The Kochen-Specker theorem as an epistemic basis for scientific perspectivism

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Abstract

Can knowledge claims be perspective-dependent while still describing the world, as a realist view of science does? This is a crucial issue raised in Massimi's (2018) perspectivism. In this thesis, we take up the challenge by investigating the implications of quantum contextuality (Bell, 1966; Kochen, Specker, 1967) on perspectival truth and reference. We employ Karakostas' analysis of the Kochen-Specker theorem (2014; Karakostas, Zafiris, 2017) which demonstrates that in quantum mechanics the truth-conditions of experimental propositions describing quantum properties are context-dependent, as well as Massimi's account of perspectival truth which contends that perspectives provide the truth-conditions for knowledge claims. Our aim is to show that quantum mechanics furnishes an epistemic basis for perspectivism and provides additional insights into the methodological role of the notion of perspective in scientific inquiry. We hope that this thesis will pave the way for the application of perspectives in forward-thinking movements both in physics and philosophy.

Dedication

To those who act in accordance with their prohairesis.

Declaration

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Contents

1 Introduction

An important motivation driving this study is the way in which philosophy and science can work together to overcome certain problems presented in the practice of physics through their embedding in the domains of philosophy of science and mathematics. Contemporary physics is characterised by the existence of partially compatible or even incompatible descriptions of reality as portrayed by the most empirically successful physical theories: quantum theory and general relativity. Such a situation in physics constitutes a major problem since the most successful descriptions of reality are not necessarily conceptually, logically and mathematically compatible. The field of quantum gravity seeks to reconcile the aforementioned descriptions of reality by developing theories where gravitational and quantum phenomena co-exist. The issue of quantum gravity will not be examined here. However, we hope that this thesis will contribute to our understanding of how a variety of not necessarily fully compatible perspectives should merge.

A first naive approach to the problems of physics could be the use of the accumulated knowledge of philosophy in order to recognize ways of moving forward. Unfortunately, such an approach is not always feasible due to the difference in terms, interests, and problems of the two fields (philosophy and physics), which is also what makes their communication difficult. For this reason, we should look at each field in its own right and not instrumentally use philosophy to meet physics' needs, effectively imposing science's values on philosophy or vice versa. Our engagement with philosophy of science such as Kuhn's philosophical framework (1962) and scientific realism (Dummett, 1982; Psillos, 1999) revealed that the problem of incompatibility of quantum theory with general relativity might be related to Kuhn's notion of incommensurability between paradigms (1962) and to how we can be realistic about scientific knowledge when it is probable that a new scientific theory will considered to be the dominant paradigm. Thus, by reflecting on similar topics in the history of science, and by comparing similar historical periods, we can begin to understand what physics is going through currently.

Philosophical examination of physical theories also provides key insights into why certain theories are good candidates for describing the world. In other words, philosophy of physics casts a critical eye on physical theories to determine under what conditions they are able to conceive reality. Philosophy's critical view can be valuable with the problem of the incompatibility of quantum theory and general relativity described in the previous paragraph, since a philosophical examination may reveal to what extent these physical theories conceive some aspect of reality, why they are compatible or not and what are the requirements of a possible combination.

We believe that this study contributes to an understanding of why quantum theory is supposed to be a successful description of micro-physical reality through the examination of the Kochen-Specker theorem (1967) and its recent developments. In this line of thinking, the Kochen-Specker theorem is investigated on the basis of the work of Karakostas (2007;2012; 2014; 2017). It revealed to us that the quantum mechanical formalism is unable to determine truth values to propositions describing properties of incompatible observables. Quantum theory, therefore, cannot make unrestricted empirical claims for everything that exists, but rather must refer to specific aspects probed by the experiment (experimental context).

Moreover, the context provides the conditions for the manifestation of properties associated only with compatible observables since it specifies in accordance with the state of the system the possible eigenvectors/eigenvalues of the subsequent measurement. Essentially, the state and the context by specifying the observable and the eigenstates through which we will examine the system represent the conditions under which we are able to cognize the system. In the absence of a context, a pure state as a superposition of probability amplitudes of all possible physical quantities is empirically meaningless. In fact, this is the primary distinction between quantum superposition and classical superposition in wave mechanics, which always specifies a pair of coordinates in phase space. As a result, the definition of a context is a prerequisite both for shaping an empirically meaningful state of the underlying micro-physical reality and for the determination of the properties of co-measurable physical quantities. One might say, at least in quantum theory, that the ontological discussion about the state of micro-physical reality cannot be separated from the epistemological discussion about the way of knowing the properties of a system.

