decompose a strict Morse tangle to simpler ones i.e. with at most one critical point. So it is sufficient to prove the convergence only for tangles with at most one critical point.

So suppose that T has only one critical point with value t_c.Then for any pairing, its coefficients in the integral of T is factors through a product of one integral which corresponds to the chords over t_c and one to the chords under t_c. The integral which corresponds to the chords over t_c converges since the integrand has no singularities. Hence we consider the chords under t_c. We take two seperate cases.

- There exists an isolated chord $(z_1, z_1{}')$ under t_c which tends to 0. In this case the isolated chord contributes nothing to the integrand due to the 1-term relation since the corresponding chord diagram D_p has an isolated chord.

- There exists a chord $(z_i, z_i{}')$ that tends to zero near t_c but the chord is not isolated, and exists at least one chord that seperates it.

Let for the second case $(z_1, z_1{}')$ the chord that tends to zero near the critical point and $(z_2, z_2{}')$ the chord that seperate it which does not tend to zero near t_c.

We have then

$$\left|\int_{t_1}^{t_c} \frac{dz_2 - dz_2'}{z_2 - z_2'}\right| \leq C\left|\int_{t_1}^{t_c} d(z_2 - z_2')\right| = C\left|(z_c - z_1) - (z_c' - z_1'')\right| \leq C'\left|z_1 - z_1'\right|$$

So this integral is of the same "order" as $z_1 - z_1'$ and this compensates the denominator corresponding to the chord (z_2, z_2'). Hence by induction one sees that this could happen for every k chords which separate (z_1, z_1'). The preceeding imply the convergence of the Kontsevich integral.

3.4 The Knizhnik–Zamolodchikov connection

The Knizhnik-Zamolodchikov equation appears in Wess-Zumino-Witten model of conformal field theory.

Let M be a smooth manifold and $\hat{A}$ an assosciative algebra. The intuition behind that is that we will use the algebra of chord diagrams but the definitions hold in general so for the moment we let $\hat{A}$ to be arbitrary or better the graded completion of a graded algebra A. We will assume also that it is an algebra over $\mathbb{C}$ with unit 1.

A connection is a 1-form whose values are in the algebra of endomorphisms of the fiber. Some results rising from the thory of connections have application to 1-forms of an arbitrary $\hat{A}$ as we discussed. So, an $\hat{A}$-valued connection is a $\hat{A}$-valued 1-form Ω on M.

Its curvature is a 2-form given in an arbitrary form $F_\Omega = d\Omega + \Omega \wedge \Omega$ The definition of the exterior differentiation and the wedge product for $\hat{A}$-valued forms is the same as in matrix-valued forms.

Let also, $\gamma : I \to M$ a piecewise-smooth curve in M, where $I = [a, b]$.

The holonomy $h_{\gamma,\Omega}$ along γ is a function $h_{\gamma,\Omega} : I \to \hat{A}$ such that $h_{\gamma,\Omega}(a) = 1$

as initial condition and satisfies the equation

$$\frac{\partial}{\partial t} h_{\gamma,\Omega}(t) = \Omega(\dot{\gamma}(t)) h_{\gamma,\Omega}(a), (t \in I)$$

In many interesting cases $h_{\gamma,\Omega}$ exists and is given by the iterated integral formula. (Note that if it exists it is unique).

$$h_{\gamma,\Omega} = 1 + \sum_{k=1}^{\infty} \int_{a \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant t} (\gamma^* \Omega)(t_k) \ldots (\gamma^* \Omega)(t_1)$$

since we take h takes values in the graded algebra $\hat{A}$ and so we can write $h = h_1 + h_2 + \ldots$

and then

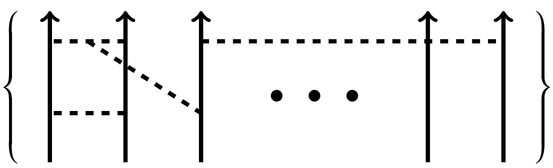
$$h_0'(t) = 0 \quad h_0(0) = 1$$

$$h_1'(t) = \Omega(t) h_0(t) \quad h_1(0) = 0$$

where $\Omega(t)$ is actually the pullback of Ω under γ. We continue inductively to take the desired formula and $h_1(t) = \int_0^t \Omega(t_1) d_{t_1}$, and respectively $h_2(t) = \int_0^t \Omega(t_2) h_1(t_2) = \int_{0<t_1<t_2<t} \Omega(t_2) \wedge \Omega(t_1) \ldots.$

Furthermore, just like in the standard theory of connections, if $F_\Omega = 0$,so Ω is flat, then the holonomy is invariant under homotopies of γ that preserve its endpoints. In the case of interest for us, The connection will be homogeneous of degree 1. Note that, in this case the kth term $h_{k,\gamma,\Omega}$ is homogeneous of degree k, and there is no problem with the convergence of the sum there. Also, as each term lives in a different degree, Chen's theory [4] implies that each term is invariant under homotopies of γ that preserve its endpoints.

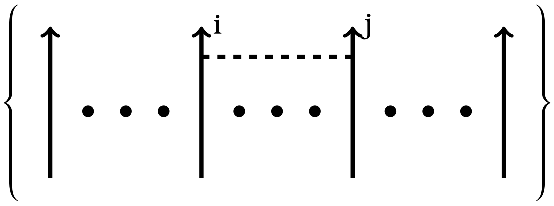
Let D_n^{KZ} be the collection of all diagrams made of n ordered upward pointing arrows, and dashed arcs and oriented vertices as in the definition of A^t, with the standard conventions about higher than trivalent vertices and about the orientation of vertices:

$D_n^{KZ} =$ 

Let A_n^{KZ} be the quotient of $D_n^{KZ}/\{STU\text{relations}\}$ Then A_n^{KZ} is a graded algebra with the composition as a product operator.

For $1 \leqslant i \leqslant j \leqslant n$ define $\Omega_{ij} \in A_n^{KZ}$ by

$\Omega_{ij} =$



Let $X_n = \{(z_1, z_2, \dots z_n) \in \mathbb{C}^n | z_i = z_j \Rightarrow i = j\}$ and let the complex 1-form on X_n defined by

$$\omega_{ij} = dlog(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j}$$

The formal Knizhnik-Zamolodchikov connection is the A_n^{KZ} -valued connection

$$\Omega_n = \sum_{1 \leqslant i < j \leqslant n} \Omega_{ij}\omega_{ij}$$

on X_n.

The Formal Knizhnik-Zamolodchikov Ω_n by a straightforward calculation can be proven to be flat, which leads to the invariance of the Kontsevich integral under horizontal deformations, small removings of needles and critical points. These are proven in detail in [1] and in [5] but are beyond the scope of this text.

3.5 Proof of the Vassiliev-Kontsevich Theorem

The first part of the theorem is already proven. We are now get ready to prove the second and difficult part of the theorem using the Kontsevich integral.

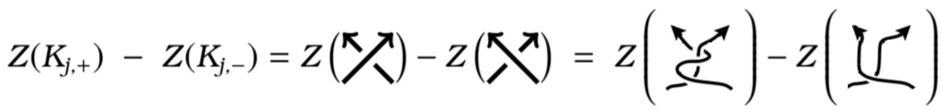
The central importance of $I(K)$ (as well as $I'(K)$) in the theory of finite-type invariants is that it is a universal Vassiliev invariant in the following sense. Consider a weight system w of order n (that is, a function on the set of chord diagrams with n chords satisfying one- and four-term relations). Applying w to the n-homogeneous part of the series $I(K)$, we get a numerical knot invariant $v := w(I(K))$. This invariant is a Vassiliev invariant of order $\leqslant n$ and $\mathrm{symb}(v) = w$. Moreover, any Vassiliev invariant is a sum of Vassiliev invariants obtained in this way. Indeed, any Vassiliev invariant v has a symbol $\mathrm{symb}(v)$ which is a weight system. So $v(K)$ and $\mathrm{symb}(v)(I(K))$ are two Vassiliev invariants with the same symbol. Their difference is of order $\leqslant (n-1)$ and we can repeat the same process by induction. This argument proves the Kontsevich theorem.

To prove that $v : K \longmapsto w(I(K))$ is a Vassiliev invariant with symbol w, it is enough to show that $I(K_D) = D +$ (terms of order > n) for a singular knot K_D representing a chord diagram D of order n. Let $p_1, \ldots p_n$ be the singular crossings of K_D.

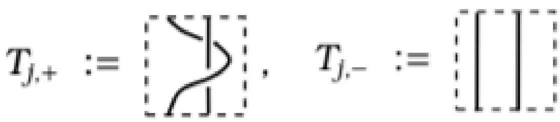
Since the denominator of $I(K)$ starts with the unit of the algebra $\widehat{\mathcal{A}}$, it is sufficient to prove that

$$Z(K_D) = D + (\text{terms of order} > n)\ .$$

By definition, $Z(K_D)$ is the alternating sum of 2^n values of Z on the complete resolutions of the singular knot K_D at all its singular crossings $p_1, \ldots p_n$. To see what happens at a single singular crossing p_j, let us look at the difference of Z on two knots $K_{j,+}$ and $K_{j,-}$ where an overpassing in $K_{j,+}$ is changed to an underpassing in $K_{j,-}$. By an isotopy we have:



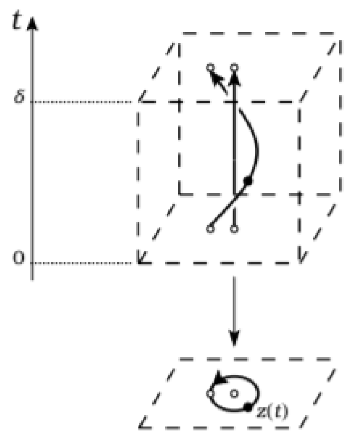
These two knots differ only in that the first knot has one string that makes a small curl around another string. We assume that this curl is very thin and enclose it in a box of height δ to consider it as a tangle $T_{j,+}$. Let $T_{j,-}$ be the similar tangle for $K_{j,-}$ consisting of two parallel vertical strings:



The difference $Z(K_{j,+}) - Z(K_{j,-})$ comes only from the difference of the Kontsevich integrals of tangles $Z(T_{j,+}) - Z(T_{j,-})$. For the tangle $T_{j,-}$, both functions

$z(t)$ and $z'(t)$ are constant, hence $Z(T_{j,-})$ consists of only one summand: the trivial chord diagram $D_-^{(0)}$ on $T_{j,-}$ (here 'trivial' means with an empty set of chords) with coefficient one. The zero degree term $D_+^{(0)}$ of $Z(T_{j,+})$ is also the trivial chord diagram on $T_{j,+}$ with coefficient one. So $D_-^{(0)} = D_+^{(0)}$ as chord diagrams. Thus the difference $Z(T_{j,+}) - Z(T_{j,-})$ starts with a first degree term which comes from the chord connecting the two strings of $T_{j,+}$.

