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Quantum nonlocality and operator space theory

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Abstract

In this study we present the connection between the theory of Quantum Nonlocality and the Operator Space Theory. Nonlocality refers to the study of natural phenomena that violate the principle of locality in physics and which, using the mathematical framework established by Bell, is reduced to the study of the so-called Bell functionals. More specifically, defining the classical and quantum value of such a function, certain inequalities arise, called Bell inequalities. The study of nonlocality focuses on the cases in which we have a violation of these inequalities, that is, when the quantum value exceeds the corresponding classical one. Using tools from Banach space theory and Operator Space Theory we correlate the values of Bell functionals with specific norms in these spaces.

We begin with the study of norms in tensor products, specifically with the injective and minimal tensor norms, which we then correlate with the norms in the spaces of bounded and completely bounded bilinear forms. We present some basic results of Operator Space Theory while we focus our attention on specific operator spaces and their norms. Next we introduce the sets of the classical and quantum probability distributions and study the values of Bell functional when acting on these sets. We also discuss these values in the context of the so-called two-player games. These values express the maximum expected value of the probability of winning a game using either "classic" resources (classic value) or "quantum" resources (quantum value).

The class of games called XOR games highlights the connection between the theory that studies the values of a game and its so-called violation of the corresponding quantum value and Operator Space Theory, as we prove that these values are now equal to injective and minimal tensor norms of their associated tensors. In this context we also prove that Grothendieck's theorem provides an upper bound on the so-called violation, that is, on the maximum possible deviation between classical and quantum value. We then present a theorem that excludes the existence of such a bound in the case of three-player games and therefore predicts an infinite possible violation.

In the last chapters of this study we extend the connection we made, to the more general two-player games class and show that the numbers of "questions" and "answers" in such a game are upper bounds for the quantum and classical value of the game. Finally, we discuss the quantification of the phenomenon called entanglement through the Schmidt rank of a Schmidt decomposition and show that the dimension of the state space is an upper bound to the maximum violation of a Bell inequality.

Περίληψη

Στη παρούσα μελέτη παρουσιάζουμε τη σχέση μεταξύ της Κβαντομηχανικής Θεωρίας της μη-τοπικότητας και της Θεωρίας Χώρων Τελεστών. Η μη-τοπικότητα αφόρα στη μελέτη φυσικών φαινομένων που παραβιάζουν την Φυσική αρχή της τοπικότητας και η οποία, κάνοντας χρήση του μαθηματικού πλαισίου που έθεσε ο Bell, ανάγεται στη μελέτη των λεγόμενων συναρτησοειδών Bell. Πιο συγκεκριμένα, ορίζοντας τη κλασική και κβαντική τιμή ενός τέτοιου συναρτησοειδούς προκύπτουν κάποιες ανισότητες που ονομάζονται ανισότητες Bell. Η μελέτη της μη-τοπικότητας εστιάζει στις περιπτώσεις στις οποίες έχουμε παραβίαση (violation) των ανισοτήτων αυτών, δηλαδή, όταν η κβαντική τιμή ξεπερνάει την αντίστοιχη κλασική. Χρησιμοποιώντας εργαλεία από τη Θεωρία Χώρων Banach και τη Θεωρία Χώρων Τελεστών συσχετίζουμε τις τιμές των συναρτησοειδών Bell με συγκεκριμένες νόρμες σε αυτούς τους χώρους.

Ξεκινάμε με τη μελέτη νορμών σε τανυστικά γινόμενα, συγκεκριμένα με τις injective και minimal νόρμες, τις οποίες στη συνέχεια συσχετίζουμε με τις νόρμες στους χώρους των φραγμένων και πλήρως φραγμένων διγραμμικών απεικονίσεων. Παρουσιάζουμε ορισμένα βασικά αποτελέσματα της θεωρίας χώρων τελεστών και εστιάζουμε σε συγκεκριμένους χώρους χώρους τελεστών και στις νόρμες τους. Στη συνέχεια εισάγουμε τα σύνολα των κλασικών και κβαντικών κατανομών πιθανότητας και μελετάμε τις τιμές που παίρνουν τα συναρτησοειδή Bell δρώντας σε αυτά τα σύνολα. Συζητάμε επίσης για τις τιμές αυτές στο πλαίσιο των λεγόμενων παιγνίων μεταξύ δύο παικτών. Οι τιμές

αυτές εκφράζουν την μέγιστη μέση τιμή της πιθανότητας νίκης ενός παιγνίου χρησιμοποιώντας είτε ¨κλασικούς¨ πόρους (κλασική τιμή) είτε ¨κβαντικούς¨ πόρους (κβαντική τιμή).

Η κλάση παιγνίων που ονομάζονται XOR παίγνια αναδεικνύει τη σχέση της θεωρίας που μελετάει τις τιμές ενός παιγνίου και την λεγόμενη παραβίασή της από την αντίστοιχη κβαντική τιμή με τη Θεωρία Χώρων Τελεστών, καθώς αποδεικνύουμε ότι οι τιμές αυτές ισούνται πλέον με injective και minimal νόρμες των επαγόμενων τανυστών. Σε αυτό το πλαίσιο αποδεικνύουμε επίσης ότι το θεώρημα του Grothendieck θέτει ένα άνω φράγμα στην λεγόμενη παραβίαση, δηλαδή, στην μέγιστη δυνατή απόκλιση μεταξύ κλασικής και κβαντικής τιμής. Έπειτα διατυπώνουμε ενα θεώρημα που αποκλείει την ύπαρξη φράγματος στην περίπτωση των παιγνίων τριών παικτών και συνεπώς προβλέπει άπειρη δυνατή παραβίαση.

Στα τελευταία κεφάλαια της παρούσας μελέτης επεκτείνουμε τη σύνδεση που κάναμε στην ευρύτερη κλάση παιγνίων δύο παικτών και δείχνουμε ότι τα πλήθη των ¨ερωτήσεων¨ και των ¨απαντήσεων¨ σε ένα τέτοιο παίγνιο αποτελούν άνω φράγματα για τον λόγο της κβαντικής και κλασικής τιμής του παιγνίου. Τέλος, μελετάμε την ποσοτικοποίηση του φαινομένου που ονομάζεται entanglement μιας κατάστασης μέσω του Schmidt rank μιας αναπαράστασης Schmidt και δείχνουμε ότι η διάσταση του χώρου των καταστάσεων αποτελεί άνω φράγμα στη μέγιστη παραβίαση μιας ανισότητας Bell.

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Introduction

In 1935 A. Einstein, B. Podolsky and N. Rosen [1] proposed an experiment whose aim was to show that Quantum Mechanics (i.e. the description of reality as given by the wave function) is not a complete physical theory. The authors introduced a condition for the reality of a physical quantity; a condition which they considered reasonable since its negation led to a definition in which the reality of a physical quantity depends on the measurement performed on its system. Such a criterion characterizes what we call *realism*. The phenomenon predicted by their experiment, i.e. the fact that when using two spatially separated particles sharing an "entangled" state (as it was later called) one can produce an immediate effect in one of them by just acting on the other one, is known as "spooky action at a distance" and was considered impossible by the authors as it violated the *local realist* view of causality. *Locality* is a principle that states that an action at one point, to have an influence at another point, needs to have something (e.g. particle or wave) to mediate the action between the points[1]. The phenomenon underlying the aforementioned paradox, known as *Quantum entanglement* is one of the most intriguing aspects of quantum mechanics.

To resolve this supposed paradox, if one admits Quantum Mechanics as a complete theory, it seems necessary to supplement the theory by additional variables, to which we do not have complete access, and hence to

[1]Together with Relativity theory, such an influence is limited by the speed of light.

assume a "hidden variable" interpretation of Quantum Mechanics. However, in 1964, J. S. Bell [2] showed that the predictions of quantum theory are incompatible with those of any physical theory satisfying the conditions of a Local Hidden Variable model. Such a model proposes that there exists a classical probability distribution to which we do not have complete access and which models our uncertainty. In particular, Bell showed that the assumption of a Local Hidden Variable model implies some inequalities, on the set of correlations obtained in a certain measurement scenario, that are violated by certain quantum correlations produced with an entangled state. Those inequalities are since then called Bell inequalities and are of great importance for the purposes of this study. The violations of Bell inequalities is a phenomenon known as Quantum Nonlocality, because it implies the existence of a quantum correlation which cannot be explained by means of a Local Hidden Variable model of nature. From A. Aspect's experiments [3] in 1982, to the recent 2015 loophole-free experiment [4], Bell's work led to experimental verifications of such a counterintuitive phenomenon, which provide evidence that nature does not obey the laws of classical physics.

We shall now discuss the notion of locality by providing it with a mathematical definition and considering it in the case of a "Bell experiment". In a typical "Bell experiment", two systems which may have previously interacted, are now spatially separated and are each measured by one of two distant observers, Alice and Bob. The spatial separation condition is needed to ensure that Alice's measurement cannot influence Bob's measurement and vice versa. Charlie, (a 'referee') prepares two particles, in whatever way he wants and sends one of the particles to Alice and the other one to Bob. Subsequently Alice performs a measurement and let x denote the measurement she makes. Similarly Bob chooses to measure y. Once the measurements are performed they yield outcomes a and b for Alice and Bob respectively. In general, the outcomes a and b may vary, even when the same choices of measurements x and y are made. We may thus assume that they are governed by a probability distribution $P(a, b|x, y)$. We further assume that Alice and Bob 'respond' according to probability distributions $P_A(a|x)$ and $P_B(b|y)$. Let us also assume that Charlie can prepare a similar pair of particles as many times as he wants. When such an experiment is performed, it will be found, in general, that

$$P(a,b|x,y) \neq P_A(a|x)P_B(b|y)$$

implying that the outcomes on both sides are not statistically independent from each other. Even though the two systems may be separated by a large distance, the existence of such correlations is nothing mysterious. In particular, it does not necessarily imply some kind of direct influence of one system on the other, for these correlations may simply reveal some dependence relation between the two systems which was established when they interacted in the past. Let us formalize now the idea of a Local Hidden Variable theory. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between the outcomes a and b. Once all such factors have been taken into account, we should be now be able to factorize :

$$P(a,b|x,y,\lambda) = P_A(a|x,\lambda)P_B(b|y,\lambda)$$

The factorizability condition simply expresses the idea that we have found a way of describing the fact that the probability of a only depends on the past variables λ and the local measurement x and not on the distant measurement and outcome, and similarly for b. We further assume that the variables λ, which take values in a set Λ, are characterized by a probability distribution $q(\lambda)$. Combined with the above factorizability condition we can thus write

$$P(a,b|x,y) = \int_{\Lambda} P_A(a|x,\lambda)P_B(b|y,\lambda)q(\lambda)d\lambda \qquad (0.0.1)$$

where we also implicitly assumed that the measurements x and y can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda|x,y) = q(\lambda)$. This decomposition now represents a precise condition for locality in the context of Bell experiments. It is now straightforward to prove that the predictions of quantum theory for certain experiments involving entangled particles do not admit a decomposition of the form 0.0.1.

Indeed, let us assume for simplicity that there are only two measurement choices for Alice and Bob $x, y \in \{0, 1\}$ and the same holds for their outcomes $a, b \in \{-1, 1\}$. Let $\langle a_x b_y \rangle = \sum_{a,b} abP(a, b|x, y)$ be the expectation value of the product ab for the measurement choices (x, y) and consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \tag{0.0.2}$$

If the probabilities satisfy the factorizability condition of locality 0.0.1 then we may write

$$\langle a_x b_y \rangle = \int \langle a_x \rangle_{\lambda} \langle b_y \rangle_{\lambda} q(\lambda) d\lambda$$

where $\langle a_x \rangle_{\lambda} = \sum_a aP_A(a|x, \lambda)$ and $\langle b_y \rangle_{\lambda} = \sum_b bP_B(b|y, \lambda)$ are the local expectations that take values in $[-1, 1]$. Subsequently, if we replace the above expressions to 0.0.2 we find that

$$S = \int \langle a_0 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} + \langle a_0 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} + \langle a_1 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} - \langle a_1 \rangle_{\lambda} \langle b_1 \rangle_{\lambda} q(\lambda) d\lambda$$

Since $\langle a_x \rangle, \langle b_y \rangle \in [-1, 1]$ for all x, y we have that

$$\begin{aligned}
&\langle a_0 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} + \langle a_0 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} + \langle a_1 \rangle_{\lambda} \langle b_0 \rangle_{\lambda} - \langle a_1 \rangle_{\lambda} \langle b_1 \rangle_{\lambda} \\
&\leq |\langle b_0 \rangle_{\lambda} + \langle b_1 \rangle_{\lambda}| + |\langle b_0 \rangle_{\lambda} - \langle b_1 \rangle_{\lambda}| \\
&= 2\langle b_0 \rangle_{\lambda} \\
&\leq 2.
\end{aligned}$$

a result that we will prove with in Section 3.1. Hence, we can see that

$$S \leq 2 \tag{0.0.3}$$

an inequality known as Clauser-Horne-Shimony-Holt (CHSH) inequality [5].

Let us assume now that Nature is explained by quantum mechanics and that the state formed by both particles is described by

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

This state is often called Einstein-Poldosky-Rosen (EPR) state. Here we considered the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the basis vectors

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the notation $|ab\rangle = |a\rangle \otimes |b\rangle$, where the above notation and notions will be discussed throughout the next chapters of these notes.

We further assume the matrices $A_0 = \sigma_x$, $B_0 = \frac{-\sigma_z - \sigma_x}{\sqrt{2}}$, $A_1 = \sigma_z$, $B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$ to be our "measurements", where here we are using the standard notation for the *Pauli matrices*:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can then calculate:

$$\begin{aligned} \langle\psi|A_0 \otimes B_0|\psi\rangle &= -\frac{1}{\sqrt{2}} \\ \langle\psi|A_0 \otimes B_1|\psi\rangle &= -\frac{1}{\sqrt{2}} \\ \langle\psi|A_1 \otimes B_0|\psi\rangle &= -\frac{1}{\sqrt{2}} \\ \langle\psi|A_1 \otimes B_1|\psi\rangle &= \frac{1}{\sqrt{2}} \end{aligned}$$

which leads us to

$$S = \langle\psi|A_0 \otimes B_0|\psi\rangle + \langle\psi|A_0 \otimes B_1|\psi\rangle + \langle\psi|A_1 \otimes B_0|\psi\rangle - \langle\psi|A_1 \otimes B_1|\psi\rangle = 2\sqrt{2} \tag{0.0.4}$$

which is in contradiction with 0.0.3 and thus with the locality condition 0.0.1. This is the content of Bell's theorem, establishing that quantum theory (and any model reproducing its predictions) cannot be explained by a Local Hidden Variable model.

We deem it necessary at this point to explain why the expression 0.0.4 is the quantum analogue of 0.0.2. In Quantum probability theory, the expectation value of an observable X (selfadjoint operator) in the state (positive operator of unit trace) ρ is defined [6] to be

$$\mathbb{E}_\rho(X) = \mathrm{Tr}(\rho X).$$

Thus, in the state $\rho = |\psi\rangle\langle\psi|$ the expected value of the observable $A \otimes B$ is $\mathbb{E}_\rho(A\otimes B) = \mathrm{Tr}((A\otimes B)\rho)$, where it is an easy exercise to verify that $\mathrm{Tr}((A\otimes B)\rho) = \langle\psi|A\otimes B|\psi\rangle$. Finally, note that the spatial separation condition of Alice and Bob, is mathematically expressed by the tensor product model of the two systems as it suitably reflects this idea.

Having established the historical background, we can now summarize the main subject of this work as the following fundamental question in the study of nonlocality:

Suppose $M = (M_{xy}) \in \mathbb{R}^{n\times m}$ (a Bell functional) is such that

$$\sup \left| \sum_{x,y} M_{xy} \int a_x(\lambda) b_y(\lambda) q(\lambda) d\lambda \right| \leq 1, \tag{0.0.5}$$

where the supremum is over all probability spaces Λ, probability measures $d\lambda$ and functions $a_x, b_y : \Lambda \to [-1,1]$.

How large can (a violation of the Bell inequality)

$$\sup \left| \sum_{x,y} M_{xy} \langle\psi|A_x \otimes B_y|\psi\rangle \right|, \tag{0.0.6}$$

be? Here the supremum is over Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, states $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and observables $A_x \in \mathcal{B}(\mathcal{H}_1)$, $B_y \in \mathcal{B}(\mathcal{H}_2)$.

The connection of the theory of nonlocality with Banach space theory arises with the observation that quantity 0.0.5 is precisely the *injective norm* of the tensor $M \in \ell_1^n(\mathbb{R}) \otimes \ell_1^m(\mathbb{R})$ or equivalently the norm of M when it is viewed as a bilinear form $\ell_\infty^n \times \ell_\infty^m \to \mathbb{R}$. But what does the second quantity 0.0.6 correspond to? The answer to this question emerges from the theory of *operator spaces*. Operator Space Theory provides us with the essential terminology. The quantity 0.0.6 is precisely the *minimal norm* of the tensor M or equivalently the *completely bounded* norm $\|\cdot\|_{cb}$ of the associated bilinear form. Thus, the fundamental question in the study of nonlocality expressed above, finds a direct reformulation in terms of Operator Space Theory as follows:

Suppose that $M : \ell_\infty^n \times \ell_\infty^m \to \mathbb{R}$ is such that $\|M\| \leq 1$.
How large can $\|M\|_{cb}$ be?

This interesting connection, introduced in 2008 by Perez-Garcia et al.[7], has led to a number of exciting developments, with techniques of Operator Space Theory used to derive new results in quantum information theory and vice versa.

We will now briefly sketch the topics discussed in each chapter:

Chapter 0: We introduce the notations that will be used throughout this work; notations that involve linear maps, matrices and isomorphisms and at last Dirac's famous Bra-ket notation that is widely used in quantum information theory. Also we give definitions and examples of norms on the space of matrices M_n.

Chapter 1: In the first section we introduce norms on the tensor products of vector spaces, Banach spaces, Hilbert spaces and operator spaces. We state and prove many results concerning explicit forms of well known tensor norms which are proven to fit perfectly in the study of nonlocality. In the second section we introduce the basic notions of Operator Space Theory. We focus on the completely bounded norms of operators and bilinear forms on certain operator spaces. We also discuss certain isometric identifications between tensor products of operator spaces and the spaces of bounded bilinear forms, connecting the injective tensor norm with the operator norm of a bilinear form and the minimal tensor norm with the completely bounded norm of a bilinear form.

Chapter 2: We begin with a brief introduction to correlation matrices and their connection to the so-called correlation Bell functionals. Next we introduce the sets of classical and quantum probability distributions, we define the Bell functionals, the Bell inequalities and the violations of the latter. We also define the notion of the classical and quantum values of a game. In the last section we introduce the notion of a multiplayer game and its values showing its tight connection with the Bell functionals.

Chapter 3: We deal with a special class of games, the XOR games. We define the classical and quantum biases of an XOR game to which we give explicit forms unveiling the tight connection of the theory of nonlocality to that of operator spaces, that is, we connect the classical and quantum biases of a game to the injective and minimal tensor norms, respectively. Also we prove the famous CHSH inequality and its violation using the framework of operator norms. Next we discuss the impact of Grothendieck's fundamental theorem to the study of nonlocality. More specifically we prove that Grothendieck's inequalities set an upper bound on the violations achieved in two player XOR games. Finally, we present certain results concerning the case of three-player games XOR games and which allow for unbounded violations.

Chapter 4: We extend the connection of the theory of nonlocality and Operator Space Theory, to the case of general two-player games. As with the XOR class , we give explicit form to the classical and entangled values of those games again as the injective and minimal tensor norms, respectively, of their associated tensors. We also prove that the numbers of "questions" and "answers" in a game, provide upper bounds on the largest possible violations of those games.

Chapter 5: In this last chapter, we quantify the notion of entanglement through the Schmidt decomposition and the Schmidt rank connecting the notion with the dimension of the state space. Finally, we give dimension dependent bounds for the largest violations achievable by states having Schmidt rank at most the dimension of the state space.

Notation and basic conventions

0.1 Notation

Throughout these notes, we use boldfont for finite sets as $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}$ and small capital letters X,Y,A,B for their cardinalities. The sets $\mathbf{X}, \mathbf{Y}$ will usually denote sets of *questions*, or *inputs*, to a game or Bell inequality, and $\mathbf{A}, \mathbf{B}$ will denote sets of *answers*, or *outputs*.

Write ℓ_p^n for the n-dimensional complex ℓ_p space and $\ell_p^n(\mathbb{R})$ for the real one. Unless specified otherwise, the space $\mathbb{C}^n$ will be endowed with the Hilbertian norm and identified with ℓ_2^n. We write $\mathrm{Ball}(X)$ for the closed unit ball of the normed space X.

Given vector spaces X, Y and Z, $L(X,Y)$ is the space of linear maps from X to Y and we write $L(X)$ for the space $L(X,X)$. $\mathrm{Id}_X \in L(X)$ is the identity map on X. $B(X \times Y; Z)$ is the space of bilinear maps from $X \times Y$ to Z. If $Z = \mathbb{C}$ we just write $B(X \times Y)$. We write $\|\cdot\|_X$ for the norm on X. When we write $\|x\|$ without explicitly specifying the norm we always mean the "natural" norm on x: the Banach space norm if $x \in X$ or the operator norm if $x \in L(X,Y)$ or $x \in B(X \times Y; Z)$.

Given Banach spaces X, Y and Z, a linear map $T \in L(X,Y)$ is called bounded if its norm $\|T\| := \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}$ is finite. We denote $\mathcal{B}(X,Y)$ the Banach space of bounded linear maps. Similarly, a bilinear map $B \in B(X \times Y; Z)$ is called bounded, if $\|B\| := \sup\{\|B(x,y)\|_Z : \|x\|_X, \|y\|_Y \leq 1\} < \infty$ and we write $\mathcal{B}(X \times Y; Z)$ for the space of such maps. If X is a Banach space we write $X^* = \mathcal{B}(X, \mathbb{C})$ for its dual, and $X^\sharp = L(X, \mathbb{C})$ for its algebraic dual.

We write $M_{n,m}$ for the set of $n \times m$ matrices with complex entries and simply M_n for the case of $m = n$.

Given a Hilbert space $\mathcal{H}$ we may often use the following notation: $\mathcal{P}(\mathcal{H})$: the set of all orthogonal projections in $\mathcal{H}$, whose elements are called *events*, $\mathcal{O}(\mathcal{H})$: the set of all selfadjoint operators in $\mathcal{H}$, whose elements are called *observables*, $\mathcal{S}(\mathcal{H})$: the set of positive operators with unit trace in $\mathcal{H}$, whose elements are called *states*, $\mathcal{B}_+(\mathcal{H})$: the set of positive operators in $\mathcal{H}$.

Let $\mathcal{A}$ be a Banach algebra with a unit. We usually denote its unit by $\mathbf{1}_{\mathcal{A}}$.

The *Kronecker delta* is defined by:

$$\delta_{i,j} = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases}$$

0.2 Matrices and operators

Given a complex vector space V and $n \in \mathbb{N}$, an n-*tuple* over V is an element of the space $V^n = V \oplus \cdots \oplus V$, where we employ the vertical display of such an element, that is, for $v = (v_i)_{i=1}^n \in V^n$,

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

where $v_i \in V$ for each $i = 1, \ldots, n$.

Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{pmatrix}$ denote the usual basis vectors of $\mathbb{C}^n$ and V a complex vector space. Then, we have the following linear identifications

$$V^n \cong \mathbb{C}^n \otimes V \cong V \otimes \mathbb{C}^n,$$

defined by

$$V^n \cong \mathbb{C}^n \otimes V \qquad\qquad V^n \cong V \otimes \mathbb{C}^n$$

$$v \mapsto \sum_{i=1}^{n} e_i \otimes v_i \qquad\qquad u \mapsto \sum_{i=1}^{n} v_i \otimes e_i.$$

Each linear map $\phi : V \to W$ between complex vector spaces, defines a linear map

$$\phi_n : V^n \to W^n$$
$$\phi_n(v) = (\phi(v_i)),$$

or equivalently

$$\phi_n = \mathrm{Id} \otimes \phi : \mathbb{C}^n \otimes V \to \mathbb{C}^n \otimes W$$
$$e_i \otimes v_i \mapsto e_i \otimes \phi(u_i).$$

Using more general indices, let $\mathcal{S}$ be an arbitrary set. We define an $\mathcal{S}$-*tuple* $u = (u_s)_{s\in\mathcal{S}}$ to be a function from $\mathcal{S}$ to V. Let $V^{\mathcal{S}}$ denote the space of $\mathcal{S}$-tuples. A bijection of index sets $f : \mathcal{S} \to \mathcal{S}'$, determines a linear isomorphism $V^{\mathcal{S}} \cong V^{\mathcal{S}'}$ via $u_s = u'_{f(s)}$.

To relate to the previous notation, we may use the integer n instead of the set $\{1,\ldots,n\}$. For example the space of n-tuples $\mathbb{C}^{\{1,\ldots,n\}}$ will be denoted by $\mathbb{C}^n$. More specifically, let Σ be an alphabet, i.e. a finite non empty set. The complex space $\mathbb{C}^\Sigma$ will be also viewed as $\mathbb{C}^n$ for $|\Sigma| = n$. Fix a bijection $f : \{1,\ldots,n\} \to \Sigma$ and associate each function $u \in \mathbb{C}^\Sigma$ to a vector in $\mathbb{C}^n$ whose k-th entry is $u(f(k))$, for all $k \in \{1,\ldots,n\}$.

Given a vector space V and integers n, m we denote the vector space of $n \times m$ matrices with entries from V by $M_{n,m}(V)$. That is, an element v of $M_{n,m}(V)$ is

$$v = \begin{pmatrix} v_{11} & \ldots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \ldots & v_{nm} \end{pmatrix}$$

or simply $v = [v_{i,j}]$, where $v_{i,j} \in V$ for all $1 \le i \le n$ and $1 \le j \le m$.

We write $M_{n,m}(\mathbb{C}) = M_{n,m}$ and $M_n = M_{n,n}$.

The *matrix units*

$$E_{i,j} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & 1_{i,j} & \vdots \\ 0 & \dots & 0 \end{pmatrix},$$

form a vector basis for $M_{n,m}$.

The *identity matrix* $\mathbf{1}_{M_n} = I_n$ of M_n is given by

$$I_n = \sum_{i=1}^{n} E_{i,i} = (\delta_{i,j})_{i,j=1}^{n}.$$

Using the matrix units we obtain the identifications

$$M_n(V) \cong M_n \otimes V \cong V \otimes M_n \tag{0.2.1}$$

by

$$M_n(V) \cong M_n \otimes V \tag{0.2.2}$$
$$v = [v_{i,j}] \mapsto \sum E_{i,j} \otimes v_{i,j} \tag{0.2.3}$$

and

$$M_n(V) \cong V \otimes M_n \tag{0.2.4}$$
$$v = [v_{i,j}] \mapsto \sum v_{i,j} \otimes E_{i,j}. \tag{0.2.5}$$

Also, let $a = [a_{i,j}] \in M_n$ and $v_0 \in V$ and then the elementary tensor $a \otimes v_0$ is given by

$$a \otimes v_0 = \sum_{i,j=1}^{n} a_{i,j} E_{i,j} \otimes v_0 = \sum_{i,j=1}^{n} E_{i,j} \otimes v_0 a_{i,j} = [a_{i,j} v_0].$$

Thus by 0.2.2, for example, an element $u \in M_n(V)$ may be represented as $u = \sum_i A_i \otimes v_i$ where $A_i \in M_n$ and $v_i \in V$, i.e. as a linear combination with matrix coefficients. Note that this is not a unique representation and it is often convenient to use other such decompositions.

Let $u \in M_{n,m}(V)$ and $v \in M_{p,q}(V)$, we define the *direct sum* $u \oplus v \in M_{n+p,m+q}(V)$ by

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_{n+p,m+q}(V).$$

If we are given $a \in M_{n,p}$, $u \in M_{p,q}(V)$ and $b \in M_{q,m}$ we can define the matrix product $aub \in M_{p,q}(V)$ by

$$aub = \Big[\sum_{k,l} a_{i,k}u_{k,l}b_{l,j}\Big]_{1\le i\le p, 1\le j\le q}.$$

Equivalently, if $v \in V$ and $c \in M_{p,q}$, we then have

$$a(c\otimes v)b = a[c_{ij}v]b = \Big[\sum_{k,l} a_{i,k}c_{k,l}b_{l,j}v\Big] = acb\otimes v.$$

Note that if we have $a \in M_m$, $b \in M_m$ and $c \in M_m(M_n)$, using the identification $M_m(M_n) \cong M_m \otimes M_n$ we obtain that

$$acb = (a\otimes I_n)c(b\otimes I_n).$$

Consider now the space $M_n(M_d(V))$, for V a complex vector space. An element $A \in M_n(M_d(V))$ of this space is of the form $A = (A_{i,j})_{i,j=1}^n$, where each $A_{i,j} \in M_d(V)$. This means that for all $i,j = 1,\dots,n$, $A_{i,j} = (a_{i,j,k,l})_{k,l=1}^d$ and $a_{i,j,k,l} \in V$ for all i,j,k,l. We can see now that by just deleting the parentheses we obtain an element of $M_{nd}(V)$. Also, if we set $B_{k,l} = (a_{i,j,k,l})_{i,j=1}^n$, then we get an element of $M_n(V)$ and thus $B = (B_{k,l})_{k,l=1}^d$ is an element of $M_d(M_n(V))$. Again, deleting the extra parentheses gives us an element of $M_{nd}(V)$. Thus, we get the isomorphisms

$$M_n(M_d(V)) \cong M_d(M_n(V)) \cong M_{nd}(V). \tag{0.2.6}$$

With these identifications A, B are unitary equivalent elements of $M_{nd}(V)$. In fact, the unitary is just a permutation matrix. Indeed, let $A \in M_n(M_d(V))$ be a matrix as above, then we can write A as an element of $M_{nd}(V)$ as $A = (c_{s,t})_{s,t=1}^{nd}$ and so $c_{s,t} = a_{i,j,k,l}$ where $s = d(i-1)+k$ and $t = d(j-1)+l$. The correspondence $\pi : \{1,\dots,n\}\times\{1,\dots,d\} \to \{1,\dots,nd\}$ where $\pi(i,k) = d(i-1)+k$ is a bijection and the same holds for the mapping $(j,l) \mapsto d(j-1)+l$. As for B, considered as an element of $M_{nd}(V)$ it is written $B = (d_{s,t})_{s,t=1}^{nd}$ and then $s = n(k-1)+i$ and $t = n(l-1)+j$. Thus, we established an one-to-one and onto correspondence between the rows and the columns of the two matrices, i.e. a permutation.

To see this more clearly, we consider for example the case of $M_2(M_3(V))$ and $M_3(M_2(V))$. The correspondence is the following:

$$A = \begin{bmatrix} \begin{bmatrix} a_{1,1,1,1} & a_{1,1,1,2} & a_{1,1,1,3} \\ a_{1,1,2,1} & a_{1,1,2,2} & a_{1,1,2,3} \\ a_{1,1,3,1} & a_{1,1,3,2} & a_{1,1,3,3} \end{bmatrix} & \begin{bmatrix} a_{1,2,1,1} & a_{1,2,1,2} & a_{1,2,1,3} \\ a_{1,2,3,1} & a_{1,2,2,2} & a_{1,2,2,3} \\ a_{1,2,3,1} & a_{1,2,3,2} & a_{1,2,3,3} \end{bmatrix} \\ \begin{bmatrix} a_{2,1,1,1} & a_{2,1,1,2} & a_{2,1,1,3} \\ a_{2,1,3,1} & a_{2,1,2,2} & a_{2,1,2,3} \\ a_{2,1,3,1} & a_{2,1,3,2} & a_{2,1,3,3} \end{bmatrix} & \begin{bmatrix} a_{2,2,1,1} & a_{2,2,1,2} & a_{2,2,1,3} \\ a_{2,2,3,1} & a_{2,2,2,2} & a_{2,2,2,3} \\ a_{2,2,3,1} & a_{2,2,3,2} & a_{2,2,3,3} \end{bmatrix} \end{bmatrix},$$

while

$$B = \begin{bmatrix} \begin{bmatrix} a_{1,1,1,1} & a_{1,2,1,1} \\ a_{2,1,1,1} & a_{2,2,1,1} \end{bmatrix} & \begin{bmatrix} a_{1,1,1,2} & a_{1,2,1,2} \\ a_{2,1,1,2} & a_{2,2,1,2} \end{bmatrix} & \begin{bmatrix} a_{1,1,1,3} & a_{1,2,1,3} \\ a_{2,1,1,3} & a_{2,2,1,3} \end{bmatrix} \\ \begin{bmatrix} a_{1,1,2,1} & a_{1,2,2,1} \\ a_{2,1,2,1} & a_{2,2,2,1} \end{bmatrix} & \begin{bmatrix} a_{1,1,2,2} & a_{1,2,2,2} \\ a_{2,1,2,2} & a_{2,2,2,2} \end{bmatrix} & \begin{bmatrix} a_{1,1,2,3} & a_{1,2,2,3} \\ a_{2,1,2,3} & a_{2,2,2,3} \end{bmatrix} \\ \begin{bmatrix} a_{1,1,3,1} & a_{1,2,3,1} \\ a_{2,1,3,1} & a_{2,2,3,1} \end{bmatrix} & \begin{bmatrix} a_{1,1,3,2} & a_{1,2,3,2} \\ a_{2,1,3,2} & a_{2,2,3,2} \end{bmatrix} & \begin{bmatrix} a_{1,1,3,3} & a_{1,2,3,3} \\ a_{2,1,3,3} & a_{2,2,3,3} \end{bmatrix} \end{bmatrix}.$$

So, if we drop the parentheses we can see that $B = P^T A P$ where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given a linear map $\phi : V \to W$, we get the corresponding map

$$\phi_n : M_n(V) \to M_n(W)$$
$$\phi_n(v) = [\phi(v_{i,j})]$$

where $v \in [v_{i,j}] \in M_n(V)$, or equivalently

$$\phi_n(a \otimes v_0) = \phi_n([a_{i,j} v_0]) = [\phi(a_{i,j} v_0)] = [a_{i,j} \phi(v_0)] = a \otimes \phi(v_0),$$

where $a = [a_{i,j}] \in M_n$ and $v_0 \in V$. Thus,

$$\phi_n = \mathrm{Id} \otimes \phi : M_n \otimes V \to M_n \otimes W.$$

Let $M_n(L(V,W))$ denote the space whose elements are $n \times n$ matrices of linear maps, $\phi = [\phi_{i,j}] \in M_n(L(V,W))$, where each $\phi_{i,j} \in L(V,W)$. We can regard such a matrix as a linear map from V to $M_n(W)$ and obtain an isomorphism between those spaces as

$$\begin{aligned} M_n(L(V,W)) &\cong L(V, M_n(W)) \\ [\phi_{i,j}] &\mapsto \big(v \mapsto [\phi_{i,j}(v)]\big). \end{aligned}$$

The associated map on the right-hand side is well defined since $v = w$ implies $\phi_{i,j}(v) = \phi_{i,j}(w)$ for all $i,j = 1, \ldots, n$ and thus $[\phi_{i,j}(v)] = [\phi_{i,j}(w)]$. Also, let $T : v \in V \mapsto [T^v_{i,j}] \in M_n(W)$ be a linear map, then for each i,j we may define the linear map $T_{i,j} : v \in V \mapsto T^v_{i,j} \in W$. This is also well defined, since $T^v_{i,j} \neq T^w_{i,j}$ for some i,j implies $T(v) \neq T(w)$ and consequently $v \neq w$.

0.3 Matrix norms

Since M_n is a vector space of dimension n^2, one can measure the size of a matrix by using any norm on $\mathbb{C}^{n^2}$. However, M_n is not just a vector space; it has a natural multiplication operation, and thus it seems necessary to introduce a norm that takes advantage of this extra structure.

Definition 0.3.1. *A function $\|\cdot\| : M_n \to \mathbb{R}$ is a ring norm on M_n if, for all A, $B \in M_n$, it satisfies the following:*

1. $\|A\| \geq 0$ *and* $\|A\| = 0$ *if and only if* $A = 0$
2. $\|c \cdot A\| = |c| \cdot \|A\|$, *for all* $c \in \mathbb{C}$
3. $\|A + B\| \leq \|A\| + \|B\|$
4. $\|A \cdot B\| \leqslant \|A\| \cdot \|B\|$

The first four properties of a ring norm are identical to the axioms for a norm. A norm on matrices that does not necessarily satisfy property (4) is a vector norm on matrices. Here are some examples of norms on the space M_n when treated as $\mathbb{C}^{n^2}$.

Definition 0.3.2. *The ℓ_1-norm defined for $A \in M_n$ by*

$$\|A\|_{\ell_1} := \sum_{i,j=1}^{n} |a_{i,j}|$$

is a ring norm.

Example 0.3.3. *The ℓ_2-norm (Frobenius norm, Hilbert-Schmidt norm) defined for $A \in M_n$ by*

$$\|A\|_{\ell_2} := |\mathrm{Tr}(AA^*)|^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{\frac{1}{2}}.$$

is a ring norm. We may also denote the Frobenius norm as $\|A\|_2$. Note that the Frobenius norm is just the Euclidean norm of A thought of as a vector in $\mathbb{C}^{n^2}$. Since tr(AA^*) is the sum of the eigenvalues of AA^*, and these eigenvalues are just the squares of the singular values $\sigma_k(A)$ of A, we have an alternative characterization of the Frobenius norm:

$$\|A\|_2 = \sqrt{\sigma_1(A)^2 + \cdots + \sigma_n(A)^2}.$$

The singular values of A are the same as those of A^*, so :

$$\|A\|_2 = \|A^*\|_2.$$

Example 0.3.4. *The ℓ_∞-norm or maximum norm defined for $A \in M_n$ by:*

$$\|A\|_{\max} := \max_{1 \le i,j \le n} |a_{i,j}|.$$

is a norm on the vector space M_n but it is not a ring norm. However $n\|A\|_{\max}$ is a ring norm.

Example 0.3.5. *One can simply define the ℓ_p-norm for all $p \ge 1$ by*

$$\|A\|_{\ell_p} := \left(\sum_{i,j=1}^{n} |a_{i,j}|^p\right)^{\frac{1}{p}}.$$

which is a norm on the vector space M_n and of course for $p = 1, 2$ it yields the cases discussed above.

Definition 0.3.6. *Let $||\cdot||'$ be a norm on $\mathbb{C}^n$. We define a norm $||\cdot||$ on M_n by*

$$||A|| = \sup_{||x||'=1} ||Ax||'.$$

and call it the ring norm induced by the vector norm $||\cdot||'$, or the operator norm on M_n, or simply the induced norm when it is clear from the context.

