Crossed products of operator spaces and applications to Harmonic Analysis of non commutative groups

Dimitrios Andreou

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National and Kapodistrian University of Athens
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Advisory Committee

Aristides Katavolos, Emeritus Professor, National and Kapodistrian University of Athens (supervisor)

Michael Anoussis, Professor, University of Aegean

Dimitrios Gatzouras, Professor, National and Kapodistrian University of Athens

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Elias G. Katsoulis, Professor, East Carolina University

Konstantinos Tyros, Assistant Professor, National and Kapodistrian University of Athens

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Introduction

The interaction between Operator Theory and Harmonic Analysis of locally compact groups has occupied mathematicians since the very beginning of the theory of operator algebras. One of the most important notions connecting the two areas is the crossed product construction for locally compact group actions by automorphisms on von Neumann algebras, originally introduced by Murray and von Neumann in order to construct examples of factors of type II or III. Since then, the notion of crossed product has been extensively studied and successfully extended to other categories such as C*-algebras.

Over the last decades, crossed products have been introduced and studied in the case of group actions on more general categories of operator spaces, e.g. non-selfadjoint operator algebras, ternary rings of operators, operator systems and even general operator spaces. For more details on these topics, the reader is referred to the work of Katsoulis and Ramsey [29], Salmi and Skalski [44], Amini, Echterhoff and Nikpey [1], Harris and Kim [21], Ng [39], Hamana [19], Uuye and Zacharias [56] and Crann and Neufang [12].

The need of introducing appropriate crossed product type constructions for group actions on general operator spaces is dictated by the fact that it is often necessary to consider more relaxed operator space structures than von Neumann algebras or C*-algebras, in order to formulate and study several concepts from different areas of mathematics using operator theoretic techniques (quantization).

Motivation and objectives

This thesis concentrates on crossed products arising from locally compact group actions on *dual* operator spaces by w*-continuous completely isometric isomorphisms. In this case, there are (at least) two natural, yet generally different, kinds of crossed products. Namely, for an action α of a locally compact group G on a dual operator space X, we have a *Fubini crossed product* $X \rtimes_\alpha^{\mathcal{F}} G$ and a *spatial crossed product* $X \overline{\rtimes}_\alpha G$, such that $X \overline{\rtimes}_\alpha G \subseteq X \rtimes_\alpha^{\mathcal{F}} G$.

Informally speaking, the Fubini crossed product $X \rtimes_\alpha^{\mathcal{F}} G$ is the appropriate object representing and also generalizing concepts from Harmonic Analysis defined by fixed point properties, e.g. (*jointly*) *harmonic operators* and *non-commutative Poisson boundaries*. On the other hand, the corresponding spatial crossed product $X\overline{\rtimes}_\alpha G$ consists of the 'tractable' elements in $X \rtimes_\alpha^{\mathcal{F}} G$, in the sense of admitting an explicit representation in terms of elements of X and translation operators. A very interesting problem concerning harmonic operators is to find at least sufficient conditions so that every operator, which is harmonic with respect to a family of measures on G, can be explicitly described using only harmonic functions (with respect to the same family) and translation operators.

Since this problem can be reduced (as we show in Chapter 4) to whether a certain Fubini crossed product coincides with the corresponding spatial one, the relation between $X\overline{\rtimes}_\alpha G$ and $X \rtimes_\alpha^{\mathcal{F}} G$ deserves to be studied more thoroughly in a more general setting.

It is already known that $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ holds when X is a von Neumann algebra (Digernes-Takesaki [53, Chapter X, Corollary 1.22]) or more generally a W*-TRO (Salmi-Skalski [44]). However, for an arbitrary dual operator space X, the equality $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ may fail even if the action α is trivial.

On the other hand, Crann and Neufang [12] have recently proved that if G has the *approximation property* (AP) of Haagerup and Kraus, then $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ holds for any dual operator space X and any G-action α on X. They have also observed that the converse is true at least under the assumption that the group G is inner amenable in the sense of Paterson [43] (e.g. discrete).

The main objectives of the present dissertation can be summarized as follows:

- Describe the relation between $X\overline{\rtimes}_\alpha G$ and $X \rtimes_\alpha^{\mathcal{F}} G$ for an arbitrary G-action α on some dual operator space X and find necessary and sufficient conditions for the equality $X\overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$.
- Prove that the converse of the theorem of Crann and Neufang mentioned above remains valid for arbitrary locally compact groups (i.e. without the assumption of inner amenability).
- Apply the general theory developed in the setting of jointly harmonic operators.

The key idea to achieve the first of the above objectives is to extend the main aspects of the classical duality theory for crossed products of von Neumann algebras by actions of locally compact groups to the setting of dual operator spaces. To this end, we shall adopt the language of comodules over Hopf-von Neumann algebras, since this provides a natural and effective framework for the study of Takesaki-duality (see section 3.3) in the setting of (not necessarily abelian) locally compact groups.

As for the second objective, i.e. the characterization of groups with the approximation property in terms of crossed products, the main idea is to prove that the approximation property for a group G can be equivalently translated into an algebraic condition (saturation) for the class of comodules over the group (Hopf-)von Neumann algebra $L(G)$. This will provide us with a link between the approximation property and the duality theory developed for the crossed products under discussion.

Last but not least, we will prove that jointly harmonic operators arise naturally as the Fubini crossed product of jointly harmonic functions by translation action, whereas the harmonic operators admitting an explicit representation in terms of jointly harmonic functions and translation operators are realized as the spatial crossed product associated with the same action. This fact will allow us to apply our general results in order to shed some light on the problem of representability (in the above sense) of harmonic operators.

Structure and results

The main body of this thesis is organized in four chapters. In this section we summarize the content and basic results of Chapters 1 to 4. For the definitions required for the summary of each chapter presented below, the reader is referred to the respective chapter.

Chapter 1 Here we present the necessary mathematical background concerning dual operator spaces. In particular, we recall the basic properties of Fubini and spatial tensor products of dual operator spaces as well as tensor product maps, since they constitute the basic tools for the study of comodules over Hopf-von Neumann algebras and, in particular, crossed products. Next we give a brief discussion on the concept of stable point w*-convergence for nets of normal completely bounded maps on a von Neumann algebra. This concept is crucial in order to establish the connection between comodules over group von Neumann algebras and the approximation property of Haagerup and Kraus. We close this chapter with a short review on the von Neumann algebras $L^\infty(G)$ and $L(G)$ associated with a locally compact group G.

Chapter 2 This chapter begins by introducing the basic notions concerning Hopf-von Neumann algebras and their associated comodules in the dual operator space framework. Our attention is focused on the notions of *non-degeneracy* and *saturation* for comodules since these two notions play a crucial rôle in characterizing groups with the approximation property as well as in extending the concept of duality for crossed products from the category of von Neumann algebras to the context of dual operator spaces.

The main results of Chapter 2, concerning non-degeneracy and saturation, are the following:

Corollary 2.2.7. *Let (M, Δ) be a Hopf-von Neumann algebra. If every M-comodule is non-degenerate, then every M-comodule is saturated.*

Proposition 2.2.9. *For a Hopf-von Neumann algebra (M, Δ) the following conditions are equivalent:*

(a) Every M-comodule is saturated;

(b) For any M-comodule (X, α) and any $x \in X$, we have $x \in \overline{M_* \cdot x}^{\mathrm{w}^*}$;

(c) There exists a net $\{\omega_i\} \subseteq M_*$, such that $\omega_i \cdot x \longrightarrow x$ in the w-topology for any M-comodule X and any $x \in X$;*

(d) There exists a net $\{\omega_i\} \subseteq M_*$, such that the net $\{(\mathrm{id}_M \otimes \omega_i) \circ \Delta\} \subseteq CB_\sigma(M)$ converges in the stable point-w-topology to the identity map id_M.*

The above prepare the ground for establishing the connection between the approximation property (AP) of Haagerup and Kraus and properties of comodules over group von Neumann algebras. Namely we get the next

Proposition 2.3.14. *For a locally compact group G the following conditions are equivalent:*

(a) G has the AP;

(b) Every $L(G)$-comodule is saturated;

(c) For any $L(G)$-comodule (Y, δ) and any $y \in Y$, we have $y \in \overline{A(G) \cdot y}^{w^*}$;

(d) There exists a net $\{u_i\}_{i \in I}$ in $A(G)$ such that for any $L(G)$-comodule (Y, δ) and any $y \in Y$ we have that $u_i \cdot y \longrightarrow y$ ultraweakly;

(e) Every $L(G)$-comodule is non-degenerate.

Furthermore, we show that the Hopf-von Neumann algebra $L^\infty(G)$ admits only non-degenerate and saturated comodules, with no further assumption on the group G.

Lemma 2.3.5. *Every $L^\infty(G)$-comodule is non-degenerate and saturated. In particular, for any $L^\infty(G)$-comodule X and any $x \in X$, we have that $x \in \overline{L^1(G) \cdot x}^{\mathrm{w}^*}$.*

Chapter 3 We give the definitions for the Fubini and spatial crossed products $X \rtimes_\alpha^{\mathcal{F}} G$ and $X \overline{\rtimes}_\alpha G$ respectively for an $L^\infty(G)$-comodule (X, α) [12, 19, 44, 56], as well as their 'dual' analogues $Y \ltimes_\delta^{\mathcal{F}} G$ and $Y \overline{\ltimes}_\delta G$ for an $L(G)$-comodule (Y, δ) [19, 44, 3].

The spaces $X\overline{\rtimes}_\alpha G$ and $X\rtimes_\alpha^{\mathcal{F}} G$ admit a natural $L(G)$-comodule structure via the dual $L(G)$-action $\widehat{\alpha}$ associated with α as well as an $L(G)$-bimodule structure. Similarly, $Y\ltimes_\delta^{\mathcal{F}} G$ and $Y\overline{\ltimes}_\delta G$ become $L^\infty(G)$-comodules via a dual $L^\infty(G)$-action $\widehat{\delta}$ and they are also $L^\infty(G)$-bimodules.

A first analysis of the aforementioned structure yields the two first main results of this chapter:

Corollary 3.1.9. *For every $L^\infty(G)$-comodule (X,α) the Fubini crossed product $(X\rtimes_\alpha^{\mathcal{F}} G,\widehat{\alpha})$ is a saturated $L(G)$-comodule and the spatial crossed product $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ is a non-degenerate $L(G)$-comodule.*

Theorem 3.2.10. *For every $L(G)$-comodule (Y,δ) we have:*

$$Y\ltimes_\delta^{\mathcal{F}} G = Y\overline{\ltimes}_\delta G.$$

It is worthwhile noticing that the proof of Theorem 3.2.10 relies firmly on the fact that $(Y\ltimes_\delta^{\mathcal{F}} G,\widehat{\delta})$ and $(Y\overline{\ltimes}_\delta G,\widehat{\delta})$ are both non-degenerate and saturated as $L^\infty(G)$-comodules (by Lemma 2.3.5). Also, Corollary 3.1.9 suggests that if the equality $X\rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$ is valid for some $L^\infty(G)$-comodule (X,α), then $(X\rtimes_\alpha^{\mathcal{F}} G,\widehat{\alpha})$ and $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ have to be both non-degenerate and saturated $L(G)$-comodules (this happens when G has the AP; Proposition 2.3.14).

For the rest of Chapter 3, we deal with the double crossed products associated with an $L^\infty(G)$-comodule (X,α) and an $L(G)$-comodule (Y,δ), namely

- $(X\overline{\rtimes}_\alpha G)\ltimes_{\widehat{\alpha}} G\subseteq (X\rtimes_\alpha^{\mathcal{F}} G)\ltimes_{\widehat{\alpha}} G$,
- $(Y\ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}} G\subseteq (Y\ltimes_\delta G)\rtimes_{\widehat{\delta}}^{\mathcal{F}} G$,

where the symbol $\ltimes$ can be interpreted either as $\ltimes^{\mathcal{F}}$ or $\overline{\ltimes}$ thanks to Theorem 3.2.10.

Recall that, in the case of von Neumann algebra crossed products, the original algebra can be recovered from its double crossed product at the cost of tensoring by $B(L^2(G))$ (Takesaki-duality). The key observation is that this happens because the von Neumann algebra structure guarantees non-degeneracy and saturation for W*-$L(G)$-comodules. In fact, it turns out that non-degeneracy is equivalent to Takesaki-duality for spatial crossed products, whereas saturation is equivalent to Takesaki-duality for Fubini crossed products. This leads to the following two results, which summarize Propositions 3.3.2, 3.3.3, 3.3.5 and 3.3.6.

Proposition. *For any $L^\infty(G)$-comodule (X,α), the double crossed products $(X\overline{\rtimes}_\alpha G)\ltimes_{\widehat{\alpha}} G$ and $(X\rtimes_\alpha^{\mathcal{F}} G)\ltimes_{\widehat{\alpha}} G$ are both canonically isomorphic to $X\overline{\otimes}B(L^2(G))$.*

Proposition. *For any $L(G)$-comodule (Y,δ), there is a w^*-continuous complete isometry $\phi\colon Y\overline{\otimes}B(L^2(G)) \to Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))$, such that:*

1. *(Y,δ) is non-degenerate if and only if*
$$\phi\left(Y\overline{\otimes}B(L^2(G))\right) = (Y \ltimes_\delta G)\,\overline{\rtimes}_{\widehat{\delta}}G;$$
2. *(Y,δ) is saturated if and only if*
$$\phi\left(Y\overline{\otimes}B(L^2(G))\right) = (Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G.$$

A first application of the above duality results yields a more transparent image of the relation between $X\overline{\rtimes}_\alpha G$ and $X \rtimes^{\mathcal{F}}_\alpha G$ for an $L^\infty(G)$-comodule (X,α).

Corollary 3.3.7. *For any $L^\infty(G)$-comodule (X,α) we have:*

(i) $(X \rtimes^{\mathcal{F}}_\alpha G) \ltimes_{\widehat{\alpha}} G = (X\overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G$;

(ii) $\mathrm{Sat}(X\overline{\rtimes}_\alpha G,\widehat{\alpha}) = \mathrm{Sat}\left(X \rtimes^{\mathcal{F}}_\alpha G,\widehat{\alpha}\right) = \widehat{\alpha}\left(X \rtimes^{\mathcal{F}}_\alpha G\right)$;

(iii) $X \rtimes^{\mathcal{F}}_\alpha G = \{y \in X\overline{\otimes}B(L^2(G)):\ A(G)\cdot y \subseteq X\overline{\rtimes}_\alpha G\}$,
where $u\cdot y = (\mathrm{id}_{X\overline{\otimes}B(L^2(G))}\otimes u)\circ(\mathrm{id}_X\otimes\delta_G)(y)$ *for* $u \in A(G)$ *and* $y \in X\overline{\otimes}B(L^2(G))$.

(iv) $X\overline{\rtimes}_\alpha G = \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot(X \rtimes^{\mathcal{F}}_\alpha G)\}$.

Theorem 3.3.8. *Let (X,α) be an $L^\infty(G)$-comodule. Then, $X\overline{\rtimes}_\alpha G$ is the largest non-degenerate $L(G)$-subcomodule of $(X \rtimes^{\mathcal{F}}_\alpha G,\widehat{\alpha})$. Also, $X \rtimes^{\mathcal{F}}_\alpha G$ is the smallest saturated $L(G)$-subcomodule of $(X\overline{\otimes}B(L^2(G)),\mathrm{id}_X\otimes\delta_G)$ containing $X\overline{\rtimes}_\alpha G$. In particular, the following conditions are equivalent:*

(a) $X \rtimes^{\mathcal{F}}_\alpha G = X\overline{\rtimes}_\alpha G$;

(b) $\left(X \rtimes^{\mathcal{F}}_\alpha G,\widehat{\alpha}\right)$ *is a non-degenerate $L(G)$-comodule;*

(c) $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ *is a saturated $L(G)$-comodule.*

As a second application of Takesaki-duality, we obtain a complete characterization of locally compact groups with the approximation property (AP) in terms of the crossed product functors.

Theorem 3.3.10. *For a locally compact group G the following conditions are equivalent:*

(i) *G has the AP;*

(ii) $(Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G = (Y \ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}}G$, *for any $L(G)$-comodule (Y,δ);*

(iii) $X \rtimes^{\mathcal{F}}_\alpha G = X\overline{\rtimes}_\alpha G$, *for any $L^\infty(G)$-comodule (X,α);*

(iv) $((Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G,\ \widehat{\widehat{\delta}}) \simeq (Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ *for any $L(G)$-comodule (Y,δ);*

(v) $((Y \ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}}G,\ \widehat{\widehat{\delta}}) \simeq (Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ *for any $L(G)$-comodule (Y,δ).*

Chapter 4 In the last chapter, we give a new perspective on the concept of jointly harmonic operators introduced by Anoussis, Katavolos and Todorov [5, 6].

Recall the canonical representations of the group measure algebra $M(G)$ and the completely bounded multiplier algebra $M_{cb}A(G)$:

$$\Theta\colon M(G) \to CB(B(L^2(G))),$$
$$\Theta(\nu)(T) = \int_G \mathrm{Ad}\rho_s(T)\, d\nu(s), \quad \nu \in M(G),\ T \in B(L^2(G))$$

and

$$\widehat{\Theta}\colon M_{cb}A(G) \to CB(B(L^2(G))),$$
$$\widehat{\Theta}(u)(\lambda_s f) = u(s)\lambda_s f, \quad u \in M_{cb}A(G),\ s \in G,\ f \in L^\infty(G).$$

For arbitrary families $\Lambda \subseteq M(G)$ and $\Sigma \subseteq M_{cb}A(G)$, we have the *jointly Λ-harmonic functions*

$$\mathcal{H}(\Lambda) := \{f \in L^\infty(G) :\ \Theta(\mu)(f) = f \text{ for all } \mu \in \Lambda\}$$

and the *jointly Λ-harmonic operators*

$$\widetilde{\mathcal{H}}(\Lambda) := \{T \in B(L^2(G)) :\ \Theta(\mu)(T) = T \text{ for all } \mu \in \Lambda\},$$

as well as the *jointly Σ-harmonic functionals*

$$\mathcal{H}_\Sigma := \{T \in L(G) :\ \widehat{\Theta}(\sigma)(T) = T \text{ for all } \sigma \in \Sigma\}$$

and the *jointly Σ-harmonic operators*

$$\widetilde{\mathcal{H}}_\Sigma := \{T \in B(L^2(G)) :\ \widehat{\Theta}(\sigma)(T) = T \text{ for all } \sigma \in \Sigma\}.$$

Note that the above subspaces of $B(L^2(G))$ are not necessarily von Neumann algebras. However, they are w*-closed subspaces of $B(L^2(G))$ and thus (concrete) dual operator spaces. Moreover, we have the obvious inclusions $\mathcal{H}(\Lambda) \subseteq \widetilde{\mathcal{H}}(\Lambda)$ and $\mathcal{H}_\Sigma \subseteq \widetilde{\mathcal{H}}_\Sigma$. Also, note that $\widetilde{\mathcal{H}}(\Lambda)$ is an $L(G)$-bimodule, whereas $\widetilde{\mathcal{H}}_\Sigma$ is an $L^\infty(G)$-bimodule (because $\Theta(\mu)$ and $\widehat{\Theta}(\sigma)$ are normal bimodule maps on $B(L^2(G))$ over $L(G)$ and $L^\infty(G)$ respectively). Thus it is natural to ask whether the bimodules generated by $\mathcal{H}(\Lambda)$ and $\mathcal{H}_\Sigma$ are sufficient in order to describe $\widetilde{\mathcal{H}}(\Lambda)$ and $\widetilde{\mathcal{H}}_\Sigma$ respectively, that is whether the equalities

$$\mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda)) = \widetilde{\mathcal{H}}(\Lambda), \quad \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma) = \widetilde{\mathcal{H}}_\Sigma \tag{R}$$

hold for arbitrary $\Lambda \subseteq M(G)$ and $\Sigma \subseteq M_{cb}A(G)$.

The key observation here is that $\widetilde{\mathcal{H}}(\Lambda)$ coincides with $\ker\Theta(J(\Lambda))$, i.e. the common kernel of the maps $\Theta(h)$ for $h \in J(\Lambda)$, where $J(\Lambda) = \mathcal{H}(\Lambda)_\perp \subseteq L^1(G)$ (the preannihilator of $\mathcal{H}(\Lambda)$). Similarly, $\widetilde{\mathcal{H}}_\Sigma = \ker\widehat{\Theta}(J_\Sigma)$ for $J_\Sigma =$

$(\mathcal{H}_\Sigma)_\perp \subseteq A(G)$. Also, $J(\Lambda)$ and J_Σ are closed (left) ideals of $L^1(G)$ and $A(G)$ respectively.

We prove that there is a canonical isomorphism, preserving both the $L^\infty(G)$-comodule and the $L^\infty(G)$-bimodule structure, such that $\ker\widehat{\Theta}(J) \simeq J^\perp \ltimes^{\mathcal{F}}_{\delta_G} G$ and $\mathrm{Bim}_{L^\infty(G)}(J^\perp) \simeq J^\perp \overline{\ltimes}_{\delta_G} G$ for any closed ideal J of $A(G)$ (see Proposition 4.3.1).

Therefore, by an immediate application of Theorem 3.2.10 we obtain the following, which summarizes Proposition 4.3.1 and Corollary 4.3.2.

Proposition. *Let G be a locally compact group. For any closed ideal J of $A(G)$, it holds*

$$\ker\widehat{\Theta}(J) = \mathrm{Bim}_{L^\infty(G)}(J^\perp) \simeq J^\perp \ltimes_{\delta_G} G.$$

In particular, for any family $\Sigma \subseteq M_{cb}A(G)$ we have

$$\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma) \simeq \mathcal{H}_\Sigma \ltimes_{\delta_G} G.$$

The above extends [4, Theorem 3.2] and [5, Corollary 2.12], since our proof (which is significantly less technical) does not assume second countability of G as in [4, 5].

As a byproduct of the relations $\ker\widehat{\Theta}(J) = \mathrm{Bim}_{L^\infty(G)}(J^\perp) \simeq J^\perp \ltimes_{\delta_G} G$, we obtain a characterization of those locally compact groups G, for which every subcomodule of $L(G)$ satisfies a 'weak Takesaki-duality' (saturation condition), in the sense that we can recover any closed ideal $J \subseteq A(G)$ as $(L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^\perp))_\perp$. More precisely, we get the next

Proposition 4.3.5. *Let G be a locally compact group. Then, the following conditions are equivalent:*

(a) G has Ditkin's property at infinity, i.e. for any $u \in A(G)$, it holds that $u \in \overline{A(G)u}^{\|\cdot\|_{A(G)}}$.

(b) Every $L(G)$-subcomodule of $(L(G), \delta_G)$ is saturated.

(c) Every $L(G)$-subcomodule of $(L(G), \delta_G)$ is non-degenerate.

(d) For every closed ideal J of $A(G)$, we have $L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^\perp) = J^\perp$.

This improves [4, Lemma 4.5] and answers a question raised by the authors in [4, Question 4.8].

Finally, we prove an analogous result (Proposition 4.3.7) for closed left ideals J of $L^1(G)$, namely that

$$\mathrm{Bim}_{L(G)}(J^\perp) \simeq J^\perp \overline{\rtimes}_{\alpha_G} G \subseteq J^\perp \rtimes^{\mathcal{F}}_{\alpha_G} G \simeq \ker\Theta(J).$$

This, in combination with the results of Chapter 3, allows us to describe in more detail the relation between $\mathrm{Bim}_{L(G)}(J^\perp)$ and $\ker\Theta(J)$ and, in particular, the relation between $\mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ and $\widetilde{\mathcal{H}}(\Lambda)$, for a family $\Lambda \subseteq M(G)$, by taking $J = J(\Lambda)$.

Proposition 4.3.8. *For any closed left ideal J of $L^1(G)$, $\mathrm{Bim}_{L(G)}(J^\perp)$ is the largest non-degenerate $L(G)$-subcomodule of $(B(L^2(G)),\delta_G)$ contained in $\ker\Theta(J)$, i.e.*

$$\mathrm{Bim}_{L(G)}(J^\perp) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{\widehat{\Theta}(u)(T):\ u \in A(G),\ T \in \ker\Theta(J)\}$$

and $\ker\Theta(J)$ is the smallest saturated $L(G)$-subcomodule of $(B(L^2(G)),\delta_G)$ containing $\mathrm{Bim}_{L(G)}(J^\perp)$, i.e.

$$\ker\Theta(J) = \{T \in B(L^2(G)):\ \widehat{\Theta}(u)(T) \in \mathrm{Bim}_{L(G)}(J^\perp),\ \forall u \in A(G)\}.$$

Thus, the following conditions are equivalent:

(a) $\mathrm{Bim}_{L(G)}(J^\perp) = \ker\Theta(J)$*;*

(b) $(\ker\Theta(J),\delta_G)$ *is a non-degenerate $L(G)$-comodule, i.e.*

$$\ker\Theta(J) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{\widehat{\Theta}(A(G))(\ker\Theta(J))\};$$

(c) $(\mathrm{Bim}_{L(G)}(J^\perp),\delta_G)$ *is a saturated $L(G)$-comodule, i.e. if $T \in B(L^2(G))$ satisfies $\widehat{\Theta}(u)(T) \in \mathrm{Bim}_{L(G)}(J^\perp)$ $\forall u \in A(G)$, then $T \in \mathrm{Bim}_{L(G)}(J^\perp)$.*

Note that if G has the AP, then every $L(G)$-comodule is saturated. Thus $\mathrm{Bim}_{L(G)}(J^\perp) = \ker\Theta(J)$ for any left closed ideal J of $L^1(G)$ and thus $\mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda)) = \widetilde{\mathcal{H}}(\Lambda)$ for any $\Lambda \subseteq M(G)$.

The equality $\mathrm{Bim}_{L(G)}(J^\perp) = \ker\Theta(J)$ was originally shown by Anoussis, Katavolos and Todorov [6] for the case where G is either abelian or compact or weakly amenable discrete and for arbitrary locally compact groups with the AP by Crann and Neufang [12].

The above analysis of the relation between $\mathrm{Bim}_{L(G)}(J^\perp)$ and $\ker\Theta(J)$ allows us now to introduce a condition on G, a priori weaker than the AP, which also guarantees the validity of $\mathrm{Bim}_{L(G)}(J^\perp) = \ker\Theta(J)$ and $\mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda)) = \widetilde{\mathcal{H}}(\Lambda)$.

Corollary 4.3.9. *If every operator $T \in B(L^2(G))$ satisfies*

$$T \in \mathrm{Bim}_{L(G)}\{\widehat{\Theta}(u)(T):\ u \in A(G)\}, \tag{1}$$

then $\ker\Theta(J) = \mathrm{Bim}_{L(G)}(J^\perp)$ for any closed left ideal J of $L^1(G)$. In particular, if condition (1) is satisfied for all $T \in B(L^2(G))$, then $\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ for any family $\Lambda \subseteq M(G)$.

Note that condition (1) is a priori much weaker than the AP. Indeed, the AP implies that there is a net $\{u_i\} \subseteq A(G)$ such that

$$\widehat{\Theta}(u_i)(T) \longrightarrow T \text{ ultraweakly for all } T \in B(L^2(G))$$

and the net $\{u_i\}$ is independent of the choice of the operator $T \in B(L^2(G))$. On the other hand, condition (1) means that an operator $T \in B(L^2(G))$ can be approximated in the ultraweak topology by linear combinations of the form

$$\sum_{j=1}^{n} x_j \widehat{\Theta}(u_j)(T) y_j, \text{ with } u_j \in A(G),\ x_j, y_j \in L(G),$$

where now the choice of the functions u_j and the translation operators x_j and y_j may depend on the choice of that particular T.

In fact, to the author's knowledge, it is unknown whether there exist groups failing condition (1).

Some of the main results of Chapters 2, 3 and 4 appear in the following works of the author:

1. D. Andreou, *Crossed products of dual operator spaces by locally compact groups*, Studia Mathematica **258** (2021), no. 3, 241-267.

2. D. Andreou, *Crossed products of dual operator spaces and a characterization of groups with the approximation property,* (submitted) arXiv.org:2004.07169 (2020)

Περίληψη

Στην παρούσα διατριβή ασχολούμαστε με σταυρωτά γινόμενα που προκύπτουν από δράσεις τοπικά συμπαγών ομάδων σε *δυϊκούς* χώρους τελεστών μέσω w*-συνεχών πλήρως ισομετρικών ισομορφισμών. Σε αυτήν την περίπτωση, υπάρχουν (τουλάχιστον) δύο φυσιολογικά, αν και γενικά διαφορετικά, είδη σταυρωτών γινομένων. Ειδικότερα, για μια δράση α μιας τοπικά συμπαγούς ομάδας G σε ένα δυϊκό χώρο τελεστών X, ορίζεται ένα *σταυρωτό γινόμενο Fubini* $X \rtimes_\alpha^{\mathcal{F}} G$ και ένα *χωρικό (spatial) σταυρωτό γινόμενο* $X \overline{\rtimes}_\alpha G$, τέτοια ώστε $X \overline{\rtimes}_\alpha G \subseteq X \rtimes_\alpha^{\mathcal{F}} G$.

Άτυπα μιλώντας, το σταυρωτό γινόμενο Fubini $X \rtimes_\alpha^{\mathcal{F}} G$ είναι το κατάλληλο αντικείμενο για την αναπαράσταση, αλλά και γενίκευση, εννοιών από την Αρμονική Ανάλυση που ορίζονται μέσω ιδιοτήτων σταθερού σημείου, όπως, για παράδειγμα, οι *από κοινού αρμονικοί τελεστές* (*jointly harmonic operators*) και τα *μη μεταθετικά σύνορα Poisson*. Εξ άλλου, το αντίστοιχο χωρικό σταυρωτό γινόμενο $X \overline{\rtimes}_\alpha G$ αποτελείται από τα ανιχνεύσιμα στοιχεία του $X \rtimes_\alpha^{\mathcal{F}} G$, δηλαδή εκείνα που μπορούν να αναπαρασταθούν χρησιμοποιώντας μόνο στοιχεία του X και τελεστές μετατόπισης.

Ένα εξαιρετικά ενδιαφέρον πρόβλημα αναφορικά με τους αρμονικούς τελεστές είναι η εύρεση τουλάχιστον ικανών συνθηκών, προκειμένου κάθε αρμονικός τελεστής ως προς μια οικογένεια μέτρων στην G να μπορεί να αναπαρασταθεί χρησιμοποιώντας μόνον αρμονικές συναρτήσεις της G (ως προς την ίδια οικογένεια) και τελεστές μετατόπισης. Εφ' όσον, λοιπόν, αυτό το πρόβλημα ανάγεται (όπως δείχνουμε στο κεφάλαιο 4) στο κατά πόσον ένα συγκεκριμένο σταυρωτό γινόμενο Fubini συμπίπτει με το αντίστοιχο χωρικό σταυρωτό γινόμενο, η σχέση μεταξύ των σταυρωτών γινομένων $X \overline{\rtimes}_\alpha G$ και $X \rtimes_\alpha^{\mathcal{F}} G$ αξίζει να μελετηθεί πιο διεξοδικά σε ένα γενικότερο πλαίσιο.

Κεφάλαιο 1: Εισαγωγικά

Στην ενότητα 1.1 παρουσιάζονται επιλεγμένα θέματα από τα [7, 14, 52] που αφορούν βασικές έννοιες της Θεωρίας Χώρων και Αλγεβρών Τελεστών. Στην επόμενη ενότητα 1.2, έχοντας ως αναφορές τις [7, 14, 19, 20, 30, 31, 52, 55], συνοψίζουμε τις βασικές ιδιότητες του χωρικού (spatial) τανυστικού γινομένου και του τανυστικού γινομένου Fubini για δυϊκούς χώρους τελεστών, τις οποίες θα χρησιμοποιήσουμε στα επόμενα. Η ενότητα 1.3 αποτελεί μια σύντομη, αλλά επαρκή για τον σκοπό της παρούσας διατριβής, παρουσίαση της έννοιας της stable point-w*-σύγκλισης για πλήρως φραγμένες w*-συνεχείς απεικονίσεις αλγεβρών von Neumann [13, 22, 32]. Τέλος, στην ενότητα 1.4, συνοψίζονται οι βασικές ιδιότητες τοπικά συμπαγών ομάδων, καθώς και των βασικών αλγεβρών von Neumann και Banach που ορίζονται από μια τοπικά συμπαγή ομάδα [15, 16, 23, 46, 47].

Κεφάλαιο 2: Γενική θεωρία συμπροτύπων

Σε αυτό το κεφάλαιο, ακολουθώντας κυρίως την ορολογία του [19], ασχολούμαστε με δυϊκούς χώρους τελεστών που είναι συμπρότυπα (comodules) πάνω από άλγεβρες Hopf-von Neumann (οι βασικοί ορισμοί δίνονται στην ενότητα 2.1). Στην ενότητα 2.2, ορίζουμε τις έννοιες *κορεσμένο* (saturated) και *μη εκφυλισμένο* (non-degenerate) για συμπρότυπα. Αφ' ενός, δείχνουμε ότι αν κάθε συμπρότυπο μιας άλγεβρας Hopf-von Neumann M είναι μη εκφυλισμένο, τότε κάθε συμπρότυπο της M είναι κορεσμένο. Αφ' ετέρου, αποδεικνύουμε ότι μια άλγεβρα Hopf-von Neumann M διαθέτει μόνο κορεσμένα συμπρότυπα αν και μόνον αν η ταυτοτική απεικόνιση id_M προσεγγίζεται στην τοπολογία της stable point-w*-σύγκλισης από απεικονίσεις της μορφής $(\mathrm{id}_M \otimes \omega) \circ \Delta$, όπου $\omega \in M_*$ και Δ το συνγινόμενο (comultiplication) της M.

Στην ενότητα 2.3, για μια τοπικά συμπαγή ομάδα G, περιγράφουμε την κανονική δομή Hopf-von Neumann των αλγεβρών $L^\infty(G)$ και $L(G)$ [46, 49]. Κατόπιν, αποδεικνύουμε ότι κάθε $L^\infty(G)$-συμπρότυπο είναι ταυτόχρονα κορεσμένο και μη εκφυλισμένο. Επί πλέον, δείχνουμε ότι κάθε $L(G)$-συμπρότυπο είναι κορεσμένο αν και μόνον αν κάθε $L(G)$-συμπρότυπο είναι μη εκφυλισμένο αν και μόνον αν η G έχει την προσεγγιστική ιδιότητα (AP) κατά Haagerup-Kraus [22].

Κεφάλαιο 3: Σταυρωτά γινόμενα

Σε αυτό το κεφάλαιο θεωρούμε σταυρωτά γινόμενα δυϊκών χώρων τελεστών. Έστω ένα $L^\infty(G)$-συμπρότυπο (X, α) και ένα $L(G)$-συμπρότυπο (Y, δ). Τα σταυρωτά γινόμενα Fubini $X \rtimes_\alpha^{\mathcal{F}} G$ και $Y \ltimes_\delta^{\mathcal{F}} G$ ορίζονται ως τα σταθερά σημεία κάποιων καταλλήλων δράσεων, ενώ τα αντίστοιχα χωρικά σταυρωτά γινόμενα $X \overline{\rtimes}_\alpha G$ και $Y \overline{\ltimes}_\delta G$ ορίζονται, το μεν πρώτο ως το w*-κλειστό $L(G)$-διπρότυπο που παράγεται από το $\alpha(X)$, το δε δεύτερο ως το w*-κλειστό $L^\infty(G)$-διπρότυπο που παράγεται από το $\delta(Y)$. Επίσης, τα μεν $X \overline{\rtimes}_\alpha G$ και $X \rtimes_\alpha^{\mathcal{F}} G$ αποκτούν δομή $L(G)$-συμπροτύπου μέσω της δυϊκής δράσης $\widehat{\alpha}$, τα δε $Y \overline{\ltimes}_\delta G$ και $Y \ltimes_\delta^{\mathcal{F}} G$ αποκτούν δομή $L^\infty(G)$-συμπροτύπου μέσω της δυϊκής δράσης $\widehat{\delta}$.

Η βασική μας ιδέα είναι ότι το $X \rtimes_\alpha^{\mathcal{F}} G$ είναι το ελάχιστο κορεσμένο $L(G)$-συμπρότυπο που περιέχει το $X \overline{\rtimes}_\alpha G$, ενώ το $X \overline{\rtimes}_\alpha G$ είναι το μέγιστο μη εκφυλισμένο $L(G)$-συμπρότυπο που περιέχεται στο $X \rtimes_\alpha^{\mathcal{F}} G$. Όπως αποδεικνύουμε, η ίδια αρχή ισχύει και για τα $L^\infty(G)$-συμπρότυπα $Y \overline{\ltimes}_\delta G$ και $Y \ltimes_\delta^{\mathcal{F}} G$ και επομένως η ισότητα $Y \overline{\ltimes}_\delta G = Y \ltimes_\delta^{\mathcal{F}} G$ είναι πάντα αληθής, αφού κάθε $L^\infty(G)$-συμπρότυπο είναι πάντοτε κορεσμένο και μη εκφυλισμένο.

Στην συνέχεια, δείχνουμε ότι μια τοπικά συμπαγής ομάδα G έχει την προσεγγιστική ιδιότητα (AP) των Haagerup-Kraus αν και μόνον αν $X \overline{\rtimes}_\alpha G = X \rtimes_\alpha^{\mathcal{F}} G$ για κάθε $L^\infty(G)$-συμπρότυπο (X, α). Η μια κατεύθυνση (ευθύ), που αποδείχθηκε αρχικά από τους Crann και Neufang [12] χρησιμοποιώντας αρκετά τεχνικά επιχειρήματα, προκύπτει άμεσα από τα αποτελέσματα που έχουμε αναπτύξει μέχρι στιγμής. Για την αντίστροφη κατεύθυνση, το βασικό εργαλείο που χρησιμοποιούμε είναι το εξής: για ένα $L(G)$-συμπρότυπο (Y, δ), έχουμε $(Y \ltimes_\delta G)\overline{\rtimes}_{\hat{\delta}}G \simeq Y\overline{\otimes}B(L^2(G))$ αν και μόνον αν το Y είναι μη εκφυλισμένο, ενώ $(Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\hat{\delta}} G \simeq Y\overline{\otimes}B(L^2(G))$ αν και μόνον αν το Y είναι κορεσμένο. Επομένως, αν $X\overline{\rtimes}_\alpha G = X \rtimes^{\mathcal{F}}_\alpha G$ για κάθε $L^\infty(G)$-συμπρότυπο (X, α), τότε $(Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\hat{\delta}} G = (Y \ltimes_\delta G)\overline{\rtimes}_{\hat{\delta}}G$ για κάθε $L(G)$-συμπρότυπο (Y, δ). Αυτό σημαίνει ότι κάθε κορεσμένο $L(G)$-συμπρότυπο είναι και μη εκφυλισμένο, το οποίο, όπως δείχνουμε με ένα απλό επιχείρημα, συνεπάγεται ότι κάθε $L(G)$-συμπρότυπο είναι μη εκφυλισμένο, δηλαδή, ότι η G έχει την προσεγγιστική ιδιότητα (AP).

Κεφάλαιο 4: Εφαρμογές στην Αρμονική Ανάλυση

Στο τελευταίο κεφάλαιο, αποδεικνύουμε ότι οι χώροι των από κοινού αρμονικών τελεστών $\widetilde{\mathcal{H}}(\Lambda)$ και $\widetilde{\mathcal{H}}_\Sigma$ [5, 6] για αυθαίρετα υποσύνολα $\Lambda \subseteq M(G)$ και $\Sigma \subseteq M_{cb}A(G)$ ταυτίζονται αντίστοιχα με τα σταυρωτά γινόμενα (Fubini) $\widetilde{\mathcal{H}}(\Lambda) \simeq \mathcal{H}(\Lambda) \rtimes^{\mathcal{F}}_{\alpha_G} G$ και $\widetilde{\mathcal{H}}_\Sigma \simeq \mathcal{H}_\Sigma \ltimes^{\mathcal{F}}_{\delta_G} G$, όπου $\mathcal{H}(\Lambda)$ οι Λ-αρμονικές συναρτήσεις στην $L^\infty(G)$ και $\mathcal{H}_\Sigma$ τα Σ-αρμονικά συναρτησοειδή στην $L(G)$. Ταυτόχρονα, μέσω των ίδιων ισομορφισμών έχουμε $\mathcal{H}(\Lambda)\overline{\rtimes}_{\alpha_G}G \simeq \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ και $\mathcal{H}_\Sigma\overline{\ltimes}_{\delta_G}G \simeq \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma)$.

Σαν εφαρμογή των παραπάνω γενικεύουμε αποτελέσματα των [4, 5] για αυθαίρετες (όχι απαραιτήτως δεύτερες αριθμήσιμες) τοπικά συμπαγείς ομάδες. Επίσης, απαντάμε στο εξής ερώτημα που έθεσαν οι Ανούσης, Κατάβολος και Todorov στο [4]: για ποιες ομάδες G, ισχύει $L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^\perp) = J^\perp$ για κάθε κλειστό ιδεώδες J της $A(G)$; Τέλος, βρίσκουμε συνθήκες, a priori ασθενέστερες της προσεγγιστικής ιδιότητας (AP) των Haagerup και Kraus, οι οποίες εξασφαλίζουν ότι η ισότητα $\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ θα ισχύει για κάθε $\Lambda \subseteq M(G)$, γενικεύοντας αντίστοιχα αποτελέσματα των [6, 12].

Contents

Chapter 1

Preliminaries

1.1 Operator spaces

We begin by presenting some basic elements and notions from Operator Space Theory and von Neumann Algebra Theory. The proofs of the results stated in this section have been omitted since the topics selected here can be found in many books such as [7, 14, 52].

For Hilbert spaces H and K, we denote by $B(K, H)$ the space of bounded operators from K to H. We also write $B(H)$ for $B(H, H)$.

We make the following notational convention. We denote the identity map on X by id_X (omitting the subscript X when implied by the context). Also, we write 1_M for the unit element of an algebra M (again omitting the subscript if it is clear). In the special case $M = B(H)$ for a Hilbert space H, we write $1_H := \mathrm{id}_H = 1_{B(H)}$.

For positive integers m and n and a vector space V, we denote by $M_{m,n}(V)$ the $m \times n$ matrices with entries in V. Also, we write $M_n(V) := M_{n,n}(V)$ and $M_{m,n} := M_{m,n}(\mathbb{C})$.