Hence, $Z(K_D)$ starts from degree n. Moreover, the term of degree n is proportional to a chord diagram whose j-th chord connects the two strings of the tangle $T_{j,+}$ corresponding to the jth singular crossing p_j in K_D. This chord diagram is precisely D. Now to calculate the order n part of $Z(K_D)$ we must compute the coefficient of D. It is equal to the product of coefficients of one chord terms in $Z(T_{j,+})$ over all j. We will show that all of them are equal to 1. Indeed, it is possible to choose the coordinates z and t is such a way that $z'(t) \equiv 0$ for one string, and for another one, the point $z(t)$ makes one complete turn around zero when t varies from 0 to δ:



So we have

$$\frac{1}{2\pi i}\int_0^\delta \frac{dz - dz'}{z - z'} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} = 1$$

by the Cauchy theorem. $\square$

Chapter 4

Finite type invariants for spherical knotoids

In this chapter we first introduce the notion of singular knotoid in any connected, oriented surface Σ. Then we introduce the notion of a linear chord diagram and we define finite type invariants for knotoids.

4.1 Singular knotoid diagrams and rigid vertex isotopy

Definition 4.1.1. A *singular knotoid diagram* is a knotoid diagram with some (finite) undeclared double points which are transversal, the singular crossings. See Fig. 4.1 for some examples.

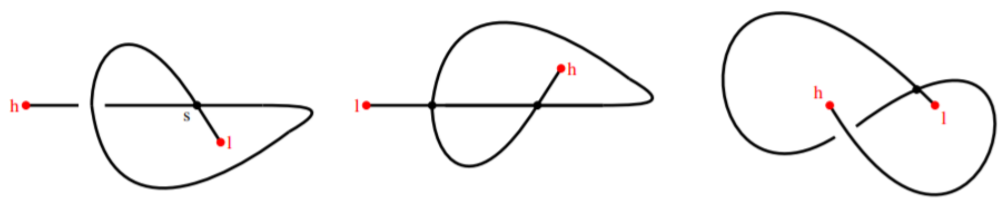


Figure 4.1: Singular knotoids

Definition 4.1.2. A *rigid vertex isotopy for singular knotoid diagrams* is generated by locally planar isotopy and the Reidemeister moves for classical knotoids (recall Fig. 1.1), extended by the rigid vertex isotopy moves involving singular crossings (recall Fig. 2.1). An isotopy class of singular knotoid diagrams is called *singular knotoid*.

Note that in the theory of singular knotoids we still have the restrictions of the forbidden moves (recall Fig. 1.2).

4.2 Types of closures

Definition 4.2.1. We call *classical closure for (classical or singular) knotoids of type 'o' (resp. 'u')* a mapping c^o (resp. c^u) from the set of (classical or singular) knotoid diagrams in a surface Σ to the set of singular knot diagrams in Σ, induced by the over- (resp. under-) closure on knotoid diagrams.

$$c^o(K) := K^o \quad \text{and} \quad c^u(K) := K^u$$

where K^o, K^u denote the singular knots in Σ obtained via the over- resp. under-closure of a knotoid K. For examples, see the top row of Fig. 4.2.

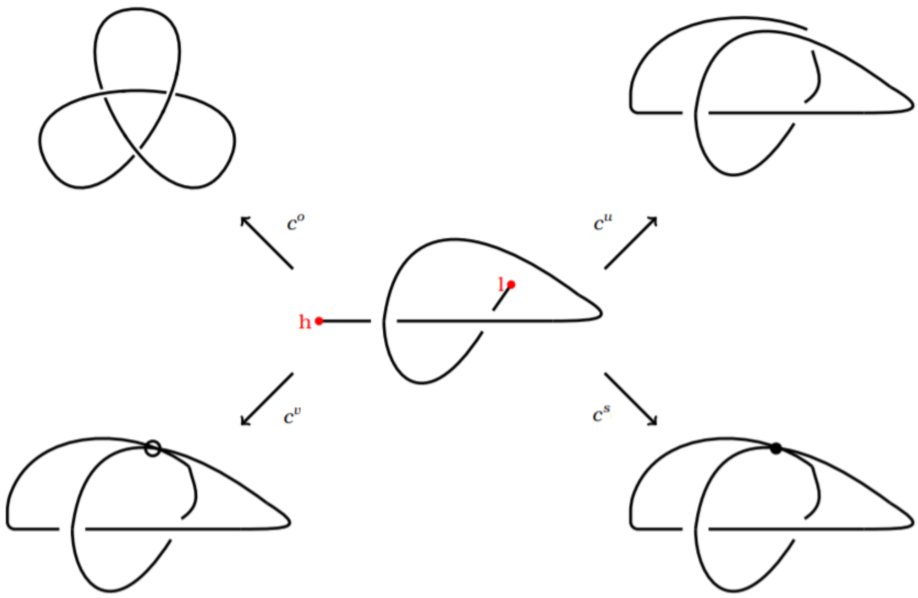


Figure 4.2: Types of closures: over- and under-closures (top row), virtual closure (bottom left), and singular closure (bottom right)

Lemma 4.2.2. Any classical closure (over or under) for spherical or planar (singular) knotoids is well-defined up to isotopy.

Proof. We want to show that a singular knotoid $K \subset S^2$ determines, via the under-closure, a unique singular knot $K_u \subset S^3$ up to isotopy.

Indeed, pick an embedded arc $a \subset S^2$ connecting the endpoints of K and otherwise meeting K transversely at a finite set of points distinct from the crossings of K (a shortcut for K) $K \cup a$ is a knot diagram declaring that a passes everywhere under K. The orientation of K from leg to head defines an orientation on $K \cup a$ The knot in S^3 represented by $K \cup a$ is denoted by K^u. We say that K represents K^u.

K^u does not depend on the choice of the shortcut a because, by the Reidemeister moves, any two shortcuts for K are isotopic in the class of embedded arcs in S^2 connecting the endpoints of K.

□

Remark 4.2.3. Intuitively, one might consider taking all possible closures by taking a diagram that realizes the height and an arc connecting leg to head and not passing through any other crossing, but creating c new crossings. Then one would declare with 2^c options which crossing is under and which over. Given two isotopic knotoid diagrams K^1,K^2 with the same height, say 2, there is no way to prove that the knot $K^1_{o,u}$ is knot-isotopic with the knot $K^2_{o,u}$ or $K^2_{u,o}$, where $K^n_{o,u}$ is the knot that we obtain if, starting from the leg of K^n and following the arc a, we declare the first extra crossing to be over-crossing and the second one to be under-crossing. Respectively, we denote $K^n_{i_1,i_2,\ldots,i_c}$ to be the closure of the knotoid K^n with height c, such that $i_1, \ldots, i_c \in \{o, u\}$ and if $i_k = o$ this means that the kth extra crossing is declared to be over, while if $i_k = u$ this means that the kth extra crossing is declared to be under, $k = 1, \ldots, c$. The above mean that $K^n_{i_1,i_2,\ldots,i_c}$ is not well-defined, as different shortcuts will possibly give non isotopic knots. For example if we take two distinct shortcuts in the knotoid illustrated in Fig. 4.3 , we will end up with different normalized bracket polynomials with the (u, o)-closure.

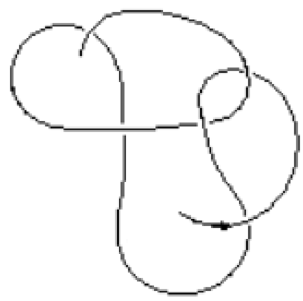


Figure 4.3: A knotoid that attains two distinct (u,o)-closures

We will now introduce the virtual closure. Virtual knot theory was introduced by L.H. Kauffman in 1998 [13]. We recall that a *virtual knot diagram* is an immersion of S^1 in the plane, containing finitely many double points, some of which are classical crossings and some are virtual crossings, which can be viewed as permutation crossings with no 'under' or 'over' specification. Virtual isotopy comprises the planar isotopy, the Reidemeister moves and the extra isotopy moves involving also virtual crossings, illustrated in Fig. 4.4. A *virtual knot* is a virtual isotopy class of virtual knot diagrams.

Definition 4.2.4. We call *virtual closure for knotoids* (classical, singular or virtual) a mapping c^v from the set of (classical, singular or virtual) knotoid diagrams in a surface Σ to the set of virtual knot diagrams in Σ, induced by the virtual closure on knotoid diagrams, whereby all extra crossings are declared to be virtual when taking a shortcut for the knotoid diagram. Namely,

$$c^v(K) := K^v$$

where K^v denotes the virtual knot in Σ obtained via the virtual closure of a knotoid K. See bottom left of Fig. 4.2.

For singular virtual knotoid diagrams we further have:

Definition 4.2.5. Two singular virtual knotoid diagrams differ by *singular virtual isotopy* if they differ by (a finite sequence of) planar isotopy, the Reidemeister moves, extended by the local moves involving virtual crossings, and by rigid vertex isotopy.

Lemma 4.2.6. Virtual closure is well-defined up to virtual isotopy.

Proof. Clearly, the closing arc may move freely, by the detour move, so the definition does not depend on the choice of shortcut. Moreover, any virtual/rigid vertex isotopy move between knotoid diagrams is also a valid isotopy move for their virtual closures. □

For (singular) knotoids of height one, we can also define a singular closure, as follows. We say that a knotoid is *prime* if it is not a product of two knotoids. Equivalently, a prime knotoid has no local knotting.