Note that the supremum in the definition above is a maximum since we deal with a finite dimensional vector space. Also, it is clear that the norm defined above is just the norm of $A \in M_n$ if we see it as an operator $A \in \mathcal{B}(\mathbb{C}^n)$, for $(\mathbb{C}^n, ||\cdot||')$ thus it is no surprise that it satisfies the following properties:

Proposition 0.3.7. *The operator norm $||\cdot||$ on M_n, induced by $||\cdot||'$ on $\mathbb{C}^n$ has the following properties:*

1. $||I_{M_n}|| = 1$.
2. $||Ax||' \le ||A|| \cdot ||x||'$ *for all* $A \in M_n$ *and* $x \in \mathbb{C}^n$.
3. $||\cdot||$ *is a ring norm on* M_n.
4. $||A|| = \sup_{||x||' \le 1} ||Ax||' = \sup_{x \ne 0} \frac{||Ax||'}{||x||'}$.

Proof. 1. Note that $||I_{M_n}|| = \max_{||x||'=1} ||I_{M_n}x||' = \max_{||x||'=1} ||x||' = 1$.

2. Let A be a matrix in M_n. The case $x = 0$ is obvious, so let $x \ne 0$. The element $\frac{x}{||x||'}$ is of norm one, thus $||A|| \ge \left\|A\frac{x}{||x||'}\right\|' \Rightarrow ||A|| \cdot ||x||' \ge ||Ax||'$.

3. We will verify the four requirements. Obviously $||A|| \ge 0$ and if $A = 0$, then $||A|| = 0$. Now if $A \ne 0$, then there exists an element x such that $Ax \ne 0$ and so $||A|| \ge ||Ax||' > 0$. For each $c \in \mathbb{C}$, we have that $||cA|| = \max_{||x||'} ||cAx||' = \max_{||x||'} |c| \, ||Ax||' = |c| \, ||A||$. For every unit vector $x \in \mathbb{C}^n$, we have that $||(A+B)x||' = ||Ax+Bx||' \le ||Ax||' + ||Bx||' \le ||A|| + ||B||$ and also that $||(AB)x||' = ||A(Bx)||' \le ||A|| \, ||Bx||' \le ||A|| \, ||B||$ which gives us the desired assertions.

4. Clearly we have $\sup_{x\neq 0} \frac{\|Ax\|'}{\|x\|'} \leq \|A\| \leq \sup_{\|x\|'\leq 1} \|Ax\|'$. Suppose that $x \neq 0$, and $\|x\|' \leq 1$. Then $\sup_{x\neq 0} \frac{\|Ax\|'}{\|x\|'} \geq \frac{\|Ax\|'}{\|x\|'} \geq \|Ax\|'$ and this finishes our proof.

□

Example 0.3.8. *The maximum column sum ring norm is defined on M_n by*

$$\|A\|_{col} := \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{i,j}|$$

where $A = [a_{i,j}] \in M_n$. This norm is induced by the ℓ_1-norm on $\mathbb{C}^n$ and hence it is a ring norm.

Example 0.3.9. *The maximum row sum ring norm is defined on M_n by*

$$\|A\|_{row} := \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{i,j}|$$

where $A = [a_{i,j}] \in M_n$. This norm is induced by the ℓ_∞-norm on $\mathbb{C}^n$ and hence it is a ring norm.

Example 0.3.10. *The spectral norm $\|\cdot\|_{S_\infty^n}$ is defined on M_n by:*

$$\|A\|_{S_\infty^n} := \max_{\|x\|_2=1} \|Ax\|_2$$

where $x \in \mathbb{C}^n$ and $\|\cdot\|_2$ denotes the Euclidean norm. We will see that $\|\cdot\|_{S_\infty^n}$ is induced by the Euclidean norm on $\mathbb{C}^n$ and hence is a ring norm.

Example 0.3.11. *The Schatten p-norms , for $1 \leq p < \infty$ are defined by*

$$\|A\|_{S_p^n} := \left(\mathrm{Tr}\left((A^*A)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}$$

and are ring norms. We denote the Schatten spaces by $S_p^n := (M_n, \|\cdot\|_p)$. The Schatten p-norm of a matrix A coincides with the ordinary ℓ_p-norm of the vector of singular values of A, that is ,

$$\|A\|_{S_p^n} = \left(\mathrm{Tr}\left((A^*A)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |\sigma_i(A)|^p\right)^{\frac{1}{p}}$$

This family includes the three most commonly used norms in quantum information theory: the spectral norm, the Frobenius norm, and the trace norm.

- $p = 1$: *(The trace norm)*

  The Schatten 1-norm is defined by

  $$\|A\|_{S_1^n} := \operatorname{Tr}\left(\sqrt{A^*A}\right)$$

  and it is called the trace norm. It is also equal to the sum of the singular values of A.

- $p = 2$: *(The Frobenius norm)*

  The Schatten 2-norm defined by

  $$\|A\|_2 := \left(\operatorname{Tr}(A^*A)\right)^{\frac{1}{2}}$$

  is obviously the Frobenius norm defined in 0.3.3.

- $p = \infty$: *(The spectral norm)*

  The Schatten ∞-norm, more commonly spectral norm, is defined by

  $$\|A\|_{S_\infty^n} := \sigma_{\max}(A)$$

  the largest singular value of the matrix A. We set $S_\infty^n := (M_n, \|\cdot\|_\infty)$ to be the corresponding Schatten space. We can easily see, using the singular value decomposition of A, that $\|\cdot\|_{S_\infty^n}$ is induced by the Euclidean norm on $\mathbb{C}^n$ and hence it is the operator norm already described in 0.3.10. Indeed, let $A = V\Sigma W^*$ be such a decomposition , where V, W are unitary, $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. Using unitary invariance and monotonicity of the Euclidean norm we have

  $$\begin{aligned}\sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \left\|V\Sigma W^* x\right\|_2 = \sup_{\|x\|_2=1} \left\|\Sigma W^* x\right\|_2 \\ = \sup_{\|Wy\|_2=1} \|\Sigma y\|_2 = \sup_{\|y\|_2=1} \|\Sigma y\|_2 \\ \leq \sup_{\|y\|_2=1} \|\sigma_1 y\|_2 = \sigma_1 \sup_{\|y\|_2=1} \|y\|_2 = \sigma_1\end{aligned}$$

  As for the converse, observe that $\|\Sigma e_1\|_2 = \sigma_1$, so we conclude that $\sup_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A)$.

The spectral norm also has the property that

$$||AA^*||_{S_\infty^n} = ||A^*A||_{S_\infty^n} = ||A||^2_{S_\infty^n}$$

for every $A \in M_n$.

From now on, we will drop the subscript of $||\cdot||_{S_\infty^n}$ and denote it just $||\cdot||$ when it comes to M_n.

0.4 Bra-ket notation

Throughout this thesis, we will be using the Dirac Bra-ket notation, commonly used in quantum mechanics. It resembles another common notation in which: $x \in \mathbb{R}^n$ is a column vector, i.e. a $n \times 1$ matrix. The transpose x^T is a row vector , or a linear functional on $\mathbb{R}^n$, xy^T is the outer product of column vectors x and y, while $x^T y$ is their inner (scalar) product.

Dirac's notation is quite similar. The vectors in a Hilbert space $\mathcal{H}$, are written as $|\psi\rangle$ (a ket vector) while the same vector but now seen as an element of $\mathcal{H}^*$ is written as $|\psi\rangle^*$ and denoted by $\langle\psi|$ (a bra vector). The bra-ket notation works seamlessly with standard operations on Hilbert spaces. The action of a functional $\langle\psi|$ on a vector $|x\rangle$ is $\langle\psi|x\rangle$, an alternative notation for the scalar product $\langle\psi, x\rangle$ (recall that the inner product is defined to be linear in the second argument). If $A \in \mathcal{B}(\mathcal{H})$ and $\psi \in \mathcal{H}$ then $A|\psi\rangle = |A\psi\rangle$ and $\langle A\psi| = (A|\psi\rangle)^* = \langle\psi|A^*$. Consequently the quantity $\langle\psi'|A|\psi\rangle$ can either be read as $\langle\psi', A\psi\rangle$ or $\langle A^*\psi', \psi\rangle$ the equality of which is a consequence of A.2.7.

Now let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and ψ_1, ψ_2 two vectors in $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. The operator $|\psi_1\rangle\langle\psi_2| : \mathcal{H}_2 \to \mathcal{H}_1$ acts on $x \in \mathcal{H}_2$ as follows

$$x \mapsto |\psi_1\rangle\langle\psi_2|x\rangle = \langle\psi_2|x\rangle|\psi_1\rangle.$$

As for the space $\mathbb{C}^n$ we can represent bras and kets as :

$$\langle a| = \begin{pmatrix} a_1^* & a_2^* & \dots & a_n^* \end{pmatrix}$$

$$|b\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

so the bra-ket $\langle a|b\rangle$ equals matrix multiplication or simply the inner product (A.2.1) on the space $\mathbb{C}^n$ while the ket-bra $|b\rangle\langle a|$ represents the matrix multiplication of a column and a row, whose outcome is an $n\times n$ matrix or equivalently, a linear operator $|b\rangle\langle a|\colon \mathbb{C}^n \to \mathbb{C}^n$. We denote the standard basis in $\mathbb{C}^n$ by $\{|1\rangle, \ldots, |n\rangle\}$, that is we set $|i\rangle = e_i$ and of course $\langle i| = e_i^*$. Also we set $|ii\rangle := |i\rangle\otimes|i\rangle$. If also $|u\rangle \in \mathcal{H}_1$ and $|v\rangle \in \mathcal{H}_2$ then $|u\rangle\otimes|v\rangle \in \mathcal{H}_1\otimes\mathcal{H}_2$ which we may denote sometimes as $|u\rangle|v\rangle$.

Tensor norms and Operator space theory

1.1 Tensor norms

Let X, Y be two normed spaces. We will define norms on the tensor product $X \otimes Y$. It is natural to require from a "tensor norm" to be "submultiplicative", i.e. to satisfy $\|x \otimes y\| \leq \|x\| \|y\|$. When a norm on $X \otimes Y$ satisfies $\|x \otimes y\| \leq \|x\| \|y\|$ we say that it is a subcross-norm, and when it satisfies $\|x \otimes y\| = \|x\| \|y\|$ we call it a cross-norm.

Supposing that we have the subcross-norm condition, a typical element $u \in X \otimes Y$ may be represented as $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$ and $y_i \in Y$ for all $i = 1, \ldots, n$ and consequently it will satisfy $\|u\| \leq \sum_i \|x_i\| \|y_i\|$, by the triangle inequality. However, such a representation isn't unique and thus applying this for all such representations we obtain that,

$$\|u\| \leq \inf \left\{ \sum_i \|x_i\| \|y_i\| : u = \sum_i x_i \otimes y_i \right\}.$$

Motivated by the previous observations we have,

Definition 1.1.1. *Let X, Y be two normed spaces. The projective norm is the function $\| \cdot \|_\pi : X \otimes Y \to \mathbb{R}_+$ defined by*

$$\|u\|_\pi = \inf \left\{ \sum_i \|x_i\| \|y_i\| : u = \sum_i x_i \otimes y_i \right\}$$

for all $u \in X \otimes Y$.

The projective norm is indeed a cross-norm.

Proposition 1.1.2. *Let X, Y be normed spaces. The projective norm $\|\cdot\|_\pi$ is a norm on $X \otimes Y$ and it satisfies*

$$\|x \otimes y\|_\pi = \|x\| \|y\|$$

for any $x \in X$ and $y \in Y$.

Proof. See [8] Proposition 2.1. □

We denote by $X \otimes_\pi Y$ the tensor product of X and Y endowed with the projective norm $\|\cdot\|_\pi$. Unless the spaces X and Y are finite dimensional, this space is not complete. We denote the completion of $X \otimes Y$ with the projective norm by $X \hat{\otimes}_\pi Y$ and call it the projective tensor product.

We shall now present a different approach to introducing a norm to the tensor product of two normed spaces. We can view elements of $X \otimes Y$ as bilinear forms on the product $X^\sharp \times Y^\sharp$, where $X^\sharp$ denotes the algebraic dual of the normed space X. Namely, if $u = \sum_i x_i \otimes y_i \in X \otimes Y$ then the associated bilinear form is defined to be

$$B_u(\varphi, \psi) = \sum_i \varphi(x_i) \psi(y_i)$$

for $\varphi \in X^\sharp$, $\psi \in Y^\sharp$. Thus we have the canonical embedding

$$X \otimes Y \hookrightarrow B(X^\sharp \times Y^\sharp).$$

Of course we have to verify that the above associated bilinear form is well defined. Define first the bilinear mapping $(x, y) \mapsto \big(B_{x,y} : (\varphi, \psi) \mapsto \varphi(x)\psi(y)\big)$ and then use the universal property (A.1.4) to yield the desired bilinear form. When the spaces in question are duals, we have a much simpler embedding. The element $\hat{u} = \sum_i \varphi_i \otimes \psi_i \in X^\sharp \otimes Y^\sharp$ corresponds to the bilinear form

$$B_{\hat{u}}(x, y) = \sum_i \varphi_i(x) \psi_i(y)$$

for $x \in X$ and $y \in Y$. Thus we have another canonical embedding

$$X^\sharp \otimes Y^\sharp \hookrightarrow B(X \times Y).$$

Let $u = \sum_i x_i \otimes y_i \in X \otimes Y$, then restricting the associated bilinear form B_u to the product $X^* \times Y^*$ of the dual spaces, we obtain a bounded bilinear form and thus we have the canonical algebraic embedding

$$X \otimes Y \hookrightarrow \mathcal{B}(X^* \times Y^*) \tag{1.1.1}$$

Motivated by the above embedding we may now define the following

Definition 1.1.3. *The injective norm $\|\cdot\|_\epsilon$ on $X \otimes Y$ is defined by*

$$\|u\|_\epsilon = \sup\left\{\left|\sum_i \varphi(x_i)\psi(y_i)\right| : \varphi \in \mathrm{Ball}(X^*), \psi \in \mathrm{Ball}(Y^*)\right\}$$

where $u = \sum_i x_i \otimes y_i \in X \otimes Y$ is any representation of u.

Again we denote by $X \otimes_\epsilon Y$ the tensor product with the injective norm, and unless the spaces are finite dimensional we take the completion $X \hat{\otimes}_\epsilon Y$ which will be called the injective tensor product.

Remark 1.1.4. *Note also that if $u = \sum_i x_i \otimes y_i \in X \otimes Y$, we may replace the balls $\mathrm{Ball}(X^*)$ and $\mathrm{Ball}(Y^*)$, in the definition of the injective norm, with norming sets.*[1] *Indeed, let $A \subseteq \mathrm{Ball}(X^*)$ and $B \subseteq \mathrm{Ball}(Y^*)$ be norming sets, then*

$$\begin{aligned}
\|u\|_\epsilon &= \sup\left\{\left|\sum_i \varphi(x_i)\psi(y_i)\right| : \varphi \in \mathrm{Ball}(X^*), \psi \in \mathrm{Ball}(Y^*)\right\} \\
&= \sup_{\psi \in \mathrm{Ball}(Y^*)} \sup_{\varphi \in \mathrm{Ball}(X^*)} \left|\varphi\Big(\sum_i x_i \psi(y_i)\Big)\right| = \sup_{\psi \in \mathrm{Ball}(Y^*)} \left\|\sum_i x_i \psi(y_i)\right\|_X \\
&= \sup_{\psi \in \mathrm{Ball}(Y^*)} \sup_{\varphi \in A} \left|\sum_i \varphi(x_i)\psi(y_i)\right| = \sup_{\varphi \in A} \left\|\sum \varphi(x_i) y_i\right\|_Y \\
&= \sup\left\{\left|\sum_i \varphi(x_i)\psi(y_i)\right| : \varphi \in A, \psi \in B\right\}
\end{aligned}$$

Corollary 1.1.5. *Since $\mathrm{Ball}(X) \subseteq \mathrm{Ball}(X^{**})$ is a norming set we have that if $u = \sum_i \varphi_i \otimes \psi_i \in X^* \otimes Y^*$, then*

$$\|u\|_\epsilon = \sup\left\{\left|\sum_i \varphi_i(x)\psi_i(y)\right| : x \in \mathrm{Ball}(X), y \in \mathrm{Ball}(Y)\right\} \tag{1.1.2}$$

[1] Recall, that a subset $A \subseteq \mathrm{Ball}(X^*)$ is called a norming set, if we have that $\|x\| = \sup\{|\varphi(x)| : \varphi \in A\}$ for all $x \in X$.

Let us also note, that for $u = \sum_i x_i \otimes y_i$ we can associate operators $L_u : X^* \to Y$ and $R_u : Y^* \to X$ by

$$L_u(\phi) = \sum_i \phi(x_i) y_i$$
$$R_u(\psi) = \sum_i \psi(y_i) x_i$$

for $\phi \in X^*$ and $\psi \in Y^*$. These operators have the same norm as the bilinear form B_u and thus this gives us two more formulas for the injective norm

$$\begin{aligned} \|u\|_\epsilon &= \sup\left\{ \left\| \sum_i \phi(x_i) y_i \right\|_Y : \phi \in \mathrm{Ball}(X^*) \right\} \\ &= \sup\left\{ \left\| \sum_i \psi(y_i) x_i \right\|_X : \psi \in \mathrm{Ball}(Y^*) \right\} \end{aligned}$$

Proposition 1.1.6. *If the normed spaces X, Y are finite dimensional, then we also have the canonical algebraic identification*

$$B(X \times Y) \cong X^* \otimes Y^*. \tag{1.1.3}$$

Proof. Indeed, let $\{e_i\}$ and $\{f_i\}$ be bases of X and Y respectively and let $\{e_i^*\}$ and $\{f_i^*\}$ be their dual bases. If $B : X \times Y \to \mathbb{C}$ is a bilinear form such that

$$B(e_i, f_j) = b_{i,j}$$

for every i, j, then its associated tensor is given by

$$\hat{B} = \sum_{i,j} b_{i,j} e_i^* \otimes f_j^* \in X^* \otimes Y^*. \tag{1.1.4}$$

Conversely, given an element $x = \sum_{s,t} x_{s,t} e_s^* \otimes f_t^* \in X^* \otimes Y^*$, its natural action on $X \times Y$ is defined by $\tilde{x}(e_i, f_j) = \sum_{s,t} x_{s,t} e_s^*(e_i) f_t^*(f_j) = x_{i,j}$. □

Remark 1.1.7. *For finite dimensional normed spaces X, Y, if we endow $X \otimes Y$ with the injective tensor norm, the isomorphism 1.1.3 becomes* <u>*isometric*</u>

$$\mathcal{B}(X \times Y) \cong X^* \otimes_\epsilon Y^*. \tag{1.1.5}$$

and thus

$$\|B\| = \|\hat{B}\|_{X^* \otimes_\epsilon Y^*}. \tag{1.1.6}$$

Proof. Let $B : X \times Y \to \mathbb{C}$ be a bounded bilinear form and $\hat{B}$ the associated tensor, as before. Then, by Corrolary 1.1.5 we have that

$$\begin{aligned}\left\|\hat{B}\right\|_{X^*\otimes_\epsilon Y^*} &= \sup\left\{\left|\sum_{i,j} b_{i,j} e_i^*(x) f_j^*(y)\right| : x \in \mathrm{Ball}(X), y \in \mathrm{Ball}(Y)\right\}\\ &= \sup\left\{\left|\sum_{i,j} b_{i,j} x_i y_j\right| : \left\|\sum_k x_k e_k\right\|_X \le 1, \left\|\sum_l y_l f_l\right\|_Y \le 1\right\}\\ &= \|B\|\end{aligned}$$

□

Proposition 1.1.8. *Let X, Y be normed spaces, then*

1. $\|u\|_\epsilon \le \|u\|_\pi$ *for every* $u \in X \otimes Y$.
2. $\|x \otimes y\|_\epsilon = \|x\|\,\|y\|$ *for every* $x \in X$, $y \in Y$.

Proof. Assertion 1. follows from the definitions and the triangle inequality. As for 2. using again triangle inequality we get $\|x \otimes y\|_\epsilon \le \|x\|\,\|y\|$ and Hahn-Banach theorem, gives us linear functionals $\varphi \in X^*, \psi \in Y^*$ such that $\|\varphi\| = 1 = \|\psi\|$, $\varphi(x) = \|x\|$ and $\psi(y) = \|y\|$. Hence, $\|x \otimes y\|_\epsilon \ge |\varphi(x)\psi(y)| = \|x\|\,\|y\|$. □

We will now consider the tensor product of operators. Let $T : X \to Y$ and $S : V \to W$, be linear operators between vector spaces. One can define the linear operator $T \otimes S : X \otimes V \to Y \otimes W$ by setting $T \otimes S(x \otimes v) = T(x) \otimes S(v)$. Endow the tensor product with a norm and we have the following:

Proposition 1.1.9. *Let $T : X \to Y$ and $S : V \to W$ bounded, linear operators between normed spaces. Then there is a unique bounded linear operator $T \otimes_\pi S : X \hat{\otimes}_\pi V \to Y \hat{\otimes}_\pi W$ such that $(T \otimes_\pi S)(x \otimes v) = T(x) \otimes S(v)$ for every $x \in X$, $y \in Y$. Furthermore, $\|T \otimes_\pi S\| = \|T\|\,\|S\|$.*

Proof. See [8] Proposition 2.3. □

as well as

Proposition 1.1.10. *Let $T : X \to Y$ and $S : V \to W$ bounded, linear operators between normed spaces. Then there is a unique bounded linear operator $T \otimes_\epsilon S : X \hat{\otimes}_\epsilon V \to Y \hat{\otimes}_\epsilon W$ such that $(T \otimes_\epsilon S)(x \otimes v) = T(x) \otimes S(v)$ for every $x \in X$, $y \in Y$. Furthermore, $\|T \otimes_\epsilon S\| = \|T\|\,\|S\|$.*

Proof. See [8] Proposition 3.2. □

Remark 1.1.11. *If in addition $T : X \to Y$ and $S : V \to W$ are linear isometric isomorphisms then so is $T \otimes_\epsilon S : X \hat{\otimes}_\epsilon V \to Y \hat{\otimes}_\epsilon W$.*

Definition 1.1.12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. We make the (algebraic) tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ into a pre- Hilbert space as follows. Define the inner product*

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{hs} := \langle v_1, v_2 \rangle_{\mathcal{H}_1} \cdot \langle w_1, w_2 \rangle_{\mathcal{H}_2}. \tag{1.1.7}$$

That is, if $x = \sum_i v_i \otimes w_i$ and $y = \sum_i v_i^{'} \otimes w_i^{'}$ then

$$\langle x, y \rangle_{hs} = \sum_{i,j} \langle v_i, v_j^{'} \rangle_{\mathcal{H}_1} \langle w_i, w_j^{'} \rangle_{\mathcal{H}_2}.$$

We denote by $\mathcal{H}_1 \otimes_{hs} \mathcal{H}_2$ or $\mathcal{H}_1 \otimes_2 \mathcal{H}_2$ the completion of the space $(\mathcal{H}_1 \otimes \mathcal{H}_2, \|\cdot\|_{hs})$, where $\|\cdot\|_{hs} := \sqrt{\langle \cdot, \cdot \rangle_{hs}}$ and call it the Hilbert space tensor product.

Comment 1.1.13. $\langle \cdot, \cdot \rangle_{hs}$ is a well defined inner product on the space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof. Fix $(v_1, w_1) \in \mathcal{H}_1 \times \mathcal{H}_2$ and note that by the universal property (A.1.4) of the tensor product,

$$B : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathbb{C} : v_2 \otimes w_2 \mapsto \langle v_1, v_2 \rangle_{\mathcal{H}_1} \cdot \langle w_1, w_2 \rangle_{\mathcal{H}_2}$$

induces a well defined linear form, such that, if $u = \sum_i x_i \otimes y_i$ then $B(u) = \sum_i \langle v_1, x_i \rangle_{\mathcal{H}_1} \cdot \langle w_1, y_i \rangle_{\mathcal{H}_2}$. Hence, we have that the map

$$u = \sum_i x_i \otimes y_i \mapsto \langle v, u \rangle_{hs} = B(u) = \sum_i \langle v_1, x_i \rangle_{\mathcal{H}_1} \cdot \langle w_1, y_i \rangle_{\mathcal{H}_2}$$

for $v = v_1 \otimes w_1$, is well defined and linear.

Now let $u = \sum_i x_i \otimes y_i \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and again by the universal property, consider the following map

$$C : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathbb{C} : v_1 \otimes w_1 \mapsto \sum_i \overline{\langle v_1, x_i \rangle_{\mathcal{H}_1} \cdot \langle w_1, y_i \rangle_{\mathcal{H}_2}}$$

which is again linear because of the complex conjugation. So, for $v = \sum_j x'_j \otimes y'_j$, $C(v) = \sum_{i,j} \overline{\langle x'_j, x_i\rangle_{\mathcal{H}_1} \cdot \langle y'_j, y_i\rangle_{\mathcal{H}_2}}$ is well defined and $v \mapsto C(v)$ is linear. Hence, for every $u = \sum_i x_i \otimes y_i \in \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$v = \sum_j x'_j \otimes y'_j \mapsto \langle v, u\rangle_{hs} = \sum_{i,j} \langle x'_j, x_i\rangle_{\mathcal{H}_1} \cdot \langle y'_j, y_i\rangle_{\mathcal{H}_2}$$

is anti-linear.

Finally, note that for every $v = \sum_j x'_j \otimes y'_j$ the map

$$u = \sum_i x_i \otimes y_i \mapsto \langle v, \sum_i x_i \otimes y_i\rangle_{hs} = \sum_j \Big(\sum_i \langle x'_j, x_i\rangle_{\mathcal{H}_1} \cdot \langle y'_j, y_i\rangle_{\mathcal{H}_2} \Big) = \sum_j \langle x'_j \otimes y'_j, u\rangle_{hs}$$

is linear because $\sum_j \langle x'_j \otimes y'_j, u\rangle_{hs} := \sum_j B_j(u)$ and all $B'_j s$ were linear. Thus, we proved that $(v, u) \mapsto \langle v, u\rangle_{hs}$ is a well defined, linear in the second argument hermitian form.

We shall now proceed to the positive definite part. Let $u = \sum_{i=1}^n x_i \otimes y_i \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and also let $\{e_k : k = 1, \ldots, m\}$ be an orthonormal basis of the subspace $\operatorname{span}\{y_i : i = 1, \ldots, n\} \subseteq \mathcal{H}_2$. Then, there exist elements $\{\xi_k\}_{k=1}^m \subseteq \mathcal{H}_1$ such that, $u = \sum_{k=1}^m \xi_k \otimes e_k$ and thus,

$$\begin{aligned} \langle u, u\rangle_{hs} &= \sum_{k,l} \langle \xi_k, \xi_l\rangle_{\mathcal{H}_1} \cdot \langle e_k, e_l\rangle_{\mathcal{H}_2} \\ &= \sum_{k,l} \langle \xi_k, \xi_l\rangle_{\mathcal{H}_1} \cdot \delta_{k,l} \\ &= \sum_k \langle \xi_k, \xi_k\rangle_{\mathcal{H}_1} \\ &= \sum_k \|\xi_k\|^2_{\mathcal{H}_1} \geq 0. \end{aligned}$$

From this expression we also have that $\langle u, u\rangle_{hs} = 0 \Leftrightarrow \sum_k \|\xi_k\|^2_{\mathcal{H}_1} = 0$ and consequently, $u = \sum_k 0 \otimes e_k = 0$.

□

Remark 1.1.14. *It is easy to verify that the norm induced by the inner product $\|\cdot\|_{hs}$ is a cross-norm. Indeed, for $h_i \in \mathcal{H}_i$, $i = 1, 2$:*

$$\|h_1 \otimes h_2\|^2_{hs} = \langle h_1 \otimes h_2, h_1 \otimes h_2\rangle_{hs} = \langle h_1, h_1\rangle_{\mathcal{H}_1} \langle h_2, h_2\rangle_{\mathcal{H}_2} = \|h_1\|^2_{\mathcal{H}_1} \|h_2\|^2_{\mathcal{H}_2}.$$

Remark 1.1.15. *Note that $\mathcal{H}_1 \otimes_2 \mathcal{H}_2 \cong \mathcal{H}_2 \otimes_2 \mathcal{H}_1$ isometrically, via the map $h \otimes k \mapsto k \otimes h$.*

Proposition 1.1.16. *Let $T : \mathcal{H} \to \mathcal{H}$ and $S : \mathcal{K} \to \mathcal{K}$ be bounded, linear operators between Hilbert spaces. Then there is a unique bounded linear operator $T \otimes S : \mathcal{H} \otimes_2 \mathcal{K} \to \mathcal{H} \otimes_2 \mathcal{K}$ such that $(T \otimes S)(h \otimes k) = T(h) \otimes S(k)$ for every $h \in \mathcal{H}$, $k \in \mathcal{K}$. Furthermore,*

1. $\|T \otimes S\| = \|T\| \|S\|$.
2. $(T_1 + \lambda T_2) \otimes S = T_1 \otimes S + \lambda (T_2 \otimes S)$.
3. $T \otimes (S_1 + \lambda S_2) = T \otimes S_1 + \lambda (T \otimes S_2)$.
4. $(T_1 \otimes S_1)(T_2 \otimes S_2) = (T_1 T_2) \otimes (S_1 S_2)$.
5. $(T \otimes S)^* = T^* \otimes S^*$.

Where $\lambda \in \mathbb{C}$, $T_i \in \mathcal{B}(\mathcal{H}_i)$ and $S_i \in \mathcal{B}(\mathcal{K}_i)$ for Hilbert spaces $\mathcal{H}_i, \mathcal{K}_i$, $i = 1, 2$.

Proof. Define the bilinear map $b : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ by $(h, k) \mapsto (Th) \otimes (Sk)$. By the universal property of the algebraic tensor product there is a unique linear map, which we denote by $T \otimes S$ such that $T \otimes S : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ and $(T \otimes S)(h \otimes k) = b(h, k) = (Th) \otimes (Sk)$. First we prove the boundedness on the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ and then extend to $\mathcal{H} \otimes_2 \mathcal{K}$ by density.

Let $x = \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{K}$, and set $L = \operatorname{span}\{y_i : i = 1, \dots, n\} \subseteq \mathcal{K}$. Then there exists an orthonormal basis $\{e_k : k = 1, \dots, m\}$ of L for which we have $x = \sum_{k=1}^{m} h_k \otimes e_k$ for some uniquely determined $\{h_k\}_{k=1}^{m} \subseteq \mathcal{H}$. Hence, it holds that:

$$\|\sum_{k=1}^{m} h_k \otimes e_k\|_{hs}^2 = \sum_{k,l=1}^{m} \langle h_k, h_l \rangle_{\mathcal{H}} \langle e_k, e_l \rangle_{\mathcal{K}} = \sum_{k=1}^{m} \|h_k\|_{\mathcal{H}}^2 .$$

Now define the linear map $T \otimes \mathrm{Id}_{\mathcal{K}}$ as above and note that

$$\begin{aligned} \left\| (T \otimes \mathrm{Id}_{\mathcal{K}})(\sum_{k=1}^{m} h_k \otimes e_k) \right\|_{hs}^2 &= \left\| \sum_{k=1}^{m} Th_k \otimes e_k \right\|_{hs}^2 = \sum_{k=1}^{m} \|Th_k\|_{\mathcal{H}}^2 \leq \\ &\leq \|T\|^2 \sum_{k=1}^{m} \|h_k\|_{\mathcal{H}}^2 = \|T\|^2 \|\sum_{k=1}^{m} h_k \otimes e_k\|_{hs}^2 \end{aligned}$$

Thus $T \otimes \mathrm{Id}_{\mathcal{K}}$ is bounded and so is $\mathrm{Id}_{\mathcal{H}} \otimes S$ if we do the same, hence $T \otimes S = (T \otimes \mathrm{Id}_{\mathcal{K}})(\mathrm{Id}_{\mathcal{H}} \otimes S)$ extends to a bounded linear operator on $\mathcal{H} \otimes_2 \mathcal{K}$. We already proved that $\|T \otimes S\| \leq \|T\| \|S\|$ and if we pick arbitrary $h \in \mathcal{H}$ and $k \in \mathcal{K}$ such that $\|h\|_{\mathcal{H}} = 1$ and $\|k\|_{\mathcal{K}} = 1$ then also $\|h \otimes k\|_{hs} = 1$ and thus $\|T \otimes S\| \geq \|(T \otimes S)(h \otimes k)\|_{hs} = \|(Th) \otimes (Sk)\|_{hs} = \|Th\|_{\mathcal{H}} \|Sk\|_{\mathcal{K}}$. Hence taking supremum over all such h and k, we have that $\|T \otimes S\| = \|T\| \cdot \|S\|$.

As for the algebraic properties 2), 3), 4) and 5), it's easy to see that they hold when restricted to $\mathcal{H} \otimes \mathcal{K}$ and hence they are valid on the space $\mathcal{H} \otimes_2 \mathcal{K}$ by continuity. $\square$

Definition 1.1.17. *Consider now two operator spaces $X \subseteq \mathcal{B}(\mathcal{H})$ and $Y \subseteq \mathcal{B}(\mathcal{K})$ for $\mathcal{H}, \mathcal{K}$ some Hilbert spaces. Then their minimal (spatial) tensor product is defined to be the completion of their algebraic tensor product with the norm induced by $\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$ as we established in the previous proposition, through the inclusion:*

$$X \otimes Y \subseteq \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$$

i.e. we set $X \otimes_{min} Y := \overline{X \otimes Y}^{norm}$ to be the minimal tensor product.

We denote the norm by $\|\cdot\|_{min}$ and call it the minimal tensor norm. The Banach space $X \otimes_{min} Y$ is obviously an operator space.

For the notions of operator spaces refer to Section 1.2. For now, an operator space X is a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For such a subspace, the operator norm on $\mathcal{B}(\mathcal{H})$ induces a sequence of matrix norms $\|\cdot\|_d$ on $M_d(X)$. Also a map $T : X \to Y$ between operator spaces, induces the maps $T_d : M_d(X) \to M_d(Y)$ defined by $T_d([x_{i,j}]) = [T(x_{i,j})]$ for every $d \geq 1$. The map T will be called completely bounded if the norm $\|T\|_{cb} := \sup_d \|T_d\|$ is finite, completely isometric if every T_d is isometric and a complete isomorphism if every T_d is an isomorphism.

Remark 1.1.18. *Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces. If $T : \mathcal{H} \to \mathcal{K}$ is an isometry, then the linear map $u_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ defined by $u_T(x) = TxT^*$ is completely isometric. Moreover, if $T : \mathcal{H} \to \mathcal{K}$ is a surjective isometry, then $u_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a completely isometric isomorphism.*

Proof. Note first that u_T is an isometry; indeed for all $x \in \mathcal{B}(\mathcal{H})$ we have $\|TxT^*\|_{\mathcal{B}(\mathcal{K})} \leq \|x\|_{\mathcal{B}(\mathcal{H})}$ and also $\|x\|_{\mathcal{B}(\mathcal{H})} = \|T^*TxT^*T\|_{\mathcal{B}(\mathcal{H})} \leq \|T^*\| \|TxT^*\|_{\mathcal{B}(\mathcal{K})} \|T\| = \|TxT^*\|_{\mathcal{B}(\mathcal{K})}$ for T is an isometry, i.e. $T^*T = \mathrm{Id}_{\mathcal{H}}$ and $\|T\| = 1$.

Now observe that T induces an isometry T_n from $\ell_2^n(\mathcal{H})$ to $\ell_2^n(\mathcal{K})$ such that $(u_T)_n(y) = T_n y T_n^*$ for any $y \in M_n(\mathcal{B}(\mathcal{K})) \cong \mathcal{B}(\ell_2^n(\mathcal{K}))$. Since $(u_T)_n$ is of the same form as u_T and n was arbitrary, then u_T is a complete isometry. □

Proposition 1.1.19. *If $X \subseteq \mathcal{B}(\mathcal{H})$ is an operator space, then we have the following completely isometric identification*

$$M_n(X) \cong M_n \otimes_{min} X. \tag{1.1.8}$$

Proof. First of all, note that $M_n \cong \mathcal{B}(\ell_2^n)$, so the minimal tensor norm on $M_n \otimes_{min} X$ is the norm induced by $\mathcal{B}(\ell_2^n \otimes_2 \mathcal{H})$ and the norm on $M_n(X)$ is the one induced by $\mathcal{B}(\ell_2^n(\mathcal{H}))$. We already know that $M_n(X) \cong M_n \otimes X$ algebraically so we will prove that $\mathcal{B}(\ell_2^n \otimes_2 \mathcal{H}) \cong \mathcal{B}(\ell_2^n(\mathcal{H}))$ completely isometrically and that the restriction of the map that makes the latter isomorphism completely isometric is the one that identifies the first one.

