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PhD Thesis

The contribution of decoherence in the dynamics of open quantum spin chains

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Contents

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Summary

In the present thesis the main goal is the construction of a consistent coherent-state path integral formalism in the continuum limit, for use in the study of both closed and open quantum systems. Under the scope of this study, the cases of bosonic, fermionic and spin coherent-state path integrals are considered.

Bosonic and spin coherent-state path integrals are studied with the use of the theory of geometric quantization. Using this formalism, and understanding how the identification of the continuum limit of coherent-state path integrals relates to the topic of Kähler quantization, it is possible not only to approach the problem of interest, but also to gain new insight in the theory of Kähler quantization. More specifically, the correct continuum limit is proven to be identifiable through an inverse procedure to that of geometric quantization, leading at the same time to the first physical argument regarding the necessity of the metaplectic correction in the context of Kähler quantization.

Fermionic coherent-state path integrals on the other hand are approached in a different context, in which the use of the Faddeev-Jackiw construction for constrained systems acts as an intermediate step for the identification of the correct continuum limit. The results of this approach are immediately applied in more complicated configurations, like the case of the spin-1/2 1D XY Heisenberg spin chain, with the goal of showcasing their correctness. This leads to many interesting results regarding the handling of correlation functions using this path integral formalism. The same formalism is later generalized in order to handle non-equilibrium dynamics, where the aforementioned construction is applied for the case of a time dependent transverse magnetic field, acting on the XY spin chain of interest.

With the use of these two approaches it becomes then possible to proceed with the generalization and application of this formalism to the case of open quantum systems. Both cases of bosonic and fermionic systems are studied in this context, where the analysis remains as general as possible. Later on, the limits of an isotropic interaction and a Markovian environment are considered, in order to compare these results to known analytical results appearing in the literature, and consequently showcasing the correctness of these arguments.

1 Introduction

The Feynman path integral formalism is the most powerful tool for taking into account quantum behaviour via classical computations [1, 2]. Ideally suited for semiclassical calculations, the path integral machinery provides a variety of analytical methods for studying the dynamics of quantum correlations in closed and open quantum systems [3,4]. The extension of path integration to the ordinary complex plane $\mathbb{C}$ through the Glauber coherent states [5], to the complex non-flat manifold $\bar{\mathbb{C}}$ through the $su(2)$ spin coherent states [6–8] and to fermionic systems through the fermionic coherent states [3,9] has allowed for the application of path integral techniques to the study of many-body systems [10, 11]. These systems are of great interest for both condensed matter physics and quantum information science, due to the fact that they naturally support entangled states. Correlations in these states have a fundamental quantum character, as they do not have a classical counterpart, and can serve both as the means for understanding quantum phase transitions and as the main tool for quantum information processing [12–14].

During the last years, there have been considerable advances in the study of the static properties and the dynamics of closed many-body quantum systems, both at experimental and theoretical level [15,16]. However, despite these advances, the usage of path integral techniques in the corresponding analysis is rather restricted. The main reason is that, currently, there is no universally accepted way to define path integration on complexified spaces, spanned by the coherent state bases, for a general system written in terms of bosonic, spin or fermionic operators [17–20]. When free from conceptual and structural issues, path integrals over coherent states can provide a wide variety of techniques, analytical and numerical, for the analysis of closed and open quantum systems. In the current thesis a step is made towards this direction, where a series of inconsistency-free methods are developed for the study of bosonic, spin and fermionic systems through the use of coherent-state path integrals. Furthermore, this method is generalized for the study of driven and open quantum systems.

The methods which shall be used in order for this to be achieved, differ tremendously for the cases of bosonic/spin and fermionic coherent states. In the former cases, a se-

ries of state-of-the-art mathematical tools coming from the field of geometric quantization will be necessary, since both bosonic and spin coherent-state path integrals can be interpreted as phase-space path integrals over Kähler manifolds [21]. Even though the theory of quantization over symplectic manifolds is a very mathematically involved topic, it will become apparent that the use of some specific results, not only allows for the solution of the issues of bosonic variable coherent-state path integrals, but also provides clarity to ambiguous topics in the field of Kähler quantization [21–24]. More specifically, the presented methodology allows for the first mathematically strict answer to the unanswered question regarding the necessity of the metaplectic correction. The case of fermionic coherent-state path integrals cannot be studied in such a way though, since fermions do not accommodate a symplectic structure, which leads to the need of a radically different procedure. The identification of such a procedure is possible following some recent results regarding the consistent path integral quantization of Majorana fermion systems [26]. A series of steps can then be proposed, leading to the identification of well defined actions for use in complex fermion path integrals, which are exactly the fermionic coherent-state path integrals. Both these solutions differ from the known ones [18–20, 28, 29], since the known methods either depend on the discrete structure, a feature which is unattractive in the context of continuum path integrals, or are extremely indirect.

The main body of the present thesis is comprised of two parts; the theoretical background, presented in section 2, and the methods and results, presented in sections 3, 4 and 5. In section 2 all necessary theoretical tools are presented, regarding both the canonical and the path integral formulation of quantum theory, while emphasis is given on the issues appearing in coherent-state path integrals, as also on the existing solutions. In section 3 an inconsistency-free method for the identification of the continuum limit for bosonic and spin coherent-state path integrals is presented, where a method for the consistent identification of the continuum limit through the inversion of the geometric quantization procedure is constructed. All results coming from this method agree with the known results found in the bibliography [18–20], giving the correct coherent-state path integrals. The strict phase-space path integral formulation used in that context is also generalized and results for higher orders and interactions are also presented. In section 4, fermionic coherent-state path integrals are addressed,

where the correct continuum limit is found via the corresponding Majorana fermion one. In the same section this construction is used for the study of the 1D XY spin chain model [30, 31], allowing for the very simple computation of a series of known results. These results act as non-trivial verifications for the consistency of the proposed procedure. This method is then generalized for the driven case, via the formulation of a Schwinger-Keldysh path integral. In section 5, all aforementioned results for the bosonic and fermionic coherent-state path integrals are used for the study of open quantum systems, where a systematic and inconsistency-free formulation of the path integral for composite systems is constructed. Even though this construction is initially general, it is later applied for the study of systems interacting with a Markovian environment. More specifically, the examples of the bosonic and fermionic harmonic oscillator cases are addressed in detail. Finally, a short epilogue follows, where a summary of the presented research, and its possible future directions, are presented. The main text is accompanied by nine appendices, containing calculations and clarifications on specific topics related to chapters 3, 4 and 5.

2 Theoretical Background

Section summary

In this section, the main goal is the presentation of a comprehensive introduction to the theory of quantization in both the canonical and path integral formulations. In this context, great focus is given on the topic of coherent-state path integrals, which until the time of writing still do not possess a definitive description at the time-continuous limit, with the identification of such a description being the main topic of study in some of the following sections. The construction of these integrals and details on their issues in the continuum limit are extensively provided, as also are the necessary tools for the use of these continuum structures.

2.1 Classical Mechanics

In the context of classical mechanics [32], a classical system is characterized by a quantity with energy units called the Lagrangian function L, and more specifically its inte-

gral

$$S = \int_{t_i}^{t_f} L dt \tag{2.1}$$

which defines the action S. The action constitutes a functional of the trajectory a system follows along its time evolution from an initial time t_i to a final time t_f, where the system trajectory is defined as the value of its corresponding fields at each time t. If for a system the action is known, then the classical trajectory is defined as that which makes the action static with respect to each field of the theory. More specifically, if the action is considered as a functional of a set of fields $\{\phi^i(t)\}$, and of their time derivatives, then according to the stationary action principle, the solution to the equation

$$\frac{\delta S(\phi^i(t), \dot{\phi}^i(t), \dots)}{\delta \phi^j(t')} = 0 \tag{2.2}$$

characterizes the classical trajectory. The symbol δ denotes the functional differentiation, which is defined with respect to the fields as

$$\frac{\delta \phi^i(t)}{\delta \phi^j(t')} = \delta^i{}_j \delta(t - t'). \tag{2.3}$$

Here, $\delta^i{}_j$ is the Kronecker's delta function, while $\delta(t - t')$ is the Dirac delta distribution. By considering the value of the fields at times t_i and t_f equal to zero, or considering the case of periodic boundary conditions when this assumption is justified, and the Lagrangian as a function only of the fields, their first derivatives, and possibly of time itself, it is possible to reduce the equation (2.2) to a more analytical form

$$\frac{\delta L(\phi^i(t), \dot{\phi}^i(t), t)}{\delta \phi^j(t)} - \frac{d}{dt}\left(\frac{\delta L(\phi^i(t), \dot{\phi}^i(t), t)}{\delta \dot{\phi}^j(t)}\right) = 0. \tag{2.4}$$

In the present thesis all the Lagrangians which will be considered will contain only derivatives up to the first order, and thus eq. (2.4) is always valid.
In this Lagrangian formulation, for every field $\phi^i(t)$ one can define its conjugate momentum $\pi_j(t)$ as

$$\pi_j(t') = \frac{\delta S(\phi^i(t), \dot{\phi}^i(t))}{\delta \dot{\phi}^j(t')}. \tag{2.5}$$

This equation can then be inverted, with the use of a Lagrange transformation of the Lagrangian, giving the Hamiltonian function

$$H(\phi^i(t), \pi_i(t), t) = \sum_{i=1}^{n} \pi_i(t)\dot{\phi}^i(t) - L(\phi^i(t), \dot{\phi}^i(t), t), \tag{2.6}$$

where n is the number of fields in the theory, and for which it can be proved that

$$\dot{\phi}^j(t) = \frac{\delta H(\phi^i(t), \pi_i(t), t)}{\delta \pi_j(t)} \tag{2.7}$$

and

$$\dot{\pi}_j(t) = -\frac{\delta H(\phi^i(t), \pi_i(t), t)}{\delta \phi^j(t)}, \tag{2.8}$$

which constitute Hamilton's equations. Using these two equations it is possible to study the time evolution of any function F of the fields and of their conjugate momenta, since the time evolution can be found to be equal to

$$\frac{dF(t, \phi^i(t), \pi_i(t))}{dt} = \frac{\partial F(t, \phi^i(t), \pi_i(t))}{\partial t} + \{F, H\}, \tag{2.9}$$

where $\{f, g\}$ defines the Poisson bracket

$$\{F, H\} = \tag{2.10}$$
$$= \sum_{j=1}^{n} \left(\frac{F(t, \phi^i(t), \pi_i(t))}{\phi^j(t)} \frac{H(\phi^i(t), \pi_i(t), t)}{\pi_j(t)} - \frac{H(\phi^i(t), \pi_i(t), t)}{\phi^j(t)} \frac{F(t, \phi^i(t), \pi_i(t))}{\pi_j(t)} \right).$$

From this result it can be seen that the Hamiltonian function controls the time evolution of all quantities in a theory. Furthermore, any smooth function of the fields $\phi^i(t)$ and their conjugate momenta $\pi_i(t)$ is considered as a classical observable.
A very important example of a Poisson bracket is the one calculated between a field $\phi^i(t)$ and a conjugate momentum $\pi_j(t)$ which gives

$$\{\phi^i(t), \phi^j(t)\} = \{\pi_i(t), \pi_j(t)\} = 0, \quad \{\phi^i(t), \pi_j(t)\} = \delta^i{}_j. \tag{2.11}$$

2.2 Canonical Quantization

Even though the quantum theory is considered as the more fundamental viewing of physical reality, due to the inability of defining it through a set of physical principles,

it can be understood only through the process of quantizing a corresponding classical theory. This quantization is possible through two seemingly equivalent methods; the canonical quantization [33] and the functional or path integral quantization [3]. Nevertheless, canonical quantization will be mainly addressed in the first subsections of the present thesis, not only because it historically precedes the path integral quantization, but also because the latter can naturally be derived from it as will be later shown.

At the Hamiltonian level, the canonical quantization of a system is possible through the replacing of the fields of a theory by their corresponding operators, acting linearly on a Hilbert space, and of the Poisson brackets by commutation relations. However, this procedure is not always well defined, since for more complicated functions of the fields and their respective conjugate momenta it leads to mathematical inconsistencies [22, 34, 35]. Nevertheless, it is a fact that in the simple case of the functions $\phi^i(t)$ and $\pi_i(t)$ themselves, the quantization procedure can consistently be performed at the operator level, by defining the corresponding operators $\hat{\phi}_i(t)$, $\hat{\pi}_j(t)$ with the following commutation relations:

$$[\hat{\phi}^i, \hat{\phi}^j] = [\hat{\pi}_i, \hat{\pi}_j] = 0, \quad [\hat{\phi}^i, \hat{\pi}_j] = i\hbar\delta^i{}_j. \tag{2.12}$$

The generalization of this commutator algebra is not straightforward though, as will be argued later. Here, $\hbar = h/2\pi$ is the reduced Planck constant.
In this context, the states of a quantum system are expressed either as vectors $|\psi\rangle$ of a Hilbert space, or equivalently as wave-functions $\psi(\vec{x})$, which are locally functions[1] of the position $\vec{x}$. The notation $|\psi\rangle$ is called Dirac's notation for a state with name ψ, while the corresponding wave-function is connected to this vector through equation

$$\psi(\vec{x}) = \langle \vec{x}|\psi\rangle\,, \tag{2.13}$$

which defines the inner product of the state vector $|\psi\rangle$ with the vector $|\vec{x}\rangle$ of the Hilbert space, the latter signifying the position on a D-dimensional space with coordinates x^μ, $\mu = 1, \ldots, D$. In the case of the usual quantum mechanics considered here, the space is considered to be the Euclidean D-dimensional flat space, and thus the aforementioned

[1] Wave-functions are not functions per se, in a mathematical sense, but sections of a complex line bundle. Nevertheless, details on this topic do not need to be addressed when the metric in the space of fields $\phi^i(t)$ is flat, as in the case of the usual quantum mechanics on the Euclidean plane.

coordinate system holds globally. The symbol $\langle\vec{x}| = (|\vec{x}\rangle)^\dagger$ defines the conjugate transpose element to the vector $|\vec{x}\rangle$, which is defined through the symbol †. Moreover, the set of position vectors $\{|\vec{x}\rangle\}$ constitutes a complete orthonormal continuous basis of Hilbert space, since

$$\langle\vec{x}_1|\vec{x}_2\rangle = \delta^D(\vec{x}_1 - \vec{x}_2), \tag{2.14}$$

where $\delta^D(\vec{x}_1 - \vec{x}_2)$ is the D-dimensional Dirac delta distribution, and there exists a resolution of the identity

$$\hat{\mathbb{1}} = \int_{\mathbb{R}^D} d^D x \, |\vec{x}\rangle \, \langle\vec{x}| \, . \tag{2.15}$$

It is easy to see then, that the inner product of two state vectors can be written as

$$\langle\psi|\phi\rangle = \int_{\mathbb{R}^D} d^D x \psi^*(\vec{x})\phi(\vec{x}), \tag{2.16}$$

while the normalization is usually chosen to be

$$\langle\psi|\psi\rangle = \int_{\mathbb{R}^D} d^D x |\psi(\vec{x})|^2 = 1. \tag{2.17}$$

It must be noted that the integral in eq. (2.17) converges only when the function $\psi(\vec{x})$ is square-integrable on $\mathbb{R}^D$ ($\psi(\vec{x})$ belongs in $\mathcal{L}^2$), which is a very important requirement for the existence of a wave-function formalism.

In order to always have real eigenvalues, the quantum observables are defined as Hermitian operators, where for an operator $\hat{A}$, Hermitianicity is defined through the inner product between two wave-functions (ψ, ϕ) as

$$(\hat{A}\psi, \phi) = (\psi, \hat{A}\phi). \tag{2.18}$$

Considering now the fields ϕ^μ as the usual position coordinates x^μ, then the corresponding conjugate momenta are none other than the physical momentum components p_μ. Then, for the quantization of these quantities, the corresponding operators acting on elements of the Hilbert space are known to be

$$\hat{X}^\mu\psi(\vec{x}) = x^\mu\psi(\vec{x}), \quad \hat{P}_\mu\psi(\vec{x}) = -i\hbar\partial_\mu\psi(\vec{x}). \tag{2.19}$$

The representation of quantum mechanics used above, in which the wave-functions are considered as functions only of the position, is called the position representation.

With the use of the Fourier transformation of these wave-functions though, it is possible to relate this representation to the momentum representation in which the wave-functions are only functions of the momenta.

The structures that have been referenced up to this point can be represented with the use of a much more strict mathematical formalism, which will be analysed in subsection 2.18. In subsection 2.18 the method of Geometric Quantization is presented, which provides a procedure for constructing the quantum equivalents of classical observable functions, and the results of which will be used in the section 3 of the present thesis.

The generalization of canonical quantization for higher order operators was first attempted by Dirac using a set of quantization constraints. According to these, a quantization map $\hat{Q}$, taking classical observables (functions) to their corresponding quantum observables (Hermitian operators), should have the following properties [22, 34]

1. $\mathbb{C}$-linearity: $\hat{Q}(rf + g) = r\hat{Q}(f) + \hat{Q}(g)$, $r \in \mathbb{C}$, f, g smooth functions.
2. $\hat{Q}(1) = \hat{\mathbb{1}}$.
3. Hermitianicity with respect to the canonical inner product

$$\int_M \mu(\vec{x})\psi_1^*[\hat{Q}(f)\psi_2] = \int_M \mu(\vec{x})[\hat{Q}(f)\psi_1]^*\psi_2, \tag{2.20}$$

   where $\mu(\vec{x})$ is the measure of integration over the manifold M on which the quantum theory is defined.

4. Define a Lie-algebra homomorphism

$$[\hat{Q}(f), \hat{Q}(g)] = i\hbar\hat{Q}[\{f, g\}]. \tag{2.21}$$

5. If $\{f_1, f_2, \ldots, f_n\}$ is a complete set of observable functions, then $\{\hat{Q}(f_1), \hat{Q}(f_2), \ldots, \hat{Q}(f_n)\}$ is a complete set of observable operators.

It has been proved though [22, 34] that it is impossible for all these constraints to hold for a general complete set of classical observables, and thus it is usual for the last two constraints to be weakened such that the map $\hat{Q}$ is restricted to act at least on a

closed Lie-subalgebra of observable functions. An example of such an inconsistency is expressed by the Groenewold-Van Hove theorem [22,34] for the quantization of the 2D Euclidean phase space, which presents the following constraint

Theorem 1 (Groenewold-Van Hove theorem) *There does not exist a consistent quantization map $\hat{Q}$ which maps the position (x) and momentum (p) observables to their respective operators, and holds for polynomials of degree equal or higher than three, with respect to x and p functions.*

A loophole which may allow for the avoidance of this issue is that one may consider eq. (2.21) as perturbative with respect to the $\hbar$ parameter, and as such

$$[\hat{Q}(f), \hat{Q}(g)] = i\hbar\hat{Q}[\{f, g\}] + \mathcal{O}(\hbar^2). \tag{2.22}$$

Nevertheless, since the form the higher orders should possess is not known, neither from a physical nor a mathematical viewpoint, one cannot use this equation for quantitative calculations, or for the study of quantum mechanics per se. In the context of this thesis the possibility of quantizing classical observables up to the second order using Dirac's constraints will be considered as a fact, since it is allowed by the Groenewold-Van Hove theorem. Using eq. (2.21) it is easy to find then, that for the position (x) and momentum (p) functions

$$\begin{gathered} \hat{Q}(x) = \hat{X}, \quad \hat{Q}(p) = \hat{P}, \\ \hat{Q}(x^2) = \hat{X}^2, \quad \hat{Q}(p^2) = \hat{P}^2, \quad \hat{Q}(xp) = \frac{1}{2}\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right). \end{gathered} \tag{2.23}$$

However, a different approach is needed for the canonical quantization of fermionic operators, which appear naturally in the study of systems obeying the Fermi-Dirac statistics. In such occasions, due to the states of identical fermions being antisymmetric under the exchange of two particles

$$|\phi\rangle\,|\psi\rangle = -\,|\psi\rangle\,|\phi\rangle\,, \tag{2.24}$$

the operator algebra should be expressed through anticommutation relations. Thus, relations (2.12) are replaced by

$$\{\hat{\phi}^i, \hat{\phi}^j\} = \{\hat{\pi}_i, \hat{\pi}_j\} = 0 \quad \{\hat{\phi}^i, \hat{\pi}_j\} = i\hbar\delta^i{}_j, \tag{2.25}$$

with $\{,\}$ denoting the anticommutator. In contrast to the bosonic case, canonical quantization of fermionic operators does not present issues, since the anticommutative nature of the quantities at the classical level as Grassmann variables, does not allow for the construction of higher order classical observables with respect to a single variable. Thus, in this case the knowledge of the quantization of the first orders is enough for the complete understanding of the theory.

2.3 Time evolution

In analogy to classical mechanics, where the Hamiltonian function controlled the evolution of a system [32], the time evolution in quantum mechanics is considered driven by a Hermitian operator $\hat{H}$ defined as the Hamiltonian operator [33]. In this case, the evolution of the system, i.e. of its states/wave-functions, is controlled by the Schroedinger's equation[2]

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}(t)\left|\psi(t)\right\rangle. \tag{2.26}$$

The solution of this equation gives the evolution of states with respect to time, being of the form

$$\left|\psi(t_f)\right\rangle = \hat{U}(t_f,t_i)\left|\psi(t_i)\right\rangle, \quad \hat{U}(t_f,t_i) = \hat{T}e^{-\frac{i}{\hbar}\int_{t_i}^{t_f}dt\hat{H}(t)}, \tag{2.27}$$

where $\hat{T}$ is the time ordering operator and

$$\begin{aligned}
&\hat{T}e^{-\frac{i}{\hbar}\int_{t_i}^{t_f}dt\hat{H}(t)} = \\
&= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}\int_{t_i}^{t_n}\int_{t_i}^{t_{n-1}}\cdots\int_{t_i}^{t_2}\hat{H}(t_n)\hat{H}(t_{n-1})\cdots\hat{H}(t_1)dt_1dt_2\ldots dt_n = \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}\int_{t_i}^{t}\int_{t_i}^{t}\cdots\int_{t_i}^{t}\hat{T}\left(\hat{H}(t_n)\hat{H}(t_{n-1})\cdots\hat{H}(t_1)\right)dt_1dt_2\ldots dt_n.
\end{aligned} \tag{2.28}$$

This foundation of quantum mechanics, through the evolution of the states of a system, is called the Schroedinger picture. Besides this, there exist two more equivalent ways to describe the time evolution, called the Heisenberg's and Interaction pictures,

[2]This equation is known to be valid only for non-relativistic cases, with the Klein-Gordon and Dirac equations being the relativistic equivalents for bosonic and spin-1/2 particles respectively.

where in the context of the present thesis only the former will be addressed. In Heisenberg's picture, the states of the theory are considered constant, while the time evolution in transferred to the operators as

$$\hat{A}(t_f) = \hat{U}^\dagger(t_f, t_i)\hat{A}(t_i)\hat{U}(t_f, t_i), \tag{2.29}$$

where

$$\hat{U}^\dagger(t_1, t_2) = \hat{T}^\dagger e^{\frac{i}{\hbar}\int_{t_1}^{t_2} dt\hat{H}(t)}, \tag{2.30}$$

with $\hat{T}^\dagger$ being the anti-time ordering operator. Finally, it is easy to prove the property

$$\hat{U}^\dagger(t_1, t_2) = \hat{U}(t_2, t_1). \tag{2.31}$$

2.4 Correlation functions

The quantities that characterize the dynamic evolution of states are the transition amplitudes, while the ones characterizing the dynamic evolution of operators are their correlation functions [33, 36]. In the Schroedinger picture, for an initial state $|\psi_i\rangle$, its transition amplitude to a state $|\psi_f\rangle$ after a time interval $t_f - t_i$ is defined as the matrix element

$$\langle\psi_f|\hat{U}(t_f, t_i)|\psi_i\rangle\,. \tag{2.32}$$

This quantity cannot be considered as an observable one, since it is complex in general, and thus the corresponding physical quantity is its measure, which corresponds to the probability amplitude of the state $|\psi_i\rangle$ transitioning to the state $|\psi_f\rangle$ after time $t_f - t_i$.

In the Heisenberg picture, where the evolution is observed on the operators of the theory, the n-point correlation functions are defined as the quantities measuring the correlation between n-operators $\{\hat{A}_j\}$, $j = 1, \ldots n$, during the time evolution from t_i to t_f as

$${}_H\langle\psi_f, t_f|\hat{T}\left(\hat{A}_1(t_1)\hat{A}_2(t_2)\cdots\hat{A}_n(t_n)\right)|\psi_i, t_i\rangle_H\,. \tag{2.33}$$

The initial and final states, characterizing the matrix element, are defined as $|\psi, t\rangle_H = \hat{U}^\dagger(t, t_i)\,|\psi\rangle$ and correspond to states on which the evolved operators under eq. (2.29)

act independently of time as

$$\hat{A}(t)\left|\psi,t\right\rangle_H = \hat{U}^\dagger(t,t_i)\hat{A}(t_i)\left|\psi\right\rangle, \tag{2.34}$$

where $|\psi\rangle$ is a time-independent state in the Heisenberg picture. The choice of these states is made due to the fact that if a state $|\psi\rangle$ is an eigenstate of a Heisenberg operator $\hat{\psi}(t_i)$ then the state $|\psi,t\rangle_H$ is the eigenstate of the evolved operator $\hat{\psi}(t)$:

$$\hat{\psi}(t_i)\left|\psi\right\rangle = \psi\left|\psi\right\rangle \Rightarrow \hat{\psi}(t)\left|\psi,t\right\rangle_H = \psi\left|\psi,t\right\rangle_H. \tag{2.35}$$

2.5 Examples of Hilbert spaces

Two very important Hilbert spaces that will be studied extensively in the context of the present thesis are those of the harmonic oscillator and of spin systems.

2.5.1 The harmonic oscillator

The quantum harmonic oscillator's evolution is driven by the Hamiltonian operator

$$\hat{H} = \left(\frac{k}{2}\hat{X}^2 + \frac{1}{2m}\hat{P}^2\right). \tag{2.36}$$

This operator constitutes an exact analogue of the corresponding classical Hamiltonian function, the canonical quantization of which is consistent under Dirac's constraints. Here, the symbol m denotes the mass of the oscillating object while k defines the oscillation constant. The study of this system can be performed much more easily through the definition of the creation and annihilation operators of the harmonic oscillator

$$\hat{a}^\dagger = \frac{m\omega}{2\hbar}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right), \quad \hat{a} = \frac{m\omega}{2\hbar}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right), \tag{2.37}$$

which obey the commutation relation

$$[\hat{a},\hat{a}^\dagger] = 1, \tag{2.38}$$

with ω being the angular frequency. In the basis of these operators it is easy to see that the Hamiltonian operator assumes the form

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \tag{2.39}$$

Defining the vacuum state $|0\rangle$ as the state on which $\hat{a}\,|0\rangle = 0$, it is easy to see that the state $\hat{a}^{\dagger n}\,|0\rangle$, $n \in \mathbb{N}$, constitutes an eigenstate of the operator $\hat{a}^\dagger \hat{a}$ with eigenvalue n. This indicates that one can define the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ and a countably infinite basis $\{|n\rangle\}$ covering the Hilbert space, for which

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{2.40}$$

This basis is orthonormal and complete, defining a resolution of the identity

$$\hat{\mathbb{1}} = \sum_{n=0}^{\infty} |n\rangle\,\langle n|\,. \tag{2.41}$$

Since the states $\{|n\rangle\}$ are the eigenstates of the number operator, they also define the quantized energy levels - modes - of the Hamiltonian (2.39). The nature of the eigenstates of the harmonic oscillator through the aforementioned countably infinite basis allows for the use of the same basis for the representation of single energy bosonic degrees of freedom. In that case, the action of the operator $\hat{N}$ on a state $|n\rangle$ provides the number of bosons n, where $|n\rangle$ is the state of n identical bosons.

In analogy to the quantum harmonic oscillator, the Hilbert space of which has an equivalent structure to that of a system with single energy bosonic degrees of freedom, it is possbile to define the fermionic quantum harmonic oscillator, of which the Hilbert space is equivalent to that of a system with single energy fermionic degrees of freedom. The fermionic harmonic oscillator is defined as

$$\hat{H}_f = \hbar\omega \left(\hat{\psi}^\dagger \hat{\psi} - \frac{1}{2} \right), \tag{2.42}$$

where the creation and annihilation operators $\hat{\psi}^\dagger$ and $\hat{\psi}$ respectively are of fermionic nature. As a result, the commutation relation (2.38) is switched to the anticommutation relation $\{\hat{\psi}, \hat{\psi}^\dagger\} = 1$, while due to the nilpotency of fermionic operators: $\hat{\psi}^n = \hat{\psi}^{\dagger n} = 0\ \forall n > 1$. This result leads to the construction of a two dimensional basis for

the fermionic Hilbert space $\{|1\rangle, |0\rangle\}$ where the elements correspond to the existence or not of a fermionic degree of freedom. This basis is compatible with the Fermi-Dirac statistics which do not allow for the simultaneous existence of two fermionic degrees of freedom with the same characteristics. In the case of the bosonic oscillator there was no such constraint, since bosons obey the Bose-Einstein statistics. The aforementioned fermionic basis is again orthonormal and complete, providing the resolution of the identity

$$\hat{\mathbb{1}} = |0\rangle \langle 0| + |1\rangle \langle 1|. \tag{2.43}$$

The analogue of position and momentum operators in the fermionic case, i.e. the Hermitian operators related to the complex $\hat{\psi}$, $\hat{\psi}^\dagger$ ones, are the Majorana operators

$$\hat{\gamma}_1 = \hat{\psi} + \hat{\psi}^\dagger, \quad \hat{\gamma}_2 = i\left(\hat{\psi}^\dagger - \hat{\psi}\right); \quad \{\hat{\gamma}_a, \hat{\gamma}_b\} = i\delta_{ab}. \tag{2.44}$$

The case of more fermionic operators $\hat{\psi}_i$, $i = 1, \ldots, N$, and their conjugates, can be generalized naturally by introducing for each couple $\hat{\psi}_i$, $\hat{\psi}_i^\dagger$ two Majorana operators $\hat{\gamma}_{2i-1}$ and $\hat{\gamma}_{2i}$.

2.5.2 Spin

The algebra of spin generators $\hat{S}_x, \hat{S}_y$ and $\hat{S}_z$, has the form

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_{k=x,y,z} \epsilon_{ijk} \hat{S}_k, \quad i, j = x, y, z, \tag{2.45}$$

which is none other than the $su(2)$ Lie-algebra, with ϵ_{ijk} being the fully antisymmetric tensor. The basis of the corresponding Hilbert space is constructed with respect to the eigenstates of an $su(2)$ element $\hat{S}_C$ generating a Cartan subalgebra, i.e. the maximal Abelian subalgebra of $su(2)$. This Cartan element is usually chosen to be the operator $\hat{S}_z$. In a spin representation with value s it is possible to define a highest weight state $|s, s\rangle$, which constitutes the eigenstate of $\hat{S}_C$ with the highest eigenvalue s, and define the ladder operators $\hat{S}_+$ and $\hat{S}_-$, obeying the following algebra

$$[\hat{S}_C, \hat{S}_\pm] = \pm\hbar \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = 2\hbar \hat{S}_C. \tag{2.46}$$

These operators construct a $(2s+1)$-dimensional basis $\{|s,j\rangle\}$ of the Hilbert space and act as

$$\hat{S}_C |s,j\rangle = \hbar j |s,j\rangle , \quad \hat{S}_\pm |s,j\rangle = \hbar\sqrt{(s\mp j)(s\pm j+1)}\,|s,j\pm 1\rangle , \tag{2.47}$$

where $\{|s,j\rangle\}$ is the eigenstates of $\hat{S}_C$ with eigenvalue j in the spin-s representation of $su(2)$. In the case that $\hat{S}_C = \hat{S}_z$ is chosen, the ladder operators adopt the form

$$\hat{S}_\pm = \hat{S}_x \pm \hat{S}_y. \tag{2.48}$$

This basis is orthonormal and complete, and thus defines a resolution of the identity

$$\hat{\mathbb{1}} = \sum_{j=-s}^{s} |s,j\rangle \langle s,j| . \tag{2.49}$$

For spin-1/2 it is easy to observe the similarity between the Hilbert spaces of spin and fermionic degrees of freedom, since both are covered by a two dimensional basis. This similarity is made formal through the Jordan-Wigner transformation [31] which identifies the states $|s,-1/2\rangle = |1\rangle$ and $|s,+1/2\rangle = |0\rangle$ and defines the mapping

$$\begin{aligned} \hat{S}_x &= \frac{1}{2}\left(\hat{\psi}^\dagger + \hat{\psi}\right), \\ \hat{S}_y &= \frac{i}{2}\left(\hat{\psi}^\dagger - \hat{\psi}\right), \\ \hat{S}_z &= \frac{1}{2} - \hat{\psi}^\dagger\hat{\psi}. \end{aligned} \tag{2.50}$$

A similar mapping [31] can be used to identify the tensor product of the Hilbert space of n fermion modes, with a lattice of n spin-1/2 degrees of freedom, and takes the form

$$\begin{aligned} \hat{S}_{xi} &= \frac{1}{2}\prod_{k=1}^{i-1}\left(\frac{1}{2} - \hat{\psi}_k^\dagger\hat{\psi}_k\right)\left(\hat{\psi}_i^\dagger + \hat{\psi}_i\right), \\ \hat{S}_{yi} &= \frac{i}{2}\prod_{k=1}^{i-1}\left(\frac{1}{2} - \hat{\psi}_k^\dagger\hat{\psi}_k\right)\left(\hat{\psi}_i^\dagger - \hat{\psi}_i\right), \\ \hat{S}_{zi} &= \frac{1}{2} - \hat{\psi}_i^\dagger\hat{\psi}_i, \end{aligned} \tag{2.51}$$

where $\hat{\psi}_l^\dagger, \hat{\psi}_l$ are the creation and annihilation operators respectively of the l-th fermionic mode, and $\hat{S}_{jl}$, $j = x,y,z$, is the $\hat{S}_j$ operator acting on the l-th subsystem, with $l =$

$1, \ldots, n$. The product prefactors appearing in eq. (2.51) are necessary, since when referring to different subsystems, products of the quantities in the r.h.s. anti-commute in the absence of these prefactors, while the quantities on the l.h.s. always commute. This issue is solved through the inclusion of these extra factors.

2.6 Coherent states

The Hilbert spaces of bosonic, fermionic and spin systems allow for the definition of an overcomplete basis, which is called the basis of coherent states [3,37].

2.6.1 Bosonic coherent states

In the case of bosonic degrees of freedom, the coherent states are defined with respect to a complex number z as

$$|z\rangle_b = e^{-\frac{|z|^2}{2}} e^{z\hat{a}^\dagger} |0\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle , \tag{2.52}$$

where the coefficient $e^{-\frac{|z|^2}{2}}$ acts as a normalization. These states are the eigenstates of the annihilation operator since

$$\hat{a} |z\rangle_b = z |z\rangle_b . \tag{2.53}$$

The set $\{|z\rangle_b\}$, with $z \in \mathbb{C}$, does not constitute an orthonormal basis, given that the inner product of the coherent states can be found to be

$${}_b\langle z_2|z_1\rangle_b = e^{-\frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + \bar{z}_2 z_1}, \tag{2.54}$$

but nevertheless constitutes an overcomplete basis, since it allows for the resolution of the identity

$$\hat{\mathbb{1}} = \frac{1}{\pi} \int_{\mathbb{C}} d^2z \, |z\rangle_b \, {}_b\langle z|. \tag{2.55}$$

The integration measure above - and whenever integration is performed over the complex plane - is denoted as $d^2z = \frac{1}{2i} dz d\bar{z} = dRe(z) dIm(z)$. In this basis, the trace of any operator can be calculated as

$$\mathrm{tr}\hat{O} = \frac{1}{\pi} \int_{\mathbb{C}} d^2z \, {}_b\langle z|\hat{O}|z\rangle_b . \tag{2.56}$$

2.6.2 Fermionic coherent states

Coherent states can be similarly defined for fermionic systems using the fermionic creation and annihilation operators, but this time with respect to a Grassmann variable θ. Grassmann variables are anticommuting variables, a property which leads - up to an arbitrary normalization - to the following statements

$$\theta^2 = 0, \quad \frac{1}{\sqrt{\pi}} \int d\theta = 0, \quad \frac{1}{\sqrt{\pi}} \int d\theta\theta = 1, \tag{2.57}$$

where θ is considered complex. The fermionic coherent states are then defined as

$$|\theta\rangle = e^{-\frac{\bar{\theta}\theta}{2}} e^{\hat{\psi}^\dagger \theta} |0\rangle = e^{-\frac{\bar{\theta}\theta}{2}} \left(|0\rangle - \theta |1\rangle\right), \tag{2.58}$$

while, in analogy to the bosonic case, they correspond to the eigenstates of the fermionic annihilation operator since

$$\hat{\psi} |\theta\rangle = \theta |\theta\rangle . \tag{2.59}$$

Their inner product can also be found to have the form

$$\langle \theta_2 | \theta_1 \rangle = e^{-\frac{\bar{\theta}_1\theta_1}{2} - \frac{\bar{\theta}_2\theta_2}{2} + \bar{\theta}_2\theta_1}, \tag{2.60}$$

which again indicates the non-orthonormal nature of this basis. The overcompleteness is again observed though the resolution of the identity that this basis allows

$$\hat{\mathbb{1}} = \frac{1}{\pi} \int d\bar{\theta} d\theta \, |\theta\rangle \langle\theta| . \tag{2.61}$$

Finally, in this case, the trace of any operator can be computed as

$$\mathrm{tr}\hat{O} = \frac{1}{\pi} \int d\bar{\theta} d\theta \, \langle -\theta | \hat{O} | \theta \rangle . \tag{2.62}$$

2.6.3 Spin coherent states

The Hilbert space of spin systems allows for the construction of the $su(2)$ coherent states

$$|z\rangle_s = \frac{1}{(1+|z|^2)^s} e^{z\hat{S}_-} |s,s\rangle = \frac{1}{(1+|z|^2)^s} \sum_{j=-s}^{s} \left[\frac{(2s)!}{(s-j)!(s+j)!}\right]^{\frac{1}{2}} z^{s-j} |s,j\rangle , \tag{2.63}$$

where the factor $1/(1+|z|^2)^s$ acts this time as the normalization, and s is the spin of the corresponding representation. Once more, these states constitute an overcomplete, non-orthonormal basis, with inner product

$$ {}_s\langle z_2|z_1\rangle_s = \frac{(1+\bar{z}_2 z_1)^{2s}}{(1+|z_1|^2)^s(1+|z_2|^2)^s} \tag{2.64} $$

and resolution of the identity

$$ \hat{\mathbb{1}}_{S^2} = \frac{2s+1}{\pi}\int_{\mathbb{C}} |z\rangle_s\, {}_s\langle z| \frac{d^2z}{(1+|z|^2)^2}. \tag{2.65} $$

The index S^2 in eq. (2.65) symbolizes the space over which the integration is performed, which is none other than the 2-Sphere. The canonical measure of integration is then analogous to $d^2z/(1+|z|^2)^2$, with z being the chart coordinate covering the whole 2-Sphere, except of a point at $|z| \to \infty$. In this case, the trace of any operator can be computed as

$$ \mathrm{tr}\hat{O} = \frac{2s+1}{\pi}\int_{\mathbb{C}} \frac{d^2z}{(1+|z|^2)^2}\, {}_s\langle z|\hat{O}\,|z\rangle_s\,. \tag{2.66} $$

2.7 Open quantum systems

Up to this point, the states were defined to be vectors of a Hilbert space. However, this formulation cannot hold true for the case of open quantum systems. To allow for the definition of mixed states and processes like decoherence, one needs to reformulate quantum mechanics using the density matrix formalism [38,39], which treats the states as operators. States that can equivalently be expressed as vectors $|\psi\rangle$, are called pure states, and for these the corresponding density matrix is defined as

$$ \hat{\rho}_{\mathrm{Pure}} = |\psi\rangle\,\langle\psi|\,. \tag{2.67} $$

In the general case though, a density matrix $\hat{\rho}$ is defined to have the following properties:

1. $\hat{\rho}$ is positive,
2. $\mathrm{tr}\hat{\rho} = 1$,

and in the case that the state $\hat{\rho}$ is pure, also $\mathrm{tr}\hat{\rho}^2 = 1$. A very important example of a mixed state is the thermal state

$$\hat{\rho}_{\mathrm{Th}} = \frac{e^{-\beta\hat{H}}}{Z}, \tag{2.68}$$

which characterizes a quantum system in thermal equilibrium, and arises as the quantum analogue of the classical canonical ensemble, with $Z = \mathrm{tr}\left[e^{-\beta\hat{H}}\right]$ being the partition function and $\beta = 1/T$ the inverse temperature[3]. In the density matrix formalism, the expectation value of an arbitrary operator $\hat{O}$, with respect to a system characterized by a state $\hat{\rho}$, can be found to be

$$\langle\hat{O}\rangle = \mathrm{tr}\left[\hat{\rho}\hat{O}\right]. \tag{2.69}$$

2.8 Phase Transitions and Critical Phenomena

A phase transition [40] is a phenomenon observed when a thermodynamic system appears to change its physical behaviour, at a given point of its parameter space. During a phase transition, certain properties of the medium change, often discontinuously, as a result of the change of external conditions, such as temperature, pressure, or others. Such phenomena are possible both in classical and quantum systems with examples ranging from phase transitions between states of matter, to quantum condensations such as the case of the Bose-Einstein condensate. A very important case is the one of quantum phase transitions [41], which become apparent at the zero temperature limit of specific quantum systems, and are immediately related to the entanglement properties of the ground-state.

Phase transitions are classified in two categories, of the first and of the second order. First order phase transitions are characterized by the absorption or release of a fixed, and typically large, amount of energy per volume. During such processes, the temperature of the corresponding system will stay constant as more heat is added, with the system remaining in a fixed-phase regime, in which some parts of the system have completed the transition and others have not. Familiar examples are the melting of ice or the boiling of water.

[3]The Boltzmann's constant has been set here $k_B = 1$ for simplicity.

On the other hand, second-order phase transitions, which are also known as "continuous phase transitions", are characterized by a divergent susceptibility, an infinite correlation length, and a power law decay of correlations near criticality. The aforementioned power law is a result of the theory becoming scale invariant at the critical point. An example of such a transition is the quantum phase transition observed in the XY spin chain, which will be addressed in the next subsection. Such transitions are characterized by the Kibble-Zurek mechanism (KZM) or the adiabatic-impulse-adiabatic approximation [42,43], which is based on the fact that the evolution of a system through such a phase transition cannot be considered adiabatic near the critical point, irrespectively of how slow the driving is. In such cases the evolution is considered initially adiabatic, becoming non-adiabatic near the critical point. The term quantum denotes transitions connecting different quantum phases of matter, which is the reason why these are in general observed at zero temperature. Such transitions are accompanied by abrupt changes in the ground-state of a many-body system due to its quantum fluctuations and thus, are immediately related to its entanglement properties.

The aforementioned singular behaviour of some quantities in a second-order phase transition can be characterized by a set of critical exponents $\{\alpha, \beta, \gamma, \delta, \nu, z, \eta, \dots\}$. The two most relevant critical exponents in the context of the present thesis are the exponent ν, corresponding to the divergence of the correlation length ξ, and the dynamic critical exponent z, corresponding to the divergence of the characteristic time τ. The divergence of the correlation length is the main reason for the power law behaviour of correlation functions near the fixed point, since

$$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle \sim \frac{e^{-\lambda\left(\frac{|x_1-x_2|}{\xi}\right)^m}}{|x_1-x_2|^k}, \quad k, \lambda, m \geq 0,$$

while it is also solely responsible for the singular contributions to thermodynamic quantities. The divergence of the characteristic time scale can be easily understood from that of the correlation length as $\tau \sim \xi^z$. Models which present the same behaviour near criticality, and as a result share the same critical exponents, are said to belong to the same universality class.

2.9 Heisenberg spin chains

Heisenberg spin chains are quantum statistical models used both in the study of phase transitions as also in quantum information processing. They represent models in which spin degrees of freedom interact successively with each other, usually in a nearest neighbour fashion. Considering now the case of spin-1/2, the general Hamiltonian for an $1D$ chain of $N+1$ such spins and only nearest neighbour interactions is

$$\hat{H} = -\sum_{j=1}^{N} [a\sigma_j^x \sigma_{j+1}^x + b\sigma_j^y \sigma_{j+1}^y + c\sigma_j^z \sigma_{j+1}^z + h\sigma_j^z]. \tag{2.70}$$

Here, σ_j^μ, $\mu = x, y, z$ are the Pauli matrices at the j-th point of the spin lattice, $j = 1, ..., N+1$. In the Hamiltonian presented in eq. (2.70) the last term has been included in order to allow the interaction of the system with an external magnetic field h directed along the z axis. Even though such models in general present many interesting critical properties at the thermodynamic limit $N \to \infty$, as also strong connections to $2D$ Quantum Field Theories, in the case of finite N these are also deemed important for the study of perfect quantum information transfer.

These models are in general named depending on the choice of the factors a, b and c, i.e. the choice $a \neq b \neq c$ defines the Heisenberg XYZ model, while the choice $a = b \neq c$ defines the Heisenberg XXZ model. A model which will be studied extensively in the context of this thesis is the transverse field 1D XY model [30, 31]

$$\hat{H} = -\sum_{j=1}^{N} \left[\frac{1+r}{2} \sigma_j^x \sigma_{j+1}^x + \frac{1-r}{2} \sigma_j^y \sigma_{j+1}^y + h\sigma_j^z \right], \tag{2.71}$$

where r measures the anisotropy between the x and y couplings. At $r = 0$ one recovers the isotropic XY limit, also known as the XX model, while at $r = 1$ the Ising limit is recovered. All anisotropic models $(0 < r \leq 1)$ belong to the same universality class, i.e. the Ising class, whereas the isotropic XX model belongs to a different universality class [31]. XY models exhibit three phases: the oscillatory, the ferromagnetic and paramagnetic phases. In contrast to the paramagnetic phase, which appears in the $h > 1$ regime, the first two are ordered phases and appear for $r^2 + h^2 < 1$ (oscillatory) and $r^2 + h^2 > 1$, $h < 1$ (ferromagnetic).