Suppose that X and Y are vector spaces and that $u\colon X \to Y$ is a linear map. For a positive integer n, we write u_n for the associated map

$$u_n\colon M_n(X) \to M_n(Y)\colon [x_{ij}] \mapsto [u(x_{ij})].$$

This may also be thought of as the map $\mathrm{id}_{M_n} \otimes u$ on the algebraic tensor product $M_n \otimes X$. Similarly one may define $u_{m,n}\colon M_{m,n}(X) \to M_{m,n}(Y)$. If each of the matrix spaces $M_n(X)$ and $M_n(Y)$ has a given norm $\|\cdot\|_n$ and if u_n is an isometry for all $n \in \mathbb{N}$, then we say that u is *completely isometric*, or is a *complete isometry*. Similarly, u is *completely contractive* if each u_n is a contraction. A map u is *completely bounded* if

$$\|u\|_{cb} := \sup\{\|[u(x_{ij})]\|_n : \|[x_{ij}]\|_n \leq 1, \forall n \in \mathbb{N}\} < \infty.$$

Compositions of completely bounded maps are completely bounded, and one has the expected relation

$$\|u \circ v\|_{cb} \leq \|u\|_{cb}\|v\|_{cb}.$$

If $u\colon X \to Y$ is a completely bounded linear bijection, and if its inverse is completely bounded too, then we say that u is a *complete isomorphism*. In this case, we say that X and Y are *completely isomorphic*.

If $m, n \in \mathbb{N}$, and K, H are Hilbert spaces, then $M_{m,n}(B(K,H))$ inherits a norm $\|\cdot\|_{m,n}$ via the natural algebraic isomorphism

$$M_{m,n}(B(K,H)) \simeq B(K^{(n)}, H^{(m)})$$

which becomes an isometry. Recall that $H^{(m)}$ is the Hilbert space direct sum of m copies of H.

A *concrete operator space* is a norm closed linear subspace X of $B(K,H)$, for Hilbert spaces H, K (since $B(K,H) \subseteq B(H \oplus K)$ it is enough to consider the case $H = K$). An *abstract operator space* is a pair $(X, \{\|\cdot\|_n\}_{n\geq 1})$, consisting of a vector space X, and a norm on $M_n(X)$ for all $n \in \mathbb{N}$, such that there exists a linear complete isometry $u\colon X \to B(K,H)$ for some Hilbert spaces H and K. In this case we call the sequence $\{\|\cdot\|_n\}_{n\geq 1}$ an *operator space structure* on the vector space X. An operator space structure on a normed space $(X, \|\cdot\|)$ means a sequence of matrix norms as above, but with $\|\cdot\| = \|\cdot\|_1$.

Clearly subspaces of operator spaces are again operator spaces. We often identify two operator spaces if they are completely isometrically isomorphic.

Operator space structures on vector spaces can be abstractly characterized by Ruan's representation theorem.

Theorem 1.1.1 (Ruan)**.** *Suppose that X is a vector space, and that for each $n \in \mathbb{N}$ we are given a norm $\|\cdot\|_n$ on $M_n(X)$. Then X is linearly completely isometrically isomorphic to a linear subspace of $B(H)$, for some Hilbert space H if and only if the following conditions hold:*

(R1) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$ *for all* $n \in \mathbb{N}$ *and all* $\alpha, \beta \in M_n$ *and* $x \in M_n(X)$;

(R2) *For all* $x \in M_m(X)$ *and* $y \in M_n(X)$, *we have*

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}.$$

Conditions (R1) and (R2) are often called Ruan's axioms.

If X, Y are operator spaces, then the space $CB(X,Y)$ of completely bounded linear maps from X to Y, is also an operator space, with matrix norms determined via the canonical isomorphism

$$M_n(CB(X,Y)) \simeq CB(X, M_n(Y)).$$

Equivalently, if $[u_{ij}] \in M_n(CB(X,Y))$, then

$$\|[u_{ij}]\|_n = \sup\{\|[u_{ij}(x_{kl})]\|_{nm} : [x_{kl}] \in M_m(X),\ \|[x_{kl}]\| \leq 1,\ m \in \mathbb{N}\} \quad (1.1)$$

where the matrix $[u_{ij}(x_{kl})]$ is indexed on rows by i and k and on columns by j and l. Applying the above with n replaced by nN, to the space of matrices $M_N(M_n(CB(X,Y))) = M_{nN}(CB(X,Y))$, yields

$$M_n(CB(X,Y)) \simeq CB(X, M_n(Y)) \quad (1.2)$$

completely isometrically.

One can verify that the norms (1.1) define an operator space structure on $CB(X,Y)$ by a direct application of Ruan's theorem.

1.1.1 Dual operator spaces

In the special case when $Y = \mathbb{C}$, for any operator space X, we obtain an operator space structure on $X^* = CB(X,\mathbb{C})$. The latter space equals $B(X,\mathbb{C})$ isometrically since any continuous functional ϕ on an operator space X is completely bounded with $\|\phi\| = \|\phi\|_{cb}$ (see e.g. [7, 1.2.6]). We call X^*, viewed as an operator space in this way, the *operator space dual* of X. By (1.2) we have

$$M_n(X^*) \simeq CB(X, M_n)$$

completely isometrically.

An operator space Y is said to be a *dual operator space* if Y is completely isometrically isomorphic to the operator space dual X^* of an operator space X. We also say that X is an *operator space predual* of Y and we write X as Y_*.

If H is a Hilbert space then the space $\mathcal{T}(H)$ of *trace class operators* on H, i.e. the space of all $T \in B(H)$ satisfying

$$\|T\|_1 := \mathrm{tr}|T| < \infty$$

is a Banach space with respect to the trace class norm $\|\cdot\|_1$. The trace tr is a contractive functional on $\mathcal{T}(H)$, and via the dual pairing $(S,T) \mapsto \mathrm{tr}(ST)$ it is well-known that $\mathcal{T}(H)^* \simeq B(H)$ isometrically. In fact, regarding $\mathcal{T}(H)$ as an operator space with the operator space structure it inherits as a subspace of $B(H)^*$, the latter isomorphism is completely isometric.

From the above it is clear that the product on $B(H)$ is separately w*-continuous. That is, if $S_i \longrightarrow S$ in the w*-topology on $B(H)$, then $S_iT \longrightarrow ST$ and $TS_i \longrightarrow TS$ in the w*-topology too. The w*-topology on $B(H)$ is also called the *σ-weak topology* or the *ultraweak topology*. A linear functional on $B(H)$ is σ-weakly continuous if and only if it is of the form

$$T \mapsto \sum_{k=1}^{\infty} \langle T\xi_k | \eta_k \rangle$$

for $\xi_k, \eta_k \in H$ with $\sum_{k=1}^{\infty} \|\xi_k\|^2$ and $\sum_{k=1}^{\infty} \|\eta_k\|^2$ finite.

So, we can identify $B(H)_*$ with the σ -weakly continuous functionals on $B(H)$.

The class of w*-closed subspaces of $B(H)$ for a Hilbert space H coincides essentially with the class of dual operator spaces. More precisely we have

Proposition 1.1.2. *Every w*-closed linear subspace of $B(H)$ is a dual operator space. Conversely, any dual operator space is completely isometrically isomorphic, via a w*-w*-homeomorphism, to a w*-closed subspace of $B(H)$, for some Hilbert space H.*

For this reason, in the sequel, the term 'dual operator space' will refer to a w*-closed subspace of $B(H)$ for some Hilbert space H since the properties that we are mainly interested in are independent of the choice of the Hilbert space H.

Also, for a w*-closed subspace X of $B(H)$ an operator space predual X_* (not necessarily unique) is the space of all σ-weakly continuous functionals on X.

Remark 1.1.3. By appealing to the Krein-Smulian theorem, one may see that if $u\colon X \to Y$ is a w*-continuous isometry between dual Banach spaces X and Y, then u has w*-closed range and it is a w*-w*-homeomorphism from X onto $u(X)$.

Therefore, if X and Y are in addition dual operator spaces and $u\colon X \to Y$ is a w*-continuous complete isometry, then $u(X)$ is a dual operator space (completely isometrically isomorphic and w*-w*-homeomorphic to X).

1.1.2 Von Neumann algebras

Let H be a Hilbert space. For a subset M of $B(H)$ we denote by M' the *commutant* of M, that is the set

$$M' := \{T \in B(H) : \; TS = ST \quad \forall S \in M\}.$$

We also denote by M'' the commutant $(M')'$ of M'. It is clear from the definition that $M \subseteq M''$.

Moreover, one can see that M' is always a subalgebra of $B(H)$ which contains the identity operator 1_H. Furthermore, from the fact that the multiplication is separately w*-continuous it follows that M' is also a w*-closed subalgebra of $B(H)$.

If M is, in addition, selfadjoint, i.e. $T^* \in M$ whenever $T \in M$, then M' is selfadjoint too.

A unital selfadjoint subalgebra M of $B(H)$, for a Hilbert space H, is called a *von Neumann algebra* (acting on H) if $M = M''$.

Since commutants are w*-closed, any von Neumann algebra is w*-closed and thus a dual operator space. The converse is also true for a unital self-adjoint subalgebra of $B(H)$; that is the well known von Neumann's double commutant theorem.

Theorem 1.1.4 (von Neumann)**.** *If M is a selfadjoint unital subalgebra of $B(H)$ for a Hilbert space H, then*

$$M'' = \overline{M}^{\mathrm{w}^*}.$$

Thus the following conditions are equivalent:

(i) M is a von Neumann algebra, i.e. $M = M''$;

(ii) M is w-closed in $B(H)$.*

1.2 Tensor products

In this section we discuss about the Fubini and spatial tensor product of (dual) operator spaces as well as tensor product maps. These are necessary tools in order to study crossed products of group actions on operator spaces.

For more details and the proofs of the results stated below, the reader is referred to [7, 14, 19, 20, 30, 31, 52, 55].

For any Hilbert spaces H and K, there exists a unique inner product $\langle\cdot|\cdot\rangle$ on the algebraic tensor product $H \odot K$ of H and K, such that for all $\xi_1, \xi_2 \in H$ and $\eta_1, \eta_2 \in K$ we have

$$\langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2 \rangle = \langle \xi_1 | \xi_2 \rangle \langle \eta_1 | \eta_2 \rangle.$$

We denote by $H \otimes K$ the *Hilbert space tensor product* of H and K, that is the completion of the algebraic tensor product $H \odot K$ with respect to the above inner product.

Also, for any operators $T_1 \in B(H)$ and $T_2 \in B(K)$ there is a unique bounded operator $T_0 \in B(H \otimes K)$, such that

$$(T_0)(\xi \otimes \eta) = (T_1\xi) \otimes (T_2\eta), \qquad \xi \in H,\ \eta \in K.$$

We denote the operator T_0 by $T_1 \otimes T_2$.

Let X and Y be respectively w*-closed subspaces of $B(H)$ and $B(K)$. The *spatial tensor product* of X and Y denoted by $X \overline{\otimes} Y$ is defined as the w*-closed subspace of $B(H \otimes K)$ spanned by the operators $x \otimes y$ for $x \in X$ and $y \in Y$, that is

$$X \overline{\otimes} Y := \overline{\mathrm{span}}^{\mathrm{w}^*}\{x \otimes y :\ x \in X,\ y \in Y\} \subseteq B(H \otimes K).$$

In the special case when $X = B(H)$ and $Y = B(K)$, we have

$$B(H) \overline{\otimes} B(K) = B(H \otimes K).$$

This follows from von Neumann's double commutant theorem and Tomita's commutation theorem for tensor products of von Neumann algebras; see for example [52, Chapter IV, Proposition 1.6 and Theorem 5.9].

Theorem 1.2.1 (Tomita)**.** *For any von Neumann algebras M and N we have*

$$(M\overline{\otimes}N)' = M'\overline{\otimes}N'.$$

For ultraweakly continuous functionals $\phi \in B(H)_*$ and $\psi \in B(K)_*$ the linear mappings

$$\phi \otimes \mathrm{id}_{B(K)} \colon B(H) \otimes B(K) \to B(K) : \ a \otimes b \mapsto \phi(a)b$$

and

$$\mathrm{id}_{B(H)} \otimes \psi \colon B(H) \otimes B(K) \to B(H) : \ a \otimes b \mapsto \psi(b)a$$

have unique ultraweakly continuous extensions to $B(H \otimes K)$ (see e.g. [14, Lemma 7.2.2]) which we denote also by $\phi\otimes\mathrm{id}_{B(K)}$ and $\mathrm{id}_{B(H)}\otimes\psi$. Following Tomiyama [55], we call $\phi\otimes\mathrm{id}_{B(K)}$ the *right slice mapping* induced by ϕ and $\mathrm{id}_{B(H)} \otimes \psi$ the *left slice mapping* induced by ψ.

We denote by $\phi \otimes \psi$ each of the compositions $\phi \circ (\mathrm{id}_{B(H)} \otimes \psi)$ and $\psi \circ (\phi \otimes \mathrm{id}_{B(K)})$ since they clearly coincide (because they coincide on the elements of the form $a \otimes b$ whose linear span is w*-dense in $B(H \otimes K)$).

The *Fubini tensor product* of X and Y is defined by

$$X\overline{\otimes}_{\mathcal{F}}Y := \{T \in B(H \otimes K) : \ (\mathrm{id}_{B(H)} \otimes \psi)(T) \in X, \\ (\phi \otimes \mathrm{id}_{B(K)})(T) \in Y, \ \forall\psi \in B(K)_*, \ \forall\phi \in B(H)_*\}.$$

It is obvious from the definitions that, for any dual operator spaces X and Y, we have

$$X\overline{\otimes}Y \subseteq X\overline{\otimes}_{\mathcal{F}}Y,$$

but the equality $X\overline{\otimes}Y = X\overline{\otimes}_{\mathcal{F}}Y$ is not always valid. For example, if H is an infinite dimensional Hilbert space, then there exists a dual operator space X such that $X\overline{\otimes}B(H)^{**} \neq X\overline{\otimes}_{\mathcal{F}}B(H)^{**}$ [31].

However, it was proved by Tomiyama [55, Theorem 2.1] that for any von Neumann algebras M and N we have

$$M\overline{\otimes}N = M\overline{\otimes}_{\mathcal{F}}N.$$

This is in fact equivalent to Tomita's commutation theorem.

Recall that a von Neumann algebra $M \subseteq B(H)$ is *injective* if there exists a norm one projection from $B(H)$ onto M (see e.g. [54, Chapter XV, §1, Definition 1.2, Corollary 1.3]). It was shown by Kraus [30, Theorem 1.9] that if M is an injective von Neumann algebra, in particular, if $M = B(K)$ for a Hilbert space K or M is abelian, then $X\overline{\otimes}M = X\overline{\otimes}_{\mathcal{F}}M$ for any dual operator space X.

Note also that from the definition of Fubini tensor product it follows easily that for w*-closed subspaces $X_i \subseteq B(H)$ and $Y_i \subseteq B(K)$, $i = 1, 2$, we have

$$(X_1 \overline{\otimes}_{\mathcal{F}} Y_1) \cap (X_2 \overline{\otimes}_{\mathcal{F}} Y_2) = (X_1 \cap X_2) \overline{\otimes}_{\mathcal{F}} (Y_1 \cap Y_2).$$

Thus, combining this with Kraus' theorem above, it follows that for dual operator spaces $X \subseteq B(H)$ and $Y \subseteq B(K)$ we have

$$X \overline{\otimes}_{\mathcal{F}} Y = (X \overline{\otimes} B(K)) \cap (B(H) \overline{\otimes} Y). \tag{1.3}$$

Let V and W be two operator spaces. For a positive integer n and an element $u \in M_n(V \otimes W)$ one can define

$$\|u\|_{\wedge,n} = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : \ u = \alpha(v \otimes w)\beta\},$$

where the infimum is taken over all possible decompositions $u = \alpha(v \otimes w)\beta$ with $v \in M_m(V)$, $w \in M_k(W)$, $\alpha \in M_{n,mk}$ and $\beta \in M_{mk,n}$ for arbitrary $m, k \in \mathbb{N}$.

The sequence $\{\|u\|_{\wedge,n}\}_{n\geq 1}$ defines an operator space structure on the tensor product $V \otimes W$. The completion $V \widehat{\otimes} W$ of $(V \otimes W, \|\cdot\|_{\wedge,1})$ is called the *projective operator space tensor product* of V and W.

If X and Y are dual operator spaces with preduals X_* and Y_* respectively, then there exist w*-homeomorphic completely isometric isomorphisms

$$X \overline{\otimes}_{\mathcal{F}} Y \simeq (X_* \widehat{\otimes} Y_*)^* \simeq CB(X_*, Y) \simeq CB(Y_*, X). \tag{1.4}$$

See, for example, [14, Corollary 7.1.5, Theorem 7.2.3].

For $i = 1, 2$, let $X_i \subseteq B(H_i)$ and $Y_i \subseteq B(K_i)$ be dual operator spaces and let $\Phi\colon X_1 \to X_2$ and $\Psi\colon Y_1 \to Y_2$ be completely bounded w*-continuous linear maps. Then, using (1.4) and the fact that $X \overline{\otimes} B(H) = X \overline{\otimes}_{\mathcal{F}} B(H)$ for any dual operator space X and Hilbert space H, one can define w*-continuous completely bounded maps

$$\Phi \otimes \mathrm{id}_{B(K_i)}\colon X_1 \overline{\otimes} B(K_i) \to X_2 \overline{\otimes} B(K_i),$$

$$\mathrm{id}_{B(H_i)} \otimes \Psi\colon B(H_i) \overline{\otimes} Y_1 \to B(H_i) \overline{\otimes} Y_2,$$

which are respectively the unique w*-continuous extensions of the linear maps $a \otimes b \mapsto \Phi(a) \otimes b$ and $c \otimes d \mapsto c \otimes \Psi(d)$. Namely, for $x \in X_1 \overline{\otimes} B(K_i)$ and $y \in B(H_i) \overline{\otimes} Y_1$, the elements $(\Phi \otimes \mathrm{id}_{B(K_i)})(x)$ and $(\mathrm{id}_{B(H_j)} \otimes \Psi)(y)$ are respectively uniquely determined by

$$\langle (\Phi \otimes \mathrm{id}_{B(K_i)})(x), f \otimes g \rangle = \langle ((f \circ \Phi) \otimes \mathrm{id}_{B(K_i)})(x), g \rangle,$$

for $f \in X_{2*}$, $g \in B(K_i)_*$ and

$$\langle (\mathrm{id}_{B(H_j)} \otimes \Psi)(y), h \otimes k \rangle = \langle (\mathrm{id}_{B(H_j)} \otimes (k \circ \Psi))(y), h \rangle,$$

for $h \in B(H_j)_*,\ k \in Y_{2*}$.

Also, one can see that the compositions $(\Phi \otimes \mathrm{id}_{B(K_2)}) \circ (\mathrm{id}_{B(H_1)} \otimes \Psi)$ and $(\mathrm{id}_{B(H_2)} \otimes \Psi) \circ (\Phi \otimes \mathrm{id}_{B(K_1)})$ coincide on $X_1 \overline{\otimes}_{\mathcal{F}} Y_1 = (X_1 \overline{\otimes} B(K_1)) \cap (B(H_1) \overline{\otimes} Y_1)$ (recall (1.3)) and define a w*-continuous completely bounded map into $X_2 \overline{\otimes}_{\mathcal{F}} Y_2$, which is the unique w*-continuous extension of the map $x_1 \otimes x_2 \mapsto \Phi(x_1) \otimes \Psi(x_2)$ for $x_1 \in X_1$, $x_2 \in X_2$. We denote this map by

$$\Phi \otimes \Psi \colon X_1 \overline{\otimes}_{\mathcal{F}} Y_1 \to X_2 \overline{\otimes}_{\mathcal{F}} Y_2.$$

Furthermore, if Φ and Ψ are completely isometric (respectively surjective or completely contractive), then so is $\Phi \otimes \Psi$.

1.3 The stable point w*-topology

Let M be a von Neumann algebra and denote by $CB_\sigma(M)$ the w*-continuous completely bounded maps on M. Also, let K be a separable infinite dimensional Hilbert space. Following [13], we say that a net $\{T_j\}$ in $CB_\sigma(M)$ converges in the *stable point-w*-topology* to $T \in CB_\sigma(M)$ if

$$(\mathrm{id}_{B(K)} \otimes T_j)(x) \longrightarrow (\mathrm{id}_{B(K)} \otimes T)(x) \ \sigma\text{-weakly for all } x \in B(K) \overline{\otimes} M.$$

The following result [22, Proposition 1.7] states that the above convergence is independent of the choice of the Hilbert space K. The proof is based on an argument similar to that used in the proof of [32, Proposition 2.3]. For the convenience of the reader we have included this proof below.

Proposition 1.3.1 ([22], Proposition 1.7)**.** *Let $M \subseteq B(H)$ be a von Neumann algebra. A net $\{T_j\}$ in $CB_\sigma(M)$ converges in the stable point-w*-topology to $T \in CB_\sigma(M)$ if and only if, for any von Neumann algebra N, $(\mathrm{id}_N \otimes T_j)(x) \longrightarrow (\mathrm{id}_N \otimes T)(x)$ σ-weakly for all $x \in N \overline{\otimes} M$.*

Proof. Let $T_j,\ T \in CB_\sigma(M)$ and suppose that, for some separable infinite dimensional Hilbert space K_1, we have that

$$(\mathrm{id}_{B(K_1)} \otimes T_j)(x) \longrightarrow (\mathrm{id}_{B(K_1)} \otimes T)(x) \ \sigma\text{-weakly} \ \forall x \in B(K_1) \overline{\otimes} M. \quad (1.5)$$

In order to show that the same is true for any von Neumann algebra N in place of $B(K_1)$, it suffices to prove it for $N = B(K)$, where K is an arbitrary Hilbert space.

If K is finite dimensional, then it is unitarily equivalent to a closed subspace of K_1 and thus the desired conclusion follows since we may consider $B(K)$ as a subspace of $B(K_1)$. Therefore, it remains to show the desired convergence for infinite dimensional K.

So, let K be an infinite dimensional Hilbert space and $\omega \in (B(K) \overline{\otimes} M)_*$. Then, since ω is a countable sum of vector functionals and each vector in $K \otimes H$ is in the closed linear span of a countable number of basis vectors, there exists a projection p in $B(K)$ with at most countably infinite dimensional range, such that

$$\langle (p \otimes 1)x(p \otimes 1), \omega \rangle = \langle x, \omega \rangle \quad \text{for all } x \in B(K)\overline{\otimes}M. \tag{1.6}$$

Let K_0 denote the range of p and let ω_0 denote the restriction of ω to $B(K_0)\overline{\otimes}M$. It follows from (1.6) that, for any $x \in B(K)\overline{\otimes}M$,

$$\begin{aligned}\langle (\mathrm{id}_{B(K)} \otimes T_j)(x), \omega \rangle &= \langle (p \otimes 1)(\mathrm{id}_{B(K)} \otimes T_j)(x)(p \otimes 1), \omega_0 \rangle \\ &= \langle (\mathrm{id}_{B(K_0)} \otimes T_j)((p \otimes 1)x(p \otimes 1)), \omega_0 \rangle\end{aligned}$$

and the last quantity converges to $\langle (\mathrm{id}_{B(K_0)} \otimes T)((p \otimes 1)x(p \otimes 1)), \omega_0 \rangle$ which by (1.6) is equal to $\langle (\mathrm{id}_{B(K)} \otimes T)(x), \omega \rangle$. This follows from (1.5) since K_0 is unitarily equivalent to a closed subspace of K_1. □

1.4 Group algebras

In this section we present the von Neumann algebras usually associated with a locally compact group and their basic properties. For the proofs of the results stated in this section the reader is referred to [15], [16] [23], [46], [47].

Let G be a locally compact Hausdorff group. It is well known that there exists a unique (up to multiplication by positive constant) non trivial positive regular Borel meausure μ on G, such that for any $s \in G$ and any Borel subset $A \subseteq G$ we have

$$\mu(sA) = \mu(A).$$

This is called a *left Haar measure* on G.

Also, there exists a continuous group homomorphism $\Delta_G \colon G \to (0, +\infty)$ satisfying

$$\mu(As) = \Delta_G(s)\mu(A)$$

for any $s \in G$ and any Borel $A \subseteq G$. This is called the *modular function* of G.

Let us denote $d\mu(s)$ by ds. Then, we have the following properties for integration with respect to the Haar measure

$$d(ts) = ds,$$

$$d(st) = \Delta_G(t)ds \qquad \forall t \in G,$$

$$ds^{-1} = \Delta_G(s)^{-1}ds.$$

From now on, G will denote always a locally compact Hausdorff group with a fixed left Haar measure ds and modular function Δ_G.

The space $L^1(G)$ of equivalence classes up to almost everywhere equality of integrable complex functions on G is an involutive Banach algebra with respect to the norm

$$\|f\|_1 = \int_G |f(s)|ds, \qquad f \in L^1(G)$$

and product (convolution) and involution respectively defined by

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)dt$$

$$f^*(s) = \Delta_G(s)^{-1}\overline{f(s^{-1})}.$$

A *locally null* set is a Borel subset A of G such that, for any compact subset K of G, the intersection $A \cap K$ is of Haar measure zero. We say that a property holds *locally almost everywhere* if it holds for any $s \in G$ exept for a locally null subset of G.

For a measurable function $f\colon G \to \mathbb{C}$ let

$$\|f\|_\infty := \inf\{M > 0 : \ \{s \in G : |f(s)| > M\} \text{ is locally null}\}$$

be the *essential supremum* of f. If $\|f\|_\infty < \infty$, then f is called *essentially bounded*. Also, let $L^\infty(G)$ be the space of equivalence classes up to locally almost everywhere equality of essentially bounded measurable functions.

With respect to the norm $\|\cdot\|_\infty$ and equipped with the pointwise multiplication

$$(fg)(s) = f(s)g(s) \qquad s \in G$$

and the involution given by complex conjugation

$$f^*(s) = \overline{f(s)} \qquad s \in G,$$

$L^\infty(G)$ is a unital C*-algebra (the unit given by the constant function 1). This means that $L^\infty(G)$ is an involutive Banach algebra with the additional property (the *C*-property*)

$$\|ff^*\|_\infty = \|f\|_\infty^2 \qquad f \in L^\infty(G).$$

Moreover, $(L^\infty(G), \|\cdot\|_\infty)$ is isometrically isomorphic to the dual Banach space of $(L^1(G), \|\cdot\|_1)$ via the isometric isomorphism

$$T\colon L^\infty(G) \to L^1(G)^*$$

$$T(\phi)(f) = \int_G f(s)\phi(s)ds, \qquad \phi \in L^\infty(G), \ f \in L^1(G).$$

Recall that the space $L^2(G)$ of equivalence classes of square-integrable functions, i.e. those functions $\xi\colon G \to \mathbb{C}$ such that

$$\|\xi\|_2 := \int_G |\xi(s)|^2 ds < \infty,$$

is a Hilbert space with respect to the inner product $\langle\cdot|\cdot\rangle$ given by

$$\langle\xi|\eta\rangle = \int_G \xi(s)\overline{\eta(s)}ds \qquad \xi,\eta \in L^2(G).$$

The linear map $\mathcal{M}\colon L^\infty(G) \to B(L^2(G))$ defined by

$$(\mathcal{M}_\phi\xi)(s) = \phi(s)\xi(s) \qquad \phi \in L^\infty(G),\ \xi \in L^2(G),\ s \in G$$

is an isometric *-homomorphism, which is a homeomorphism with respect to the $\sigma(L^\infty(G), L^1(G))$-topology and the ultraweak topology on $B(L^2(G))$.

Because of the above, we will always identify $L^\infty(G)$ with the von Neumann subalgebra of $B(L^2(G))$ consisting of all multiplication operators $\mathcal{M}_\phi$ with $\phi \in L^\infty(G)$. Furthermore, in the following we supress the map $\mathcal{M}$ using the same symbol for an element of $L^\infty(G)$ and the repsective multiplicaton operator.

Another important von Neumann algebra associated with the group G is the *left von Neumann algebra* of G, that is the von Neumann algebra

$$L(G) := \lambda(G)'' \subseteq B(L^2(G))$$

generated by the *left regular representation*

$$\lambda\colon G \to B(L^2(G))$$

$$\lambda_s\xi(t) = \xi(s^{-1}t), \qquad \xi \in L^2(G),\ s,t \in G.$$

Note that λ is a unitary representation, i.e. λ is a strongly continuous (the map $s \mapsto \lambda_s\xi$ is continuous for all $\xi \in L^2(G)$) group homomorphism into the group of unitary operators on $L^2(G)$.

As usual, for $f \in L^1(G)$ we denote by

$$\lambda(f) = \int_G f(t)\lambda_t\, dt$$

the unique (convolution) operator in $B(L^2(G))$ satisfying

$$\langle\lambda(f)\xi|\eta\rangle = \int_G f(t)\langle\lambda_t\xi|\eta\rangle\, dt, \qquad \xi,\eta \in L^2(G).$$

Such an operator exists since the map $(\xi,\eta) \mapsto \int_G f(t)\langle\lambda_t\xi|\eta\rangle\, dt$ is clearly a bounded sesquilinear form on $L^2(G) \times L^2(G)$. Moreover, $\lambda(f) \in L(G)$, because if $x \in L(G)'$, then $\lambda(f)x = x\lambda(f)$ and thus $\lambda(f) \in \lambda(G)'' = L(G)$.

Following Eymard [15] we denote by $A(G)$ the space of those continuous bounded functions $u\colon G \to \mathbb{C}$ of the form

$$u(s) = \langle \lambda_s \xi | \eta \rangle, \qquad s \in G$$

for some $\xi, \eta \in L^2(G)$. The set $A(G)$ equipped with the pointwise multiplication of functions and the norm

$$\|u\| = \inf\{\|\xi\|\|\eta\| : \ \xi, \eta \in L^2(G), \ u(s) = \langle \lambda_s \xi | \eta \rangle \ \forall s \in G\}$$

is a Banach algebra (this is due to Eymard; see [15, Proposition (3.4)]), called the *Fourier algebra* of G.

The following theorem also proved by Eymard [15, Théorème (3.10)] states that the predual of $L(G)$ can be identified with the Fourier algebra $A(G)$.

Theorem 1.4.1 (Eymard). *For any operator $T \in L(G)$ there exists a unique bounded functional $\phi_T \in A(G)^*$, such that, if $u \in A(G)$ and $u(s) = \langle \lambda_s \xi | \eta \rangle$ for $\xi, \eta \in L^2(G)$, then we have*

$$\langle u, \phi_T \rangle = \langle T\xi | \eta \rangle.$$

The map $T \mapsto \phi_T$ is an isometric isomorphism from $L(G)$ onto $A(G)^*$ which is a homeomorphism with respect to the ultraweak topology on $L(G)$ and the $\sigma(A(G)^*, A(G))$-topology; the ultraweakly continuous fuctionals on $L(G)$ (i.e. the elements of the predual $L(G)_*$ of $L(G)$) are exactly those of the form $T \mapsto \langle u, \phi_T \rangle$ for $u \in A(G)$.

Apart from the left regular representation λ one may define the so called *right regular representation* of G, that is the unitary representation

$$\rho\colon G \to B(L^2(G))$$

given by

$$(\rho_s \xi)(t) = \Delta_G(s)^{1/2} \xi(ts) \qquad s, t \in G, \ \xi \in L^2(G).$$

The von Neumann algebra $R(G) := \rho(G)''$ generated by the unitaries $\rho_s, \ s \in G$, is called the *right von Neumann algebra* of G.

Using respectively the fact that $L^\infty(G)$ is abelian and that for any $s, t \in G$ we have $\lambda_s \rho_t = \rho_t \lambda_s$, one can directly verify the relations $L^\infty(G) \subseteq L^\infty(G)'$ and $R(G) \subseteq L(G)'$. In fact, these inclusions are actually equalities, that is

$$L^\infty(G)' = L^\infty(G), \tag{1.7}$$

and

$$L(G)' = R(G). \tag{1.8}$$

The commutation relations (1.7) and (1.8) above follow from the commutation theorem for left Hilbert algebras (see [47, 10.1] for the definition of a left Hilbert algebra) according to which the left and right von Neumann algebras associated to a left Hilbert algebra are commutant to each other; see [47, 10.4 (2)].

Indeed, (1.7) follows from the commutation theorem [47, 10.4 (2)] since $L^\infty(G) \cap L^2(G)$ with the operations of pointwise multiplication and complex conjugation is a Hilbert algebra and the associated left and right von Neumann algebras both coincide with $L^\infty(G)$.

Similarly, the space $C_c(G)$ of compactly supported continuous complex functions on G is a left Hilbert algebra with the operations of convolution and involution

$$(\xi * \eta)(s) = \int_G \xi(t)\eta(t^{-1}s)dt, \quad \xi^*(s) = \Delta_G(s)^{-1}\overline{\xi(s^{-1})},$$

for $\xi, \eta \in C_c(G)$ and $s \in G$. In this case, the associated left and right von Neumann algebras are $L(G)$ and $R(G)$ respectively. Hence (1.8) follows again from [47, 10.4 (2)].

Note that for any $s \in G$ and $\phi \in L^\infty(G)$ (regarded as a multiplication operator) we have

$$\lambda_s \phi \lambda_s^* = l_s\phi, \quad \rho_s \phi \rho_s^* = r_s\phi, \tag{1.9}$$

where $(l_s\phi)(t) := \phi(s^{-1}t)$ and $(r_s\phi)(t) = \phi(ts)$ for $t \in G$.

Hence, if the multiplication operator associated with $\phi \in L^\infty(G)$ commutes with $L(G)$ (repsectively with $R(G)$), then ϕ is left (respectively right) translation invariant and thus ϕ is (locally almost everywhere) a constant. Therefore, using (1.7) and (1.8), we get

$$B(L^2(G)) = (L^\infty(G) \cup L(G))'' = (L^\infty(G) \cup R(G))''. \tag{1.10}$$

Moreover, from von Neumann's double commutant theorem and (1.9) it follows that (1.10) can be rewritten as

$$B(L^2(G)) = \overline{\text{span}}^{\text{w}^*}\{L^\infty(G)L(G)\} = \overline{\text{span}}^{\text{w}^*}\{L^\infty(G)R(G)\}.^1 \tag{1.11}$$

[1]We denote by AB the set $\{ab : \ a \in A, \ b \in B\}$, for any subsets $A, \ B$ of some algebra.

Chapter 2

General theory of comodules

2.1 Hopf-von Neumann algebras and comodules

Hopf-von Neumann algebras and the associated comodules generalize locally compact groups and the associated dynamical systems (see e.g. Propositions 2.3.3 and 2.3.4) providing a natural framework for the development of a nice duality theory for group actions on operator spaces even in the case where the acting group is non abelian (see Section 3.3).

Here we list the basic definitions and properties regarding Hopf-von Neumann algebras and comodules in the dual operator space setting. Our terminology is based on that of [19] with the difference that we only consider comodules in the category of dual operator spaces with w*-continuous completely contractive linear maps as morphisms.

Definition 2.1.1. A *Hopf-von Neumann algebra* is a pair (M, Δ), where M is a von Neumann algebra and $\Delta\colon M \to M\overline{\otimes}M$ is a w*-continuous unital *-monomorphism, which is *coassociative*, i.e. it holds that

$$(\Delta \otimes \mathrm{id}_M) \circ \Delta = (\mathrm{id}_M \otimes \Delta) \circ \Delta.$$

Equivalently, the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\quad\Delta\quad} & M\overline{\otimes}M \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes\mathrm{id}_M} \\ M\overline{\otimes}M & \xrightarrow{\mathrm{id}_M\otimes\Delta} & M\overline{\otimes}M\overline{\otimes}M \end{array}$$

The map Δ is called the *comultiplication* of M.

Definition 2.1.2. Let (M, Δ) be a Hopf-von Neumann algebra. An *M-comodule* is a pair (X, α) consisting of a dual operator space X and a w*-continuous complete isometry $\alpha\colon X \to X\overline{\otimes}_{\mathcal{F}}M$ which is *coassociative over Δ*, i.e. it holds that

$$(\alpha \otimes \mathrm{id}_M) \circ \alpha = (\mathrm{id}_X \otimes \Delta) \circ \alpha.$$

In other words, we have the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\alpha\quad} & X\overline{\otimes}_{\mathcal{F}}M \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha\otimes\mathrm{id}_M} \\ X\overline{\otimes}_{\mathcal{F}}M & \xrightarrow{\mathrm{id}_X\otimes\Delta} & X\overline{\otimes}_{\mathcal{F}}M\overline{\otimes}_{\mathcal{F}}M \end{array}$$

In this case, we say that α is an *action* of M on X or an M*-action* on X.

A w*-closed subspace Y of X is called an M*-subcomodule* of X if $\alpha(Y) \subseteq Y\overline{\otimes}_{\mathcal{F}}M$. In this case we write $Y \leq X$ and Y is indeed an M-comodule for the action $\alpha|_Y$.

An M*-comodule morphism* between two M-comodules (X, α) and (Y, β) is a w*-w*-continuous complete contraction $\phi\colon X \to Y$, such that

$$\beta \circ \phi = (\phi \otimes \mathrm{id}_M) \circ \alpha.$$

An M-comodule morphism is called an M*-comodule monomorphism* (respectively isomorphism) if it is a complete isometry (resp. surjective complete isometry) and we write $X \simeq Y$ for isomorphic M-comodules.

If N is a von Neumann algebra, then an M-action $\pi\colon N \to N\overline{\otimes}M$ on N that is additionally a w*-continuous unital *-monomorphism will be called a W*-M*-action* on N (or a W*-*action* of M on N) and (N, π) will be called a W*-M*-comodule*. The terms W*-M*-subcomodule*, W*-M*-comodule morphism* etc, are defined analogously.

If Y is any dual operator space, then the Fubini tensor product $Y\overline{\otimes}_{\mathcal{F}}M$ becomes an M-comodule, called a *canonical* M-comodule, with the action

$$\mathrm{id}_Y \otimes \Delta\colon Y\overline{\otimes}_{\mathcal{F}}M \to Y\overline{\otimes}_{\mathcal{F}}M\overline{\otimes}_{\mathcal{F}}M.$$

More generally, for any dual operator space Y and any M-comodule (X, α), the map $\mathrm{id}_Y \otimes \alpha$ defines an M-action on the Fubini tensor product $Y\overline{\otimes}_{\mathcal{F}}X$. Similarly, the map $(\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_Y)$ is an M-action on $X\overline{\otimes}_{\mathcal{F}}Y$ (where $\sigma\colon M\overline{\otimes}_{\mathcal{F}}Y \to Y\overline{\otimes}_{\mathcal{F}}M$ is the flip isomorphism).

Remark 2.1.3. Let (M, Δ) be a Hopf-von Neumann algebra. Every M-comodule (X, α) is isomorphic to an M-subcomodule of a canonical M-comodule, which may be taken to be of the form $(B(H)\overline{\otimes}M, \mathrm{id}_{B(H)} \otimes \Delta)$ for some Hilbert space H.

Indeed, the image $\alpha(X)$ of X under the action α is an M-subcomodule of the canonical M-comodule $X\overline{\otimes}_{\mathcal{F}}M$, since we have:

$$(\mathrm{id}_X \otimes \Delta) \circ \alpha(X) = (\alpha \otimes \mathrm{id}_M) \circ \alpha(X) \subseteq \alpha(X)\overline{\otimes}_{\mathcal{F}}M$$

and α is an M-comodule isomorphism from X onto $\alpha(X)$ and thus

$$X \simeq \alpha(X) \leq X\overline{\otimes}_{\mathcal{F}}M.$$

Furthermore, we may suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H and thus $X\overline{\otimes}_{\mathcal{F}}M \leq B(H)\overline{\otimes}M$.

Remark 2.1.4. For any Hopf-von Neumann algebra (M, Δ), the predual M_* of M with its Banach space structure and product defined by

$$\omega\varphi = (\omega \otimes \varphi) \circ \Delta, \qquad \omega, \varphi \in M_*$$

becomes a Banach algebra. Indeed, using the coassociativity of Δ, for any $\omega, \phi, \psi \in M_*$ we have

$$\begin{aligned}(\omega\phi)\psi &= ((\omega\phi) \otimes \psi) \circ \Delta \\ &= [((\omega \otimes \phi) \circ \Delta) \otimes \psi] \circ \Delta \\ &= (\omega \otimes \phi \otimes \psi) \circ (\Delta \otimes \mathrm{id}_M) \circ \Delta \\ &= (\omega \otimes \phi \otimes \psi) \circ (\mathrm{id}_M \otimes \Delta) \circ \Delta \\ &= [\omega \otimes ((\phi \otimes \psi) \circ \Delta)] \circ \Delta \\ &= \omega(\phi\psi)\end{aligned}$$

thus the above product is associative. Also, the norm on M_* is submultiplicative, i.e.

$$\|\omega\phi\| \leq \|\omega\|\|\phi\|, \qquad \omega, \phi \in M_*$$

since Δ is an isometry and $\|\omega \otimes \phi\| \leq \|\omega\|\|\phi\|$ for any $\omega,\ \phi \in M_*$ (see for example [14, Theorems 7.1.1 and 7.2.4]).

Similarly, an M-comodule (X, α) with the module operation defined by

$$\omega \cdot x = (\mathrm{id}_X \otimes \omega) \circ \alpha(x), \quad \omega \in M_*,\ x \in X$$

becomes an M_*-Banach module (see [19, Lemma 2.3 (i)] for the details).

In fact, the M-subcomodules of X are exactly the M_*-submodules of X with respect to the above M_*-module action (see [19, Lemma 2.3 (ii)]).

Also, a w*-continuous complete contraction $\phi \colon X \to Y$ between two M-comodules X and Y is an M-comodule morphism if and only if ϕ is an M_*-module morphism (see [19, Proposition 2.2 and Lemma 2.3 (iii)]).

The notion of fixed points is of great importance in the study of comodules of Hopf-von Neumann algebras and in the study of crossed products as we will see in the next sections.

Definition 2.1.5. Let (X, α) be an M-comodule over a Hopf-von Neumann algebra (M, Δ). The *fixed point subspace* of X is the operator space

$$X^\alpha = \{x \in X : \ \alpha(x) = x \otimes 1_M\}.$$

Note that X^α is obviously an M-subcomodule of X.

A very useful observation is the following: for an M-comodule (X, α) and a dual operator space Y, the fixed point spaces of the M-comodules $(Y \overline{\otimes}_{\mathcal{F}} X, \mathrm{id}_Y \otimes \alpha)$ and $(X \overline{\otimes}_{\mathcal{F}} Y, (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_Y))$ are given by

$$(Y \overline{\otimes}_{\mathcal{F}} X)^{\mathrm{id}_Y \otimes \alpha} = Y \overline{\otimes}_{\mathcal{F}} X^\alpha$$

and

$$(X\overline{\otimes}_{\mathcal{F}}Y)^{(\mathrm{id}_X\otimes\sigma)\circ(\alpha\otimes\mathrm{id}_Y)} = X^{\alpha}\overline{\otimes}_{\mathcal{F}}Y.$$

Furthermore, any M-comodule isomorphism $\phi\colon X \to Y$ between two M-comodules (X,α) and (Y,β) maps X^{α} onto Y^{β}.