Definition 4.2.7. We call *singular closure for prime knotoids of height one* (classical, singular or virtual) a mapping c^s from the set of prime knotoid diagrams (classical, singular or virtual) in a surface Σ to the set of singular knot diagrams in Σ, induced by the singular closure on knotoid diagrams which realize the height, whereby the extra crossing is declared to be singular when taking a shortcut for the knotoid diagram. Namely,

$$c^s(K) := K^s$$

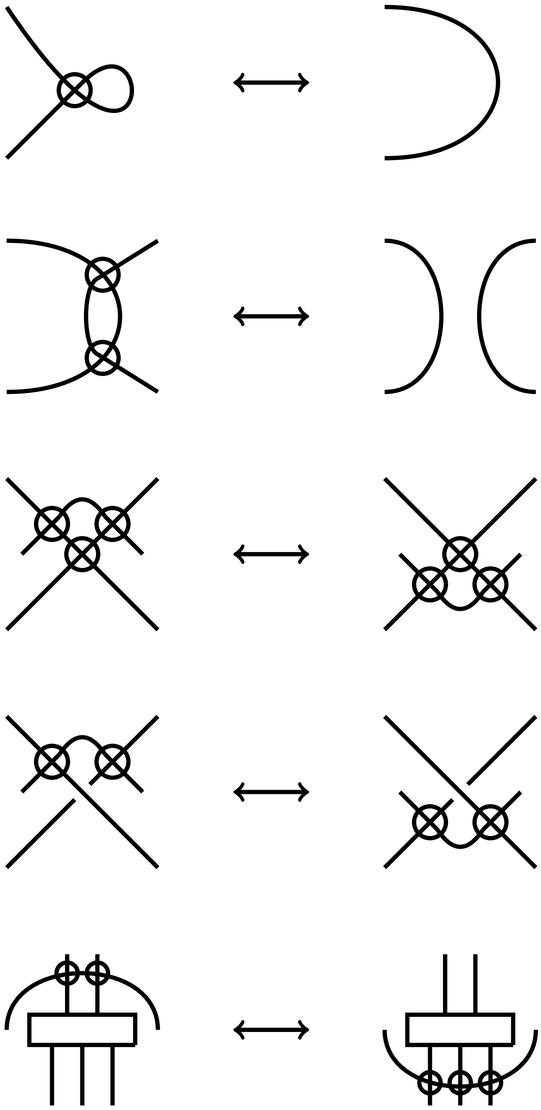


Figure 4.4: Isotopy moves involving virtual crossings. The last one is the so-called *detour move*. It generalizes the local detour move just above, and it is derived by the local virtual moves.

where K^s denotes the singular knot diagram in Σ obtained via the singular closure of the knotoid K. For an example, see the bottom right of Fig. 4.2.

Lemma 4.2.8. Singular closure for prime knotoids of height one is well-defined up to rigid vertex isotopy.

Proof. We will first argue that the definition of singular closure does not depend on the choice of shortcut. Indeed, since our diagram realizes the height one, there is at least one place where the leg is separated by the head by just one boundary arc. If there is only one such place, we are done. Suppose there are more places. We consider two neighbouring. Then the union of the associated shortcuts makes a simple closed curve. This closed curve bounds a disc,and this disc does not contain any knotting, since it would be by definition local so the two shortcuts differ by rigid vertex isotopy.

Moreover, any virtual/rigid vertex isotopy move between knotoid diagrams is also a valid isotopy move for their virtual closures. □

Remark 4.2.9. Singular closure is well-defined only for height-one prime knotoids and, moreover, on diagrams that realize the height of the knotoid they represent. The requirement for prime knotoids becomes apparent in the proof of Lemma 4.2.8. For bigger height, take, for instance, the following counterexample of height two, shown in Fig. 4.5.

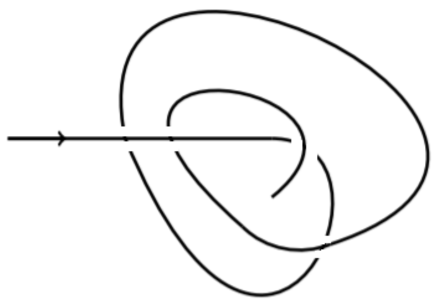


Figure 4.5: A knotoid diagram that admits two distinct singular closures

Indeed, calculating its affine polynomial we obtain

$$P_K = t^2 + t^1 + t^{-1} + t^{-2} - 4$$

so the knotoid is indeed of height 2. Yet, it admits two distinct singular closures, as indicated in Fig. 4.6: the closure s_1 illustrated by the blue arc corresponds via an oriented smoothing of the singular crossings to a non-trivial link of 3 components. However, the closure s_2 illustrated by the red arc corresponds to the trefoil knot via an oriented smoothing of the singular crossings. So, the two singular closures are not isotopic.

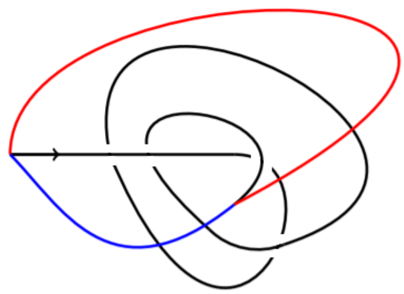


Figure 4.6: Two distinct singular closures

For the rest of the text we mainly focus on knotoids (classical, singular, virtual, etc) in S^2. Occasionally we will be making more general definitions and comments, stating explicitly the generalization.

4.3 Finite type invariants obtained by closures

Definition 4.3.1. Let v be any finite type invariant for classical knots. We call *classical closure finite type invariant for (classical) knotoids related to* v a mapping v^c from the set of (classical) knotoid diagrams to R^2, where R is a commutative ring, defined in the following way:

$$v^c(K) := (v(K^u), v(K^o))$$

where K^u, K^o are respectively the knot diagrams obtained via under- and over-closure, respectively, of a knotoid diagram K.

What we want to prove is that given a finite type invariant for knots v induces an isotopy invariant for knotoids using the classical closure. For isotopic knotoids K_1, K_2, and for any (knot) finite type invariant v, we want $v^c(K_1) = v^c(K_2)$, or equivalently we want $v(K_1^u) = v(K_2^u)$ and $v(K_1^o) = v(K_2^o)$ but this is true because v is an isotopy invariant for knots and a knotoid $K \subset S^2$ determines (via the under-closure) a unique knot $K_u \subset S^3$ (resp. $K_o \subset S^3$) up to isotopy from Lemma 4.2.2.

Definition 4.3.2. Let v be any finite type invariant for virtual knots. We call *virtual closure finite type invariant for knotoids related to* v a mapping v^v from the set of (virtual) knotoid diagrams to a commutative ring R, by computing the invariant on the resulting virtual knot. Namely,

$$v^v(K) := v(K^v)$$

where K is a (virtual) knotoid diagram and K^v is the virtual knot diagram obtained by the virtual closure.

For finite type invariants of virtual knots, see for example [13],[10], [7].

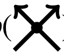
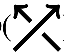
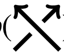
Definition 4.3.3. Let v be any finite type invariant for singular knots. We call *singular closure finite type invariant for prime knotoids of height one related to* v a mapping v^s from the set of (singular) prime knotoid diagrams realizing the height to a commutative ring R, by computing the invariant on the resulting singular knot. Namely,

$$v^s(K) := v(K^s)$$

where K is a (singular) knotoid diagram realizing the height and K^s is the singular knot diagram obtained by the singular closure. One could then say that this knotoid invariant is of order (or type) $\leqslant n$ if it vanishes on all singular knotoids with more than $n-1$ singular crossings.

4.4 Finite type invariants defined directly on knotoids

As in the case of knots, the principal idea of the combinatorial approach to the theory of finite type invariants is to extend a knotoid invariant v to singular knotoids with finitely many singular crossings, by making use of the Vassiliev skein relation:

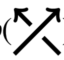
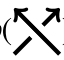
$v($$) = v($$) - v($$)$

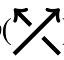
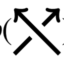
So we define:

Definition 4.4.1. A knotoid invariant, v, is said to be a *finite type invariant of type* $\leqslant n$ if its extension on singular knotoids vanishes on all singular knotoids with more than n singular crossings. Furthermore, v is said to be of type n if it is of type $\leqslant n$ and not of type $\leqslant n-1$.

Definition 4.4.2. Two singular knotoid diagrams K_1, K_2 in a connected, oriented surface Σ are said to be *singular equivalent* if one can be deformed to the other by rigid vertex isotopy moves (recall Figs. 1.1 and 2.1) and real crossing switches.

For any finite type invariant v for knotoids we consider the top row singular knotoid diagrams up to singular equivalence. Now, as in the classical case, any finite type invariant remains unchanged under crossing switches on top row diagrams, due to the Vassiliev skein relation. Indeed, if K is a top row diagram and has a positive (resp. negative) crossing we can switch this crossing by invoking the relation

$v($$) = v($$) + v($$).$

But $v($$) = 0$ because this diagram has $n+1$ singular crossings, since K is a top row diagram. Hence $v($$) = v($$)$. From the above, a top row singular knotoid may be represented by an appropriately chosen 'regular' representative of its singular equivalence class.

4.5 Linear chord diagrams

Since a spherical knotoid diagram is an immersion from $[0,1]$ to S^2 we must think of chord diagrams as 'chords' joining points of an open-ended smooth curve in the sphere, that is, a connected, oriented 1-manifold with boundary two endpoints, which correspond to the two endpoints of the singular knotoid diagram. Each chord corresponds to precisely one singular crossing of the singular knotoid diagram, and the chords respect the forbidden moves.

Definition 4.5.1. A *linear chord diagram of order n* is an oriented closed interval with a distinguished set of n disjoint pairs of distinct points being connected with an immersed unknotted curve, a 'chord', in S^2, considered up to orientation preserving diffeomorphisms of the interval, and up to isotopy of the immersed curve which fixes the endpoints. The set of all chord diagrams of order n will be denoted by L_n.