Indeed, we have that $\ell_2^n(\mathcal{H}) \cong \ell_2^n \otimes_2 \mathcal{H}$ isometrically by

$$\begin{aligned} S : \ell_2^n(\mathcal{H}) &\to \ell_2^n \otimes_2 \mathcal{H} \\ (h_i) &\mapsto \sum_i e_i \otimes h_i \end{aligned}$$

and the adjoint S^* acts on elements of $\ell_2^n \otimes_2 \mathcal{H}$ by $S^* : (u_i) \otimes h \mapsto (u_i h)$. Thus, by Remark 1.1.18, the map $u_S : \mathcal{B}(\ell_2^n(\mathcal{H})) \to \mathcal{B}(\ell_2^n \otimes_2 \mathcal{H})$ defined by $u_S(x) = SxS^*$ is a completely isometric isomorphism. Now, let $x = [x_{i,j}] \in M_n(X) \subseteq M_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\ell_2^n(\mathcal{H}))$ then $u_S([x_{i,j}]) = S[x_{i,j}]S^* \in \mathcal{B}(\ell_2^n \otimes_2 \mathcal{H})$ and recall that the algebraic identification $M_n(X) \cong M_n \otimes X$ is done via the map $\sigma : [x_{i,j}] \mapsto \sum_{i,j}^n E_{i,j} \otimes x_{i,j}$. Then for every elementary tensor $u \otimes h \in \ell_2^n \otimes_2 \mathcal{H}$ where

$u = (u_i)_{i=1}^n \in \ell_2^n$ it holds that:

$$
\begin{aligned}
u_S([x_{i,j}])(u \otimes h) &= S[x_{i,j}]S^*(u \otimes h) \\
&= S[x_{i,j}] \begin{pmatrix} u_1 h \\ \vdots \\ u_n h \end{pmatrix} = S \begin{pmatrix} \sum_j x_{1,j}(u_j h) \\ \vdots \\ \sum_j x_{n,j}(u_j h) \end{pmatrix} \\
&= \sum_i e_i \otimes \sum_j x_{i,j}(u_j h) \\
&= \sum_{i,j} e_i u_j \otimes x_{i,j}(h) \\
&= \sum_{i,j} E_{i,j}(u) \otimes x_{i,j}(h) \\
&= \sum_{i,j} E_{i,j} \otimes x_{i,j}(u \otimes h) = \sigma([x_{i,j}])(u \otimes h).
\end{aligned}
$$

Therefore they can both be completely isometrically identified with the same subspace of $\mathcal{B}(\ell_2^n \otimes_2 \mathcal{H}) \cong \mathcal{B}(\ell_2^n(\mathcal{H}))$ and the algebraic identification $M_n(X) \cong M_n \otimes X$ is completely isometric. ◻

Remark 1.1.20 (Associativity of the minimal tensor product). *Let $X \subseteq \mathcal{B}(\mathcal{H})$, $Y \subseteq \mathcal{B}(\mathcal{K})$ and $Z \subseteq \mathcal{B}(\mathcal{L})$ be operator spaces. Clearly*

$$(\mathcal{H} \otimes_2 \mathcal{K}) \otimes_2 \mathcal{L} \cong \mathcal{H} \otimes_2 (\mathcal{K} \otimes_2 \mathcal{L}) \cong \mathcal{H} \otimes_2 \mathcal{K} \otimes_2 \mathcal{L}.$$

Hence by Remark 1.1.18 we have completely isometrically

$$(X \otimes_{min} Y) \otimes_{min} Z \cong X \otimes_{min} (Y \otimes_{min} Z) \cong X \otimes_{min} Y \otimes_{min} Z.$$

Remark 1.1.21 (Commutativity of the minimal tensor space). *Let $X \subseteq \mathcal{B}(\mathcal{H})$, $Y \subseteq \mathcal{B}(\mathcal{K})$ be operator spaces. Since we have $\mathcal{H} \otimes_2 \mathcal{K} \cong \mathcal{K} \otimes_2 \mathcal{H}$, again by Remark 1.1.18 we have completely isometrically*

$$X \otimes_{min} Y \cong Y \otimes_{min} X$$

via $x \otimes y \mapsto y \otimes x$.

Remark 1.1.22 (Injectivity of the minimal tensor norm). *Let $E_1 \subseteq E_2 \subseteq \mathcal{B}(\mathcal{H})$ and $G_1 \subseteq G_2 \subseteq \mathcal{B}(\mathcal{K})$ be operator spaces so that $E_1 \otimes G_1 \subseteq E_2 \otimes G_2$. Then for any $x \in E_1 \otimes G_1$ we have*

$$\|x\|_{E_1 \otimes_{min} G_1} = \|x\|_{E_2 \otimes_{min} G_2}.$$

Consider now a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_n \subseteq \mathcal{H}$ be an n-dimensional subspace and $P_{\mathcal{H}_n} : \mathcal{H} \to \mathcal{H}_n$ the orthogonal projection. Using an orthonormal basis we can identify $\mathcal{H}_n$ with the n-dimensional Hilbert space ℓ_2^n and consequently $\mathcal{B}(\mathcal{H}_n) \cong M_n$ and recall that $M_n \otimes_{min} Y \cong M_n(Y)$. Let $v : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_n) \cong M_n$ be the map $a \mapsto P_{\mathcal{H}_n} a|_{\mathcal{H}_n}$ and $\mathcal{C}_n$ the collection of all such mappings with $\mathcal{H}_n$ arbitrary n-dimensional. Also let $X \subseteq \mathcal{B}(\mathcal{H})$, $Y \subseteq \mathcal{B}(\mathcal{K})$ be operator spaces. Then it is easy to show that for any $x = \sum_i a_i \otimes b_i \in X \otimes Y$ we have

$$\|x\|_{X \otimes_{min} Y} = \sup_{n \in \mathbb{N}, v \in \mathcal{C}_n} \left\| \sum v(a_i) \otimes b_i \right\|_{M_n(Y)}. \tag{1.1.9}$$

Indeed, we may write

$$\|x\|_{min} = \sup \left\{ |\langle t, xs \rangle| : s, t \in \mathrm{Ball}(\mathcal{H} \otimes_2 \mathcal{K}) \right\}.$$

By density, we restrict the supremum to $\mathcal{H} \otimes \mathcal{K}$ and for some finite dimensional subspace $\mathcal{H}_n \subseteq \mathcal{H}$, we have $s, t \in \mathcal{H}_n \otimes \mathcal{K}$. Hence, if v is the map defined above, we write

$$\langle t, xs \rangle = \left\langle t, (\sum_i a_i \otimes b_i) s \right\rangle = \left\langle t, (\sum_i v(a_i) \otimes b_i) s \right\rangle$$

thus we have that

$$\|x\|_{X \otimes_{min} Y} \leq \sup_{n \in \mathbb{N}, v \in \mathcal{C}_n} \left\| \sum v(a_i) \otimes b_i \right\|_{M_n(Y)}.$$

The reverse inequality is obvious thus we obtain the relation (1.1.9).

This shows that the norm on $X \otimes_{min} Y$ does not depend on the particular embedding of Y but rather on the "abstract" operator space structure of Y. Using the same arguments we can obtain the same for X.

Proposition 1.1.23. *Let $u : X \to Y$ be a c.b. map between operator spaces. Then for any other operator space G the mapping $\mathrm{Id}_G \otimes u : G \otimes X \to G \otimes Y$ extends to a bounded operator between the spaces $G \otimes_{min} X$ and $G \otimes_{min} Y$. Moreover, if we denote by $u_G : G \otimes_{min} X \to G \otimes_{min} Y$ the extension of $\mathrm{Id}_G \otimes u$ we have*

$$\|u\|_{cb} = \sup_G \|u_G\| = \sup_G \|u_G\|_{cb}$$

where the suprema run over all possible operator spaces G.

Proof. First, observe that by picking $G = M_n$ we immediately confirm that

$$\sup_G \|u_G\| \geq \|u\|_{cb}.$$

To prove the reverse inequality, assume that $G \subseteq \mathcal{B}(\mathcal{K})$ and apply the relation (1.1.9) to $G \otimes_{min} X$ and $G \otimes_{min} Y$, hence we see that

$$\left\|(\mathrm{Id}_G \otimes u)(\sum_i a_i \otimes b_i)\right\|_{G \otimes_{min} Y} = \sup_{n \in \mathbb{N}, v \in C_n} \left\|\sum_i v(a_i) \otimes u(b_i)\right\|_{M_n(Y)}.$$

Thus for all $x = \sum_i a_i \otimes b_i \in G \otimes X$, $n \in \mathbb{N}$ and $v \in C_n$

$$\left\|\sum v(a_i) \otimes u(b_i)\right\|_{M_n(Y)} = \left\|u_n(\sum v(a_i) \otimes b_i)\right\|_{M_n(Y)} \leq \|u_n\| \left\|\sum v(a_i) \otimes b_i\right\|_{M_n(X)}.$$

Therefore we have that

$$\|(\mathrm{Id}_G \otimes u)(x)\|_{min} \leq \|u\|_{cb} \|x\|_{min}$$

and consequently $\sup_G \|u_G\| \leq \|u\|_{cb}$. Finally, since G is arbitrary, we can replace it with $M_n(G)$ and use the identification

$$M_n(G \otimes_{min} X) \cong M_n(G) \otimes_{min} X$$

it follows that

$$\sup_G \|u_G\| = \sup_G \|u_G\|_{cb}.$$

□

Proposition 1.1.24. *Let $X_1, X_2, Y_1.Y_2$ be operator spaces and $u_1 \in \mathcal{CB}(X_1, Y_1)$ and $u_2 \in \mathcal{CB}(X_2, Y_2)$, then $u_1 \otimes u_2$ continuously extends by density to a completely bounded map*

$$u_1 \otimes u_2 : X_1 \otimes_{min} X_2 \to Y_1 \otimes_{min} Y_2.$$

Moreover, we have

$$\|u_1 \otimes u_2\|_{cb} = \|u_1\|_{cb} \|u_2\|_{cb}.$$

Proof. Decompose $u_1 \otimes u_2 = (u_1 \otimes \mathrm{Id}_{Y_2})(\mathrm{Id}_{X_1} \otimes u_2)$ and then it follows from Proposition 1.1.23 that

$$\|u_1 \otimes u_2\|_{cb} \leq \left\|u_1 \otimes \mathrm{Id}_{Y_2}\right\|_{cb} \left\|\mathrm{Id}_{X_1} \otimes u_2)\right\|_{cb} = \|u_1\|_{cb}\|u_2\|_{cb}\,.$$

For the converse , recall that for every $x_1 \in X_1$ and $x_2 \in X_2$ we have $\|x_1 \otimes x_2\|_{min} = \|x_1\|_{X_1}\|x_2\|_{X_2}$ and then $\|u_1 \otimes u_2\|_{cb} \geq \|u_1\|\|u_2\|$. If we repeat this but now for $\mathrm{Id}_{M_n} \otimes u_1$ and $\mathrm{Id}_{M_d} \otimes u_2$ and take the supremum over all n, d we obtain

$$\sup_{n,d} \left\|(\mathrm{Id}_{M_n} \otimes u_1) \otimes (\mathrm{Id}_{M_d} \otimes u_2)\right\| \geq \|u_1\|_{cb}\|u_2\|_{cb}\,.$$

Taking advantage of associativity and commutativity and of course of the identification $M_n \otimes M_d \cong M_{nd}$ we can also obtain

$$\left\|(\mathrm{Id}_{M_n} \otimes u_1) \otimes (\mathrm{Id}_{M_d} \otimes u_2)\right\|_{cb} = \left\|\mathrm{Id}_{M_{nd}} \otimes (u_1 \otimes u_2)\right\|_{cb}$$

where again by the previous Proposition it holds

$$\left\|\mathrm{Id}_{M_{nd}} \otimes (u_1 \otimes u_2)\right\|_{cb} \leq \|u_1 \otimes u_2\|_{cb}\,.$$

□

Remark 1.1.25. *If $u_1 : X_1 \to Y_1$ and $u_2 : X_2 \to Y_2$ are completely isometric isomorphisms then $u_1 \otimes u_2 : X_1 \otimes_{min} X_2 \to Y_1 \otimes_{min} Y_2$ is a completely isometric isomorphism as well.*

Now we are ready to generalize the property of the minimal tensor norm seen in equality (1.1.9).

Proposition 1.1.26. *For any $x = \sum a_1 \otimes b_i \in X \otimes Y$ we have*

$$\|x\|_{min} = \sup_n \left\{ \left\| \sum v(a_i) \otimes b_i \right\|_{M_n(Y)} : v \in \mathcal{CB}(X, M_n), \|v\|_{cb} \leq 1 \right\} \tag{1.1.10}$$

Furthermore

$$\|x\|_{min} = \sup \left\{ \left\| \sum v(a_i) \otimes w(b_i) \right\|_{M_{nm}} \right\} \tag{1.1.11}$$

where the supremum runs over $n, m \geq 1$ and all pairs of $v \in \mathrm{Ball}(\mathcal{CB}(X, M_n))$ and $w \in \mathrm{Ball}(\mathcal{CB}(Y, M_m))$.

Proof. By Proposition (1.1.23) we have that $\|v \otimes \mathrm{Id}_Y\| \leq \|v\|_{cb}$. Thus, for each $n \geq 1$ and $v \in \mathrm{Ball}(\mathcal{CB}(X, M_n))$

$$\begin{aligned}\|(v \otimes \mathrm{Id}_Y)(x)\|_{M_n(Y)} &\leq \|v \otimes \mathrm{Id}_Y\| \, \|x\|_{min} \\ &\leq \|v\|_{cb} \|x\|_{min} \\ &\leq \|x\|_{min}\end{aligned}$$

Consequently

$$\|x\|_{min} \geq \sup_n \left\{ \left\| \sum v(a_i) \otimes b_i \right\|_{M_n(Y)} : v \in \mathcal{CB}(X, M_n), \|v\|_{cb} \leq 1 \right\}$$

The converse inequality comes from formula (1.1.9) again. As for the assertion 1.1.11, it suffices to use commutativity of the minimal tensor product and apply equality (1.1.10) again.

□

Grothendieck introduced the notion of "reasonable" tensor norm which in the case of Banach spaces corresponds to the following: A *tensor norm* α assigns to each pair (X, Y) of Banach spaces, a norm $\|\cdot\|_{X \otimes_\alpha Y}$ on the algebraic tensor product $X \otimes Y$ so that $X \otimes_\alpha Y$ can become a Banach space and such that

1. α is *reasonable*: $\epsilon \leq \alpha \leq \pi$

2. α satisfies the metric mapping property : For Banach spaces X, Y, Z, W and $T \in L(X, Y)$, $S \in L(Z, W)$ the following holds

$$\|T \otimes S : X \otimes_\alpha Z \to Y \otimes_\alpha W\| \leq \|T\| \, \|S\|$$

In particular ϵ and π are the "extreme" reasonable tensor norms.

If X, Y are finite dimensional vector spaces, then motivated by the identification $X \otimes Y \cong (X^* \otimes Y^*)^*$ we may define the following. Given a tensor norm α we define its *dual tensor norm* α^*, for every pair of finite dimensional Banach spaces X, Y

$$\|u\|_{X \otimes_{\alpha^*} Y} := \sup\left\{ |\langle\langle u, v \rangle\rangle| : \|v\|_{X^* \otimes_\alpha Y^*} \leq 1 \right\}$$

Recall that the duality between $X\otimes Y$ and $X^*\otimes Y^*$ works in the following way: if $u=\sum_i x_i\otimes y_i\in X\otimes Y$ and $v=\sum_i \varphi_i\otimes\psi_i\in X^*\otimes Y^*$ then $\langle\langle u,v\rangle\rangle = \sum_{i,j}\varphi_j(x_i)\psi_j(y_i)$

We will need to consider another tensor norm which is very important for the purpose of these notes, as we may verify later. Let X,Y be Banach spaces, define the γ_2 *tensor norm* of $z\in X\otimes Y$ by

$$\|z\|_{X\otimes_{\gamma_2}Y} := \inf_{z=\sum x_i\otimes y_i}\left\{\sup_{x^*\in\mathrm{Ball}(X^*)}\Big(\sum_{i=1}^n |x^*(x_i)|^2\Big)^{\frac{1}{2}}\sup_{y^*\in\mathrm{Ball}(Y^*)}\Big(\sum_{i=1}^n |y^*(y_i)|^2\Big)^{\frac{1}{2}}\right\}$$

One can check that γ_2 defines indeed a tensor norm, its dual will be denoted γ_2^*.

We will now prove a lemma that will turn out to be very useful for this study. Recall that $(\ell_\infty^N)^*=\ell_1^N$.

Lemma 1.1.27. *Let $z\in\mathbb{C}^N\otimes\mathbb{C}^N$. Its γ_2 tensor norm for the space $\ell_\infty^N\otimes\ell_\infty^N$ is given by*

$$\|z\|_{\ell_\infty^N\otimes_{\gamma_2}\ell_\infty^N} = \inf\left\{\sup_k\|u_k\|_2\sup_l\|v_l\|_2 : z_{k,l}=\langle\overline{u}_k,v_l\rangle\right\}$$

where $z=\sum_{k,l}z_{k,l}e_k\otimes e_l$ and u_k,v_l are vectors in some $\mathbb{C}^r$ for all $k,l=1,\dots,N$

Proof. Let $\|z\|_{\ell_\infty^N\otimes_{\gamma_2}\ell_\infty^N}=d$, for every $\varepsilon>0$ there exist $(x_i)_{i=1}^r$ and $(y_i)_{i=1}^r$ in ℓ_∞^N such that

$$\sup_{x^*\in\mathrm{Ball}(\ell_1^N)}\Big(\sum_{i=1}^n |x^*(x_i)|^2\Big)^{\frac{1}{2}}\sup_{y^*\in\mathrm{Ball}(\ell_1^N)}\Big(\sum_{i=1}^n |y^*(y_i)|^2\Big)^{\frac{1}{2}}\le d+\varepsilon$$

and $z=\sum_{i=1}^r x_i\otimes y_i$. We have that $x_i=\sum_{k=1}^N x_i(k)e_k$ and $y_i=\sum_{l=1}^N y_i(l)e_l$ where $\{e_k\}_k$ is the canonical basis in $\mathbb{C}^N$ and hence

$$z=\sum_{i=1}^r x_i\otimes y_i=\sum_{i=1}^r\Big(\sum_{k=1}^N x_i(k)e_k\Big)\otimes\Big(\sum_{l=1}^N y_i(l)e_l\Big)=\sum_{k,l=1}^N\Big(\sum_{i=1}^r x_i(k)y_i(l)\Big)e_k\otimes e_l$$

so if we set $u_k=(x_i(k))_{i=1}^r$ and $v_l=(y_i(l))_{i=1}^r$, these sequences are elements of $\mathbb{C}^r$ and for $z_{k,l}:=\langle\overline{u}_k,v_l\rangle$, we obtain $z=\sum_{k,l}z_{k,l}e_k\otimes e_l$. Now observe that

$$\|u_k\|_2=\Big(\sum_{i=1}^r|x_i(k)|^2\Big)^{\frac{1}{2}}=\Big(\sum_{i=1}^r|\langle e_k,x_i\rangle|^2\Big)^{\frac{1}{2}}\le\sup_{x^*\in\mathrm{Ball}(\ell_1^N)}\Big(\sum_{i=1}^n|x^*(x_i)|^2\Big)^{\frac{1}{2}}$$

for every $k = 1,\ldots,N$. Applying the same for every v_l leads to

$$\sup_k \|u_k\|_2 \sup_l \|v_l\|_2 \le d+\varepsilon$$

Hence, it suffices to prove that $d \le \sup_k \|u_k\|_2 \sup_l \|v_l\|_2$. Suppose that $z_{k,l} = \langle \overline{u}_k, v_l\rangle$ for sequences $u_k = (u_i(k))_i$ and $v_l = (v_i(l))$ in $\mathbb{C}^r$ for some r. Then we can write

$$z = \sum_{k,l=1}^{N} z_{k,l} e_k \otimes e_l = \sum_{k,l=1}^{N} \Big(\sum_{i=1}^{r} u_i(k)v_i(l)\Big) e_k \otimes e_l = \sum_{i=1}^{r}\Big(\sum_{k=1}^{N} u_i(k)e_k\Big)\otimes\Big(\sum_{l=1}^{N} v_i(l)e_l\Big)$$

By defining $x_i = \sum_{k=1}^{N} u_i(k)e_k$ and $y_i = \sum_{l=1}^{N} v_i(l)e_l$ for every $i = 1,\ldots,N$ we have $z = \sum_i x_i \otimes y_i$. Moreover,

$$\sup_{x^*\in \mathrm{Ball}(\ell_1^N)} \Big(\sum_{i=1}^{n} \big|x^*(x_i)\big|^2\Big)^{\frac{1}{2}} \le \sup_k \Big(\sum_{i=1}^{n} \big|e_k^*(x_i)\big|^2\Big)^{\frac{1}{2}} = \sup_k \Big(\sum_{i=1}^{n} |u_i(k)|^2\Big)^{\frac{1}{2}} = \sup_k \|u_k\|_2$$

since every $x^* \in \ell_1^N$ is written as $x^* = \sum_j x_j^* e_j^*$ with $\sum_j |x_j^*| \le 1$, where $\{e_j^*\}$ is the dual basis $\{e_j\}$. Using the same arguments for $(y_i)_i$ we deduce that

$$\sup_{x^*\in \mathrm{Ball}(\ell_1^N)} \Big(\sum_{i=1}^{n} \big|x^*(x_i)\big|^2\Big)^{\frac{1}{2}} \sup_{y^*\in \mathrm{Ball}(\ell_1^N)} \Big(\sum_{i=1}^{n} \big|y^*(y_i)\big|^2\Big)^{\frac{1}{2}} \le \sup_k \|u_k\|_2 \sup_l \|v_l\|_2$$

which completes the proof.

□

This lemma provides us with a very simple and useful formula for the γ_2^* tensor norm on the space $\ell_1^N(\mathbb{C}) \otimes \ell_1^N(\mathbb{C})$.

Corollary 1.1.28. *The γ_2^* tensor norm for the space $\ell_1^N \otimes \ell_1^N$ is given by*

$$\|z\|_{\ell_1^N \otimes_{\gamma_2^*} \ell_1^N} = \sup\left\{ \left| \sum_{i,j=1}^{N} z_{i,j} \left\langle \overline{u_i}, v_j \right\rangle \right| : r \in \mathbb{N}, u_i, v_j \in \mathrm{Ball}(\mathbb{C}^r)\right\}$$

where $z = \sum_{k,l} z_{k,l} e_k \otimes e_l$.

Proof. Recall first that $(\ell_1^N)^* = \ell_\infty^N$, so we write the elements $w \in \ell_\infty^N \otimes \ell_\infty^N$ as $w = \sum_{k,l=1}^{N} w_{k,l} e_k^* \otimes e_l^*$. Hence, if we let $z = \sum_{i,j=1}^{N} z_{i,j} e_i \otimes e_j \in \ell_1^N \otimes \ell_1^N$, its dual action on w gives us $\langle\langle z, w\rangle\rangle = \sum_{i,j=1}^{N} z_{i,j} w_{i,j}$. If we now let an $r \in \mathbb{N}$,

$w_{i,j} = \left\langle \overline{u_i}, v_j \right\rangle$, $\|w\|_{\ell_\infty^N \otimes_{\gamma_2} \ell_\infty^N} \le 1$ where $u_i, v_j \in \mathbb{C}^r$ for all $i,j = 1,\dots,N$ and recall the definition of the dual tensor norm, we obtain that

$$\|z\|_{\ell_1^N \otimes_{\gamma_2^*} \ell_1^N} = \sup\left\{ \left| \sum_{i,j=1}^N z_{i,j} \left\langle \overline{u_i}, v_j \right\rangle \right| : r \in \mathbb{N}, w_{i,j} = \left\langle \overline{u_i}, v_j \right\rangle, \|w\|_{\ell_\infty^N \otimes_{\gamma_2} \ell_\infty^N} \le 1 \right\}$$

Finally, note that we can restrict the supremum above to the case where u_i, v_j are unit vectors, and "forget" about w. This can be done by setting $u_i' = \frac{u_i}{\sup_k \|u_k\|_2}$ and $v_j' = \sup_k \|u_k\|_2 \, v_j$. □

Note that the same arguments apply to the case where our spaces are real and thus the same formulas hold. In fact, if we choose a tensor z with real, non-negative coefficients $z_{i,j} \in \mathbb{R}$ then we can just restrict the supremum to the case where the unit vectors are real:

Remark 1.1.29. *Let $z = \sum_{i,j=1}^N z_{i,j} e_i \otimes e_j \in \ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$, where $z_{i,j} \in \mathbb{R}$ are such that $z_{i,j} \ge 0$ for all $i,j = 1,\dots,N$. Then,*

$$\|z\|_{\ell_1^N(\mathbb{C}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{C})} = \|z\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}$$

Proof. Obviously $\|z\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})} \le \|z\|_{\ell_1^N(\mathbb{C}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{C})}$. As for the reverse part, let $r \in \mathbb{N}$, and $u_i, v_j \in \mathrm{Ball}(\mathbb{C}^r)$ and denote $|u_i| := \sum_{k=1}^r |u_i(k)| \, e_k \in \mathrm{Ball}(\mathbb{R}^r)$, $\left| v_j \right| := \sum_{k=1}^r \left| v_j(k) \right| e_k \in \mathrm{Ball}(\mathbb{R}^r)$ for all $i,j = 1,\dots,N$. Now write

$$\begin{aligned}
\left| \sum_{i,j=1}^N z_{i,j} \left\langle \overline{u_i}, v_j \right\rangle \right| &= \left| \sum_{i,j=1}^N z_{i,j} \sum_{k=1}^r u_i(k) v_j(k) \right| \\
&\le \sum_{i,j=1}^N z_{i,j} \sum_{k=1}^r |u_i(k)| \left| v_j(k) \right| \\
&= \sum_{i,j=1}^N z_{i,j} |u_i| \cdot |v_j| \\
&\le \sup\left\{ \left| \sum_{i,j=1}^N z_{i,j} u_i \cdot v_j \right| : r \in \mathbb{N}, u_i, v_j \in \mathrm{Ball}(\mathbb{R}^r) \right\}
\end{aligned}$$

and thus the assertion is proved. □

1.2 Operator spaces

In this Section, we introduce all the basic notions as well as the results needed from Operator space theory. For a more in depth discussion of the subject we refer to the standard literature [9], [10] and [11].

1.2.1 Operator space structures

Definition 1.2.1. *A (concrete) operator space is a subspace $X \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

For any such subspace, the operator norm on $\mathcal{B}(\mathcal{H})$ induces a sequence of matrix norms $\|\cdot\|_d$ on $M_d(X)$ which will be called an *operator space structure* (o.s.s.). To see how this is done, let $\mathcal{A}$ be a C^*-algebra and $M_d(\mathcal{A})$ the $d \times d$ matrices with entries from $\mathcal{A}$. We'll denote a typical element of $M_d(\mathcal{A})$ by $[a_{i,j}]$.

There is a natural way of making $M_d(\mathcal{A})$ into a $*$-algebra. Namely, for $[a_{i,j}]$ and $[b_{i,j}]$ in $M_d(\mathcal{A})$ we set

$$[a_{i,j}] \cdot [b_{i,j}] = \left[\sum_{k=1}^{d} a_{i,k} b_{k,j} \right]$$

and

$$[a_{i,j}]^* = [a_{j,i}^*]$$

Apart from being a $*$-algebra, $M_d(\mathcal{A})$ can also become a C^*-algebra by introducing a norm on it. Let's start with the most basic of all C^*-algebras, $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^{(d)}$ denote the direct sum of d copies of $\mathcal{H}$, then there is a natural norm and inner product on $\mathcal{H}^{(d)}$ that makes it into a Hilbert space. Namely

$$\left\| \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} \right\|_{\mathcal{H}^{(d)}}^2 = \|h_1\|_{\mathcal{H}}^2 + \cdots + \|h_d\|_{\mathcal{H}}^2$$

and

$$\left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix} \right\rangle_{\mathcal{H}^{(\mathrm{d})}} = \langle h_1, k_1 \rangle_{\mathcal{H}} + \cdots + \langle h_d, k_d \rangle_{\mathcal{H}}$$

where

$$\begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix} \in \mathcal{H}^{(d)}$$

This Hilbert space is also often denoted $\ell_2^d(\mathcal{H})$. As we will see, it is useful to regard elements of $\mathcal{H}^{(d)}$ as column vectors.

There is a natural way to regard an element of $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ as a linear operator on $\mathcal{H}^{(d)}$, by using the ordinary rules for matrix products. That is, we set

$$(T_{i,j}) \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} := \begin{pmatrix} \sum_{j=1}^d T_{1,j}(h_j) \\ \vdots \\ \sum_{j=1}^d T_{d,j}(h_j) \end{pmatrix}$$

for $T = (T_{i,j})$ in $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ and $\begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix}$ in $\mathcal{H}^{(d)}$.

Equivalently, this corresponds to the action of $M_d \otimes \mathcal{B}(\mathcal{H})$ on $\mathbb{C}^{\mathrm{d}} \otimes \mathcal{H}$ determined by

$$(A \otimes T)(u \otimes h) = Au \otimes Th$$

where $A \in M_d$, $T \in \mathcal{B}(\mathcal{H})$, $u \in \mathbb{C}^{\mathrm{d}}$ and $h \in \mathcal{H}$.

It is easy to see that every element of $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ defines a bounded linear operator on $\mathcal{H}^{(d)}$ with the operator norm

$$\left\|(T_{i,j})\right\|_{\mathrm{M_d}(\mathcal{B}(\mathcal{H}))} := \left\|(T_{i,j})\right\|_{\mathcal{B}(\mathcal{H}^{(\mathrm{d})})} = \sup\left\{ \Big(\sum_{i=1}^d \Big\| \sum_{j=1}^d T_{i,j}(h_j) \Big\|_{\mathcal{H}}^2 \Big)^{\frac{1}{2}} : h_j \in \mathcal{H}, \sum_{j=1}^d \|h_j\|_{\mathcal{H}}^2 \le 1 \right\}$$

and that this correspondence yields a one-to-one $*$-isomorphism between the spaces $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ and $\mathcal{B}(\mathcal{H}^{(d)})$. Indeed, if we pick $(h_j)_{j=1}^d \in \mathcal{H}^{(d)}$ such that $\sum_{j=1}^{d}\|h_j\|_{\mathcal{H}}^2 \leq 1$, then

$$\begin{aligned}
\sum_{i=1}^{d}\left\|\sum_{j=1}^{d} T_{i,j}(h_j)\right\|_{\mathcal{H}}^2 &\leq \sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left\|T_{i,j}(h_j)\right\|_{\mathcal{H}}\right)^2 \\
&\leq \sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left\|T_{i,j}\right\|_{\mathcal{B}(\mathcal{H})}\left\|h_j\right\|_{\mathcal{H}}\right)^2 \\
&\leq \sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left\|T_{i,j}\right\|_{\mathcal{B}(\mathcal{H})}^2\right)\left(\sum_{j=1}^{d}\left\|h_j\right\|_{\mathcal{H}}^2\right) \\
&= \left(\sum_{j=1}^{d}\left\|h_j\right\|_{\mathcal{H}}^2\right)\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\left\|T_{i,j}\right\|_{\mathcal{B}(\mathcal{H})}^2\right) \\
&\leq \sum_{i,j=1}^{d}\left\|T_{i,j}\right\|_{\mathcal{B}(\mathcal{H})}^2
\end{aligned}$$

Thus,

$$\left\|(T_{i,j})\right\|_{\mathcal{B}(\mathcal{H}^{(d)})}^2 \leq \sum_{i,j=1}^{d}\|T_{i,j}\|_{\mathcal{B}(\mathcal{H})}^2$$

so every matrix in $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ gives a well defined bounded linear map on $\mathcal{B}(\mathcal{H}^{(d)})$. Conversely given a $T \in \mathcal{B}(\mathcal{H}^{(d)})$ one obtains a matrix $(T_{i,j})$ in $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ by setting

$$T_{i,j} = V_i^* T V_j$$

where

$$V_j : \mathcal{H} \to \mathcal{H}^{(d)} \; : h \mapsto \begin{pmatrix} 0 \\ \vdots \\ h \\ \vdots \\ 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 \\ \vdots \\ h \\ \vdots \\ 0 \end{pmatrix} = h \otimes e_j$ is the column with h in the j-th row and zero elsewhere.

On the other hand,

$$V_j^* : \mathcal{H}^{(d)} \to \mathcal{H} : \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} \mapsto h_j.$$

Now if we let $\phi : \mathrm{M_d}(\mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H}^{(d)})$ denote the map through which we interpreted the matrices as bounded linear operators, we can see that $\phi((T_{i,j})) = T$. Thus the identification $\mathrm{M_d}(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(d)})$ gives us a norm that makes $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ into a C^*-algebra.

Now given a C^*-algebra $\mathcal{A}$, we can choose a one-to-one $*$-representation of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$ so that $\mathcal{A}$ can be identified as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. This allows us to identify $M_d(\mathcal{A})$ as a $*$-subalgebra of $\mathrm{M_d}(\mathcal{B}(\mathcal{H}))$ and it is now straightforward to verify that under this representation $M_d(\mathcal{A})$ becomes a C^*-algebra.

At this point, recall the isomorphisms 0.2.6 that we established in Section 0.2. In the case of C^*-algebras these isomorphisms have the interesting property of preserving norm and positivity. Indeed, let $\mathcal{A}$ be a C^* algebra, then the operation, through which we pass from $M_n(M_d(\mathcal{A}))$ to $M_d(M_n(\mathcal{A}))$ is a permutation and the identified elements of the latter spaces are unitary (*permutation*) equivalent elements of the C^*-algebra $M_{nd}(\mathcal{A})$. Thus, it is a $*$-isomorphism which we will refer to as the canonical shuffle. The canonical shuffle will play an important role since, as a $*$-isomorphism, it is isometric and maps positive elements to positive elements.

Tensor product notation is widely used throughout these notes, we therefore consider it appropriate to understand the canonical shuffle in this "language". Again we will show that $M_n(M_d(\mathcal{A})) \cong M_d(M_n(\mathcal{A}))$ via a $*$-isomorphism. Note first that $M_n(\mathcal{A})$ and $M_n \otimes \mathcal{A}$ are $*$-isomorphic via the map $[a_{i,j}] \mapsto \sum_{i,j=1}^n E_{i,j} \otimes a_{i,j}$, where $\{E_{i,j}\}$ are the matrix units in M_n. If we also let $\{F_{i,j}\}$ be the matrix units in M_d, then the set

$$\left\{E_{i,j} \otimes F_{k,l} : i,j = 1,\dots,n \quad k,l = 1,\dots,d\right\}$$

is a basis for the $*$-algebra $M_n \otimes M_d$. Observe now that $M_n \otimes M_d$ and $M_d \otimes M_n$

are $*$-isomorphic via the map

$$\sum_{i,j,k,l} a_{i,j,k,l} E_{i,j} \otimes F_{k,l} \mapsto \sum_{i,j,k,l} a_{i,j,k,l} F_{k,l} \otimes E_{i,j}.$$

and since $M_n(M_d)$ and $M_d(M_n)$ are also $*$-isomorphic, our assertion follows from the sequence of $*$-isomorphisms:

$$\begin{aligned} M_n(M_d(\mathcal{A})) &\cong M_n(M_d \otimes \mathcal{A}) \\ &\cong M_n \otimes (M_d \otimes \mathcal{A}) \\ &\cong (M_n \otimes M_d) \otimes \mathcal{A} \\ &\cong (M_d \otimes M_n) \otimes \mathcal{A} \\ &\cong M_d \otimes (M_n \otimes \mathcal{A}) \\ &\cong M_d(M_n \otimes \mathcal{A}) \cong M_d(M_n(\mathcal{A})). \end{aligned}$$

So if we consider $A \in M_n(M_d(\mathcal{A}))$ then its image under the above string of $*$-isomorphisms is an element B of $M_d(M_n(\mathcal{A}))$ with the same norm as A.

So far we defined the notion of a concrete operator space and proved that its inclusion in a $\mathcal{B}(\mathcal{H})$ yields a sequence of matrix norms on the space $M_d(X)$. Ruan's theorem provides an alternative definition of an operator space as a complex vector space V equipped with a sequence of matrix norms $(M_d(V), \|\cdot\|_d)$ satisfying two requirements that we will see shortly.

Definition 1.2.2. *An <u>abstract operator space</u> is a (complex) vector space V together with a sequence of matrix norms $\|\cdot\|_n$ on $M_n(V)$ such that*

- ***M1*** $\left\| \begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix} \right\|_{c+d} = \|u \oplus w\|_{c+d} = \max\{\|u\|_c, \|w\|_d\}$
- ***M2*** $\|aub\|_d \leq \|a\| \|u\|_c \|b\|$

for all $u \in M_c(V)$, $w \in M_d(V)$, with $c, d \geq 1$ and $a \in M_{d,c}$, $b \in M_{c,d}$.

Conditions **M1** and **M2** are often called *Ruan's axioms*. Ruan's theorem asserts that these conditions characterise operator space structures on a vector space.

Theorem 1.2.3 (Ruan [12])**.** *Any abstract operator space is "completely" isometrically isomorphic to a concrete operator space, i.e. if* $(V, \{\|\cdot\|_d : d \geq 1\})$ *is an abstract operator space, then there exists a Hilbert space* $\mathcal{H}$ *and a linear map* $J : V \to \mathcal{B}(\mathcal{H})$ *such that, for every* $d \geq 1$ *and* $v = [v_{i,j}] \in M_d(V)$*,*

$$\left\|[J(v_{i,j})]\right\|_{\mathrm{M_d}(\mathcal{B}(\mathcal{H}))} = \|v\|_{M_d(V)}\,.$$

In fact, the matrix norms on $\mathcal{B}(\mathcal{H})$ satisfy Ruan's axioms.

Proposition 1.2.4. *The o.s.s. on* $\mathcal{B}(\mathcal{H})$ *satisfies Ruan's axioms.*

Proof. Recall that if $T_i \in \mathcal{B}(\mathcal{H}_i)$ and $\sup_i \|T_i\| < \infty$ then $T = \bigoplus_i T_i$ defines a bounded linear operator on $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ such that

$$\|T\|_{\mathcal{B}(\mathcal{H})} = \sup_j \left\|T_j\right\|_{\mathcal{B}(\mathcal{H}_j)}$$

Then **M1** follows as a special case.

As for **M2** notice that, if $a \in M_{d,c}$, $b \in M_{c,d}$ and $u \in M_c(\mathcal{B}(\mathcal{H}))$ then aub corresponds to $(a \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})})u(b \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})})$ and thus

$$\|aub\| = \|(a \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})})u(b \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})})\| \leq$$
$$\leq \|a \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})}\|\,\|u\|_c\,\|b \otimes \mathrm{Id}_{\mathcal{B}(\mathcal{H})}\| = \|a\|\,\|u\|_c\,\|b\|$$

where the equality in the end comes from the fact that, if $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ then the operator

$$T \otimes S : \mathcal{H} \otimes_2 \mathcal{K} \to \mathcal{H} \otimes_2 \mathcal{K}$$

satisfies

$$\|T \otimes S\| = \|T\|\,\|S\|\,.$$

(See Proposition 1.1.16). □

A sequence of matrix norms satisfying Ruan's axioms, equivalently the inclusion of X into $\mathcal{B}(\mathcal{H})$, which yields such a sequence, is called an *operator space structure* (o.s.s.) on X.

Remark 1.2.5. *Notice that, given any such structure* $(M_d(X), \|\cdot\|_d)$*:*

1. *It follows from* ***M1*** *that the mapping*
$$u \mapsto u \oplus 0 = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}$$
is an isometry from $M_d(X)$ *into* $M_{d+1}(X)$

2. *From* ***M2*** *we see that, if* $u \in M_d(X)$ *and* $a \in M_d$ *unitary then*
$$\|au\| \leq \|u\| = \|a^* au\| \leq \|a^*\| \cdot \|au\| = \|au\|.$$
By symmetry the same applies to right multiplication, and we conclude that we may permute rows and columns of u without affecting its norm since such an operation corresponds to multiplication on the left or right by a permutation matrix (which is unitary).

