The mean curvature flow of entire graphs

Konstantinos Ganotis
Department of Natural Sciences
School of Mathematics

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Supervisor: **Panagiotis Gianniotis**

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Abstract

In this thesis we present a classic result in the theory of mean curvature flow, due to Ecker-Huisken. In particular, we discuss the behaviour of entire graphs under the mean curvature flow. In their work, Ecker-Huisken proved that, in the case of entire graphs, the flow exists for all time. Furthermore, if the initial graph is asymptotically conical then, after suitable rescaling, the flow converges to a self-similar expanding solution of mean curvature flow with the same asymptotically conical behaviour.

to my uncle
who drew geometry on the sand
and took the forest with him

0 Introduction

A geometric flow is the gradient flow associated to a functional on a manifold which has a geometric interpretation, usually associated with some extrinsic or intrinsic curvature. They can be interpreted as flows on a moduli space (for intrinsic flows) or a parameter space (for extrinsic flows).

These are of fundamental interest in the calculus of variations, and include several famous problems and theories. Particularly interesting are their critical points.

Extrinsic geometric flows are flows on embedded submanifolds, or more generally immersed submanifolds. In general they change both the Riemannian metric and the immersion.

The mean curvature flow $\big(MCF\big)$ is an **extrinsic** geometric flow of hypersurfaces in a Riemannian manifold (for example, smooth surfaces in 3-dimensional Euclidean space). Intuitively, a hypersurface evolves under mean curvature flow if **the normal component of the velocity** of which a point on the surface moves, is given by the **mean curvature of the surface**.

For example, a round **sphere** evolves under mean curvature flow by **shrinking inward uniformly** (since the mean curvature vector of a sphere points inward). Except in special cases, **the mean curvature flow develops singularities**.

The most familiar example of mean curvature flow is in the evolution of soap films. A similar 2-dimensional phenomenon is oil drops on the surface of water, which evolve into disks (circular boundary).

Mean curvature flow is the generalization of **curve-shortening flow** to n-dimensions. In the case of a compact hypersurface and under the restriction that the enclosed volume remains constant (under scaling) we have the **surface tension flow**. MCF **extremalizes surface area**, and *minimal surfaces* are the **critical points for the mean curvature flow**

One important result (due to G. Huisken) is that compact, convex hypersurfaces converge to round points.

In this thesis we are mainly interested in the behaviour, under MCF, of **entire graphs**. The main result is that:

Hypersurfaces that are initially Lipschitz and asymptotically conical converge to ***expanding solitons****.*

In the first chapter we discuss the general theory of MCF and derive MCF as the gradient flow of the area functional.

The second chapter is devoted to the computation of the evolution equations for the basic geometric quantities under MCF.

In the third chapter we begin engaging particularly with entire graphs. We prove the monotonicity formula which is the main tool for estimating quantities such as *the height, the gradient and the norm of the second fundamental form.*

In the fourth chapter we prove that, starting with an embedding which is an entire graph, the solution of MCF exists for all time and the hypersurfaces converge, as $t \to \infty$ to a limit M_∞

In the fifth and final chapter we talk about rescaling. We discuss how the various quantities and operators scale and give the proof of our main theorem.

1 Preliminaries

1.1 Hypersurfaces in the Euclidean space

We examine **n-dimensional manifolds** M immersed (but mostly embedded) **isometrically** in $\mathbb{R}^{n+1}$. These are called *hypersurfaces.*

We will be interchanging between M and φ to denote the *n-dimensional manifold* M immersed in $\mathbb{R}^{n+1}$ by the **immersion** φ:

$$M \xrightarrow[iso]{\varphi} \mathbb{R}^{n+1}$$

Subsequently we will view M ***both as a subset*** of $\mathbb{R}^{n+1}$ with the identification $M \leftrightarrow \varphi(M) \subset \mathbb{R}^{n+1}$ and as a ***n-dimensional Riemannian manifold on its own*** with the metric g induced by the immersion φ.

More accurately

$$(M, g) \underset{isom}{\leftrightarrow} (\varphi(M), \bar{g}|_{\varphi(M)}) \subset (\mathbb{R}^{n+1}, \bar{g})$$

$$g := \varphi^* \bar{g}$$

$$\Leftrightarrow d\varphi(g) = \bar{g}|_M$$

that is

$$\langle X, Y \rangle_M := \langle X, Y \rangle_g \overset{\varphi:}{\underset{isom}{=}} \langle d\varphi(X), d\varphi(Y) \rangle_{d\varphi(g)} = \langle d\varphi(X), d\varphi(Y) \rangle_{\mathbb{R}^{n+1}}$$

for every $X, Y \in TM$. Sometimes we will write $\langle \cdot | \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle$ for the inner product

Identifying TM with its *(isometric)* image under $d\varphi$, we observe that for every point $q = \varphi(p) \in \varphi(M)$ the tangent space of $\mathbb{R}^{n+1}$ at q splits:

$$T_q\mathbb{R}^{n+1} = T_pM \oplus M^{\perp}|_q$$

where $M^{\perp}|_q$ is the one-dimensional subspace spanned by $\nu|_q$ (line through q with direction given by $\overrightarrow{\nu}|_q$, the unit normal at q).

For a $p \in M$ we cant think of the point $\varphi(p, t) = q \in \mathbb{R}^{n+1}$ as a (n+1)-tuple

$$\overrightarrow{\varphi}(p, t) = (y_1(p, t), ..., y_{n+1}(p, t))$$

where the *scalars* y_i are the **coordinates** of the vector $\overrightarrow{\varphi}(t) \in \mathbb{R}^{n+1}$. Here we view $\mathbb{R}^{n+1}$ as a ***vector space***, and O its origin, rather than the *ambient Euclidean space*

We have

$$D\varphi(p,t) := (Dy_1(p,t), ..., Dy_{n+1}(p,t)) \in \mathbb{R}^{n\times(n+1)} \quad \forall t$$

and, after a choice of coordinates for a chart of M

$$D\varphi(x_1,...,x_n,t) = (Dy_1(x_1,...,x_n,t), ..., Dy_{n+1}(x_1,...,x_n,t))$$

that is

$$D\varphi(x_1,...,x_n;t) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & ... & \frac{\partial y_1}{\partial x_n} \\ . & . & & . \\ . & & & . \\ . & & . & . \\ \frac{\partial y_{n+1}}{\partial x_1} & \frac{\partial y_{n+1}}{\partial x_2} & ... & \frac{\partial y_{n+1}}{\partial x_n} \end{pmatrix} : T_pM \mapsto T_q\mathbb{R}^{n+1}$$

Since our objects of study will be families of ***n-dimensional hypersurfaces***
parametrized by t, we require at each t that

$$\dim[\varphi_t(M)] := \mathrm{rank}[D\varphi](\,\cdot\,,t) = n$$

Stated differently: the row vectors $\overrightarrow{\frac{\partial\varphi}{\partial x_i}}$, $\quad i = 1,...,n$ form a basis for the domain of $D\varphi$ which is T_pM and the identification $T_pM = d\varphi(T_pM)$ suggests that

$$T_pM = \mathrm{span}\left\{ \left.\overrightarrow{\frac{\partial\varphi}{\partial x_1}}\right|_p, ..., \left.\overrightarrow{\frac{\partial\varphi}{\partial x_n}}\right|_p \right\}$$

Adding the vector $\nu|_{\varphi(p)}$ we can complete this set to a basis for $T_{\varphi(p)}\mathbb{R}^{n+1}$.

1.2 The extrinsic geometry of a hypersurface in $\mathbb{R}^{n+1}$

Choose a local chart $(x_1, ..., x_n)$ for M. In this chart we have

$$d\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial \varphi}{\partial x_i}$$

which are vector fields of $\mathbb{R}^{n+1}$ tangent to $\varphi(M)$, so
(considering φ to be an embedding)

$$g_{ij} = \left\langle \overrightarrow{\frac{\partial \varphi}{\partial x_i}} \middle| \overrightarrow{\frac{\partial \varphi}{\partial x_j}} \right\rangle_{\mathbb{R}^{n+1}}$$

This will be widely used in the computations later.

We also define the **Second fundamental form** of M

$$A(\cdot, \cdot) = h_{ij} dx^i dx^j$$

where h_{ij} is defined by

$$h_{ij} := \left\langle \nu \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle_{\mathbb{R}^{n+1}}$$

So the second fundamental form is clearly a **symmetric** $(2,0)-$tensor field on M

We express the Riemann curvature tensor of M_t by means of its second fundamental form using the Gauss equation:

$$R_{ijkl} = \left\langle \nabla^2_{ji} \frac{\partial}{\partial x_k} - \nabla^2_{ij} \frac{\partial}{\partial x_k} \middle| \frac{\partial}{\partial x_l} \right\rangle = h_{ik} h_{jl} - h_{il} h_{jk} = A * A$$

And the Codazzi equations give the symmetries of ∇A

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}$$

These imply the important Simons' identity

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{ls} h_{sj} - |A|^2 h_{ij}$$

See [4] for detailed discussion and proofs

We also want the Gauss - Weingarten relations:

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k} + h_{ij}\nu, \quad \frac{\partial}{\partial x_j}\nu = -h_{jl} g^{ls} \frac{\partial \varphi}{\partial x_s} \tag{1}$$

which essentially express the extrinsic geometry of M in terms of the intrinsic geometry of M and the geometry of the ambient space $\mathbb{R}^{n+1}$. In other words the fact that: $\nabla^M = \nabla^{\mathbb{R}^{n+1}} - A\nu$.

We denote
$\mathfrak{X}(M)$: the set of all vector fields tangent to M,
$\overline{\mathfrak{X}}(M)$: the set of all vector fields of $\mathbb{R}^{n+1}$ attached to M

We call the next one "our favourite identity" because we will be using it in great extent:

$$\Delta f g = f \Delta g + g \Delta f + 2 \langle \nabla f, \nabla g \rangle \tag{2}$$

for scalar functions on M and

$$\Delta \langle S, T \rangle = \langle S, \Delta T \rangle + \langle S, \Delta T \rangle + 2 \langle \nabla S, \nabla T \rangle \tag{3}$$

for arbitrary tensors.

Last but not least we will be dealing with quantities that are functions of space and time $f(x_1, ..., x_n; t)$. So we are interested in their spatial as well as their temporal derivatives. We define the **box operator**

$$\Box f := \left(\frac{d}{dt} - \Delta \right) f$$

1.3 First Variation of the Area

Given now an immersion $\varphi : M \to \mathbb{R}^{n+1}$ of a hypersurface in $\mathbb{R}^{n+1}$, we consider the Area functional

$$Area(\varphi) := \int_M d\mu = \int_{\varphi(M)\subset\mathbb{R}^{n+1}} d\mathcal{L}^n$$

where μ is the measure on M and $\mathcal{L}$ is the n-dimensional Lebesgue measure for hypersurfaces of $\mathbb{R}^{n+1}$.

We will analyze the first variation of the Area functional, in other words the first linear approximation of it.

We consider a variation of M as a one-parameter family of immersions

$$\varphi_t : M \to \mathbb{R}^{n+1}$$

with $t \in (-\varepsilon, \varepsilon)$ and $\varphi_0 = \varphi$, such that, **outside of a compact set** $K \subset M$, we have $\varphi_t(p) = \varphi(p)$ for every $t \in (-\varepsilon, \varepsilon)$.

Defining the vector field $\overrightarrow{X := \frac{\partial \varphi_t}{\partial t}}\Big|_{t=0}$ along M namely

$$\frac{\partial \varphi}{\partial t} : \quad M \to \mathbb{R}^{n+1}$$

We see that X is zero outside K. We call such a field the *infinitesimal generator* of the variation φ_t.

Choose now normal coordinates around an arbitrary point p of M and compute

$$\frac{\partial}{\partial t} g_{ij}\Big|_{t=0} = \frac{\partial}{\partial t}\left\langle \frac{\partial \varphi_t}{\partial x_i}\middle| \frac{\partial \varphi_t}{\partial x_j}\right\rangle\Big|_{t=0} = \left\langle \frac{\partial X}{\partial x_i}\middle| \frac{\partial \varphi}{\partial x_j}\right\rangle + \left\langle \frac{\partial \varphi}{\partial x_i}\middle| \frac{\partial X}{\partial x_j}\right\rangle$$

$$= \frac{\partial}{\partial x_i}\left\langle X\middle| \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle \frac{\partial \varphi}{\partial x_i}\middle| X\right\rangle - 2\left\langle X\middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle$$

$$\stackrel{*}{=} \frac{\partial}{\partial x_i}\left\langle X^\top\middle| \frac{\partial \varphi}{\partial x_j}\right\rangle + \frac{\partial}{\partial x_j}\left\langle \frac{\partial \varphi}{\partial x_i}\middle| X^\top\right\rangle - 2\Gamma_{ij}^k\left\langle X^\top\middle| \frac{\partial \varphi}{\partial x_k}\right\rangle - 2h_{ij}\langle X|\nu\rangle \qquad ,$$

where $X^\top$ is the tangent component of the field X, regarded as a vector field of $\mathbb{R}^{n+1}$. In $(*)$ we used the Gauss-Weingarten relations (1).