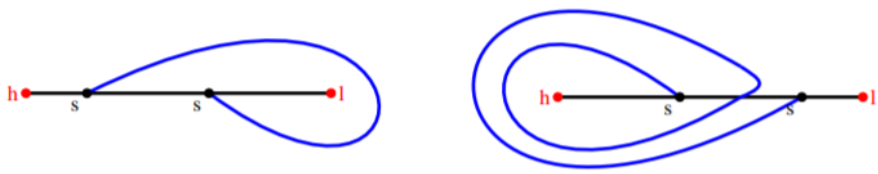


Figure 4.7: Linear chord diagrams

It is clear that there are many ways that two points can be connected by a chord, even being considered up to diffeomorphisms of the interval. Namely, a chord may wrap around the leg or/and the head a number of times, in the clockwise or counterclockwise orientation. Fig. 4.7 shows two examples in L_1, where s denotes the singular crossing in the knotoid diagram. So, contrary to the classical chord diagrams, even the set L_1 is far from being trivial.

Note that there are two trivial chord diagrams, in which the chord does not wind around the leg or the head: the one that the chord lies under the interval and the one that the chord lies over the interval, denoted by 1_u, 1_o respectively. Of course they both correspond to knot-type singular knotoids and hence they can be identified with the classical chord diagram with one chord. So, $1_u = 1_o$. Furthermore, thinking of finite type invariants, the one-term relation applies on such a trivial chord diagram, yielding zero evaluation of the invariant.

For the rest of this work we will restrict ourselves to the case $n = 1$. Our goal is to classify all different chord diagrams of order 1. With this in mind, we first consider two generators a, b: The generator a means a wrapping of the chord once around the leg with the counterclockwise orientation, starting from the point in the interval closer to the leg with destination the point closer to the head. The generator b means a wrapping of the chord around the head with the counterclockwise orientation, starting from the point in the interval closer to the leg with destination the point closer to the head.

We will now define an operation between two chords. Multiplication $x \cdot y$ means that we locate all four points in the interval such that the two involving x are both before the ones involving y, with respect to the natural orientation of the interval. Then we apply the following procedure in order to obtain a new chord out of the chords x and y. All the intersections of x and y are declared to be flat (of no importance), so we take a tubular neighbourhood of the sub-interval starting from the destination of x and ending at the start of y. Inside there we connect the destination of x with the start of y by a simple 'concatenating' arc, which is unique up to planar isotopy, so that it is transversal to the interval at both points. The result is a new chord. Note that the trivial chord diagram plays the role of the unit: $1_o \cdot x = x \cdot 1_o = x = x \cdot 1_u = 1_u \cdot x$, as it only extends the chord x by a small arc. We also have inverses with respect to the product. Indeed, for the clockwise orientations we will use the symbol a^{-1}(respectively b^{-1}) for the following reason.

Lemma 4.5.2. $a \cdot a^{-1} = 1 \quad b \cdot b^{-1} = 1.$

Proof. The idea of the proof is clearly illustrated in Fig. 4.8, and the result is an immediate consequence of the definition of the multiplication, where with $x = a$ and $y = a^{-1}$ it is clear that $a \cdot a^{-1}$ is flat isotopic to the trivial chord diagram.

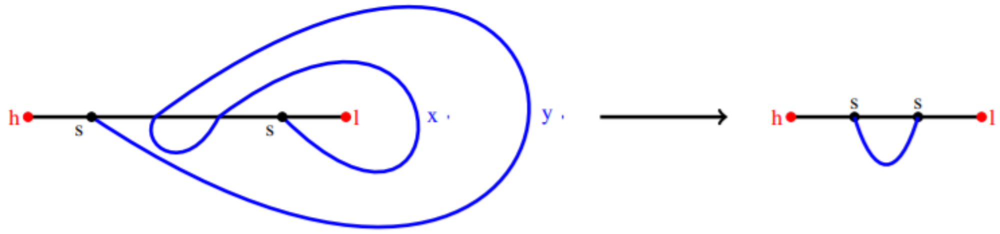


Figure 4.8: $a \cdot a^{-1} = 1$

□

Lemma 4.5.3. $a \cdot b = 1 = b \cdot a$ in S^2.

Proof. Now the idea is this. Take $x = a$ and $y = b$. Connect the endpoints as the definition suggests, such as both arcs cross the interval transversely. Then Fig. 4.9 shows that in S^2 $a \cdot b$ is trivial.

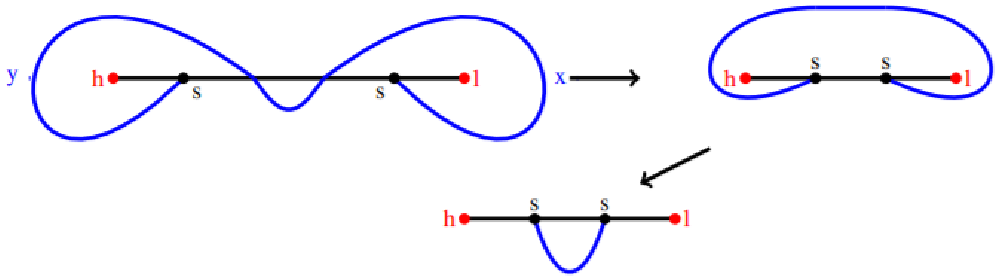


Figure 4.9: $a \cdot b = 1$

□

Corollary 4.5.4. This leaves us with only one generator in our construction, say a, and so we can assume that every chord diagram, when speaking about spherical knotoids, has its head in the outer region.

From the above, it is clear that we can rethink of a linear chord diagram with one chord, representing a singular knotoid with exactly one singularity, as a closed interval with a loop winding (only) around the leg w times.

So, the two points s can collapse into a single point and the loop can be abstracted to a circle. See Fig. 4.10.

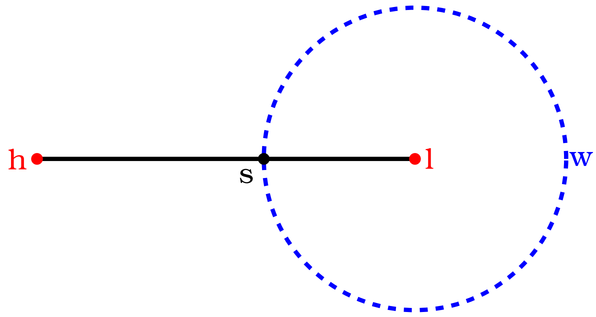


Figure 4.10: The abstraction of a chord diagram with winding number $w \in \mathbb{Z}$.

Hence, we have proven the following proposition:

Proposition 4.5.5. In S^2 for any linear chord diagram $C \in L_1$ there exists a $w \in \mathbb{Z}$ such that $C = a^w$. In other words, as a group, L_1 is infinite cyclic.

Note that in $\mathbb{R}^2$ this technique wouldn't work, since, for example, the second diagram in Fig. 4.9 is locked and non-isotopic to the third one. So, if one tries to extend this theory to $\mathbb{R}^2$ or a surface Σ, one would have to deal with more complexities.

4.6 From a singular knotoid to its chord diagram

In knot theoretical finite type invariants there is a natural way to encode the order of the singularities in a circular diagram and then join the two points of the circle with a simple interior arc. There are many non-pairwise isotopic ways to join two points of the interval with a simple arc. This is not a technicality but a first attempt to understand the complexity of the problem since chord diagrams are closely connected with singular equivalence classes of knotoids.

For example if we tried to have a convention which allows us to join two points without winding around l or h we would have the same chord diagram for the knotoids showed in Fig. 4.11

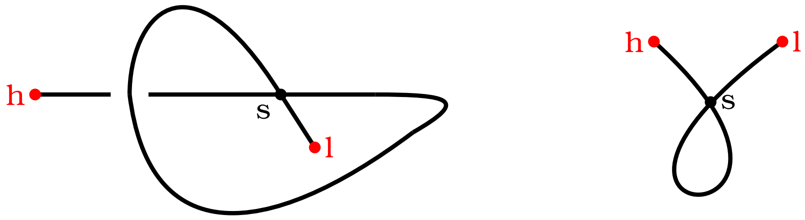


Figure 4.11: Two non singular equivalent knotoid diagram

The first attempt to overpass this obstacle is to observe that joining the points winding once around the leg counter-clockwisely would correspond to a knotoid diagram which winds once around the leg from the first time that passes form the singular crossing until the second time.

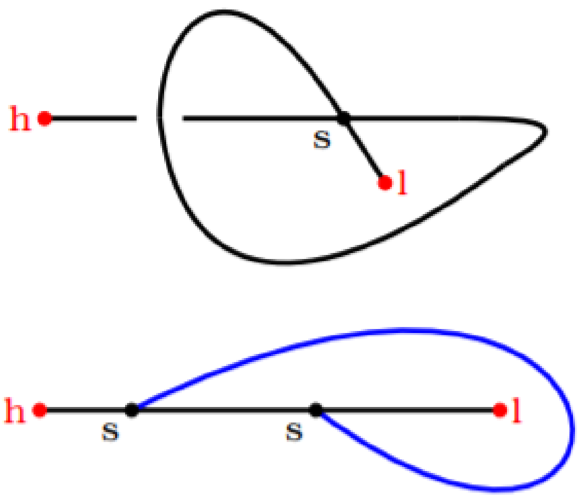


Figure 4.12: A knotoid diagram with its corresponding chord diagram

So, we end up with the following construction. First of all there is no point on considering paths that wind around the head since we know that this would correspond to a path winding around the leg the same amount of times but with reserved orientations. Take a knotoid diagram K. Split the knotoid into 3 paths. The path joining leg to singular point ($l \to s$), the singular loop, and the path joining the singular loop with the head($s \to h$). Erase $l \to s$ and $s \to h$, but keep track of the points l, h. What remains is a loop that winds some integer number of times around l. The integer is conventionally positive if the winding is counterclockwise. We now construct a chord diagram of class a^w with w being the winding number we just discussed. One can compute directly w by the usual isomorphism $\pi_1(D^2 \setminus \{l\}, s) \cong \mathbb{Z}$ and so $w \in \mathbb{Z}$ corresponds to the class of $\gamma(s) \in \pi_1(D^2 \setminus \{l\}, s)$