3. *Combining the above observations we deduce that adding (or dropping) rows of zeros and columns of zeros does not change the norm of a matrix of operators. To see this, note that, through permutations, we can suppose all the zero rows are at the bottom and all zero columns at the right of the matrix. But then Ruan's axioms tell us that the norm is unchanged if we remove those zero rows and columns.*

Remark 1.2.6. *If* X *is an operator space, then via the canonical shuffle we see that the algebraic identifications*
$$M_n(M_d(X)) \cong M_d(M_n(X)) \cong M_{nd}(X)$$
are isometric.

Remark 1.2.7. *Let* $a_1, \dots, a_n$ *and* $b_1, \dots, b_n$ *be in* $\mathcal{B}(\mathcal{H})$*. Let* $a \in M_n(\mathcal{B}(\mathcal{H}))$ *be the matrix that has* $a_1, \dots, a_n$ *as its first column and zero elsewhere, and* $b \in M_n(\mathcal{B}(\mathcal{H}))$ *the one that has* $b_1, \dots, b_n$ *as its first row and zero elsewhere. That is, we have*
$$a = \begin{pmatrix} a_1 & \\ \vdots & \bigcirc \\ a_n & \end{pmatrix} \qquad\qquad b = \begin{pmatrix} b_1 & \dots & b_n \\ & \bigcirc & \end{pmatrix}$$

Then

$$\|a\| = \left\|\sum_{i=1}^{n} a_i^* a_i\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}} \qquad \|b\| = \left\|\sum_{i=1}^{n} b_i b_i^*\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}} \tag{1.2.1}$$

Moreover, we have $\|ba\| \le \|b\|\|a\|$*, and hence*

$$\left\|\sum_i b_i a_i\right\| \le \left\|\sum_{i=1}^{n} b_i b_i^*\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}} \left\|\sum_{i=1}^{n} a_i^* a_i\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}}$$

More generally, for any $x = [x_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ *we have* $\|bxa\| \le \|b\|\|x\|\|a\|$ *and hence*

$$\left\|\sum_{i,j} b_i x_{ij} a_j\right\| \le \left\|\sum_{i=1}^{n} b_i b_i^*\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}} \|x\|_{M_n(\mathcal{B}(\mathcal{H}))} \left\|\sum_{i=1}^{n} a_i^* a_i\right\|_{\mathcal{B}(\mathcal{H})}^{\frac{1}{2}}$$

An o.s.s. on a normed space is not unique.

Example 1.2.8. *The space* $\ell_2^n(\mathbb{C})$ *can be endowed with two o.s.s.'s called* R_n *and* C_n*.*

Proof. The space $\ell_2^n(\mathbb{C}) = \ell_2^n$ can be viewed as a subspace R_n of M_n via the map $e_i \mapsto E_{1i} = |1\rangle\langle i|, i = 1, \dots, n$, which means that every *column* vector $u \in \ell_2^n$ can be interpreted as a matrix A_u that has that vector as its first row and zero elsewhere. To see this more clearly, consider an element $u \in \ell_2^n$, then the operation is the following:

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \ell_2^n \mapsto A_u = \begin{pmatrix} u_1 & \dots & u_n \\ & \bigcirc & \end{pmatrix} \in M_n.$$

It's easy to see that $\|u\|_2 = \|A_u\|$, so the embedding is isometric. Note that the norm on M_n we considered is the spectral norm $\|\cdot\|_{S_\infty^n}$.

On the other hand, we can also use the map $e_i \mapsto E_{i1} = |i\rangle\langle 1|, i = 1, \dots, n$, and identify ℓ_2^n with the subspace C_n of M_n. Through this map every $u \in \ell_2^n$ maps to a matrix B_u with that vector as its first column and zero elsewhere. Namely,

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \ell_2^n \mapsto B_u = \begin{pmatrix} u_1 & & \\ \vdots & \bigcirc & \\ u_n & & \end{pmatrix} \in M_n.$$

Again, we can verify that $\|u\|_2 = \|B_u\|$.

An element $A \in M_d(R_n)$ is a $d \times d$ matrix whose entries are $n \times n$ matrices that are zero except for their first row. Alternatively, if we use the canonical shuffle, A can be seen as an $n \times n$ matrix whose first row is made of $d \times d$ matrices $A_1, \ldots, A_n$ and all other entries are zero. Namely,

$$A = \begin{pmatrix} A_1 & \ldots & A_n \\ & \bigcirc & \end{pmatrix} \in M_n(M_d), \quad \text{where} \quad A_1, \ldots, A_n \in M_d.$$

Then, via Remark 1.2.7 we see that

$$\|A\|_{M_d(R_n)} = \left\| \sum_{i=1}^{n} A_i \otimes |1\rangle\langle i| \right\|_{M_d(R_n)} = \left\| \sum_{i=1}^{n} A_i A_i^* \right\|_{M_d}^{\frac{1}{2}}$$

Similarly, for $M_d(C_n)$, we have

$$\|A\|_{M_d(C_n)} = \left\| \sum_{i=1}^{n} A_i \otimes |i\rangle\langle 1| \right\|_{M_d(C_n)} = \left\| \sum_{i=1}^{n} A_i^* A_i \right\|_{M_d}^{\frac{1}{2}}$$

Through these two expressions we see that the two o.s.s.'s defined on ℓ_2^n can be very different. Consider for example the space ℓ_2^2. Now compute the two norms defined above for the element $A = \sum_{i=1}^{2} |i\rangle\langle 1| \otimes e_i \in M_2(\ell_2^2)$. It's easy to verify that

$$\|A\|_{M_2(R_2)} = \left\| \sum_{i=1}^{2} E_{i1} E_{i1}^* \right\|^{\frac{1}{2}} = \|I_2\|^{\frac{1}{2}} = 1$$

while

$$\|A\|_{M_2(C_2)} = \left\| \sum_{i=1}^{2} E_{i1}^* E_{i1} \right\|^{\frac{1}{2}} = \left\| \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\|^{\frac{1}{2}} = \sqrt{2}.$$

□

The following simple lemma will be used very often in these notes.

Lemma 1.2.9. *Let X be an operator space. Then for every natural number N we have the following isometric identifications*

$$\ell_\infty^N(X) \cong \ell_\infty^N \otimes_\epsilon X \cong \ell_\infty^N \otimes_{min} X \qquad (1.2.2)$$

Where $\ell_\infty^N(X)$ is the space $\mathbb{C}^N \otimes X$ equipped with the norm

$$\left\| \sum_i^N e_i \otimes x_i \right\|_{\ell_\infty^N(X)} = \sup_{i=1,\dots,N} \|x_i\|_X$$

Proof. The first identification follows easily from definition of the injective tensor norm. Indeed, for $z = \sum_i^N e_i \otimes x_i \in \ell_\infty^N \otimes X$ we have that

$$\begin{aligned}
\left\| \sum_i^N e_i \otimes x_i \right\|_\epsilon &= \sup \left\{ \left\| \sum_i^N \varphi(e_i) x_i \right\|_X : \varphi \in \mathrm{Ball}((\ell_\infty^N)^*) \right\} \\
&= \sup \left\{ \left\| \sum_i^N \varphi(i) x_i \right\|_X : \varphi \in \mathrm{Ball}(\ell_1^N) \right\} \\
&= \sup_{i=1,\dots,N} \|x_i\|_X
\end{aligned}$$

where $\varphi(i)$ is the i-th coefficient of the vector φ. Indeed, let $\varphi \in \mathrm{Ball}(\ell_1^N)$, then

$$\begin{aligned}
\left\| \sum_i^N \varphi(i) x_i \right\|_X &\leq \sum_i^N |\varphi(i)| \|x_i\|_X \\
&\leq \sum_i^N |\varphi(i)| \sup_i \|x_i\|_X \\
&\leq \sup_i \|x_i\|_X
\end{aligned}$$

and for the reverse inequality note that

$$\sup \left\{ \left\| \sum_i^N \varphi(i) x_i \right\|_X : \varphi \in \mathrm{Ball}(\ell_1^N) \right\} \geq \|x_i\|_X$$

for all $i = 1, \dots, N$, since $e_i = (\delta_j^i)_j \in \mathrm{Ball}(\ell_1^N)$ for all $i = 1, \dots, N$, (where $\delta_j^i = 1$ if $j = i$ and 0 if not).

To verify the identification $\ell_\infty^N(X) \cong \ell_\infty^N \otimes_{min} X$, recall the isometric embedding $\ell_\infty^N \hookrightarrow \mathcal{B}(\ell_2^N)$ as diagonal operators and let $X \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus the minimal tensor norm of an element $z = \sum_i^N e_i \otimes x_i \in \ell_\infty^N \otimes X$ is given by

$$\left\| \sum_i^N e_i \otimes x_i \right\|_{\ell_\infty^N \otimes_{min} X} := \left\| \sum_i^N |i\rangle\langle i| \otimes x_i \right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))} = \sup_i \|x_i\|_X .$$

Where the last equality follows from the fact that the diagonal operator $\sum_i^N |i\rangle\langle i| \otimes x_i \in \mathcal{B}(\ell_2^N(\mathcal{H}))$ has operator norm given by

$$\left\| \sum_i^N |i\rangle\langle i| \otimes x_i \right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))} = \sup\left\{ \left\| \Big(\sum_i^N |i\rangle\langle i| \otimes x_i \Big)(h) \right\|_{\ell_2^N(\mathcal{H})} : \|h\|_{\ell_2^N(\mathcal{H})} \leq 1 \right\},$$

for which we have that, for all $h = (h_i)_{i=1}^N \in \mathrm{Ball}(\ell_2^N(\mathcal{H}))$:

$$\begin{aligned}
\left\| \Big(\sum_i^N |i\rangle\langle i| \otimes x_i \Big)(h) \right\|^2_{\ell_2^N(\mathcal{H})} &= \left\| (x_i h_i)_{i=1}^N \right\|^2_{\ell_2^N(\mathcal{H})} \\
&= \sum_{i=1}^N \|x_i h_i\|^2_{\mathcal{H}} \\
&\leq \sum_{i=1}^N \|x_i\|^2_{\mathcal{B}(\mathcal{H})} \|h_i\|^2_{\mathcal{H}} \\
&\leq (\sup_i \|x_i\|_X)^2 \sum_{i=1}^N \|h_i\|^2_{\mathcal{H}} \\
&\leq (\sup_i \|x_i\|_X)^2.
\end{aligned}$$

Now if we take the supremum over all such $(h_i)_{i=1}^N \in \mathrm{Ball}(\ell_2^N(\mathcal{H}))$ we obtain the inequality

$$\left\| \sum_i^N |i\rangle\langle i| \otimes x_i \right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))} \leq \sup_i \|x_i\|_X .$$

As for the reverse inequality, pick any $h \in \mathrm{Ball}(\mathcal{H})$ and $j = 1, \ldots, N$, and note that we have $e_j \otimes h \in \mathrm{Ball}(\ell_2^N(\mathcal{H}))$, hence

$$\begin{aligned}
\left\| \sum_i^N |i\rangle\langle i| \otimes x_i \right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))} &\geq \left\| \Big(\sum_i^N |i\rangle\langle i| \otimes x_i \Big)(e_j \otimes h) \right\|^2_{\ell_2^N(\mathcal{H})} \\
&= \left\| (\delta_i^j x_j h)_{i=1}^N \right\|^2_{\ell_2^N(\mathcal{H})} \\
&= \left\| x_j h \right\|_{\mathcal{H}}
\end{aligned}$$

and by taking the supremum over all such $h \in \mathrm{Ball}(\mathcal{H})$ we obtain

$$\left\| \sum_i^N |i\rangle\langle i| \otimes x_i \right\|_{\mathcal{B}(\ell_2^N(\mathcal{H}))} \geq \left\| x_j \right\|_X$$

for every $j = 1, \ldots, N$ which completes the proof. $\square$

Example 1.2.10. *Another example of a space with "natural" o.s.s., besides $\mathcal{B}(\mathcal{H})$ itself, is $\ell_\infty^n = (\mathbb{C}^n, \|\cdot\|_\infty)$, whose o.s.s. is obtained by using the map $e_i \mapsto E_{ii} = |i\rangle\langle i|$, i.e., the embedding of an element of ℓ_∞^n as the diagonal of an n-dimensional matrix, which is an isometric identification of ℓ_∞^n with a subspace of $\mathcal{B}(\ell_2^n)$. Namely, through the embedding $\ell_\infty^n \hookrightarrow \mathcal{B}(\ell_2^n)$ where we map $u \mapsto D_u$ and $\|D_u\| = \|u\|_\infty = \sup|u_i|$.*

Through this embedding, which we already used in the previous Lemma, ℓ_∞^n becomes an operator space and consequently, gets its o.s.s.. Indeed, by the previous lemma, the commutativity of the minimal tensor norm and Remark 1.1.19 we have that

$$\ell_\infty^N(M_d) \cong \ell_\infty^N \otimes_{min} M_d \cong M_d \otimes_{min} \ell_\infty^N \cong M_d(\ell_\infty^N)$$

isometrically for all $d \in \mathbb{N}$. Thus, we obtain the sequence of norms

$$\left\| \sum_{i=1}^n A_i \otimes |i\rangle\langle i| \right\|_{M_d(\ell_\infty^n)} = \sup_{i \in \{1,\ldots,n\}} \|A_i\|_{M_d}.$$

In the “isometric” theory of Banach spaces, two Banach spaces X,Y are identified if they are isometrically isomorphic, i.e. there exists a bounded linear map $T : X \to Y$ which is an isomorphism and $\|Tx\| = \|x\|$ for each $x \in X$. However, in the case of operator spaces we need morphisms that keep track of the extra information gained by the sequence of matrix norms defined by the o.s.s..

Definition 1.2.11. *Let X,Y be operator spaces and $T : X \to Y$ a linear map between them. Also let T_d denote the linear map*

$$T_d : M_d(X) \to M_d(Y)$$
$$[u_{i,j}]_{i,j} \mapsto [T(u_{i,j})]_{i,j}$$

for each $d \in \mathbb{N}$, then:

- $\|T\|_{cb} := \sup_d \|T_d\|$ *is called the completely bounded norm.*

- *If also* $\|T\|_{cb} < \infty$ *then* T *is called* *completely bounded. The set of all completely bounded maps from* X *to* Y *with the above norm, is called the space of all c.b. maps from* X *to* Y *and is denoted by* $C\mathcal{B}(X, Y)$.

- *A map* $T : X \to Y$ *is called a* *complete isometry* *if each* T_d *is an isometry. Moreover* T *is called a* *complete contraction* *if* $\|T\|_{cb} \leq 1$.

- *Two operator spaces* X, Y *are said to be* *completely isomorphic* *whenever there exists a linear isomorphism* $T : X \to Y$ *such that* T *and* T^{-1} *are c.b.*

- *Two operator spaces* X, Y *are said to be* *completely isometrically isomorphic* *or just completely isometric whenever there exists a linear isomorphism* $T : X \to Y$ *such that* T_d *is an isometry for all* $d \geq 1$ *(or, equivalently, that satisfies* $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$*).*

- *For* C^**-algebras* $\mathcal{A}, \mathcal{B}$ *a linear map* $T : \mathcal{A} \to \mathcal{B}$ *is called* *completely positive* *if* $T_d(x)$ *is a positive element of* $M_d(\mathcal{B})$ *for every* d *and every positive element* $x \in M_d(\mathcal{A})$. *If also* $\mathcal{A}, \mathcal{B}$ *have units then* T *is called* *unital* *if* $T(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$.

Notice that the case $d = 1$ is just $T : X \to Y$, since 1×1 matrices of X are just elements of X. So trivially if T is completely bounded, it is also bounded and if T is completely positive then it is also positive; similarly for a complete isometry. In general "completely" stands for a property that T_d enjoys for every d.

We will also make use of the tensor notation for such linear maps. That is, if we identify $M_d(X)$ with $M_d \otimes X$ via the linear map that sends each $[v_{i,j}] \otimes x \in M_d \otimes X$ to the matrix $[v_{i,j}x] \in M_d(X)$, then since $T_d([v_{i,j}x]) = [T(v_{i,j}x)] = [v_{i,j}T(x)]$, the map T_d is identified with $\mathrm{Id}_{M_d} \otimes T$, where Id_{M_d} is the identity map on M_d.

However it is not necessarily true that a bounded map between operator spaces is completely bounded, an isometry is a complete isometry and that a positive map is completely positive. Indeed, consider the following examples.

Example 1.2.12. *The transpose map $T : R_n \to C_n$, between the spaces R_n and C_n introduced in Example 1.2.8, is isometric but not completely isometric.*

Proof. Let $T : R_n \to C_n$ denote the transpose map, whose action is the following: let $u_1, \dots, u_n \in \mathbb{C}$

$$A_u = \begin{pmatrix} u_1 & \dots & u_n \\ & \bigcirc & \end{pmatrix} \in R_n \mapsto T(A_u) = B_u = \begin{pmatrix} u_1 & & \\ \vdots & \bigcirc & \\ u_n & & \end{pmatrix} \in C_n.$$

We know that $\|T(A_u)\| = \|u\|_2 = \|A_u\|$, where $u = (u_i)_{i=1}^n \in \ell_2^n$ and hence T is isometric. Recall also (see Example 1.2.8) that elements $A \in M_d(R_n)$ and $B \in M_d(C_n)$ can be written as $A = \sum_{i=1}^n A_i \otimes |1\rangle\langle i|$ and $B = \sum_{i=1}^n B_i \otimes |i\rangle\langle 1|$ respectively, for $A_i, B_i \in M_d$, and have norms given by

$$\|A\|_{M_d(R_n)} = \left\| \sum_{i=1}^n A_i \otimes |1\rangle\langle i| \right\|_{M_d(R_n)} = \left\| \sum_{i=1}^n A_i A_i^* \right\|_{M_d}^{\frac{1}{2}}$$

and

$$\|B\|_{M_d(C_n)} = \left\| \sum_{i=1}^n B_i \otimes |i\rangle\langle 1| \right\|_{M_d(C_n)} = \left\| \sum_{i=1}^n B_i^* B_i \right\|_{M_d}^{\frac{1}{2}}.$$

So, if we consider the element $A = \sum_{i=1}^n E_{i,1} \otimes |1\rangle\langle i| \in M_d(R_n)$, where $E_{i,j}$ are the matrix units in M_d, then T_d acts on A giving the element $T_d(A) = \sum_{i=1}^n E_{i,1} \otimes |i\rangle\langle 1| \in M_d(C_n)$, namely,

$$A = \begin{pmatrix} E_{1,1} & \dots & E_{i,1} & \dots & E_{n,1} \\ & & \bigcirc & & \end{pmatrix} \mapsto T_d(A) = \begin{pmatrix} E_{1,1} & & \\ \vdots & \bigcirc & \\ E_{n,1} & & \end{pmatrix}.$$

Now,

$$\|A\|_{M_d(R_n)} = \left\| \sum_{i=1}^n E_{i,1} E_{i,1}^* \right\|_{M_d}^{\frac{1}{2}} = \left\| \sum_{i=1}^n E_{i,1} E_{1,i} \right\|_{M_d}^{\frac{1}{2}} = \left\| \sum_{i=1}^n E_{i,i} \right\|_{M_d}^{\frac{1}{2}} = \|I_n\|_{M_d}^{\frac{1}{2}} = 1$$

while

$$\|T_d(A)\|_{M_d(C_n)} = \left\| \sum_{i=1}^n E_{i,1}^* E_{i,1} \right\|_{M_d}^{\frac{1}{2}} = \left\| \sum_{i=1}^n E_{1,i} E_{i,1} \right\|_{M_d}^{\frac{1}{2}} = \left\| \sum_{i=1}^n E_{1,1} \right\|_{M_d}^{\frac{1}{2}} = \sqrt{n}.$$

Therefore, $\|T_d(A)\|_{M_d(C_n)} \neq \|A\|_{M_d(R_n)}$, and then T is isometric but not completely isometric.

$\square$

Example 1.2.13. *Let $\{E_{i,j}\}_{i,j=1}^2$ denote the matrix units on M_2. Let $T : M_2 \to M_2$ again denote the transpose map. We shall see that T is a positive map but not completely positive. It is easy to verify that the transpose of a positive matrix is positive. So T is positive. Now consider $T_2 : M_2(M_2) \to M_2(M_2)$ and note that*

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is positive, but

$$T_2\left(\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is not positive. Thus T is a positive map but not completely positive.

Example 1.2.14. *We shall now consider the case of a bounded but not completely bounded map. To this end, consider a separable, infinite-dimensional Hilbert space $\mathcal{H}$ with a countable, orthonormal basis $\{e_i\}_{i=1}^\infty$. Every bounded, linear operator A on $\mathcal{H}$ defines an $\infty \times \infty$ matrix $[\langle e_i, Ae_j\rangle]$. Define then a map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by the transpose with respect to the basis. Then T will be an isometry, but $\|T_d\| \geq d$, i.e. it is a bounded but not completely bounded map.*

Proof. Note that we can write $T(A) = JA^*J$ for every $A \in \mathcal{B}(\mathcal{H})$, where $J \in \mathcal{B}(\mathcal{H})$ is the conjugate linear map such that $J(x) = J(\sum_i \langle e_i, x\rangle e_i) = \sum_i \overline{\langle e_i, x\rangle} e_i$. Note also that J is an isometry since $\|x\|^2 = \sum_{i=1}^\infty |\langle e_i, x\rangle|^2 = \|J(x)\|^2$ for every $x \in \mathcal{H}$ and clearly $J^2 = \mathrm{Id}_{\mathcal{H}}$. Thus we can easily verify that for every $A \in \mathcal{B}(\mathcal{H})$, $\|JA^*J\| = \|A\|$ which means that T is also isometric.

Now let $\{E_{i,j}\}_{i,j=1}^{\infty}$ be the matrix units on $\mathcal{H}$ and fix an integer $d \in \mathbb{N}$. The matrix units $E_{i,j}$ are defined on the basis by

$$E_{i,j}e_k = \begin{cases} e_i & \text{if} \quad k = j \\ 0 & \text{if} \quad k \neq j. \end{cases}$$

It also holds that $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$ and $E_{i,j}^* = T(E_{i,j}) = E_{j,i}$. So for $A = [E_{j,i}]_{i,j=1}^{d} \in \mathrm{M_d}(\mathcal{B}(\mathcal{H}))$, which is the element whose (i,j)th entry is the matrix $E_{j,i}$, it holds that $\|A\| = 1$. Indeed,

$$\begin{aligned} \|A\|^2 = \left\|AA^*\right\| = \left\|[E_{j,i}] \cdot [E_{j,i}]^*\right\| &= \left\|[E_{j,i}] \cdot [E_{j,i}]\right\| \\ &= \left\|[\sum_{k=1}^{d} E_{k,i}E_{j,k}]\right\| \\ &= \left\|[\sum_{k=1}^{d} \delta_{i,j}E_{k,k}]\right\| \\ &= \left\|\mathrm{diag}(\sum_{k=1}^{d} E_{k,k})\right\| = \left\|\sum_{k=1}^{d} E_{k,k}\right\| = 1. \end{aligned}$$

However,

$$\begin{aligned} \|T_d(A)\|^2 = \left\|[T(E_{j,i})]\right\|^2 &= \left\|[E_{i,j}]\right\|^2 \\ &= \left\|[E_{i,j}] \cdot [E_{i,j}]^*\right\| \\ &= \left\|[E_{i,j}] \cdot [E_{i,j}]\right\| \\ &= \left\|[dE_{i,j}]\right\| = d\left\|[E_{i,j}]\right\|. \end{aligned}$$

Where from the equalities above we conclude that $\left\|[E_{i,j}]\right\| = d$ and thus $\|T_d(A)\| = d$. Therefore $\|T_d\| \geq d$. □

We will now prove a very standard proposition that will be used often, but first we need the following lemma.

Lemma 1.2.15. *Let $\mathcal{A}$ be a C^*-algebra with unit and $a \in \mathcal{A}$. Then,*

$$\|a\| \leq 1 \Leftrightarrow \begin{pmatrix} \mathbf{1}_{\mathcal{A}} & a \\ a^* & \mathbf{1}_{\mathcal{A}} \end{pmatrix} \geq 0$$

(meaning that it is positive in $M_2(\mathcal{A})$).

Proof. Represent $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ through the map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and set $\pi(a) = A$. If $\|A\| \leq 1$, then for all x, $y \in \mathcal{H}$

$$\left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \mathbf{1} & A \\ A^* & \mathbf{1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \langle x, x\rangle + \langle x, Ay\rangle + \langle Ay, x\rangle + \langle y, y\rangle$$
$$\geq \|x\|^2 - 2\|A\| \cdot \|y\| \cdot \|x\| + \|y\|\|^2 \geq (\|x\| - \|y\|)^2 \geq 0$$

Conversely, if $\|A\| > 1$, then there exist vectors $x, y \in \mathcal{H}$ such that $\langle Ay, x\rangle < -1$ so the above inner product will be negative. □

Proposition 1.2.16. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with units $T : \mathcal{A} \to \mathcal{B}$ be a completely positive and unital map. Then,*

$$\|T\|_{cb} = \|T\| = 1$$

Proof. For any unital map, it holds that, $1 = \|T(\mathbf{1}_{\mathcal{A}})\| \leq \|T\| \leq \|T\|_{cb}$. As for the converse suppose that we have $x \in M_d(\mathcal{A})$ such that $\|x\| \leq 1$. From the lemma above we deduce the positivity of

$$\begin{pmatrix} \mathbf{1}_{M_d(\mathcal{A})} & x \\ x^* & \mathbf{1}_{M_d(\mathcal{A})} \end{pmatrix} \in M_2(M_d(\mathcal{A}))$$

and from the fact that T is completely positive and unital we conclude that

$$T_{2d}\left(\begin{pmatrix} \mathbf{1}_{M_d(\mathcal{A})} & x \\ x^* & \mathbf{1}_{M_d(\mathcal{A})} \end{pmatrix} \right) = \begin{pmatrix} \mathbf{1}_{M_d(\mathcal{B})} & T_d(x) \\ T_d(x)^* & \mathbf{1}_{M_d(\mathcal{B})} \end{pmatrix}$$

is a positive element of $M_2(M_d(\mathcal{B}))$. Use the lemma again to conclude that $\|T_d(x)\| \leq 1$, and so $\|T_d\| \leq 1$ for every $d \in \mathbb{N}$. Thus $\|T\|_{cb} \leq 1$. □

1.2.2 Dual spaces

We shall now provide an o.s.s. to the dual of an operator space X, namely, $X^* = \mathcal{B}(X, \mathbb{C})$. The goal is to introduce an appropriate norm to the space $M_d(X^*)$. First though, we are going to need the following results from operator space theory.

Proposition 1.2.17. *Let X be an operator space and $T : X \to M_d$ be a linear map. Then $\|T\|_{cb} = \|T_d\|$.*

Proof. See [11] Proposition 2.2.2. □

Corollary 1.2.18. *Let X be an operator space and $\varphi : X \to \mathbb{C}$ be a linear map. Then $\|\varphi\|_{cb} = \|\varphi\|$.*

Remark 1.2.19. *As we've already seen, completely bounded maps are bounded, so for operator spaces X, Y we have that, $\mathcal{CB}(X,Y) \subseteq \mathcal{B}(X,Y)$. However, as we see in Corollary 1.2.18, in the case of linear functionals the converse is also true, so*

$$X^* = \mathcal{B}(X,\mathbb{C}) = \mathcal{CB}(X,\mathbb{C}) \tag{1.2.3}$$

Finally, using the isomorphism

$$M_n(\mathcal{CB}(X,Y)) \cong \mathcal{CB}(X, M_n(Y)) \tag{1.2.4}$$

$$[\varphi_{ij}] \mapsto \Phi : \Big(x \mapsto [\varphi_{ij}](x) := [\varphi_{ij}(x)]\Big) \tag{1.2.5}$$

we define the sequence of norms on $M_n(\mathcal{CB}(X,Y))$ to be

$$\|[\varphi_{ij}]\|_n := \|\Phi\|_{\mathcal{CB}(X,M_n(Y))} = \sup\Big\{\|[\varphi_{ij}(x_{kl})]\|_{nm} : [x_{kl}] \in \mathrm{Ball}(M_m(X)), m \in \mathbb{N}\Big\} \tag{1.2.6}$$

making the identification $M_n(\mathcal{CB}(X,Y)) \cong \mathcal{CB}(X, M_n(Y))$ isometric. The sequence of matrix norms $(M_n(\mathcal{CB}(X,Y)), \|\cdot\|_n)$ satisfy the conditions **M1** and **M2** and thus by Ruan's theorem turns $\mathcal{CB}(X,Y)$ into an operator space.

In particular, the case of $Y = \mathbb{C}$ yields an o.s.s. for the topological dual of an operator space X. As we already saw $X^* = \mathcal{CB}(X,\mathbb{C})$, so X^* is an operator space called the <u>operator space dual</u> of X.

One can also consider the tensor notation to get a simpler realisation of the associated c.b. map. We know that $M_d \otimes X^* \cong M_d(X^*)$; the previous identification associates a $\varphi = \sum_i a_i \otimes x_i^* \in M_d(X^*)$ to the map T^φ given by

$$M_d(X^*) \cong \mathcal{CB}(X, M_d) \tag{1.2.7}$$

$$\varphi \mapsto \Big(T^\varphi : v \mapsto \sum_i x_i^*(v) a_i\Big) \tag{1.2.8}$$

This leads to the sequence of norms

$$\|\varphi\|_{M_d(X^*)} = \|T^\varphi\|_{cb}\,. \tag{1.2.9}$$

But since $T^\varphi : X \to M_d$, by Proposition 1.2.17 we have

$$\|\varphi\|_{M_d(X^*)} = \|T_d^\varphi\|. \tag{1.2.10}$$

An interesting fact about the duality of operator spaces is that they turn the canonical inclusion map to the second dual into a complete isometry. Recall that the canonical inclusion $\tau : X \hookrightarrow X^{**}$ is specified by its action:

$$x \mapsto \left(f \in X^* \mapsto \tau[x](f) = f(x)\right)$$

Proposition 1.2.20. *If $X \subseteq \mathcal{B}(\mathcal{H})$ is an operator space, then the canonical inclusion*

$$\tau : X \hookrightarrow X^{**}$$

is completely isometric.

Proof. Fix an $n \in \mathbb{N}$ and $[x_{i,j}] \in M_n(X)$. Then $[\tau(x_{i,j})] \in M_n(X^{**})$. Recall relation (1.2.6)

$$\begin{aligned}\left\|[\tau(x_{i,j})]\right\|_n &= \sup\left\{\left\|[\tau(x_{i,j})(f_{k,l})]\right\|_{nm} : [f_{kl}] \in \mathrm{Ball}(M_m(X^*)), m \in \mathbb{N}\right\}\\ &= \sup\left\{\left\|[f_{k,l}(x_{i,j})]\right\|_{nm} : [f_{kl}] \in \mathrm{Ball}(M_m(X^*)), m \in \mathbb{N}\right\}\\ &\leq \left\|[x_{i,j}]\right\|_n\end{aligned}$$

Now since $\left\|[\tau(x_{i,j})]\right\|_n$ equals the supremum above and $M_m(X^*) \cong \mathcal{CB}(X, M_m)$ we can see that

$$\left\|[\tau(x_{i,j})]\right\|_n = \sup\left\{\left\|[f(x_{i,j})]\right\|_{nm} : f \in \mathrm{Ball}(\mathcal{CB}(X, M_m)), m \in \mathbb{N}\right\}$$

hence it suffices to prove the following claim:

Given $n \in \mathbb{N}$, $\varepsilon > 0$ and $[x_{i,j}] \in M_n(X)$, there exists an integer $m \in \mathbb{N}$ and a completely contractive $u : X \to M_m$ such that $\left\|[u(x_{i,j})]\right\| \geq \left\|[x_{i,j}]\right\| - \varepsilon$.

Proof. As we know, if $X \subseteq \mathcal{B}(\mathcal{H})$, then $[x_{i,j}] \in M_n(X) \subseteq \mathcal{B}(\mathcal{H}^{(n)})$; thus we can write its norm as

$$\left\|[x_{i,j}]\right\|_n = \sup\left\{\left|\left\langle [x_{i,j}]y, z\right\rangle\right| : y, z \in \mathrm{Ball}(\mathcal{H}^{(n)})\right\}.$$

So if we are given an $\varepsilon > 0$ there exists $y, z \in \mathrm{Ball}(\mathcal{H}^{(n)})$ such that

$$\left|\left\langle [x_{i,j}]y, z\right\rangle\right| \geq \left\|[x_{i,j}]\right\|_n - \varepsilon.$$

If $y = (y_k)$ and $z = (z_k)$, where $y_k, z_k \in \mathcal{H}$ then this inequality becomes

$$\Big|\sum_{i,j}\Big\langle x_{i,j}y_j, z_i\Big\rangle\Big| \geq \big\|[x_{i,j}]\big\|_n - \varepsilon.$$

Now let $K = span\{y_1,\dots,y_n,z_1,\dots,z_n\}$ in $\mathcal{H}$. Since K is finite dimensional there is an isometric $*$-isomorphism $\pi : \mathcal{B}(K) \to M_m$ where $m = dim(K)$. We know from the theory of C^*-algebras that π is completely isometric. Let $P_K : \mathcal{H} \to \mathcal{H}$ the projection onto K and $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(K)$ the map $T(x) = P_K x|_K$ which we can verify is also completely contractive. Let $u = \pi \circ T$ and note that u is completely contractive too. We have $\Big\langle [T(x_{i,j})]y, z\Big\rangle = \sum_{i,j}\Big\langle T(x_{i,j})y_j, z_i\Big\rangle$, so

$$\big\|[T(x_{i,j})]\big\|_n \geq \Big|\sum_{i,j}\Big\langle T(x_{i,j})y_j, z_i\Big\rangle\Big| = \Big|\sum_{i,j}\Big\langle P_K x_{i,j}y_j, z_i\Big\rangle\Big| = \Big|\sum_{i,j}\Big\langle x_{i,j}y_j, z_i\Big\rangle\Big|$$

since both y_i and z_i belong to K. Thus,

$$\big\|[u(x_{i,j})]\big\|_n = \big\|[\pi(T(x_{i,j}))]\big\|_n = \big\|[T(x_{i,j})]\big\|_n \geq \Big|\sum_{i,j}\Big\langle x_{i,j}y_j, z_i\Big\rangle\Big| \geq \big\|[x_{i,j}]\big\|_n - \varepsilon.$$

□

Corollary 1.2.21. *Let X be a finite dimensional operator space. Then,*

$$M_n(X) \cong M_n(X^{**})$$

isometrically for all $n \in \mathbb{N}$, via the canonical inclusion $\tau : X \to X^{**}$ (which is an isomorphism).

Example 1.2.22. *Duality allows us to introduce a natural o.s.s. on the space ℓ_1^n as the dual of ℓ_∞^n: Let $A = \sum_i A_i \otimes e_i^*$ denote an element of $M_d(\ell_1^n) \cong CB(\ell_\infty^n, M_d)$, where $\{e_i^*\}_{i=1}^n$ is the basis of ℓ_1^n dual to the usual basis $\{e_i\}_{i=1}^n$ of ℓ_∞^n, and $A_i \in M_d$. Then by identifying again as in the relation 1.2.7 using the analogous map with 1.2.8 we have*

$$M_d \otimes (\ell_\infty^n)^* \cong CB(\ell_\infty^n, M_d)$$
$$A \mapsto \Big(T^A : v \mapsto \sum_i A_i v_i\Big)$$

where $v \in \ell_\infty^n$ is written as $v = \sum_{k=1}^n v_k e_k$. Then again $\|T^A\|_{cb} = \|T_d^A\|$ where

$$T_d^A = \mathrm{Id}_{M_d} \otimes T^A : M_d(\ell_\infty^n) \to M_{d^2}$$
$$\sum_i B_i \otimes e_i \mapsto \sum_i B_i \otimes A_i$$

So we have

$$\begin{aligned}\|A\|_{M_d(\ell_1^n)} &= \left\| \sum_i A_i \otimes e_i^* \right\|_{M_d(\ell_1^n)} \\ &= \left\|T_d^A\right\| := \sup\left\{ \left\| T_d^A(\sum_i B_i \otimes e_i) \right\|_{M_{d^2}} : \left\| \sum_i B_i \otimes e_i \right\|_{M_d(\ell_\infty^n)} = 1 \right\} \\ &= \sup\left\{ \left\| \sum_i B_i \otimes A_i \right\|_{M_{d^2}} : \left\| \sum_i B_i \otimes e_i \right\|_{M_d(\ell_\infty^n)} = 1 \right\}\end{aligned}$$

and since $\left\|\sum_i B_i \otimes e_i\right\|_{M_d(\ell_\infty^n)} = \sup_i \|B_i\|_{M_d}$ as we saw in example 1.2.10, we conclude that:

$$\|A\|_{M_d(\ell_1^n)} = \sup\left\{ \left\| \sum_i B_i \otimes A_i \right\|_{M_{d^2}} : \{B_i\}_i \in M_d, sup_i \|B_i\| \le 1 \right\}$$

1.2.3 Bilinear forms and tensor products

As in the case of linear maps between operator spaces, we would like to introduce some of the notions to bilinear forms.

Let X, Y be two operator spaces and $B : X \times Y \to \mathbb{C}$ a bilinear form. For every $d \in \mathbb{N}$ define a bilinear operator

$$B_d : M_d(X) \times M_d(Y) \to M_{d^2}$$
$$(a \otimes x, b \otimes y) \mapsto B(x, y) a \otimes b$$

For every $a, b \in M_d$, $x \in X$, $y \in Y$. We say that B is completely bounded if its completely bounded norm is finite

$$\|B\|_{cb} := \sup_{d \in \mathbb{N}} \|B_d\| < \infty$$

more specifically

$$\sup_{d\in\mathbb{N}} \|B_d\| = \sup\{\|B_d(A,B)\|_{M_{d^2}} : d \in \mathbb{N}, A \in \mathrm{Ball}(M_d(X)), B \in \mathrm{Ball}(M_d(Y))\}.$$

We denote by $\mathcal{CB}(X\times Y)$ the space of all completely bounded bilinear forms from $X \times Y$, equipped with the c.b. norm.

