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Introduction to Derived Categories

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Εισαγωγή

Η θεωρία των παραγόμενων συναρτητών (derived functors) σε αβελιανές κατηγορίες αποτελεί μια θεμελιώδη έννοια τη ομολογικής άλγεβρας η οποία διατρέχει όλα τα σύγχρονα μαθηματικά, ειδικότερα την αλγεβρική γεωμετρία, ακόμα και την θεωρητική φυσική. Ως επόμενο βήμα, η θεωρία των παραγόμενων (derived) κατηγοριών είναι ένα ισχυρό εργαλείο που απλοποιεί σημαντικά την ομολογική άλγεβρα. Η μελέτη, λοιπόν, μετατοπίζεται από τα αντικείμενα μιας κατηγορίας στην μελέτη των chain complexes, που αποτελούνται από αντικείμενα της καητηγορίας, εφοδιασμένα με μία ισχυρότερη έννοια ισοδυναμίας (αυτή των quasi-ismoprhisms). Η θεωρία αυτή, έχει τις ρίζες της στην δουλειά [Tohoku] των Alexander Grothendieck και, του μαθητή του, Jean-Louis Verdier, η οποία μας δίνει την δυνατότητα να εκφράσουμε έννοιες με απλό και περισσότερο κατανοητό τρόπο αποφεύγοντας την χρήση περίπλοκων φασματικών ακολουθιών.

Η ιδέα πίσω από τις derived categories είναι η εξής: *«Τα complexes είναι καλά, ενώ η ομολογία των complexes είναι κακή»*. Όπως αναφέρθηκε και παραπάνω, θα θέλαμε τα chain complexes ως ειδοποιό - αμετάβλητο στοιχείο των χώρων, επειδή έχουν όλη την πληροφορία που μας είναι χρήσιμη για την ομοτοπία ενός χώρου, - η ομολογία περιέχει λιγότερη πληροφορία - οπότε έτσι προήλθε η παραπάνω ιδέα. Αφού, λοιπόν, τα στοιχεία της derived category είναι τα chain complexes, χρειαζόμαστε έναν τρόπο να ταυτίζουμε αυτά τα οποία είναι ισομορφικά, οδηγούμαστε έτσι, στην ένοια των quasi-isomorphisms οι οποίοι παίζουν έναν σημαντικό ρόλο στην κατασκευή της νέας αυτής κατηγορίας, δίνοντάς μας την ισχυρότερη έννοια ισοδυναμίας που αναφέραμε. Ειδικότερα, ομοτοπικά complexes είναι quasi-ισομορφικά.

Ο κύριος σκοπός αυτής της διπλωματικής εργασίας είναι η απόδειξη του θεωρήματος της φασματκής ακολουθίας του Grothendieck 5.2.2 η οποία υπολογίζει τους derived functors της σύνθεσης δύο συναρτητών, γνωρίζοντας μόνο τους derived functors καθένα συναρτητή. Η φασματική ακολουθία του Leray 5.5.8 και η φασματική ακολουθία των Lyndon-Hochschild-Serre είναι δύο εκ των πολλών άλλων ειδικών περιπτώσεων της φασματικής ακολουθίας του Grothendieck.

Στο τελευταίο κεφάλαιο της εργασίας θα δούμε δύο αποδείξεις αυτού του αποτελέσματος, η πρώτη χρησιμοποιώντας μόνον γνώσεις από την θεωρία φασματικών ακολουθιών και η δεύτερη, πιο άμεση και απλοποιημένη, στη γλώσσα των derived categories. Δεν θα ήταν λάθος να συμπεράνουμε πως οι derived categories μας παρέχουν έναν πιο απλό τρόπο να κάνουμε υπολογισμούς οι οποίοι θα ήταν πολυπλοκότεροι αν γινόντουσαν μέσω φασματικών ακολουθιών.

Πιο συγκεκριμένα, στο πρώτο κεφάλαιο θυμόμαστε κάποιους βασικούς ορισμούς και κατασκευές όπως είναι η (συν)ομολογία, η σχέση ομοτοπίας, τα ακριβή τρίγωνα και οι τριγωνίσιμες κατηγορίες, θέτοντας έτσι τα θεμέλια για τα ακόλουθα κεφάλαια. Στο δεύτερο κεφάλαιο κατασκευάζουμε τους αριστερούς derived functors (και δυϊκά τους δεξιούς), δείχνουμε πως είναι καλά ορισμένοι και οπλιζόμαστε με τις κατάλληλες προτάσεις και πορίσματα. Εν συνεχεία, στο τρίτο κεφάλαιο, παίρνουμε μια ιδέα από τον κόσμο των φασματικών ακολουθιών, καταλαβαίνοντας πώς ορίζονται και συγκλίνουν ενώ παράλληλα βλέπουμε - ως παραδείγματα - πώς κάποια ήδη γνωστά αποτελέσματα (όπως το Snake Lemma) αποδεικνύονται εύκολα με χρήση αυτού του νέου εργαλείου. Στο τέταρτο κεφάλαιο, τοπικοποιούμε την κατηγορία ομοτοπίας μιας αβελιανής κατηγορίας ως προς τους quasi-isomorphisms παίρνοντας έτσι την derived category, όπου εξηγούμε και πώς αυτή λειτουργεί, δηλαδή, ποιά είναι τα αντικείμενα, οι μορφισμοί και η σύνθεση μορφισμών της. Τελικά, στο πέμπτο κεφάλαιο, αποδεικνύουμε το θεώρημα του Grothendieck, όπως περιγράφηκε παραπάνω, και εν συνεχεία κάνουμε μια σύντομη αναφορά στην φασματική ακολουθία του Leray τονίζοντας την άμεση επαγωγή της από αυτήν του πρώτου.

Αθήνα, 30 Μαρτίου 2021

Introduction

The theory involving derived functors on abelian categories is a fundamental concenpt of homological algebra that has a lot of applications on modern mathematics, especially on algebraic geometry, and even on theoretical physics. Derived functors, being an important tool, it was necessary to take the next step and extend our theory, introducing ourselves to derived categories, which simplify a lot of homological algebra. Instead of looking at the objects of a category, we study their chain complexes which are equipped with a stronger concept of equivalence (that of quasi-isomorphisms). All this has its roots in [Tohoku], the work of Alexander Grothendieck and, his student, Jean-Louis Verdier, which provided us with essential tools to avoid using the, more complex, spectral sequences.

The motto - idea, one will find in every paper concerning the derived categories, is *"Complexes Good, Homology of Complexes Bad"*. As stated above, we needed chain complexes as a natural invariant because they have all the information we want about the homotopy of a space - homology holds little information about that - thus the motto. With this, quasi-isomorphisms, which are isomorphisms of the induced maps on homology, come in to play a crucial role into the construction of derived categories, identifying the isomorphic complexes in our new category, which is the stronger equivalence relation we needed. Namely, homotopic complexes are also quasi-isomorphic.

The main goal of this thesis is to prove Grothendieck's Spectral Sequence Theorem 5.2.2 which computes the derived functors of the composition of two functors, by knowing the derived functors of each functor. The Leray spectral sequence 5.5.8 and the Lyndon-Hochschild-Serre spectral sequence are just a couple out of the many special cases of Grothendieck's spectral sequence.

We will see two proofs of this result. The first time using some basic knowledge about spectral sequences and hypercohomology and the second time, a more direct - simplified proof, using derived category language. It wouldn't be false to assume that the derived category has provided us with a simpler way of doing calculations which are rather complicated if done using the spectral sequence formula.

More precisely, in Chapter 1 we are reminded of some basic definitions and structures such as (co)homology, homotopy, exact triangles and triangulated categories, thus setting the stage for the following chapters. In Chapter 2 we construct the left derived functors (and dually the right ones), we show that they are well defined and equip ourselves with the needed propositions. In Chapter 3 we take a quick glance into the spectral sequences world, understanding how they are defined, how they converge as well as seeing some examples of already known results (like the Snake Lemma) being easily proven using our new tool.

Next, we localize the homotopy category of an abelian category with respect to quasi-isomorphisms, thus obtaining the Derived Category in Chapter 4, where we also explain how this works (object, morphism, composition - wise). Finally, in Chapter 5, we have all the tools we need to prove Grothendieck's Spectral Sequence Theorem, using both spectral sequences and derived categories, the later making the proof as easy as a corollary. As a direct application of this, we take a quick look at Leray's Spectral Sequence.

Athens, April 23, 2021

Chapter 1

Basic Definitions

1.1 Complexes, Homology & Functors

Definition 1.1.1. *Let $\mathscr{A}$ be an abelian category. A **complex** is a sequence of objects (called terms) and morphisms (called differentials),*

$$\ldots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \ldots \tag{1.1}$$

with the composite of adjacent morphisms being 0:

$$d_n \circ d_{n+1} = 0 \ , \quad for\ all\ n \in \mathbb{Z}$$

We write $(A_\bullet, d_\bullet)$ (or simply $\boldsymbol{A}$) for the sequence described above.

In the category of left R-Modules ${}_R\mathbf{Mod}$ the condition $d_n d_{n+1} = 0$ is equivalent to $\mathrm{Im} d_{n+1} \subseteq \mathrm{Ker} d_n$.

Definition 1.1.2. *Let $\mathscr{A}$ be an abelian category. We will define the category $\boldsymbol{Ch}(\mathscr{A})$ of complexes in $\mathscr{A}$ as the category which has as objects complexes $(A_\bullet, d_\bullet)$ whose terms and differentials are in $\mathscr{A}$, as morphisms chain maps*

$$f = (f_n) : (A_\bullet, d_\bullet) \longrightarrow (A'_\bullet, d'_\bullet)$$

making the following diagram commute:

$$\begin{tikzcd}
\ldots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \ldots \\
\ldots \arrow[r] & A'_{n+1} \arrow[r, "d'_{n+1}"] & A'_n \arrow[r, "d'_n"] & A'_{n-1} \arrow[r] & \ldots
\end{tikzcd} \tag{1.2}$$

(i.e. $f_n \circ d_{n+1} = d'_{n+1} \circ f_{n+1}$ for all $n \in \mathbb{Z}$)
and as composition $(g_n) \circ (f_n) = (g_n \circ f_n)$ (i.e. coordinatewise composition).

Remark 1.1.3. (*i*) ***Ch***$(\mathscr{A})$ *is an abelian category when $\mathscr{A}$ is. The proof is really simple, i.e. the zero object is the complex $\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$ and the kernel of a morphism is the complex whose terms are the* $\mathrm{Ker} f_n,\ n \in \mathbb{Z}$.

(*ii*) *Every exact sequence is a complex, for the equalities* $\mathrm{Im} d_{n+1} = \mathrm{Ker} d_n$ *imply* $d_{n+1}d_n = 0$.

Since $\mathbf{Ch}(\mathscr{A})$ is an abelian category when $\mathscr{A}$ is, its Homomorphism sets are abelian groups, i.e. addition is given by

$$f + g = (f_n + g_n), \quad \textit{where} \quad f = (f_n) \quad \textit{and} \quad g = (g_n).$$

Example 1.1.4.

- *If* $A \in Ob(\mathscr{A})$ *and* $k \in \mathbb{Z}$ *is a fixed integer, then the sequence* $\rho^k(A)$ *whose* k*-th term is* A *and other terms are* 0 *and differentials are zero maps is a complex, called* ***concentrated in degree*** k.
- *Every morphism* $f : A \to B$ *is a differential and the induced complex is called* ***concentrated in degrees*** $(k, k-1)$.
- *A short exact sequence can be made into a complex by adding* 0*s to the left and right:*

$$\cdots \to 0 \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to 0 \to \dots$$

  We assume that A *is term 2,* B *is term 1 and* C *is term 0.*
- *Every sequence of objects* (M_n) *occurs in a complex, namely,* $(M_\bullet, d_\bullet)$ *in which all the differentials* d_n *are* 0.

Now we observe that the idea of describing a module by generators and relations gives rise to a complex. Remember that every R-module M is a quotient of a free R-module, thus $M = F/K$, where F is free and K is the submodule of relations, that is, $0 \to K \to F \to M \to 0$ is exact; i.e. $K = \mathrm{Ker}(F \to M)$. If X is a basis of F then, $(X|K)$ is called a **presentation** of M. Basically, $(X|K)$ is a complete description of M up to isomorphism but, in practice it gives us little information. A presentation though, allows us to treat equations in M as if they were equations in the free R-module F. Computations in F, especially those involved in whether elements of F lie in K, become much simpler when K is also free and thus it has a basis (submodules of free modules need not be free). However, if R is principal ideal domain then, every submodule of a free module is free, and so K has a basis Y. In that case we say that $(X|Y)$ is a presentation.

For a general ring R we can iterate the idea of presentations in the following form: if $M \simeq F/K$, where F is a free, then $K = F_1/K_1$ for some free F_1 (thus K_1 is the relations among the relations). Now $0 \to K_1 \to F_1 \to K \to 0$ is exact. Splicing this to the earlier exact sequence gives exactness of

$$0 \to K_1 \to F_1 \xrightarrow{d} F \to M \to 0$$

where $d : F_1 \to F$ is the composite $F_1 \to K \subseteq F$ and $\mathrm{Im} d = K = \mathrm{Ker}(F \to M)$.

Now, we can repeat this and get $K_1 \simeq F_2/K_2$ for some free F_2. Continuing this construction gives an infinitely long exact sequence of free modules and homomorphisms, called a **resolution** of M, which serves as a generalized presentation, it is a way of treating equations in M by a sequence of equations in free modules. Keep in mind that resolutions are exact sequences, and exact sequences are complexes.

Definition 1.1.5. *A **projective resolution** of $A \in Ob(\mathscr{A})$, where $\mathscr{A}$ is an abelian category, is an exact sequence $\boldsymbol{P}$*

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

*in which each P_n is projective (see A). If $\mathscr{A}$ is ${}_R\textbf{Mod}$ or $\textbf{Mod}_R$, then a **free resolution** of a module A is a projective resolution in which each P_n is free; a **flat resolution** is an exact resolution in which each P_n is flat.*

*If $\boldsymbol{P}$ is a projective resolution of A, then its **deleted projective resolution** is the complex $\boldsymbol{P}_A$*

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0.$$

A projective resolution (or free or flat) is a complex if we assume that it has been lengthened by adding 0s to the right. Of course, a deleted resolution is no longer exact if $A \neq 0$, for $\mathrm{Im}d_1 = \mathrm{Ker}\varepsilon \neq \mathrm{Ker}(P_0 \to 0) = P_0$.

Deleting A loses no information: $A \cong \mathrm{Coker}d_1$; the inverse operation, restoring A to $\mathbf{P}_A$, is called **augmenting**. Deleted resolutions should be regarded as glorified representations.

Proposition 1.1.6. *Every (left or right) R-module A has a free resolution (which is necessarily a projective resolution and a flat resolution).*

Proof. There are a free module F_0 and an exact sequence

$$0 \to K_1 \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} A \to 0.$$

Similarly, there are a free module F_1, a surjection $\varepsilon_1 : F_1 \to K_1$, and an exact sequence

$$0 \to K_2 \xrightarrow{i_2} F_1 \xrightarrow{\varepsilon_1} K_1 \to 0.$$

Splice these together: define $d_1 : F_1 \to F_0$ to be the composite $i_1\varepsilon_1$. It is plain that $\mathrm{Im}d_1 = K_1 = \mathrm{Ker}\varepsilon$ and $\mathrm{Ker}d_1 = K_2$, yielding the exact row

$$\begin{tikzcd}
 & & F_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & F_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
0 \arrow[r] & K_2 \arrow[ur] & & K_1 \arrow[ur, "i_1"'] & & &
\end{tikzcd}$$

This construction can be iterated for all $n \geq 0$, and the ultimate exact sequence is infinitely long.

Now for the parenthetical statement: free $\Rightarrow$ projective $\Rightarrow$ flat.

□

We have proven more.

Corollary 1.1.7. *If $\mathscr{A}$ is an abelian category with enough projectives, then every object has a projective resolution.*

Remark 1.1.8. *An abelian category $\mathscr{A}$ is said to have enough projectives if, for every object $A \in \mathscr{A}$ there is a projective object $P \in \mathscr{A}$ and an epimorphism $P \to A$, or equivalently, a short exact sequence*

$$0 \to B \to P \to A \to 0.$$

Definition 1.1.9. *An **injective resolution** of $A \in Ob(\mathscr{A})$, where $\mathscr{A}$ is an abelian category, is an exact sequence **E***

$$0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \dots$$

in which each E^n is injective (see A).

*If **E** is an injective resolution of A, then its **deleted injective resolution** is the complex $\mathbf{E}^A$*

$$0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \dots$$

Deleting A loses no information, for $A \cong \mathrm{Ker} d^0$.

Proposition 1.1.10. *Every (left or right) R-module A has an injective resolution.*

Proof. Similar to the previous proof, but instead of kernels we use cokernels. □

Deleted injective resolutions should be regarded as duals of presentations.

Corollary 1.1.11. *If $\mathscr{A}$ is an abelian category with enough injectives, then every object has an injective resolution.*

Remark 1.1.12.

- *Every sheaf with values in $\mathscr{A}$ has an injective resolution.*
- *Most categories of sheaves do not have enough projectives.*
- *We may lengthen an injective resolution by adding 0s to the left, but this does not yet make it a complex, for the definition of complexes the indices must decrease if we go to the right. The simplest way to satisfy the definition is to use negative indices: define $C_{-n} = E^n$, and*

  $$0 \to A \to C_0 \to C_{-1} \to C_{-2} \to \dots$$

  is a complex.

Definition 1.1.13. *Given a projective resolution **P** in an abelian category $\mathscr{A}$,*

$$\dots \to P_n \xrightarrow{d_n} P_{n-1} \to \dots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0,$$

*define $K_0 = \mathrm{Ker}\varepsilon$ and $K_n = \mathrm{Ker} d_n$, for $n \geq 1$. We call the K_n the n**th syzygy** of **P**.*

*Given an injective resolution **E***

$$0 \to A \xrightarrow{\eta} E_0 \xrightarrow{d^0} E^1 \to \dots \to E^n \xrightarrow{d^n} E^{n+1} \to \dots$$

*define $V^0 = \mathrm{Coker}\eta$ and $V^n = \mathrm{Coker} d^{n-1}$, for $n \geq 1$. We call V^n the n**th cosygyzy** of **E**.*

Example 1.1.14. *Let $\mathscr{A}$ be an abelian category*

(i) *Let $F : \mathscr{A} \to \boldsymbol{Ab}$ (the category of abelian groups) be a covariant additive functor, and let $\boldsymbol{A}$*
$$\cdots \to A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$
be a complex. Then $(F\boldsymbol{A}, Fd) = F\boldsymbol{A}$ is
$$\cdots \to F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \to \ldots$$
is also a complex, for $0 = F(0) = F(d_n d_{n+1}) = F(d_n)F(d_{n+1})$ (the equation $0 = F(0)$ holds because F is additive). Note that even if the original complex is exact, the functored complex $F\boldsymbol{A}$ may not be exact.

(ii) *If F is a contravariant additive functor, it is also true that $F\boldsymbol{A}$ is a complex but we have to arrange notation so that differentials lower indices.*

Definition 1.1.15. *If $\boldsymbol{A}$ is a complex in $\boldsymbol{Ch}(\mathscr{A})$, where $\mathscr{A}$ is an abelian category, define*
$$n - \boldsymbol{chains} = A_n$$
$$n - \boldsymbol{cycles} = Z_n(\boldsymbol{A}) = \mathrm{Ker} d_n$$
$$n - \boldsymbol{boundaries} = B_n(\boldsymbol{A}) = \mathrm{Im} d_{n+1}.$$
Notice that A_n, Z_n, and B_n all lie in $\mathscr{A}$.

In ${}_R\mathbf{Mod}$ the equation $d_n d_{n+1} = 0$ in a complex is equivalent to the condition $\mathrm{Im} d_{n+1} \subseteq \mathrm{Ker} d_n$, hence $B_n(\mathbf{A}) \subseteq Z_n(\mathbf{A})$ for every complex $\mathbf{A}$. This is also true in an abelian category:

$$\begin{tikzcd}
B_n \arrow[d, dashed, "\theta"'] \arrow[dr, "j"] \arrow[drr, "0"] & & \\
Z_n \arrow[r, "i"'] & A_n \arrow[r, "d_n"'] & A_{n-1}.
\end{tikzcd}$$

Definition 1.1.16. *If $\boldsymbol{A}$ is a complex in $\boldsymbol{Ch}(\mathscr{A})$, where $\mathscr{A}$ is an abelian category, and $n \in \mathbb{Z}$, its **n-th homology** is*
$$H_n(\boldsymbol{A}) = Z_n(\boldsymbol{A})/B_n(\boldsymbol{A}).$$
*Now $H_n(\boldsymbol{A})$ lies in $Ob(\mathscr{A})$ if quotients are viewed as objects. However, if we recognise $\mathscr{A}$ as a full subcategory of $\boldsymbol{Ab}$, then an element of $H_n(\boldsymbol{A})$ is a coset $z + B_n(\boldsymbol{A})$, which we call a **homology class**, and often denote it by $cls(z)$.*

Example 1.1.17.

(i) *A complex $\boldsymbol{A}$ is an exact sequence if and only if $H_n(\boldsymbol{A}) = 0$ for all n. Thus, homology measures the deviation of a complex from being an exact sequence. An exact sequence is often called an **acyclic complex**; which means no cycles that are boundaries.*

(ii) *There are two **fundamental exact sequences** arising from a complex $\boldsymbol{A}$ for each $n \in \mathbb{Z}$,*
$$0 \to B_n \xrightarrow{i_n} Z_n \to H_n(\boldsymbol{A}) \to 0$$

and

$$0 \to Z_n \xrightarrow{j_n} A_n \xrightarrow{d'_n} B_{n-1} \to 0,$$

where i_n, j_n are inclusions and $j_{n-1}i_{n-1}d'_n = d_n$; that is d'_n is just d_n with its target changed from A_{n-1} to $\mathrm{Im} d_n = B_{n-1}$.

(iii) *If $\boldsymbol{A}$ is a complex with all $d_n = 0$, then $H_n(\boldsymbol{A}) = A_n$ for all $n \in \mathbb{Z}$, for*

$$H_n(\boldsymbol{A}) = \mathrm{Ker} d_n / \mathrm{Im} d_{n+1} = \mathrm{Ker} d_n = A_n.$$

In particular, the subcomplexes $\boldsymbol{Z}$ of cycles and $\boldsymbol{B}$ of boundaries have all differentials 0, and so $H_n(\boldsymbol{Z}) = Z_n$ and $H_n(\boldsymbol{B}) = B_n$

Metatheorem 1.1.18. *Let $\mathscr{A}$ be an abelian category.*

(i) *If a statement is of the form "p implies q", where p and q are categorical statements about a diagram in $\mathscr{A}$, and if the statement is true in* ***Ab****, then the statement is true in $\mathscr{A}$.*

(ii) *Consider a statement of the form "p implies q", where p is a categorical statement concerning a diagram in $\mathscr{A}$, and q states that additional morphisms exist between certain objects in the diagram and that some categorical statement is true of the extended diagram. If the statement can be proved in* ***Ab*** *by constructing the additional morphisms through diagram chasing, then the statement is true in $\mathscr{A}$.*

Proposition 1.1.19. *If $\mathscr{A}$ is an abelian category, then $H_n : \boldsymbol{Ch}(\mathscr{A}) \to \mathscr{A}$ is an additive functor for each $n \in \mathbb{Z}$.*

Proof. In light of the **Metatheorem**, it suffices to prove this proposition when $\mathscr{A} = \mathbf{Ab}$. We have defined H_n on objects, it remains to define H_n on morphisms. If $f : \mathbf{A} \to \mathbf{A}'$ is a chain map, define $H_n(f) : H_n(\mathbf{A}) \to H_n(\mathbf{A}')$ by

$$H_n(f) : cls(z_n) \mapsto cls(f_n z_n).$$

We must show that $f_n z_n$ is a cycle and that $H_n(f)$ is independent of the choice of cycle z_n; both of these follow from f being a chain map, that is, from commutativity of the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & A'_{n+1} \arrow[r, "d'_{n+1}"] & A'_n \arrow[r, "d'_n"] & A'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

First, let z be an n-cycle in $Z_n(\mathbf{A})$, so that $d_n z = 0$. Then commutativity of the diagram gives $d'_n f_n z = f_{n-1} d_n z = 0$, so that $f_n z$ is an n-cycle.