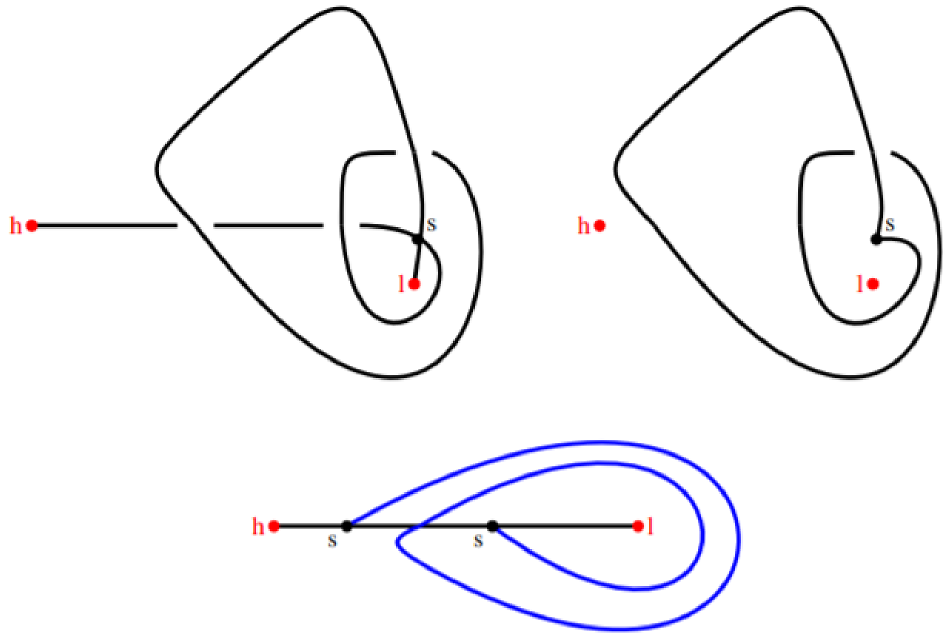


Figure 4.13: The singular loop and the corresponding chord diagram

4.7 From a chord diagram to a singular knotoid

It is a natural question to ask whether the construction of a chord diagram corresponds to a representative of a singular equivalence class and here we will follow the classical construction of surgery along the two endpoints. The intuition behind the technique is that a chord diagram illustrates the nodal structure of the singularities, and specially in pure knotoids every chord illustrates the winding number around the leg as you traverse in a singular loop.

Given a linear chord diagram D with precisely one chord c which has endpoints a, b with a being closer to l and b closer to h. The technique is simple but yet accurate. Take two small open subintervals each one containing an endpoint and cut it out from the interval, say $(a - \varepsilon, a + \varepsilon), (b - \delta, b + \delta)$. The chord c now has a new initial point from $a + \varepsilon$. (That is a convention, we can follow the same procedure if the new initial point is $a - \varepsilon$.

Take a new arc with initial point $a - \varepsilon$, say γ which crosses transversely c in an ε' neighbourhood of a, and has no other common points in the neighbourhood with c. Then γ traverses next to c (in a small thickened band following c without crossing it, (except of course the cases that c has self-intersections but these cases can be excluded in spherical knotoids), until the thickened band enters a small δ' neighbourhood of b, where either γ should be linked with $b - \delta$ and c with $b + \delta$ or the other way, so we are forced to choose the only connectivity pattern that leads to a knotoid and not a multi-knotoid. This construction may or may not create an extra flat crossing in the δ' neighbourhood of b. Actually, in spherical knotoids the pattern is always the same, and it always create an extra flat crossing, but this generality could solve some future ambiguities.

Note, again, that all the self crossings of c, the crossings of c with γ and c, γ with the interval are flat since we are trivially in a top row diagram so we regard singular-equivalence classes of knotoids with one singularity or equivalently isotopy classes of flat knotoids with one singularity.

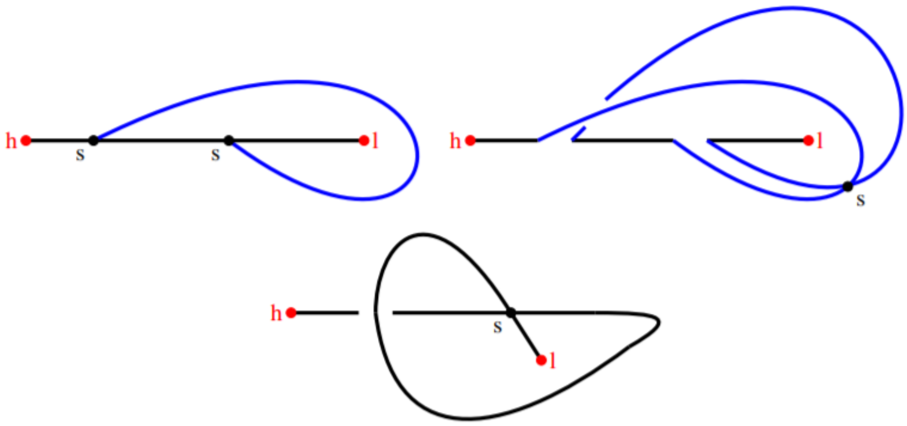


Figure 4.14: Recovering a singular knotoid from a chord diagram

Chapter 5

Rail representations for knotoids

5.1 The rail representation for planar knotoids

One thing that is worth mentioning now for our purpose is a special way to represent planar knotoids. Intuitively speaking, one could take a knot in the thickened surface $\Sigma \times F$ out of knotoid diagram that lives in a surface Σ. For $K(\mathbb{R}^2)$ there is something more specific. Identify the plane of the planar knotoid K with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. K can be embedded into $\mathbb{R}^3$ by pushing the overpasses of the diagram into the upper half-space and the underpasses into the lower half-space in the vertical direction. This creates an embedding of $[0, 1]$ in $\mathbb{R}^3$. The leg and the head of the diagram are attached to the two lines, $\{l\} \times \mathbb{R}$ and $\{h\} \times \mathbb{R}$ that pass through the leg and the head, respectively and are perpendicular to the plane of the diagram. Moving the endpoints of K along these lines gives rise to open oriented curves embedded in $\mathbb{R}^3$ with two endpoints lying on each line. Such a curve is said to be the rail representation of a knotoid diagram. An example of a knotoid K_2 and its rail representation K_1 is illustrated in Fig.5.1.

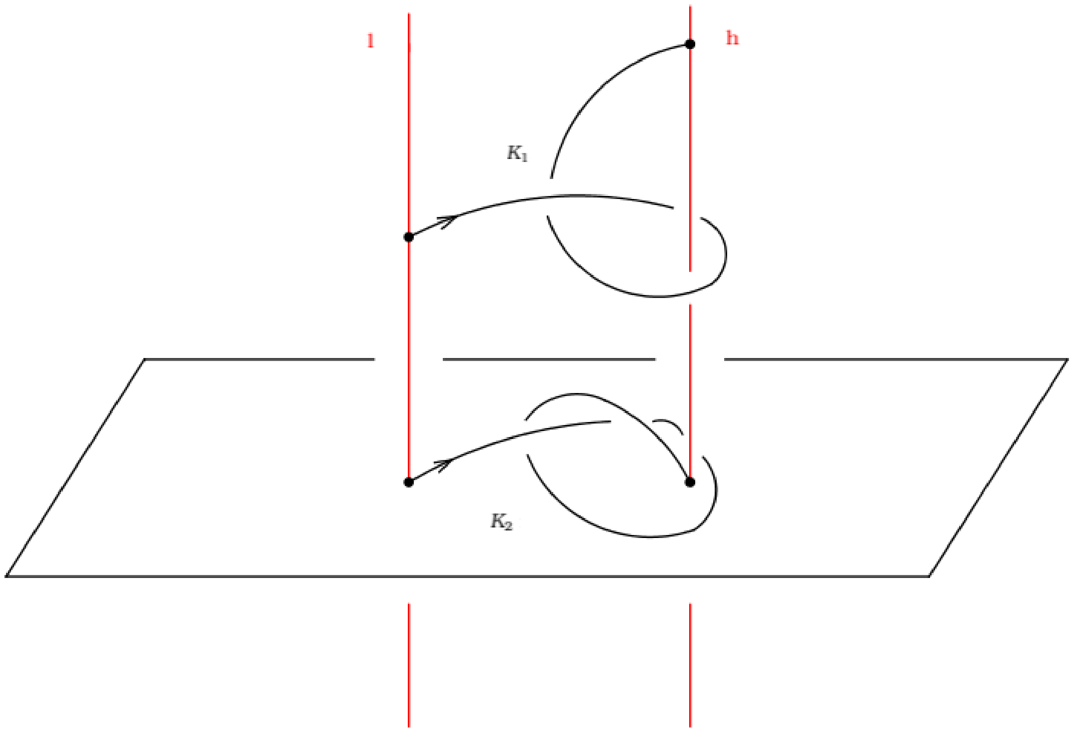


Figure 5.1: Example of a knotoid and its rail representation

Definition 5.1.1. Two smooth open oriented curves embedded in $\mathbb{R}^3$ with the endpoints that are attached to two special lines, are said to be *line isotopic* if there is a smooth ambient isotopy of the pair $(\mathbb{R}^3/\{t\times\mathbb{R}, h\times\mathbb{R}\}, t\times\mathbb{R}\cup h\times\mathbb{R})$, taking one curve to the other curve in the complement of the lines, taking endpoints to endpoints, and lines to lines.

In [8] N.Gügümcü and L.H. Kauffman prove the following:

Theorem 5.1.2. There is a one-to-one correspondence between the set of knotoids in $\mathbb{R}^2$ and the set of line-isotopy classes of smooth open oriented curves in $\mathbb{R}^3$ with two endpoints attached to lines that pass through the endpoints and perpendicular to the *xy*-plane, and generally Two open oriented curves embedded in $\mathbb{R}^3$ that are both generic to a given plane, are line isotopic (with respect to the lines determined by the endpoints of the curves and the plane) if and only if the projections of the curves to that plane are equivalent knotoid diagrams in the plane.