This model has a quantum second-order phase transition from its ferromagnetic to its paramagnetic phase at $h = 1$, when it is studied in the thermodynamic limit $N \to \infty$. As referenced above, the behaviour of this model at the critical point differs between the cases of $r = 0$ and $r \neq 0$. In the case of the isotropic XX model, it is known that the correlation length critical exponent is $\nu = 1/2$, and the dynamic critical exponent is $z = 2$. On the contrary, for $r \neq 0$ the model belongs in the Ising universality class, in which $\nu = 1$ and $z = 1$. It must be noted that even for r very close to, but not exactly zero, the system still shows Ising behaviour.

As referenced in subsection 2.5, the spin-1/2 degrees of freedom can be very easily mapped to fermionic degrees of freedom via the Jordan-Wigner transformation. This method allows for a much more simple study of the quantum Hamiltonian (2.71), but also presents a few new subtleties. One such subtlety arises after imposing periodic boundary conditions for the chain, which are made formal through the constraint $\sigma_N^\mu \sigma_{N+1}^\mu = \sigma_N^\mu \sigma_1^\mu$, $\mu = x, y, z$ and not the intuitive choice $\sigma_{N+1}^\mu = \sigma_1^\mu$. This is necessary, since the latter choice would not return the correct spectrum and eigenstates for arbitrary N, when working with the fermionic degrees of freedom [31]. On the contrary, the new constraint has two solutions, one being the case of odd number of fermions and $\hat{\psi}_{N+1} = \hat{\psi}_1$ and the other the case of even number of fermions and $\hat{\psi}_{N+1} = -\hat{\psi}_1$. Thus, for $\hat{\psi}'s$ that are periodic the fermion number is odd, whereas when these are antiperiodic the fermion number is even.

It is easy to see then that the parity operator

$$\hat{P} = \prod_{j=1}^{N} (1 - 2\hat{\psi}_j^\dagger \hat{\psi}_j) = \prod_{j=1}^{N} \sigma_j^z, \tag{2.72}$$

is a symmetry of the Hamiltonian (2.71), and thus the Hilbert space of the model is split in two sectors, corresponding to the even or odd number of fermions. To incorporate the aforementioned boundary conditions on the $\hat{\psi}_j$ operators it is usually preferable working with their Fourier transformations,

$$\hat{\psi}_j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i\frac{2\pi}{N} j(m+b)} \hat{c}_m^{(b)}, \tag{2.73}$$

where $b = 0$ for periodic $\hat{\psi}_j$'s and $b = 1/2$ for anti-periodic $\hat{\psi}_j$'s.

2.10 The Schwinger-Keldysh Formalism

In the case of non-equilibrium dynamics, some methods used in the equilibrium case fail [44], and thus new methods need to be considered for the understanding of such systems. This failure becomes apparent both in the breakdown of the conventional Time-Ordered perturbation theory, as also in the inability of regaining information for the dynamics through the Partition Function and its subsequent Wick's rotation (see subsection 2.15). An appropriate method for the study of systems out of equilibrium is based on the definition of a time contour on the complex plane, along which perturbation theory is indeed valid.

The Schwinger-Keldysh formalism [44–47], which is a basic tool for the study of systems out of equilibrium, refers to the construction of a specific path in the complex-time plane for the system to follow, in order to compute expectation values of operators like the one in eq. (2.69) as functions of time. Since the study can be performed in the same fashion either in the Schroedinger or the Heisenberg picture, it suffices to refer to the latter. Considering the time evolution performed only to the operators of the system as $\hat{O}(t) = \hat{U}^{\dagger}(t, t_{in})\hat{O}(t_{in})\hat{U}(t, t_{in})$ with t_{in} being an initial reference time, and the system initially being at a state $\hat{\rho}_{in} = \hat{\rho}(t_{in})$:

$$\langle \hat{O}(t) \rangle = \mathrm{tr}\left[\hat{\rho}_{in}\hat{O}(t)\right] = \mathrm{tr}\left[\hat{\rho}_{in}\hat{U}^{\dagger}(t, t_{in})\hat{O}(t_{in})\hat{U}(t, t_{in})\right]. \tag{2.74}$$

By introducing a new time instance T through the equation $\hat{U}^{\dagger}(T, t_{in})\hat{U}(T, t_{in}) = \hat{\mathbb{1}}$ and using the property $\hat{U}^{\dagger}(t_1, t_2) = \hat{U}(t_2, t_1)$ eq. (2.74) can be brought to the following form

$$\langle \hat{O}(t) \rangle = \mathrm{tr}\left[\hat{\rho}_{in}\hat{U}(t_{in}, T)\hat{U}(T, t)\hat{O}(t_{in})\hat{U}(t, t_{in})\right]. \tag{2.75}$$

In the case that the system is initially in thermal equilibrium with inverse temperature β and initial Hamiltonian $\hat{H}_{in}$

$$\hat{\rho}_{in} = \frac{e^{-\beta\hat{H}_{in}}}{\mathrm{tr}\left[e^{-\beta\hat{H}_{in}}\right]}, \tag{2.76}$$

it is also possible to consider the operator $e^{-\beta\hat{H}_{in}}$ as an evolution operator $\hat{U}(t_{in} - i\beta, t_{in})$ with respect to a constant Hamiltonian $\hat{H}_{in}$ in the direction of imaginary time. Then, by defining the partition function of the system as $Z_{in} = \text{tr}\left[e^{-\beta\hat{H}_{in}}\right]$ eq. (2.74) further reduces to

$$\langle\hat{O}(t)\rangle = \frac{1}{Z_{in}}\text{tr}\left[\hat{U}(t_{in} - i\beta, t_{in})\hat{U}(t_{in}, T)\hat{U}(T, t)\hat{O}(t_{in})\hat{U}(t, t_{in})\right]. \tag{2.77}$$

This result can be easily translated as the evolution of a system with respect to time defined in the complex plane, where the specific path can be read from eq. (2.77) from right to left:

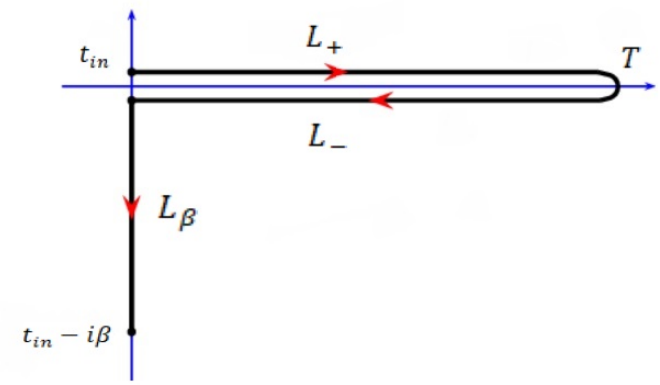
- The system starts at time $t_{in} + i0$ and evolves above the real axis, until the time instance $t_+ = t + i0$, in which the operator $\hat{O}(t_{in})$ acts.
- The system continues to evolve above the real axis until the time $T_+ = T + i0$. The complete evolution above the real time axis defines the line L_+.
- The system evolves backwards in time below the real axis until it returns to the initial time $t_{in-} = t_{in} - i0$. The evolution below the time axis defines the line L_-.
- The system evolves parallel to the imaginary axis, from t_{in-} to $t_{in-} - i\beta$. The corresponding line in the complex plane is defined as L_β.

For simplicity and without loss of generality the choice $t_{in} = -T$ will be used later, to reduce the number of constants.
The complete path in the complex plane denotes the Keldysh contour, which is presented in figure (1) and is denoted as C, while the real part consisting only of the lines L_+ and L_- is denoted as P. Furthermore, the existence of the trace in eq. (2.77) defines periodic boundary conditions for the fields of the theory, in the case that these are bosonic, and anti-periodic if these are fermionic.

2.11 Feynman's Path Integral

In addition to the canonical quantization presented in the previous subsections, it is possible to construct a new method of quantization through the study of the correlation function. This method is the so called Path Integral quantization [3, 48], which

Figure 1: The Keldysh Contour



was firstly introduced by Feynman [1,2], who costructed the phase space path integral in the Euclidean plane. In the following, his original construction is presented, for a one-dimensional system with Hamiltonian operator $\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{X})$, corresponding to a particle of mass m moving in the presence of a potential $V(\hat{X})$. The same procedure directly generalizes to more dimensions.

By discretizing the time evolution of the correlation function (2.32) in the Schroedinger picture, with initial and final states being elements of the position basis $\{|x\rangle\}$

$$\langle x_f|\hat{T}e^{-\frac{i}{\hbar}\int_{t_i}^{t_f} dt\hat{H}(t)}|x_i\rangle = \lim_{\epsilon\to 0}\lim_{N\to\infty} \langle x_f|\hat{T}\prod_{n=1}^{N} e^{-\frac{i}{\hbar}\epsilon\hat{H}(t_i+n\epsilon)}|x_i\rangle\,, \tag{2.78}$$

where $(N+1)\epsilon = t_f - t_i$, and using the resolution of the identity in the space of position and momentum between the infinitesimal time evolution operators, it is easy to find [36]

$$\begin{aligned}\langle x_f|\hat{T}e^{-\frac{i}{\hbar}\int_{t_i}^{t_f} dt\hat{H}(t)}|x_i\rangle =& \lim_{\epsilon\to 0}\lim_{N\to\infty}\left[\prod_{k=1}^{N-1}\int_{-\infty}^{+\infty} dx_k\right]\left[\prod_{k=0}^{N-1}\int_{-\infty}^{+\infty}\frac{dp_k}{2\pi\hbar}\right]\times \\ &\times\exp\left(\frac{i}{\hbar}\epsilon\sum_{k=0}^{N-1}[p_k\frac{(x_{k+1}-x_k)}{\epsilon} - H_{cl}(p_k,x_k)]\right),\end{aligned} \tag{2.79}$$

with $x_0 = x_i$, $x_N = x_f$ and $H_{cl}(p_k, x_k) = \langle x_k, p_k|\hat{H}|x_k, p_k\rangle = \frac{p_k^2}{2} + V(x_k)$. This quantity is considered to have a well defined continuum limit

$$\langle x_f|\hat{T}e^{-\frac{i}{\hbar}\int_{t_i}^{t_f} dt' \hat{H}(t')}|x_i\rangle = \int \mathcal{D}p \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x \exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt[p\dot{q} - H_{cl}(q,p)]\right), \tag{2.80}$$

where

$$\int \mathcal{D}x = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left[\prod_{k=1}^{N-1}\int_{-\infty}^{+\infty} dx_k\right] \tag{2.81}$$

and

$$\int \mathcal{D}p = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left[\prod_{k=0}^{N-1}\int_{-\infty}^{+\infty} \frac{dp_k}{2\pi\hbar}\right]. \tag{2.82}$$

It must be noted that, even though the above measures of integration can be well defined as integrals over the paths[4], this is not the case for path integrals over more complicated structures, as is the case of path integrals over fields in quantum field theories. In such cases, there exists an inability in defining a general functional measure for the integration, and thus the use of path integrals is not generally well defined in a strict mathematical sense, despite the fact that it remains the usual practice. For this reason, it is necessary to refer to the well defined discrete form seen in eq. (2.79), for the handling of such cases. Nevertheless, this should also be the standard method even when the continuum limit seems to be well defined, since the correct form of this limit may be highly non-trivial and not immediately emerging from the discrete one, as will be noted later.

The same construction can be used for the study of the correlation functions (2.33) in the Heisenberg picture [36]. In what follows, the procedure in the $2D$ phase space (1 spacial dimension) is presented, while the generalization to higher dimensions is immediate. It is then possible to express the correlation function as

$$\begin{aligned} &{}_H\left\langle x_f, t_f|\hat{T}\left(\hat{X}_1(t_1)\hat{X}_2(t_2)\cdots\hat{X}_n(t_n)\right)|x_i, t_i\right\rangle_H = \\ &= \langle x_f|\hat{U}(t_f - t_n)\hat{X}_n\hat{U}(t_n - t_{n-1})\ldots\hat{X}_1\hat{U}(t_1 - t_i)|x_i\rangle\,, \end{aligned} \tag{2.83}$$

where the time ordering $t_n > t_{n-1} > \cdots > t_1$ is considered, and the operators with no time argument on the r.h.s. are defined at an arbitrary time t_0. In this form, all physical

[4]This will also prove to be the case for all path integrals over well defined phase-spaces.

quantities, i.e. the operators $\hat{X}_j$, $j = 1, \dots n$, and the initial and final states $|x_i\rangle$, $|x_f\rangle$ respectively, have been returned to the form they had at a reference time t_0. In the above relation, one can again perform the aforementioned discretization procedure and use the resolution of the identity between the ordered exponentials, leading to the following result

$$
\begin{aligned}
&{}_H\langle x_f, t_f|\hat{T}\left(\hat{X}_1(t_1)\hat{X}_2(t_2)\cdots\hat{X}_n(t_n)\right)|x_i, t_i\rangle_H = \\
&= \int \mathcal{D}p \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x x(t_n)x(t_{n-1})\dots x(t_1)\exp\left(\frac{i}{\hbar}\int_{t_i}^{t_f} dt[p\dot{q} - H_{cl}(q,p)]\right).
\end{aligned}
\tag{2.84}
$$

Since eq. (2.84) has the same form for all choices of the time ordering of $t_1, \dots, t_n$, this path integral by construction takes all possible time orderings into account. All the above arguments can be trivially generalized for correlation functions of arbitrary operators, that are functions of $\hat{X}$ and $\hat{P}$. Due to arguments related to symplectic geometry, the path integral of eq. (2.84) can again be considered well defined in the continuum. Nevertheless, caution should be exercised with its use, for the reason stated previously. The most important characteristic of this result is that the fields $x(t_i)$ now appear as functions and not as operators, while the time ordering is included in the construction itself. As will be pointed out later, the time-continuous form allows for the use of classical methods for the calculation of quantum results, which is the most desirable advantage this form has, in contrast to the discretized one. Since the use of such functional methods can lead to enormous simplifications in complicated problems, the identification of the exact continuum limit of such quantities becomes very important, and is a major topic of the present thesis.

The path integral construction analyzed above paved the way for the foundation of path integral quantization, which was later implemented in both many-body systems through Coherent-state path integrals [3–11] and Quantum Field theory/Statistical Mechanics leading to the theory of the Renormalization Group [49,50]. In that context and for all the cases encountered in the present thesis, path integration is considered over a set of complex fields $\{\phi_i\}$, $i = 1, \dots, M$ and the functional integrals that will be

calculated are of the following form

$$\int\limits_{BC}\left[\prod_{i=1}^{M}\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\right]\exp\left\{i\int_{t_i}^{t_f}dt\sum_{i,j=1}^{M}\bar{\phi}_i(\delta_{ij}i\partial_t-V_{ij})\phi_j\right\}, \tag{2.85}$$

where the index BC denotes the boundary conditions of the quantity the functional integral corresponds to. In the previous quantity and from this point onward $\hbar$ is set to $\hbar=1$ for simplicity. Such functional integrals will appear in the study of Coherent-State path integrals.

2.12 Coherent-state path integrals

The aforementioned discretization procedure and the subsequent use of the resolution of the identity with respect to a given basis can also be performed using coherent state bases [3–11]. A correlation function in the space of coherent states can be discretized in analogy to eq. (2.78) as

$$\langle z_f|\hat{T}e^{-i\int_{t_i}^{t_f}dt\hat{H}(t)}|z_i\rangle=\lim_{\epsilon\to 0}\lim_{N\to\infty}\langle z_f|\hat{T}\prod_{n=1}^{N}e^{-i\epsilon\hat{H}(t_i+n\epsilon)}|z_i\rangle\,, \tag{2.86}$$

where $(N+1)\epsilon=t_f-t_i$. Here, even though coherent states are denoted as $|z\rangle$, the arguments do not necessarily refer to some specific case, but to all three aforementioned cases of bosonic, fermionic and spin coherent states. Using the resolution of the identity in a coherent state basis, it is then easy to proceed with the discretization procedure as

$$\begin{aligned}
&\langle z_f|\hat{T}e^{-i\int_{t_i}^{t_f}dt\hat{H}(t)}|z_i\rangle=\\
&=\lim_{\epsilon\to 0}\lim_{N\to\infty}\left[\prod_{n=1}^{N}\int_{\mathbb{C}}\mu(z_n,\bar{z}_n)\right]\langle z_f|\,e^{-i\epsilon\hat{H}_N}\,|z_N\rangle\,\langle z_1|\,e^{-i\epsilon\hat{H}_0}\,|z_i\rangle\prod_{n=1}^{N-1}\langle z_{n+1}|\,e^{-i\epsilon\hat{H}_n}\,|z_n\rangle=\\
&=\lim_{\epsilon\to 0}\lim_{N\to\infty}\left[\prod_{l=1}^{N}\int_{\mathbb{C}}\mu(z,\bar{z})\right]\langle z_f|z_N\rangle\,\langle z_1|z_i\rangle\left[\prod_{n=1}^{N-1}\langle z_{n+1}|z_n\rangle\right]\times\\
&\quad\times e^{-i\epsilon\left(\sum_{n=1}^{N-1}H_n(z_n,\bar{z}_{n+1})+H_N(z_N,\bar{z}_f)+H_0(z_i,\bar{z}_1)\right)},
\end{aligned} \tag{2.87}$$

where $H_n(z_n,\bar{z}_{n+1})=\langle z_{n+1}|\hat{H}_n|z_n\rangle/\langle z_{n+1}|z_n\rangle$, $\hat{H}_n=\hat{H}(t_i+n\epsilon)$ and $\mu(z,\bar{z})$ is the measure of integration of the corresponding set of coherent states. The usual parametriza-

tion of this quantity can be performed through the introduction of two multiplicative terms as

$$\langle z_f|z_N\rangle \langle z_1|z_i\rangle = \frac{\langle z_f|z_N\rangle \langle z_1|z_i\rangle \langle z_{N+1}|z_N\rangle \langle z_1|z_0\rangle}{\langle z_{N+1}|z_N\rangle \langle z_1|z_0\rangle}, \tag{2.88}$$

where $z_0 = z_i$, $\bar{z}_{N+1} = \bar{z}_f$, and by redefining the path integral as

$$\begin{aligned} &\langle z_f|\hat{T}e^{-i\int_{t_i}^{t_f} dt\hat{H}(t)}|z_i\rangle = \\ &= \lim_{\epsilon\to 0 N\to\infty} lim \left[\prod_{n=1}^{N}\int_{\mathbb{C}} \mu(z_n,\bar{z}_n)\right] e^{-\Gamma_{fi}} \left[\prod_{n=0}^{N}\langle z_{n+1}|z_n\rangle\right] e^{-i\epsilon\sum_{n=0}^{N} H_n(z_n,\bar{z}_{n+1})}, \end{aligned} \tag{2.89}$$

where $\Gamma_{fi} = \ln\left\{\langle z_f|z_N\rangle \langle z_1|z_i\rangle / \langle z_{N+1}|z_N\rangle \langle z_1|z_0\rangle\right\}$ is the boundary term. This term is not necessary for the construction per se, but plays a practical role, since it is crucial in avoiding discontinuities in the classical paths during the calculations.

Even though the continuum limit of these quantities is not uniquely defined, due to the asymmetry presented in the Hamiltonian term at the discrete level, it can still be used for the qualitative study of systems. In the three cases of coherent states presented in subsection 2.6, the continuum form of the corresponding path integrals is the following:

- For bosonic coherent states [3,5]

$$\begin{aligned} {}_b\langle z_f|\hat{T}e^{-i\int_{t_i}^{t_f} dt\hat{H}(t)}|z_i\rangle_b = &\int_{z(t_i)=z_i}^{\bar{z}(t_f)=\bar{z}_f} \left[\prod_{t\in[t_i,t_f]}\frac{d^2z(t)}{\pi}\right] e^{-\Gamma_{fi}}\times \\ &\times \exp\left(i\int_{t_i}^{t_f} dt\left[\frac{i}{2}\left(\bar{z}\dot{z}-\dot{\bar{z}}z\right)-H_{cl}(z,\bar{z})\right]\right), \end{aligned} \tag{2.90}$$

  with $\Gamma_{fi} = \frac{1}{2}\left(|z_f|^2+|z_i|^2\right) - \frac{1}{2}\left(\bar{z}_f z(t_f) + \bar{z}(t_i)z_i\right)$.

- For fermionic coherent states [3,9]

$$\begin{aligned} \langle\theta_f|\hat{T}e^{-i\int_{t_i}^{t_f} dt\hat{H}(t)}|\theta_i\rangle = &\int_{\theta(t_i)=\theta_i}^{\bar{\theta}(t_f)=\bar{\theta}_f} \left[\prod_{t\in[t_i,t_f]}\frac{d^2\theta(t)}{\pi}\right] e^{-\Gamma_{fi}}\times \\ &\times \exp\left(i\int_{t_i}^{t_f} dt\left[\frac{i}{2}\left(\bar{\theta}\dot{\theta}-\dot{\bar{\theta}}\theta\right)-H_{cl}(\theta,\bar{\theta})\right]\right), \end{aligned} \tag{2.91}$$

with $\Gamma_{fi} = \frac{1}{2}\left(|\theta_f|^2 + |\theta_i|^2\right) - \frac{1}{2}\left(\bar{\theta}_f\theta(t_f) + \bar{\theta}(t_i)\theta_i\right)$, and $|\theta|^2 = \bar{\theta}\theta$.

- For spin coherent states [6–8]

$$
{}_s\langle z_f|\hat{T}e^{-i\int_{t_i}^{t_f}dt\hat{H}(t)}|z_i\rangle_s = \int_{z(t_i)=z_i}^{\bar{z}(t_f)=\bar{z}_f}\left[\prod_{t\in[t_i,t_f]}\frac{2s+1}{\pi}\frac{d^2z(t)}{(1+|z|^2)^2}\right]e^{-\Gamma_{fi}}\times \\
\times\exp\left(i\int_{t_i}^{t_f}dt\left[is\frac{\bar{z}\dot{z}-\dot{\bar{z}}z}{1+|z|^2}-H_{cl}(z,\bar{z})\right]\right), \tag{2.92}
$$

with $\Gamma_{fi} = s\ln\frac{(1+\bar{z}_f z(t_f))(1+\bar{z}(t_i)z_i)}{(1+|z_f|^2)(1+|z_i|^2)}$.

As noted previously, the safest route when using these quantities is to always consider them at the discrete level, where the Hamiltonian term has an asymmetric form. As will be commented shortly, the use of these quantities in the continuum fails when it comes to exact calculations, since the most logical choice of Hamiltonian symbol H_{cl} is not the correct one. In this case, further corrections are needed, which must be identified through other methods.

Since in all cases presented above, path integration was performed over a set of complex fields, for reasons of simplicity the symbol ϕ will be used instead of z and θ whenever methods or arguments hold for all coherent state bases. Furthermore, the general case of M such sets of complex fields ϕ_i, $i = 1,\ldots,M$ will be considered whenever possible, since such cases will be encountered later, when more than one subsystems are involved.

2.13 Methods for the computation of quadratic path integrals in the continuum

Quadratic path integrals in the form of eq. (2.85) can be computed in the continuum [3] through the calculation of the functional determinant of the operator $\delta_{ij}i\partial_t - V_{ij}(t)$

$$
\int_{BC}\left[\prod_{i=1}^{M}\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\right]\exp\left\{i\int_{t_i}^{t_f}dt\sum_{i,j=1}^{M}\bar{\phi}_i(\delta_{ij}i\partial_t - V_{ij})\phi_j\right\} = \frac{1}{\mathrm{Det}\left[i\partial_t - V(t)\right]}, \tag{2.93}
$$

where the operator acts on the space of differentiable functions, in the interval $t \in [t_i, t_f]$, with the appropriate boundary conditions. In this form the Γ_{fi} factors are not

included, since the boundary conditions have already been taken into account in the structure of the function space on which the operator acts. Equation (2.93) is true only for bosonic variables, since in the case that ϕ_i's are Grassmann, the determinant appears in the numerator. To compute this functional determinant the following property can be used $\mathrm{Det}A = \exp\{\mathrm{Trlog}A\}$, which is valid for an arbitrary kernel A. One can then proceed by introducing the corresponding determinant of the free theory

$$\mathrm{Det}\left[i\partial_t - V(t)\right] = \mathrm{Det}\left[i\partial_t\right] \frac{\mathrm{Det}\left[i\partial_t - V(t)\right]}{\mathrm{Det}\left[i\partial_t\right]}. \tag{2.94}$$

The quantity which needs to be calculated then is

$$\frac{\mathrm{Det}\left[i\partial_t - V(t)\right]}{\mathrm{Det}\left[i\partial_t\right]} = \exp\left\{\mathrm{Trlog}\left[i\partial_t - V(t)\right] - \mathrm{Trlog}\left[i\partial_t\right]\right\}. \tag{2.95}$$

From this point onward the symbol Tr denotes the functional trace over both time and matrix indices, and the symbol tr denotes the trace over only matrix indices. Using the identity

$$-\int_0^1 d\lambda \frac{V(t)}{i\partial_t - \lambda V(t)} = \log\left[i\partial_t - V(t)\right] - \log\left[i\partial_t\right] \tag{2.96}$$

one then finds[5]

$$\frac{\mathrm{Det}\left[i\partial_t - V(t)\right]}{\mathrm{Det}\left[i\partial_t\right]} = \exp\left\{-\int_0^1 d\lambda \mathrm{Tr}\frac{V(t)}{i\partial_t - \lambda V(t)}\right\}. \tag{2.97}$$

It is easy to see that the previous quantity reduces to

$$\begin{aligned}\frac{\mathrm{Det}\left[i\partial_t - V(t)\right]}{\mathrm{Det}\left[i\partial_t\right]} &= \exp\left\{-\int_0^1 d\lambda \mathrm{Tr}V(t)G_\lambda(t,t')\right\} = \\ &= \exp\left\{-\int_0^1 d\lambda \mathrm{tr}\int_{t_i}^{t_f} dt V(t)G_\lambda(t,t)\right\},\end{aligned} \tag{2.98}$$

where $G_\lambda(t,t')$ is the Green's function satisfying the equation $\left(i\partial_t - \lambda V(t)\right)G_\lambda(t,t') = \hat{\mathbb{1}}\delta(t-t')$ with the appropriate boundary conditions at t_i and t_f. The calculation then proceeds for each case by solving these Green's equations and substituting the solutions into eq. (2.98). The computation of the factor $\mathrm{Det}[i\partial_t]$ can be performed with the use of dimensional regularization [3], but this determinant can be considered just

[5]While this equality holds true in the strict sense only for specific values of $V(t)$, it can be analytically continued outside of its radius of convergence or, in the case that it is used for general $V(t)$, the result can be regularized through appropriate methods.

as a normalization, since its contribution does not contain information regarding the potential $V(t)$.

2.14 The generating functional

The form given to the path integrals of subsection 2.12 allows for the definition of a new quantity, the generating functional:

$$Z[\phi_f, \phi_i; J] = \tag{2.99}$$
$$= \int_{\phi_j(t_i)=\phi_{ji}}^{\bar{\phi}_j(t_f)=\bar{\phi}_{jf}} \left[\prod_{i=1}^{M} \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\right] e^{-\Gamma_{fi}} \exp\left\{ i \int_{t_i}^{t_f} dt \sum_{i,j=1}^{M} \left[\bar{\phi}_i K_{ij} \phi_j + \bar{J}_i(t)\phi_i(t) + \bar{\phi}_i(t) J_i(t)\right] \right\},$$

with $K_{ij} = \delta_{ij} i\partial_t - V_{ij}$ for the case of the previous subsection, and $J_i(t)$, $\bar{J}_i(t)$ being external currents. Using this quantity it is possible to construct correlation functions as

$$\begin{aligned}(-i)^n \frac{\delta^n Z[\phi_f, \phi_i; J]}{\delta \bar{J}(t_n) \dots \delta \bar{J}(t_1)}\bigg|_{J=0} = \int_{\phi_j(t_i)=\phi_{ji}}^{\bar{\phi}_j(t_f)=\bar{\phi}_{jf}} \left[\prod_{i=1}^{M} \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i\right] e^{-\Gamma_{fi}} \phi_n(t_n) \dots \phi_1(t_1) \times \\ \times \exp\left\{ i \int_{t_i}^{t_f} dt \sum_{i,j=1}^{M} \left[\bar{\phi}_i K_{ij} \phi_j\right] \right\}\end{aligned} \tag{2.100}$$

and similarly for conjugate fields $\bar{\phi}_i$ and mixed cases. The same procedure can be used even if the initial action weighing the path integral is more complicated. It must be noted that even if ϕ_i and $\bar{\phi}_i$ are the projections of the operators $\hat{\phi}_i$ and $\hat{\phi}_i^\dagger$ respectively on a basis of states, the previous path integral is not necessarily equal to ${}_H\langle \phi_f, t_f | \hat{T}\left(\hat{\phi}_1(t_1)\hat{\phi}_2(t_2)\cdots\hat{\phi}_n(t_n)\right)|\phi_i, t_i\rangle_H$ as will be shown later in the main text. On

the contrary, it must be considered that

$$
\begin{aligned}
&{}_H\left\langle \phi_f, t_f|\hat{T}\left(\hat{\phi}_1(t_1)\hat{\phi}_2(t_2)\cdots\hat{\phi}_n(t_n)\right)|\phi_i, t_i\right\rangle_H = \\
&= (-i)^n F\left(\frac{\delta}{\delta\bar{J}_1(t_1)}\right)\ldots F\left(\frac{\delta}{\delta\bar{J}_n(t_n)}\right) Z[\phi_f, \phi_i; J]\Big|_{J=0} = \qquad (2.101)\\
&= \int\limits_{\phi_j(t_i)=\phi_{ji}}^{\bar{\phi}_j(t_f)=\bar{\phi}_{jf}} \left[\prod_{i=1}^{M}\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\right] e^{-\Gamma_{fi}} F(\phi_1(t_1))\ldots F(\phi_n(t_n))\exp\left\{i\int_{t_i}^{t_f} dt \sum_{i,j=1}^{M}\left[\bar{\phi}_i K_{ij}\phi_j\right]\right\},
\end{aligned}
$$

where $F\left(\frac{\delta}{\delta\bar{J}_i(t_i)}\right)$ is a function of the functional derivatives with respect to the external current $\bar{J}_i(t_i)$ and $F(\phi_i(t_i))$ the analogue for the function $\phi_i(t_i)$. As will be made clear later, even though the choice $F(\phi_i(t_i)) = \phi_i(t_i)$ may work in the case of a real space, i.e. where the fields $\hat{X}_i$ are not complex, in the case of coherent states inconsistencies will become apparent and will make the identification of the correct function F non-trivial. Nevertheless, when no inconsistencies appear or in the case that these are resolved, such functional differentiations can produce a large variety of quantum quantities. For this reason, the generating functional of eq. (2.99) can be expressed in a more practical form through the change of variables

$$
\phi_i(t) \to \phi_i(t) - \sum_{j=1}^{M}\int_{t_i}^{t_f} dt' K_{ij}^{-1}(t,t')J_j(t'), \qquad (2.102)
$$

where for reasons of simplicity the case of periodic boundary conditions is presented[6]:

$$
\begin{aligned}
&Z[\mathrm{Per}; J] = \qquad (2.103)\\
&= Z[\mathrm{Per}; 0]\int\limits_{(+)}\left[\prod_{i=1}^{M}\mathcal{D}\phi_i\mathcal{D}\bar{\phi}_i\right]\exp\left\{-i\int_{t_i}^{t_f} dt\int_{t_i}^{t_f} dt' \sum_{i,j=1}^{M}\bar{J}_i(t)K_{ij}^{-1}(t,t')J_j(t')\right\}.
\end{aligned}
$$

[6]In this equation and from now on, the Γ_{fi} factor is dropped when considering periodic/anti-periodic path integrals (partition functions), since the classical paths with periodic/anti-periodic boundary conditions are continuous. On the other hand, this factor could be necessary when computing such quantities as

$$
Z = \int_{\mathbf{C}}\mu(\phi,\bar{\phi})\left\langle \pm\phi|\hat{T}e^{-i\int_{t_i}^{t_f} dt\hat{H}(t)}|\phi\right\rangle,
$$

in which case, depending on the method being used, it may or may not be needed in the calculation of the matrix element. The upper symbol in $\pm$ denotes the bosonic case while the lower denotes the fermionic case.

The abbreviation Per inside the Z functionals denotes the periodic boundary conditions, while the symbol $(+)$ denotes the same at the level of the path integral. In a more general case, boundary terms originating from the aforementioned change of variables may appear, along with any pre-existing boundary term Γ_{fi}.

In the case of quadratic actions of the form

$$S = \int_{t_i}^{t_f} dt \sum_{i,j=1}^{M} \bar{\phi}_i (\delta_{ij} i\partial_t - V_{ij}) \phi_j, \tag{2.104}$$

the kernel K_{ij}^{-1} appearing in eq. (2.103) is none other than the solution to the Green's equation

$$\sum_{j=1}^{M} (\delta_{ij} i\partial_t - V_{ij}(t)) G_{jk}(t, t') = \delta_{ik} \delta(t - t'), \tag{2.105}$$

with the appropriate boundary conditions.

2.15 Path integrals in imaginary time and the Wick rotation

The path integral method can be used also for the study of thermal systems [3]. In such systems, which are defined through the thermal state (2.68), the partition function

$$Z = \mathrm{tr} \left[e^{-\beta \hat{H}} \right] \tag{2.106}$$

can be represented as a path integral with periodic/anti-periodic boundary conditions, defined with respect to the imaginary time $\tau = -i\hbar\beta$. The case of periodic boundary conditions corresponds to bosonic systems, while the case of anti-periodic conditions corresponds to fermionic ones. The result, found in the case of bosonic coherent states has the form

$$Z = \int\limits_{(+)} \left[\prod_{t \in [0,\beta]} \frac{d^2 z(t)}{\pi} \right] \exp\left(- \int_0^\beta dt \left[\frac{1}{2} (\bar{z}\dot{z} - \dot{\bar{z}} z) + H_{cl}(z, \bar{z}) \right] \right), \tag{2.107}$$

with the generalization to the fermionic and spin systems being straightforward. Again, the $\hbar$ constant has been set equal to 1. As expected, one can also introduce external currents as in the real-time case, the functional derivatives with respect to which, in this case, provide the expectation values of operators. The results can be related to

the corresponding real-time quantities through the Wick rotation, which will be introduced in the next paragraph, and thus present an immediate relation between the dynamic evolution of a system in real-time and the properties of the same system at thermal equilibrium.

To see what such a change towards the real-time results provides, it is vital to see how the partition function behaves in the zero-temperature limit. Considering the eigenstates of the Hamiltonian constituting a complete and orthonormal basis, the partition function can be expressed as

$$Z = \langle G.S.|e^{-\beta\hat{H}}|G.S.\rangle + \langle E_1|e^{-\beta\hat{H}}|E_1\rangle + \dots, \tag{2.108}$$

where the vector $|G.S.\rangle$ denotes the ground state of the system, which in this case is considered non-degenerate and with energy E_0, while the vector $|E_n\rangle$ denotes the n-th excited state with energy E_n. For $\beta \to \infty$, it is easy to see that

$$\begin{aligned} \lim_{\beta\to\infty} Z = \lim_{\beta\to\infty} \left(e^{-\beta E_0} + e^{-\beta E_1} + \dots\right) = \lim_{\beta\to\infty} e^{-\beta E_0}\left(1 + e^{-\beta(E_1-E_0)} + \dots\right) \simeq \\ \simeq e^{-\beta E_0} = \langle G.S.|e^{-\beta\hat{H}}|G.S.\rangle\,. \end{aligned} \tag{2.109}$$

In case of degeneracy in the ground state, the result at the zero-temperature limit would just be generalized to an equiprobable mixture of all degenerate ground states. In the result of eq. (2.109), one can see that with a simple change of variable $\beta = iT$ ($\hbar = 1$), with T defining an imaginary time, the result becomes just a ground state - ground state correlation function. Similarly, when taking the zero-temperature limit of expectation values of operators, and after using this change of variables, their ground state - ground state correlators are recovered. This method is a valid way to study the dynamic properties of systems with specific Hamiltonians using their thermal properties at equilibrium. Nevertheless, when the evolution of a system is expressed through non-equilibrium dynamics this equivalence breaks down, and the study of dynamics needs to be performed using the Swinger-Keldysh formalism (see subsection 2.10).

The definition of the imaginary time can be made formal through the analytic continuation of time [51] as

$$\tau = te^{i\epsilon}. \tag{2.110}$$

The constraint for this procedure to be valid is for the rotation of the time axis in the complex plane - during the change of ϵ - to not lead to singularities. This is true for $\epsilon \in [0, \pi]$, and subsequently the choice $\tau = it$ is possible. The use of the Wick rotation is sometimes the best way to compute real world quantities, since there exist occasions like in Quantum Field Theory, where the only way for the path integral to make sense is in imaginary time, where the weight of the paths is real.

2.16 Inconsistencies of the continuum limit

As referenced previously, computations of path integrals in the continuum not always lead to correct results [17–20]. To showcase this issue, the method presented in subsection 2.13 for the calculation of path integrals in the continuum will be addressed again, in order to present the issue for all coherent state cases referenced previously. For simplicity, the case of periodic boundary conditions will be studied, while similar arguments can be used for the case of more general boundary conditions.

Consider the bosonic variable path integral

$$\int\limits_{(+)} \left[\prod_{i=1}^{M} \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i \right] e^{iS}, \tag{2.111}$$

where S is the action characterizing the time evolution and ϕ_i, $\bar{\phi}_i$, $i = 1, \dots N$, are the fields of the theory and their conjugates. In the case where the action can be parametrized as

$$S = \int_{t_i}^{t_f} dt \sum_{i,j=1}^{M} \left\{ \bar{\phi}_i (\delta_{ij} i\partial_t - V_{ij}) \phi_j \right\}, \tag{2.112}$$

the calculation has been shown to be possible through the computation of the functional determinant as

$$\begin{aligned} \int\limits_{(+)} \left[\prod_{i=1}^{M} \mathcal{D}\phi_i \mathcal{D}\bar{\phi}_i \right] e^{iS} &= \frac{1}{\mathrm{Det}[i\partial_t]} \frac{\mathrm{Det}[i\partial_t]}{\mathrm{Det}\,[i\partial_t - V]} = \\ &= \frac{1}{\mathrm{Det}[i\partial_t]} e^{-\mathrm{trlog}[i\partial_t - V]^{-1} + \mathrm{trlog}[i\partial_t]^{-1}}. \end{aligned} \tag{2.113}$$

In the above equation, $(i\partial_t - V)^{-1} \equiv G$ is the Green's function satisfying equation $(i\partial_t - V)\, G(t-t') = \delta(t-t')$ with periodic boundary conditions, while $(\partial_t)^{-1} \equiv G_0$ is the respective Green's function for the free theory, with respect to which the normalization is performed. The calculation in the continuum then takes as granted a symmetric discrete form for the term involving V

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\int\limits_{(+)}\left[\prod_{n=1}^{N}\prod_{i=1}^{M}\int_{\mathbb{C}}\mu(\phi_{i,n},\bar{\phi}_{i,n})\right]\exp\left(i\sum_{n=0}^{N}\left[i\bar{\phi}_{in}\nabla_n\phi_{in}-V(\phi_{i,n},\bar{\phi}_{i,n})\right]\right), \quad (2.114)$$

where $\nabla_n\phi_{in} = (\phi_{i,n+1} - \phi_{in})/\epsilon$ is the lattice derivative, and $\mu(\phi_{i,n},\bar{\phi}_{i,n})$ is the measure of integration in the space of ϕ_i and $\bar{\phi}_i$ at time $t = t_i + n\epsilon$. It is easy to see that this is not the quantity under study in the case that the fields ϕ_i correspond to the eigenvalues of coherent states, since the Hamiltonian term is of the asymmetric form $H_{cl}(z_n, \bar{z}_{n+1})$.

Nevertheless, if one proceeds with the calculation of the coherent-state path integral through the functional determinant, and thus assumes the above symmetric structure, the results will be wrong. Furthermore, the same wrong results would be recovered if one computes the discrete determinant of coherent-state path integrals with the symmetric discretization of eq. (2.114) and then performs the limit $N \to \infty$. On the contrary, the correct result can be found by calculating the discrete determinant for the asymmetric Hamiltonian term and then performing the limit $N \to \infty$. Three examples of the wrong results that this naive use of the symmetric calculation can lead to are the following:

- In the study of the harmonic oscillator $\hat{H} = \omega\left(\hat{a}^\dagger\hat{a} + 1/2\right)$ through bosonic coherent states, if the Hamiltonian symbol is chosen to be ${}_b\langle z|\hat{H}\,|z\rangle_b = \omega\left(|z|^2 + 1/2\right)$:

$$\int\limits_{(+)}\left[\prod_{t\in[t_i,t_f]}\frac{d^2z(t)}{\pi}\right]\exp\left(i\int_{t_i}^{t_f}dt\left[\frac{i}{2}\left(\bar{z}\dot{z}-\dot{\bar{z}}z\right)-\omega\left(|z|^2+\frac{1}{2}\right)\right]\right) = \quad (2.115)$$

$$= e^{-\frac{i}{2}\omega(t_f-t_i)}\sum_{n=0}^{\infty}e^{-i\omega(t_f-t_i)(n+\frac{1}{2})} \neq \sum_{n=0}^{\infty}e^{-i\omega(t_f-t_i)\left(n+\frac{1}{2}\right)} = \mathrm{tr}\left[e^{-i(t_f-t_i)\omega(\hat{a}^\dagger\hat{a}+1/2)}\right].$$

- In the study of the fermionic harmonic oscillator $\hat{H} = \omega\left(\hat{\psi}^\dagger\hat{\psi} - 1/2\right)$ through fermionic coherent states, if the Hamiltonian symbol is chosen to be

$\langle\theta|\hat{H}|\theta\rangle = \omega\left(\bar{\theta}\theta - 1/2\right)$[7]:

$$\int\limits_{(-)}\left[\prod_{t\in[t_i,t_f]}\frac{d^2\theta(t)}{\pi}\right]\exp\left(i\int_{t_i}^{t_f}dt\left[\frac{i}{2}\left(\bar{\theta}\dot{\theta}-\dot{\bar{\theta}}\theta\right)-\omega\left(\bar{\theta}\theta-\frac{1}{2}\right)\right]\right)= \tag{2.116}$$

$$= 2e^{\frac{i}{2}\omega(t_f-t_i)}\cos\left(\omega\frac{t_f-t_i}{2}\right)\neq 2\cos\left(\omega\frac{t_f-t_i}{2}\right)=\mathrm{tr}\left[e^{-i(t_f-t_i)\omega(\hat{\psi}^\dagger\hat{\psi}-1/2)}\right].$$

- In the study of the spin operator $\omega\hat{S}_z$ through spin coherent states, if the Hamiltonian symbol is chosen to be ${}_s\langle z|\hat{H}\,|z\rangle_s = s\omega\frac{1-|z|^2}{1+|z|^2}$:

$$\int\limits_{(+)}\left[\prod_{t\in[t_i,t_f]}\frac{2s+1}{\pi}\frac{d^2z(t)}{(1+|z|^2)^2}\right]\exp\left(i\int_{t_i}^{t_f}dt\left[is\frac{\bar{z}\dot{z}-\dot{\bar{z}}z}{1+|z|^2}-s\omega\frac{1-|z|^2}{1+|z|^2}\right]\right)=$$

$$= e^{\frac{i}{2}\omega(t_f-t_i)}\sum_{j=-s}^{s}e^{ij\omega(t_f-t_i)}\neq\sum_{j=-s}^{s}e^{ij\omega(t_f-t_i)}=\mathrm{tr}\left[e^{-i(t_f-t_i)\omega\hat{S}_z}\right]. \tag{2.117}$$

Even though these results differ from the correct ones just by a phase factor, for higher order operators it is easy to see that the changes would be much more severe. It must be emphasized again that if the same calculations were performed in the discrete level using the original asymmetric form of the Hamiltonian symbol, and the subsequent limit $N\to\infty$, the results would indeed be the correct ones. Even though the solution to this issue is not obvious, the mistake which leads to it is clearly the wrong translation of the countably infinite product of integrals, in the original discretized path integral, to an uncountably infinite one.

2.17 Proposals for the solution of the issues

Even though an immediate method for the solution of the inconsistencies of coherent-state path integrals has not been invented until today, there have been attempts of at least recovering the correct results through the study of the continuum limit. In this subsection two such cases will be addressed [18–20, 28, 29].