Besides quantum contextuality, in this study we examine another line of thought called scientific perspectivism, which reaches analogous conclusions about scientific process as the analysis of the Kochen-Specker theorem. Although, scientific perspectivism refers to perspectival nature of knowledge in line with contextuality of quantum theory, it originates from a totally different outset by drawing examples from the history of science. First introduced by Giere (2006) and further developed by Massimi (2018), this view supports a realistic view of scientific knowledge, albeit not of the traditional kind. Giere defines scientific perspectivism as the space between objectivist realism and constructivism and attempts the creative combination of the two views in three acts: firstly, by the articulation of a scheme of scientific theorising that locates the perspectivity of science in theory-building, secondly, by the comparison of perspectivism with the practice of cartography in which we highlight the similarity of Riemann's geometry with perspectivism, and thirdly, by the definition of perspectival truth as that which is "relative to perspective". Following that, we present Massimi's view on the question of how we ensure that perspectives conceive reality, while they are bound to change, as well as the consequences of Kochen-Specker's theorem on the kinds of perspectival truth that she defines.

The chapter on scientific perspectivism concludes by introducing Karakostas and Zafiris' perspectivist methodology, which describes a way of relating, combining and managing contexts in quantum theory. In spite of the fact that the Kochen-Specker theorem reveals the contextual nature of quantum theory, it does not address the connection of contexts in quantum systems. Karakostas and Zafiris' perspectivist methodology, through the use of category theory, provides the means of relating contexts into structures of increasing complexity by showing that a quantum object-system "is the colimit (or inductive limit) of all perspectives directed towards it and jointly covering it in a compatible manner" (Karakostas, Zafiris, 2021, p. 16). Essentially, the different perspectives on a quantum system are glued together based on colimits, which can be seen as the binding factors between them. Colimits are key to the combination of perspectives as they ensure that the synthesis produced is a coherent whole, rather than an amalgam of disordered scattered viewpoints.

The goal of this thesis is to demonstrate that even though scientific perspectivism has a wider and different origin than quantum theory, the two lines of thought (philosophical and scientific) meet and produce very insightful views about reality and the cognitive process. In particular, the relationship between quantum theory and scientific perspectivism can be expressed in a few points: first, any cognitive process in quantum theory or generally in contemporary science is always inextricably associated with the specification of a context/perspective; second, the context provides the conditions under which our claims can be assigned truth-values; third, the states of affairs are not simply found out there ready to verify our claims but are curved out by our contexts; and fourth, the truth-conditions provided by a context are not necessarily material in nature. These conditions relate to the cognitive faculties of our understanding, as they refer to the observable defining the context as well as to how we make empirically accessible the quantum state of the system.

Through this study, our hope is to show that we can only comprehend micro-physical reality by contextualizing a system based on a variety of appropriate sets of co-measurable observables. This conclusion involves two claims concerning realism: on the one hand, we can be realists about empirical propositions based on conditions compatible with the context; and on the other hand, a naive realist view about empirical propositions based on conditions independent of the context is bound to fail. Also, we can be realists about theoretical claims referring to quantum theory as a whole (e.g., the non-separability of quantum systems) (Karakostas, 2007). If we take into consideration the conclusions about the realism of quantum mechanics and the truth-conditions, the question about the realism of perspectives (whether the perspectives conceive something real since they are bound to change) is transformed into the question about the conditions of a perspective. In other words, a perspective is more than just a fact-mediator or fact-verifier; it actively shapes the deductive determination of empirical quantum reality.

Based on these conclusions, we inspire in a future work to generalise the way in which these conditions are found in a given perspective. Particularly, we take the view that Karakostas and Zafiris' approach can provide us with a systematic way of tackling the question of truth-conditions at the level of physical theories. The mathematics of category theory offers the opportunity for generalisation outside the quantum domain and more precisely on a "trans-perspectival" level of inquiry in order to study possible combinations of perspectives like quantum theory and general relativity (Karakostas, Zafiris, 2021, p. 31), as well as to articulate a framework of theory-change. Karakostas and Zafiris' perspectivist methodology enables us to combine partially compatible perspectives through the categorical theoretic concept of colimit that creates objects of inquiry where the perspectives coexist. As objects-systems that exhibit simultaneously quantum and relativistic features are considered the holy grail of theoretical physics, since they could supply the basis for the construction of a theory of quantum gravity, the concept of colimit can surely give us tremendous insights.