Indeed, since $\beta\circ\phi = (\phi\otimes\mathrm{id})\circ\alpha$, for any $x\in X^{\alpha}$ we get

$$\begin{aligned}\beta(\phi(x)) &= (\phi\otimes\mathrm{id})(\alpha(x))\\ &= (\phi\otimes\mathrm{id})(x\otimes 1)\\ &= \phi(x)\otimes 1\end{aligned}$$

that is $\phi(x)\in Y^{\beta}$ and thus $\phi(X^{\alpha})\subseteq Y^{\beta}$.

Conversely, if $y\in Y^{\beta}$, then since ϕ is onto there exists an $x\in X$ such that $\phi(x)=y$. Therefore, we have

$$\begin{aligned}(\phi\otimes\mathrm{id})(\alpha(x)) &= \beta(\phi(x))\\ &= \beta(y)\\ &= y\otimes 1\\ &= \phi(x)\otimes 1\\ &= (\phi\otimes\mathrm{id})(x\otimes 1)\end{aligned}$$

and therefore $\alpha(x) = x\otimes 1$, because $\phi\otimes\mathrm{id}$ is an isometry (since ϕ is an isometry too). This shows the inclusion $Y^{\beta}\subseteq\phi(X^{\alpha})$.

Another important notion concerning actions of Hopf-von Neumann algebras is *commutativity of actions*:

Definition 2.1.6. Let (M_1,Δ_1) and (M_2,Δ_2) be two Hopf-von Neumann algebras and α_1, α_2 be actions of M_1 and M_2 respectively on the same operator space X. We say that α_1 and α_2 *commute* if

$$(\alpha_1\otimes\mathrm{id}_{M_2})\circ\alpha_2 = (\mathrm{id}_X\otimes\sigma)\circ(\alpha_2\otimes\mathrm{id}_{M_1})\circ\alpha_1,$$

where $\sigma\colon M_2\overline{\otimes}M_1\to M_1\overline{\otimes}M_2 :\ x\otimes y\mapsto y\otimes x$ is the flip isomorphism.

The next lemma due to Hamana states that the fixed point space X^{α} of a given comodule (X,α) becomes a comodule with respect to any other action on X that commutes with α.

Lemma 2.1.7 ([19], Lemma 5.2)**.** *If α_1 and α_2 are respectively commuting actions on the same operator space X (Definition 2.1.6) of two Hopf-von Neumann algebras M_1 and M_2, then the fixed point subspace X^{α_1} is an M_2-subcomodule of (X,α_2), i.e. the restriction $\alpha_2|_{X^{\alpha_1}}$ is an action of M_2 on X^{α_1}.*

2.2 Saturated and non-degenerate comodules

In this section we examine the notions of *non-degeneracy* and *saturation* for general M-comodules for a Hopf-von Neumann algebra M. As we will see in Section 3.3 these two notions are equivalent forms of Takesaki-duality for the crossed products of comodules over $L^\infty(G)$ and $L(G)$.

For the use of these two notions in the classical duality theory of crossed products of von Neumann algebras the reader is referred to [35], [37], [38], [48], [49], [50], [51] and [57].

The term saturation was introduced in [48] for W*-comodules, while the term non-degeneracy was probably first used in [34, page 256].

Definition 2.2.1. Let (M, Δ) be a Hopf-von Neumann algebra acting on a Hilbert space K and (X, α) be an M-comodule with X being a w*-closed subspace of $B(H)$ for a Hilbert space H. Then, (X, α) is called *non-degenerate* if

$$X \overline{\otimes} B(K) = \overline{\operatorname{span}}^{\mathrm{w}^*}\{(1_H \otimes b)\alpha(x) : \ x \in X, \ b \in B(K)\}.$$

Remark 2.2.2. Suppose that (M, Δ) and (X, α) are as in Definition 2.2.1 and let (Y, β) be an M-comodule with Y being a w*-closed subspace of $B(L)$ for some Hilbert space L. If $\phi\colon X \to Y$ is an M-comodule isomorphism and X is non-degenerate, then Y is non-degenerate too.

Indeed, since $\phi\colon X \to Y$ is a w*-bicontinuous completely isometric isomorphism, so is the map $\psi := \phi \otimes \mathrm{id}\colon X \overline{\otimes} B(K) \to Y \overline{\otimes} B(K)$ and clearly ψ satisfies the following:

$$\psi((1_H \otimes b)z) = (1_L \otimes b)\psi(z), \qquad \text{for any } z \in X \overline{\otimes} B(K) \text{ and } b \in B(K).$$

Also, since $\beta \circ \phi = (\phi \otimes \mathrm{id}) \circ \alpha$ and $\phi(X) = Y$ it follows that $\psi(\alpha(X)) = \beta(Y)$. Thus if X is non-degenerate, then so is Y.

In particular, the non-degeneracy of (X, α) does not depend on the Hilbert space H on which X is represented.

Proposition 2.2.3. *If (M, Δ) is a Hopf-von Neumann algebra acting on a Hilbert space K and (X, α) is a non-degenerate M-comodule, then*

$$X = \overline{\operatorname{span}}^{\mathrm{w}^*}\{M_* \cdot X\}.$$

Proof. Let $\phi \in X_*$, such that $\phi(\omega \cdot x) = 0$, for all $\omega \in M_*$ and $x \in X$. Then, we have:

$$\begin{aligned}
&\phi \circ (\mathrm{id}_X \otimes \omega) \circ \alpha(x) = 0, \quad \forall \omega \in M_*, \ \forall x \in X \\
\Longrightarrow\ &\omega \circ (\phi \otimes \mathrm{id}_{B(K)}) \circ \alpha(x) = 0, \quad \forall \omega \in M_*, \ \forall x \in X \\
\Longrightarrow\ &(\phi \otimes \mathrm{id}_{B(K)}) \circ \alpha(x) = 0, \quad \forall x \in X \\
\Longrightarrow\ &b(\phi \otimes \mathrm{id}_{B(K)}) \circ \alpha(x) = 0, \quad \forall b \in B(K), \ \forall x \in X \\
\Longrightarrow\ &(\phi \otimes \mathrm{id}_{B(K)})((1_H \otimes b)\alpha(x)) = 0, \quad \forall b \in B(K), \ \forall x \in X.
\end{aligned}$$

Since (X,α) is non-degenerate, the last condition implies that

$$(\phi \otimes \mathrm{id}_{B(K)})(y) = 0 \text{ for any } y \in X \overline{\otimes} B(K),$$

thus $\phi(x)1 = (\phi \otimes \mathrm{id}_{B(K)})(x \otimes 1) = 0$ for any $x \in X$ and hence $\phi = 0$. So the desired conclusion follows from the Hahn-Banach theorem. □

We do not know whether the converse of Proposition 2.2.3 above holds for general Hopf-von Neumann algebras. However, it is true at least when the Hopf-von Neumann algebra under discussion is either $(L^\infty(G), \alpha_G)$ or $(L(G), \delta_G)$ for a locally compact group G (see Section 2.3 for the definitions); this follows from Lemma 2.3.5 and Corollary 2.3.8.

Definition 2.2.4. Let (M,Δ) be a Hopf-von Neumann algebra and (X,α) be an M-comodule. The *saturation space* of (X,α) is the space

$$\mathrm{Sat}(X,\alpha) := \{y \in X \overline{\otimes}_{\mathcal{F}} M : \ (\mathrm{id}_X \otimes \Delta)(y) = (\alpha \otimes \mathrm{id}_M)(y)\}.$$

Obviously, $\alpha(X) \subseteq \mathrm{Sat}(X,\alpha)$. We say that (X,α) is *saturated* if $\alpha(X) = \mathrm{Sat}(X,\alpha)$.

Proposition 2.2.5. *Let (M,Δ) be a Hopf-von Neumann algebra and (X,α) be an M-comodule. Then the following hold:*

(i) The saturation space $\mathrm{Sat}(X,\alpha)$ is an M-subcomodule of the canonical M-comodule $(X \overline{\otimes}_{\mathcal{F}} M, \mathrm{id}_X \otimes \Delta)$;

(ii) For the M_*-module action on $\mathrm{Sat}(X,\alpha)$ defined by the canonical M-action $\mathrm{id}_X \otimes \Delta$, we have $M_* \cdot \mathrm{Sat}(X,\alpha) \subseteq \alpha(X)$;

(iii) The M-comodule $(\mathrm{Sat}(X,\alpha), \mathrm{id}_X \otimes \Delta)$ is non-degenerate if and only if (X,α) is non-degenerate and saturated.

Proof. (i) Let $y \in \mathrm{Sat}(X,\alpha)$. Then, $(\mathrm{id}_X \otimes \Delta)(y) = (\alpha \otimes \mathrm{id}_M)(y) \in \alpha(X) \overline{\otimes}_{\mathcal{F}} M \subseteq \mathrm{Sat}(X,\alpha) \overline{\otimes}_{\mathcal{F}} M$. Thus, $\mathrm{Sat}(X,\alpha)$ is an M-subcomodule of $(X \overline{\otimes}_{\mathcal{F}} M, \mathrm{id}_X \otimes \Delta)$.

(ii) Let $\omega \in M_*$ and $y \in \mathrm{Sat}(X,\alpha)$. Since $\mathrm{Sat}(X,\alpha) \subseteq X \overline{\otimes}_{\mathcal{F}} M$, we have that $(\mathrm{id}_X \otimes \omega)(y) \in X$. Therefore, we get:

$$\begin{aligned}\omega \cdot y &= (\mathrm{id}_X \otimes \mathrm{id}_M \otimes \omega) \circ (\mathrm{id}_X \otimes \Delta)(y) \\ &= (\mathrm{id}_X \otimes \mathrm{id}_M \otimes \omega) \circ (\alpha \otimes \mathrm{id}_M)(y) \\ &= \alpha \circ (\mathrm{id}_X \otimes \omega)(y) \in \alpha(X),\end{aligned}$$

thus $M_* \cdot \mathrm{Sat}(X,\alpha) \subseteq \alpha(X)$.

(iii) Suppose that $(\mathrm{Sat}(X,\alpha), \mathrm{id}_X \otimes \Delta)$ is non-degenerate. Then, by Proposition 2.2.3 and Proposition 2.2.5 (ii), it follows immediately that

$$\mathrm{Sat}(X,\alpha) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{M_* \cdot \mathrm{Sat}(X,\alpha)\} \subseteq \alpha(X),$$

therefore $\mathrm{Sat}(X,\alpha) = \alpha(X)$, i.e. (X,α) is saturated. On the other hand, (X,α) is isomorphic to $(\alpha(X), \mathrm{id}_X \otimes \Delta)$, which is non-degenerate since $\mathrm{Sat}(X,\alpha) = \alpha(X)$. Thus, (X,α) is non-degenerate.

Conversely, suppose that (X,α) is non-degenerate and saturated. Then, since $(X,\alpha) \simeq (\alpha(X), \mathrm{id}_X \otimes \Delta) = (\mathrm{Sat}(X,\alpha), \mathrm{id}_X \otimes \Delta)$, it follows that $(\mathrm{Sat}(X,\alpha), \mathrm{id}_X \otimes \Delta)$ is non-degenerate. □

Remark 2.2.6. Let (M,Δ) be a Hopf-von Neumann algebra and let (Y_i, δ_i) for $i = 1, 2$ be two M-comodules. Also, let $\phi\colon Y_1 \to Y_2$ be an M-comodule isomorphism. Then, the map $\phi \otimes \mathrm{id}_M \colon Y_1 \overline{\otimes}_{\mathcal{F}} M \to Y_2 \overline{\otimes}_{\mathcal{F}} M$ is an M-comodule isomorphism for the canonical actions $\mathrm{id}_{Y_i} \otimes \Delta$, $i = 1, 2$. Furthermore, $\phi \otimes \mathrm{id}_M$ maps $\mathrm{Sat}(Y_1, \delta_1)$ onto $\mathrm{Sat}(Y_2, \delta_2)$. Therefore, saturation is preserved by comodule isomorphisms.

Indeed, for any $x \in Y_1 \overline{\otimes}_{\mathcal{F}} M$, we have:

$$\begin{aligned}
& (\phi \otimes \mathrm{id}_M)(x) \in \mathrm{Sat}(Y_2, \delta_2) \\
\iff & (\delta_2 \otimes \mathrm{id}_M) \circ (\phi \otimes \mathrm{id}_M)(x) = (\mathrm{id}_{Y_2} \otimes \Delta) \circ (\phi \otimes \mathrm{id}_M)(x) \\
\iff & ((\delta_2 \circ \phi) \otimes \mathrm{id}_M)(x) = (\phi \otimes \mathrm{id}_M \otimes \mathrm{id}_M) \circ (\mathrm{id}_{Y_1} \otimes \Delta)(x) \\
\iff & [((\phi \otimes \mathrm{id}_M) \circ \delta_1) \otimes \mathrm{id}_M]\,(x) = (\phi \otimes \mathrm{id}_M \otimes \mathrm{id}_M) \circ (\mathrm{id}_{Y_1} \otimes \Delta)(x) \\
\iff & (\phi \otimes \mathrm{id}_M \otimes \mathrm{id}_M) \circ (\delta_1 \otimes \mathrm{id}_M)(x) = \\
& (\phi \otimes \mathrm{id}_M \otimes \mathrm{id}_M) \circ (\mathrm{id}_{Y_1} \otimes \Delta)(x) \\
\iff & (\delta_1 \otimes \mathrm{id}_M)(x) = (\mathrm{id}_{Y_1} \otimes \Delta)(x) \\
\iff & x \in \mathrm{Sat}(Y_1, \delta_1).
\end{aligned}$$

Since $\phi \otimes \mathrm{id}_M$ is onto $Y_2 \overline{\otimes}_{\mathcal{F}} M$, the above equivalences show that it maps $\mathrm{Sat}(Y_1, \delta_1)$ onto $\mathrm{Sat}(Y_2, \delta_2)$.

The following corollary follows immediately from Proposition 2.2.5 (iii).

Corollary 2.2.7. *Let (M,Δ) be a Hopf-von Neumann algebra. If every M-comodule is non-degenerate, then every M-comodule is saturated.*

Remark 2.2.8. Note that it does not follow from Corollary 2.2.7 or its proof that any non-degenerate M-comodule is necessarily saturated and indeed this is not true in general. For example, if G is a locally compact group, then the group von Neumann algebra $L(G)$ admits a comultiplication δ_G (see Section 2.3 below). If in addition G fails the approximation property in the sense of Haagerup-Kraus, then there exist non-degenerate $L(G)$-comodules which are not saturated as well as saturated $L(G)$-comodules which are not non-degenerate (see Proposition 2.3.14). More precisely, such examples arise as crossed products of groups without the approximation property acting on dual operator spaces (see Corollary 3.1.9 and Theorems 3.3.8 and 3.3.10).

The next result states that, for a Hopf-von Neumann algebra (M,Δ), the condition that every M-comodule is saturated (which is an algebraic condition by definition) is equivalent to the existence of a (not necessarily norm bounded) net $\{\omega_i\}$ in M_*, such that, for any M-comodule (X, α), any element $x \in X$ is the w*-limit of the net $\{\omega_i \cdot x\}$, where the M_*-module operation on X is given by

$$(\omega, x) \mapsto \omega \cdot x = (\mathrm{id}_X \otimes \omega)(\alpha(x))$$

(Proposition 2.2.9). In particular, it follows that the net $\{\omega_i\}$ above is a weak approximate unit for M_* regarded as a Banach algebra with the product

$$(\omega, \phi) \mapsto \omega\phi = (\omega \otimes \phi) \circ \Delta.$$

Proposition 2.2.9. *For a Hopf-von Neumann algebra (M, Δ) the following conditions are equivalent:*

(a) Every M-comodule is saturated;

(b) For any M-comodule (X, α), any M-subcomodule Z of X and any $x \in X$, the following implication holds:

$$\alpha(x) \in Z\overline{\otimes}_{\mathcal{F}} M \implies x \in Z;$$

(c) For any M-comodule (X, α) and any $x \in X$, we have $x \in \overline{M_* \cdot x}^{\mathrm{w}^*}$;

(d) There exists a net $\{\omega_i\} \subseteq M_*$, such that $\omega_i \cdot x \longrightarrow x$ in the w-topology for any M-comodule X and any $x \in X$;*

(e) There exists a net $\{\omega_i\} \subseteq M_*$, such that the net $\{(\mathrm{id}_M \otimes \omega_i) \circ \Delta\} \subseteq CB_\sigma(M)$ converges in the stable point-w-topology to the identity map id_M.*

Moreover, if any of the above conditions is satisfied, then M_* regarded as a Banach algebra with the product induced by Δ has a (right) weak approximate unit. That is, there exists a net $\{\omega_i\} \subseteq M_*$, such that

$$\langle x, \omega\omega_i \rangle \longrightarrow \langle x, \omega \rangle \qquad \text{for all } \omega \in M_* \text{ and } x \in M.$$

Therefore, for any $\omega \in M_*$, it holds that $\omega \in \overline{\omega M_*}^{\|\cdot\|}$.

Proof. (a) $\implies$ (b): Suppose that every M-comodule is saturated. Let (X, α) be an M-comodule, Z an M-subcomodule of X and $x \in X$ such that $\alpha(x) \in Z\overline{\otimes}_{\mathcal{F}} M$. By assumption, α restricts to an M-action on Z and since $\alpha(x) \in Z\overline{\otimes}_{\mathcal{F}} M$ and $(\alpha \otimes \mathrm{id})(\alpha(x)) = (\mathrm{id} \otimes \Delta)(\alpha(x))$ it follows that $\alpha(x) \in \mathrm{Sat}(Z, \alpha|_Z)$. But $(Z, \alpha|_Z)$ is saturated by hypothesis and therefore $\alpha(x) \in \alpha(Z)$. Thus, $x \in Z$, because α is an isometry.

(b) $\implies$ (c): Let (X, α) be an M-comodule and $x \in X$ and put $Z := \overline{M_* \cdot x}^{\mathrm{w}^*}$. Then, by Remark 2.1.4, it follows that Z is an M-subcomodule of X since it is an M_*-module by definition.

Also, we have that $\alpha(x) \in Z\overline{\otimes}_{\mathcal{F}}M$. Indeed, by the definition of the Fubini tensor product, the condition $\alpha(x) \in Z\overline{\otimes}_{\mathcal{F}}M$ is equivalent to the following

$$\omega \cdot x = (\mathrm{id} \otimes \omega)(\alpha(x)) \in Z, \quad \forall \omega \in M_*,$$

which is true by the definition of Z. Therefore, the assumption that (b) holds implies that $x \in Z$, that is $x \in \overline{M_* \cdot x}^{\mathrm{w}^*}$.

(c) $\Longrightarrow$ (a): Suppose that (c) is true and take an M-comodule (Y, β). Consider the M-comodule (X, α) with $X := \mathrm{Sat}(Y, \beta)$ and $\alpha = \mathrm{id}_Y \otimes \Delta$. Then, by (c), it follows that $z \in \overline{M_* \cdot z}^{\mathrm{w}^*} \subseteq \overline{M_* \cdot \mathrm{Sat}(Y, \beta)}^{\mathrm{w}^*}$, for all $z \in \mathrm{Sat}(Y, \beta)$. But from Proposition 2.2.5 (ii), we have that $M_* \cdot \mathrm{Sat}(Y, \beta) \subseteq \beta(Y)$ and therefore $z \in \beta(Y)$, for all $z \in \mathrm{Sat}(Y, \beta)$, that is (Y, β) is saturated.

(d) $\Longrightarrow$ (c): This is clearly obvious.

(e) $\Longrightarrow$ (d): Let (X, α) be an M-comodule with X being a w*-closed subspace of $B(H)$ for some Hilbert space H. First, observe that, for any $\omega \in M_*$, we have the following:

$$(\mathrm{id}_X \otimes \Phi_\omega) \circ \alpha = \alpha \circ (\mathrm{id}_X \otimes \omega) \circ \alpha, \tag{2.1}$$

where $\Phi_\omega := (\mathrm{id}_M \otimes \omega) \circ \Delta$.

Indeed, since

$$\alpha \circ (\mathrm{id}_X \otimes \omega) = (\mathrm{id}_X \otimes \mathrm{id}_M \otimes \omega) \circ (\alpha \otimes \mathrm{id}_M),$$

and

$$(\alpha \otimes \mathrm{id}_M) \circ \alpha = (\mathrm{id}_X \otimes \Delta) \circ \alpha,$$

we get:

$$\begin{aligned}
\alpha \circ (\mathrm{id}_X \otimes \omega) \circ \alpha &= (\mathrm{id}_X \otimes \mathrm{id}_M \otimes \omega) \circ (\alpha \otimes \mathrm{id}_M) \circ \alpha \\
&= (\mathrm{id}_X \otimes \mathrm{id}_M \otimes \omega) \circ (\mathrm{id}_X \otimes \Delta) \circ \alpha \\
&= [\mathrm{id}_X \otimes ((\mathrm{id}_M \otimes \omega) \circ \Delta)] \circ \alpha \\
&= (\mathrm{id}_X \otimes \Phi_\omega) \circ \alpha.
\end{aligned}$$

From Proposition 1.3.1 and the assumption that (e) holds it follows that, for any $x \in X$ we have:

$$(\mathrm{id}_X \otimes \Phi_{\omega_i})(\alpha(x)) \longrightarrow \alpha(x) \quad \text{ultraweakly},$$

because $\alpha(X) \subseteq X\overline{\otimes}_{\mathcal{F}}M \subseteq B(H)\overline{\otimes}M$. Thus, (2.1) implies that

$$\alpha \circ (\mathrm{id}_X \otimes \omega_i) \circ \alpha(x) \longrightarrow \alpha(x) \quad \text{ultraweakly}.$$

On the other hand, α is a w*-continuous isometry, therefore it is a w*-w*-homeomorphism from X onto $\alpha(X)$ (recall 1.1.3) and thus

$$\omega_i \cdot x = (\mathrm{id}_X \otimes \omega_i) \circ \alpha(x) \longrightarrow x \quad \text{ultraweakly}.$$

(c) $\Longrightarrow$ (e): Assume that, for any M-comodule (X, α) and any $x \in X$, we have that $x \in \overline{M_* \cdot x}^{\mathrm{w}^*}$. Let H be a Hilbert space. Thus taking $X = B(H)\overline{\otimes}M$ and $\alpha = \mathrm{id}_{B(H)} \otimes \Delta$ yields that, for any $x \in B(H)\overline{\otimes}M$, there exists a net $\{\omega_i\}$ in M_* which may depend on the choice of x, such that

$$(\mathrm{id}_{B(H)} \otimes \mathrm{id}_M \otimes \omega_i) \circ (\mathrm{id}_{B(H)} \otimes \Delta)(x) \longrightarrow x \text{ ultraweakly.}$$

Therefore, since

$$\begin{aligned}(\mathrm{id}_{B(H)} \otimes \mathrm{id}_M \otimes \omega_i) \circ (\mathrm{id}_{B(H)} \otimes \Delta) &= \mathrm{id}_{B(H)} \otimes ((\mathrm{id}_M \otimes \omega_i) \circ \Delta) \\ &= \mathrm{id}_{B(H)} \otimes \Phi_{\omega_i},\end{aligned}$$

where $\Phi_\omega := (\mathrm{id}_M \otimes \omega_i) \circ \Delta$ for $\omega \in M_*$, it follows that for any Hilbert space H and any $x \in B(H)\overline{\otimes}M$ there exists a net $\{\omega_i\}$ in M_*, such that $(\mathrm{id}_{B(H)} \otimes \Phi_{\omega_i})(x) \longrightarrow x$ ultraweakly.

Now, consider a separable infinite dimensional Hilbert space K and let $F = \{x_1, \ldots, x_n\}$ be a finite subset of $B(K)\overline{\otimes}M$. Then, $x = x_1 \oplus \cdots \oplus x_n$ may be viewed as an element of $B(K^{(n)})\overline{\otimes}M$, where $K^{(n)}$ is the direct sum of n copies of K. Hence, applying the above argument for $K^{(n)}$ in place of H, we get that there exists a net $\{\omega_i\}$ in M_*, such that $(\mathrm{id}_{B(K^{(n)})} \otimes \Phi_{\omega_i})(x) \longrightarrow x$ ultraweakly and thus it follows that $(\mathrm{id}_{B(K)} \otimes \Phi_{\omega_i})(y) \longrightarrow y$ ultraweakly for all $y \in F$. Therefore, if F is a finite subset of $B(K)\overline{\otimes}M$ and $\mathfrak{N}$ is an ultraweak neighborhood of 0, then there is an element $\omega_{(F,\mathfrak{N})} \in M_*$, such that

$$(\mathrm{id}_{B(K)} \otimes \Phi_{\omega_{(F,\mathfrak{N})}})(y) \in y + \mathfrak{N}, \quad \forall y \in F.$$

So, the set of all pairs $(F, \mathfrak{N})$ becomes a directed set with the partial order defined by $(F_1, \mathfrak{N}_1) \leq (F_2, \mathfrak{N}_2)$ if $F_1 \subseteq F_2$ and $\mathfrak{N}_2 \subseteq \mathfrak{N}_1$ and it is clear that

$$(\mathrm{id}_{B(K)} \otimes \Phi_{\omega_{(F,\mathfrak{N})}})(y) \longrightarrow y \;\text{ ultraweakly for all } y \in B(K)\overline{\otimes}M,$$

that is the net $\{\Phi_{\omega_{(F,\mathfrak{N})}}\}$ converges in the stable point-w*-topology to id_M.

For the last statement of the proposition, note that if, for example, condition (e) holds, then for any $x \in M$ and $\omega \in M_*$ we have

$$\langle x, \omega\omega_i \rangle = \langle \Delta(x), \omega \otimes \omega_i \rangle = \langle (\mathrm{id}_M \otimes \omega_i)(\Delta(x)), \omega \rangle \longrightarrow \langle x, \omega \rangle.$$

Thus, since ωM_* is a linear subspace of M_* (and hence convex), from the Hahn-Banach theorem it follows that $\omega \in \overline{\omega M_*}^{\|\cdot\|}$. □

The next two lemmas describe two basic ways of constructing new saturated comodules.

Lemma 2.2.10. *Let (M, Δ) be a Hopf-von Neumann algebra and (Y, β) be a saturated M-comodule. Then, $(X\overline{\otimes}_{\mathcal{F}}Y, \mathrm{id}_X \otimes \beta)$ is a saturated M-comodule for any dual operator space X.*

Proof. Let X be a dual operator space. First we have to check that $(X\overline{\otimes}_{\mathcal{F}}Y, \mathrm{id}_X \otimes \beta)$ is an M-comodule. Indeed, we have:

$$\begin{aligned}(\mathrm{id}_X \otimes \beta \otimes \mathrm{id}_M) \circ (\mathrm{id}_X \otimes \beta) &= \mathrm{id}_X \otimes [(\beta \otimes \mathrm{id}_M) \circ \beta] \\ &= \mathrm{id}_X \otimes [(\mathrm{id}_Y \otimes \Delta) \circ \beta] \\ &= (\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \Delta) \circ (\mathrm{id}_X \otimes \beta).\end{aligned}$$

Now, take $z \in \mathrm{Sat}(X\overline{\otimes}_{\mathcal{F}}Y, \mathrm{id}_X \otimes \beta)$. We have to prove that

$$z \in (\mathrm{id}_X \otimes \beta)(X\overline{\otimes}_{\mathcal{F}}Y) = X\overline{\otimes}_{\mathcal{F}}\beta(Y).$$

Indeed, since $z \in \mathrm{Sat}(X\overline{\otimes}_{\mathcal{F}}Y, \mathrm{id}_X \otimes \beta)$ we have:

$$(\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \Delta)(z) = (\mathrm{id}_X \otimes \beta \otimes \mathrm{id}_M)(z).$$

Therefore, for any $\omega \in X_*$, we get:

$$(\omega \otimes \mathrm{id}_{Y\overline{\otimes}_{\mathcal{F}}M\overline{\otimes}_{\mathcal{F}}M}) \circ (\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \Delta)(z) = (\omega \otimes \mathrm{id}_{Y\overline{\otimes}_{\mathcal{F}}M\overline{\otimes}_{\mathcal{F}}M}) \circ (\mathrm{id}_X \otimes \beta \otimes \mathrm{id}_M)(z)$$

that is

$$(\mathrm{id}_Y \otimes \Delta) \circ (\omega \otimes \mathrm{id}_{Y\overline{\otimes}_{\mathcal{F}}M})(z) = (\beta \otimes \mathrm{id}_M) \circ (\omega \otimes \mathrm{id}_{Y\overline{\otimes}_{\mathcal{F}}M})(z).$$

Thus $(\omega \otimes \mathrm{id}_{Y\overline{\otimes}_{\mathcal{F}}M})(z) \in \mathrm{Sat}(Y, \beta) = \beta(Y)$ for all $\omega \in X_*$ and hence $z \in X\overline{\otimes}_{\mathcal{F}}\beta(Y)$. □

Lemma 2.2.11. *Let M_1 and M_2 be two Hopf-von Neumann algebras and let α_1 and α_2 be actions of M_1 and M_2 respectively on the same dual operator space X. Suppose that (X, α_2) is a saturated M_2-comodule and that α_1 and α_2 commute, i.e.*

$$(\alpha_1 \otimes \mathrm{id}_{M_2}) \circ \alpha_2 = (\mathrm{id}_X \otimes \sigma) \circ (\alpha_2 \otimes \mathrm{id}_{M_1}) \circ \alpha_1,$$

where $\sigma\colon M_2\overline{\otimes}M_1 \to M_1\overline{\otimes}M_2 : \ x \otimes y \mapsto y \otimes x$ is the flip isomorphism. Then, the fixed point space $(X^{\alpha_1}, \alpha_2|_{X^{\alpha_1}})$ is a saturated M_2-comodule.

Proof. Since the actions α_1 and α_2 commute, X^{α_1} is an M_2-subcomodule of (X, α_2) by Lemma 2.1.7. Also, since (X, α_2) is saturated, i.e. $\mathrm{Sat}(X, \alpha_2) = \alpha_2(X)$, we get

$$\begin{aligned}\mathrm{Sat}(X^{\alpha_1}, \alpha_2|_{X^{\alpha_1}}) &= (X^{\alpha_1}\overline{\otimes}_{\mathcal{F}}M) \cap \mathrm{Sat}(X, \alpha_2) \\ &= (X^{\alpha_1}\overline{\otimes}_{\mathcal{F}}M) \cap \alpha_2(X).\end{aligned}$$

Thus it suffices to show that $(X^{\alpha_1}\overline{\otimes}_{\mathcal{F}}M_2) \cap \alpha_2(X) \subseteq \alpha_2(X^{\alpha_1})$.

Take $y \in (X^{\alpha_1}\overline{\otimes}_{\mathcal{F}}M_2) \cap \alpha_2(X)$. Then $y = \alpha_2(x)$ for some $x \in X$ and so we only need to prove that $x \in X^{\alpha_1}$, i.e. $\alpha_1(x) = x \otimes 1$. Indeed, since $y \in X^{\alpha_1}\overline{\otimes}_{\mathcal{F}}M_2$ it follows that

$$(\mathrm{id}_X \otimes \sigma) \circ (\alpha_1 \otimes \mathrm{id}_{M_2})(y) = y \otimes 1$$

and therefore

$$\begin{aligned}(\alpha_2 \otimes \mathrm{id}_{M_1})(x \otimes 1) &= \alpha_2(x) \otimes 1 \\ &= y \otimes 1 \\ &= (\mathrm{id}_X \otimes \sigma) \circ (\alpha_1 \otimes \mathrm{id}_{M_2})(y) \\ &= (\mathrm{id}_X \otimes \sigma) \circ (\alpha_1 \otimes \mathrm{id}_{M_2})(\alpha_2(x)) \\ &= (\alpha_2 \otimes \mathrm{id}_{M_1})(\alpha_1(x)),\end{aligned}$$

where the last equality follows from the commutativity of the actions α_1 and α_2. Since $\alpha_2 \otimes \mathrm{id}_{M_1}$ is an isometry it follows that $\alpha_1(x) = x \otimes 1$ and the proof is complete. □

2.3 Group Hopf-von Neumann algebras

2.3.1 Notation and basic properties

Let G be a locally compact (Hausdorff) group with left Haar measure ds and modular function Δ_G.

For any $\xi, \eta \in L^2(G)$, we identify $\xi \otimes \eta \in L^2(G \times G)$ with the function $(s,t) \mapsto \xi(s)\eta(t),\ s,t \in G$. This identification yields an isomorphism between the Hilbert spaces $L^2(G) \otimes L^2(G)$ and $L^2(G \times G)$.

Thus one also obtains a canonical (unital w*-continuous) *-isomorphism

$$B(L^2(G))\overline{\otimes}B(L^2(G)) = B(L^2(G) \otimes L^2(G)) \simeq B(L^2(G \times G)),$$

which restricts to a *-isomorphism $L^\infty(G)\overline{\otimes}L^\infty(G) \simeq L^\infty(G \times G)$. That is, for any $f, g \in L^\infty(G)$ the above isomorphism identifies $f \otimes g$ with the multiplication operator on $L^2(G \times G)$ given by the function $(s,t) \mapsto f(s)g(t),\ s,t \in G$.

Consider the (fundamental) unitary operators on $L^2(G \times G)$ defined by the formulas

$$V_G\xi(s,t) = \xi(t^{-1}s,t),$$

$$W_G\xi(s,t) = \xi(s,st),$$

$$U_G\xi(s,t) = \Delta_G(t)^{1/2}\xi(st,t),$$

for $\xi \in L^2(G \times G)$ and $s,t \in G$.

The fact that V_G, W_G and U_G are unitaries (equivalently that they preserve inner products) follows directly from the Fubini theorem (see [9, Theorem 7.6.7, Lemma 9.4.2]) and the fact that the Haar measure satisfies $d(ts) = ds$ and $d(st) = \Delta_G(t)^{-1}ds$. For example, for $\xi, \eta \in L^2(G \times G)$, we have

$$\begin{aligned}\langle V_G\xi | V_G\eta\rangle &= \iint V_G\xi(s,t)\overline{V_G\eta(s,t)}\, ds\, dt \\ &= \iint \xi(t^{-1}s,t)\overline{\eta(t^{-1}s,t)}\, ds\, dt \\ &= \iint \xi(s,t)\overline{\eta(s,t)}\, ds\, dt \qquad (d(t^{-1}s) = ds) \\ &= \langle \xi|\eta\rangle.\end{aligned}$$

Similarly we get that W_G and U_G are unitaries too.

Furthermore, using the commutation relations $L^\infty(G)' = L^\infty(G)$ and $L(G)' = R(G)$ (recall (1.7), (1.8)) one may see that

$$V_G \in L(G)\overline{\otimes}L^\infty(G),\ W_G \in L^\infty(G)\overline{\otimes}L(G),\ U_G \in L^\infty(G)\overline{\otimes}R(G),$$

viewing V_G, W_G and U_G as elements of $B(L^2(G))\overline{\otimes}B(L^2(G))$.

The map $\alpha_G\colon L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G)$ defined by

$$\alpha_G(f) = V_G^*(f\otimes 1)V_G, \qquad f\in L^\infty(G),$$

is a comultiplication on $L^\infty(G)$. Note that $\alpha_G(f)$ is the multiplication operator on $L^2(G\times G)$ given by the function

$$\alpha_G(f)(s,t) = f(ts), \qquad s,t\in G.$$

Also, one can check that α_G is indeed a comultiplication by observing that the co-associativity rule

$$(\alpha_G\otimes \mathrm{id})\circ\alpha_G = (\mathrm{id}\otimes\alpha_G)\circ\alpha_G$$

is actually equivalent to the associativity of the multiplication on G (i.e. $(st)r = s(tr)$ for any $s,t,r\in G$).

In addition, for any $h,k\in L^1(G)$ and $f\in L^\infty(G)$, we have

$$\langle \alpha_G(f), h\otimes k\rangle = \langle f, k*h\rangle,$$

where

$$(k*h)(t) = \int_G k(s)h(s^{-1}t)\, ds, \quad t\in G,$$

is the usual convolution on $L^1(G)$.

Therefore the product defined by the comultiplication α_G on the predual $L^1(G)\simeq L^\infty(G)_*$, i.e. $hk = (h\otimes k)\circ\alpha_G$ for $h,\ k\in L^\infty(G)_*$ (see Remark 2.1.4) coincides with the (opposite) convolution on $L^1(G)$

$$hk = k*h, \quad \forall h,k\in L^1(G).$$

If $\sigma\colon B(L^2(G))\overline{\otimes}B(L^2(G)) \to B(L^2(G))\overline{\otimes}B(L^2(G)) : \ x\otimes y \mapsto y\otimes x$ is the flip isomorphism, then we have

$$\alpha'_G(f) := \sigma\circ\alpha_G(f) = U_G(f\otimes 1)U_G^*, \qquad f\in L^\infty(G).$$

Equivalently, $\alpha'_G(f)$ is the multiplication operator on $L^2(G\times G)$ given by the function $(s,t)\mapsto f(st)$ for $s,t\in G$. Moreover, the above map $\alpha'_G := \sigma\circ\alpha_G$ is another comultiplication on $L^\infty(G)$ (called the *opposite* of α_G).

Comment 2.3.1. There is no essential difference in working with either α_G or α'_G on $L^\infty(G)$. For instance, every (right) action $\alpha\colon X\to X\overline{\otimes}L^\infty(G)$ of $(L^\infty(G),\alpha_G)$ on X is determined uniquely by the left action $\beta := \sigma\circ\alpha\colon X\to L^\infty(G)\overline{\otimes}X$ of $(L^\infty(G),\alpha'_G)$ on X. By the term left action we mean that β satisfies the condition $(\alpha'_G\otimes\mathrm{id})\circ\beta = (\mathrm{id}\otimes\beta)\circ\beta$, which is clearly equivalent to $(\mathrm{id}\otimes\alpha_G)\circ\alpha = (\alpha\otimes\mathrm{id})\circ\alpha$ using $\beta=\sigma\circ\alpha$ and the definition of the flip map σ.

Similarly, the group von Neumann algebra $L(G)$ is also a Hopf-von Neumann algebra with comultiplication $\delta_G\colon L(G)\to L(G)\overline{\otimes}L(G)$ defined by

$$\delta_G(x) = W_G^*(x\otimes 1)W_G, \qquad x\in L(G).$$

This is indeed a comultiplication since, for any $s\in G$, we have

$$\delta_G(\lambda_s) = \lambda_s\otimes\lambda_s$$

and thus the composites $(\delta_G\otimes\mathrm{id})\circ\delta_G$ and $(\mathrm{id}\otimes\delta_G)\circ\delta_G$ both map λ_s to $\lambda_s\otimes\lambda_s\otimes\lambda_s$. Therefore $(\delta_G\otimes\mathrm{id})\circ\delta_G = (\mathrm{id}\otimes\delta_G)\circ\delta_G$ because $L(G) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{\lambda(G)\}$.

The pointwise product on $A(G)$ coincides with that induced on the predual $L(G)_*$ by the comultiplication δ_G of $L(G)$ (see Remark 2.1.4), because:

$$\langle\lambda_s, uv\rangle = u(s)v(s) = \langle\lambda_s,u\rangle\langle\lambda_s,v\rangle = \langle\lambda_s\otimes\lambda_s, u\otimes v\rangle = \langle\delta_G(\lambda_s), u\otimes v\rangle.$$

The definitions of α_G and δ_G can be extended respectively to W*-actions of the Hopf-von Neumann algebras $(L^\infty(G),\alpha_G)$ and $(L(G),\delta_G)$ on $B(L^2(G))$ still denoted by

$$\alpha_G\colon B(L^2(G))\to B(L^2(G))\overline{\otimes}L^\infty(G)$$

and

$$\delta_G\colon B(L^2(G))\to B(L^2(G))\overline{\otimes}L(G),$$

namely

$$\alpha_G(x) = V_G^*(x\otimes 1)V_G, \qquad x\in B(L^2(G))$$

and

$$\delta_G(x) = W_G^*(x\otimes 1)W_G, \qquad x\in B(L^2(G)).$$

We also have a W*-action of $(L^\infty(G), \alpha_G)$ on $B(L^2(G))$ induced by the unitary U_G, namely

$$\beta_G\colon B(L^2(G)) \to B(L^2(G))\overline{\otimes} L^\infty(G),$$

$$\beta_G(x) = U_G^*(x\otimes 1)U_G, \qquad x \in B(L^2(G)).$$

Since $L^\infty(G)' = L^\infty(G)$ and $L(G)' = R(G)$ one can easily verify the following

$$B(L^2(G))^{\alpha_G} = R(G),$$

$$B(L^2(G))^{\delta_G} = L^\infty(G),$$

$$B(L^2(G))^{\beta_G} = L(G).$$

Let us prove, for example, the relation $B(L^2(G))^{\alpha_G} = R(G)$. First, since $V_G \in L(G)\overline{\otimes} L^\infty(G) = (R(G)\overline{\otimes} L^\infty(G))'$ it follows that $V_G^*(\rho_t\otimes 1)V_G = V_G^*V_G(\rho_t\otimes 1) = \rho_t \otimes 1$ and thus we have the inclusion $R(G) \subseteq B(L^2(G))^{\alpha_G}$.

We show now that $B(L^2(G))^{\alpha_G} \subseteq R(G)$. Suppose that $x \in B(L^2(G))^{\alpha_G}$, that is $V_G(x\otimes 1) = (x\otimes 1)V_G$. For any $\xi, \eta, \phi, \psi \in C_c(G)$, we have

$$\begin{aligned}
&\langle V_G(x\otimes 1)(\xi\otimes\eta)|\phi\otimes\psi\rangle = \langle (x\otimes 1)V_G(\xi\otimes\eta)|\phi\otimes\psi\rangle \\
\Rightarrow &\langle (x\otimes 1)(\xi\otimes\eta)|V_G^*(\phi\otimes\psi)\rangle = \langle V_G(\xi\otimes\eta)|(x^*\phi)\otimes\psi\rangle \\
\Rightarrow &\iint (x\xi)(s)\eta(t)\overline{\phi(ts)\psi(t)}\; ds\; dt = \iint \xi(t^{-1}s)\eta(t)\overline{(x^*\phi)(s)\psi(t)}\; ds\; dt \\
\Rightarrow &\int\left(\int (x\xi)(t^{-1}s)\overline{\phi(s)}\; ds\right)\eta(t)\overline{\psi(t)}\; dt = \\
&= \int\left(\int \xi(t^{-1}s)\overline{(x^*\phi)(s)}\; ds\right)\eta(t)\overline{\psi(t)}\; dt \\
\Rightarrow &\int \langle \lambda_t(x\xi)|\phi\rangle\eta(t)\overline{\psi(t)}\; dt = \int \langle \lambda_t\xi|x^*\phi\rangle\eta(t)\overline{\psi(t)}\; dt.
\end{aligned}$$

Therefore we get

$$\langle \lambda_t(x\xi)|\phi\rangle = \langle x\lambda_t\xi|\phi\rangle \quad \forall \xi,\phi \in C_c(G),\ \forall t\in G$$

and since $C_c(G)$ is dense in $L^2(G)$ it follows that $x\lambda_t = \lambda_t x$ for all $t \in G$. Thus $x \in L(G)' = R(G)$.