Letting ω be the 1-form $\omega_X := \langle X^\top | \cdot \rangle$,which acts on a vector field Y like this:

$$\omega_X(Y) := \bar{g}(D\varphi(X^\top), Y) = \langle X^\top | Y \rangle_{\mathbb{R}^{n+1}}$$

this formula can be rewritten as

$$\frac{\partial}{\partial t} g_{ij}\bigg|_{t=0} = \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma^k_{ij}\omega_k - 2h_{ij}\langle X|\nu\rangle = \nabla_i \omega_j + \nabla_j \omega_i - 2h_{ij}\langle X|\nu\rangle .$$

Hence, using the formula: $\partial_t \mathrm{det} A(t) = \mathrm{det} A(t) \cdot \mathrm{Trace}[A^{-1}(t)\partial_t A(t)]$, we get

$$\frac{\partial}{\partial t}\sqrt{\det(g_{ij})}\bigg|_{t=0} = \frac{\sqrt{\det(g_{ij})} g^{ij} \frac{\partial}{\partial t} g_{ij}|_{t=0}}{2}$$

$$= \frac{\sqrt{\det(g_{ij})} g^{ij}(\nabla_i \omega + \nabla_j \omega - 2h_{ij}\langle X|\nu\rangle)}{2}$$

$$= \sqrt{\det(g_{ij})}(\mathrm{div} X^\top - H\langle X|\nu\rangle .$$

If the Area of the immersion φ is finite, the same holds for all the φ_t, as they are **compact deformations** of φ. Assuming that the compact K is contained in a single coordinate chart, we have

$$\frac{\partial}{\partial t}\mathrm{Area}(\varphi_t)\bigg|_{t=0} = \frac{\partial}{\partial t}\int_K d\mu_t\bigg|_{t=0} = \frac{\partial}{\partial t}\int_K \sqrt{\det(g_{ij}(t))}\, d\mathcal{L}^n\bigg|_{t=0}$$

$$= \int_K \frac{\partial}{\partial t}\sqrt{\det(g_{ij}(t))}\bigg|_{t=0} d\mathcal{L}^n$$

$$= \int_K (\mathrm{div} X^\top - H\langle X|\nu\rangle)\sqrt{\det(g_{ij}(t))}\, d\mathcal{L}^n$$

$$= \int_K (\mathrm{div} X^\top - H\langle X|\nu\rangle) d\mu \quad = \quad -\int_K H\langle X|\nu\rangle\, d\mu$$

where we used the fact that X is zero outside K and in the last step we applied the divergence theorem for $X^\top$(or we could just ignore it as tangential perturbations of φ as they don't change the geometric picture of the immersion).

Notice that all the integrals are well defined because we are actually integrating on the compact set K.

If K is contained in several charts, the same conclusion follows from a standard argument using a partition of unity.

Proposition 1.1. *The first variation of the Area functional depends only on the **normal component** of the infinitesimal generator $X = \frac{\partial \varphi_t}{\partial t}\big|_{t=0}$ of the variation φ_t, precisely:*

$$\frac{\partial}{\partial t} Area(\varphi_t) = -\int_M H \langle X|\nu\rangle \, d\mu.$$

Clearly the dependence is linear:

If ψ_t is a family of embeddings such that

$$\frac{\partial \psi_t}{\partial t} = X + \lambda Y$$

and φ_t^X and φ_t^Y are families of embeddings with infinitesimal generators X and Yrespectively, then

$$\frac{\partial}{\partial t}\text{Area}(\psi_t) = -\int_M H \langle X + \lambda Y|\nu\rangle \, d\mu = \frac{\partial}{\partial t}\text{Area}(\varphi_t^X) + \lambda \frac{\partial}{\partial t}\text{Area}(\varphi_t^Y)$$

This can be: $\psi_t(p) = \psi(p) + t(X|_p + \lambda Y|_p)$ considering $\psi_0(p)$ a point in $\mathbb{R}^{n+1}$ and $X|_p + \lambda Y|_p$ a vector starting at $\psi(p)$. For M_0 smooth and t small enough we can be sure that ψ_t remains a smooth immersion.

Summarizing:

$$\text{Area}(\cdot) : \{\varphi_t | \text{1-parameter families of } M \underset{iso}{\hookrightarrow} \mathbb{R}^{n+1}\} \to \mathbb{R}$$

$$\frac{\partial}{\partial t}\text{Area}(\cdot) : \{X : \text{vector fields along M}\} \to \mathbb{R}$$

Thus the ***linearization*** of the ***nonlinear*** Area functional is given by Proposition 1.1 and in view of

$$\frac{\partial}{\partial t}\text{Area}(\cdot) = -\int_M H \left\langle \frac{\partial}{\partial t}\Big(\cdot\Big) \Big| \nu \right\rangle d\mu$$

we can interpret the **linear** functional

$$\mathcal{F} : X \mapsto -\int_M H \langle \nu|X\rangle \, d\mu = -\langle H\nu|X\rangle_{L^2(M)}$$

as the "gradient" of Area:

$$\nabla\text{Area}(\cdot) := \mathcal{F}(\cdot) = -\langle H\nu|\cdot\rangle_{L^2(M)}$$

The last equation describes this **linear functional** as the inner product with the vector field $-H\nu$ which (as a vector field along M) is an element of the same domain. Regarding $\mathcal{F}$ as a covector field we see that $\mathcal{F}$ and $-H\nu$ are ***dual*** and they act on vector fields in the same way. Now, as $\mathcal{F}$ is the gradient of the Area functional, it must indicate the direction of ***steepest decay***. So, when looking for the *perturbation that* ***decreases*** *the area the most*, we should consider the ones with infinitesimal generator given by the field

$$-(\nabla Area)^* = -\mathcal{F}^* = \langle H\nu| \cdot \rangle^* = H\nu$$

This particular vector filed is determined **exclusively by geometric properties** of our the hypersurface and specifically on the **extrinsic** ones: the **mean curvature** H and the **unit normal** ν. So the flow it generates should ***not depend on the parametrization*** of the surface at any time.

Remark 1. *Two hypersurfaces that are* ***reparametrizations of one another*** *have the same area (globally and locally). That means they* ***give rise to the same Area functional*** *and hence to the* ***same first variation of it***.

We can consider the motion of a hypersurface by minus this gradient, that is, the ***mean curvature flow*** (next section). So, one looks at hypersurfaces moving with velocity $\left(\frac{\partial}{\partial t}\varphi_t\right)$ equal to $H\nu$ at every point. This means choosing, among all the velocity functions with fixed $L^2(\mu)$-norm equal to $\left(\int_M H^2 d\mu\right)^{1/2}$, the one such that the Area decreases most rapidly.

The above analysis gives an immediate characterization of the critical points of the Area functional:

$$\nabla \text{Area}(\cdot) = 0 \Leftrightarrow \int_M H \langle X|\nu\rangle \, d\mu = 0$$

for every field X with compact support $\Leftrightarrow H = 0$ everywhere on M.

This is the well known definition of the so called *minimal surfaces*.

A **second variation** formula should indicate the direction in which $\nabla\text{Area}(\cdot)$ decreases and therefore Area approaches a critical point.

The Mean Curvature Flow

Definition 1. *Let M^n be a hypersurface in $\mathbb{R}^{n+1}$. The* ***mean curvature flow*** *of M is a family of immersions*

$$\varphi_t : M \to \mathbb{R}^{n+1}, \quad t \in [0, T)$$

satisfying the equation:

$$\frac{d}{dt}\varphi(p;t) = H(p;t)\nu|_{(p;t)} \tag{4}$$

Recall that $H = \text{trace}(h_{ij})$ and $h_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle$ so, even if ν is defined up to a sign, the field $\overrightarrow{H} := H(p,t)\overrightarrow{\nu}(p,t)$ is independent of such choice. (The declaration of a direction for the unit normal decides the sign of the mean curvature scalar)

The equation (4) can be regarded either as a ***geometric equation*** in the **(n+1)-*dimensional*** Euclidean space or a ***system of n+1 scalar equations***

1.4 Mean curvature flow as the gradient flow of the Area functional

The kernel of $\mathcal{F}$ is precisely the subspace of vector fields that are **everywhere perpendicular to** ν. These are the fields **everywhere tangent to** M

$$\ker\mathcal{F} = \{X \in \overline{\mathfrak{X}}(M) : X \perp \nu\} := \overline{\mathfrak{X}}^{\top}(M)$$

The orthogonal complement of the kernel [in $\overline{\mathfrak{X}}(M)$] forms the set

$$\overline{\mathfrak{X}}^{\perp}(M) := \{X \in \overline{\mathfrak{X}}(M) : X \parallel \nu\} = \{X \in \overline{\mathfrak{X}}(M) : X = f \cdot \nu\}$$

which can be identified with the set of all scalar functions on M

$$\{f \cdot \overrightarrow{\nu} \mid f : M \to \mathbb{R}\} \underset{\langle \cdot | \nu \rangle}{\leftrightarrow} \{f : M \to \mathbb{R}\}$$

So starting with an initial hypersurface and a 1-parameter family of deformations φ_t with infinitesimal generator X_t we can compute the corresponding infinitesimal variation of the Area

$$\mathcal{F}(X_t) = -\langle H\nu|\, X\rangle_{L^2(M)} = -\left\langle H\nu|\, X^\top\right\rangle_{L^2(M)} = -\langle H\nu|\, f\nu\rangle_{L^2(M)}$$

$$= -\int_M Hf d\mu_t$$

where $f(t;p) = \langle X_t|\nu\rangle \left(= \left\langle X_t|_{\varphi(p)}, \nu|_{\varphi(p)}\right\rangle_{T_p\mathbb{R}^{n+1}}\right)$ is the normal component of X_t and the only one that interests us because the others lie in ker$\mathcal{F}$.

We are interested in the **geometric** properties of the evolving hypersurfaces, i.e. the properties that remain invariant under reparametrizations. Here we prove that the flow is indeed a **geometric flow** (immersions that differ by a reparametrization give the same flow).

Invariance under Tangential Perturbations

Proposition 1.2. *If a family of immersions $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ satisfies the system of PDEs*

$$\frac{\partial}{\partial t}\varphi(p,t) = H(p,t)\nu|_{(p,t)} + X(p,t)$$

with initial conditions

$$\varphi(p,0) = \varphi_0(p)$$

*where X is a time-dependent vector field of $\mathbb{R}^{n+1}$ **tangent** on M, more formally:*

$$X(p,t) \in D\varphi_t(T_pM) \subset T_{\varphi_t(p)}\mathbb{R}^{n+1} \quad \forall t \in [0,T), \forall p \in M$$

*Then, there exists a family ψ_t of reparametrizations (infinitesimally generated by the **tangent** field X) of the immersions φ_t which is satisfies the equation of mean curvature flow.*

A proof can be found in [3]. We only want to mention that the above flow differs to the usual mean curvature flow only by a $X_t \in TM_t = \ker\mathcal{F}$ therefore induce the same variation to the Area functional and, hence, the same geometric picture.

2 Evolution of geometric quantities

2.1 Evolution equations

Here we derive the equations describing how the basic *geometric quantities* **evolve**. We assume that φ is an embedding of the smooth hypersurface M into $\mathbb{R}^{n+1}$ and that M moves by the mean curvature flow.

Lemma 2.1. *The evolution equations for* g, ν, Γ_{ij}^k, $A = h_{ij}$, *and* H *are*

- $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$
- $\frac{\partial}{\partial t} g^{ij} = 2Hh^{ij}$
- $\frac{\partial}{\partial t} \nu = -\nabla H$
- $\frac{\partial}{\partial t} \Gamma_{ij}^k = \nabla H * A + H * \nabla A = \nabla A * A$

Proof. The first one has been computed in Section 1.2, just substitute X with $H\nu$ For the second, take

$$
\begin{aligned}
0 = \frac{\partial}{\partial t}\delta_i^j &= \frac{\partial}{\partial t} g_{is} g^{sj} \\
&= g_{is} \frac{\partial}{\partial t} g^{sj} + g^{sj} \frac{\partial}{\partial t} g_{is} \\
&= g_{is} \frac{\partial}{\partial t} g^{sj} - 2Hh_{is} g^{sj} \\
&= g_{is} \frac{\partial}{\partial t} g^{sj} - 2Hh_i^j \\
\Rightarrow \frac{\partial}{\partial t} g^{sj} &= 2g^{is} H h_i^j = 2Hh^{js}.
\end{aligned}
$$

Evolution of the unit normal comes from:

$$\begin{aligned}
\left\langle \frac{\partial \nu}{\partial t} \middle| \frac{\partial \varphi}{\partial x_i} \right\rangle &= \frac{\partial}{\partial t} \cancel{\left\langle \nu \middle| \frac{\partial \varphi}{\partial x_i} \right\rangle} - \left\langle \nu \middle| \frac{\partial}{\partial t} \frac{\partial \varphi}{\partial x_i} \right\rangle \\
&= - \left\langle \nu \middle| \frac{\partial}{\partial x_i} \frac{\partial \varphi}{\partial t} \right\rangle \\
&= - \left\langle \nu \middle| \frac{\partial}{\partial x_i} (H\nu) \right\rangle \\
&= - \frac{\partial H}{\partial x_i} - H \cancel{\left\langle \nu \middle| \frac{\partial \nu}{\partial x_i} \right\rangle}
\end{aligned}$$

as $\langle \nu | \nu \rangle = 1$ and

$$\begin{aligned}
\left\langle \nu \middle| \frac{\partial \nu}{\partial x_i} \right\rangle &= \cancel{\frac{\partial}{\partial x_i} \langle \nu | \nu \rangle} - \left\langle \frac{\partial \nu}{\partial x_i} \middle| \nu \right\rangle \\
\Rightarrow 2 \left\langle \nu \middle| \frac{\partial \nu}{\partial x_i} \right\rangle &= 0
\end{aligned}$$

and because

$$\begin{aligned}
\left\langle \nu \middle| \frac{\partial \nu}{\partial t} \right\rangle &= \cancel{\frac{\partial}{\partial t} \langle \nu | \nu \rangle} - \left\langle \frac{\partial \nu}{\partial t} \middle| \nu \right\rangle \\
\Rightarrow 2 \left\langle \nu \middle| \frac{\partial \nu}{\partial t} \right\rangle &= 0
\end{aligned}$$

we conclude that $\frac{\partial \nu}{\partial t} \in M^{\top}$. So $\frac{\partial \nu}{\partial t} = -\nabla H$

Now hang on for the terrible computation of the evolution of Christoffel symbols:

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}g_{kl} + \frac{\partial}{\partial x_k}\frac{\partial}{\partial t}g_{jl} - \frac{\partial}{\partial x_l}\frac{\partial}{\partial t}g_{jk}\right\} \\
&+ \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} - \frac{\partial}{\partial x_l}g_{jk}\right\} \\
&= \frac{1}{2}g^{il}\left\{\nabla_j\frac{\partial}{\partial t}g_{kl} + \nabla_k\frac{\partial}{\partial t}g_{jl} - \nabla_l\frac{\partial}{\partial t}g_{jk}\right\} \\
&+ \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial t}g_{kz}\Gamma^z_{jl} + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl}\right. \\
&\quad \left. + \frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - \frac{\partial}{\partial t}g_{jz}\Gamma^z_{kl} - \frac{\partial}{\partial t}g_{jz}\Gamma^z_{jl}\right\} \\
&- \frac{1}{2}g^{is}\frac{\partial}{\partial t}g_{sz}g^{lz}\left\{\frac{\partial}{\partial x_j}g_{kl} + \frac{\partial}{\partial x_k}g_{jl} + \frac{\partial}{\partial x_l}g_{jk}\right\} \\
&= \frac{1}{2}g^{il}\left\{\nabla_j\frac{\partial}{\partial t}g_{kl} + \nabla_k\frac{\partial}{\partial t}g_{jl} - \nabla_l\frac{\partial}{\partial t}g_{jk}\right\} \\
&+ g^{il}\frac{\partial}{\partial t}g_{lz}\Gamma^z_{jk} - g^{is}\frac{\partial}{\partial t}g_{sz}\Gamma^z_{jk} \\
&= \frac{1}{2}g^{il}\left\{\nabla_j\frac{\partial}{\partial t}g_{kl} + \nabla_k\frac{\partial}{\partial t}g_{jl} - \nabla_l\frac{\partial}{\partial t}g_{jk}\right\} \\
&= g^{il}\{\nabla_j(Hh_{kl}) + \nabla_k(Hh_{jl}) - \nabla_l(Hh_{jk})\} \\
&= -h^i_k\nabla_j H - h^i_j\nabla_k H + h_{jk}\nabla^i H - H(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk} \\
&= A * \nabla H + H * \nabla A \\
&= A * \nabla A
\end{aligned}$$

□

Lemma 2.2. *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{ls}h_{sj} + |A|^2 h_{ij}.$$

It follows that

$$\frac{\partial}{\partial t}h_i^j = \Delta h_i^j + |A|^2 h_i^j,$$

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$$

and

$$\frac{\partial}{\partial t}H = \Delta H + H|A|^2$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \nu \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&= \left\langle \nu \middle| \frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)\right\rangle + \left\langle \frac{\partial}{\partial t}\nu \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&\overset{(1)}{=} \left\langle \nu \middle| \frac{\partial^2}{\partial x_i x_j}(H\nu)\right\rangle - \left\langle \nabla H \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&\overset{(2)}{=} \frac{\partial^2 H}{\partial x_i x_j}\langle \nu|\nu\rangle + H\left\langle \nu \middle| \frac{\partial^2}{\partial x_i x_j}\nu\right\rangle - \left\langle \nabla H \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle \\
&\overset{(3)}{=} \frac{\partial^2 H}{\partial x_i x_j} - H\left\langle \nu \middle| \frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial \varphi}{\partial x_s}\right)\right\rangle - \left\langle \frac{\partial H}{\partial x_l}\underbrace{\frac{\partial \varphi}{\partial x_s}}_{\in M^\top} g^{ls} \middle| \Gamma_{ij}^k \frac{\partial \varphi}{\partial x_k} + \cancel{h_{ij}\nu}^{\perp}\right\rangle \\
&\overset{(4)}{=} \frac{\partial^2 H}{\partial x_i x_j} - Hh_{il}g^{ls}\left\langle \nu \middle| \frac{\partial^2 \varphi}{\partial x_i \partial x_s}\right\rangle - \Gamma_{ij}^k \frac{\partial H}{\partial x_k} \\
&= \nabla_i \nabla_j H - Hh_{il}g^{ls}h_{sj}.
\end{aligned}$$

∎

(1):Exchange of derivatives, $\partial_t \nu = -\nabla H$, MCF
(2):$\left\langle \frac{\partial}{\partial x_i}\nu|\nu\right\rangle = \cancel{\frac{\partial}{\partial x_i}\langle\nu|\nu\rangle} - \left\langle \nu|\frac{\partial}{\partial x_i}\nu\right\rangle \Rightarrow 2\left\langle \frac{\partial}{\partial x_i}\nu|\nu\right\rangle = 0$
(3):Gauss - Weingarten again, for ∇H lower the index of ∂_{x_s}
(4): $\left\langle \nu, (...)\frac{\partial \varphi}{\partial x_s}\right\rangle = 0$ and definition of h_{ij}

There is Simmons' identity for the Laplacian of A in coordinates:

$$\Delta h_{ij} = \nabla_i \nabla_j H + Hh_{il}g^{ls}h_{sj} - |A|^2 h_{ij}$$

Using this we easily get

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{ls}h_{sj} + |A|^2 h_{ij}.$$

The other equations follow from straightforward computations, since

$$\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij}$$

□

2.2 Relations between ∂_t and ∇

We compute the commutator of spatial and temporal derivatives

Lemma 2.3. *Temporal and spatial derivatives of a tensor T* ***do not commute****. Instead we have the formula*

$$\frac{\partial}{\partial t}\nabla T = \nabla \frac{\partial}{\partial t}T + T * A * \nabla A \tag{5}$$

Proof. We prove the lemma for a covariant tensor $T = T_{i_1 \dots i_k}$. The general case is analogous, as it will be clear by the following computation:

$$\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} = \frac{\partial}{\partial t}\left(\frac{\partial T_{i_1\dots i_k}}{\partial x_j} - \sum_{s=1}^{k} \Gamma^l_{ji_s} T_{i_1\dots i_{s-1},l,i_{s+1}\dots i_k}\right)$$

$$= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1},l,i_{s+1}\dots i_k}}{\partial t} - \sum_{s=1}^{k}\frac{\partial\left(\Gamma^l_{ji_s}\right)}{\partial t}T_{i_1\dots i_{s-1},l,i_{s+1}\dots i_k}$$

$$= \nabla_j \frac{\partial}{\partial t}T_{i_1\dots i_k} - \sum_{s=1}^{k}\left(A * \nabla A\right)^l_{ji_s} T_{i_1\dots i_{s-1},l,i_{s+1}\dots i_k}$$

which is the formula we wanted.

In the last equation we just used the formula for the time derivative of the Christoffel symbols. □

For the k'th covariant derivative of the (particularly interesting tensor) **second fundamental form** we have the following formula

Proposition 2.4. *[A formula for $\left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k A$]*

$$\frac{\partial}{\partial t}\nabla^k A = \Delta\nabla^k A + \sum_{p+q+r=k}\nabla^p A * \nabla^q A * \nabla^r A$$

Proof. We will do induction on k. We already saw the case $k = 0$ in the elementary evolution equations. Now for the induction step suppose our formula is true for $k-1$. We have, by the previous lemma:

$$
\begin{aligned}
\frac{\partial}{\partial t}\nabla^k A &= \nabla \frac{\partial}{\partial t}\nabla^{k-1}A + \nabla^{k-1}A * \nabla A * A \\
&= \nabla\left(\Delta\nabla^{k-1}A + \sum_{p+q+r=k-1} \nabla^p A * \nabla^q A * \nabla^r A\right) + \nabla^{k-1}A * \nabla A * A \\
&= \nabla\Delta\nabla^{k-1}A + \sum_{p+q+r=k} \nabla^p A * \nabla^q A * \nabla^r A.
\end{aligned}
$$

It would be nice if $[\nabla, \Delta]$ was zero but in general this is not the case since for any tensor T

$$[\nabla, \Delta]T \;=\; Rm * \nabla T + \nabla(Rm * T) \;=\; Rm * \nabla T + \nabla Rm * T$$

Explicitly

$$
\begin{aligned}
\nabla_k \Delta T - \Delta \nabla_k T &= g^{ij}(\nabla_k\nabla_i\nabla_j T - \nabla_i\nabla_j\nabla_k T) \\
&= g^{ij}([\nabla_k, \nabla_i]\nabla_j T + \nabla_i\nabla_k\nabla_j T - \nabla_i\nabla_j\nabla_k T) \\
&= g^{ij}([\nabla_k, \nabla_i]\nabla_j T + \nabla_i([\nabla_k, \nabla_j]T)).
\end{aligned}
$$

Recalling though that $Rm = A * A$, which is a form of the Gauss - Codazzi equations, we can see that

$$[\nabla, \Delta]\nabla^{k-1}A = A * A * \nabla\nabla^{k-1}A + \nabla(A * A) * \nabla^{k-1}A$$

So all extra terms are of the form $A * A * \nabla^k A$ and $A * \nabla A * \nabla^{k-1} A$ and fall in the summation. □

Proposition 2.5. *[Evolution of $|\nabla^k A|$] The following formula holds*

$$\frac{\partial}{\partial t}|\nabla^k A|^2 = \Delta|\nabla^k A|^2 - 2|\nabla^{k+1}A|^2 + \sum_{p+q+r=k} \nabla^p A * \nabla^q A * \nabla^r A * \nabla^k A \quad (6)$$

Proof. We compute

$$
\begin{aligned}
&\tfrac{\partial}{\partial t} g(\nabla^k A, \nabla^k A) = 2g(\nabla^k A, \tfrac{\partial}{\partial t}\nabla^k A) + \frac{\partial g}{\partial t} * \nabla^k A * \nabla^k A \\
&= 2g\left(\nabla^k A,\ \Delta \nabla^k A + \sum_{p+q+r=k} \nabla^p A * \nabla^q A * \nabla^r A\right) + (A * A) * \nabla^k A * \nabla^k A \\
&= 2g\left(\nabla^k A, \Delta \nabla^k A\right) + \sum_{p+q+r=k} 2g\left(\nabla^k A, \nabla^p A * \nabla^q A * \nabla^r A\right) + A * A * \nabla^k A * \nabla^k A \\
&= \Delta |\nabla^k A|^2 - 2|\nabla^{k+1} A|^2 + \sum_{p+q+r=k} \nabla^k A * \nabla^p A * \nabla^q A * \nabla^r A
\end{aligned}
$$

In the last equation we used that

$$\Delta \langle S, S \rangle = 2 \langle S, \Delta S \rangle + 2|\nabla S|^2$$

which we discuss in the appendix, and the fact that both

$$\sum_{p+q+r=k} g\left(\nabla^k A, \nabla^p A * \nabla^q A * \nabla^r A\right) \text{ and } A * A * \nabla^k A * \nabla^k A$$

fall in the sum

$$\sum_{p+q+r=k} \nabla^k A * \nabla^p A * \nabla^q A * \nabla^r A$$

□

If in this formula we substitute $k = 0$ we get

$$\frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 \leq \Delta |A|^2 + 2|A|^4 \tag{7}$$

Note the term of order four in the right side. We will soon find out that this is annoying and we will come up with a way to neutralize it.

Proposition 2.6. *If the second fundamental form of a* ***closed, compact hypersurface*** *is bounded up to time $T < \infty$, then all its covariant derivatives are also bounded up to time T.*

Proof. From the previous proposition (2.5) we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k A|^2 &= \Delta|\nabla^k A|^2 - |\nabla^{k+1}A|^2 + \sum_{p+q+r=k} \nabla^p A * \nabla^q A * \nabla^r A * \nabla^k A \\ &\leq \Delta|\nabla^k A|^2 + |\nabla^k A|^2 \cdot \mathcal{P}(|A|, |\nabla A|, ..., |\nabla^{k-1}A|) \\ &\qquad + \mathcal{Q}(|A|, |\nabla A|, ..., |\nabla^{k-1}A|),\end{aligned}$$

where $\mathcal{P}, \mathcal{Q}$ are polynomials on $|A|, |\nabla A|, ...$ **up to order** $(k-1)$.

This is because in the terms $\nabla^k A * \nabla^p A * \nabla^q A * \nabla^r A$ there can only be up to two occurrences of $\nabla^k A$. We examine the two cases.

• If there are two, let's say $p = k$, then it has to be $q = r = 0$ and we estimate

$$|A * A * \nabla^k A * \nabla^k A| \leq |A|^2 \cdot |\nabla^k A|^2.$$

• If there is only one we use the inequality

$$|S * T| \leq |S| \cdot |T| \leq \frac{|S|^2 + |T|^2}{2}$$

to get

$$\begin{aligned}|\nabla^k A * \nabla^p A * \nabla^q A * \nabla^r A| &\leq |\nabla^k A| \cdot |\nabla^p A * \nabla^q A * \nabla^r A| \\ &\leq \frac{|\nabla^k A|^2}{2} + \frac{|\nabla^p A * \nabla^q A * \nabla^r A|^2}{2}\end{aligned}$$

We will, proceed, once more, with induction on k, with equation (7) being the case $k = 0$.

For the induction, we assume that all the covariant derivatives of A up to order $k-1$ are bounded up to time T, so the polynomials $\mathcal{P}$ and $\mathcal{Q}$ are also bounded, say by the (nonnegative numbers) C and D respectively. Thus

$$\frac{\partial}{\partial t}|\nabla^k A|^2 \leq \Delta|\nabla^k A|^2 + C|\nabla^k A|^2 + D$$

By the maximum principle, this implies

$$\frac{d}{dt}|\nabla^k A|^2_{max} \leq C|\nabla^k A|^2_{max} + D \tag{8}$$

For any time t, the quantity

$$|\nabla^k A|_{max} := \max_{p \in M_t} |\nabla^k A(p;\cdot)|$$

is a **time-dependent** function. We can call

$$u(t) := |\nabla^k A|_{max} : [0,T) \to \mathbb{R}$$

Then (8) becomes the *ordinary differential inequality* $u' \leq Cu + D$ imposing an **exponential bound** on u. Integrating now over the (finite) interval $[0,T)$ implies that u is bounded up to time T. □

A proof for the maximum principle can be found in [3].
Later in Section 3 we prove an **alternative for entire graphs** which will be our basic tool as the **ordinary maximum principle** is only applicable in the **compact** case.