Next, assume that $z + B_n(\mathbf{A}) = y + B_n(\mathbf{A})$, hence $z - y \in B_n(\mathbf{A})$;

$$z - y = d_{n+1} a$$

for some $a \in A_{n+1}$. Applying f_n gives

$$f_n z - f_n y = f_n d_{n+1} a = d'_n f_{n+1} a \in B_n(\mathbf{A}').$$

Thus, $cls(f_n z) = cls(f_n y)$, and $H_n(f)$ is well-defined.

Let us now see that H_n is a functor. It is obvious that $H_n(1_{\mathbf{A}})$ is the identity. If f and g are chain maps whose composite gf is defined, then for every n-cycle z, we have (with obvious abbreviations)

$$\begin{aligned} H_n(gf) : cls(z) &\mapsto (gf)_n cls(z) \\ &= g_n f_n(cls(z)) \\ &= H_n(g)(cls(f_n z)) \\ &= H_n(g)H_n(f)(cls(z)). \end{aligned}$$

Finally, H_n is additive: if $f, g : \mathbf{A} \to \mathbf{A}'$ are chain maps, then

$$\begin{aligned} H_n(f+g) : cls(z) &\mapsto (f_n + g_n)cls(z) \\ &= cls(f_n z + g_n z) \\ &= (H_n(f) + H_n(g))cls(z). \end{aligned}$$

□

The previous proposition says that if $\mathbf{A}$ is a complex in an abelian category $\mathscr{A}$, then $H_n(\mathbf{A}) \in Ob(\mathscr{A})$ for all $n \in \mathbb{Z}$. In particular, if $\mathscr{A}$ is the category of all sheaves of abelian groups over a space X, then $H_n(\mathbf{A})$ is a sheaf.

Definition 1.1.20. *We call $H_n(f)$ the **induced map**, and we usually denote it by f_{*_n} or simply f_*.*

The following construction is fundamental. It gives a relation between different homologies. The proof is a series of diagram chases, which is legitimate because of the **Metatheorem**.

Proposition 1.1.21. *Let $\mathscr{A}$ be an abelican category. If*

$$0 \to \boldsymbol{A}' \xrightarrow{i} \boldsymbol{A} \xrightarrow{p} \boldsymbol{A}'' \to 0$$

is an exact sequence in $\boldsymbol{Ch}(\mathscr{A})$, then, for each $n \in \mathbb{Z}$, there is a moprhism in $\mathscr{A}$

$$\partial_n : H_n(\boldsymbol{A}'') \to H_{n-1}(\boldsymbol{A}')$$

defined by

$$\partial_n : cls(z_n'') \mapsto cls(i_{n-1}^{-1} d_n p_n^{-1} z_n'').$$

Definition 1.1.22. *The morphisms $\partial_n : H_n(\boldsymbol{A}'') \to H_{n-1}(\boldsymbol{A}')$ are called **connecting homomorphisms**.*

The first question we ask is what homology functors do to a short exact sequence of complexes. The next theorem is also proved by diagram chasing.

Theorem 1.1.23. ***(Long Exact Sequence)*** *Let $\mathscr{A}$ be an abelian category. If*

$$0 \to \boldsymbol{A}' \xrightarrow{i} \boldsymbol{A} \xrightarrow{p} \boldsymbol{A}'' \to 0$$

is an exact sequence in $\boldsymbol{Ch}(\mathscr{A})$, then, there is an exact sequence in $\mathscr{A}$

$$\cdots \to H_{n+1}(\boldsymbol{A}'') \xrightarrow{\partial_{n+1}} H_n(\boldsymbol{A}') \xrightarrow{i_*} H_n(\boldsymbol{A}) \xrightarrow{p_*} H_n(\boldsymbol{A}'') \xrightarrow{\partial_n} H_{n-1}(\boldsymbol{A}') \to \dots$$

Proof.

□

The Long Exact Sequence theorem is often called the **exact triangle** because of the diagram

$$\begin{tikzcd} H_\bullet(\mathbf{A}') \arrow[rr, "i_*"] & & H_\bullet(\mathbf{A}) \arrow[dl, "p_*"] \\ & H_\bullet(\mathbf{A}'') \arrow[ul, "\partial"] & \end{tikzcd}$$

Corollary 1.1.24. ***(Snake Lemma)*** *Let $\mathscr{A}$ be an abelian category. Given a commutative diagram in $\mathscr{A}$ with exact rows,*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g"] & A' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

there is an exact sequence in $\mathscr{A}$

$$0 \to \mathrm{Ker} f \to \mathrm{Ker} g \to \mathrm{Ker} h \to \mathrm{Coker} f \to \mathrm{Coker} g \to \mathrm{Coker} h \to 0.$$

Proof. If we view each of the vertical maps f, g and h as a complex concentrated diagram in degrees 1, 0, then the given commutative diagram can be viewed as a short exact sequence of complexes. The homology of each of these complexes has only two nonzero terms ($H_1 = \mathrm{Ker} f,\ H_0 = \mathrm{Coker} f$ and the other $H_n = 0$). The lemma now follows from the long exact sequence.

□

Theorem 1.1.25. ***(Naturality of** ∂**)*** *Let $\mathscr{A}$ be an abelian category. Given a commutative diagram in $\boldsymbol{Ch}(\mathscr{A})$ with exact rows,*

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{A}' \arrow[r, "i"] \arrow[d, "f"] & \boldsymbol{A} \arrow[r, "p"] \arrow[d, "g"] & \boldsymbol{A}'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \boldsymbol{B}' \arrow[r, "j"'] & \boldsymbol{B} \arrow[r, "q"'] & \boldsymbol{B}'' \arrow[r] & 0 \end{tikzcd}$$

there is a commutative diagram in $\mathscr{A}$ with exact rows

$$\begin{tikzcd} \dots \arrow[r] & H_n(\boldsymbol{A}') \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(\boldsymbol{A}) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(\boldsymbol{A}'') \arrow[r, "\partial"] \arrow[d, "h_*"] & H_{n-1}(\boldsymbol{A}') \arrow[r] \arrow[d, "f_*"] & \dots \\ \dots \arrow[r] & H_n(\boldsymbol{B}') \arrow[r, "j_*"'] & H_n(\boldsymbol{B}) \arrow[r, "q_*"'] & H_n(\boldsymbol{B}'') \arrow[r, "\partial'"'] & H_{n-1}(\boldsymbol{B}') \arrow[r] & \dots \end{tikzcd}$$

Proof. Exactness of the rows is the Long Exact Theorem, while commutativity of the first two squares follows from H_n being a functor. To prove commutativity of the square involving the connecting homomorphisms, let us first display the chain maps and differentials in one (3D) diagram:

$$\begin{tikzcd}
 & 0 \arrow[rr] & & A'_n \arrow[rr, "i"] \arrow[dl, "d'"'] \arrow[dd, dashed, "f_*"] & & A_n \arrow[rr, "p"] \arrow[dl, "d"'] \arrow[dd, dashed, "g_*"] & & A''_n \arrow[r] \arrow[dl, "d''"'] \arrow[dd, dashed, "h_*"] & 0 \\
0 \arrow[rr] & & A'_{n-1} \arrow[rr, "i"] \arrow[dd, "f_*"'] & & A_{n-1} \arrow[rr, "p"] \arrow[dd, "g_*"'] & & A''_{n-1} \arrow[rr] \arrow[dd, "h_*"'] & & 0 \\
 & 0 \arrow[rr, dashed] & & B'_n \arrow[rr, dashed, "j"] \arrow[dl, dashed, "\delta'"'] & & B_n \arrow[rr, dashed, "q"] \arrow[dl, dashed, "\delta"'] & & B''_n \arrow[r, dashed] \arrow[dl, dashed, "\delta''"'] & 0 \\
0 \arrow[rr] & & B'_{n-1} \arrow[rr, "j"'] & & B_{n-1} \arrow[rr, "q"'] & & B''_{n-1} \arrow[rr] & & 0
\end{tikzcd}$$

If $cls(z'') \in H_n(\mathbf{A}'')$, we must show that $f_*\partial cls(z'') = \partial h_* cls(z'')$. Let $a \in A_n$ be a lifting of z'', that is, $pa = z''$. Now $\partial cls(z'') = cls(z')$, where $iz' = da$. Hence, $f_*\partial cls(z'') = clscls(fz')$. On the other hand, since h is a chain map, we have $qga = hpa = hz''$. In computing $\partial' cls(hz'')$, we choose ga as the lifting of hz''. Hence, $\partial' cls(hz'') = cls(u')$, where $ju' = \delta ga$. But $jfz' = giz' = gda = \delta ga = ju'$, and so $fz' = u'$, because j is injective.

□

1.2 Homotopic Chain Maps

There are interresting maps of complexes that are not chain maps.

Definition 1.2.1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be complexes, and let $p \in \mathbb{Z}$. A* ***map of degree*** *p, denoted by $s : \boldsymbol{A} \to \boldsymbol{B}$, is a sequence $s = (s_n)$ with $s_n : A_n \to B_{n+p}$ for all n.*

For example, a chain map is a map of degree 0, while the differentials of $\mathbf{A}$ form a map $d : \mathbf{A} \to \mathbf{A}$ of degree -1.

We know introduce a notion that arises in topology.

Definition 1.2.2. *A chain map $f : \boldsymbol{A} \to \boldsymbol{A}'$ is* ***null-homotopic****, denoted by $f \simeq 0$ (where 0 is the zero chain map) if , for all n, there is a map $s = (s_n) : \boldsymbol{A} \to \boldsymbol{A}'$ of degree $+1$ with*

$$f_n = d'_{n+1}s_n + s_{n-1}d_n.$$

$$\begin{tikzcd}
\ldots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "s_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"] & \ldots \\
\ldots \arrow[r] & A'_{n+1} \arrow[r, "d'_{n+1}"'] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \ldots
\end{tikzcd}$$

Definition 1.2.3. *Two chain maps $f, g : \boldsymbol{A} \to \boldsymbol{A}'$ are* ***homotopic****, denoted by $f \simeq g$, if, $f - g \simeq 0$.*

Theorem 1.2.4. *Homotopic chain maps induce the same morphism in homology: if $f, g : \boldsymbol{A} \to \boldsymbol{A}'$ are chain maps and $f \simeq g$, then for all n,*

$$f_{*_n} = g_{*_n} : H_n(\boldsymbol{A}) \to H_n(\boldsymbol{A}')$$

Proof. If z is an n-cycle, then $d_n z = 0$ and

$$f_n z - g_n z = d'_{n+1} s_n z + s_{n-1} d_n z = d'_{n+1} s_n z.$$

Therefore, $f_n z - g_n z \in B_n(\mathbf{A}')$, and so $f_{*_n} = g_{*_n}$.

□

Definition 1.2.5. *A complex* $\boldsymbol{A}$ *has a* ***contracting homotopy*** *if its identity* $1_{\boldsymbol{A}}$ *is null-homotopic. A complex* $\boldsymbol{A}$ *is* ***contractible*** *if its identity* $1 = 1_{\boldsymbol{A}}$ *is null-homotopic; that is, there is* $s : \boldsymbol{A} \to \boldsymbol{A}$ *of degree* $+1$ *with* $1 = sd + ds$.

Proposition 1.2.6. *A complex* $\boldsymbol{A}$ *having a contracting homotopy is acyclic, that is, it is an exact sequence.*

Proof. $1_{\mathbf{A}} : H_n(\mathbf{A}) \to H_n(\mathbf{A})$ is the identity map, while $0_* : H_n(\mathbf{A}) \to H_n(\mathbf{A})$ is the zero map. Since $1_{\mathbf{A}} \simeq 0$, however, these maps are the same. It follows that $H_n(\mathbf{A}) = \{0\}$ for all n, that is, $\mathrm{Ker} d_n = \mathrm{Im} d_{n+1}$ for all n, and this is the definition of exactness.

□

Definition 1.2.7. *The* ***homotopy category*** *of complexes* $\boldsymbol{K}(\mathscr{A})$ *is the category whose objects are the objects of* $\boldsymbol{Ch}(\mathscr{A})$*, i.e.* $Ob(\boldsymbol{K}(\mathscr{A})) = Ob(\boldsymbol{Ch}(\mathscr{A}))$*, and morphisms* $Hom_{\boldsymbol{K}(\mathscr{A})}(\boldsymbol{A}, \boldsymbol{A}') = Hom_{\boldsymbol{Ch}(\mathscr{A})}(\boldsymbol{A}, \boldsymbol{A}')/\sim$.

That definition makes sense, i.e. that the composition is well-defined in $\mathbf{K}(\mathscr{A})$, follows from the following assertions which are easily verified.

Proposition 1.2.8.

(i) Homotopy equivalence between morphisms $\boldsymbol{A} \longrightarrow \boldsymbol{A}'$ *of complexes is an equivalence relation.*

(ii) Homotopically trivial morphisms form an ideal in the ring of morphisms of $\boldsymbol{Ch}(\mathscr{A})$.

(iii) If $f \simeq g : \boldsymbol{A} \longrightarrow \boldsymbol{A}'$*, then* $H_n(f) = H_n(g)$ *for all* i.

(iv) $f : \boldsymbol{A} \longrightarrow \boldsymbol{A}'$ *and* $g : \boldsymbol{A}' \longrightarrow \boldsymbol{A}$ *are given such that* $f \circ g \simeq id_B$ *and* $g \circ f \simeq id_A$ *, then* f *and* g *are quasi-isomorphisms (which means that the induced morphisms* $H_n(\boldsymbol{A}) \xrightarrow{f_{*_n}} H_n(\boldsymbol{A}') \xrightarrow{g_{*_n}} H_n(\boldsymbol{A})$ *of homology groups, of* f *and* g*, are isomorphisms for all* n*, see 4.1.1) and actually,* $H_n(f^{-1}) = H_n(g)$.

1.3 Cohomology As a Dual to Homology

The Cohomology is dual to homology in the sense that it is obtained by reindexing with superscripts. Namely, we take the so called **cochain complex** with rising indices by: $A^n = A_{-n}$.

Definition 1.3.1. *Let* $\mathscr{A}$ *be an abelian category. A cochain complex* $\boldsymbol{A}$ *in* $\mathscr{A}$ *is a sequence of objects (terms)* $A^n \in \mathscr{A}$ *and morphisms (differentials):*

$$\cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \dots$$

with the composite of adjacent morphisms being 0.

Morphisms, which now are called **cochain maps** (and later on the quasi-isomorphisms, see 4.1.1) are defined exactly as for chain complexes. All the cochain complexes of $\mathscr{A}$ form the category $\mathbf{Co}(\mathscr{A})$ of cochain complexes in $\mathscr{A}$. Now we have the following for an cochain complex $\mathbf{A}$:

(i) $Z^n(\mathbf{A}) = \mathrm{Ker}(d^n)$ are the n-cocycles,

(ii) $B^n(\mathbf{A}) = \mathrm{Im}(d^{n-1})$ are the n-coboundaries,

(iii) and the subquotient $H^n(\mathbf{A}) = Z^n(\mathbf{A})/B^n(\mathbf{A})$ is the n-cohomology of A.

1.4 Triangulated Categories

One of the most important things that we lost in passing to the homotopy category ($\mathbf{K}(\mathscr{A})$ = the category whose objects are complexes of objects in $\mathscr{A}$ and morphisms are chain maps modulo the homotopy equivalence relation) is the ability to say that a sequence of morphisms is exact: we no longer have notions of kernel and cokernel, since it is not an abelian category. Verdier′s initial contribution to the development of derived category was the observation that a form of exactness is still preserved, in the notion of exact triangles.

Definition 1.4.1. *Let $\mathscr{D}$ be an additive category. The structure of a triangulated category is given by an additive equivalence*

$$T : \mathscr{D} \longrightarrow \mathscr{D},$$

the shift functor, and a set of distinguished (or "exact") triangles

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

subject to the axioms TR1 - TR4 below:

Before we actually explain the axioms TR, we'll introduce the notation $A[1] := T(A)$ for any $A \in Ob(\mathscr{D})$ and $f[1] = T(f) \in Hom(A[1], B[1])$, for any $f \in Hom(A, B)$. Similarly, one writes $A[n] := T^n(A)$ and $f[n] := T^n(f)$ for $n \in \mathbb{Z}$. Thus a triangle will also be denoted by $A \to B \to C \to A[1]$.

A morphism between two triangles is given by a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "T(f)"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

It is an isomorphism if f, g, h are isomorphisms.

Now we need to define what a distinguised triangle actually is. For this, we need to start by explaining how the autofunctor $[n] : \mathbf{Co}(\mathscr{A}) \to \mathbf{Co}(\mathscr{A})$ works. It is defined by:

$$[n]A^\bullet := A^{\bullet+n} \text{ and } [n]d^\bullet_{A^\bullet} := (-1)^n d^{\bullet+n}_{A^\bullet},$$

for $(A^\bullet, d^\bullet_{A^\bullet}) \in \mathbf{Co}(\mathscr{A})$.

We usually write $A[n]$ and $d_A[n]$ meaning $[n]A^\bullet$ and $[n]d^\bullet_{A^\bullet}$ respectively.

Example 1.4.2. *For* $[1] : \boldsymbol{Co}(\mathscr{A}) \to \boldsymbol{Co}(\mathscr{A})$*, a morphism* $f : A \to B$ *in* $\boldsymbol{Ch}(\mathscr{A})$*:*

$$\begin{tikzcd}
\ldots \arrow[r] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & B \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & A^{n+2} \arrow[r] \arrow[d, "f^{n+2}"] & \ldots \\
\ldots \arrow[r] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_b^{n+1}"'] & B^{n+2} \arrow[r] & \ldots
\end{tikzcd}$$

$[1]f = f[1] : A[1] \to B[1]$, *becomes:*

$$\begin{tikzcd}
\ldots \arrow[r] & A^{n+1} \arrow[r, "-d_A^{n+1}"] \arrow[d, "f^{n+1}"] & A^{n+2} \arrow[r, "-d_A^{n+2}"] \arrow[d, "f^{n+2}"] & A^{n+3} \arrow[r] \arrow[d, "f^{n+3}"] & \ldots \\
\ldots \arrow[r] & B^{n+1} \arrow[r, "-d_B^{n+1}"'] & B^{n+2} \arrow[r, "-d_b^{n+2}"'] & B^{n+3} \arrow[r] & \ldots
\end{tikzcd}$$

Let $f : A \to B$ and $g : B \to C$ be two morphisms in $\mathbf{K}(\mathscr{A})$. Then we have $A \xrightarrow{f} B \xrightarrow{g} C$. If there is a morphism $h : C \to A[1]$ in $\mathbf{K}(\mathscr{A})$, then the sequence:

$$A \to B \to C \to A[1]$$

is said to be a triangle in $\mathbf{K}(\mathscr{A})$ and we sometimes write such triangle as:

$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\
& C \arrow[ul, "h", "(1)"'] &
\end{tikzcd}$$

A morphism of triangles $(\alpha, \beta, \gamma, \alpha[1])$ is the commutative diagram of the top and bottom triangles:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha(1)"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'_f \arrow[r, "\alpha"'] & A'[1]
\end{tikzcd}$$

in $\mathbf{K}(\mathscr{A})$. When α, β, γ are isomorphisms of $\mathbf{K}(\mathscr{A})$, then the triangles are said to be **isomorphic triangles**.

For an arbitrary given morphism $f : A \to B$ of complexes, we can construct a complex C_f and morphisms β and α so that

$$A \xrightarrow{f} B \xrightarrow{\beta} C_f \xrightarrow{\alpha} A[1]$$

may become a triangle. Define the complex C_f by

$$C_f^n := A^{n+1} \oplus B^n = A[1]^n \oplus B^n$$

and $d_C^n : C_f^n \to C_f^{n+1}$ by

$$d_C^n \begin{pmatrix} x^{n+1} \\ y_n \end{pmatrix} = \begin{bmatrix} d_A[1]^n & 0 \\ f[1]^n & d_B^n \end{bmatrix} \begin{pmatrix} x^{n+1} \\ y_n \end{pmatrix} = \begin{pmatrix} -d_A^{n+1}(x^{n+1}) \\ f^{n+1}(x^{n+1}) + d_b^n(y^n) \end{pmatrix} \in C_f^{n+1}.$$

Then we have

$$d_C^{n+1} \circ d_C^n \begin{pmatrix} x^{n+1} \\ y_n \end{pmatrix} = \begin{pmatrix} -d_A^{n+2} \circ d_A^{n+1}(x^{n+1}) \\ f^{n+2}(-d_A^{n+1}(x^{n+1})) + d_B^{n+1}(f^{n+1}(x^{n+1}) + d_B^n(y^n)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in C_f^{n+2}$$

from the commutativity of the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & A^{n+2} \arrow[r] \arrow[d, "f^{n+2}"] & \cdots \\
\cdots \arrow[r] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"'] & B^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

We have proven tha C_f is a complex. For $C_f = A[1] \oplus B$, define $\beta : B \to C_f$ and $\alpha : C_f \to A[1]$ in $A \to B \xrightarrow{\beta} C_f \xrightarrow{\alpha} A[1]$, by $\beta := \begin{bmatrix} 0 \\ 1_B \end{bmatrix}$ and $\alpha := [1_{A[1]}, 0]$. Then $A \to B \xrightarrow{\beta} C_f \xrightarrow{\alpha} A[1]$ becomes a triangle. Notice that $0 \to A \to B \xrightarrow{\beta} C_f \xrightarrow{\alpha} A[1] \to 0$ is an exact sequence in $\mathbf{Ch}(\mathscr{A})$.

A triangle $A \to B \to C \to A[1]$ is said to be a **distinguished triangle** when for a morphism $A' \xrightarrow{f'} B'$ of complexes there is an isomorphism of triangles

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\simeq"] & B \arrow[r, "g"] \arrow[d, "\simeq"] & C \arrow[r, "h"] \arrow[d, "\simeq"] & A[1] \arrow[d, "T(f)"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "\beta"'] & C_{f'} \arrow[r, "\alpha"'] & A'[1]
\end{tikzcd}$$

in $\mathbf{K}(\mathscr{A})$.

The complex C_f is said to be the **mapping cone** of $f : A \to B$. Notice that C_f depends upon the homotopy equivalence classes. Namely, if we have $f_1 \simeq f_2$, then there is an isomorphism $C_{f_1} \simeq C_{f_2}$ in $\mathbf{K}(\mathscr{A})$.

We will write "(f_1, g_1, h_1) on (A, B, C) " meaning the distinguished triangle $A \xrightarrow{f_1} B \xrightarrow{g_1} C \xrightarrow{h_1} A[1]$.

Now here are the **Axioms** for a triangulated category:

TR1 (i) Any triangle of the form

$$A \xrightarrow{id} A \to 0 \to A[1]$$

is distinguished.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \to B$ can be completed into a distinguished triangle:

$$A \xrightarrow{f} B \to C_f \to A[1]$$

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

TR3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & A[1] \arrow[d, "T(f)"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1] \end{tikzcd}$$

Then the diagram can be completed into a commutative one, i.e. to a morphism (f, g, h) of triangles, by a (not necessarily) unique $h : C \to C'$.