The relations $B(L^2(G))^{\delta_G} = L^\infty(G)$ and $B(L^2(G))^{\beta_G} = L(G)$ can be proved similarly.

In the following, $L^\infty(G)$ and $L(G)$ will always be considered as Hopf-von Neumann algebras with respect to α_G and δ_G respectively.

2.3.2 $L^\infty(G)$-comodules

In this section we present two very interesting properties of the Hopf-von Neumann algebra $(L^\infty(G), \alpha_G)$, which both generalize well known facts about W*-dynamical systems.

The first is that the class of dynamical systems given by actions of G on dual operator spaces by w*-continuous completely isometric automorphisms coincides with the class of $L^\infty(G)$-comodules. Therefore, the notion of a Hopf-von Neumann algebra and the associated comodules provide a natural framework for the study of dynamical systems.

Secondly, we show that every $L^\infty(G)$-comodule is non-degenerate and saturated (see Lemma 2.3.5 below). This is the key ingredient in the proofs of some of our main results appearing in the following (see e.g. Proposition 3.2.9, Theorem 3.2.10 and Propositions 3.3.2 and 3.3.5).

Let us begin with some terminology. A *G-dynamical system* is a triple (X, G, γ) where X is a dual operator space and $\gamma\colon G \to \mathrm{Aut(X)}$ is a *G-action* on X. That is γ is a group homomorphism from G into the group Aut(X) of w*-continuous completely isometric automorphisms of X, i.e.

$$\gamma_s \circ \gamma_t = \gamma_{st} \qquad \forall\ s, t \in G$$

and for any $\omega \in X_*$ and any $x \in X$ the function

$$s \mapsto \langle \gamma_s(x), \omega \rangle, \quad s \in G,$$

is continuous. A w*-closed subspace Y of X is called *G-invariant* if $\gamma_s(Y) \subseteq Y$ for all $s \in G$.

In the case where X is a von Neumann algebra, we will assume that, for any $s \in G$, the automorphism γ_s is additionally a unital *-homomorphism. Then, (X, G, γ) is called a W*-*dynamical system* and γ is called a W*-*G-action* on X.

The proofs of the next three results, i.e. Lemma 2.3.2 and Propositions 2.3.3 and 2.3.4, are more or less standard at least in the context of W*-dynamical systems (see for instance [46, Lemma 1/§13.1, §18.6]). However, we have included these proofs (with some changes) for the reader's convenience and in order to make it clear that the validity of the statements does not rely on the von Neumann algebra structure.

The following is needed for the proof of Proposition 2.3.3.

Lemma 2.3.2. *If X is a dual operator space and $F\colon G \to X$ is a w*-continuous and norm-bounded function, then there exists a unique element $T \in X \overline{\otimes} L^\infty(G)$, such that*

$$\langle T, \omega \otimes h \rangle = \int_G \langle F(s), \omega \rangle h(s)\ ds, \quad \forall \omega \in X_*,\ \forall h \in L^1(G).$$

Proof. Since $L^\infty(G)$ is abelian we have

$$X\overline{\otimes}L^\infty(G) = X\overline{\otimes}_{\mathcal{F}}L^\infty(G) \simeq (X_*\widehat{\otimes}L^1(G))^* \simeq CB(X_*, L^\infty(G)).$$

Consider the linear map $\Phi\colon X_* \to L^\infty(G)$ defined by

$$\Phi(\omega) = \omega \circ F, \quad \omega \in X_*$$

and observe that Φ is completely bounded since F is norm-bounded. Hence, since the isomorphism $\varphi\colon (X_*\widehat{\otimes}L^1(G))^* \to CB(X_*, L^\infty(G))$ is given by

$$\langle \varphi(u)(\omega), h\rangle = \langle u, \omega \otimes h\rangle, \quad \omega \in X_*,\ h \in L^1(G),\ u \in (X_*\widehat{\otimes}L^1(G))^*,$$

it follows that there exists a unique element $T \in X\overline{\otimes}L^\infty(G) \simeq (X_*\widehat{\otimes}L^1(G))^*$ such that, for any $\omega \in X_*$ and $h \in L^1(G)$, we have

$$\langle T, \omega \otimes h\rangle = \langle \Phi(\omega), h\rangle = \langle \omega \circ F, h\rangle = \int_G \langle F(s), \omega\rangle h(s)\ ds.$$

□

Proposition 2.3.3. *Let (X, G, γ) be a dynamical system. For every $x \in X$ there is a unique element $\pi_\gamma(x) \in X\overline{\otimes}L^\infty(G)$ such that*

$$\langle \pi_\gamma(x), \omega \otimes h\rangle = \int_G \langle \gamma_s^{-1}(x), \omega\rangle h(s)\ ds, \quad \forall \omega \in X_*,\ \forall h \in L^1(G). \tag{2.2}$$

The map $\pi_\gamma\colon X \to X\overline{\otimes}L^\infty(G)$ is an $L^\infty(G)$-action on X, i.e. π_γ is a w-continuous complete isometry and satisfies*

$$(\pi_\gamma \otimes \mathrm{id}_{L^\infty(G)}) \circ \pi_\gamma = (\mathrm{id}_X \otimes \alpha_G) \circ \pi_\gamma. \tag{2.3}$$

Also, we have

$$X^{\pi_\gamma} = \{x \in X :\ \gamma_s(x) = x,\ \forall s \in G\} \tag{2.4}$$

and the $L^\infty(G)$-subcomodules of X are exactly the G-invariant w-closed subspaces of X. Moreover, if X is a w*-closed subspace of $B(H)$, then π_γ satisfies the so called covariance relations, i.e.*

$$\pi_\gamma(\gamma_s(x)) = (1_H \otimes \lambda_s)\pi_\gamma(x)(1_H \otimes \lambda_s^{-1}), \qquad s \in G,\ x \in X. \tag{2.5}$$

Finally, if (X, G, γ) is a W-dynamical system, then (X, π_γ) is a W*-$L^\infty(G)$ -comodule.*

Proof. For any $x \in X$, the function $s \mapsto \gamma_{s^{-1}}(x)$ is w*-continuous and bounded by $\|x\|$ and thus from Lemma 2.3.2 it follows that there is a unique $\pi_\gamma(x) \in X\overline{\otimes}L^\infty(G)$ satisfying (2.2).

Using the definition of π_γ, i.e. (2.2), it follows easily that π_γ is w*-continuous and completely isometric and if X is, in addition, a von Neumann algebra and each γ_s is a *-automorphism of X, then π_γ is moreover a unital *-homomorphism.

We now show that (2.3) holds. Indeed, for $x \in X$, $\omega \in X_*$ and $k, h \in L^1(G)$, we have

$$\begin{aligned}\langle(\pi_\gamma \otimes \mathrm{id}_{L^\infty(G)})(\pi_\gamma(x)), \omega \otimes h \otimes k\rangle &= \langle \pi_\gamma(x), ((\omega \otimes h) \circ \pi_\gamma) \otimes k\rangle \\ &= \int \langle \pi_\gamma(\gamma_t^{-1}(x)), \omega \otimes h\rangle k(t)\ dt \\ &= \iint \langle \gamma_s^{-1}(\gamma_t^{-1}(x)), \omega\rangle h(s)k(t)\ ds\ dt \\ &= \iint \langle \gamma_{ts}^{-1}(x), \omega\rangle h(s)k(t)\ ds\ dt \\ &= \iint \langle \gamma_s^{-1}(x), \omega\rangle h(t^{-1}s)k(t)\ ds\ dt\end{aligned}$$

and

$$\begin{aligned}\langle(\mathrm{id}_X \otimes \alpha_G)(\pi_\gamma(x)), \omega \otimes h \otimes k\rangle &= \langle \pi_\gamma(x), \omega \otimes (h \otimes k) \circ \alpha_G\rangle \\ &= \langle \pi_\gamma(x), \omega \otimes (hk)\rangle \\ &= \langle \pi_\gamma(x), \omega \otimes (k * h)\rangle \\ &= \int \langle \gamma_s^{-1}(x), \omega\rangle (k * h)(s)\ ds \\ &= \iint \langle \gamma_s^{-1}(x), \omega\rangle k(t)h(t^{-1}s)\ dt\ ds\end{aligned}$$

and the last two integrals are equal by Fubini's theorem.

Next we prove (2.4), i.e. that for $x \in X$ we have

$$\pi_\gamma(x) = x \otimes 1 \iff \gamma_s(x) = x\ \forall s \in G.$$

Indeed, if $\gamma_s(x) = x\ \forall s \in G$, then for any $\omega \in X_*$ and $h \in L^1(G)$ we have

$$\langle \pi_\gamma(x), \omega \otimes h\rangle = \int \langle x, \omega\rangle h(s)\ ds = \langle x, \omega\rangle\langle 1, h\rangle = \langle x \otimes 1, \omega \otimes h\rangle$$

and thus $\pi_\gamma(x) = x \otimes 1$. Conversely, if $\pi_\gamma(x) = x \otimes 1$, then for any $\omega \in X_*$ and $h \in L^1(G)$ we have

$$\int \langle \gamma_s^{-1}(x), \omega\rangle h(s)\ ds = \int \langle x, \omega\rangle h(s)\ ds$$

and hence the continuous and bounded function $s \mapsto \langle \gamma_s^{-1}(x), \omega\rangle$ is almost everywhere equal to the constant $\langle x, \omega\rangle$. Therefore, $\langle \gamma_s^{-1}(x), \omega\rangle = \langle x, \omega\rangle$ for all $s \in G$ and $\omega \in X_*$, that is $\gamma_s(x) = x$ for all $s \in G$.

To prove (2.5), first note that, for $h \in L^\infty(G)_* \simeq L^1(G)$, the element $h \circ \mathrm{Ad}\lambda_s \in L^\infty(G)_*$ regarded as an element of $L^1(G)$ can be written

$$(h \circ \mathrm{Ad}\lambda_s)(t) = h(st), \qquad t \in G$$

and thus, for $\omega \in X_*$ and $h \in L^1(G)$, we have

$$\begin{aligned}\langle (1_H \otimes \lambda_s)\pi_\gamma(x)(1_H \otimes \lambda_s^{-1}), \omega \otimes h\rangle &= \langle \pi_\gamma(x), \omega \otimes (h \circ \mathrm{Ad}\lambda_s)\rangle \\ &= \int_G \langle \gamma_t^{-1}(x), \omega\rangle h(st)\ dt \\ &= \int_G \langle \gamma_{t^{-1}s}(x), \omega\rangle h(t)\ dt \\ &= \int_G \langle \gamma_t^{-1}(\gamma_s(x)), \omega\rangle h(t)\ dt \\ &= \langle \pi_\gamma(\gamma_s(x)), \omega \otimes h\rangle.\end{aligned}$$

Finally, if Y is a G-invariant (w*-closed) subspace of X, then it is clear from the definition of π_γ that $\pi_\gamma(Y) \subseteq Y\overline{\otimes}L^\infty(G)$. Conversely, if Y is an $L^\infty(G)$-subcomodule of (X, π_γ), then $L^1(G) \cdot Y \subseteq Y$, where $h \cdot y = (\mathrm{id} \otimes h) \circ \pi_\gamma(y)$ for $h \in L^1(G)$, $y \in Y$. Now, for $y \in Y$, $h \in L^1(G)$ and $s \in G$, from (2.5) it follows

$$\begin{gathered}\gamma_s(h \cdot y) = \gamma_s((\mathrm{id} \otimes h) \circ \pi_\gamma(y)) = (\mathrm{id} \otimes h) \circ \gamma_s \circ \pi_\gamma(y) \\ = (\mathrm{id} \otimes h)\left((1 \otimes \lambda_s)\pi_\gamma(y)(1 \otimes \lambda_s^{-1})\right) = (\mathrm{id} \otimes (\Delta_G(s) r_s h)) \circ \pi_\gamma(y) \\ = \Delta_G(s)((r_s h) \cdot y) \in L^1(G) \cdot Y \subseteq Y,\end{gathered}$$

where $r_s h(t) := h(ts)$. Thus $\gamma_s(L^1(G) \cdot y) \subseteq Y$ and since $y \in \overline{L^1(G) \cdot y}^{\mathrm{w}^*}$ (see Lemma 2.3.5 below), it follows that $\gamma_s(y) \in Y$ for any $s \in G$. Hence Y is G-invariant. □

Proposition 2.3.4. *For any $L^\infty(G)$-action $\alpha\colon X \to X\overline{\otimes}L^\infty(G)$ on a dual operator space X there exists a unique G-action $\gamma\colon G \to \mathrm{Aut(X)}$, such that $\alpha = \pi_\gamma$. In particular, $\gamma_s = \alpha^{-1} \circ (\mathrm{id}_X \otimes \mathrm{Ad}\lambda_s) \circ \alpha$ for $s \in G$.*

Proof. Suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H.

First observe that the $L^\infty(G)$-action

$$\alpha_G\colon B(L^2(G)) \to B(L^2(G))\overline{\otimes}L^\infty(G)$$

coincides with $\pi_{\mathrm{Ad}\lambda}$, where

$$\mathrm{Ad}\lambda\colon G \to \mathrm{Aut(B(L^2(G)))},$$

$$\mathrm{Ad}\lambda_s(T) = \lambda_s T \lambda_s^{-1}.$$

Indeed, for $x \in R(G)$ we have $\alpha_G(x) = x \otimes 1 = \pi_{\mathrm{Ad}\lambda}(x)$. Also, for $f \in L^\infty(G)$ and $k, h \in L^1(G)$, by the definitions of $\pi_{\mathrm{Ad}\lambda}$ and α_G, we have:

$$\langle \pi_{\mathrm{Ad}\lambda}(f), k \otimes h\rangle = \int_G \langle \lambda_t^{-1} f \lambda_t, k\rangle h(t)\ dt$$

$$= \int_G \left(\int_G f(ts)k(s)\ ds \right) h(t)\ dt = \langle \alpha_G(f), k \otimes h \rangle.$$

Thus we get that the $L^\infty(G)$-action

$$\mathrm{id}_X \otimes \alpha_G \colon X \overline{\otimes} B(L^2(G)) \to X \overline{\otimes} B(L^2(G)) \overline{\otimes} L^\infty(G)$$

is equal to $\pi_{\mathrm{id}_X \otimes \mathrm{Ad}\lambda}$, where $(\mathrm{id}_X \otimes \mathrm{Ad}\lambda)_s(T) = (1_H \otimes \lambda_s)T(1_H \otimes \lambda_s^{-1})$ for $s \in G$ and $T \in X \overline{\otimes} B(L^2(G))$.

Secondly, if an $L^\infty(G)$-comodule (Y, β) is isomorphic to an $L^\infty(G)$-comodule of the form (Z, π_γ), then β will be also of the form $\pi_{\gamma'}$. Indeed, if $\phi \colon Y \to Z$ is an isomorphism for the comodules (Y, β) and (Z, π_γ), one can take $\gamma'_s := \phi^{-1} \circ \gamma_s \circ \phi$ for $s \in G$.

Finally, since (X, α) is isomorphic to $(\alpha(X), \mathrm{id} \otimes \alpha_G)$ (see Remark 2.1.3) which is an $L^\infty(G)$-subcomodule of $(X \overline{\otimes} B(L^2(G)), \mathrm{id} \otimes \alpha_G)$ and since any $L^\infty(G)$-subcomodule is G-invariant (by Proposition 2.3.3), it follows from all of the above, that $\alpha = \pi_\gamma$, where γ is the G-action on X given by $\gamma_s = \alpha^{-1} \circ (\mathrm{id}_X \otimes \mathrm{Ad}\lambda_s) \circ \alpha,\ s \in G$.

For the uniqueness part note that if γ and γ' are two G-actions on X with $\pi_\gamma = \pi_{\gamma'}$, then using (2.5), for any $x \in X$ and $s \in G$, we have:

$$\begin{aligned}\pi_\gamma(\gamma_s(x)) = (1_H \otimes \lambda_s)\pi_\gamma(x)(1_H \otimes \lambda_s^{-1}) &= (1_H \otimes \lambda_s)\pi_{\gamma'}(x)(1_H \otimes \lambda_s^{-1}) \\ &= \pi_{\gamma'}(\gamma'_s(x)) = \pi_\gamma(\gamma'_s(x))\end{aligned}$$

and thus $\gamma_s(x) = \gamma'_s(x)$ since π_γ is injective. Therefore $\gamma = \gamma'$. □

Lemma 2.3.5. *Every $L^\infty(G)$-comodule is non-degenerate and saturated. In particular, for any $L^\infty(G)$-comodule X and any $x \in X$, we have that $x \in \overline{L^1(G) \cdot x}^{\mathrm{w}^*}$.*

Proof. Let (X, α) be an $L^\infty(G)$-comodule with X a w*-closed subspace of $B(H)$ for some Hilbert space H.

By Remark 2.1.3, we have that $\alpha(X)$ is an $L^\infty(G)$-subcomodule of the W*-$L^\infty(G)$-comodule (N, β) with $N = B(H) \overline{\otimes} L^\infty(G)$ and $\beta = \mathrm{id}_{B(H)} \otimes \alpha_G$.

Consider the w*-continuous *-injections $\pi_1,\ \pi_2 \colon N \to N \overline{\otimes} L^\infty(G)$ given by:

$$\langle \pi_1(y), \omega \otimes f \rangle = \int_G \langle \mathrm{Ad}(1_H \otimes \lambda_s^{-1})(y), \omega \rangle f(s)\ ds,$$
$$\langle \pi_2(y), \omega \otimes f \rangle = \int_G \langle \mathrm{Ad}(1_H \otimes \lambda_s)(y), \omega \rangle f(s)\ ds,$$

for $y \in N$, $\omega \in N_*$ and $f \in L^1(G)$. It is easy to verify that for any $y \in N$ we have:

$$\pi_1(y) = (1_H \otimes V_G^*)(y \otimes 1)(1_H \otimes V_G) = \beta(y);$$
$$\pi_2(y) = (1_H \otimes V_G)(y \otimes 1)(1_H \otimes V_G^*).$$

Therefore, since $\pi_1(\alpha(X)) = \beta(\alpha(X)) \subseteq \alpha(X)\overline{\otimes}L^\infty(G)$, it follows that for any $s \in G$ we have

$$\mathrm{Ad}(1_H \otimes \lambda_s^{-1})(\alpha(X)) = \alpha(X),$$

that is

$$\mathrm{Ad}(1_H \otimes \lambda_t)(\alpha(X)) = \alpha(X), \text{ for all } t \in G$$

and thus

$$\pi_2(\alpha(X)) \subseteq \alpha(X)\overline{\otimes}L^\infty(G).$$

Hence, since $1_H \otimes V_G \in \mathbb{C}1_H\overline{\otimes}L(G)\overline{\otimes}L^\infty(G)$ and $L^\infty(G)' = L^\infty(G)$, it follows that both $\mathrm{Ad}(1_H \otimes V_G^*)$ and $\mathrm{Ad}(1_H \otimes V_G)$ map $\alpha(X)\overline{\otimes}L^\infty(G)$ into $\alpha(X)\overline{\otimes}L^\infty(G)$ and so the restriction of $\mathrm{Ad}(1_H \otimes V_G^*)$ to $\alpha(X)\overline{\otimes}L^\infty(G)$ is a completely isometric automorphism of $\alpha(X)\overline{\otimes}L^\infty(G)$.

It follows from the above that the map $\theta\colon X\overline{\otimes}L^\infty(G) \to X\overline{\otimes}L^\infty(G)$ defined by

$$\theta = (\alpha^{-1} \otimes \mathrm{id}_{L^\infty(G)}) \circ \mathrm{Ad}(1_H \otimes V_G^*) \circ (\alpha \otimes \mathrm{id}_{L^\infty(G)})$$

is a well defined w*-continuous completely isometric automorphism of $X\overline{\otimes}L^\infty(G)$. Also, using the definition of θ, we get that

$$\theta(x \otimes 1) = \alpha(x), \qquad x \in X$$

and

$$\theta((1_H \otimes f)y) = (1_H \otimes f)\theta(y), \qquad f \in L^\infty(G),\ y \in X\overline{\otimes}L^\infty(G).$$

Therefore, it follows that

$$X\overline{\otimes}L^\infty(G) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_H \otimes f)\alpha(x) :\ x \in X,\ f \in L^\infty(G)\},$$

which using (1.11) implies that

$$X\overline{\otimes}B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_H \otimes b)\alpha(x) :\ x \in X,\ b \in B(L^2(G))\},$$

that is (X, α) is non-degenerate.

So, we have proved that every $L^\infty(G)$-comodule is non-degenerate. Thus it follows (from Corollary 2.2.7) that every $L^\infty(G)$-comodule is saturated and non-degenerate.

Now, since every $L^\infty(G)$-comodule X is saturated it follows (from Proposition 2.2.9) that $x \in \overline{L^1(G) \cdot x}^{\mathrm{w}^*}$ for all $x \in X$. □

2.3.3 $L(G)$-comodules and the approximation property

In this section we study the class of $L(G)$-comodules with respect to the notions of non-degeneracy and saturation.

First, we show that an $L(G)$-comodule (Y, δ) is non-degenerate if and only if Y is the w*-closed linear span of $A(G) \cdot Y$, i.e. the converse of Proposition 2.2.3 is true for the Hopf-von Neumann algebra $(L(G), \delta_G)$.

Using the above result, it follows that every $L(G)$-comodule is saturated if and only if every $L(G)$-comodule is non-degenerate, that is the converse of Corollary 2.2.7 is valid for $(L(G), \delta_G)$.

For a Hopf-von Neumann algebra (M, Δ), recall that if any M-comodule is saturated, then M_* has a weak approximate unit (Proposition 2.2.9). We prove here that the existence of an a priori stronger notion of approximate unit in the Fourier algebra $A(G)$ introduced by Haagerup and Kraus [22] (see Definition 2.3.11) is necessary and sufficient in order to have that every $L(G)$-comodule is saturated.

The next two results, that is Lemma 2.3.6 and Corollary 2.3.7, were originally proved in [50] for W*-$L(G)$-comodules (see Lemma II.1.4 and Corollary II.1.5 in [50]). However, exactly the same arguments as in the proofs of Lemma II.1.4 and Corollary II.1.5 in [50] work for the case of $L(G)$-comodules which are not necessarily von Neumann algebras. We have included the proofs both for the sake of completeness and in order to make it apparent for the reader that the von Neumann algebra structure is redundant.

Lemma 2.3.6. *Let $\delta\colon Y \to Y\overline{\otimes}_{\mathcal{F}}L(G)$ be an $L(G)$-action on a w*-closed subspace Y of $B(H)$ for some Hilbert space H. For any $y \in Y$ and any $f, k \in A(G)$ with compact support, we have*

$$\int_G \Delta_G(s)^{-1}(1_H \otimes \lambda_s)\delta((f_s k) \cdot y)\, ds = (k \cdot y) \otimes \lambda(\Delta_G^{-1} f), \tag{2.6}$$

where $f_s(t) = f(st)$ and the integral is understood in the w-topology of $Y\overline{\otimes}_{\mathcal{F}}L(G) \simeq (Y_* \widehat{\otimes} A(G))^*$.*

Proof. First, observe that the function

$$G \ni s \mapsto \Delta_G(s)^{-1}(1_H \otimes \lambda_s)\delta((f_s k) \cdot y) \in Y\overline{\otimes}_{\mathcal{F}}L(G)$$

is w*-continuous and has compact support because the function $f_s k$ is zero outside the set $(\mathrm{supp(f)})(\mathrm{supp(k)})^{-1}$ which is compact. Therefore, the integral in (2.6) is well defined and represents a (unique) element of $Y\overline{\otimes}_{\mathcal{F}}L(G)$.

Also, for any $\phi \in Y_*$ and any $h \in A(G)$, we have:

$$\begin{aligned}
&\left\langle \int_G \Delta_G(s)^{-1}(1_H \otimes \lambda_s)\delta((f_s k)\cdot y)\ ds, \phi \otimes h \right\rangle = \\
&= \int_G \langle (1_H \otimes \lambda_s)\delta((f_s k)\cdot y), \phi \otimes h\rangle \Delta_G(s)^{-1}\ ds \\
&= \int_G \langle \delta((f_s k)\cdot y), \phi \otimes h_s\rangle \Delta_G(s)^{-1}\ ds \\
&= \int_G \langle h_s \cdot ((f_s k)\cdot y), \phi\rangle \Delta_G(s)^{-1}\ ds \\
&= \int_G \langle (h_s f_s k)\cdot y), \phi\rangle \Delta_G(s)^{-1}\ ds \\
&= \int_G \langle \delta(y), \phi \otimes (hf)_s k\rangle \Delta_G(s)^{-1}\ ds \\
&= \left\langle \delta(y), \phi \otimes \int_G (hf)_s k \Delta_G(s)^{-1}\ ds \right\rangle.
\end{aligned}$$

The integral $\int_G (hf)_s k\Delta_G(s)^{-1}\ ds$ is understood in the $\sigma(A(G), L(G))$-topology and defines a unique element $b \in A(G)$. For any $t \in G$, we have:

$$\begin{aligned}
b(t) = \langle \lambda_t, b\rangle &= \left\langle \lambda_t, \int_G (hf)_s k\Delta_G(s)^{-1}\ ds \right\rangle \\
&= \int_G \langle \lambda_t, (hf)_s k\rangle \Delta_G(s)^{-1}\ ds \\
&= \int_G h(st)f(st)k(t)\Delta_G(s)^{-1}\ ds \\
&= \left(\int_G h(s)f(s)\Delta_G(s)^{-1}\ ds \right) k(t) \\
&= \langle \lambda(f\Delta_G^{-1}), h\rangle k(t),
\end{aligned}$$

thus $b = \langle \lambda(f\Delta_G^{-1}), h\rangle k$ and the first chain of equalities may be continued as follows

$$\begin{aligned}
\langle \delta(y), \phi \otimes k\rangle \langle \lambda(\Delta_G^{-1} f), h\rangle &= \langle k \cdot y, \phi\rangle \langle \lambda(\Delta_G^{-1} f), h\rangle \\
&= \langle (k\cdot y) \otimes \lambda(\Delta_G^{-1} f), \phi \otimes h\rangle
\end{aligned}$$

and the proof is complete. □

Corollary 2.3.7. *If $\delta\colon Y \to Y \overline{\otimes}_{\mathcal{F}} L(G)$ is an $L(G)$-action on a w*-closed subspace Y of $B(H)$ for some Hilbert space H, then for any $y \in Y$ and any $k \in A(G)$, we have:*

$$(k\cdot y) \otimes 1_{L^2(G)} \in \overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_H \otimes \lambda_s)\delta((hk)\cdot y) :\ s \in G,\ h \in A(G)\}, \quad (2.7)$$

where $k \cdot y = (\mathrm{id}_Y \otimes k)(\delta(y))$, for $k \in A(G)$ and $y \in Y$.

Proof. From Lemma 2.3.6 it follows that

$$(k \cdot y) \otimes \lambda(\Delta_G^{-1} f) \in \overline{\operatorname{span}}^{w^*}\{(1_H \otimes \lambda_s)\delta((hk) \cdot x) : \; s \in G, \; h \in A(G)\} \quad (2.8)$$

for any $y \in Y$ and any $k, f \in A(G)$ both with compact support. By [15, Lemme (3.2)], we have that $L^1(G)$ contains a bounded approximate identity of the form $\{\Delta_G^{-1} f_i\}_{i \in I}$, where $f_i \in A(G)$ are functions of compact support. Therefore, $\lambda(\Delta_G^{-1} f_i) \longrightarrow 1_{L^2(G)}$ ultraweakly and thus using (2.8), we get (2.7) for $k \in A(G)$ with compact support. Since the functions in $A(G)$ with compact support are norm dense in $A(G)$ (by [15, Proposition (3.26)]), we get (2.7) for all $k \in A(G)$. □

Using the above we can prove that the converse of Proposition 2.2.3 is true for the Hopf-von Neumann algebra $(L(G), \delta_G)$.

Corollary 2.3.8. *Let (Y, δ) be an $L(G)$-comodule where Y is a w^*-closed subspace of $B(H)$ for some Hilbert space H. Then, the following are equivalent:*

(i) $Y = \overline{\operatorname{span}}^{w^*}\{h \cdot y : \; h \in A(G), \; y \in Y\}$;

(ii) (Y, δ) *is non-degenerate;*

(iii) $Y \overline{\otimes} B(L^2(G)) = \overline{\operatorname{span}}^{w^*}\{(1_H \otimes b)\delta(y)(1_H \otimes c) : y \in Y, b, c \in B(L^2(G))\}$,

where $h \cdot y = (\mathrm{id}_Y \otimes h)(\delta(y))$, *for* $h \in A(G)$ *and* $y \in Y$.

Proof. (iii) $\Longrightarrow$ (i): Let $\phi \in Y_*$, such that $\phi(h \cdot y) = 0$, for all $h \in A(G)$ and $y \in Y$. Then, we have:

$$\begin{aligned}
&\phi \circ (\mathrm{id}_Y \otimes h) \circ \delta(x) = 0, \quad \forall h \in A(G), \; \forall y \in Y \\
\Longrightarrow\; &\langle (\phi \otimes \mathrm{id}_{B(K)}) \circ \delta(x), h \rangle = 0, \quad \forall h \in A(G), \; \forall y \in Y \\
\Longrightarrow\; &(\phi \otimes \mathrm{id}_{B(K)}) \circ \delta(x) = 0, \quad \forall y \in Y \\
\Longrightarrow\; &b(\phi \otimes \mathrm{id}_{B(K)})(\delta(x))c = 0, \quad \forall b, c \in B(K), \; \forall y \in Y \\
\Longrightarrow\; &(\phi \otimes \mathrm{id}_{B(K)})\left((1_H \otimes b)\delta(x)(1_H \otimes c)\right) = 0, \quad \forall b, c \in B(K), \; \forall y \in Y.
\end{aligned}$$

Since (iii) holds, the last condition implies that

$$(\phi \otimes \mathrm{id}_{B(L^2(G))})(z) = 0, \qquad \forall z \in Y \overline{\otimes} B(L^2(G)),$$

thus $\phi(y)1 = (\phi \otimes \mathrm{id}_{B(L^2(G))})(y \otimes 1) = 0$ for any $y \in Y$ and hence $\phi = 0$. Thus the desired conclusion, i.e. condition (i), follows from the Hahn-Banach theorem.

(ii) $\Longrightarrow$ (iii): This follows from the obvious inclusion

$$\begin{aligned}
&\{(1_H \otimes b)\delta(y) : \; y \in Y, \; b \in B(L^2(G))\} \subseteq \\
&\subseteq \{(1_H \otimes b)\delta(y)(1_H \otimes c) : \; y \in Y, \; b, c \in B(L^2(G))\}.
\end{aligned}$$

(i) $\implies$ (ii): Suppose that $Y = \overline{\text{span}}^{w^*}\{h \cdot y : \ h \in A(G), \ y \in Y\}$. From Corollary 2.3.7 above it follows that, for any $z \in Y$, we have

$$z \otimes 1_{L^2(G)} \in \overline{\text{span}}^{w^*}\{(1_H \otimes b)\delta(y) : \ b \in B(L^2(G)), \ y \in Y\}.$$

Therefore, for any $z \in Y$ and $c \in B(L^2(G))$, we get that

$$z \otimes c = (1_H \otimes c)(z \otimes 1_{L^2(G)}) \in \overline{\text{span}}^{w^*}\{(1_H \otimes b)\delta(y) : \ b \in B(L^2(G)), \ y \in Y\},$$

because the multiplication in $B(H)\overline{\otimes}B(L^2(G))$ is separately w*-continuous. Thus, we have that

$$Y\overline{\otimes}B(L^2(G)) \subseteq \overline{\text{span}}^{w^*}\{(1_H \otimes b)\delta(y) : \ b \in B(L^2(G)), \ y \in Y\}$$

and thus we get that Y is non-degenerate since the reverse inclusion is trivial. □

Comment 2.3.9. Note that, for an $L(G)$-comodule (Y, δ), the subspace

$$Z := \overline{\text{span}}^{w^*}\{A(G) \cdot Y\} \subseteq Y$$

is the largest non-degenerate $L(G)$-subcomodule of Y.

Indeed, Z is clearly an $L(G)$-subcomodule of Y because it is an $A(G)$-submodule of Y. Also, Z is non-degenerate by Corollary 2.3.7 (using a similar argument as in the proof of the implication (i) $\implies$ (ii) in Corollary 2.3.8). Finally, if Z_0 is a non-degenerate $L(G)$-subcomodule of Y, then $Z_0 = \overline{\text{span}}^{w^*}\{A(G) \cdot Z_0\}$ (by Proposition 2.2.3) and thus we have that

$$Z_0 = \overline{\text{span}}^{w^*}\{A(G) \cdot Z_0\} \subseteq \overline{\text{span}}^{w^*}\{A(G) \cdot Y\} = Z.$$

Following [8], [11] and [22], we have the following definitions.

Definition 2.3.10. A complex-valued function $u: G \to \mathbb{C}$ is called a *multiplier* for the Fourier algebra $A(G)$ if the linear map $m_u(v) = uv$ maps $A(G)$ into $A(G)$. In this case, a straightforward application of the closed graph theorem shows that m_u is a bounded operator. For a multiplier u we denote by $M_u: L(G) \to L(G)$ the adjoint map m_u^* of m_u. The function u is called a *completely bounded multiplier* if M_u is completely bounded. The space of all completely bounded multipliers is denoted by $M_{cb}A(G)$.

It is known (see e.g. [8]) that $M_{cb}A(G)$ is a Banach algebra with the norm $||u||_{M_{cb}} = ||M_u||_{cb}$ and pointwise multiplication. Moreover, $A(G) \subseteq M_{cb}A(G)$ (see [8]) and $M_{cb}A(G)$ is the dual Banach space of the Banach space $Q(G)$, which is defined to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_Q = \sup\left\{\left|\int_G f(s)u(s)ds\right| : \ u \in M_{cb}A(G), \ \|u\|_{M_{cb}} \leq 1\right\}.$$

Definition 2.3.11 (Haagerup-Kraus, [22])**.** We say that a locally compact group G has *the approximation property* (or the AP) if there is a net $\{u_i\}_{i\in I}$ in $A(G)$, such that $u_i \to 1$ in the $\sigma(M_{cb}A(G), Q(G))$-topology.

Note that the approximation property is a weaker notion than amenability since amenability means exactly that the Fourier algebra $A(G)$ has a (norm) bounded approximate identity (see [36]).

For more details on completely bounded multipliers and the approximation property see for example [8], [11] and [22].

We will need the following theorem due to Haagerup and Kraus (see [22, Theorem 1.9]).

Theorem 2.3.12 (Haagerup-Kraus)**.** *For a locally compact group G, the following conditions are equivalent:*

(i) G has the AP;

(ii) There is a net $\{u_i\} \subseteq A(G)$, such that the net $\{M_{u_i}\} \subseteq CB_\sigma(L(G))$ converges in the stable point-w-topology to $\mathrm{id}_{L(G)}$.*

Remark 2.3.13. Observe that for any $u, h \in A(G)$ and $y \in L(G)$ we have:

$$\begin{aligned}\langle M_u(y), h\rangle &= \langle y, m_u(h)\rangle \\ &= \langle y, hu\rangle \\ &= \langle \delta_G(y), h \otimes u\rangle \\ &= \langle (\mathrm{id}_{L(G)} \otimes u) \circ \delta_G(y), h\rangle,\end{aligned}$$

therefore

$$M_u = (\mathrm{id}_{L(G)} \otimes u) \circ \delta_G, \quad \text{for all } u \in A(G). \tag{2.9}$$

Proposition 2.3.14. *For a locally compact group G the following conditions are equivalent:*

(a) G has the AP;

(b) Every $L(G)$-comodule is saturated;

(c) For any $L(G)$-comodule (Y, δ), any $L(G)$-subcomodule Z of Y and any $y \in Y$, we have that $\delta(y) \in Z\overline{\otimes}_{\mathcal{F}} L(G)$ implies $y \in Z$;

(d) For any $L(G)$-comodule (Y, δ) and any $y \in Y$, we have $y \in \overline{A(G) \cdot y}^{w^*}$;

(e) There exists a net $\{u_i\}_{i\in I}$ in $A(G)$ such that for any $L(G)$-comodule (Y, δ) and any $y \in Y$ we have that $u_i \cdot y \longrightarrow y$ ultraweakly;

(f) Every $L(G)$-comodule is non-degenerate.

Proof. The equivalence of conditions (a) to (e) follows immediately from Proposition 2.2.9, Theorem 2.3.12 and the relation (2.9). The implication (f) $\Longrightarrow$ (b) follows from Corollary 2.2.7 whereas the implication (d) $\Longrightarrow$ (f) follows from Corollary 2.3.8. $\square$

Remark 2.3.15. According to Proposition 2.3.14, if G has the AP, then every $L(G)$-comodule is saturated and non-degenerate.

On the other hand, if G does not have the AP, then Proposition 2.3.14 guarantees the existence of $L(G)$-comodules which are not saturated and the existence of $L(G)$-comodules that are not non-degenerate. However, the author does not know of any example of a group G (necessarily without the AP), such that there exists a single $L(G)$-comodule which is neither saturated nor non-degenerate.

Chapter 3

Crossed products

3.1 Crossed products of $L^\infty(G)$-comodules

Before we proceed to the study of crossed products of $L^\infty(G)$-comodules, let us recall some known results from the theory of crossed products of von Neumann algebras.

Let M be a von Neumann algebra and let $\gamma\colon G \to \mathrm{Aut}(M)$ be a W*-G-action on M, i.e. a group homomorphism from G to the group of unital w*-continuous *-automorphisms of M, such that the function

$$G \ni s \mapsto \gamma_s(x) \in M$$

is w*-continuous for any $x \in M$. Then, by Proposition 2.3.3, we have a W*-$L^\infty(G)$-action $\alpha\colon M \to M\overline{\otimes}L^\infty(G)$ given by

$$\langle \alpha(x), \omega \otimes f \rangle = \int_G \langle \gamma_{s^{-1}}(x), \omega \rangle f(s)\, ds, \quad x \in M,\ \omega \in M_*, f \in L^1(G).$$

Recall that the fixed points of the action γ are exactly the fixed point subspace M^α, that is an $x \in M$ satisfies $\alpha(x) = x \otimes 1$ if and only if $\gamma_s(x) = x$ for all $s \in G$ (see Proposition 2.3.3).

The (usual) *crossed product* $M \rtimes_\alpha G$ (or $M \rtimes_\gamma G$) is defined as the von Neumann subalgebra of $M\overline{\otimes}B(L^2(G))$ generated by $\alpha(M)$ and $\mathbb{C}1\overline{\otimes}L(G)$, which by von Neumann's double commutant theorem is given by

$$M \rtimes_\alpha G = \left(\alpha(M) \cup (\mathbb{C}1\overline{\otimes}L(G))\right)''.$$

According to the Digernes-Takesaki theorem (see for example [53, Chapter X, Corollary 1.22]) we have that $M \rtimes_\alpha G$ is equal to the fixed point algebra of the W*-G-action β on $M\overline{\otimes}B(L^2(G))$ defined by

$$\beta_s = \gamma_s \otimes \mathrm{Ad}\rho_s, \quad s \in G.$$

Moreover, (see also [38] page 9) the W*-$L^\infty(G)$-action corresponding to the G-action $\beta = \gamma \otimes \mathrm{Ad}\rho$, which we denote by $\widetilde{\alpha}$, is given directly by α via the formula:

$$\widetilde{\alpha} = (\mathrm{id}_M \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_M \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$$

where σ is the flip isomorphism on $B(L^2(G))\overline{\otimes}B(L^2(G))$, i.e. $\sigma(x \otimes y) = y \otimes x$. Therefore, the Digernes-Takesaki theorem may be rephrased as

$$M \rtimes_\alpha G = \big(M\overline{\otimes}B(L^2(G))\big)^{\widetilde{\alpha}}.$$

Taking into consideration all of the above, Hamana [19] suggested Definitions 3.1.1 and 3.1.3 below. However, for the sake of consistency, our terminology and symbols are slightly different from Hamana's (see Remark 3.1.5).

Definition 3.1.1. For an $L^\infty(G)$-comodule (X, α), we define the map

$$\widetilde{\alpha} \colon X\overline{\otimes}B(L^2(G)) \to X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)$$

by

$$\widetilde{\alpha} = (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))}),$$

where σ is the flip isomorphism on $B(L^2(G))\overline{\otimes}B(L^2(G))$.

$$\begin{array}{ccc}
X\overline{\otimes}B(L^2(G)) & \xrightarrow{\alpha\otimes\mathrm{id}_{B(L^2(G))}} & X\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G)) \\
 & & \big\downarrow{\scriptstyle \mathrm{id}_X\otimes\sigma} \\
 & & X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G) \\
 & \searrow{\scriptstyle \widetilde{\alpha}} & \big\downarrow{\scriptstyle \mathrm{id}_X\otimes\mathrm{Ad}U_G^*} \\
 & & X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)
\end{array}$$

The next result is essentially the same as [19, Lemma 5.3 (i)] with the appropriate modifications since Hamana considers on $L^\infty(G)$ the (opposite) comultiplication $\sigma \circ \alpha_G$ and uses the right group von Neumann algebra $R(G)$ instead of $L(G)$ as the dual object of $L^\infty(G)$.

Proposition 3.1.2 (Hamana [19])**.** *If (X, α) is an $L^\infty(G)$-comodule, then $\widetilde{\alpha}$ is an $L^\infty(G)$-action on $X\overline{\otimes}B(L^2(G))$ which commutes with the $L(G)$-action $\mathrm{id}_X \otimes \delta_G$ on $X\overline{\otimes}B(L^2(G))$.*

Proof. By Remark 2.1.3 we may suppose that X is a w*-closed subspace of a von Neumann algebra N of the form $N = B(H)\overline{\otimes}L^\infty(G)$ for some Hilbert space H and $\alpha = \varepsilon|_X$, where $\varepsilon = \mathrm{id}_{B(H)} \otimes \alpha_G$. Then obviously $\widetilde{\alpha} = \widetilde{\varepsilon}|_{X\overline{\otimes}B(L^2(G))}$ and $\widetilde{\varepsilon}$ is a W*-$L^\infty(G)$-action on $N\overline{\otimes}B(L^2(G))$. Since $\widetilde{\varepsilon}$ is a w*-continuous *-monomorphism, the latter fact can be easily verified by checking the relation

$$(\widetilde{\varepsilon}\otimes \mathrm{id}_{L^\infty(G)})\circ\widetilde{\varepsilon} = (\mathrm{id}_N\otimes \mathrm{id}_{B(L^2(G))}\otimes\alpha_G)\circ\widetilde{\varepsilon}$$

on the generators of $N\overline{\otimes}B(L^2(G))$, that is on the elements of the form $z\otimes 1$, $1\otimes 1\otimes f$ and $1\otimes 1\otimes\lambda_s$ for $z\in N$, $f\in L^\infty(G)$ and $s\in G$, because $B(L^2(G))$ is generated by $L(G)$ and $L^\infty(G)$.