3 Estimates for entire graphs

Our main object of study will be **entire graph hypersurfaces**. Because these are non-compact, the standard maximum principle doesn't work and we need to develop another method, the **Monotonicity formula**. We then proceed with some estimates that will be important in the next chapters. We estimate the **height, "gradient"** and all the derivatives of the **curvature** ($|\nabla^m A|$) and obtain bounds for all time.

3.1 Entire graphs

We come now to our main object of study, which is **entire graphs**. These are hypersurfaces that are graphs of functions $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined in the entire $\mathbb{R}^n$.

Fix a unit vector $\omega \in \mathbb{R}^{n+1}$ and a hyperplane Π orthogonal to ω. We declare an **origin** O on Π. We define the ***height*** of M_t with respect to the hyperplane Π by

$$u = \langle y, \omega \rangle$$

Remark 2. *We want to clarify that u and y_{n+1} differ by a choice of coordinates of $\mathbb{R}^{n+1}$*

$$u = u(x) = \langle \overrightarrow{\varphi}(p), \omega \rangle = \langle (y_1, ..., y_{n+1}) | (\omega_1, ..., \omega_{n+1}) \rangle$$

which in general is different from y_{n+1}. So, working in coordinates we will have

$$M \ni p = x \overset{\varphi}{\mapsto} y = \varphi(x) \in \mathbb{R}^{n+1}$$

where x and y are interpreted as coordinates.
*We will be using u instead of the last coordinate of φ in the y_i- base of $\mathbb{R}^{n+1}$ because it is **geometrically defined** and hence **coordinate invariant**.*

In the following we shall identify the image $\varphi(p, t)$ of a point $p \in M$ and its coordinate vector $y = \varphi(p, t)$.

3.2 The Monotonicity formula

Our basic tool will be the Monotonicity formula. We are forced to develop such a technology because the classic Maximum principle doesn't apply on non-compact domains.

For a fixed point $(y_0, t_0) \in \mathbb{R}^{n+1}$ we define the "backward heat kernel" $\rho = \rho(x, t)$ by

$$\rho(x,t) = (4\pi(t_0 - t))^{-n/2}\, exp\left(\frac{-|y_0 - y|^2}{4(t_0 - t)}\right), \quad t > t_0,$$

$y = \varphi(x)$ such that

$$\frac{d}{dt}\rho = -\Delta\rho + \rho \cdot \left(\frac{\langle y_0 - y, H\rangle}{(t_0 - t)} - \frac{1}{4}\frac{|(y_0 - y)^{\perp}|^2}{t_0 - t)^2}\right)$$

One can find in [3] how this implies the monotonicity formula

$$\frac{d}{dt}\int_{M_t} \rho\, d\mu_t = -\int_{M_t} \rho \left| H + \frac{1}{2\tau}(y - y_0)^{\perp}\right| d\mu_\tau. \tag{9}$$

where $d\mu_t$ is the measure on M_t and $\tau = t_0 - t$. Proceeding as in [2] or [3] we obtain more generally for a function $f = f(x, t)$ on M that

$$\frac{d}{dt}\int_{M_t} f\rho\, d\mu_t = \int_{M_t}\left(\frac{d}{dt}f - \Delta f\right)\rho\, d\mu_t - \int_{M_t} f\rho\left|H + \frac{1}{2\tau}(y - y_0)^{\perp}\right|^2 d\mu_\tau. \tag{10}$$

Remark 3. *If $\frac{d}{dt}\int_M \rho = 0$ then the manifold shrinks to y_0. In fact, in this situation we have a* ***shrinking soliton****.*

All integrals are finite and integration by parts is permitted for the surfaces and functions we are going to consider in the sequel.

Corollary 3.1. *Suppose the function $f = f(x, t)$ satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) f \leq \overrightarrow{\alpha} \cdot \overrightarrow{\nabla f} \tag{11}$$

for some vector field α, where ∇ denotes the tangential gradient on M. If $\alpha_0 := \sup_{M\times[0,t_1]} |\alpha| < \infty$ for some $t_1 > 0$, then

$$\sup_{M_t} f \leq \sup_{M_0} f$$

for all $t \in [0, t_1]$.

Proof. Let $k := \sup_{M_0} f$ and define $f_k(\cdot, t) = \max\Big(0, f(\cdot, t) - k\Big)$. We aim to prove that $\forall t \in [0, t_1] :\ f_k(\cdot, t) = 0$, so that

$$f(\cdot, t) \leq k = \sup_{M_0} f,\ \forall t \in [0, t_1]$$

which is what we want.

Note that, by definition,

$$f_k(p, 0) = \max\Big(0, f(p, 0) - k\Big) = 0 \quad \forall p \in M_0$$

since $k := \sup_{M_0} \geq f(p, 0) \quad \forall p \in M_0$. So we have the result for $t = 0$ and we want to "push it" through time.

From (11) and our favourite identity (2) for f_k^2, we derive

$$\left(\frac{d}{dt} - \Delta\right) f_k^2 \leq 2 f_k \overrightarrow{\alpha} \cdot \overrightarrow{\nabla f_k} - 2|\nabla f_k|^2.$$

Using Young's inequality (with $q = \sqrt{2}$) we obtain that

$$2 f_k \alpha \ \cdot\ \nabla f_k \ \leq\ \left(\frac{1}{\sqrt{2}}\right)^2 f_k^2 |\alpha|^2 \ +\ (\sqrt{2})^2 |\nabla f_k|^2 \ \leq\ \frac{1}{2} f_k^2 \alpha_0^2 \ +\ 2|\nabla f_k|^2$$

$$\left(\frac{d}{dt} - \Delta\right) f_k^2 \leq \frac{1}{2} \alpha_0^2 f_k^2.$$

We may now employ (10) with f_k^2 instead of f and choose $t_0 > t$, and x_0 arbitrary in the definition of ρ to conclude

$$\frac{d}{dt} \int f_k^2 \rho \, d\mu_t \leq \frac{1}{2} \alpha_0^2 \int f_k^2 \rho \, d\mu_t$$

since $f_k^2 \rho \geq 0$. If we now call

$$F(t) := \int f_k^2 \rho \, d\mu_t$$

the previous inequality yields the *ordinary differential inequality*

$$F'(t) \leq \frac{1}{2} \alpha_0^2 F(t)$$

which, integrated over $[0, t]$ for any t in the **finite** interval $[0, t_1]$ yields

$$F(t) \leq F(0)e^{ct}$$

But

$$f(\cdot, 0) = 0 \Rightarrow F(0) = \int_{M_0} f_k(\cdot, 0)^2 \rho \, d\mu_t = 0$$

and as F is non-negative it must be zero for all time implying

$$\int_{M_t} f_k(\cdot, t)^2 \rho \, d\mu_t = 0$$

this completes the proof since, by definition, $\rho > 0$. □

3.3 Height estimates

We need to ensure that our manifold doesn't escape to infinity.

Recall that the *height* of M_t is defined by

$$u = \langle y, \omega \rangle$$

and observe

$$\Delta\varphi = \Delta y = H\nu \Rightarrow \Delta \langle y, \omega \rangle = \langle H\nu, \omega \rangle$$

Since ω is fixed in both time and space we have

$$\left(\frac{d}{dt} - \Delta\right) u = 0 \tag{12}$$

Lemma 3.2. *i) The function $\eta_1 = \eta_1(x,t) := |y|^2 + 2nt$ satisfies*

$$\eta_1 = \left(\frac{d}{dt} - \Delta\right) \eta_1 = 0$$

ii) The function η_2 defined by

$$\eta_2(x,t) := 1 + |y|^2 - u^2 + 2nt$$

satisfies for any $p > 0$

$$\left(\frac{d}{dt} - \Delta\right) \eta_2^2 = -p(p-1)|\nabla\eta_2|^2\eta_2^{p-2} + 2p\eta_2^{p-1}|\nabla u|^2$$

Proof. Because of mean curvature flow we have

$$\frac{d}{dt}\eta_1 = 2\langle y, H\nu \rangle + 2n$$

and the first identity then follows from

$$\Delta\eta_1 = 2\overrightarrow{\varphi} \cdot \overrightarrow{\Delta\varphi} + 2|\nabla\varphi|^2 = 2\langle y, H\nu \rangle + 2n$$

because

$$\begin{aligned} |\nabla\varphi|^2 = g(\nabla\varphi, \nabla\varphi) &= g\left(\frac{\partial\varphi}{\partial x_i}dx^i, \frac{\partial\varphi}{\partial x_j}dx^j\right) \\ &= g\left(\frac{\partial}{\partial x_i} \cdot dx^i, \frac{\partial}{\partial x_j} \cdot dx^j\right) = g_{ij}g^{ij} = tr(g) = n \end{aligned}$$

Now by (12) and our favourite identity (2)

$$\left(\frac{d}{dt} - \Delta\right)u^2 = -2|\nabla u|^2$$

we have using (i)

$$\left(\frac{d}{dt} - \Delta\right)\eta_2 = 2|\nabla u|^2 \tag{13}$$

Now we use a more general form of our favourite identity (2) that is

$$\begin{aligned}\Delta f^p = \operatorname{Tr}(\nabla^2 f^p) = \operatorname{Tr}\Big(\nabla(pf^{p-1}\nabla f)\Big)\\ = \operatorname{Tr}\Big(p(p-1)f^{p-2}\nabla f \otimes \nabla f + pf^{p-1}\nabla^2 f\Big)\\ = p(p-1)f^{p-2}|\nabla f|^2 + pf^{p-1}\Delta f\end{aligned} \tag{14}$$

we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)f^p = \frac{d}{dt}f^p - \Delta f^p\\ = pf^{p-1}\frac{d}{dt}f - p(p-1)f^{p-2}|\nabla f|^2 - pf^{p-1}\Delta f\\ = -p(p-1)|\nabla f|^2 f^{p-2} + pf^{p-1}\left(\frac{d}{dt}f - \Delta f\right)\end{aligned}$$

where substituting f with η_2 and using (13) we have the result. □

Our motivation for the above choices for η_1 and η_2 comes from the observation that $\sqrt{|y|^2 - u^2}$ measures the distance $dist(O, x)$ where $x \in \Pi$ above which $y = \varphi(x) \in M \subset \mathbb{R}^{n+1}$ is graphed.
From now on we will call by the same symbol both $x \in M$ and $y = \varphi(x) \in \varphi(M)$. It is totally confusing but it is used in the literature.

Definition 2. *We say that some quantity $\mathcal{Q}$ "grows at most polynomially in space" with degree p if the $\mathcal{Q} \leq C\left(\sqrt{|y|^2 - u^2}\right)^p$ for some $C \geq 0$*

Now we assume that $u(\cdot, 0)$ grows at most polynomially in space and we show that $u(\cdot, t)$ satisfies the same polynomial growth estimate.

$$M_0 \text{ grows at most polynomially} \;\Rightarrow\; \text{the same holds for } M_t$$

Proposition 3.3. *If for some $c_0 < \infty, p \geq 0$, the inequality*

$$u^2 \leq c_0(1 + |y|^2 - u^2)^p$$

is satisfied on M_0, i.e. for $t = 0$ then for all $t > 0$

$$u^2 \leq c_0\left(1 + |y|^2 - u^2 + (2n + 4(p-1)t)\right)^p,$$

where $u = u(x, t)$.

Proof. Notice that the desired inequality, for $t = 0$ is the give one. Again, we will use our basic tool to prove that this initial bound is preserved through time. For this we introduce the new function

$$\eta = \eta(x, t) := 1 + |y|^2 - u^2 + \left(2n + 4(p-1)\right)t$$

Observe that

$$u^2 \leq c_0\eta^p \Leftrightarrow u^2\eta^{-p} \leq c_0.$$

and compute the evolution equation for $u^2\eta^{-p}$:

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)u^2\eta^{-p} = -2\eta^{-p}|\nabla u|^2 - p(p+1)\eta^{-p-2}|\nabla\eta|^2u^2 \\ - 2p\eta^{-p-1}u^2|\nabla u|^2 - 4(p-1)p\eta^{-p-1}u^2 \\ - 4p\eta^{-p-1}u\nabla u \cdot \nabla\eta\end{aligned}$$

Using Young's inequality we obtain

$$|4p\eta^{-p-1}u\nabla u \cdot \nabla\eta| \leq 2\eta^{-p}|\nabla u|^2 + 2p^2u^2\eta^{-p-2}|\nabla\eta|^2.$$

Let us call $e_i := \frac{\partial}{\partial x_i}$
Now observe that $\nabla_i u = \langle e_i, \omega\rangle$ implies

$$\nabla_i\eta = 2\langle e_i, x - \langle x, \omega\rangle\,\omega\rangle$$

because (we identify x and y again)

$$\begin{aligned}\nabla_i\eta &= 2\left\langle y, \frac{\partial\varphi}{\partial x_i}\right\rangle - 2\langle y, \omega\rangle\langle e_i, \omega\rangle \\ &= 2\langle y, e_i\rangle - 2\langle e_i, \langle y, \omega\rangle\,\omega\rangle \\ &= \langle e_i, y - \langle y, \omega\rangle\,\omega\rangle\end{aligned}$$

which yields

$$|\nabla\eta|^2 \leq 4\eta.$$

as

$$\begin{aligned}
|\nabla\eta|^2 &= 2g(\nabla_i\eta, \nabla_j\eta) \\
&= 4\left\langle e_i, y - \langle y, \omega\rangle\, \omega\right\rangle \left\langle e_j, y - \langle y, \omega\rangle\, \omega\right\rangle \\
&\leq 4\left(y - \langle y, \omega\rangle\, \omega\right) \cdot \left(y - \langle y, \omega\rangle\, \omega\right) \\
&\leq 4|y|^2 - \langle y, \omega\rangle^2 \\
&= 4(|y|^2 - u^2) \\
&\leq 4\eta
\end{aligned}$$

(think geometrically)
Thus we derive:

$$\left(\frac{d}{dt} - \Delta\right)u^2\eta^{-p} \leq 0$$

and the result follows from corollary (3.1). $\square$

3.4 Gradient estimates

To ensure that M_t stays a graph for all times we have to estimate $\upsilon := \langle \nu, \omega \rangle$ from below or equivalently

$$\upsilon := \langle \nu, \omega \rangle^{-1}$$

from above. One can consider υ to indicate whether M is "graphable". Let $A = h_{ij}$ be the **second fundamental form**.