Overall, an inquiry of how we put together partially compatible perspectives in order to understand the whole can be seen as a new way of practising science. Until now the main approach to physics is reduction, that is to analyse the whole into parts and trying to reduce any behaviour of a system to its subsystems (reductionism). Karakostas and Zafiris' perspectivist methodology is a reverse approach to reductionism, since the whole is the result of the combination of partial perspectives into higher theoretical structures, which are able to transcend the usual synthesis based on the existence of supposedly autonomous self-contained parts. Complex systems are examples of a similar culture of scientific practice, where scientists are trying to make sense of emergent phenomena. Provided that Karakostas and Zafiris' perspectivist methodology has built in the quantum mechanical relationship between the whole and the parts, it could shed light on the problem of emergence and especially under what conditions complex systems exhibit emergent phenomena. In this way, our first intuition about the usefulness of the interaction of philosophy and science in overcoming problems could be proved fruitful.

2 The Kochen-Specker theorem and quantum contextuality

In contemporary science, particularly in quantum mechanics, the way we think about observation and theorization changed dramatically throughout the last century. This shift was captured by Bell (1966) and independently by Simon Kochen and Ernst Specker (1967), who proposed a theorem that views experimentation as determining the contextual basis for the manifestation of properties of quantum systems while rejecting the previously held position of experiment as the revelation of a predetermined reality. Kochen-Specker theorem is a no-go theorem meaning that it expresses what quantum theory is not by generating a contradiction if we assume certain assumptions about quantum reality. Kochen and Specker arrive at this conclusion without anchoring their argumentation on the notion of measurement and relying solely on the algebra of projection operators, ensuring that we can interpret quantum theory by overcoming instrumentalist accounts of science, which are based only on experimental operations to explain quantum theory.

According to the Kochen-Specker theorem, there is no assignment of definite values to all projection operators associated with a system. This partial determination naturally leads to the view that each experimental environment, interconnected with the underlying micro-physical reality, allows certain properties to manifest. In other words, the experimental context and the quantum state of the system are critical for the manifestation of quantum properties as well as the provision of conditions for knowledge claims (Karakostas, 2014, p. 14). Our aim in this essay is to understand the indeterminacy of empirical knowledge in quantum mechanics (Karakostas, 2007, Zafiris, Karakostas, 2013, Karakostas, 2014, Karakostas, Zafiris, 2017), and suggest under what conditions is possible an appropriate combination of contexts (Karakostas, Zafiris, 2021). This analysis will include the subsidiary claim that we can realistically interpret our empirical claims about quantum reality only within a context. To this end, in the following sections, we will discuss the Kochen-Specker theorem (1967) and its implications for the assignment of certain values to projection operators and experimental propositions referring to the eigenvalues of observables.

2.1 The status of properties in physics

One of the most fundamental assumptions of physics is that objects always have definite properties defined as numerical values that fall within specific ranges (Isham, 1995, p. 67). In classical physics, we speak of systems that possess definite values for their observables (physical quantities) at any time, while the experimental propositions referring to these properties have determinate truth-values. For example, the proposition "This desk is two metres wide", has a definite truth-value and reflects the fact that properties are attributes of objects. This realistic viewpoint holds that all physical quantities associated with a system have specific numerical values. Simply put, the system has its properties prior to measurement, and the measurement reveals them to us.

However, the situation of properties in quantum mechanics is different. The Kochen-Specker theorem (1967), although first established for the exclusion of a certain class of hidden variable theories in quantum mechanics, holds that not all observables in a system have or "possess" definite properties at all times (Isham, 1995, p.194). Therefore, semantically speaking, we cannot assign truth values to every proposition specifying whether the system possesses a property (Karakostas, Zafiris, 2017, p. 851). This seems counter-intuitive on behalf of quantum theory, as it is incompatible with our daily experiences and objects. We think of objects as defined by their properties, and therefore it is difficult for us to understand how quantum systems gradually manifest them. But before we delve into how a quantum system manifests its properties, let's first look at how observables, properties, projections, and empirical propositions coexist in quantum theory.

2.2 Observables, properties and projections

To understand what it means that the properties are not fully manifested or possessed by the system, we must first examine their definition. In quantum mechanics, the mathematical object of self-adjoint operator represents any observable A of a system S. In the simplest case, the self-adjoint operator has a discrete non-degenerate spectrum of eigenvalues $\{a_i\}$ for $i = 1, 2, \ldots, m$ that represents the properties of the observable. Each observable has a collection of projections $\{P_{a_1}, P_{a_2}, \ldots, P_{a_m}\}$ defined as $P_{a_i} = |a_i >< a_i|$ corresponding to the proposition 'the observable A has a certain eigenvalue a_i' for $i = 1, \ldots, m$ (ibid, pp. 66, 102). In light of the definition, we can see that in quantum mechanics for each property a_i there is a projection and a proposition, both of which obtain its value (0/false or 1/true) according to whether the system has the property (i.e., 1/true) or not (i.e., 0/false). If the quantum system S is not measured and is in a superposition of states, then the projections of the system are not determined. As a result, we say that in the absence of measurement the properties have not yet been manifested, or the system does not possess them.