TR4 (Octahedron Axiom). Suppose that there are three distinguished triangles

$$(f_1, g_1, h_1) \;\; on \;\; (A, B, C'),$$

$$(f_2, g_2, h_2) \;\; on \;\; (B, C, A'),$$

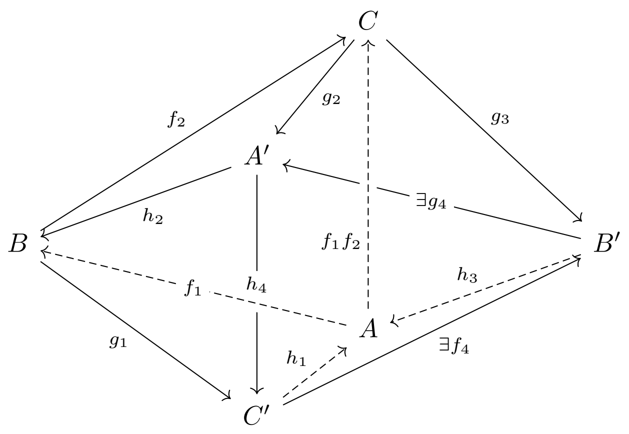
$$(f_1 f_2, g_3, h_3) \;\; on \;\; (A, C, B').$$

Then there is a forth one

$$(f_4, g_4, h_4) \;\; on \;\; (C', B', A')$$

such that in the following octahedron we have,

1. the four distinguished triangles form four of the faces,
2. the remaining four faces commute, that is $h_1 = h_3 f_4 : C' \to B \to A[1]$ and $g_2 = g_4 g_3 : C \to B' \to A'$,
3. $h_3 f_2 = f_4 g_2 : B \to B'$,
4. $f_1 h_3 = h_2 g_4 : B' \to B$.



The first two axioms TR1 and TR2 seem very natural. Essentially, they are saying that the set of distinguished triangles is preserved under shift and isomorphisms and that there are enough distinguished triangles available. The third one, TR3, seems a little less so, due to the non-uniqueness of the completing morphism. Note, a priori we have not required that in a triangle $A \to B \to C \to A[1]$ the composition $A \to C$ is zero. But this can be easily deduced by combining TR1 and TR3.

Definition 1.4.3. *An additive functor*

$$F : \mathscr{D} \longrightarrow \mathscr{D}'$$

between triangulated categories $\mathscr{D}$ and $\mathscr{D}'$ is called exact if the following two conditions are satisfied:

(i) There is a natural transformation

$$F \circ T_{\mathscr{D}} \xrightarrow{\sim} T_{\mathscr{D}'} \circ F.$$

(ii) Any distinguished triangle $A \to B \to C \to A[1]$ in $\mathscr{D}$ is mapped to a distinguished triangle $F(A) \to F(B) \to F(C) \to F(A)[1]$ in $\mathscr{D}'$, where $F(A[1])$ is identified with $F(A)[1]$ via the functor isomorphism in (i).

Chapter 2

Derived Functors

Derived Functors is the source we need to get short exact sequences of complexes. The main idea is to replace every module by a deleted resolution of it. Given a short exact sequence of modules, we shall see that this replacement gives short exact sequence of complexes. We then apply either Hom or $\otimes$, and the resulting homology modules are called Ext or Tor.

2.1 The Comparison Theorem

We know that a module has many presentations; since resolutions are generalized presentations, the next result is foundamental.

Theorem 2.1.1. ***(Comparison Theorem).*** *Let $\mathscr{A}$ be an abelian category. Given a morphism $f : A \to A'$ in $\mathscr{A}$ consider the diagram*

$$\begin{tikzcd} \dots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\hat{f}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\hat{f}_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "\hat{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \dots \arrow[r] & P'_2 \arrow[r, "d'_2"'] & P'_1 \arrow[r, "d'_1"'] & P'_0 \arrow[r, "\varepsilon'"'] & A' \arrow[r] & 0 \end{tikzcd}$$

where the rows are complexes. If each P_n in the top row is projective, and if the bottom row is exact, then there exists a chain map $\hat{f} : P_A \to P'_{A'}$ making the completed diagram commute. Moreover, any two such chain maps are homotopic.

Remark 2.1.2. *The dual of the theorem is also true. Given a morphism $g : A' \to A$, consider the diagram of the negative complexes*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & E^0 \arrow[r] & E^1 \arrow[r] & E^2 \arrow[r] & \dots \\ 0 \arrow[r] & A' \arrow[r] \arrow[u, "g"] & X^0 \arrow[r] \arrow[u, dashed] & X^1 \arrow[r] \arrow[u, dashed] & X^2 \arrow[r] \arrow[u, dashed] & \dots \end{tikzcd}$$

If the bottom row is exact and each E^n in the top row is injective, then there exists a chain map $X^{A'} \to E^A$ making the completed diagram commute.

Proof. It suffices to prove the result for $\mathscr{A} = \mathbf{Ab}$.

(i) We prove the existence of $\hat{f}_n$ by induction on $n \geq 0$.

For the base step $n = 0$, consider the diagram

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "\hat{f}_0"'] \arrow[d, "f\varepsilon"] & \\ P_0' \arrow[r, "\varepsilon'"'] & A' \arrow[r] & 0. \end{tikzcd}$$

Since ε' is surjective and P_0 is projective, there is a map $\hat{f}_0 : P_0 \to P_0'$ with $\varepsilon' \hat{f}_0 = f\varepsilon$.

For the inductive step, consider the diagram

$$\begin{tikzcd} P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"] & P_{n-1} \arrow[d, "\hat{f}_{n-1}"] \\ P_{n+1}' \arrow[r, "d_{n+1}'"'] & P_n' \arrow[r, "d_n'"'] & P_{n-1}'. \end{tikzcd}$$

If $\mathrm{Im}\hat{f}_n d_{n+1} \subseteq \mathrm{Im}d_{n+1}'$, then we have the diagram

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, "\hat{f}_{n+1}"'] \arrow[d, "\hat{f}_n d_{n+1}"] & \\ P_{n+1}' \arrow[r, "d_{n+1}'"'] & \mathrm{Im}d_{n+1}' \arrow[r] & 0, \end{tikzcd}$$

and projectivity of P_{n+1} gives $\hat{f}_{n+1} : P_{n+1} \to P_{n+1}'$ with $d_{n+1}'\hat{f}_{n+1} = \hat{f}_n d_{n+1}$. To check that this holds, note the exactness of P_n' of the bottom row of the original diagram gives $\mathrm{Im}d_{n+1}' = \mathrm{Ker}d_n'$, and so it suffices to prove that $d_n'\hat{f}_n d_{n+1} = 0$. But $d_n'\hat{f}_n d_{n+1} = \hat{f}_{n-1} d_n d_{n+1} = 0$.

(ii) We prove the uniqueness of $\hat{f}$ up to homotopy. If $h : P_A \to P_A'$ is another chain map with $\varepsilon' h_0 = f\varepsilon$, we construct the terms $s_n : P_n \to P_n'$ of a homotopy s by unduction on $n \geq -1$. That is we will show that

$$h_n - \hat{f}_n = d_{n+1}' s_n + s_{n+1} d_n.$$

For the base step, first view A and 0 as being terms -1 and -2 in the top complex, and define $d_0 = \varepsilon$ and $d_{-1} = 0$. Also view A' and 0 as being terms -1 and -2 in the bottom complex, and define $d_0' = \varepsilon'$ and $d_{-1} = 0$. Finally, define $\hat{f}_{-1} = f = h_{-1}$ and $s_{-2} = 0$.

$$\begin{tikzcd} \dots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0=\varepsilon"] \arrow[dl, dashed, "s_0"'] \arrow[d, shift right, "\hat{f}_0"'] \arrow[d, shift left, "h_0"] & A \arrow[r, "d_{-1}"] \arrow[dl, "s_{-1}"] \arrow[d, "f"] & 0 \arrow[dl, "s_{-2}"] \\ \dots \arrow[r] & P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "d_0'=\varepsilon"'] & A' \arrow[r, "d_{-1}'"'] & 0. \end{tikzcd}$$

With this notation, defining $s_{-1} = 0$ gives $h_{-1} - \hat{f}_{-1} = f - f = 0 = d_0' s_{-1} + s_{-2} d_{-1}$.

For the inductive step, it suffices to prove, for all $n \geq -1$, that

$$\mathrm{Im}(h_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}) \subseteq \mathrm{Im} d'_{n+2},$$

for then we have a diagram with exact row

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, "s_{n+1}"'] \arrow[d, "h_{n+1}-\hat{f}_{n+1}-s_n d_{n+1}"] & \\ P'_{n+2} \arrow[r, "d'_{n+2}"'] & \mathrm{Im} d'_{n+2} \arrow[r] & 0, \end{tikzcd}$$

and projectivity of P'_{n+1} gives a map $s_{n+1} : P_{n+1} \to P'_{n+2}$ satisfying the desired equation. As in the proof of part (i), exactness of the bottom row of the original diagram gives $\mathrm{Im} d'_{n+2} = \mathrm{Ker} d'_{n+1}$, and so it suffices to prove $d'_{n+1}(h_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}) = 0$. But

$$d'_{n+1}(h_{n+1} - \hat{f}_{n+1} - s_n d_{n+1}) = d'_{n+1}(h_{n+1} - \hat{f}_{n+1}) - d'_{n+1} s_n d_{n+1}$$

$$= d'_{n+1}(h_{n+1} - \hat{f}_{n+1}) - (h_n - \hat{f}_n - s_{n-1} d_n) d_{n+1}$$

$$= d'_{n+1}(h_{n+1} - \hat{f}_{n+1}) - (h_n - \hat{f}_n) d_{n+1}$$

and the last term is zero because h and $\hat{f}$ are chain maps.

□

We introduce a term to describe the chain map $\hat{f}$ just constructed.

Definition 2.1.3. *If $f : A \to A'$ is a morphism and P_A and $P_{A'}$ are deleted projective resolutions of A and A', respectively, then a chain map $\hat{f} : P_A \to P'_{A'}$ is said to be* ***over*** *f if $f\varepsilon = \varepsilon' \hat{f}_0$.*

$$\begin{tikzcd} \dots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\hat{f}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\hat{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \dots \arrow[r] & P'_2 \arrow[r, "d'_2"'] & P'_1 \arrow[r, "d'_1"'] & P'_0 \arrow[r, "\varepsilon'"'] & A' \arrow[r] & 0. \end{tikzcd}$$

Given a morphism $f : A \to A'$, the comparison theorem implies that a chain map over f exists between deleted projective resolutions of A and A'. Moreover, such a chain map is unique up to homotopy.

2.2 Left Derived Functors

Given an additive covariant functor $T : \mathscr{A} \to \mathscr{B}$ between abelian categories, where $\mathscr{A}$ has enough projectives, we construct the *left derived functors* $L_nT : \mathscr{A} \to \mathscr{B}$, for all $n \in \mathbb{Z}$. We will define it firstly on objects and then on morphisms.

Choose a projective resolution $\mathbf{P}$ (we will later prove that the definition doesn't depend on the choice of projective resolution)

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

for every object A. From the deleted resolution $\mathbf{P}_A$, form the complex $T\mathbf{P}_A$, take homology, and define

$$(L_nT)A = H_n(T\mathbf{P}_A).$$

Let $f : A \to A'$ be a morphism. By the comparison theorem, there is a chain map $\hat{f} : \mathbf{P}_A \to \mathbf{P}_{A'}$ over f. Then $T\hat{f} : T\mathbf{P}_A \to T\mathbf{P}_{A'}$ is also a chain map, and we define $(L_nT)f : (L_nT)A \to (L_nT)A'$ by

$$(L_nT)f = H_n(T\hat{f}) = (T\hat{f})_{*_n}.$$

In more detail, if $z \in \mathrm{Ker} Td_n$, then

$$(L_nT)f : z + \mathrm{Im} Td_{n+1} \mapsto (T\hat{f}_n)z + \mathrm{Im} Td'_{n+1},$$

that is,

$$(L_nT)f : cls(z) \mapsto cls(T\hat{f}_n z).$$

In pictures, look at the chosen projective resolutions:

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "f"] & 0 \\
\dots \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & A' \arrow[r] & 0.
\end{tikzcd}$$

Fill in a chain map $\hat{f}$ over f, delete A and A' and apply T to this diagram, then take the map induced by $T\hat{f}$ in homology.

Theorem 2.2.1. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough projectives, then $L_nT : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor for every $n \in \mathbb{Z}$.*

Lemma 2.2.2. *If $f, g : \mathbf{A} \to \mathbf{A'}$ are chain maps, and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. If $f \simeq g$, then $Ff \simeq Fg$.*

Proof. We will prove that L_nT is well defined on morphisms, it is then routine to check that is an additive covariant functor (H_n is an additive covariant functor $\mathbf{Ch}(\mathscr{A}) \to \mathscr{A}$).

If $h : \mathbf{P}_A \to \mathbf{P}_{A'}$ is another chain map over f, then the comparison theorem says that $h \simeq \hat{f}$, therefore, $Th \simeq T\hat{f}$, and so $H_n(Th) = H_n(T\hat{f})$ by Theorem 1.2.4

□

Definition 2.2.3. *Given an additive covariant functor $T : \mathscr{A} \to \mathscr{B}$ between abelian categories, where $\mathscr{A}$ has enough projectives, the functors L_nT are called the **left derived functors** of T.*

Proposition 2.2.4. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, then $(L_nT)A = 0$ for all negative n and for all A.*

Proof. $(L_nT)A = 0$ because for all negative n the n-th term of $\mathbf{P}_A$ is 0. $\square$

The functors L_nT are called left derived functors because of the last proposition. Since $L_nT = 0$ on the right, that is, for all negative n, these functors are of interest only on the left; that is, for $n \geq 0$.

Definition 2.2.5. *If B is a left R-module and $T = \square \otimes_R B$, define*

$$Tor_n^R(\square, B) = L_nT.$$

Thus if $\mathbf{P}$

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

is the chosen projective resolution of a right R-module A, then

$$Tor_n^R(A, B) = H_n(\mathbf{P}_A \otimes_R B) = \frac{\operatorname{Ker}(d_n \otimes 1_B)}{\operatorname{Im}(d_{n+1} \otimes 1_B)}.$$

The domain of $Tor_n^R(\square, B)$ is $\mathbf{Mod}_R$ and its target is $\mathbf{Ab}$. In particular if R is commutative, then $A \otimes B$ is an R-module, and so the values of $Tor_n^R(\square, B)$ lie in ${}_R\mathbf{Mod}$.

We can also form the left derived functors $A \otimes_R \square$, obtaining functors ${}_R\mathbf{Mod} \to \mathbf{Ab}$.

Definition 2.2.6. *If A is a right R-module and $T = A \otimes_R \square$, define*

$$tor_n^R(A, \square) = L_nT.$$

Thus, if $\mathbf{Q}$

$$\cdots \to Q_2 \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{\eta} B \to 0$$

is the chosen projective resolution of a left R-module B, then

$$tor_n^R(A, B) = H_n(A \otimes_R \mathbf{Q}_B) = \frac{\operatorname{Ker}(1_A \otimes d_n)}{\operatorname{Im}(1_A \otimes d_{n+1})}.$$

One known result of Homological Algebra which we will later prove on 2.2.21 is:

Theorem 2.2.7. *If A is a left R-module and B is a right R-module, then, for all $n \geq 0$*

$$Tor_n^R(A, B) \cong tor_n^R(A, B).$$

Now we show that L_nT is independent of the choice of projective resolution.

Proposition 2.2.8. *Let $\mathscr{A}$ be an abelian category with enough projectives. Assume that new choices $\hat{\mathbf{P}}_A$ of deleted projective resolutions have been made, and denote the left derived functors arising from these new choices by $\hat{L}_nT$.*

If $T : \mathscr{A} \to \mathscr{C}$ is an additive covariant functor, where $\mathscr{C}$ is an abelian category, then the functors L_nT and $\hat{L}_nT$, for each $n \geq 0$, are naturally isomorphic. In particular, for all A, the objects

$$(L_nT)A \cong (\hat{L}_nT)A$$

are independent of the choice of projective resolution of A.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\dots \arrow[r] & \hat{P}_2 \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

where the top row is the chosen projective resolution of A used to define L_nT and the bottom is that used to define $\hat{L}_nT$. By the comparison theorem, there is a chain map $\iota : \mathbf{P}_A \to \hat{\mathbf{P}}_A$ over 1_A. Applying T gives a chain map $T\iota : T\mathbf{P}_A \to T\hat{\mathbf{P}}_A$ over $T1_A = 1_{TA}$. This last chain map induces morphisms for each n,

$$\tau_A = (T\iota)_* : (L_nT)A \to (\hat{L}_nT)A.$$

We now prove that each τ_A is an isomorphism (proving the last statement in the theorem) by constructing its inverse.

Turn the preceding diagram upside down, so that the chosen projective resolution $\mathbf{P}_A$ is now the bottom row. Again, the comparison theorem gives a chain map, say, $\kappa : \mathbf{P}_A \to \hat{\mathbf{P}}_A$. Now the composite $\kappa\iota$ is a chain map from $\mathbf{P}_A$ to itself over 1_A. By uniqeness in the statement of the comparison theorem $\kappa\iota \simeq 1_{\mathbf{P}_A}$. Similarly, $\iota\kappa \simeq 1_{\hat{\mathbf{P}}_A}$. It follows that, $T(\kappa\iota) \simeq 1_{T\mathbf{P}_A}$ and $T(\iota\kappa) \simeq 1_{T\hat{\mathbf{P}}_A}$. Hence, $1_{(\hat{L}_nT)A} = (T\iota\kappa)_* = (T\iota)_*(T\kappa)_*$ and $1_{(L_nT)A} = (T\kappa\iota)_* = (T\kappa)_*(T\iota)_*$. Therefore, $\tau_A = (T\iota)_*$ is an isomorphism. We now prove that the isomorphisms τ_A constitute a natural isomorphism; that is, if $f : A \to B$ is a morphism, then the following diagram commutes:

$$\begin{tikzcd}
(L_nT)A \arrow[r, "\tau_A"] \arrow[d, "(L_nT)f"'] & (\hat{L}_nT)A \arrow[d, "(\hat{L}_nT)f"] \\
(L_nT)B \arrow[r, "\tau_B"'] & (\hat{L}_nT)B
\end{tikzcd}$$

To evaluate the clockwise direction, consider the diagram

$$\begin{tikzcd}
\dots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\dots \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & 0 \\
\dots \arrow[r] & \hat{Q}_1 \arrow[r] & \hat{Q}_0 \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

where the bottom is the new chosen projective resolution of B. The comparison theorem gives a chain map $\mathbf{P}_A \to \hat{\mathbf{Q}}_B$ over $f1_A = f$. Going counterclockwise, the picture will now have the original chosen projective resolution of B as its middle row, and we get a chain map $\mathbf{P}_A \to \hat{\mathbf{Q}}_B$ over $1_Bf = f$. The uniqueness statement in the comparison theorem tells us that these two chain maps are homotopic, and so they induce the same morphism in homology. Thus, the appropriate diagram commutes, showing that $\tau : L_nT \to \hat{L}_nT$ is a natural isomorphism.

□

Corollary 2.2.9. *The modules $Tor_n^R(A,B)$ are independant of the choice of projective resolution of A and the modules $tor_n^R(A,B)$ are independent of the choice of projective resolution of B.*

Corollary 2.2.10. *Let $T :_R \textbf{\textit{Mod}} \to_S \textbf{\textit{Mod}}$ be an additive covariant functor. If P is a projective module then $(L_nT)P = \{0\}$ for all $n \geq 1$. In particular if A and P are right R-modules with P projective, and if B and Q are left R-modules with Q projective, then for all $n \geq 1$,*

$$Tor_n^R(P,B) = \{0\} \quad \textit{and} \quad tor_n^R(A,Q) = \{0\}.$$

Proof. Since P is projective, a projective resolution is $\mathbf{P}$, the complex with 1_P concentrated in degrees 0, -1. The corresponding deleted projective resolution $\mathbf{P}_P$ is the complex with P concentrated in degree 0. Hence, $T\mathbf{P}_P$ has the nth term $\{0\}$ for all $n \geq 1$, and so $(L_nT)P = H_n(T\mathbf{P}_P) = \{0\}$ for all $n \geq 1$.

□

Corollary 2.2.11. *Let $\mathscr{A}$ be an abelian category with enough projectives. Let* $\textbf{\textit{P}}$

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

be a projective resolution of $A \in Obj\mathscr{A}$. Define $K_0 = \mathrm{Ker}\varepsilon$ and $K_n = \mathrm{Ker}d_n$ for all $n \geq 1$. Then

$$(L_{n+1}T)A \cong (L_nT)K_0 \cong (L_{n-1}T)K_1 \cong \ldots \cong (L_1T)K_{n-1}$$

In particular if $\mathscr{A} = \textbf{\textit{Mod}}_R$ and B is a left R-module,

$$Tor_{n+1}^R(A,B) \cong Tor_n^R(K_0,B) \cong \ldots \cong Tor_1^R(K_{n-1},B).$$

Similarly, if A is a left R-module and $\textbf{\textit{P}}'$

$$\cdots \to P_2' \xrightarrow{d_2'} P_1' \xrightarrow{d_1'} P_0' \xrightarrow{\varepsilon'} B \to 0$$

be a projective resolution of a left R-module B, and define $V_0 = \mathrm{Ker}\varepsilon'$ and $V_n = \mathrm{Ker}d_n'$ for all $n \geq 1$. Then,

$$tor_{n+1}^R(A,B) \cong tor_n^R(A,V_0) \cong \ldots \cong tor_1^R(A,V_{n-1}) \quad .$$

Proof. By exactness of $\mathbf{P}$, we have $K_0 = \mathrm{Ker}\varepsilon = \mathrm{Im}d_1$, and so $\mathbf{Q}$

$$\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} K_0 \to 0$$

is a projective resolution of K_0 if we relabel the indices (replace each n by $n-1$ and define $Q_n = P_{n+1}$ and $\delta_n = d_{n+1}$ for all $n \geq 0$). Since the value of L_nT on a module is independent of the choice of projective resolution, we have

$$(L_nT)K_0 = H_n(T\mathbf{Q}_{K_0}) = \frac{\mathrm{Ker}T\delta_n}{\mathrm{Im}T\delta_{n+1}} = \frac{\mathrm{Ker}Td_{n+1}}{\mathrm{Im}Td_{n+2}} = H_{n+1}(T\mathbf{P}_A) \cong (L_{n+1}T)A.$$

The remaining isomorphisms are obtained by iteration.

□

We are now going to show that there is a long exact sequence of left derived functors. We begin with a useful lemma; it says that if we are given a short exact sequence $0 \to A' \to A \to A'' \to 0$ as well as projective resolutions of A' and A'' , then we can "fill in the horseshoe"; that is, there is a projective resolution of A that fits in the middle.

Proposition 2.2.12. ***(Horseshoe Lemma).*** *Given a diagram in an abelian category $\mathscr{A}$ with enough projectives,*

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_1' \arrow[d] & & P_1'' \arrow[d] & \\
 & P_0' \arrow[d] & & P_0'' \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "i"'] & A \arrow[r, "q"'] & A'' \arrow[r] & 0,
\end{tikzcd}$$

where the columns are projective resolutions and the row is exact, then there exist a projective resolution of A and chain maps so that the three columns form an exact sequence of complexes.

Remark 2.2.13. *The dual theorem, in which projective resolutions are replaced by injective resolutions, is also true.*

Proof. We show first that there are a projective Q_0 and a commutative 3×3 diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0' \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & K_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "i_0"] \arrow[d, "\varepsilon'"'] & Q_0 \arrow[r, "q_0"] \arrow[d, "\varepsilon"'] & P_0'' \arrow[r] \arrow[d, "\varepsilon''"] \arrow[dl, dashed, "\sigma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"'] \arrow[d] & A \arrow[r, "q"'] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Define $Q_0 = P_0' \bigoplus P_0''$, it is projective because both P_0' and P_0'' are projective. Define $i_0 : P_0' \to P_0' \bigoplus P_0''$ by $x' \to (x', 0)$ and define $q_0 : P_0' \bigoplus P_0'' \to P_0''$ by $(x', x'') \to x''$. It is clear that

$$0 \to P_0' \xrightarrow{i_0} Q_0 \xrightarrow{q_0} P_0'' \to 0$$

is exact. Since P_0'' is projective, there exists a map $\sigma : P_0'' \to A$ with $q\sigma = \varepsilon''$. Now define $\varepsilon : Q_0 \to A$ by $\varepsilon : (x', x'') \mapsto i\varepsilon' x' + \sigma x''$ (the map σ makes the square with base $A \xrightarrow{q} A''$ commute). Surjectivity of ε follows from the Five Lemma (which will be proven later on as an example on spectral sequences 3.4.2). It is a routine exercise that if $V_0 = \mathrm{Ker}\varepsilon$, then there are maps $K'_0 \to K_0$ and $K_0 \to K''_0$ (where $K'_0 = \mathrm{Ker}\varepsilon'$ and $K''_0 = \mathrm{Ker}\varepsilon''$), so that the resulting 3×3 diagram commutes. Exactness of the top row is also a simple exercise and its omited.