Lemma 3.4. *The quantity υ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) \upsilon = -|A|^2 \upsilon - 2\upsilon^{-1} |\nabla \upsilon|^2.$$

Proof. Recall that, for any function f and any $p \in \mathbb{R}$

$$\left(\frac{d}{dt} - \Delta\right) f^p = -p(p-1)|\nabla f|^2 f^{p-2} + p f^{p-1} \left(\frac{d}{dt} - \Delta\right) f$$

Substituting $p = -1$ and $f = \upsilon^{-1} = \langle \nu, \omega \rangle$ we get

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) \upsilon = \left(\frac{d}{dt} - \Delta\right) f^{-1} &= -2|\nabla f|^2 f^{-3} - f^{-2} \left(\frac{d}{dt} - \Delta\right) f \\
&= -2|\nabla(\upsilon^{-1})|^2 \upsilon^3 - \upsilon^2 \left(\frac{d}{dt} - \Delta\right) \langle \nu, \omega \rangle \\
&= -2\upsilon^{-1} |\nabla \upsilon|^2 - \upsilon^2 \left\langle \left(\frac{d}{dt} - \Delta\right) \nu, \omega \right\rangle \\
&= -2\upsilon^{-1} |\nabla \upsilon|^2 - \upsilon^2 \left\langle |A|^2 \nu, \omega \right\rangle \\
&= -2\upsilon^{-1} |\nabla \upsilon|^2 - \upsilon |A|^2
\end{aligned}$$

Because $\Delta \nu = -\nabla H - |A|^2 \nu$ and $\frac{d}{dt} \nu = -\nabla H$ □

Remark 4 ($\upsilon > 0$)**.**

$$\begin{aligned}
\upsilon^{-1} &= \langle \nu, \omega \rangle \leq |\nu| \cdot |\omega| = 1 \\
&\Rightarrow \upsilon = \langle \nu, \omega \rangle^{-1} \geq 1 > 0
\end{aligned}$$

Since all the terms in the right of the above equality are non positive we can apply corollary (3.1) to conclude with

Corollary 3.5. *If υ is bounded at time $t = 0$, it remains bounded by the same constant for all time.*

The following proposition proves that polynomial bounds for the gradient function υ are preserved.

Proposition 3.6. *If for some* $c_1 < \infty, p \geq 0$, *we have*

$$\upsilon \leq c_1(1 + |y|^2 - u^2)^p$$

at time $t = 0$, *then for all* $t > 0$ *the inequality*

$$\upsilon \leq c_1(1 + |y|^2 - u^2 + 2nt)^p$$

holds for $\upsilon = \upsilon(x, t)$

Proof.

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) \upsilon\eta_2^{-p} &= \upsilon \left(\frac{d}{dt} - \Delta\right) \eta_2^{-p} + \eta_2^{-p} \left(\frac{d}{dt} - \Delta\right) \upsilon + 2\nabla\eta_2^{-p} \cdot \nabla\upsilon \\ &= -|A|^2\upsilon\eta_2^{-p} - 2\upsilon^{-1}|\nabla\upsilon|^2\eta_2^{-p} \\ &\quad - \upsilon\Big(p(p+1)|\nabla\eta_2|^2\eta_2^{-p-2} - 2p\eta_2^{-p-1}|\nabla\upsilon|^2\Big) \\ &\quad - 2p\,\eta_2^{p-1}\nabla\upsilon \cdot \nabla\eta_2\end{aligned}$$

and the term $-2p\eta_2^{p-1}\nabla\upsilon \cdot \nabla\eta_2$ cannot beat the negative terms so the right-hand side is non-positive. This is because

$$\begin{aligned}|2p\eta_2^{-p-1}\nabla\upsilon \cdot \nabla\eta_2| &= 2|\sqrt{2}\upsilon^{-1/2}\eta_2^{-p/2}\nabla\upsilon \cdot \tfrac{1}{\sqrt{2}}p\upsilon^{1/2}\eta_2^{-p/2}\eta_2^{-1}\nabla\eta_2| \\ &\overset{\text{Young}}{\leq} 2\upsilon^{-1}|\nabla\upsilon|^2\eta_2^{-p} + \frac{1}{2}p^2\upsilon\eta_2^{-p-2}|\nabla\eta_2|^2\end{aligned}$$

with $p \geq 0$. □

3.5 Curvature estimates

Not all hypersurfaces that are entire graphs behave the same way under the mean curvature flow. In this section we prove that, under condition (15) the surfaces "flatten out" as $t \to \infty$. We cannot conjecture the same for graphs violating this condition, as there are specific counterexamples. There exist **stable minimal graphs** that are **non-flat** (see [5]). These will be **equilibrium points** of the mean curvature flow.

From now on we shall only consider the case of **linear** growth (proposition (3.6) with $p = 0$), i.e. we assume that for some fixed constant $c_1 \geq 1$ the inequality

$$\upsilon \leq c_1 \tag{15}$$

holds everywhere on M_0. Corollary 3.5 then ensures that (15) remains valid for all $t > 0$.

To guarantee longtime existence of a solution for the mean curvature flow, it is crucial to obtain **a priori bounds** for the **second fundamental form** on M_t. In Theorem 3.10 we derive **uniform** estimates for the curvature, and all its derivatives, that allow us to prove the existence of a longtime smooth solution to the flow for *sufficiently good* initial data M_0.

Lemma 3.7. *The curvature satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 \upsilon^2 \leq -2\upsilon^{-1} \nabla \upsilon \cdot \nabla \left(|A|^2 \upsilon^2\right).$$

Proof. From Section 2, Lemma 2.2, 3rd equation for $|A|$ and Kato's inequality $\left(\big|\nabla |A|\big| \geq |\nabla A|\right)$ we have:

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4$$

$$\leq -2\big|\nabla |A|\big|^2 + 2|A|^4$$

Using our estimates for the derivatives of υ (see gradient estimates, Lemma 3.4) and our favourite identity we have:

$$\left(\frac{d}{dt} - \Delta\right) \upsilon^2 = -2|A|^2\upsilon^2 - 6|\nabla \upsilon|^2 \tag{16}$$

And again by our favourite identity we have:

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 v^2 \leq -2\Big|\nabla|A|\Big|^2 v^2 - 6|\nabla v|^2|A|^2 - 2\nabla|A|^2 \cdot \nabla v^2 \tag{17}$$

Which we want to be ≤ 0, only the last term is bothering us. To confront it, Young's inequality could be a firs idea. Unfortunately, if we simply try the Young inequality we would end up with

$$-2\nabla|A|^2 \cdot \nabla v^2 \leq \Big|\nabla|A|^2\Big|^2 + |\nabla v^2|^2$$

for which nothing can be done.

We examine this annoying term using the wonderful trick which is not only Young inequality, but also splitting the terms (which will also be used later)

So we split in two:

$$-2\nabla|A|^2 \cdot \nabla v^2 = -\nabla|A|^2 \cdot \nabla v^2 - 4|A|v\nabla|A| \cdot \nabla v$$

And treat each term differently.
We want to make $\nabla(|A|^2v^2)$ appear multiplied by a vector $\overrightarrow{\xi}$ in order to use the corollary (3.1)
Thus we compute:

$$\nabla(|A|^2v^2) = |A|^2\nabla v^2 + v^2\nabla|A|^2$$

We would also like in (17) to exploit the term

$$-6|\nabla v|^2|A|^2$$

So we might consider taking the inner product of $\nabla(|A|^2v^2)$ with ∇v^2 to get:

$$\begin{aligned}
\nabla(|A|^2v^2) \cdot \nabla v^2 &= |A|^2\nabla v^2 \cdot \nabla v^2 + v^2\nabla|A|^2 \cdot \nabla v^2 \\
&= |A|^2|\nabla v^2|^2 + v^2\nabla|A|^2 \cdot \nabla v^2 \\
&= |A|^2|2v\nabla v|^2 + v^2\nabla|A|^2 \cdot \nabla v^2 \\
&= 4|A|^2v^2|\nabla v|^2 + v^2\nabla|A|^2 \cdot \nabla v^2
\end{aligned}$$

We are now in position to handle the **inner product term of unknown sign,** in (17)

Splitting $2\nabla|A|^2 \cdot \nabla \upsilon^2$ in two terms:

- $\nabla|A|^2 \cdot \nabla\upsilon^2 = \upsilon^{-2}\nabla(|A|^2\upsilon^2) \cdot \nabla\upsilon^2 - 4|A|^2|\nabla\upsilon|^2$
- $\nabla|A|^2 \cdot \nabla\upsilon^2 = 2|A|\nabla|A| \cdot 2\upsilon\nabla\upsilon = -2\ (-\sqrt{2}\upsilon)\nabla|A| \cdot (\sqrt{2}|A|)\nabla\upsilon$

Where the strange $\pm\sqrt{2}$ is introduced so that Young can help us once more.

We now derive for (17):

$$\begin{aligned} -2\nabla|A|^2 \cdot \nabla\upsilon^2 &= -\Big(\upsilon^{-2}\ \nabla\big(|A|^2\upsilon^2\big) \cdot \nabla\upsilon^2 - 4|A|^2|\nabla\upsilon|^2\Big) + 2\big(-\sqrt{2}\upsilon\big)\nabla|A| \cdot \big(\sqrt{2}|A|\big)\nabla\upsilon \\ {}_{\text{Young}} &\leq -\Big(\upsilon^{-2}\nabla\big(|A|^2\upsilon^2\big) \cdot \nabla\upsilon^2 - 4|A|^2|\nabla\upsilon|^2\Big) + 2\upsilon^2\big|\nabla|A|\big|^2 + 2|A|^2|\nabla\upsilon|^2 \\ &= -\upsilon^{-2}\nabla\big(|A|^2\upsilon^2\big) \cdot \nabla\upsilon^2 + 2\upsilon^2\big|\nabla|A|\big|^2 + 6|A|^2|\nabla\upsilon|^2 \qquad \blacksquare \end{aligned}$$

So (17) becomes:

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right)|A|^2\upsilon^2 &\leq -2\big|\nabla|A|\big|^2\upsilon^2 - 6|\nabla\upsilon|^2|A|^2 - 2\nabla|A|^2 \cdot \nabla\upsilon^2 \\ &\leq \cancel{-2|\nabla|A||^2\upsilon^2} - \cancel{6|\nabla\upsilon|^2|A|^2} - \upsilon^{-2}\nabla\big(|A|^2\upsilon^2\big) \cdot \nabla\upsilon^2 \\ &\quad + \cancel{2\upsilon^2|\nabla|A||^2} + \cancel{6|A|^2|\nabla\upsilon|^2} \\ &= -\upsilon^{-2}\nabla\big(|A|^2\upsilon^2\big) \cdot \nabla\upsilon^2 \\ &= -2\upsilon^{-1}\nabla\upsilon \cdot \nabla\big(|A|^2\upsilon^2\big) \end{aligned}$$

which gives the result. □

Corollary 3.8. *If M_t is a smooth solution of the mean curvature flow with <u>bounded gradient</u> and <u>bounded curvature</u> on each M_t, then there is the **a priori estimate***

$$\sup_{M_t}|A|^2\upsilon^2 \leq \sup_{M_0}|A|^2\upsilon^2.$$

Proof. We use $\nabla\nu = A$ along with the Cauchy - Schwartz inequality to compute

$$|\nabla\upsilon| = |\upsilon^{-2}\left\langle\nabla\nu, \omega\right\rangle| \leq \upsilon^{-2}|\nabla\nu| \cdot |\omega| = \upsilon^{-2}|A| \Rightarrow \upsilon|\nabla\upsilon| \leq |A|\upsilon.$$

Since $|A|$ and υ are bounded, so is $\upsilon^{-1}|\nabla\upsilon|$, and we can proceed using corollary (3.1) with $\alpha = -2\upsilon^{-1}\nabla\upsilon$ □

Here we will discuss the boundedness of the **derivatives of any order** of A

Proposition 3.9. *If the second fundamental form of an* ***entire graph*** *is bounded up to time T, then all its covariant derivatives are also bounded up to time T.*

Proof. Exactly as in Section 2, Proposition 2.6. Only, we cannot use the maximum principle here, we use Corollary 3.1 instead. □

Proposition 3.10. *Let M_t be a smooth solution satisfying (15). Then for each $m \in \mathbb{N}$ there is a constant $C(m)$ depending only on c_1, n and m (c_1 is the linear bound of υ) such that*

$$t^{m+1}|\nabla^m A|^2 \leq C(m) \tag{18}$$

Proof. The proof is based on a form of recursion, like the one we saw in Proposition (2.6) of Chapter 2

To establish the case $m = 0$ we compute from Lemma 3.7

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)\left(2t|A|^2\upsilon^2 + \upsilon^2\right) &\leq -2\upsilon^{-1}\nabla\upsilon \cdot \nabla\left(2t|A|^2\upsilon^2\right) - 6|\nabla\upsilon|^2 \\ &\leq -2\upsilon^{-1}\nabla\upsilon \cdot \nabla\left(2t|A|^2\upsilon^2 + \upsilon^2\right).\end{aligned}$$

Again Corollary (3.1) yields that the estimate

$$2t|A|^2\upsilon^2 \leq c_1^2$$

holds uniformly on M_t. We now proceed by induction on m in a way similar to that in Chapter 2, Proposition (2.4).