When we make a measurement on observable A, the superposition of states $|\psi>$ collapses into a single eigenstate $|a_i>$ of A determining at the same time the projections' values. Only in this case, we are legitimized to say that the system S has or 'possesses' the defined value/property a_i and to assign the proper truth-value to the corresponding proposition ($P_{a_i} = 1$). In such a circumstance the remaining projections of the observable A have the definite value of 0, and the propositions describing them are false, indicating that the system does not possess these properties.

Apart from observable A (the observable to be measured), physical quantities associated with the quantum system S are divided into two categories: those that are compatible/commutative/co-measurable with A ($[B,A] = 0$) and those that are not ($[C,A] \neq 0$) (von Neumann, 1932/2018). Before measurement, the projections of the two categories have no values, and we say that the properties of the system are latent or potential. Provided that their projection values are not 1 or 0, their properties are neither manifested nor non-manifested, but exhibiting potentially their manifestation under appropriate conditions (Isham, 1995, pp. 83-84; Karakostas, 2007). Popper describes this type of properties as *propensities* (1956). After the measurement of observable A, the projections associated with the co-measurable observables of the system obtain their values, while the projections of non-co-measurable observables do not. Consequently, during the measurement, the latent/potential properties of the co-measurable observables are transformed into definite properties possessed by the system.

However, observables that are not co-measurable with A still remain undefined and require a different experimental arrangement in order to be manifested by the system. For example, in the Stern-Gerlach experiment[1] (1922a, 1922b), when we prepare the system for the measurement of its spin property towards the x-direction, we cannot know the properties of the spin in the z or y direction. The non-co-measurable physical quantities are repre-

[1]For a recent presentation see Sakurai, 1994, pp. 2-6.

sented by non-commutative operators, and their projections remain undetermined. For example, a measurement of spin in the direction of x makes the system to possess one of the two properties (either $-\hbar/2$ or $+\hbar/2$), while the values of projections describing the properties of spin in z and y directions remain unknown and the propositions related to these remain undecidable (without truth-value).

Indeed, the Kochen-Specker theorem systematizes the underdetermination of properties in quantum theory by proving the impossibility of assigning truth-values to every proposition pertaining to the system independently of the choice of co-measurable observables. However, before we can grasp the Kochen-Specker theorem, we should first look at some definitions.

In standard quantum mechanics, the space associated to a physical system is formulated by a separable, complex Hilbert space $\mathscr{H}$. The aforementioned projections $\{P_i\}$ and empirical propositions describing system's properties belong in the Hilbert space $\mathscr{H}$ or equivalently in the closed linear subspace $\mathscr{H}_{P_i}$ of $\mathscr{H}$ upon which the projection operator P_i projects.

Definition 2.1. Let $B(\mathscr{H})$ be the algebra of bounded operators on some Hilbert space $\mathscr{H}$. For example, the element $A \in B(\mathscr{H})$ represents a quantum observable.

Definition 2.2. A valuation on $B(\mathscr{H})$ is a function

$$\lambda : B(\mathscr{H}) \to \mathbb{R} \tag{1}$$

to the real numbers, satisfying two conditions:

- value rule: the value $\lambda(A)$ belongs to the set of eigenvalues of A;
- functional composition principle (FUNC): for any pair of self-adjoint operators A, B such that $B = f(A)$ for some real-valued function f we have $\lambda(B) = f(\lambda(A))$.

Observe that if A_1 and A_2 commute, then it follows from the spectral theorem that there exists an operator C and continuous functions f_1 and f_2 such that $A_1 = f_1(C)$ and $A_2 = f_2(C)$. Then the FUNC principle implies that a valuation satisfies

$$\lambda(A_1 + A_2) = \lambda(A_1) + \lambda(A_2) \tag{2}$$

and

$$\lambda(A_1 A_2) = \lambda(A_1)\lambda(A_2) \tag{3}$$

Theorem 1 (Kochen-Specker theorem)**.** *If the dimension of Hilbert space is greater than two ($dim(\mathscr{H}) > 2$), there are no valuations on $B(\mathscr{H})$ for all operators that satisfy the above properties.*

Furthermore, the one-to-one correspondence between the set of all closed linear subspaces of $\mathscr{H}$ and the set of all projection operators L_H ensures us that the lattice structure of the Hilbert subspaces is inherited into the algebra of projections L_H (Varadarajan, 2007). Provided that the elements of the algebra of projections L_H are the orthogonal projection operators $\{P_i\}$ of the algebra of bounded operators $B(\mathscr{H})$, an equivalent form of the Kochen-Specker theorem (Karakostas, Zafiris, 2017) can be posed as follows:

Theorem 2 (Kochen-Specker theorem)**.** *For any quantum system associated to a Hilbert space of dimension greater than two ($dim(\mathscr{H}) > 2$), there does not exists a global (for all projections), truth-functional assignment $h : L_H \to \{0,1\}$ from the algebra of projections L_H [2] to a two-valued set.*

The Kochen-Specker theorem proves geometrically that it is impossible to assign values 0 and 1 to all projection operators $\{P_i\}$ regardless of the system's specific features. In this way, the formalism of standard quantum mechanics rules out the possibility of assigning values to all projections/propositions and thus definite properties to every observable pertaining to the quantum system.

What factors, however, influence which propositions and projections have definite values? The state of the system and the experimenter's choice of co-measurable physical quantities determine the answer to this question. For instance, if the state is a superposition of multiple eigenstates $|\psi >= \sum_i c_i|\psi_i >$, the values of the projections are completely undetermined because their definition depends on a single eigenstate. A superposition of

[2]A quantum event algebra L is an orthomodular $\sigma-$orthoposet (e.g., Dalla Chiara et al., 2004), that is supplied with a maximal element 1 and with an operation of orthocomplementation $[-]^* : L \to L$, which satisfy, for all $l \in L$, the following conditions: (i) $l \leq 1$, (ii) $l^{**} = l$, (iii) $l \vee l^* = 1$, (iv) $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, (v) $l \perp l \vee \acute{l} \in L$, (vi) for $l, \acute{l} \in L$, $l \leq \acute{l}$ implies that l and $\acute{l}$ are compatible, where $0 := 1^*$, $l \perp \acute{l} := l \leq \acute{l}^*$ and the operations of join $\vee$ and meet $\wedge$ are defined as usually. In view of the fact that the quantum event algebra L is isomorphic with the orthocomplemented lattice of orthogonal projections P_i on a complex Hilbert space, denoted by L_H, we can safely assume that the quantum observables are in bijective (one-to-one) correspondence with the Hermitian operators of the Hilbert space. As a result, for the sake of our discussion we can think of the quantum event algebra L as being equivalent to the algebra of projections and propositions L_H pertaining to the system.

states is not empirically accessible as it refers to all possible measurement contexts and all observables (Karakostas, 2007, pp. 284-285, 295).

Instead, if we prepare the system to be in mixed state $\rho = \sum_{i=1}^{n} c_i^2 |a_i >< a_i|$, the only determined projections are those of the observable A and of all co-measurable observables. In this case, we can deduce which properties are manifested, to calculate the projections' values, and make knowledge claims based on empirical propositions with a definite truth-value. Consequently, we need to contextualize the system based on a certain observable and state in order to be cognizable (empirically accessible), as it will become clearer in the sequel.

According to Karakostas (ibid, p. 293), contextualization occurs before measurement when scientists prepare the system's state. Modern experiments achieve this by using equipment that amplifies the aspects of the system that we are interested in while reducing the irrelevant observables and degrees of freedom. Scientists follow very specific recipes to ensure that such procedures result in the same prepared system (Peres, 2002, p. 12), while sometimes the experimenter's creativity is critical in preparing a system. Although preparation procedures are too primitive for quantum mechanics to have a theoretical counterpart, we can observe three theoretical consequences. The first is the specification of an orthonormal basis set for the state-space as the eigenstates $\{|a_i>\}$ of the observable A to be measured, which is equivalent to choosing the set of co-measurable observables and their common eigenvalues/eigenstates. This is followed by a restriction of the system's state-space that results in the space spanned by the eigenstates $|a_i>$ for $i = 1, 2, \ldots, m$, defined as $\mathscr{H}_A = span\{|a_i>\}$. The third theoretical consequence is the value-determination of the set of projections operators $P_{a_i} = |a_i >< a_i|$ corresponding to the properties of A and those of the co-measurable observables. Given these consequences, contextualization can be summarized as a re-description of the system from the general initial state $|\psi>$ to the mixed state $W_a = \sum_{i=1}^{n} c_i^2 |a_i >< a_i|$ (Karakostas, 2007, p. 295).

As a result of the determined values of projections $P_{a_i} = |a_i >< a_i|$, we can say that quantum systems have well-defined properties only within a context. The choice of the set of co-measurable observables and the state of the system co-determine which properties will manifest. Hence, empirical claims in quantum mechanics are always expressed and constrained by the context of co-measurable observables (Karakostas, Zafiris, 2017, p. 879). In the next section, we will discuss the implications of contextuality for knowledge claims about quantum systems.