We now prove, by induction on $n \geq 0$, that the bottom n rows of the desired diagram can be constructed. For the inductive step, assume that the first n steps have been filled in, and let $V_n = \mathrm{Ker}(Q_n \to Q_{n-1})$, while $K'_n = \mathrm{Ker}d'_n$ and $K''_n = \mathrm{Ker}d''_n$. As in the base step, there is a commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'_{n+1} \arrow[r] \arrow[d] & V_{n+1} \arrow[r] \arrow[d] & K''_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d'_{n+1}"'] & Q_{n+1} \arrow[r, "q_{n+1}"] \arrow[d, "\delta_{n+1}"] & P''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & K'_n \arrow[r] \arrow[d] & V_n \arrow[r] \arrow[d] & K''_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now splice this diagram to the nth diagram by defined $\delta_{n+1} : Q_{n+1} \to Q_n$ as the composite $Q_{n+1} \to V_n \to Q_n$.

□

Corollary 2.2.14. *Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence of left R-modules. If both A' and A'' are finitely presented, then A is finitely presented.*

Proof. There are exact sequences $0 \to K'_0 \to P'_0 \to A' \to 0$ and $0 \to K''_0 \to P''_0 \to A'' \to 0$, where P'_0, P''_0, K'_0, K''_0 are finitely generated and P'_0, P''_0 are projective. As in the begining of the proof of the previous proposition, there is a 3×3 commutative diagram, with Q_0 projective, whose rows and columns are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K'_0 \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & K''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Both Q_0 and V_0 are finitely generated, being extensions of finitely generated modules, and so A is finitely generated.

□

Theorem 2.2.15. *Given a commutative diagram of right R-modules having exact rows,*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{p} & A'' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & C' & \xrightarrow[j]{} & C & \xrightarrow[q]{} & C'' & \longrightarrow & 0 \end{array}$$

there is a commutative diagram with exact rows for every left R-module B,

$$\begin{array}{ccccccc} Tor_n^R(A',B) & \xrightarrow{i_*} & Tor_n^R(A,B) & \xrightarrow{p_*} & Tor_n^R(A'',B) & \xrightarrow{\partial_n} & Tor_{n-1}^R(A',B) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f_*} \\ Tor_n^R(C',B) & \xrightarrow[j_*]{} & Tor_n^R(C,B) & \xrightarrow[q_*]{} & Tor_n^R(C'',B) & \xrightarrow[\partial'_n]{} & Tor_{n-1}^R(C',B) \end{array}$$

The similar statement for $tor_n^R(A, \square)$ is also true.

Proof. Exactness of $0 \to A' \to A \to A'' \to 0$ gives exactness of the sequence of deleted complexes $0 \to \mathbf{P}_{A'} \to \mathbf{P}_A \to \mathbf{P}_{A''} \to 0$. If $T = \square \otimes_R B$, then $0 \to T\mathbf{P}_{A'} \to T\mathbf{P}_A \to T\mathbf{P}_{A''} \to 0$ is still exact, for every row splits because each term of $\mathbf{P}_{A''}$ is projective. Therefore the naturality of connecting homomorphisms ∂ applies at once.

□

We now show that a short exact sequence gives a long exact sequence of left derived functors.

Theorem 2.2.16. *Let $\mathscr{A}$ be an abelian category with enough projectives. If $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ is an exact sequence in $\mathscr{A}$ and $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor, where $\mathscr{B}$ is an abelian category, then there is a long exact sequence in $\mathscr{B}$*

$$\cdots \to (L_nT)A' \xrightarrow{(L_nT)i} (L_nT)A \xrightarrow{(L_nT)p} (L_nT)A'' \xrightarrow{\partial_n}$$

$$(L_{n-1}T)A' \xrightarrow{(L_{n-1}T)i} (L_{n-1}T)A \xrightarrow{(L_{n-1}T)p} (L_{n-1}T)A'' \xrightarrow{\partial_{n-1}} \ldots$$

which ends with

$$\cdots \to (L_0T)A' \to (L_0T)A \to (L_0T)A'' \to 0.$$

Proof. Let $\mathbf{P}'$ and $\mathbf{P}''$ be the chosen projective resolutions of A' and A'', respectively. By the Horseshoe Lemma, there is a projective resolution $\hat{\mathbf{P}}$ of A with

$$0 \to \mathbf{P}'_{A'} \xrightarrow{j} \hat{\mathbf{P}}_A \xrightarrow{q} \mathbf{P}''_{A''} \to 0.$$

Here, j is a chain map over i and q is a chain map over p. Applying T gives the sequence of complexes

$$0 \to T\mathbf{P}'_{A'} \xrightarrow{Tj} T\hat{\mathbf{P}}_A \xrightarrow{Tq} T\mathbf{P}''_{A''} \to 0.$$

This sequence is exact, for each row $0 \to P'_m \xrightarrow{j_n} \hat{P}_n \xrightarrow{q_n} P''_n \to 0$ is a split exact sequence (because P''_n is projective, and additive functors preserve split exact sequences). Thus, there is a long exact sequence

$$\cdots \to H_n(T\mathbf{P}'_{A'}) \xrightarrow{(Tj)_*} H_n(T\hat{\mathbf{P}}'_A) \xrightarrow{(Tq)_*} H_n(T\mathbf{P}''_{A''}) \xrightarrow{\partial_n} H_{n-1}(T\mathbf{P}'_{A'}) \xrightarrow{(Tj)_*} \ldots$$

that is, there is an exact sequence

$$\cdots \to (L_nT)A' \xrightarrow{(Tj)_*} (\hat{L_nT})A \xrightarrow{(Tq)_*} (L_nT)A'' \xrightarrow{\partial_n} (L_{n-1}T)A' \xrightarrow{(Tj)_*} \ldots$$

The sequence does terminate with 0, for $L_{-1}T$ is zero for all negative n, by Proposition 2.2.4.

We do not know that $\hat{\mathbf{P}}_A$ arises from the projective resolution of A originally chosen, and so we must change it into the sequence we seek. There are chain maps $\kappa : \mathbf{P}_A \to \hat{\mathbf{P}}_A$ and $\lambda : \hat{\mathbf{P}}_A \to \mathbf{P}_A$, where both κ, λ are chain maps over 1_A in opposite direction. Indeed, as in the proof of Proposition 2.2.8, $T\kappa T\lambda$ and $T\lambda T\kappa$ are chain maps over 1_{TA} in opposite directions, whose induced maps in homology are isomorphisms, in fact $(T\lambda)_* : \hat{L}_nT \to L_nT$ is the inverse of $(T\kappa)_*$. Now $\hat{i}$ is a chain map over i and $\hat{p}$ is a chain map over p, while κ, λ are chain maps over 1_A.

The diagram displaying these chain maps is not commutative.

$$\begin{tikzcd}
\mathbf{P}'_{A'} \arrow[r, "j"] \arrow[dr, "\hat{i}"'] & \hat{\mathbf{P}}_A \arrow[r, "q"] \arrow[d, shift left, "\lambda"] & \mathbf{P}''_{A''} \\
& \mathbf{P}_A \arrow[u, shift left, "\kappa"] \arrow[ur, "\hat{p}"'] &
\end{tikzcd}$$

Consider the diagram after applying T and taking homology.

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(T\mathbf{P}'_{A'}) \arrow[r, "(Tj)_*"] \arrow[dr, "(T\hat{i})_*"'] & H_n(T\hat{\mathbf{P}}_A) \arrow[r, "(Tq)_*"] \arrow[d, shift left, "(T\hat{\lambda})_*"] & H_n(T\mathbf{P}''_{A''}) \arrow[r] & \ldots \\
& & H_n(T\mathbf{P}_A) \arrow[u, shift left, "(T\kappa)_*"] \arrow[ur, "(T\hat{p})_*"'] & &
\end{tikzcd}$$

The noncommutative diagram remains noncommutative after applying T , but the last diagram is commutative. Now, $T\lambda Tj \simeq T\hat{i}$ because both are chain maps $T\mathbf{P}'_{A'} \to T\mathbf{P}_A$ over Ti, hence, $(T\lambda Tj)_* = (T\hat{i})_*$, because homotopic chain maps induce the same homomorphism in homology. But $(T\lambda Tj)_* = (T\lambda)_*(Tj)_*$, and so

$$(T\lambda)_*(Tj)_* = (T\hat{i})_* = (L_nT)i.$$

Similarly, $(Tq)_*(T\kappa)_* = (T\hat{p})_* = (L_nT)p$.

The proof that

$$(L_nT)A' \xrightarrow{(L_nT)i} (L_nT)A \xrightarrow{(L_nT)p} (L_nT)A''$$

is exact can be completed easily.

□

Corollary 2.2.17. *If $T :_R$ **Mod** $\to_S$ **Mod** is an additive covariant functor, then the functor L_0T is right exact.*

Proof. If $A \to B \to C \to 0$ is exact then so is $(L_0T)A \to (L_0T)B \to (L_0T)C \to 0$.

□

Theorem 2.2.18.

(i) If an additive covariant functor $T : \mathscr{A} \to \mathscr{B}$ is right exact, where $\mathscr{A}$, $\mathscr{B}$ are abelian categories and $\mathscr{A}$ has enough projectives, then T is naturally isomorphic to L_0T .

(ii) The functor $\square \otimes_R B$ is naturally isomoprhic to $Tor_0^R(\square, B)$, and the functor $A \otimes_R \square$ is naturally isomorphic to $tor_0^R(A, \square)$. Hence for all right R-modules A and left R-modules B, there are isomorphisms

$$Tor_0^R(A, B) \cong A \otimes_R B \cong tor_0^R(A, B).$$

Proof.

(i) Let $\mathbf{P}$

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

be the chosen projective resolution of A. By definition $(L_0T)A = \mathrm{Coker}Td_1$. But right exactness of T gives a right exact sequence

$$TP_1 \xrightarrow{Td_1} TP_0 \xrightarrow{T\varepsilon} TA \to 0.$$

Now $T\varepsilon$ induces an isomorphism $\sigma_A = \mathrm{Coker}Td_1 \to TA$, by the First Isomorphism Theorem, that is,

$$\mathrm{Coker}Td_1 = TP_0/\mathrm{Im}Td_1 = TP_0/\mathrm{Ker}T\varepsilon \xrightarrow{\sigma_A} \mathrm{Im}T\varepsilon = TA.$$

It is easy to prove that $\sigma = (\sigma_A)_{A \in Obj(\mathbf{Mod}_R)} : L_0T \to T$ is a natural isomorphism.

(ii) Immediate from part (i), for both $\square \otimes_R B$ and $A \otimes_R \square$ are additive covariant right exact functors.

□

Corollary 2.2.19. *If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of right R-modules, then there is a long exact sequence for every left R-module B,*

$$\begin{aligned} \cdots \to & Tor_2^R(A', B) \to Tor_2^R(A, B) \to Tor_2^R(A'', B) \\ & \to Tor_1^R(A', B) \to Tor_1^R(A, B) \to Tor_1^R(A'', B) \\ & \qquad \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to 0. \end{aligned}$$

The similar statement holds for $tor_n^R(\square, B)$.

Thus the Tor sequence repairs the loss of exactess after tensoring a short exact sequence.

We prove now that Tor and tor are the same, and we begin with a variation of the snake lemma.

Lemma 2.2.20. *Given the commutative diagram with exact rows and columns in an abelian category $\mathscr{A}$*

$$\begin{tikzcd}
 & \mathrm{Ker}f \arrow[r] \arrow[d, "i"] & 0 \arrow[d] & \mathrm{Ker}h \arrow[d] & \\
\mathrm{Ker}\alpha \arrow[r, "j"] \arrow[d] & L' \arrow[r, "\alpha"] \arrow[d, "f"] & M' \arrow[r] \arrow[d, "g"] & N' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\mathrm{Ker}b \arrow[r] & L'' \arrow[r, "b"] \arrow[d] & M'' \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

then $\mathrm{Ker}f \simeq \mathrm{Ker}\alpha$ *and* $\mathrm{Ker}h \simeq \mathrm{Ker}b$*.*

Proof. It is a routine exercise (using a variation of the Snake Lemma) to obtain exactness of

$$\mathrm{Ker}g \to \mathrm{Ker}h \to \mathrm{Coker}f \to \mathrm{Coker}g.$$

Now $\mathrm{Ker}g = \{0\}$, $\mathrm{Coker}f = L''$, $\mathrm{Coker}g = M''$, and we may assume $\mathrm{Coker}f \to \mathrm{Coker}g$ is b. Thus, $0 \to \mathrm{Ker}h \to L'' \xrightarrow{b} M''$ is exact, and we conclude that $\mathrm{Ker}h \simeq \mathrm{Ker}b$.

We may assume that i and j are inclusions. Commutativity of the square with coker corner $\mathrm{Ker}\alpha$ gives $fj = 0$, that is $\mathrm{Ker}\alpha = \mathrm{Im}\ j \subseteq \mathrm{Ker}f = \mathrm{Im}\ i$. Commutativity of the square with corner $\mathrm{Ker}f$ gives $\alpha i = 0$, that is, $\mathrm{Ker}f = \mathrm{Im}\ i \subseteq \mathrm{Ker}\alpha = \mathrm{Im}\ j$. Therefore $\mathrm{Im}\ i = \mathrm{Im}\ j$ and $\mathrm{Ker}f = \mathrm{Ker}\alpha$.

□

Theorem 2.2.21. *Let A be a right R-module and B be a left R-module and $\boldsymbol{P}$*

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$$

and $\boldsymbol{Q}$

$$\cdots \to Q_1 \xrightarrow{d'_1} Q_0 \xrightarrow{\varepsilon'} B \to 0$$

be projective resolutions. Then $H_n(\boldsymbol{P}_A \otimes_R B) \simeq H_n(A \otimes_R \boldsymbol{Q}_B)$ for all $n \geq 0$, that is,

$$Tor_n^R(A, B) \simeq tor_n^R(A, B).$$

Proof. The proof is by inclusion on $n \geq 0$.

The base step $n = 0$ is true, by Theorem 2.2.18 (ii). Let us display the syzygies of $\mathbf{P}$ by "factoring" it into short exact sequences:

$$\begin{tikzcd}[column sep=small]
P_3 \arrow[rr] \arrow[dr] & & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & A \arrow[r] & 0. \\
 & K_2 \arrow[ur] & & K_1 \arrow[ur] & & K_0 \arrow[ur] & & &
\end{tikzcd}$$

There are exact sequences $0 \to K_i \to P_i \to K_{i-1} \to 0$ for all $i \geq 0$ if we write $A = K_{-1}$ (so that $0 \to K_0 \to P_0 \to A \to 0$ has the same notation as the others). Similarly we display the syzygies of $\mathbf{Q}$ by factoring it into short exact sequences $0 \to V_j \to Q_j \to V_{j-1} \to 0$ for all $j \geq 0$. Since tensor is a functor of two variables, the following diagram commutes for each $i, j \geq 0$.

$$\begin{tikzcd}
 & X \arrow[r] \arrow[d] & 0 \arrow[d] & W \arrow[d] & \\
Y \arrow[r] \arrow[d] & K_i \otimes V_j \arrow[r] \arrow[d] & K_i \otimes Q_j \arrow[r] \arrow[d] & K_i \otimes V_{j-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_i \otimes V_j \arrow[r] \arrow[d] & P_i \otimes Q_j \arrow[r] \arrow[d] & P_i \otimes V_{j-1} \arrow[r] \arrow[d] & 0 \\
Z \arrow[r] & K_{i-1} \otimes V_j \arrow[r] \arrow[d] & K_{i-1} \otimes Q_j \arrow[r] \arrow[d] & K_{i-1} \otimes V_{j-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows and columns are exact because tensor is right exact; the modules X, Y, Z, W are, by definition, kernels of obvious arrows. Zeros flank the middle row and column because P_i and Q_j are flat (they are even projective). Now $W = Tor_1(K_{i-1}, V_{j-1})$, $X = Tor_1(K_{i-1}, V_j)$, $Y = tor_1(K_i, V_{j-1})$ and $Z = tor_1(K_{i-1}, V_{j-1})$. By the previous Lemma we conclude that, for all $i, j \geq -1$,

$$Tor_1(K_{i-1}, V_{j-1}) \simeq tor_1(K_{i-1}, V_{j-1}).$$

If $i = 0 = j$, then $Tor_1(A, B) \simeq tor_1(A, B)$ because $K_{-1} = A$ and $V_{-1} = B$. The theorem has been proved for $n = 1$.

We now prove the inductive step. Corollary 2.2.11 gives

$$tor_{n+1}(A, B) \simeq tor_1(A, V_{n-1}) = tor_1(K_{-1}, V_{n-1}),$$

$$Tor_{n+1}(A, B) \simeq Tor_1(K_{n-1}, B) = Tor_1(K_{n-1}, V_{-1}).$$

Use these isomorphisms and the isomorphisms $X \simeq Y$, i.e.

$$Tor_1(K_{i-1}, V_j) = tor_1(K_i, V_{j-1}).$$

To go from any equation to the one below it, use the theorem for $n = 1$:

$$\begin{aligned}
tor_{n+1}(A, B) &\simeq tor_1(K_{-1}, V_{n-1}), \\
Tor_1(K_{-1}, V_{n-1}) &\simeq tor_1(K_0, V_{n-2}), \\
Tor_1(K_0, V_{n-2}) &\simeq tor_1(K_1, V_{n-3}), \\
&\cdots \\
Tor_1(K_{n-2}, V_0) &\simeq tor_1(K_{n-1}, V_{-1}), \\
Tor_1(K_{n-1}, V_{-1}) &\simeq Tor_{n+1}(A, B).
\end{aligned}$$

□

2.3 Right Derived Functors

Right derived functors is the dual notion of left derived functors and so every-hting that holds true for the left derived funtors so does for the right derived functors. This paragraph's goal is to give an idea about the right derived functors and to see how they are actually dual to the left ones and so all the proofs will be omitted.

Given an additive covariant functor $T : \mathscr{A} \to \mathscr{B}$ between abelian categories where $\mathscr{A}$ has enough injectives, we construct the *right derived functors* $R^nT : \mathscr{A} \to \mathscr{B}$, for all $n \in \mathbb{Z}$.

Choose an injective resolution $\mathbf{E}$

$$0 \to B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} \dots$$

for every object B. Form the complex $T\mathbf{E}^B$, where $\mathbf{E}^B$ is the deleted injective resolution, and take homology:

$$(R^nT)B = H^n(T\mathbf{E}^B) = \frac{\mathrm{Ker}Td^n}{\mathrm{Im}Td^{n-1}}.$$

The definition of $(R^nT)f$, where $f : B \to B'$ is a homomorphism, is similar to that for the left derived functors:

By the dual of the comparison theorem, there is a chain map $\hat{f} : \mathbf{E}^B \to \mathbf{E}^{B'}$ over f, unique to homotopy, and so there is a well-defined map $(R^nT)f : H^n(T\mathbf{E}^B) \to H^n(T\mathbf{E}^{B'})$ induced in homology, namely, $(T\hat{f})_{n*}$.

In pictures, look at the chosen injective resolutions:

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] & E'^0 \arrow[r] & E'^1 \arrow[r] & \dots \\
0 \arrow[r] & B \arrow[r] \arrow[u, "f"] & E^0 \arrow[r] \arrow[u, dashed] & E^1 \arrow[r] \arrow[u, dashed] & \dots
\end{tikzcd}$$

Fill in a chain map $\hat{f}$ over f, then apply T to this diagram and then take the map induced by $T\hat{f}$ in homology.

Theorem 2.3.1. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough injectives, then $R^nT : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor for every $n \in \mathbb{Z}$.*

Definition 2.3.2. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough injectives, the functors R^nT are called the right derived functors of T.*

Proposition 2.3.3. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough injectives, then $(R^nT)B = 0$ for all negative n and for all B.*

Assume that new choises $\hat{E}$ of injective resolutions have been made; denote the right derived functors arising from these choise by $\hat{R}^nT$.

Proposition 2.3.4. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough injectives, then the functors R^nT and $\hat{R}^nT$ are naturally isomorphic for each n. In particular for all $B \in Obj(\mathscr{A})$,*

$$(R^nT)B \cong (\hat{R}^nT)B,$$

and so these objects are independent of the choise of their injective resolutions.

Theorem 2.3.5. *If $0 \to B \xrightarrow{i} B \xrightarrow{p} B'' \to 0$ is an exact sequence in an abelian category $\mathscr{A}$ with enough injectives, and if $T\mathscr{A} \to \mathscr{B}$ is an additive covariant functor, where $\mathscr{B}$ is an abelian category, then there exists a long exact sequence*

$$\cdots \to (R^nT)B' \xrightarrow{(R^nT)i} (R^nnT)B \xrightarrow{(R^nT)p} (R^nnT)B'' \xrightarrow{\partial_n}$$

$$(R^{n+1}T)B' \xrightarrow{(R^{n+1}T)i} (R^{n+1}T)B \xrightarrow{(R^{n+1}T)p} (R^{n+1}T)B'' \xrightarrow{\partial^{n+1}} \dots$$

that begins with

$$0 \to (R^0T)B' \to (R^0T)B \to (R^0T)B'' \to \dots.$$

Corollary 2.3.6. *If $T : \mathscr{A} \to \mathscr{B}$ is an additive covariant functor between abelian categories, where $\mathscr{A}$ has enough injectives, then the functor R^0T is left exact.*

Theorem 2.3.7. *If an additive covariant functor $T : \mathscr{A} \to \mathscr{B}$ is left exact, where $\mathscr{A}$ and $\mathscr{B}$ are abelian categories and $\mathscr{A}$ has enough injectives, then T is naturally isomorphic to R^0T.*

Chapter 3

Spectral Sequences

Spectral sequences are a powerful book-keeping tool for proving things involving complicated commutative diagrams [RV]. Our first goal will be to find the cohomology of the double complex, and for that we will need the spectral sequence which is a recipe for computing some information regarding it.

In this section we will be working in Mod_R of R-modules over some ring A.

3.1 Double Complexes

Definition 3.1.1. *A* ***double complex*** *is a collection of R-modules $(E^{p,q})_{p,q\in\mathbb{Z}}$ along with a set of "rightward" morphisms $d_{\to}^{p,q} : E^{p,q} \to E^{p+1,q}$ and "upward" morphisms $d_{\uparrow}^{p,q} : E^{p,q} \to E^{p,q+1}$ satisfying the following conditions:*

(i) $d_{\to}^{p,q} \circ d_{\to}^{p+1,q} = d_{\to}^2 = 0,$

(ii) $d_{\uparrow}^{p,q} \circ d_{\uparrow}^{p,q+1} = d_{\uparrow}^2 = 0,$

(iii) either $d_{\to}d_{\uparrow} = d_{\uparrow}d_{\to}$ (all the squares commute) or $d_{\to}d_{\uparrow} + d_{\uparrow}d_{\to} = 0$ (all the squares anticommute).

Remark 3.1.2. *In the superscript, the first entry denotes the column number (the "x-coordinate"), and the second entry denotes the row number (the "y-coordinate"). This is opposite to the convetion for matrices. The subscript is meant to suggest the direction of arrows. We will write $d_{\to}$ and $d_{\uparrow}$ and ignore the superscript.*

About condition (iii), both cases come up in nature and we can switch from one to the other by replacing $d_{\uparrow}^{p,q}$ with $(-1)^p d_{\uparrow}^{p,q}$. So from now on we will assume that all the squares anticommute, knowing how to turn the commuting case into this one. There is no differece in the "recipe", basically because the image and kernel of homomorphism f equal the image and kernel respectively of $-f$.

$$\begin{tikzcd}
E^{p,q+1} \arrow[r, "d_{\rightarrow}^{p,q+1}"] & E^{p+1,q+1} \\
E^{p,q} \arrow[u, "d_{\uparrow}^{p,q}"] \arrow[r, "d_{\rightarrow}^{p,q}"'] \arrow[ur, phantom, "\textit{anticommutes}"] & E^{p+1,q} \arrow[u, "d_{\uparrow}^{p+1,q}"']
\end{tikzcd}$$

From the double complex we construct the corresponding **single complex** $E^{\bullet} = Tot^{\bullet}_{\oplus}$ with $E^k = \bigoplus_i E^{i,k-i}$, with $d = d_{\rightarrow} + d_{\uparrow}$. So, when there is a single superscript k, we mean the sum of the k-th antidiagonal of the double complex. This single complex is also called the **direct sum total complex**. Note that $d^2 = (d_{\rightarrow} + d_{\uparrow})^2 = d^2_{\rightarrow} + (d_{\rightarrow}d_{\uparrow} + d_{\uparrow}d_{\rightarrow}) + d^2_{\uparrow} = 0$, so $E^{\bullet}$ is indeed a complex.