We have for arbitrary $l \geq 0$ the inequality

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)\left(t^{l+1}|\nabla^{l+1}A|^2\right) &\leq -2t^{l+1}|\nabla^{l+1}A|^2 + (l+1)t^l|\nabla^l A|^2 \\ &\quad C(l,n)t^{l+1}\sum_{i+j+k=l}|\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^l A|.\end{aligned}$$

Suppose (18) is established up to $(m-1)$. Then we estimate

$$\begin{aligned}&t^{l+1}\sum_{i+j+k=l}|\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^l A| \\ &\leq Ct^{l+1}\sum_{i+j+k=l}t^{-1-i/2-j/2}\,|\nabla^k A|\,|\nabla^l A| \leq Ct^{l/2}\sum_{k\leq l}t^{k/2}|\nabla^k A|\,|\nabla^l A| \\ &\leq C\sum_{k\leq l}t^k|\nabla^k A|^2\end{aligned}$$

with constants C depending only on l, n and c_1.

The last inequality has a tricky point, let's examine this using

$$(*) \quad t^{k/2}|\nabla^k A| \le t^{l/2}|\nabla^l A| \Rightarrow t^{k/2}|\nabla^k A| t^{l/2}|\nabla^l A| \le t^l|\nabla^l A|^2$$
$$= t^k|\nabla^k A|^2 \,\Big\{k = l\Big\}$$

For $k \le l$ we split the sum in two

$$\begin{aligned} & t^{l/2}\sum_{k\le l}^{*} t^{k/2}|\nabla^k A|\,|\nabla^l A| + t^{l/2}\sum_{k\le l}^{\text{not }*} t^{k/2}|\nabla^k A|\,|\nabla^l A| \\ & \le l\cdot t^l|\nabla^l A|^2 + \sum_{k\le l} t^k|\nabla^k A|^2 \\ & \le (l+1)\cdot t^l|\nabla^l A|^2 \end{aligned}$$

Thus we obtain for all $l \le m$ the inequality

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{l+1}|\nabla^l A|^2\right) \le -2t^{l+1}|\nabla^{l+1}A|^2 + C\sum_{k\le l} t^k|\nabla^k A|^2.$$

Which yields a recursive argument as follows:

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{m+1}|\nabla^m A|^2\right) \le -2t^{m+1}|\nabla^{m+1}A|^2 + C\sum_{k\le m} t^k|\nabla^k A|^2$$
$$\left(\frac{d}{dt} - \Delta\right)\left(t^{m}|\nabla^{m-1} A|^2\right) \le -2t^{m}|\nabla^{m}A|^2 + \quad C'\sum_{k\le m-1} t^k|\nabla^k A|^2$$

Adding *enough* (k_1) of the second inequality to the first gives

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right)&\left(t^{m+1}|\nabla^m A|^2 + k_1 t^m|\nabla^{m-1}A|^2\right) \\ & \le -2t^{m+1}|\nabla^{m+1}A|^2 + \quad C'\sum_{k\le m-1} t^k|\nabla^k A|^2 \\ & \le \quad C'\sum_{k\le m-1} t^k|\nabla^k A|^2 \end{aligned}$$

and the order of the right hand side has decreased by one. *Let's do recursion!* We can continue this process choosing each time $k_2, k_3, ...k_{m+1}$ such that finally

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{m+1}|\nabla^m A|^2 + k_1 t^m |\nabla^{m-1} A|^2 + k_2 t^{m-2}|\nabla^{m-2}A|^2 + \ldots \right.$$
$$\left. \ldots + k_{m-1}t^2|\nabla A|^2 + k_m t|A|^2\right) \le |\nabla^0 A|^2 = \widetilde{C}|A|^2 \quad (19)$$

And how do we deal with $|A|^2$?

We have from equation (16) of Lemma 3.7

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)\upsilon^2 &= -2|A|^2\upsilon^2 - 6|\nabla\upsilon|^2 \\ &\le -2|A|^2\upsilon^2 \\ &\le -2|A|^2\end{aligned}$$

(remember: $\upsilon \ge 0$).
So we can use υ^2 to fight $|A|^2$.

Adding enough (k_{m+1}) of this inequality to (19) we end up with

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{m+1}|\nabla^m A|^2 + \ldots + t|A|^2 + \upsilon^2\right) \le 0$$

and using Corollary 3.1 we obtain

$$t^{m+1}|\nabla^m A|^2 + t^m|\nabla^{m-1}A|^2 + \ldots + t|A|^2 + \upsilon^2 \le C$$

uniformly in time as C depends only on the order m and the initial ***curvature*** and ***gradient*** bounds.

The result follows since $t^m|\nabla^{m-1}A|^2 + \ldots + t|A|^2 + \upsilon^2 \ge 0$. □

4 Longtime existence and Convergence

Here we prove that, as long as the second fundamental form tensor is bounded, the flow cannot develop a singularity.

Theorem 4.1. *Suppose φ_t is a smooth solution of the mean curvature flow in the interval $[0, T)$ with $T < \infty$. If $|A_t|$ is bounded for $t \in [0, T)$ then T cannot be a singular time.*

Proof. Since $\partial_t \varphi = H \cdot \overrightarrow{\nu}$, with $\overrightarrow{\nu}$: unit we have:

$$|\varphi(p,t) - \varphi(p,s)| = \left| \int_s^t \partial_t \varphi(p,\xi) d\xi \right| = \left| \int_s^t H(p,\xi) \overrightarrow{\nu} d\xi \right| \leq \int_s^t |H(p,\xi)| d\xi$$

for every $0 \leq s \leq t < T$ This is because H is the trace of the bounded tensor A so there exists $C < \infty$ such that $|H(p,t)| \leq C \quad \forall t < T$
(The terms in $|\cdot|$ are integrals of vector quantities but this doesn't affect the validity of our calculations)

This inequality says that φ_t converges. Let's see why:

Using our favourite Cauchy sequence $\{t_n\}$ we obtain:

$$|\varphi(\cdot, t_n) - \varphi(\cdot, t_m)| \leq C|t_n - t_m| < \varepsilon$$

for n_0 large enough and $m > n > n_0$

For example:

$$\left| \varphi\left(\cdot, T - \frac{1}{n}\right) - \varphi\left(\cdot, T - \frac{1}{m}\right) \right| \leq C \left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon$$

Therefore the sequence $\varphi_n := \varphi(\,\cdot\,, T - 1/n)$ is Cauchy in the (Banach) space $C(M, \mathbb{R}^{n+1})$ and thus converges to a φ_T as $n \to +\infty$. That is φ_t converges, as $t \to T$, to a function φ_T of unknown (yet) features. We will prove that φ_T is a C^∞ embedding but, for now, we can only know that it is continuous as the uniform limit of the continuous functions $\varphi_t(\cdot)$.

Let's examine the properties of this limit map (which essentially is the embedding of the limit manifold (M_T, g_T) in $\mathbb{R}^{n+1}$) and justify the writing $\varphi_t \to \varphi_T$

At this point we want to remind the reader that we are looking at M_t's as Riemannian submanifolds of $\mathbb{R}^{n+1}$ with the identifications, for all $t > 0$:

$$M_t := \{\varphi_t(M)\} \underset{isom}{\hookrightarrow} \mathbb{R}^{n+1}$$

To be more precise $M_t := \{\varphi_t(\cdot) : (M, g_t) \underset{isom}{\hookrightarrow} (\mathbb{R}^{n+1}, \bar{g})\}$

so $\varphi_t(\cdot)$ evolves according to the evolution of g_t on M
and isometric embedding means that $\bar{g}|_{M_t} = g_t$

We will show that $M_t \overset{C_c^\infty}{\to} M_T$

We will prove simultaneously that φ_T is C^∞ and $\partial^k \varphi_t \to \partial^k \varphi_T$ for each $k \in \mathbb{N}$ in the compact sense. We will deduce this applying the Ascoli-Arzela theorem on the family $\{\partial^k \varphi_t(\cdot)\}_{t \geq 0}$ for each k. For this we only need every such set S_k to be equibounded that is:

$$|\partial^k \varphi_t| < C_k \quad \forall t < T,$$

where C_k depends on T, which we will be proving subsequently.

Just a word before we go on: the inequalities are uniform in time as we saw, but also in space:

$$|\partial^k \varphi_t(p)| < C_k \quad \forall t < T, \quad \forall p \in M$$

in other words:

$$||\partial^k \varphi_t(\cdot)||_\infty < C_k \quad \forall t < T$$

where by $|| \cdot ||_\infty$ we mean the supremmum over all $p \in M$

The tricky point is that our hypersurfaces are not compact so we take closed balls $\overline{B}_r$ centered at the origin with increasing radii and cover all of $\mathbb{R}^n$. In each such ball we can use the Ascoli - Arzela theorem.

For each $0 < t < T$ and each $p \in M_t$ there is a ball $B_{r(p)}$ such that

$$|\partial^k \varphi_t(p)| < C_k \quad \forall t < T, \quad \forall p \in B_r(p)$$

Proving for each k that $|\partial^k \varphi_t| < C_k \quad \forall t < T$ (on every compact set)

To do so we will compare these k'th derivatives of the C^∞ maps $\varphi_t : M \hookrightarrow \mathbb{R}^{n+1}$ to the respective k'th covariant derivatives which are bounded as we know. (In fact they decrease to zero)

This means that we need to bound some tensor quantities so let S_t be a tensor on M_t. What does it mean that S_t is bounded? It means that $|S_t|_{g(t)} \le C$. A question arises: if we have such a bound for $t = 0$ can we hope to maintain it during the evolution?

To answer this we observe that there are two distinct evolving quantities, namely: S_t and g_t. So let's at first consider a simpler case where the tensor is merely a *fixed* vector $\upsilon \in T_pM$ (in the sense of $\upsilon := \upsilon_t = (D\varphi_t)(\upsilon) \in T_{\varphi_t(p)}\varphi_t(M)$) and look at the evolution of its norm $|\upsilon|_{g(t)}$.

(Note that υ **does not vary with time**. It appears changing only because **its image under the immersions vary**. It is an extrinsic viewpoint: υ_t's differ inside $\mathbb{R}^{n+1}$ but they are the images of the same vector $\upsilon \in T_pM$)

We compute:

$$\frac{d}{dt} log|\upsilon|^2_{g(t)} = \frac{\frac{\partial g_{ij}}{\partial t}\upsilon^i\upsilon^j}{|\upsilon|^2_{g(t)}} = \frac{-2Hh_{ij}\upsilon^i\upsilon^j}{|\upsilon|^2_{g(t)}} \le C\frac{|A|^2|\upsilon|^2_{g(t)}}{|\upsilon|^2_{g(t)}} \le C$$

Because $H = \langle A, id\rangle \underset{Cauchy-Swchartz}{\le} |A||id_n| = n|A|$

$$-2Hh_{ij}\upsilon^i\upsilon^j \le 2H|A||\upsilon|^2 \le 2n|A|^2|\upsilon|^2$$

Integrating we get, for every $0 \le s \le t < T$:

$$\left| log\frac{|\upsilon|^2_{g(t)}}{|\upsilon|^2_{g(s)}} \right| \le \int_s^t \left| \frac{d}{d\xi} log|\upsilon|^2_{g(\xi)} d\xi \right| \le C(t-s) \le C \cdot T$$

which implies: $-CT \le log\frac{|\upsilon|^2_{g(t)}}{|\upsilon|^2_{g(s)}} \le CT$, that is

$$|\upsilon|^2_{g(s)}e^{-CT} \le |\upsilon|^2_{g(t)} \le |\upsilon|^2_{g(s)}e^{CT}$$

and suggests that, up to time T, all the norms are equivalent.
Specially for $s = 0$

$$|\upsilon|^2_{g(t)} \le |\upsilon|^2_{g(0)}e^{CT}$$

And letting $t \to T$ we conclude that $|v|_{g(T)} \leq |v|_{g(0)} e^{CT/2}$ for any arbitrarily chosen $v \in T_pM$.

So the limit norm is equivalent to the finite ones as well.

The most important consequence of such equivalence is that we can use any of the (finite time) norms to bound a tensor at $t = T$. So without loss of generality we will simply write $|\cdot|$ in our estimates.

Now by the evolution equation for the Christoffel symbols *(to compare tensor and coordinate expressions as we are working extrinsically)* we see that

$$|\Gamma^k_{ij}(t)| \leq |\Gamma^k_{ij}(0)| + \int_0^t \left| \frac{d}{d\xi} \Gamma^k_{ij}(\xi) \right| d\xi \leq C + \int_0^T |A * \nabla A| d\xi \leq C + DT$$

for some constants depending only on the initial hypersurface. This passes to the limit and we have

$$|\Gamma^k_{ij}(T)| \leq C + DT$$

Thus, after fixing a (any local chart, we see the Christoffel symbols are uniformly bounded in time. This implies for every tensor S,

$$\left| \left| \frac{\partial S}{\partial x_i} \right| - |\nabla_i S| \right| \leq C|S| \tag{20}$$

where we take the partial derivative of the *scalar part* of the tensor S for instance

$$\nabla S = \frac{\partial S}{\partial x_i} dx^i \otimes (...) + S \otimes \nabla (...)$$

That means, the derivatives in coordinates differ by the relative covariant ones by eqibounded terms. So **if a tensor is bounded then its coordinate derivative is bounded iff the respective covariant derivative is bounded.**

In the rest of the proof, for simplicity, we will denote by ∂ the coordinate derivatives and by ∇ the covariant ones

As the time derivative of the Christoffel symbols is a tensor of the form $A * \nabla A$, we have

$$|\partial_t \partial^s \Gamma^k_{ij}| = |\partial^s \partial_t \Gamma^k_{ij}| \leq |\partial^s (A * \nabla A)|,$$

($[\partial_t, \partial_i] = 0$ as ordinary derivatives)
hence, by induction on the order s and integration as above, one can show that $|\partial^s \Gamma^k_{ij}| \le C$ for every $s \in \mathbb{N}$

We provide here the induction step, namely:

$$|\partial^s \Gamma^k_{ij}| = |\partial^s (A * \nabla A)| \le C \Rightarrow |\partial^{s+1}(A * \nabla A)| = |\partial^{s+1}\Gamma^k_{ij}| \le \tilde{C}$$

To see this we estimate:

$$\begin{aligned}
|\partial^{s+1}(A * \nabla A)| &= |\partial(\partial^s A * \nabla A)| \\
&= |\partial\big(\partial^s (A * \nabla A)\big) - \nabla\big(\partial^s A * \nabla A\big) + \nabla\big(\partial^s (A * \nabla A)\big)| \\
&\le \big|\partial(\partial^s A * \nabla A) - \nabla\big(\partial^s (A * \nabla A)\big)\big| + |\nabla\big(\partial^s (A * \nabla A)\big)| \\
{\scriptstyle by (20)} &\le C_1 |\partial^s (A * \nabla A)| + C_2 \\
{\scriptstyle \text{by inductive hypothesis}} &\le C_1 C + C_2
\end{aligned}$$

(the tensor quantities are considered bounded and we want to bound the partials)

Then, again by induction, the following formula relating the iterated covariant and coordinate derivatives of the tensor S holds:

$$\begin{aligned}
||\nabla^s S| - |\partial^s S|| &\le |\nabla^s S - \partial^s S| \\
&\le \sum_{i=1}^{s} \sum_{j_1 + \ldots + j_i + k \le s-1} |\partial^{j_1}\Gamma \ldots \partial^{j_i}\Gamma \partial^k S| \le C \sum_{k=1}^{s-1} |\partial^k S|
\end{aligned}$$

This implies that if a tensor has all its covariant derivatives bounded, also all the coordinate derivatives are bounded. In particular this holds for the tensor A, that is, $|\partial^k A| \le C_k$. Moreover, by induction, as $\nabla^k g = 0$ all the coordinate derivatives of the metric tensor g are equibounded.