The natural one-to-one correspondence between bilinear forms and tensor products will play an important role throughout this thesis. It is obtained through the identifications

$$\mathrm{B}(X\times Y) \cong (X\otimes Y)^\sharp \qquad\qquad \mathrm{B}(X\times Y) \cong L(X.Y^\sharp)$$

$$B \mapsto \Big(x\otimes y \mapsto B(x,y)\Big) \qquad\qquad B \mapsto \Big(T_B : \langle\langle T_B(x), y\rangle\rangle = B(x,y)\Big)$$

and if the spaces X, Y are finite dimensional, one has the natural algebraic identification (1.1.3)

$$B(X\times Y) \cong X^* \otimes Y^*.$$

where the latter identification or can be made isometric by introducing the injective norm (Definition 1.1.3) for normed spaces X, Y.

Recall that the injective norm of $z \in X\otimes Y$ is defined by

$$\|z\|_{X\otimes_\epsilon Y} := \sup\left\{\Big|\langle\langle z, x^*\otimes y^*\rangle\rangle\Big| : x^* \in \mathrm{Ball}(X^*), y^* \in \mathrm{Ball}(Y^*)\right\},$$

and thus, if X, Y are finite dimensional, then (Remark 1.1.7) for every bilinear form $B : X\times Y \to \mathbb{C}$, we have

$$\|B\| = \|\hat{B}\|_{X^*\otimes_\epsilon Y^*},$$

where $\hat{B} \in X^*\otimes Y^*$ is the tensor associated to the bilinear form B, given by $\hat{B} = \sum_{i,j} B(e_i, f_j) e_i^* \otimes f_j^* \in X^*\otimes Y^*$.

The correspondence can be extended to the case of operator spaces, through the minimal tensor norm (Definition 1.1.17). However one may use the equivalent definition which we derived in Proposition 1.1.26: Let $z \in X\otimes Y$, then the minimal tensor product of z will be

$$\|z\|_{X\otimes_{min} Y} = \sup_d \|z\|_{X\otimes_{min_d} Y} \tag{1.2.11}$$

where

$$\|z\|_{X\otimes_{min_d} Y} = \sup\left\{\|(T\otimes S)(z)\|_{M_{d^2}} : T\in \mathrm{Ball}(\mathcal{CB}(X,M_d)), S\in \mathrm{Ball}(\mathcal{CB}(Y,M_d))\right\} \tag{1.2.12}$$

With this definition, it is now easy to verify that whenever X and Y are finite dimensional we have another isometric identification between the space of bilinear forms and the tensor product of duals.

Remark 1.2.23. *Let X, Y be finite dimensional Banach spaces, then if we endow $X^*\otimes Y^*$ with the minimal tensor norm and $B(X\times Y)$ with the completely bounded norm, the identification $B(X\times Y)\cong X^*\otimes Y^*$ becomes isometric:*

$$\mathcal{CB}(X\times Y)\cong X^*\otimes_{min} Y^* \tag{1.2.13}$$

and thus

$$\|B\|_{cb} = \|\hat{B}\|_{X^*\otimes_{min} Y^*}. \tag{1.2.14}$$

Proof. Note first that since X is finite dimensional we canonically isometrically identify it with X^{**} (it is reflexive) and thus we also identify isometrically $M_d(X)$ with $M_d(X^{**})$ by Corollary 1.2.21. Thus, if $a\otimes x\in M_d(X)$, recall that from the correspondence in identification 1.2.7, its norm can obtained from the c.b. norm of the associated map

$$\varphi\in X^*\mapsto T^{a\otimes x}(\varphi) = ax(\varphi) = a\varphi(x)\in M_d.$$

Hence, $\|a\otimes x\|_{M_d(X)} = \left\|T^{a\otimes x}\right\|_{cb}$.

Now let $d\geq 1$, $x=\sum_k a_k\otimes x_k\in \mathrm{Ball}(M_d(X))$ and $y=\sum_l b_l\otimes y_l\in \mathrm{Ball}(M_d(Y))$ where $a_k, b_l\in M_d$, $x_k\in X$ and $y_l\in Y$. For each k,l we have that $x_k = \sum_i e_i^*(x_k)e_i$ and $y_l=\sum_j f_j^*(y_l)f_j$, then if we denote by $S^{\sum_l b_l\otimes y_l}$ the associated map to $y = \sum_l b_l \otimes y_l$ we have

$$\begin{aligned}
\left\| B_d\Big(\sum_k a_k \otimes x_k, \sum_l b_l \otimes y_l \Big) \right\|_{M_{d^2}} &= \left\| \sum_{k,l} B(x_k, y_l) a_k \otimes b_l \right\|_{M_{d^2}} \\
&= \left\| \sum_{i,j} b_{i,j} \Big(\sum_k e_i^*(x_k) a_k \Big) \otimes \Big(\sum_l f_j^*(y_l) b_l \Big) \right\|_{M_{d^2}} \\
&= \left\| \sum_{i,j} b_{i,j} \Big(T^{\sum_k a_k \otimes x_k}(e_i^*) \Big) \otimes \Big(S^{\sum_l b_l \otimes y_l}(f_j^*) \Big) \right\|_{M_{d^2}} \\
&= \left\| (T^{\sum_k a_k \otimes x_k} \otimes S^{\sum_l b_l \otimes y_l})(\hat{B}) \right\|_{M_{d^2}} \\
&\leq \left\| \hat{B} \right\|_{X^* \otimes_{min d} Y^*}
\end{aligned}$$

Taking the supremum over all $d \geq 1$ and $x \in \mathrm{Ball}(M_d(X))$, $y \in \mathrm{Ball}(M_d(Y))$ we obtain $\|B\|_{cb} \leq \left\| \hat{B} \right\|_{X^* \otimes_{min} Y^*}$.

As for the reverse inequality, notice that by the correspondence 1.2.7 again we identify a map $T \in \mathcal{CB}(X^*, M_d)$ with an element $\sum_i a_i \otimes T_i \in M_d(X^{**})$ and again see it as an element of $M_d(X)$ which implies that we can write each $T_i = \sum_k e_k^*(T_i) e_k = \sum_k T_i(e_k^*) e_k$ and consequently $\sum_i a_i \otimes T_i = \sum_i a_i \otimes (\sum_k T_i(e_k^*) e_k)$. All these identifications are isometric, thus we can proceed as follows: For $d \geq 1$, $T \in \mathrm{Ball}(\mathcal{CB}(X^*, M_d))$ and $S \in \mathrm{Ball}(\mathcal{CB}(Y^*, M_d))$

$$\begin{aligned}
\|B_d\| &\geq \left\| B_d\Big(\sum_i a_i \otimes (\sum_k T_i(e_k^*) e_k), \sum_j b_j \otimes (\sum_l S_j(f_l^*) f_l) \Big) \right\|_{M_{d^2}} \\
&= \left\| \sum_{k,l} B(e_k, f_l) \Big(\sum_i a_i T_i(e_k^*) \Big) \otimes \Big(\sum_j b_j S_j(f_l^*) \Big) \right\|_{M_{d^2}} \\
&= \left\| \sum_{k,l} b_{k,l} \Big(T(e_k^*) \Big) \otimes \Big(S(f_l^*) \Big) \right\|_{M_{d^2}} \\
&= \left\| \Big(T \otimes S \Big)(\hat{B}) \right\|_{M_{d^2}}
\end{aligned}$$

Since $d \geq 1$ and the operators $T \in \mathrm{Ball}(\mathcal{CB}(X^*, M_d))$ and $S \in \mathrm{Ball}(\mathcal{CB}(Y^*, M_d))$ were arbitrary we finally have

$$\|B\|_{cb} \geq \left\| \hat{B} \right\|_{X^* \otimes_{min} Y^*}.$$

and hence the equality is proved.

$\square$

Correlation matrices, Bell functionals and games

2.1 Correlation matrices

Let us consider again the Alice and Bob scenario discussed in the Introduction. We can generalize to the case of more than two measurements, that is, Alice can perform N different measurements $A_1, \ldots, A_N$ and similarly Bob performs $B_1, \ldots, B_N$, each with possible outcomes ± 1. Let us also write

$$\gamma_{x,y} = \mathbb{E}[A_x B_y], \quad \text{for every} \quad x, y = 1, \ldots, N \tag{2.1.1}$$

Here, $\mathbb{E}[A_x B_y]$ denotes the expected value of the product of the outputs of A_x and B_y for every x, y.

Definition 2.1.1. *The matrices $\gamma = (\gamma_{x,y})_{x,y=1}^N$, where $\gamma_{x,y} = \mathbb{E}[A_x B_y]$ for all $x, y = 1, \ldots, N$ are called correlation matrices.*

In a Local Hidden Variable model of Nature, correlation matrices are of the form

$$\gamma_{x,y} = \int_\Lambda A_x(\lambda) B_y(\lambda) d\mathbb{P}(\lambda) \tag{2.1.2}$$

where $(\Lambda, \mathbb{P})$ is the "hidden" probability space, and if we fix one of these states $\lambda \in \Lambda$, $A_x(\lambda) = +1 or -1$ and similarly for $B_y(\lambda)$, for every x, y. We call these matrices, *classical correlation matrices* and denote by $\mathcal{L}_N$ the set of classical correlation matrices of size N. Note that the elements of $\mathcal{L}_N$ are those matrices whose entries $\mathbb{E}[A_x B_y]$ given by the expected value of the product of the outcomes of the binary measurements A_x and B_y when we describe the corresponding measurement procedure by using a Local Hidden Variable model as we did in the Introduction.

It is straightforward to verify that each classical correlation matrix is a convex combination of elements of the form $(t_x s_y)_{x,y=1}^N$ where $t_x = \pm 1$ and $s_y = \pm 1$, for every $x, y = 1, \ldots, N$. Thus,

$$\mathcal{L}_N = \mathrm{Conv}\Big\{(t_x s_y)_{x,y=1}^N : t_x = \pm 1, s_y = \pm 1, \quad \text{for every} \quad x, y = 1, \ldots, N\Big\}$$

Hence, $\mathcal{L}_N$ is a polytope in $\mathbb{R}^{N^2}$.

In Quantum Information Theory, a measurement process on a bipartite system usually involves two finite dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ corresponding to Alice and Bob respectively, and a *state* $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Furthermore, each of Alice's measurement outputs A_x is described by the positive operators $\{E_x, \mathrm{Id} - E_x\} \subseteq \mathcal{B}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$ where E_x is the operator associated with the output 1 while $\mathrm{Id} - E_x$ is associated with -1; similarly Bob's outcomes B_y are described by $\{F_y, \mathrm{Id} - F_y\} \subseteq \mathcal{B}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with F_y associated with 1 and $\mathrm{Id} - F_y$ with -1. Then, if Alice and Bob perform measurements A_x and B_y, the corresponding table of probabilities is the following:

$$P(x,y) = \begin{cases} \mathrm{Tr}\Big((E_x \otimes F_y)\rho\Big) & \text{is the probability of outputs 1 and 1} \\ \mathrm{Tr}\Big((E_x \otimes (\mathrm{Id} - F_y))\rho\Big) & \text{is the probability of outputs 1 and -1} \\ \mathrm{Tr}\Big(((\mathrm{Id} - E_x) \otimes F_y)\rho\Big) & \text{is the probability of outputs -1 and 1} \\ \mathrm{Tr}\Big(((\mathrm{Id} - E_x) \otimes (\mathrm{Id} - F_y))\rho\Big) & \text{is the probability of outputs -1 and -1} \end{cases}$$

So we can compute

$$\begin{aligned} \gamma_{x,y} = \mathbb{E}[A_x B_y] &= [P(1,1|x,y) + P(-1,-1|x,y)] - [P(-1,1|x,y) + P(1,-1|x,y)] \\ &= \mathrm{Tr}\Big((E_x \otimes F_y + (\mathrm{Id} - E_x) \otimes (\mathrm{Id} - F_y) - E_x \otimes (\mathrm{Id} - F_y) - (\mathrm{Id} - E_x) \otimes F_y)\rho\Big) \\ &= \mathrm{Tr}\Big(((\mathrm{Id} - 2E_x) \otimes (\mathrm{Id} - 2F_y))\rho\Big) \end{aligned}$$

Note also that if we write $A_x = \mathrm{Id} - 2E_x$, this is a selfadjoint operator of norm $\|A_x\| \leq 1$ and every selfadjoint operator $\|A_x\| \leq 1$ can be written as $A_x = \mathrm{Id} - 2E_x$ where E_x is a positive operator of norm less than or equal to one (see Section 3.1). Similarly, if we write $B_y = \mathrm{Id} - 2F_y$ for every y, we reach the following definition.

Definition 2.1.2. *We say that $\gamma = (\gamma_{x,y})_{x,y=1}^N$ is a quantum correlation matrix if there exist selfadjoint operators $A_1, \dots, A_N$ and $B_1, \dots, B_N$ acting on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, with $\max_{x,y}\{\|A_x\|, \|B_y\|\} \leq 1$ and a state ρ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$, such that*

$$\gamma_{x,y} = \mathrm{Tr}\Big((A_x \otimes B_y)\rho\Big) \quad \textit{for every} \quad x, y = 1, \dots, N.$$

We denote by $\mathcal{Q}_N$ the set of quantum correlation matrices of order N.

The extreme points of the set of states $\mathcal{S}(\mathcal{H})$ are called *pure states*. These extreme points are exactly the rank-one projections of the form $|\psi\rangle\langle\psi|$. Indeed, just note that any positive, trace-one operator ρ admits a spectral decomposition of the form $\rho = \sum_{i=1}^r \lambda_i |u_i\rangle\langle u_i|$ where $\lambda_i \geq 0$ are its eigenvalues and $\{u_i\}$ is an orthonormal set of its eigenvectors, $\sum_{i=1}^r \lambda_i = 1$ since its a trace equals to one and $r = \mathrm{rank}(\rho)$. However we will often refer to a unit vector $|\psi\rangle$ itself as a pure state or a state vector.

As we will see, in the definition of the set quantum correlation matrices, the states can always be assumed pure. This comes as a consequence of the process called state purification. Namely,

Definition 2.1.3 (Purification)**.** *Let $\rho \in \mathcal{S}(\mathcal{H})$ be a state. There exists a unit vector $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}'$, where $\mathcal{H}'$ is simply an auxiliary Hilbert space, such that $\mathrm{Tr}_{\mathcal{H}'}(|\psi\rangle\langle\psi|) := (\mathrm{Id}_{\mathcal{H}} \otimes \mathrm{Tr})(|\psi\rangle\langle\psi|) = \rho$. We say that the vector $|\psi\rangle$ in the extended Hilbert space, is a purification of the state ρ.*

Such a purification indeed exists:

Remark 2.1.4. *The states in the Definition 2.1.2 the states can be purified.*

Proof. In order to obtain such a purification, suppose that we have a state $\rho \in \mathcal{B}(\mathcal{H})$. Being positive, ρ admits a spectral decomposition

$$\rho = \sum_{i=1}^{r} \lambda_i |u_i\rangle\langle u_i|$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{r} \lambda_i = 1$, since ρ has trace one. Then a purification of ρ is given by

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}|u_i\rangle \otimes |u_i\rangle$$

where $|\psi\rangle \in \mathcal{H}\otimes\mathcal{H}'$ and the auxiliary Hilbert space $\mathcal{H}'$ is of dimension r. Finally, if we further replace the selfadjoint operator $\|A\| \leq 1$ with $\tilde{A} := A\otimes \mathrm{Id}_{\mathcal{H}'}$ we obtain an operator acting on the extended Hilbert space $\mathcal{H}\otimes\mathcal{H}'$ that is also selfadjoint and $\left\|\tilde{A}\right\| = \|A\|$ and such that

$$\mathrm{Tr}(A\rho) = \mathrm{Tr}\left(\tilde{A}|\psi\rangle\langle\psi|\right)$$

So the probabilities are preserved, and hence by increasing the dimension of the original Hilbert space, we can restrict our attention to the case of pure states. □

As in the case of $\mathcal{L}_N$, we can see that Q_N is also convex.

Proposition 2.1.5. *The set Q_N of quantum correlation matrices is convex.*

Proof. Indeed, let $\gamma, \gamma' \in Q_N$ and $t \in [0,1]$. Then, $\gamma_{x,y} = \mathrm{Tr}\left(A_x \otimes B_y \rho\right)$ and $\gamma'_{x,y} = \mathrm{Tr}\left(A'_x \otimes B'_y \rho'\right)$, where $A_x, A'_x, B_y, B'_y, \rho, \rho'$ are operators on $\mathcal{H}$ as in the definition and we will show that $t\gamma + (1-t)\gamma' \in Q_N$. To this end, we define the following elements

$$\tilde{A}_x = \begin{pmatrix} A_x & 0 \\ 0 & A'_x \end{pmatrix}, \quad \tilde{B}_y = \begin{pmatrix} B_y & 0 \\ 0 & B'_y \end{pmatrix} \quad \text{and} \quad \tilde{\rho} = \begin{pmatrix} t\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-t)\rho' \end{pmatrix}$$

where $\tilde{A}_x, \tilde{B}_y$ are selfadjoint operators of norm lower than or equal to one, $\tilde{\rho}$ is a state. Observe that

$$\tilde{A}_x \otimes \tilde{B}_y = \begin{pmatrix} A_x & 0 \\ 0 & A'_x \end{pmatrix} \otimes \tilde{B}_y = \begin{pmatrix} A_x \otimes \tilde{B}_y & \bigcirc \\ \bigcirc & A'_x \otimes \tilde{B}_y \end{pmatrix} = \begin{pmatrix} A_x \otimes \begin{pmatrix} B_y & 0 \\ 0 & B'_y \end{pmatrix} & \bigcirc \\ \bigcirc & A'_x \otimes \begin{pmatrix} B_y & 0 \\ 0 & B'_y \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} A_x \otimes B_y & 0 \\ 0 & A_x \otimes B'_y \end{pmatrix} & \bigcirc \\ \bigcirc & \begin{pmatrix} A'_x \otimes B_y & 0 \\ 0 & A'_x \otimes B'_y \end{pmatrix} \end{pmatrix}.$$

So we conclude that

$$\mathrm{Tr}\left(\tilde{A}_x \otimes \tilde{B}_y \tilde{\rho}\right) = t\,\mathrm{Tr}\left(A_x \otimes B_y \rho\right) + (1-t)\mathrm{Tr}\left(A'_x \otimes B'_y \rho'\right)$$

for every $x, y = 1, \ldots, N$ and this finishes the proof. □

It is not hard to verify that $\mathcal{L}_N \subseteq \mathcal{Q}_N$, in fact we prove a similar result in Remark 2.2.10.

We shall now move towards the so called *Bell inequalities*. Every matrix $M = (M_{x,y})_{x,y=1}^N \in M_N(\mathbb{R})$, defines a functional M acting on the set of correlation matrices by means of the duality:

$$\langle\langle M, \gamma \rangle\rangle = \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y}$$

We call *correlation Bell functional* any matrix $M = (M_{x,y})_{x,y=1}^N$ with real coefficients. To any such matrix $M = (M_{x,y})_{x,y=1}^N$ we can associate an inequality

$$\left| \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} \right| \leq \omega(M)$$

where

$$\omega(M) := \sup\left\{ \left| \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} \right| : \gamma = (\gamma_{x,y})_{x,y=1}^N \in \mathcal{L}_N \right\}$$

$$= \sup\left\{ \left| \sum_{x,y=1}^{N} M_{x,y} t_x s_y \right| : t_x = \pm 1, s_y = \pm 1, \quad \text{for every} \quad x, y \right\}$$

is called the *classical value* of M. Here, the last inequality follows by convexity of the set $\mathcal{L}_N$.

A *(correlation) Bell inequality* is an upper bound on the quantity $\omega(M)$. Actually, we have already seen a Bell inequality. Indeed, the CHSH inequality derived in 0.0.3 in the Introduction, corresponds, using this framework, to the Bell functional:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

where, as we showed, there exists a certain quantum correlation matrix $\gamma \in \mathcal{Q}_N$ for which

$$\sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} = 2\sqrt{2}\,.$$

In such a case, we say that the correlation γ violates the corresponding correlation Bell inequality or that we have a *Bell inequality violation.*

We can also define the *quantum value* of a Bell functional M as

$$\omega^*(M) := \sup\left\{ \left| \sum_{x,y=1}^{N} M_{x,y}\gamma_{x,y} \right| : \gamma = (\gamma_{x,y})_{x,y=1}^{N} \in \mathcal{Q}_N \right\}.$$

We will be studying these values throughout these notes due to their significance for the theory of nonlocality. In the next section we extend the notions discussed into a more general context, however, many of the results can be also applied to the case of correlation matrices.

2.2 Quantum nonlocality: The general case

We have already encountered Bell functionals in the previous Section. As we saw, a *correlation* Bell functional was a functional acting on correlation matrices. In order to fully unveil the connection between the theory of nonlocality and that of operator spaces we need to extend these notions, starting by establishing the more general sets of conditional distributions. Given finite sets $\mathbf{X}$ and $\mathbf{A}$ denote by $\mathcal{P}(\mathbf{A}|\mathbf{X})$ the set

$$\mathcal{P}(\mathbf{A}|\mathbf{X}) = \left\{ P = (P(a|x))_{x,a} \in \mathbb{R}_+^{AX} : \forall x \in \mathbf{X}, \sum_{a\in\mathbf{A}} P(a|x) = 1 \right\}$$

In the case of bipartite conditional distributions we will use notation $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ instead of $\mathcal{P}(\mathbf{A}\times\mathbf{B}\,|\,\mathbf{X}\times\mathbf{Y})$.

Definition 2.2.1 (Bell functional)**.** *A* <u>*Bell functional*</u> *is simply a linear form on* $\mathbb{R}^{ABXY}$*. Any such functional is specified by a family of coefficients* $M = (M_{x,y}^{a,b})_{x,y;a,b} \in \mathbb{R}^{\boldsymbol{A}\times\boldsymbol{B}\times\boldsymbol{X}\times\boldsymbol{Y}}$ *and its action on* $\mathcal{P}(\boldsymbol{AB}|\boldsymbol{XY})$ *is given by*

$$P \in \mathcal{P}(\boldsymbol{AB}|\boldsymbol{XY}) \mapsto \omega(M;P) := \sum_{x,y;a,b} M_{x,y}^{a,b} P(a,b|x,y) \in \mathbb{R}. \tag{2.2.1}$$

We will refer to X and A (respectively, Y and B) as the number of *inputs* and *outputs* to and from the first (respectively, second) system acted on by the Bell functional.

A *Bell inequality* is an upper bound on the largest value that expression 2.2.1 can take when restricted to the subset of $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ consisting of *classical* conditional distributions which correspond to the convex hull of product distributions:

Definition 2.2.2. *We define the set of* <u>*Classical probability distributions*</u> *to be*

$$\mathcal{P}_C(\boldsymbol{AB}|\boldsymbol{XY}) = \mathrm{Conv}\Big\{(P(a|x)Q(b|y))_{x,y;a,b} : P \in \mathcal{P}(\boldsymbol{A}|\boldsymbol{X}), Q \in \mathcal{P}(\boldsymbol{B}|\boldsymbol{Y})\Big\}.$$

It follows from the definition that $\mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$ is a closed convex set of $\mathbb{R}^{ABXY}$.

Definition 2.2.3 (Bell inequality)**.** *Let M be a Bell functional. A* <u>*Bell inequality*</u> *is an upper bound on the quantity*

$$\omega(M) := \sup_{P\in\mathcal{P}_C(\boldsymbol{AB}|\boldsymbol{XY})} |\omega(M;P)|. \tag{2.2.2}$$

We refer to $\omega(M)$ *as the* <u>*classical value*</u> *of the functional M.*

The second value associated to a Bell functional is its <u>quantum value</u> (or *entangled value*), which corresponds to its supremum over the subset of $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$ consisting of those distributions that can be implemented locally using measurements on a bipartite quantum state:

Definition 2.2.4. *We define the set of Quantum probability distributions to be*

$$\mathcal{P}_Q(\boldsymbol{AB}|\boldsymbol{XY}) = \Big\{ \Big(\langle \psi | A_x^a \otimes B_y^b | \psi \rangle \Big)_{x,y;a,b} : d \in \mathbb{N}, \|\psi\|_{\mathbb{C}^d \otimes_2 \mathbb{C}^d} = 1, A_x^a, B_y^b \in \mathcal{B}_+(\mathbb{C}^d), \\ \sum_a A_x^a = \sum_b B_y^b = \mathrm{Id}_{\mathbb{C}^d}, \forall (x,y) \in \boldsymbol{X} \times \boldsymbol{Y} \Big\}$$

The constraints $A_x^a \in \mathcal{B}_+(\mathbb{C}^d)$, $\sum_a A_x^a = \mathrm{Id}_{\mathbb{C}^d}$ for every x, correspond to the general notion of measurement called *positive operator-valued measure* (POVM) in quantum information.

Definition 2.2.5 (POVM). *A positive operator-valued measure on a Hilbert space $\mathcal{H}$ is a set $\{L_i\}_{i=1}^m \subseteq \mathcal{B}(\mathcal{H})$ with $L_i \geq 0$, for every i and $\sum_{i=1}^m L_i = \mathrm{Id}_{\mathcal{H}}$.*

It has been quite a problem for many years, whether the set of quantum probability distributions (Definition 2.2.4) is closed or not, it is now known that in certain cases it is not! (See [13], [14]).

The notion of a POVM is actually closely related to the theory of operator spaces, in fact we will prove the following result.

Proposition 2.2.6. *Let $\{A^a\}_{a=1}^A$ be a POVM on $\mathbb{C}^d$, then the linear map*

$$T : \ell_\infty^A \to M_d$$
$$e_a \mapsto A^a$$

is completely positive and unital and thus by Proposition 1.2.16, it holds that $\|T\|_{cb} = 1$. Conversely, any completely positive unital map $T : \ell_\infty^A \to M_d$ defines a POVM.

First we are going to need the following Lemma

Lemma 2.2.7. *If $x = [x_{kj}] \in M_n$ is a positive matrix and $a \in \mathcal{B}$ is a positive element of a C^* algebra, then $x \otimes a = [x_{kj}a]$ is a positive element of the C^* algebra $M_n(\mathcal{B})$.*

Proof. Observe that the the matrix $x \otimes a = [x_{kj}a]$ is just the matrix product $[x_{kj}]\mathrm{diag}(a)$ (where $\mathrm{diag}(a) \in M_n(\mathcal{B})$ has a in each diagonal entry and zeroes elsewhere). Note, that these two matrices commute. Now, there exists $b \in \mathcal{B}_+$ such that $b^2 = a$ and so $\text{diag}(b)^2 = \text{diag}(a)$. Thus $\text{diag}(b)$ also commutes with x and therefore

$$[x_{kj}]\text{diag}(a) = [x_{kj}]\text{diag}(b)^2 = \text{diag}(b)[x_{kj}]\text{diag}(b)$$

which is a positive element of the C* algebra $M_n(\mathcal{B})$, since it is of the form $cuu^*c = (cu)(cu)^*$ where $c = \text{diag}(b)$ and $[x_{kj}] = uu^*$ since it is positive.

□

We can now proceed to the proof.

Proof. (Proposition 2.2.6) Notice first that if $x = (x_a) \in \ell_\infty^A$ then $x = \sum x_a e_a$ and thus the map is defined as $T(x) = \sum_a x_a A^a$. This map is clearly positive, because $x \in \ell_\infty^A$ is positive iff all its coordinates x_a are non-negative.

We claim that T is completely positive.

Indeed, suppose $x \in M_n(\ell_\infty^A)$ is positive; we have to prove that $T_n(x) \in M_n(M_d)$ is positive. We may write $x = \sum_{a=1}^A x_a \otimes e_a$ where $e_a \in \ell_\infty^A$ are the elements of the usual basis and $x_a \in M_n$.

Then
$$T_n(x) = (\text{Id}_{M_n} \otimes T)(x) = \sum_{a=1}^{A} x_a \otimes T(e_a) = \sum_{a=1}^{A} x_a \otimes A^a$$
and so it suffices to prove that each term $x_a \otimes A^a$ is positive which is indeed verified by the Lemma above. Finally, we have that T_n is positive for all $n \in \mathbb{N}$ and thus T is completely positive. The unital part comes easily since $\sum_a e_a = \mathbf{1}_{\ell_\infty^A}$ and consequently $T(\mathbf{1}_{\ell_\infty^A}) = T(\sum_a e_a) = \sum_a T(e_a) = \sum_a A^a = I_d$. □

Remark 2.2.8. *In fact the following result of Stinespring holds: A positive linear map from any abelian unital C* algebra $C(X)$ into any C* algebra is necessarily completely positive (the proof uses partitions of the identity, see [9], Theorem 3.11).*

With the definition of the set $\mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})$ that we have,

Definition 2.2.9. *The entangled value of M is defined as*

$$\omega^*(M) := \sup_{P \in \mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})} |\omega(M;P)|$$

Remark 2.2.10. *Every classical probability distribution is also a quantum one, namely:*

$$\mathcal{P}_C(\boldsymbol{AB}|\boldsymbol{XY}) \subseteq \mathcal{P}_Q(\boldsymbol{AB}|\boldsymbol{XY})$$

and as consequence, it holds in general that

$$\omega(M) \leq \omega^*(M).$$

Proof. Let $P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$ be an arbitrary element. Then there exists a $K \in \mathbb{N}$ and $\{t_i\}_{i=1}^K$ with $t_i \geq 0$, $\sum_{i=1}^K t_i = 1$ such that $P = \sum_{i=1}^K t_i \Big(P^i(a|x)Q^i(b|y)\Big)_{x,y;a,b} = \Big(\sum_{i=1}^K t_i P^i(a|x)Q^i(b|y)\Big)_{x,y;a,b}$. For every $a \in \mathbf{A}$ and $x \in \mathbf{X}$ we set

$$P(a|x) = \begin{pmatrix} P^1(a|x) & & & & \bigcirc \\ & \ddots & & & \\ & & P^i(a|x) & & \\ & & & \ddots & \\ \bigcirc & & & & P^K(a|x) \end{pmatrix} \in M_K$$

and also, for every $b \in \mathbf{B}$ and $y \in \mathbf{Y}$

$$Q(b|y) = \begin{pmatrix} Q^1(b|y) & & & & \bigcirc \\ & \ddots & & & \\ & & Q^i(b|y) & & \\ & & & \ddots & \\ \bigcirc & & & & Q^K(b|y) \end{pmatrix} \in M_K$$

Clearly, for every $x \in \mathbf{X}$ and $y \in \mathbf{Y}$, $\{P(a|x)\}_a$ and $\{Q(b|y)\}_b$ define POVM's on $\mathbb{C}^K$. Consider the vector $|\psi\rangle = \sum_{i=1}^K \sqrt{t_i}|ii\rangle \in \mathbb{C}^K \otimes \mathbb{C}^K$. This is a pure state, indeed:

$$\begin{aligned} \|\psi\| = \left\langle \sum_{i=1}^K \sqrt{t_i}|ii\rangle, \sum_{j=1}^K \sqrt{t_j}|jj\rangle \right\rangle &= \sum_{i,j=1}^K \sqrt{t_i}\sqrt{t_j}\Big\langle e_i \otimes e_i, e_j \otimes e_j\Big\rangle \\ &= \sum_{i,j=1}^K \sqrt{t_i}\sqrt{t_j}\Big\langle e_i, e_j\Big\rangle\Big\langle e_i, e_j\Big\rangle = \sum_i t_i = 1\,. \end{aligned}$$

Finally, note that $(P(a|x) \otimes Q(b|y))|\psi\rangle = \sum_i \sqrt{t_i} P^i(a|x) e_i \otimes Q^i(b|y) e_i$ and thus, for each $a \in \mathbf{A}$, $x \in \mathbf{X}$, $b \in \mathbf{B}$ and $y \in \mathbf{Y}$

$$\begin{aligned}\langle\psi|(P(a|x) \otimes Q(b|y))|\psi\rangle &= \Big\langle \sum_j \sqrt{t_j} e_j \otimes e_j, \sum_i \sqrt{t_i} P^i(a|x) e_i \otimes Q^i(b|y) e_i \Big\rangle \\ &= \sum_{i,j} \sqrt{t_i}\sqrt{t_j} \Big\langle e_j, P^i(a|x) e_i \Big\rangle \Big\langle e_j, Q^i(b|y) e_i \Big\rangle \\ &= \sum_i t_i P^i(a|x) Q^i(b|y).\end{aligned}$$

So we have

$$P = \Big(\sum_{i=1}^K t_i P^i(a|x) Q^i(b|y) \Big)_{x,y;a,b} = \Big(\langle\psi|(P(a|x) \otimes Q(b|y))|\psi\rangle \Big)_{x,y;a,b}.$$

□

Hence we always have that $\omega(M) \leq \omega^*(M)$. We say that we have a Bell inequality violation when the inequality is strict: when we have an M such that $\omega(M) < \omega^*(M)$. The existence of such Bell functionals shows that the set of quantum conditional distributions is strictly larger than the classical one.

2.2.1 Connection with correlation matrices

According to the definition of the sets $\mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$ and $\mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})$, we can understand the setting of correlation Bell functionals studied in the previous section as a particular case of the setting considered in this section. More specifically, let us consider the situation where $\mathbf{A} = \mathbf{B} = \{-1, 1\}$ and $\mathbf{X} = \mathbf{Y} = \{1, \ldots, N\}$ and compute the correlations of a given probability distribution $P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY})$. That is, for every x, y we have

$$\gamma_{x,y} = \mathbb{E}(a_x b_y) = P(1,1|x,y) + P(-1,-1|x,y) - P(-1,1|x,y) - P(1,-1|x,y)$$

where we can see that the set of correlation matrices $(\gamma_{x,y})_{x,y=1}^N$ written in this form coincides with the set of classical correlation matrices $\mathcal{L}_N$ when we restrict to $P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$ and also coincides with the set of quantum correlation matrices $\mathcal{Q}_N$ when we restrict to $P \in \mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})$.

2.3 Multiplayer games

Let's begin with the simple two-player one-round games $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$, which are specified by finite sets $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}$, a probability distribution $\pi : \mathbf{X} \times \mathbf{Y} \to [0, 1]$, and a payoff function $V : \mathbf{X} \times \mathbf{Y} \times \mathbf{A} \times \mathbf{B} \to [0, 1]$. One could think of the game as the following process: Suppose that there are two players, Alice and Bob. The finite sets $\mathbf{X}$ and $\mathbf{Y}$ represent the questions addressed to Alice and Bob respectively, and the finite sets $\mathbf{A}$ and $\mathbf{B}$ represent the answers that Alice and Bob give, respectively, according to the questions given to them. The questions are addressed to the players by a referee, who is selecting the pairs of questions $(x, y) \in \mathbf{X} \times \mathbf{Y}$ according to the distribution π. During the game Alice and Bob are separated in the sense that they are so far away that information, which travels at a finite speed, cannot be exchanged between them until they produce the answers. That is, they cannot communicate with each other and come up with a mutual strategy. This is a one-round game: the game begins, the players are provided with a pair of questions, they give their answers back and the referee declares that the players "win" the game with probability $V(a, b, x, y)$; alternatively we may say that the players are attributed with a "payoff" $V(a, b, x, y)$ for their answers. The players give their answers using a probabilistic "strategy", which is an element of $\mathcal{P}(\mathbf{AB}|\mathbf{XY})$. The value of the game is the highest probability with which the players can win the game, where the probability is taken over the choice of questions according to the distribution π, each players' strategy and the payoff function. Alternatively, the value of the game can be interpreted as the maximum expected payoff than can be achieved by the players.

Multiplayer games are the sub-class of Bell functionals M such that all the coefficients $M_{x,y}^{a,b}$ are non-negative and satisfy the normalization condition $\sum_{x,y;a,b} M_{x,y}^{a,b} = 1$. Indeed any game induces a Bell functional $G_{x,y}^{a,b} = \pi(x, y)V(a, b, x, y)$ for every a, b, x, y. This connection allows us now to extend the definitions of classical and entangled values given earlier, to the corresponding quantities for games. More specifically, given a game G, we define

$$\omega(G) := \sup_{P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})} \left| \sum_{x,y;a,b} \pi(x,y) V(a,b,x,y) P(a,b|x,y) \right| \tag{2.3.1}$$

$$\omega^*(G) := \sup_{P \in \mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})} \left| \sum_{x,y;a,b} \pi(x,y) V(a,b,x,y) P(a,b|x,y) \right| \tag{2.3.2}$$

The values above are precisely the highest probability of winning the game G when the players are allowed to use classical resources in the first case and quantum resources in the second case.

Conversely, any Bell functional with non-negative coefficients that satisfies the normalization condition, can be made into a game by setting $\pi(x,y) = \sum_{a,b} M_{x,y}^{a,b}$ and $V(a,b,x,y) = M_{x,y}^{a,b}/\pi(x,y)$.

XOR games

3.1 Prologue

XOR games are the simplest and more comprehensible class of two-player one-round games that are interesting from the point of view of Quantum Nonlocality. The characteristic properties of XOR games provide us with numerous results that underline the connection of nonlocality with the theory of Banach space tensor products and operator space tensor products.

Two-player XOR games correspond to the restricted family of games for which the answer alphabets **A** and **B** are binary. More specifically, $\mathbf{A} = \mathbf{B} = \{0, 1\}$ and the payoff function $V(a, b, x, y)$ depends only on x, y and the parity[1] of a and b. We will restrict our attention to functions of the form $V(a, b, x, y) = \frac{1}{2}(1 + (-1)^{a \oplus b \oplus c_{xy}})$, for some $c_{xy} \in \{0, 1\}$, where $a \oplus b$ is the "Exclusive or" (XOR) [2]. We will further restrict to the case where $\mathbf{X} = \mathbf{Y} = \{1, \ldots, N\}$, but the general situation is completely analogous.

In general, a strategy for the players is specified by an element $P \in \mathcal{P}(\mathbf{AB}|\mathbf{XY})$ which gives the probability that Alice and Bob answer a and b when they are asked questions x and y respectively. Given one such

[1]In Boolean algebra, a parity function is a Boolean function whose value is 1 if and only if the input vector has an odd number of ones.