The first method attempts to solve the issues at the level of the computation of the functional determinant, using the fact that the quantity $G_\lambda(t,t')$ in equation (2.98)

[7] The symbol $(-)$ denotes the antiperiodic boundary conditions of the corresponding path integral.

presents a discontinuity at the value $t' = t$. This can be seen from its exact form, since the solution of eq. $(i\partial - \lambda V(t))\, G_\lambda(t,t') = \delta(t-t')$, in the case of periodic boundary conditions and for one set of complex fields ϕ, $\bar{\phi}$, is

$$G_\lambda(t,t') = -i\left[\Theta(t-t') - \left(1 - e^{i\lambda\int_{t_i}^{t_f} V(t_1)dt_1}\right)^{-1}\right] e^{-i\lambda\int_{t'}^{t} V(t_1)dt_1}, \tag{2.118}$$

where the discontinuity appears due to the step function Θ. This shows that different choices for $\Theta(0)$ lead to different results, with the Heaviside Theta prescription $\Theta(0) = 1/2$ leading to the wrong result presented in the previous subsection.
The proposal that has been made in this context [28,29], in order to recover the correct result, is to take into account the discrete ancestor of the continuous Hamiltonian and use the limit form $G_\lambda(t,t) = i\left(1 - e^{i\lambda\int_{t_i}^{t_f} V(t_1)dt_1}\right)^{-1}$ for $t' = t$ in eq. (2.98). This way the calculation of the path integral yields the correct result and the correct answer for the partition function is recovered.
Even though this prescription is enough to cure the inconsistences presented in the computations of (2.115), (2.116) and (2.117), which refer to quadratic path integrals, this is not the case for less trivial systems such as the Bose-Hubbard model or the spin system $\hat{S}_z^2$. In the context of fermionic systems, it is a fact that terms of higher orders cannot appear, and thus one may argue that the problem has been solved. Nevertheless this is not true, since for more complex systems, where the classical Hamiltonian may need further manipulations (e.g. a diagonalization), these are only possible - and mathematically consistent - when the underlying discrete time lattice is symmetric. This is due to the fact that the asymmetric discrete action is not invariant under canonical transformations, a fact that contradicts the physical demand for the path integral to share this invariance with classical mechanics [3].

A less immediate procedure, which though circumvents the issue altogether, was presented in [18,20]. In these works the Hamiltonian operator of a bosonic system under study is firstly projected on an orthonormal basis interpreted as the set of position and momentum eigenstates of the Hilbert space, before performing the discretization procedure and the subsequent leap to the continuum limit. The resulting limit then, being inheretly symmetric, yields a consistently defined classical action, which is a

function of the corresponding position and momentum variables, identified as the eigenvalues of the previous eigenstates. By virtue of invariance of classical mechanics under canonical transformations, the system could then be returned to the space of complex coordinates, which are then identified as the eigenvalues of the annihilation and creation operators on the set of coherent states. This way, the correct action is recovered, in the sense that the functional determinant computed with the use of the symmetric slicing hypothesis (and thus of the Heaviside Theta $\Theta(0) = 1/2$ limit for the discontinuity) leads to the correct result. For the case of the harmonic oscillator $\hat{H} = \omega(\hat{a}^\dagger \hat{a} + 1/2)$, the correct Hamiltonian function for use in path integrals was thus identified to be

$$H_{harm}(z, \bar{z}) = \omega |z|^2. \tag{2.119}$$

The same technique was later generalized [19] for spin systems, where the Hamiltonian operator $\hat{S}_z$ was mapped to a function of the the harmonic oscillator's number operator, restricted on a subset of the corresponding Hilbert space. The dimension of this subset was chosen equal to the dimension of the representation of su(2) used in the problem. This mapping was possible through the Holstein-Primakoff transformation [52], which for $\hat{S}_z$ gives in the spin s representation of $su(2)$

$$\hat{S}_z = s - \hat{a}^\dagger \hat{a}. \tag{2.120}$$

Using the respective position and momentum basis of this harmonic oscillator representation, the correct continuum limit was again recovered. In the case of the $\omega \hat{S}_z$ operator, the correct Hamiltonian symbol for use in path integrals was found to be

$$H_{\omega \hat{S}_z}(z, \bar{z}) = \omega \left[s \frac{1 - |z|^2}{1 + |z|^2} + \frac{1}{2} \right]. \tag{2.121}$$

This indicated that the discretization procedure successfully reproduces the correct continuum limit, when performed on a basis that leads to a symmetric discrete structure. This procedure, used both in bosonic and spins systems, was successfully implemented for higher order Hamiltonian operators, where it again led to the recovery of the correct results.

2.18 Geometric Quantization

The classical and quantum formalisms, defined in the previous subsections, can be described very naturally through the use of symplectic geometry and the theory of vector bundles [21–24, 53].

The mathematical formalism of classical mechanics [53] is based on the definition of the phase space of a theory as a symplectic manifold M, i.e. a C^∞-manifold of even number of dimensions $2n$, on which a non-degenerate closed 2-form ω can be defined. The properties of this space M allow always for the local identification of a set of coordinates q^i and p_i, called the Darboux coordinates, which parametrize the 2-form ω as

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i, \tag{2.122}$$

with $i = 1, \dots n$. The vector fields ξ on the manifold M, which preserve ω along their flow, are called Hamiltonian vector fields. The effect of a vector field's flow on k-forms is defined through the action of the Lie-derivative

$$\mathcal{L}_\xi \omega = \left(i_\xi d + d i_\xi\right) \omega, \tag{2.123}$$

where $d = \left(\frac{\partial}{\partial x^\mu}\right) dx^\mu$ is the exterior derivative and $i_\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$ is the interior derivative with respect to a vector field ξ. In these equations it must be noted that dx^μ and $\frac{\partial}{\partial x^\mu}$, which can be found on the right of the exterior and the interior derivative respectively, are nothing more than the coordinates of these quantities in the space of 1-forms $\Omega^1(M)$ and vector fields T^1M respectively, and act on each other as

$$dx^\mu \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} dx^\mu = \delta^\mu{}_\nu. \tag{2.124}$$

The closedness of ω: $d\omega = 0$, leads to a very important relation between the vector fields that preserve ω and the set of C^∞-functions on the manifold M, since from eq. (2.123) it can be observed that to each Hamiltonian vector field ξ corresponds a C^∞-function H, and vice-versa:

$$dH = -i_\xi \omega \Rightarrow \partial_\mu H = -\xi^\nu \omega_{\nu\mu}. \tag{2.125}$$

It can be easily seen at this point that the definition of the Poisson bracket can be naturally reproduced as the change of a function, with respect to time t, under the flow of a Hamiltonian vector field ξ

$$\partial_t f \equiv \mathcal{L}_\xi f = \xi^\mu \partial_\mu f = \omega^{\#\mu\nu} \partial_\nu H \partial_\mu f \equiv \{f, H\}, \tag{2.126}$$

where $\omega^{\#\mu\nu}$ is the inverse tensor of $\omega_{\nu\mu}$, and the function f is not immediately dependent on time. In the following the index # is dropped, and the inverse property is identified by the upper position of the space indices. In the local set of Darboux coordinates, the aforementioned quantity assumes the form $\partial_t f = \sum_{i=1}^{n} \left(\partial_{q^i} f \partial_{p_i} H - \partial_{q^i} H \partial_{p_i} H \right)$ as expected. The general form of eq. (2.126) has the advantage of not needing the use of a local coordinate patch, parametrized as the "position" and "momentum" coordinates, but allows for the study of more complicated phase-spaces, which as will be emphasized later have very important roles in quantum mechanics.

For the construction of the quantum theory [21–24] the next step is the identification of the proper formalism for the study of wave-functions and of the operators acting on those. The intuitive idea that wave-functions are functions though proves not to be mathematically consistent in the general case. This is due to the fact that the presence of curvature on a manifold makes it necessary for all derivatives appearing inside the operators (e.g. the momentum) to be substituted by their covariant analogues, the latter acting by definition on sections of vector bundles [53]. The most formal way of defining wave-functions is thus as sections of a $\mathbb{C}$-line bundle over a manifold M. The $\mathbb{C}$-line bundle structure is simply the definition of a complex space $\mathbb{C}$ over each point of a given manifold, and a section is a specific choice of element from the complex space over each point. In the case where the manifold is symplectic, it is very easy to define a proper inner product for the sections of the aforementioned bundle

$$(\Psi_1, \Psi_2) = \int \mu(\vec{x}) \Psi_1^*(\vec{x}) \Psi_2(\vec{x}), \tag{2.127}$$

with the measure of integration being the canonical symplectic induced one

$$\mu(\vec{x}) = \frac{\omega}{(2n)!} = \sqrt{\det \omega} d^{2n} x. \tag{2.128}$$

The Hermitianicity condition of an operator $\hat{H}$ in turn becomes

$$\int d^{2n}x\sqrt{\det\omega}\Psi_1^*[\hat{H}\Psi_2] = \int d^{2n}x\sqrt{\det\omega}[\hat{H}\Psi_1]^*\Psi_2. \tag{2.129}$$

For the preservation of the covariance, in the case of a curved symplectic space, the covariant derivative is defined as

$$D_\mu = \partial_\mu + iA_\mu, \tag{2.130}$$

where A is the symplectic potential, locally satisfying eq. $\omega = -dA$.

Since in the context of this formalism wave-functions can locally be considered as functions of all the coordinates of the phase-space, they define a reducible representation of the quantum theory. In order to reduce the dimensions of the representations by half and in turn construct irreducible representations (e.g. the position and momentum representations) one has to define a polarization for the wave-functions. This polarization is performed through the choice of an n-dimensional sheet P of the $2n$-dimensional phase space M, where the flow of vector fields ξ oriented along P preserve the polarized wave-functions

$$\xi^\mu D_\mu \Psi = 0. \tag{2.131}$$

In the case where M is chosen as the usual Euclidean phase space, a polarization with respect to the momentum directions would provide the position representation of quantum mechanics and vice-versa. In the case of complex manifolds, which will be studied later, the polarization is performed along the antiholomorphic directions.

In the context of geometric quantization it is then possible to identify an operator $\hat{P}(f)$ for all classical C^∞-functions f, of the form

$$\hat{P}(f) = -i(\xi^\mu D_\mu + if). \tag{2.132}$$

This operator acts on wave-functions and satisfies the weakened set of Dirac's constraints presented in subsection 2.2, such that it is at least valid for a subset of the space of smooth functions. Here, ξ is the Hamiltonian vector field corresponding to the function f, as defined in eq. (2.125). In the case where a real polarization is con-

sidered, the Hermitianicity property of an operator does not hold after the choice of polarization. To see this, one should consider the case of the aforementioned momentum polarization. For the reduced phase-space, spanned only by the position coordinates, the integration measure cannot remain $\sqrt{\omega}d^n x d^n p$, since, if wave-functions depend only on $\vec{x}$, all inner products would diverge due to the integrations over the momenta. Even though in the case of the Euclidean space the new measure over the reduced phase-space can be chosen to be $d^n x$, in a curved space, like the 2-Sphere, the reduced phase-space would have a much more complicated measure of integration $J(x^\mu)$. This measure though would lack in general the properties of the canonical measure of integration $\omega/(2n)!$. As a result, this would lead to the breakdown of Hermitianicity

$$\int J d^n x \Psi_1^*(\hat{P}(f)\Psi_2) - \int J d^n x (\hat{P}(f)\Psi_1)^* \Psi_2 = i \int J d^n x [\partial_\mu \xi^\mu + \xi^\mu \partial_\mu \ln J] \Psi_1^* \Psi_2. \quad (2.133)$$

For this reason the operator $\hat{P}(f)$ is called the pre-quantum one, at this point in the construction. This issue can be avoided through the redefinition of wave-functions not only as sections of a $\mathbb{C}$-line bundle, but as a tensor product of such sections with sections of the half-form bundle δ. The sections of the latter bundle have the property

$$\delta_1 \otimes \delta_2 \in \Omega(M), \quad (2.134)$$

where $\Omega(M)$ is the space of k-forms over a manifold M and δ_1, δ_2 are elements of the half-form bundle over the same manifold. Wave-functions are thus defined as elements which are locally represented in the form

$$\Psi(x)\sqrt{J(x)}, \quad (2.135)$$

with $\Psi(x)$ being the old polarized wave-function and $\sqrt{J(x)}$ the new polarized half-form contribution. The action of the operator, which is now defined as the quantum one, is then found to be

$$\hat{Q}(f)[\Psi(x)\sqrt{J(x)}] = [-i(\xi^\mu D_\mu + if)\Psi(x)]\sqrt{J(x)} - i\Psi(x)\mathcal{L}_\xi \sqrt{J(x)}, \quad (2.136)$$

where

$$\mathcal{L}_\xi \sqrt{J(x)} = \left(\frac{1}{2}\partial_\mu \xi^\mu + \xi^\mu \partial_\mu\right)\sqrt{J(x)} \quad (2.137)$$

is the Lie-derivative with respect to the Hamiltonian vector field ξ of the C^∞-function f, acting on the half-form $\sqrt{J(x)}$. The index μ in the above equation corresponds to the coordinates of the reduced phase-space only. The new term appearing in (2.136, containing the Lie-derivative, defines the metaplectic correction. This procedure is considered mathematically and physically consistent, but is valid only in the case where the vector field ξ, the flow of which drives the time evolution, preserves the polarization, i.e. for a polarized vector field $V \in T^1M/T^1P$:

$$\mathcal{L}_\xi V \in T^1M/T^1P. \tag{2.138}$$

If this condition is not met it is necessary to introduce further mathematical structure during the construction of the operator and its action, in the form of the BKS kernels [21]. These, in theory, act by correcting the possible change made to the polarization, under the flow of a vector field ξ. The topic of BKS kernels though falls out of the scope of the present thesis, since geometric quantization will be used only for the study of polarization preserving vector fields. Nevertheless, this construction leads to the major result that path integrals can emerge naturally as limits of the BKS construction. Such phase-space path integrals are defined through the lifting of the integration to the space of paths, with boundary conditions depending on the quantity under investigation, and assume the form

$$\int\limits_{BC} \left[\prod_{t\in[t_i,t_f]} \frac{\omega^n(x^\nu(t))}{(2n)!} \right] e^{i\int_{t_i}^{t_f} dt\left\{A_\mu(x^\nu(t))\dot{x}^\mu(t) - H(x^\nu(t))\right\}}. \tag{2.139}$$

Here, A_μ defines the symplectic potential on the manifold the classical action is defined on, x^μ is a set of coordinates on the phase-space, and

$$\prod_{t\in[t_i,t_f]} \frac{\omega^n(x^\nu(t))}{(2n)!} = \lim_{N\to\infty} \prod_{j=1}^{N} \frac{\omega^n(x_j^\nu)}{(2n)!} \tag{2.140}$$

is the functional measure. The path integral presented above corresponds to a functional integral over the whole phase-space and not over a reduced one produced after the choice of a polarization. The Hamiltonian function H in this formula represents the classical observable that controls the evolution of wave-functions of the Hilbert space, through the flow of its corresponding Hamiltonian vector field.

While in the case of real polarizations issues appear in the identification of a new well defined measure $J(x^\mu)$ for the reduced phase-space, the same is not true in the case of complex polarizations. In the case of Kähler manifolds [54], i.e. Hermitian manifolds on which the Kähler form is closed, one can keep using the old symplectic-induced measure of integration of the complete phase-space, since the inner products continue to converge even after the holomorphic polarization is applied. To understand this statement an introduction to the theory of complex manifolds must be made.

A $2n$-dimensional complex manifold M is a manifold on which, for each point $p \in M$, one can find an open neighbourhood which can be charted by a complex set of coordinates $\{z^i, \bar{z}^i\}$, with $i = 1, \ldots, n$. On such a manifold, both the space of vector fields and the space of 1-forms are separated in disjoint unions of spaces, spanned by the holomorphic and anti-holomorphic elements respectively. One can then easily define a tensor J_p acting on vector fields as

$$J_p \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial z^\mu}, \quad J_p \frac{\partial}{\partial \bar{z}^\mu} = -i \frac{\partial}{\partial \bar{z}^\mu}, \tag{2.141}$$

which is called the almost complex structure. A Hermitian manifold is a complex manifold, on which a metric g_p can be defined on the coordinate patch around each point $p \in M$, and has the following compatibility property:

$$g_p(J_p X, J_p Y) = g_p(X, Y), \quad X, Y \in T_p M. \tag{2.142}$$

Here, the index p is used to signify the local nature of this argument. Using this metric one can also define the corresponding Kähler form Ω_p, which acts on vector fields as

$$\Omega_p(X, Y) = g_p(J_p X, Y), \quad X, Y \in T_p M. \tag{2.143}$$

This 2-form has the important property that it provides a nowhere vanishing top-form

$$\Omega \wedge \cdots \wedge \Omega, \tag{2.144}$$

where the index p has been dropped in order to denote that this property holds over the whole manifold M. This top-form then is well-fit for use as a measure of integra-

tion over the manifold M. It must be noted that a manifold with the aforementioned properties is not necessarily symplectic, which is the case only when $d\Omega = 0$. Complex manifolds on which a closed Kähler form can be defined are called Kähler manifolds.

To show why the holomorphic polarization does not lead to the need of a new measure of integration, it suffices to study the case of a 2-dimensional Kähler manifold M. On such a manifold, the holomorphic polarization is defined through equation

$$D_{\bar{z}}\Phi(z,\bar{z}) = 0 \implies \left(\frac{\partial}{\partial \bar{z}} + iA_{\bar{z}}\right)\Phi(z,\bar{z}) = 0, \tag{2.145}$$

where the connection A is the symplectic potential on the underlying symplectic manifold and Φ a wave-function in the usual sense, i.e. a section of a $\mathbb{C}$-line bundle. The sign in front of $A_{\bar{z}}$ must be such that this equation has a solution $\Phi(z,\bar{z}) = \phi(z)\exp\left[-Y(z,\bar{z})/2\right]$, where Y is a function with the property $\mathrm{Re}\left[Y(z,\bar{z})\right] \to +\infty$ sufficiently fast for $|z| \to +\infty$. Then, a canonical inner product can be defined [24] for the polarized wave-functions as

$$(\Phi_1, \Phi_2) = \int_M \phi_1^*(\bar{z})\phi_2(z)e^{-Y(z,\bar{z})}\mu(z,\bar{z}), \tag{2.146}$$

again using the canonical measure of the 2-dimensional Kähler manifold $\mu(z,\bar{z})$. The aforementioned inner product defines a structure that coincides with a Bargmann space of holomorphic functions on the complex plane, with measure $e^{-Y(z,\bar{z})}\frac{\mu(z,\bar{z})}{dz\wedge d\bar{z}}$[8]. This shows that complex irreducible representations of quantum mechanics can exist without the introduction of the half-form structure.

Nevertheless, one could perform the same construction after introducing the half-form structure, in which case the holomorphicaly polarized wave-functions are locally of the form

$$\Phi(z,\bar{z}) = \phi(z)e^{-\frac{Y(z,\bar{z})}{2}}\sqrt{dz}. \tag{2.147}$$

Here, the polarization of the function part remained the same as before, and the condition of holomorphic polarization for the half-form part was also taken into account. The canonical inner product for this half-form improved structure is defined through

[8] In what follows, the symbol $\frac{a}{b}$ with a and b differential forms of the same degree is the function for which $\left(\frac{a}{b}\right) \times b = a$.

the BKS pairing [24]

$$(\Phi_1, \Phi_2) = k \int_M \left(\frac{(\Phi_1^* \otimes \Phi_1^*) \wedge (\Phi_2 \otimes \Phi_2)}{\mu(z, \bar{z})} \right)^{\frac{1}{2}} \mu(z, \bar{z}), \tag{2.148}$$

where k is a normalization constant. By substituting eq. (2.147) in (2.148) this expression simplifies to

$$(\Phi_1, \Phi_2) = k \int_M \phi_1^*(\bar{z}) \phi_2(z) e^{-Y(z,\bar{z})} \left(\frac{dz \wedge d\bar{z}}{\mu(z, \bar{z})} \right)^{\frac{1}{2}} \mu(z, \bar{z}), \tag{2.149}$$

which once again coincides with a Bargmann space of holomorphic functions on the complex plane, but this time with measure

$$e^{-Y(z,\bar{z})} \left(\frac{\mu(z, \bar{z})}{dz \wedge d\bar{z}} \right)^{\frac{1}{2}}, \tag{2.150}$$

which in general may differ from the non half-form corrected result. In the case of Kähler manifolds, both structures - categorized by the inclusion or not of the half-form structure - are mathematically consistent, and thus the inclusion or not of the metaplectic correction in the construction of the quantum operators has more of a physical than a mathematical significance. Even though it is considered that the metaplectic correction leads to more physically correct results [23], its significance has not yet been quantitatively studied. As will be argued in section 3, this study can be easily performed through the application of geometric quantization in the construction of coherent-state path integrals in the continuum.

To understand how the metaplectic correction contributes to the resulting operators, it is essential to continue with the study of 2D Kähler manifolds. In that case, if the holomorphic polarization is considered, the half-form part of the polarized wave-function (2.147) can be chosen to be $\sqrt{dz}$, since any extra coefficient can be absorbed in the holomorphic function part $\phi(z)$. As a result

$$\mathcal{L}_{\xi} \sqrt{dz} = \frac{1}{2} \partial_z \xi^z \sqrt{dz}. \tag{2.151}$$

The action of the operator on the polarized wave-functions can then be expressed as

$$\hat{Q}(f)[\Phi(z,\bar{z})] = -i\left(\xi^z D_z + if + \frac{1}{2}\partial_z\xi^z\right)\Phi(z,\bar{z}). \tag{2.152}$$

By writing explicitly the form of the holomorphic wave-functions (2.147), one finds the action of the quantum operator on the function part $\phi(z)$ to be

$$\hat{Q}(f)\phi(z) = \left(-i\xi^z\partial_z - \frac{i}{2}\left(\partial_z + 4iA_z\right)\xi^z + f\right)\phi(z). \tag{2.153}$$

The metaplectic correction then appears in the action of $\hat{Q}(f)$ as the $-\frac{i}{2}\partial_z\xi^z$ contribution.

2.19 WKB approximation

The WKB (Wentzel-Kramers-Brillouin) approximation is commonly used in the context of path integrals in order to compute the semiclassical approximation of quantum quantities. In that context, one expands the action of the exponential, acting as the weight of path integration, around its saddle point. Since this saddle point is identified by demanding the functional derivative of the action with respect to the fields to be zero, the field solutions are exactly the classical configurations of the theory. Considering the fields ϕ_i - not necessarily complex - of a theory, this expansion then takes the following form

$$S = S|_{\phi_i=\phi_{icl}} + \sum_{i,j}\left.\frac{\delta^2 S}{\delta\phi_i\delta\phi_j}\right|_{\phi_i=\phi_{icl}}\delta\phi_i\delta\phi_j + \ldots, \tag{2.154}$$

where $\phi_i = \phi_{icl} + \delta\phi_i$. Truncating the aforementioned expansion after the second term, the path integral reduces to the form of a Gaussian one

$$\begin{aligned}\int_{BC}\prod_i \mathcal{D}\phi_i e^{iS} &= e^{iS|_{\phi_i=\phi_{icl}}}\int_{\phi_i(t_i)=0}^{\phi_i(t_f)=0}\prod_i \mathcal{D}\delta\phi_i e^{i\sum_{i,j}\left.\frac{\delta^2 S}{\delta\phi_i\delta\phi_j}\right|_{\phi_i=\phi_{icl}}\delta\phi_i\delta\phi_j} = \\ &= e^{iS|_{\phi_i=\phi_{icl}}}\,\mathrm{Det}\left[\left.\frac{\delta^2 S}{\delta\phi_i\delta\phi_j}\right|_{\phi_i=\phi_{icl}}\right]^{\mp\frac{1}{2}},\end{aligned} \tag{2.155}$$

where the negative exponent corresponds to the bosonic case and the positive exponent to the fermionic one. When path integration is considered over a symplectic phase-space, where the fields are the phase-space coordinates, there exist cases where this approximation can truly be exact due to the path integral generalization of the Duistermaat-Heckman theorem [55]. These are the cases where the Hamiltonian flow, driving the evolution of a system, leaves the metric of the underlying phase-space invariant, i.e. when the Lie-derivative of the metric with respect to the Hamiltonian vector field is equal to zero [56]

$$\mathcal{L}_{\xi} g = 0. \tag{2.156}$$

2.20 Faddeev-Jackiw method for path integrals over Majorana variables

The path integral representation of Majorana fermions has proved to be a highly non-trivial problem, since their complicated canonical quantization does not allow for the usual discretization construction. The proposal for the consistent path integral quantization was presented in [26], where the system of Majorana fermions was considered as a constrained system of complex fermions. This was achieved by working at the canonical level, in the Hilbert space generated by the fermionic operators $\hat{\psi}_a^\dagger$ and $\hat{\psi}_a$, with the constraint $\hat{\Omega} \equiv \hat{\psi}_a^\dagger - \hat{\psi}_a = 0$ being applied. To make the transition from this construction to the path integral representation of Majorana variables, the Faddeev-Jackiw method was used.

The Faddeev-Jackiw formalism [25] for a constrained system - which will not be addressed further in the context of the present thesis - is a method for obtaining a classical theory, consistent with the algebra of the quantum problem. Once this classical field theory is obtained, quantization can then proceed via both path integral and canonical quantization. In [26] the extension of this procedure to fermionic degrees of freedom was used to achieve exactly this in the case of Majorana fermions, where the correct Lagrangian, fit for use in path integration over real Grassmann variables, was identified. This procedure led to the result that for the identification of the classical Hamiltonian from the quantum one, it is enough just to substitute all Majorana operators $\hat{\gamma}_i$ with their Grassmann variable counterparts[9]. As will be argued in section 4,

[9] This is in contrast to the complex fermion case, in which the action weighing path integration can't be identified that easily, similarly to the bosonic and spin cases (see subsections 2.16 and 2.17).

a solution for the issues encountered in fermionic coherent-state path integrals can be easily found through this result.

3 Coherent state path integrals in the continuum via Geometric de-quantization

Section summary

In this section, bosonic variable path integrals will be studied, as is the case for bosonic and spin systems, where with the use of geometric quantization it will be possible to identify the correct continuum limit. Details on the topic of Kähler quantization are also extensively studied, with great focus given on the importance of the metaplectic correction in quantization theory. This section is based on the works presented in [57,58].

Section introduction

Before proceeding, it is necessary to understand that the issues inherited at the continuum limit are reflected only on the Hamiltonian term, since the kinematical one, being a consequence of the inner product (see eq. (2.89)), is independent of the slicing procedure; it depends only on the choice of the basis. On the contrary, the Hamiltonian term not only depends on this procedure, but also, its discrete form, which is not always symmetric with respect to the time slicing [3], does not translate uniquely to the continuum. Such an asymmetric structure cannot be invariant under canonical transformations - as should be the case with classical mechanics - and consequently does not represent the discrete form of a well defined classical Hamiltonian. As a result, the continuum limit should not be confused with a classical action and any direct calculation using functional techniques cannot be considered as legitimate[10]. In order to identify this limit with a classical action, strict conditions must be met to ensure its validity in the context of Hamiltonian mechanics. In appendix A the dependence of this limit on the slicing procedure is addressed. Furthermore, as has been shown

[10]Quantum field theory methods based on the computation of a functional determinant necessarily imply a symmetric underlying discrete structure.

in [18–20] and referenced in subsection 2.17, the correct continuum limit can be recovered if the discretization procedure leads to a symmetric discrete form, since each slice is invariant under canonical transformations in its own right.

In the present section this issue is approached from the point of view of half-form quantization, allowing for a more formal understanding of the continuum limit. As explained in subsection 2.18, path integrals emerge naturally in the study of geometric quantization as limits of the BKS construction [21]. This procedure leads to mathematically well defined phase-space path integrals, which share a lot of common traits with the ones constructed through the usual path integral construction [59–68]. Nevertheless, there has been no consideration of using geometric quantization to confront the inconsistencies plaguing the latter. This task is undertaken here, where a map is identified with the use of half-form quantization, taking operators to their corresponding correct Hamiltonian symbols for use in coherent-state path integrals. For this reason, the contribution of the metaplectic correction to all results is extensively studied and its necessity in the context of path integration is proved. This section is accompanied by two appendices: in appendix A a series of comments regarding the dependence of the continuum limit on the slicing procedure is presented; in appendix B some known no-go theorems that have been proved in the context of quantization theory are addressed.

3.1 Half-form quantization on 2D Kähler manifolds

As mentioned in subsection 2.18, geometric quantization defines a procedure through which one can identify the formal structure of the Hilbert space constructed over a given symplectic manifold, while providing at the same time a consistent quantization map for the classical observables. In the case of Kähler manifolds it was further argued how the formalism of the Hilbert space did not necessarily need the introduction of half-forms, but nevertheless that this was also possible. The quantum operator corresponding to a classical observable function f, identified under the context of half-form quantization, was then found to be

$$\hat{Q}(f) = -i\bar{\xi}^z\partial_z - \frac{i}{2}\left(\partial_z + 4iA_z\right)\xi^z + f, \tag{3.1}$$

where $\xi^\mu = \omega^{\mu\nu}\partial_\nu f$ is the Hamiltonian vector field corresponding to the smooth function f. In this form, the operator (3.1) acts only on the holomorphic function $\phi(z)$ of the half-form corrected polarized wave-functions (2.147) and is Hermitian [24] under the inner product (2.148). The presence of the half-form formalism can be seen from the inclusion of the metaplectic correction term $-\frac{i}{2}\partial_z\xi^z$ of eq. (3.1). In the context of geometric quantization of Kähler manifolds this correction, even if not mathematically necessary, appears to be needed in order to construct the desirable quantum mechanics, even in the simple case of the harmonic oscillator [23].
A very strict constraint this construction must adhere to refers to the Hamiltonian vector field ξ, the flow of which induces the action of the respective operator. This field must preserve the polarization [21–24], i.e. its Lie Derivative acting on any polarized vector field $P = \phi(z)\frac{\partial}{\partial z}$ must lie in the space of polarized vector fields

$$\mathcal{L}_\xi(P) = [\xi, P] = \left(\xi^z\partial_z\phi - \phi\partial_z\xi^z\right)\partial_z, \quad \forall z \in \mathbb{C}, \tag{3.2}$$

with the coefficient of the antiholomorphic direction $-\phi\partial_z\xi^{\bar{z}}$ being equal to zero. Thus, it suffices to have

$$\partial_z\xi^{\bar{z}} = 0. \tag{3.3}$$

If this condition is not met the operator is not legitimate, since the preservation of the polarization is a strict condition for the validity of the whole procedure. In the case where polarization is not preserved, one needs to include further structure during the construction, provided by the BKS kernels [69]. It must be also noted that the procedure leading to (3.1) is expected to hold only for operators which contain derivatives up to the first order, since only these are linearly related to vector fields. In subsection 3.3 it will be argued that the extension of this procedure to higher powers of such operators can be performed with the use of functional techniques, without the introduction of extra mathematical structure.

In the case of Kähler manifolds, two maximally symmetric spaces, the 2D Euclidean plane and the 2-Sphere, have proved to be of fundamental importance in the study of physical systems. These manifolds appear in the study of bosonic [70] and spin systems [71,72] respectively, while the complex coordinates used to chart them enter as the complex variables in the definitions of the coherent states. The relation of these

systems with the aforementioned symplectic manifolds arises naturally in both cases, since the same algebra is shared both at the level of the generators of the quantum algebra and of the isometries over the respective manifold.

The main idea, advocated in the next subsection, is to consider the quantum operator as known and use eq. (3.1) as a first order differential equation with respect to the function f. In this case, the condition (3.3) is required to be fulfilled by the Hamiltonian vector field corresponding to the function f, both if the function is considered connected to the operator through half-form and simple quantization, i.e. with and without the metaplectic correction. This way the aim is to understand how the action of that operator is represented as a classical observable on a given symplectic manifold and use this reasoning to identify the correct Hamiltonian symbol, weighing time-continuous coherent-state path integrals. The choice of the symplectic structure during this procedure, along with the necessity of the metaplectic correction, are addressed in detail in the next subsection, where the connection of the aforementioned results with the definition of the continuum limit of path integration is explicitly studied. Even though, the significance of the metaplectic correction has been argued extensively [23] in the context of canonical quantization, with the aforementioned inverse procedure the aim is to prove its unavoidable necessity in the context of path integrals.

It must be emphasized that in the proposed procedure the fundamental quantities will be the quantum operators, and thus no pre-existing knowledge is considered regarding the classical observables and their corresponding vector fields. For this reason, by considering a specific operator and reversing half-form quantization, the Hamiltonian vector field corresponding to the identified classical observable, could prove not to preserve the polarization. This is not the case in original half-form quantization, where functions were considered as the fundamental quantities, and thus all functions were chosen initially to share this property. In that case, the final operators differed depending on the inclusion or not of the half-form structure.

3.2 Path integral construction in the continuum

In this subsection it will be showcased how half-form quantization can be used to identify the correct continuum limit for the Hamiltonian symbol appearing in path integrals. To connect geometric quantization with the usual Dirac bra/ket notation the set of coherent states $|z\rangle$ has to be expressed through their more abstract mathematical definition [24]. These are elements of the Hilbert space, the inner product of which with an arbitrary state $|\phi\rangle$ gives the value of the corresponding holomorphic wavefunction at a point z

$$\phi(z) = \langle z|\phi\rangle \,. \tag{3.4}$$

This definition allows eq. (2.149) to be rewritten as

$$(\Phi_1, \Phi_2) = \langle \phi_1| \left\{ k \int_M |z\rangle \, \langle z| \, e^{-Y(z,\bar{z})} \left(\frac{dz \wedge d\bar{z}}{\mu(z,\bar{z})} \right)^{\frac{1}{2}} \mu(z,\bar{z}) \right\} |\phi_2\rangle \,, \tag{3.5}$$

through which the resolution of the identity can be formally defined

$$\hat{\mathbb{1}} = k \int_M |z\rangle \, \langle z| \, e^{-Y(z,\bar{z})} \left(\frac{dz \wedge d\bar{z}}{\mu(z,\bar{z})} \right)^{\frac{1}{2}} \mu(z,\bar{z}). \tag{3.6}$$

The existence of the resolution of the identity allows for the use of coherent states during the discretization of - real or imaginary - time evolution, through which the path integral representation of the generating functional or the partition function is constructed [3,70–72] as shown in section 2. Sets of coherent states were presented for both the cases of bosonic and of spin systems in subsection 2.6 and possess very interesting physical properties as states of each Hilbert space. In this subsection though, it will be more appropriate to express these in their not normalized, holomorphic form, for reasons that will become apparent later. In the bosonic case then, the holomorphic coherent states have the following form [3]

$$|z\rangle_b^{(0)} = e^{z\hat{a}^\dagger} \, |0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, |n\rangle \,, \tag{3.7}$$

with resolution of the identity [64,65,67]

$$\hat{\mathbb{1}}_{E_2} = \frac{1}{2\pi i} \int_{\mathbb{C}} |z\rangle_b^{(0)} \, {}_b^{(0)}\langle z| e^{-|z|^2} dz \wedge d\bar{z}, \tag{3.8}$$

where the upper index (0) denotes the lack of normalization. Similarly, for a spin system in the highest weight s representation of $su(2)$, the corresponding overcomplete set of coherent states [37] is given by

$$|z\rangle_s^{(0)} = e^{z\hat{S}_-}\,|s,s\rangle = \sum_{j=-s}^{s}\left[\frac{(2s)!}{(s-j)!(s+j)!}\right]^{\frac{1}{2}} z^{s-j}\,|s,j\rangle\,, \tag{3.9}$$

where the resolution of the identity is [56, 65]

$$\hat{\mathbb{1}}_{S_2} = \frac{2s+1}{2\pi i}\int_{\mathbb{C}} |z\rangle_s^{(0)}\,{}_s^{(0)}\langle z|e^{-2s\ln[1+|z|^2]}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}. \tag{3.10}$$

It is easy to see though, that using the definitions of the normalized coherent states (2.52) and (2.63)

$$|z\rangle_b = e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle\,, \tag{3.11}$$

$$|z\rangle_s = \frac{1}{(1+|z|^2)^s}\sum_{j=-s}^{s}\left[\frac{(2s)!}{(s-j)!(s+j)!}\right]^{\frac{1}{2}} z^{s-j}\,|s,j\rangle\,, \tag{3.12}$$

the resolution of the identity in both systems, presented in eqs. (2.55) and (2.65), takes the form

$$\hat{\mathbb{1}} \sim \int_M \mu^{(j)}(z,\bar{z})\,|z\rangle_j\,{}_j\langle z|, \tag{3.13}$$

where $\mu^{(j)}(z,\bar{z}) \sim \omega^{(j)}(z,\bar{z})_{z\bar{z}}dz\wedge d\bar{z}$ is the canonical measure of integration, on the 2D Euclidean plane for bosonic systems and on the 2-Sphere for spin systems. The index $j = b, s$ signifies the bosonic and the spin coherent states respectively and will be dismissed in the following segment for simplicity. Furthermore, in what follows the measure will be denoted as $d^2z_\wedge = dz\wedge d\bar{z}$, in order to simplify the integral expressions.

As mentioned in subsection 2.16, the discrete construction of a path integral is indeed well defined, as long as it is considered as a product of countably infinite terms. Considering the case of the partition function, the usual slicing procedure [3, 70] was shown (see eq. (2.89)) to lead to the following result

$$\mathrm{tr}\left[\hat{T}e^{-i\int_{t_i}^{t_f}dt\hat{H}}\right] = \lim_{\epsilon\to 0}\lim_{N\to\infty}\int_{(+)}\prod_{j=0}^{N}\mu(z_j,\bar{z}_j)\prod_{j=0}^{N}\langle z_{j+1}|z_j\rangle\,e^{-i\epsilon H(\bar{z}_{j+1},z_j)}, \tag{3.14}$$

where

$$H(\bar{z}_{j+1}, z_j) = \frac{\langle z_{j+1}|\hat{H}_j|z_j\rangle}{\langle z_{j+1}|z_j\rangle} + \mathcal{O}(\epsilon)$$

is the discrete form of the Hamiltonian symbol weighing the paths, $|z_{N+1}\rangle = |z_0\rangle$ is the periodicity condition and $\hat{H}_j$ is the Hamiltonian operator at time $t = \epsilon j$. Nevertheless, the standard assumption $\lim_{\epsilon\to 0} \langle z_{j+1}|\hat{H}_j|z_j\rangle \,/\, \langle z_{j+1}|z_j\rangle \;\to\; \langle z|\hat{H}|z\rangle$ was shown not to produce correct results [17, 18]. In the approach presented in this section, the limit of the aforementioned countably infinite product to an uncountably infinite one is demanded to be well defined in the context of Hamiltonian mechanics, a demand which is vital for the existence of a consistent time-continuous path integral. Next, this demand can be quantified by requiring the continuum limit of the discrete expression (3.14) to coincide with the mathematically well-defined phase-space path integral produced as the result of geometric quantization (see eq. (2.139)). The partition function is then defined to be of the form [21, 55]

$$\mathrm{tr}\left[\hat{T}e^{-i\int_{t_i}^{t_f} dt\hat{H}}\right] = \tag{3.15}$$

$$= \int\limits_{(+)} \mathcal{N}\left[\prod_{t\in[t_i,t_f]} d^2z_\wedge(t)\sqrt{\det||\omega(z(t),\bar{z}(t))||}\right] e^{i\int_{t_i}^{t_f} dt\left\{A_\mu(z(t),\bar{z}(t))\dot{x}^\mu(t) - H(z(t),\bar{z}(t))\right\}},$$

where A_μ defines the symplectic potential of the manifold, on which the classical action is defined, $x^\mu = \{z, \bar{z}\}$ is the set of complex coordinates, and

$$\prod_{t\in[t_i,t_f]} d^2z_\wedge(t)\sqrt{\det||\omega(z(t),\bar{z}(t))||} = \lim_{N\to\infty}\prod_{j=0}^{N} d^2z_{j\wedge}\sqrt{\det\omega(z_j,\bar{z}_j)} \tag{3.16}$$

is the functional measure. The Hamiltonian function H in this formula represents the classical observable that controls the evolution of wave-functions of the Hilbert space, through the flow of its corresponding Hamiltonian vector field. At this point it is important to note that the discrete structure which supports the integral (3.15) is symmetric by construction. The "normalization factor" $\mathcal{N}$ comes from the fact that the measure of integration $\prod_{t\in[t_i,t_f]}\mu(z(t),\bar{z}(t)) = \lim\limits_{N\to\infty}\prod_{j=0}^{N}\mu(z_j,\bar{z}_j)$, originating from (3.14), may not match perfectly with the symplectic structure appearing in the action[11]

[11] In the sense that $\mu(z,\bar{z}) \neq \omega(z,\bar{z}) = -dA$.

and can be modified as

$$\mu(z(t),\bar{z}(t)) = \frac{\mu(z(t),\bar{z}(t))}{\sqrt{\det||\omega(z(t),\bar{z}(t))||}d^2z_\wedge(t)}\sqrt{\det||\omega(z(t),\bar{z}(t))||}d^2z_\wedge(t) \equiv$$
$$\equiv \mathcal{N}\sqrt{\det||\omega(z(t),\bar{z}(t))||}d^2z_\wedge(t),$$

in order to bring forth the canonical measure of the symplectic manifold. This matching becomes necessary when one wants to perform semi-classical calculations at the continuum limit. If this factor is constant, it can just be factored out, as in the case of bosonic and spin systems [55] where

$$\mathcal{N}_b = \lim_{N\to\infty}\prod_{j=0}^{N}\frac{-1}{2\pi}, \quad \mathcal{N}_S = \lim_{N\to\infty}\prod_{j=0}^{N}\frac{s+1/2}{s}\frac{-1}{2\pi} \tag{3.17}$$

respectively. Otherwise, this factor must be implemented to the action as a functional determinant similarly to the Faddeev-Poppov procedure of non-Abelian Gauge theories [49]. Formula (3.15) will be the formal definition of a time-continuous coherent-state path integral.

The first step in the construction, before changing the point of view to geometric quantization, is the identification of the symplectic manifold over which the path integration occurs. The symplectic potential A, defining the underlying symplectic structure in the continuum limit (3.15), can be easily found from the limit of the countably infinite product of inner product terms in eq. (3.14) as

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\prod_{j=0}^{N}\langle z_{j+1}|z_j\rangle \equiv e^{i\int_{t_i}^{t_f}dtA_\mu(z(t),\bar{z}(t))\dot{x}^\mu(t)}. \tag{3.18}$$

This limit can be easily seen to be well defined, since it is independent of the slicing procedure. At the same time, the symplectic potential A can be interpreted as an emergent connection on the curved classical manifold, since it dictates how the time slices are sewed together at the continuum limit.

Due to the interpretation of the continuum limit of (3.14) as (3.15), the limit of the Hamiltonian term is now expected to represent the classical observable, the quantization of which generates the time evolution of wave-functions on the manifold with

symplectic potential A. Formally, for operators represented as first order differential ones and which correspond to polarization preserving Hamiltonian vector fields, this limit should be the smooth function f satisfying eq. (3.1). The inclusion of the metaplectic correction in this equation will prove to be the only ingredient needed to correct the problematic Hamiltonian symbols encountered in the literature, making eq. (3.15) computationally true and not an "up to a phase" proportionality, or worse in the case of higher order operators.

To proceed, the classical function appearing in eq. (3.1) can be defined as the inverse of $\hat{Q}$ acting on the operator $\hat{H}$, while the operators for which this procedure is valid are defined as $\hat{Q}^{-1}$ de-quantizable. The continuum limit of the Hamiltonian symbol can then be defined for the set of $\hat{Q}^{-1}$ de-quantizable operators (labelled by the index $d.q.$) as

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} H_{d.q.}(\bar{z}_{j+1}, z_j) \equiv \hat{Q}^{-1}(\hat{H}_{d.q.})(z(t), \bar{z}(t)). \tag{3.19}$$

It must be noted that if eq. (3.1) is to be used, the representation of an operator as a differential one should be identified though its action on the purely holomorphic form of the coherent states, i.e. when these are not normalized. This operator representation is not expected to be a representation in the Lie-algebra sense, since the coordinate induced forms these operators take are just the result of their specific action on the corresponding set of coherent states. It is also easy to see that the Hamiltonian symbol corresponding to the identity operator is trivially $\hat{Q}^{-1}(\hat{\mathbb{1}}) = 1$ over all symplectic manifolds.

This procedure is not expected to be valid when the Hamiltonian vector field corresponding to an operator does not preserve the polarization, i.e. for operators that do not belong in the set of $\hat{Q}^{-1}$ de-quantizable ones. This case will not be discussed in the context of this thesis.

Bosonic coherent states

The identification (3.18) in the basis of bosonic coherent states (3.7) yields the following result

$$A_\mu \dot{x}^\mu = \frac{i}{2}(\bar{z}\dot{z} - z\dot{\bar{z}}). \tag{3.20}$$

This identifies the symplectic potential of the induced classical mechanics to be $A = \frac{i}{2}(\bar{z}dz - zd\bar{z})$. The coordinate induced forms for the annihilation and creation operators on the Kähler manifold are[12]

$$\hat{a}\,|z\rangle_b^{(0)} = z\,|z\rangle_b^{(0)} \quad \text{and} \quad \hat{a}^\dagger\,|z\rangle_b^{(0)} = \frac{\partial}{\partial z}\,|z\rangle_b^{(0)} \tag{3.21}$$

respectively, while for the number operator

$$\hat{a}^\dagger\hat{a}\,|z\rangle_b^{(0)} = z\frac{\partial}{\partial z}\,|z\rangle_b^{(0)}\,. \tag{3.22}$$

Plugging these into equation (3.1) it is easy to find the respective smooth functions

$$\hat{Q}^{-1}(z) = z, \quad \hat{Q}^{-1}\left(\frac{\partial}{\partial z}\right) = \bar{z} \tag{3.23}$$

and

$$\hat{Q}^{-1}\left(z\frac{\partial}{\partial z}\right) = |z|^2 - \frac{1}{2}, \tag{3.24}$$

which indeed are known to provide correct results [3,18] when used as classical Hamiltonians in the respective time-continuous coherent-state path integrals. More specifically, through this method the result of [18], presented in eq. (2.119), was reproduced. In all considerations a symmetric underlying discrete structure is implied. In this example, the metaplectic correction contributed only in the case of the number operator, appearing as the extra $-\frac{1}{2}$ term in eq. (3.24). In the opposite point of view, it is considered [23] that the correct quantum physics for the observable $|z|^2$ are provided by the operator $z\frac{\partial}{\partial z} + \frac{1}{2}$ and not by $z\frac{\partial}{\partial z}$. This is based on arguments related to the zero point energy of the harmonic oscillator and the commutator algebra appearing after the quantization. Nevertheless, no direct or mathematically robust argument could be given until now, regarding why the correction should be included. In this example it was proved that the de-quantization procedure gives such a reason, since this correction was vital for the computationally exact mapping between the canonical and path integral quantization of this system.