Thus, in order to prove that $\widetilde{\alpha}$ is an $L^\infty(G)$-action on $X\overline{\otimes}B(L^2(G))$, we only need to show that $\widetilde{\varepsilon}(X\overline{\otimes}B(L^2(G)))\subseteq X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)$. Indeed, we have

$$(\varepsilon\otimes \mathrm{id}_{B(L^2(G))})(X\overline{\otimes}B(L^2(G)))\subseteq X\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G))$$

and thus

$$(\mathrm{id}_X\otimes\sigma)\circ(\varepsilon\otimes \mathrm{id}_{B(L^2(G))})(X\overline{\otimes}B(L^2(G)))\subseteq X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G).$$

Since $U_G\in R(G)\overline{\otimes}L^\infty(G)$ and $X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)$ is a $\mathbb{C}1_H\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)$-bimodule, we get:

$$(\mathrm{id}_X\otimes \mathrm{Ad}U_G^*)\circ(\mathrm{id}_X\otimes\sigma)\circ(\varepsilon\otimes \mathrm{id}_{B(L^2(G))})(X\overline{\otimes}B(L^2(G))) \subseteq X\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G).$$

On the other hand, in order to prove that $\widetilde{\alpha}$ and $\mathrm{id}_X\otimes\delta_G$ commute, it suffices to verify that $\mathrm{id}_{B(H)}\otimes \mathrm{id}_{L^\infty(G)}\otimes\delta_G$ and $\widetilde{\varepsilon}$ commute, where $\varepsilon = \mathrm{id}_{B(H)}\otimes\alpha_G$. Because $\widetilde{\varepsilon}$ and $\mathrm{id}_{B(H)}\otimes \mathrm{id}_{L^\infty(G)}\otimes\delta_G$ act identically on the first factor $B(H)$, we only need to prove that $\mathrm{id}_{L^\infty(G)}\otimes\delta_G$ and $\widetilde{\alpha_G}$ commute, that is:

$$\begin{aligned}(\widetilde{\alpha_G}\otimes \mathrm{id}_{L(G)})\circ(\mathrm{id}_{L^\infty(G)}\otimes\delta_G) &= \\ =(\mathrm{id}_{L^\infty(G)}\otimes \mathrm{id}_{B(L^2(G))}\otimes\sigma)\circ(\mathrm{id}_{L^\infty(G)}&\otimes\delta_G\otimes \mathrm{id}_{L^\infty(G)})\circ\widetilde{\alpha_G}\end{aligned}\tag{3.1}$$

Let S denote the unitary on $L^2(G)\otimes L^2(G)$ with $S(\xi\otimes\eta)=\eta\otimes\xi$. Thus, the flip isomorphism σ on $B(L^2(G))\overline{\otimes}B(L^2(G))$ is written as $\sigma = \mathrm{Ad}S$. If $a\in L^\infty(G)$ and $b\in B(L^2(G))$, then by applying the left and right hand sides of (3.1) on $a\otimes b$, we get respectively:

$$\begin{aligned}&(\widetilde{\alpha_G}\otimes \mathrm{id}_{L(G)})\circ(\mathrm{id}_{L^\infty(G)}\otimes\delta_G)(a\otimes b) = \\ &\mathrm{Ad}[(1\otimes U_G^*\otimes 1)(1\otimes S\otimes 1)(V_G^*\otimes 1\otimes 1)(1\otimes S\otimes 1)(1\otimes 1\otimes S) \\ &(1\otimes W_G^*\otimes 1)](a\otimes b\otimes 1\otimes 1)\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&(\mathrm{id}_{L^\infty(G)}\otimes \mathrm{id}_{B(L^2(G))}\otimes\sigma)\circ(\mathrm{id}_{L^\infty(G)}\otimes\delta_G\otimes \mathrm{id}_{L^\infty(G)})\circ\widetilde{\alpha_G}(a\otimes b) = \\ &\mathrm{Ad}[(1\otimes 1\otimes S)(1\otimes W_G^*\otimes 1)(1\otimes 1\otimes S)(1\otimes U_G^*\otimes 1)(1\otimes S\otimes 1) \\ &(V_G^*\otimes 1\otimes 1)(1\otimes S\otimes 1)](a\otimes b\otimes 1\otimes 1)\end{aligned}\tag{3.3}$$

Consider the unitaries in the square brackets in (3.2) and (3.3):

$A = (1 \otimes U_G^* \otimes 1)(1 \otimes S \otimes 1)(V_G^* \otimes 1 \otimes 1)(1 \otimes S \otimes 1)(1 \otimes 1 \otimes S)(1 \otimes W_G^* \otimes 1)$

and

$B = (1\otimes 1\otimes S)(1\otimes W_G^*\otimes 1)(1\otimes 1\otimes S)(1\otimes U_G^*\otimes 1)(1\otimes S\otimes 1)(V_G^*\otimes 1\otimes 1)(1\otimes S\otimes 1).$

Then, (3.1) is equivalent to

$A(a\otimes b\otimes 1\otimes 1)A^* = B(a\otimes b\otimes 1\otimes 1)B^*,$ for all $a \in L^\infty(G)$ and $b \in B(L^2(G))$,

which in turn is equivalent to the condition:

$$A^*B \in L^\infty(G)\overline{\otimes}\mathbb{C}1\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)).$$

The last condition is verified by computing

$$A^*B = 1 \otimes 1 \otimes V_G S \in \mathbb{C}1\overline{\otimes}\mathbb{C}1\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)).$$

□

Definition 3.1.3. Let (X, α) be an $L^\infty(G)$-comodule. The *Fubini crossed product* of X by α is defined to be the $L(G)$-comodule $(X \rtimes_\alpha^{\mathcal{F}} G,\ \widehat{\alpha})$, where

$$X \rtimes_\alpha^{\mathcal{F}} G := (X\overline{\otimes}B(L^2(G)))^{\widetilde{\alpha}}$$

and

$$\widehat{\alpha} := (\mathrm{id}_X \otimes \delta_G)|_{X\rtimes_\alpha^{\mathcal{F}}G}.$$

The $L(G)$-action $\widehat{\alpha}\colon X \rtimes_\alpha^{\mathcal{F}} G \to (X \rtimes_\alpha^{\mathcal{F}} G)\overline{\otimes}_{\mathcal{F}}L(G)$ is called the *dual action* of α.

By Proposition 3.1.2 and Lemma 2.1.7 we get that $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ is indeed an $L(G)$-subcomodule of $(X\overline{\otimes}B(L^2(G)), \mathrm{id}_X \otimes \delta_G)$.

Definition 3.1.4. Let (X, α) be an $L^\infty(G)$-comodule and suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H. The *spatial crossed product* of X by α is defined to be the space

$$\begin{aligned} X\overline{\rtimes}_\alpha G &:= \overline{\operatorname{span}}^{w^*}\{(1_H \otimes \lambda_s)\alpha(x)(1_H \otimes \lambda_t) :\ s,t \in G,\ x \in X\} \\ &\subseteq B(H)\overline{\otimes}B(L^2(G)). \end{aligned}$$

Note that Definition 3.1.4 is naturally dictated by the fact that if M is a von Neumann algebra, γ is a G-action on M and α is the W*-$L^\infty(G)$-action on M corresponding to γ as above, then the crossed product $M \rtimes_\alpha G$ is equal to the w*-closed $\mathbb{C}1\overline{\otimes}L(G)$-bimodule generated by $\alpha(M)$. This follows immediately from the well known covariance relations (recall (2.5)):

$$\alpha(\gamma_s(x)) = (1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_s^{-1}), \quad s \in G,\ x \in M.$$

Remark 3.1.5. From the discussion above, it follows that if (M,α) is a W*-$L^\infty(G)$-comodule, then $M \ltimes_\alpha^{\mathcal{F}} G = M \overline{\ltimes}_\alpha G = M \ltimes_\alpha G$, where $M \ltimes_\alpha G = (\alpha(M) \cup (\mathbb{C}1 \overline{\otimes} L(G)))''$ is the usual von Neumann algebra crossed product. Interestingly, we will prove later that this is not true in general for arbitrary $L^\infty(G)$-comodules unless G has the approximation property of Haagerup and Kraus (see Theorem 3.3.10).

Note that if (X,α) is an $L^\infty(G)$-comodule with α trivial, that is $\alpha(x) = x \otimes 1$ for all $x \in X$, then for $x \in X$ and $b \in B(L^2(G))$ we have

$$\begin{aligned}\widetilde{\alpha}(x \otimes b) &= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})(x \otimes b) \\ &= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma)(x \otimes 1 \otimes b) \\ &= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*)(x \otimes b \otimes 1) \\ &= (\mathrm{id}_X \otimes \beta_G)(x \otimes b)\end{aligned}$$

and thus $\widetilde{\alpha} = \mathrm{id}_X \otimes \beta_G$. Since $B(L^2(G))^{\beta_G} = L(G)$ it follows that

$$\begin{aligned}X \ltimes_\alpha^{\mathcal{F}} G &= (X \overline{\otimes} B(L^2(G)))^{\widetilde{\alpha}} \\ &= (X \overline{\otimes} B(L^2(G)))^{\mathrm{id}_X \otimes \beta_G} \\ &= X \overline{\otimes}_{\mathcal{F}} (B(L^2(G)))^{\beta_G} \\ &= X \overline{\otimes}_{\mathcal{F}} L(G).\end{aligned}$$

This actually explains the term 'Fubini crossed product' which was first used in [56]. We should note here that Hamana had already considered the notion of Fubini crossed products in [19] but he did not use the same term.

On the other hand, it is obvious that $X \overline{\ltimes}_\alpha G = X \overline{\otimes} L(G)$ when α is trivial and thus the term 'spatial crossed product' is similarly justified.

Also, for a locally compact group (even in the discrete case) it is not necessarily true that $X \overline{\otimes} L(G) = X \overline{\otimes}_{\mathcal{F}} L(G)$ for any dual operator space X. Indeed, if we take G to be any discrete group failing the approximation property (for example $G = SL(3,\mathbb{Z})$, see [33]), then, by [22, Theorem 2.1], it follows that there is a dual operator space X such that $X \overline{\otimes} L(G) \neq X \overline{\otimes}_{\mathcal{F}} L(G)$. Therefore, in this case, the equality $X \ltimes_\alpha^{\mathcal{F}} G = X \overline{\ltimes}_\alpha G$ is not valid for all $L^\infty(G)$-comodules (X,α) in contrast to the von Neumann algebra case. Thus the distinction between Fubini and spatial crossed products seems to be necessary in the setting of general dual operator spaces.

It was shown by Crann and Neufang [12] that if G is a locally compact group with the AP, then $X \ltimes_\alpha^{\mathcal{F}} G = X \overline{\ltimes}_\alpha G$ for any $L^\infty(G)$-comodule (X,α) [12, Corollary 4.8]. We warn the reader that Crann and Neufang consider G-invariant subspaces of von Neumann algebras instead of general $L^\infty(G)$-comodules, but this is not restrictive at all. Indeed, every $L^\infty(G)$-comodule is isomorphic to a subcomodule of a W*-$L^\infty(G)$-comodule (see Remark 2.1.3 and Proposition 3.1.8), that is, a G-invariant subspace of a von Neumann algebra, since every W*-$L^\infty(G)$-action comes from a pointwise G-action as pointed out above.

Also, in [12] Crann and Neufang define the Fubini crossed product of a G-invariant subspace X of a von Neumann algebra M using an appropriate operator valued weight (see [12, Definition 3.1]). However, their definition is equivalent to Definition 3.1.3. Indeed, from [12, Proposition 3.2] it follows that the Fubini crossed product of X in the sense of Crann-Neufang is the intersection

$$(M \rtimes_\alpha G) \cap (X \overline{\otimes} B(L^2(G))).$$

Since $M \rtimes_\alpha G = (M \overline{\otimes} B(L^2(G)))^{\widetilde{\alpha}}$ (by the Digernes-Takesaki theorem), it follows that the above intersection is equal to $(X \overline{\otimes} B(L^2(G)))^{\widetilde{\alpha}}$ that is the Fubini crossed product $X \rtimes_\alpha^{\mathcal{F}} G$ according to Definition 3.1.3.

Later, using a generalized version of Takesaki-duality and its relation with the AP, we will give an alternative proof of the aforementioned result of Crann and Neufang (i.e. [12, Corollary 4.8]), avoiding the use of operator valued weights. Moreover, we are going to prove that its converse is also true (see Theorem 3.3.10).

Remark 3.1.6. Let H, K be Hilbert spaces, $X \subseteq B(H)$ be a w*-closed subspace and $b,\ c \in B(K)$. Then, we have

$$(1_H \otimes b)(X \overline{\otimes} B(K))(1_H \otimes c) \subseteq X \overline{\otimes} B(K).$$

As a consequence, if (X, α) is an $L^\infty(G)$-comodule, then

$$X \overline{\rtimes}_\alpha G \subseteq X \overline{\otimes} B(L^2(G)),$$

because $\alpha(X) \subseteq X \overline{\otimes} L^\infty(G) \subseteq X \overline{\otimes} B(L^2(G))$.

Also, if in addition Y is a w*-closed subspace of $B(L)$ for some Hilbert space L and $\phi\colon X \to Y$ is a w*-continuous completely bounded map, then $\phi \otimes \mathrm{id}_{B(K)}\colon\ X \overline{\otimes} B(K) \to Y \overline{\otimes} B(K)$ is a w*-continuous $B(K)$-bimodule map in the sense that

$$(\phi \otimes \mathrm{id}_{B(K)})((1_H \otimes a)x(1_H \otimes b)) = (1_L \otimes a)(\phi \otimes \mathrm{id}_{B(K)})(x)(1_L \otimes b),$$

for all $a,\ b \in B(K)$ and $x \in X \overline{\otimes} B(K)$.

Proposition 3.1.7. *Let (X, α) be an $L^\infty(G)$-comodule and suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H. Then, $X \rtimes_\alpha^{\mathcal{F}} G$ is an $L(G)$-bimodule, i.e.*

$$(1_H \otimes \lambda_s) y (1_H \otimes \lambda_t) \in X \rtimes_\alpha^{\mathcal{F}} G, \qquad s, t \in G,\ y \in X \rtimes_\alpha^{\mathcal{F}} G$$

and

$$\alpha(X) \subseteq X \rtimes_\alpha^{\mathcal{F}} G.$$

Therefore, we have:

$$X\overline{\rtimes}_\alpha G \subseteq X \rtimes_\alpha^{\mathcal{F}} G.$$

Furthermore, $\widehat{\alpha}(X\overline{\rtimes}_\alpha G) \subseteq (X\overline{\rtimes}_\alpha G)\overline{\otimes}_{\mathcal{F}} L(G)$, *that is* $X\overline{\rtimes}_\alpha G$ *is an* $L(G)$-*subcomodule of* $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$.

Proof. Let $s \in G$ and $y \in X \rtimes_\alpha^{\mathcal{F}} G$. Then, by Remark 3.1.6 we have that $(1_H \otimes \lambda_s)y \in X\overline{\otimes}B(L^2(G))$ and $\widetilde{\alpha}(y) = y \otimes 1$, by Definition 3.1.3. Also, by Remark 3.1.6, we have that

$$(\alpha \otimes \mathrm{id}_{B(L^2(G))})((1_H \otimes \lambda_s)y) = (1_H \otimes 1_{L^2(G)} \otimes \lambda_s)(\alpha \otimes \mathrm{id}_{B(L^2(G))})(y).$$

Thus, we have:

$$\begin{aligned}
&\widetilde{\alpha}((1_H \otimes \lambda_s)y) = \\
&= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})((1_H \otimes \lambda_s)y) \\
&= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \big((1_H \otimes 1_{L^2(G)} \otimes \lambda_s)(\alpha \otimes \mathrm{id}_{B(L^2(G))})(y)\big) \\
&= \big[(\mathrm{id}_{B(H)} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_{B(H)} \otimes \sigma)((1_H \otimes 1_{L^2(G)} \otimes \lambda_s))\big]\, \widetilde{\alpha}(y) \\
&= \big[(1_H \otimes U_G^*)(1_H \otimes \lambda_s \otimes 1_{L^2(G)})(1_H \otimes U_G)\big]\,(y \otimes 1_{L^2(G)}) \\
&= (1_H \otimes \lambda_s)y \otimes 1_{L^2(G)},
\end{aligned}$$

where the third equality above follows from the fact that $(\mathrm{id}_{B(H)} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_{B(H)} \otimes \sigma)$ is a *-homomorphism and thus multiplicative, while the last equality is because $U_G \in R(G)\overline{\otimes}L^\infty(G)$ and $R(G) = L(G)'$. Therefore, $(1_H \otimes \lambda_s)y \in X \rtimes_\alpha^{\mathcal{F}} G$. Similarly, we get $y(1_H \otimes \lambda_t) \in X \rtimes_\alpha^{\mathcal{F}} G$ for all $t \in G$ and $y \in X \rtimes_\alpha^{\mathcal{F}} G$.
On the other hand, if $x \in X$, then:

$$\begin{aligned}
\widetilde{\alpha}(\alpha(x)) &= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})(\alpha(x)) \\
&= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\mathrm{id}_X \otimes \alpha_G)(\alpha(x)) \\
&= (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \alpha_G')(\alpha(x)) \\
&= (1_H \otimes U_G^*)(1_H \otimes U_G)(\alpha(x) \otimes 1_{L^2(G)})(1_H \otimes U_G^*)(1_H \otimes U_G) \\
&= \alpha(x) \otimes 1_{L^2(G)},
\end{aligned}$$

because $\alpha_G' = \sigma \circ \alpha_G$ and $\alpha_G'(f) = U_G(f \otimes 1)U_G^*$, for all $f \in L^\infty(G)$ (see subsection 2.3.1). Hence, $\alpha(X) \subseteq X \rtimes_\alpha^{\mathcal{F}} G$.
Finally, for $x \in X$ and $s \in G$, we have:

$$\begin{aligned}
\widehat{\alpha}((1_H \otimes \lambda_s)\alpha(x)) &= (\mathrm{id}_{B(H)} \otimes \delta_G)((1_H \otimes \lambda_s)\alpha(x)) \\
&= (\mathrm{id}_{B(H)} \otimes \delta_G)(1_H \otimes \lambda_s))(\mathrm{id}_{B(H)} \otimes \delta_G)(\alpha(x)) \\
&= (1_H \otimes \delta_G(\lambda_s))(1_H \otimes W_G^*)(\alpha(x) \otimes 1_{L^2(G)})(1_H \otimes W_G) \\
&= (1_H \otimes \lambda_s \otimes \lambda_s)(\alpha(x) \otimes 1_{L^2(G)}),
\end{aligned}$$

because $1_H \otimes W_G \in \mathbb{C}1_H \overline{\otimes} L^\infty(G) \overline{\otimes} L(G)$ commutes with $\alpha(x) \otimes 1_{L^2(G)} \in B(H) \overline{\otimes} L^\infty(G) \overline{\otimes} \mathbb{C}1_{L^2(G)}$. Therefore, we get:

$$\widehat{\alpha}((1_H \otimes \lambda_s)\alpha(x)) = ((1_H \otimes \lambda_s)\alpha(x)) \otimes \lambda_s$$

and it follows that $\widehat{\alpha}(X \overline{\rtimes}_\alpha G) \subseteq (X \overline{\rtimes}_\alpha G) \overline{\otimes}_{\mathcal{F}} L(G)$. □

The next result proves that, for any $L^\infty(G)$-comodule X, both the Fubini crossed product and the spatial crossed product are unique up to comodule isomorphisms and thus independent of the Hilbert space on which X is represented.

Proposition 3.1.8 (Uniqueness of crossed product)**.** *Let (X, α) and (Y, β) be two $L^\infty(G)$-comodules and suppose that X and Y are w*-closed subspaces of $B(H)$ and $B(K)$ respectively. Let $\Phi\colon X \to Y$ be an $L^\infty(G)$-comodule isomorphism.*

Then the map $\Psi := \Phi \otimes \mathrm{id}_{B(L^2(G))}\colon X \overline{\otimes} B(L^2(G)) \to Y \overline{\otimes} B(L^2(G))$ is an $L^\infty(G)$-comodule isomorphism from $(X \overline{\otimes} B(L^2(G)), \widetilde{\alpha})$ onto $(Y \overline{\otimes} B(L^2(G)), \widetilde{\beta})$, which maps $X \rtimes_\alpha^{\mathcal{F}} G$ onto $Y \rtimes_\beta^{\mathcal{F}} G$ and $X \overline{\rtimes}_\alpha G$ onto $Y \overline{\rtimes}_\beta G$. Also, $\Psi|_{X \rtimes_\alpha^{\mathcal{F}} G}$ is an $L(G)$-comodule isomorphism from $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ onto $(Y \rtimes_\beta^{\mathcal{F}} G, \widehat{\beta})$ and $\Psi|_{X \overline{\rtimes}_\alpha G}$ is an $L(G)$-comodule isomorphism from $(X \overline{\rtimes}_\alpha G, \widehat{\alpha})$ onto $(Y \overline{\rtimes}_\beta G, \widehat{\beta})$.

Furthermore, Ψ is an $L(G)$-bimodule map, i.e. $\Psi((1_H \otimes \lambda_s)x(1_H \otimes \lambda_t)) = (1_K \otimes \lambda_s)\Psi(x)(1_K \otimes \lambda_t)$, for all $s, t \in G$ and $x \in X \overline{\otimes} B(L^2(G))$.

Proof. First, since Φ is a comodule morphism we have that $\beta \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \alpha$ and hence:

$$\begin{aligned}
\widetilde{\beta} \circ \Psi &= (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma) \circ (\beta \otimes \mathrm{id}) \circ (\Phi \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma) \circ ((\beta \circ \Phi) \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma) \circ [((\Phi \otimes \mathrm{id}) \circ \alpha) \otimes \mathrm{id}] \\
&= (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma) \circ (\Phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\alpha \otimes \mathrm{id}) \\
&= (\Phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma) \circ (\alpha \otimes \mathrm{id}) \\
&= (\Psi \otimes \mathrm{id}) \circ \widetilde{\alpha}.
\end{aligned}$$

Thus Ψ is an $L^\infty(G)$-comodule isomorphism from $(X \overline{\otimes} B(L^2(G)), \widetilde{\alpha})$ onto $(Y \overline{\otimes} B(L^2(G)), \widetilde{\beta})$. This implies that Ψ maps the fixed point subspace $X \rtimes_\alpha^{\mathcal{F}} G$ of $\widetilde{\alpha}$ onto the fixed point subspace $Y \rtimes_\beta^{\mathcal{F}} G$ of $\widetilde{\beta}$. On the other hand, the relation $\beta \circ \Phi = (\Phi \otimes id) \circ \alpha$ yields that

$$\Psi(\alpha(X)) = (\Phi \otimes \mathrm{id})(\alpha(X)) = \beta(\Phi(X)) = \beta(Y)$$

and since Ψ is an $L(G)$-bimodule map (see Remark 3.1.6) it follows that Ψ maps $X \overline{\rtimes}_\alpha G$ onto $Y \overline{\rtimes}_\beta G$. It remains to show that

$$\widehat{\beta} \circ \Psi = (\Psi \otimes \mathrm{id}) \circ \widehat{\alpha}.$$

Indeed:

$$\begin{aligned}\widehat{\beta}\circ\Psi &= (\mathrm{id}_Y\otimes\delta_G)\circ(\Phi\otimes\mathrm{id}_{B(L^2(G))})\\ &= (\Phi\otimes\mathrm{id}_{B(L^2(G))}\otimes\mathrm{id}_{L(G)})\circ(\mathrm{id}_X\otimes\delta_G)\\ &= (\Psi\otimes\mathrm{id}_{L(G)})\circ\widehat{\alpha}.\end{aligned}$$

□

Corollary 3.1.9. *For any $L^\infty(G)$-comodule (X,α) the Fubini crossed product $(X\rtimes^{\mathcal{F}}_\alpha G,\widehat{\alpha})$ is a saturated $L(G)$-comodule and the spatial crossed product $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ is a non-degenerate $L(G)$-comodule.*

Proof. Suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H and let $K := H\otimes L^2(G)$.

First we show that $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ is a non-degenerate $L(G)$-comodule. We have that $(X\overline{\rtimes}_\alpha G)\overline{\otimes}B(L^2(G))$ is a $\mathbb{C}1_K\overline{\otimes}B(L^2(G))$-bimodule. Thus, since $\widehat{\alpha}(X\overline{\rtimes}_\alpha G)\subseteq(X\overline{\rtimes}_\alpha G)\overline{\otimes}_{\mathcal{F}}L(G)\subseteq(X\overline{\rtimes}_\alpha G)\overline{\otimes}B(L^2(G))$, we have the inclusion

$$(X\overline{\rtimes}_\alpha G)\overline{\otimes}B(L^2(G))\supseteq\overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_K\otimes b)\widehat{\alpha}(y):\ b\in B(L^2(G)),\ y\in X\overline{\rtimes}_\alpha G\}.$$

For the reverse inclusion, observe that for any $s,t\in G$, $x\in X$ and $b\in B(L^2(G))$, we have

$$\begin{aligned}&((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t))\otimes b=\\ &=(1_H\otimes 1_{L^2(G)}\otimes b\lambda_t^{-1}\lambda_s^{-1})(((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t))\otimes\lambda_{st})\\ &=(1_H\otimes 1_{L^2(G)}\otimes b\lambda_{st}^{-1})(1_H\otimes\lambda_s\otimes\lambda_s)(\alpha(x)\otimes 1_{L^2(G)})(1_H\otimes\lambda_t\otimes\lambda_t)\\ &=(1_H\otimes 1_{L^2(G)}\otimes b\lambda_{st}^{-1})(\mathrm{id}_{B(H)}\otimes\delta_G)((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t))\\ &=(1_H\otimes 1_{L^2(G)}\otimes b\lambda_{st}^{-1})\widehat{\alpha}((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t)),\end{aligned}$$

since

$$(\mathrm{id}_{B(H)}\otimes\delta_G)(1_H\otimes\lambda_s)=1_H\otimes\lambda_s\otimes\lambda_s \text{ and } (\mathrm{id}_{B(H)}\otimes\delta_G)(\alpha(x))=\alpha(x)\otimes 1.$$

Therefore, we get

$$\begin{aligned}&((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t))\otimes b\in\\ &\in\overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_K\otimes c)\widehat{\alpha}(y):\ c\in B(L^2(G)),\ y\in X\overline{\rtimes}_\alpha G\}.\end{aligned}$$

Since $(X\overline{\rtimes}_\alpha G)\overline{\otimes}B(L^2(G))$ is in the w*-closed linear span of the elements of the form $((1_H\otimes\lambda_s)\alpha(x)(1_H\otimes\lambda_t))\otimes b$, we obtain the desired inclusion and so $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ is non-degenerate.

For the Fubini crossed product, note that by Lemma 2.2.10 the $L(G)$-comodule $(X\overline{\otimes}B(L^2(G)),\mathrm{id}_X\otimes\delta_G)$ is saturated, because $(B(L^2(G)),\delta_G)$

is saturated (see Remark 3.3.4). Also, the actions $\widetilde{\alpha}$ and $\mathrm{id}_X \otimes \delta_G$ on $X\overline{\otimes}B(L^2(G))$ commute by Proposition 3.1.2. Thus, it follows from Lemma 2.2.11 that $(X \rtimes^{\mathcal{F}}_{\alpha} G, \widehat{\alpha})$ is saturated, because $X \rtimes^{\mathcal{F}}_{\alpha} G = \big(X\overline{\otimes}B(L^2(G))\big)^{\widetilde{\alpha}}$ and $\widehat{\alpha} = (\mathrm{id}_X \otimes \delta_G)|_{X \rtimes^{\mathcal{F}}_{\alpha} G}$ by definition. $\square$

Proposition 3.1.10. *For any $L^\infty(G)$-comodule (X, α), we have*

$$(X \rtimes^{\mathcal{F}}_{\alpha} G)^{\widehat{\alpha}} = (X\overline{\rtimes}_{\alpha}G)^{\widehat{\alpha}} = \alpha(X) = \mathrm{Sat}(X, \alpha) = (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}}.$$

Proof. We prove first that $\mathrm{Sat}(X, \alpha) = (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}}$. Indeed, for any $x \in X\overline{\otimes}L^\infty(G)$, we have:

$$\begin{aligned} x \in (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}} &\iff \widetilde{\alpha}(x) = x \otimes 1 \\ &\iff (\mathrm{id}_X \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_X \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})(x) = x \otimes 1 \\ &\iff (\alpha \otimes \mathrm{id}_{L^\infty(G)})(x) = (\mathrm{id}_X \otimes \sigma)\left((1_H \otimes U_G)(x \otimes 1)(1_H \otimes U_G^*)\right) \\ &\iff (\alpha \otimes \mathrm{id}_{L^\infty(G)})(x) = (\mathrm{id}_X \otimes \alpha_G)(x) \\ &\iff x \in \mathrm{Sat}(X, \alpha). \end{aligned}$$

For the fourth equivalence above we used the fact that

$$\sigma \circ \alpha_G(f) = U_G(f \otimes 1)U_G^*$$

for any $f \in L^\infty(G)$.

Now, we prove that $\big(X \rtimes^{\mathcal{F}}_{\alpha} G\big)^{\widehat{\alpha}} = (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}}$. Indeed, since the actions $\mathrm{id}_X \otimes \delta_G$ and $\widetilde{\alpha}$ commute (see Proposition 3.1.2) and $\widehat{\alpha} = (\mathrm{id}_X \otimes \delta_G)|_{X \rtimes^{\mathcal{F}}_{\alpha} G}$ it follows:

$$\begin{aligned} \big(X \rtimes^{\mathcal{F}}_{\alpha} G\big)^{\widehat{\alpha}} &= \left(\big(X\overline{\otimes}B(L^2(G))\big)^{\widetilde{\alpha}}\right)^{\mathrm{id}_X \otimes \delta_G} \\ &= \left(\big(X\overline{\otimes}B(L^2(G))\big)^{\mathrm{id}_X \otimes \delta_G}\right)^{\widetilde{\alpha}} \\ &= \left(X\overline{\otimes}_{\mathcal{F}} \big(B(L^2(G))\big)^{\delta_G}\right)^{\widetilde{\alpha}} \\ &= (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}}. \end{aligned}$$

The last equality follows from the fact that $B(L^2(G))^{\delta_G} = L^\infty(G)$.

By Lemma 2.3.5 we have that $\mathrm{Sat}(X, \alpha) = \alpha(X)$ and thus we get

$$(X \rtimes^{\mathcal{F}}_{\alpha} G)^{\widehat{\alpha}} = \alpha(X) = \mathrm{Sat}(X, \alpha) = (X\overline{\otimes}L^\infty(G))^{\widetilde{\alpha}}.$$

So it remains to show that $(X\overline{\rtimes}_{\alpha}G)^{\widehat{\alpha}} = \alpha(X)$. Indeed, since $(X\overline{\rtimes}_{\alpha}G, \widehat{\alpha})$ is an $L(G)$-subcomodule of $(X \rtimes^{\mathcal{F}}_{\alpha} G, \widehat{\alpha})$ it follows that

$$\begin{aligned} (X\overline{\rtimes}_{\alpha}G)^{\widehat{\alpha}} &= (X \rtimes^{\mathcal{F}}_{\alpha} G)^{\widehat{\alpha}} \cap (X\overline{\rtimes}_{\alpha}G) \\ &= \alpha(X) \cap (X\overline{\rtimes}_{\alpha}G) \\ &= \alpha(X), \end{aligned}$$

since $\alpha(X) \subseteq X\overline{\rtimes}_{\alpha}G$. $\square$

3.2 Crossed products of $L(G)$-comodules

Here we consider the analogues of the Fubini and the spatial crossed products in the category of $L(G)$-comodules.

The main and most interesting difference between $L^\infty(G)$-comodules and $L(G)$-comodules is that for any $L(G)$-comodule (Y, δ) the associated Fubini and spatial crossed products are equal without any further assumption on the group G or the space Y (Theorem 3.2.10). The reason behind this is that the Fubini and the spatial crossed product of an $L(G)$-comodule admit a natural $L^\infty(G)$-comodule structure and thus they are always non-degenerate and saturated by Lemma 2.3.5. This will be clear from the use of Lemma 2.3.5 in the proof of Proposition 3.2.9 below, from which Theorem 3.2.10 follows.

The Definitions 3.2.1 and 3.1.3 below follow [19] with slight changes in terminology and symbols for the sake of consistency.

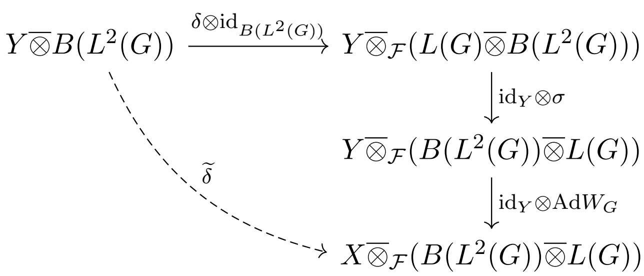
Definition 3.2.1. For an $L(G)$-comodule (Y, δ), we define the map

$$\widetilde{\delta}\colon Y\overline{\otimes}B(L^2(G)) \to Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))\overline{\otimes}_{\mathcal{F}}L(G)$$

by

$$\widetilde{\delta} = (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))}),$$

where σ is the flip isomorphism on $B(L^2(G))\overline{\otimes}B(L^2(G))$.



The next proposition, essentially the same as [19, Lemma 5.3 (ii)], is the analogue of Proposition 3.1.2 for $L(G)$-comodules. Note that the proof is the same as that of Proposition 3.1.2 with the appropriate modifications.

Proposition 3.2.2 (Hamana [19])**.** *For an $L(G)$-comodule (Y, δ), the map $\widetilde{\delta}$ is an $L(G)$-action on $Y\overline{\otimes}B(L^2(G))$ commuting with the $L^\infty(G)$-action $\mathrm{id}_Y \otimes \beta_G$ on $Y\overline{\otimes}B(L^2(G))$.*

Proof. By Remark 2.1.3 we may suppose that Y is a w*-closed subspace of a von Neumann algebra N of the form $N = B(H)\overline{\otimes}L(G)$ for some Hilbert space H and $\delta = \varepsilon|_Y$, where $\varepsilon = \mathrm{id}_{B(H)} \otimes \delta_G$. Then obviously $\widetilde{\delta} = \widetilde{\varepsilon}|_{Y\overline{\otimes}B(L^2(G))}$ and $\widetilde{\varepsilon}$ is a W*-$L(G)$-action on $N\overline{\otimes}B(L^2(G))$. Since $\widetilde{\varepsilon}$ is a w*-continuous *-monomorphism, the latter fact can be easily verified by checking the relation

$$(\widetilde{\varepsilon}\otimes \mathrm{id}_{L(G)})\circ\widetilde{\varepsilon}=(\mathrm{id}_N\otimes \mathrm{id}_{B(L^2(G))}\otimes\delta_G)\circ\widetilde{\varepsilon}$$

on the generators of $N\overline{\otimes}B(L^2(G))$, that is on the elements of the form $z\otimes 1$, $1\otimes 1\otimes f$ and $1\otimes 1\otimes\rho_s$ for $z\in N$, $f\in L^\infty(G)$ and $s\in G$, because $B(L^2(G))$ is generated by $R(G)$ and $L^\infty(G)$.

Thus, in order to prove that $\widetilde{\delta}$ is an $L(G)$-action on $Y\overline{\otimes}B(L^2(G))=Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))$, we only need to show that

$$\widetilde{\varepsilon}(Y\overline{\otimes}_{\mathcal{F}}B(L^2(G)))\subseteq Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))\overline{\otimes}_{\mathcal{F}}L(G).$$

Indeed, we have

$$(\varepsilon\otimes\mathrm{id}_{B(L^2(G))})(Y\overline{\otimes}_{\mathcal{F}}B(L^2(G)))\subseteq Y\overline{\otimes}_{\mathcal{F}}L(G)\overline{\otimes}_{\mathcal{F}}B(L^2(G))$$

and thus

$$(\mathrm{id}_Y\otimes\sigma)\circ(\varepsilon\otimes\mathrm{id}_{B(L^2(G))})(Y\overline{\otimes}_{\mathcal{F}}B(L^2(G)))\subseteq Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))\overline{\otimes}_{\mathcal{F}}L(G).$$

Since $W_G\in L^\infty(G)\overline{\otimes}L(G)$ and $Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))\overline{\otimes}_{\mathcal{F}}L(G)$ is a $\mathbb{C}1_H\overline{\otimes}B(L^2(G))\overline{\otimes}L(G)$-bimodule, it follows that

$$(\mathrm{id}_Y\otimes\mathrm{Ad}W_G)\circ(\mathrm{id}_Y\otimes\sigma)\circ(\varepsilon\otimes\mathrm{id}_{B(L^2(G))})(Y\overline{\otimes}_{\mathcal{F}}B(L^2(G)))$$
$$\subseteq Y\overline{\otimes}_{\mathcal{F}}B(L^2(G))\overline{\otimes}_{\mathcal{F}}L(G).$$

On the other hand, in order to prove that $\widetilde{\delta}$ and $\mathrm{id}_Y\otimes\beta_G$ commute, it suffices to verify that $\mathrm{id}_{B(H)}\otimes\mathrm{id}_{L(G)}\otimes\beta_G$ and $\widetilde{\varepsilon}$ commute, where $\varepsilon=\mathrm{id}_{B(H)}\otimes\delta_G$. Because $\widetilde{\varepsilon}$ and $\mathrm{id}_{B(H)}\otimes\mathrm{id}_{L(G)}\otimes\beta_G$ act identically on the first factor $B(H)$, we only need to prove that $\mathrm{id}_{L(G)}\otimes\beta_G$ and $\widetilde{\delta_G}$ commute, that is:

$$\begin{aligned}(\widetilde{\delta_G}\otimes\mathrm{id}_{L^\infty(G)})\circ(\mathrm{id}_{L(G)}\otimes\beta_G)=\\ =(\mathrm{id}_{L(G)}\otimes\mathrm{id}_{B(L^2(G))}\otimes\sigma)\circ(\mathrm{id}_{L(G)}\otimes\beta_G\otimes\mathrm{id}_{L(G)})\circ\widetilde{\delta_G}\end{aligned}\tag{3.4}$$

Let S denote the unitary on $L^2(G)\otimes L^2(G)$ with $S(\xi\otimes\eta)=\eta\otimes\xi$. Thus, the flip isomorphism σ on $B(L^2(G))\overline{\otimes}B(L^2(G))$ is written as $\sigma=\mathrm{Ad}S$. If $a\in L(G)$ and $b\in B(L^2(G))$, then by applying the left and right hand sides of (3.4) on $a\otimes b$, we get respectively:

$$\begin{aligned}&(\widetilde{\delta_G}\otimes\mathrm{id}_{L^\infty(G)})\circ(\mathrm{id}_{L(G)}\otimes\beta_G)(a\otimes b)=\\ &\mathrm{Ad}[(1\otimes W_G\otimes 1)(1\otimes S\otimes 1)(W_G^*\otimes 1\otimes 1)(1\otimes S\otimes 1)(1\otimes 1\otimes S)\\ &(1\otimes U_G^*\otimes 1)](a\otimes b\otimes 1\otimes 1)\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}&(\mathrm{id}_{L(G)} \otimes \mathrm{id}_{B(L^2(G))} \otimes \sigma) \circ (\mathrm{id}_{L(G)} \otimes \beta_G \otimes \mathrm{id}_{L(G)}) \circ \widetilde{\delta_G}(a \otimes b) = \\ &\mathrm{Ad}[(1 \otimes 1 \otimes S)(1 \otimes U_G^* \otimes 1)(1 \otimes 1 \otimes S)(1 \otimes W_G \otimes 1)(1 \otimes S \otimes 1) \\ &(W_G^* \otimes 1 \otimes 1)(1 \otimes S \otimes 1)](a \otimes b \otimes 1 \otimes 1)\end{aligned} \tag{3.6}$$

Consider the unitaries in the square brackets in (3.5) and (3.6):

$$A = (1 \otimes W_G \otimes 1)(1 \otimes S \otimes 1)(W_G^* \otimes 1 \otimes 1)(1 \otimes S \otimes 1)(1 \otimes 1 \otimes S)(1 \otimes U_G^* \otimes 1)$$

and

$$B = (1\otimes 1\otimes S)(1\otimes U_G^*\otimes 1)(1\otimes 1\otimes S)(1\otimes W_G\otimes 1)(1\otimes S\otimes 1)(W_G^*\otimes 1\otimes 1)(1\otimes S\otimes 1)$$

Then, (3.4) is equivalent to

$$A(a\otimes b\otimes 1\otimes 1)A^* = B(a\otimes b\otimes 1\otimes 1)B^*, \text{ for all } a \in L(G) \text{ and } b \in B(L^2(G))$$

and this is equivalent to the following condition:

$$A^*B \in R(G)\overline{\otimes}\mathbb{C}1\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)),$$

which is true since a computation shows that A^*B acts identically on the first two variables, i.e.

$$A^*B \in \mathbb{C}1\overline{\otimes}\mathbb{C}1\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)).$$

□

Definition 3.2.3. Let (Y,δ) be an $L(G)$-comodule. The *Fubini crossed product* of Y by δ is defined to be the $L^\infty(G)$-comodule $(Y \ltimes_\delta^{\mathcal{F}} G,\ \widehat{\delta})$, where

$$Y \ltimes_\delta^{\mathcal{F}} G := (Y\overline{\otimes}B(L^2(G)))^{\tilde{\delta}}$$

and

$$\widehat{\delta} := (\mathrm{id}_X \otimes \beta_G)|_{Y \ltimes_\delta^{\mathcal{F}} G}.$$

The $L^\infty(G)$-action $\widehat{\delta}\colon Y \ltimes_\delta^{\mathcal{F}} G \to (Y \ltimes_\delta^{\mathcal{F}} G)\overline{\otimes}L^\infty(G)$ is called the *dual action* of δ.

By Proposition 3.2.2 and Lemma 2.1.7, $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$ is indeed an $L^\infty(G)$-subcomodule of $(Y\overline{\otimes}B(L^2(G)), \mathrm{id}_Y \otimes \beta_G)$.