[2]The Exclusive or (XOR), denoted by $p \oplus q$, is the logical operation : $p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$.

strategy the value achieved by P in G can be expressed as

$$\begin{aligned}\omega(G;P) &= \sum_{x,y=1}^{N}\sum_{a,b}\pi(x,y)V(a,b,x,y)P(a,b|x,y)\\ &= \sum_{x,y=1}^{N}\sum_{a,b}\pi(x,y)\frac{1+(-1)^{a\oplus b\oplus c_{xy}}}{2}P(a,b|x,y)\\ &= \frac{1}{2}+\frac{1}{2}\sum_{x,y=1}^{N}\pi(x,y)(-1)^{c_{xy}}[P(0,0|x,y)+P(1,1|x,y)-P(0,1|x,y)-P(1,0|x,y)].\end{aligned}$$

This last expression motivates the introduction of the *bias* of an XOR game,

$$\beta(G;P) := 2\omega(G;P) - 1 \in [-1,1],$$

a quantity that will prove more convenient to work with than the value of the game. Hence,

Definition 3.1.1. *Let G be a two-player XOR game, then its bias is defined to be*

$$\begin{aligned}\beta(G;P) &:= 2\omega(G;P) - 1\\ &= \sum_{x,y=1}^{N}\pi(x,y)(-1)^{c_{xy}}[P(0,0|x,y)+P(1,1|x,y)-P(0,1|x,y)-P(1,0|x,y)]\end{aligned}$$

Optimizing over classical strategies, we obtain the classical bias $\beta(G)$ of an XOR game:

$$\begin{aligned}\beta(G) &:= \sup_{P\in\mathcal{P}_C(\mathbf{AB}|\mathbf{XY})}|\beta(G;P)|\\ &= \sup_{P^1\in\mathcal{P}(\mathbf{A}|\mathbf{X}),P^2\in\mathcal{P}(\mathbf{B}|\mathbf{Y})}\left|\sum_{x,y=1}^{N}\pi(x,y)(-1)^{c_{xy}}\Big(P^1(0|x)-P^1(1|x)\Big)\Big(P^2(0|y)-P^2(1|y)\Big)\right|\end{aligned}$$

With the above definition of the classical bias, one can obtain the following simpler form:

Proposition 3.1.2. *Let G be a two-player XOR game, then its classical bias is given by*

$$\beta(G) = \sup_{A\in\mathrm{Ball}(\ell_\infty^N(\mathbb{R})),B\in\mathrm{Ball}(\ell_\infty^N(\mathbb{R}))}\left|\sum_{x,y=1}^{N}\pi(x,y)(-1)^{c_{xy}}A_xB_y\right|,$$

where $A \in \mathrm{Ball}(\ell_\infty^N(\mathbb{R}))$ is in the unit ball of $\ell_\infty^N(\mathbb{R})$, that is, $A = (A_x) \in \mathbb{R}^N$ and $\|A\|_\infty \leq 1$.

Proof. To verify the above equality, note first that the quantity $A_x := P^1(0|x) - P^1(1|x)$ is indeed in $\mathrm{Ball}(\ell_\infty^N(\mathbb{R}))$ since $P^1(0|x) + P^1(1|x) = 1$ for all $1 \leq x \leq N$. So the classical bias $\beta(G)$ is smaller that the quantity on the right-hand side. As for the reverse inequality, let $A \in \mathrm{Ball}(\ell_\infty^N(\mathbb{R}))$. For each $A_x \in \mathbb{R}$ with $|A_x| \leq 1$ we can write $A_x = \frac{A_x+1}{2} - \frac{1-A_x}{2}$ where we have that $A_x(0) := \frac{A_x+1}{2} \geq 0$, $A_x(1) := \frac{1-A_x}{2} \geq 0$ and $A_x(0) + A_x(1) = 1$ which confirms the equality between the suprema since $(A_x(a))_{x,a} \in \mathcal{P}(\mathbf{A}|\mathbf{X})$. □

To the game G, we may associate a bilinear form, which we also denote by $G : \ell_\infty^N(\mathbb{R}) \times \ell_\infty^N(\mathbb{R}) \to \mathbb{R}$ (slightly abusing the notation), defined on basis elements by $G(e_x, e_y) = \pi(x,y)(-1)^{c_{xy}}$. Let also $\hat{\mathrm{G}}$ be the associated tensor defined in 1.1.4, namely,

$$\hat{\mathrm{G}} = \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} e_x \otimes e_y \in \ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R}).$$

Thus, the classical bias we defined above can be related to the norm of the associated bilinear form and the injective tensor norm of its associated tensor:

Corollary 3.1.3. *Let the XOR game G be associated with the bilinear form $G \in \mathcal{B}(\ell_\infty^N(\mathbb{R}) \times \ell_\infty^N(\mathbb{R}), \mathbb{R})$ and the tensor $\hat{\mathrm{G}} \in \ell_1^N(\mathbb{R}) \otimes \ell_1^N(\mathbb{R})$, then*

$$\beta(G) = \|G\|_{\mathcal{B}(\ell_\infty^N(\mathbb{R}) \times \ell_\infty^N(\mathbb{R}))}. \tag{3.1.1}$$

Moreover, by relation 1.1.6 we also have

$$\beta(G) = \left\|\hat{\mathrm{G}}\right\|_{\ell_1^N(\mathbb{R}) \otimes_\epsilon \ell_1^N(\mathbb{R})}. \tag{3.1.2}$$

The corollary above marks our first connection between the theory of games and that of Banach spaces and allows us to compute the bias of a game in terms of tensor norms. Note that the norm in 3.1.1 is taken over the real Banach spaces $\ell_\infty^N(\mathbb{R})$. In case we need to consider the complex ones, we have the following result.

Lemma 3.1.4. *Let $G \in \mathcal{B}(\ell_\infty^N(\mathbb{R}) \times \ell_\infty^N(\mathbb{R}))$ be a real bilinear form. Then,*

$$\|G\|_{\mathcal{B}(\ell_\infty^N(\mathbb{R})\times\ell_\infty^N(\mathbb{R}))} \leq \|G\|_{\mathcal{B}(\ell_\infty^N(\mathbb{C})\times\ell_\infty^N(\mathbb{C}))} \leq \sqrt{2}\|G\|_{\mathcal{B}(\ell_\infty^N(\mathbb{R})\times\ell_\infty^N(\mathbb{R}))}$$

Moreover, for each inequality, there exists a G for which it is an equality.

The first inequality in the lemma is obvious; for the second one, see [15]. A bilinear form, for which the second inequality becomes an equality is the following:

$$G_{CHSH} : (e_x, e_y) \mapsto \frac{1}{4}(-1)^{x\wedge y} \tag{3.1.3}$$

for all $x, y \in \{0,1\}$. This bilinear form corresponds to the famous CHSH inequality that we already encountered in the Introduction, and for which Quantum Mechanics predicts a noticeable violation. We shall deal with this violation later after having established an explicit form for the entangled bias of an XOR game.

Let's consider first the entangled bias of an XOR game by means of Definition 3.1.1 :

$$\begin{aligned}\beta^*(G) &:= \sup_{P\in\mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})} |\beta(G;P)| \\ &= \sup_{d;\|\psi\|=1,\{A_x^a\},\{B_y^b\}POVM's} \left| \sum_{x,y=1}^N \pi(x,y)(-1)^{c_{xy}} \langle\psi|\left(A_x^0 - A_x^1\right)\otimes\left(B_y^0 - B_y^1\right)|\psi\rangle \right|.\end{aligned}$$

Where the supremum above is taken over integers d, unit vectors $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and POVM's on $\mathbb{C}^d$.

Similarly to the classical bias, we can relate the entangled bias to a certain operator space norm, namely:

Proposition 3.1.5. *Let G be a two-player XOR game, then its entangled bias is given by*

$$\beta^*(G) = \sup\left\{\left\|\sum_{x,y=1}^N \pi(x,y)(-1)^{c_{xy}} A_x \otimes B_y\right\|\right\},$$

where the supremum is taken over all integers $d \in \mathbb{N}$ and selfadjoint elements $A, B \in \ell_\infty^N(M_d)$ of norms less than or equal to 1.

Proof. Fix an integer d and a unit vector $|\psi\rangle \in \mathbb{C}^d$. If we set $A_x = A_x^0 - A_x^1$ then this is selfadjoint since A_x^0, A_x^1 are selfadjoint. Also, $\mathrm{Id} \geq A_x^0 + A_x^1 \geq A_x^a$, for both $a = 0, 1$ since A_x^a positive, which leads us to $\langle h, A_x^a h\rangle \leq \|h\|^2$ for each $h \in \mathbb{C}^d$. By Proposition A.2.32, taking the supremum over all $\|h\| = 1$ assures us that $\left\|A_x^a\right\| = \sup_{\|h\|=1} \left|\langle h, A_x^a h\rangle\right| \leq 1$. Hence,

$$\beta^*(G) \leq \sup_{d;\|\psi\|=1, A,B\in \mathrm{Ball}(\ell_\infty^N(M_d)): A_x = A_x^*, B_y = B_y^*} \left\{ \left| \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} \langle \psi | A_x \otimes B_y | \psi\rangle \right| \right\}.$$

To prove the reverse inequality, let A_x be a selfadjoint element in M_d with $\|A_x\| \leq 1$ and notice that we can write $A_x = (A_x + \mathrm{Id})/2 - (\mathrm{Id} - A_x)/2$. Now set $A_x^0 = (A_x + \mathrm{Id})/2$ and $A_x^1 = (\mathrm{Id} - A_x)/2$ and observe that they add up to the identity and they are positive: Indeed, let $h \in \mathbb{C}^d$,

$$\left\langle h, A_x^0 h\right\rangle = \left\langle h, \frac{(A_x + \mathrm{Id})}{2} h\right\rangle = \frac{1}{2}\langle h, h\rangle + \frac{1}{2}\langle h, A_x h\rangle.$$

If $\langle h, A_x h\rangle \geq 0$ then, $\left\langle h, A_x^0 h\right\rangle \geq 0$. If $\langle h, A_x h\rangle \leq 0$, then $-\langle h, A_x h\rangle = |\langle h, A_x h\rangle| \leq \|h\|^2 = \langle h, h\rangle$. In a similar fashion we see that $A_x^1 \geq 0$ and thus we get the equality.

Finally, we have that

$$\begin{aligned}
&\sup_{d;|\psi\rangle\in \mathrm{Ball}(\mathbb{C}^d\otimes\mathbb{C}^d), \{A_x^a\}, \{B_y^b\} POVM's} \left| \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} \langle \psi | \left(A_x^0 - A_x^1\right) \otimes \left(B_y^0 - B_y^1\right) | \psi\rangle \right| \\
&= \sup_{d;\|\psi\|=1, A,B\in \mathrm{Ball}(\ell_\infty^N(M_d)): A_x = A_x^*, B_y = B_y^*} \left\{ \left| \langle \psi | \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} A_x \otimes B_y | \psi\rangle \right| \right\} \\
&= \sup_{d; A,B\in \mathrm{Ball}(\ell_\infty^N(M_d)): A_x = A_x^*, B_y = B_y^*} \left\| \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} A_x \otimes B_y \right\|.
\end{aligned}$$

where the last equality comes from the fact that if A_x and B_y are selfadjoint operators on $\mathbb{C}^d$, then $A_x \otimes B_y$ is selfadjoint on $\mathbb{C}^d \otimes \mathbb{C}^d$ and consequently so is $\sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} A_x \otimes B_y$. □

Note that in order to compute the entangled bias of an XOR game, we have to take the supremum over selfadjoint operators. However, if we use the map

$$A \mapsto \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

which associates every operator $A \in \mathcal{B}(\mathcal{H})$ to a selfadjoint one in $M_2(\mathcal{B}(\mathcal{H}))$ with the same norm and take advantage of the "freedom" the unrestricted dimension provides us with, we can prove that the supremum is left unchanged. Indeed the assertion follows from this lemma.

Lemma 3.1.6. *Let $c_{xy} \in \mathbb{R}$ for all x, y in some finite sets, we define*

$$a := \sup\left\{ \left| \sum_{x,y} c_{x,y}\langle \xi_1 | A_x \otimes B_y | \xi_2 \rangle \right| : d \in \mathbb{N}, A_x, B_y \in \mathrm{Ball}(\mathcal{B}(\ell_2^{\mathrm{d}})), \xi_i \in \mathrm{Ball}(\ell_2^{\mathrm{d}} \otimes \ell_2^{\mathrm{d}}) \right\}$$

Then, a can be achieved by taking the supremum only over selfadjoint operators.

Proof. Since the space is finite dimensional, the sup is attained (it is a max). So, there exist an integer $d \in \mathbb{N}$, operators $A_x, B_y \in \mathrm{Ball}(\mathcal{B}(\ell_2^{\mathrm{d}}))$ and vectors $\xi_1, \xi_2 \in \mathrm{Ball}(\ell_2^{\mathrm{d}} \otimes \ell_2^{\mathrm{d}})$ so that

$$a = \left| \sum_{x,y} c_{x,y}\langle \xi_1 | A_x \otimes B_y | \xi_2 \rangle \right|$$

Note first that we can pick $\|\xi_1\| = \|\xi_2\| = 1$. Indeed just let $\xi_i' = \frac{\xi_i}{\|\xi_i\|}$ and check that

$$a \geq \left| \sum_{x,y} c_{x,y}\langle \xi_1' | A_x \otimes B_y | \xi_2' \rangle \right| = \frac{1}{\|\xi_1\|}\frac{1}{\|\xi_2\|} \left| \sum_{x,y} c_{x,y}\langle \xi_1 | A_x \otimes B_y | \xi_2 \rangle \right| \geq a$$

Now we rewrite

$$a = \left| \sum_{x,y} c_{x,y}\langle \xi_1 | A_x \otimes B_y | \xi_2 \rangle \right| = \left| \langle \xi_1 | \sum_{x,y} c_{x,y} A_x \otimes B_y | \xi_2 \rangle \right| = |\langle \xi_1 | \Gamma_c | \xi_2 \rangle|$$

where $\Gamma_c := \sum_{x,y} c_{x,y} A_x \otimes B_y$. We also write $\langle \xi_1 | \Gamma_c | \xi_2 \rangle = e^{is} |\langle \xi_1 | \Gamma_c | \xi_2 \rangle|$ and so we get $|\langle \xi_1 | \Gamma_c | \xi_2 \rangle| = e^{-is}\langle \xi_1 | \Gamma_c | \xi_2 \rangle$. Also define

$$D_x := e^{-i\frac{s}{2}} A_x \qquad D_x' := \begin{pmatrix} 0 & D_x^* \\ D_x & 0 \end{pmatrix}$$

$$E_y := e^{-i\frac{s}{2}} B_y \qquad E_y' := \begin{pmatrix} 0 & E_y^* \\ E_y & 0 \end{pmatrix}$$

obviously D_x has the same norm with A_x and E_y with B_y but also the same holds for D'_x and E'_y because applying a permutation and using Ruan's axioms gives $\left\|D'_x\right\| = \max\{\|D_x\|, \left\|D^*_x\right\|\} = \|D_x\| = \|A_x\|$. If we suppose now that $\xi_1 = \sum_i x_i \otimes y_i \in \ell_2^{\mathrm{d}} \otimes \ell_2^{\mathrm{d}}$ and $\xi_2 = \sum_j z_j \otimes w_j \in \ell_2^{\mathrm{d}} \otimes \ell_2^{\mathrm{d}}$ are unit vectors then

$$\xi'_1 = \sum_i \begin{pmatrix} 0 \\ x_i \end{pmatrix} \otimes \begin{pmatrix} 0 \\ y_i \end{pmatrix} \in \ell_2^{2d} \otimes \ell_2^{2d} \qquad \xi'_2 = \sum_j \begin{pmatrix} z_j \\ 0 \end{pmatrix} \otimes \begin{pmatrix} w_j \\ 0 \end{pmatrix} \in \ell_2^{2d} \otimes \ell_2^{2d}$$

are also norm one vectors, in fact we have

$$\begin{aligned}
\left\|\xi'_2\right\|^2 &= \langle \xi'_2, \xi'_2 \rangle \\
&= \sum_{i,j} \left\langle \begin{pmatrix} z_i \\ 0 \end{pmatrix}, \begin{pmatrix} z_j \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} w_i \\ 0 \end{pmatrix}, \begin{pmatrix} w_j \\ 0 \end{pmatrix} \right\rangle \\
&= \sum_{i,j} \left\langle z_i, z_j \right\rangle \left\langle w_i, w_j \right\rangle \\
&= \langle \xi_2, \xi_2 \rangle \\
&= \|\xi_2\|^2 = 1
\end{aligned}$$

and of course $\left\|\xi'_1\right\| = \|\xi_1\|$. Now observe that

$$\langle \xi_1 | D_x \otimes E_y | \xi_2 \rangle = \sum_{i,j} \left\langle x_i \otimes y_i | D_x \otimes E_y | z_j \otimes w_j \right\rangle = \sum_{i,j} \left\langle x_i, D_x z_j \right\rangle \left\langle y_i, E_y w_j \right\rangle$$

Finally,

$$\begin{aligned}
\langle \xi'_1 | D'_x \otimes E'_y | \xi'_2 \rangle &= \sum_{i,j} \langle \begin{pmatrix} 0 \\ x_i \end{pmatrix} \otimes \begin{pmatrix} 0 \\ y_i \end{pmatrix} | \begin{pmatrix} 0 & D^*_x \\ D_x & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & E^*_y \\ E_y & 0 \end{pmatrix} | \begin{pmatrix} z_j \\ 0 \end{pmatrix} \otimes \begin{pmatrix} w_j \\ 0 \end{pmatrix} \rangle \\
&= \sum_{i,j} \left\langle \begin{pmatrix} 0 \\ x_i \end{pmatrix} \otimes \begin{pmatrix} 0 \\ y_i \end{pmatrix}, \begin{pmatrix} 0 \\ D_x z_j \end{pmatrix} \otimes \begin{pmatrix} 0 \\ E_y w_j \end{pmatrix} \right\rangle \\
&= \sum_{i,j} \left\langle \begin{pmatrix} 0 \\ x_i \end{pmatrix}, \begin{pmatrix} 0 \\ D_x z_j \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ y_i \end{pmatrix}, \begin{pmatrix} 0 \\ E_y w_j \end{pmatrix} \right\rangle \\
&= \sum_{i,j} \left\langle x_i, D_x z_j \right\rangle \left\langle y_i, E_y w_j \right\rangle \\
&= \langle \xi_1 | D_x \otimes E_y | \xi_2 \rangle
\end{aligned}$$

and if we sum for all x's and y's and multiply by the coefficients c_{xy} we obtain

$$\sum_{x,y} c_{xy}\langle \xi_1'|D_x'\otimes E_y'|\xi_2'\rangle = \sum_{x,y} c_{xy}\langle \xi_1|D_x\otimes E_y|\xi_2\rangle = \sum_{x,y} c_{xy}\langle \xi_1|e^{-is}A_x\otimes B_y|\xi_2\rangle$$
$$= \langle \xi_1|e^{-is}\sum_{x,y} c_{xy}A_x\otimes B_y|\xi_2\rangle = \langle \xi_1|e^{-is}\Gamma_c|\xi_2\rangle = |\langle \xi_1|\Gamma_c|\xi_2\rangle| = \alpha$$

□

Again, for any XOR game G recall its associated bilinear form

$$G \in \mathcal{B}(\ell_\infty^N(\mathbb{R})\times \ell_\infty^N(\mathbb{R}), \mathbb{R})$$

defined by $G(e_x, e_y) = \pi(x,y)(-1)^{c_{xy}}$ and its associated tensor

$$\hat{G} = \sum_{x,y=1}^{N} \pi(x,y)(-1)^{c_{xy}} e_x\otimes e_y \in \ell_1^N(\mathbb{R})\otimes \ell_1^N(\mathbb{R}),$$

and so we end up with the following result:

Corollary 3.1.7. *Let G be a two-player XOR game, then its entangled values is given by*

$$\beta^*(G) = \|G\|_{cb(\ell_\infty^N(\mathbb{C})\times\ell_\infty^N(\mathbb{C}))}, \tag{3.1.4}$$

and by the identity 1.2.14 we also have

$$\beta^*(G) = \left\|\hat{G}\right\|_{\ell_1^N\otimes_{min}\ell_1^N} \tag{3.1.5}$$

Proof. Indeed, by the previous lemma we have that

$$\beta^*(G) = \sup_{d;A,B\in \mathrm{Ball}(\ell_\infty^N(M_d))}\left\{\left\|\sum_{x,y=1}^{N}\pi(x,y)(-1)^{c_{xy}}A_x\otimes B_y\right\|\right\},$$

so by the isometric identifications $\ell_\infty^N(M_d)\cong \ell_\infty^N\otimes_{min} M_d \cong M_d(\ell_\infty^N)$ we can see that this is exactly the completely bounded norm of its associated bilinear form.

□

Let's return now to the CHSH inequality. This corresponds, in a more general fashion than the original inequality, to the value of the classical bias $\beta(G_{CHSH}) = \|G_{CHSH}\|_{\mathcal{B}(\ell^2_\infty(\mathbb{R})\times\ell^2_\infty(\mathbb{R}),\mathbb{R})}$ where $G_{CHSH}(e_x,e_y) = \frac{1}{4}(-1)^{x\wedge y}$ for $x,y \in \{0,1\}$. Thus,

$$\begin{aligned}
\beta(G_{CHSH}) &= \sup_{A_x,B_y\in \mathrm{Ball}(\mathbb{R})} \left| \sum_{x,y=0}^{1} \frac{1}{4}(-1)^{x\wedge y} A_x B_y \right| \\
&= \sup_{|A_x|,|B_y|\le 1, x,y=0,1} \left| \frac{1}{4}\Big(A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1\Big) \right| \\
&\le \frac{1}{4} \sup_{|A_x|,|B_y|\le 1, x,y=0,1} |A_0B_0 + A_0B_1| + |A_1B_0 - A_1B_1|.
\end{aligned}$$

Proposition 3.1.8 (CHSH)**.** *The CHSH inequality states that*

$$\beta(G_{CHSH}) \le \frac{1}{2}.$$

In fact, we have an equality, since we can chose $A_x = B_y = 1$ for all x, y.

The inequality above follows from the lemma:

Lemma 3.1.9. *For any four numbers $a, b, c, d \in [-1, 1]$, the following inequalities hold*

1. $|a \pm c| \le 1 \pm ac$
2. $|ab \pm cb| \le 1 \pm ac$
3. $|ab - cb| + |ad + dc| \le 2$

Proof. 1. Since $1 \pm ac \ge 0$ for every $a, c \in [-1, 1]$, the claim is equivalent to

$$|a \pm c|^2 = a^2 + c^2 \pm 2ac \le (1 \pm ac)^2 = 1 + a^2c^2 \pm 2ac$$

and this is equivalent to

$$a^2(1 - c^2) \le 1 - c^2$$

which is true since $1 - c^2 \ge 0$. Moreover, equality holds iff $a^2 = 1$ i.e. iff $a = \pm 1$, equivalently, iff $c = \pm 1$ since we can replace a with c.

2. For every $b \in [-1,1]$ we have

$$|ab \pm cb| = |b||a \pm c| \leq |a \pm c| \leq 1 \pm ac$$

from 1).

3. For every $a,b,c,d \in [-1,1]$, using 2) above we have

$$|ac - bc| \leq 1 - ac$$
$$|ad + dc| \leq 1 + ac$$

adding those two gives us the desired inequality. □

The CHSH inequality states that classical bias of the XOR game corresponding to the bilinear form G_{CHSH} is at most $\frac{1}{2}$. However, quantum mechanics predicts a noticeable violation

$$\beta^*(G_{CHSH}) = \sqrt{2}/2.$$

We first prove *Tsirelson's bound*, namely

$$\beta^*(G_{CHSH}) \leq \frac{\sqrt{2}}{2}.$$

Recall that the quantum bias is given by

$$\beta^*(G_{CHSH}) = \sup_{d;A_x,B_y \in \mathrm{Ball}(M_d),x,y=0,1} \left\| \frac{1}{4}\Big(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1\Big) \right\|$$

where the supremum can be restricted to selfadjoint operators.

Here is the original proof of Tsirelson [16]:

Proposition 3.1.10. *Let A_0, A_1, B_0, B_1 be selfadjoint operators of norm less than or equal to 1, such that $[A_x, B_y] = 0$ for every $x, y = 0, 1$. Then*

$$A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \leq 2\sqrt{2}\,\mathrm{Id}$$

Proof. Let A_0, A_1, B_0, B_1 be as in the statement then

$$\begin{aligned}
A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 &= \frac{1}{\sqrt{2}}(A_0^2 + A_1^2 + B_0^2 + B_1^2) \\
&- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_0 - B_0) + A_1 - B_1\Big)^2 \\
&- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_0 - B_1) - A_1 - B_0\Big)^2 \\
&- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_1 + B_0) + A_1 + B_1\Big)^2 \\
&- \frac{\sqrt{2}-1}{8}\Big((\sqrt{2}+1)(A_1 + B_1) - A_1 - B_1\Big)^2 \\
&\leq \frac{1}{\sqrt{2}}(A_0^2 + A_1^2 + B_0^2 + B_1^2) \\
&\leq 2\sqrt{2}\,\mathrm{Id}
\end{aligned}$$

□

If we are given selfadjoint operators A_0, A_1, B_0, B_1 of norm lower than or equal to 1, then the operators $A_0 \otimes \mathrm{Id}, A_1 \otimes \mathrm{Id}, \mathrm{Id} \otimes B_0, \mathrm{Id} \otimes B_1$ satisfy the requirements of the previous proposition and hence,

$$\|A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1\| \leq 2\sqrt{2}$$

which shows Tsirelson's bound.

However, in order to obtain a violation we have to prove the equality. For this, we will use this definition of the quantum bias:

$$\beta^*(G_{CHSH}) = \sup_{d;A,B \in \mathrm{Ball}(\ell_\infty^2(M_d)):A_x = A_x^*, B_y = B_y^*} \left| \sum_{x,y=0}^{1} \frac{1}{4}(-1)^{x \wedge y} \langle \psi | A_x \otimes B_y | \psi \rangle \right|.$$

So, if we recall the relation 0.0.4 that we derived in the Introduction, this leads us to

$$\begin{aligned}
\beta^*(G_{CHSH}) &\geq \frac{1}{4}\Big|\langle \psi | A_0 \otimes B_0 | \psi \rangle + \langle \psi | A_0 \otimes B_1 | \psi \rangle + \langle \psi | A_1 \otimes B_0 | \psi \rangle - \langle \psi | A_1 \otimes B_1 | \psi \rangle\Big| \\
&= \frac{1}{4}\left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right| \\
&= \frac{\sqrt{2}}{2}.
\end{aligned}$$

Finally, $\beta(G_{CHSH}) \leq \frac{1}{2} < \frac{\sqrt{2}}{2} = \beta^*(G_{CHSH})$ and hence the bell functional G_{CHSH} serves as a "witness" to the fact that quantum conditional distributions are strictly more than the classical ones, i.e.

$$\mathcal{P}_C(\mathbf{AB}|\mathbf{XY}) \subsetneq \mathcal{P}_Q(\mathbf{AB}|\mathbf{XY}).$$

Also, recall that the value of a game G classical or quantum, is the maximum probability that Alice and Bob win against the referee, for some particular strategy. Thus, in the CHSH game we see that

$$\omega(G) = \frac{1}{2} + \frac{\beta(G)}{2} \leq \frac{3}{4} = 0.75$$

i.e. the probability is at most 0.75 using classical resources, while

$$\omega^*(G) = \frac{1}{2} + \frac{\beta^*(G)}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4} \approx 0.853$$

i.e. the probability is at most approximately 0.853 using "quantum" resources.

To conclude the prologue, we observe that there is no difference of substance between the viewpoints of Bell functionals and of games. Indeed, we turned XOR games into Bell functionals and used the framework of the latter to come to conclusions concerning the values of the games. However this correspondence goes both ways, to any tensor $G \in \ell_1^N \otimes \ell_1^N$ with real coefficients that satisfies the normalization condition $\sum_{x,y} \left| G_{x,y} \right| = 1$, we may associate an XOR game by defining $\pi(x,y) = \left| G_{x,y} \right|$ and $(-1)^{c_{xy}} = sign(G_{x,y})$. In particular, any Bell functional $M : \ell_\infty^N \times \ell_\infty^N \to \mathbb{R}$ can, up to normalization, be made into an equivalent XOR game.

3.2 Grothendieck's theorem as a fundamental limit of nonlocality

In his "Résumé" [17] Grothendieck proved a theorem that he called “Théorème fondamental de la théorie des espaces métriques". That theorem plays an important role in the study of Quantum Nonlocality as it provides us with an upper bound on how large the violation of a bipartite Bell inequality can be. Tsirelson was the first to point out this connection. For an extensive discussion of the Grothendieck's theorem and its generalizations, we refer to the survey [18] . As for our case, recall the explicit form that we gave to the γ_2^* tensor norm for the space $\ell_1^N(\mathbb{C})\otimes\ell_1^N(\mathbb{C})$ (see 1.1.28)

$$\left\|\sum_{x,y=1}^{N} c_{xy} e_x \otimes e_y\right\|_{\ell_1^N\otimes_{\gamma_2^*}\ell_1^N} = \sup\left\{\left|\sum_{x,y=1}^{N} c_{xy} a_x\cdot b_y\right| : d\in\mathbb{N}, a_x, b_y\in \mathrm{Ball}(\mathbb{C}^d)\right\}$$

where, without loss of generality, we can restrict the supremum on d to $d\le N$ and $a_x\cdot b_y = \left\langle \bar{a}_x, b_y\right\rangle$. Also, we define the real γ_2^* tensor norm for the space $\ell_1^N(\mathbb{R})\otimes\ell_1^N(\mathbb{R})$ similarly. Grothendieck's theorem states the following.

Theorem 3.2.1 (Grothendieck's inequality)**.** *There exist universal constants $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ such that for any integers N,M and $C_1\in\mathbb{R}^N\otimes\mathbb{R}^M$, $C_2\in\mathbb{C}^N\otimes\mathbb{C}^M$,*

- $\|C_1\|_{\ell_1^N(\mathbb{R})\otimes_{\gamma_2^*}\ell_1^M(\mathbb{R})} \le K_G^{\mathbb{R}}\|C_1\|_{\ell_1^N(\mathbb{R})\otimes_\epsilon\ell_1^M(\mathbb{R})}$
- $\|C_2\|_{\ell_1^N(\mathbb{C})\otimes_{\gamma_2^*}\ell_1^M(\mathbb{C})} \le K_G^{\mathbb{C}}\|C_2\|_{\ell_1^N(\mathbb{C})\otimes_\epsilon\ell_1^M(\mathbb{C})}$

The precise values of $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are unknown, although it is known that

$$1 < K_G^{\mathbb{C}} < K_G^{\mathbb{R}} < \frac{\pi}{2(\log(1+\sqrt{2}))} \approx 1.782$$

Recalling the relation between the classical bias and the injective tensor norm (3.1.2), namely,

$$\beta(G) = \left\|\hat{G}\right\|_{\ell_1^N(\mathbb{R})\otimes_\epsilon\ell_1^N(\mathbb{R})}$$

already gives us a hint on how to proceed, that is, relating the entangled bias (which equals to the minimal tensor norm by 3.1.5) to the γ_2^* tensor norm. It is Tsirelson again, who made that crucial observation. Tsirelson [16] further showed that the inequality above becomes an equality for the case of real tensors $\hat{G}$. We will present here a proof based on Tsirelson's argument in his original work.

Let $\hat{G}$ the associated tensor of a game G, then

$$\|\hat{G}\|_{\ell_1^N \otimes_{min} \ell_1^N} = \sup_{d; A_x, B_y \in \mathrm{Ball}(M_d(\mathbb{C}))} \left\| \sum_{x,y} \hat{G}_{xy} A_x \otimes B_y \right\|$$
$$= \sup_{d; |\psi\rangle \in \mathrm{Ball}(\mathbb{C}^d \otimes \mathbb{C}^d), A_x, B_y \in \mathrm{Ball}(M_d(\mathbb{C}))} \left| \sum_{x,y} \hat{G}_{xy} \langle \psi | A_x \otimes B_y | \psi \rangle \right|$$

We have the following observation:

Proposition 3.2.2. *Let G be a two-player XOR game and $\hat{G}$ its associated tensor, then*

$$\|\hat{G}\|_{\ell_1^N \otimes_{min} \ell_1^N} \leq \sup_{d \in \mathbb{N}, a_x, b_y \in \mathrm{Ball}(\mathbb{R}^d)} \left| \sum_{x,y=1}^{N} \hat{G}_{xy} a_x \cdot b_y \right| = \|\hat{G}\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}.$$

Proof. Fix an integer $d \in \mathbb{N}$, and let $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be a unit vector and matrices $A_x, B_y \in \mathrm{Ball}(M_d(\mathbb{C}))$ for every x, y. Observe that the vectors

$$|a_x\rangle = A_x \otimes \mathrm{Id}\, |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \quad \text{and} \quad |b_y\rangle = \mathrm{Id} \otimes B_y |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$$

satisfy $\langle a_x, b_y \rangle = \langle \psi | A_x \otimes B_y | \psi \rangle$. It is clear that these vectors are in the unit ball of $\mathbb{C}^d \otimes \mathbb{C}^d$. However, they could be complex vectors. The key point is that we know that $\langle \psi | A_x \otimes B_y | \psi \rangle \in \mathbb{R}$ due to the fact that A_x, B_y can be chosen to be selfadjoint. So, if we define

$$|\tilde{a_x}\rangle = \mathrm{Re}(|a_x\rangle) \oplus \mathrm{Im}(|a_x\rangle) \quad \text{and} \quad |\tilde{b_y}\rangle = \mathrm{Re}(|b_y\rangle) \oplus \mathrm{Im}(|b_y\rangle)$$

we obtain real unit vectors such that

$$\tilde{a_x} \cdot \tilde{b_y} = \langle \tilde{a_x}, \tilde{b_y} \rangle = \mathrm{Re}(\langle a_x, b_y \rangle) = \langle a_x, b_y \rangle$$

Finally, note that $|a_x\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \cong \mathbb{C}^{d^2}$ and thus $|\tilde{a_x}\rangle \in \mathbb{R}^{d^2} \oplus \mathbb{R}^{d^2} \cong \mathbb{R}^{2d^2}$, so we take advantage of the unrestricted dimension and obtain the inequality. □

Definition 3.2.3. *A family of $n \times n$ selfadjoint unitary matrices that anticommute will be called a family of Clifford unitaries. That is, a set of matrices $\{X_1, \ldots, X_m\}$ is a family of Clifford unitaries if $X_i^* = X_i$, $\forall i = 1, \ldots, m$ and $X_i X_j = -X_j X_i$, $\forall i \neq j$.*

Example 3.2.4. *Such a set indeed exists. Let*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so we have

$$XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad ZX = -XZ$$

This is a set of $m = 2$ Clifford unitaries. We will now construct a set of $m > 2$ such matrices. We set

$$\begin{aligned}
C_1 &= Z \otimes I_2 \otimes \cdots \otimes I_2 \in \otimes_{i=1}^m M_2 \\
C_2 &= X \otimes Z \otimes I_2 \otimes \cdots \otimes I_2 \in \otimes_{i=1}^m M_2 \\
&\vdots \\
C_i &= \otimes_{k=1}^{i-1} X \otimes Z \otimes \otimes_{k=1}^{m-i} I_2 \in \otimes_{i=1}^m M_2
\end{aligned}$$

Then we have that $C_i^* = C_i$ and $C_i^2 = \mathrm{Id}$ for all $i = 1, \ldots, m$ and also notice that if $i < j$

$$\begin{aligned}
C_i C_j &= X^2 \otimes \cdots \otimes X^2 \otimes ZX \otimes X \otimes \cdots \otimes X \otimes I_2 \otimes \cdots \otimes I_2 \\
C_j C_i &= X^2 \otimes \cdots \otimes X^2 \otimes XZ \otimes X \otimes \cdots \otimes X \otimes I_2 \otimes \cdots \otimes I_2
\end{aligned}$$

hence $C_i C_j = -C_j C_i$ whenever $i \neq j$. Note that all the C_i's are real matrices, so $C_i = C_i^T$. To conclude, we can choose m such real matrices, each in $\otimes_{i=1}^m M_2 \cong M_{2^m}$.

Theorem 3.2.5. *Let $a_x, b_y \in \mathbb{R}^d$ with $\|a_x\| \leq 1$ and $\left\|b_y\right\| \leq 1$ for all $x, y = 1, \ldots, N$, then there exists an integer $n \in \mathbb{N}$, a state $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^n \otimes \mathbb{C}^n)$ and observables $A_x, B_y \in \mathrm{Ball}(M_n)$ such that $a_x \cdot b_y = \langle \psi | A_x \otimes B_y | \psi \rangle$. In particular the dimension is $n = 2^d$ and the state is maximally entangled.*

Proof. Write $a_x = (a_x(1), \ldots, a_x(d))$ and $b_y = (b_y(1), \ldots, b_y(d))$ and so we have $a_x \cdot b_y = \sum_{i=1}^d a_x(i) b_y(i)$. Now pick d Clifford unitaries, as constructed above, $\{C_1, \ldots, C_d\} \subseteq M_n$ (actually $n = 2^d$) and set $A_x = \sum_{i=1}^d a_x(i)C_i$, $B_y = \sum_{i=1}^d b_y(i)C_i$. Obviously $A_x^* = A_x$, $B_y^* = B_y$ and

$$A_x^2 = \sum_{i,j=1}^d a_x(i)a_x(j)C_iC_j = \sum_{i=1}^d a_x(i)^2C_i^2 + \sum_{i<j}(a_x(i)a_x(j) - a_x(i)a_x(j))C_iC_j = \|a_x\|^2 \,\mathrm{Id}$$

hence $A_x^2 \leq \mathrm{Id}$ so that $\|A_x\| \leq 1$. The same applies for every B_y. Now note that

$$A_xB_y^T = \sum_{i,j=1}^d a_x(i)b_y(j)C_iC_j^T = \sum_{i=1}^d a_x(i)b_y(i)C_iC_i^T + \sum_{i\neq j} a_x(i)b_y(j)C_iC_j^T$$

and also that $\mathrm{Tr}\left(C_iC_j\right) = \mathrm{Tr}\left(C_jC_i\right)$ and since $C_iC_j = -C_jC_i$ we have that $\mathrm{Tr}\left(C_iC_j\right) = 0$ whenever $i \neq j$. Taking into account that $C_i = C_i^T$ we obtain

$$\mathrm{Tr}\left(A_xB_y^T\right) = \mathrm{Tr}\left(\sum_{i=1}^d a_x(i)b_y(i)C_i^2\right) = \sum_{i=1}^d a_x(i)b_y(i)\mathrm{Tr}\,(\mathrm{Id}) = n \cdot \sum_{i=1}^d a_x(i)b_y(i).$$

Finally, pick the maximally entangled state $\psi_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n |ii\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ and observe that

$$\langle \psi_n | A_x \otimes B_y | \psi_n \rangle = \frac{1}{n}\mathrm{Tr}\left(A_xB_y^T\right).$$

Hence,

$$a_x \cdot b_y = \sum_{i=1}^d a_x(i)b_y(i) = \frac{1}{n}\mathrm{Tr}\left(A_xB_y^T\right) = \langle \psi_n | A_x \otimes B_y | \psi_n \rangle.$$

□

We thus reached the following conclusion:

Corollary 3.2.6. *Let G be a two-player XOR game and $\hat{G}$ its associated tensor, then*

$$\left\|\hat{G}\right\|_{\ell_1^N \otimes_{min} \ell_1^N} = \left\|\hat{G}\right\|_{\ell_1^N(\mathbb{R}) \otimes_{\gamma_2^*} \ell_1^N(\mathbb{R})}.$$

Combining the correspondence between the tensor norms and biases, as expressed by the relations 3.1.2 and 3.1.5 with Grothendieck's fundamental inequality gives us the following results for nonlocality.