For the bosonic system at hand, the canonical to path integral de-quantization mapping is thus possible for all the generators of the algebra. Due to the linearity of this

[12]These forms can be found explicitly by acting with the operators on the set of coherent states, and as explained previously do not constitute a representation of the operator algebra.

map it is possible to also deduce the correct Hamiltonian symbol corresponding to any linear function of the previous three operators

$$\hat{F}(k,l,m,d) = k\hat{a}^\dagger\hat{a} + l\hat{a} + m\hat{a}^\dagger + d\hat{\mathbb{1}}, \tag{3.25}$$

where $k,l,m,d \in \mathbb{C}$. This is found to be

$$\hat{Q}^{-1}\left(\hat{F}(k,l,m,d)\right) = k\left(|z|^2 - \frac{1}{2}\right) + lz + m\bar{z} + d. \tag{3.26}$$

As expected, if the metaplectic correction is not taken into account, these results become the expectation values of the respective operators in the coherent-state basis, which if used in time-continuous path integration lead to wrong results [17,18,20].

Spin coherent states

For the spin coherent states (3.9), the corresponding construction defines the kinematic term [71,72]

$$A^{(s)}_\mu \dot{x}^\mu = is\frac{(\bar{z}\dot{z} - z\dot{\bar{z}})}{1+|z|^2}, \tag{3.27}$$

which identifies the connection as $A^{(s)} = s\frac{i}{1+|z|^2}(\bar{z}dz - zd\bar{z})$. The coordinate induced forms of the $su(2)$ generators $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ in the highest weight s representation, can be found from their action on the coherent states to be

$$\hat{S}_x |z\rangle^{(0)}_s = \left[\frac{1-z^2}{2}\frac{\partial}{\partial z} + sz\right]|z\rangle^{(0)}_s, \tag{3.28}$$

$$\hat{S}_y |z\rangle^{(0)}_s = \left[i\frac{1+z^2}{2}\frac{\partial}{\partial z} - isz\right]|z\rangle^{(0)}_s, \tag{3.29}$$

$$\hat{S}_z |z\rangle^{(0)}_s = \left[-z\frac{\partial}{\partial z} + s\right]|z\rangle^{(0)}_s. \tag{3.30}$$

For these operators now eq. (3.1) can be used to compute the corresponding classical Hamiltonians.
Plugging $\hat{S}_z$ in eq. (3.1) one can easily find

$$\hat{Q}^{-1}\left(\hat{S}_z\right) = s\frac{1-|z|^2}{1+|z|^2} + \frac{1}{2}. \tag{3.31}$$

After this result, the partition function for the simple system $\hat{H} = \omega \hat{S}_z$ assumes the following form

$$Z = \mathrm{tr}\left[e^{-iT\omega\hat{S}_z}\right] = \tag{3.32}$$
$$= \mathcal{N}_S \int\limits_{(+)} \left\{ \prod_{t\in[0,T]} 2si \frac{d^2 z_\wedge(t)}{(1+|z(t)|^2)^2} \right\} e^{i\int_0^T dt \left\{ is\frac{\bar{z}(t)\dot{z}(t) - z(t)\dot{\bar{z}}(t)}{1+|z(t)|^2} - \omega\left(s\frac{1-|z(t)|^2}{1+|z(t)|^2}+\frac{1}{2}\right)\right\}}.$$

The measure of integration is constructed such that

$$\mathcal{N}_S \int \left\{ \prod_{t\in[0,T]} 2si \frac{d^2 z_\wedge(t)}{(1+|z(t)|^2)^2} \right\} = \lim_{N\to\infty} \prod_{j=0}^{N} \frac{2s+1}{2\pi i} \int \frac{d^2 z_{j\wedge}}{(1+|z_j|^2)^2} \tag{3.33}$$

and gives the canonical measure of integration coming from the coherent states. Note that, the discrete form of the integral (3.32) is again defined through the symmetric slicing $|z|^2 \leftrightarrow \bar{z}_j z_j$, as it was also in the bosonic case. The integration in (3.32) is easily performed [18] with the use of the WKB approximation (see subsection 2.19), yielding the correct result $Z = \sum_{j=-s}^{s} e^{-i\omega T j} = \sin[\omega T(s+1/2)]/\sin[\omega T/2]$. In this example the WKB approximation returned the complete result, since the Hamiltonian vector field corresponding to the $\hat{S}_z$ operator is an isometry, and thus the approximation is exact due to the generalization of Duistermaat-Heckman theorem. The metaplectic correction, appearing as the $+\frac{1}{2}$ term in eq. (3.31), was once again exactly the term needed for the correct partition function to be recovered. It must be emphasized that this result coincides with the result found in [19] and presented in eq. (2.121).
For the $\hat{S}_x$ and $\hat{S}_y$ operators one finds

$$\hat{Q}^{-1}\left(\hat{S}_x\right) = s\frac{z+\bar{z}}{1+|z|^2} + \frac{z}{2}, \tag{3.34}$$

$$\hat{Q}^{-1}\left(\hat{S}_y\right) = -is\frac{z-\bar{z}}{1+|z|^2} - i\frac{z}{2}, \tag{3.35}$$

which, due to the metaplectic correction, correspond to classical observables with polarization changing Hamiltonian vector fields. This fact renders them untrustworthy for use. Once again, if no metaplectic correction is taken into account, the usual spin coherent-state expectation values of the previous operators are retained, which are known though to provide wrong results [17,18]. Nevertheless, it is interesting that the not half-form corrected results indeed lead to polarization preserving Hamiltonian

vector fields, even though they are wrong for use in path integration. The problematic metaplectic correction appearing in eqs. (3.34) and (3.35) can be traced back to the de-quantization of the operator $\hat{S}_+$, since the linear combination $\hat{S}_- = \hat{S}_x - i\hat{S}_y$ admits no correction.

From eqs. (3.23), (3.24) and (3.31) it can be deduced that if the classical observable of a quantum operator corresponds to a polarization preserving vector field (with and without its metaplectic correction) then the corrected observable can be used as the Hamiltonian term in the corresponding time-continuous path integral. In the context of path integral quantization thus, the metaplectic correction appears to be a consistent and immediate way to arrive at correct results in the continuum, making its importance unambiguous. However, in cases like (3.34), (3.35), where the polarization is not preserved, the procedure fails and no useful information can be gained for the continuum limit without the introduction of extra mathematical structure.

At this point a comment is in order. It is interesting to see that the study of the resolution of the identity on the 2-Sphere would give through the use of eq. (3.6)

$$\hat{\mathbb{1}}_{S^2} = k' \int_{\mathbb{C}} |z\rangle_s^{(0)}\, {}_s^{(0)}\langle z| e^{-Y_{S^2}(z,\bar{z})} \frac{dz \wedge d\bar{z}}{1+|z|^2}. \tag{3.36}$$

If this expression is demanded to coincide with the one appearing in eq. (3.10), the function in the exponent should be $Y_{S^2} = 2\left(s+\frac{1}{2}\right)\ln\left(1+|z|^2\right)$. If one had made the connection with geometric quantization at this level of structure, i.e. during the construction of the functional measure, a natural choice would be to identify Y_{S^2} as above. Taking into account that Y_{S^2} is also defined through the solution of eq. (2.145) one would then get, for the underlying symplectic structure, the 1-form $A' = \left(s+\frac{1}{2}\right)\frac{i}{1+|z|^2}(\bar{z}dz - zd\bar{z})$, which differs from the connection A that appears in eq. (3.27). As can be readily checked, the use of A' instead of A in eq. (3.1) yields a Hamiltonian function which leads to a wrong result for the partition function (3.32). As a consequence, the symplectic structures defined from the functional measure and those defined from the kinematic term of the action differ in the half-form corrected construction. This was not the case with bosonic path integrals for which both symplectic structures coincide. Nevertheless, in the procedure proposed in the present thesis, the symplectic structure must be necessarily defined through the kinematic

term, since this way it is immediately related to the classical mechanics defined by the action.

Returning to the topic at hand, it was shown that for the proper de-quantization of the $\hat{S}_x$ and $\hat{S}_y$ operators, extra structure is needed during the construction of the continuum path integral, since neither correspond to polarization preserving vector fields. While this is indeed a drawback, it is not totally unexpected, since during the path integral construction a choice has been made for the operator $\hat{S}_z$ to act as the generator of the Cartan subalgebra of the $su(2)$ algebra. From that point onward, a specific representation of $su(2)$ was used, with respect to which the coherent states were constructed. This issue can be bypassed for any operator that is a constant linear combination of the spin operators over $\mathbb{R}$, by defining through it a new generator of the Cartan subalgebra and using it as a redefined $\hat{S}'_z$.

This formal redefinition is possible through the rotation

$$\hat{H} = a\hat{S}_x + b\hat{S}_y + c\hat{S}_z \equiv \omega\hat{S}'_z, \quad a, b, c \in \mathbb{R}, \tag{3.37}$$

where $\omega = \sqrt{a^2 + b^2 + c^2}$. Any functional integration then proceeds the same way as with the usual $\hat{S}_z$ operator, by considering a path integral constructed in the basis of the $\hat{S}'_z$ induced coherent states. It must be noted that spin-1/2 systems can also be addressed through fermionic path integrals [73], for which a procedure for the identification of the correct continuum limit will be presented in section 4.

3.3 Higher orders and interactions

3.3.1 Higher orders

The generalization of the previous results for some higher order operators can proceed with no need of additional mathematical structure, since the rigorous formalism of functional integrals allows for power reducing manipulations. Through these, path integration naturally defines a more general de-quantization map, which provides the proper Hamiltonian symbols for operators that are not necessarily $\hat{Q}^{-1}$ de-quantizable. By definition, this map is expected to share a lot of common traits with

the inverse of a consistent - under Dirac's constraints - quantization map [22, 34]. These similarities and possible differences will be addressed in depth, while details on Dirac's constraints and no-go theorems in quantization theory can be found in appendix B.

So far, a consistent de-quantization procedure has been proposed, which maps first order differential operators to their respective Hamiltonian symbols through the action of $\hat{Q}^{-1}$. Nevertheless, if path integration is to be considered as a useful technique for actual systems, a generalization of this procedure to include higher order operators and interactions must be defined. This procedure can be initially defined through the action of a more general de-quantization map $\mathcal{Q}^{-1}$, this time taking arbitrary operators to their respective Hamiltonian symbols in the continuum

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} H(\bar{z}_{j+1}, z_j) \equiv \mathcal{Q}^{-1}(\hat{H}(z(t), \bar{z}(t))). \tag{3.38}$$

The action of this map on $\hat{Q}^{-1}$ de-quantizable operators is defined as in the previous subsection

$$\mathcal{Q}^{-1}|_{d.q.} = \hat{Q}^{-1}, \tag{3.39}$$

where $\hat{Q}^{-1}$ is the de-quantization map constructed via half-form quantization. Nevertheless, the action of $\mathcal{Q}^{-1}$ on higher order operators cannot be understood that simply through geometric quantization[13] and for this reason the corresponding study will be performed with the use of functional techniques. While in the definition of $\mathcal{Q}^{-1}$ the exponent -1 was used, in analogy to the case of $\hat{Q}^{-1}$, it is not guaranteed that there exists a corresponding unique quantization map $\mathcal{Q}$ such that $(\mathcal{Q}^{-1})^{-1} = \mathcal{Q}$. Thus, the map $\mathcal{Q}^{-1}$ is not considered to be invertible and the -1 exponent is used only as an index to suggest its de-quantization property. In the same context the map $\mathcal{Q}^{-1}$ is neither considered to be necessarily linear on the space of operators, since during the usual slicing procedure the Hamiltonian symbol may acquire non-trivial contributions from commutator terms. In the following, the action of $\mathcal{Q}^{-1}$ on polynomials of $\hat{Q}^{-1}$ de-quantizable operators will be extensively studied.

In what follows, all arguments will refer strictly to constant Hamiltonian operators

[13] Unless one introduces extra mathematical structure.

and thus for simplicity the time ordering operator $\hat{T}$ is dropped from the expressions. The functional identity which will be used to identify the Hamiltonian symbol in higher orders is

$$\text{tr}\left[e^{\pm iT\hat{H}^2}\right] \sim \text{tr}\left[\int \mathcal{D}\xi e^{\left\{\mp\frac{i}{4}\int_0^T dt\xi^2 + i\int_0^T dt\xi\hat{H}\right\}}\right] = \int \mathcal{D}\xi e^{\mp\frac{i}{4}\int_0^T dt\xi^2}\text{tr}\left[e^{i\int_0^T dt\xi\hat{H}}\right] \quad (3.40)$$

and its generalization for an arbitrary positive integer power k

$$\text{tr}\left[e^{\pm iT\hat{H}^k}\right] = \text{tr}\left[e^{\pm iT\left\{\frac{1}{2}\left(\hat{H}+\hat{H}^{k-1}\right)^2 - \frac{1}{2}\hat{H}^2 - \frac{1}{2}\hat{H}^{2k-2}\right\}}\right] \sim \quad (3.41)$$

$$\sim \text{tr}\left[\int \mathcal{D}\xi_1 \int \mathcal{D}\xi_2 \int \mathcal{D}\xi_3 e^{\mp\frac{i}{4}\int_0^T dt\left\{\xi_1^2 - \xi_2^2 - \xi_3^2\right\}} e^{\frac{i}{\sqrt{2}}\int_0^T dt(\xi_1+\xi_2)\hat{H} + \frac{i}{\sqrt{2}}\int_0^T dt(\xi_1+\xi_3)\hat{H}^{k-1}}\right],$$

both of which are valid for an operator $\hat{H}$ which has a complete set of eigenstates. Through recursive use of eq. (3.41), it is easy to prove the power mapping property

$$\mathcal{Q}^{-1}\left(\hat{H}^k\right) = \left(\hat{Q}^{-1}\left(\hat{H}\right)\right)^k,\ k \in \mathbb{N}, \quad (3.42)$$

for all constant Hermitian $\hat{Q}^{-1}$ de-quantizable operators $\hat{H}$. This provides the Hamiltonian symbol corresponding to the operator $\hat{H}^k$ in time-continuous coherent-state path integration. More specifically, starting with $\text{tr}\left[e^{-iT\hat{H}^k}\right]$ and lowering the power of the operator to its first order, through the recursive use of eq. (3.41), one can map the trace involving the first order operator to its path integral representation in the way showcased in subsection 3.2. Then, one easily arrives at eq. (3.42), after integrating out the auxiliary fields ξ_i. This result can be generalized for the case of time dependent Hamiltonians, when the time dependence factors out as $\hat{H}(t) = f(t)\hat{H}$, where $\hat{H}$ is a constant Hermitian operator and $f(t)$ is a real smooth function.

Moving to the case of linear combinations of operators, where the de-quantization of each one is considered known independently, one cannot be sure that $\mathcal{Q}^{-1}$ will act linearly on these and for a deeper understanding of its behaviour more sophisticated methods should be considered. Nevertheless, linearity is true for the case when all operators appearing in the linear combination commute, since the previous construction can proceed independently for each operator. This property can be checked by

studying the action of $\mathcal{Q}^{-1}$ on the quantum operator

$$\hat{H} = \sum_{n=0}^{N} c_n (\hat{a}^\dagger \hat{a})^n, \quad c_n \in \mathbb{R}, \tag{3.43}$$

the de-quantization of which according to the previous arguments gives

$$H_{cl} = \mathcal{Q}^{-1}\left(\sum_{n=0}^{N} c_n (\hat{a}^\dagger \hat{a})^n\right) = \sum_{n=0}^{N} c_n \left(|z|^2 - \frac{1}{2}\right)^n. \tag{3.44}$$

In cases like this, the factors c_n can be considered time dependent, since the operators do not mix and continue to commute at all times. For the sake of simplicity though, these will be considered constant in the following. The calculation of the partition function

$$Z = \int\limits_{(+)} \mathcal{D}^2 z(t) e^{i\int_0^T dt\left(\frac{i}{2}(\bar{z}\dot{z} - \dot{\bar{z}}z) - \sum_{n=0}^{N} c_n \left(|z|^2 - \frac{1}{2}\right)^n\right)}, \tag{3.45}$$

where
$\int \mathcal{D}^2 z(t) = \mathcal{N} \int \left\{\prod_{t\in[0,T]} i d^2 z(t)\right\} = \lim_{N\to\infty} \prod_{j=0}^{N} \frac{1}{2\pi i} \int d^2 z_j$, proceeds then by introducing the identity [74–77]

$$1 = \int \mathcal{D}\zeta \delta[\zeta - |z|^2] = \int \mathcal{D}\zeta \int \mathcal{D}\sigma e^{-i\int_0^T dt \sigma(\zeta - |z|^2)} \tag{3.46}$$

in eq. (3.45). As a result, the partition function (3.45) can be recasted into the form

$$Z = \int \mathcal{D}\zeta \int \mathcal{D}\sigma e^{-i\int_0^T dt\left(\sigma\zeta + \sum_{n=0}^{N} c_n \left(\zeta - \frac{1}{2}\right)^n\right)} F(\zeta, \sigma), \tag{3.47}$$

where

$$F(\zeta, \sigma) = \int\limits_{(+)} \mathcal{D}^2 z e^{i\int_0^T dt\left(\frac{i}{2}(\bar{z}\dot{z} - \dot{\bar{z}}z) + \sigma|z|^2\right)}. \tag{3.48}$$

The last integral can be calculated by standard means [3], giving
$F(\zeta, \sigma) = \sum_{m=0}^{\infty} e^{i\int_0^T dt \sigma\left(m + \frac{1}{2}\right)}$ and consequently

$$Z = \sum_{m=0}^{\infty} \int \mathcal{D}\zeta \int \mathcal{D}\sigma e^{-i\int_0^T dt\left(\sigma\left(\zeta - m - \frac{1}{2}\right) + \sum_{n=0}^{N} c_n \left(\zeta - \frac{1}{2}\right)^n\right)}. \tag{3.49}$$

The integration over the field σ yields the functional delta $\delta[\zeta - m - \frac{1}{2}]$ and thus fixes the ζ variable to the values $m + \frac{1}{2}$, leading to

$$Z = \sum_{m=0}^{\infty} e^{-iT\sum_{n=0}^{N} c_n m^n}, \tag{3.50}$$

which is the correct result.

At this point an important observation is in order. In the aforementioned construction, no consideration regarding the ordering was made. This is because the choice of the continuum limit depends only on the de-quantization of the first order operator, which is performed through eq. (3.1). If in the previous example a different ordering was initially used for the operator (3.43) (normal, anti-normal or Weyl) the resulting continuum limit arising from the aforementioned procedure would again provide the correct Hamiltonian symbol for use in each case. This is a result of the preservation of the $\hat{Q}^{-1}$ de-quantizable operators' commutation properties at the level of the path integral, since the property

$$\left(\hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\right)|z\rangle = |z\rangle \tag{3.51}$$

leads to

$$\hat{Q}^{-1}\left(\hat{a}\hat{a}^{\dagger}\right) = \hat{Q}^{-1}\left(\hat{a}^{\dagger}\hat{a}\right) + 1. \tag{3.52}$$

It is easy to understand then, that the process of expressing the Hamiltonian (3.43) with respect to the operators $\hat{a}\hat{a}^{\dagger}$, and then mapping it through $\hat{Q}^{-1}\left(\hat{a}\hat{a}^{\dagger}\right) = |z|^2 + \frac{1}{2}$ commutes with the process of mapping the initial form with $\hat{Q}^{-1}\left(\hat{a}^{\dagger}\hat{a}\right) = |z|^2 - \frac{1}{2}$. Thus, the aforementioned construction naturally provides the correct and uniquely defined Hamiltonian symbol for each case, at the continuum limit of path integration.

For the study of general Hamiltonian operators that are products of equal powers of $\hat{a}$ and $\hat{a}^{\dagger}$ or linear combinations of such terms, the correct mapping can then be identified by firstly rewriting the Hamiltonian in the form of (3.43) and then de-quantizing it as showcased above. This argument can be generalized even more, to state that whenever a Hamiltonian operator can be parametrized as a polynomial of a Hermitian $\hat{Q}^{-1}$ de-quantizable one, the mapping to the continuum limit can be performed linearly using eq. (3.42). In the case of the harmonic oscillator, this argument can thus be applied for the study of higher powers of the general Hermitian operator

$\hat{H} = k\hat{a}^\dagger\hat{a} + c\hat{a} + \bar{c}\hat{a}^\dagger + d$, where $k, d \in \mathbb{R}$ and $c \in \mathbb{C}$, which is indeed a $\hat{Q}^{-1}$ dequantizable operator.

Summarizing the results up to this point, for the bosonic case the action of the dequantization map $\mathcal{Q}^{-1}$ on operators of the form

$$\hat{F} = \sum_{n=0}^{N} l_n \left(k\hat{a}^\dagger\hat{a} + c\hat{a} + \bar{c}\hat{a}^\dagger + d\right)^n, \tag{3.53}$$

was identified to be

$$\mathcal{Q}^{-1}\left(\hat{F}\right) = \sum_{n=0}^{N} l_n \left[k\left(|z|^2 - \frac{1}{2}\right) + cz + \bar{c}\bar{z} + d\right]^n, \tag{3.54}$$

providing the corresponding Hamiltonian symbols in the continuum. Here, $k, d \in \mathbb{R}$ and $c \in \mathbb{C}$ are constant, but the $l_n \in \mathbb{R}$ factors can in general be time dependent $\forall n = 1, \ldots, N$. More formally, it was shown that for the aforementioned $k, d \in \mathbb{R}$ and $c \in \mathbb{C}$, the map $\mathcal{Q}^{-1}$ takes elements linearly, from the set

$$N_q^{(b)} = \text{span}_{\mathbb{R}} \left\{ \left(k\hat{a}^\dagger\hat{a} + c\hat{a} + \bar{c}\hat{a}^\dagger + d\right)^n | n \geq 0 \right\} \tag{3.55}$$

to elements of the set

$$N_{cl}^{(b)} = \text{span}_{\mathbb{R}} \left\{ \left[k\left(|z|^2 - \frac{1}{2}\right) + cz + \bar{c}\bar{z} + d\right]^n | n \geq 0 \right\}. \tag{3.56}$$

When acting on the subset of operators (3.55), $\mathcal{Q}^{-1}$ can be seen to be invertible. Furthermore, its inverse shares a lot of similarities with a quantization map subjected to Dirac's constraints and as long as it only connects (3.56) with (3.55), it does not violate any no-go theorems presented in quantization theory [22, 34]. It must be noted that while operators of the form (3.55) correspond to a large class of physical systems, they still span a relatively restricted set. However, extensions of this set may be possible through different functional methods downgrading the complexity of quantum operators, or the inversion of more general geometric quantization methods.

In the case of spin systems, the classical Hamiltonian corresponding to the generator $\hat{S}_z$ in the highest weight s representation of $su(2)$ appears in (3.31). The dequantization of its higher powers can be identified again, through the recursive use

of eq. (3.41). In the functional integration formalism the action of $\mathcal{Q}^{-1}$ can then be identified, this time taking elements linearly from the set

$$N_q^{(s)} = \text{span}_{\mathbb{R}} \left\{ \hat{S}_z^n | n \geq 0 \right\} \tag{3.57}$$

to elements of the set

$$N_{cl}^{(s)} = \text{span}_{\mathbb{R}} \left\{ \left(s \frac{1-|z|^2}{1+|z|^2} + \frac{1}{2} \right)^n | n \geq 0 \right\}, \tag{3.58}$$

providing the correct Hamiltonian symbols. The validity of these symbols will be checked through a highly non-trivial example in the next subsection. As expected, acting on the subset of operators (3.57), $\mathcal{Q}^{-1}$ is invertible and its inverse acting on the corresponding subset of functions (3.58) is again valid under Dirac's constraints [22, 34], since the algebra in both subsets is Abelian.

The aforementioned compatibility of the inverse of $\mathcal{Q}^{-1}$ with Dirac's constraints was expected due to the method used for its construction. To clarify this statement, some important steps in the aforementioned procedure must be addressed. At the level of algebra generators, the identification of the Hamiltonian symbol proceeded through the inversion of eq. (3.1), where for the quantum operator $\hat{Q}$ the following property holds

$$[\hat{Q}(f), \hat{Q}(g)] = i\hat{Q}(\{f, g\}), \tag{3.59}$$

for all the consistently de-quantizable cases studied, i.e. for the operators $\hat{Q} = \hat{a}$, $\hat{a}^\dagger$, $\hat{a}^\dagger \hat{a}$ and $\hat{S}_z$. This property is one of Dirac's constraints for the canonical quantization of classical observables. It is known that (3.59) cannot hold for general sets of quantizable classical functions $\{f, g, \dots\}$, since the inclusion of higher order classical observables in many occasions leads to inconsistencies [22,34,35]. When the observables commute this property is trivially true, since both the Poisson brackets and the commutation relations of all elements in eq. (3.59) are zero. In the previous cases, all the operator/function subsets connected through the general de-quantization map $\mathcal{Q}^{-1}$: (3.55)/(3.56) and (3.57)/(3.58), shared this property. These subsets were identified by solving eq. (3.1) for a first order operator with respect to its classical observable and generalizing this result for higher powers through functional methods. This procedure produced Abelian subsets on which $\mathcal{Q}^{-1}$ is invertible and $\left(\mathcal{Q}^{-1}\right)^{-1}$ satisfies eq.

(3.59), making it compatible with Dirac's constraints. Nevertheless, even though geometric quantization provided an invertible map for the above examples, $\mathcal{Q}^{-1}$ may not be invertible in general and thus bare no relation to an inverse theory of quantization.

3.3.2 Interactions

Through the previously proposed methods it is also possible to study some very general classes of interactions analytically. These are the cases involving linear combinations of tensor products of operators, which can be simultaneously de-quantized and used as Hamiltonian symbols in functional integrals. Such systems appear in the form

$$\hat{\mathcal{O}}_{int} = \sum_{l=1}^{N} \omega_l \hat{\mathcal{O}}^{l1} \otimes \hat{\mathcal{O}}^{l2} \otimes \cdots \otimes \hat{\mathcal{O}}^{lk}, \tag{3.60}$$

where $\hat{\mathcal{O}}^{lj}$ represents the operator of the j-th system participating in the l-th interaction term.

To approach such systems, it is firstly necessary to understand how the map $\mathcal{Q}^{-1}$ acts on tensor products of operators. Any consistent path integration map, providing the Hamiltonian symbols in the continuum limit, should map operators acting on different Hilbert spaces independently. This property can be derived naturally from the discretization construction. By definition then, $\mathcal{Q}^{-1}$ has this property which quantitatively is expressed as

$$\mathcal{Q}^{-1}\left(\hat{H}_1 \otimes \hat{H}_2\right) = \mathcal{Q}^{-1}\left(\hat{H}_1\right) \cdot \mathcal{Q}^{-1}\left(\hat{H}_2\right). \tag{3.61}$$

Then, regarding operators of the form (3.60), the map $\mathcal{Q}^{-1}$ acts on these as

$$\mathcal{Q}^{-1}\left(\hat{\mathcal{O}}_{int}\right) = \sum_{l=1}^{N} \omega_l \mathcal{Q}^{-1}\left(\hat{\mathcal{O}}^{l1}\right) \mathcal{Q}^{-1}\left(\hat{\mathcal{O}}^{l2}\right) \cdots \mathcal{Q}^{-1}\left(\hat{\mathcal{O}}^{lk}\right), \tag{3.62}$$

as long as all N operators corresponding to each subsystem can be simultaneously de-quantized. Of course, this is trivially true when all N operators commute. In bosonic systems, the previous arguments allow for the consistent study of interaction terms between k subsystems of the form:

1. Interactions where all subsystems take part through a single first order operator

and its powers:

$$\begin{aligned}\hat{\mathcal{O}}_{int1}^{(bos)} &= \sum_{l=1}^{N} \omega_l \otimes_{j=1}^{k} \left(k_j \hat{a}_j^\dagger \hat{a}_j + c_j \hat{a}_j + \bar{c}_j \hat{a}_j^\dagger + d_j \hat{\mathbb{1}}\right)^{n_{lj}}, \\ \omega_l, k_j, d_j &\in \mathbb{R}, \quad c_j \in \mathbb{C}.\end{aligned} \tag{3.63}$$

Here, the index j refers to the subsystems, l to the interaction term and $n_{lj} \in \mathbb{N}$ is the power of the j-th subsystem operator in the l-th interaction term. The parameters k_j, d_j and c_j are considered constant, but ω_l may be time-dependent.

2. Interactions where at least one subsystem takes part through different, but simultaneously de-quantizable operators:

$$\hat{\mathcal{O}}_{int2}^{(bos)} = \hat{F}^{(1)} \otimes \hat{a}_j^\dagger \hat{a}_j + \hat{F}^{(2)} \otimes \hat{a}_j + \hat{F}^{(2)*} \otimes \hat{a}_j^\dagger + \hat{F}^{(3)} \otimes \hat{\mathbb{1}}_j. \tag{3.64}$$

Here, the index j refers to the subsystem with the aforementioned property, and $\hat{F}^{(1)}, \hat{F}^{(3)} \in \mathbb{R}$, $\hat{F}^{(2)} \in \mathbb{C}$ are functions of the operators of all the other subsystems, which also have to be simultaneously de-quantizable.

In the same fashion, in spin systems one can study interactions of the form

$$\hat{\mathcal{O}}_{int}^{(spin)} = \sum_{l=1}^{N} \omega_l \hat{S}_{C_1}^{n_{l1}} \otimes \hat{S}_{C_2}^{n_{l2}} \otimes \cdots \otimes \hat{S}_{C_k}^{n_{lk}}, \quad n_{lj} \in \mathbb{N}, \tag{3.65}$$

where all the indices have the same meaning as in the aforementioned bosonic case. In this context, the operator $\hat{S}_{C_j}^m$ identifies the m-th power of a generator of the j-th subsystem's Cartan subalgebra, where $\hat{S}_{C_j}$ can be decomposed as (3.37). The path integral construction then proceeds as if considering $\hat{S}_{C_j}^m \to \hat{S}_{z_j}^m$ in a rotated frame.

To showcase the validity of this mapping, the general N-system interaction Hamiltonian containing $\hat{S}_z$ operators up to a power k will be studied

$$\hat{H}_k = \sum_{a_1=1}^{k_1} \cdots \sum_{a_N=1}^{k_N} c_{a_1 \ldots a_N} \hat{S}_{z_1}^{a_1} \otimes \cdots \otimes \hat{S}_{z_N}^{a_N}, \tag{3.66}$$

where $a_j \in \mathbb{N}$, $\sum_{j=1}^{N} k_j = k$ and $c_{a_1 \ldots a_N} \in \mathbb{R}$. For this operator the de-quantization procedure leads to

$$H_{cl}^{(s)}\left(|z_1|,\ldots,|z_N|\right) = \mathcal{Q}^{-1}\left(\sum_{a_1=1}^{k_1}\cdots\sum_{a_N=1}^{k_N} c_{a_1\ldots a_N}\hat{S}_{z_1}^{a_1}\otimes\cdots\otimes\hat{S}_{z_N}^{a_N}\right) = \tag{3.67}$$

$$= \sum_{a_1=1}^{k_1}\cdots\sum_{a_N=1}^{k_N} c_{a_1\ldots a_N}\left(s\frac{1-|z_1|^2}{1+|z_1|^2}+\frac{1}{2}\right)^{a_1}\cdots\left(s\frac{1-|z_N|^2}{1+|z_N|^2}+\frac{1}{2}\right)^{a_N}.$$

The calculation of the partition function

$$Z = \int\limits_{(+)}\left(\prod_{i=1}^{N}\mathcal{D}^2\mu(z_i)\right)e^{i\sum_{i=1}^{N}\int_0^T dt\left\{is\frac{\bar{z}_i\dot{z}_i - z_i\dot{\bar{z}}_i}{1+|z_i|^2} - H_{cl}^{(s)}\left(|z_1|,\ldots,|z_N|\right)\right\}}, \tag{3.68}$$

where $\int\mathcal{D}^2\mu(z) = \mathcal{N}_S\int\left\{\prod_{t\in[0,T]} 2si\frac{d^2z(t)}{(1+|z(t)|^2)^2}\right\} = \lim\limits_{N\to\infty}\prod_{j=0}^{N}\frac{2s+1}{2\pi i}\int\frac{d^2z_j}{(1+|z_j|^2)^2}$, proceeds again by introducing the identity [74–77]

$$\begin{aligned} 1 &= \int\left(\prod_{i=1}^{N}\mathcal{D}\zeta_i\right)\delta\left[\zeta_i - \left(s\frac{1-|z_i|^2}{1+|z_i|^2}+\frac{1}{2}\right)\right] = \\ &= \int\left(\prod_{i=1}^{N}\mathcal{D}\zeta_i\right)\int\left(\prod_{i=1}^{N}\mathcal{D}\sigma_i\right)e^{-i\int_0^T dt\sigma_i\left(\zeta_i - \left(s\frac{1-|z_i|^2}{1+|z_i|^2}+\frac{1}{2}\right)\right)} \end{aligned} \tag{3.69}$$

in eq. (3.68). As a result, the partition function (3.68) can be recasted into the form

$$Z = \int\left(\prod_{i=1}^{N}\mathcal{D}\zeta_i\right)\int\left(\prod_{i=1}^{N}\mathcal{D}\sigma_i\right)e^{-i\sum_{i=1}^{N}\int_0^T dt\left(\sigma_i\zeta_i + H_{cl}^{(s)}(\zeta_1,\ldots,\zeta_N)\right)}\prod_{i=1}^{N}F(\zeta_i,\sigma_i), \tag{3.70}$$

where

$$F(\zeta_i,\sigma_i) = \int\limits_{(+)}\mathcal{D}^2\mu(z_i)e^{i\int_0^T dt\left\{is\frac{\bar{z}_i\dot{z}_i - z_i\dot{\bar{z}}_i}{1+|z_i|^2} + \sigma_i\left(s\frac{1-|z_i|^2}{1+|z_i|^2}+\frac{1}{2}\right)\right\}} \tag{3.71}$$

and

$$H_{cl}^{(s)}\left(\zeta_1,\ldots,\zeta_N\right) = \sum_{a_1=1}^{k_1}\cdots\sum_{a_N=1}^{k_N} c_{a_1\ldots a_N}\zeta_1^{a_1}\cdots\zeta_N^{a_N}. \tag{3.72}$$

The last integral can be calculated by standard means [19, 29, 56, 65] and the result reads $F(\zeta_i, \sigma_i) = \sum_{m_i=-s}^{s} e^{i m_i \int_0^T dt \sigma_i}$, leading to

$$Z = \sum_{m_1=-s}^{s} \cdots \sum_{m_N=-s}^{s} \int \left(\prod_{i=1}^{N} \mathcal{D}\zeta_i \right) \int \left(\prod_{i=1}^{N} \mathcal{D}\sigma_i \right) \times \tag{3.73}$$
$$\times \exp \left[-i \sum_{i=1}^{N} \int_0^T dt \left(\sigma_i \left(\zeta_i - m_i \right) + H_{cl}^{(s)} \left(\zeta_1, \ldots, \zeta_N \right) \right) \right].$$

Finally, the integrations over the fields σ_i yield N functional delta distributions $\delta \left[\zeta_i - m_i \right]$, fixing the ζ_i variables to the values m_i and thus give

$$Z = \sum_{m_1=-s}^{s} \cdots \sum_{m_N=-s}^{s} e^{-iT \sum_{a_1=1}^{k_1} \cdots \sum_{a_N=1}^{k_N} c_{a_1 \ldots a_N} m_1^{a_1} \cdots m_N^{a_N}}, \tag{3.74}$$

which is the correct result.

4 Fermionic path integrals and correlation dynamics in an 1D XY system

Section summary

In this section, the case of fermionic path integrals is considered, and the results and methods are based on the work [73]. For such integrals neither the symplectic method presented in section 3, nor the leap through Hermitian operators presented in subsection 2.17 generalize, since Grassmann manifolds do not allow for a symplectic description and the canonical quantization of the real Majorana operators is not straightforward. Thus, a different approach should be considered. This will be the use of the result found through the Faddeev-Jackiw procedure, which was presented in subsection 2.20. This construction is later applied to the case of an 1D XY spin chain, with both constant and time-dependent transverse magnetic fields.

Section introduction

To show how this construction is possible, the known example of the simple fermionic oscillator will be presented

$$\hat{H} = \omega \left(\hat{\psi}^{\dagger} \hat{\psi} - \frac{1}{2} \right), \tag{4.1}$$

which is connected to the spin Hamiltonian $\hat{H} = -\omega \hat{S}_z$ via the Jordan-Wigner transformation. The partition function in this case can be trivially computed without any reference to path integration: $\text{tr}\left[e^{-\beta \hat{H}}\right] = e^{\omega\beta/2} + e^{-\omega\beta/2} = 2\cosh(\omega\beta/2)$, as referenced in section 2.

By introducing the Majorana operators of eq. (2.44), the Hamiltonian (4.1) assumes the form $\hat{H}_{cl} = -i\frac{\omega}{2}\hat{\gamma}_2\hat{\gamma}_1$. The construction of the corresponding path integral proceeds then via the Faddeev-Jackiw method and dictates [26] the form $H_{cl} = -i\frac{\omega}{2}\gamma_2\gamma_1$ for the classical function which weighs the path integration, with $\{\gamma_a, \gamma_b\} = 0$. The integral constructed in this way represents the partition function of the system as a path integral over real Majorana Grassmann variables. It is then an inevitable demand for this integral to be connected with the corresponding integral over complex Grassmann variables through the canonical transformation $\gamma_1 = \zeta + \bar{\zeta}$, $-i\gamma_2 = \bar{\zeta} - \zeta$. This approach yields the Hamiltonian $H_M = \omega\bar{\zeta}\zeta$ as the proper weight for the integration over fermionic paths

$$Z = \int\limits_{(-)} \mathcal{D}\bar{\zeta}\mathcal{D}\zeta \exp\left\{ -\int_{-\beta/2}^{\beta/2} d\tau \bar{\zeta} \left(\partial_\tau + \omega\right) \zeta \right\}. \tag{4.2}$$

It is worth noting that for the above mentioned canonical transformation to be valid, the discretization prescription underlying the continuous form must be the symmetric one $\bar{\zeta}_n\zeta_n \underset{N\to\infty}{\longrightarrow} \bar{\zeta}\zeta$. Thus, for the calculation of the above integral one must use the symmetric limit value $G_\lambda(0) = \frac{1}{2} - \left(1 + e^{\beta\lambda\omega}\right)^{-1}$ for the Green's function in (2.98). In this way, the correct result is produced.

The calculation presented above can be summarized in a simple proposal: in order to use the path integral formalism for a system, the quantum Hamiltonian of which is given in terms of fermionic creation and annihilation operators, the first step is to rewrite it in terms of Majorana operators. By replacing these with the corresponding

real Grassmann variables, according to the Faddeev-Jackiw procedure, one then gets the well defined, unique action for the real Grassmann variable path integral. Finally, the canonical change of variables to complex Grassmann fields provides the classical Hamiltonian that must weigh the paths over fermionic coherent states. This whole construction fixes the discrete ancestor of the continuous expressions to be the symmetric one.

4.1 Spin systems through fermionic path integrals

To demonstrate the general form of the quantum Hamiltonians of interest, consider the spin Hamiltonian

$$\hat{H} = -\sum_{j=1}^{N} \left[a_j \sigma_j^x \sigma_{j+1}^x + b_j \sigma_j^y \sigma_{j+1}^y + c_j \sigma_j^z \sigma_{j+1}^z + h_j \sigma_j^z \right]. \tag{4.3}$$

By applying the Jordan-Wigner transformation (2.51), the Hamiltonian operator (4.3) can be re-expressed in terms of fermionic creation and annihilation operators as

$$\begin{aligned} \hat{H} = -\sum_{j=1}^{N} \Bigg[& a_j \left(\hat{\psi}_j^\dagger - \hat{\psi}_j \right) \left(\hat{\psi}_{j+1}^\dagger + \hat{\psi}_{j+1} \right) + b_j \left(\hat{\psi}_{j+1}^\dagger - \hat{\psi}_{j+1} \right) \left(\hat{\psi}_j^\dagger + \hat{\psi}_j \right) + \\ & + c_j \left(1 - 2\hat{\psi}_j^\dagger \hat{\psi}_j \right) \left(1 - 2\hat{\psi}_{j+1}^\dagger \hat{\psi}_{j+1} \right) + h_j \left(1 - 2\hat{\psi}_j^\dagger \hat{\psi}_j \right) \Bigg]. \end{aligned} \tag{4.4}$$

Introducing the Majorana operators

$$\hat{\gamma}_{2j-1} = \hat{\psi}_j^\dagger + \hat{\psi}_j, \quad -i\hat{\gamma}_{2j} = \hat{\psi}_j^\dagger - \hat{\psi}_j; \quad \{\hat{\gamma}_j, \hat{\gamma}_k\} = \delta_{jk}, \tag{4.5}$$

the Hamiltonian (4.3) assumes the form

$$\hat{H} = i \sum_{j=1}^{N} \left(a_j \hat{\gamma}_{2j} \hat{\gamma}_{2j+1} + b_j \hat{\gamma}_{2j+2} \hat{\gamma}_{2j-1} + i c_j \hat{\gamma}_{2j-1} \hat{\gamma}_{2j} \hat{\gamma}_{2j+1} \hat{\gamma}_{2j+2} + h_j \hat{\gamma}_{2j-1} \hat{\gamma}_{2j} \right). \tag{4.6}$$

Using the Faddeev-Jackiw procedure one can now identify the classical Hamiltonian weighing the functional integral over Majorana variables, by replacing the Majorana operators with classical real Grassmann variables γ_j. By changing these back to the

complex Grassmann variables via the canonical transformation

$$\gamma_{2j-1} = \bar{\zeta}_j + \zeta_j, \quad -i\gamma_{2j} = \bar{\zeta}_j - \zeta_j; \quad \{\zeta_j, \zeta_k\} = \{\bar{\zeta}_j, \zeta_k\} = \{\bar{\zeta}_j, \bar{\zeta}_k\} = 0, \tag{4.7}$$

the Hamiltonian weighing the functional integral over complex Grassmann variables is recovered

$$H_{cl} = \sum_{j=1}^{N} \Big[a_j \left(\bar{\zeta}_{j+1} + \zeta_{j+1}\right)\left(\bar{\zeta}_j - \zeta_j\right) + b_j \left(\bar{\zeta}_j + \zeta_j\right)\left(\bar{\zeta}_{j+1} - \zeta_{j+1}\right) - \\ - 4c_j|\zeta_j|^2|\zeta_{j+1}|^2 + 2h_j|\zeta_j|^2 \Big], \tag{4.8}$$

which is also the Hamiltonian weighing the corresponding fermionic coherent-state path integral. In the present section, the focus will be on the dynamics of ground state correlators in a spin-chain system described by the XY model. Besides the exact evaluation of correlators' time-dependence, some known results, that have been evaluated by different means, shall be recovered, in order to perform a series of non-trivial checks regarding the proposed procedure. As a concrete example, a simple calculation pertaining to the two-spin system $\hat{H} = -\omega \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ is presented in appendix C.

4.2 Time dependent correlations

The study of entanglement in 1D, spin-1/2 chain models, is of great interest not only in the field of condensed matter physics, but also in quantum information science, where entangled states are of fundamental importance in information processing. As explained in subsection 2.9, the XY model is a well-known and exactly solvable model that exhibits a quantum phase transition. This transition signals the onset of long-range correlations in the ground state of the system, and is of purely quantum mechanical nature, as it is connected to the entanglement properties of the ground state [12–14]. Thus, the XY model constitutes the ideal stage for the application of the aforementioned path integral formalism, which in turn provides very practical tools for the study of the dynamics of vacuum correlation functions.

The couplings in the anisotropic XY model are defined [12–14] to be $a_j = (1+r)/2$, $b_j = (1-r)/2$ and $h_j = h$, $c_j = 0\ \forall j$, while the spin chain is considered to have an even

number of sites N and periodic boundary conditions for the spin interactions. Thus, the Hamiltonian (4.8) reads as follows

$$H_{XYcl} = \sum_{j=1}^{N} \left[r\left(\zeta_j\zeta_{j+1} - \bar{\zeta}_j\bar{\zeta}_{j+1}\right) + \left(\zeta_j\bar{\zeta}_{j+1} - \bar{\zeta}_j\zeta_{j+1}\right) + 2h\bar{\zeta}_j\zeta_j \right]. \tag{4.9}$$

For the purposes of this chapter it is natural to introduce Grassmann sources, redefining the generating functional as

$$Z[J] = \int\limits_{(-)} \mathcal{D}\bar{\zeta}\mathcal{D}\zeta \exp\left\{ -\int_{-\beta/2}^{\beta/2} d\tau \left[\sum_{j=1}^{N} \bar{\zeta}_j\dot{\zeta}_j + H_{XYcl} - i\sum_{j=1}^{N}(\bar{J}_j\zeta_j + \bar{\zeta}_j J_j) \right] \right\}. \tag{4.10}$$

The functional derivatives of this integral generate the expectation values of operators as

$$\left.\frac{\delta^2 \ln Z[J]}{\delta J_b(\tau_2)\delta\bar{J}_a(\tau_1)}\right|_{J=0} = \langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c . \tag{4.11}$$

Here $\hat{T}$ signifies the time ordering of the operators, which is implied by the path integral procedure, and the index c denotes the connected part of the expectation values. This equation is not true though when $b = a$ and $\tau_2 = \tau_1$ simultaneously since, when all indices (time and site) are equal, one should expect similar issues as those appearing during the identification of the correct classical Hamiltonian weighing the path integration. To identify the correct way to handle such cases, it is enough to understand how the simple spin operators of the original system are mapped in this context.