Working in compact sets, we ensure that $|\varphi|$ is bounded and $|\partial\varphi| = |e_i| = 1$, then by the Gauss - Weingarten relations,

$$\partial^2\varphi = \Gamma\partial\varphi + A\nu, \quad \partial\nu = A * \partial\varphi$$

we get

$$\begin{aligned}
|\partial^k\varphi| &= \left|\sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\Gamma\partial^{i+1}\varphi + \sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}A\partial^i\nu\right| \\
{\scriptstyle |\partial^*\Gamma|\le C} &\le C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + \tilde{C}\sum_{i=0}^{k-2}|\partial^{k-2-i}A\partial^{i-1}(A*\partial\varphi)| \\
{\scriptstyle |\partial^*A|\le\tilde{C}} &\le C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + \tilde{C}\sum_{i=1}^{k-2}|\partial^{i-1}(A*\partial\varphi)| + \tilde{\tilde{C}} \\
&\le C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + \tilde{C}\sum_{i=1}^{k-2}\left|\sum_{p+q+r=i-1}\partial^pA*\partial^qg*\partial^{r+1}\varphi\right| + \tilde{\tilde{C}} \\
{\scriptstyle |\partial^*g|\cdot|\partial^*A|\le\tilde{C}} &\le C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + \tilde{C}\sum_{i=1}^{k-2}\sum_{r=0}^{i-1}\left|\partial^{r+1}\varphi\right| + C \\
&\le C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + \tilde{C}\sum_{i=1}^{k-2}|\partial^i\varphi| + C \\
&\le C\sum_{i=0}^{k-1}|\partial^i\varphi|
\end{aligned}$$

(only caring about the **boundedness** of constants)

This gives the tool for an (obvious) induction argument which yields

$$|\partial^k\varphi| < C_k \quad \text{time independent}$$

for $t \in [0, T)$, in all compact subsets of M.

By the Ascoli - Arzela theorem we see that for every $t \in [0, T)$ the limit

$$\varphi_t(\cdot) \to \varphi_T(\cdot)$$

is in C_c^∞ and conclude that $\varphi_T : M \to \mathbb{R}^{n+1}$ is a C^∞ embedding.

We can do the same computations with $\partial_t \varphi$ and find out that

$$|\partial_t^s \partial_x^k \varphi| \leq C_{s,k}$$

and smoothly pass to the limit. So the convergence is also C^∞ in time. Thus the domain extends to the *temporal boundary* of $M \times [0, T)$ and $\lim_{t \to T} \varphi_t(\cdot) = \varphi_T(\cdot)$.

Now by the short time existence theory we can "restart" the flow past time T which contradicts to its maximality. $\square$

5 Asymptotic behavior

In this section we study the behavior of solutions M_t for large times t in the case of linear growth. For simplicity we shall additionally assume that the initial surface M_0 has bounded curvature. We saw in Proposition 3.10 that the surfaces M_t "flatten out" as $t \to \infty$, and *if they do not diverge to* ∞ (e.g. if u is bounded *in time*), then they must converge to a plane.

However, in general the surfaces *do diverge to infinity*, in fact at speed proportional to $t^{-1/2}$, and Proposition 3.10 does not yield any information about their global shape.

To study the global shape of M_t for $t \to \infty$ we will now *rescale* the surfaces in such a way that they do not diverge to infinity. We then examine the properties of the rescaled manifolds and retain a bound on their curvature.

5.1 Rescaling

We define

$$\tilde{\varphi}(s) := \frac{1}{\sqrt{2t+1}}\varphi(t) \tag{21}$$

where the new "time" variable s is given by

$$s = \frac{1}{2}\log(2t+1), \quad s \in [0, +\infty)$$

The basic geometric quantities rescale as follows:

$$\begin{aligned}
\tilde{g}_{ij} &= \left\langle \frac{\partial \tilde{\varphi}}{\partial x_i}, \frac{\partial \tilde{\varphi}}{\partial x_j} \right\rangle = \left(\frac{1}{\sqrt{2t+1}} \right)^2 \left\langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle = \frac{1}{2t+1} g_{ij} \\
\tilde{h}_{ij} &= \left\langle \frac{\partial \tilde{\nu}}{\partial x_i}, \frac{\partial \tilde{\varphi}}{\partial x_j} \right\rangle = \frac{1}{\sqrt{2t+1}} \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle = \frac{1}{\sqrt{2t+1}} h_{ij} \\
\tilde{H} &= \tilde{g}^{ij}\tilde{h}_{ij} = \sqrt{2t+1}\, g^{ij} h_{ij} = \sqrt{2t+1}\, H \\
\tilde{\nu} &= \nu \quad \text{because } \left\{ \tilde{\nu} \perp \frac{\partial \tilde{\varphi}}{\partial x_k},\ |\tilde{\nu}| = 1 \right\}
\end{aligned}$$

Notice that we adopted another convention for h_{ij} than previously. This will help with the proofs that follow throughout this last chapter. We cite here the corresponding formulae affected by this convention.

- $\dfrac{\partial}{\partial t}\nu = -\nabla H$ becomes $\dfrac{\partial}{\partial t}\nu = \nabla H$
- $\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} + 2H h_{il} g^{ls} h_{sj} - |A|^2 h_{ij}$
- $\frac{\partial}{\partial t}H = \Delta H + H|A|^2$
- The MCF equation doesn't change

The normalized equation then becomes

$$\frac{d}{dt}\tilde{\varphi} = \tilde{H}\tilde{\nu} - \tilde{\varphi} \tag{22}$$

Indeed

$$\begin{aligned}
\tilde{\varphi} = \frac{1}{\sqrt{2t+1}}\varphi \\
\Rightarrow \frac{\partial \tilde{\varphi}}{\partial s} &= \left(\frac{ds}{dt}\right)^{-1} \frac{\partial}{\partial t}\left(\frac{\varphi}{\sqrt{2t+1}}\right) \\
&= (2t+1)\left(\frac{(2t+1)^{1/2} H\nu - (2t+1)^{-1/2}\varphi}{2t+1}\right) \\
&= \big((2t+1)^{1/2} H\nu - \tilde{\varphi}\big) \\
&= \tilde{H}\tilde{\nu} - \tilde{\varphi}.
\end{aligned}$$

The estimates from Proposition 3.3, Corollary 3.5,Corollary 3.8 and Proposition 3.10 translate to **estimates for the rescaled** embedding

$$\begin{aligned}
\tilde{u}^2(x,s) &\leq \tilde{c}_0\big(1 + |\tilde{y}|^2 - \tilde{u}^2(x,s)\big), \\
\tilde{v}(x,s) &\leq c_1, \\
|\tilde{A}|^2(x,s) &\leq c_2,
\end{aligned}$$

with constants depending only on the initial bounds for the respective quantities on M_0.

Indeed:

$$\tilde{u}\big(\tilde{y}(x),s\big) = \langle \tilde{y}, \omega \rangle = \frac{1}{\sqrt{2t+1}} \langle y(s), \omega \rangle = \frac{1}{\sqrt{2t+1}} u\big(y(x),s\big) \tag{23}$$

♦ Notice that we count in the new time scale: s instead of t ♦

So the rescaled *height estimate* is

$$
\begin{aligned}
\tilde{u}^2 &= \frac{1}{2t+1}u^2 \\
&\leq \frac{1}{2t+1}c_0\big(1+|y|^2-u^2+2nt\big) \\
{\scriptstyle t\to\infty} &= c_0\big(n+|\tilde{y}|^2-\tilde{u}^2\big)
\end{aligned}
$$

The new gradient estimate:

$$
\tilde{\upsilon}(x,s) = \langle\tilde{\nu},\omega\rangle^{-1} = \langle\nu,\omega\rangle^{-1} = \upsilon(x,s) \leq c_1
$$

since $\tilde{\nu}=\nu$
As for the rescaled curvature:

$$
\begin{aligned}
|\tilde{A}|^2 = \tilde{g}(\tilde{A},\tilde{A}) &= \tilde{g}^{jk}\tilde{g}^{il}\tilde{h}_{ij}\tilde{h}_{kl} \\
&= (2t+1)^2\left(\frac{1}{\sqrt{2t+1}}\right)^2 g^{jk}g^{il}h_{ij}h_{kl} \\
&= (2t+1)|A|^2 \\
&= \frac{2t+1}{t}t|A|^2 \\
{\scriptstyle (t|A|^2\leq c_2)} &\leq \Big(2+\frac{1}{t}\Big)c_2 \\
{\scriptstyle (t\to\infty)} &\leq c_2'
\end{aligned}
$$

For the rescaled surfaces, $\tilde{M}_s=\tilde{\varphi}_s(M)$ we then establish the following result concerning asymptotic convergence.

Theorem 5.1. *Suppose M_0 satisfies the linear growth condition (15) and has bounded curvature. If in addition the estimate*

$$
\langle y,\nu\rangle \leq c_3\big(1+|y|^2\big)^{1-\delta} \tag{24}
$$

is valid on M_0 for some constants c_3, $\delta>0$. Then the solution $\tilde{M}_s$ of the normalized equation (22) converges for $s\to\infty$ to a limiting surface $\tilde{M}_\infty$ satisfying the equation

$$
\langle\tilde{y},\tilde{\nu}\rangle = \widetilde{H} \tag{25}
$$

We refer the reader to [1] for an explicit counterexample proving that condition (24) is indeed necessary.

The result follows from the estimate

$$\sup_{\tilde{M}_s} \frac{\big(\tilde{H} - \langle \tilde{y}, \tilde{\nu} \rangle\big)^2 \tilde{v}^2}{\big(1 + \alpha |\tilde{y}|^2\big)^{1-\varepsilon}} \le e^{-\beta s} \sup_{\tilde{M}_0} \frac{\big(H - \langle y, \nu \rangle\big)^2 v^2}{\big(1 + \alpha |y|^2\big)^{1-\varepsilon}} \tag{26}$$

which we derive for all $\varepsilon < \delta$ with some constants $\alpha > 0, \beta > 0$ depending only on ε, n, c_1 and c_2.

The right supremmum is finite on every compact subset of $\tilde{M}_0$. This implies, in particular, exponentially fast convergence **on compact subsets**

Remarks
i) In view of the interior estimates in Proposition 3.10 the conclusion of Theorem 5.1 remains valid for Lipschitz initial data provided condition (24) is satisfied for some $t_0 > 0$.

ii) Any initial surface M_0 given by $\varphi_0 : M \to \mathbb{R}^{n+1}$ satisfying (25) gives rise to an expanding selfsimilar solution of the mean curvature flow in the sense that

$$\varphi_t = \sqrt{2t+1}\varphi_0$$

satisfies

$$\left(\frac{d}{dt}\varphi\right)^\perp = H.$$

Theorem 5.1 then says that M_t becomes asymptotically selfsimilar.

In the one-dimensional case an example for "curves of constant shape" evolving from a corner was numerically obtained by Brakke in [6]. It is an open problem to understand and possibly classify solutions of equation (25). We show in the appendix that the equation

$$\varphi^\perp = -H\nu$$

(characterizing *contracting* selfsimilar solutions of the mean curvature flow) has only trivial solutions in the class of entire graphs of polynomial growth.

We begin the proof of the theorem with the following lemma.