Furthermore, in [18] D. Kodokostas and S. Lambropoulou develop the theory of rail knotoids.

5.2 A rail representation for spherical knotoids

Definition 5.2.1. The spherical knotoid diagrams which have the head in the point at infinity are called *semi-special*. The semi-special knotoid diagrams which have the leg in the point 0 are called *special*.

Lemma 5.2.2. Every spherical knotoid $K \in \mathcal{K}(S^2)$ is isotopic to another $\tilde{K} \in \mathcal{K}(S^2)$ such that there is a path joining the head of the knotoid with the point at infinity ∞ and does not intersect any other part of the knotoid, or $\tilde{K}$ is semi-special.

Proof. Take a point $p \neq h$ in the connected component of h and by the fact that S^2 is rotationally symmetric p can be thought as the point at infinity via isometry. One could also think that given h and ∞, we are not allowed by the forbidden moves to pass branches of the knotoid K through h, but we can pass them through ∞ and so if in the initial position we had k intersection points when joining h, ∞, by passing k branches to the other side of ∞ we have the desired semi-special diagram only by using locally-planar isotopy. □

Lemma 5.2.3. The spherical knotoids $\mathcal{K}(S^2)$ are in bijection with the planar knotoids whose head is in the non-bounded connected component of $\mathbb{R}^2 \setminus K$ if K is the planar knotoid ($\mathcal{K}'(\mathbb{R}^2)$).

$$\mathcal{K}(S^2) \longrightarrow K'(\mathbb{R}^2)$$

We could equivalently think of these knotoids as "long knots in one direction", meaning that the leg is trapped but following the knotoid diagram we can approach the head with a straight line outside a compact region.

Proof. Let $K \in \mathcal{K}(S^2)$.From Lemma 5.2.2 we can assume without loss of generality that K is semi-special. Technically we need the head to be "very close" to ∞.

By the decompactification of S^2 using the usual stereographic projection the initial diagram corresponds uniquely to a $\tilde{K} \in K(\mathbb{R}^2)$ since

Then the knotoid diagram K except a neighbourhood of h, is in a compact subset B which does not contain ∞.

Let

$$\pi : S^2 \setminus \{\infty\} \longrightarrow \mathbb{R}^2$$

be the stereographic projection.

Then we have that $\pi(B)$ is compact and $\pi(h) \in \pi(S^2 \setminus B)$ is not non-bounded and has no common point with the knotoid other than the small path near $\pi(h)$. Correspondingly we can assume that outside a compact $F \supseteq B$ the knotoid reaches the head with a straight line, and from our hypothesis that h is "very close" to the ∞, we get $\pi(h)$ is "very far" from the non-linear part of the knotoid. One could determine exactly the metrics that make clear this "very close" situation but this is easy and irrelevant to our goal.

The converse is easy the same since we begin with a knotoid $K \in K(\mathbb{R}^2)$ which is inside a compact F except a straight line from the boundary of F to h, which has a "big length". Applying π^{-1} we will get a semi-special knotoid which is what we want.

$\square$

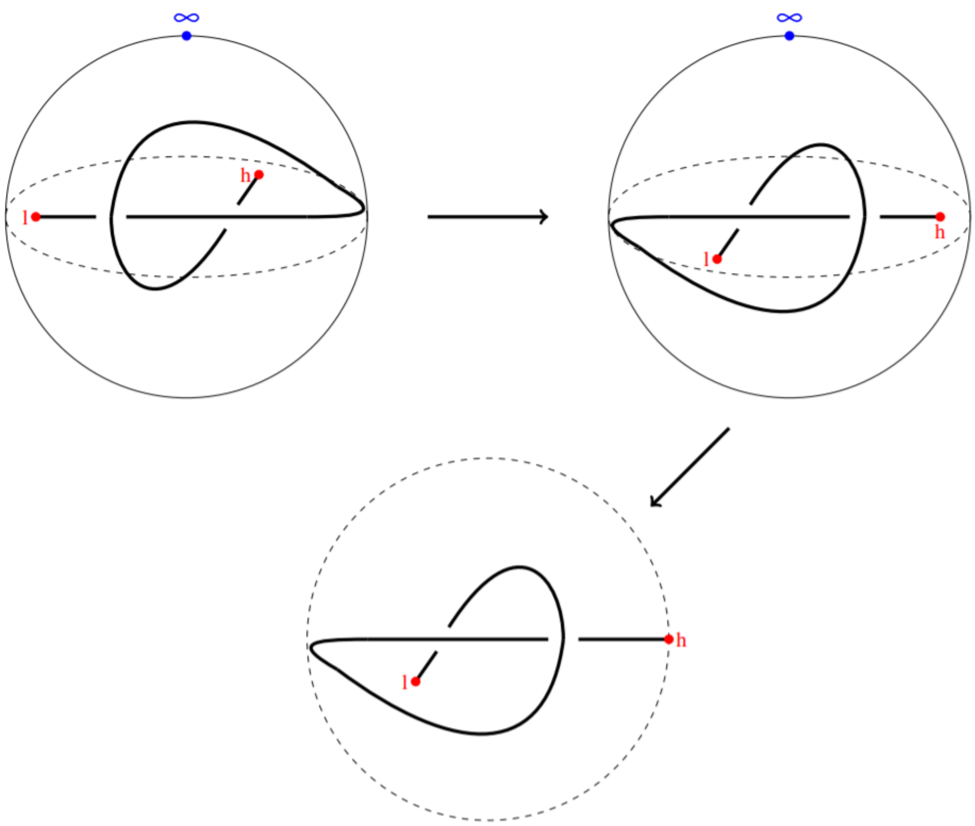


Figure 5.2: Transformation to a semi-special knotoid diagram and then to a planar

Lemma 5.2.4. Every descending classical spherical knotoid diagram is isotopic to the trivial knotoid.

Proof. Since in any isotopy class of spherical knotoids there is a semi-special representative, we choose without loss of generality K to be a descending semi-special spherical knotoid diagram. In S^2 take a disc D_1 centered at ∞ which intersects the diagram only at a single segment, starts with the head and intersecting the disc at exactly one point.

Take now the disc D^2, possibly with application of some planar isotopies centered at leg such that D_1 is tangent to D_2. This disc encloses the rest of the knotoid diagram except for a segment a which links the head with the tangential point.

Applying the stereographic projection of lemma 5.2.3 yields in the plane the knotoid in which the head is outside a compact containing all the rest of the knotoid except a small neighbourhood of h.

Then take the rail lifting in $\mathbb{R}^3$. This gives rise to a solid cylinder with central axis the rail of l, denoted by L. Denote also the rail of h, by H. H is allowed to move in the boundary of the solid cylinder. Our knotoid now corresponds to a curve c in $\mathbb{R}^3$ for which if two points $p_1 = c(t_1), p_2 = c(t_2)$ with $t_1 < t_2$ with $p_1 = (x, y, z_1), p_2 = (x, y, z_2)$, then $z_1 > z_2$. Here we use the convention that L, H are perpendicular to the x, y plane.

if the leg at the point $(x, y, 1)$ on L. The head is pulled down until K becomes a helical curve. The curve c does not wind around H since K is a semi-special knotoid diagram. Then the curve c is line isotopic to a curve with h fixed on the boundary after winding around L, where l is fixed. The part of the c that winds around the L together with L, both oriented downwards, is a 2-braid. The parts that correspond to a braid word $\sigma_i \sigma_i^{-1}$ are cancelled so we have a braid word of the form σ_i^n. Now we can unwind c from the bottom to the top by consecutive rotation of π in the counterclockwise direction around L for n-times. Then the projection of c to the plane is the trivial knotoid diagram. □

Chapter 6

Type-1 invariants for spherical knotoids

What is of crucial importance here is the notion of the regular diagrams which gives rise to our main result for the v_1

6.1 Regular diagrams and the classification theorem

Definition 6.1.1. A *regular diagram* is a knotoid diagram K with 1 singular crossing which has the minimal crossing number , is descending and the singularity is located "next to" the the leg of the knotoid i.e. There exist no other crossings between the leg and the singular crossing. Finally, there exists no local knotting and the real crossing number is odd.

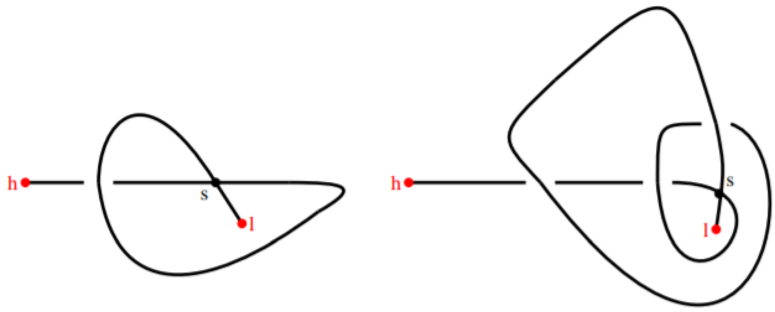


Figure 6.1: Two regular diagrams

Our goal is to prove the following theorem

Theorem 6.1.2. Every knotoid diagram with 1 singularity is singular equivalent to one and only one regular diagram.

With this in hand we will be ready to show that we can handle the complexity occurring by the forbidden moves and the rigidity of the singularity just by the winding number of the so-called singular loop.

Definition 6.1.3. The *singular loop* is the oriented closed path that starts from the singular crossing, as it is first encountered from the leg, follows the knotoid diagram and ends at the singular crossing.

Lemma 6.1.4. In a spherical singular knotoid that has 1 exactly singular point, the singular loop contains all the knotedness up to singular knotoid isotopy. This means that the path joining the head with the singularity contains only crossings that get involved in the singular loop and all others can be excluded. The same for the path joining the leg with the singularity.

Proof. We can consider the knotoid to be descending as crossing switches do not change the singular equivalence class of the knotoid.

We consider the rail representation for the spherical knotoid using the isomorphism of Lemma 5.2.3. The knotoid can be thought to live in the interior of $C_R \setminus C_1$ where C_n is the solid cylinder of radius n centered at l except a small unknotted segment that links l with the boundary of C_1 and the head h lives in the boundary of C_R.