Spin-spin correlators of the form $\langle \sigma_i^\alpha \sigma_j^\beta \rangle_c = \langle \sigma_i^\alpha \sigma_j^\beta \rangle - \langle \sigma_i^\alpha \rangle \langle \sigma_j^\beta \rangle, (\alpha, \beta = x, y, z)$ are physically quite important as they probe the entanglement content of the ground state [12]. Such types of correlators can be produced by applying on the generating integral (4.10) the appropriate functional derivatives:

$$\langle \hat{T}\left(\sigma_i^\alpha(\tau_2)\sigma_j^\beta(\tau_1)\right)\rangle_c = D_i^\alpha(\tau_2)D_j^\beta(\tau_1)\ln Z[J]\big|_{J=0}. \tag{4.12}$$

To determine the form of these operations one can begin from the defining relation

$$\langle \sigma_j^\alpha \rangle = \frac{1}{Z(\beta)} \mathrm{tr}\left(e^{-\beta\hat{H}(\hat{\psi}^\dagger,\hat{\psi})}\sigma_j^\alpha(\hat{\psi}^\dagger,\hat{\psi}) \right), \tag{4.13}$$

where the operators $\sigma_j^{\alpha}(\hat{\psi}^{\dagger}, \hat{\psi})$ are expressed through the Jordan-Wigner transformation (2.51)[14]. The next step is to interpret eq. (4.13) as a path integral over fermionic coherent states. In the standard formulation, presented in section 2.16, the operator $\sigma_j^z = 1 - 2\hat{\psi}_j^{\dagger}\hat{\psi}_j$ is interpreted by the classical function $1 - 2\bar{\zeta}_j\zeta_j$, while the Faddeev-Jackiw approach, presented in section 4.1, yields the function $-2\bar{\zeta}_j\zeta_j$. Since the σ_j^z operator enters non-trivially in each and every spin correlation function, it is evident that the two prescriptions produce different results. Given that the static correlation functions for the XY model are known, they can serve as a criterion for distinguishing between the two approaches. In the following, it is proved that the correct results are produced through the Faddeev-Jackiw construction, which yields

$$\begin{aligned}
\langle \sigma_j^x \rangle &= \frac{1}{Z(\beta)} \int\limits_{(-)} \mathcal{D}\bar{\zeta}\mathcal{D}\zeta e^{-S(\bar{\zeta},\zeta)} \left(\prod_{k=1}^{j-1} 2\zeta_k\bar{\zeta}_k \right) (\bar{\zeta}_j + \zeta_j), \\
\langle \sigma_j^y \rangle &= \frac{i}{Z(\beta)} \int\limits_{(-)} \mathcal{D}\bar{\zeta}\mathcal{D}\zeta e^{-S(\bar{\zeta},\zeta)} \left(\prod_{k=1}^{j-1} 2\zeta_k\bar{\zeta}_k \right) (\bar{\zeta}_j - \zeta_j), \\
\langle \sigma_j^z \rangle &= \frac{2}{Z(\beta)} \int\limits_{(-)} \mathcal{D}\bar{\zeta}\mathcal{D}\zeta e^{-S(\bar{\zeta},\zeta)} \zeta_j\bar{\zeta}_j.
\end{aligned} \tag{4.14}$$

Thus, the operations in eq. (4.12) are defined as

$$\begin{aligned}
D_j^x(\tau) &\equiv \left(\prod_{k=1}^{j-1} 2\frac{\delta^2}{\delta\bar{J}_k(\tau)\delta J_k(\tau)} \right) \left(\frac{\delta}{\delta J_j(\tau)} + \frac{\delta}{\delta\bar{J}_j(\tau)} \right), \\
D_j^y(\tau) &\equiv i \left(\prod_{k=1}^{j-1} 2\frac{\delta^2}{\delta\bar{J}_k(\tau)\delta J_k(\tau)} \right) \left(\frac{\delta}{\delta J_j(\tau)} - \frac{\delta}{\delta\bar{J}_j(\tau)} \right), \\
D_j^z(\tau) &\equiv 2\frac{\delta^2}{\delta\bar{J}_j(\tau)\delta J_j(\tau)}.
\end{aligned} \tag{4.15}$$

This result also indicates the correct prescription for the equal time fermionic correlation functions[15] to be

$$\langle \hat{\psi}_b^{\dagger}(\tau)\hat{\psi}_a(\tau) \rangle_c = \left. \frac{\delta^2 \ln Z[J]}{\delta J_b(\tau)\delta\bar{J}_a(\tau)} \right|_{J=0} + \frac{1}{2}\delta_{ab}. \tag{4.16}$$

[14]For spin-1/2 the spin operators $\hat{S}_\mu$ and Pauli matrices σ_μ, $\mu = x, y, z$, are related through the equation $\hat{S}_\mu = \sigma_\mu/2$.

[15]In the case that both operators are holomorphic or antiholomorphic the equal time/site limit is trivially 0, due to the nilpotency of fermionic operators.

If the ground state of the system is unique, eq. (4.12) produces at the limit $\beta \to \infty$ the ground state expectation value of the operator. In the case of degeneracy, the zero-temperature limit projects on an equiprobable mixture of the degenerate ground states. At the limit $\beta \to \infty$ and after the Wick rotation $\tau \to it$ the operation (4.12) generates the time dependent, two-point connected, vacuum correlation function $\langle \hat{T}\left(\sigma_i^a(t_2)\sigma_j^b(t_1)\right)\rangle_c$ [51]. The study of the static entanglement entropy has been based [12, 13] on the equal time version of the above defined correlation functions. In the current section the aforementioned path integral technique will be used to investigate the dynamics of ground-state correlation functions, while a series of known results will be confirmed.

To deal with path integration weighted by (4.9), the usual [12–14, 31] tactic of separating the Hilbert space of the system into two independent sectors (see subsection 2.9), corresponding to periodic (odd fermion number) or anti-periodic (even fermion number) boundary conditions for the fermionic degrees of freedom, will be followed. Consequently, this also allows for the path integral study of each sector independently, as explained in detail in appendix D. In the following, the case of anti-periodic condition $\zeta_{N+1} = -\zeta_1$ will be studied, which is defined through the Fourier transform of the Grassmann fields (see eq. (2.73) for the operator analogues)

$$\zeta_j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}(m+\frac{1}{2})j} c_m, \quad J_j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}(m+\frac{1}{2})j} \lambda_m. \tag{4.17}$$

In turn, this is connected to a chain of even number of fermions. For the sector corresponding to the odd fermion number case, the change $m + 1/2 \to m$ is required [31]. It must be noted that expectation values found with the use of sector specific generating functionals (as is the case in the following), which correspond to the independent contribution of a given sector, are valid only for operators that preserve the sectors. Further details on the sector representation in the context of path integrals can be found in appendix D. From now on, the upper indices $(e)/(o)$ are used to denote quantities representing the even/odd fermion number sectors respectively.

Inserting (4.17) in (4.9), the classical Hamiltonian corresponding to the generating

functional of the even sector $Z^{(e)}[J]$ adopts the form

$$H^{(e)}_{XYcl} = \sum_{m=0}^{N/2-1} H^{(e)}_m, \tag{4.18}$$

$$H^{(e)}_m = 2 \begin{pmatrix} \bar{c}_m & c_{N-m-1} \end{pmatrix} \begin{pmatrix} k_m & -il_m \\ il_m & -k_m \end{pmatrix} \begin{pmatrix} c_m \\ \bar{c}_{N-m-1} \end{pmatrix},$$

where the following abbreviations have been used

$$k_m = h - \cos\frac{2\pi}{N}\left(m+\frac{1}{2}\right) \quad l_m = r\sin\frac{2\pi}{N}\left(m+\frac{1}{2}\right). \tag{4.19}$$

The crucial observation here is that the interactions connect only the fields $\bar{c}_m$ with c_m and $\bar{c}_{N-m-1}$, and the fields c_m with $\bar{c}_m$ and c_{N-m-1}. Thus, the generating functional can be factorized as

$$Z^{(e)}[J] = \prod_{m=0}^{N/2-1} Z^{(e)}_m[J], \tag{4.20}$$

$$Z^{(e)}_m[J] = \int_{(-)} \mathcal{D}\bar{c}_m \mathcal{D}c_m \mathcal{D}\bar{c}_{N-m-1} \mathcal{D}c_{N-m-1} e^{-S^{(e)}_m[J]},$$

where

$$S^{(e)}_m[J] = \int_{-\beta/2}^{\beta/2} d\tau \Big\{ (\bar{c}_m \dot{c}_m + \bar{c}_{N-m-1}\dot{c}_{N-m-1}) + H^{(e)}_m - \tag{4.21}$$

$$- i\left(\bar{\lambda}_m c_m + \bar{c}_m \lambda_m + \bar{\lambda}_{N-m-1} c_{N-m-1} + \bar{c}_{N-m-1}\lambda_{N-m-1}\right) \Big\}.$$

The Hamiltonian $H^{(e)}_m$, defined in (4.18), can be easily diagonalized through a unitary Bogoliubov transformation

$$H^{(e)}_m = U_m \begin{pmatrix} \epsilon_m & 0 \\ 0 & -\epsilon_m \end{pmatrix} U^{\dagger}_m, \quad U_m = \begin{pmatrix} \cos\theta_m & i\sin\theta_m \\ i\sin\theta_m & \cos\theta_m \end{pmatrix}. \tag{4.22}$$

In this expression

$$\epsilon_m = 2\sqrt{\left(h - \cos\frac{2\pi}{N}\left(m+\frac{1}{2}\right)\right)^2 + \left(r\sin\frac{2\pi}{N}\left(m+\frac{1}{2}\right)\right)^2} \tag{4.23}$$

and

$$\tan(2\theta_m) = \frac{r\sin\frac{2\pi}{N}\left(m+\frac{1}{2}\right)}{h - \cos\frac{2\pi}{N}\left(m+\frac{1}{2}\right)}. \tag{4.24}$$

By making the change of variables

$$\begin{pmatrix} c_m \\ \bar{c}_{N-m-1} \end{pmatrix} = U_m \begin{pmatrix} \xi_m \\ \bar{\xi}_{N-m-1} \end{pmatrix}, \tag{4.25}$$

the action (4.21) can then be written in the following form

$$S_m^{(e)}[J] = \int_{-\beta/2}^{\beta/2} d\tau \left(\bar{\eta}_m D_m \eta_m - i\bar{\mu}_m \eta_m - i\bar{\eta}_m \mu_m \right), \tag{4.26}$$

where

$$\eta_m = \begin{pmatrix} \xi_m \\ \bar{\xi}_{N-m-1} \end{pmatrix}, \quad D_m = \begin{pmatrix} \partial_\tau + \epsilon_m & 0 \\ 0 & \partial_\tau - \epsilon_m \end{pmatrix} \tag{4.27}$$

and

$$\bar{\mu}_m = \begin{pmatrix} \bar{\lambda}_m & -\lambda_{N-m-1} \end{pmatrix} U_m. \tag{4.28}$$

Before proceeding, it is worth noting that the change of variables (4.25) (and the subsequent diagonalization) is permitted by the symmetric form of the discrete time lattice structure which defines the path integral. On the contrary, if the asymmetric discrete form had been kept, this change would not be possible. Written in this form, the integrals in (4.20) can be easily calculated.

In the limit $\beta \to \infty$, the generating functional (4.20) becomes

$$Z_m^{(e)}[J] = Z_m^{(e)}[0] \exp\left\{ -\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \bar{\mu}_m(\tau) G_m(\tau - \tau') \mu_m(\tau') \right\}, \tag{4.29}$$

with

$$G_m \equiv D_m^{-1} = \begin{pmatrix} G_m^{(+)} & 0 \\ 0 & G_m^{(-)} \end{pmatrix}$$

and

$$G_m^{(+)}(\tau - \tau') = \Theta(\tau - \tau') e^{-(\tau - \tau')\epsilon_m}, \quad G_m^{(-)}(\tau - \tau') = -\Theta(\tau' - \tau) e^{-(\tau' - \tau)\epsilon_m}. \tag{4.30}$$

The Green's function $G_m^{(+)}$, which propagates the m modes, has been chosen to obey casuality: $G^{(+)}(\tau - \tau') = 0$ for $\tau - \tau' < 0$. It is, in fact, the antiperiodic function $G_m^{(+)}(\tau - \tau') = \left[\Theta(\tau - \tau') - \left(1 + e^{\beta\epsilon_m}\right)^{-1} \right] e^{-(\tau - \tau')\epsilon_m}$ at the limit $\beta \to \infty$. The advanced function $G_m^{(-)}(\tau - \tau')$ propagates the $N - m - 1$ conjugate modes backwards, and obeys the boundary condition $G_m^{(-)}(\tau - \tau') = 0$ for $\tau - \tau' > 0$. As expected, it is

the $\beta \to \infty$ limit of the antiperiodic Green's function
$G_m^{(-)}(\tau - \tau') = \left[\left(1 + e^{\beta\epsilon_m}\right)^{-1} - \Theta(\tau' - \tau)\right] e^{-(\tau'-\tau)\epsilon_m}$. Note that, according to the aforementioned prescription, the Θ function appearing in (4.30) is the Heaviside step function, for which $\Theta(0) = 1/2$. In eq. (4.29), the system's partition function $Z_m^{(e)}[0]$, corresponding to the even fermion number sector, appears as a normalization factor. The path integral computation of this quantity can be found in appendix D, where the known result [78] is recovered. Nevertheless, as long as one is interested in the study of a single sector (the even one in this case), this factor proves to be irrelevant in the computation of $\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(e)}$.

By acting with the functional derivatives on the generating functional (4.29), it is an easy task to compute the following expressions, that are the basis for all correlation functions in the even sector:

$$\langle \hat{T}\left(\hat{\psi}_b(\tau_2)\hat{\psi}_a^\dagger(\tau_1)\right)\rangle_c^{(e)} = \tag{4.31}$$
$$= \frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}\left(m+\frac{1}{2}\right)(b-a)}\left(\cos^2\theta_m G_m^{(+)}(\tau_2 - \tau_1) + \sin^2\theta_m G_m^{(-)}(\tau_2 - \tau_1)\right),$$

$$\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(e)} = \tag{4.32}$$
$$= -\frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}\left(m+\frac{1}{2}\right)(b-a)}\left(\cos^2\theta_m G_m^{(+)}(\tau_1 - \tau_2) + \sin^2\theta_m G_m^{(-)}(\tau_1 - \tau_2)\right),$$

$$\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a^\dagger(\tau_1)\right)\rangle_c^{(e)} = \tag{4.33}$$
$$= \frac{i}{2N}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}\left(m+\frac{1}{2}\right)(b-a)}\sin(2\theta_m)\left(G_m^{(+)}(\tau_1 - \tau_2) - G_m^{(-)}(\tau_1 - \tau_2)\right),$$

$$\langle \hat{T}\left(\hat{\psi}_b(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(e)} = \tag{4.34}$$
$$= -\frac{i}{2N}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}\left(m+\frac{1}{2}\right)(b-a)}\sin(2\theta_m)\left(G_m^{(+)}(\tau_2 - \tau_1) - G_m^{(-)}(\tau_2 - \tau_1)\right).$$

Note that, by interchanging τ_1, τ_2 and a, b in (4.32) and comparing the result with (4.31) it can be seen that $\langle \hat{T}\left(\hat{\psi}_b(\tau_2)\hat{\psi}_a^\dagger(\tau_1)\right)\rangle_c^{(e)} = -\langle \hat{T}\left(\hat{\psi}_a^\dagger(\tau_1)\hat{\psi}_b(\tau_2)\right)\rangle_c^{(e)}$, which is anticipated due to the fermionic nature of the operators. Furthermore, it is easy to see that $\langle \hat{T}\left(\hat{\psi}_b(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(e)*} = \langle \hat{T}^\dagger\left(\hat{\psi}_a^\dagger(\tau_1)\hat{\psi}_b^\dagger(\tau_2)\right)\rangle_c^{(e)}$. In the equal time case, due to eq. (4.16), the correlation functions (4.31) and (4.32) receive an extra $+\frac{1}{2}\delta_{ab}$ contribution.

Due to the quadratic nature of the Hamiltonian at hand, these correlation functions contain all the information needed for the analysis of the system. As a first example the transverse magnetization $\langle\sigma^z\rangle$ of the even sector is presented, which is a site-independent quantity due to the translational invariance of the system

$$\langle\sigma^z\rangle^{(e)} = 2\frac{\delta^2 \ln Z^{(e)}[J]}{\delta\bar{J}_j(\tau)\delta J_j(\tau)}\bigg|_{J=0} = \frac{1}{N}\sum_{m=0}^{N-1}\cos(2\theta_m). \tag{4.35}$$

At the thermodynamic limit $N \to \infty$ this result reduces to the full expectation value of the operator, coinciding with the known [79] result

$$\langle\sigma^z\rangle = \frac{1}{\pi}\int_0^\pi d\phi \frac{|h-\cos\phi|}{\sqrt{(h-\cos\phi)^2+(r\sin\phi)^2}}, \tag{4.36}$$

which confirms that the correct functional operations representing the spin operators are those given in eq. (4.15). From the physical point of view, more interesting is the connected time dependent correlation function

$$\langle \hat{T}\left(\sigma_j^z(\tau_2)\sigma_k^z(\tau_1)\right)\rangle_c = 4\frac{\delta^4 \ln Z[J]}{\delta\bar{J}_j(\tau_2)\delta J_j(\tau_2)\delta\bar{J}_k(\tau_1)\delta J_k(\tau_1)}\bigg|_{J=0}, \tag{4.37}$$

where in order to find the real time result, one just needs to perform the Wick rotation $\tau \to it$. The calculation of the even sector contribution to the correlator (4.37) is then quite simple and yields the following result:

$$\langle \hat{T}\left(\sigma_{j+l}^z(t_2)\sigma_j^z(t_1)\right)\rangle_c^{(e)} = A_l(|t_2-t_1|) + B_l(|t_2-t_1|), \tag{4.38}$$

where in the last expression the following abbreviations are used

$$A_l(|t|) = \left\{ \frac{2}{N} \sum_{m=0}^{N-1} e^{-i|t|\epsilon_m} \cos^2\theta_m \cos\left(\phi_m l\right) \right\} \left\{ \frac{2}{N} \sum_{m=0}^{N-1} e^{-i|t|\epsilon_m} \sin^2\theta_m \cos\left(\phi_m l\right) \right\},$$

$$B_l(|t|) = \left\{ \frac{1}{N} \sum_{m=0}^{N-1} e^{-i|t|\epsilon_m} \sin 2\theta_m \sin\left(\phi_m l\right) \right\}^2, \quad \phi_m = \frac{2\pi}{N}\left(m + \frac{1}{2}\right) \tag{4.39}$$

By taking into account that $G_m^{(+)}(0) = -G_m^{(-)}(0) = 1/2$ and setting $r = 1$, for reasons of comparison, the static result is recovered

$$\langle \sigma_{j+l}^z \sigma_j^z \rangle_c^{(e)} = -\Sigma(l)\Sigma(-l), \quad \Sigma(l) = hR(l) - R(l+1), \tag{4.40}$$

with

$$R(l) = \frac{1}{N} \sum_{m=0}^{N-1} \frac{\cos(\phi_m l)}{\sqrt{(h - \cos\phi_m)^2 + \sin^2\phi_m}} \underset{N\to\infty}{=} \frac{1}{\pi} \int_0^\pi d\phi \frac{\cos(\phi l)}{\sqrt{(h - \cos\phi)^2 + \sin^2\phi}}, \tag{4.41}$$

which in the thermodynamic limit again represents the complete correlation function. Despite the fact that the result (4.40) has been known for a long time [79][16], it was re-derived here because it is a strong indication that the correct way to define path integration over fermionic coherent states is through the Faddeev-Jackiw method.

For a further check, at the thermodynamic limit $N \to \infty$, the functions (4.39) forming the correlator (4.38) adopt the following form

$$\begin{aligned} A_l(|t|) &= \frac{1}{\pi^2} \int_0^\pi d\phi \int_0^\pi d\phi' \frac{e^{-i|t|(\epsilon(\phi)+\epsilon(\phi'))}}{\epsilon(\phi)\epsilon(\phi')} \times \\ &\times \left[\epsilon(\phi) + 2(h - \cos\phi)\right] \left[\epsilon(\phi') - 2(h - \cos\phi')\right] \cos(\phi l)\cos(\phi' l), \\ B_l(|t|) &= \frac{4}{\pi^2} \int_0^\pi d\phi \int_0^\pi d\phi' \frac{e^{-i|t|(\epsilon(\phi)+\epsilon(\phi'))}}{\epsilon(\phi)\epsilon(\phi')} \sin\phi \sin\phi' \sin(\phi l)\sin(\phi' l). \end{aligned} \tag{4.42}$$

[16]The sign difference in h is due to the sign difference of the magnetic field used in the Hamiltonian of the cited paper.

The integrations in eq. (4.42) can then be reorganized as follows:

$$\begin{aligned} A_l(|t|) =& \frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\phi e^{-i|t|\epsilon(\phi)+il\phi}\left[1+\frac{2(h-\cos\phi)}{\epsilon(\phi)}\right]\times \\ &\times\int_{-\pi}^{\pi} d\phi' e^{-i|t|\epsilon(\phi')+il\phi'}\left[1-\frac{2(h-\cos\phi')}{\epsilon(\phi')}\right] \end{aligned} \tag{4.43}$$

and

$$B_l(|t|) = -\left[\frac{1}{\pi}\int_{-\pi}^{\pi} d\phi e^{-i|t|\epsilon(\phi)+il\phi}\frac{\sin\phi}{\epsilon(\phi)}\right]^2. \tag{4.44}$$

By using the continuum version of eq. (4.24): $\tan(2\theta(\phi)) = \sin\phi/(h-\cos\phi)$, it is easy to find that

$$\cos(2\theta(\phi)) = 2(h-\cos\phi)/\epsilon(\phi), \quad \sin(2\theta(\phi)) = 2\sin\phi/\epsilon(\phi) \tag{4.45}$$

and consequently

$$\begin{aligned} &\langle\hat{T}\left(\sigma_{j+l}^z(t_2)\sigma_j^z(t_1)\right)\rangle_c = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{il\phi-i|t|\epsilon(\phi)}\right]^2 - \\ &-\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{il\phi-i|t|\epsilon(\phi)}\cos(2\theta(\phi))\right]^2 - \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi e^{il\phi-i|t|\epsilon(\phi)}\sin(2\theta(\phi))\right]^2. \end{aligned} \tag{4.46}$$

This result coincides with the zero-temperature limit of the exact result that has been found by Th. Niemeijer [80]. Here, $\epsilon(\phi)$ is the continuum limit of the discrete energy ϵ_m

$$\epsilon(\phi) = 2\sqrt{(h-\cos\phi)^2+(r\sin\phi)^2}. \tag{4.47}$$

As it is obvious, the correlator (4.38) and consequently the entanglement, disappear at the limit $t\to\infty$ due to strong oscillations. However, this argument does not hold at the critical point, where $|h-\cos\phi| = \lambda << 1$ and $\phi\sim\lambda$. Thus, the time after which correlations are strongly diminished scales as $t\sim 1/\lambda$. Even though the result of eq. (4.38) was produced for the even fermion number sector, the corresponding result for the odd sector could be immediately recovered through the change $m+\frac{1}{2}\to m$. An important note is that while this method takes advantage of the sector separation presented in eq. (D.2) of appendix D, it remains valid only for operators that preserve the sector. For operators that do not have this property though this method should be modified, in order to include the possible sector interpolation (as for example in the case where the expectation value of the single σ_j^x operator is considered).

It is also easy to compute the real time correlator of the Majorana operators (4.5)

$$iB_{ba}(t_2,t_1) \equiv \langle \hat{T}\left(\hat{\gamma}_{2b}(t_2)\hat{\gamma}_{2a-1}(t_1)\right)\rangle_c \,. \tag{4.48}$$

Inserting the Wick rotated correlators (4.31)-(4.34) into eq. (4.48), one gets the even sector, real time correlator

$$B^{(e)}_{ba}(t_2,t_1) = -\frac{1}{N}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}\left(m+\frac{1}{2}\right)(b-a)-2i\theta_m} e^{-i|t_2-t_1|\epsilon_m}, \tag{4.49}$$

which at the thermodynamic limit $N \to \infty$ reduces to the complete correlation function

$$B_l(t) \equiv B_{ba}(t_2,t_1) = -\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{il\phi-2i\theta(\phi)} e^{-i|t|\epsilon(\phi)}, \quad l=b-a, \quad t=t_2-t_1. \tag{4.50}$$

In this expression, the function $\theta(\phi)$ is defined as

$$2\theta(\phi) = \begin{cases} \arctan\frac{r\sin\phi}{h-\cos\phi} & , \quad h-\cos\phi > 0, \\ \arctan\frac{r\sin\phi}{h-\cos\phi}+\pi & , \quad h-\cos\phi < 0. \end{cases} \tag{4.51}$$

For the XX model ($r=0$) and $|h| \leq 1$ the integral (4.50) simplifies to

$$\begin{aligned} B_l(t) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{il\phi}\text{sign}(\cos\phi - h) e^{-2i|t||\cos\phi-h|} = \\ &= \frac{1}{\pi}\int_{-\phi_h}^{\phi_h} d\phi e^{il\phi}\cos\left[2t(h-\cos\phi)\right], \end{aligned} \tag{4.52}$$

where ϕ_h is defined through the equation $cos\phi_h = h$.
The evaluation of this integral yields the result

$$B_l(t) = \frac{2}{\pi}\sum_{k=-\infty}^{\infty} J_k(2t)\frac{\sin\left[(l+k)\phi_h\right]}{l+k}\cos\left(2ht - k\frac{\pi}{2}\right), \tag{4.53}$$

with J_k being the Bessel functions. For the static case, one does not have to take into account eq. (4.16) and repeat all computations, since when identifying the weighted quantity corresponding to $\hat{T}\left(\hat{\gamma}_{2b}(t)\hat{\gamma}_{2a-1}(t)\right)$, through the Faddeev-Jackiw prescription, all extra $1/2$ contributions cancel, giving the function $\gamma_{2b}(t)\gamma_{2a-1}(t)$. Thus, the limit $t_1 = t_2$ of eq. (4.53) can be performed trivially, in which case only the $k=0$ term

survives, giving

$$B_{ba,eq}(t) = \frac{2}{\pi}\frac{\sin(l\phi_h)}{l}. \tag{4.54}$$

Here, the abbreviation used for the equal time correlation function is $B_{ba,eq}(t) \equiv B_{ba}(t,t)$. At the vicinity of the critical value $h = 1$ one can then find $\phi_h \simeq \sqrt{2\lambda}$ where $\lambda \equiv 1 - h \to +0$. This value sets the scale after which strong oscillations diminish correlations. Thus, the correlation length behaves as $\xi \sim \lambda^{-\frac{1}{2}} \sim \lambda^{-\nu}$ giving the corresponding critical exponent $\nu = 1/2$ [41]. As $|t| \to \infty$, the correlator (4.53) goes to zero as $B_l \sim |t|^{-1/2}$, except for the critical vicinity where it oscillates as

$$B_l(t) \simeq \frac{2}{\pi l}\sin(\phi_h l)\cos(2t\lambda). \tag{4.55}$$

The time scale after which correlations are turned off due to the strong oscillations can easily be seen from the last equation being: $t \sim \lambda^{-1} \sim \xi^2$. Consequently, the dynamic critical exponent defined through equation $t \sim \xi^z$ is $z = 2$ [42]. Therefore, the entanglement entropy is expected [12–14] to scale as $S \sim \frac{1}{3}\log\xi \sim \frac{1}{6}\log t$ at the critical vicinity.

4.3 Driven correlations

In this subsection the case of a time dependent transverse field $h = h(t)$ that drives the evolution of an XY spin chain will be considered. For the quantum Ising model ($r = 1$) and for a field linearly dependent on time, the entanglement dynamics have been extensively studied [15, 16, 81–83], mainly by numerical methods. The present work is a contribution to the analytical methods available for the study of spin systems out of equilibrium. The quantities of interest are of the form

$$\begin{aligned}\langle \hat{T}\left(\hat{O}_{1H}(t_1)\hat{O}_{2H}(t_2)\cdots\hat{O}_{NH}(t_N)\right)\rangle \equiv \\ \mathrm{tr}\left[\hat{\rho}_{in}\hat{T}\left(\hat{O}_{1H}(t_1)\hat{O}_{2H}(t_2)\cdots\hat{O}_{NH}(t_N)\right)\right],\end{aligned} \tag{4.56}$$

where $\hat{\rho}_{in} = \hat{\rho}(t_{in})$ is the initial density matrix of the system at time t_{in} and $\hat{O}_{nH}(t_n)$ is a Heisenberg operator at time t_n:

$$\hat{O}_{nH}(t_n) = \hat{U}(t_{in}, t_n)\hat{O}_n(t_{in})\hat{U}(t_n, t_{in}), \quad \hat{U}(t, t_{in}) \equiv \hat{T}e^{-i\int_{t_{in}}^{t} dt'\hat{H}(t')}. \tag{4.57}$$

In the following, the system is assumed to be initially in thermal equilibrium of temperature β

$$\hat{\rho}_{in} = \frac{e^{-\beta \hat{H}_{in}}}{Z_{in}(\beta)}, \tag{4.58}$$

where $Z_{in}(\beta)$ is the partition function with respect to $\hat{H}_{in}$. In this expression, the symbol $\hat{H}_{in}$ identifies the Hamiltonian operator at time t_{in}, where the magnetic field has its initial value $h_{in} = h(t_{in})$. By taking the zero temperature limit $\beta \to \infty$, and as long as the ground state is unique, the density matrix reduces to $\hat{\rho}_{in} = |0_{in}\rangle \langle 0_{in}|$. If the ground state though isn't unique, the zero-temperature limit produces an equiprobable mixture of degenerate states. In the following, the case of interest is that of two Heisenberg operators $\hat{O}_1 = \hat{\psi}_b^\dagger$ and $\hat{O}_2 = \hat{\psi}_a$, where the a and b indices indicate the fermionic modes.

To deal with the out-of-equilibrium dynamics of the system, the expectation value of the Heisenberg operator $\hat{\psi}_b^\dagger(t_2)\hat{\psi}_a(t_1)$ is re-expressed in the following form:

$$\begin{aligned} G_{ba}(\rho; t_2, t_1) &= \mathrm{tr}\left[\hat{\rho}(t_{in})\hat{\psi}_b^\dagger(t_2)\hat{\psi}_a(t_1)\right]\Theta(t_2 - t_1) - \mathrm{tr}\left[\hat{\rho}(t_{in})\hat{\psi}_a(t_1)\hat{\psi}_b^\dagger(t_2)\right]\Theta(t_1 - t_2) = \\ &= \mathrm{tr}\left[\hat{\rho}(t_{in})\hat{U}(t_{in}, t_2)\hat{\psi}_b^\dagger(t_{in})\hat{U}(t_2, t_1)\hat{\psi}_a(t_{in})\hat{U}(t_1, t_{in})\right]\Theta(t_2 - t_1) - \\ &- \mathrm{tr}\left[\hat{\rho}(t_{in})\hat{U}(t_{in}, t_1)\hat{\psi}_a(t_{in})\hat{U}(t_1, t_2)\hat{\psi}_b^\dagger(t_{in})\hat{U}(t_2, t_{in})\right]\Theta(t_1 - t_2), \end{aligned} \tag{4.59}$$

for $t_2 \neq t_1$, with Θ being the Heaviside Theta function. The $-$ symbol in the above equation is a consequence of the fermionic nature of the operators, while, as will be made clear in section 5, in the bosonic case it would be replaced by a $+$ symbol.

The forward-backward time structure of such correlators has been examined a long time ago through the Schwinger-Keldysh formalism [3,45–47] presented in subsection 2.10. The basic ingredient of this formalism is that the time variable is defined on the complex plane along a time-closed contour P, which encircles the real t axis, running from $t_{in+} \equiv t_{in} + i0$ to $t_{in-} \equiv t_{in} - i0$. The contour consists of two straight lines. The first one, denoted as L_+, joins the points $t_{in} + i0$ and $T + i0$, where T is an arbitrary time instance, up to which the evolution of the system is considered. Along this line the time variable is denoted as t_+. The second line joins the points $T - i0$ and $t_{in} - i0$. Along this line, defined as L_-, time is denoted as t_-. In the case of thermal initial

states this contour can be further extended by a complex time line, running parallel to the imaginary axis, from t_{in-} to $t_{in-} - i\beta$, where β is the inverse temperature of the corresponding thermal state. This third line on the complex time plane is denoted as L_β, while the extended contour as C. A natural ordering arises in this configuration (as did already in P) since times along L_+ are before times along L_-, which also are considered to be before times along L_β (see Fig. (1)).

The introduction of the resolution of the identity in the space of coherent states $|\vec{\zeta}\rangle$, between the operators in eq. (4.59), leads to the re-writing of the correlation function (4.59) as a series of matrix elements

$$G_{ba}(\rho; t_2, t_1) = \int d^2\vec{\zeta} d^2\vec{\zeta}' \, \langle -\vec{\zeta}|\hat{\rho}(t_{in})|\vec{\zeta}'\rangle \, F_{ba}\left(\vec{\zeta}', \vec{\zeta}; t_2, t_1\right), \tag{4.60}$$

where

$$\begin{aligned} F_{ba}\left(\vec{\zeta}', \vec{\zeta}; t_2, t_1\right) = \langle\vec{\zeta}'|\hat{U}(t_{in}, t_2)\hat{\psi}_b^\dagger(t_{in})\hat{U}(t_2, t_1)\hat{\psi}_a(t_{in})\hat{U}(t_1, t_{in})|\vec{\zeta}\rangle \, \Theta(t_2 - t_1) - \\ - \langle\vec{\zeta}'|\hat{U}(t_{in}, t_1)\hat{\psi}_a(t_{in})\hat{U}(t_1, t_2)\hat{\psi}_b^\dagger(t_{in})\hat{U}(t_2, t_{in})|\vec{\zeta}\rangle \, \Theta(t_1 - t_2), \end{aligned} \tag{4.61}$$

which can be handled through the path integral method according to the rules presented in section 4.1. Here, the vector symbol denotes the set of all degrees of freedom for the system. Leaving the details for appendix E - where both the cases of bosons and fermions are considered - the path integral form of the aforementioned correlation function is found to be

$$G_{ba}(\rho; t_2, t_1) = \frac{1}{Z_{in}(\beta)} \int\limits_{(-)} \mathcal{D}^2\vec{\zeta}(t_C) e^{i\int_C dt_C \left[i\vec{\bar{\zeta}}\dot{\vec{\zeta}} - H_{XYcl}(\vec{\bar{\zeta}}, \vec{\zeta})\right]} \bar{\zeta}_b(t_{2+})\zeta_a(t_{1+}), \tag{4.62}$$

which allows for the definition of the Keldysh contour generating functional for the out-of-equilibrium expectation values

$$Z[J] = \frac{1}{Z_{in}(\beta)} \int\limits_{(-)} \mathcal{D}^2\vec{\zeta}(t_C) e^{i\int_C dt_C \left[i\vec{\bar{\zeta}}\dot{\vec{\zeta}} - H_{XYcl}(\vec{\bar{\zeta}}, \vec{\zeta})\right] - i\int_C dt_C \vec{J}\vec{\zeta} - i\int_C dt_C \vec{\bar{\zeta}}\vec{J}}. \tag{4.63}$$

This integral has been constructed in exactly the same way as the corresponding imaginary time generating functional presented in section 4.1. In the present case, the time variable runs along the Keldysh contour C, where the fields $\vec{\zeta}(t_\pm) \equiv \vec{\zeta}_\pm(t)$ are treated as independent. The magnetic field is considered to be the same along $L_\pm$:

$h_+(t) = h_-(t)$, while along L_β it retains its initial value h_{in}. This way, correlation functions can be retrieved via functional differentiation of eq. (4.63). Without loss of generality, for the rest of this section it is considered that $t_{in} = -T$.

Apart from the complications of the Schwinger-Keldysh formalism, the calculation proceeds by following the same steps as in the imaginary time case. It must be noted that once again the study will be performed on the even fermion number sector of the system. The basic calculational difficulty then resides in the fact that both the eigenvalues ϵ_m and the matrix U_m, that diagonalizes the Hamiltonian in eq. (4.18), become time dependent quantities, due to the time dependence of the magnetic field. As a consequence, the extra off-diagonal contribution along the P contour

$$\begin{gathered} U_m^\dagger i\partial_{t_P} U_m = -\dot{\theta}_m \sigma^x, \\ \dot{\theta}_m = -\dot{h}\frac{2r}{\epsilon_m^2}\sin\left[\frac{2\pi}{N}\left(m+\frac{1}{2}\right)\right] = -\dot{h}\frac{2r}{\epsilon_m^2}\sin\phi_m \quad (r \neq 0) \end{gathered} \tag{4.64}$$

appears in the kinematic term of the Keldysh analogue of (4.26), a fact that makes the calculation more involved. However, due to the quadratic nature of the action, the calculation can be carried out.

The generating functional in eq. (4.20) is then transcribed to the Keldysh language as

$$Z^{(e)}[J] = \prod_{m=0}^{N/2-1} Z_m^{(e)}[J], \quad Z_m^{(e)}[J] = \int\limits_{(-)} \mathcal{D}\bar{\eta}_m \mathcal{D}\eta_m e^{iS_m^{(e)}[J]}, \tag{4.65}$$

with

$$S_m^{(e)}[J] = \int_C dt_C \left[\bar{\eta}_m \left(D_m - \dot{\theta}_m \sigma^x\right)\eta_m - \bar{\mu}_m \eta_m - \bar{\eta}_m \mu_m\right]. \tag{4.66}$$

Here, the operator D_m is the real time version of the corresponding operator in eq. (4.27)

$$D_m = \begin{pmatrix} i\partial_{t_C} - \epsilon_m(t_C) & 0 \\ 0 & i\partial_{t_C} + \epsilon_m(t_C) \end{pmatrix}. \tag{4.67}$$

The generating functional can be calculated exactly, according to the Keldysh contour algebra rules presented in appendix F, and the result reads as follows

$$Z_m^{(e)}[J] = Z_m^{(e)}[0]\exp\left[-i\int_C dt_C \int_C dt'_C \bar{\mu}_m(t_C)\tilde{G}_m(t_C, t'_C)\mu_m(t'_C)\right], \tag{4.68}$$

where the Green's function $\tilde{G}_m$ is defined as the antiperiodic solution of the Green's equation

$$\left(D_m - \dot{\theta}_m \sigma^x\right) \tilde{G}_m = \hat{\mathbb{1}}, \tag{4.69}$$

along the contour C. The solution to this equation and the computation of the normalization factor of eq. (4.68) are presented in appendix G.

Regarding $\tilde{G}_m$, it is shown that it can be expressed as

$$\tilde{G}_m = G_m + M_m, \tag{4.70}$$

where M_m is a convergent series in $\dot{\theta}_m$, which can be systematically computed order by order

$$M_m = G_m \dot{\theta}_m \sigma^x G_m + \mathcal{O}\left(\dot{\theta}_m^2\right). \tag{4.71}$$

Here, G_m is the antiperiodic Green's function satisfying the equation $D_m G_m = \hat{\mathbb{1}}$ over C, the solution to which can be found in appendix F. Now, all the needed correlation functions can be deduced through the corresponding functional differentiations, as for example is the case of

$$\langle \hat{T}\left(\hat{\psi}_b^\dagger(t_2)\hat{\psi}_a(t_1)\right)\rangle_c = \left.\frac{\delta^2 Z[J]}{\delta J_b(t_{2+})\delta \bar{J}_a(t_{1+})}\right|_{J=0}, \tag{4.72}$$

which, as noted in section 4.1, receives an additive factor in the limit where $t_2 = t_1$ and $b = a$ simultaneously. In the following, the equal time case will be considered. The contribution of the even fermion number sector to the equal time correlator, up to the first order in $\dot{\theta}_m$, is then found to be

$$\langle \hat{\psi}_b^\dagger(t)\hat{\psi}_a(t)\rangle_c^{(e)} = -\frac{i}{2N}\sum_{m=0}^{N-1} e^{-i\frac{2\pi}{N}\left(m+\frac{1}{2}\right)(b-a)} \times \tag{4.73}$$

$$\times \operatorname{tr}\left(\left[(\mathbb{1} + G_m\dot{\theta}_m\sigma^x)G_m\right](t,t)\left[\cos(2\theta_m(t))\sigma^z - \sin(2\theta_m(t))\sigma^y\right]\right) + \frac{1}{2}\delta_{ba} + \mathcal{O}(\dot{\theta}_m^2).$$

The case of a magnetic field $h = h(\omega t)$ was further examined in appendix G, in the limit $T \to \infty$ and $\beta \to \infty$, where it was proved that at the adiabatic limit, the contribution of $M_m = G_m\dot{\theta}_m\sigma^x G_m + \mathcal{O}(\dot{\theta}_m^2)$ in eq. (4.73) is strongly suppressed as $\omega \to 0$. In

this case, the dominant term in eq. (4.73) is the one coming from G_m and thus

$$\langle\hat{\psi}_b^\dagger(t)\hat{\psi}_a(t)\rangle_c^{(e)} \overset{\omega\to 0}{\simeq} -\frac{i}{2N}\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}\left(m+\frac{1}{2}\right)(b-a)}\cos\left(2\theta_m(t)\right)\mathrm{tr}\left[G_m(t,t)\sigma^z\right]+\frac{1}{2}\delta_{ba} \underset{\beta\to\infty}{=}$$

$$\underset{\beta\to\infty}{=} -\frac{1}{2N}\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}\left(m+\frac{1}{2}\right)(b-a)}\cos\left(2\theta_m(t)\right)+\frac{1}{2}\delta_{ba}, \tag{4.74}$$

where the property $G^{(-)}(t_C,t'_C)=-G^{(+)}(t'_C,t_C)$ was used. From this result it is easy to also compute the even sector expectation value of the Heisenberg operator $\sigma_i^z(t)$, as $\langle\sigma_i^z(t)\rangle_c = 1-2\langle\hat{\psi}_i^\dagger(t)\hat{\psi}_i(t)\rangle_c$. A comparison with the results obtained for a time-independent field in eqs. (4.32) and (4.35), leads to the conclusion that at the adiabatic limit the main change in the correlation function is the replacement $\theta_m\to\theta_m(t)$. However, as proved in appendix G, this statement holds only away from the critical region $|h-1|\sim\mathcal{O}(\omega^{1/2})$. In that neighbourhood, irrespective of how slow the driving is, the contribution of M_m in eq. (4.73) becomes equally important to that of G_m. This is in accordance with the so-called Kibble-Zurek mechanism (KZM) or the adiabatic-impulse-adiabatic approximation [42,43], presented in subsection 2.8. It is easy to see then that, even if the evolution of the system is considered adiabatic away from the critical point, when near that point the evolution becomes non-adiabatic, as the energy gap changes with a rate comparable to the energy gap itself: $|\dot{h}|/|h-1|\sim\mathcal{O}(\omega^{1/2})$. In such a case, the entanglement entropy is expected [16] to behave as $S\simeq\frac{1}{12}\log_2\frac{1}{\omega}$.

5 Reduced dynamics for bosonic and fermionic systems

Section summary

In this section, a path integral method for the study of dynamics of bosonic and fermionic systems embedded in thermal quantum environments will be presented. The methodology will be presented simultaneously for the bosonic and fermionic cases, since the differences in the context of path integrals are remarkably subtle. It must be noted that an analogous methodology to the fermionic one can be also used for the study of spin-1/2 systems, which relate to fermionic systems via the Jordan-Wigner transformation [31]. This section is based on the upcoming work [84].

5.1 The Reduced Generating Functional

In order to study the reduced dynamics of a system S, coupled to a thermal environment E, the compound system will be considered to be driven by the usual quantum Hamiltonian

$$\hat{H} = \hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{H}_I(t). \tag{5.1}$$

In the following, it will be taken as granted that all the parts of the total Hamiltonian are expressed in terms of the corresponding creation and annihilation (bosonic or fermionic) operators: $\hat{H}_S = \hat{H}_S(\hat{a}^\dagger, \hat{a})$, $\hat{H}_E = \hat{H}_E(\hat{b}^\dagger, \hat{b})$, $\hat{H}_I = \hat{H}_I(\hat{a}^\dagger, \hat{a}; \hat{b}^\dagger, \hat{b})$. Here, the notation $\hat{a}$ and $\hat{a}^\dagger$ is used to denote the operators of the system, and the notation $\hat{b}$ and $\hat{b}^\dagger$ to denote the operators of the environment.
For the analysis, the following physical picture shall be adopted: Up to a moment t_{in} the environment and the system are independent of each other $\left(\hat{H}_I(t < t_{in}) = 0\right)$ and the initial density operator can be expressed in the product form $\hat{\rho} = \hat{\rho}_E \otimes \hat{\rho}_S$. Furthermore, prior to the moment t_{in}, the environment will be assumed to be in equilibrium at a temperature $T = 1/\beta$, meaning that $\hat{\rho}_E = e^{-\beta \hat{H}_E}/Z_E(\beta)$, with $Z_E(\beta)$ being the corresponding partition function. No prior information is considered regarding the system's initial state, but this specific topic will play an important role later.