2.3 Principle of bivalence and the logical structure of quantum propositions

In this section, we will look at how the Kochen-Specker theorem influences the logical structure of empirical propositions that describe the system's properties. In fact, a realistic interpretation of quantum theory significantly depends on the logical structure of propositions. According to Dummett (1982, p. 101), a realistic interpretation of a given class of propositions necessitates the obtaining of certain conditions that explain the truth-values of those propositions. This feature of realistic interpretations is called objectivist semantics, and guarantees that the terms of propositions, through their conditions, refer to an objective, independent reality (Dummett, 1982, pp. 62-63). With this in mind, as long as is demonstrated that the conditions for the truth-values obtain, we can interpret quantum propositions realistically.

As previously stated, each projection operator P_{a_i} corresponds to the proposition "observable A has the value/property a_i," and thus, following the preparation procedure, the propositions associated with observable A have definite truth-values. In fact, the most important feature of the preparation is conditioning some of the system's values (Karakostas, 2007, p. 293). In other words, what enables the manifestation of properties is not the system alone, but its connection with an apparatus capable of expressing the specific property. Thus, in the future, we can be sure that the same preparation procedure will cause the same manifestation of properties. The preparation procedure does not serve as a pre-stage for the verification but as a precondition for the manifestation of properties and the truth-values of the quantum propositions describing them (Karakostas, 2014, p. 14).

Specifically, every time we prepare the system in the same state, the context provides the same conditions for the same set of propositions. Such a situation reveals a contextual/quantum definition of causality because the same manifestation of properties is caused by the same conditions provided by the context. The conditions for the manifestation of properties are not obtained by some observer's intangible consciousness but by the micro-physical reality involving the system itself and the physical environment surrounding it. For this reason, we could say that since the conditions for propositions with definite values obtain, a realistic interpretation of those propositions is possible, whereas it is impossible for propositions that remain undecidable because the conditions for them are not met. Simply put, we can only be realists about propositions whose truth-values are decided by the context.

However, the Kochen-Specker theorem prevents us from assigning truth-values to all propositions describing the properties of a system. As a consequence, we cannot apply the principle of bivalence and certain propositions remain undecidable (Karakostas, Zafiris, 2017, p. 7). These types of empirical propositions are associated with observables that are incompatible/non-co-measurable/non-commuting with the observable A for which the system is prepared to be measured. A realistic interpretation of these is impossible, since the preparation of A does not fix any conditions for the truth-values of incompatible propositions. This situation is represented mathematically if we consider the observable C which is incompatible with the context of A ($[C, A] \neq 0$) and whose projections are $P_{c_i} = |c_i >< c_i|$. The values of C-projections $< P_{c_i} >= Tr(\rho P_{c_i})$ for $\rho = \sum_{i=1}^{n} w_i^2 |a_i >< a_i|$, depend on the product $0 < w_i^2 |< c_i|a_i >|^2 < 1$ which leaves undefined the proposition P_{c_i}. Hence the empirical propositions related to C do not have definite truth-values.

Although, the manifestation of a quantum system is not as absolute as in classical physics, the dependence of the same partial manifestation of the system on certain conditions through the preparation procedure can be a step towards the formulation of a quantum version of objectivity (Karakostas, 2014, p. 14). In the quantum case, the objectivity of the determined propositions is not based on some unconditioned reality that happened to be observed or on the subjectivity of the observer. Instead, objectivity is ensured by the preparation procedure that constitutes or objectifies the states of affairs addressed by the definite propositions. Or, semantically, the preparation procedure establishes the correspondence between the propositional content and the appropriate states of affairs. Hence, the objectivity of empirical claims is based on the fact that the same set of propositions reflects the same objective reality (quantum states of affairs) formed during the same preparation procedure.

In technical terms, we refer to the objective reality formed by the quantum system's connection with the preparation apparatus as the truth-maker of a specific set of propositions. The context, in conjunction with quantum states of affairs, provides truth conditions that contribute to meaning, i.e., comprehension of the proposition's content, by establishing a clear reference between propositional terms and objective reality (truth-makers). This objective reality, however, is not 'out there' independent of anything else, as in classical mechanics, but has been formed in an experimental environment as a result of a preparation process that objectifies certain aspects of the quantum system (ibid, pp. 13-14). For this reason, the facts are contextual since they are formed during a certain preparation procedure, and thus the quantum propositions that refer to them are also contextual. As we read from Karakostas (ibid, p. 14) "Truth contextuality follows naturally from the contextuality of facts". Contextual truth-assignment is not in accordance with relativistic accounts of truth because the truth-values of knowledge claims are linked to the fulfilment of certain conditions represented by the context and are not simply relative to it (Karakostas, Zafiris, 2017, p. 879).