We could have - instead of direct sum - taken the product $Tot^k_{\prod} = \prod_i E^{i,k-1}$ (and $d = d_{\rightarrow} + d_{\uparrow}$ like above) giving us the **product total complex** of the double complex.

The cohomology of the single complex is called the **hypercohomology** of the double complex.

3.2 Spectral Sequences

Definition 3.2.1. *A* ***spectral sequence*** *with* ***rightward orientation*** *is a sequence of tables or* ***pages*** ${}_{\rightarrow}E_0^{p,q}$, ${}_{\rightarrow}E_1^{p,q}$, ${}_{\rightarrow}E_2^{p,q}$, $\ldots$ *for* $p, q \in \mathbb{Z}$, *where* ${}_{\rightarrow}E_0^{p,q} = E^{p,q}$, *along with a differential*

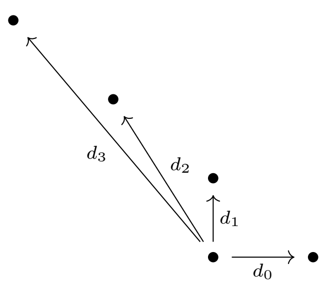
$${}_{\rightarrow}d_r^{p,q} : {}_{\rightarrow}E_r^{p,q} \rightarrow {}_{\rightarrow}E_r^{p-r+1,q+r}$$

with ${}_{\rightarrow}d_r^{p,q} \circ_{\rightarrow} d_r^{p+r-1,q-r} = 0$, *and with an isomorphism of the cohomology of* ${}_{\rightarrow}d_r$ *at* ${}_{\rightarrow}E_r^{p,q}$ *(i.e.* $\mathrm{Ker}_{\rightarrow}d_r^{p,q}/\mathrm{Im}_{\rightarrow}d_r^{p+r-1,q-r}$ *) with* ${}_{\rightarrow}E_{r+1}^{p,q}$

The orientation indicates that our 0-th differential is the rightward one $d_0 = d_{\rightarrow}$. The left subscript "$\rightarrow$" is usually ommited.

We write ${}_{\rightarrow}E_{\bullet}^{\bullet,\bullet}$ and we mean the rightward oriented spectral sequence with differential ${}_{\rightarrow}d_r^{p,q}$.

The order of the morphisms is best understood visually:



each of the morphisms applies to different pages.

Before we continue with the complete defintion of ${}_{\rightarrow}E_{\bullet}^{\bullet,\bullet}$ and its differential, we describe d_0, d_1, d_2 to better understand how this construction works.

Note that $E_r^{p,q}$ is always a subquotient of the corresponding term on the i-th page $E_i^{p,q}$ for all $i < r$. In particular, if $E^{p,q} = E_0^{p,q} = 0$ then $E_r^{p,q} = 0$ for all r.

Suppose now that $E^{\bullet,\bullet}$ is a **first quadrant double complex**, i.e. $E^{p,q} = 0$ if $q < 0$ or $p < 0$. So $E_r^{p,q} = 0$ for all r unless $p, q \in \mathbb{N} \bigcup \{0\}$. Then for any fixed pair (p, q) once r is sufficiently large, $E_{r+1}^{p,q}$ is computed from $(E_r^{\bullet,\bullet}, d_r)$ using the complex:

$$\begin{tikzcd}[column sep=small] 0 & & \\ & E_r^{p,q} \arrow[ul, "d_r^{p,q}"'] & \\ & & 0 \arrow[ul, "d_r^{p+r-1,q-r}"'] \end{tikzcd}$$

and thus we have canonical isomorphisms

$$E_r^{p,q} \simeq E_{r+1}^{p,q} \simeq E_{r+2}^{p,q} \simeq \dots .$$

We denote this module $E_\infty^{p,q}$. The same idea works in other circumstances, for example if the double complex is only nonzero in a finite number of rows ($E^{p,q} = 0$ for all $q \in [q_0, q_1]$. This will come up for example in the mapping cones.

Now we are ready to describe the first few pages of the spectral sequence explicitly. As stated above, the differential d_0 on $E_0^{\bullet,\bullet} = E^{\bullet,\bullet}$ is defined to be $d_\to$, the rows are complexes:

$$\textit{The 0th page } E_0 : \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

and so E_1 is just the table of cohomologies of the rows. There are now vertical maps $d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$ of the row cohomology groups, induced by $d_\uparrow$, and these make the columns into complexes (which is essentially that a map of complexes induces a map on homology). We have "used up the horizontal morphisms, but the vertical differentials live on".

$$\textit{The 1st page } E_1 : \quad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \\ \bullet \arrow[u] & \bullet \arrow[u] & \bullet \arrow[u] \end{tikzcd}$$

Now, we take, again, cohomology of d_1 on E_1, giving us a new table $E_2^{p,q}$. It turns out that there are natural morphisms from each entry to the entry two above and one to the left, and that the composition of these two is 0.

$$\textit{The 2nd page } E_2 : \quad \begin{tikzcd} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet \arrow[uul] & \bullet \arrow[uul] \end{tikzcd}$$

This is the begining of the pattern and we are able to see by now that the morphisms of the 3rd page won't fit in our 3 x 3 diagram.

Definition 3.2.2. *A cohomology spectral sequence is said to be* ***bounded*** *if for each $n \in \mathbb{Z}$ there are finitely many non-zero terms of total degree n in $E_r^{p,q}$, i.e. there are finitely many $E_r^{p,q} \neq 0$ with $p+q=n$. If so, then for each p and q there is an r_0 such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq r_0$. We write $E_\infty^{p,q}$ for this stable value of $E_r^{p,q}$.*

We say that a bounded spectral sequence ${}_{\rightarrow}E_\bullet^{\bullet,\bullet}$ ***converges*** *to $H^\bullet(E^\bullet) = H^n$ if we are given a family of objects H^n each having a finite filtration*

$$0 = F^t H^n \subseteq \cdots \subseteq F^{p+1} H^n \subseteq F^p H_n \subseteq F^{p-1} H^n \subseteq \cdots \subseteq F^s H^n = H^n$$

and we are given isomorphisms $E_\infty^{p,q} \cong F^p H^n / F^{p+1} H^n$.

We write $E_r^{p,q} \Rightarrow H^{p+q}$ for the bounded convergence.

If a first quadrant cohomology spectral sequence converges to $H^\bullet$, then each H^n has a finite filtration of length $n+1$:

$$0 = F^{n+1} H^n \subseteq^{n+1} H^n \subseteq \cdots \subseteq F^1 H^n \subseteq F^0 H^n = H^n.$$

The bottom peace $F^n H^n \simeq E_\infty^{n,0}$ is located on the $x-$axis (called the **base term**) and the top quotient $H^n / F^1 H^n \simeq E_\infty^{0,n}$ is located on the $y-axis$ (called the **fiber term**). Note that each arrow leaving the $y-$axis is zero and each arrow landing on the $x-$axis is also zero. The resulting maps $E_r^{0,n} \to E_\infty^{0,n} \subset H^n$ and $H^n \to E_\infty^{n,0} \subset E_r^{n,0}$ are called the **edge homomorphisms**.

Theorem 3.2.3. *There is a filtration of $H^n(E^\bullet)$ by $E_\infty^{p,q}$ where $p+q=n$. More precicely, there is a filtration*

$$E_\infty^{0,n} \xhookrightarrow{E_\infty^{1,n-1}} \textbf{?} \xhookrightarrow{E_\infty^{2,n-2}} \ldots \xhookrightarrow{E_\infty^{n,0}} H^n(E^\bullet), \tag{3.1}$$

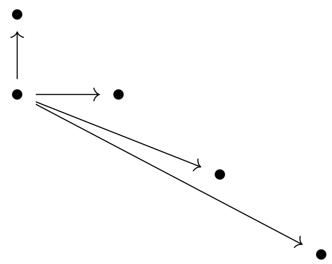
where the quotients are displayed above each inclusion.

A tip for remember which way the quotients are supposed to go. The differentials on later and later pages point deeper and deeper into the filtration. Thus the entries in the direction of the later arrowheads are the subobjects, and the entries in the direction of the later "arrowtails" are quotients. This tip has the advantage of being independent of the details of the spectral sequence, e.g., the "quadrant" or the orientation.

Although the filtration gives only partial information about $H^\bullet(E^\bullet)$, sometimes one can find $H^\bullet(E^\bullet)$ precisely. For example, if all $E_\infty^{i,k-i} = 0$ or if all but one of them are zero (e.g. $E_r^{\bullet,\bullet}$ has precisely one non-zero row or column). Another example is in the category of vector spaces over a field, in which case we can find the dimension of $H^k(E^\bullet)$. Also, in "lucky circumstances", E_2 (or some other small page) already equals E_∞.

Remark 3.2.4. *The Other Orientation*

We could as well have done everything in the opposite direction, i.e. reversing the roles of horizontal and vertical morphisms.Then the sequence of arrows giving the spectral sequence would look like this:



This spectral sequence is denoted by ${}_{\uparrow}E_\bullet^{\bullet,\bullet}$ *(with the "**upward orientation**"). Then we would again get pieces of a filtration of* $H^\bullet(E^\bullet)$ *(where we would have to be a bit careful with the order with which* ${}_{\uparrow}E_\infty^{p,q}$ *corresponds to the subquotients - it is the opposite order of that of eq.(3.1) for* ${}_{\rightarrow}E_\infty^{p,q}$*. In general there is no isomorphism between* ${}_{\uparrow}E_\infty^{p,q}$ *and* ${}_{\rightarrow}E_\infty^{p,q}$*.*

Whichever map we choose, either the horizontal or the vertical one, both algorithms compute information about the same thing $(H^\bullet(E^\bullet))$*.*

3.3 The Completion of the Definition of Spectral Sequences

To complete the definition of spectral sequences we have yet to describe the pages and the diffential of the spectral sequence explicitly, and prove that they behave the way we want them to. More precisely, we want:

(i) describe $E_r^{p,q}$ and consequently verify that $E_0^{p,q} = E^{p,q}$,

(ii) describe d_r and verify that $d_r^2 = 0$,

(iii) verify that $E_{r+1}^{p,q}$ is given by cohomology using d_r,

(iv) verify that $H^k(E^\bullet)$ is filtered by $E_\infty^{p,k-p}$ as in *eq.*(3.1).

Remark 3.3.1. *Spectral sequences are actually spectral functors. It is useful to notice that the proof implies that spectral sequences are functorial in the 0-th page: the spectral sequence formalism has good functorial properties in the double complex.*

We say that an element of $E^{\bullet,\bullet}$ is a (p,q)-**strip** if it is an element of $\oplus_{i\neq 0} E^{p-i,q+i}$. Its non-zero entries lie on an "upper leftwards" semi-infinite intidiagonal starting with position (p,q). We say that the (p,q)-**entry** (the projection to $E^{p,q}$ is the **leading term** of the (p,q)-strip. Let $S^{p,q} \subset E^{\bullet,\bullet}$ be the submodule of all the (p,q)-strips. Clearly, $S^{p,q} \subset E^{p+q}$ and $S^{k,0} = E^k$.

$$\begin{array}{ccccc}
\ddots & 0 & 0 & 0 & 0 \\
0 & *^{p-2,q+2} & 0 & 0 & 0 \\
0 & 0 & *^{p-1,q+1} & 0 & 0 \\
0 & 0 & 0 & *^{p,q} & 0 \\
0 & 0 & 0 & 0 & 0^{p+1,q-1}
\end{array}$$

Note that the differential $d = d_\uparrow + d_\rightarrow$ sends a (p,q)-strip x to a $(p+1,q)$-strip dx. If dx is futhermore a $(p-r+1, q+r)$-strip $r \neq 0$, then we say that x is an r-**closed** (p,q)-strip - "the differential knocks x atleast r terms deepre into the filtration". We denote the set of all r-closed (p,q)-strips $S_r^{p,q}$. For example, $S_0^{p,q} = S^{p,q}$ and $S_0^{k,0} = E^k$.
An element of $S_r^{p,q}$ may be depicted as:

$$\begin{tikzcd}
\ddots \arrow[r] & ? & & & \\
& *^{p+2,q-2} \arrow[u] \arrow[r] & 0 & & \\
& & *^{p+1,q-1} \arrow[u] \arrow[r] & 0 & \\
& & & *^{p,q} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

We are now ready to give a first definition of $E_r^{p,q}$, which by construction should be a subquotient of $E^{p,q}$. We describe it as such by describing two submodules $Y_r^{p,q} \subset X_r^{p,q} \subset E^{p,q}$, and defining

$$E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}}.$$

Let $X_r^{p,q}$ be the elements of $E^{p,q}$ that are leading terms of r-closed (p,q)-strips. Note that, by definition, d sends $(r-1)$-closed $(p+(r-1)-1, q-(r-1))$-strips to (p,q)-strips.
Let $Y_r^{p,q}$ be the leading (p,q)-terms of the differential d of $(r-1)$-closed $(p+(r-1)-1, q-(r-1))$-strips (where the differential is considered as a (p,q)-strip).

Remark 3.3.2. *It is easy to verify that $E_0^{p,q}$ is (canonically isomorphic to) $E^{p,q}$.*

Now, for the defintion of the differential d_r of such an element $x \in X_r^{p,q}$, we take any r-closed (p,q)-strip with leading term x. Its differential d is a

$(p-r+1, q+r)$-strip, and we take its leading term. The choise of r-closed (p,q)-strip, means that is not a well-defined element of $E^{p,q}$. But it is well-defined modulo the differentials of the $(r-1)$-closed $(p-1, q+1)$-strips, and hence gives a map $E_r^{p,q} \to E_r^{p-r+1,q+r}$.

This definition is fairly short, but not much fun to work with, so we will forget it, and instead dive into a snakes' nest of subscripts and superscripts.

Remark 3.3.3. *The following are easily verified.*

(i) $S^{p,q} = S^{p-1,q+1} \oplus E^{p,q}$.

(ii) Any closed (p,q)-strip is r-closed for all r, i.e. any element $x \in S^{p,q} = S_0^{p,q}$ that is a cycle ($dx = 0$), is automatically in $S_r^{p,q}$ for all r. For exapmple, this holds when x is a boundary (i.e. of the form dy).

(iii) For fixed p, q

$$S_0^{p,q} \supset S_1^{p,q} \supset \cdots \supset S_r^{p,q} \supset \ldots$$

stabilizes for $r >> 0$. (i.e. $S_r^{p,q} = S_{r+1}^{p,q} = \ldots$). Denote the stabilized module $S_\infty^{p,q}$. Now, $S_\infty^{p,q}$ is the set of closed (p,q)-strips (those strips annihilated by d, i.e. the cycles). In particular, $S_\infty^{k,0}$ is the set of cycles in E^k.

Definition 3.3.4. *Define*

$$X_r^{p,q} = \frac{S_r^{p,q}}{S_{r-1}^{p-1,q+1}}$$

and

$$Y_r^{p,q} = \frac{dS_{r-1}^{p+(r-1)-1,q-(r-1)} + S_{r-1}^{p-1.q+1}}{S_{r-1}^{p-1,q+1}}.$$

Then $Y_r^{p,q} \subset X_r^{p,q}$.

We define

$$E_r^{p,q} = \frac{X_r^{p,q}}{Y_r^{p,q}} = \frac{S_r^{p,q}}{dS_{r-1}^{p+(r-1)-1,q-(r-1)} + S_{r-1}^{p-1.q+1}}.$$

We have completed our first request **(i)**.

Corollary 3.3.5. *$E_\infty^{p,q}$ gives subquotients of $H^k(E^\bullet)$: $E_r^{p,q}$ stabilizes as $r \to \infty$. For $r >> 0$, interpret $S_r^{p,q}/dS_{r-1}^{p+(r-1)-1,q-(r-1)}$ as the cycles in $S_\infty^{p,q} \subset E^{p+q}$ modulo those boundary elements of dE^{p+q-1} contained in $S_\infty^{p,q}$. Finally, we can show that $H^k(E^\bullet)$ is indeed filtered (as described in the previous paragraph).*

We have completed our forth request **(iv)**.

For the definition of the map $d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$ we notice that it is induced by our differential d: d sends r-closed (p,q)-strips $S_r^{p,q}$ to $(p-r+1, q+r)$-strips $S^{p-r+1,q+r}$, by the definition "r-closed", whose image lies in $S_r^{p-r+1,q+r}$.

Again, we can verify that d sends

$$dS_{r-1}^{p+(r-1)-1,q-(r-1)} + S_{r-1}^{p-1,q+1} \to dS_{r-1}^{(p-r+1)+(r-1)-1,(q+r)-(r-1)} + S_{r-1}^{(p-r+1)-1,(q+r)-1}.$$

The first term on the left goes to 0 from $d^2 = 0$ and the second term on the left goes to the first term on the right.

Definition 3.3.6.

$$d_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$$

where

$$E_r^{p,q} = \frac{S_r^{p,q}}{dS_{r-1}^{p+(r-1)-1,q-(r-1)} + S_{r-1}^{p-1,q+1}}$$

and

$$E^{p-r+1,q+r} = \frac{S_r^{p-r+1,q+r}}{dS_{r-1}^{p-1,q+1} + S_{r-1}^{p-r,q+r+1}}$$

and clearly $d_r^2 = 0$ (as we may interpret it as taking an element of $S_r^{p,q}$ and applying d twice).

We have completed our second request **(ii)**.

Now, to verify the cohomology of d_r at $E_r^{p,q}$ is $E_{r+1}^{p,q}$:

$$\frac{S_r^{p+r-1,q-r}}{dS_{r-1}^{p+2r-3,q-2r+1} + S_{r-1}^{p+r-2,q-r+1}} \xrightarrow{d_r} \frac{S_r^{p,q}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}} \xrightarrow{d_r} \frac{S_r^{p-r+1,q+r}}{dS_{r-1}^{p-1,q+1} + S_{r-1}^{p-r,q+r+1}} \tag{3.2}$$

is naturally identified with

$$\frac{S_{r+1}^{p,q}}{dS_r^{p+r-1,q-r} + S_r^{p-1,q+1}}.$$

We begin by understanding the kernel of the right map on *eq.*(3.2). Suppose that $a \in S_r^{p,q}$ is mapped to 0. This means that $da = db + c$, where $b \in S_{r-1}^{p-1,q+1}$. If $u = a - b$, then $u \in S^{p,q}$, while $du = c \in S_{r-1}^{p-r,q+r+1} \subset S^{p-r,q+r+1}$, from which u is $(r+1)$-closed, i.e. $u =\in S_{r+1}^{p,q}$. Thus, $a = b + u \in S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}$. Conversely, any $a \in S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}$ satisfies

$$da \in dS_{r-1}^{p-1,q+1} + dS_{r+1}^{p,q} \subset dS_{r-1}^{p-1,q+1} + S_{r-1}^{p-r,q+r+1}$$

using $dS_{r+1}^{p,q} \subset S_0^{p-r,q+r+1}$ and **Remark** 3.3.3 **(ii)**. So any such a is indeed the kernel of

$$S_r^{p,q} \to \frac{S_r^{p-r+1,q+r}}{dS_{r-1}^{p-1,q+1} + S_{r-1}^{p-r,q+r+1}}.$$

Hence, the kernel of the right map of *eq.*(3.2) is

$$\mathrm{Ker} = \frac{S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}.$$

Next, the image of the left map of *eq.*(3.2) is immediately

$$\mathrm{Im} = \frac{dS_r^{p+r-1,q-r} + dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}} = \frac{dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}}{dS_{r-1}^{p+r-2,q-r+1} + S_{r-1}^{p-1,q+1}}$$

as $S_r^{p+r-1,q-r}$ contains $S_{r-1}^{p+r-2,q-r+1}$.

Thus the cohomology of *eq.*(3.2) is:

$$\mathrm{Ker}/\mathrm{Im} = \frac{S_{r-1}^{p-1,q+1} + S_{r+1}^{p,q}}{dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}} = \frac{S_{r+1}^{p,q}}{S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1})}$$

where the equality on the right uses the fact that $dS_r^{p+r-1,q-r} \subset S_{r+1}^{p,q}$ and an isomorphism theorem. Thus we must show that

$$S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}) = dS_r^{p+r-1,q-r} + S_r^{p-1,q+1}.$$

However,

$$S_{r+1}^{p,q} \cap (dS_r^{p+r-1,q-r} + S_{r-1}^{p-1,q+1}) = dS_r^{p+r-1,q-r} + S_{r+1}^{p,q} \cap S_{r-1}^{p-1,q+1}$$

and $S_{r+1}^{p,q} \cap S_{r-1}^{p+1,q-1}$ consists of $(p-1,q+1)$-strips whose differential vanishes up to row $p+r$, from which $S_{r+1}^{p,q} \cap S_{r-1}^{p-1,q+1} = S_r^{p-1,q+1}$ as desired.

We have completed our third and final request **(iii)**.

3.4 Examples

We are now ready to see how this is useful. The moral of these examples is the following. In the past, we may have proved various facts involving various sorts of diagrams, by chasing elements around. Now, we will just plug them into a spectral sequence, and let the spectral sequence machinery do your chasing for us.

Example 3.4.1. *Proving the Snake Lemma.*

Consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the rows are exact in the middle (at A, B, C, D , E and F) and the squares commute. Normally the Snake Lemma is described with the vertical arrows pointing downwards, but we want to fit this into our Spectral Sequence conventions. We wish to show that there is an exact sequence:

$$0 \to \mathrm{Ker}\alpha \to \mathrm{Ker}\beta \to \mathrm{Ker}\gamma \to \mathrm{Coker}\alpha \to \mathrm{Coker}\beta \to \mathrm{Coker}\gamma \to 0. \tag{3.3}$$

We "plug" this into our spectral sequence machinery. We first compute the cohomology using the rightward orientation, then, because the rows are exact, $E_1^{p,q} = 0$, so the spectral sequence has already converged: $E_\infty^{p,q} = 0$

We next compute this "0" in another way, by computing the spectral sequence using the upward orientation. Then ${}_\uparrow E_1^{\bullet,\bullet}$ (with its differentials) is:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Coker}\alpha & \longrightarrow & \mathrm{Coker}\beta & \longrightarrow & \mathrm{Coker}\gamma & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & \mathrm{Ker}\alpha & \longrightarrow & \mathrm{Ker}\beta & \longrightarrow & \mathrm{Ker}\gamma & \longrightarrow & 0 \end{array}$$

Then, ${}_{\uparrow}E_2^{\bullet,\bullet}$ *is of the form:*

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ? \arrow[drr] & ? & 0 & \\
& 0 & ? & ? \arrow[drr] & ?? \arrow[drr] & 0 & \\
& & & & & 0 & 0
\end{tikzcd}$$

We see that after ${}_{\uparrow}E_2^{\bullet,\bullet}$ *all the terms will stabilize except for the double-question-marks - all the maps to and from the single-question-marks are to and from 0 - entries. And after* ${}_{\uparrow}E_3^{\bullet,\bullet}$*, even these two double-question-mark terms will stabilize. But in the end our complex must be the 0 complex. This means that in* ${}_{\uparrow}E_2^{\bullet,\bullet}$ *all entries must be 0, except for the two double-question-marks, and these two must be isomorphic. That means that* $0 \to \mathrm{Ker}\alpha \to \mathrm{Ker}\beta \to \mathrm{Ker}\gamma$ *and* $\mathrm{Coker}\alpha \to \mathrm{Coker}\beta \to \mathrm{Coker}\gamma \to 0$ *are both exact - which comes from the vanishing of the single question marks - and*

$$\mathrm{Coker}(\mathrm{Ker}\beta \to \mathrm{Ker}\gamma) \simeq \mathrm{Ker}(\mathrm{Coker}\alpha \to \mathrm{Coker}\beta)$$

is an isomorphism - that comes from the equality of the double question marks. Taken together we have proven the exactness of eq.(3.3) and hence the Snake Lemma.