After the initialization of the interaction, the two interacting systems become entangled and the evolution of the subsystem is in general highly non-trivial. To probe it, the objects of interest are again going to be averages of the form:

$$G_{ij}^{(T)}(\rho; t_2, t_1) = \langle \hat{T}\left(\hat{a}_{iS}^\dagger(t_2)\hat{a}_{jS}(t_1)\right)\rangle_\rho = \mathrm{tr}\left[\hat{\rho}(t_{in})\hat{T}\left(\hat{a}_{iS}^\dagger(t_2)\hat{a}_{jS}(t_1)\right)\right]. \tag{5.2}$$

In this expression $\hat{a}_{iS}(t)$ is a Heisenberg operator that refers to the subsystem at a specific time t, the subscript i is a site/space or mode index, $\hat{\rho}(t_{in}) = \hat{\rho}_{in}$ is the initial density operator of the compound system, $\hat{T}$ is the time ordering operator and the trace operation refers to both the environment's and the subsystem's degrees of freedom. In this subsection the formalism will be kept as general as possible, in order to present the structure of the method, leaving for appendices E, H and I the details of the calculations, and skipping for the next subsection the discussion of the examples of the bosonic and the fermionic harmonic oscillators. The generalization to n point correlators is, at least in principle, straightforward.

By writing the Heisenberg operators in eq. (5.2) as

$$\hat{a}_{iS}(t) = \left[\hat{T}^{\dagger} e^{i\int_{t_{in}}^{t} dt' \hat{H}(t')}\right] \hat{a}_{iS}(t_{in}) \left[\hat{T} e^{-i\int_{t_{in}}^{t} dt' \hat{H}(t')}\right] \equiv \hat{U}(t_{in}, t)\hat{a}_{iS}\hat{U}(t, t_{in}), \tag{5.3}$$

where $\hat{T}^{\dagger}$ is the anti-time ordering operator, it is easy to proceed exactly as in the case of subsection 4.3 and express the correlator (5.2) in the following form:

$$\begin{aligned} G_{ij}^{(T)}(\rho; t_2, t_1) &= \mathrm{Tr}\left[\hat{\rho}(t_{in})\hat{a}_{iS}^{\dagger}(t_2)\hat{a}_{jS}(t_1)\right]\Theta(t_2 - t_1) \pm \mathrm{Tr}\left[\hat{\rho}(t_{in})\hat{a}_{jS}(t_1)\hat{a}_{iS}^{\dagger}(t_2)\right]\Theta(t_1 - t_2) = \\ &= \mathrm{Tr}\left[\hat{\rho}(t_{in})\hat{U}(t_{in}, t_2)\hat{a}_{iS}^{\dagger}(t_{in})\hat{U}(t_2, t_1)\hat{a}_{jS}(t_{in})\hat{U}(t_1, t_{in})\right]\Theta(t_2 - t_1) \pm \\ &\pm \mathrm{Tr}\left[\hat{\rho}(t_{in})\hat{U}(t_{in}, t_1)\hat{a}_{jS}(t_{in})\hat{U}(t_1, t_2)\hat{a}_{iS}^{\dagger}(t_{in})\hat{U}(t_2, t_{in})\right]\Theta(t_1 - t_2), \end{aligned} \tag{5.4}$$

for $t_2 \neq t_1$, with Θ being the Heaviside Theta function. In the $\pm$ symbol appearing above and in what follows, the upper symbol corresponds to the bosons and the lower to fermions. Once again, the Keldysh contour parametrization will be used to construct the appropriate path integral.

To analyse correlators of the type (5.4) it is possible again to express them as path integrals over the space spanned by the bosonic or fermionic coherent states. In the following, the notation $|\vec{z}_S\rangle \equiv |z_1, \ldots\rangle = \otimes_{j\in S} |z_j\rangle$ and $|\vec{\zeta}_E\rangle \equiv |\zeta_1, \ldots\rangle = \otimes_{\mu\in E} |\zeta_\mu\rangle$ will be used for the overcomplete bases pertaining to the subsystem and the environment respectively. For the composite system the notation $|\vec{Z}\rangle \equiv |\vec{z}_S, \vec{\zeta}_E\rangle = |\vec{z}_S\rangle \otimes |\vec{\zeta}_E\rangle$ will be used, while the completeness will be expressed in the abbreviated form:

$$\int d^2\vec{Z}\, |\vec{Z}\rangle \langle\vec{Z}| = \prod_{j\in S} \int \frac{d\bar{z}_j dz_j}{\pi} |z_j\rangle \langle z_j| \otimes \prod_{\mu\in E} \int \frac{d\bar{\zeta}_\mu d\zeta_\mu}{\pi} |\zeta_\mu\rangle \langle\zeta_\mu| = \hat{\mathbb{1}}. \tag{5.5}$$

with $d^2z \equiv dRe(z)dIm(z)$ for z of bosonic nature and $d^2z \equiv d\bar{z}dz$ for z of Grassmann one. In this basis, the correlator (5.4) can be written as

$$G_{ij}(\rho; t_2, t_1) = \int d^2\vec{Z} d^2\vec{Z}'\, \langle \pm\vec{Z}|\hat{\rho}(t_{in})|\vec{Z}'\rangle\, F_{ij}\left(\vec{Z}', \vec{Z}; t_2, t_1\right), \tag{5.6}$$

where

$$F_{ij}\left(\vec{Z}',\vec{Z};t_2,t_1\right)=\langle\vec{Z}'|\hat{U}(t_{in},t_2)\hat{a}^{\dagger}_{iS}(t_{in})\hat{U}(t_2,t_1)\hat{a}_{jS}(t_{in})\hat{U}(t_1,t_{in})|\vec{Z}\rangle\,\Theta(t_2-t_1)\pm \\ \pm\langle\vec{Z}'|\hat{U}(t_{in},t_1)\hat{a}_{jS}(t_{in})\hat{U}(t_1,t_2)\hat{a}^{\dagger}_{iS}(t_{in})\hat{U}(t_2,t_{in})|\vec{Z}\rangle\,\Theta(t_1-t_2). \tag{5.7}$$

In appendix E it was proved that the last amplitude can be interpreted as a path integral over the Keldysh contour P (see fig. (1)) as follows:

$$F_{ij}(\vec{Z}',\vec{Z};t_2,t_1)= \tag{5.8}$$
$$=\int_{\vec{Z}_P(t_{in+})=\vec{Z}}^{\vec{Z}_P(t_{in-})=\vec{Z}'}\mathcal{D}^2\vec{Z}_P(t_P)e^{-\Gamma_{in-,in+}(\vec{Z}',\vec{Z})+i\int_P dt_P\left[\frac{i}{2}(\vec{Z}_P\cdot\dot{\vec{Z}}_P-\dot{\vec{Z}}_P\cdot\vec{Z}_P)-H(\vec{Z}_P,\vec{Z}_P)\right]}\bar{z}_{Pi}(t_{2+})z_{Pj}(t_{1+}).$$

The time variable t_P, that parametrizes the paths under integration, is defined along the P contour, at the end of which the boundary conditions are set. The factor in the last integral is defined as

$$\Gamma_{in-,in+}(\vec{\bar{Z}}',\vec{Z};\vec{\bar{Z}}_P,\vec{Z}_P)=\frac{1}{2}\left(|\vec{Z}'|^2+|\vec{Z}|^2\right)-\frac{1}{2}\left(\vec{\bar{Z}}'\cdot\vec{Z}_P(t_{in-})+\vec{\bar{Z}}_P(t_{in+})\cdot\vec{Z}\right)= \\ =\Gamma^{S}_{in-,in+}(\vec{\bar{z}}',\vec{z};\vec{\bar{z}}_P,\vec{z}_P)+\Gamma^{E}_{in-,in+}(\vec{\bar{\zeta}}',\vec{\zeta};\vec{\bar{\zeta}}_P,\vec{\zeta}_P), \tag{5.9}$$

and corresponds to the usual boundary term, used for the consistent calculation of coherent-state path integrals, with the two terms of the second line referring to the contribution of the subsystem and the environment respectively. In appendix E the result (5.8) was further modified, for the case where $\vec{Z}$ corresponded to an isolated thermal system, in order to represent an evolution not only along the contour P, but also extended on a line L_β parallel to the imaginary axis. In the present case though, only the environment will be considered being initially in a thermal state, and thus this extension is impossible for all degrees of freedom.

By inserting eq. (5.8) into eq. (5.6), the latter can be recasted as follows,

$$G_{ij}^{(T)}(\rho;t_2,t_1) = \int d^2z d^2z' \,\langle \pm\vec{z}|\hat{\rho}_S(t_{in})|\vec{z}'\rangle \times$$
$$\times \int\limits_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{z}_P(t_{in-})=\vec{z}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{z}',\vec{z};\vec{z}_P,\vec{z}_P)+i\int_P dt_P\left[\frac{i}{2}(\vec{z}_P\cdot\dot{\vec{z}}_P-\dot{\vec{z}}_P\cdot\vec{z}_P)-H_S(\vec{z}_P,\vec{z}_P)\right]} \times \tag{5.10}$$
$$\times\, \bar{z}_{Pi}(t_{2+}) z_{Pj}(t_{1+}) I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}_P(t_P)\right] \equiv G_{ij,++}(\rho;t_2,t_1),$$

where the last factor

$$I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}_P(t_P)\right] = \int d^2\zeta d^2\zeta' \,\langle \pm\vec{\zeta}|\hat{\rho}_E(t_{in})|\vec{\zeta}'\rangle \times \tag{5.11}$$
$$\times \int\limits_{\zeta_P(t_{in+})=\vec{\zeta}}^{\vec{\bar{\zeta}}_P(t_{in-})=\vec{\zeta}'} \mathcal{D}^2\vec{\zeta}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{\zeta}',\vec{\zeta};\vec{\bar{\zeta}}_P,\vec{\zeta}_P)+i\int_P dt_P\left[\frac{i}{2}(\vec{\bar{\zeta}}_P\cdot\dot{\vec{\zeta}}_P-\dot{\vec{\bar{\zeta}}}_P\cdot\vec{\zeta}_P)-H_E(\vec{\bar{\zeta}}_P,\vec{\zeta}_P)-H_I(\vec{\bar{Z}}_P,\vec{Z}_P)\right]}$$

corresponds to the influence functional of the Feynman-Vernon's technique and incorporates the influence of the environment on the correlation function. By following the same steps, similar expressions can be derived for all correlation functions of the subsystem. Thus, it is useful to define the reduced generating functional

$$Z_S[\vec{J},\vec{\bar{J}}] = \int d^2z d^2z' \,\langle \pm\vec{z}|\hat{\rho}_S(t_{in})|\vec{z}'\rangle \times \tag{5.12}$$
$$\times \int\limits_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{z}_P(t_{in-})=\vec{z}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{z}',\vec{z};\vec{\bar{z}}_P,\vec{z}_P)+i\int_P dt_P\left[\frac{i}{2}(\vec{\bar{z}}_P\cdot\dot{\vec{z}}_P-\dot{\vec{\bar{z}}}_P\cdot\vec{z}_P)-H_S(\vec{\bar{z}}_P,\vec{z}_P)\right]-i\int_P dt_P\left(\vec{J}\cdot\vec{z}_P+\vec{\bar{z}}_P\cdot\vec{J}\right)} \times$$
$$\times\, I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}_P(t_P)\right],$$

with the help of which all correlation functions of the system, $G_{ij,\pm\pm}$, can be derived through the appropriate functional differentiations. Note that the fields $\vec{z}_P(t_+) \equiv \vec{z}_+(t)$ and $\vec{z}_P(t_-) \equiv \vec{z}_-(t)$ are independent fields and must be integrated separately. The r.h.s. notation for these fields is meant to signify the complex-time line (L_+ or L_- respectively) on which they are supported on.

The matrix element of the thermal environment's density matrix, appearing in eq. (5.11), can again be expressed as a path integral with respect to fields parametrized

along the lower imaginary axis, running from t_{in-} to $t_{in-} - i\beta$, as

$$\langle \pm\vec{\zeta}|\hat{\rho}_E(t_{in})|\vec{\zeta}'\rangle = \tag{5.13}$$
$$= \frac{1}{Z_E(\beta)} \int_{\vec{\zeta}(t_{in-})=\vec{\zeta}'}^{\vec{\zeta}(t_{in-}-i\beta)=\pm\vec{\zeta}} \mathcal{D}^2\vec{\zeta}_{im}(t_{im}) e^{-\Gamma_{\beta,in-}(\vec{\bar{\zeta}},\zeta',\vec{\bar{\zeta}}_{im},\vec{\zeta}_{im}) + i\int_{t_{in-}}^{t_{in-}-i\beta} dt_{im}\left[\frac{i}{2}(\vec{\bar{\zeta}}_{im}\cdot\dot{\vec{\zeta}}_{im} - \dot{\vec{\bar{\zeta}}}_{im}\cdot\vec{\zeta}_{im}) - H_E(\vec{\bar{\zeta}}_{im},\vec{\zeta}_{im})\right]},$$

where

$$\Gamma_{\beta,in-}(\vec{\bar{\zeta}},\zeta',\vec{\bar{\zeta}}_{im},\vec{\zeta}_{im}) = \frac{1}{2}\left(|\vec{\zeta}|^2 + |\vec{\zeta}'|^2\right) - \frac{1}{2}\left(\vec{\bar{\zeta}}\cdot\vec{\zeta}_{im}(t_{in}-i\beta) + \vec{\bar{\zeta}}_{im}(t_{in-})\cdot\vec{\zeta}'\right). \tag{5.14}$$

Combining the expressions (5.11) and (5.13) one then arrives at a path integral parametrized along the Keldysh contour C (see Fig. (1))

$$I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}_P(t_P)\right] = \tag{5.15}$$
$$= \frac{1}{Z_E(\beta)} \int_{(\pm)} \mathcal{D}^2\vec{\zeta}_C(t_C) e^{i\int_C dt_C\left[i\vec{\bar{\zeta}}_C\cdot\dot{\vec{\zeta}}_C - H_E(\vec{\bar{\zeta}}_C,\vec{\zeta}_C)\right] - i\int_P dt_P H_I(\vec{\bar{Z}}_P,\vec{Z}_P)}.$$

The subsript $(\pm)$ on the path integral denotes the periodic or antiperiodic boundary conditions, induced by the trace operation for bosonic and fermionic systems respectively. Regarding the integral over P alone, which appears in the exponential of the previous equation, the symbol $\vec{Z}_P$ has been used (and subsequently $\vec{z}_P$ and $\vec{\zeta}_P$) for the fields. This is just a choice of notation, in order to show that the degrees of freedom contained in the last integral of the exponential are only those over P, and thus it is considered $\zeta_C(t_P) \equiv \zeta_P(t_P)$. Path integration of eq. (5.15) is thus performed with respect to $\vec{\zeta}_C$, which is parametrized along the whole Keldysh contour C. This will also be the case for the fields of the subsystem S, in the case that its initial state is considered thermal with some temperature β_S.

Regarding the matrix element $\langle \pm\vec{z}|\hat{\rho}_S(t_{in})|\vec{z}'\rangle$ appearing in eqs. (5.10) and (5.12), and considering the case where the system was not initially in a thermal state[17], this matrix

[17]Such a thermal case allows for this matrix element to be represented as an imaginary time evolution. In that case, this term together with the path integral over P can be sewed, in order to provide a periodic/anti-periodic path integral on a complete Keldysh contour C_S

$$Z_S[\vec{J},\vec{\bar{J}}] \sim \int_{(\pm)} \mathcal{D}^2\vec{z}_C(t_C) e^{i\int_{C_S} dt_C\left[i\vec{\bar{z}}_C\cdot\dot{\vec{z}}_C - H_S(\vec{\bar{z}}_C,\vec{z}_C)\right] - i\int_P dt_P\left(\vec{\bar{J}}\cdot\vec{z}_P + \vec{\bar{z}}_P\cdot\vec{J}\right)} I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}(t_P)\right],$$

element is bound to provide a function $f(\pm\vec{z},\vec{z}') = \langle \pm\vec{z}|\hat{\rho}_S(t_{in})|\vec{z}'\rangle$ of $\vec{z}$ and $\vec{z}'$. This results to the reduced generating functional simplifying to

$$Z_S[\vec{J},\vec{\bar{J}}] = \int d^2z d^2z' f(\pm\vec{z},\vec{z}')\times \tag{5.16}$$

$$\times \int\limits_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{z}_P(t_{in-})=\pm\vec{z}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{z}',\vec{z};\vec{z}_P,\vec{z}_P)+i\int_P dt_P\left[\frac{i}{2}(\vec{z}_P\cdot\dot{\vec{z}}_P-\dot{\vec{z}}_P\cdot\vec{z}_P)-H_S(\vec{z}_P,\vec{z}_P)\right]-i\int_P dt_P\left(\vec{\bar{J}}\cdot\vec{z}_P+\vec{z}_P\cdot\vec{J}\right)}$$

$$\times I_E\left[\vec{z}_P(t_P),\vec{z}(t_P)\right],$$

which represents a path integral over fields parametrized only over P. A case which can enormously simplify calculations is when the initial state of the system is considered pure and of the form $\prod_{j\in S}\otimes|0_j\rangle$, for which $f(\pm\vec{z},\vec{z}') = 1$. In that case, (5.16) becomes a path integral over the whole complex phase-space, from the initial Keldysh time t_{in+} to the final instance t_{in-}. This property will be used extensively in the following, while the methodology for recovering results for other initial states will be addressed in detail.

Up to this point, the analysis has been completely general, except for the product form that has been adopted for the initial density operator. In general, a closed-form result for the influence functional (5.15) is not possible. However, the embodiment of the environment in a path integral with periodic/anti-periodic boundary conditions offers a variety of techniques for a systematic approximation of the environment's influence on the system. In the present investigation, the environment and its interaction with the system will be considered to correspond to a solvable model. More specifically, the environment will be assumed to be simulated by a collection of bosonic/fermionic oscillators

$$\hat{H}_E = \sum_{\mu\in E} E_\mu\left(\hat{b}_\mu^\dagger\hat{b}_\mu \pm \frac{1}{2}\right). \tag{5.17}$$

with the operators $\hat{b}_\mu$ and $\hat{b}_\mu^\dagger$ denoting the annihilation and creation operators respectively, of the environment modes. The methodology would be similar to the presented one for any other choice of quadratic Hamiltonian corresponding to the environment. In the case that higher order Hamiltonians are considered, for which though a consistent continuum limit must first be identified, the same methodology could be gener-

similar to the path integral pertaining to the environment in eq. (5.15).

alized at least at the semiclassical level. The interaction term $\hat{H}_I$ is chosen to be of the linear form

$$\hat{H}_I = \sum_{j\in S,\mu\in E} \left(g_{j\mu}\hat{b}^\dagger_\mu \hat{a}_j + g^*_{j\mu}\hat{a}^\dagger_j \hat{b}_\mu \right). \tag{5.18}$$

The influence functional (5.15) for the bosonic and fermionic Hamiltonians presented above can then be parametrized as

$$\begin{aligned} &I_E\left[\vec{\bar{z}}_P(t_P), \vec{z}_P(t_P)\right] = \\ &\quad = \frac{1}{Z_E(\beta)} \prod_{\mu\in E} \int_{(\pm)} \mathcal{D}^2\zeta_{C\mu}(t_C) e^{i\int_C dt_C \left[i\bar{\zeta}_{C\mu}\cdot\dot{\zeta}_{C\mu} - E_\mu \bar{\zeta}_{C\mu}\zeta_{C\mu} - \bar{r}_\mu \zeta_{C\mu} - \bar{\zeta}_{C\mu} r_\mu \right]}, \end{aligned} \tag{5.19}$$

where the following abbreviation has been used for the interaction term $r_\mu(t_C) = \sum_{j\in S} z_{Cj}(t_C) g_{j\mu}(t_C)$, with the understanding that $g_{j\mu}(t_C) = 0$ along the imaginary part L_β of the Keldysh contour. In the last equation, the classical function $H_E = \sum_{\mu\in E} E_\mu \bar{\zeta}_\mu \zeta_\mu$, which represents the quantum Hamiltonian (5.17) at the level of path integration, has been constructed according to the rules derived in sections 3 and 4, for the bosonic and fermionic cases respectively. Due to its quadratic structure, the functional integral (5.19) can be calculated exactly. Leaving the details of the calculation for appendix H, here it is enough to quote the result

$$\begin{aligned} &I_E\left[\vec{\bar{z}}_P(t_P), \vec{z}_P(t_P)\right] = \\ &\quad = \tilde{C}_E^{-1} \exp\left[-\int_P dt_P \int_P dt'_P \sum_{j,k\in S} \bar{z}_{Pj}(t_P) \Delta_{P,jk}(t_P - t'_P) z_{Pk}(t'_P) \right], \end{aligned} \tag{5.20}$$

with the kernel $\Delta_{P,jk}(t_P - t'_P)$ being non-local in both time and space indices

$$\Delta_{P,jk}(t_P - t'_P) = \sum_{\mu\in E} g^*_{j\mu} g_{k\mu} \left[\Theta_P(t_P - t'_P) - \left(1 \mp e^{\beta E_\mu}\right)^{-1} \right] e^{-i(t_P - t'_P)E_\mu} \tag{5.21}$$

and $\Theta_P(t_P - t'_P)$ being the step function defined with respect to the Keldysh contour P. The normalization constant $\tilde{C}_E^{-1} = C_E^{-1} Z_E^{-1}(\beta)$ is also computed in appendix H.

Thus, the generating functional (5.16) for the reduced correlators can be written in

the following form

$$Z_S[\vec{J},\vec{\bar{J}}] = \tilde{C}_E^{-1} \int d^2z d^2z' f(\pm\vec{z},\vec{z}') \int_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{z}_P(t_{in-})=\vec{z}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{z}',\vec{z};\vec{z}_P,\vec{z}_P)+iS(\vec{z}_P,\vec{z}_P)}, \quad (5.22)$$

where the action weighing path integration is

$$S(\vec{z}_P,\vec{z}_P) = \int_P dt_P \left[\frac{i}{2}(\vec{z}_P\cdot\dot{\vec{z}}_P - \dot{\vec{z}}_P\cdot\vec{z}_P) - H_S(\vec{z}_P,\vec{z}_P)\right] + \quad (5.23)$$

$$+ i\int_P dt_P \int_P dt'_P \sum_{j,k=1}^{N} \bar{z}_{Pj}(t_P)\Delta_{P,jk}(t_P - t'_P) z_{Pk}(t'_P) - \int_P dt_P \sum_{j=1}^{N} \left(\bar{J}_j z_{Pj} + \bar{z}_{Pj} J_j\right).$$

It is easy to see that the introduction of the $\Delta_{P,jk}$ factor, originating from the integration of the environmental degrees of freedom, leads to two inter-related results: the first one is that the initially independent fields $\vec{z}_P(t_+) \equiv \vec{z}_+(t)$ and $\vec{z}_P(t_-) \equiv \vec{z}_-(t)$ are now connected. The second one is that the kernel (5.21) leads to the appearance of a non-zero imaginary part in the action (5.23), which signals the onset of diffusion for the system.

If the system's Hamiltonian is quadratic and diagonalizable, the generating functional can now, in principle, be calculated. In such a case, the system's connected correlation functions $G_{ij,\pm\pm}(\rho;t_2,t_1)$ can be systematically derived by applying the proper functional derivatives on the generating functional, where as an example

$$G_{ij}^{(T)}(\rho;t_2,t_1) \equiv G_{ij,++}(\rho;t_2,t_1) = \mp \frac{\delta^2 \ln Z[\vec{J},\vec{\bar{J}}]}{\delta J_i(t_{1+})\delta\bar{J}_j(t_{2+})}\bigg|_{J=0}. \quad (5.24)$$

Furthermore, when the Hamiltonian is not diagonalizable, the generating functional (5.22) can be used as the basis of a systematic perturbative approach to the system's dynamics. For reasons of comparison and for the sake of practical calculations, the influence functional (5.20) is re-represented, in appendix H, in terms of the conventional time notation.

In the equal time and index case eq. (5.24) accepts an extra $-1/2$ factor in the case of fermionic systems as stated in section 4 [73] and an $+1/2$ factor in the case of bosonic systems; the latter case will be confirmed through an example later on. In both cases,

these extra $\pm 1/2$ factors come exactly from the identification of the correct functions corresponding to the operators at the continuum limit, now though at the level of path integrated quantities and not at the level of the action. In the present construction, the equal time limit of the system's reduced correlation function (5.10) becomes then

$$\begin{aligned} G_{ij}^{(T)}(\rho;t,t) &= \int d^2z d^2z' \, \langle \pm\vec{z}|\hat{\rho}_S(t_{in})|\vec{z}'\rangle \times \\ &\times \int_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{z}_P(t_{in-})=\vec{z}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{z}',\vec{z};\vec{z}_P,\vec{z}_P)+i\int_P dt_P\left[\frac{i}{2}(\vec{z}_P\cdot\dot{\vec{z}}_P-\dot{\vec{z}}_P\cdot\vec{z}_P)-H_S(\vec{z}_P,\vec{z}_P)\right]} \times \\ &\times \left(\bar{z}_{Pj}(t_+)z_{Pk}(t_+) \pm \frac{1}{2}\delta_{jk}\right) I_E\left[\vec{\bar{z}}_P(t_P),\vec{z}_P(t_P)\right], \end{aligned} \tag{5.25}$$

which leads to the need of an extra $\pm\frac{1}{2}\delta_{jk}$ factor for the computation of connected correlation functions, after the application of the appropriate functional derivatives on the generating functional.

At this point, it is important to present the method for handling the $f(\pm\vec{\bar{z}},\vec{z}')$ term. Starting with the case of the initial state $\hat{\rho}_S(t_{in}) = \prod_{j\in S}\otimes|0_j\rangle\langle 0_j|$, it is easy to see that the reduced generating functional (5.22) simplifies tremendously, since $f(\pm\vec{\bar{z}},\vec{z}')=1$, to the form

$$Z_S^{(0)}[\vec{J},\vec{\bar{J}}] = \tilde{C}_E^{-1}\int \mathcal{D}^2\vec{z}_P(t_P) e^{iS(\vec{\bar{z}}_P,\vec{z}_P)}, \tag{5.26}$$

where the action weighing the path integral remains the same. The path integral presented above has simplified, and now can be computed as an integral over the whole space of fields z_P and $\bar{z}_P$, from the instance t_{in+} to the instance t_{in-} on the Keldysh contour P. The boundary term is not present in this case, since the path integral in this form does not present any specific boundary conditions at t_{in+} or t_{in-}. The choice of such an initial state for the system can in many cases be ideal, especially when the time scales in which one is interested in are typically large, and thus the initial conditions are not expected to have an impact. More importantly though, this choice can also serve as a first step for the computation of correlation functions for a system with initial state $\hat{\rho}_S(t_{in}) = \prod_{j\in S}\otimes|n_j\rangle\langle n_j|$, where n_j is the number of particles in the $j-th$ subsystem.

In the case of the initial state $\hat{\rho}_S(t_{in}) = \prod_{j\in S} \otimes |n_j\rangle \langle n_j|$ it is easy to compute

$$f(\pm\vec{\bar{z}}, \vec{z}') = (-1)^{\sum_{j\in S} n_j} \prod_{j\in S} \left(\bar{z}_j z'_j\right)^{n_j}, \tag{5.27}$$

leading to the generating functional (5.22) reducing to the form

$$Z_S[\vec{J}, \vec{\bar{J}}] = \tag{5.28}$$

$$= (-1)^{\sum_{j\in S} n_j} \tilde{C}_E^{-1} \int d^2\vec{z} d^2\vec{\bar{z}}' \prod_{j\in S} \left(\bar{z}_j z'_j\right)^{n_j} \int_{\vec{z}_P(t_{in+})=\vec{z}}^{\vec{\bar{z}}_P(t_{in-})=\vec{\bar{z}}'} \mathcal{D}^2\vec{z}_P(t_P) e^{-\Gamma_{in-,in+}(\vec{\bar{z}}',\vec{z};\vec{\bar{z}}_P,\vec{z}_P)+iS(\vec{\bar{z}}_P,\vec{z}_P)}.$$

Even though the $\Gamma_{in-,in+}(\vec{\bar{z}}', \vec{z}; \vec{\bar{z}}_P, \vec{z}_P)$ term is included in the above integral, it exists for calculational reasons only, since with its use the classical orbits become continuous. Nevertheless, the path integral defined here corresponds to a path integral from the initial variables $\vec{z}$, $\vec{\bar{z}}$ at t_{in+} to the final variables $\vec{z}'$, $\vec{\bar{z}}'$ at t_{in-}, even if the classical orbit used for the calculation satisfies only the initial constraint for the holomorphic variable, and the final constraint for the anti-holomorphic one. Thus, the variables $\bar{z}_j = \bar{z}_{Pj}(t_{in+})$ and $z'_j = z_{Pj}(t_{in-})$ can easily be identified with the action of the functional derivatives $\pm i\frac{\delta}{\delta J(t_{in+})}$ and $i\frac{\delta}{\delta \bar{J}(t_{in-})}$, respectively, on the zero-particle initial state generating functional (5.26). This way, it becomes possible to compute correlation functions for more complicated initial configurations, using boundary functional derivatives on the zero-particle initial state generating functional $Z_S^{(0)}[\vec{J}, \vec{\bar{J}}]$. This specific methodology will be addressed in more detail after the computation of the zero-particle result.

There are some simplifying assumptions that can be made in the following, according to the physical problem of interest. The first one refers to the coupling strengths $g_{j\mu}$. A rather common choice is to consider them as equal both in magnitude and phase for all sites: $\forall j \in S : g_{j\mu} = \gamma_\mu$, $\mu \in E$. In this way, the presence of the environment induces interactions all over the system's range. The structure of the environment can be then specified by its spectral density

$$D(E) = \sum_{\mu\in E} |\gamma_\mu|^2 \delta(E - E_\mu). \tag{5.29}$$

At the continuum limit, the usually adopted form [39] for this function reads as follows

$$D(E) = \lambda E \left(\frac{E}{E_c}\right)^{n-1} e^{-E/E_c}, \quad (E > 0). \tag{5.30}$$

where λ is a dimensionless coupling constant, and E_c is an exponential cut-off. Depending on the value of n, the environment is classified as sub-ohmic ($0 < n < 1$), ohmic ($n = 1$) and super-ohmic ($n > 1$) [39]. The second simplification, which is mandatory for the Lindblad equation, refers to the case of a Markovian environment. To quantify the Markovian limit, the fields corresponding to the system can be redefined in eq. (5.22) as $z_{Pj}(t_P) \to z_{Pj}(t_P)e^{-i\epsilon_S t_P}$, where $\epsilon_S \sim 1/\tau_S$ sets the shortest time scale τ_S that characterizes significant changes in the system. In this way, the kernel (5.21) is recasted to the form

$$\begin{aligned}\Delta_P(t_P, t'_P) &= \sum_{\mu\in E} |\gamma_\mu|^2 \left[\theta_P(t_P - t'_P) - \left(1 \mp e^{\beta E_\mu}\right)^{-1}\right] e^{-i(t_P - t'_P)(E_\mu - \epsilon_S)} = \\ &= \int_0^\infty dE D(E) \left[\theta_P(t_P - t'_P) - \left(1 \mp e^{\beta E}\right)^{-1}\right] e^{-i(t_P - t'_P)(E - \epsilon_S)},\end{aligned} \tag{5.31}$$

which now is a scalar with respect to the field indices, due to the equal coupling strength choice. The Markovian environment corresponds to the case in which the system's time scale τ_S is much larger than the corresponding environmental characteristic scale τ_E and, consequently, the environmental memory effects are absent. This is a natural expectation when the environment is stochastic and much larger than the system itself. In appendix H it is taken into account that, at the aforementioned Markovian limit, integrands like the one appearing in eq. (5.20) can be assumed to be very fast decaying functions of the time difference, thus allowing for the expansion $z_{Pj_\pm}(t') \simeq z_{Pj_\pm}(t) + \mathcal{O}(t - t')$. The functional (5.20) then assumes the form

$$I_E\left[\vec{\bar{z}}_P(t_P), \vec{z}_P(t_P)\right] = \tilde{C}_E^{-1} \exp\left[-\int_P dt_P \int_P dt'_P \sum_{j,k\in S} \bar{z}_{Pj}(t_P)\tilde{\Delta}_P(t_P - t'_P) z_{Pk}(t'_P)\right], \tag{5.32}$$

where

$$\tilde{\Delta}_P(t_\mu - t'_\nu) = \Delta_{\mu\nu}\delta(t - t'), \quad \mu, \nu = +, -, \tag{5.33}$$

$$\Delta_{++} = -i\delta E + \Gamma\left(\frac{1}{2} - b\right), \quad \Delta_{--} = i\delta E + \Gamma\left(\frac{1}{2} - b\right),$$
$$\Delta_{+-} = -\Gamma b, \quad \Delta_{-+} = \Gamma(1-b) \tag{5.34}$$

and

$$\Gamma = 2\pi D(\epsilon_s), \quad \delta E = Pr.\int_0^\infty dE \frac{D(E)}{E-\epsilon_s}, \quad b = \frac{1}{1 \mp e^{\beta\epsilon_S}}. \tag{5.35}$$

Here, $\delta(t-t')$ is the conventional delta function, which connects to the contour delta function on P (see appendix H) as $\delta(t-t') = \delta_P(t_+ - t'_+) = -\delta_P(t_- - t'_-)$.

By using the influence functional (5.32), and restoring the fields that have been rescaled, the generating functional (5.22) simplifies through the replacement of the kernel Δ_P with $\tilde{\Delta}_P$.

In the next subsection the present methodology is developed for all quadratic and diagonalizable Hamiltonians, and its application is presented on the characteristic examples of the bosonic and fermionic harmonic oscillators, as a proof-of-concept.

5.2 Computing the generating functional

The zero-particle initial state generating functional

Proceeding with the case of the zero-particle initial state, the Hamiltonian operators under consideration will be of the following form

$$\hat{H}_S = \sum_{j,k=1}^{N} \hat{a}_j^\dagger K_{jk} \hat{a}_k + \hat{H}_{nl} + C, \tag{5.36}$$

where the term $\hat{H}_{nl}$ is defined to include all higher order terms, which are bound to lead to non-linearities in the equations of motion. It is easy to see that the classical Hamiltonian corresponding to the operator (5.36) is

$$H_{S,cl} = \sum_{j,k\in S} \bar{z}_j K_{jk} z_k \pm \frac{1}{2}\sum_{j\in S} K_{jj} + H_{nl}(\vec{\bar{z}}, \vec{z}) + C, \tag{5.37}$$

where the quadratic term was identified according to the rules defined in sections 3 and 4 [18–20,26,27,73], and a classical function H_{nl} must in general also be consistently

identified for the higher order terms. Considering $C = \mp\frac{1}{2}\sum_{j\in S} K_{jj}$ for simplicity, it is easy to see that the zero-particle initial state generating functional, for the system's reduced correlation functions, simplifies to the following form

$$Z_S^{(0)}[\vec{J},\vec{\bar{J}}] = \tilde{C}_E^{-1}\int \mathcal{D}^2\vec{z}_P(t_P)e^{iS(\vec{\bar{z}}_P,\vec{z}_P)}, \tag{5.38}$$

where the action weighing path integration is

$$\begin{aligned} S(\vec{\bar{z}}_P,\vec{z}_P) = &\int_P dt_P\left[\vec{\bar{z}}_{Pj}(t_P)(i\delta_{jk}\partial_{t_P} - K_{jk})z_{Pk}(t_P) - H_{nl}(\vec{\bar{z}},\vec{z})\right] + \\ &+ i\int_P dt_P\int_P dt'_P\sum_{j,k=1}^{N}\bar{z}_{Pj}(t_P)\Delta_{P,jk}(t_P-t'_P)z_{Pk}(t'_P) - \int_P dt_P\sum_{j=1}^{N}\left(\bar{J}_j z_{Pj} + \bar{z}_{Pj}J_j\right). \end{aligned} \tag{5.39}$$

At the Markovian limit, the only change in the above result is the substitution of the Δ_P kernel by the $\tilde{\Delta}_P$ one of eq. (5.33). It is easy to see that the real contribution of the $i\Delta_P$ term produces a shift to the energy scales of the system, which appears in both equal site and interaction quadratic terms, in some cases even leading to the appearance of interactions between degrees of freedom that initially were not communicating. At the Markovian limit this shift was computed explicitly in the previous section, and it was found to be the contribution of the $-\delta E$ term of eq. (5.35) to all entries of the K matrix. Proceeding with the Markovian case, and in order to consistently take into account the energy shift, the renormalized matrix K can be defined, with elements $K_{ij}^{(R)} = K_{ij} - \delta E$, as also the $\tilde{\Delta}'_P$ kernel representing the initial $\tilde{\Delta}_P$ kernel without the δE terms. Thus, the action now can be seen to be of the form

$$\begin{aligned} S(\vec{\bar{z}}_P,\vec{z}_P) = &\int_P dt_P\left[\vec{\bar{z}}_{Pj}(t_P)(i\delta_{jk}\partial_{t_P} - K_{jk}^{(R)})z_{Pk}(t_P) - H_{nl}(\vec{\bar{z}},\vec{z})\right] + \\ &+ i\int_P dt_P\int_P dt'_P\sum_{j,k=1}^{N}\bar{z}_{Pj}(t_P)\tilde{\Delta}'_P(t_P-t'_P)z_{Pk}(t'_P) - \int_P dt_P\sum_{j=1}^{N}\left(\bar{J}_j z_{Pj} + \bar{z}_{Pj}J_j\right). \end{aligned} \tag{5.40}$$

In the action above, the $\tilde{\Delta}'_P$ kernel includes only the diffusion effect of the environment, and all dynamic effects have been included in the renormalized matrix $K^{(R)}$.

It is a known fact that in the case of a non-zero $H_{nl}(\vec{\bar{z}},\vec{z})$ term the equations of mo-

tion corresponding to this specific action adopt the following non-linear form

$$\left(i\partial_{t_P}\delta_{jk} - K_{jk}\right) z_k^{cl}(t_P) - \partial_{\bar{z}_j} H_{nl} + i\sum_{k\in S}\int_P dt'_P \Delta_{jk}(t_P - t'_P) z_k^{cl}(t'_P) = J_j(t_P) \tag{5.41}$$

$$\left(i\partial_{t_P}\delta_{jk} + K_{jk}\right) \bar{z}_j^{cl}(t_P) \pm \partial_{z_j} H_{nl} - i\sum_{k\in S}\int_P dt'_P \bar{z}_k^{cl}(t'_P)\Delta_{kj}(t'_P - t_P) = -\bar{J}_j(t_P), \tag{5.42}$$

which in general can be solved only perturbatively with respect to J. This allows for the computation of the classical solutions as series of $\vec{J}$ and $\vec{\bar{J}}$, which afterwards can be used at the semiclassical level to produce semiclassical correlation functions.

For the computation of the zero-particle initial state reduced correlation functions it is also possible to proceed with the use of the Effective Action $\Gamma_S^{(0)}[\vec{z},\vec{\bar{z}}]$ [3, 99], which is defined as the Legendre transformation of the generating functional of connected correlation functions $W_S^{(0)}[\vec{J},\vec{\bar{J}}] = -i\ln Z_S^{(0)}[\vec{J},\vec{\bar{J}}]$:

$$\Gamma_S^{(0)}[\vec{z},\vec{\bar{z}}] = \vec{\bar{J}}\cdot\vec{z} + \vec{\bar{z}}\cdot\vec{J} - W_S^{(0)}[\vec{J},\vec{\bar{J}}]. \tag{5.43}$$

In the above definition of the Effective action the fields $\vec{z}$ and $\vec{\bar{z}}$ are the background fields, which at the semiclassical limit are equal to the classical configurations of the physical fields. The Effective action is usually used in order to compute 1-Particle Irreducible Feynman diagrams, but it also simplifies tremendously the calculations of connected two-point functions, since the property

$$\frac{\delta^2\Gamma_S^{(0)}[\vec{z},\vec{\bar{z}}]}{\delta z_j(t_P)\delta\bar{z}_k(t'_P)} = \frac{\delta J_k(t'_P)}{\delta z_j(t_P)}, \tag{5.44}$$

combined with

$$\pm iG_{jk}^{(c;T)}(t_P,t'_P) \equiv \frac{\delta^2 W_S^{(0)}[\vec{J},\vec{\bar{J}}]}{\delta J_j(t_P)\delta\bar{J}_k(t'_P)} = \frac{\delta z_k(t'_P)}{\delta J_j(t_P)}, \tag{5.45}$$

leads to

$$\frac{\delta^2\Gamma_S^{(0)}[\vec{z},\vec{\bar{z}}]}{\delta\bar{z}_j(t_P)\delta z_k(t'_P)} = \mp iG^{(c;T)}{}_{jk}^{-1}(t_p,t'_P), \tag{5.46}$$

where $G_{jk}^{(c;T)}(t_P,t'_P) = \langle\hat{T}\left(\hat{a}_j^\dagger(t_P)\hat{a}_k(t'_P)\right)\rangle_\rho^{(c)}$ is the 2-point connected correlator for the system. Similar methods can be also used for the computation of higher order connected correlation functions.

At the aforementioned semiclassical limit the Effective action reduces to the form

$$\Gamma_S^{(0)}[\vec{z},\vec{\bar{z}}] = S(\vec{z},\vec{\bar{z}})\big|_{J=0} \pm i\mathrm{trlog}\left[\frac{\delta^2 S(\vec{z},\vec{\bar{z}})}{\delta\bar{z}_j(t)\delta z_k(t')}\right], \tag{5.47}$$

where $\hbar$ has been set to $\hbar = 1$. In the quadratic case though, which will be the case under study in the following, the second term reduces to a constant which is absorbed by the measure of path integration.

In the following, the case of a quadratic and diagonalizable Hamiltonian will be presented, since it serves as a prime example for the structure of the approach. For this case it is possible to redefine the fields using a unitary transformation $\hat{U}$, which diagonalizes the matrix $K^{(R)}$

$$K^{(R)} = \hat{U}D\hat{U}^\dagger, \quad D_{jk} = \epsilon_j^{(R)}\delta_{jk}. \tag{5.48}$$

Here, $\epsilon_j^{(R)}$ are the renormalized eigenvalues, which include the contribution coming from the energy shift. Through a redefinition of the fields as $\eta_j = \sum_{k\in S}\hat{U}_{jk}^\dagger z_k$ and $\bar{\eta}_j = \sum_{k\in S}\bar{z}_k\hat{U}_{kj}$, the reduced Effective action can be then brought to the very simple form

$$\begin{aligned}\Gamma_S^{(0)}\left[\vec{\bar{\eta}}_P,\vec{\eta}_P\right] = &\int_P dt_P \sum_{j\in S}\bar{\eta}_{Pj}(t_P)(i\partial_{t_P} - \epsilon_j^{(R)})\eta_{Pj}(t_P) + \\ &+ i\int_P dt_P\int_P dt'_P \sum_{j,k=1}^{N}\bar{\eta}_{Pj}(t_P)\tilde{\Delta}'_P(t_P - t'_P)\eta_{Pk}(t'_P),\end{aligned} \tag{5.49}$$

thus leading to the identification of the following Green's equation for the reduced 2-point connected correlation function of the $\vec{\eta}$ and $\vec{\bar{\eta}}$ fields

$$\sum_{k\in S}\int_P d\tilde{t}_P\left[\left(i\partial_{t_P} - \epsilon_j^{(R)}\right)\delta_{jk}\delta_P(t_P - \tilde{t}_P) + i\tilde{\Delta}'_P(t_P,\tilde{t}_P)\right]L_{kl}(\tilde{t}_P,t'_P) = \delta_{jl}\delta_P(t_P - t'_P), \tag{5.50}$$

where $L_{kl}(\tilde{t}_P,t'_P) = \mp iG_{kl}^{(c;T)}(\tilde{t}_P,t'_P)$. These equations call for some explanations. As already discussed these equations can be easily solved. The functions $\tilde{\Delta}'_P$, which are present exclusively due to the interaction with the environment, are responsible for the decoherence and dissipation effects induced on the system's evolution. From this point of view, the corresponding terms share a lot of similarities with the dissipator

function in the Lindblad equation. It must be noted however, that eq. (5.50) is not equivalent to the Lindblad equation in many aspects. First of all, eq. (5.50) holds even if memory effects are present for a non-Markovian environment, alas with the Δ'_P kernel now being the real part of (5.31); the imaginary part is considered again absorbed inside the renormalized energies. Secondly, its corresponding Green's equation yields the full time-dependent correlator and not its equal-time version only, as would be the case if we had relied on the Lindblad equation alone.
It must be also noted that even though the renormalized energy parametrization above appears to be the most physical one, it is still a valid choice to work with and diagonalize the initial non-renormalized Hamiltonian. This just means keeping the initial $i\tilde{\Delta}_P$ kernel, containing both real and imaginary contributions, and not including the $-\delta E$ factor to the Hamiltonian prior to its diagonalization. This may sometimes simplify calculations, as will be the case presented later in the example. The method which will now be followed can hold in an analogous form even in the case of non-Markovian dynamics [100].

It is easy to see how equation (5.50) can be solved, using the corresponding Fourier transformations for all elements of the Green's operator

$$L_{ij}(t_P, t'_P) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(t_P - t'_P)} L^F_{ij;PP'}(k), \tag{5.51}$$

where the indices P, P' take the values $+/-$ depending on which of the lines L_+ and L_- the element L_{ij} is supported on, in each case. The appropriate solution for this Green's equation has to satisfy the constraint $L(t_{in+}, t'_P) = 0$ at the initial instance t_{in+}, where the boundary conditions for the holomorphic propagating fields would be considered in the classical equations of motion. This constraint must be set even if no boundary condition exists for the field η, as is the case in this configuration, in order to consistently define the Green's function in the Keldysh contour P. Nevertheless, the propagator identified via the Fourier transformation may differ from the desirable one up to a solution of the homogeneous equation and must be modified accordingly.