Lemma 5.2. *The quantity $\langle y, \nu \rangle$ satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) \langle y, \nu \rangle = |A|^2 \langle y, \nu \rangle + 2H.$$

Proof. Remember that $y = \varphi(x)$
From the equation $(d/dt)\nu = \nabla H$ we compute

$$\begin{aligned}\frac{d}{dt}\langle y, \nu \rangle &= \langle H\nu, \nu \rangle + \langle y, \nabla H \rangle \\ &= H + \langle y, \nabla H \rangle\end{aligned}$$

while

$$\begin{aligned}\Delta \langle \varphi, \nu \rangle &= \langle \Delta\varphi, \nu \rangle + \langle \varphi, \Delta\nu \rangle + 2 \langle \nabla\varphi, \nabla\nu \rangle \\ * \quad &= H - \left\langle \varphi, |A|^2\nu - \nabla H \right\rangle - 2H \\ &= -H - |A|^2 \langle \varphi, \nu \rangle + \langle \varphi, \nabla H \rangle\end{aligned}$$

So

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) \langle y, \nu \rangle &= H + \langle y, \nabla H \rangle + H + |A|^2 \langle y, \nu \rangle - \langle y, \nabla H \rangle \\ &= 2H + |A|^2 \langle y, \nu \rangle\end{aligned}$$

$$(*): \quad \langle \nabla\varphi, \nabla\nu \rangle = \nabla \cancel{\left\langle \underbrace{\nabla\varphi}_{\in M^\top}, \nu \right\rangle} - \langle \Delta\varphi, \nu \rangle = -H$$

We can now show that up to a time-dependent factor, condition (24) is preserved for all $s > 0$. □

Lemma 5.3. *Suppose M_0 satisfies the assumptions of Theorem 5.1. Then on $\tilde{M}_s$ we have the estimate*

$$\langle \tilde{y}, \nu \rangle^2 \leq C(s)(1 + |\tilde{y}|^2)^{1-\delta}$$

with a constant depending on s and c_2.

Proof. Since the constant in the estimate is allowed to depend on time it is sufficient to look at the **unnormalized** flow. From the previous lemma we infer

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\langle y,\nu\rangle^2 &= 2\langle y,\nu\rangle\left(\frac{d}{dt}-\Delta\right)\langle y,\nu\rangle - 2|\nabla\langle y,\nu\rangle|^2\\ &= 2|A|^2\langle y,\nu\rangle^2 + 4H\langle y,\nu\rangle - 2|\nabla\langle y,\nu\rangle|^2\\ &\leq C\Big(\langle y,\nu\rangle^2+1\Big) - 2|\nabla\langle y,\nu\rangle|^2 \qquad (27)\end{aligned}$$

Because we consider $|A|^2 \leq \widetilde{C}$, therefore $H \leq \widetilde{C}$, and using Young on $4H\langle y,\nu\rangle \leq 2H^2 + 2\langle y,\nu\rangle^2$ we get

$$2(\sqrt{2}H)(\sqrt{2}\langle y,\nu\rangle) \leq 2H^2 + 2\langle y,\nu\rangle^2$$

$$\begin{aligned}2\langle y,\nu\rangle^2|A|^2 + 4H\langle y,\nu\rangle &\leq 2\widetilde{C}\langle y,\nu\rangle^2 + 2\widetilde{C} + 2\langle y,\nu\rangle^2\\ &\leq 2\big(\widetilde{C}+1\big)\langle y,\nu\rangle^2 + 2\widetilde{C} + 2\\ &\underset{C:=2(\widetilde{C}+1)}{=} C\big(\langle y,\nu\rangle^2+1\big)\end{aligned}$$

From now on we denote all constants depending only on c_2 and s by C. We now write $f = \langle y, \nu \rangle$, multiply the above equation by a test function ρ and estimate

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) f^2\rho &= \rho\Big(C(f^2+1) - 2|\nabla f|^2\Big) + f^2\left(\frac{d}{dt} - \Delta\right)\rho - 2\left\langle \nabla f, \nabla\rho\right\rangle \\
&= \rho\Big(Cf^2 + C - 2|\nabla f|^2\Big) + \rho^{-1}\rho f^2\left(\frac{d}{dt} - \Delta\right)\rho - 2\left\langle \nabla f, \nabla\rho\right\rangle \\
&= f^2\rho\left(C + \rho^{-1}\left(\frac{d}{dt} - \Delta\right)\right) + C\rho - 2\rho|\nabla f|^2 - 2\left\langle \nabla f, \nabla\rho\right\rangle \\
&\overset{\text{Young}}{\leq} \left(C + \rho^{-1}\left(\frac{d}{dt} - \Delta\right)\right) f^2\rho + C\rho
\end{aligned}$$

Choosing $\rho = \eta_1^{\delta-1}$ where $\eta_1 = 1 + |x|^2 + 2nt$ we derive from Lemma 3.2,

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\rho &= \left(\frac{d}{dt} - \Delta\right)\eta_1^{\delta-1} \\
&= -(\delta-1)(\delta-2)|\nabla\eta_1|^2\eta_1^{\delta-3} + (...)\cancel{\left(\frac{d}{dt} - \Delta\right)}\eta_1 \\
&= -(\delta-1)(\delta-2)|\nabla\eta_1|^2\eta_1^{\delta-3} \\
&\leq 0 \quad \text{since } \delta < 1
\end{aligned}$$

Furthermore,

$$\rho^{-2}|\nabla\rho|^2 = (1-\delta)^2\eta_1^{-2}|\nabla\eta_1|^2 \leq 4(1-\delta)^2\eta_1 \leq 4$$

since $|\nabla\eta_1|^2 \leq 4\eta_1$.

Altogether we conclude

$$\left(\frac{d}{dt} - \Delta\right) f^2\rho \leq C\Big(f^2\rho + 1\Big)$$

such that by Corollary 3.1, $f^2\rho$ can at most grow exponentially in time. This implies the result. Let's see this for the rescaled flow:

$$f^2\rho \leq e^t \Rightarrow f^2 \leq \rho^{-1}e^{-t}$$

i.e.

$$\begin{aligned}
&\langle y,\nu\rangle^2 \le e^{-t}(1+|y|^2+2nt)^{1-\delta}\\
{}_{y=\sqrt{2t+1}\tilde y} &\Rightarrow (2t+1)\langle \tilde y,\nu\rangle^2 \le C(t)\Big(1+(2t+1)|\tilde y|^2+2nt\Big)^{1-\delta}\\
&\Leftrightarrow \langle \tilde y,\nu\rangle^2 \le C(t)\frac{\Big((2t+1)|\tilde y|^2+2nt+1\Big)^{1-\delta}}{2t+1}\\
&\Leftrightarrow \langle \tilde y,\nu\rangle^2 \le C(t)\Big(|\tilde y|^2+c(\delta)\Big)^{1-\delta}.
\end{aligned}$$

□

Lemma 5.4. *The normalized quantity $\tilde H - \langle \tilde y,\nu\rangle$ satisfies the evolution equation*

$$\left(\frac{d}{ds}-\tilde\Delta\right)\Big(\tilde H-\langle \tilde y,\nu\rangle\Big) = \Big(|\tilde A|^2-1\Big)\Big(\tilde H-\langle \tilde y,\nu\rangle\Big) \tag{28}$$

Proof. We first compute

$$\tilde\Delta f = \tilde g^{ij}\nabla_{ij} f = (2t+1)g^{ij}\nabla_{ij} f = (2t+1)\Delta f$$

$$\frac{\partial}{\partial s} f = \left(\frac{ds}{dt}\right)^{-1}\frac{\partial}{\partial t} f = (2t+1)\frac{\partial}{\partial t} f$$

Very helpfully then

$$\tilde\Box f = (2t+1)\Box f$$

for any f.
From the evolution of H (see chapter 2) and Lemma 5.2 we compute the identities

$$\begin{aligned}
\left(\frac{d}{ds}-\tilde\Delta\right)\tilde H &= |\tilde A|^2\tilde H+\tilde H\\
\left(\frac{d}{ds}-\tilde\Delta\right)\langle \tilde y,\nu\rangle &= |\tilde A|^2\langle \tilde y,\nu\rangle + 2\tilde H - \langle \tilde y,\nu\rangle .
\end{aligned}$$

The computation, recalling that $\widetilde H = \sqrt{2t+1}H$, is

$$\begin{aligned}
\tilde\Box\tilde H &= (2t+1)\Box\sqrt{2t+1}H\\
&= (2t+1)\big(\sqrt{2t+1}\Box H + H\Box\sqrt{2t+1}\big)\\
&= H|A|^2(2t+1)^{3/2}+H(2t+1)^{1/2}\\
&= \tilde H|\tilde A|^2+\tilde H
\end{aligned} \tag{29}$$

Furthermore

$$
\begin{aligned}
\widetilde{\Box} \langle \tilde{y}, \nu \rangle &= (2t+1)\Box(2t+1)^{-1/2} \langle y, \nu \rangle \\
&= (2t+1)\big[(2t+1)^{-1/2}\Box \langle y, \nu \rangle + \langle y, \nu \rangle \Box(2t+1)^{-1/2}\big] \\
&= (2t+1)^{1/2}\Box \langle y, \nu \rangle - (2t+1)\cdot(2t+1)^{-3/2} \\
&= (2t+1)\cdot(2t+1)^{-1/2}|A|^2 \langle y, \nu \rangle + 2\tilde{H} - (2t+1)^{-1/2} \langle y, \nu \rangle \\
&= |\tilde{A}|^2 \langle \tilde{y}, \nu \rangle + 2\tilde{H} - \langle \tilde{y}, \nu \rangle
\end{aligned}
$$

□

Proof of the main theorem

From (28) we infer The normalized gradient $\tilde{\upsilon}$ satisfies the equation

$$
\begin{aligned}
\Big(\frac{d}{ds} - \tilde{\Delta}\Big)\tilde{\upsilon}^2 &= (2t+1)\Big(\frac{d}{dt} - \Delta\Big)\upsilon^2 \\
&= (2t+1)\Big(- |A|^2\upsilon^2 - 6|\nabla\upsilon|^2\Big) \\
&= -|\tilde{A}|^2\tilde{\upsilon}^2 - 6|\nabla\tilde{\upsilon}|^2_{\tilde{g}}
\end{aligned}
$$

Observe in the last equality that

$$
|\cdot|_{\tilde{g}} = \frac{1}{2t+1}|\cdot|_g
$$

for vectors but

$$
|\cdot|_{\tilde{g}} = (2t+1)|\cdot|_g
$$

for covectors, such as $\nabla\tilde{\upsilon}$

We may then proceed exactly as in the proof of Lemma 3.7 (split and Young) to obtain for $f^2 = \big(\tilde{H} - \langle \tilde{y}, \nu \rangle\big)^2\tilde{\upsilon}^2$ the inequality

$$
\Big(\frac{d}{ds} - \tilde{\Delta}\Big)f^2 \leq -2f^2 - 2\tilde{\upsilon}^{-1}\nabla\tilde{\upsilon}\cdot\nabla f^2.
$$

Multiplying by a test function ρ we compute

$$
\begin{aligned}
\Big(\frac{d}{ds} - \tilde{\Delta}\Big)\rho f^2 &\leq -2\rho f^2 - 2\rho\tilde{\upsilon}^{-1}\nabla\tilde{\upsilon}\cdot\nabla f^2 \\
&\quad + f^2\Big(\frac{d}{ds} - \tilde{\Delta}\Big)\rho - 2\nabla\rho\cdot\nabla f^2
\end{aligned}
\tag{30}
$$

Now let $0 < \varepsilon < \delta$ and define $\rho(\tilde{y}, s) = \eta_\alpha^{\varepsilon-1}(\tilde{y}) e^{\beta s}$ with $\eta_\alpha(\tilde{y}) = 1 + \alpha|\tilde{y}|^2$ where α, β are small positive constants to be determined later. Then the normalized equation (22) implies

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\eta_\alpha = -2\alpha\big(|\tilde{y}|^2 + n\big)$$

and therefore

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\rho \leq \big(\beta + 2(1-\varepsilon)(\alpha n + 1)\big)\rho \tag{31}$$

Moreover, (same trick as in 3.7)

$$\begin{aligned} -2\rho\tilde{\upsilon}^{-1}\nabla\tilde{\upsilon}\cdot\nabla f^2 - 2\nabla\rho\cdot\nabla f^2 = \\ & -2\big(\tilde{\upsilon}^{-1}\nabla\tilde{\upsilon}\cdot\rho^{-1}\nabla\rho\big)\cdot\nabla\big(f^2\rho\big) \qquad (32) \\ & +2|\nabla\rho|^2 f^2\rho^{-1} + 2f^2\tilde{\upsilon}^{-1}b\nabla\tilde{\upsilon}\cdot\nabla\rho. \end{aligned}$$

and we obtain from $|\nabla\eta_\alpha|^2 \leq 4\alpha\eta_\alpha$ the estimate

$$|\nabla\rho| \leq 2\alpha^{1/2}\rho.$$

Combining now (30),(31) and (32) and using the fact that $|\nabla\tilde{\upsilon}| \leq |\tilde{A}|\tilde{\upsilon}^2$ (which is scale invariant) we derive for $g = f^2\rho$ the inequality

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)g \leq \xi\cdot\nabla g + \big(\beta + c\alpha^{1/2} - 2\varepsilon\big)g$$

where $\xi = -2\big(\tilde{\upsilon}^{-1}\nabla\tilde{\upsilon} + \rho^{-1}\nabla\rho\big)$ (which is clearly bounded) and c depends on c_1, c_2 and n. Choosing then α, β suitably small depending on ε and c we see that

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)g \leq \xi\cdot\nabla g$$

for all $s \geq 0$. Lemma (5.3) ensures that g **vanishes at infinity** (because f^2 does since $\delta < 1$) and enables us to apply the 3.1 to conclude that g is **uniformly bounded** by its initial data. This proves estimate (26) and completes the proof of Theorem 5.1.

6 Appendix

Proposition 6.1. *If M is an entire graph of at most polynomial growth satisfying the equation $\varphi^\perp = H\nu$ or equivalently*

$$\langle\varphi, \nu\rangle = H$$

then M is a plane.

Proof. This equation gives $\nabla_i H = \left\langle \varphi, \frac{\partial}{\partial x_l} \right\rangle h_{il}$ and hence

$$\Delta \upsilon = |A|^2 \upsilon + 2\upsilon^{-1} |\nabla \upsilon|^2 + \left\langle \varphi, \frac{\partial}{\partial x_i} \right\rangle \nabla_i \upsilon.$$

We multiply this equation by $\rho = e^{\frac{-|x|^2}{2}}$ which after integration by parts leads to

$$\int_M |A|^2 \upsilon \rho d\mu + 2 \int_M \upsilon^{-1} |\nabla \upsilon|^2 \rho d\mu = 0,$$

thus implying the result. □

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