Definition 3.2.4. Let (Y,δ) be an $L(G)$-comodule and suppose that Y is w*-closed in $B(K)$ for some Hilbert space K. The *spatial crossed product* of Y by δ is defined to be the space

$$\begin{aligned}Y\overline{\ltimes}_\delta G &:= \overline{\mathrm{span}}^{\mathrm{w}^*}\{(1_K \otimes f)\delta(y)(1_K \otimes g) :\ f,g \in L^\infty(G),\ y \in Y\} \\ &\subseteq B(K)\overline{\otimes}B(L^2(G)).\end{aligned}$$

Proposition 3.2.5. *Let (Y, δ) be an $L(G)$-comodule and suppose that Y is a w^*-closed subspace of $B(K)$ for some Hilbert space K. Then, $Y \ltimes_\delta^{\mathcal{F}} G$ is an $L^\infty(G)$-bimodule, i.e.*

$$(1_K \otimes f)y(1_K \otimes g) \in Y \ltimes_\delta^{\mathcal{F}} G, \qquad f, g \in L^\infty(G),\ y \in Y \ltimes_\delta^{\mathcal{F}} G$$

and $\delta(Y) \subseteq Y \ltimes_\delta^{\mathcal{F}} G$. Therefore, we have:

$$Y \overline{\ltimes}_\delta G \subseteq Y \ltimes_\delta^{\mathcal{F}} G.$$

In addition, $\widehat{\delta}(Y \overline{\ltimes}_\delta G) \subseteq (Y \overline{\ltimes}_\delta G) \overline{\otimes} L^\infty(G)$, that is $Y \overline{\ltimes}_\delta G$ is an $L^\infty(G)$-subcomodule of $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$.

Proof. Let $f \in L^\infty(G)$ and $y \in Y \ltimes_\delta^{\mathcal{F}} G$. Then, by Remark 3.1.6 we have that $(1_K \otimes f)y \in Y \overline{\otimes} B(L^2(G))$ and $\widetilde{\delta}(y) = y \otimes 1$, by Definition 3.2.3. Also, by Remark 3.1.6, we have that

$$(\delta \otimes \mathrm{id}_{B(L^2(G))})((1_K \otimes f)y) = (1_K \otimes 1_{L^2(G)} \otimes f)(\delta \otimes \mathrm{id}_{B(L^2(G))})(y).$$

Thus, it follows:

$$\begin{aligned}
\widetilde{\delta}((1_H \otimes f)y) &= (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})((1_K \otimes f)y) \\
&= (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \sigma) \big((1_K \otimes 1_{L^2(G)} \otimes f)(\delta \otimes \mathrm{id}_{B(L^2(G))})(y)\big) \\
&= \big[(\mathrm{id}_{B(K)} \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_{B(K)} \otimes \sigma)((1_K \otimes 1_{L^2(G)} \otimes f))\big]\, \widetilde{\delta}(y) \\
&= \big[(1_K \otimes W_G)(1_K \otimes f \otimes 1_{L^2(G)})(1_K \otimes W_G^*)\big]\, (y \otimes 1_{L^2(G)}) \\
&= (1_K \otimes f)y \otimes 1_{L^2(G)},
\end{aligned}$$

where the third equality above holds since $(\mathrm{id}_{B(K)} \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_{B(K)} \otimes \sigma)$ is a $*$-homomorphism and thus multiplicative, while the last equality is true because $W_G \in L^\infty(G) \overline{\otimes} L(G)$ and thus $W_G(f \otimes 1)W_G^* = f \otimes 1$. The fourth equality follows from the assumption $\widetilde{\delta}(y) = y \otimes 1$. Therefore,

$$(1_K \otimes f)y \in Y \ltimes_\delta^{\mathcal{F}} G.$$

Similarly, we get $y(1_K \otimes g) \in Y \ltimes_\delta^{\mathcal{F}} G$ for all $g \in L^\infty(G)$ and $y \in Y \ltimes_\delta^{\mathcal{F}} G$.

On the other hand, if $x \in Y$, then:

$$\begin{aligned}
\widetilde{\delta}(\delta(x)) &= (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})(\delta(x)) \\
&= (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\mathrm{id}_Y \otimes \delta_G)(\delta(x)) \\
&= (\mathrm{id}_Y \otimes \mathrm{Ad}W_G) \circ (\mathrm{id}_Y \otimes \delta_G)(\delta(x)) \\
&= (1_K \otimes W_G)(1_K \otimes W_G^*)(\delta(x) \otimes 1_{L^2(G)})(1_K \otimes W_G)(1_K \otimes W_G^*) \\
&= \delta(x) \otimes 1_{L^2(G)},
\end{aligned}$$

where the third equality holds because $\sigma \circ \delta_G = \delta_G$. Hence, $\delta(Y) \subseteq Y \ltimes_\delta^{\mathcal{F}} G$.

Let $x \in Y$ and $f \in L^\infty(G)$. Since $\beta_G(z) = z \otimes 1$ for any $z \in L(G)$ and $\delta(x) \in Y \overline{\otimes}_{\mathcal{F}} L(G)$, it follows that $(\mathrm{id}_{B(K)} \otimes \beta_G)(\delta(x)) = \delta(x) \otimes 1$. Thus we get:

$$\begin{aligned}\widehat{\delta}((1_K \otimes f)\delta(x)) &= (\mathrm{id}_{B(K)} \otimes \beta_G)((1_K \otimes f)\delta(x)) \\ &= (1_K \otimes \beta_G(f))(\mathrm{id}_{B(K)} \otimes \beta_G)(\delta(x)) \\ &= (1_K \otimes \beta_G(f))(\delta(x) \otimes 1) \in (Y \overline{\ltimes}_\delta G) \overline{\otimes} L^\infty(G),\end{aligned}$$

because $\beta_G(f) \in L^\infty(G) \overline{\otimes} L^\infty(G)$ and $\delta(x) \in Y \overline{\ltimes}_\delta G$. Therefore $(Y \overline{\ltimes}_\delta G, \widehat{\delta})$ is an $L^\infty(G)$-subcomodule of $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$. □

Proposition 3.2.6 (Uniqueness of crossed product)**.** *Let (Y, δ) and (Z, ε) be two $L(G)$-comodules and suppose that Y and Z are w^*-closed subspaces of $B(H)$ and $B(K)$ respectively. Let $\Phi\colon Y \to Z$ be an $L(G)$-comodule isomorphism.*

Then the map $\Psi := \Phi \otimes \mathrm{id}_{B(L^2(G))}\colon Y \overline{\otimes} B(L^2(G)) \to Z \overline{\otimes} B(L^2(G))$ is an $L(G)$-comodule isomorphism from $(Y \overline{\otimes} B(L^2(G)), \widetilde{\delta})$ onto $(Z \overline{\otimes} B(L^2(G)), \widetilde{\varepsilon})$, which maps $Y \ltimes_\delta^{\mathcal{F}} G$ onto $Z \ltimes_\varepsilon^{\mathcal{F}} G$ and $Y \overline{\ltimes}_\delta G$ onto $Z \overline{\rtimes}_\varepsilon G$. Also, $\Psi|_{Y \ltimes_\delta^{\mathcal{F}} G}$ is an $L^\infty(G)$-comodule isomorphism from $(Y \ltimes_\delta^{\mathcal{F}} G, \widehat{\delta})$ onto $(Z \ltimes_\varepsilon^{\mathcal{F}} G, \widehat{\varepsilon})$ and $\Psi|_{Y \overline{\ltimes}_\delta G}$ is an $L^\infty(G)$-comodule isomorphism from $(Z \overline{\ltimes}_\delta G, \widehat{\delta})$ onto $(Z \overline{\ltimes}_\varepsilon G, \widehat{\varepsilon})$.

Moreover, Ψ is an $L^\infty(G)$-bimodule map, i.e. $\Psi((1_H \otimes f)x(1_H \otimes g)) = (1_K \otimes f)\Psi(x)(1_K \otimes g)$, for all $f, g \in L^\infty(G)$ and $x \in Y \overline{\otimes} B(L^2(G))$.

Proof. First, since Φ is a comodule morphism we have that $\varepsilon \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \delta$ and hence:

$$\begin{aligned}\widetilde{\varepsilon} \circ \Psi &= (\mathrm{id} \otimes \mathrm{Ad} W_G) \circ (\mathrm{id} \otimes \sigma) \circ (\varepsilon \otimes \mathrm{id}) \circ (\Phi \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \mathrm{Ad} W_G) \circ (\mathrm{id} \otimes \sigma) \circ ((\varepsilon \circ \Phi) \otimes \mathrm{id}) \\ &= (\mathrm{id} \otimes \mathrm{Ad} W_G) \circ (\mathrm{id} \otimes \sigma) \circ [((\Phi \otimes \mathrm{id}) \circ \delta) \otimes \mathrm{id}] \\ &= (\mathrm{id} \otimes \mathrm{Ad} W_G) \circ (\mathrm{id} \otimes \sigma) \circ (\Phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id}) \\ &= (\Phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{Ad} W_G) \circ (\mathrm{id} \otimes \sigma) \circ (\delta \otimes \mathrm{id}) \\ &= (\Psi \otimes \mathrm{id}) \circ \widetilde{\delta}\end{aligned}$$

and thus Ψ is an $L(G)$-comodule isomorphism from $(Y \overline{\otimes} B(L^2(G)), \widetilde{\delta})$ onto $(Z \overline{\otimes} B(L^2(G)), \widetilde{\varepsilon})$. This implies that Ψ maps the fixed point subspace $Y \ltimes_\delta^{\mathcal{F}} G$ of $\widetilde{\delta}$ onto the fixed point subspace $Z \ltimes_\varepsilon^{\mathcal{F}} G$ of $\widetilde{\varepsilon}$.

On the other hand, the relation $\varepsilon \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \delta$ yields that

$$\Psi(\delta(Y)) = (\Phi \otimes \mathrm{id})(\delta(Y)) = \varepsilon(\Phi(Y)) = \varepsilon(Z)$$

and since Ψ is an $L^\infty(G)$-bimodule isomorphism (see Remark 3.1.6) it follows that Ψ maps $Y \overline{\ltimes}_\delta G$ onto $Z \overline{\ltimes}_\varepsilon G$.

Finally, we have

$$\begin{aligned}\widehat{\varepsilon} \circ \Psi &= (\mathrm{id}_Z \otimes \beta_G) \circ (\Phi \otimes \mathrm{id}_{B(L^2(G))}) \\ &= (\Phi \otimes \mathrm{id}_{B(L^2(G))} \otimes \mathrm{id}_{L^\infty(G)}) \circ (\mathrm{id}_Y \otimes \beta_G) \\ &= (\Psi \otimes \mathrm{id}_{L^\infty(G)}) \circ \widehat{\delta}.\end{aligned}$$

□

Note that until now everything seems to work in complete analogy to the case of $L^\infty(G)$-comodules. However, from now on the differences between $L^\infty(G)$-comodules and $L(G)$-comodules will start to become apparent.

Proposition 3.2.7. *For any $L(G)$-comodule (Y, δ) we have:*

$$\delta(Y) \subseteq \left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}} = \mathrm{Sat}(Y, \delta) = (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}}.$$

Proof. Since $\delta(Y) \subseteq \mathrm{Sat}(Y, \delta)$ is obvious (see Definition 2.2.4) we only have to show the equalities $\left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}} = \mathrm{Sat}(Y, \delta) = (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}}$. Suppose that Y is a w*-closed subspace of $B(H)$ for some Hilbert space H.

We prove first that $\mathrm{Sat}(Y, \delta) = (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}}$. Indeed, for any $x \in Y \overline{\otimes}_{\mathcal{F}} L(G)$, we have:

$$\begin{aligned} x \in (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}} &\iff \widetilde{\delta}(x) = x \otimes 1 \\ &\iff (\mathrm{id}_Y \otimes \mathrm{Ad} W_G) \circ (\mathrm{id}_Y \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})(x) = x \otimes 1 \\ &\iff (\delta \otimes \mathrm{id}_{L(G)})(x) = (\mathrm{id}_Y \otimes \sigma)\left((1_H \otimes W_G^*)(x \otimes 1)(1_H \otimes W_G)\right) \\ &\iff (\delta \otimes \mathrm{id}_{L(G)})(x) = (\mathrm{id}_Y \otimes \sigma) \circ (\mathrm{id}_Y \otimes \delta_G)(x) \\ &\iff (\delta \otimes \mathrm{id}_{L(G)})(x) = (\mathrm{id}_Y \otimes \delta_G)(x) \\ &\iff x \in \mathrm{Sat}(Y, \delta), \end{aligned}$$

where for the fourth equivalence above we used the fact that $\sigma \circ \delta_G = \delta_G$ since $\delta_G(\lambda_s) = \lambda_s \otimes \lambda_s$ for all $s \in G$.

It remains to prove that $\left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}} = (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}}$. Indeed, since the actions $\mathrm{id}_Y \otimes \beta_G$ and $\widetilde{\delta}$ commute (see Proposition 3.2.2) and $\widehat{\delta} = (\mathrm{id}_Y \otimes \beta_G)|_{Y \ltimes_\delta^{\mathcal{F}} G}$ it follows that:

$$\begin{aligned}\left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}} &= \left(\left(Y \overline{\otimes} B(L^2(G))\right)^{\widetilde{\delta}}\right)^{\mathrm{id}_Y \otimes \beta_G} \\ &= \left(\left(Y \overline{\otimes} B(L^2(G))\right)^{\mathrm{id}_Y \otimes \beta_G}\right)^{\widetilde{\delta}} \\ &= \left(Y \overline{\otimes}_{\mathcal{F}} \left(B(L^2(G))\right)^{\beta_G}\right)^{\widetilde{\delta}} \\ &= (Y \overline{\otimes}_{\mathcal{F}} L(G))^{\widetilde{\delta}}.\end{aligned}$$

The last equality follows from the fact that $B(L^2(G))^{\beta_G} = L(G)$. □

Remark 3.2.8. We denote by Λ the operator $\Lambda \in B(L^2(G))$ given by

$$\Lambda\xi(s) = \Delta_G(s)^{-1/2}\xi(s^{-1}), \qquad s \in G,\ \xi \in L^2(G).$$

One can see that Λ is clearly a selfadjoint unitary, such that $\Lambda R(G)\Lambda = L(G)$ and more precisely, we have:

$$\Lambda\rho_t\Lambda = \lambda_t, \qquad t \in G.$$

Indeed, for any $s, t \in G$ and $\xi \in L^2(G)$, we have:

$$\begin{aligned}(\Lambda\rho_t\Lambda\xi)(s) &= \Delta_G(s)^{-1/2}(\rho_t\Lambda\xi)(s^{-1}) \\ &= \Delta_G(s)^{-1/2}\Delta_G(t)^{1/2}(\Lambda\xi)(s^{-1}t) \\ &= \Delta_G(s^{-1}t)^{1/2}\Delta_G(s^{-1}t)^{-1/2}\xi(t^{-1}s) = \lambda_t\xi(s).\end{aligned}$$

Also, we put

$$W_\Lambda := (1 \otimes \Lambda)W_G,$$

that is

$$W_\Lambda\xi(s,t) = \Delta_G(t)^{-1/2}\xi(s, st^{-1}) \qquad s, t \in G,\ \xi \in L^2(G \times G).$$

Note that $W_\Lambda \in L^\infty(G)\overline{\otimes}B(L^2(G))$, because $W_G \in L^\infty(G)\overline{\otimes}L(G)$ and W_Λ satisfies

$$U_G W_\Lambda S = W_G,$$

where $S\xi(s,t) = \xi(t,s)$ is the flip operator on $L^2(G \times G)$.

Indeed, for $\xi \in L^2(G \times G)$, we have

$$\begin{aligned}(U_G W_\Lambda S\xi)(s,t) &= \Delta_G(t)^{1/2}(W_\Lambda S\xi)(st,t) \\ &= \Delta_G(t)^{1/2}\Delta_G(t)^{-1/2}(S\xi)(st, stt^{-1}) \\ &= S\xi(st,t) = \xi(s,st) = W_G\xi(s,t).\end{aligned}$$

Proposition 3.2.9. *Let (Y,δ) be an $L(G)$-comodule and suppose that Y is a w*-closed subspace of $B(H)$ for some Hilbert space H. Then we have:*

$$Y \ltimes_\delta^{\mathcal{F}} G = \overline{\operatorname{span}}^{\mathrm{w}^*}\left\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))\left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}}\right\}.$$

Proof. First put $K := H \otimes L^2(G)$ and $X := Y \ltimes_\delta^{\mathcal{F}} G$. Then $(X, \widehat{\delta})$ is an $L^\infty(G)$-subcomodule of $(B(K), \alpha)$, where $\alpha := \mathrm{id}_{B(H)} \otimes \beta_G \colon B(K) \to B(K)\overline{\otimes}L^\infty(G)$.

Consider the $L^\infty(G)$-actions

$$\widetilde{\alpha},\ \bar{\alpha} \colon B(K)\overline{\otimes}B(L^2(G)) \to B(K)\overline{\otimes}B(L^2(G))\overline{\otimes}L^\infty(G)$$

defined by

$$\widetilde{\alpha} = (\mathrm{id}_{B(K)} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id}_{B(K)} \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$$

and

$$\bar{\alpha} = (\mathrm{id}_{B(K)} \otimes \sigma) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))}).$$

Recall the unitary W_Λ with $W_\Lambda \xi(s,t) = \Delta_G(t)^{-1/2}\xi(s, st^{-1})$ and put

$$W := 1_H \otimes W_\Lambda.$$

Since $W_\Lambda \in L^\infty(G)\overline{\otimes}B(L^2(G))$, it follows that

$$W \in \mathbb{C}1_H\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G)) \subseteq B(K)\overline{\otimes}B(L^2(G)).$$

Claim: The w*-continuous *-automorphism

$$\mathrm{Ad}W \colon B(K)\overline{\otimes}B(L^2(G)) \to B(K)\overline{\otimes}B(L^2(G))$$

is an $L^\infty(G)$-comodule isomorphism from $(B(K)\overline{\otimes}B(L^2(G)), \widetilde{\alpha})$ onto $(B(K)\overline{\otimes}B(L^2(G)), \bar{\alpha})$, that is:

$$\bar{\alpha} \circ \mathrm{Ad}W = (\mathrm{Ad}W \otimes \mathrm{id}_{L^\infty(G)}) \circ \widetilde{\alpha}. \tag{3.7}$$

Proof of the Claim: In order to prove (3.7) we show first the following

$$\bar{\alpha}(W) = (W \otimes 1_{L^2(G)})(1_K \otimes U_G^*). \tag{3.8}$$

Let $S \in B(L^2(G))\overline{\otimes}B(L^2(G))$ denote the flip operator, i.e. $S(\xi \otimes \eta) = \eta \otimes \xi$ and thus $\mathrm{Ad}S = \sigma$. For any $a \in B(H)$ and $b, c \in B(L^2(G))$ we have:

$$\bar{\alpha}(a \otimes b \otimes c) = (\mathrm{id}_{B(K)} \otimes \sigma)(\alpha(a \otimes b) \otimes c)$$

$$= (\mathrm{id}_{B(K)} \otimes \sigma)(a \otimes \beta_G(b) \otimes c) = (\mathrm{id}_{B(K)} \otimes \sigma)(a \otimes (U_G^*(b \otimes 1)U_G) \otimes c)$$

$$= (1 \otimes 1 \otimes S)(1 \otimes U_G^* \otimes 1)(a \otimes b \otimes 1 \otimes c)(1 \otimes U_G \otimes 1)(1 \otimes 1 \otimes S) =$$

$$(1\otimes1\otimes S)(1\otimes U_G^*\otimes1)(1\otimes1\otimes S)(a\otimes b\otimes c\otimes1)(1\otimes1\otimes S)(1\otimes U_G\otimes1)(1\otimes1\otimes S)$$

therefore we get

$$\bar{\alpha}(W) = (1\otimes1\otimes S)(1\otimes U_G^*\otimes1)(1\otimes1\otimes S)(W\otimes1)(1\otimes1\otimes S)(1\otimes U_G\otimes1)(1\otimes1\otimes S)$$

$$= (1\otimes1\otimes S)(1\otimes U_G^*\otimes1)(1\otimes1\otimes S)(1\otimes W_\Lambda\otimes1)(1\otimes1\otimes S)(1\otimes U_G\otimes1)(1\otimes1\otimes S)$$

and thus (3.8) is equivalent to the following

$$(1 \otimes S)(U_G^* \otimes 1)(1 \otimes S)(W_\Lambda \otimes 1)(1 \otimes S)(U_G \otimes 1)(1 \otimes S) = (W_\Lambda \otimes 1)(1 \otimes U_G^*),$$

which can be easily checked by computation. Thus (3.8) is proved.

Now (3.7) follows from (3.8) since, for any $T \in B(K)\overline{\otimes}B(L^2(G))$, we have

$$\begin{aligned}(\bar{\alpha} \circ \mathrm{Ad}W)(T) &= \bar{\alpha}(W)\bar{\alpha}(T)\bar{\alpha}(W)^* \\ &= (W \otimes 1)(1 \otimes U_G^*)\bar{\alpha}(T)(1 \otimes U_G)(W^* \otimes 1) \\ &= (\mathrm{Ad}W \otimes \mathrm{id}_{L^\infty(G)}) \circ (\mathrm{id}_{B(K)} \otimes \mathrm{Ad}U_G^*) \circ \bar{\alpha}(T) \\ &= (\mathrm{Ad}W \otimes \mathrm{id}_{L^\infty(G)}) \circ \widetilde{\alpha}(T).\end{aligned}$$

Thus the **Claim** is proved.

By Corollary 2.3.5, (X, α) is non-degenerate, that is:

$$X\overline{\otimes}B(L^2(G)) = \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_K\overline{\otimes}B(L^2(G)))\alpha(X)\}$$

and therefore we get:

$$\begin{aligned}X\overline{\otimes}B(L^2(G)) &\subseteq \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_K\overline{\otimes}L^\infty(G))(\mathbb{C}1_K\overline{\otimes}L(G))\alpha(X)\} \\ &\subseteq \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_K\overline{\otimes}L^\infty(G))(X \rtimes_\alpha^{\mathcal{F}} G)\},\end{aligned}$$

since $B(L^2(G)) = \overline{\operatorname{span}}^{\mathrm{w}^*}\{L^\infty(G)L(G)\}$ and $(\mathbb{C}1_K\overline{\otimes}L(G))\alpha(X) \subseteq X\overline{\rtimes}_\alpha G \subseteq X \rtimes_\alpha^{\mathcal{F}} G$.

Since $W \in \mathbb{C}1_H\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G))$ and X is a $\mathbb{C}1_H\overline{\otimes}L^\infty(G)$-module, we have that $\mathrm{Ad}W$ maps $X\overline{\otimes}B(L^2(G))$ onto itself and therefore we get:

$$\begin{aligned}X\overline{\otimes}B(L^2(G)) &= W(X\overline{\otimes}B(L^2(G)))W^* \\ &\subseteq \overline{\operatorname{span}}^{\mathrm{w}^*}\{W(\mathbb{C}1_K\overline{\otimes}L^\infty(G))W^*W(X \rtimes_\alpha^{\mathcal{F}} G)W^*\}.\end{aligned}$$

Also, $X\overline{\otimes}B(L^2(G))$ is an $L^\infty(G)$-subcomodule of both $(B(K)\overline{\otimes}B(L^2(G)), \widetilde{\alpha})$ and $(B(K)\overline{\otimes}B(L^2(G)), \bar{\alpha})$ and thus it follows from (3.7) and the **Claim** that the restriction of $\mathrm{Ad}W$ to $X\overline{\otimes}B(L^2(G))$ is an $L^\infty(G)$-comodule isomorphism from $(X\overline{\otimes}B(L^2(G)), \widetilde{\alpha})$ onto $(X\overline{\otimes}B(L^2(G)), \bar{\alpha})$. Therefore, it maps the fixed point subspace $X \rtimes_\alpha^{\mathcal{F}} G = (X\overline{\otimes}B(L^2(G)))^{\widetilde{\alpha}}$ onto $(X\overline{\otimes}B(L^2(G)))^{\bar{\alpha}} = X^\alpha\overline{\otimes}B(L^2(G))$.

On the other hand, we have

$$W(\mathbb{C}1_K\overline{\otimes}L^\infty(G))W^* \subseteq \mathbb{C}1_H\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G)),$$

since $W \in \mathbb{C}1_H\overline{\otimes}L^\infty(G)\,\overline{\otimes}B(L^2(G))$. Therefore, it follows that:

$$\begin{aligned}X\overline{\otimes}B(L^2(G)) &\subseteq \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G)\overline{\otimes}B(L^2(G)))(X^\alpha\overline{\otimes}B(L^2(G)))\} \\ &\subseteq \left(\overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))X^\alpha\}\right)\overline{\otimes}B(L^2(G)).\end{aligned}$$

Since the reverse inclusion is obvious we get that

$$X\overline{\otimes}B(L^2(G)) = \left(\overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))X^\alpha\}\right)\overline{\otimes}B(L^2(G))$$

and therefore $X = \overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H \overline{\otimes} L^\infty(G))X^\alpha\}$, that is:

$$Y \ltimes_\delta^{\mathcal{F}} G = \overline{\operatorname{span}}^{\mathrm{w}^*} \left\{ (\mathbb{C}1_H \overline{\otimes} L^\infty(G)) \left(Y \ltimes_\delta^{\mathcal{F}} G\right)^{\widehat{\delta}} \right\}.$$

□

Now, we are able to prove the following important theorem.

Theorem 3.2.10. *For every $L(G)$-comodule (Y, δ) we have:*

$$Y \ltimes_\delta^{\mathcal{F}} G = Y \overline{\ltimes}_\delta G.$$

Proof. Since $Y \overline{\ltimes}_\delta G \subseteq Y \ltimes_\delta^{\mathcal{F}} G$ it suffices to prove that $Y \ltimes_\delta^{\mathcal{F}} G \subseteq Y \overline{\ltimes}_\delta G$. Suppose that Y is a w*-closed subspace of $B(H)$ for some Hilbert space H and consider the following W*-$L(G)$-action on $B(H) \overline{\otimes} B(L^2(G))$:

$$\varepsilon := \mathrm{id}_{B(H)} \otimes \delta_G \colon B(H) \overline{\otimes} B(L^2(G)) \to B(H) \overline{\otimes} B(L^2(G)) \overline{\otimes} L(G)$$

that is

$$\varepsilon(x) = (1_H \otimes W_G^*)(x \otimes 1)(1_H \otimes W_G), \qquad x \in B(H) \overline{\otimes} B(L^2(G))$$

For any $x \in \mathrm{Sat}(Y, \delta)$ and $f \in L^\infty(G)$ we have:

$$\begin{aligned} \varepsilon((1_H \otimes f)x) &= (\mathrm{id}_{B(H)} \otimes \delta_G)((1_H \otimes f)x) \\ &= (1_H \otimes f \otimes 1_{L^2(G)})(\mathrm{id}_{B(H)} \otimes \delta_G)(x) \\ &= (1_H \otimes f \otimes 1_{L^2(G)})(\delta \otimes \mathrm{id}_{L(G)})(x), \end{aligned}$$

where the last equality is obtained by the definition of $\mathrm{Sat}(Y, \delta)$.

Since $\mathrm{Sat}(Y, \delta) \subseteq Y \overline{\otimes}_{\mathcal{F}} L(G)$, it follows that

$$(\delta \otimes \mathrm{id}_{L(G)})(x) \in \delta(Y) \overline{\otimes}_{\mathcal{F}} L(G) \subseteq (Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$$

and thus $(1_H \otimes f \otimes 1_{L^2(G)})(\delta \otimes \mathrm{id}_{L(G)})(x) \in (Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$, because $(Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$ is a $\mathbb{C}1_H \overline{\otimes} L^\infty(G) \overline{\otimes} L(G)$-bimodule since $Y \overline{\ltimes}_\delta G$ is in turn a $\mathbb{C}1_H \overline{\otimes} L^\infty(G)$-bimodule.

Thus we have proved that ε maps $\overline{\operatorname{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H \overline{\otimes} L^\infty(G))\mathrm{Sat}(Y, \delta)\}$ into the Fubini tensor product $(Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$ and so Proposition 3.2.9 and Proposition 3.2.7 imply that

$$\varepsilon\left(Y \ltimes_\delta^{\mathcal{F}} G\right) \subseteq (Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$$

that is

$$\left(Y \ltimes_\delta^{\mathcal{F}} G\right) \overline{\otimes} \mathbb{C}1 \subseteq (1_H \otimes W_G) \left((Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)\right) (1_H \otimes W_G^*).$$

Now, note that $1_H \otimes W_G \in \mathbb{C}1_H \overline{\otimes} L^\infty(G) \overline{\otimes} L(G)$ and $(Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$ is a $\mathbb{C}1_H \overline{\otimes} L^\infty(G) \overline{\otimes} L(G)$-bimodule. Therefore we get that

$$\left(Y \ltimes_\delta^{\mathcal{F}} G\right) \overline{\otimes} \mathbb{C}1 \subseteq (Y \overline{\ltimes}_\delta G) \overline{\otimes}_{\mathcal{F}} L(G)$$

and thus $Y \ltimes_\delta^{\mathcal{F}} G \subseteq Y \overline{\ltimes}_\delta G$. □

Theorem 3.2.10 allows us to simplify our notation:

Definition 3.2.11. For an $L(G)$-comodule (Y, δ), we will write $Y \ltimes_\delta G$ instead of $Y \ltimes_\delta^{\mathcal{F}} G$ or $Y \overline{\ltimes}_\delta G$ (since they coincide).

3.3 Duality theory and applications

The following theorem was first proved by Takesaki for abelian groups [51]. The general case (i.e. for arbitrary locally compact group) was later proved independently in [35], [37], [49] and [50].

Theorem 3.3.1 (Takesaki-duality). *Let (M, α) be a W^*-$L^\infty(G)$-comodule and (N, δ) be a W^*-$L(G)$-comodule. The map $(\mathrm{id}_M \otimes \mathrm{Ad}V_G) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$ defines a W^*-$L^\infty(G)$-comodule isomorphism*

$$(M \overline{\otimes} B(L^2(G)), \widetilde{\alpha}) \simeq \Big((M \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G, \widehat{\widehat{\alpha}} \Big)$$

and the map $(\mathrm{id}_N \otimes \mathrm{Ad}W_\Lambda) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})$ defines a W^*-$L(G)$-comodule isomorphism

$$(N \overline{\otimes} B(L^2(G)), \widetilde{\delta}) \simeq \left((N \ltimes_\delta G) \rtimes_{\widehat{\delta}} G, \widehat{\widehat{\delta}} \right).$$

Recall that an $L^\infty(G)$-comodule (X, α) is non-degenerate and saturated (by Lemma 2.3.5). Using the non-degeneracy we will obtain the Takesaki-duality for the spatial crossed product, i.e.

$$(X \overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G \simeq X \overline{\otimes} B(L^2(G)),$$

whereas the saturation of (X, α) yields the corresponding result for the Fubini crossed product, that is

$$\big(X \rtimes_\alpha^{\mathcal{F}} G\big) \ltimes_{\widehat{\alpha}} G \simeq X \overline{\otimes} B(L^2(G))$$

(see Propositions 3.3.2 and 3.3.5 below).

The same ideas can be used to show that an $L(G)$-comodule (Y, δ) is non-degenerate if and only if

$$(Y \ltimes_\delta G) \overline{\rtimes}_{\widehat{\delta}} G \simeq Y \overline{\otimes} B(L^2(G)),$$

whereas (Y, δ) is saturated if and only if

$$(Y \ltimes_\delta G) \rtimes_{\widehat{\delta}}^{\mathcal{F}} G \simeq Y \overline{\otimes} B(L^2(G))$$

(see Propositions 3.3.3 and 3.3.6).

As a consequence we get two of the main results of this chapter. The first one (Theorem 3.3.8) states that for a fixed $L^\infty(G)$-comodule (X, α) the equality $X \rtimes_\alpha^{\mathcal{F}} G = X \overline{\rtimes}_\alpha G$ holds if and only if $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ is non-degenerate if and only if $(X \overline{\rtimes}_\alpha G, \widehat{\alpha})$ is saturated. The second one (Theorem 3.3.10) states that the locally compact group G has the AP if and only if $X \rtimes_\alpha^{\mathcal{F}} G = X \overline{\rtimes}_\alpha G$ holds for any $L^\infty(G)$-comodule (X, α).

3.3.1 Duality for spatial crossed products

Here we show that Theorem 3.3.1 can be directly generalized for double spatial crossed products of $L^\infty(G)$-comodules and non-degenerate $L(G)$-comodules (non-degeneracy is necessary) using similar arguments as in the case of crossed products of von Neumann algebras. Compare, for example, the proof of Theorem 3.3.1 presented in [38, Chapter I, Theorems 2.5 and 2.7].

Proposition 3.3.2. *Let (X, α) be an $L^\infty(G)$-comodule and consider the map*

$$\pi \colon X\overline{\otimes}B(L^2(G)) \to X\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)),$$

defined by $\pi := (\mathrm{id}_X \otimes \mathrm{Ad}V_G) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$. Then, π is an $L^\infty(G)$-comodule isomorphism from $(X\overline{\otimes}B(L^2(G)), \widetilde{\alpha})$ onto $((X\overline{\rtimes}_\alpha G) \ltimes_\delta G,\ \widehat{\delta})$, where $\delta = \widehat{\alpha} = (\mathrm{id}_X \otimes \delta_G)|_{X\overline{\rtimes}_\alpha G}$. In addition, π satisfies

$$\pi(X\overline{\rtimes}_\alpha G) = \delta(X\overline{\rtimes}_\alpha G).$$

Proof. Suppose that X is a w*-closed subspace of $B(H)$ for some Hilbert space H and let $K := H \otimes L^2(G)$. Since (X, α) is non-degenerate (by Lemma 2.3.5) and

$$B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{L^\infty(G)L(G)\} = \overline{\mathrm{span}}^{\mathrm{w}^*}\{L(G)L^\infty(G)\}$$

we have:

$$\begin{aligned} X\overline{\otimes}B(L^2(G)) &= \overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))(\mathbb{C}1_H\overline{\otimes}L(G))\alpha(X)(\mathbb{C}1_H\overline{\otimes}L(G)) \\ &\qquad (\mathbb{C}1_H\overline{\otimes}L^\infty(G))\} \\ &= \overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))(X\overline{\rtimes}_\alpha G)(\mathbb{C}1_H\overline{\otimes}L^\infty(G))\}. \end{aligned} \tag{3.9}$$

On the other hand, by the definition of the crossed product, we have:

$$(X\overline{\rtimes}_\alpha G) \ltimes_\delta G = \overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_K\overline{\otimes}L^\infty(G))\delta(X\overline{\rtimes}_\alpha G)(\mathbb{C}1_K\overline{\otimes}L^\infty(G))\} \tag{3.10}$$

Since the map π is clearly a w*-continuous complete isometry, by the equalities (3.9) and (3.10), in order to prove that π is an $L^\infty(G)$-comodule isomorphism onto $(X\overline{\rtimes}_\alpha G) \ltimes_\delta G$, it suffices to verify the following conditions:

$$\pi(X\overline{\rtimes}_\alpha G) = \delta(X\overline{\rtimes}_\alpha G), \tag{3.11}$$

$$\pi((1 \otimes f)y) = (1 \otimes 1 \otimes f)\pi(y), \quad \text{for } f \in L^\infty(G) \text{ and } y \in X\overline{\otimes}B(L^2(G)), \tag{3.12}$$

and

$$\widehat{\delta} \circ \pi = (\pi \otimes \mathrm{id}) \circ \widetilde{\alpha} \tag{3.13}$$

For any $x \in X$ and $s, t \in G$ we have:

$$\pi((1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_t)) = (\mathrm{id} \otimes \mathrm{Ad}V_G)((\alpha \otimes \mathrm{id})((1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_t)))$$

$$= (\mathrm{id} \otimes \mathrm{Ad}V_G)((1 \otimes 1 \otimes \lambda_s)(\alpha \otimes \mathrm{id})(\alpha(x))(1 \otimes 1 \otimes \lambda_t))$$
$$= (\mathrm{id} \otimes \mathrm{Ad}V_G)((1 \otimes 1 \otimes \lambda_s)(\mathrm{id} \otimes \alpha_G)(\alpha(x))(1 \otimes 1 \otimes \lambda_t))$$
$$= (1 \otimes V_G)(1 \otimes 1 \otimes \lambda_s)(\mathrm{id} \otimes \alpha_G)(\alpha(x))(1 \otimes 1 \otimes \lambda_t)(1 \otimes V_G^*)$$
$$= (1 \otimes V_G)(1 \otimes 1 \otimes \lambda_s)(1 \otimes V_G^*)(\alpha(x) \otimes 1)(1 \otimes V_G)(1 \otimes 1 \otimes \lambda_t)(1 \otimes V_G^*) =$$
$$(1 \otimes SW_G^* S)(1 \otimes 1 \otimes \lambda_s)(1 \otimes SW_G S)(\alpha(x) \otimes 1)(1 \otimes SW_G^* S)(1 \otimes 1 \otimes \lambda_t)(1 \otimes SW_G S)$$
$$= (1 \otimes SW_G^*)(1 \otimes \lambda_s \otimes 1)(1 \otimes W_G S)(\alpha(x) \otimes 1)(1 \otimes SW_G^*)(1 \otimes \lambda_t \otimes 1)(1 \otimes W_G S)$$
$$= (1 \otimes S)(1 \otimes \lambda_s \otimes \lambda_s)(1 \otimes S)(\alpha(x) \otimes 1)(1 \otimes S)(1 \otimes \lambda_t \otimes \lambda_t)(1 \otimes S)$$
$$= (1 \otimes \lambda_s \otimes \lambda_s)(\alpha(x) \otimes 1)(1 \otimes \lambda_t \otimes \lambda_t)$$
$$= (\mathrm{id} \otimes \delta_G)((1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_t))$$

and thus the equality (3.11) is proved.

On the other hand, for any $f, g \in L^\infty(G)$ and $y \in X\overline{\otimes}B(L^2(G))$, we have:

$$\pi((1 \otimes f)y(1 \otimes g)) = (\mathrm{id} \otimes \mathrm{Ad}V_G) \circ (\alpha \otimes \mathrm{id})((1 \otimes f)y(1 \otimes g))$$
$$= (\mathrm{id} \otimes \mathrm{Ad}V_G)((1 \otimes 1 \otimes f)(\alpha \otimes \mathrm{id})(y)(1 \otimes 1 \otimes g))$$
$$= (1 \otimes V_G)(1 \otimes 1 \otimes f)(1 \otimes V_G^*)\pi(y)(1 \otimes V_G)(1 \otimes 1 \otimes g)(1 \otimes V_G^*)$$
$$= (1 \otimes 1 \otimes f)\pi(y)(1 \otimes 1 \otimes g),$$

because $V_G \in L(G)\overline{\otimes}L^\infty(G)$ and therefore it commutes with $1 \otimes f$ and $1 \otimes g$. Hence we have proved (3.12).

Since (X, α) is non-degenerate, we have

$$X\overline{\otimes}B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}L^\infty(G))(\mathbb{C}1_H\overline{\otimes}L(G))\alpha(X)\}.$$

Thus it remains verify (3.13) for elements of the form $y = (1 \otimes f)(1 \otimes \lambda_s)\alpha(x)$, where $f \in L^\infty(G)$, $s \in G$ and $x \in X$. Indeed, we have

$$\widetilde{\alpha}((1 \otimes f)(1 \otimes \lambda_s)\alpha(x)) = (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma)\,[(\alpha \otimes \mathrm{id})((1 \otimes f\lambda_s)\alpha(x))]$$
$$= (\mathrm{id} \otimes \mathrm{Ad}U_G^*) \circ (\mathrm{id} \otimes \sigma)\,[(1 \otimes 1 \otimes f\lambda_s)(\mathrm{id} \otimes \alpha_G)(\alpha(x))]$$
$$= (\mathrm{id} \otimes \mathrm{Ad}U_G^*)\,[(1 \otimes f\lambda_s \otimes 1)(\mathrm{id} \otimes \mathrm{Ad}U_G)(\alpha(x) \otimes 1)]$$
$$= (\mathrm{id} \otimes \mathrm{Ad}U_G^*)(1 \otimes f \otimes 1)(\mathrm{id} \otimes \mathrm{Ad}U_G^*)(1 \otimes \lambda_s \otimes 1)(\alpha(x) \otimes 1)$$
$$= (1 \otimes \beta_G(f))(1 \otimes \lambda_s \otimes 1)(\alpha(x) \otimes 1)$$

and therefore we get

$$(\pi \otimes \mathrm{id}) \circ \widetilde{\alpha}((1 \otimes f)(1 \otimes \lambda_s)\alpha(x)) =$$
$$= (\pi \otimes \mathrm{id})((1 \otimes \beta_G(f))(1 \otimes \lambda_s \otimes 1)(\alpha(x) \otimes 1))$$

$$
\begin{aligned}
&= (1 \otimes 1 \otimes \beta_G(f))\,(\pi((1 \otimes \lambda_s)\alpha(x)) \otimes 1)\\
&= (1 \otimes 1 \otimes \beta_G(f))(\delta((1 \otimes \lambda_s)\alpha(x)) \otimes 1)\\
&= (1 \otimes 1 \otimes \beta_G(f))\widehat{\delta}(\delta((1 \otimes \lambda_s)\alpha(x)))\\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \beta_G)((1 \otimes 1 \otimes f)\delta((1 \otimes \lambda_s)\alpha(x)))\\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \beta_G)((1 \otimes 1 \otimes f)\pi((1 \otimes \lambda_s)\alpha(x)))\\
&= \widehat{\delta} \circ \pi((1 \otimes f)(1 \otimes \lambda_s)\alpha(x)).
\end{aligned}
$$

□

Proposition 3.3.3. *Let (Y, δ) be an $L(G)$-comodule and consider the map*

$$\pi \colon Y \overline{\otimes} B(L^2(G)) \to Y \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G)),$$

given by $\pi := (\mathrm{id}_Y \otimes \mathrm{Ad} W_\Lambda) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})$. Then, π satisfies

$$\pi(Y \ltimes_\delta G) = \widehat{\delta}(Y \ltimes_\delta G)$$

and the following conditions are equivalent:

(i) (Y, δ) is non-degenerate;

(ii) The map π is an $L(G)$-comodule isomorphism from $(Y \overline{\otimes} B(L^2(G)), \widetilde{\delta})$ onto the double spatial crossed product $((Y \ltimes_\delta G)\overline{\rtimes}_\alpha G,\ \widehat{\alpha})$, where $\alpha = \widehat{\delta} = (\mathrm{id}_Y \otimes \beta_G)|_{Y \ltimes_\delta G}$.