Corollary 3.2.7. *Let G be a two-player XOR game. The largest bias achievable by entangled players is bounded as follows*

$$\beta^*(G) \leq K_G^{\mathbb{R}} \beta(G).$$

Recall that the CHSH violation predicts a violation of size

$$\frac{\beta^*(G_{CHSH})}{\beta(G_{CHSH})} = \sqrt{2} \approx 1.414 < 1.782$$

which is close to being optimal.

Tsirelson's characterization of the entangled bias leads to a quantitative understanding (see section 5.1) of the amount of entanglement needed to play optimally, in an XOR game. That is, using that the supremum in the entangled bias is achieved by unit vectors a_x, b_y of dimension $d \leq N$, Proposition 3.2.5 says that an optimal strategy for the players can be achieved by observables of dimension 2^d together with a maximally entangled state of the same dimension. It is worth noting that Tsirelson's original version actually uses Clifford matrices of dimension $2^{\lfloor d/2 \rfloor}$ [3] which can be constructed using the Pauli matrices. For such a construction and proof of the result see [19].

3.3 Three-player XOR games: Unbounded violations

Grothendieck's fundamental theorem, together with the correspondence established in the previous section, between the classical and entangled biases of a two-player XOR game, and the bounded and completely bounded norm of the associated bilinear form, respectively, sets an upper bound on how large the violation of an XOR game between two players, can be. Motivated by this crucial result for nonlocality, and its potential extension

[3] Where for every positive real number r, $\lfloor r \rfloor$ denotes the largest natural number z such that $z \leq r$.

in the tripartite case, Perez-Garcia et al. [7] proved that in the case of three-player XOR games, the following result holds:

Theorem 3.3.1. *For every integer n, there exists an N and a trilinear form* $T : \ell_\infty^{2^{n^2}} \otimes \ell_\infty^{2^{N^2}} \otimes \ell_\infty^{2^{N^2}} \to \mathbb{C}$ *such that*

$$\|T\|_{cb(\ell_\infty^{2^{n^2}} \times \ell_\infty^{2^{N^2}} \times \ell_\infty^{2^{N^2}})} \geq \left\|T \otimes \mathrm{Id}_{M_n} \otimes \mathrm{Id}_{M_N} \otimes \mathrm{Id}_{M_N}\right\| = \Omega(\sqrt{n})\|T\|_{\mathcal{B}(\ell_\infty^{2^{n^2}} \times \ell_\infty^{2^{N^2}} \times \ell_\infty^{2^{N^2}})}.$$

By $f(n) = \Omega(g(n))$ we mean that for large n values, $f(n)$ is at least a constant multiple of $g(n)$. One can verify that we can extend the connections, made for bilinear forms, between the bounded and completely bounded norms and the injective and minimal tensor norms, respectively, to the case of multilinear forms. This means that the separation between the injective and the minimal tensor norms can be interpreted as the absence of a tripartite generalization of Grothendieck's inequality. Furthermore, J.Briët and T.Vidick improved the above result by giving a much more explicit construction (Theorem 3.3.1 only guarantees its existence) of a family of three-player XOR games which achieve a large quantum-classical gap. Here is the result, that improves the parameters, stated in terms of the biases of three-player XOR games.

Theorem 3.3.2 ([20]). *For any integer n and* $N = 2^n$ *there exists a three-player XOR game* G_N*, with* N^2 *questions per player, such that*

$$\beta^*(G_N) \geq \Omega(\sqrt{N}\log^{-\frac{5}{2}} N)\beta(G_N).$$

Moreover, there is an entangled strategy which achieves a bias of

$$\Omega(\sqrt{N}\log^{-\frac{5}{2}} N)\beta(G_N),$$

uses an entangled state of local dimension N per player, and in which the players observables are tensor products of n Pauli matrices.

Thus these results imply that tripartite Bell inequalities can be violated by unbounded amounts in quantum mechanics. Both theorems extend to any number $k \geq 3$ of players, giving a violation of order $\tilde{\Omega}(n^{\frac{k-2}{2}})$, where $\tilde{\Omega}$ ignores polylogarithmic factors, i.e. $\tilde{\Omega}(h(n)) = \Omega(h(n)\log^k(h(n))$ for some k.

Nevertheless, the violations cannot grow arbitrarily fast, as J.Briët and T.Vidick [20] also proved, it cannot exceed the product of the square roots of the dimension of each space.

Proposition 3.3.3. *If G is a r-player XOR game in which at least r-2 of the players have at most Q possible questions each, then*

$$\beta^*(G) \leq O(\sqrt{Q^{r-2}})\beta(G).$$

Measuring nonlocality via tensor norms

In the previous chapter, we explored the class of XOR games and especially their convenient connection between the biases of the games and certain Banach space and operator space norms. The goal now is to extend these connections for the general setting of two-player games, further demonstrating that operator spaces provide a natural framework for the study of their nonlocal properties. Finally, in the last section we describe some upper and lower bounds on the maximum ratio ω^*/ω, both for the case of games and of Bell functionals.

4.1 Two-player games

Given a two-player game $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$, a classical strategy for the players is described by an element of $\mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$ (see 2.2.2), that is, the convex hull of the set of product distributions, where $\mathbf{A}$ and $\mathbf{B}$ denote the sets of answers and $\mathbf{X}$ and $\mathbf{Y}$ the sets of questions. For simplicity we assume that $\mathbf{A} = \mathbf{B} = \{1, \ldots, K\}$ and that $\mathbf{X} = \mathbf{Y} = \{1, \ldots, N\}$. In this senario we can denote the classical and quantum sets of distributions by $\mathcal{P}_C(K|N)$ and $\mathcal{P}_Q(K|N)$ respectively [1].

The normalization condition $\sup_x \sum_a |P(a|x)| \leq 1$ suggests the introduction of the space $\ell_\infty^N(\ell_1^K)$.

[1]Many authors use instead the notation $C_{loc}(N,K)$ and $C_q(N,K)$.

Definition 4.1.1. *The Banach space* $\ell_\infty^N(\ell_1^K)$ *is defined as* $(\mathbb{C}^{NK}, \|\cdot\|_{\infty,1})$ *equipped with the norm*

$$\left\|(R(x,a))_{x,a}\right\|_{\infty,1} := \sup_{x=1,\dots,N} \sum_{a=1}^{K} |R(x,a)|\,.$$

In fact, the correspondence between bilinear forms and tensors invites us to consider the dual space $(\ell_\infty^N(\ell_1^K))^* = \ell_1^N(\ell_\infty^K)$,

Definition 4.1.2. *The Banach space* $\ell_1^N(\ell_\infty^K)$ *is the space* $(\mathbb{C}^{NK}, \|\cdot\|_{1,\infty})$, *where*

$$\left\|(R(x,a))_{x,a}\right\|_{1,\infty} := \sum_{x=1}^{N} \sup_{a=1,\dots,K} |R(x,a)|\,.$$

If we now recall the definition of Bell functionals and their action on classical conditional distributions, we see that a game can be interpreted as a bilinear form $G : \ell_\infty^N(\ell_1^K) \times \ell_\infty^N(\ell_1^K) \to \mathbb{C}$ defined by

$$G(P,Q) = \sum_{x,y;a,b} G_{x,y}^{a,b} P(x,a) Q(y,b),$$

where $G_{x,y}^{a,b} = \pi(x,y)V(a,b,x,y)$, while its norm is given by

$$\|G\| = \sup\left\{ \left| \sum_{x,y;a,b} G_{x,y}^{a,b} P(x,a) Q(y,b) \right| : \|P\|_{\infty,1} \leq 1, \|Q\|_{\infty,1} \leq 1 \right\}. \tag{4.1.1}$$

So it's natural to consider the associated tensor

$$\hat{G} = \sum_{x,y;a,b} G_{x,y}^{a,b} (e_x \otimes e'_a) \otimes (e_y \otimes e'_b) \in \ell_1^N(\ell_\infty^K) \otimes \ell_1^N(\ell_\infty^K)$$

which according to the relation 1.1.6 satisfies

$$\|G\| = \|\hat{G}\|_{\ell_1^N(\ell_\infty^K) \otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Here, $\{e_i\}_{i=1}^N$ denotes the canonical basis in ℓ_1^N and $\{e'_j\}_{j=1}^K$ denotes the canonical basis in ℓ_∞^K.

While formula 4.1.1 and the definition of the classical value of a game G (see 2.3.1)

$$\omega(G) := \sup_{P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})} \left| \sum_{x,y;a,b} G_{x,y}^{a,b} P(a,b|x,y) \right|$$

make it clear that $\omega(G) \leq \|G\|$, the equality between these quantities does not hold in general since the space $\ell_\infty^N(\ell_1^K)$ allows for elements with complex coefficients. Note that this could indeed happen for general Bell functionals M; however, for the case of a game G these quantities coincide.

Proposition 4.1.3. *Given a two-player game G,*

$$\omega(G) = \left\|\hat{G}\right\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}.$$

Proof. As we know $\|G\| = \|\hat{G}\|_{\ell_1^N(\ell_\infty^K)\otimes_\epsilon \ell_1^N(\ell_\infty^K)}$ and we can easily verify that $\omega(G) \leq \|G\|$. Thus, it suffices to prove that $\|G\| \leq \omega(G)$. So let P, Q such that $\|P\|_{\infty,1} \leq 1, \|Q\|_{\infty,1} \leq 1$ and write

$$\left|\sum_{x,y;a,b} G_{x,y}^{a,b} P(x,a)Q(y,b)\right| \leq \sum_{x,y;a,b} G_{x,y}^{a,b}|P(x,a)||Q(y,b)| \leq \omega(G)$$

where the last inequality follows from the fact that $\left\|(|P(x,a)|)_{x,a}\right\|_{\infty,1} \leq 1$, $\left\|(|Q(y,b)|)_{y,b}\right\|_{\infty,1} \leq 1$ and that G has non-negative coefficients. □

Remark 4.1.4. *One might object that, in the last inequality, we used elements $(P(x,a))_{x,a} \in \ell_\infty^N(\ell_1^K)$ that satisfy the condition $\sup_x \sum_a |P(x,a)| \leq 1$ instead of $\sum_a |P(x,a)| = 1$ for every x, as the definition of conditional distributions dictates. However, we can fix this inconvenience by proving the following:*

$$\|G\| = \sup\left\{\left|\sum_{x,y;a,b} G_{x,y}^{a,b} P(x,a)Q(y,b)\right| : \sum_a |P(x,a)| = 1, \sum_b |Q(y,b)| = 1, \forall x,y\right\}$$

Proof. Let $(P(x,a))_{x,a}$ and $(Q(y,b))_{y,b}$ be non zero elements of $\ell_\infty^N(\ell_1^K)$, such that $\sup_x \sum_a |P(x,a)| \leq 1$ and $\sup_y \sum_b |Q(y,b)| \leq 1$. We set $M_x^P := \sum_a |P(x,a)|$, $M_y^Q := \sum_b |Q(y,b)|$ and note that $M_x^P \leq \sup_x \sum_a |P(x,a)| \leq 1$ and also $M_y^Q \leq \sup_y \sum_b |Q(y,b)| \leq 1$. If we also define $P'(x,a) = \frac{1}{M_x^P}|P(x,a)| \in \mathbb{R}$, $Q'(y,b) =$

$\frac{1}{M_y^Q}|Q(y,b)| \in \mathbb{R}$, then obviously $\sum_a P'(x,a) = 1$ and $\sum_b Q'(y,b) = 1$. Now write

$$\begin{aligned}
\left|\sum_{x,y;a,b} G_{x,y}^{a,b} P(x,a)Q(y,b)\right| &= \left|\sum_{x,y;a,b} M_y^Q M_x^P G_{x,y}^{a,b}\frac{1}{M_x^P}P(x,a)\frac{1}{M_y^Q}Q(y,b)\right| \\
&\leq \sum_{x,y;a,b} M_y^Q M_x^P G_{x,y}^{a,b}\frac{1}{M_x^P}|P(x,a)|\frac{1}{M_y^Q}|Q(y,b)| \\
&= \sum_{x,y;a,b} M_y^Q M_x^P G_{x,y}^{a,b} P'(x,a)Q'(y,b) \\
&\leq \sum_{x,y;a,b} G_{x,y}^{a,b} P'(x,a)Q'(y,b)
\end{aligned}$$

Where the desired assertion follows from the fact that G, P' and Q' have non-negative coefficients. □

We proceed to analyze quantum strategies for the players, i.e. the set $\mathcal{P}_Q(\mathbf{AB}|\mathbf{XY})$ (see 2.2.4), and their relation to the completely bounded norm of $G : \ell_\infty^N(\ell_1^K) \times \ell_\infty^N(\ell_1^K) \to \mathbb{C}$. Towards this end, we need to introduce an o.s.s for the space $\ell_\infty^N(\ell_1^K)$. Using the o.s.s. of ℓ_1^K introduced in 1.2.22 together with Lemma 1.2.9 one can verify that

$$\left\|\sum_{x,a} T_x^a \otimes (e_x \otimes e_a)\right\|_{M_d(\ell_\infty^N(\ell_1^K))} = \sup_x \left\|\sum_a T_x^a \otimes e_a\right\|_{M_d(\ell_1^K)}, d \geq 1 \tag{4.1.2}$$

defines a suitable o.s.s. on $\ell_\infty^N(\ell_1^K)$. Moreover, a corresponding o.s.s. can be placed on $\ell_1^N(\ell_\infty^K) = (\ell_\infty^N(\ell_1^K))^*$ using duality.

Remark 4.1.5. *With the above structures in hand we may express the completely bounded norm of G as*

$$\|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))} := \sup_d \left\|G \otimes \mathrm{Id}_{M_d} \otimes \mathrm{Id}_{M_d}\right\|_{\mathcal{B}(M_d(\ell_\infty^N(\ell_1^K))\times M_d(\ell_\infty^N(\ell_1^K)), M_{d^2})} \tag{4.1.3}$$

$$= \sup \left\|\sum_{x,y;a,b} G_{x,y}^{a,b} T_x^a \otimes S_y^b\right\|_{M_{d^2}} \tag{4.1.4}$$

where the supremum is taken over all integers $d \geq 1$ and matrices $T_x^a, S_y^b \in M_d$ such that

$$\max\left\{\left\|\sum_{x,a} T_x^a \otimes (e_x \otimes e_a)\right\|_{M_d(\ell_\infty^N(\ell_1^K))}, \left\|\sum_{y,b} S_y^b \otimes (e_y \otimes e_b)\right\|_{M_d(\ell_\infty^N(\ell_1^K))}\right\} \leq 1.$$

Using the completely isometric correspondence described in 1.2.14, this norm coincides with the minimal norm of the associated tensor $\hat{G}$

$$\|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))} = \|\hat{G}\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)}.$$

Similarly to the classical value case, it turns out that the entangled value of a two-player game has the "desired" property to equal the minimal tensor norm of the corresponding tensor.

Proposition 4.1.6. *Given a two-player game G,*

$$\omega^*(G) = \|\hat{G}\|_{\ell_1^N(\ell_\infty^K)\otimes_{min}\ell_1^N(\ell_\infty^K)}$$

Proof. Recall first the definition of the entangled value of G

$$\begin{aligned}\omega^*(G) &= \sup_{d,\|\psi\|=1,\{A_x^a\},\{B_y^b\}POVM's}\left|\sum_{x,y;a,b} G_{x,y}^{a,b}\langle\psi|A_x^a\otimes B_y^b|\psi\rangle\right| \\ &= \sup_{d,\{A_x^a\},\{B_y^b\}POVM's}\left\|\sum_{x,y;a,b} G_{x,y}^{a,b} A_x^a \otimes B_y^b\right\|_{M_{d^2}}.\end{aligned}$$

We will show that it equals the completely bounded norm of the associated bilinear form, and consequently, the minimal tensor product of its associated tensor.

- $\omega^*(G) \leq \|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))}$.

Given a family of POVM's $\{A_x^a\}_{a=1}^K$ in M_d, it suffices to prove that for every $x = 1\ldots,N$

$$\left\|\sum_{x,a} A_x^a \otimes (e_x \otimes e_a)\right\|_{M_d(\ell_\infty^N(\ell_1^K))} = 1.$$

According to the equality 4.1.2, this will follow from the fact that $\left\|\sum_a A_x^a \otimes e_a\right\|_{M_d(\ell_1^K)} = 1$, for every x. Indeed, define the map $T^x : \ell_\infty^K \to M_d$, by

$$T^x : v \in \ell_\infty^K \mapsto \sum_a \langle e_a, v\rangle A_x^a \in M_d$$

and recall from 1.2.9 that

$$\left\|\sum_a A_x^a \otimes e_a\right\|_{M_d(\ell_1^K)} = \left\|T^x : \ell_\infty^K \to M_d\right\|_{cb}$$

Finally, the map T^x satisfies $T^x(e_a) = A_x^a$ and so $\|T^x\|_{cb} = 1$ by Proposition 2.2.6 . Using the same arguments for Bob's POVM $\{B_y^b\}_{b=1}^K$ we deduce that

$$\omega^*(G) \leq \|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))}\,.$$

It remains to show the reverse inequality:

- $\omega^*(G) \geq \|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))}\,.$

By the supremum in 4.1.4, given an $\varepsilon > 0$, there exists an integer $d \geq 1$, matrices $T_x^a, S_y^b \in M_d$ satisfying

$$\left\|\sum_{x,a} T_x^a \otimes (e_x \otimes e_a)\right\|_{M_d(\ell_\infty^N(\ell_1^K))} \leq 1, \quad \left\|\sum_{y,b} S_y^b \otimes (e_y \otimes e_b)\right\|_{M_d(\ell_\infty^N(\ell_1^K))} \leq 1$$

and unit vectors $|u\rangle, |v\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ such that

$$\left|\langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} T_x^a \otimes S_y^b |v\rangle\right| > \|G\|_{cb(\ell_\infty^N(\ell_1^K)\times\ell_\infty^N(\ell_1^K))} - \varepsilon.$$

As we already discussed, the condition

$$\left\|\sum_{x,a} T_x^a \otimes (e_x \otimes e_a)\right\|_{M_d(\ell_\infty^N(\ell_1^K))} = \sup_x \left\|\sum_a T_x^a \otimes e_a\right\|_{M_d(\ell_1^K)} \leq 1$$

is equivalent to

$$\left\|T^x : \ell_\infty^K \to M_d\right\|_{cb} \leq 1 \quad \text{for every} \quad x,$$

where $T^x(e_a) = T_x^a$ for every a. The same bound applies to the operators S_y^b. However the operators T_x^a, S_y^b are not necessarily positive or even Hermitian. In order to recover a proper quantum strategy, we appeal to the following.

Theorem 4.1.7. *Let $\mathcal{A}$ be a C^*-algebra with unit and let $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely bounded. Then there exist completely positive maps $\psi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, with $\|\psi_i\| = \|T\|_{cb}$ for $i = 1,2$ such that the map $\Psi : \mathcal{A} \to M_2(\mathcal{B}(\mathcal{H}))$ given by*

$$\Psi(a) = \begin{pmatrix} \psi_1(a) & T(a) \\ T^*(a) & \psi_2(a) \end{pmatrix}, \quad a \in \mathcal{A}$$

is completely positive. Moreover, if $\|T\|_{cb} \le 1$ then we may take ψ_1 , ψ_2 to be unital.

Theorem 4.1.7 is a direct consequence of [9, Theorem 8.3.], where the same statement is proved but with the map Ψ replaced by the map $\eta : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ given by

$$\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \psi_1(a) & T(b) \\ T^*(c) & \psi_2(d) \end{pmatrix}.$$

In fact, the complete positivity of η is equivalent to the complete positivity of Ψ [9, Exercise 8.9].

In our setting, $\mathcal{A} = \ell_\infty^K$ and since this is an abelian C^*-algebra, a map $T : \ell_\infty^K \to \mathcal{B}(\mathcal{H})$ is completely positive if and only if it is positive, that is, $T(a) \in \mathcal{B}(\mathcal{H})$ is positive for every positive element $a \in \ell_\infty^K$. For every $x = 1, \ldots, N$, we apply Theorem 4.1.7 to the map $T^x : \ell_\infty^K \to M_d$ defined previously, and find completely positive and unital maps $\psi_x^i : \ell_\infty^K \to M_d$, $i = 1,2$ such that the map $\Psi_x : \ell_\infty^K \to M_2(M_d)$ defined by

$$\Psi_x(a) = \begin{pmatrix} \psi_x^1(a) & T^x(a) \\ (T^x)^*(a) & \psi_x^2(a) \end{pmatrix}, \quad a \in \ell_\infty^K$$

is completely positive. Similarly, for every $y = 1, \ldots, N$, we define $S^y : \ell_\infty^K \to M_d$ and find completely positive and unital $\varphi_y^i : \ell_\infty^K \to M_d$, $i = 1,2$ and $\Phi_y : \ell_\infty^K \to M_2(M_d)$. Since these maps are positive, the element

$$\Gamma = \sum_{x,y;a,b} G_{x,y}^{a,b} \Psi_x(e_a) \otimes \Phi_y(e_b) \in M_2(M_d) \otimes M_2(M_d)$$

is also positive. Assume that $|u\rangle = \sum_i x_i \otimes y_i$ and $|v\rangle = \sum_j z_j \otimes w_j$ and consider the unit vectors $\tilde{u} = \sum_i \begin{pmatrix} x_i \\ 0 \end{pmatrix} \otimes \begin{pmatrix} y_i \\ 0 \end{pmatrix} \in \mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$ and $\tilde{v} = \sum_j \begin{pmatrix} 0 \\ z_j \end{pmatrix} \otimes \begin{pmatrix} 0 \\ w_j \end{pmatrix} \in \mathbb{C}^{2d} \otimes \mathbb{C}^{2d}$, we have that

$$\left| \langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} T_x^a \otimes S_y^b |v\rangle \right| = |\langle \tilde{u}|\Gamma|\tilde{v}\rangle| \leq |\langle \tilde{u}|\Gamma|\tilde{u}\rangle|^{\frac{1}{2}} \, |\langle \tilde{v}|\Gamma|\tilde{v}\rangle|^{\frac{1}{2}}$$

$$= \left| \langle u| \sum_{x,y;a,b} G_{x,y}^{a,b} \psi_x^1(e_a) \otimes \varphi_y^1(e_b) |u\rangle \right|^{\frac{1}{2}} \left| \langle v| \sum_{x,y;a,b} G_{x,y}^{a,b} \psi_x^2(e_a) \otimes \varphi_y^2(e_b) |v\rangle \right|^{\frac{1}{2}}$$

$$\leq \omega^*(G)$$

where the first inequality follows from the Cauchy-Schwartz inequality and the positivity of the element Γ and the second from the fact that the maps ψ_x^i, φ_y^i are completely positive and unital, for every $x, y = 1, \ldots, N$ and $i = 1, 2$ and thus by the converse part of Proposition 2.2.6 they define POVM's. □

4.2 Bounds on the largest violations achievable in two-player games

As we mentioned previously, the results discussed in Section 4.1 can be analogously stated and proved in the general case where $G = (\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}, \pi, V)$ without assuming the same number of inputs and outputs for Alice and Bob. In particular, given one such game where X,Y,A,B are the cardinalities of $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}$ respectively, we will have

$$\omega(G) = \left\|\hat{G}\right\|_{\ell_1^X(\ell_\infty^A) \otimes_\epsilon \ell_1^Y(\ell_\infty^B)} \quad \text{and} \quad \omega^*(G) = \left\|\hat{G}\right\|_{\ell_1^X(\ell_\infty^A) \otimes_{min} \ell_1^Y(\ell_\infty^B)} \tag{4.2.1}$$

where

$$\hat{G} = \sum_{x,y;a,b} G_{x,y}^{a,b} (e_x \otimes e_a) \otimes (e_y \otimes e_b) \in \mathbb{R}^X \otimes \mathbb{R}^A \otimes \mathbb{R}^Y \otimes \mathbb{R}^B.$$

Considering here this general context is interesting because it will allow us to study the dependence of the upper bounds on the quantity ω^*/ω as a function of the parameters X,A,Y,B.

The proof of the following proposition is a good example of the application of estimates from the theory of operator spaces to bounds on the entangled and classical values of a multiplayer game.

Proposition 4.2.1. *The following inequalities hold for any two-player game G:*

1. $\omega^*(G) \leq \min\{X, Y\}\omega(G)$
2. $\omega^*(G) \leq K_G^{\mathbb{C}} \sqrt{AB}\ \omega(G)$

where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant.

Proof. The proof of each item is based on a different way of bounding the norm of the identity map

$$\mathrm{Id}\otimes\mathrm{Id} : \ell_1^X(\ell_\infty^A)\otimes_\epsilon \ell_1^Y(\ell_\infty^B) \to \ell_1^X(\ell_\infty^A)\otimes_{min} \ell_1^Y(\ell_\infty^B) \tag{4.2.2}$$

Using the relations 4.2.1, any such bound implies the same bound for the ratio ω^*/ω.

1. Without loss of generality we may assume that $X \leq Y$. Recall also that $\ell_1^N(\ell_\infty^K) = (\mathbb{C}^{NK}, \|\cdot\|_{1,\infty})$. The identity map defined above can be decomposed as the sequence

$$\ell_1^X(\ell_\infty^A)\otimes_\epsilon \ell_1^Y(\ell_\infty^B) \to \ell_\infty^{XA}\otimes_\epsilon \ell_1^Y(\ell_\infty^B) \to \ell_\infty^{XA}\otimes_{min} \ell_1^Y(\ell_\infty^B) \to \ell_1^X(\ell_\infty^A)\otimes_{min} \ell_1^Y(\ell_\infty^B)$$

where all arrows correspond to the identity. By Lemma 1.2.9 it follows that the second arrow has norm 1. For the first arrow we write

$$\mathrm{Id}_{XA}\otimes_\epsilon \mathrm{Id}_{YB} : \ell_1^X(\ell_\infty^A)\otimes_\epsilon \ell_1^Y(\ell_\infty^B) \to \ell_\infty^{XA}\otimes_\epsilon \ell_1^Y(\ell_\infty^B)$$

the corresponding identity map. Proposition 1.1.10, tells us that

$$\|\mathrm{Id}_{XA}\otimes_\epsilon \mathrm{Id}_{YB}\| = \left\|\mathrm{Id}_{XA} : \ell_1^X(\ell_\infty^A) \to \ell_\infty^{XA}\right\| \left\|\mathrm{Id}_{YB} : \ell_1^Y(\ell_\infty^B) \to \ell_1^Y(\ell_\infty^B)\right\|$$

and since it is clear that $\|\mathrm{Id}_{YB}\| = 1$ it suffices to check the norm of Id_{XA}. Let $(z_{x,a}) \in \mathbb{C}^{XA}$ with $\left\|(z_{x,a})\right\|_{1,\infty} = \sum_x \sup_a |z_{x,a}| = 1$, it is clear that $\left\|(z_{x,a})\right\|_\infty =$

$\sup_{x,a}|z_{x,a}| \leq \sum_x \sup_a |z_{x,a}| = 1$. Hence, $\left\|\mathrm{Id}_{XA} : \ell_1^X(\ell_\infty^A) \to \ell_\infty^{XA}\right\| \leq 1$. Now lets move on to the third arrow. Let's also denote the third arrow by

$$\mathrm{Id}_{XA} \otimes_{min} \mathrm{Id}_{YB} : \ell_\infty^{XA} \otimes_{min} \ell_1^Y(\ell_\infty^B) \to \ell_1^X(\ell_\infty^A) \otimes_{min} \ell_1^Y(\ell_\infty^B)$$

Again, Proposition 1.1.24 tells us that

$$\|\mathrm{Id}_{XA} \otimes_{min} \mathrm{Id}_{YB}\|_{cb} = \left\|\mathrm{Id}_{XA} : \ell_\infty^{XA} \to \ell_1^X(\ell_\infty^A)\right\|_{cb} \left\|\mathrm{Id}_{YB} : \ell_1^Y(\ell_\infty^B) \to \ell_1^Y(\ell_\infty^B)\right\|_{cb}$$

and since it is clear again that $\left\|\mathrm{Id}_{YB} : \ell_1^Y(\ell_\infty^B) \to \ell_1^Y(\ell_\infty^B)\right\|_{cb} = 1$, we will examine the norm $\left\|\mathrm{Id}_{XA} : \ell_\infty^{XA} \to \ell_1^X(\ell_\infty^A)\right\|_{cb} = \sup_d \left\|\mathrm{Id}_{M_d} \otimes \mathrm{Id}_{XA}\right\|$. Let $d \in \mathbb{N}$, and recall that

$$\left\|\mathrm{Id}_{M_d} \otimes \mathrm{Id}_{XA}\right\| = \sup\{\|u\|_{M_d(\ell_1^X(\ell_\infty^A))} : \|u\|_{M_d(\ell_\infty^{XA})} = 1\}$$

Suppose that $u = \sum_{x,a} A_x^a \otimes (e_x \otimes e_a) \in M_d(\ell_\infty^{XA})$ with

$$\|u\|_{M_d(\ell_\infty^{XA})} = \sup_{x,a} \left\|A_x^a\right\| = 1 \tag{4.2.3}$$

Recall also (1.2.9) that

$$\|u\|_{M_d(\ell_1^X(\ell_\infty^A))} = \|u\|_{M_d((\ell_\infty^X(\ell_1^A))^*)} = \left\|T^u : \ell_\infty^X(\ell_1^A) \to M_d\right\|_{cb} = \left\|T_d^u\right\|_{M_{d^2}}$$

where

$$T^u : v_1 \otimes v_2 \mapsto \sum_{x,a} A_x^a e_x(v_1) e_a(v_2)$$

Thus we need to compute the operator norm of the map

$$T_d^u = \mathrm{Id}_{M_d} \otimes T^u : M_d(\ell_\infty^X(\ell_1^A)) \to M_{d^2}$$

Suppose that we have $\sum_{x,a} E_{x,a} \otimes e_x \otimes e_a \in M_d(\ell_\infty^X(\ell_1^A))$ with norm

$$\left\|\sum_{x,a} E_{x,a} \otimes e_x \otimes e_a\right\|_{M_d(\ell_\infty^X(\ell_1^A))} = \sup_x \left\|\sum_a E_{x,a} \otimes e_a\right\|_{M_d(\ell_1^A)} = 1$$

which we defined in Example 1.2.22 to be

$$\sup_x \left\|\sum_a E_{x,a} \otimes e_a\right\|_{M_d(\ell_1^A)} = \sup \left\|\sum_a E_{x,a} \otimes B_a\right\|_{M_{d^2}}$$

where the supremum runs all over $x \in \mathbf{X}$ and matrices $\{B_a\}_a \subseteq M_d$ such that $\sup_a \|B_a\| \leq 1$. Finally,

$$\begin{aligned}\left\|\mathrm{Id}_{M_d} \otimes T^u(\sum_{x,a} E_{x,a} \otimes (e_x \otimes e_a))\right\|_{M_{d^2}} &= \left\|\sum_{x,a} E_{x,a} \otimes A_x^a\right\|_{M_{d^2}} \\ &\leq \sum_x \left\|\sum_a E_{x,a} \otimes A_x^a\right\|_{M_{d^2}} \\ &\leq \mathrm{X} \sup_x \sup_{\{B_a\}} \left\|\sum_a E_{x,a} \otimes B_a\right\|_{M_{d^2}} = \mathrm{X}\end{aligned}$$

where we used that $\{A_x^a\}_a \subseteq M_d$ satisfy $\sup_a \left\|A_x^a\right\| \leq 1$ which is true by relation 4.2.3.

To complete the proof, just compose all the three maps to get the desired estimate.

2. The proof of the second item makes use of the Fourier transform

$$\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N, \qquad \mathcal{F}_N : e_j \mapsto \sum_{k=1}^N e^{\frac{2\pi i}{N} jk} e_k \quad \forall j = 1, \ldots, N$$

Note that the inverse map is

$$\mathcal{F}_N^{-1} : \mathbb{C}^N \to \mathbb{C}^N, \qquad \mathcal{F}_N^{-1} : e_j \mapsto \frac{1}{N} \sum_{k=1}^N e^{-\frac{2\pi i}{N} jk} e_k \quad \forall j = 1, \ldots, N$$

Then, the identity map 4.2.2 can be decomposed as

$$\ell_1^X(\ell_\infty^A) \otimes_\epsilon \ell_1^Y(\ell_\infty^B) \xrightarrow{\iota_1} \ell_1^{XA} \otimes_\epsilon \ell_1^{YB} \xrightarrow{\iota_2} \ell_1^{XA} \otimes_{min} \ell_1^{YB} \xrightarrow{\iota_3} \ell_1^X(\ell_\infty^A) \otimes_{min} \ell_1^Y(\ell_\infty^B)$$

where

$$\begin{aligned}\iota_1 &= (\mathrm{Id}_X \otimes \mathcal{F}_A) \otimes (\mathrm{Id}_Y \otimes \mathcal{F}_B), \\ \iota_2 &= \mathrm{Id} \\ \iota_3 &= (\mathrm{Id}_X \otimes \mathcal{F}_A^{-1}) \otimes (\mathrm{Id}_Y \otimes \mathcal{F}_B^{-1}).\end{aligned}$$

Here the norms of the first and the third maps are at most $(\mathrm{AB})^{3/2}$ and $(\mathrm{AB})^{-1}$ respectively. Indeed, we can verify that the following estimates hold $\left\|\mathcal{F}_N : \ell_\infty^N \to \ell_1^N\right\| \leq N^{3/2}$ and $\left\|\mathcal{F}_N^{-1} : \ell_1^N \to \ell_\infty^N\right\| \leq N^{-1}$.

- $\left\|\mathcal{F}_N : \ell_\infty^N \to \ell_1^N\right\| \leq N^{3/2}$: Suppose that $\|x\|_\infty = \sup_j |x_j| = 1$ and denote $\mathcal{F}_N^k(x) = \sum_{j=1}^N x_j e^{\frac{2\pi i}{N}jk}$ the coefficients of its image under the transform $\mathcal{F}_N(x)$, for each $k = 1, \ldots, N$. So, we have that

$$\begin{aligned} \|\mathcal{F}_N(x)\|_{\ell_1^N} = \sum_{k=1}^N \left|\mathcal{F}_N^k(x)\right| &\overset{(cs)}{\leq} \sqrt{N}\Big(\sum_{k=1}^N \left|\mathcal{F}_N^k(x)\right|^2\Big)^{1/2} \\ &\overset{(p)}{=} \sqrt{N}\Big(\sum_{j=1}^N \left|x_j\right|^2\Big)^{1/2} \\ &\leq \sqrt{N}\sum_{j=1}^N \left|x_j\right| \\ &\leq \sqrt{N}\cdot N = N^{3/2} \end{aligned}$$

  where inequality (cs) follows from the Cauchy-Schwarz inequality and (p) is due to the Lemma 4.2.2 that we prove later.

- $\left\|\mathcal{F}_N^{-1} : \ell_1^N \to \ell_\infty^N\right\| \leq N^{-1}$: This is quite clear since $\sum_j |x_j| = 1$ implies that

$$\left\|\mathcal{F}_N^{-1}(x)\right\|_{\ell_\infty^N} = \sup_k \left|\frac{1}{N}\sum_{j=1}^N x_j e^{-\frac{2\pi i}{N}jk}\right| \leq \frac{1}{N}\sum_{j=1}^N |x_j| = \frac{1}{N}$$

Finally, the first map $(\mathrm{Id}_X \otimes \mathcal{F}_A) \otimes (\mathrm{Id}_Y \otimes \mathcal{F}_B)$ indeed has norm at most $(AB)^{3/2}$ due to the metric mapping property of the ϵ-norm, the third map $(\mathrm{Id}_X \otimes \mathcal{F}_A^{-1}) \otimes (\mathrm{Id}_Y \otimes \mathcal{F}_B^{-1})$ has norm less than or equal to $(AB)^{-1}$ because of the metric mapping property of the minimal tensor norm and the fact that the c.b. norm of the Fourier inverse equals its operator norm (see Theorem 3.9 [9]). Finally, Grothendieck's theorem says that the second map has norm at most $K_G^{\mathbb{C}}$. Composing the three estimates proves the assertion. □

Lemma 4.2.2. *Let $\{e_j : j = 1, \ldots, N\}$ be the usual o.n.basis of ℓ_2^N and consider the linear map*

$$\mathcal{F} = \mathcal{F}_N : \ell_2^N \to \ell_2^N : e_j \mapsto f_j := \frac{1}{\sqrt{N}}\sum_{k=1}^N \omega^{jk} e_k, \quad j = 1, \ldots, N$$

where $\omega := \exp(\frac{2\pi i}{N})$.

The map $\mathcal{F}$ is a unitary operator, and hence it preserves the ℓ_2 norm. To see this, it is enough to verify that the family $\{f_j\}$ is orthonormal (hence it will be a basis of ℓ_2^N).

Proof. Indeed, we have

$$\langle f_j, f_{j'}\rangle = \frac{1}{N}\sum_{k,k'=1}^{N}\langle \omega^{jk}e_k, \omega^{j'k'}e_{k'}\rangle = \frac{1}{N}\sum_{k=1}^{N}\omega^{-jk}\omega^{j'k}$$

This is 1 if $j = j'$. And if $j \neq j'$, then, writing ω_1 for $\omega^{j'-j}$ we have

$$\frac{1}{N}\sum_{k=1}^{N}\omega^{-jk}\omega^{j'k} = \frac{1}{N}\sum_{k=1}^{N}\omega_1^k = 0$$

because $\omega_1 \neq 1$ and

$$(1-\omega_1)\sum_{k=1}^{N}\omega_1^k = \omega_1 - \omega_1^{N+1} = 0$$

since ω_1 is an N-th root of unity.