It should be noted that in general the conditions provided by the context of an observable A and the system's state are not necessary material. Sure, these conditions may have a physical aspect, but in principle that aspect does not clarify their being. We can think of them as conditions for the possibility of experience of the quantum events. These are implicit assumptions about the nature of quantum events, which through devices and theories establish the quantum reality as empirically accessible to us. Unless and until these conditions are met in a preparation procedure, we are unable to comprehend any quantum event. In classical mechanics, an analogy can be found for example in the concept of space, which is a prerequisite for the possibility of experiencing three-dimensional objects and was first emphasized by Kant in his "Critique of Pure Reason" (1998). In the case of incompatible quantum observables, the preconditions for their experience are not met during the preparation procedure, so we cannot experience their properties and must be agnostic about knowledge claims referring to them.

Another significant consequence of the theorem is that even though we might try our best to improve our experimental arrangements, it is impossible to determine the properties of the incompatible observables and that is why the overall logical structure of the system will always be non-Boolean. We have to accept the undecidability of incompatible properties as an immanent feature of quantum existence. At the same time, we can be realists only for the propositions regarding properties compatible with the context. Rather than a broad claim for everything that exists at the quantum level, any empirical claim for quantum existence must be asserted through a context that provides the truth-conditions. The Kochen-Specker theorem, therefore, shows that the cognitive process in quantum mechanics cannot be all-encompassing or a "God's eye view" that is independent of any condition (ibid, p. 877).

In light of the inherent indeterminacy of quantum properties, we see that the concept of context is a necessary tool for the interpretation of the quantum mechanics because it provides a means of reference to the conditions involved in the epistemic process as well as on how the quantum state is re-described in order to become empirically relevant to us. As previously stated, the context is defined by the observable under investigation and together with the quantum state co-determine the properties of co-measurable observables. Since it is impossible to attribute properties to quantum systems prior to measurement, the quantum object is not found in nature ready for inspection, but is contextual as it is gradually determined by the context. As a result, one could argue that the metaphysical position of scientific realism must be restated in order to take into account the findings of the Kochen-Specker theorem. This is not to say that we should reject a realistic interpretation of quantum mechanics but the metaphysical position of traditional scientific realism[3] should be separated between the independence of existence of quantum reality and how it becomes knowable to us. In the latter case, as this chapter hopefully clarified, quantum reality (states of affairs) is not simply discovered by our instruments or independent of the epistemic process, but rather must be embedded in a context in order to manifest its properties. In the following chapter, we will see how philosophy of science, based on various events from the history of science, comes to similar conclusions about knowledge while it attempts to maintain a realistic interpretation of science.

[3] see Psillos, 1999

3 Scientific perspectivism

During the twentieth century, rapid advances in science led to a widespread production of philosophical approaches intended to help make sense of the situation. This resulted in the creation of a complex and specialized field, called philosophy of science, that takes on the task of explaining the issues raised by scientific research. Philosophy of science emerged as a distinct academic subject during the previous century.

The central theme of this discipline is to comprehend science as a cognitive activity. This can be accomplished by asking specific questions such as how empirical facts relate to physical theories, what the scientific method is, and how science achieves objectivity. Last but not least, how we can describe the development and change of scientific theories. Throughout the first half of the previous century, philosophy of science was dominated by the formal framework of logical empiricism/positivism, which provided answers to some of the aforementioned questions by viewing science as a linguistic system. Demonstrating how mathematical theories can be used in empirical sciences was central to logical empiricism.

In the 1960s, the historical and naturalistic turns undermined logical empiricism. While the aforementioned questions remained unchanged, historiography and the examination of scientific practice provided further answers. The main conclusion of these studies was the "incommensurability" thesis, which stated scientists' inadequacy to compare paradigms from different historical periods. Historians were led to believe that the meaning of concepts like truth, objectivity, natural laws, and so on is constrained by current social-economic relations and is not universally accepted across all historical periods. After the 1980s, constructivism emerged as the major proponent of the historical turn, arguing in its extreme form that scientific notions are social constructs unable to characterize our knowledge being historically contingent (Psillos, 2007, introduction).

Therefore, today's philosophy of science can be divided into two main schools: constructivism, having emerged from the naturalistic and historical turn of the 1960s, emphasizes historical and social aspects of scientific knowledge and is usually associated with anti-realism; and realism that stresses the formal aspects of science and its accuracy in depicting reality as evidence for its validity. Scientific perspectivism is a collection of viewpoints that attempt to overcome the aforementioned dichotomy.