Proof. Suppose that Y is a w*-closed subspace of $B(H)$ for some Hilbert space H and put $X := Y \ltimes_\delta G$.

We claim that, for any $s, t \in G$, $f, g \in L^\infty(G)$ and $y \in Y$, we have:

$$\pi((1 \otimes \rho_t f)\delta(y)(1 \otimes g\rho_s)) = (1 \otimes 1 \otimes \lambda_t)\alpha((1 \otimes f)\delta(y)(1 \otimes g))(1 \otimes 1 \otimes \lambda_s). \tag{3.14}$$

In order to prove (3.14), first observe the following:

$$
\begin{aligned}
W_\Lambda(1 \otimes \rho_t f)W_\Lambda^* &= W_\Lambda(1 \otimes \rho_t)W_\Lambda^* W_\Lambda(1 \otimes f)W_\Lambda^*\\
&= (1 \otimes \Lambda)W_G(1 \otimes \rho_t)W_G^*(1 \otimes \Lambda)W_\Lambda S(f \otimes 1)SW_\Lambda^*\\
&= (1 \otimes \Lambda)(1 \otimes \rho_t)W_G W_G^*(1 \otimes \Lambda)W_\Lambda S\delta_G(f)SW_\Lambda^*\\
&= (1 \otimes \Lambda)(1 \otimes \rho_t)(1 \otimes \Lambda)W_\Lambda SW_G^*(f \otimes 1)W_G SW_\Lambda^*\\
&= (1 \otimes \lambda_t)U_G^*(f \otimes 1)U_G\\
&= (1 \otimes \lambda_t)\beta_G(f)
\end{aligned}
$$

and similarly $W_\Lambda(1 \otimes g\rho_s)W_\Lambda^* = \beta_G(g)(1 \otimes \lambda_s)$. Also, we have

$$\alpha(\delta(y)) = (\mathrm{id} \otimes \beta_G)(\delta(y)) = \delta(y) \otimes 1.$$

Thus we get:

$$\begin{aligned}
&\pi((1\otimes\rho_t f)\delta(y)(1\otimes g\rho_s)) = \\
&= (1\otimes W_\Lambda)(\delta\otimes\mathrm{id})((1\otimes\rho_t f)\delta(y)(1\otimes g\rho_s))(1\otimes W_\Lambda *)\\
&= (1\otimes W_\Lambda)(1\otimes 1\otimes\rho_t f)(\delta\otimes\mathrm{id})(\delta(y))(1\otimes g\rho_s)(1\otimes W_\Lambda^*)\\
&= (1\otimes W_\Lambda)(1\otimes 1\otimes\rho_t f)(\mathrm{id}\otimes\delta_G)(\delta(y))(1\otimes 1\otimes g\rho_s)(1\otimes W_\Lambda^*)\\
&= (1\otimes W_\Lambda)(1\otimes 1\otimes\rho_t f)(1\otimes W_G^*)(\delta(y)\otimes 1)(1\otimes W_G)(1\otimes 1\otimes g\rho_s)(1\otimes W_\Lambda^*)\\
&= (1\otimes W_\Lambda)(1\otimes 1\otimes\rho_t f)(1\otimes W_G^*)(1\otimes 1\otimes\Lambda)(\delta(y)\otimes 1)(1\otimes 1\otimes\Lambda)(1\otimes W_G)\\
&\qquad (1\otimes 1\otimes g\rho_s)(1\otimes W_\Lambda^*)\\
&= (1\otimes W_\Lambda)(1\otimes 1\otimes\rho_t f)(1\otimes W_\Lambda^*)(\delta(y)\otimes 1)(1\otimes W_\Lambda)(1\otimes 1\otimes g\rho_s)(1\otimes W_\Lambda^*)\\
&= (1\otimes 1\otimes\lambda_t)\left[(\mathrm{id}\otimes\beta_G)((1\otimes f)\delta(y)(1\otimes g))\right](1\otimes 1\otimes\lambda_s)\\
&= (1\otimes 1\otimes\lambda_t)\alpha((1\otimes f)\delta(y)(1\otimes g))(1\otimes 1\otimes\lambda_s)
\end{aligned}$$

and hence (3.14) is proved. The equality $\pi(X) = \alpha(X)$ follows easily from (3.14).

(i) $\Longrightarrow$ (ii): Suppose that (Y,δ) is non-degenerate. Since

$$B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{R(G)L^\infty(G)\} = \overline{\mathrm{span}}^{\mathrm{w}^*}\{L^\infty(G)R(G)\},$$

we have

$$\begin{aligned}
Y\overline{\otimes}B(L^2(G)) &= \overline{\mathrm{span}}^{\mathrm{w}^*}\left\{\left(\mathbb{C}1_H\overline{\otimes}B(L^2(G))\right)\delta(Y)\left(\mathbb{C}1_H\overline{\otimes}B(L^2(G))\right)\right\}\\
&= \overline{\mathrm{span}}^{\mathrm{w}^*}\left\{(1\otimes\rho_t f)\delta(y)(1\otimes g\rho_s)) : s,t\in G,\ f,g\in L^\infty(G),\ y\in Y\right\}.
\end{aligned}\tag{3.15}$$

Clearly, the equality $\pi(Y\overline{\otimes}B(L^2(G))) = X\overline{\rtimes}_\alpha G$ follows from (3.14) and (3.15). It remains to prove that π is an $L(G)$-comodule isomorphism from $(Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ onto $(X\overline{\rtimes}_\alpha G,\ \widehat{\alpha})$. Since π is completely isometric and onto $X\overline{\rtimes}_\alpha G$, it suffices to prove that

$$\widehat{\alpha}\circ\pi(x) = (\pi\otimes\mathrm{id})\circ\widetilde{\delta}(x),\quad \forall x\in Y\overline{\otimes}B(L^2(G)).\tag{3.16}$$

Since (Y,δ) is non-degenerate, we have

$$Y\overline{\otimes}B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\left\{\left(\mathbb{C}1_H\overline{\otimes}B(L^2(G))\right)\delta(Y)\right\}$$

and thus we only have to verify (3.16) for $x = (1\otimes\rho_t f)\delta(y)$, where $t\in G$, $f\in L^\infty(G)$ and $y\in Y$. Indeed, this follows immediately from the calculations below:

$$\begin{aligned}
\widehat{\alpha}\circ\pi((1\otimes\rho_t f)\delta(y)) &= \widehat{\alpha}((1\otimes 1\otimes\lambda_t)\alpha((1\otimes f)\delta(y)))\\
&= [(1\otimes 1\otimes\lambda_t)\alpha((1\otimes f)\delta(y))]\otimes\lambda_t\\
&= [(1\otimes 1\otimes\lambda_t)\pi((1\otimes f)\delta(y))]\otimes\lambda_t\\
&= \pi((1\otimes\rho_t f)\delta(y))\otimes\lambda_t\\
&= (\pi\otimes\mathrm{id})([(1\otimes\rho_t f)\delta(y)]\otimes\lambda_t)
\end{aligned}$$

and on the other hand we have

$$\begin{aligned}
\widetilde{\delta}((1\otimes\rho_t f)\delta(y)) =&(\mathrm{id}\otimes\mathrm{Ad}W_G\circ\sigma)\circ(\delta\otimes\mathrm{id})((1\otimes\rho_t f)\delta(y))\\
=&(\mathrm{id}\otimes\mathrm{Ad}W_G\circ\sigma)((1\otimes 1\otimes\rho_t f)(\mathrm{id}\otimes\delta_G)(\delta(y)))\\
=&(\mathrm{id}\otimes\mathrm{Ad}W_G)(1\otimes\rho_t f\otimes 1)\,[\delta(y)\otimes 1]\\
=&(1\otimes W_G(\rho_t\otimes 1)W_G^*)(1\otimes W_G(f\otimes 1)W_G^*)(\delta(y)\otimes 1)\\
=&(1\otimes\rho_t\otimes\lambda_t)(1\otimes f\otimes 1)(\delta(y)\otimes 1)\\
=&[(1\otimes\rho_t f)\delta(y)]\otimes\lambda_t.
\end{aligned}$$

(ii) $\Longrightarrow$ (i): Recall that $(X\overline{\rtimes}_\alpha G,\widehat{\alpha})$ is non-degenerate for any $L^\infty(G)$-comodule (X,α) by Corollary 3.1.9. Therefore, $((Y\overline{\ltimes}_\delta G)\overline{\rtimes}_\alpha G,\ \widehat{\alpha})$ is always non-degenerate and since it is isomorphic to $(Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ (by assumption), it follows that $(Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ is non-degenerate too. That is:

$$Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))=\overline{\mathrm{span}}^{\mathrm{w}^*}\{N\widetilde{\delta}(Y\overline{\otimes}B(L^2(G)))N\},$$

where $N:=\mathbb{C}1_H\overline{\otimes}\mathbb{C}1_{L^2(G)}\overline{\otimes}B(L^2(G))$. Also put $M:=\mathbb{C}1_H\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))$. Thus, we get:

$$\begin{aligned}
&Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))=\\
&=\overline{\mathrm{span}}^{\mathrm{w}^*}\{N(1_H\otimes W_G S)(\delta\otimes\mathrm{id}_{B(L^2(G))})(Y\overline{\otimes}B(L^2(G)))(1_H\otimes SW_G^*)N\}\\
&\subseteq\overline{\mathrm{span}}^{\mathrm{w}^*}\{M(\delta(Y)\overline{\otimes}B(L^2(G)))M\}\\
&\subseteq\left(\overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}B(L^2(G)))\delta(Y)(\mathbb{C}1_H\overline{\otimes}B(L^2(G)))\}\right)\overline{\otimes}B(L^2(G))
\end{aligned}$$

and therefore

$$Y\overline{\otimes}B(L^2(G))=\overline{\mathrm{span}}^{\mathrm{w}^*}\{(\mathbb{C}1_H\overline{\otimes}B(L^2(G)))\delta(Y)(\mathbb{C}1_H\overline{\otimes}B(L^2(G)))\},$$

which means that (Y,δ) is non-degenerate (by Corollary 2.3.8). □

3.3.2 Duality for Fubini crossed products

As already mentioned, Fubini crossed products were first considered by Hamana [19, Definition 5.4] for comodule actions of $L^\infty(G)$ on complete operator spaces (not necessarily dual operator spaces). That is, in Hamana's context an $L^\infty(G)$-comodule is a norm closed subspace of $B(H)$ for some Hilbert space H with a complete isometry $\alpha\colon X\to X\overline{\otimes}_{\mathcal{F}}L^\infty(G)$ satisfying

$$(\alpha\otimes\mathrm{id})\circ\alpha=(\mathrm{id}\otimes\alpha_G)\circ\alpha.$$

Since at least one factor is w*-closed, Fubini tensor products and tensor product maps are still well behaved in this context (see [19, Section 1]).

Moreover, Hamana proved (see [19, Proposition 5.7]) that, for a pair (X,α) as above, $(X \rtimes_{\alpha}^{\mathcal{F}} G) \ltimes_{\widehat{\alpha}}^{\mathcal{F}} G$ is canonically completely isometrically isomorphic to $X\overline{\otimes}_{\mathcal{F}}B(L^2(G))$ if and only if X is G-complete (see [19, Definition 5.5]).

On the other hand, it is not hard to see that an $L^\infty(G)$-comodule is G-complete in the sense of Hamana ([19, Definition 5.5]) if and only if X is saturated. In our context, all $L^\infty(G)$-comodules are assumed to be dual operator spaces with w*-continuous comodule actions and thus they are saturated (i.e. G-complete) by Lemma 2.3.5. Therefore, Proposition 3.3.5 below is an immediate consequence of [19, Proposition 5.7] and Lemma 2.3.5. However, we have included a full proof of Proposition 3.3.5 for the sake of completeness.

Also, using the same idea in the case of $L(G)$-comodules, we get Proposition 3.3.6.

Remark 3.3.4. Let (M,Δ) be a Hopf-von Neumann algebra with M acting on a Hilbert space K. If Z_0 is an M-subcomodule of an M-comodule (Z,β), then clearly we have

$$\mathrm{Sat}(Z_0,\beta|_{Z_0}) = (Z_0\overline{\otimes}_{\mathcal{F}}M)\cap \mathrm{Sat}(Z,\beta) = (Z_0\overline{\otimes}B(K))\cap \mathrm{Sat}(Z,\beta).$$

The first equality follows from Definition 2.2.4 and the fact that Z_0 is a subcomodule of Z, whereas the second one holds since $\mathrm{Sat}(Z,\beta)\subseteq Z\overline{\otimes}_{\mathcal{F}}M$ and $Z_0\overline{\otimes}_{\mathcal{F}}M = (Z_0\overline{\otimes}B(K))\cap(Z\overline{\otimes}_{\mathcal{F}}M)$.

If, in addition, Z is saturated, then Z_0 will be saturated if and only if $\beta(Z_0) = (Z_0\overline{\otimes}_{\mathcal{F}}M)\cap\beta(Z)$ or, equivalently, $\beta(Z_0) = (Z_0\overline{\otimes}B(K))\cap\beta(Z)$.

Hence, for any $L^\infty(G)$-subcomodule X_0 of an $L^\infty(G)$-comodule (X,α), we have

$$\alpha(X_0) = (X_0\overline{\otimes}B(L^2(G)))\cap\alpha(X),$$

because all $L^\infty(G)$-comodules are saturated (see Lemma 2.3.5).

On the other hand, an $L(G)$-comodule may fail to be saturated, e.g. if G fails the AP (see Proposition 2.3.14). However, it is known that every W*-$L(G)$-comodule is saturated (see e.g. [50, Proposition II.1.1]). Thus, if Y is an $L(G)$-subcomodule of a W*-$L(G)$-comodule (N,δ), then Y will be saturated if and only if

$$\delta(Y) = (Y\overline{\otimes}_{\mathcal{F}}L(G))\cap\delta(N)$$

or, equivalently,

$$\delta(Y) = (Y\overline{\otimes}B(L^2(G)))\cap\delta(N).$$

Proposition 3.3.5 (Hamana [19])**.** *Let (X,α) be an $L^\infty(G)$-comodule and let*

$$\pi\colon X\overline{\otimes}B(L^2(G)) \to X\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))$$

be the map $\pi := (\mathrm{id}_X \otimes \mathrm{Ad}V_G) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$. *Then,* π *is an* $L^\infty(G)$-*comodule isomorphism from* $(X \overline{\otimes} B(L^2(G)), \widetilde{\alpha})$ *onto* $((X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_\delta G,\ \widetilde{\widehat{\delta}})$, *where* $\delta = \widehat{\alpha} = (\mathrm{id}_X \otimes \delta_G)|_{X \rtimes_\alpha^{\mathcal{F}} G}$. *In addition,* π *satisfies*

$$\pi\left(X \rtimes_\alpha^{\mathcal{F}} G\right) = \delta\left(X \rtimes_\alpha^{\mathcal{F}} G\right).$$

Proof. By Remark 2.1.3 and Remark 2.2.6 we may assume that (X, α) is an $L^\infty(G)$-subcomodule of some W*-$L^\infty(G)$-comodule M, i.e. M is a von Neumann algebra such that X is a w*-closed subspace of M and α extends to a W*-$L^\infty(G)$-action on M, which we still denote by α for simplicity. Also, the map π extends to the map $(\mathrm{id}_M \otimes \mathrm{Ad}V_G) \circ (\alpha \otimes \mathrm{id}_{B(L^2(G))})$ which gives the $L^\infty(G)$-comodule isomorphism between $\left((M \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G, \widetilde{\widehat{\alpha}}\right)$ and $(M \overline{\otimes} B(L^2(G)), \widetilde{\alpha})$. It follows that π is an $L^\infty(G)$-comodule monomorphism from $(X \overline{\otimes} B(L^2(G)), \widetilde{\alpha})$ into $(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G)),\ \mathrm{id}_X \otimes \mathrm{id}_{B(L^2(G))} \otimes \beta_G)$.

Thus it suffices to show that π maps $X \overline{\otimes} B(L^2(G))$ onto $(X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_\delta G$. First observe that

$$\begin{aligned} X \rtimes_\alpha^{\mathcal{F}} G &= \left(X \overline{\otimes} B(L^2(G))\right)^{\widetilde{\alpha}} \\ &= \left(X \overline{\otimes} B(L^2(G))\right) \cap \left(M \overline{\otimes} B(L^2(G))\right)^{\widetilde{\alpha}} \\ &= \left(X \overline{\otimes} B(L^2(G))\right) \cap (M \rtimes_\alpha G) \end{aligned}$$

and thus

$$\begin{aligned} \left(X \rtimes_\alpha^{\mathcal{F}} G\right) \overline{\otimes} B(L^2(G)) &= \left[\left(X \overline{\otimes} B(L^2(G))\right) \cap (M \rtimes_\alpha G)\right] \overline{\otimes} B(L^2(G)) \\ &= \left(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G))\right) \cap \left[(M \rtimes_\alpha G) \overline{\otimes} B(L^2(G))\right]. \end{aligned}$$

From the above equality and since $\pi(M \overline{\otimes} B(L^2(G))) = (M \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G$ (by Theorem 3.3.1) we get:

$$\begin{aligned} (X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_\delta G &= \left(\left(X \rtimes_\alpha^{\mathcal{F}} G\right) \overline{\otimes} B(L^2(G))\right)^{\widetilde{\delta}} \\ &= \left(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G))\right) \cap \left[(M \rtimes_\alpha G) \overline{\otimes} B(L^2(G))\right]^{\widetilde{(\widehat{\alpha})}} \\ &= \left(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G))\right) \cap \left[(M \rtimes_\alpha G) \ltimes_{\widehat{\alpha}} G\right] \\ &= \left(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G))\right) \cap \pi\left(M \overline{\otimes} B(L^2(G))\right). \end{aligned}$$

Therefore, the equality

$$\pi\left(X \overline{\otimes} B(L^2(G))\right) = (X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_\delta G$$

is equivalent to

$$\pi\left(X \overline{\otimes} B(L^2(G))\right) = \left(X \overline{\otimes} B(L^2(G)) \overline{\otimes} B(L^2(G))\right) \cap \pi\left(M \overline{\otimes} B(L^2(G))\right). \tag{3.17}$$

Since

$$\pi\left(X\overline{\otimes}B(L^2(G))\right) = (1\otimes V_G)(\alpha(X)\overline{\otimes}B(L^2(G)))(1\otimes V_G^*)$$

and

$$\pi\left(M\overline{\otimes}B(L^2(G))\right) = (1\otimes V_G)(\alpha(M)\overline{\otimes}B(L^2(G)))(1\otimes V_G^*),$$

the equality (3.17) is equivalent to

$$\begin{aligned}\alpha(X)\overline{\otimes}B(L^2(G)) &= \\ &= \left[X\overline{\otimes}V_G^*\left(B(L^2(G))\overline{\otimes}B(L^2(G))\right)V_G\right]\cap\left(\alpha(X)\overline{\otimes}B(L^2(G))\right) \\ &= \left(X\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right)\cap\left(\alpha(M)\overline{\otimes}B(L^2(G))\right) \\ &= \left[\left(X\overline{\otimes}B(L^2(G))\right)\cap\alpha(M)\right]\overline{\otimes}B(L^2(G)),\end{aligned}$$

or equivalently,

$$\alpha(X) = \left(X\overline{\otimes}B(L^2(G))\right)\cap\alpha(M),$$

which is true by Remark 3.3.4.

For the last statement, note that since π is a comodule isomorphism it maps the fixed point subspace $X\rtimes_\alpha^{\mathcal{F}} G = (X\overline{\otimes}B(L^2(G)))^{\widetilde{\alpha}}$ onto the fixed point subspace $((X\rtimes_\alpha^{\mathcal{F}} G)\ltimes_\delta G)^{\widehat{\delta}} = \mathrm{Sat}(X\rtimes_\alpha^{\mathcal{F}} G,\widehat{\alpha}) = \widehat{\alpha}(X\rtimes_\alpha^{\mathcal{F}} G)$ (recall Proposition 3.2.7 and Corollary 3.1.9). □

Proposition 3.3.6. *Let (Y,δ) be an $L(G)$-comodule and consider the map*

$$\pi\colon Y\overline{\otimes}B(L^2(G))\to Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))$$

given by $\pi := (\mathrm{id}_Y\otimes\mathrm{Ad}W_\Lambda)\circ(\delta\otimes\mathrm{id}_{B(L^2(G))})$. Then, the following are equivalent:

- *(i) (Y,δ) is saturated;*
- *(ii) The map π is an $L(G)$-comodule isomorphism from $(Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ onto the double Fubini crossed product $((Y\ltimes_\delta G)\rtimes_\alpha^{\mathcal{F}} G,\ \widehat{\alpha})$, where $\alpha = \widehat{\delta} = (\mathrm{id}_Y\otimes\beta_G)|_{Y\ltimes_\delta G}$.*

Proof. By Remark 2.1.3 and Remark 2.2.6 we may assume that (Y,δ) is an $L(G)$-subcomodule of some W*-$L(G)$-comodule N, i.e. N is a von Neumann algebra such that Y is a w*-closed subspace of N and δ extends to a W*-$L(G)$-action on N, which we still denote by δ for simplicity. Also, the map π extends to the map $(\mathrm{id}_N\otimes\mathrm{Ad}W_\Lambda)\circ(\delta\otimes\mathrm{id}_{B(L^2(G))})$ which gives the $L(G)$-comodule isomorphism between $\left((N\ltimes_\delta G)\rtimes_{\widehat{\delta}} G,\widehat{\widehat{\delta}}\right)$ and $(N\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ (see Theorem 3.3.1). It follows that π is an $L(G)$-comodule monomorphism from $(Y\overline{\otimes}B(L^2(G)),\widetilde{\delta})$ into $(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G)),\ \mathrm{id}_Y\otimes\mathrm{id}_{B(L^2(G))}\otimes\delta_G)$.

Thus it suffices to show that π maps $Y\overline{\otimes}B(L^2(G))$ onto $(Y \ltimes_\delta G) \rtimes_\alpha^{\mathcal{F}} G$ if and only if (Y, δ) is saturated.

First observe that

$$\begin{aligned} Y \ltimes_\delta G &= \left(Y\overline{\otimes}B(L^2(G))\right)^{\widetilde{\delta}} \\ &= \left(Y\overline{\otimes}B(L^2(G))\right) \cap \left(N\overline{\otimes}B(L^2(G))\right)^{\widetilde{\delta}} \\ &= \left(Y\overline{\otimes}B(L^2(G))\right) \cap (N \ltimes_\delta G) \end{aligned}$$

and thus

$$\begin{aligned} &(Y \ltimes_\delta G)\,\overline{\otimes}B(L^2(G)) = \\ &= \left[\left(Y\overline{\otimes}B(L^2(G))\right) \cap (N \ltimes_\delta G)\right]\overline{\otimes}B(L^2(G)) \\ &= \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \left[(N \ltimes_\delta G)\,\overline{\otimes}B(L^2(G))\right]. \end{aligned}$$

From the above equality and Theorem 3.3.1 we get:

$$\begin{aligned} (Y \ltimes_\delta G) \rtimes_\alpha^{\mathcal{F}} G &= \left((Y \ltimes_\delta G)\,\overline{\otimes}B(L^2(G))\right)^{\widetilde{\alpha}} \\ &= \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \left[(N \ltimes_\delta G)\,\overline{\otimes}B(L^2(G))\right]^{\widetilde{(\widehat{\delta})}} \\ &= \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \left[(N \ltimes_\delta G) \rtimes_{\widehat{\delta}} G\right] \\ &= \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \pi\left(N\overline{\otimes}B(L^2(G))\right). \end{aligned}$$

Therefore, the equality

$$\pi\left(Y\overline{\otimes}B(L^2(G))\right) = (Y \ltimes_\delta G) \rtimes_\alpha^{\mathcal{F}} G$$

is equivalent to

$$\pi\left(Y\overline{\otimes}B(L^2(G))\right) = \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \pi\left(N\overline{\otimes}B(L^2(G))\right). \tag{3.18}$$

Since

$$\pi\left(Y\overline{\otimes}B(L^2(G))\right) = (1 \otimes W_\Lambda)(\delta(Y)\overline{\otimes}B(L^2(G)))(1 \otimes W_\Lambda^*)$$

and

$$\pi\left(N\overline{\otimes}B(L^2(G))\right) = (1 \otimes W_\Lambda)(\delta(N)\overline{\otimes}B(L^2(G)))(1 \otimes W_\Lambda^*),$$

the equality (3.18) is equivalent to

$$\begin{aligned} &\delta(Y)\overline{\otimes}B(L^2(G)) = \\ &= \left[Y\overline{\otimes}W_\Lambda^*\left(B(L^2(G))\overline{\otimes}B(L^2(G))\right)W_\Lambda\right] \cap \left(\delta(N)\overline{\otimes}B(L^2(G))\right) \\ &= \left(Y\overline{\otimes}B(L^2(G))\overline{\otimes}B(L^2(G))\right) \cap \left(\delta(N)\overline{\otimes}B(L^2(G))\right) \\ &= \left[\left(Y\overline{\otimes}B(L^2(G))\right) \cap \delta(N)\right]\overline{\otimes}B(L^2(G)), \end{aligned}$$

or equivalently,

$$\delta(Y) = \left(Y\overline{\otimes}B(L^2(G))\right) \cap \delta(N),$$

which, by Remark 3.3.4, is true if and only if (Y, δ) is saturated. □

3.3.3 Applications of duality theory

The next simple corollary essentially states that, for an $L^\infty(G)$-comodule (X,α), the saturation space of the spatial crossed product $X\overline{\rtimes}_\alpha G$ is isomorphic to the Fubini crossed product $X \rtimes_\alpha^{\mathcal{F}} G$.

Corollary 3.3.7. *For any $L^\infty(G)$-comodule (X,α) we have:*

(i) $(X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_{\widehat{\alpha}} G = (X\overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G$;

(ii) $\mathrm{Sat}(X\overline{\rtimes}_\alpha G, \widehat{\alpha}) = \mathrm{Sat}\left(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha}\right) = \widehat{\alpha}\left(X \rtimes_\alpha^{\mathcal{F}} G\right)$;

(iii) $X \rtimes_\alpha^{\mathcal{F}} G = \{y \in X\overline{\otimes}B(L^2(G)) :\ A(G)\cdot y \subseteq X\overline{\rtimes}_\alpha G\}$, *where* $u\cdot y = (\mathrm{id}_{X\overline{\otimes}B(L^2(G))}\otimes u)(\mathrm{id}_X\otimes\delta_G)(y)$ *for* $u \in A(G)$ *and* $y \in X\overline{\otimes}B(L^2(G))$.

(iv) $X\overline{\rtimes}_\alpha G = \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot(X \rtimes_\alpha^{\mathcal{F}} G)\}$.

Proof. Statement (i) is an obvious consequence of Propositions 3.3.2 and 3.3.5.

For statement (ii), we have

$$\begin{aligned}\mathrm{Sat}(X\overline{\rtimes}_\alpha G, \widehat{\alpha}) &= ((X\overline{\rtimes}_\alpha G) \ltimes_{\widehat{\alpha}} G)^{\widehat{\widehat{\alpha}}}\\ &= \left((X \rtimes_\alpha^{\mathcal{F}} G) \ltimes_{\widehat{\alpha}} G\right)^{\widehat{\widehat{\alpha}}}\\ &= \mathrm{Sat}(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})\\ &= \widehat{\alpha}\left(X \rtimes_\alpha^{\mathcal{F}} G\right),\end{aligned}$$

where both the first and the third equalities follow from Proposition 3.2.7, the second equality follows from statement (i) and the fourth equality holds because $\left(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha}\right)$ is a saturated $L(G)$-comodule by Corollary 3.1.9.

In order to show (iii), observe that an element $y \in X\overline{\otimes}B(L^2(G))$ satisfies $u\cdot y \in X\overline{\rtimes}_\alpha G$ for all $u \in A(G)$ if and only if

$$(\mathrm{id}_{X\overline{\otimes}B(L^2(G))}\otimes u)((\mathrm{id}_X\otimes\delta_G)(y)) \in X\overline{\rtimes}_\alpha G$$

for all $u \in A(G)$. This is equivalent to

$$\begin{aligned}(\mathrm{id}_X\otimes\delta_G)(y) \in ((X\overline{\rtimes}_\alpha G)\overline{\otimes}_{\mathcal{F}} L(G)) \cap \widehat{\alpha}(X\overline{\otimes}B(L^2(G))) = \mathrm{Sat}(X\overline{\rtimes}_\alpha G, \widehat{\alpha}) =\\ = \widehat{\alpha}(X \rtimes_\alpha^{\mathcal{F}} G) = (\mathrm{id}_X\otimes\delta_G)(X \rtimes_\alpha^{\mathcal{F}} G),\end{aligned}$$

which in turn is equivalent to $y \in X \rtimes_\alpha^{\mathcal{F}} G$, because $\mathrm{id}_X\otimes\delta_G$ is injective. Hence (iii) is proved.

Finally, from (iii) it follows that $A(G)\cdot(X \rtimes_\alpha^{\mathcal{F}} G) \subseteq X\overline{\rtimes}_\alpha G$ and thus $\overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot(X \rtimes_\alpha^{\mathcal{F}} G)\} \subseteq X\overline{\rtimes}_\alpha G$. On the other hand, since $X\overline{\rtimes}_\alpha G$ is a non-degenerate $L(G)$-comodule (see Corollary 3.1.9), by Proposition 2.2.3, we get

$$X\overline{\rtimes}_\alpha G = \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot(X\overline{\rtimes}_\alpha G)\} \subseteq \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot(X \rtimes_\alpha^{\mathcal{F}} G)\}$$

and thus (iv) holds. ∎

Theorem 3.3.8. *Let (X,α) be an $L^\infty(G)$-comodule. Then, $X\overline{\rtimes}_\alpha G$ is the largest non-degenerate $L(G)$-subcomodule of $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$. Also, $X \rtimes_\alpha^{\mathcal{F}} G$ is the smallest saturated $L(G)$-subcomodule of $(X\overline{\otimes}B(L^2(G)), \mathrm{id}_X \otimes \delta_G)$ containing $X\overline{\rtimes}_\alpha G$. In particular, the following conditions are equivalent:*

(a) $X \rtimes_\alpha^{\mathcal{F}} G = X\overline{\rtimes}_\alpha G$;

(b) $\left(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha}\right)$ *is a non-degenerate $L(G)$-comodule;*

(c) $(X\overline{\rtimes}_\alpha G, \widehat{\alpha})$ *is a saturated $L(G)$-comodule.*

Proof. Recall that $(X\overline{\rtimes}_\alpha G, \widehat{\alpha})$ is non-degenerate and $(X \rtimes_\alpha^{\mathcal{F}} G, \widehat{\alpha})$ is saturated by Corollary 3.1.9.

Let Y be an $L(G)$-subcomodule of $(X\overline{\otimes}B(L^2(G)), \mathrm{id}_X \otimes \delta_G)$.

If Y is non-degenerate and $Y \subseteq X \rtimes_\alpha^{\mathcal{F}} G$, then by Corollaries 2.3.8 and 3.3.7 we have

$$Y = \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot Y\} \subseteq \overline{\mathrm{span}}^{\mathrm{w}^*}\{A(G)\cdot (X \rtimes_\alpha^{\mathcal{F}} G)\} = X\overline{\rtimes}_\alpha G.$$

On the other hand, if Y is saturated and $X\overline{\rtimes}_\alpha G \subseteq Y$, then again Corollary 3.3.7 yields that

$$\widehat{\alpha}(X \rtimes_\alpha^{\mathcal{F}} G) = \mathrm{Sat}(X\overline{\rtimes}_\alpha G, \widehat{\alpha}) \subseteq \mathrm{Sat}(Y, \mathrm{id}_X \otimes \delta_G) = (\mathrm{id}_X \otimes \delta_G)(Y)$$

that is $(\mathrm{id}_X \otimes \delta_G)(X \rtimes_\alpha^{\mathcal{F}} G) \subseteq (\mathrm{id}_X \otimes \delta_G)(Y)$ and therefore $X \rtimes_\alpha^{\mathcal{F}} G \subseteq Y$.

Thus the first statement of the theorem is proved and so the equivalence of conditions (a), (b) and (c) is obvious. □

Lemma 3.3.9. *(i) For any $L(G)$-comodule (Y,δ), the saturation space $(\mathrm{Sat}(Y,\delta), \mathrm{id}_Y \otimes \delta_G)$ is a saturated $L(G)$-comodule;*

(ii) If every saturated $L(G)$-comodule is non-degenerate, then G has the AP.

Proof. (i) Take an $L(G)$-comodule (Y,δ) and suppose that Y is a w*-closed subspace of $B(K)$ for some Hilbert space K. Since $(Y,\delta) \simeq (\delta(Y), \mathrm{id}_Y \otimes \delta_G)$ (see Remark 2.1.3), it follows that

$$(\mathrm{Sat}(Y,\delta), \mathrm{id}_Y \otimes \delta_G) \simeq \left(\mathrm{Sat}\left(\delta(Y), \mathrm{id}_Y \otimes \delta_G\right), \mathrm{id}_{\delta(Y)} \otimes \delta_G\right)$$

(by Remark 2.2.6) and thus (again by Remark 2.2.6) it suffices to prove that $\left(\mathrm{Sat}\left(\delta(Y), \mathrm{id}_Y \otimes \delta_G\right), \mathrm{id}_{\delta(Y)} \otimes \delta_G\right)$ is saturated. Indeed, by Remark 3.3.4, we have:

$$\mathrm{Sat}\left(\delta(Y), \mathrm{id}_Y \otimes \delta_G\right) = (\mathrm{id}_{B(K)} \otimes \delta_G)(B(K)\overline{\otimes}L(G)) \cap (\delta(Y)\overline{\otimes}_{\mathcal{F}}L(G)).$$

Now, observe that $(\mathrm{id}_{B(K)} \otimes \delta_G)(B(K)\overline{\otimes}L(G))$ is a W*-$L(G)$-subcomodule of $(B(K)\overline{\otimes}L(G)\overline{\otimes}L(G), \mathrm{id}_{B(K)} \otimes \mathrm{id}_{L(G)} \otimes \delta_G)$, hence it is saturated. Also, since $(L(G), \delta_G)$ is saturated (as a W*-$L(G)$-comodule), $\delta(Y)\overline{\otimes}_{\mathcal{F}}L(G)$ is a saturated $L(G)$-subcomodule of $(B(K)\overline{\otimes}L(G)\overline{\otimes}L(G), \mathrm{id}_{B(K)\overline{\otimes}L(G)} \otimes \delta_G)$ by Lemma 2.2.10. Therefore, the desired conclusion follows from the fact that the intersection of saturated subcomodules is clearly saturated too.

(ii) If every saturated $L(G)$-comodule is non-degenerate, then from (i) it follows that, for any $L(G)$-comodule (Y, δ), $(\mathrm{Sat}(Y, \delta), \mathrm{id}_Y \otimes \delta_G)$ is non-degenerate and thus (Y, δ) is non-degenerate by Proposition 2.2.5 (iii). Hence every $L(G)$-comodule will be non-degenerate which means that G has the AP (see Proposition 2.3.14). □

We are now in position to prove the following functorial characterization of locally compact groups with the approximation property (Theorem 3.3.10) in terms of the crossed product functors. This result along with Proposition 2.3.14 complement recent work of Crann and Neufang [12, Theorem 4.1 and Corollary 4.8].

Theorem 3.3.10. *For a locally compact group G the following conditions are equivalent:*

(i) *G has the AP;*

(ii) $(Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G = (Y \ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}}G$, *for any $L(G)$-comodule (Y, δ);*

(iii) $X \rtimes^{\mathcal{F}}_{\alpha} G = X\overline{\rtimes}_\alpha G$, *for any $L^\infty(G)$-comodule (X, α);*

(iv) $((Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G,\ \widehat{\widehat{\delta}}) \simeq (Y\overline{\otimes}B(L^2(G)), \widetilde{\delta})$ *for any $L(G)$-comodule (Y, δ);*

(v) $((Y \ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}}G,\ \widehat{\widehat{\delta}}) \simeq (Y\overline{\otimes}B(L^2(G)), \widetilde{\delta})$ *for any $L(G)$-comodule (Y, δ),*

where the isomorphism in both (iv) and (v) is $(\mathrm{id}_Y \otimes \mathrm{Ad}W_\Lambda)\circ(\delta\otimes\mathrm{id}_{B(L^2(G))})$.

Proof. The implication (iii) $\Longrightarrow$ (ii) is obvious and the implication (i) $\Longrightarrow$ (iii) is a direct consequence of Proposition 2.3.14 and Theorem 3.3.8. Also, the equivalences (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) follow immediately from Propositions 2.3.14, 3.3.3 and 3.3.6. Therefore, it suffices to prove that (ii) implies (i).

Assume that condition (ii) is satisfied. Then, every saturated $L(G)$-comodule is non-degenerate. Indeed, if (Y, δ) is a saturated $L(G)$-comodule, then Proposition 3.3.6 yields that

$$(Y \ltimes_\delta G)\overline{\rtimes}_{\widehat{\delta}}G = (Y \ltimes_\delta G) \rtimes^{\mathcal{F}}_{\widehat{\delta}} G = \pi(Y\overline{\otimes}B(L^2(G))),$$

where $\pi = (\mathrm{id}_Y \otimes \mathrm{Ad}W_\Lambda) \circ (\delta \otimes \mathrm{id}_{B(L^2(G))})$ and thus (Y, δ) is non-degenerate by Proposition 3.3.3.

Therefore, condition (ii) implies that all saturated $L(G)$-comodules are non-degenerate and thus G has the AP by Lemma 3.3.9 (ii). □

Chapter 4

Applications to Harmonic Analysis

In this chapter we investigate the relation between crossed products of dual operator spaces and the jointly harmonic operators defined by Anoussis, Katavolos and Todorov in [5, 6] as a natural generalization of the harmonic operators introduced in [25] by Jaworski and Neufang and in [41] by Neufang and Runde.

Let μ be a probability measure on G. A function $f \in L^\infty(G)$ is called *μ-harmonic* if it is a fixed point of the map $P_\mu \colon L^\infty(G) \to L^\infty(G)$ given by

$$(P_\mu f)(s) = \int_G f(st)\ d\mu(t). \tag{4.1}$$

That is, a function $f \in L^\infty(G)$ is μ-harmonic if $P_\mu f = f$.

Harmonic functions have played an important role in the study of random walks on discrete groups and in harmonic analysis of locally compact groups (see [17]). The non commutative analogue (quantization) of this concept can be obtained either by passing from functions to operators, i.e. from elements of $L^\infty(G)$ to elements of $B(L^2(G))$ (see [25, 42]), or by replacing $L^\infty(G)$ with $L(G)$ (see [10, 41]).

4.1 Representations of $M(G)$ and $M_{cb}A(G)$

Before we proceed in describing (jointly) harmonic operators and their connection to crossed products, it is important to understand how the action of a measure μ on $L^\infty(G)$ via the map P_μ can be extended to $B(L^2(G))$.

Let $M(G)$ be the measure algebra of G, that is the set of regular complex Borel measures on G. This is a Banach algebra with respect to the product given by the usual convolution of measures. Recall that, for $\mu, \nu \in M(G)$, the convolution $\mu * \nu$ is the element in $M(G)$ satisfying

$$\int_G f(x)\ d(\mu * \nu)(x) = \int_G \int_G f(st)\ d\mu(s)\ d\nu(t)$$

for every bounded Borel function f on G. As usual, we identify $L^1(G)$ with the ideal of $M(G)$ consisting of all absolutely continuous measures with respect to the Haar measure. For more details on $M(G)$ see e.g. [9, Section 9.4].

Also, $M(G)$ has a natural dual operator space structure since it is isometrically isomorphic to $C_0(G)^*$ (by the Riesz-Markov-Kakutani representation theorem).

There exists a w*-w*-continuous completely isometric representation

$$\Theta\colon M(G) \to CB_\sigma(B(L^2(G))),$$

with

$$\Theta(\nu)(T) = \int_G \mathrm{Ad}\rho_s(T)\ d\nu(s), \qquad \nu \in M(G),\ T \in B(L^2(G)),$$

the integral being understood in the w*-topology.

Observe that, for every $\mu \in M(G)$, the map $\Theta(\mu)$ is a w*-continuous $L(G)$-bimodule map on $B(L^2(G))$ mapping $L^\infty(G)$ onto itself (since $L^\infty(G)$ is translation invariant). In fact, for any $f \in L^\infty(G)$, we have

$$\Theta(\mu)(f) = \int_G f(st)\ d\mu(t)$$

and thus $\Theta(\mu)$ extends the map P_μ by (4.1). For more details on the representation Θ the reader is referred to [18], [40], [42] and [45].

On the other hand, $M_{cb}A(G)$ has also a natural operator space structure via the identification $u \in M_{cb}A(G) \mapsto M_u \in CB(L(G))$ (recall Definition 2.3.10). Again, there is a w*-w*-continuous representation [42, Theorem 4.3]

$$\widehat{\Theta}\colon M_{cb}A(G) \to CB_\sigma(B(L^2(G))),$$

such that for any $u \in M_{cb}A(G)$, $s \in G$ and $f \in L^\infty(G)$ we have

$$\widehat{\Theta}(u)(\lambda_s f) = u(s)\lambda_s f$$

and each $\widehat{\Theta}(u)$ is a completely bounded w*-continuous $L^\infty(G)$-bimodule map on $B(L^2(G))$ extending the completely bounded multiplier $M_u\colon L(G) \to L(G)$.

Note that $\widehat{\Theta}$ is the non commutative analogue of Θ since if G is an abelian group with dual group $\widehat{G}$, then $A(G) \simeq L^1(\widehat{G})$ and $M_{cb}A(G) \simeq M(\widehat{G})$.

4.2 Harmonic operators

For $\mu \in M(G)$ (not necessarily a probability measure) let $\mathcal{H}(\mu)$ be the space of μ-harmonic functions, i.e.

$$\mathcal{H}(\mu) := \{f \in L^\infty(G) : \; P_\mu(f) = f\}.$$

Since P_μ is the restriction of the map $\Theta(\mu)$ on $L^\infty(G)$, Jaworski and Neufang [25] defined the *μ-harmonic operators* as the space

$$\widetilde{\mathcal{H}}(\mu) := \{T \in B(L^2(G)) : \; \Theta(\mu)(T) = T\}$$

thus extending the notion of harmonicity in a non commutative setting.

Also, Jaworski and Neufang [25, Proposition 6.3] proved that $\widetilde{\mathcal{H}}(\mu)$ can be realized as a crossed product (generalizing work of Izumi for discrete countable groups [24]). More precisely, they proved that if G is second countable and μ is a probability measure, then $\widetilde{\mathcal{H}}(\mu)$ and $\mathcal{H}(\mu)$ admit a certain von Neumann algebra product, generally different from that of $B(L^2(G))$, such that $\widetilde{\mathcal{H}}(\mu)$ is the crossed product of $\mathcal{H}(\mu)$ by the left translation action of G. Note that $\mathcal{H}(\mu)$ is indeed a left translation invariant subspace of $L^\infty(G)$ since it is the set of fixed points of the w*-continuous $L(G)$-bimodule map P_μ.