We notice that in the end we didn't really care about the double complex, we just needed it as a prop to prove the Snake Lemma.

Example 3.4.2. *The Five Lemma.*

Suppose

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\varepsilon"]
\end{tikzcd}$$

where the rows are exact and the squares commute.

Suppose that $\alpha, \beta, \delta, \varepsilon$ *are isomorphisms. We will show that* γ *is an isomorphism.*

We first compute the cohomology of the total complex using the rightward orientation. We choose this because we see that we will get a lot of zeros.

Then, ${}_{\rightarrow}E_1^{\bullet,\bullet}$ *looks like this:*

$$\begin{tikzcd}
? & 0 & 0 & 0 & ? \\
? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u]
\end{tikzcd}$$

Then, ${}_{\rightarrow}E_2^{\bullet,\bullet}$ *looks similar, and the sequence will converge by* E_2*, as well as never get any arrows between two nonzero entries in a table thereafter. We can't conclude that the cohomology of the total complex vanishes, but we can note that it vanishes in all but four degrees - and most important, it vanishes in* *the two degrees corresponding to the entries C and H (the source and target of γ).*

We next compute this using the upward orientation, then, ${}_{\uparrow}E_1^{\bullet,\bullet}$ looks like this:

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow ? \longrightarrow 0 \longrightarrow 0$$

and the spectral sequence converges at this step. We wish to show that these two question marks are zero. But they are precisely the cohomology groups of the total complex that we just showed were zero.

3.5 Spectral Sequence of a Double Complex

There are two filtrations associated to every double complex $E^{\bullet,\bullet}$ (one by columns and one by rows), resulting to two spectral sequences related to the homology of $Tot(E)$.

Definition 3.5.1. *(Filtration by Columns) If $E = E^{\bullet,\bullet}$ is a double complex, we may filter the (product or direct sum) total complex $Tot(E)$ by the columns of E, letting ${}^{I}F_nTot(E)$ be the total complex of the double subcomplex of C:*

$$({}^{I}\tau_{\leq n}E)^{p,q} = E^{p,q}, p \leq n, \text{ and } 0 \text{ otherwise.}$$

$$\begin{array}{ccccc} \dots & * & * & 0 & 0 \\ \\ \dots & * & * & 0 & 0 \\ \\ \dots & * & * & 0 & 0 \end{array}$$

This gives rise to a spectral sequence ${}^{I}E_r^{p,q}$ starting with the 0-page being the double complex itself and the differentials being its "rightward" morphisms:

$${}^{I}E_0^{p,q} = E^{p,q} \text{ and } d_0 =_{\rightarrow} d.$$

So the first page -which is the essentially the cohomology induced by the "rightward" morphisms of the double complex- and its maps -which are induced on cohomology from the "upward" morphisms- are:

$${}^{I}E_1^{p,q} = H^q(E^{p,\bullet}) \text{ and } d_1 =_{\uparrow} d_* : H^q(E^{p,\bullet}) \to H^q(E^{p+1,\bullet}).$$

So now, continuing this construction, we get:

$${}^{I}E_2^{p,q} = H^p(H^q(E)).$$

Remark 3.5.2. *If E is a first quadrant double complex, i.e. $p \geq 0, q \geq 0$, the filtration is canonically bounded, and we have a convergent spectral sequence:*

$${}^{I}E_2^{p,q} = H^p(H^q(E)) \Rightarrow H^{p+q}(Tot(E)).$$

Filtering the double complex by rows (by doing the same thing but interchanging the roles of p and q in the notation and also the roles of "upward" and "rightward" morphisms), gives rise to a spectral sequence ${}^{II}E_r^{p,q}$ with its 0-page being the double complex itself and differentials being the "upward" morphisms:

$$ {}^{II}E_0^{p,q} = E^{q,p} \textit{ and } d_0 =_{\uparrow} d. $$

The next two pages are:

$$ {}^{II}E_1^{p,q} = H^q(E^\bullet, p) \;\; \textit{and} \;\; d_1 =_{\rightarrow} d_* : H^q(E^{\bullet,p}) \to H^q(E^{\bullet,p+1}), $$
$$ {}^{II}E_2^{p,q} = H^p(H^q(E)). $$

Of course, if E is a first quadrant double complex, this filtration is canonically bounded, and the spectral sequence converges to to $H^{p+q}(Tot(E))$.

Remark 3.5.3. *Interchange of p and q in the notation should not be surprising, since interchanging the roles of p and q converts the filtration by rows into the filtration by columns and interchanges the spectral sequences ${}^{I}E$ and ${}^{II}E$.*

Chapter 4

Derived Categories

Behind the construction of the derived category there is a general procedure, called localization. Roughly, one constructs the localization of a category with respect to a localizing class of morphisms. In our case, these are the quasi-isomorphisms. It turns out that quasi-isomorphisms indeed form a localizing class in $\mathbf{K}(\mathscr{A})$ (but not in $\mathbf{Ch}(\mathscr{A})$). For now we won't bother with that, we will instead skip that part and assume that derived categories exist in "our universe" (**Theorem 4.1.2**). In the following chapter (**4.1**) we see what is a morphism in derived category and how composition of morphisms has meaning. In chapter (**4.2**) we will do all the work necessary to explain why derived categories exist through localizing the class of quasi-ismorphisms.

4.1 A Thorough Introduction to Derived Categories

Definition 4.1.1. *A morphism of complexes $f : A^\bullet \longrightarrow B^\bullet$ is called a quasi-isomorphism (or qis, for short) if for all $i \in \mathbb{Z}$ the induced map $H^i(f) : H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ is an isomoprhism.*

With this definition we can reveal that the central idea for the definition of the derived category is this: quasi-isomorphic complexes should become isomorphic objects in the derived category. We shall begin with the following existence theorem. Details of the construction are provided in the next chapter.

Theorem 4.1.2. *Let $\mathscr{A}$ be an abelian category and let $\boldsymbol{Ch}(\mathscr{A})$ be its category of complexes. Then, there exists a category $\boldsymbol{D}(\mathscr{A})$, the derived category of $\mathscr{A}$, and a functor*

$$Q : \boldsymbol{Ch}(\mathscr{A}) \longrightarrow \boldsymbol{D}(\mathscr{A})$$

such that:

(i) If $f : A^\bullet \longrightarrow B^\bullet$, is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $\boldsymbol{D}(\mathscr{A})$.

(ii) Any functor $F : \boldsymbol{Ch}(\mathscr{A}) \longrightarrow \mathscr{D}$ satisfying property (i) factorizes uniquely over $Q : \boldsymbol{Ch}(\mathscr{A}) \longrightarrow \boldsymbol{D}(\mathscr{A})$, i.e. there exists a unique functor (up to

isomorphism) $G : \boldsymbol{D}(\mathscr{A}) \longrightarrow \mathscr{D}$ *with* $F \simeq G \circ Q$ *:*

$$\begin{tikzcd} \boldsymbol{Ch}(\mathscr{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & \boldsymbol{D}(\mathscr{A}) \arrow[dl, "G"] \\ & \mathscr{D} & \end{tikzcd}$$

The theorem is a pure existence result. In order to be able to work with the derived category, we need to understand which objects become isomorphic under $Q : \mathbf{Ch}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$ and how to represent morphisms in the derived category. Explaining this, will provide the proof for the above theorem. Moreover, we shall observe the following facts:

Corollary 4.1.3.

(i) Under the functor $Q : \boldsymbol{Ch}(\mathscr{A}) \longrightarrow \boldsymbol{D}(\mathscr{A})$ *the objects of the two categories are identified.*

(ii) The cohomology objects $H^i(A^\bullet)$ *of an object* $A^\bullet \in \boldsymbol{D}(\mathscr{A})$ *are well-defined objects of the abelian category* $\mathscr{A}$*.*

(iii) Viewing any object in $\mathscr{A}$ *as a complex concentrated in degree zero yields an equivalence between* $\mathscr{A}$ *and the full subcategory of* $\boldsymbol{D}(\mathscr{A})$ *that consists of complexes* $A^\bullet$ *with* $H^i(A^\bullet) = 0$ *for* $i \neq 0$*.*

Remark 4.1.4. *The term "concentrated in degree zero" refers to mapping an object* $A \in \mathscr{A}$ *to the complex* $\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \ldots$ *which identifies* $\mathscr{A}$ *with a full subcategory of* $\boldsymbol{Ch}(\mathscr{A})$*.*

Unlike the category of complexes $\mathbf{Ch}(\mathscr{A})$, the derived category $\mathbf{D}(\mathscr{A})$ is not always abelian, but it is always triangulated. The shift functor descends to $\mathbf{D}(\mathscr{A})$ and a natural class of distinguished triangles can be found.

Suppose we have a quasi-isomorphism $C^\bullet \longrightarrow A^\bullet$ in $\mathbf{Ch}(\mathscr{A})$. As the derived category is to be constructed in a way that any quasi-isomorphism (in $\mathbf{Ch}(\mathscr{A})$) becomes an isomorphism (in $\mathbf{D}(\mathscr{A})$), any morphism of complexes $C^\bullet \longrightarrow B^\bullet$ will have to count as a morphism $A^\bullet \longrightarrow B^\bullet$ in the derived category. This leads to the definition of morphism in the derived category as diagrams of the form:

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

In order to make this an actual definition of the morphism in the derived category, one has to explain when two such "roofs" are considered equal and how to define their composition. The natural context for both problems is the homotopy category of complexes $\mathbf{K}(\mathscr{A})$. This will provide an intermediate step in passing from $\mathbf{Ch}(\mathscr{A})$ to $\mathbf{D}(\mathscr{A})$:

$$\begin{tikzcd} \mathbf{Ch}(\mathscr{A}) \arrow[rr] \arrow[dr, bend right] & & \mathbf{D}(\mathscr{A}) \\ & \mathbf{K}(\mathscr{A}) \arrow[ur, bend right] & \end{tikzcd}$$

By abuse of notation we shall again write $Q : \mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$ for the natural functor.

Let us now recall some useful stuff from **Paragraph** 1.2. Two morphisms of complexes

$$f, g : A^\bullet \longrightarrow B^\bullet$$

are called homotopically equivalent ($f \sim q$) if there exists a collection of homomorphisms $h^i : A^i \longrightarrow B^{i-1},\ i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^i \circ h^i.$$

The homotopy category of complexes $\mathbf{K}(\mathscr{A})$ is the category whose objects are the objects of $\mathbf{Ch}(\mathscr{A})$, i.e. $Ob(\mathbf{K}(\mathscr{A})) = Ob(\mathbf{Ch}(\mathscr{A}))$, and morphisms $Hom_{\mathbf{K}(\mathscr{A})}(A^\bullet, B^\bullet) = Hom_{\mathbf{Ch}(\mathscr{A})}(A^\bullet, B^\bullet)/\sim$.

That definition makes sense, i.e. that the composition is well-defined in $\mathbf{K}(\mathscr{A})$, follows from the following assertions which are easily verified.

Proposition 4.1.5.

(i) *Homotopy equivalence between morphisms $A^\bullet \longrightarrow B^\bullet$ of complexes is an equivalence relation.*

(ii) *Homotopically trivial morphisms form an ideal in the morphisms of $\boldsymbol{Ch}(\mathscr{A})$.*

(iii) *If $f \simeq g : A^\bullet \longrightarrow B^\bullet$, then $H^i(f) = H^i(g)$ for all i.*

(iv) *$f : A^\bullet \longrightarrow B^\bullet$ and $g : B^\bullet \longrightarrow A^\bullet$ are given such that $f \circ g \simeq id_B$ and $g \circ f \simeq id_A$, then f and g are quasi-isomorphisms (and actually, $H^i(f^{-1}) = H^i(g)$.*

Now, for the definition of derived category, we have all the tools we need.

(1) The objects of the derived category come naturally as follows:

$$Ob(\mathbf{D}(\mathscr{A})) = Ob(\mathbf{K}(\mathscr{A})) = Ob(\mathbf{Ch}(\mathscr{A}))$$

(2) The set of morphisms $Hom_{\mathbf{D}(\mathscr{A})}(A^\bullet, B^\bullet)$ viewed as objects in $\mathbf{D}(\mathscr{A})$ is the set of equivalent classes of diagrams of the form:

$$\begin{tikzcd}
 & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

Two such diagrams are equivalent if they are dominated in the homotopy category $\mathbf{K}(\mathscr{A})$ by a third one of the same short, which means that the following diagram is commutative:

$$\begin{tikzcd}
 & & C^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dashed, bend right, "qis"'] & & \\
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "qis"'] \arrow[dr] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

Now two things we need to point is that composition of quasi-isomorphisms is a quasi-isomorphism (i.e. $C^\bullet \to C_1^\bullet \to A^\bullet$ is a qis) and that the compositions $C^\bullet \to C_1^\bullet \to A^\bullet$ and $C^\bullet \to C_2^\bullet \to A^\bullet$ are homotopy equivalent, so since the first one is a qis, so is the second one.

(3) Now we have to define the composition of two morphisms. If two morphisms:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

are given, we want the composition of both be given by a commutative diagram (in the homotopy category $\mathbf{K}(\mathscr{A})$) of the form

$$\begin{tikzcd} & & C_0^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd} \tag{4.1}$$

Now we have to make sure that (1) such diagram exists and that (2) is unique up to isomorphism. Both things hold true but first we need to remember the concept of mapping cone from **Paragraph** 1.4 in order to explain why will it also play the central role in the definition of triangulated structure on $\mathbf{K}(\mathscr{A})$ and $\mathbf{D}(\mathscr{A})$.

Definition 4.1.6. *Let $f : A^\bullet \longrightarrow B^\bullet$ be a complex morphism. Its mapping cone is the complex C_f with*

$$C_f^i = A^{i+1} \oplus B^i \quad \textit{and} \quad d_{C_f}^i = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

It is easy to verify that mapping cone is a complex. Moreover, there exist two natural complex morphisms:

$$\beta : B^\bullet \longrightarrow C_f \quad \textit{and} \quad \alpha : C_f \longrightarrow A^\bullet[1]$$

given by the natural injection $B^\bullet \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively. The composition $B^\bullet \to C_f \to A^\bullet[1]$ is trivial and the composition $A^\bullet \to B^\bullet \to C_f$ is homotopic to the trivial map. Moreover, $B^\bullet \to C_f \to A^\bullet[1]$ is a short exact sequence of complexes, so we obtain a long exact cohomology sequence:

$$\cdots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C_f) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \dots$$

Also, by construction, any commutative diagram can be completed as follows:

$$\begin{tikzcd} A_1^\bullet \arrow[r, "f_1"] \arrow[d] & B_1^\bullet \arrow[r] \arrow[d] & C_{f_1} \arrow[r] \arrow[d, dashed] & A_1^\bullet[1] \arrow[d] \\ A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r] & C_{f_2} \arrow[r] & A_2^\bullet[1] \end{tikzcd}$$

(the first square commutes).

Proposition 4.1.7. *Let $f : A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes and let C_f be its mapping cone. Then there exists a complex morphism $g : A^\bullet[1] \longrightarrow C_\beta$ which is an isomorphism in $\boldsymbol{K}(\mathscr{A})$ and such that the following diagram is commutative in $\boldsymbol{K}(\mathscr{A})$:*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\beta"] \arrow[d, "="] & C_f \arrow[r, "\alpha"] \arrow[d, "="] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dashed, "g"] & B^\bullet[1] \arrow[d, "="] \\
B^\bullet \arrow[r, "\beta"] & C_f \arrow[r, "\beta_\beta"] & C_\beta \arrow[r, "\alpha_\beta"] & B^\bullet[1]
\end{tikzcd}$$

Proof. We define the morphism $g : A^\bullet[1] \longrightarrow C_\beta$ as follows. Let:

$$A^\bullet[1] = A^{i+1} \longrightarrow C^i_\beta = B^{i+1} \oplus C^i_f = B^{i+1} \oplus A^{i+1} \oplus B^i$$

be the map $(-f^{i+1}, id, 0)$ which is easy to verify that is indeed a complex morphism.
The inverse g^{-1} in $\mathbf{K}(\mathscr{A})$ can be given as the projection from the middle factor. The commutativity (in $\mathbf{K}(\mathscr{A})$) of the square:

$$\begin{tikzcd}
A^\bullet[1] \arrow[r, "-f"] \arrow[d, "g"'] & B^\bullet[1] \arrow[d, "="] \\
C_\beta \arrow[r, "\alpha_\beta"] & B^\bullet[1]
\end{tikzcd}$$

is trivial. The remaining square:

$$\begin{tikzcd}
C_f \arrow[r, "\alpha"] \arrow[d, "="'] & A^\bullet[1] \arrow[d, "g"] \\
C_f \arrow[r, "\beta_\beta"] & C_\beta
\end{tikzcd}$$

does not commute in $\mathbf{Ch}(\mathscr{A})$, but it does commute up to homotopy. To prove this we need to check that $g \circ g^{-1}$ is homotopic to the identity and then use it to check that $g^{-1} \circ \beta_\beta = \alpha$.

□

Now we'll see how to use the construction of the mapping cone in order to compose two morphisms in the derived category.

Proposition 4.1.8. *Let $f : A^\bullet \longrightarrow B^\bullet$ be a quasi-isomorphism and $g : C^\bullet \longrightarrow B^\bullet$ be an arbitrary morphism. There exists a commutative diagram in $\boldsymbol{K}(\mathscr{A})$:*

$$\begin{tikzcd}
C_0^\bullet \arrow[r, "qis"] \arrow[d, "="'] & C^\bullet \arrow[d, "g"] \\
A^\bullet \arrow[r, "f", "qis"'] & B^\bullet
\end{tikzcd} .$$

Proof. Note that the existence of a commutative diagram (even in $\mathbf{Ch}(\mathscr{A})$ and even without $A^\bullet \longrightarrow B^\bullet$ being a qis) is trivial. The difficulty consists in constructing it such that $C_0^\bullet \longrightarrow C^\bullet$ is a qis.
The idea is to make use of a commutative diagram of the form:

$$\begin{tikzcd} C_{\beta\circ g}[1] \arrow[r, "\beta"] \arrow[d, dashed] & C^\bullet \arrow[r] \arrow[d, "g"] & C_f \arrow[r] \arrow[d, "="] & C_{\beta\circ g} \arrow[d, dashed] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\beta"] & C_f \arrow[r] & A^\bullet[1] \end{tikzcd}$$

Due to the previous proposition (4.1.7) we know that $B^\bullet \xrightarrow{\beta} C_f \longrightarrow A^\bullet[1]$ in $\mathbf{K}(\mathscr{A})$ is isomorphic to the triangle $B^\bullet \xrightarrow{\beta} C_f \longrightarrow C_\beta$. Now we can just use the natural morphism $C_{\beta\circ g} \longrightarrow C_\beta$.
Using the long exact cohomology sequence we can prove that the morphism $C_0^\bullet := C_{\beta\circ g}[-1] \longrightarrow C^\bullet$ is a quasi-isomorphism.

□

Now we get the *corollary* we worked for:

Corollary 4.1.9. *The composition of mophisms in the derived category as proposed in eq.(4.1) exists and is well defined.*

Proof. Apply Proposition (4.1.8) to

$$\begin{tikzcd} & C_1^\bullet \arrow[d] \\ C_2^\bullet \arrow[r, "qis"'] & B^\bullet \end{tikzcd} .$$

in *eq.*(4.1). □

4.2 Localizing into the the Derived Category

The derived category $\mathbf{D}(\mathscr{A})$ is defined to be the localization $S^{-1}\mathbf{K}(\mathscr{A})$ of the homotopy category of complexes $\mathbf{K}(\mathscr{A})$ at the collection S of quasi-isomorphisms, in the sense of the following definition:

Definition 4.2.1. *Let S be a collection of morphisms in a category $\mathscr{C}$. A localization of $\mathscr{C}$ with repsect to S is a category $S^{-1}(\mathscr{C})$ together with a functor $q : \mathscr{C} \longrightarrow S^{-1}(\mathscr{C})$ such that:*

(i) $q(s)$ is an isomorphism in $S^{-1}(\mathscr{C})$ for every $s \in S$,

(ii) any functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ such that $F(s)$ is an isomorphisms for all $s \in S$ factors uniquely through q.

Remark 4.2.2. *(1) Compare this to the* ***Theorem** 4.1.2* *and see the obvious connection.*

(2) Again like in the ***Theorem** 4.1.2* ***(ii)*** *the localization is unique up to isomorphism.*

Example 4.2.3. *(i) Let S be the collection of chain homotopy equivalences in $\boldsymbol{Ch}(\mathscr{A})$. The universal property of $\boldsymbol{Ch}(\mathscr{A}) \longrightarrow \boldsymbol{K}(\mathscr{A})$ (every other functor from $\boldsymbol{Ch}(\mathscr{A})$ to some category that sends chain homotopy equivalences to isomorphisms, factors uniquely through $\boldsymbol{K}(\mathscr{A})$) shows that $\boldsymbol{K}(\mathscr{A})$ is the localization $S^{-1}\,\boldsymbol{Ch}(\mathscr{A})$.*

(ii) Let $\hat{S}$ be the collection of all quasi-isomorphisms in $\boldsymbol{Ch}(\mathscr{A})$. Since $\hat{S}$ contains S of part (i), it follows that

$$\hat{S}^{-1}\,\boldsymbol{Ch}(\mathscr{A}) = \hat{S}^{-1}(S^{-1}\,\boldsymbol{Ch}(\mathscr{A})) = S^{-1}\boldsymbol{K}(\mathscr{A}) = \boldsymbol{D}(\mathscr{A}).$$

Therefore we could have defined the derived category to be the localization $\hat{S}^{-1}\mathbf{Ch}(\mathscr{A})$. However, in order to prove that $\hat{S}^{-1}\mathbf{Ch}(\mathscr{A})$ exists we must first prove that $\hat{S}^{-1}\mathbf{K}(\mathscr{A})$ exists, by giving an explicit description of the morphisms.

Remark 4.2.4. *If $\mathscr{C}$ is a small category, every localization $S^{-1}\mathscr{C}$ exists. It is also not hard to see that $S^{-1}\mathscr{C}$ exists when the class S is a set. However, when the class S is not a set, the existence of localization is a delicate set-theoritic question.*

The issue of whether or not $S^{-1}\mathscr{C}$ exists in our universe is important to some schools of thought, and in particular to topologists who need to localize with respect to homology theories. In this section we shall consider a special case in which $S^{-1}\mathscr{C}$ can be constructed "within our universe", the case in which S is a locally small multiplicative system. Later we will see that the class of quasi-isomorphisms in $\mathbf{K}(\mathscr{A})$ is a localy small multiplicative system when $\mathscr{A}$ is either $\mathbf{Mod}_R$ of right R-modules or **Sheaves(X)** of sheaves on a topological space X.

Definition 4.2.5. *A collection S of morphisms in a category $\mathscr{C}$ is called a multiplicative system in $\mathscr{C}$ if it satisfies the following self-dual axioms:*

(1) S is closed under composition and contains all identity morphisms.