Finally, even though the reduced 2-point connected correlation function can be easily computed via the above arguments for a zero-particle initial state, it is also important to present the generating functional $Z_S^{(0)}[\vec{J}, \vec{J}]$ in its final form, at least for the quadratic

case. This is because the $Z_S^{(0)}[\vec{J},\vec{\bar{J}}]$ functional can then be used for the construction of the non-zero-particle initial state generating functionals as stated in subsection 5.1. To do this, it suffices to compute the classical configurations for the $\vec{\eta}$ and $\vec{\bar{\eta}}$ fields, which are found to be the following

$$\eta_j^{cl} = \int_P dt'_P L_{jk}(t_P, t'_P)\lambda_k(t'_P), \quad \bar{\eta}_j^{cl} = \int_P dt'_P \bar{\lambda}_k(t'_P) L_{kj}(t'_P, t_P), \tag{5.52}$$

with $\lambda_j = \sum_{k\in S} \hat{U}_{jk} J_k$ and $\bar{\lambda}_j = \sum_{k\in S} \bar{J}_k \hat{U}^{\dagger}_{kj}$ being the redefined currents after the diagonalization. Thus, by perturbing the action (5.23) - with $H_{nl}(\vec{z},\vec{\bar{z}}) = 0$ - around its classical configuration, one can identify the resulting zero-particle initial state reduced generating functional for the system's correlation functions to be

$$Z_S^{(0)}[\vec{J},\vec{\bar{J}}] \sim \exp\left[-i \sum_{j,k\in S} \int_P dt_P \int_P dt'_P \bar{\lambda}_j(t_P) L_{jk}(t_P, t'_P)\lambda_k(t'_P)\right]. \tag{5.53}$$

The similarity symbol used above identifies the existence of a series of irrelevant normalization terms. The use of this generating functional allows now for the study of zero and non-zero particle initial states, in the latter case through the use of appropriate boundary functional derivatives on (5.53), as discussed in the previous subsection. In order not to burden the text, the application of these functional derivatives will be showcased in the context of the following example.

The harmonic oscillator case

In this subsection the aforementioned method will be applied for the cases of the bosonic and fermionic harmonic oscillators, coupled to a Markovian environment, since this setup can serve as the ideal example for testing the presented methodology. It will be made apparent that by using the above theoretical tools, one can consistently compute all correlation functions for the system, and at the equal time limit recover exactly the results that have been found with the use of the Lindblad equation. The Hamiltonian operator corresponding to the dynamics of the harmonic oscillator is given as

$$\hat{H} = \epsilon\left(\hat{a}^{\dagger}\hat{a} \pm \frac{1}{2}\right), \tag{5.54}$$

where the energy scale ϵ is the unique scale of the system, and thus it is also the energy ϵ_S with respect to which the Markovian approximation is performed. The classical

Hamiltonian can be easily identified then to be

$$H_S = \epsilon \bar{z} z, \tag{5.55}$$

and thus the Green's equation (5.50) reduces to the following form

$$\int_P d\tilde{t}_P \left[(i\partial_{t_P} - \epsilon^{(R)}) \delta_P(t_P - \tilde{t}_P) + i\tilde{\Delta}'_P(t_P, \tilde{t}_P) \right] L(\tilde{t}_P, t'_P) = \delta_P(t_P - t'_P), \tag{5.56}$$

where $L = \mp i G^{(c;T)}$ and $\epsilon^{(R)} = \epsilon - \delta E(\epsilon)$. The solution to this Green's equation along the Keldysh contour P is found in appendix I to be the following

$$L(t_P, t'_P) = \tilde{L}(t_P, t'_P) + L_{hom}(t_P, t'_P), \tag{5.57}$$

where

$$\tilde{L}(t_+, t'_+) = -i\Theta(t - t') e^{-i\epsilon^{(R)}(t-t') - \Gamma(t-t')/2} + ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{5.58}$$

$$\tilde{L}(t_-, t'_+) = -i(1-b) e^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{5.59}$$

$$\tilde{L}(t_-, t'_-) = -i\Theta(t' - t) e^{-i\epsilon^{(R)}(t-t') - \Gamma(t'-t)/2} + ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{5.60}$$

$$\tilde{L}(t_+, t'_-) = ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2}, \tag{5.61}$$

and

$$L_{hom}(t_P, t'_P) = -ibe^{+\Gamma\left(t_{in+} - \frac{t_P}{2} - \frac{t'_P}{2}\right) - i\epsilon^{(R)}(t_P - t'_P)}. \tag{5.62}$$

In this result, the $L_{hom}(t_P, t'_P)$ term is the solution of the homogeneous analogue of the Green's equation (5.56), which is found by the constraint for the Green's function $L(t_P, t'_P)$ to vanish as its first argument goes to t_{in+}. Note that, by ignoring the presence of the environment ($b = 0, \Gamma = 0$) the zero temperature limit of the correlators appears. It is easy to compute then the correlation functions of the system where, as an example, the time ordered case is presented

$$\langle \hat{T}\left(\hat{a}^\dagger(t_2)\hat{a}(t_1)\right)\rangle_\rho^{(c;0)} = \mp \frac{\delta^2 \ln Z_S^{(0)}}{\delta J(t_{2+}) \delta \bar{J}(t_{1+})}\bigg|_{J=0} = \pm i L_{++}(t_1, t_2), \tag{5.63}$$

which, as noted in subsection 5.1, admits an extra additive $\mp 1/2$ factor when the time instances are equal. It is very interesting to study this equal time limit, which then

gives

$$\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_\rho^{(c;0)} = \mp \frac{\delta^2 \ln Z_S^{(0)}}{\delta J(t_+)\delta \bar{J}(t_+)}\bigg|_{J=0} \mp \frac{1}{2} = \frac{1}{e^{\beta\epsilon} \mp 1}\left[1 - e^{-\Gamma(t-t_{in})}\right]. \tag{5.64}$$

This result shows that the mean occupation number is initially ($t = t_{in}$) zero, as the system begins from its vacuum state. When the system becomes coupled to a finite temperature environment, particles from the environment can be absorbed by the system and emitted back. This transaction leads, at $t \to \infty$, to a mean occupation number which is equal to $\left(e^{\beta\epsilon} \mp 1\right)^{-1}$, meaning that the system becomes thermalized, approaching the temperature of the environment. If this temperature is zero, $\langle \hat{a}^\dagger \hat{a}\rangle = 0$ as the system remains in its ground state.

It is also straightforward to consider the case of the initial state being $\hat{\rho}_S(t_{in}) = |1\rangle\langle 1|$, in which case the corresponding generating functional can be easily related to the zero-particle initial state one as

$$\begin{aligned} Z_S^{(1)}[\vec{J},\vec{\bar{J}}] &\sim \mp \frac{\partial}{\partial J(t_{in+})}\frac{\partial}{\partial \bar{J}(t_{in-})} Z_S^{(0)}[\vec{J},\vec{\bar{J}}] \sim \frac{\partial}{\partial J(t_{in+})}\frac{\partial}{\partial \bar{J}(t_{in-})} e^{-i\int_P dt_P \int_P dt'_P \bar{J}(t_P) L(t_P,t'_P) J(t'_P)} = \\ &= \left[iL(t_{in-},t_{in+}) + \int_P dt_P \int_P dt'_P L(t_{in-},t_P) L(t'_P,t_{in+}) J(t_P)\bar{J}(t'_P)\right] Z_S^{(0)}[\vec{J},\vec{\bar{J}}] = \\ &= \left[1 \pm \int_P dt_P \int_P dt'_P L(t_{in-},t_P) L(t'_P,t_{in+})\bar{J}(t'_P) J(t_P)\right] Z_S^{(0)}[\vec{J},\vec{\bar{J}}]. \end{aligned} \tag{5.65}$$

Thus, all the 2-point connected correlation functions can be computed from the logarithm of this generating functional. As an example we present here the equal time result

$$\begin{aligned} \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle_\rho^{(c;1)} &= \mp \frac{\delta^2 \ln Z_S^{(1)}[\vec{J},\vec{\bar{J}}]}{\delta J(t_+)\delta \bar{J}(t_+)}\bigg|_{J=0} \mp \frac{1}{2} = \\ &= \mp \frac{\delta^2 \ln Z_S^{(0)}[\vec{J},\vec{\bar{J}}]}{\delta J(t_+)\delta \bar{J}(t_+)} \mp \frac{1}{2} - L(t_{in-},t_+)L(t_+,t_{in+}) = \\ &= e^{-\Gamma(t-t_{in})} + \frac{1}{e^{\beta\epsilon} \mp 1}\left[1 - e^{-\Gamma(t-t_{in})}\right]. \end{aligned} \tag{5.66}$$

The same result can be easily found using the Lindlad equation [88], thus showcasing the validity of the presented procedure. Nevertheless, the most important advantage

of the above method is that it further allows for the computation of unequal-time correlation functions.

It is simple to also tackle the generalization of the above system to the case of N initially de-coupled harmonic oscillators, with equal energy ϵ. In this case, as was commented in the previous subsection, it is simpler to work with the initial Hamiltonian and use the complex $i\tilde{\Delta}$ kernel instead of the purely imaginary one $i\tilde{\Delta}'$. This leads to the equation for the propagator (5.50) reducing to the form

$$\sum_{k\in S}\int_P d\tilde{t}_P \left[(i\partial_{t_P}-\epsilon)\,\delta_{jk}\delta_P(t_P-\tilde{t}_P)+i\tilde{\Delta}_P(t_P,\tilde{t}_P)\right] L_{kl}(\tilde{t}_P,t'_P)=\delta_{jl}\delta_P(t_P-t'_P). \quad (5.67)$$

The $\tilde{\Delta}$ kernel here is the initial one found in (5.34). The simplification of using this parametrization is that by summing the N differential equations one gets

$$\int_P d\tilde{t}_P \left[(i\partial_{t_P}-\epsilon)\,\delta_P(t_P-\tilde{t}_P)+iN\tilde{\Delta}_P(t_P,\tilde{t}_P)\right] L^{(N)}(t_P,t'_P)=\delta_P(t_P-t'_P), \quad (5.68)$$

where $L^{(N)}(t_P,t'_P)=\sum_{j\in S}L_{jk}(\tilde{t}_P,t'_P)$. In the previous Green's function no site index is used, since it is a site independent quantity. At this point it can be seen that by adding the real part of $iN\tilde{\Delta}_P$ to the energy of the system, the corresponding Green's equation becomes equivalent to (5.56), with the substitutions $\Gamma\to N\Gamma$ and $\delta E\to N\delta E$. Thus, the value of $L^{(N)}(t_P,t'_P)$ follows accordingly. Substituting the value of $\sum_{j\in S}L_{jk}(\tilde{t}_P,t'_P)$ in eq. (5.68) with the solution $L^{(N)}(t_P,t'_P)$ leads to the following result for the propagator of the fields z_j over the P contour

$$L_{ij}(t_P,t'_P)=G^{(0)}(t_P,t'_P)\delta_{ij}-iF(t_P,t'_P), \quad (5.69)$$

where

$$G^{(0)}(t_P,t'_P)=-i\Theta_P(t_P-t'_P)e^{-i\epsilon(t_P-t'_P)} \quad (5.70)$$

is the free propagator for a field of the system and

$$F(t_P,t'_P)=\int_P dt_{1P}\int_P dt_{2P}G^{(0)}(t_P,t_{1P})\tilde{\Delta}_P(t_{1P}-t_{2P})L^{(N)}(t_{2P},t'_P) \quad (5.71)$$

is the complete contribution of the environment. Using the result of eq. (5.69) it is possible then to compute many quantities corresponding to the system of N initially independent harmonic oscillators in the vacuum state. As an example, the resulting

expectation value of the number operator of each subsystem is presented

$$\langle \hat{a}_j^\dagger(t)\hat{a}_j(t)\rangle_\rho^{(c;0)} = \frac{1}{N}\frac{1}{e^{\beta\epsilon}\mp 1}\left[1 - e^{-N\Gamma(t-t_{in})}\right]. \tag{5.72}$$

This showcases the very interesting result that even though the harmonic oscillators start as independent, they end up thermalizing as a unique harmonic oscillator due to their interaction with the environment. It must be noted that correlation functions for non-zero-particle initial states can again be identified using the appropriate generating functionals constructed via boundary functional derivatives of the zero-particle initial state one.

6 Epilogue

In the present thesis, the main goal was the construction of a consistent coherent-state path integral formalism, which could provide, in a direct way, the correct continuum functional integrals for finite quantum systems. Even though this undertaking was entirely successful in the context of fermionic coherent-state path integrals, it led only to a partial solution in the bosonic and general spin cases. Nevertheless, in the latter case, the presented construction was enough to provide a definite answer regarding the need for the metaplectic quantization in Kähler manifolds, a topic in mathematical physics which, for decades now, hasn't received a straightforward answer.
The consistency of the aforementioned formalism allowed not only for the application of the coherent-state path integral formalism in driven and open quantum systems, but proved to significantly simplify their study. More specifically, the application on the driven XY chain in subsection 4.3 led to highly non-trivial results with tremendously little effort, such as the manifestation of the Kibble-Zurek mechanism. Furthermore, the application of the formalism in open quantum systems proved not only to be possible, but also to provide new tools such as the differential relations between the generating functionals of systems with different initial states and the easily solvable Green's equations pertaining to the connected two point correlation functions, which proved to be the generalization of the Linblad equation.

At this point, two future directions can be understood to be of major importance. The first one is the understanding of the geometric de-quantization theory for the cases of operators corresponding to non-polarization-preserving Hamiltonian vector fields. For this, the formalism should touch on the topic of BKS kernels, which though is a highly non-trivial topic, due to the inability of providing a closed solution for all operators. In the case that progress is made in that sense, many major results are expected to be presented in the context of quantization theory. The second direction is to proceed with the application of the existing formalism on more complicate systems, such as Heisenberg spin chains coupled to a thermal environment, which would provide important insight in the physics of such systems.

A Connection with overcompleteness

The dependence of the continuum limit on the discretization procedure can be easily verified from the simple case of the quantum harmonic oscillator. As indicated in the main text, the standard discretization procedure in the space of coherent states leads to the classical Hamiltonian $H_{cl} = |z|^2 + \frac{1}{2}$, which in turn gives the wrong value for the partition function $Z = \sum_{n=0}^{\infty} e^{-iT(n+1)}$. Using a different manipulation where, for each slice, a symmetric form

$$e^{-i\epsilon\hat{H}} = \frac{1}{2\pi i} \int |z_j\rangle \, e^{-i\epsilon H_{cl}^j(\bar{z}_j, z_j)} d^2 z_j \, \langle z_j| \tag{A.1}$$

is considered, the partition function can be expressed as
$Z = \int \prod_{j=0}^{N} \frac{1}{2\pi i} d^2 z_j \, \langle z_{j+1}|z_j\rangle \, e^{-i\epsilon H_{cl}^j}$, where $|z_{N+1}\rangle = |z_0\rangle$. For each slice it is now possible to employ the Fock resolution of the exponential operator, from which the H_{cl}^j of this construction can be found. Equation

$$\langle z_k|e^{-i\epsilon\hat{H}}|z_k\rangle = \frac{1}{2\pi i} \int \langle z_k|z_j\rangle \, e^{-i\epsilon H_{cl}^j} d^2 z_j \, \langle z_j|z_k\rangle \tag{A.2}$$

can be solved, up to the first order in ϵ, through the choice of the Ansatz $H_{cl}^j = A|z_j|^2 + Bz_j + C\bar{z}_j + D$, giving the factors to be

$$A = 1, \quad B = 0, \quad C = 0, \quad D = -\frac{1}{2}. \tag{A.3}$$

Thus, using the usual convergence rule $z_{j+1} \simeq z_j + \epsilon \nabla_j z_j$, the continuum limit of the classical Hamiltonian can be found to be $H_{cl}^j = |z_j|^2 - \frac{1}{2}$. The calculation of the partition function in the continuum, with the use of this weight, yields the wrong result $Z = \sum_{n=0}^{\infty} e^{-iTn}$. It is interesting that both methods not only lead to wrong results, but also to results that differ from each other. Inconsistencies such as these have been observed only in cases where overcomplete sets of states are used during the construction. On the contrary, in the usual Feynman procedure, the continuum limit appears to be uniquely defined [3].

B Obstruction results in quantization theory

Along the lines of quantization theory a great number of no-go theorems have been proved, defining very strict bounds for the subsets of observables which can be consistently mapped to operators. In this context, consistency is defined as the compatibility of a mapping under the maximum possible number of Dirac's quantization constraints presented in subsection 2.2 of the main text. As mentioned there, it has been proved impossible keeping all of these properties at the same time, with the last two usually being weakened, such that the map is restricted only to a subspace of observable functions. Groenewold-Van Hove theorem for the 2D Euclidean plane [22, 34] (see subsection 2.2) and its generalization for the 2-Sphere [35] give the two simplest cases of this obstruction.

The proof for this theorem is very simple for the case of the 2D Euclidean phase space, and can be summed up in only a few lines. Let $\hat{Q}$ be a homomorphism between the algebras of classical observables and operators - as indicated in (2.21) - acting on the space of smooth functions in the usual way: $\hat{Q}(1) = \hat{I}$, $\hat{Q}(x) = \hat{X}$ and $\hat{Q}(p) = \hat{P}$. Through the use of [2.21] one can identify the action of this map on second order operators

$$\hat{Q}(x^2) = \hat{X}^2, \quad \hat{Q}(p^2) = \hat{P}^2, \quad \hat{Q}(xp) = \frac{1}{2}(\hat{X}\hat{P} + \hat{P}\hat{X}), \tag{B.1}$$

which is found to be unique. Up to this point the subsets of observables and their operator images are closed and thus $\hat{Q}$ is indeed a homomorphism. On the contrary, the extension of this subset through the inclusion of higher order observables fails to remain consistent. To show this, firstly one has to extend the previous calculation and compute

$$\begin{gathered} \hat{Q}(x^3) = \hat{X}^3, \quad \hat{Q}(p^3) = \hat{P}^3, \\ \hat{Q}(x^2p) = \frac{1}{2}(\hat{X}^2\hat{P} + \hat{P}\hat{X}^2), \quad \hat{Q}(xp^2) = \frac{1}{2}(\hat{X}\hat{P}^2 + \hat{P}^2\hat{X}). \end{gathered} \tag{B.2}$$

These results do not behave well under the closedness condition, since the calculation of $\{x^3, p^3\}$ and $\{x^2p, xp^2\}$ through [2.21] gives rise to a false equality

$$\frac{1}{3}[\hat{X}^3, \hat{P}^3] = \frac{1}{4}[\hat{X}^2\hat{P} + \hat{P}\hat{X}^2, \hat{X}\hat{P}^2, \hat{P}\hat{X}^2]. \tag{B.3}$$

This result renders the higher order generalization of the map $\hat{Q}$ invalid. Thus, the maximal quantizable subalgebra of the space of observables, containing the subset $\{1, x, p\}$, is $\{1, x, p, x^2, p^2, xp\}$ with $\hat{Q}$ defined from the previous mappings.

Through the canonical transformation

$$z = \frac{1}{\sqrt{2}}(x + ip), \quad \bar{z} = \frac{1}{\sqrt{2}}(x - ip), \tag{B.4}$$

this can be rephrased, such that the maximal quantizable subalgebra of observables containing $\{1, z, \bar{z}\}$ is $\{1, z, \bar{z}, z^2, \bar{z}^2, \bar{z}z\}$, mapping these observables 1-1 to the subset of operators

$$\left\{1, \hat{a}, \hat{a}^\dagger, \hat{a}^2, \hat{a}^{\dagger 2}, \hat{a}^\dagger a + \frac{1}{2}\right\}. \tag{B.5}$$

The generalization of Groenewold's theorem for S^2 is also presented, albeit without the proof, since it is highly more involved [35]:

Theorem 2 (Groenewold's theorem for S^2) *Let S_1, S_2, S_3 be observable functions on the 2-Sphere satisfying $\{S_i, S_j\} = \sum_{k=1}^{3} \epsilon_{ijk} S_k$. Then, the maximal Poisson subalgebra on the 2-Sphere, which contains $\{1, S_1, S_2, S_3\}$, and can be consistently quantized, is just that generated by $\{1, S_1, S_2, S_3\}$ itself.*

C Functional integration of $\hat{H} = -\omega \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$

In this appendix, a simple system described by the Hamiltonian $\hat{H} = -\omega \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$ is considered, using the method presented in section 4. The partition function of this system is known to be

$$Z = \mathrm{tr}\left[e^{-\beta \hat{H}}\right] = e^{-3\beta\omega/4} + 3e^{\beta\omega/4}. \tag{C.1}$$

Using the Jordan-Wigner transformation, the Hamiltonian at hand assumes the form

$$\begin{aligned} \hat{H} = -\frac{\omega}{4}\Big[&\left(\hat{\psi}_1^\dagger - \hat{\psi}_1\right)\left(\hat{\psi}_2^\dagger + \hat{\psi}_2\right) + \left(\hat{\psi}_2^\dagger - \hat{\psi}_2\right)\left(\hat{\psi}_1^\dagger + \hat{\psi}_1\right) + \\ &+ \left(1 - 2\hat{\psi}_1^\dagger \hat{\psi}_1\right)\left(1 - 2\hat{\psi}_2^\dagger \hat{\psi}_2\right)\Big], \end{aligned} \tag{C.2}$$

which in terms of Majorana operators reads as

$$\hat{H} = i\frac{\omega}{4}\left(\hat{\gamma}_2\hat{\gamma}_3 + \hat{\gamma}_4\hat{\gamma}_1 - i\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\hat{\gamma}_4\right). \tag{C.3}$$

According to the Faddeev-Jackiw quantization scheme, the classical function entering the Majorana path integral representation is the classical counterpart of (C.3)

$$H_M = i\frac{\omega}{4}\left(\gamma_2\gamma_3 + \gamma_4\gamma_1 - i\gamma_1\gamma_2\gamma_3\gamma_4\right), \tag{C.4}$$

which in turn translates to the complex Grassmann variables as

$$H = -\frac{\omega}{2}\left(\bar{\zeta}_1\zeta_2 + \bar{\zeta}_2\zeta_1 + 2|\zeta_1|^2|\zeta_2|^2\right). \tag{C.5}$$

Thus, the integral to be evaluated is

$$Z = \left(\prod_{j=1}^{2}\int_{(-)}\mathcal{D}\bar{\zeta}_j\mathcal{D}\zeta_j e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\bar{\zeta}_j\dot{\zeta}_j}\right)\exp\left[\frac{\omega}{2}\int_{-\beta/2}^{\beta/2}d\tau\left(\bar{\zeta}_1\zeta_2 + \bar{\zeta}_2\zeta_1 + 2|\zeta_1|^2|\zeta_2|^2\right)\right]. \tag{C.6}$$

It is convenient to perform a change of variables, induced through the unitary transformation

$$\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}. \tag{C.7}$$

After this change, the integral (C.6) is recasted to the form

$$Z = \left(\prod_{j=1}^{2}\int_{(-)}\mathcal{D}\bar{\eta}_j\mathcal{D}\eta_j e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\bar{\eta}_j\dot{\eta}_j}\right)\exp\left[\frac{\omega}{2}\int_{-\beta/2}^{\beta/2}d\tau\left(|\eta_1|^2 - |\eta_2|^2 + 2|\eta_1|^2|\eta_2|^2\right)\right]. \tag{C.8}$$

It is easy now to perform the integration over the first Grassmann field

$$\begin{aligned} Z_1 &= \int_{(-)}\mathcal{D}\bar{\eta}_1\mathcal{D}\eta_1\exp\left[-\int_{-\beta/2}^{\beta/2}d\tau\bar{\eta}_1\left(\partial_\tau - \frac{\omega}{2} - \omega|\eta_2|^2\right)\eta_1\right] = \\ &= 2\cosh\left(\frac{\beta\omega}{4} + \frac{\omega}{2}\int_{-\beta/2}^{\beta/2}d\tau|\eta_2|^2\right). \end{aligned} \tag{C.9}$$

To arrive at the last result, the symmetric prescription for the underlying lattice structure was considered. By inserting eq. (C.9) into eq. (C.8), the correct quantum result

is recovered

$$Z = e^{\beta\omega/4} \int\limits_{(-)} \mathcal{D}\bar{\eta}_2 \mathcal{D}\eta_2 e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \bar{\eta}_2 \dot{\eta}_2} + e^{-\beta\omega/4} \int\limits_{(-)} \mathcal{D}\bar{\eta}_2 \mathcal{D}\eta_2 e^{-\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \bar{\eta}_2 (\partial_\tau + \omega) \eta_2} =$$

$$= e^{-3\beta\omega/4} + 3e^{\beta\omega/4}. \tag{C.10}$$

D The path integral representation of the fermion number sector decomposition

In this appendix details regarding the sector separation in the context of path integrals are presented. In this context, the partition function pertaining to the even sector of the 1D XY spin chain, appearing in eq. (4.29), is computed.

In the main text, path integration was only considered over the even fermion number sector. This was allowed by the sector separation of the Hilbert space, which can also be easily observed in the context of path integration. When considering the partition function, its functional integral form can be separated in two independent path integrals, each one of which is defined with respect to fields abiding under the corresponding fermionic boundary conditions

$$Z = \mathrm{tr}\left[e^{-\beta\hat{H}}\right] = \frac{1}{2}\mathrm{tr}[(\hat{\mathbb{1}} + e^{i\pi\hat{N}})e^{-\beta\hat{H}^{(e)}}] + \frac{1}{2}\mathrm{tr}[(\hat{\mathbb{1}} - e^{i\pi\hat{N}})e^{-\beta\hat{H}^{(o)}}] \equiv Z^{(e)} + Z^{(o)}. \tag{D.1}$$

Here, $\hat{N} = \sum_{j=1}^{N} \hat{\psi}_j^\dagger \hat{\psi}_j$ is the number operator expressed in the space of fermionic degrees of freedom, and the parity operator $\hat{P} = \prod_{j=1}^{N} \sigma_j^z = e^{i\pi\hat{N}}$ is a symmetry of the model. This symmetry leads to the separation of the Hilbert space in two independent sectors, corresponding to the eigenvalues ± 1 of $\hat{P}$ respectively. Eq. (D.1) can then easily be transcribed to the path integral language, with the constraint that the $\hat{\mathbb{1}} \pm e^{i\pi\hat{N}}$ factors are absorbed in the functional measures, in order to be taken into account during all subsequent functional integrations. The same argument can further be used when considering expectation values of operators which preserve the sectors, where as an example the complete correlation function of $\hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)$ can be computed

as

$$\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c = \frac{Z^{(e)}}{Z}\frac{\delta^2 \ln Z^{(e)}[J]}{\delta J_b(\tau_2)\delta \bar{J}_a(\tau_1)}\bigg|_{J=0} + \frac{Z^{(o)}}{Z}\frac{\delta^2 \ln Z^{(o)}[J]}{\delta J_b(\tau_2)\delta \bar{J}_a(\tau_1)}\bigg|_{J=0}. \quad \text{(D.2)}$$

In the main text the study focused only on the first term, where

$$\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(e)} = \frac{\delta^2 \ln Z^{(e)}[J]}{\delta J_b(\tau_2)\delta \bar{J}_a(\tau_1)}\bigg|_{J=0}, \quad \text{(D.3)}$$

which can also provide the result for the corresponding $\langle \hat{T}\left(\hat{\psi}_b^\dagger(\tau_2)\hat{\psi}_a(\tau_1)\right)\rangle_c^{(o)}$ factor through the change of $m + 1/2 \to m$ in the Fourier transformation (4.17). In the thermodynamic limit $N \to \infty$, both these factors are expected to reduce to the full correlation function. Nevertheless, the decomposition (D.2) of the generating functionals is valid only in the case that the operator under study preserves the sector. In the opposite case a different approach should be considered.

Regarding the partition function (D.1), the path integral computation becomes a bit more involved when considering each sector separately, exactly due to the $\hat{\mathbb{1}} \pm e^{i\pi\hat{N}}$ factors which must be taken into account during all calculations. Considering the case of $Z^{(e)}$, the computation of the corresponding functional integral can proceed using two methods. The first one is to continue with the aforementioned construction and compute the functional determinants of the operators $\partial_\tau + \epsilon_m$, restricted to the Hilbert space spanned by the even sector. This procedure though is far more involved than the following one, which amounts to taking into account the projection factor by including it in the Hamiltonian of the system and not by absorbing it into the measure.

In the main text the $Z^{(e)}$ term was studied, in which the Hamiltonian is restricted only on the even fermion number sector. This allowed for the use of the Fourier transformations (4.17) and the subsequent Bogoliubov transformation, which diagonalized the Hamiltonian function as given in eq. (4.22). For the functional derivation of $Z^{(e)}$ one then has to compute two functional integrals corresponding to tr $\left[e^{-\beta\hat{H}^{(e)}}\right]$ and to

$\text{tr}\left[e^{i\pi\hat{N}}e^{-\beta\hat{H}^{(e)}}\right]$. The first trace can be immediately found to be

$$\text{tr}\left[e^{-\beta\hat{H}^{(e)}}\right] = \prod_{m=0}^{N/2-1}\int_{(-)}\mathcal{D}\bar{\eta}_m\mathcal{D}\eta_m e^{\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}}d\tau\bar{\eta}_m D_m\eta_m}, \tag{D.4}$$

in analogy to the main text. This integral differs from the normalization of (4.29), which needs to be computed, since in eq. (D.4) the computation of the functional determinant has no sector constraint. The even sector constraint will be taken into account through the calculation of the second trace. The computation proceeds then as usual [3] giving

$$\text{tr}\left[e^{-\beta\hat{H}^{(e)}}\right] = \prod_{m=0}^{N-1}2\cosh(\beta\epsilon_m/2). \tag{D.5}$$

For the second trace, it is far more simple to perform the Fourier and the subsequent Bogoliubov transformations on the operators $\hat{\psi}_j$ of the quantum Hamiltonian, instead of performing them in the fields of the path integral. This leads to the very simple rewriting of the parity operator $\hat{P} = e^{i\pi\hat{N}}$ in the space of the $\hat{\xi}_m$ Bogoliubov modes

$$\begin{aligned}\hat{P} = e^{i\pi\sum_{j=1}^{N}\hat{\psi}_j^\dagger\hat{\psi}_j} = e^{i\pi\sum_{m=0}^{N-1}\hat{c}_m^\dagger\hat{c}_m} = \prod_{m=0}^{N/2-1}\left(1-2\hat{c}_m^\dagger\hat{c}_m\right)\left(1-2\hat{c}_{N-m-1}^\dagger\hat{c}_{N-m-1}\right) = \\ = \prod_{m=0}^{N/2-1}\left(1-2\hat{\xi}_m^\dagger\hat{\xi}_m\right)\left(1-2\hat{\xi}_{N-m-1}^\dagger\hat{\xi}_{N-m-1}\right) = e^{i\pi\sum_{m=0}^{N-1}\hat{\xi}_m^\dagger\hat{\xi}_m} \equiv e^{i\pi\hat{N}_q}.\end{aligned} \tag{D.6}$$

At the same time, the Hamiltonian operator of the even sector assumes the form

$$\hat{H}^{(e)} = \sum_{m=0}^{N-1}\epsilon_m\left(\hat{\xi}_m^\dagger\hat{\xi}_m - \frac{1}{2}\right), \tag{D.7}$$

with ϵ_m given in eq. (4.23). The transition to path integration then proceeds smoothly by combining the operators $\hat{N}_q$ and $\hat{H}^{(e)}$ in a single exponential,

$$\begin{aligned}e^{i\pi\hat{N}_q}e^{-\beta\hat{H}^{(e)}} &= \exp\left\{i\pi\sum_{m=0}^{N-1}\hat{\xi}_m^\dagger\hat{\xi}_m - \beta\sum_{\mu=0}^{N-1}\epsilon_m\left(\hat{\xi}_m^\dagger\hat{\xi}_m - \frac{1}{2}\right)\right\} = \\ &= e^{i\pi\frac{N}{2}}\exp\left\{-\beta\sum_{m=0}^{N-1}\left(\epsilon_m - i\frac{\pi}{\beta}\right)\left(\hat{\xi}_m^\dagger\hat{\xi}_m - \frac{1}{2}\right)\right\},\end{aligned} \tag{D.8}$$

which is possible since $[\hat{N}_q, \hat{H}^{(e)}] = 0$, and define a new Hamiltonian operator

$$\hat{\tilde{H}}^{(e)} = \sum_{\mu=0}^{N-1} \left(\epsilon_m - i\frac{\pi}{\beta}\right)\left(\hat{\xi}_m^\dagger \hat{\xi}_m - \frac{1}{2}\right). \tag{D.9}$$

It is easy to see that by mapping the trace of this operator to its path integral representation one ends up with

$$\mathrm{tr}\left[e^{-\beta\hat{\tilde{H}}^{(e)}}\right] = e^{i\pi\frac{N}{2}} \prod_{m=0}^{N/2-1} \int_{(-)} \mathcal{D}\bar{\eta}_m \mathcal{D}\eta_m e^{\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \bar{\eta}_m D'_m \eta_m}, \tag{D.10}$$

where the matrix D'_m is now defined as

$$D'_m = \begin{pmatrix} \partial_\tau + \epsilon_m - i\frac{\pi}{\beta} & 0 \\ 0 & \partial_\tau - \epsilon_m + i\frac{\pi}{\beta} \end{pmatrix}. \tag{D.11}$$

This path integral can be computed with the usual method [3] leading to

$$\mathrm{tr}\left[e^{-\beta\hat{\tilde{H}}^{(e)}}\right] = e^{i\pi\frac{N}{2}} \prod_{m=0}^{N-1} \left(e^{\beta\epsilon_m/2 - i\pi/2} + e^{-\beta\epsilon_m/2 + i\pi/2}\right) = \prod_{m=0}^{N-1} 2\sinh\left(\beta\epsilon_m/2\right). \tag{D.12}$$

The complete $Z^{(e)}$ can then be recovered as

$$\begin{aligned} Z^{(e)} &= \frac{1}{2}\left(\mathrm{tr}\left[e^{-\beta\hat{H}^{(e)}}\right] + \mathrm{tr}\left[e^{-\beta\hat{\tilde{H}}^{(e)}}\right]\right) = \\ &= \frac{1}{2}\left(\prod_{m=0}^{N-1} 2\cosh\left(\beta\epsilon_m/2\right) + \prod_{m=0}^{N-1} 2\sinh\left(\beta\epsilon_m/2\right)\right) \end{aligned} \tag{D.13}$$

and coincides with the known result [30]. The computation of $Z^{(o)}$ can be performed using the same method, giving

$$Z^{(o)}(\beta) = \frac{1}{2}\left(\prod_{m=0}^{N-1} 2\cosh\left(\beta\epsilon_m/2\right) - \prod_{m=0}^{N-1} 2\sinh\left(\beta\epsilon_m/2\right)\right), \tag{D.14}$$

with $m + 1/2 \to m$. It is easy to see that in both of these results the second factor becomes irrelevant in the thermodynamic limit $N \to \infty$, where eqs. (D.13) and (D.14) become equal

$$\lim_{N\to\infty} Z^{(e)}(\beta) = \lim_{N\to\infty} Z^{(o)}(\beta) = \frac{1}{2}\prod_{m=0}^{N-1} 2\cosh(\beta\epsilon_m/2). \tag{D.15}$$

Since for finite N the generating functionals in the even and the odd sectors are weighted by different normalization factors, as seen in eq. (D.2), this normalization has to be taken into account for the complete correlation function. Nevertheless, these factors prove to be irrelevant when one is interested in the study of a single sector, as pointed out in the main text.

E Path integral formulation and the Schwinger-Keldysh formalism

In this appendix, eqs. (4.62) and (5.8) of sections 4 and 5 of the main text will be derived.

The main idea is to consider a set of degrees of freedom - denoted $\vec{Z}$ - which may correspond to an isolated system, as in section 4, or may be separated to system and environment ones, as in section 5. Since the construction of the formalism initially does not depend on this specific nature of the degrees of freedom, the following formulation will proceed as generally as possible and specific cases will be considered only when necessary.

The first step towards these proofs is to interpret [18,20] the coherent-state basis matrix elements of the time evolution operator as in eqs. (2.90), (2.91)

$$\langle \vec{Z}_b | \hat{U}(t_b, t_a) | \vec{Z}_a \rangle = \int_{\vec{Z}(t_a)=\vec{Z}_a}^{\vec{\bar{Z}}(t_b)=\vec{\bar{Z}}_b} \mathcal{D}^2 \vec{Z} e^{-\Gamma_{ba} + i \int_{t_a}^{t_b} dt \left[\frac{i}{2} \left(\vec{\bar{Z}} \dot{\vec{Z}} - \dot{\vec{\bar{Z}}} \vec{Z} \right) - H_{cl}(\vec{\bar{Z}}, \vec{Z}) \right]}, \tag{E.1}$$

with

$$\Gamma_{ba} = \frac{1}{2} \left(|\vec{Z}_b|^2 + |\vec{Z}_a|^2 \right) - \frac{1}{2} \left(\vec{\bar{Z}}_b \vec{Z}(t_b) + \vec{\bar{Z}}(t_a) \vec{Z}_a \right). \tag{E.2}$$

The key identity which shall be used then, is the one that refers to the measure of integration in coherent-state path integrals:

$$\int d^2\vec{Z}_b \int_{\vec{Z}(t_b)=\vec{Z}_b}^{\vec{Z}(t_c)=\vec{Z}_c} \mathcal{D}^2\vec{Z}(t) e^{-\Gamma_{cb}+i\int_{t_b}^{t_c} dt L(\vec{Z},\vec{Z})} \int_{\vec{Z}(t_a)=\vec{Z}_a}^{\vec{Z}(t_b)=\vec{Z}_b} \mathcal{D}^2\vec{Z}'(t) e^{-\Gamma_{ba}+i\int_{t_a}^{t_b} dt L(\vec{Z}',\vec{Z}')} = \\ = \int_{\vec{Z}(t_a)=\vec{Z}_a}^{\vec{Z}(t_c)=\vec{Z}_c} \mathcal{D}^2\vec{Z}(t) e^{-\Gamma_{ca}+i\int_{t_a}^{t_c} dt L(\vec{z},\vec{Z})}. \tag{E.3}$$

This identity, which can be easily proved using the discrete version of the relevant integrals, represents the fundamental property of the time evolution operator

$$\hat{U}(t_c,t_b)\hat{U}(t_b,t_a) = \hat{U}(t_c,t_a), \quad t_a < t_b < t_c, \tag{E.4}$$

translated in the language of coherent-state path integrals.

This identity can then be applied repeatedly if the first term of eq. (4.61) - and (5.7) in the more general case - is written in a form parametrized along the Keldysh contour:

$$\begin{aligned} &F_{ij}^{(t_2>t_1)}\left(\vec{Z}',\vec{Z};t_2,t_1\right) = \\ &= \langle\vec{Z}'|\hat{U}(t_{in-},T_-)\hat{U}(T_+,t_{2+})\hat{a}_i^\dagger(t_{in})\hat{U}(t_{2+},t_{1+})\hat{a}_j(t_{in})\hat{U}(t_{1+},t_{in+})|\vec{Z}\rangle\,\Theta(t_2-t_1) = \\ &= \int d^2\vec{Z}_1 \int d^2\vec{Z}_2 \int d^2\vec{Z}_3\, \langle\vec{Z}'|(t_{in-},T_-)|\vec{Z}_3\rangle\, \langle\vec{Z}_3|(T_+,t_{2+})|\vec{Z}_2\rangle \times \\ &\times \langle\vec{Z}_2|\hat{a}_i^\dagger(t_{in})\hat{U}(t_{2+},t_{1+})\hat{a}_j(t_{in})|\vec{Z}_1\rangle\, \langle\vec{Z}_1|\hat{U}(t_{1+},t_{in+})|\vec{Z}\rangle\,\Theta(t_2-t_1), \end{aligned} \tag{E.5}$$

where an arbitrary time instance T has been introduced, which will act as the limit between the time and anti-time ordered phases of the time evolution of the system. By interpreting each one of the factors as a path integral, the following expression can

then be found:

$$F_{ij}^{(t_2>t_1)}\left(\vec{Z}',\vec{Z};t_2,t_1\right)=\int d^2\vec{Z}_1\int d^2\vec{Z}_2\int d^2\vec{Z}_3\int\limits_{\vec{m}(T_-)=\vec{Z}_3}^{\vec{m}(t_{in-})=\vec{Z}'}\mathcal{D}^2\vec{m}(t_-)e^{-\Gamma_{in-,3}+i\int_{T_-}^{t_{in-}}dt_-L(\vec{m},\vec{m}))}\times$$

$$\times\int\limits_{\vec{n}(t_{2+})=\vec{Z}_2}^{\vec{n}(T_+)=\vec{Z}_3}\mathcal{D}^2\vec{n}(t_+)e^{-\Gamma_{32}+i\int_{t_{2+}}^{T_+}dt_-L(\vec{n},\vec{n})}\int\limits_{\vec{k}(t_{1+})=\vec{Z}_1}^{\vec{k}(t_{2+})=\vec{Z}_2}\mathcal{D}^2\vec{k}(t_+)e^{-\Gamma_{21}+i\int_{t_{1+}}^{t_{2+}}dt_+L(\vec{k},\vec{k})}\bar{k}_i(t_{2+})k_j(t_{1+})\times$$

$$\times\int\limits_{\vec{\lambda}(t_{in+})=\vec{Z}}^{\vec{\lambda}(t_{1+})=\vec{Z}_1}\mathcal{D}^2\vec{\lambda}(t_+)e^{-\Gamma_{1,in+}+i\int_{t_{in+}}^{t_{1+}}dt_+L(\vec{\lambda},\vec{\lambda})}\Theta(t_2-t_1).\tag{E.6}$$

The combining identity (E.3) can be then applied along the Keldysh contour P to arrive at the result

$$F_{ij}^{(t_2>t_1)}(\vec{Z}',\vec{Z};t_2,t_1)=\tag{E.7}$$

$$=\int\limits_{\vec{Z}(t_{in+})=\vec{Z}}^{\vec{Z}(t_{in-})=\vec{Z}'}\mathcal{D}^2\vec{Z}(t_P)e^{-\Gamma_{in-,in+}+i\int_P dt_P L(\vec{z},\vec{z},\vec{\zeta},\vec{\zeta})}\bar{Z}_i(t_{2+})Z_j(t_{1+})\Theta(t_2-t_1).$$

Following the same reasoning, the second term in eqs. (4.61) and (5.7) can also be parametrized along the contour P to get

$$F_{ij}^{(t_2>t_1)}\left(\vec{Z}',\vec{Z};t_2,t_1\right)=\tag{E.8}$$

$$=\langle\vec{Z}'|\hat{U}(t_{in-},T_-)\hat{U}(T_+,t_{1+})\hat{a}_j(t_{in})\hat{U}(t_{1+},t_{2+})\hat{a}_i^\dagger(t_{in})\hat{U}(t_{2+},t_{in+})|\vec{Z}\rangle\,\Theta(t_1-t_2),$$

which can be represented as a path integral of the form

$$F_{ij}^{(t_1>t_2)}(\vec{Z}',\vec{Z};t_2,t_1)=\tag{E.9}$$

$$=\int\limits_{\vec{Z}(t_{in+})=\vec{Z}}^{\vec{Z}(t_{in-})=\vec{Z}'}\mathcal{D}^2\vec{Z}(t_P)e^{-\Gamma_{in-,in+}+i\int_P dt_P L(\vec{z},\vec{z},\vec{\zeta},\vec{\zeta})}\bar{Z}_i(t_{2+})Z_j(t_{1+})\Theta(t_1-t_2).$$

The exponential factor appearing in the last expression assumes the standard form:

$$\Gamma_{in-,in+}=\frac{1}{2}\left(|\vec{Z}|^2+|\vec{Z}'|^2\right)-\frac{1}{2}\left(\vec{\bar{Z}}(t_{in+})\cdot\vec{Z}+\vec{\bar{Z}}'\cdot\vec{Z}(t_{in-})\right).\tag{E.10}$$

By combining then eqs. (E.7) and (E.9), eq. (5.8) of the main text is recovered, in which case the fields $\bar{Z}_i$ and Z_j correspond only to a specific subsystem.

In the following segment of the present appendix, the degrees of freedom $\vec{Z}$ are considered corresponding to an isolated thermal quantum system of inverse temperature β, which is the case of subsection 4.3. This allows for the extension of the P contour by a straight line, denoted L_β, running parallel to the imaginary axis from t_{in-} to $t_{in-} - i\beta$. This differs from the case analyzed in section 5, where the degrees of freedom are separated to system and environment ones and the procedure may be generalized such that only the environment is in a thermal state.