On the other hand, Chu and Lau [10], replacing $L^\infty(G)$ with $L(G)$ and probability measures with normalized positive definite functions in the Fourier-Stieltjes algebra $B(G)$ (see [15] for more details on $B(G)$), introduced and studied another non commutative analogue of harmonicity. In particular, for a normalized positive definite function $\sigma \in B(G)$ they defined the *σ-harmonic operators* in $L(G)$ as the space

$$\mathcal{H}_\sigma := \{T \in L(G) : \; \sigma \cdot T = T\},$$

where $\sigma \cdot T$ is the operator in $L(G) \simeq A(G)^*$ defined by $\langle \sigma \cdot T, u\rangle = \langle T, u\sigma\rangle$ for all $u \in A(G)$ (note that $A(G)$ is an ideal in $B(G)$ [15] and thus $A(G)\sigma \subseteq A(G)$). They proved that $\mathcal{H}_\sigma$ is a von Neumann subalgebra of $L(G)$ ([10, Proposition 3.2.10]).

Note that $\mathcal{H}_\sigma$ is indeed the non commutative analogue of $\mathcal{H}(\mu)$ since, if G is abelian with dual group $\widehat{G}$, then there are isometric isomorphisms $L(G) \simeq L^\infty(\widehat{G})$ and $M_{cb}A(G) = B(G) \simeq M(\widehat{G})$ (implemented respectively by the Fourier and Fourier-Stieltjes transforms).

The definition of $\mathcal{H}_\sigma$ obviously makes sense for any $\sigma \in M_{cb}A(G)$ since $A(G)$ is an ideal of $M_{cb}A(G)$. Moreover, since M_σ is the adjoint of the map $u \mapsto \sigma u$ on $A(G)$, for any $T \in L(G)$, $u \in A(G)$ and $\sigma \in M_{cb}A(G)$, we have

$$\langle \sigma \cdot T, u\rangle = \langle T, u\sigma\rangle = \langle M_\sigma(T), u\rangle = \langle \widehat{\Theta}(\sigma)(T), u\rangle,$$

i.e. $\sigma \cdot T = \widehat{\Theta}(\sigma)(T)$ and thus the $M_{cb}A(G)$-module action on $L(G)$ extends to the whole of $B(L^2(G))$ via the representation $\widehat{\Theta}$ of $M_{cb}A(G)$ on $B(L^2(G))$.

This led Neufang and Runde [41] to define the *σ-harmonic operators* in $B(L^2(G))$ by
$$\widetilde{\mathcal{H}}_\sigma := \{T \in B(L^2(G)) :\ \widehat{\Theta}(\sigma)(T) = T\}.$$
They also proved that, under certain hypotheses, $\widetilde{\mathcal{H}}_\sigma$ is the von Neumann algebra generated by $L^\infty(G)$ and $\mathcal{H}_\sigma$ (see [41, Theorem 4.8]). That is the analogue of the result of Jaworski and Neufang.

It should be noted that all of the above have been extended and unified in the locally compact quantum group setting by Junge, Neufang and Ruan in [26] and by Kalantar, Neufang and Ruan in [27].

On the other hand, Anoussis, Katavolos and Todorov [5, 6] have also extended the concept of harmonic operators, but towards a different direction. More precisely, instead of considering harmonic operators with respect to a single element in $M(G)$ or $M_{cb}A(G)$, they studied operators which are harmonic with respect to any element of an arbitrary subset of $M(G)$ or $M_{cb}A(G)$.

For an arbitrary family $\Lambda \subseteq M(G)$ (not necessarily consisting of probability measures), we have the *jointly Λ-harmonic functions*

$$\mathcal{H}(\Lambda) := \{f \in L^\infty(G) :\ \Theta(\mu)(f) = f \text{ for all } \mu \in \Lambda\}$$

and the *jointly Λ-harmonic operators*

$$\widetilde{\mathcal{H}}(\Lambda) := \{T \in B(L^2(G)) :\ \Theta(\mu)(T) = T \text{ for all } \mu \in \Lambda\}.$$

Similarly, for a family $\Sigma \subseteq M_{cb}A(G)$, they define the *jointly Σ-harmonic functionals*

$$\mathcal{H}_\Sigma := \{T \in L(G) :\ \widehat{\Theta}(\sigma)(T) = T \text{ for all } \sigma \in \Sigma\}$$

and the *jointly Σ-harmonic operators*

$$\widetilde{\mathcal{H}}_\Sigma := \{T \in B(L^2(G)) :\ \widehat{\Theta}(\sigma)(T) = T \text{ for all } \sigma \in \Sigma\}.$$

Note that jointly harmonic operators with respect to arbitrary subsets of $M(G)$ or $M_{cb}A(G)$ may not admit a von Neumann algebra structure.

Let us fix some additional notation. For a subset $\mathcal{U} \subseteq B(L^2(G))$ we let

$$\mathrm{Bim}_{L^\infty(G)}(\mathcal{U}) := \overline{\mathrm{span}}^{\mathrm{w}^*}\{xTy :\ x, y \in L^\infty(G),\ T \in \mathcal{U}\}$$

and
$$\mathrm{Bim}_{L(G)}(\mathcal{U}) := \overline{\mathrm{span}}^{\mathrm{w}^*}\{xTy :\ x, y \in L(G),\ T \in \mathcal{U}\},$$
i.e. the w*-closed sub-bimodules of $B(L^2(G))$ over respectively $L^\infty(G)$ and $L(G)$ generated by $\mathcal{U}$.

It was proved by Anoussis, Katavolos and Todorov in [5] that $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma)$ for any family $\Sigma \subseteq M_{cb}A(G)$ at least in the case where G

is a locally compact second countable group thus generalizing the result of Neufang and Runde [41, Theorem 4.8].

Similarly, in [6], for a locally compact group G (not necessarily second countable), the authors showed that $\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ for any family $\Lambda \subseteq M(G)$, when G is either abelian or compact or weakly amenable discrete. This was recently generalized by Crann and Neufang [12] for any locally compact group with the AP.

In the sequel, we will prove that, for any locally compact group G and arbitrary families $\Lambda \subseteq M(G)$ and $\Sigma \subseteq M_{cb}A(G)$, the spaces $\widetilde{\mathcal{H}}(\Lambda)$ and $\widetilde{\mathcal{H}}_\Sigma$ respectively arise as Fubini crossed products of the dual operator spaces $\mathcal{H}(\Lambda)$ and $\mathcal{H}_\Sigma$, whereas the associated spatial crossed products of $\mathcal{H}(\Lambda)$ and $\mathcal{H}_\Sigma$ can be respectively identified with the bimodules $\mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ and $\mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma)$. This provides a more conceptual perspective of jointly harmonic operators, which could possibly be extended to the setting of locally compact quantum groups. The advantage of this approach is that the realization of jointly harmonic operators as Fubini crossed products does not require the use of a von Neumann algebra product (perhaps different from that on $B(L^2(G))$) or imposing some additional condition on the group G.

As applications, we generalize the aforementioned results of [5] and [12]. In particular, we give an alternative (less technical) proof of the equality $\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma)$, removing the assumption that G is second countable. Also, we show that the equality $\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ holds for any family $\Lambda \subseteq M(G)$ at least when G satisfies a condition a priori weaker than the AP.

Finally, we prove that, for a locally compact group G, the equality $\mathrm{Bim}_{L^\infty(G)}(J^\perp) \cap L(G) = J^\perp$ holds for any closed ideal J of $A(G)$ if and only if G has Ditkin's property at infinity [28, Remark 5.1.8 (2)] thus answering to a question raised by the authors in [4] (see [4, Question 4.8]).

4.3 Crossed products and harmonic operators

For subsets $A \subseteq M(G)$ and $B \subseteq M_{cb}A(G)$ we let

$$\ker \Theta(A) := \bigcap \{\ker \Theta(a) : \ a \in A\}$$

and similarly

$$\ker \widehat{\Theta}(B) := \bigcap \{\ker \widehat{\Theta}(b) : \ b \in B\}.$$

Also, for families $\Lambda \subseteq M(G)$ and $\Sigma \subseteq M_{cb}A(G)$ let

$$J(\Lambda) := \overline{\mathrm{span}\{h * \mu - h : \ h \in L^1(G), \ \mu \in \Lambda\}}^{\|\cdot\|_{L^1(G)}}$$

and

$$J_\Sigma := \overline{\mathrm{span}\{u\sigma - u : \ u \in A(G), \ \sigma \in \Sigma\}}^{\|\cdot\|_{A(G)}}.$$

Then, clearly, $J(\Lambda)$ is a closed left ideal in $L^1(G)$ and J_Σ is a closed ideal in $A(G)$, such that

$$\mathcal{H}(\Lambda) = J(\Lambda)^\perp \text{ and } \mathcal{H}_\Sigma = J_\Sigma^\perp,$$

$$\widetilde{\mathcal{H}}(\Lambda) = \ker \Theta(J(\Lambda)) \text{ and } \widetilde{\mathcal{H}}_\Sigma = \ker \widehat{\Theta}(J_\Sigma).$$

Therefore, the study of jointly harmonic operators leads naturally to the study of ideals of $L^1(G)$ and $A(G)$ and their annihilators respectively in $L^\infty(G)$ and $L(G)$. For this reason, let us begin with a short discussion on ideals and annihilators.

Suppose that (M, Δ) is a Hopf-von Neumann algebra and recall that its predual M_* has a natural Banach algebra structure with respect to the product given by the preadjoint of Δ, that is

$$\omega\phi = (\omega \otimes \phi) \circ \Delta, \qquad \omega, \phi \in M_*.$$

Note that a closed subspace I of M_* is a right ideal of M_* if and only if its annihilator in M, i.e. the space

$$I^\perp = \{x \in M : \ \langle x, \omega \rangle = 0, \ \forall \omega \in I\},$$

is an M-subcomodule of (M, Δ), that is $\Delta(I^\perp) \subseteq I^\perp \overline{\otimes}_{\mathcal{F}} M$. Indeed, by the definition of Fubini tensor product, the inclusion $\Delta(I^\perp) \subseteq I^\perp \overline{\otimes}_{\mathcal{F}} M$ is equivalent to

$$(\mathrm{id} \otimes \omega)(\Delta(x)) \in I^\perp \quad \text{ for all } x \in I^\perp, \ \omega \in M_*,$$

equivalently

$$\langle x, \phi\omega \rangle = \langle (\mathrm{id} \otimes \omega)(\Delta(x)), \phi \rangle = 0 \quad \text{ for all } x \in I^\perp, \ \omega \in M_*, \ \phi \in I,$$

which, by the Hahn-Banach theorem, is equivalent to $\phi\omega \in I$ for all $\omega \in M_*$ and $\phi \in I$.

Similarly, a w*-closed subspace X of M is an M-subcomodule of M if and only if its preannihilator in M_*, i.e. the space

$$X_\perp = \{\omega \in M_* : \ \langle x, \omega \rangle = 0, \ \forall x \in X\},$$

is a right ideal of M_*. Thus the M-subcomodules of a Hopf-von Neumann algebra M are in a one to one correspondence with the right ideals of M_* by taking annihilators and preannihilators.

4.3.1 Ideals of $A(G)$ and $\widetilde{\mathcal{H}}_\Sigma$

Let J be a closed ideal of the Fourier algebra $A(G)$ and let $J^\perp$ be its annihilator in $L(G)$. As we already explained, $J^\perp$ is an $L(G)$-subcomodule of $(L(G), \delta_G)$, that is $\delta_G(J^\perp) \subseteq J^\perp \overline{\otimes}_{\mathcal{F}} L(G)$. In fact, every $L(G)$-subcomodule of $(L(G), \delta_G)$ arises in this way by taking its preannihilator in $A(G)$.

According to the next result (Proposition 4.3.1), for any closed ideal J of $A(G)$, the spaces $\mathrm{Bim}_{L^\infty(G)}(J^\perp)$ and $\ker \widehat{\Theta}(J)$ coincide since they are both canonically isomorphic to the crossed product $J^\perp \ltimes_{\delta_G} G$ of the $L(G)$-comodule $(J^\perp, \delta_G)$. Thus we obtain an alternative proof of [4, Theorem 3.2] as well as [5, Corollary 2.12], which is less technical and does not rely on the second countability of G.

Proposition 4.3.1. *The w*-continuous *-monomorphism*

$$\Phi\colon B(L^2(G)) \to B(L^2(G)) \overline{\otimes} B(L^2(G))$$

defined by

$$\Phi(T) = SW_G^*(T \otimes 1)W_G S, \quad T \in B(L^2(G)), \tag{4.2}$$

is an $L^\infty(G)$-comodule isomorphism from $(B(L^2(G)), \beta_G)$ onto $(L(G) \ltimes_{\delta_G} G, \widehat{\delta_G})$. Also, if J is a closed ideal of $A(G)$, then

$$\Phi(\mathrm{Bim}_{L^\infty(G)}(J^\perp)) = J^\perp \overline{\ltimes}_{\delta_G} G$$

and

$$\Phi(\ker \widehat{\Theta}(J)) = J^\perp \ltimes_{\delta_G}^{\mathcal{F}} G.$$

Therefore, $\mathrm{Bim}_{L^\infty(G)}(J^\perp) = \ker \widehat{\Theta}(J)$.

Proof. First note that

$$\Phi(\lambda_s) = SW_G^*(\lambda_s \otimes 1)W_G S = S(\lambda_s \otimes \lambda_s)S = \lambda_s \otimes \lambda_s = \delta_G(\lambda_s),$$

for any $s \in G$.

Also, since $\delta_G(f) = f \otimes 1$ for any $f \in L^\infty(G)$ we get:

$$\Phi(f) = SW_G^*(f \otimes 1)W_G S = S(f \otimes 1)S = 1 \otimes f,$$

for all $f \in L^\infty(G)$.

From the above calculations it is obvious that $\Phi(\mathrm{Bim}_{L^\infty(G)}(J^\perp)) = J^\perp \overline{\ltimes}_{\delta_G} G$. Also, $\Phi(B(L^2(G))) = L(G) \ltimes_{\delta_G} G$, because $B(L^2(G))$ is the w*-closed linear span of $L^\infty(G)L(G)$.

Since $B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{L^\infty(G)L(G)\}$, in order to prove that Φ is an $L^\infty(G)$-comodule isomorphism with respect to the $L^\infty(G)$-actions β_G and $\mathrm{id}_{B(L^2(G))} \otimes \beta_G$ it suffices to verify the equality

$$(\mathrm{id}_{B(L^2(G))} \otimes \beta_G) \circ \Phi(x) = (\Phi \otimes \mathrm{id}_{L^\infty(G)}) \circ \beta_G(x)$$

for $x = \lambda_s$, $s \in G$, and for $x = f \in L^\infty(G)$. Indeed, for $s \in G$ and $f \in L^\infty(G)$, we have:

$$\begin{aligned}(\mathrm{id}_{B(L^2(G))} \otimes \beta_G) \circ \Phi(\lambda_s) &= (\mathrm{id}_{B(L^2(G))} \otimes \beta_G)(\lambda_s \otimes \lambda_s)\\ &= \lambda_s \otimes \lambda_s \otimes 1\\ &= \Phi(\lambda_s) \otimes 1\\ &= (\Phi \otimes \mathrm{id})(\lambda_s \otimes 1)\\ &= (\Phi \otimes \mathrm{id})(\beta_G(\lambda_s))\end{aligned}$$

On the other hand, since $\Phi(g) = 1 \otimes g$ for all $g \in L^\infty(G)$ it follows that $(\Phi \otimes \mathrm{id})(y) = 1 \otimes y$ for any $y \in L^\infty(G) \overline{\otimes} L^\infty(G)$. Therefore, when $f \in L^\infty(G)$ we get:

$$\begin{aligned}(\mathrm{id}_{B(L^2(G))} \otimes \beta_G) \circ \Phi(f) &= (\mathrm{id}_{B(L^2(G))} \otimes \beta_G)(1 \otimes f)\\ &= 1 \otimes \beta_G(f)\\ &= (\Phi \otimes \mathrm{id}_{L^\infty(G)})(\beta_G(f)),\end{aligned}$$

because $\beta_G(f) \in L^\infty(G) \overline{\otimes} L^\infty(G)$.

It remains to show that $\Phi(\ker \widehat{\Theta}(J)) = J^\perp \ltimes^{\mathcal{F}}_{\delta_G} G$. To this end, one first observes the following:

$$\widehat{\Theta}(u) = (u \otimes \mathrm{id}_{B(L^2(G))}) \circ \Phi, \quad \text{for all } u \in A(G). \tag{4.3}$$

Indeed, if $s \in G$ and $f \in L^\infty(G)$, then we have:

$$\begin{aligned}\widehat{\Theta}(u)(f\lambda_s) &= u(s) f \lambda_s\\ &= (u \otimes \mathrm{id})(\lambda_s \otimes (f\lambda_s))\\ &= (u \otimes \mathrm{id})((1 \otimes f)(\lambda_s \otimes \lambda_s))\\ &= (u \otimes \mathrm{id})(\Phi(f)\Phi(\lambda_s))\\ &= (u \otimes \mathrm{id})(\Phi(f\lambda_s))\end{aligned}$$

and hence (4.3) follows because $B(L^2(G)) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{L^\infty(G)L(G)\}$.

Next, we get:

$$
\begin{aligned}
J^{\perp} \ltimes^{\mathcal{F}}_{\delta_G} G &= \Big(J^{\perp}\overline{\otimes}B(L^2(G))\Big)^{\widetilde{\delta_G}} \\
&= \big(L(G)\overline{\otimes}B(L^2(G))\big)^{\widetilde{\delta_G}} \cap \Big(J^{\perp}\overline{\otimes}B(L^2(G))\Big) \\
&= (L(G) \ltimes_{\delta_G} G) \cap \Big(J^{\perp}\overline{\otimes}B(L^2(G))\Big) \\
&= \Phi(B(L^2(G))) \cap \Big(J^{\perp}\overline{\otimes}B(L^2(G))\Big) \\
&= \Big\{T \in \Phi(B(L^2(G))) :\ (\mathrm{id} \otimes \omega)(T) \in J^{\perp},\ \forall \omega \in B(L^2(G))_*\Big\} \\
&= \big\{T \in \Phi(B(L^2(G))) :\ \langle(\mathrm{id} \otimes \omega)(T), u\rangle = 0,\ \forall \omega \in B(L^2(G))_*,\ \forall u \in J\big\} \\
&= \big\{T \in \Phi(B(L^2(G))) :\ \langle(u \otimes \mathrm{id})(T), \omega\rangle = 0,\ \forall \omega \in B(L^2(G))_*,\ \forall u \in J\big\} \\
&= \big\{T \in \Phi(B(L^2(G))) :\ (u \otimes \mathrm{id})(T) = 0,\ \forall u \in J\big\} \\
&= \Phi(\ker \widehat{\Theta}(J)),
\end{aligned}
$$

where the last equality follows from (4.3). Thus $J^{\perp} \ltimes^{\mathcal{F}}_{\delta_G} G = \Phi(\ker \widehat{\Theta}(J))$.

Finally, since $J^{\perp} \ltimes^{\mathcal{F}}_{\delta_G} G = J^{\perp}\overline{\rtimes}_{\delta_G} G$ (by Theorem 3.2.10), it follows that $\mathrm{Bim}_{L^\infty(G)}(J^{\perp}) = \ker \widehat{\Theta}(J)$. □

The next corollary follows immediately from Proposition 4.3.1.

Corollary 4.3.2. *For any family $\Sigma \subseteq M_{cb}A(G)$ we have that $\mathcal{H}_\Sigma$ is an $L(G)$-subcomodule of $L(G)$ and*

$$\widetilde{\mathcal{H}}_\Sigma = \mathrm{Bim}_{L^\infty(G)}(\mathcal{H}_\Sigma) \simeq \mathcal{H}_\Sigma \ltimes_{\delta_G} G.$$

Corollary 4.3.3. *For any closed ideal J of $A(G)$ the following are equivalent:*

(i) $(J^{\perp}, \delta_G)$ *is a saturated $L(G)$-comodule;*

(ii) $L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^{\perp}) = J^{\perp}$.

Proof. Let J be a closed ideal of $A(G)$. By Proposition 4.3.1 we get that $(\mathrm{Bim}_{L^\infty(G)}(J^{\perp}), \beta_G)$ is an $L^\infty(G)$-comodule, which is isomorphic to the $L^\infty(G)$-comodule $(J^{\perp} \ltimes_{\delta_G} G, \widehat{\delta_G})$ via the isomorphism Φ (4.2). Therefore, Φ maps the fixed point subspace $(\mathrm{Bim}_{L^\infty(G)}(J^{\perp}))^{\beta_G}$ onto $\big(J^{\perp} \ltimes_{\delta_G} G\big)^{\widehat{\delta_G}} = \mathrm{Sat}(J^{\perp}, \delta_G)$ (see Proposition 3.2.7 and Theorem 3.2.10). Also,

$$
\begin{aligned}
(\mathrm{Bim}_{L^\infty(G)}(J^{\perp}))^{\beta_G} &= B(L^2(G))^{\beta_G} \cap \mathrm{Bim}_{L^\infty(G)}(J^{\perp}) \\
&= L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^{\perp})
\end{aligned}
$$

therefore $\Phi(L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^{\perp})) = \mathrm{Sat}(J^{\perp}, \delta_G)$. Since $J^{\perp} \subseteq L(G)$, we have $\Phi(J^{\perp}) = \delta_G(J^{\perp})$ and thus $(J^{\perp}, \delta_G)$ will be a saturated $L(G)$-comodule if and only if $L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^{\perp}) = J^{\perp}$. □

In [4] Anoussis, Katavolos and Todorov proved that if $A(G)$ admits an approximate unit (not necessarily bounded), then

$$L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^\perp) = J^\perp$$

for any closed ideal J of $A(G)$ [4, Lemma 4.5]. They asked whether the same conclusion holds for an arbitrary group G. Clearly, from Corollary 4.3.3, this question is equivalent to asking whether every $L(G)$-subcomodule of $(L(G), \delta_G)$ is saturated. Using this point of view, we prove below (Proposition 4.3.5) that a condition which is a priori weaker than the existence of a (possibly unbounded) approximate unit in $A(G)$ is necessary and sufficient. This improves [4, Lemma 4.5].

Definition 4.3.4. Let G be a locally compact group. Following [28, Remark 5.1.8 (2)], we say that G has *Ditkin's property at infinity* (or *property* D_∞), if

$$u \in \overline{A(G)u}^{\|\cdot\|}, \quad \forall u \in A(G).$$

Also, following [15], we say that an element $x \in L(G)$ satisfies *condition (H)* if

$$x \in \overline{A(G) \cdot x}^{\mathrm{w}^*}.$$

Although the equivalence between statements (a) to (c) in the next result is already known (see e.g. [15]), we have included its proof for the sake of completeness.

Proposition 4.3.5. *Let G be a locally compact group. Then, the following conditions are equivalent:*

(a) G has property D_∞.

(b) Every $x \in L(G)$ satisfies condition (H).

(c) For any $x \in L(G)$ and $h \in A(G)$, if $h \cdot x = 0$, then $\langle x,\ h\rangle = 0$.

(d) For any $L(G)$-subcomodule Y of $(L(G), \delta_G)$ and any $x \in L(G)$ we have

$$\delta_G(x) \in Y \overline{\otimes}_{\mathcal{F}} L(G) \iff x \in Y.$$

(e) Every $L(G)$-subcomodule of $(L(G), \delta_G)$ is saturated.

(f) Every $L(G)$-subcomodule of $(L(G), \delta_G)$ is non-degenerate.

(g) For every closed ideal J of $A(G)$, we have $L(G) \cap \mathrm{Bim}_{L^\infty(G)}(J^\perp) = J^\perp$.

Proof. (b) $\Longrightarrow$ (a): Suppose that every element in $L(G)$ satisfies condition (H) and that there exists $u \in A(G)$, such that $u \notin \overline{A(G)u}^{\|\cdot\|}$. Then, there exists $x \in L(G)$, such that $\langle x, u\rangle \neq 0$ and $\langle x, vu\rangle = 0$, for all $v \in A(G)$. This means that $\langle v \cdot x, u\rangle = 0$, for all $v \in A(G)$ and since x satisfies condition (H) it is implied that $\langle x, u\rangle = 0$, a contradiction.

(c) $\Longrightarrow$ (b): Suppose that for any $x \in L(G)$ and $h \in A(G)$, $h \cdot x = 0$ implies that $\langle x,\ h\rangle = 0$. If there exists an $x \in L(G)$, such that $x \notin \overline{A(G) \cdot x}^{w^*}$, then there must be an $h \in A(G)$, such that $\langle x,\ h\rangle \neq 0$ and $\langle u \cdot x, h\rangle = 0$, for any $u \in A(G)$. But $\langle u \cdot x, h\rangle = \langle x, hu\rangle = \langle x, uh\rangle = \langle h \cdot x, u\rangle$, therefore we get that $\langle h \cdot x, u\rangle = 0$, for all $u \in A(G)$ and thus $h \cdot x = 0$, which implies that $\langle x, h\rangle = 0$, by hypothesis. Hence, we have a contradiction.

(a) $\Longrightarrow$ (c): Assume that G has D_∞ and there exist $x \in L(G)$ and $h \in A(G)$, such that $h \cdot x = 0$ and $\langle x, h\rangle \neq 0$. Then, $\langle h \cdot x, u\rangle = 0$, for all $u \in A(G)$, that is $\langle x, uh\rangle = 0$, for all $u \in A(G)$. But, since G has D_∞, we have that there is a net (u_i) in $A(G)$, such that $u_i h \longrightarrow h$. Therefore, $\langle x, h\rangle = \lim\langle x, u_i h\rangle = 0$, which is a contradiction.

(e) $\Longrightarrow$ (d): Let Y be an $L(G)$-subcomodule of $(L(G), \delta_G)$ and let $x \in L(G)$, with $\delta_G(x) \in Y \overline{\otimes}_{\mathcal{F}} L(G)$. Then, by the co-associativity of δ_G, we have that $(\delta_G \otimes \mathrm{id}_{L(G)})(\delta_G(x)) = (\mathrm{id}_Y \otimes \delta_G)(\delta_G(x))$. Thus, $\delta_G(x) \in \mathrm{Sat}(Y, \delta_G) = \delta_G(Y)$, since Y is saturated. Therefore, $x \in Y$, because δ_G is isometric.

(d) $\Longrightarrow$ (b): Take an $x \in L(G)$ and put $Y := \overline{A(G) \cdot x}^{w^*}$. Then, Y is clearly a subcomodule of $(L(G), \delta_G)$ (because it is an $A(G)$-submodule) and $\delta_G(x) \in Y \overline{\otimes}_{\mathcal{F}} L(G)$. Indeed, if not, then there must be $h,\ u \in A(G)$, such that $\langle y, u\rangle = 0$, for all $y \in Y$, and $\langle \delta_G(x), u \otimes h\rangle \neq 0$. But $\langle \delta_G(x), u \otimes h\rangle \neq 0$ implies that $\langle h \cdot x, u\rangle \neq 0$, while u annihilates Y and $h \cdot x \in Y$ by definition, which is a contradiction. Therefore, $\delta_G(x) \in Y \overline{\otimes}_{\mathcal{F}} L(G)$ and (d) implies that $x \in Y$.

(b) $\Longrightarrow$ (f): This follows immediately from Corollary 2.3.8.

(f) $\Longrightarrow$ (e): Suppose that every $L(G)$-subcomodule of $(L(G), \delta_G)$ is non-degenerate. Let Y be an $L(G)$-subcomodule of $L(G)$. If we put

$$Y_1 := \{x \in L(G) :\ A(G) \cdot x \subseteq Y\},$$

then clearly Y_1 is an $L(G)$-subcomodule of $L(G)$ which contains Y. Furthermore, it is clear by the definition of Y_1 that

$$\delta_G(Y_1) = (Y \overline{\otimes}_{\mathcal{F}} L(G)) \cap \delta_G(L(G)) = \mathrm{Sat}(Y, \delta_G).$$

Since Y_1 is non-degenerate by assumption, we get that

$$Y_1 = \overline{\mathrm{span}}^{w^*}\{A(G) \cdot Y_1\} \subseteq Y$$

and therefore $Y = Y_1$, that is Y is saturated because $\delta_G(Y_1) = \mathrm{Sat}(Y, \delta_G)$.

(e) $\Longleftrightarrow$ (g): This follows from Corollary 4.3.3 since the map $J \mapsto J^\perp$ is clearly a bijection between the set of all closed ideals of $A(G)$ and the set of all $L(G)$-subcomodules of $(L(G), \delta_G)$. □

Remark 4.3.6. Note that if G has the AP, then clearly G has property D_∞. However, to the author's knowledge, whether there exist groups failing property D_∞ is still open.

Also, Proposition 4.3.5 implies that a closed ideal J of the Fourier algebra $A(G)$ can be recovered from $\mathrm{Bim}_{L^\infty(G)}(J^\perp)$, i.e. the map $J \mapsto \mathrm{Bim}_{L^\infty(G)}(J^\perp)$ is one to one, at least when G has property D_∞. It is unknown whether the injectivity of this map follows without assuming property D_∞.

4.3.2 Ideals of $L^1(G)$ and $\widetilde{\mathcal{H}}(\Lambda)$

Suppose that J is a closed left ideal of $L^1(G)$ with annihilator $J^\perp \subseteq L^\infty(G)$. Recall that the product induced on $L^1(G)$ by the comultiplication α_G of $L^\infty(G)$ is given by the opposite convolution:

$$kh = (k \otimes h) \circ \alpha_G = h * k \qquad h, k \in L^1(G)$$

and thus $\alpha_G(J^\perp) \subseteq J^\perp \overline{\otimes} L^\infty(G)$, that is $J^\perp$ is an $L^\infty(G)$-subcomodule of $L^\infty(G)$, since J is a right ideal with respect to the above product on $L^1(G)$.

Proposition 4.3.7. *The w*-continuous *-monomorphism*

$$\Psi\colon B(L^2(G)) \to B(L^2(G)) \overline{\otimes} B(L^2(G))$$

given by

$$\Psi(T) = V_G^* \delta_G(T) V_G = V_G^* W_G^* (T \otimes 1) W_G V_G, \quad T \in B(L^2(G))$$

is a W-$L(G)$-comodule isomorphism from $(B(L^2(G)), \delta_G)$ onto $(L^\infty(G) \rtimes_{\alpha_G} G, \widehat{\alpha_G})$. Also, for any closed left ideal J of $L^1(G)$, we have*

$$\Psi(\mathrm{Bim}_{L(G)}(J^\perp)) = J^\perp \overline{\rtimes}_{\alpha_G} G$$

and

$$\Psi(\ker \Theta(J)) = J^\perp \rtimes_{\alpha_G}^{\mathcal{F}} G.$$

Proof. First, one has to verify

$$\Psi(f) = \alpha_G(f), \quad f \in L^\infty(G) \tag{4.4}$$

and

$$\Psi(\lambda_s) = 1 \otimes \lambda_s, \quad s \in G. \tag{4.5}$$

Indeed, for $f \in L^\infty(G)$ and $s \in G$, we have respectively

$$\Psi(f) = V_G^* \delta_G(f) V_G = V_G^* (f \otimes 1) V_G = \alpha_G(f)$$

and

$$\Psi(\lambda_s) = V_G^* (\lambda_s \otimes \lambda_s) V_G = S W_G S (\lambda_s \otimes \lambda_s) S W_G^* S$$

$$= SW_G(\lambda_s \otimes \lambda_s)W_G^* S = S(\lambda_s \otimes 1)S = (1 \otimes \lambda_s).$$

Obviously, since $B(L^2(G)) = \overline{\text{span}}^{w^*}\{L^\infty(G)L(G)\}$, (4.4) and (4.5) imply that $\Psi(B(L^2(G))) = L^\infty(G) \rtimes_{\alpha_G} G$ and $\Psi(\text{Bim}_{L(G)}(J^\perp)) = J^\perp \overline{\rtimes}_{\alpha_G} G$, for any closed left ideal J of $L^1(G)$.

Also, we have

$$\widehat{\alpha_G} \circ \Psi = (\Psi \otimes \text{id}_{L(G)}) \circ \delta_G$$

since

$$\widehat{\alpha_G} \circ \Psi(f\lambda_s) = (\text{id} \otimes \delta_G)(\alpha_G(f(1 \otimes \lambda_s)) = (\alpha_G(f) \otimes 1)(1 \otimes \lambda_s \otimes \lambda_s)$$

$$= \Psi(f\lambda_s) \otimes \lambda_s = (\Psi \otimes \text{id})((f\lambda_s) \otimes \lambda_s) = (\Psi \otimes \text{id})(\delta_G(f\lambda_s)),$$

for any $s \in G$ and $f \in L^\infty(G)$ and therefore, Ψ is a W*-$L(G)$-comodule isomorphism from $(B(L^2(G)), \delta_G)$ onto $(L^\infty(G) \rtimes_{\alpha_G} G, \widehat{\alpha_G})$.

It remains to show that $\Psi(\ker \Theta(J)) = J^\perp \rtimes_{\alpha_G}^{\mathcal{F}} G$. Indeed, note that

$$\begin{aligned} J^\perp \rtimes_{\alpha_G}^{\mathcal{F}} G &= (J^\perp \overline{\otimes} B(L^2(G)))^{\widetilde{\alpha_G}} \\ &= (L^\infty(G) \overline{\otimes} B(L^2(G)))^{\widetilde{\alpha_G}} \cap (J^\perp \overline{\otimes} B(L^2(G))) \\ &= (L^\infty(G) \rtimes_{\alpha_G} G) \cap (J^\perp \overline{\otimes} B(L^2(G))), \end{aligned}$$

since $(L^\infty(G)\overline{\otimes}B(L^2(G)))^{\widetilde{\alpha_G}} = (L^\infty(G)\overline{\rtimes}_{\alpha_G} G)$ by the Digernes-Takesaki theorem. Therefore if $y \in L^\infty(G)\overline{\rtimes}_{\alpha_G} G$, then

$$y \in J^\perp \rtimes_{\alpha_G}^{\mathcal{F}} G \iff (h \otimes \text{id}_{B(L^2(G))})(y) = 0, \ \forall h \in J.$$

Since $\ker \Theta(J)$ is the intersection of the kernels of the maps $\Theta(h)$ for $h \in J$ and $J^\perp \rtimes_{\alpha_G}^{\mathcal{F}} G$ is the intersection of the kernels of the maps $(h \otimes \text{id}_{B(L^2(G))})$ restricted to the image of Ψ for $h \in J$, it suffices to prove that

$$\Theta(h) = (h \otimes \text{id}_{B(L^2(G))}) \circ \Psi, \quad \forall h \in L^1(G). \tag{4.6}$$

Observe that for $f \in L^\infty(G)$ and $h \in L^1(G)$, we have $\Theta(h)(f) = f_h$, where $f_h(t) = \int_G h(s)f(ts)ds$. Therefore, for any $k \in L^1(G)$,

$$\langle \Theta(h)(f),\ k\rangle = \langle f_h, k\rangle = \int_G f_h(t)k(t)dt = \iint_{G\times G} h(s)f(ts)k(t)dsdt$$

$$= \langle \alpha_G(f), h \otimes k\rangle = \langle (h \otimes \text{id})(\alpha_G(f)), k\rangle = \langle (h \otimes \text{id})(\Psi(f)), k\rangle$$

that is $\Theta(h)(f) = (h \otimes \text{id})(\Psi(f))$ for all $f \in L^\infty(G)$.

Thus, since for any $y \in L(G)$ we have

$$\Theta(h)(fy) = \Theta(h)(f)y$$

and

$$(h \otimes \text{id})(\Psi(fy)) = (h \otimes \text{id})(\Psi(f)(1 \otimes y)) = (h \otimes \text{id})(\Psi(f))y,$$

(4.6) follows from the w*-continuity of $\Theta(h)$ and $(h \otimes \text{id}_{B(L^2(G))}) \circ \Psi$ and the fact that $B(L^2(G)) = \overline{\text{span}}^{w^*}\{L^\infty(G)L(G)\}$. □

By Proposition 4.3.7 above, for any closed left ideal J of $L^1(G)$, the $L(G)$-bimodules $\mathrm{Bim}_{L(G)}(J^\perp)$ and $\ker\Theta(J)$ are, in addition, $L(G)$-subcomodules of $(B(L^2(G)), \delta_G)$ and respectively canonically isomorphic to $J^\perp \overline{\rtimes}_{\alpha_G} G$ and $J^\perp \rtimes^{\mathcal{F}}_{\alpha_G} G$. Therefore, we can describe the relation between $\mathrm{Bim}_{L(G)}(J^\perp)$ and $\ker\Theta(J)$ using Corollary 3.3.7 and Theorem 3.3.8.

First, note that the $A(G)$-module action on $B(L^2(G))$ induced by δ_G is given by the representation $\widehat{\Theta}$, i.e. for any $u \in A(G)$ and $T \in B(L^2(G))$, we have

$$\widehat{\Theta}(u)(T) = (\mathrm{id}_{B(L^2(G))} \otimes u)(\delta_G(T)) = u \cdot T. \tag{4.7}$$

Indeed, since the maps $\widehat{\Theta}(u)$ and $(\mathrm{id} \otimes u) \circ \delta_G$ are both w*-continuous extensions of $M_u \colon L(G) \to L(G)$ (recall Remark 2.3.13), we only need to show that $(\mathrm{id} \otimes u) \circ \delta_G$ is an $L^\infty(G)$-bimodule map. This is true because $B(L^2(G))^{\delta_G} = L^\infty(G)$ and thus we have

$$(\mathrm{id} \otimes u) \circ \delta_G(fT) = (\mathrm{id} \otimes u)((f \otimes 1)\delta_G(T)) = f(\mathrm{id} \otimes u)(\delta_G(T)),$$

for any $f \in L^\infty(G)$ and $T \in B(L^2(G))$.

Proposition 4.3.8. *For any closed left ideal J of $L^1(G)$, $\mathrm{Bim}_{L(G)}(J^\perp)$ is the largest non-degenerate $L(G)$-subcomodule of $(B(L^2(G)), \delta_G)$ contained in $\ker\Theta(J)$, i.e.*

$$\mathrm{Bim}_{L(G)}(J^\perp) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{\widehat{\Theta}(u)(T) : \ u \in A(G), \ T \in \ker\Theta(J)\}$$

and $\ker\Theta(J)$ is the smallest saturated $L(G)$-subcomodule of $(B(L^2(G)), \delta_G)$ containing $\mathrm{Bim}_{L(G)}(J^\perp)$, i.e.

$$\ker\Theta(J) = \{T \in B(L^2(G)) : \ \widehat{\Theta}(u)(T) \in \mathrm{Bim}_{L(G)}(J^\perp), \ \forall u \in A(G)\}.$$

Thus, the following conditions are equivalent:

(a) $\mathrm{Bim}_{L(G)}(J^\perp) = \ker\Theta(J)$*;*

(b) $(\ker\Theta(J), \delta_G)$ *is a non-degenerate $L(G)$-comodule, i.e.*

$$\ker\Theta(J) = \overline{\mathrm{span}}^{\mathrm{w}^*}\{\widehat{\Theta}(A(G))(\ker\Theta(J))\};$$

(c) $(\mathrm{Bim}_{L(G)}(J^\perp), \delta_G)$ *is a saturated $L(G)$-comodule, i.e. if $T \in B(L^2(G))$ satisfies $\widehat{\Theta}(u)(T) \in \mathrm{Bim}_{L(G)}(J^\perp)$ $\forall u \in A(G)$, then $T \in \mathrm{Bim}_{L(G)}(J^\perp)$.*

Proof. The proof follows immediately by combining Proposition 4.3.7 with Corollary 3.3.7 and Theorem 3.3.8, since

$$\widehat{\Theta}(u)(T) = (\mathrm{id}_{B(L^2(G))} \otimes u)(\delta_G(T)) = u \cdot T,$$

for any $u \in A(G)$ and $T \in B(L^2(G))$ as noted above (see (4.7)). □

Corollary 4.3.9. *If every operator $T \in B(L^2(G))$ satisfies*

$$T \in \mathrm{Bim}_{L(G)}\{\widehat{\Theta}(u)(T) : \ u \in A(G)\}, \tag{4.8}$$

then $\ker\Theta(J) = \mathrm{Bim}_{L(G)}(J^{\perp})$ *for any closed left ideal J of $L^1(G)$. In particular, if condition (4.8) is satisfied for all $T \in B(L^2(G))$, then $\widetilde{\mathcal{H}}(\Lambda) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda))$ for any family $\Lambda \subseteq M(G)$.*

Proof. Suppose that condition (4.8) is satisfied for any operator in $B(L^2(G))$ and let J be a closed left ideal of $L^1(G)$. If $T \in B(L^2(G))$ satisfies $\widehat{\Theta}(u)(T) \in \mathrm{Bim}_{L(G)}(J^{\perp})$ for all $u \in A(G)$, then clearly $T \in \mathrm{Bim}_{L(G)}(J^{\perp})$ by (4.8). Thus the equivalence of (a) and (c) in Proposition 4.3.8 yields that $\ker\Theta(J) = \mathrm{Bim}_{L(G)}(J^{\perp})$.

Also, if $\Lambda \subseteq M(G)$ and (4.8) holds for any operator in $B(L^2(G))$, then we have

$$\widetilde{\mathcal{H}}(\Lambda) = \ker\Theta(J(\Lambda)) = \mathrm{Bim}_{L(G)}(J(\Lambda)^{\perp}) = \mathrm{Bim}_{L(G)}(\mathcal{H}(\Lambda)).$$

□

Remark 4.3.10. Condition (4.8) means that we can recover the operator $T \in B(L^2(G))$ from its images under the maps $\widehat{\Theta}(u)$ for $u \in A(G)$ by multiplying with elements of $L(G)$ and taking w*-limits and linear combinations.

Note that from Proposition 2.3.14 it follows that if G has the AP, then there exists a net $(u_i)_{i\in I}$ in $A(G)$, such that

$$\widehat{\Theta}(u_i)(T) = u_i \cdot T \longrightarrow T, \text{ ultraweakly for all } T \in B(L^2(G)).$$

This condition is a priori stronger than (4.8). Therefore, Corollary 4.3.9 generalizes [12, Theorem 5.5], which states that $\ker\Theta(J) = \mathrm{Bim}_{L(G)}(J^{\perp})$ for any closed left ideal J of $L^1(G)$ if G has the AP.

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List of symbols

Index of terms