Here is the matrix of $\mathcal{F}_N$ wrt $\{e_j : j = 1, \ldots, N\}$:

$$\frac{1}{\sqrt{N}}\begin{bmatrix} \omega & \omega^2 & \ldots & \omega^{N-1} & 1 \\ \omega^2 & \omega^4 & \ldots & \ldots & 1 \\ \vdots & \vdots & \ldots & \ldots & \vdots \\ \omega^{N-1} & \vdots & \ldots & \ldots & 1 \\ 1 & \omega^{2(N-1)} & \ldots & \ldots & 1 \end{bmatrix}.$$

$\square$

Entanglement in nonlocal games

The study of multiplayer games in quantum information theory is motivated by the desire to develop a quantitative understanding of the nonlocal properties of entanglement. In the previous chapters we focused on the ratio of the entangled and classical values of games as a measure of nonlocality. In this chapter we refine the notion by investigating how other measures of entanglement such as Schmidt rank or entropy of entanglement, are reflected in the properties of nonlocal games. In the first section we establish a more firm ground for the understanding of the notion of entanglement while in the next section we provide upper bounds for the ratio ω^*/ω as a function of the Schmidt rank of the entangled state.

5.1 Quantifying entanglement

In this section we will briefly present some of the notions concerning the formalization and the quantitative understanding of entanglement. For a more extensive approach we refer to [21] and [22].

In quantum probability theory, *states* are the positive operators of unit trace on a Hilbert space $\mathcal{H}$. The space of all states is often denoted by $\mathcal{S}(\mathcal{H})$. States are also called *density operators* or even *density matrices*. The extreme points of this space are the one-dimensional projections, which are called *pure states*. For any unit vector $|u\rangle$, the operator $|u\rangle\langle u|$ is a one-dimensional projection. Up to multiplication by a scalar of unit modulus, the unit vector $|u\rangle$ is uniquely determined by the pure state $|u\rangle\langle u|$ and hence we often refer to $|u\rangle$ itself as the pure state. Note that $|u\rangle\langle u|$ is an operator of unit trace, usually called the density operator corresponding to the state $|u\rangle$. We first state and prove an important property of pure states in a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We may also denote by $\mathcal{O}(\mathcal{H})$ the set of selfadjoint operators on $\mathcal{H}$ which we also call *observables*.

Theorem 5.1.1 (Schmidt Decomposition)**.** *Every pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be written in terms of non-negative real coefficients $\{\lambda_k\}_{k=1}^d$, where $d = \min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}$, and two orthonormal sets $\{|\phi_k^A\rangle\} \subseteq \mathcal{H}_A$ and $\{|\psi_k^B\rangle\} \subseteq \mathcal{H}_B$ as,*

$$|\psi\rangle = \sum_{k=1}^{d} \lambda_k |\phi_k^A\rangle |\psi_k^B\rangle$$

where $\sum_{k=1}^{d} \lambda_k^2 = 1$.

Proof. Write $d_A = \dim(\mathcal{H}_A)$ and $d_B = \dim(\mathcal{H}_B)$ and assume that $d_A \geq d_B$. We can write a vector $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ in terms of orthonormal bases as

$$|\psi\rangle = \sum_{i,j=1}^{d_A, d_B} a_{i,j} |i_A\rangle |j_B\rangle$$

Let $E = [a_{i,j}] \in M_{d_A, d_B}$ be the corresponding matrix, then, by singular value decomposition, there exist unitaries $U \in M_{d_A}$, $V \in M_{d_B}$ and a positive diagonal $\Sigma \in M_{d_B}$ whose entries $\{\lambda_k\}_{k=1}^{d_B}$ are the singular values of E, such that

$$E = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^*$$

Thus,

$$\begin{aligned} |\psi\rangle &= \sum_{i,j,k} u_{i,k} \lambda_k v_{k,j} |i_A\rangle |j_B\rangle \\ &= \sum_{k=1}^{d_B} \lambda_k |\phi_k^A\rangle |\psi_k^B\rangle \end{aligned}$$

where the vectors $\{|\phi_k^A\rangle\}$ consitute an orthonormal set in $\mathcal{H}_A$ and $\{|\psi_k^B\rangle\}$ in $\mathcal{H}_B$ due to the fact that U,V are unitary. Moreover, since $|\psi\rangle$ is unit vector, the corresponding matrix has Frobenius norm equal to one: $\|E\|_2 = \sum_{i,j}|a_{i,j}|^2 = 1$ which in turn implies that $\sum_k \eta_k^2 = 1$. □

Remark 5.1.2. *In fact, for a vector* $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ *we write*

$$|\psi\rangle = \sum_{k=1}^{r} \eta_k |\phi_k^A\rangle|\psi_k^B\rangle$$

where $r = \mathrm{rank}(E)$ *is the rank of the corresponding matrix, since* $\mathrm{rank}(E) = \mathrm{rank}(\Sigma)$*, where the latter equals the number of non zero diagonal entries of* Σ.

Definition 5.1.3 (Partial trace). *Let* $\mathcal{H}_A, \mathcal{H}_B$ *be Hilbert spaces. The partial trace over* $\mathcal{H}_B$ *is the map*

$$\mathrm{Tr}_{\mathcal{H}_B} := \mathrm{Id}_{\mathcal{H}_A} \otimes \mathrm{Tr} : \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$$

Its action is given by

$$\mathrm{Tr}_{\mathcal{H}_B}(A \otimes B) = A(\mathrm{Tr}(B))$$

for $A \in \mathcal{B}(\mathcal{H}_A)$ *and* $B \in \mathcal{B}(\mathcal{H}_B)$*. Similarly, we define the partial trace over* $\mathcal{H}_A$ *as* $\mathrm{Tr}_{\mathcal{H}_A} := \mathrm{Tr} \otimes \mathrm{Id}_{\mathcal{H}_B}$.

Given a density operator $\rho_{AB} \in \mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$, the *reduced state* or *marginal state* on $\mathcal{H}_A$ denoted as ρ_A, is obtained by tracing over $\mathcal{H}_B : \rho_A = \mathrm{Tr}_B(\rho_{AB})$. Similarly we obtain the reduced state on $\mathcal{H}_B : \rho_B = \mathrm{Tr}_A(\rho_{AB})$.

Now, consider the marginals of a pure state $|\psi\rangle \in \mathcal{H}_{AB}$. It is a simple exercise to show that the marginals of $|\psi\rangle$ are not pure in general, they are *mixed states*.

Lemma 5.1.4. *The reduced states of the density operator* $|\psi\rangle\langle\psi|$ *corresponding to the pure state* $|\psi\rangle = \sum_{k=1}^{r} \eta_k |\phi_k^A\rangle|\psi_k^B\rangle$ *are given by*

$$\rho_A = \sum_{k=1}^{r} \eta_k^2 |\phi_k^A\rangle\langle\phi_k^A|, \quad \rho_B = \sum_{k=1}^{r} \eta_k^2 |\psi_k^B\rangle\langle\psi_k^B|$$

Proof. Note first that

$$|\psi\rangle\langle\psi| = \sum_{k,l} \lambda_k \lambda_l |\phi_k^A\rangle\langle\phi_k^A| \otimes |\psi_k^B\rangle\langle\psi_k^B|$$

and also that $\mathrm{Tr}_B = \mathrm{Id}_A \otimes \mathrm{Tr}$ where its action on product operators is given by $\mathrm{Tr}_B(T \otimes S) = (\mathrm{Tr}(S))T$. Hence,

$$\begin{aligned}
\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) &= \sum_{k,l} \lambda_k \lambda_l |\phi_k^A\rangle\langle\phi_l^A| \cdot \mathrm{Tr}(|\psi_k^B\rangle\langle\psi_l^B|) \\
&= \sum_{k,l} \lambda_k \lambda_l |\phi_k^A\rangle\langle\phi_l^A| \cdot \left\langle \psi_k^B | \psi_l^B \right\rangle \\
&= \sum_{k,l} \lambda_k \lambda_l |\phi_k^A\rangle\langle\phi_l^A| \cdot \delta_l^k \\
&= \sum_{k=1}^{r} \lambda_k^2 |\phi_k^A\rangle\langle\phi_k^A|
\end{aligned}$$

where we used the fact that $\mathrm{Tr}(|u\rangle\langle v|) = \langle v|u\rangle$. □

Thus, the marginals of a pure state are no longer pure, they are mixed states, contrary to the classical case, where the marginals of the extreme points of the set of joint distributions are in fact extreme points of the set of distributions over the individual sample spaces. This important departure from classical probability theory, leads naturally to the notion of entanglement.

According to the Schmidt decomposition, to any pure state $|\psi\rangle$, we can associate a probability distribution $\{\lambda_1^2, \ldots, \lambda_r^2\}$.

The *Shannon entropy* of a classical probability distribution $p = \{p_1, \ldots, p_k\}$ is defined as

$$H(p) = -\sum_{1 \le i \le k, p_i > 0} p_i \log p_i.$$

Analogously, the *von Neumann entropy* of a quantum state ρ is defined as

$$S(\rho) := H(\lambda(\rho)),$$

where $\lambda(\rho)$ is the vector of eigenvalues of ρ. The von Neumann entropy can be also expressed as

$$S(\rho) = -\mathrm{Tr}\left(\rho \log \rho\right),$$

where $\rho \log \rho$ refers to the operator obtained by extending the function

$$f(t) = \begin{cases} t \log t, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

to the case of positive semidefinite operators by defining the map

$$f(T) = \sum_{k=1}^{m} f(\lambda_k) E_k$$

where λ_k are the eigenvalues and E_k are projections on the eigenspace according to the spectral theorem.

Now, given a joint distribution over two sample spaces A,B, the Shannon entropy of the joint distribution is always greater than the Shannon entropy of the marginals, that is, $H(AB) \geq H(A)$. But in quantum systems, while the pure states of a joint system will always have zero entropy, it is not the case in general for their reduced states will have non-zero entropy. Indeed, for a pure state $|\psi\rangle$ in a joint system $\mathcal{H}_{AB}$ we have $S(|\psi\rangle\langle\psi|) = 0$, whereas for the marginals $S(\rho_A) = S(\rho_B) = -\sum_k \lambda_k^2 \log \lambda_k^2$. In fact, for pure states of two systems, A and B, the entropy of the reduced density matrix of either A or B is a measure of entanglement.

Furthermore, for a bipartite pure state $|\psi\rangle$, the Schmidt decomposition provides a way of quantifying the deviation of $|\psi\rangle$ away from a product pure state. The number of product states in the Schmidt decomposition and the relative weights assigned to the different product states, quantify the entanglement of state $|\psi\rangle$.

Definition 5.1.5 (Schmidt rank). *Given a bipartite pure state $|\psi\rangle \in \mathcal{H}_{AB}$ with a Schmidt decomposition $|\psi\rangle = \sum_{k=1}^{r} \lambda_k |\phi_k^A\rangle|\psi_k^B\rangle$,*

1. *The number r of non-zero coefficients λ_k is defined to be the **Schmidt rank** of the state $|\psi\rangle$.*

2. *A bipartite pure state $|\psi\rangle$ is said to be **entangled** if it has Schmidt rank greater than one.*

3. *$S(\rho_A)$ $(= S(\rho_B))$ is a measure of the entanglement of the pure state $|\psi\rangle$.*

If a pure state $\rho \in \mathcal{H}_{AB}$ is not entangled, i.e. it is written as $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. So, the Schmidt rank is $r = 1$ and also $\lambda_1 = 1$, which means that $S(\rho_A) = S(\rho_B) = 0$ and thus the von Neumann entropy is indeed a legitimate way to distinguish between entangled and not entangled pure states.

For a d-dimensional space, the von Neumann entropy of any state is bounded from above by $\log d$. Therefore, the maximum entanglement of a bipartite pure state $|\psi\rangle \in \mathcal{H}_{AB}$ is $\log\min(d_A, d_B)$.

Remark 5.1.6. *If $d = \min(d_A, d_B)$, the **maximally entangled state** in $\mathcal{H}_{AB}$ is*

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i_A\rangle|i_B\rangle$$

where $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

5.2 Dimension-dependent bounds for two-player games

To study the Schmidt rank, for any Bell functional M and integer d, define

$$\omega_d^*(M) = \sup \left| \sum_{x,y;a,b} M_{x,y}^{a,b} \langle\psi| A_x^a \otimes B_y^b |\psi\rangle \right|$$

where the supremum is taken over all $r \leq d$, vectors $|\psi\rangle \in \mathrm{Ball}(\mathbb{C}^r \otimes \mathbb{C}^r)$ and families of POVM's $\{A_x^a\}_a$ and $\{B_y^b\}_b$ in M_r. Clearly $(\omega_d^*(M))_d$ forms an increasing sequence that converges to $\omega^*(M)$ as $d \to \infty$. Thus, the quantity

$$\sup_M \frac{\omega_d^*(M)}{\omega(M)}$$

expresses the largest violation of a Bell inequality achievable by states of Schmidt rank at most d. In this section we describe known bounds for this quantity.

Proposition 5.2.1. *Let G a two-player game and M a Bell functional then for every $d \geq 1$,*

1. $\omega_d^*(G) \leq d\ \omega(G)$,

2. $\omega_d^*(M) \leq 2d\ \omega(M)$.

Proof. 1. Consider families of POVM's $\{A_x^a\}_{x,a}$, $\{B_y^b\}_{y,b}$ in M_d and a pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. By the Schmidt decomposition, we can write $|\psi\rangle = \sum_{k=1}^d \lambda_k |\phi_k\rangle \otimes |\psi_k\rangle$ where $\sum_{k=1}^d |\lambda_k|^2 = 1$. Thus,

$$\begin{aligned}
\sum_{x,y;a,b} G_{x,y}^{a,b} \langle \psi | A_x^a \otimes B_y^b | \psi \rangle &= \sum_{k,l} \lambda_k \lambda_l \sum_{x,y;a,b} G_{x,y}^{a,b} \langle \phi_k | A_x^a | \phi_l \rangle \langle \psi_k | B_y^b | \psi_l \rangle \\
&\leq d \max_{k,l} \left| \sum_{x,y;a,b} G_{x,y}^{a,b} \langle \phi_k | A_x^a | \phi_l \rangle \langle \psi_k | B_y^b | \psi_l \rangle \right| \qquad (5.2.1)
\end{aligned}$$

where the inequality above comes from the fact that $\left| \sum_{k,l} \lambda_k \lambda_l \right| \leq d$. Indeed, recall that $\|x\|_1 \leq \sqrt{n} \|x\|_2$, for all $x \in \mathbb{R}^n$, so for $x = (\lambda_k)_{k=1}^d$ we have that $\left| \sum_{k,l} \lambda_k \lambda_l \right| = \left| \sum_k \lambda_k \right|^2 \leq d \sum_k |\lambda_k|^2$ and since $\sum_{k=1}^k |\lambda_k|^2 = 1$ we have the assertion. Now fix k, l and an $x \in \mathbf{X}$ and note that

$$\begin{aligned}
\sum_a \left| \langle \phi_k | A_x^a | \phi_l \rangle \right| &\leq \sum_a \left| \langle \phi_k | A_x^a | \phi_k \rangle \right|^{\frac{1}{2}} \left| \langle \phi_l | A_x^a | \phi_l \rangle \right|^{\frac{1}{2}} \\
&\leq \Big(\sum_a \left| \langle \phi_k | A_x^a | \phi_k \rangle \right| \Big)^{\frac{1}{2}} \Big(\sum_a \left| \langle \phi_l | A_x^a | \phi_l \rangle \right| \Big)^{\frac{1}{2}} \\
&= 1
\end{aligned}$$

where the first two inequalities follow from the positivity of the $\{A_x^a\}$ and the Cauchy-Schwarz inequality while the equality in the end comes from the fact that again each A_x^a is positive and add up to the identity, $\sum_a A_x^a = \mathrm{Id}$ and the set $\{\phi_k\}_k$ is orthonormal. Indeed, check that

$$\sum_a \left| \langle \phi_k | A_x^a | \phi_k \rangle \right| = \sum_a \langle \phi_k | A_x^a | \phi_k \rangle = \langle \phi_k | \sum_a A_x^a | \phi_k \rangle = \|\phi_k\|^2 = 1$$

Applying the same arguments for Bob's POVM and using that G has non-negative coefficients, we get that for every k, l,

$$\left| \sum_{x,y;a,b} G_{x,y}^{a,b} \langle \phi_k | A_x^a | \phi_l \rangle \langle \psi_k | B_y^b | \psi_l \rangle \right| \leq \sum_{x,y;a,b} G_{x,y}^{a,b} \left| \langle \phi_k | A_x^a | \phi_l \rangle \right| \left| \langle \psi_k | B_y^b | \psi_l \rangle \right| \leq \omega(G)$$

Together with 5.2.1, this proves the first estimate.

2. For the second estimate, proceed as above to obtain again 5.2.1. It thus suffices to show that for every k,l

$$\left| \sum_{x,y;a,b} M_{x,y}^{a,b} \langle \varphi_k | A_x^a | \varphi_l \rangle \langle \psi_k | B_y^b | \psi_l \rangle \right| \leq 2\omega(M)$$

Fix k,l and decompose the rank-one operator $\rho = |\varphi_l\rangle\langle\varphi_k|$ as $\rho = \rho_1 + i\rho_2$, where $\rho_j, j = 1,2$ are selfadjoint operators with $\left\|\rho_j\right\|_{S_1^d} \leq 1$. This follows from the fact that the operator ρ has one non-zero singular value and that $\|\rho\| = \sigma_{\max}(\rho) = 1$, so $\|\rho\|_{S_1^d} = 1$ and as a consequence $\left\|\rho_j\right\|_{S_1^d} \leq \|\rho\|_{S_1^d} = 1$. By the spectral theorem for the Hermitian operators ρ_j, each ρ_j can be expressed as $\rho_j = \sum_{s=1}^d \alpha_s^j |f_s^j\rangle\langle f_s^j|$, where $\alpha_s^j \in \mathbb{R}$, $\sum_{s=1}^d |\alpha_s^j| \leq 1$ and $\{|f_s^j\rangle\}$ is an orthonormal basis of $\mathbb{C}^d$ for $j = 1,2$. Now observe that $\langle\varphi_k|A_x^a|\varphi_l\rangle = \mathrm{Tr}\left(A_x^a|\varphi_l\rangle\langle\varphi_k|\right)$ and thus, by linearity of the trace,

$$\langle \varphi_k | A_x^a | \varphi_l \rangle = \sum_{s=1}^d \alpha_s^j \langle f_s^1 | A_x^a | f_s^1 \rangle + i \sum_{s=1}^d \alpha_s^j \langle f_s^2 | A_x^a | f_s^2 \rangle.$$

Hence,

$$\begin{aligned} \left| \sum_{x,y;a,b} M_{x,y}^{a,b} \langle \varphi_k | A_x^a | \varphi_l \rangle \langle \psi_k | B_y^b | \psi_l \rangle \right| &\leq \sum_{j=1}^2 \sum_{s=1}^d |\alpha_s^j| \left| \sum_{x,y;a,b} M_{x,y}^{a,b} \langle f_s^j | A_x^a | f_s^j \rangle \langle \psi_k | B_y^b | \psi_l \rangle \right| \\ &\leq 2 \sup_{P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})} \left| \sum_{x,y;a,b} M_{x,y}^{a,b} P(a|x) \langle \psi_k | B_y^b | \psi_l \rangle \right| \end{aligned}$$

where in the last inequality we used that $(P(a|x))_{x,a} = (\langle f|A_x^a|f\rangle)_{x,a}$ is a classical probability distribution for every unit vector $|f\rangle$. Since the coefficients $M_{x,y}^{a,b}$ are real, then for every $P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$, the operator $\sum_{x,y;a,b} M_{x,y}^{a,b} P(a|x) B_y^b$

is a selfadjoint operator in M_d. So for any $P \in \mathcal{P}_C(\mathbf{AB}|\mathbf{XY})$, its norm satisfies

$$\begin{aligned}\left|\sum_{x,y;a,b} M^{a,b}_{x,y} P(a|x)\langle\psi_k|B^b_y|\psi_l\rangle\right| &= \left|\langle\psi_k|\sum_{x,y;a,b} M^{a,b}_{x,y} P(a|x) B^b_y|\psi_l\rangle\right| \\ &\leq \left\|\sum_{x,y;a,b} M^{a,b}_{x,y} P(a|x) B^b_y\right\| \\ &= \sup_{|\psi\rangle\in \mathrm{Ball}(\mathbb{C}^d)} \left|\sum_{x,y;a,b} M^{a,b}_{x,y} P(a|x)\langle\psi|B^b_y|\psi\rangle\right| \\ &\leq \omega(M).\end{aligned}$$

□

Appendix A

A.1 Tensor products

We present here the definition and the main theorem concerning tensor products of vector spaces. For proof of the following results see [23]. For a different approach see [24].

Let E, F and M be complex vector spaces. Let also $\varphi : E \times F \to M$ a bilinear map.

Definition A.1.1. *We say that E and F are φ-linearly disjoint if the following holds:*

Let $\{x_1, \ldots, x_n\}$ be a finite subset of E, and $\{y_1, \ldots, y_n\}$ a finite subset of F, satisfying

$$\sum_{i=1}^{n} \varphi(x_i, y_i) = 0.$$

Then, if $x_1, \ldots, x_n$ are linearly independent, $y_1 = \cdots = y_n = 0$,
and if $y_1, \ldots, y_n$ are linearly independent, $x_1 = \cdots = x_n = 0$.

Definition A.1.2. *A tensor product of E and F is a pair (M, φ), consisting of a vector space M and a bilinear map $\varphi : E \times F \to M$ such that:*

1. *The image of $E \times F$ under φ, spans the space M.*
2. *E and F are φ-linearly disjoint.*

We will first prove the existence of the tensor product and the well-known "universal property" which states that tensor products are unique up to isomorphisms. Let's see first an equivalent definition of the φ-linearly disjointness.

Proposition A.1.3. *Let E, F and M vector spaces and φ a bilinear map from $E \times F$ into M. Then the following are equivalent"*

1. *E and F are φ-linearly disjoint*
2. *Let $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ be linearly independent sets of vectors in E and F respectively. Then $\{\varphi(x_i, y_j)\}_{i,j}$ are linearly independent vectors in M.*

Theorem A.1.4. *Let E,F be two vector spaces.*

1. *(Existence) There exists a tensor product of E and F.*
2. *(Universal property) Let (M, φ) be a tensor product of E and F. Let G be any vector space, and $b : E \times F \to G$ any bilinear map. There exists a unique linear map $B : M \to G$ such that the following diagram*

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "\varphi"'] & G \\ & M \arrow[u, "B"'] \end{tikzcd}$$

   is commutative.

3. *(Uniqueness) If (M_1, φ_1) and (M_2, φ_2) are two tensor products of E and F, there exists a linear isomorphism π from M_1 into M_2 such that the following diagram*

$$\begin{tikzcd} E \times F \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_1"'] & M_2 \\ & M_1 \arrow[u, "\pi"'] \end{tikzcd}$$

   is commutative.

The tensor product of E and F will be denoted by $E \otimes F$ and the bilinear map φ of $E \times F$ into $E \otimes F$ will be denoted by

$$(x, y) \mapsto x \otimes y$$

rather than φ.

Remark A.1.5. *To rephrase the universal property, if G is a vector space and $b : E \times F \to G$ a bilinear map, then there exists a unique linear map $B : E \otimes F \to G$ such that $B(x \otimes y) = b(x, y)$. That is, the space of bilinear maps $E \times F \to G$ is isomorphic to the space of linear maps $E \otimes F \to G$:*

$$B(E \times F, G) \cong L(E \otimes F, G).$$

A.2 Operator theory

Definition A.2.1 (Inner product)**.** *Let $\mathcal{V}$ be a vector space over the field $\mathbb{C}$. An inner product on $\mathcal{V}$ is a map*

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

such that:

1. *It is linear in the second argument:*

   $$\langle x, \lambda y_1 + \mu y_2 \rangle = \lambda \langle x, y_1 \rangle + \mu \langle x, y_2 \rangle$$

   for all $x, y_1, y_2 \in \mathcal{V}$ and λ, $\mu \in \mathbb{C}$

2. *It is Hermitian:*

   $$\overline{\langle x, y \rangle} = \langle y, x \rangle$$

   for all $x, y \in \mathcal{V}$

3. *It is positive definite:*

   $$\langle x, x \rangle \geq 0$$

   for all $x \in \mathcal{V}$ with equality if and only of $x = 0$.

Definition A.2.2. *If $\mathcal{H}$ is a complex vector space and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{H}$, then we call the pair $(\mathcal{H},\langle\cdot,\cdot\rangle)$ a pre-Hilbert space. A pre-Hilbert space becomes a normed space with the norm induced by the inner product $\|\cdot\|_{\mathcal{H}} := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.*

Definition A.2.3. *A complete pre-Hilbert space is called a Hilbert space.*

Example A.2.4. *The space $\mathbb{C}^n$ equipped with the inner product $\langle x,y\rangle := \sum_{i=1}^{n} \overline{x_i} y_i$ is a Hilbert space, equipped with the Euclidean norm $\|x\|_2 := \sqrt{\langle x,x\rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}$. When referring to $\mathbb{C}^n$ we will usually mean $(\mathbb{C}^n, \|\cdot\|_2)$ which will be also denoted by $\ell_2^n = (\mathbb{C}^n, \|\cdot\|_2)$.*

Example A.2.5. *If $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, then so is $\mathcal{H}_1 \oplus \mathcal{H}_2$ with inner product*

$$\langle (v_1,v_2),(w_1,w_2)\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2} := \langle v_1,w_1\rangle_{\mathcal{H}_1} + \langle v_2,w_2\rangle_{\mathcal{H}_2}$$

Similarly $\mathcal{H}_1 \otimes \mathcal{H}_2$ becomes a Hilbert space, with inner product

$$\langle v_1\otimes v_2, w_1\otimes w_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} := \langle v_1,w_1\rangle_{\mathcal{H}_1} \cdot \langle v_2,w_2\rangle_{\mathcal{H}_2}$$

and taking the completion as we already discussed in Section 1.1.

Remark A.2.6. *Note that if $T \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ where $\mathcal{H}_1,\mathcal{H}_2$ Hilbert spaces,*

$$\begin{aligned}\|T\| &= \sup\{\|Tx\|_2 : x \in \mathrm{Ball}(\mathcal{H}_1)\} \\ &= \sup\{|\langle y,Tx\rangle_2| : x \in \mathrm{Ball}(\mathcal{H}_1), y \in \mathrm{Ball}(\mathcal{H}_2)\}\end{aligned}$$

Theorem A.2.7 (adjoint)**.** *Let $\mathcal{H}_1$,$\mathcal{H}_2$ be two Hilbert spaces and $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator, then there exists a unique bounded operator $T^* : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\langle T^* x_2, x_1\rangle_{\mathcal{H}_1} = \langle x_2, Tx_1\rangle_{\mathcal{H}_2}$$

for all $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{H}_2$

The operator $T^* : H_1 \to H_2$ is called the adjoint of T and it is a bounded operator for which $\|T^*\| = \|T\|$.

Remark A.2.8. *When $T = T^*$, then T is called selfadjoint or Hermitian.*

Definition A.2.9 (Involution)**.** *Let $\mathcal{A}$ be a complex algebra. We call involution a map $\mathcal{A} \to \mathcal{A} : a \mapsto a^*$ with the following properties :*

1. $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$
2. $a^{**} = a$
3. $(ab)^* = b^* a^*$

for each $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$
We call such an algebra a $*$-algebra.

Remark A.2.10. *The map $A \mapsto A^*$ on $\mathcal{B}(\mathcal{H})$ is an involution.*

Definition A.2.11. *A Banach algebra $\mathcal{A}$, is an associative algebra equipped with a norm, such that*

$$||x \cdot y|| \leq ||x|| \cdot ||y||$$

for all $x, y \in \mathcal{A}$ and that is also a complete space (Banach) with respect to that norm.

Definition A.2.12 (C^*-algebra)**.** *Let $(\mathcal{A}, *)$ be a Banach algebra with involution. We say that $(\mathcal{A}, *)$ is a C^*-algebra if its norm satisfies the C^*-property:*

$$||a^* a|| = ||a||^2.$$

Definition A.2.13. *A linear map $\phi : \mathcal{A} \to \mathcal{B}$, between C^*-algebras is called a $*$-homomorphism if it preserves the products and the adjoints. If in addition ϕ is a bijection, it is called $*$-isomorphism.*

Proposition A.2.14. *If $\phi : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism between Cs-algebras, then ϕ is isometric.*

Proposition A.2.15. *If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.*

Let $\mathcal{A}$ and $\mathcal{B}$ be two $*$-algebras. One can make their algebraic tensor product $\mathcal{A}\otimes\mathcal{B}$ a $*$-algebra, by defining the operations

$$(a\otimes b)(c\otimes d) = ac\otimes bd$$

and

$$(a\otimes b)^* = a^*\otimes b^*$$

for all $a, c\in\mathcal{A}$ and $b, d\in\mathcal{B}$.

If we also equip $\mathcal{A}\otimes\mathcal{B}$ with a norm $\|\cdot\|_\gamma$ that satisfies the C^*-algebra axioms

$$\|xy\|_\gamma \leq \|x\|_\gamma \|y\|_\gamma \qquad\qquad \left\|x^* x\right\|_\gamma = \|x\|_\gamma^2$$

for all $x, y\in\mathcal{A}\otimes\mathcal{B}$, then the completion with respect to such a norm, of the algebraic tensor product $\mathcal{A}\otimes\mathcal{B}$ is a C^*-algebra, which we denote by $\mathcal{A}\hat{\otimes}_\gamma\mathcal{B}$ or $\mathcal{A}\otimes_\gamma\mathcal{B}$ or simply $\mathcal{A}\otimes\mathcal{B}$ whenever it is clear from the context what we mean.

Remark A.2.16. *Let $(E,\langle\cdot,\cdot\rangle)$, $(F,\langle\cdot,\cdot\rangle)$ be two finite dimensional pre-Hilbert spaces and let also $\{e_1,\dots,e_n\}\subseteq E$ and $\{f_1,\dots,f_k\}\subseteq F$ be two orthonormal bases. Then,*

$$\mathcal{B}(E,F)\cong M_{k,n}(\mathbb{C}).$$

Indeed, every linear map $T: E\to F$ defines a matrix $[a_{i,j}]\in M_{k,n}(\mathbb{C})$ by setting $a_{i,j} = \langle f_i, Te_j\rangle$ and every matrix $A = [a_{i,j}]$ defines an operator $T_A : E\to F$ by acting on column vectors of coefficients with matrix multiplication, that is: if $x = \sum_{i=1}^n x_i e_i \in E$, then $T_A(x) = \sum_{j=1}^k y_j f_j$ where

$$\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} = [a_{i,j}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1,j}x_j \\ \vdots \\ \sum_{j=1}^n a_{k,j}x_j. \end{bmatrix}$$

Corollary A.2.17. *Let $M_n(\mathbb{C})$ be the space of $n\times n$ matrices, we define the selfadjoint of a matrix $A = [a_{i,j}]\in M_n$ to be $A^* = [\overline{a_{j,i}}]$. Then by the above lemma*

$$\mathcal{B}(\mathbb{C}^n,\mathbb{C}^n)\cong M_n(\mathbb{C}).$$

Moreover, the isomorphism preserves linearity, multiplication and the adjoints .

Corollary A.2.18. *As $\ell_2^n = (\mathbb{C}^n, \|\cdot\|_2)$ is a Hilbert space, then $\mathcal{B}(\ell_2^n)$ is a C^*-algebra. Using the identification $M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$ we conclude that $M_n(\mathbb{C})$ is also a C^*-algebra if we use the induced operator norm on matrices. More specifically the operator norm induced by the Euclidean norm, i.e., the spectral norm (0.3.10).*

Definition A.2.19 (Trace)**.** *Let $\mathcal{H}$ be finite dimensional Hilbert space with $\dim(\mathcal{H}) = n$. We define the Trace of an operator $T : \mathcal{H} \to \mathcal{H}$, to be the trace of its matrix representation. That is, if T defines a matrix $[T_{i,j}]$ by $T_{i,j} = \langle e_i|T|e_j\rangle$, then its trace is defined by*

$$\mathrm{Tr}(T) := \mathrm{Tr}\Big([T_{i,j}]\Big) = \sum_{i=1}^{n} \langle e_i|T|e_i\rangle.$$

Note that the definition of the trace does not depend on the choice of basis.

Remark A.2.20. *A $\|\cdot\|$–closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra iff it is selfadjoint, i.e., is it satisfies $A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$.*

Theorem A.2.21 (Gelfand-Naimark)**.** *Every C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a closed C^*-subalgebra of a $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, i.e., there exists a Hilbert space $\mathcal{H}$ and a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ that preserves the algebraic structure and the norm.*

Definition A.2.22. *Suppose $\mathcal{A}$ is a Banach algebra with a unit $\mathbf{1}$. An element $x \in \mathcal{A}$ is called invertible if there exists an $x^{-1} \in \mathcal{A}$ such that $xx^{-1} = x^{-1}x = \mathbf{1}$. We denote the set of these x by* $\mathrm{Inv}(\mathcal{A})$.

Definition A.2.23 (Spectrum)**.** *Suppose $\mathcal{A}$ is a Banach algebra with a unit $\mathbf{1}$ and an element $x \in \mathcal{A}$. The spectrum $\sigma(x)$ of x is the set*

$$\sigma(x) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - x \notin \mathrm{Inv}(\mathcal{A})\}.$$

Definition A.2.24. *If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$ then:*

1. *If $a^*a = aa^*$, then a is called normal.*

2. *If $a = a^*$, then a is called selfadjoint.*

3. *If also $\mathcal{A}$ has a unit $\mathbf{1}$, an $x \in \mathcal{A}$ is called unitary if $a^*a = \mathbf{1}$ and $aa^* = \mathbf{1}$.*

4. *If $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$, then a is called positive.*

Remark A.2.25. *If $\mathcal{A}$ is a C^*-algebra, we may write $a \geq 0$ when $a \in \mathcal{A}$ is positive. We also say $a \geq b$ for $a, b \in \mathcal{A}$ if $a - b \in \mathcal{A}_+$.*

Remark A.2.26. *Denote by $\mathcal{A}_+$ the set of all positive elements of $\mathcal{A}$. One can see that $\mathcal{A}_+ = \{b^*b : b \in \mathcal{A}\}$ and it is also a norm-closed cone.*

Definition A.2.27.

1. *We say that an operator $A \in \mathcal{B}(\mathcal{H})$ is a positive operator, if $\langle x, Ax \rangle \geq 0$ for every $x \in \mathcal{H}$. We denote the set of all positive operators on $\mathcal{H}$ by $\mathcal{B}_+(\mathcal{H})$.*

2. *If also $A, B \in \mathcal{B}_+(\mathcal{H})$, we write $A \geq B$ if $\langle x, Ax \rangle \geq \langle x, Bx \rangle$ for all $x \in \mathcal{H}$, that is, if $A - B \in \mathcal{B}_+(\mathcal{H})$.*

Theorem A.2.28. *Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. Then,*

$$\mathcal{H} = M \oplus M^{\perp}.$$

So, for every $x \in \mathcal{H}$ we write $x = P_M(x) + P_{M^{\perp}}(x)$, where $P_M(x) \in M$ and $P_{M^{\perp}}(x) \in M^{\perp}$. The map $P_M : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator called the projection on M.

Proposition A.2.29. *An operator $P \in \mathcal{B}(\mathcal{H})$ is a projection if and only if it satisfies $P = P^2 = P^*$.*

Proposition A.2.30. *An operator $U \in \mathcal{B}(\mathcal{H})$ is unitary (i.e. $UU^* = U^* = \mathrm{Id}_{\mathcal{H}}$) if and only if U is isometric and surjective.*

Theorem A.2.31. *Let $T \in \mathcal{B}(\mathcal{H})$. The following properties are equivalent:*

1. *T is a positive operator*

2. *There exists $B \in \mathcal{B}(\mathcal{H})$ positive, such that $T = B^2$*

3. *There exists $S \in \mathcal{B}(\mathcal{H})$, such that $T = S^* S$*

4. *$T = T^*$ and $\sigma(T) \subseteq \mathbb{R}_+$*

Proposition A.2.32. *If $T \in \mathcal{B}(\mathcal{H})$ is selfadjoint, then*

$$\|T\| = \sup\{|\langle h, Th\rangle| : \|h\| = 1\}$$

Theorem A.2.33 (Spectral theorem)**.** *Let $\mathcal{H}$ be a finite dimensional Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then, there exist a positive integer m, distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and nonzero projections $E_1, \ldots, E_m \in \mathcal{B}(\mathcal{H})$ with $\sum_{k=1}^{m} E_k = \mathrm{Id}_{\mathcal{H}}$ such that,*

$$T = \sum_{k=1}^{m} \lambda_k E_k.$$

Each scalar λ_k is an eigenvalue of T and E_k is the projection operator onto the eigenspace corresponding to the eigenvalue λ_k.

Theorem A.2.34 (Spectral decomposition)**.** *Let $\mathcal{H}$ be an n-dimensional Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ a normal operator. Then, there exist an orthonormal basis $\{|\psi_k\rangle\}_{k=1}^{n}$ of $\mathcal{H}$ such that*

$$T = \sum_{k=1}^{n} \lambda_k |\psi_k\rangle\langle\psi_k|$$

where $\{\lambda_1, \cdots \lambda_n\}$ are the eigenvalues of T.

Remark A.2.35. *Note that in the spectral decomposition of a selfadjoint operator T, the eigenvalues $\{\lambda_k\}$ are real. Moreover, if the operator is positive, then its eigenvalues are non-negative.*

Theorem A.2.36 (Singular value decomposition)**.** *Let $A \in M_{n,m}$ be a matrix with complex coefficients, let $q = \min\{n, m\}$ and suppose that $\mathrm{rank}(A) = r$. Then,* *There exist unitary matrices $V \in M_n$ and $W \in M_m$ such that*

$$A = V\Sigma W,$$

where

$$\Sigma = \begin{cases} \Sigma_q & \text{if} \quad m = n, \\ \begin{bmatrix} \Sigma_q & 0 \end{bmatrix} \in M_{n,m} & \text{if} \quad m > n, \\ \begin{bmatrix} \Sigma_q \\ 0 \end{bmatrix} \in M_{n,m} & \text{if} \quad m < n, \end{cases}$$

in which Σ_q is the diagonal matrix

$$\Sigma_q = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_q \end{bmatrix}$$

where $\sigma_1 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$.

If we order the non-zero eigenvalues of the positive matrix A^*A as $\lambda_1(A^*A) \leq \cdots \leq \lambda_r(A^*A)$, then the parameters $\sigma_1, \ldots, \sigma_r$ are exactly the positive square roots of these eigenvalues. That is, $\sigma_i = \sqrt{\lambda_i(A^*A)}$ for all $i = 1, \ldots, r$. The scalars $\sigma_1, \ldots, \sigma_r$ are called the <u>singular values</u> of A.

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