(2) (Ore Condition) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathscr{C}$ there is a commutative diagram in $\mathscr{C}$ "$gs = tf$", $s \in S$:

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[g]{} & Y \end{array} .$$

(3) (Cancellation) If $f, g : X \to Y$ are parallel morphisms (i.e. they have the same source and target) in $\mathscr{C}$ then the following two conditions are equivalent:

$sf = sg$ for some $s \in S$ with source Y

$ft = gt$ for some $t \in S$ with target X

Example 4.2.6. *(Localizations of rings) An associative ring R with unit may be considered as an additive category $\mathscr{R}$ with one object $\bullet$ via $R = End_{\mathscr{R}}(\bullet)$. Let S be a subset of R closed under multiplication and containing 1. If R is commutative, or more generally if S is in the center of R, then S is always* *a multiplicative system in $\mathscr{R}$. The usual ring of fractions $S^{-1}R$ is also the localization $S^{-1}\mathscr{R}$ of the category $\mathscr{R}$.*

If S is not central, then S is a multiplicative system in $\mathscr{R}$ if and only if it is a "2-sided denominator set" (we won't bother with its definition) in R. This is the tool we need to construct the classical ring of fractions $S^{-1}R$ which is the prototype we will use in the construction of the localization $S^{-1}C$. Each element in the ring of fraction is being represented as either fs^{-1} or $t^{-1}g$ $(f, g \in R$ and $s, t \in S)$ (remember the Ore Condition above, $gs = tf$), and again $S^{-1}R$ is the localization of the category $\mathscr{R}$.

We call a chain in $\mathscr{C}$ of the form:

$$fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

a **(left) fraction** if $s \in S$.

Now we define two such fractions fs^{-1} and gt^{-1} : $X \xleftarrow{t} X_2 \xrightarrow{g} Y$ to be equivalent if a third fraction $X \longleftarrow X_3 \longrightarrow Y$ exists, fitting into a commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

It is easy to verify that this is an equivalence relation. We will write $Hom_S(X, Y)$ for the family of the equivalence classes of such fractions. Unfortunately, there is no reason for this to be a set, unless S is *locally small* in the sense of the following definition.

Definition 4.2.7. *A multiplicative system S is called localy small (on the left) if for each $X \in Ob(\mathscr{C})$ there exists a set of morphisms S_X in S, all having target X, such that for every $X_1 \to X$ in S there is a map $X_2 \to X_1$ in $\mathscr{C}$ so that the composite $X_2 \to X_1 \to X$ is in S_X.*

If S is locally small, then to see that $Hom_S(X, Y)$ is a set for every X, Y we make S_X the objects of a small category, a morphism from $X_1 \xrightarrow{s} X$ to $X_2 \xrightarrow{t} X$ being a map $X_2 \longrightarrow X_1$ in $\mathscr{C}$ so that t is $X_2 \longrightarrow X_1 \xrightarrow{s} X$. The Ore Condition says that by enlarging S_X slightly we can make it a filtered category. There is a functor $Hom_{\mathscr{C}}(\ , Y)$ from S_X to **Sets** sendind s to the set of fraction fs^{-1}, and $Hom_S(X, Y)$ to the colimit of this functor.

Composition of fractions is defined as follows:
To compose $X \longleftarrow X' \xrightarrow{g} Y$ with $Y \xleftarrow[t]{} Y' \longrightarrow Z$ we use the Ore condition to find a diagram

$$\begin{tikzcd} & W \arrow[r, "f"] \arrow[d, "s"] & Y' \arrow[r] \arrow[d, "t"] & Z \\ X & X' \arrow[l] \arrow[r, "g"] & Y & \end{tikzcd}$$

with $s \in S$. The composite is the class of the fraction $X \longleftarrow W \longrightarrow Z$ in $Hom_S(X, Z)$. It is not hard to verify that the equivalence class of the composite is independent of the choice of X' and Y', so that we have well-defined a pairing

$$Hom_S(X, Y) \times Hom_S(Y, Z) \longrightarrow Hom_S(X, Z).$$

It is clear from the construction that composition is associative. The $Hom_S(X, Y)$ (if they are sets) form the morphisms of a category having the same objects as $\mathscr{C}$; it will be our localization $S^{-1}\mathscr{C}$.

Theorem 4.2.8. *Let S be a locally small multiplicative system of morphisms in a category $\mathscr{C}$. Then the category $S^{-1}\mathscr{C}$ constructed above exists and is a localization of $\mathscr{C}$ with respect to S. The universal functor $q : \mathscr{C} \longrightarrow S^{-1}\mathscr{C}$ send $f : X \longrightarrow Y$ to the sequence $X \xleftarrow{=} X \xrightarrow{f} Y$.*

Proof. To see that $q : C \longrightarrow S^{-1}C$ is a functor, observe that the composition of $X \xleftarrow{=} X \xrightarrow{f} Y$ and $Y \xleftarrow{=} Y \xrightarrow{h} Z$ is $X \xleftarrow{=} X \xrightarrow{hf} Z$ since we can choose $t = id_X$ and $f = g$. If $s \in S$ then $q(s)$ is an isomorphism because the composition $X \xleftarrow{=} X \xrightarrow{s} Y$ and $Y \xleftarrow{s} X \xrightarrow{=} X$ is $X \xleftarrow{=} X \xrightarrow{=} X$ (take $W = X$). Finally, suppose that $F : \mathscr{C} \to \mathscr{D}$ is another functor sendind S to isomorphisms. Define $S^{-1}F : S^{-1}\mathscr{C} \to \mathscr{D}$ by sending the fraction fs^{-1} to $F(f)F(s)^{-1}$. Given g and t, the equality $gs = tf$ in C shows that $F(g)F(s) = F(t)F(f) \iff F(t^{-1}g) = F(fs^{-1})$; it follows that $S^{-1}F$ respects composition and is a functor. It is clear that $F = S^{-1}F \circ q$ and that this factorization is unique.

□

Corollary 4.2.9. *$S^{-1}\mathscr{C}$ can be constructed using equivalence classes of "right fractions" $t^{-1}g : X \xrightarrow{g} Y' \underset{t}{\longleftarrow} Y$. provided that S is locally small on the right (the dual notion of locally small, involving maps $Y \to Y'$ in S).*

Proof. S^{op} is a multiplicative system in $\mathscr{C}^{op}$. Since $\mathscr{C}^{op} \to (S^{op})^{-1}\mathscr{C}^{op}$ is a localizaion, so is its dual $\mathscr{C} \to [(S^{op})^{-1}(\mathscr{C}^{op})]^{op}$, but this is constructed using the fractions $t^{-1}g$.

□

Corollary 4.2.10. *Two parallel maps $f, g : X \to Y$ in $\mathscr{C}$ become identified in $S^{-1}\mathscr{C}$ if and only if $sf = sg$ for some $s : X_3 \to X,\ \ x \in S$.*

Corollary 4.2.11. *Suppose that $\mathscr{C}$ has a zero object. The for every X in C :*

$$q(0) \cong 0 \textit{ in } S^{-1} \iff S \textit{ contains the zero map } X \xrightarrow{0} X.$$

Proof. Since $q(0)$ is a zero object in $S^{-1}\mathscr{C}$, $q(0) \cong 0$ if and only if the parallel maps $0, id_X : X \to X$ become identified in $S^{-1}\mathscr{C}$, that is, if and only if $0 = s \circ 0 = s$ for some s.

□

Corollary 4.2.12. *If $\mathscr{C}$ is an additive category, then so is $S^{-1}C$, and q is an additive functor.*

Proof. If $\mathscr{C}$ is an additive category, we can add fractions from X to Y as follows. Given fractions $f_1 s_1^{-1}$ and $f_2 s_2^{-1}$, we use the Ore condition to find an $s : X_2 \to X$ in S and $f_1', f_2' : X_2 \to Y$, so that $f_1 s_1^{-1} \sim f_1' s^{-1}$ and $f_2 s_2^{-1} \sim f_2' s^{-1}$; the sum $(f_1' + f_2')s^{-1}$ is well defined up to equivalence. Since $q(X \times Y) \cong q(X) \times q(Y)$ (if the product $X \times X$ exists in $\mathscr{C}$ the " $\cong$ " holds) in $S^{-1}\mathscr{C}$, it follows that $S^{-1}\mathscr{C}$ is an additive category and that q is an additive functor.

□

Now we are ready to show that $\mathbf{D}(\mathscr{A})$ is a triangulated category and that it exists, at least if $\mathscr{A}$ is $\mathbf{Mod}_R$ or $\mathbf{Sheaves}(X)$. For this we generalize slightly. Let $\mathbf{K}$ be a triangulated category. The system S arising from a cohomological functor $H : \mathbf{K} \to \mathscr{A}$ is the collection of all morphisms s in $\mathbf{K}$ such that $H^i(s)$ is an isomorphism for all $i \in \mathbb{Z}$. For example, the quasi-isomorphisms $\hat{S}$ arise from the cohomological functor H^0.

Proposition 4.2.13. *If S arises from a cohomological functor, then*

(1) S is a multiplicative system.

(2) $S^{-1}\boldsymbol{K}$ is a triangulated category, and $\boldsymbol{K} \to S^{-1}\boldsymbol{K}$ is a morphism of triangulated categories (in any universe containing $S^{-1}\boldsymbol{K}$).

Proof. We first show that the system S is multiplicative, by checking that the axioms hold.

(1) The first Axiom (S is closed under composition and contains all identity morphisms) is trivial.

(2) To prove this, let $f : X \to Y$ and $s : Z \to Y$ be given. Embed s in an exact triangle (s, u, δ) on (Z, Y, C) using (TR1). Complete $uf : X \to C$ into an exact triangle (t, uf, v) on (W, X, C). By axiom (TR3) there is a morphism g such that

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r, "v"] \arrow[d, "="] & W[-1] \arrow[d, "Tg"] \\ Z \arrow[r, "s"] & Y \arrow[r, "u"] & C \arrow[r, "\delta"] & Z[-1] \end{tikzcd}$$

is a morphism of triangles. If $H^\bullet(s)$ is an isomorphism, $H^\bullet(C) = 0$. Applying this to the long exact sequence of the other triangle, we see that $H^\bullet(t)$ is also an isomorphism. The symmetric assertion may be proven similarly, or by appeal to $\mathbf{K}^{op} = \mathscr{A}^{op}$.

(3) To verify this, we consider the difference $h = f - g$. Given $s : Y \to Y'$ in S with $sf = sg$, embed s in an exact triangle (u, s, δ) on (Z, Y, Y'). Note that $H^\bullet(Z) = 0$. Since $Hom_{\mathbf{K}}(X, \ )$ is a cohomological functor,

$$Hom_{\mathbf{K}}(X, Z) \xrightarrow{u} Hom_{\mathbf{K}}(X, Y) \xrightarrow{s} Hom_{\mathbf{K}}(X, Y')$$

is exact. Since $s(f - g) = 0$, there is a $g : X \to Z$ in $\mathbf{K}$ such that $f - g = ug$. Embed g in an exact triangle (t, g, w) on (X', X, Z). Since $gt = 0$, $(f - g)t = ugt = 0$, whence $ft = gt$. And since $H^\bullet(X') \cong H^\bullet(X)$, that is, $t \in S$. The other implication of axiom (3) is analogous and may be deducted from the above by appeal to $\mathbf{K}^{op} = \mathscr{A}^{op}$.

Now suppose that $S^{-1}\mathbf{K}$ exists. The formula $T(fs^{-1}) = T(f)T(s)^{-1}$ defines a translation functor T on $S^{-1}\mathbf{K}$. To show that $S^{-1}\mathbf{K}$ is triangulated, we need to define exact triangles. Given $us_1^{-1} : A \to B$, $vs_2^{-1} : B \to C$ and $ws_3^{-1} : C \leftarrow C' \to T(A)$, the Ore condition for S yields morphisms $t_1 : A' \to A$ and $t_2 : B' \to B$ in S and $u' : A' \to B'$, $v' : B' \to C'$ in C so that $us_1^{-1} \cong t_2u't_1^{-1}$ and $vs_2^{-1} \cong s_3v't_2^{-1}$. We say that $(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$ is an exact triangle in $S^{-1}\mathbf{K}$ just in case (u', v', w) is an exact triangle in $\mathbf{K}$.
The verification that $S^{-1}\mathbf{K}$ is triangulated is omitted being straightforward but lengthy; one uses the fact that $Hom_S(X, Y)$ may also be calculated using fractions of the form $t^{-1}g$.

$\square$

Corollary 4.2.14. ***(Universal Property)*** *Let $F : \boldsymbol{K} \to \boldsymbol{L}$ be a morphism of triangulated categories such that $F(s)$ is an isomorphism for all $s \in S$, where S arises from a cohomological functor. Since $q : \boldsymbol{K} \to S^{-1}\boldsymbol{K}$ is a localization, there is a unique functor $F' : S^{-1}\boldsymbol{K} \to \boldsymbol{L}$ such that $F = F' \circ q$. In fact, F' is a morphism of triagnulated categories.*

Corollary 4.2.15. *$\boldsymbol{D}(\mathscr{A})$ is a triangulated category.*

Proposition 4.2.16. *Let R be a ring. Then $\boldsymbol{D}(\mathscr{A})$ exists and is a triangulated category if $\mathscr{A}$ is the category $\boldsymbol{Mod}_R$ or either of*

- ***Presheaves****(X), presheaves of R-modules on a topological space X,*
- ***Sheaves****(X), sheaves of R-modules on a topological space X.*

Proof. We will use A instead of $A^\bullet$ for cochain complexes in this proof in order to make it more coherent and readable.

We have to prove that the multiplicative system $\hat{S}$ is locally small. Given a fixed cochain complex of R-modules A, choose an infinite cardinal number κ larger than the cardinality of the sets underlying the A^i and R. Call a cochain complex B petite if its underlying sets have cardinality $< \kappa$; there is a set of isomorphism classes of petite cochain complexes, hence a set S_X of isomorphism classes of quasi-isomorphisms $A' \to A$ with A' petite.

Given a quasi-ismorphism $B \to A$ it suffices to show that B contains a petite subcomplex B' quasi-isomorphic to A. Since $H^\bullet(A)$ has cardinality $< \kappa$, there is a petite subcomplex B_0 of B such that the map $f_0 : H^\bullet(B_0) \to H^\bullet(A)$ is onto. Since $\mathrm{Ker}(f_0)$ has cardinality $< \kappa$, we can enlarge B_0 to a petite subcomplex B_1 such that $\mathrm{Ker}(f_0)$ vanishes in $H^\bullet(B_1)$. Inductively, we can construct an increasing sequence of petite subcomplexes B_n of B such that the kernel of $H^\bullet(B_n) \to H(A)$ vanishes in $H^\bullet(B_{n+1})$, but then their union $B' = \bigcup_{n\in\mathbb{N}} B_n$ is a petite subcomplex of B with

$$H^\bullet(B') \cong \varinjlim H^\bullet(B_n) \cong H^\bullet(A).$$

The proof for presheaves is identical, except that κ must bound the number of open subsets U as well as the cardinality of $A(U)$ for every open subset U of X. The proof for sheaves is similar, using the following three additional facts:

(1) if κ bounds the cardinal of $A(U)$ for all U and the number of such U, then κ also bounds the cardinality of the stalks A_x for $x \in X$,

(2) a map $B \to A$ is a quasi-isomorphism in **Sheaves**(X) if and only if every map of stalks $B_x \to A_x$ is a quasi-isomorphism,

(3) for every directed system of sheaves, we have

$$H^\bullet(\varinjlim B_n) = \varinjlim H^\bullet(B_n)\ .$$

$\square$

Chapter 5

Grothendieck's Spectral Sequence Theorem

In his classic paper, [Tohoku], Grothendieck introduced a spectral sequence associated to the composition of two functors. Today it is one of the organizational principles of Homological Algebra. In this chapter our main goal is to prove **Grothendieck's Spectral Sequence Theorem** 5.2.2 using spectral sequences and then simplify this result using derived categories. In a nutshell this very powerful tool Grothendieck has provided us with, is a spectral sequence that computes the derived functors of the composition of two functors solely by the knowledge of their derived functors. Before doing all that we will need to define what a **hyper-derived functor** is, which will play a crucial role in the following chapter.

5.1 Hypercohomology

Definition 5.1.1. *Let $\mathscr{A}$ be an abelian category that has enough projectives. A (left) Cartan - Eilenberg resolution $P_{\bullet,\bullet}$ of a chain complex $\boldsymbol{A}_\bullet$ in $\boldsymbol{Ch}(\mathscr{A})$ is an upper half-plane double complex (i.e. $P_{p,q} = 0$ if $q < 0$), consisting of projective objects of $\mathscr{A}$ together with a chain map, called augmentation $P_{\bullet,0} \xrightarrow{\varepsilon} \boldsymbol{A}_\bullet$ such that for every p*

(1) If $A_p = 0$, the column $P_{p,\bullet}$ is zero.

(2) The maps on boundaries and homology

$$B_p(\varepsilon) : B_p(P, d_\rightarrow) \rightarrow B_p(\boldsymbol{A})$$

$$H_p(\varepsilon) : H_p(P, d_\rightarrow) \rightarrow H_p(\boldsymbol{A})$$

are projective resolutions in $\mathscr{A}$. Here $B_p(P, d_\rightarrow)$ denotes the horizontal boundaries in the (p, q) spot, that is, the chain complex whose q^{th} term is $d_\rightarrow(P_{p+1,q})$. The chain complexes $Z_p(P, d_\rightarrow)$ and $H_p(P, d_\rightarrow) = Z_p(P, d_\rightarrow)/B_p(\varepsilon)$ are defined similarly.

Lemma 5.1.2. *Every chain complex $\boldsymbol{A}_\bullet$ has a Cartan - Eilenberg resolution $P_{\bullet,\bullet} \rightarrow \boldsymbol{A}_\bullet$.*

Definition 5.1.3. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor and $\mathscr{A}$ be an abelian category with enough projectives. We define the left hyper-derived functors $\mathbb{L}_i F$ as follows:*

If $\boldsymbol{A}$ is a chain complex in $\boldsymbol{Ch}(\mathscr{A})$ and $P \to \boldsymbol{A}$ is Cartan - Eilenberg resolution, then $\mathbb{L}_i F(\boldsymbol{A}) = H_i Tot^{\oplus}(F(P))$.

If $f : \boldsymbol{A} \to \boldsymbol{B}$ is a chain map and $\hat{f} : P \to Q$ is a map of Cartan - Eilenberg resolutions over f, then $\mathbb{L}_i F(f) = H_i Tot(\hat{f})$ from $\mathbb{L}_i F(\boldsymbol{A})$ to $\mathbb{L}_i F(\boldsymbol{B})$.

Remark 5.1.4. *We can show that $\mathbb{L}F(\boldsymbol{A})$ is independent of the choise of P. Basically if $f, g : \boldsymbol{A} \to \boldsymbol{B}$ are homotopic chain maps and $\hat{f}, \hat{g} : P \to Q$ are Cartain-Eilenberg resolutions lying over them, then $\hat{f}$ is chain homotopic to $\hat{g}$. Then we can show that any two Cartan - Eilenberg resolutions P , Q of $\boldsymbol{A}$ are chain homotopy equivalent. With this we can conclude that for any additive functor F the chain complexes $H_i Tot^{\oplus}(F(P))$ and $H_i Tot^{\oplus}(F(Q))$ are chain homotopy equivalent.*

Lemma 5.1.5. *If $0 \to \boldsymbol{A} \to \boldsymbol{B} \to \boldsymbol{C} \to 0$ is a short exact sequence of bounded below complexes, there is a long exact sequence*

$$\cdots \to \mathbb{L}_{i+1} F(\boldsymbol{C}) \xrightarrow{\delta} \mathbb{L}_i F(\boldsymbol{A}) \to \mathbb{L}_i F(\boldsymbol{B}) \to \mathbb{L}_i F(\boldsymbol{C}) \xrightarrow{\delta} \cdots .$$

Proposition 5.1.6. *There is always a convergent spectral sequence*

$$^{II}E^2_{p,q} = (L_p F)(H_q(\boldsymbol{A})) \Rightarrow \mathbb{L}_{p+q} F(\boldsymbol{A}).$$

If $\boldsymbol{A}$ is bounded below, there is a convergent spectral sequence

$$^{I}E^2_{p,q} = H_p(L_q F(\boldsymbol{A})) \Rightarrow \mathbb{L}_{p+q} F(\boldsymbol{A}).$$

Corollary 5.1.7.

(1) If $\boldsymbol{A}$ is exact, $\mathbb{L}_i F(\boldsymbol{A}) = 0$, for all i.

(2) Any quasi-isomorphism $f : \boldsymbol{A} \to \boldsymbol{B}$ induces isomorphisms

$$\mathbb{L}_\bullet F(\boldsymbol{A}) \cong \mathbb{L}_\bullet F(\boldsymbol{B}).$$

(3) If each A_p is F-acyclic, that is, $L_q F(A_p) = 0$, for all $q \neq 0$, and $\boldsymbol{A}$ is bounded below, then

$$\mathbb{L}_p F(\boldsymbol{A}) = H_p(F(\boldsymbol{A})), \text{ for all } p.$$

As always we can define and get the dual cohomological variant of all the above:

Hypercohomology Spectral Sequence 5.1.8. *Let $\mathscr{A}$ be an abelian category with enough injectives. A (right) Cartan - Eilenberg resolution of a cochain complex $\boldsymbol{A}^\bullet$ in $\boldsymbol{Co}(\mathscr{A})$ is an upper half-plane complex $I^{\bullet,\bullet}$ of injective objects of $\mathscr{A}$, together with an augmentation $\boldsymbol{A}^\bullet \to I^{\bullet,0}$ such that the maps on coboundaries and cohomology are injective resolutions of $B^p(\boldsymbol{A})$ and $H^p(\boldsymbol{A})$. Every cochain complex has a Cartan - Eilenberg resolution $\boldsymbol{A} \to I$. If $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor, we define $\mathbb{R}^i F(\boldsymbol{A})$ to be $H^i Tot^{\Pi}(F(I))$, at least when $Tot^{\Pi}(F(I))$ exists in $\mathscr{B}$. The $\mathbb{R}^i F$ are called the right hyper-derived functors of F.*

If $\boldsymbol{A}$ *is in* $\boldsymbol{Co}(\mathscr{A})$*, the two spectral sequences arising from the upper half plane double cochain complex* $F(I)$ *becomes*

$$^{II}E_2^{p,q} = (R^pF)(H^q(\boldsymbol{A})) \Rightarrow \mathbb{R}^{p+q}F(\boldsymbol{A}),\ \textit{weakly convergent},$$

and

$$^{I}E_2^{p,q} = H^p(R^qF(\boldsymbol{A}) \Rightarrow \mathbb{R}^{p+q}F(\boldsymbol{A}),\ \textit{if } \boldsymbol{A} \textit{ is bounded below.}$$

5.2 Grothendieck's Spectral Sequence

In this chapter we will be working in $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ which are abelian categories such that both $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. We are given left exact functors $G : \mathscr{A} \to \mathscr{B}$ and $F : \mathscr{B} \to \mathscr{C}$.

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "G"] \arrow[dr, "FG"'] & & \mathscr{B} \arrow[dl, "F"] \\ & \mathscr{C} & \end{tikzcd}$$

Definition 5.2.1. *Let* $F : \mathscr{B} \to \mathscr{C}$ *be a left exact functor. An object* $B \in Ob(\mathscr{B})$ *is called* F*-acyclic if the derived functors of* F *vanish on* B*, that is,* $R^iF(B) = 0$ *for* $i \neq 0$.

Grothendieck's Spectral Sequence Theorem 5.2.2. *Given the above cohomological setup, suppose that* G *sends injective objects of* $\mathscr{A}$ *to* F*-acyclic objects of* $\mathscr{B}$*. Then, there exists a first quadrant cohomological sequence of each* $A \in Ob(\mathscr{A})$*:*

$$^{I}E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

The edge maps in spectral sequence are the natural maps

$$(R^pF)(GA) \to R^p(FG)(A) \textit{ and } R^q(FG)A \to F(R^qG(A)).$$

The exact sequence of low term is

$$0 \to (R^1F)(GA) \to R^1(FG)(A) \to F(R^1G(A)) \to (R^2F)(GA) \to R^2(FG)A.$$

Proof. Choose an injective resolution $A \to I$ of A in $\mathscr{A}$ and apply G to get a cochain complex $G(I)$ in $\mathscr{B}$. Using a first quadrant Cartain-Eilenberg resolution of $G(I)$, form the hyper-derived functors $\mathbb{R}^nF(G(I))$. There are two spectral sequences converging to hyper-derived functors. The first spectral sequence is

$$^{I}E_2^{p,q} = H^p((R^qF)(GI)) \Rightarrow (\mathbb{R}^{p+q}F)(GI).$$

By hypothesis, each $G(I^p)$ is F-acyclic, so $(R^qF)(G(I^p)) = 0$ for $q \neq 0$. Therefore this spectral sequence collapses to yield

$$(\mathbb{R}^pF)(GI) \cong H^p(FG(I)) = R^p(FG)(A).$$

The second spectral sequence is therefore

$$^{II}E_2^{p,q} = (R^pF)H^q(G(I)) \Rightarrow R^{p+q}(FG)(A).$$

Since $H^q(G(I)) = R^qG(A)$, it is Grothendieck's spectral sequence.