First of all because the knotoid is descending, the subpath joining l with s is a classical spherical descending knotoid and hence a trivial that lays over the rest of the diagram, with the technique that we described earlier by unwinding the arc around the L rail.

The same argument will show that the subpath joining s to h is trivial and underpasses all the rest of the diagram.

So every crossing that is not involved in the singular loop can be excluded via singular equivalence.

□

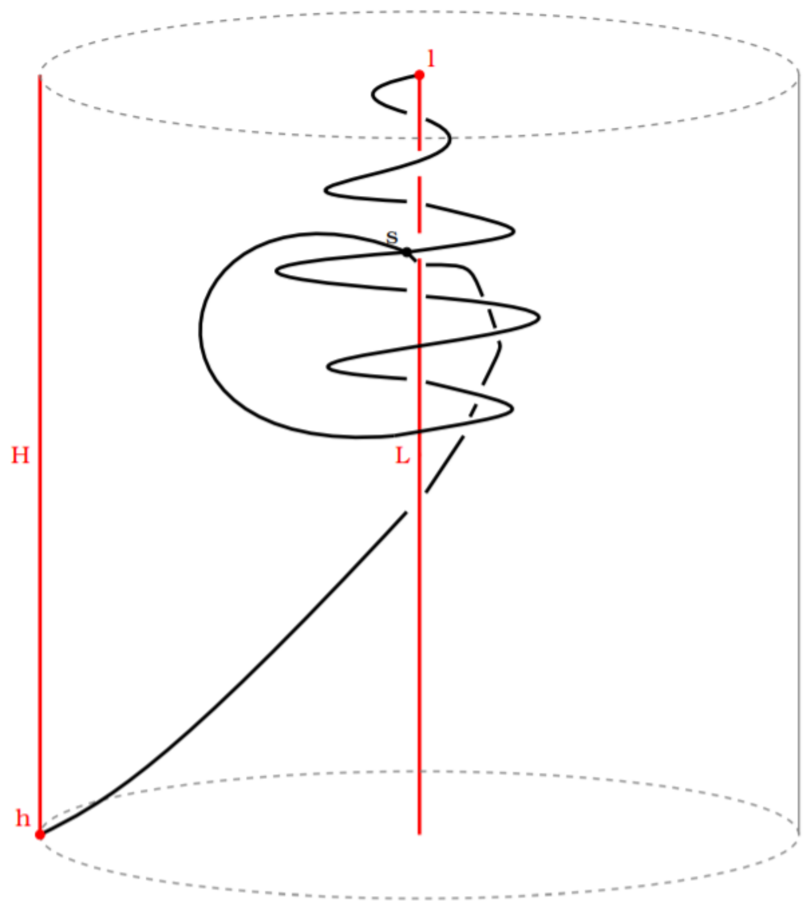


Figure 6.2: Rail representation of a regular knotoid

Proof of the Classification Theorem

Since the only thing that matters is the singular loop we have the following considerations.

1. The cylindrical anulus $C_R \setminus C_1$ deformation retracts to a disc without a point $D^2 \setminus \{l\}$.

2. Two regular diagrams have two singular loops in a bijection as we saw, so they are classified by $\pi_1(D^2 \setminus \{l\}, s)$. so the regular diagrams are different among each other.

3. There is a singular equivalence making the singular loop wrap around the point grading the radii of the winding from the bigger to the smaller. We then have $w - 1$ self intersections i.e crossings where w is this winding number.

4. We then must join s with h with an underarc. This will create w extra under crossings projecting again the knotoid to the plane, applying w times the Jordan curve theorem.

This constructive proof gives us the result that any knotoid diagram is singular equivalent to exactly one regular diagram.

Q.E.D.

6.2 The singular height

Definition 6.2.1. We define the *singular height* of a knotoid diagram K to be the minimal height of its singular equivalence class. $sh(K) = min\{h(K')|K' \sim K\}$, where $\sim$ means singular equivalence.

Obviously $sh(K) \leqslant h(K)$ and there exists a regular diagram in the singular equivalence class of K which realizes singular height. Of course every regular diagram realizes singular height.

Thus, it is natural to say that since a regular diagram realizes height and a linear chord diagram corresponds to exactly one regular diagram, chord diagrams realize singular height. The question here is simple: In planar knotoids where the construction of regular singular knotoids is not so easy can we detect singular height, just by seeing the chord diagram.

The farther we can reach for the moment is conjecture the formula that holds in this occasion and search it further in the future. First of all recall that in planar knotoids b is a free generator as well as a. We don;t have that $ab = 1$ as in Lemma 4.5.3 so we should think linear chord diagrams fro planar knotoids as words of $F_2[a, b]$.

So let $c \in F_2[a, b]$, then $c = a^{n_1} b^{m_1} \dots a^{n_k} b^{m_k}, k < \infty, n_i, m_i \in \mathbb{Z}$

- $m_i \neq 0 \quad \forall i \leqslant k - 1$
- $n_i \neq 0 \quad \forall i \geqslant 2$
- if $m_1 = 0 \Rightarrow m_i = 0$ for every i and then $c = a^r$ for some $r \in \mathbb{Z}$

We search the first two consecutive exponents which have the same sign.

1. If the first such pair is $n_j > 0, m_j > 0$ for some j, then let d be the number of (consecutive) pairs n_i, m_i where $n_i > 0, m_i > 0$ added by the number of (consecutive) pairs m_i, n_{i+1} where $m_i < 0, m_{i+1} < 0$
2. If it is $m_j < 0, m_{j+1} < 0$ let d be the same as in (1).
3. If the first such pair is $n_j < 0, m_j < 0$ for some j, then let d be the number of (consecutive) pairs n_i, m_i where $n_i < 0, m_i < 0$ added by the number of (consecutive) pairs m_i, n_{i+1} where $m_i > 0, m_{i+1} > 0$
4. If it is $m_j > 0, m_{j+1} > 0$ let d be the same as in (3)

Conjecture 6.2.2. *Given a linear chord diagram for planar knotoids D, with one chord, then the singular height of the corresponding knotoid diagram is given by the formula*

$$sh(K) = \left[\sum_{i=1}^{k} (|n_i| + |m_i|) \right] - 2d$$

6.3 The integration theorem

With the proof of the classification theorem, some natural questions arise. First of all, we have the following algorithm. Given a weight system W i.e.: a function $W : LCD \to R$ where R is a commutative ring, we can construct an invariant on knotoids by this way. Starting from the leg of a knotoid diagram K, and the first time that we see a crossing that under-passes the first time, or the first crossing that the knotoid fails to be descending. Call that crossing c. Change c using the Vassiliev skein relation so we get the under crossing ± the evaluation of the weight system in the regular diagram which is singular equivalent to K_c. K_c is the given knotoid with c nodified, and ± corresponds to the sign of c. In the end, continuing with the same process we will have a descending diagram which will be trivial and so contributes nothing to our sum.

So this algorithm says that

$$v_1(K) = \sum_{c \in CR(K)} \delta_c sgn(c) W(K_c)$$

where

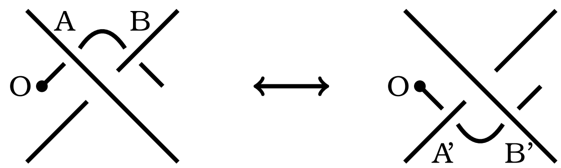
$$\delta_c = \begin{cases} 0, & \text{c is an over crossing the first time} \\ 1, & \text{otherwise} \end{cases}$$

$sgn(c)$ is the sign of the crossing c and $W(K_c)$ is the value of W in the knotoid diagram K with c nodified.

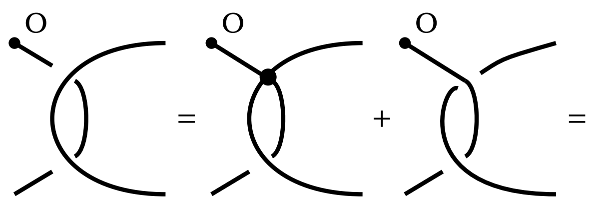
Theorem 6.3.1. *Every function $W : LCD \to R$ that has zero evaluation in the trivial linear chord diagram, using the Vassiliev skein relation gives rise to a knotoid invariant where LCD is the set of equivalence classes of linear chord diagrams, or equivalently the sum that we defined is stable in the isotopy class of the knotoid diagram.*

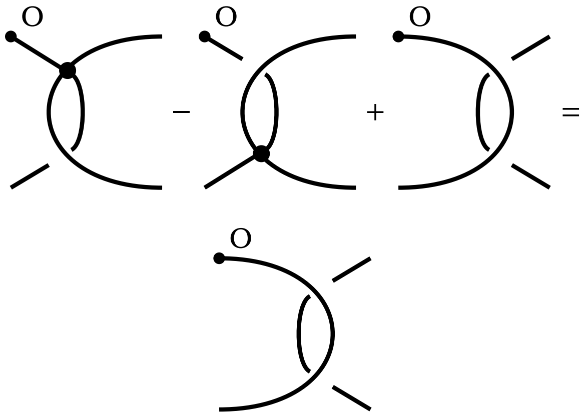
Proof. First of all, any Reidemeister I move would create a trivial curl and a crossing in which the weight system with the crossing nodified would be 0 so Reidemeister I moves contribute nothing to our sum.