In the following, the case of subsection 4.3 is addressed, yet in more generality, since both cases of bosons and fermions are considered. Furthermore, the symbol $\vec{Z}$ is retained for the fields, despite that in sections 4 and 5 different symbols may be used to denote each specific case. In the case that the initial state of the system is considered isolated and thermal with inverse temperature β, the first matrix element of eq. (4.60), corresponding to the density matrix, can also be written as a path integral. More specifically, its corresponding path integral form

$$\begin{aligned} &\langle \pm\vec{Z}|\hat{\rho}(t_{in})|\vec{Z}'\rangle = \\ &= \frac{1}{Z_{in}(\beta)} \int\limits_{\vec{Z}(t_{in}-i0)=\vec{Z}'}^{\bar{\vec{Z}}(t_{in}-i\beta)=\pm\vec{Z}} \mathcal{D}^2\vec{Z}(t_{im}) e^{-\Gamma_{\beta 0}+i\int_{t_{in}-i0}^{t_{in}-i\beta} dt_{im}\left[\frac{i}{2}\left(\bar{\vec{Z}}\dot{\vec{Z}}-\dot{\bar{\vec{Z}}}\vec{Z}\right)-H_{in,cl}(\bar{\vec{Z}},\vec{Z})\right]}, \end{aligned} \tag{E.11}$$

is such that, the paths are parametrized along the lower imaginary axis, running from $t_{in} - i0$ to $t_{in} - i\beta$. Here and in the following, the upper symbol in $\pm$ denotes the bosonic case, while the lower denotes the fermionic one. The reason for the unconventional parametrization of eq. (E.11) is that, combining all the path integral expressions in eq. (4.60) and using the identity (E.3), one gets a functional integral parametrized along the Keldysh contour C (see fig. (1))

$$G_{ij}(\rho; t_2, t_1) = \frac{1}{Z_{in}(\beta)} \int\limits_{(\pm)} \mathcal{D}^2\vec{Z}(t_C) e^{i\int_C dt_C\left[i\bar{\vec{Z}}\dot{\vec{Z}}-H_{XYcl}(\bar{\vec{Z}},\vec{Z})\right]} \bar{Z}_i(t_{2+}) Z_j(t_{1+}). \tag{E.12}$$

The parametrization in this integral has been defined along the Keldysh contour and can be read from expressions (E.5) and (E.8), beginning from the right end: in the case of eq. (E.5) $(t_2 > t_1)$ the contour begins from the point $t_{in} + i0$ and continues till the point $t_1 + i0 \equiv t_{1+}$, where the operator $\hat{a}_j(t_1)$ acts. Then, it continues till the point $t_2 + i0$, where the operator $\hat{a}_j^\dagger(t_2)$ acts and afterwards proceeds to a point $T + i0$ completing the contour part above the real axis, which in its entirety is denoted as L_+. Afterwards, the contour continues along the line L_-, which joins the points $T - i0$ and $t_{in} - i0$, below the real axis. Finally, the contour continues parallel to the imaginary axis, joining the point $t_{in} - i0$ with $t_{in} - i\beta$, with the corresponding contour line denoted L_β. The trace operation in eq. (4.56) induces the periodic/anti-periodic boundary conditions, which are symbolically denoted as $(\pm)$ in the path integral. These boundary conditions are responsible for the disappearance of the boundary terms that appear in each of the individual integrals. This is an important simplification, as will be confirmed both in subsection 4.3 and in section 5, where in the latter the more general case of a system coupled to a thermal environment is addressed. The case of eq. (E.8) $(t_1 > t_2)$ follows naturally. In eq. (4.63) one can also find the corresponding generating functional, through which all correlation functions can be produced, with an example being the two point functions

$$G_{ij,\mu\nu}(\rho; t_2, t_1) = \frac{1}{Z_{in}(\beta)} \int\limits_{(\pm)} \mathcal{D}^2\vec{Z}(t_C) e^{i\int_C dt_C \left[i\vec{\bar{Z}}\dot{\vec{Z}} - H_{XYcl}(\vec{\bar{Z}},\vec{Z})\right]} \bar{Z}_i(t_{2\mu}) Z_j(t_{1\nu}), \qquad \text{(E.13)}$$

where $\mu, \nu = +, -$.

F Calculations along the Keldysh Contour

For calculations along the Keldysh contour, it is important to take into account the natural ordering defined on it. In this context, it is natural to introduce the contour step function $\Theta_C(t_C - t'_C)$, which becomes 1 only when the time instance t_C follows the time instance t'_C along the Keldysh contour C. More specifically, it is easy to see that along L_+: $\Theta_C(t_+ - t'_+) \equiv \Theta_{++}(t - t') = \Theta(t - t')$, along L_-: $\Theta_C(t_- - t'_-) \equiv \Theta_{--}(t - t') = \Theta(t' - t)$, while after a Wick rotation $t_{im} = t_{in} - i\tau$ one gets $\Theta_C(t_{im} - t'_{im}) \equiv \Theta(\tau - \tau')$, along the imaginary axis L_β. In accordance, the contour delta function

$\delta_C(t_C - t'_C)$ can be easily derived through equation

$$\int_C dt'_C f(t'_C)\delta_C(t'_C - t_C) = f(t_C). \tag{F.1}$$

This equation gives that $\delta_C(t_C - t'_C)$ is zero when the time instances t_C and t'_C belong in different points of C, while $\delta_C(t_+ - t'_+) = \delta(t - t')$, $\delta_C(t_- - t'_-) = -\delta(t - t')$ and $\delta_C(t_{im} - t'_{im}) = i\delta(\tau - \tau')$. As expected, the following distribution property holds:

$$\partial_{t_C}\Theta_C(t_C - t'_C) = \delta_C(t_C - t'_C). \tag{F.2}$$

Using these rules for the algebra, one can study integrals like the ones in eq. (4.65) exactly like their real time analogues, just by substituting all functions/distributions appearing along the way with their Keldysh contour equivalents. This procedure, used specifically for the integrals in (4.65), leads to the result (4.68). A strict method to derive this algebra is presented in appendix H, where the empirical procedure of subsection 4.3 is performed more strictly for the calculation of the influence functional (5.15), along the Keldysh contour.

As a concrete example for the use of the aforementioned algebra, the solution to the Green's equation (4.69) without the $\dot{\theta}_m$ term follows. This Green's function will prove very useful in the main text

$$\begin{pmatrix} i\partial_{t_C} - \epsilon_m(t_C) & 0 \\ 0 & i\partial_{t_C} + \epsilon_m(t_C) \end{pmatrix} \begin{pmatrix} G_m^{(+)}(t_C, t'_C) & 0 \\ 0 & G_m^{(-)}(t_C, t'_C) \end{pmatrix} = \hat{\mathbb{1}}\delta_C(t_C - t'_C). \tag{F.3}$$

According to the conventions used here for the magnetic field, the energy does not distinguish the L_+ and L_- lines and thus $\epsilon_m(t_+) = \epsilon_m(t_-)$, while it remains constant along the imaginary axis $\epsilon_m = \epsilon_m(h_{in})$. The function $G_m^{(+)}$, which propagates the ζ modes (ζ being the field symbol used in section 4), obeys the antiperiodic boundary conditions and assumes the form

$$G_m^{(+)}(t_C, t'_C) = -i\left[\Theta_C(t_C - t'_C) - \left(1 + e^{\beta\epsilon_m(h_{in})}\right)^{-1}\right] e^{-i\int_{t'_C}^{t_C} dt''_C \epsilon_m(t''_C)}. \tag{F.4}$$

The corresponding function for the $N - m - 1$ modes, which again obeys the antiperiodic boundary conditions assumes the form

$$G_m^{(-)}(t_C, t'_C) = -i\left[\left(1 + e^{\beta\epsilon_m(h_{in})}\right)^{-1} - \Theta_C(t'_C - t_C)\right] e^{i\int_{t'_C}^{t_C} dt''_C \epsilon_m(t''_C)}. \tag{F.5}$$

These functions, at the zero-temperature limit, are the Keldysh analogues of the functions $G_m^{(\pm)}$ in eq. (4.30) of the main text. Finally, it is easy to see that $G_m^{(-)}(t_C, t'_C) = -G_m^{(+)}(t'_C, t_C)$.

G Calculation of equal time correlation functions

In this appendix one can find the proofs for the basic relations of subsection 4.3.

The normalization factor of eq. (4.68) can be expressed in the form

$$Z_m^{(e)}[0] = \mathrm{Det}\left(D_m - \dot{\theta}_m \sigma^x\right) = \mathrm{Det}\left(D_m\right)\mathrm{Det}\left(\mathbb{1} - K_m\right), \tag{G.1}$$

with $K_m = G_m \dot{\theta}_m \sigma^x$[18] and thus

$$Z^{(e)}[0] = Z_0^{(e)}(\beta)\left(\prod_{m=0}^{N/2-1}\mathrm{Det}\left(\mathbb{1} - K_m\right)\right), \tag{G.2}$$

with $Z_0^{(e)}(\beta) = Z_{in}^{(e)}(\beta) = \prod_{m=0}^{N/2-1}\mathrm{Det}\left(D_m\right)$. In the previous equation the Green's function G_m has been introduced, satisfying eq. $D_m G_m = \hat{\mathbb{1}}$, the solution of which was found in appendix F. Repeating the arguments of appendix D, the first factor in eq. (G.2) is immediately found to be

$$Z_0^{(e)}(\beta) = \frac{1}{2}\left\{\prod_{m=0}^{N-1} 2\cosh\left(\beta\epsilon_m(h_{in})/2\right) + \prod_{m=0}^{N-1} 2\sinh\left(\beta\epsilon_m(h_{in})/2\right)\right\}. \tag{G.3}$$

For the calculation of the second term one can rewrite

$$\mathrm{Det}\left(\hat{\mathbb{1}} - K_m\right) = \mathrm{Det}\left(\hat{\mathbb{1}} - \lambda K_m\right)\big|_{\lambda=1}, \tag{G.4}$$

in order to take advantage of the fact that the $\mathrm{Det}\left(\hat{\mathbb{1}} - \lambda K_m\right)$ can be expanded as a convergent power series with respect to λ [85]

$$\mathrm{Det}\left(\hat{\mathbb{1}} - \lambda K_m\right) = \sum_{n=0}^{\infty}\frac{1}{n!}a_n^{(m)}\lambda^n, \tag{G.5}$$

[18] In index notation

$$\left(K_m(t_C - t'_C)\right)^a{}_b = \left(G_m(t_C - t'_C)\right)^a{}_c\,(\sigma^x)^c{}_b\,\dot{\theta}_m(t'_C).$$

with

$$a_n^{(m)} = (-1)^n \det \begin{pmatrix} \mathrm{Tr}K_m & n-1 & & \\ \mathrm{Tr}K_m^2 & \mathrm{Tr}K_m & n-2 & \\ . & . & . & . \\ . & . & . & . \\ \mathrm{Tr}K_m^n & \mathrm{Tr}K_m^{n-1} & . & . \ \mathrm{Tr}K_m \end{pmatrix}, \quad n \geq 1 \tag{G.6}$$

and $a_0 = 1$. In this expression, the trace symbol denotes tracing over time and matrix indices:

$$\mathrm{Tr}\,(\dots) = \mathrm{tr} \int_C dt_C \, \langle t_C|(\dots)|t_C\rangle = \int_C dt_C \sum_a \langle t_C|(\dots)_{aa}|t_C\rangle \,. \tag{G.7}$$

This series is convergent $\forall\ \lambda$ as long as K_m is well behaved, i.e. when $||K_m||^2 < \infty$, which is true for the Keldysh contour as long as $\beta < \infty$ and $T < \infty$. In any case though, one can compute the result in the region where the series converges and then analytically continue it outside of it. Thus, each order can be systematically understood from this expansion as a series in powers of $\dot{\theta}_m$. It can be easily confirmed that $\mathrm{Tr}K_m^{2\nu-1} = 0$, $\nu = 1, 2, \dots$. Thus, in the coefficients (G.6) only the even powers of K_m contribute. Taking then the limit of $\lambda = 1$, the functional determinant can be brought to the form

$$\begin{aligned} &\mathrm{Det}\left(\hat{\mathbb{1}} - K_m\right) = \exp\left\{\mathrm{logDet}\left(\hat{\mathbb{1}} - K_m\right)\right\} = \\ &= \exp\left\{\log\left[1 - \frac{1}{2}\mathrm{Tr}K_m^2 - \frac{1}{4}\left(\mathrm{Tr}K_m^4 - \frac{1}{2}\left(\mathrm{Tr}K_m^2\right)^2\right) + \dots\right]\right\} = \\ &= \exp\left\{-\frac{1}{2}\mathrm{Tr}K_m^2 - \frac{1}{4}\mathrm{Tr}K_m^4 - \dots\right\}. \end{aligned} \tag{G.8}$$

As a result, the normalization factor (G.1) can be recasted to the form

$$Z^{(e)}[0] = Z_0^{(e)}(\beta) e^{-\frac{1}{2}\sum_{m=0}^{N-1} E_m}, \tag{G.9}$$

where

$$E_m = \frac{1}{2}\mathrm{Tr}K_m^2 + \frac{1}{4}\mathrm{Tr}K_m^4 + \dots. \tag{G.10}$$

Regarding the solution to the Green's equation

$$\tilde{D}_m \tilde{G}_m = \left(D_m - \dot{\theta}_m \sigma^x\right) \tilde{G}_m = \hat{\mathbb{1}}, \tag{G.11}$$

it can be seen that

$$\tilde{G}_m = \frac{\hat{\mathbb{1}}}{\hat{\mathbb{1}} - K_m} G_m, \quad K_m = G_m \dot{\theta}_m \sigma^x, \tag{G.12}$$

which can be easily handled using the Hemholtz technique [86]. Through this technique, $\tilde{G}_m$ can be interpreted in the following form:

$$\tilde{G}_m = \frac{1}{\mathrm{Det}(\hat{\mathbb{1}} - K_m)} \left(\sum_{k=0}^{\infty} \frac{1}{k!} b_k^{(m)} \right) G_m, \tag{G.13}$$

where the coefficients $b_k^{(m)}$ can be found by substituting $\mathrm{Tr}K_m^n \to \mathrm{Tr}K_m^n - K_m^n$ in $a_k^{(m)}$ of eq. (G.6).

To express this result in the form of a single sum, the abbreviation $A_n^{(m)}$ can be used for the term involving all terms of the form $K_m^k \mathrm{Tr}K_m^l$ for which $k + l = n$. The previous result can then symbolically be written as the expansion

$$\tilde{G}_m = \sum_{n=0}^{\infty} A_n^{(m)} G_m = G_m + M_m. \tag{G.14}$$

Considering the case of a magnetic field $h = h(\omega t)$, one can easily estimate the contribution of the terms $M_m = K_m G_m + \mathcal{O}(\dot{\theta}_m^2)$, by performing the rescaling $\omega t_C \to t_C$. Since the function $\dot{\theta}_m$ appears always tied to a time integration (in the expansions of (G.8) and (G.14)), the rescaling leaves intact the combination $\int_C dt_C \dot{\theta}_m(t_C)$ at the limit $\beta \to \infty$ and $T \to \infty$. Thus, the frequency ω appears only in the exponential factors of the Green's functions as[19]

$$\exp\left(i \int_{t'_C}^{t_C} dt''_C \epsilon(t''_C) \right) \to \exp\left(i \int_{t'_C}^{t_C} dt''_C \epsilon(t''_C)/\omega \right). \tag{G.15}$$

In the $\beta \to \infty$ limit then, one can use the following identity repeatedly

$$\exp\left(i \int_{t'_C}^{t_C} dt''_C \epsilon(t''_C)/\omega \right) = -i \frac{\omega}{\epsilon_m(t_C)} \partial_{t_C} \exp\left(i \int_{t'_C}^{t_C} dt''_C \epsilon(t''_C)/\omega \right) \tag{G.16}$$

and perform the necessary partial integrations, in all terms appearing inside M_m. Assuming that the boundary terms go sufficiently fast to zero as $T \to \infty$, this method leads to a series expansion in powers of ω, which justifies the conclusion presented in the main text, that the contribution of M_m in eq. (4.73) is suppressed at the adiabatic limit. However, this argument fails if $\int_{t'_C}^{t_C} dt''_C \epsilon(t''_C) \simeq \omega \to 0$. This can only hap-

[19] The limits of integration appear unchanged, since the corresponding time variables have also been rescaled.

pen when the driven system approaches its critical neighbourhood: in the region of width $\delta t \sim \omega^{1/2}$, where $\epsilon \sim |h-1| \sim \omega^{1/2}$, the exponential factor in (G.15) is almost constant and the adiabatic expansion is not valid, in accordance to the KZ mechanism.

H Computation of the Influence Functional

In this appendix the integral

$$I_E\left[\vec{\bar{z}}_s(t_P), \vec{z}_s(t_P)\right] = \frac{1}{Z_E(\beta)} \int_{(\pm)} \mathcal{D}^2\vec{\bar{\zeta}}(t_C) e^{i\sum_{\mu\in E}\int_C dt_C\left(i\bar{\zeta}_\mu \partial_{t_C}\zeta_\mu - E_\mu\bar{\zeta}_\mu\zeta_\mu - \bar{\zeta}_\mu r_\mu - \bar{r}_\mu\zeta_\mu\right)} \tag{H.1}$$

is calculated.

Calculations along the C contour can be understood by introducing the parametrization

$$t_C = t_C(\lambda), \quad \lambda \in [0,1], \quad t_C(0) = t_{in+}, \quad t_C(1) = t_{in} - i\beta, \tag{H.2}$$

which allows for the change of variable

$$\int_C dt_C(...) = \int_0^1 d\lambda \dot{t}_C(...). \tag{H.3}$$

By performing the change of variables to the λ variable, the action in eq. (H.1) adopts the following form

$$S_E = \int_0^1 d\lambda \left(i\bar{\zeta}_\mu \partial_\lambda \zeta_\mu - \dot{t}_C E_\mu \bar{\zeta}_\mu \zeta_\mu - \dot{t}_C \bar{\zeta}_\mu r_\mu - \dot{t}_C \bar{r}_\mu \zeta_\mu\right). \tag{H.4}$$

Due to the quadratic form of the Hamiltonian, the integral can be exactly calculated just by minimizing the relevant action, leading to the following equations of motion for the fields ζ_μ and $\bar{\zeta}_\mu$

$$(i\partial_\lambda - \dot{t}_C E_\mu)\zeta_\mu = \dot{t}_C r_\mu, \quad (i\partial_\lambda + \dot{t}_C E_\mu)\bar{\zeta}_\mu = -\dot{t}_C \bar{r}_\mu. \tag{H.5}$$

The corresponding solutions with periodic/anti-periodic boundary conditions assume the form

$$\zeta_\mu^{(cl)} = \int_0^1 d\lambda' G_\mu(\lambda,\lambda')\dot{t}_C(\lambda') r_\mu(\lambda'), \quad \bar{\zeta}_\mu^{(cl)} = -\int_0^1 d\lambda' \bar{G}_\mu(\lambda,\lambda')\dot{t}_C(\lambda')\bar{r}_\mu(\lambda'), \tag{H.6}$$

where the Green's functions G_μ and $\bar{G}_\mu$ propagating the ζ_μ and $\bar{\zeta}_\mu$ modes are defined through the equations

$$\left(i\partial_\lambda - \dot{t}_C(\lambda)E_\mu\right) G_\mu(\lambda,\lambda') = \delta(\lambda-\lambda'), \quad \left(i\partial_\lambda + \dot{t}_C(\lambda)E_\mu\right) \bar{G}_\mu(\lambda,\lambda') = \delta(\lambda-\lambda'). \tag{H.7}$$

The solutions of eqs. (H.7) with periodic/anti-periodic boundary conditions can be found to be

$$G_\mu(\lambda,\lambda') = -i\left[\theta(\lambda-\lambda') - \left(1 \mp e^{\beta E_\mu}\right)^{-1}\right] e^{-iE_\mu \int_\lambda^{\lambda'} d\lambda'' \dot{t}_C(\lambda'')}, \tag{H.8}$$

$$\bar{G}_\mu(\lambda,\lambda') = -i\left[\left(1 \mp e^{\beta E_\mu}\right)^{-1} - \theta(\lambda'-\lambda)\right] e^{-iE_\mu \int_{\lambda'}^{\lambda} d\lambda'' \dot{t}_C(\lambda'')}. \tag{H.9}$$

Performing then the changes of variables $\zeta_\mu \to \zeta_\mu^{(cl)} + n_\mu$, $\bar{\zeta}_\mu \to \bar{\zeta}_\mu^{(cl)} + \bar{n}_\mu$ in the integral (H.1), where $n(0) = \pm n(1)$, $\bar{n}(0) = \pm \bar{n}(1)$, one finds

$$I_E\left[\vec{\bar{z}}_s(t_P), \vec{z}_s(t_P)\right] = \tilde{C}_E^{-1} e^{-i\sum_{\mu\in E}\int_0^1 d\lambda \int_0^1 d\lambda' \dot{t}_C(\lambda)\bar{r}_\mu(\lambda)G_\mu(\lambda,\lambda')r_\mu(\lambda')\dot{t}_C(\lambda')}, \tag{H.10}$$

where $\tilde{C}_E^{-1} = C_E^{-1} Z_E^{-1}(\beta)$ and

$$\begin{aligned} C_E^{-1} &= \int\limits_{(\pm)} \mathcal{D}^2 n(t_C) e^{i\sum_{\mu\in E}\int_C dt_C \bar{n}_\mu(t_C)(i\partial_{t_C} - E_\mu)n_\mu(t_C)} = \\ &= \prod_{\mu\in E}\left[\mathrm{Det}\left(i\partial_{t_C} - E_\mu\right)\right]^{\mp 1} = \prod_{\mu\in E} e^{\mp\frac{\beta E_\mu}{2}}\left(1 \mp e^{-\beta E_\mu}\right)^{\mp 1} \end{aligned} \tag{H.11}$$

is the normalization factor. One can see that the return to the variables $t_C = t_C(\lambda)$ leads to the Keldysh contour Green's functions

$$G_\mu(t_C, t_C') = -i\left[\Theta_C(t_C - t_C') - \left(1 \mp e^{\beta E_\mu}\right)^{-1}\right] e^{-iE_\mu(t_C - t_C')}, \tag{H.12}$$

$$\bar{G}_\mu(t_C, t_C') = -i\left[\left(1 \mp e^{\beta E_\mu}\right)^{-1} - \Theta_C(t_C' - t_C)\right] e^{-iE_\mu(t_C' - t_C)}, \tag{H.13}$$

where the contour step function is defined as

$$\Theta_C(t_C - t_C') = \Theta_C\left(t_C(\lambda) - t_C(\lambda')\right) = \Theta(\lambda - \lambda'), \tag{H.14}$$

coinciding with the Keldysh contour Θ presented in appendix H. As a result, the corresponding Keldysh contour delta distribution is defined accordingly.

With respect to the natural time indices t_C, the influence functional then becomes

$$I_E\left[\vec{\bar{z}}(t_P),\vec{z}(t_P)\right] = \tilde{C}_E^{-1} e^{-\sum_{j,k\in S}\int_P dt_P \int_P dt'_P \bar{z}_{sj}(t_P)\Delta_{j,k}(t_P,t'_P)z_{sk}(t'_P)}, \tag{H.15}$$

with

$$\Delta_{j,k}(t_P,t'_P) = \sum_{\mu\in E} g^*_{j\mu}g_{k\mu}\left[\Theta_C(t_P-t'_P) - \left(1\mp e^{\beta E_\mu}\right)^{-1}\right] e^{-i(t_P-t'_P)E_\mu}. \tag{H.16}$$

The integrals in the last exponential are only over the P contour due to the couplings $g_{k\mu}$ and $g^*_{j\mu}$ being zero in the L_β line and coming only from the interaction terms.

Restoring the parametrization along the contours allows for the definition of the following quantities

$$\begin{gathered} w_\mu(t) = \sum_{j\in S} g_{j\mu}z_j(t_+) = \sum_{j\in S} g_{j\mu}z_{j+}(t), \quad y_\mu(t) = \sum_{j\in S} g_{j\mu}z_j(t_-) = \sum_{j\in S} g_{j\mu}z_{j-}(t), \\ \tilde{w}_\mu = \int_{L_+} dt e^{itE_\mu}w_\mu(t), \quad \tilde{y}_\mu = \int_{L_-} dt e^{itE_\mu}y_\mu(t), \end{gathered} \tag{H.17}$$

where the contour parametrization of the lines L_+ and L_- is absorbed into the definition of the complex fields z (and their conjugates), such that the fields z_+ and z_- are defined only over L_+ and L_- respectively. It is then easy to find

$$\begin{aligned} I_E\left[\vec{\bar{z}}(t_P),\vec{z}(t_P)\right] = \tilde{C}_E^{-1} &\prod_{\mu\in E}\exp\left[-\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt'\bar{w}_\mu(t)e^{-i(t-t')E_\mu}w_\mu(t')\right]\times \\ &\times\prod_{\mu\in E}\exp\left[-\int_{-\infty}^{\infty}dt\int_{t}^{\infty}dt'\bar{y}_\mu(t)e^{-i(t-t')E_\mu}y_\mu(t')\right]\times \\ &\times\prod_{\mu\in E}\exp\left[-\bar{\tilde{y}}_\mu\tilde{w}_\mu + \frac{|\tilde{y}_\mu+\tilde{w}_\mu|^2}{(1\mp e^{\beta E_\mu})}\right]. \end{aligned} \tag{H.18}$$

Furthermore, in order to perform the Markovian approximation, the rescaling $z_{Sj}(t_P) \to z_{Sj}(t_P)e^{-i\epsilon_s t_P}$ may be performed on the system's fields, where ϵ_S sets the shortest time scale $\tau_s \sim 1/\epsilon_s$ that characterizes significant changes to the system. As pointed out in the main text, it is useful to consider an isotropic interaction of the environment with the subsystem's degrees of freedom, with $g_{j\mu} = \gamma_\mu$, and define the spectral density for the environment as $D(E) = \sum_{\mu\in E}|\gamma_\mu|^2\delta(E-E_\mu)$, which can later be extended in

the continuum limit. The integrals can then be easily studied in the Markovian limit, where the integrands can be assumed to be very fast decaying functions of the time difference [87–89], thus allowing for the expansion $z_{Sj_\pm}(t') \simeq z_{Sj_\pm}(t) + \mathcal{O}(t-t')$. In the same context, the integration over the t' variable, acting only on the Green's function of the environment, can be further extended from $-\infty$ to $+\infty$ [39], again by taking into account the speed of its oscillations. The integrals in eq. (H.18) can then be easily computed, where in the first two lines the integral representation of the Heaviside step function, and the Sokhotski-Plemelj theorem of complex calculus can be used to find

$$\sum_{\mu\in E}\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt'\bar{w}_\mu(t)e^{-i(t-t')E_\mu}w_\mu(t') \simeq \left(-i\delta E+\frac{\Gamma}{2}\right)\int_{-\infty}^{\infty}dt\Big|\sum_{j\in S}z_{+j}(t)\Big|^2, \quad \text{(H.19)}$$

$$\sum_{\mu\in E}\int_{-\infty}^{\infty}dt\int_{t}^{\infty}dt'\bar{y}_\mu(t)e^{-i(t-t')E_\mu}y_\mu(t') \simeq \left(i\delta E+\frac{\Gamma}{2}\right)\int_{-\infty}^{\infty}dt\Big|\sum_{j\in S}z_{-j}(t)\Big|^2, \quad \text{(H.20)}$$

with

$$\Gamma = 2\pi D(\epsilon_s), \quad \delta E = Pr.\int_0^\infty dE\frac{D(E)}{E-\epsilon_s}, \quad \text{(H.21)}$$

while for the rest of the terms

$$\sum_{\mu\in E}\bar{\tilde{y}}_\mu\tilde{w}_\mu = -\Gamma\int_{-\infty}^{\infty}dt\sum_{j,k\in S}\bar{z}_{-j}(t)z_{+k}(t), \quad \text{(H.22)}$$

$$\sum_{\mu\in E}\frac{|\tilde{y}_\mu+\tilde{w}_\mu|^2}{\left(1\mp e^{\beta E_\mu}\right)} = \Gamma b(\epsilon_s)\int_{-\infty}^{\infty}dt\sum_{j,k\in S}(\bar{z}_{+j}(t)-\bar{z}_{-j}(t))(z_{+k}(t)-z_{-k}(t)), \quad \text{(H.23)}$$

where $b(\epsilon_s) = \left(1\mp e^{\beta\epsilon_s}\right)^{-1}$.
Combining all previous results it can be seen that

$$\begin{aligned} &I_E\left[\vec{\bar{z}}(t_P),\vec{z}(t_P)\right] = \quad \text{(H.24)}\\ &= \tilde{C}_E^{-1}\exp\Bigg[-\int_{-\infty}^{\infty}dt\sum_{j,k\in S}\begin{pmatrix}\bar{z}_{+k}(t) & \bar{z}_{-j}(t)\end{pmatrix}\begin{pmatrix}\Delta_{++} & -\Delta_{+-}\\ -\Delta_{-+} & \Delta_{--}\end{pmatrix}\begin{pmatrix}z_{+k}(t)\\ z_{-k}(t)\end{pmatrix}\Bigg], \end{aligned}$$

with

$$\begin{aligned} \Delta_{++} = -i\delta E+\Gamma\left(\frac{1}{2}-b(\epsilon_s)\right), \quad \Delta_{--} = i\delta E+\Gamma\left(\frac{1}{2}-b(\epsilon_s)\right)\\ \Delta_{+-} = -\Gamma b(\epsilon_s), \quad \Delta_{-+} = \Gamma\left(1-b(\epsilon_s)\right). \end{aligned} \quad \text{(H.25)}$$

It is useful to re-express this result along the P contour, by writing

$$\int_{-\infty}^{\infty} dt \sum_{j,k\in S} \begin{pmatrix} \bar{z}_{+j}(t) & \bar{z}_{-j}(t) \end{pmatrix} \begin{pmatrix} \Delta_{++} & -\Delta_{+-} \\ -\Delta_{-+} & \Delta_{--} \end{pmatrix} \begin{pmatrix} z_{+k}(t) \\ z_{-k}(t) \end{pmatrix} =$$
$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \delta(t-t') \sum_{j,k\in S} \begin{pmatrix} \bar{z}_{+j}(t) & \bar{z}_{-j}(t) \end{pmatrix} \begin{pmatrix} \Delta_{++} & -\Delta_{+-} \\ -\Delta_{-+} & \Delta_{--} \end{pmatrix} \begin{pmatrix} z_{+k}(t') \\ z_{-k}(t') \end{pmatrix}, \tag{H.26}$$

which allows for the use of the time parametrization $z_j(t_+), z_j(t_-)$ instead of the field parametrization $z_{+j}(t), z_{-j}(t)$ respectively. The Δ kernel is then given as

$$\tilde{\Delta}(t_\mu - t'_\nu) = \Delta_{\mu\nu}\delta(t-t'), \quad \mu,\nu = +,-, \tag{H.27}$$

where in this case the delta distribution is the usual one, which refers only to the real coefficients of t_P and t'_P and differs from the Keldysh contour delta defined in (F.1). The result (H.24) then reduces to the form

$$I_E\left[\vec{\bar{z}}(t_P), \vec{z}(t_P)\right] = \tilde{C}_E^{-1} \exp\Bigg[- \int_P dt_P \int_P dt'_P \sum_{j,k\in S} \bar{z}_j(t_P)\tilde{\Delta}(t_P - t'_P) z_k(t'_P)\Bigg]. \tag{H.28}$$

I The harmonic oscillator's Green's function

In this appendix we present the solution to the equation

$$\int_P d\tilde{t}_P \left[\left(i\partial_{t_P} - \epsilon^{(R)}\right)\delta_P(t_P - \tilde{t}_P) + i\tilde{\Delta}'_P(t_P, \tilde{t}_P)\right] \tilde{L}_P(\tilde{t}_P, t'_P) = \delta_P(t_P - t'_P). \tag{I.1}$$

We use here the notation $\tilde{L}$ instead of simply L for the Green's function, in order to note that the solution identified using the Fourier transformation method may need an extra contribution from a solution of the homogeneous equation, to provide the appropriate propagator. By introducing the Fourier transforms

$$\tilde{L}_P(t_P, t'_P) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(t_P - t'_P)} \tilde{L}^F_{PP'}(k), \tag{I.2}$$

we can analyze eq. (I.1) along the branches of the Keldysh contour to find that

$$\begin{aligned} &(k-\epsilon_+)\tilde{L}^F_{++}(k) + i\Gamma b\tilde{L}^F_{-+}(k) = 1, \quad (k-\epsilon_-)\tilde{L}^F_{-+}(k) + i\Gamma(1-b)\tilde{L}^F_{++}(k) = 0, \\ &(k-\epsilon_-)\tilde{L}^F_{--}(k) + i\Gamma(1-b)\tilde{L}^F_{+-}(k) = -1, \quad (k-\epsilon_+)\tilde{L}^F_{+-}(k) + i\Gamma b\tilde{L}^F_{--}(k) = 0. \end{aligned} \tag{I.3}$$

In these equations we used the abbreviations

$$\epsilon_+ = \epsilon^{(R)} - i\Gamma\left(\frac{1}{2} - b\right), \quad \epsilon_- = \epsilon^{(R)} + i\Gamma\left(\frac{1}{2} - b\right) = \bar{\epsilon}_+. \tag{I.4}$$

The system's (I.3) equations can be easily solved, giving

$$\begin{aligned} \tilde{L}_{++}(k) &= \frac{k - \epsilon_-}{D}, \quad \tilde{L}_{-+}(k) = -i\frac{\Gamma(1-b)}{D}, \\ \tilde{L}_{--}(k) &= -\frac{k - \epsilon_+}{D}, \quad \tilde{L}_{+-}(k) = i\frac{\Gamma b}{D}, \end{aligned} \tag{I.5}$$

with

$$D = (k - k_+)(k - k_-), \quad k_\pm = \epsilon \pm i\frac{\Gamma}{2}. \tag{I.6}$$

By taking the inverse Fourier transforms of these results we find

$$\tilde{L}_P(t_+, t'_+) = \tilde{L}_{++}(t, t') = -i\Theta(t - t')e^{-i\epsilon^{(R)}(t-t') - \Gamma(t-t')/2} + ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{I.7}$$

$$\tilde{L}_P(t_-, t'_+) = \tilde{L}_{-+}(t, t') = -i(1-b)e^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{I.8}$$

$$\tilde{L}_P(t_-, t'_-) = \tilde{L}_{--}(t, t') = -i\Theta(t' - t)e^{-i\epsilon^{(R)}(t-t') - \Gamma(t'-t)/2} + ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2} \tag{I.9}$$

$$\tilde{L}_P(t_+, t'_-) = \tilde{L}_{+-}(t, t') = ibe^{-i\epsilon^{(R)}(t-t') - \Gamma|t-t'|/2}. \tag{I.10}$$

To identify then the true Green's function, one needs to add a solution $L_{hom}(t_P, t'_P)$ of the homogeneous analogue of (I.1) to the above result, in such a way that $L(t_{in+}, t'_P) + L_{hom}(t_{in+}, t'_P) = 0$. This function can be identified, through the above constraint and by imposing its continuity at the arbitrary time instance $+T$, to be of the form

$$L_{hom}(t_P, t'_P) = -ibe^{+\Gamma\left(t_{in} - \frac{t_P}{2} - \frac{t'_P}{2}\right) - i\epsilon^{(R)}(t_P - t'_P)}. \tag{I.11}$$

The true Green's function over the Keldysh contour P is thus given as

$$L(t_P, t'_P) = \tilde{L}(t_P, t'_P) + L_{hom}(t_P, t'_P). \tag{I.12}$$

References

[1] R.P. Feynman, Rev. Mod. Phys. 20,367 (1948)

[2] R.P. Feynman, Phys. Rev. 84, 108 (1951)

[3] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, 2006).

[4] U. Weiss, *Quantum Dissipative Systems* (World Scientific, 2008).

[5] R.J. Glauber, Phys. Rev. 131 (1963) 2766.

[6] A. Berezin, Sov. Phys. Usp. 23, (1980) 763.

[7] G. Junker, J.R. Klauder, Eur. Phys. J. C4 (1998) 173.

[8] J.R. Klauder, Phys. Rev. D19 (1979) 2349.

[9] J. Kurchan P. Leboeuf and M. Saraceno, Phys. Rev. A40 (1979) 6800.

[10] G. Kordas, D. Witthaut, P. Buonsante, A. Vezzani, R. Burion, A.I. Karanikas and S. Wimberger , Eur. Phys. J ST 224 (2015) 2127.

[11] G. Kordas, G. Pavlou and A.I. Karanikas, Phys. Rev. A98 (2018) 013637.

[12] T.J. Osborne and M.A. Nielsen , Phys. Rev. A66 (2002) 032110.

[13] G. Vidal, J.I. Latorre, E. Rico and A. Kitaev, Phys. Rev. Lett. 90 (2003) 227902.

[14] J.I. Lattore and A. Riera, J. Phys. A Math. Theor. 42 (2009) 504002.

[15] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, Rev. Mod. Phys. 83 (2011) 863.

[16] E. Canovi, E. Ercolessi P. Naldesi, L. Taddia and D. Vodola, Phys. Rev. B89 (2014) 104303.

[17] J.H. Wilson and V. Galitski, Phys. Rev. Lett. 106 (2011) 11401.

[18] G. Kordas, S.I. Mistakidis and A.I. Karanikas, Phys. Rev. A 90 (2014) 032104.

[19] G. Kordas, D. Kalantzis and A.I. Karanikas Ann.Physics 372 (2016) 226.

[20] G. Kordas, S.I. Mistakidis and A.I. Karanikas, Phys. Rev. A99 (2019) 026102.

[21] N.M.J. Woodhouse: *Geometric Quantization, Second Edition* Oxford University Press Inc., New York

[22] M. Blau, *Lecture Notes: Symplectic Geometry and Geometric Quantization* (1992)

[23] V. P. Nair, arXiv:1606.06407 [hep-th] (2016)

[24] Brian C. Hall: *Quantum Theory for Mathematicians* Springer Science+Business Media New York 2013

[25] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692.

[26] A. Greco, J. Phys. A Math. Theor. 49 (2016) 155004.

[27] E. Kochetov, Phys. Rev. A99, 026101 (2019)

[28] E. A. Kochetov, J. Phys. A Math. Gen. 31 (1998) 4473.

[29] A. Garg, E. A. Kochetov, K. Park, and M. Stone, J. Math Phys. 44, 48 (2003)

[30] O. Nachtmann, *Perturbative and non-Perturbative Aspects of Quantum Field Theory*. H. Latal, N. Schwinger (Eds) Springer-Verlag, Berlin, Heidelberg (1997).

[31] T. C. Wei, S. Vishveshwara, P. M. Goldbart, Quantum Inf. Comput. 11, 0326-0354 (2011)

[32] H. Goldstein, C. P. Poole Jr., J. L. Safko, *Classical Mechanics (3rd Edition)* (1992), Addison Wesley

[33] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory (3rd Edition)* (1991), Pergamon Press

[34] M. J. Gotay, H. B. Grundling, G. M. Tuynman, Journal of Nonlinear Science, J. Nonlinear Sci. Vol. 6: pp. 469-498 (1996)

[35] M. J. Gotay, H. Grundling, C.A. Hurst, Trans. A.M.S. 348 (1996) 1579-1597 dg-ga/9502008

[36] M. Blau, *Notes on (Semi-)Advanced Quantum Mechanics: The Path Integral Approach to Quantum Mechanics* (2019)

[37] J. M. Radcliffe, J. Phys. A: Gen. Phys. Vol. 4 (1971)

[38] J. Audretsch *Entangled Systems: New Directions in Quantum Physics*, Wiley-VCH (2007)

[39] H. P. Breuer, F. Petruccione *The Theory of Open Quantum Systems*, Oxford University Press (2003)

[40] P. Papon, J. Leblond, P. H. E. Meijer, *The Physics of Phase Transitions, Concepts and Applications* Second Revised Edition, Springer (2006)

[41] S. Sachdev *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).

[42] T.W.B. Kibble, J. Phys. A9 (1976) 1387; Phys. Rep. 67 (1980) 183.

[43] L. H. Zurek, Nature (London) 317 (1985) 505; Acta Phys. Pol. B24(1993) 1301; Phys. Rep. 276 (1996) 177.

[44] J. Maciejko, *An Introduction to Nonequilibrium Many-Body Theory*, Springer (2007)

[45] J. Schwinger, J. Math. Phys. 2 (1961) 407

[46] L.V. Keldysh, Sov. Phys. JETP 20 (1965) 1018

[47] L.M. Sieberer, M. Buchhold and S. Diehl Reports on Progress in Physics 79 (2016)

[48] J.R. Klauder, *Path Integrals and their Applications in Quantum, Statistical and Solid State Physics* (Plenum Press, Antwerb, Belgium 1977). Edited by G. Papadopoulos and J. Devreese.

[49] M. Srednicki, *Quantum Field Theory*, Cambridge: Cambridge University Press. doi:10.1017/CBO9780511813917 (2006)

[50] M. Kardar, Statistical Mechanics II: Statistical Physics of Fields, MIT Open Courseware,
https://ocw.mit.edu/courses/physics/8-334-statistical-mechanics-ii-statistical-physics-of-fields-spring-2014/

[51] U. Mosel, *Path Integrals in Field Theory. An Introduction.* (Springer, 2004).

[52] J. Holstein, H. Primakoff Phys. Rev. 58, 1113 (1940)

[53] F. Schuller *Lectures of the Geometric Anatomy of Theoretical Physics* (2014)

[54] M. Nakahara *Geometry, Topology and Physics*, IOP Publishing Ltd (2003)

[55] R. J. Szabo, *Equivariant Cohomology and Localization of Path Integrals* Springer-Verlag Berlin Heidelberg (2000)

[56] E. A. Kochetov, J. Math. Phys. 36, 4667 (1995)

[57] P. Lykourgias, I. Lyris, A. I. Karanikas, arXiv:2006.13720 (2020).

[58] I. Lyris, P. Lykourgias, A. I. Karanikas (2021) (To be published in Letters in Mathematical Physics)

[59] J. R. Klauder, *Continuous Representations and Path Integrals*, Revisited, in G. J. Papadopoulos and J. T. Devreese, editors, Path Integrals, NATO Advanced Study Institute, Series B: Physics, page 5, New York, (1978). Plenum.

[60] J. R. Klauder, Phys. Rev. D 19(8), 2349 (1979).

[61] Y. Weissman, J. Chem. Phys. 76, 4067 (1982).

[62] J. R. Klauder and B. S. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics*, World Scientific, Singapore, (1985)

[63] J. R. Klauder, *Some Recent Results on Wave Equations,Path Integrals and Semiclassical Approximations*, in G.Papanicolaou, editor, Random Media, Random Media. Springer, (1987).

[64] A. L. Xavier Jr. and M.A.M de Aguiar, Phys. Rev. A 54 1808 (1996).

[65] E. A. Kochetov, J. Phys. A: Math. Gen. 31 4473 (1998).

[66] M. Stone, K. Park, and A. Garg, J. Math. Phys. (N.Y) 41 8025 (2000).

[67] M. Baranger, M.A.M. de Aguiar, F. Keck, H. J. Korsch and B. Schellhaab, J. Phys. A. Math. Gen. 34 7227 (2001).

[68] J. R. Klauder , *The Feynman Path Integral: A Historical Slice*, in A Garden of Quanta, Eds. Arafume J. et al. (World Scientific Singapore) (2003), arXiv:quant-ph/0303034.

[69] Śniatycki J. (1980) *Blattner-Kostant-Sternberg Kernels.* In: Geometric Quantization and Quantum Mechanics. Applied Mathematical Sciences, vol 30. Springer, New York, NY

[70] R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[71] J. R. Klauder, *Path Integrals and their Applications in Quantum,Statistical and Solid State Physics*, edited by G. Papadopoulos and J. Devreese (Plenum Press, Antwerb, Belgium 1977)

[72] F. A. Berezin, Sov. Phys. Usp. 23, 763 (1980).

[73] I. Lyris, P. Lykourgias, A. I. Karanikas, Annals of Physics, Vol. 421 (2020).

[74] R. Stratonovich, Sov. Phys. Docl. 2 416 (1958).

[75] J. Hubbard, Phys. Rev. Lett. 3 77 (1959).

[76] M. B. Halpern, Nucl. Phys. B 173 504 (1980).

[77] A. Jevicki and B. Sakita, Nucl. Phys. B18589 (1981).

[78] M. Takahashi *Thermodynamics of One Dimensional Solvable Models* (Cambridge University Press, 1999)

[79] P. Pleufty, Annals of Physics 57,79-90, (1970)

[80] Th. Niemeijer, Physica 36, 377-419 (1967)

[81] W.H. Zurek, U. Dorner and P. Zoller, Phys. Rev. Lett. 95 (2005) 10570.

[82] L. Cincio, J. Dziarmaga, M.M. Rams and W.H. Zurek, Phys. Rev. A75 (2007) 052321.

[83] T. Caneva, R. Fazio and G.E. Santoro, Phys. Rev. B78 (2008) 104426.

[84] I. Lyris, P. Lykourgias, A. I. Karanikas, Forthcoming (2021).

[85] L. Brown, *Quantum Field Theory*, Cambridge University Press

[86] Morse and Feshbach *Methods of Theoretical Physics* part II, Mc Graw-Hill Book Company INC

[87] J. H. An, M. Feng and W. M. Zhang, Quanum Information & Computation 9, 0317 (2009).

[88] G. Kordas, D. Witthaut, P. Buonsante, A. Vezzani, R. Burioni, A. I. Karanikas, and S. Wimberger, Eur. Phys. J. Special Topics 224, 2127-2171 (2015).

[89] G. Kordas, G. E. Pavlou, A. I. Karanikas, Phys. Rev. A 98, 013637 (2018).

[90] H. G. Solari, J. Math. Phys.28, 1097 (1987).

[91] F. Bruckmann, J. D. Urbina, arXiv:1807.10462v1 [quant-ph] (2018)

[92] H. Karatsuji, and Y. Mizobushi, J. Math. Phys. 22, 757 (1981).

[93] M. Enz, and R. Scilling, J. Phys. C 19 1765 (1986).

[94] A. Garg, and G.-H. Kim, Phys. Rev. B 45, 12921 (1992).

[95] V. I. Belinicher, C. Providencia, and J. da Providencia, J. Phys. A 30, 5633 (1997).

[96] J. Shibata, and S. Takagi, Int. J. Mod. Phys. B 13, 107 (1999).

[97] E. Lieb, T. Schultz and D. Mattis, Ann. Physics. 16 (1961) 407.

[98] P. Jordan and E. Wigner, Z. Phys. 47 (1928) 631.

[99] H. Kleinert, Journal of Physics B: Atomic, Molecular and Optical Physics, 46 Vol. 17 (2013)

[100] G. Kordas, G.E. Pavlou and A.I. Karanikas, Phys. Rev. A98 (1) 013637, 2015