□

Corollary 5.2.3. *(Homology Spectral Sequence) Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories such that both $\mathscr{A}$ and $\mathscr{B}$ have enough projectives. Suppose given right exact functors $G : \mathscr{A} \to \mathscr{B}$ and $F : \mathscr{B} \to \mathscr{C}$ such that G sends projective objects of $\mathscr{A}$ to F-acyclic objects of $\mathscr{B}$. Then there is a convergent first quadrant homology spectral sequence for each $A \in Ob(\mathscr{A})$:*

$$E^2_{p,q} = (L_pF)(L_qG)(A) \Rightarrow L_{p+q}(FG)(A).$$

The exact sequence of low degree term is

$$L_2(FG)A \to (L_2F)(GA) \to F(L_1G(A)) \to L_1(FG)A \to (L_1F)(GA) \to 0.$$

5.3 Morphisms of Triangulated Cateogories

This will set the stage for the next paragraph: we will state some known results which will later be useful to us. The proofs will be omitted.

Definition 5.3.1. *A morphism $F : \boldsymbol{K} \to \boldsymbol{K}'$ of triangulated categories is a (covariant) additive functor that commutes with the translation functor T and sends exact triangles to exact triangles.*

For example, suppose we are given a morphism $F : \mathscr{A} \to \mathscr{B}$ between two abelian categories. Since F preserves chain homotopy equivalences, it extends to additive functors $\mathbf{Ch}(\mathscr{A}) \to \mathbf{Ch}(\mathscr{B})$ and $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$. Since F commutes with translation of chain complexes, it preserves mapping cones and exact triangles. Thus $F : \mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ is a morphism of triangulated categories.

We would like to extend F to a functor $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$. If $F : \mathscr{A} \to \mathscr{B}$ is exact, this is easy. However, if F is not exact, the functor $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$ will not preserve quasi-isomorphisms, and this may not be possible. The thing to expect is that if F is left or right exact, then the derived functors of F will be needed to extend something like the hyper-derived functors of F.

Let $\mathbf{K}$ denote $\mathbf{K}^+(\mathscr{A})$ the category of bounded bellow cochain complexes or any other localizing triangulated subcategory of $\mathbf{K}(\mathscr{A})$, and let $\mathbf{D}$ denote the full subcategory of the derived category $\mathbf{D}(\mathscr{A})$ corresponding to $\mathbf{K}$.

Definition 5.3.2. *Let $F : \boldsymbol{K} \to \boldsymbol{K}(\mathscr{B})$ be a morphism of triangulated categories. A (total) right derived functor of F on $\boldsymbol{K}$ is a morphism $\boldsymbol{R}F : \boldsymbol{D} \to \boldsymbol{D}(\mathscr{B})$ of triangulated categories, together with a natural transformation ξ:*

$$\begin{array}{rcl} qF : \boldsymbol{K} \longrightarrow & \boldsymbol{K}(\mathscr{B}) & \longrightarrow \boldsymbol{D}(\mathscr{B}) \\ & \Downarrow \xi & \\ (\boldsymbol{R}F)q : \boldsymbol{K} \longrightarrow & \boldsymbol{D} & \longrightarrow \boldsymbol{D}(\mathscr{B}) \end{array}$$

which is universal in the sense that if $G : \boldsymbol{D} \to \boldsymbol{D}(\mathscr{B})$ is another morphism equipped with a natural transformation $\zeta : qF \Rightarrow Gq$, then there exists a unique natural transformation $\eta : \boldsymbol{R}F \Rightarrow G$ so that $\zeta_A = \eta_{qA} \circ \xi_A$ for every A in $\boldsymbol{D}$.

This universal property guarantees that if $\mathbf{R}F$ exists, it is unique up to natural isomoporphism and that if $\mathbf{K}' \subset \mathbf{K}$, then there is a natural transformation from the right derived functor $\mathbf{R}'F$ on $\mathbf{D}'$ to the restriction of $\mathbf{R}F$ to $\mathbf{D}'$.

Similarly we can define the left derived functor of F as a morphism $\mathbf{L}F : \mathbf{D} \to \mathbf{D}(\mathscr{B})$ together with a natural transformation $\xi : (\mathbf{L}F)q \Rightarrow qF$ satisfying the dual universal propery (G factors uniquely through $\eta : G \Rightarrow \mathbf{L}F$). Sine $\mathbf{L}F$ is $\mathbf{R}(F^{op})^{op}$, where $F^{op} : \mathbf{K}^{op} \to \mathbf{K}(\mathscr{B}^{op})$ we can translate any statement about $\mathbf{R}F$ into a dual statement about $\mathbf{L}F$.

Remark 5.3.3. *If $F : \mathscr{A} \to \mathscr{B}$ is an exact functor, F preserves quasi-isomorphisms. Hence F extends trivially to $F : \boldsymbol{D}(\mathscr{A}) \to \boldsymbol{D}(\mathscr{B})$. In effect, F is its own left derived functor.*

Theorem 5.3.4. *(Existence Theorem) Let $F : \boldsymbol{K}^{+}(\mathscr{A}) \to \boldsymbol{K}(\mathscr{B})$ be a morphism of triangulated categories. If $\mathscr{A}$ has enough injectives, then the right derived functor $\boldsymbol{R}^{+}F$ exists on $\boldsymbol{D}^{+}(\mathscr{A})$, and if I is a bounded bellow complex of injectives, then*

$$\boldsymbol{R}^{+}F(I) \cong qF(I).$$

Corollary 5.3.5. *Let $F : \mathscr{A} \to \mathscr{B}$ is an additive functor between abelian categories. If $\mathscr{A}$ has enough injectives, then the hyper-derived functors $\mathbb{R}^{i}F(X)$ are the cohomology of $\boldsymbol{R}F(X)$: $\mathbb{R}^{i}F(X) \cong H^{i}\boldsymbol{R}^{+}F(X)$ for all i.*

Theorem 5.3.6. *(Generalized Existence Theorem) Suppose that $\boldsymbol{K}'$ is a triangulated subcategory of $\boldsymbol{K}$ such that:*

(1) Every X in $\boldsymbol{K}$ has a quasi-isomorphism $X \to X'$ to an object of $\boldsymbol{K}'$.

(2) Every exact complex in $\boldsymbol{K}'$ is F-acylic.

Then $\boldsymbol{D}' \xrightarrow{\simeq} \boldsymbol{D}$ and $\boldsymbol{R}F : \boldsymbol{D} \simeq \boldsymbol{D}' \xrightarrow{\boldsymbol{R}'F} \boldsymbol{D}(\mathscr{B})$ is a right derived functor of F.

5.4 Replacing Spectral Sequences

Theorem 5.4.1. *(Composition Theorem) Let $\boldsymbol{K} \subset \boldsymbol{K}(\mathscr{A})$ and $\boldsymbol{K}' \subset \boldsymbol{K}(\mathscr{B})$ be localizing triangulated subcategories, and suppose given two morphisms of triangulated categories $G : \boldsymbol{K} \to \boldsymbol{K}'$ and $F : \boldsymbol{K}' \to \boldsymbol{K}(\mathscr{C})$. Assume that $\boldsymbol{R}F$, $\boldsymbol{R}G$ and $\boldsymbol{R}(FG)$ exist, with $\boldsymbol{R}F(\boldsymbol{D}) \subseteq \boldsymbol{D}'$. Then:*

(1) There is a unique natural transformation $\zeta = \zeta_{F,G} : \boldsymbol{R}(FG) \Rightarrow \boldsymbol{R}F \circ \boldsymbol{R}G$, such that the following diagram commutes in $\boldsymbol{D}(\mathscr{C})$ for each A in $\boldsymbol{K}$.

$$\begin{tikzcd} qFG(A) \arrow[r, "\xi_F"] \arrow[d, "\xi_{FG}"] & (\boldsymbol{R}F)(qGA) \arrow[d, "\xi_G"'] \\ \boldsymbol{R}(FG)(qA) \arrow[r, "\zeta_{qA}"'] & (\boldsymbol{R}F)(\boldsymbol{R}G)(qA) \end{tikzcd}$$

(2) Suppose that there are triangulated subcategories $\boldsymbol{K}_0 \subseteq \boldsymbol{K}$ and $\boldsymbol{K}'_0 \subseteq \boldsymbol{K}'$ satisfying the hypothesis of the Generalized Existance Theorem 5.3.6 for G and F, and suppose that G sends $\boldsymbol{K}_0$ to $\boldsymbol{K}'_0$. Then ζ is an isomorphism

$$\zeta : \boldsymbol{R}(FG) \cong (\boldsymbol{R}F) \circ (\boldsymbol{R}G).$$

Proof. The first part follows from the universal property 5.3.2 of $\mathbf{R}(FG)$. For the second one, it suffices to observe that if A is in $\mathbf{K}_0$, then

$$\mathbf{R}(FG)(qA) = qFG(A) \cong \mathbf{R}F(q(GA)) \cong \mathbf{R}F(\mathbf{R}G(qA)).$$

$\square$

Corollary 5.4.2. *(Grothendieck's Spectral Sequences) Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories that both $\mathscr{A}$ and $\mathscr{B}$ have enough injectives, and suppose we are given left exact functors $G : \mathscr{A} \to \mathscr{B}$ and $F : \mathscr{B} \to \mathscr{C}$. If G sends injective objects of $\mathscr{A}$ to F-acyclic objects of $\mathscr{B}$, then*

$$\zeta : \boldsymbol{R}^+(FG) \cong (\boldsymbol{R}^+F) \circ (\boldsymbol{R}^+G).$$

There is also a convergent spectral sequence for all A:

$$E_2^{p,q} = (R^pF)(\mathbb{R}^qG)(A) \Rightarrow \mathbb{R}^{p+q}(FG)(A).$$

If A is an object of $\mathscr{A}$, this is Grothendieck's Spectral Sequence of 5.2.2.

Proof. The hypercohomology spectral sequence 5.1.8 converging to $(\mathbb{R}^{p+q}F)(\mathbf{R}G(A))$ has $E_2^{p,q}$ term $(R^pF)H^q(\mathbf{R}G(A)) = (R^pF)(\mathbb{R}^qG(A))$. □

Conceptually, the composition of functors $\mathbf{R}(FG) \cong (\mathbf{R}F) \circ (\mathbf{R}G)$ is much simpler than the original spectral sequence.

5.5 The Leray Spectral Sequence

We will, now, state some useful facts in the form of reminders (for proofs see [CW]). The goal of this section is to see a direct application of Grothendieck's Spectral Sequence 5.2.2, the Leray Spectral Sequence 5.5.8.

Definition 5.5.1. *We say that a pair $L : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ of additive functors between abelian categories are* ***adjoint****, if there is a natural isomorphism*

$$\tau : Hom_{\mathscr{B}}(L(A), B) \xrightarrow{\simeq} Hom_{\mathscr{A}}(A, R(B)).$$

Remark 5.5.2. *In this case, L is right exact and R is left exact.*

Definition 5.5.3. *Now, let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}V)$ for every open V on Y.*

Remark 5.5.4. *$f_*\mathcal{F}$ is a sheaf.*

Definition 5.5.5. *For any sheaf $\mathcal{G}$ on Y, we can define the inverse sheaf $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf sending an open set U in X to the direct limit $\varinjlim \mathcal{G}(V)$ over the poset of all open sets V in Y containing $f(U)$.*

We can show that there is a natural map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, for every sheaf $\mathcal{F}$ on X, and a natural map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$, for every sheaf $\mathcal{G}$ on Y. This means that f^{-1} and f_* are adjoint to each other, so f^{-1} preserves projectives and f_* preserves injectives. Moreover, f^{-1} is right exact and f_* is left exact.

Definition 5.5.6. *The derived functors R^if_* are called the* ***higher direct image sheaf functors****.*

Definition 5.5.7. *The* ***global sections*** *functor Γ from $Sheaves(X)$ to $\boldsymbol{Ab}$ is the functor $\Gamma(\mathcal{F}) = \mathcal{F}(X)$.*

The global sections functor Γ is right adjoint to the constant sheaves functor, so Γ is left exact.

The right derived functors of Γ are the **cohomology functors** on X:

$$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F}).$$

Leray Spectral Sequence 5.5.8. *In this setup, if $\mathcal{F}$ is a sheaf of abelian groups on X, the global sections of $f_*\mathcal{F}$ is the group*

$$(f_*\mathcal{F})(Y) = \mathcal{F}(f^{-1}Y) = \mathcal{F}(X):$$

$$\begin{tikzcd} Sheaves(X) \arrow[rr, "f_*"] \arrow[dr, "\Gamma"'] & & Sheaves(Y) \arrow[dl, "\Gamma"] \\ & \boldsymbol{Ab} & \end{tikzcd}$$

The Grothendieck Spectral Sequence 5.2.2 in this case is called the ***Leray Spectral Sequence****: since $R^p\Gamma$ is sheaf cohomology, it is actually written as*

$$E^2_{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

This spectral sequence is a tool to much of modern algebraic geometry.

Appendix A

Projective And Injective Modules

Definition A.0.1. *A covariant functor $T : R_{\boldsymbol{Mod}} \to \boldsymbol{Ab}$ is an* ***exact functor*** *if, for every exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

the sequence

$$0 \to T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C) \to 0$$

is also exact.
A contravariant functor is an ***exact functor*** *if there is always exactness of*

$$0 \to T(C) \xrightarrow{T(p)} T(B) \xrightarrow{T(i)} T(A) \to 0$$

For the next theorem, take note that every left module is a quotient of a free left module.

Theorem A.0.2. *Let F be a free left R-module. If $p : A \to A''$ is surjective, then for every $h : F \to A''$, there exists an R-homomorphism g, making the following diagram commute:*

$$\begin{tikzcd} & F \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ A \arrow[r, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

The proof for this is easy and it's emitted.

Definition A.0.3. *A* ***lifting*** *of a map $h : C \to A''$ is a map $g : C \to A$ with $pg = h$*

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[d, "h"] \\ A \arrow[r, "p"'] & A'' \end{tikzcd}$$

That g is a lifting of h says that $h = p_*(g)$

If C is any, not necessarily free, module, then a lifting, if one exists, need not be unique. Exactness of

$$0 \to \text{Ker}(p) \xrightarrow{i} A \xrightarrow{p} A''$$

where i is the inclusion, gives $pi = 0$. Any other lifting has the form $g + if$ for $f : C \to \text{Ker}p$; this follows from exactness of

$$0 \to Hom(C, \text{Ker}p) \xrightarrow{i_*} Hom(C, A) \xrightarrow{p_*} Hom(C, A'')$$

for any two liftings of h differ by a map $if \in \text{Im}i_* = \text{Ker}p_*$.

Definition A.0.4. *A left R-module P is* ***projective*** *if, whenever p is surjective and h is any map, there exists a lifting g, that is, there exists a map g making the following diagram commute:*

$$\begin{tikzcd} & P \arrow[d, "h"] \arrow[dl, dashed, "g"'] & \\ A \arrow[r, "p"'] & A'' \arrow[r] & 0 \end{tikzcd}$$

Theorem A.0.2 says that every free left R-module is projective. Though, not every projective R-module is free, it depends on the ring R.

Projective modules arise in a natural way: We know that the Hom functors are left exact; that is, for any module P, applying $Hom_R(P, \square)$ to an exact sequence

$$0 \to A' \xrightarrow{i} A \xrightarrow{p} A''$$

gives an exact sequence

$$0 \to Hom_R(P, A') \xrightarrow{i_*} Hom_R(P, A) \xrightarrow{p_*} Hom_R(P, A'').$$

Proposition A.0.5. *A left R-module P is projective if and only if $Hom_R(P, \square)$ is an exact functor.*

Remark A.0.6. *Since $Hom_R(P, \square)$ is a left exact functor, the point of the proposition is that p_* is surjective whenever p is surjective.*

Proof. If P is projective, then for some $h : P \to A''$, there exists a lifting $g : P \to A$ with $pg = h$. Thus, if $h \in Hom_R(P, A'')$, then $h = pg = p_*(g) \in \text{Im}p_*$, and so p_* is surjective. Hence, $Hom_R(P, \square)$ is an exact functor.

For the converse, assume that $Hom(P, \square)$ is an exact functor, so that p_* is surjective: if $h \in Hom_R(P, A'')$, there exists $g \in Hom_R(P, A)$ with $h = p_*(g) = pg$. This says that given p and h, there exists a lifting g making the diagram commute. That is, P is projective.

□

Proposition A.0.7. *A left R-module P is surjective if and only if every exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} P \to 0$ splits.*

The proof is emitted.

Definition A.0.8. *A left R-module E is* ***injective*** *if, whenever i is an injection, there exists a map g making the following diagram commute:*

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "i"'] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

Proposition A.0.9. *A left R-module E is injective if and only if $Hom(\square, E)$ is an exact functor.*

Proof. If

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence, we must prove exactness of

$$0 \to Hom_R(C, E) \xrightarrow{p_*} Hom_R(B, E) \xrightarrow{i_*} Hom_R(A, E) \to 0.$$

Since $Hom(\square, E)$ is a left exact contravariant functor, the point of the proposition is that the induced map i_* is injective. If $f \in Hom_R(A, E)$, there exists $g \in Hom_R(A, E)$, then $f = gi = i_*(g) \in \mathrm{Im} i_*$, and so the induced map i_* is surjective. Therefore, $Hom(\square, E)$ is an exact functor.

$\square$

Proposition A.0.10. *If a left R-module E is injective, then every short exact sequence $0 \to E \xrightarrow{i} B \xrightarrow{p} C \to 0$ splits.*

Proof.

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "1_E"] \arrow[r, "i"'] & B \arrow[ul, dashed, "g"'] \end{tikzcd}$$

Since E is injective, there exists $g : B \to E$ making the diagram commute.

$\square$

Appendix B

Flat Modules

Definition B.0.1. *If R is a ring, then a right R-module A is **flat** if $A \bigotimes_R \square$ is an exact functor, that is, whenever*

$$0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$$

is an exact sequence of left R-modules, then

$$0 \to A \otimes_R B' \xrightarrow{1_A \otimes i} A \otimes_R B \xrightarrow{1_A \otimes p} A \otimes_R B'' \to 0$$

is an exact sequence of abelian groups. Flatness of left R-modules is defined similarly.

The functors $A \bigotimes_R \square : \mathbf{Mod} \to \mathbf{Ab}$ being right exact, gives us that a right R-module A is flat, if and only if, whenever $i : B' \to B$ is an injection, then $1_A \otimes i : A \otimes_R B' \to A \otimes_R B$ is also an injection.

Proposition B.0.2. *Let R be an arbitrary ring.*

(i) The right R-module R is a flat right R-module.

(ii) A direct sum $\bigoplus_j M_j$ of right R-modules if flat, if and only if each M_j is flat.

(iii) Every projective right R-module is flat.

Lemma B.0.3. *Let $0 \to A \xrightarrow{i} B$ be an exact sequence of left R-modules, and let M be a right R-module. If $u \in \mathrm{Ker}(1_M \otimes i)$, then there are a finitely generated submodule $N \subseteq M$ and an element $u' \in N \otimes_R A$ such that:*

(i) $u' \in \mathrm{Ker}(1_N \otimes i)$,

(ii) $u = \kappa \otimes 1_A$, where $\kappa : N \to M$ is the inclusion.

Proposition B.0.4. *If every finitely generated submodule of a right R-module M is flat, then M is flat.*

Proof. It suffices to prove the exactness of $0 \to A \xrightarrow{i} B$ gives exactness of $0 \to M \otimes_R A \xrightarrow{1_M \otimes i} M \otimes_R B$. If $u \in \mathrm{Ker}(1_M \otimes i)$, then the lemma provides a finitely generated submodule $N \subseteq M$ and an element $u' \in N \otimes_R A$ with $u' \in \mathrm{Ker}(1_N \otimes i)$ and $u = (\kappa \otimes 1_A)(u')$. Now $1_N \otimes i$ is injective, by hypothesis, so that $u' = 0$; moreover, $u = (\kappa \otimes 1_A)(u') = 0$. Therefore, $1_M \otimes i$ is an injection and M is flat. $\square$

Appendix C

Limits

We say that a category is a **small category** if the objects and the morphisms are sets. Now, if $\mathscr{F}$ is any small category and $\mathscr{C}$ is any category, then a functor $F : \mathscr{F} \to \mathscr{C}$ (with an onject $A_i \in \mathscr{C}$ for every $i \in \mathscr{F}$, and appropriate commuting morphisms dictated by $\mathscr{F}$) is said to be a **diagram indexed** by $\mathscr{F}$. We call $\mathscr{F}$ an index category.

Example C.0.1. *(i) One useful indexed category is a partially ordered set, in which there is at most one morphism between any two objects.*

(ii) Another index category is the following square:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

If $\mathscr{C}$ is another category, the functor from square to $\mathscr{C}$ is precicely the data of a commuting diagram.

Definition C.0.2. *The **limit of the diagram** is an object $\varprojlim_{\mathscr{F}} A_i$ of $\mathscr{C}$ along with morphisms $f_j : \varprojlim_{\mathscr{F}} A_i \to A_j$ for each $j \in \mathscr{F}$, such that if $m : j \to k$ is a morphism in $\mathscr{F}$, then the following diagram commutes:*

$$\begin{tikzcd} \varprojlim_{\mathscr{F}} A_i \arrow[d, "f_j"'] \arrow[dr, "f_k"] & \\ A_j \arrow[r, "F(m)"'] & A_k \end{tikzcd}$$

and this object and maps to each A_i are universal with respect to this property. More precisely, given any other object W alogn with maps $g_i : W \to A_i$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in $\mathscr{F}$, then $g_k = F(m) \circ g_j$), then there is a unique map

$$g : W \to \varprojlim_{\mathscr{F}} A_i$$

so that $g_i = f_i \circ g$ for all i.

Remark C.0.3. *If the limit exists, it is unique up to isomorphism.*

Appendix D

Filtration

The filtration is the generalization of the **normal series** of groups.

Definition D.0.1. *A* ***filtration*** *of a module M is a family $(M_p)_{p\in\mathbb{Z}}$ of submodules of M such that*

$$\cdots \subseteq M_{p-1} \subseteq M_p \subseteq M_{p+1} \subseteq \ldots.$$

The ***factor modules*** *of this filtration form the grades module $(M_p/M_{p-1})_{p\in\mathbb{Z}}$.*

One can define filtrations of objects in any abelian category. In particular, a filtration of a complex $\mathbf{C}$ is a family of subcomplexes $(F^p\mathbf{C})_{p\in\mathbb{Z}}$ with

$$\cdots \subseteq F^{p-1}\mathbf{C} \subseteq F^p\mathbf{C} \subseteq F^{p+1}\mathbf{C} \ldots.$$

In more detail, a filtration of $\mathbf{C}$ is a commutative diagram such that, for each n, the nth column is a filtration of C_n.

$$\begin{tikzcd}
\ldots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] & C_{n-1} \arrow[r] & \ldots \\
\ldots \arrow[r] & F^p_{n+1} \arrow[r] \arrow[u] & F^p_n \arrow[r] \arrow[u] & F^p_{n-1} \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & F^{p-1}_{n+1} \arrow[r] \arrow[u] & F^{p-1}_n \arrow[r] \arrow[u] & F^{p-1}_{n-1} \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

Remark D.0.2. *Filtrations need not be ascending, they can be descending too (with re-indexed submodules). Either case is a filtration.*

Filtrations may have only finitely many terms; if we have $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n$, we can define $M_i = M_0$ for all $i < 0$ and $M_j = M_n$ for all $j > n$. Moreover, the "endpoints" (if there are any) of a filtration of M need not be $\{0\}$ or M; that is, neither $\{0\}$ nor M must equalt to M_p for some p.

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