The Reidemeister III move does not create any crossings and it does not hide any crossings involved. Furthermore, performing the Reidemeister III move there are some possible ways because of the orientations. In any such choice, we will see the contribution of this change is equal 0. An interesting case is when all three arcs as illustrated below are oriented from left to right and so both crossings change signs.The central crossing contributes to the sum the same as it contributes after the Reidemeister III move. Let A the left crossing and B the right one. Then their corresponding crossings after the Reidemeister move are A', B' respectively, and the knotoid diagram has initial point O. But then $W(K_A) = W(K_{B'}), W(B) = W(K_{A'})$,since we regard singular equivalence classes of singular (flat) knotoids and course neither the leg nor the head are in the closed region between the arcs, or else we performed a forbidden move. Moreover, $\delta_A = \delta_{B'}, \delta_B = \delta_{A'}$ since the same arcs are involved and of course $sgn(A) = sgn(B'), sgn(B) = sgn(A')$. Hence again no contribution to our sum. What we describe here is illustrated in the following picture



The last thing we must check is the Reidemeister II move. The idea is again to perform such a move and show that it contributes nothing to the sum.We will sum over diagrams for simplicity and of course mean the evaluations on such diagrams. Choose in the diagrams following the orientation from the top to the bottom and the understrand appears first in the knotoid diagram, with initial point O as described in the following picture.





which contributes nothing more than the two parallel lines to the sum since both crossings are first overcrossings. The proof for all other choices is identical except possibly some opposite signs. □

6.4 Examples of non-trivial v_1

6.4.1 The affine index polynomial

Proposition 6.4.1. The affine index polynomial is a Vassiliev invariant for knotoids of type 1.

Proof. We have to prove that if we extend the affine index polynomial by the Vassiliev skein relation to singular knotoids. We have that

$$P_K(t) = \sum_{c \in Cr(K)} sgn(c)(t^{\omega_K(c)} - 1)$$

So for every singularity inserted in a knot diagram we have the following contribution to the sum

$$P(\times) = P(\times) - P(\times) = t^{\omega_+(c)} - 1 + (t^{\omega_-(c)} - 1) = t^{\omega_+} + t^{\omega_-} - 2$$

The −2 is also justified by the change of writhe by 2 when one changes a positive crossing to a negative.

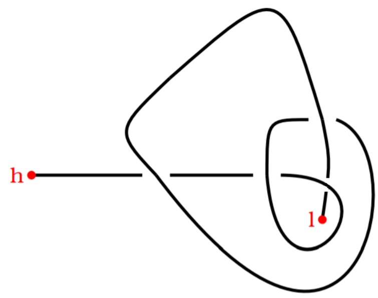
Take a knotoid diagram with two singularities $.c, d$ with corresponding ω_+ and ω'_+ This yields the following contributions.

$$P\left[\times_c, \times_d\right] = P_{++} - P_{+-} - P_{-+} + P_{--} =$$

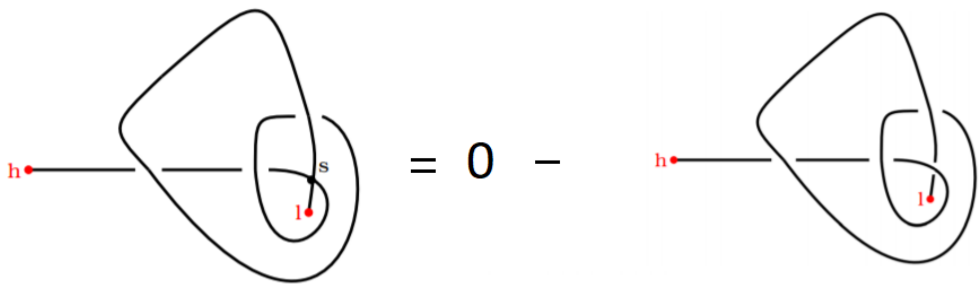
$$(t^{\omega_+}-1)+(t^{\omega'_+}-1)-(-t^{\omega_-}+1)-(t^{\omega'_+}-1)-(t^{\omega_+}-1)-(-t^{\omega'_-}+1)+(-t^{\omega_-}+1)+(-t^{\omega'_-}+1) = 0$$

So P_K is a Vassiliev invariant of knotoids of type 1. $\square$

Now recall the calculation in 1.4.2 where we calculated the affine index polynomial of this knotoid



It was $P_K = -(t^2 + t^{-2} - 2)$. We also have that (omitting the $P_K(t)$)



So the regular diagram of order 2 has an affine index polynomial $P_K = t^2 + t^{-2} - 2$ which of course was what we expected due to the calculation of the affine index polynomial on an abstract knotoid diagram with one singularity.

6.4.2 The invariants coming from Turaev extended bracket

Proposition 6.4.2. *Recall the Turaev's extended bracket $T_K(A, u)$. Making the substitution $A = e^x$. We hence represent the Turaev polynomial of a* *knotoid K as a power series*

$$T(K) = \sum_{k=0}^{\infty} \sum_{l=-n_1}^{m_1} t_{k,l} u^l x^k$$

Then each $t_{k,l}$ is a Vassiliev invariant of type k for all $l = \{-n_1, -n_1+1, \ldots, m_1\}$

Proof. We got the skein realation

$$-A^4 T\left(\times\right) + A^{-4} T\left(\times\right) = (A^2 - A^{-2}) T\left(\asymp\right) =$$

$$-\left(1 + 4x + \frac{16x^2}{2} + \ldots\right) T\left(\times\right) + \left(1 - 4x + \frac{16x^2}{2} - \ldots\right) T\left(\times\right) =$$

$$= ((1 + 2x + \ldots) - (1 - 2x + \ldots))\, T\left(\asymp\right)$$

Hence, by the Vassiliev skein relation

$$T\left(\times\right) = T\left(\times\right) - T\left(\times\right) = 4x(\text{some mess})$$

So $T\left(\times\right)$ is divisible by x.

Hence, if a knotoid has $n > k$ singularities then the coefficient of x^k vanishes.

So we have the following results:

1. $\sum_{l=-n_1}^{m_1} t_{k,l} u^l$ is a Vassiliev invariant of type k

2. The coefficient of x^k vanishes means that in the knotoids with $n > k$ singularities $\sum_{l=-n_1}^{m_1} t_{k,l} u^l = 0 \quad \forall u$ means that $t_{k,l} = 0 \quad \forall l$. So every $t_{k,l}$ is a Vassiliev invariant of type $\leqslant k$

$\square$

Bibliography

[1] D. Bar-Natan, "On the vassiliev knot invariants," *Topology*, vol. 34, no. 2, pp. 423 – 472, 1995.

[2] D. Bar-Natan, T. Le, and D. Thurston, "Two applications of elementary knot theory to lie algebras and vassiliev invariants," *Geometry and Topology*, vol. 7, 05 2002.

[3] J. S. Birman and X.-S. Lin, "Knot polynomials and vassiliev's invariants." *Inventiones mathematicae*, vol. 111, no. 2, pp. 225–270, 1993.

[4] K.-T. Chen, "Iterated path integrals and generalized paths," *Bull. Amer. Math. Soc.*, vol. 73, no. 6, pp. 935–938, 11 1967.

[5] S. Chmutov, S. Duzhin, and J. Mostovoy, "Introduction to vassiliev knot invariants," *Introduction to Vassiliev Knot Invariants*, 03 2011.

[6] D. Goundaroulis, J. Dorier, and A. Stasiak, "A systematic classification of knotoids on the plane and on the sphere," *arXiv: Geometric Topology*, 2019.

[7] M. Goussarov, M. Polyak, and O. Viro, "Finite-type invariants of classical and virtual knots," *Topology*, vol. 39, pp. 1045–1068, 2000.

[8] N. Gügümcü and L. H. Kauffman, "New invariants of knotoids," *European Journal of Combinatorics*, vol. 65, pp. 186 – 229, 2017.

[9] N. Gügümcü, L. H. Kauffman, and S. Lambropoulou, "A survey on knotoids, braidoids and their applications," *arXiv: Geometric Topology*, 2016.

[10] A. Henrich, “A sequence of degree one vassiliev invariants for virtual knots,” *Journal of Knot Theory and Its Ramifications*, vol. 19, no. 04, pp. 461–487, 2010.

[11] V. F. R. Jones, “A polynomial invariant for knots via von neumann algebras,” *Bull. Amer. Math. Soc. (N.S.)*, vol. 12, no. 1, pp. 103–111, 01 1985.

[12] L. H. Kauffman, “State models and the jones polynomial,” *Topology*, vol. 26, no. 3, pp. 395 – 407, 1987.

[13] ——, “Virtual knot theory,” *European Journal of Combinatorics*, vol. 20, no. 7, pp. 663 – 691, 1999.

[14] ——, *Knots and Physics*, 3rd ed. World Scientific, 2001.

[15] ——, “Knot diagrammatics,” *Handbook of Knot Theory*, 11 2004.

[16] ——, “An affine index polynomial invariant of virtual knots,” *Journal of Knot Theory and Its Ramifications*, vol. 22, no. 04, p. 1340007, 2013.

[17] V. Knizhnik and A. Zamolodchikov, “Current algebra and wess-zumino model in two dimensions,” *Nuclear Physics B*, vol. 247, no. 1, pp. 83 – 103, 1984.

[18] D. Kodokostas and S. Lambropoulou, “Rail knotoids,” *Journal of Knot Theory and Its Ramifications*, vol. 28, no. 13, p. 1940019, 2019.

[19] T. Kohno, “Monodromy representations of braid groups and yang-baxter equations,” *Ann. Inst. Fourier*, vol. 37, pp. 139–160, 1987.

[20] M. Kontsevich, “Vassiliev’s knot invariants,” *Adv. in Soviet Math.*, vol. 16, pp. 137–150, 1993.

[21] D. Kutluay, “Winding homology of knotoids,” *arXiv: Geometric Topology*, 2020.

[22] M. Polyak and O. Viro, “Gauss diagram formulas for vassiliev invariants,” *IMRN. International Mathematics Research Notices*, vol. 1994, 01 1994.

[23] T. Stanford, “Finite-type invariants of knots, links, and graphs,” *Topology*, vol. 35, no. 4, pp. 1027 – 1050, 1996.

[24] ——, “Some computational results on mod 2 finite-type invariants of knots and string links, invariants of knots and 3-manifolds,” *Geom. Topol.Monogr.*, vol. 4, pp. 363–376, 2004.

[25] V. Turaev, “Knotoids,” *Osaka J. Math.*, vol. 49, no. 1, pp. 195–223, 03 2012.

[26] V. Vassiliev, “Cohomology of knot spaces,” *Theory of Singularities and Its Applications (ed. V. I. Arnold), Adv. Soviet Math.*, vol. 1, pp. 23–70, 1990.