3.1 Between scientific realism and constructivism

As an introductory remark, we can define perspectivism as a collection of philosophical views that support the idea of scientific knowledge describing the world, while acknowledging scientific knowledge's historical and cultural context (Giere, 2006; Massimi, 2018). The first part of the previous view is a realistic commitment, whereas the second is motivated by the effects of social constructivism in science. It is for this reason that perspectivism is sometimes motivated by the synthesis of scientific realism and social constructivism.

In its extreme form, objectivist realism proposes that science can make universal and objectively true claims about the world, and that these claims, once verified, are permanent parts of our knowledge, regardless of future theory changes.

For instance, according to Steven Weinberg (2001, p. 126): "What drives us onward in the work of science is precisely the sense that there are truths out there to be discovered, truths that once discovered will form a permanent part of human knowledge".

This excerpt reveals an extreme form of realism that supports the idea that the scientific process produces claims that can never be falsified. Perspectivism considers this position as excessive while acknowledging realism to be correct in pointing out that unfalsified scientific claims grasp something real about the world (Giere, 2006, p. 4). Perspectivism seeks to restore the notion of truth adopted by realism, one that, according to Giere (2006, p. 81), is "relative to a perspective" or, according to Massimi (2018, p. 343), meets certain "performance-adequacy standards" shared across multiple perspectives. By introducing the concept of perspective, perspectivism, or perspectival realism for Massimi, preserves the correspondence of scientific statements with reality while rejecting objectivist realism's thesis that our scientific theories hold a permanent and objectively true account of the world, "a view from nowhere".

Perspectivism also considers the picture of social constructivism, which sees science as a phenomenon of social interactions and institutions, rather than as the result of an isolated scientific cogito (Giere, 2006, pp. 3, 6-9; Massimi, 2018, p. 1). The fields of history and sociology of science observe that antagonism found in the context of social interactions affects the status of scientific theories and thus what may be true or false.

Another aspect that perspectivism shares with social constructivism is the position of epistemological pluralism (Giere, 2006, p. 92; Massimi, 2018, p. 344). In such a case, one maintains that there are multiple ways of gaining knowledge and several descriptions of the same phenomena. The existence of multiple perspectives on a given phenomenon supports the fact that scientific claims about it are made from a particular point of view, and broadens the range of explainability. Additionally, the multiplicity of explanations for a phenomenon can affect how fundamental we consider it to be, since the same explanan is involved in many different explanans (Giere, 2006, p. 93). This new formulation of fundamentality is based on multiple perspectives, but it remains perspectival and subject to revision in the future.

Toward this end, it is instructive to examine the origins of scientific perspectivism, beginning with Giere's (2006) formulation, which takes ideas from color vision and uses them to articulate a general scheme of scientific practice, and progressing to Massimi's concept of perspectival truth.

3.2 Contingency thesis, experimentation, and scientific theorizing

Giere develops the epistemic basis for his approach in chapters two and three of "Scientific Perspectivism" (2006). Such a philosophical position is motivated by the way we make sense of the world, which is mainly through human vision and scientific instruments. The fact that observation is a many-to-one relationship is used to demonstrate the perspectival nature of experimentation and scientific theorizing (ibid., p. 42). Specifically, the human eye and measuring apparatus select a very narrow type of input in contrast to the diversity of stimuli in nature. For Giere, selectivity in observation sets the limits and forms the perspectival nature of our knowledge.

Physical theories are also perspectival due to the way they are validated and constructed. The instruments used in modern experiments are built using older theories than those currently being tested, which have even more limitations. These older theories, in turn, prevailed based on observations that were perspectival in and of themselves due to the use of more primitive instruments, and so on. As a result, the perspectival nature of instruments blends into with the verification of physical theories, both of which end up being perspectival in a stronger sense. Consequently, Giere's contingency thesis proposes that the scientific process is always subject to some degree of contingency and demonstrates the alignment between perspectivism and constructivism (2006, p. 7). Particularly, the contingency thesis claims that human judgments and values distort reality and experimental methods, introducing uncertainty into the scientific process. The experimental method is founded on previously validated theoretical and observational perspectives that influence the outcome and interpretation of the experiment. These previously validated perspectives, in turn, are founded on misguided daily values, meanings, and experiences. Therefore, the product of scientific practice is a mixture of perspectives that is still biased and not objective in a stronger sense.

The possibility of another model or subsequent experiment to fit the object under observation better than the current one is a common conclusion regarding the status of an experiment. Every model or experiment can be improved by a subsequent one that turns out to be deficient in other areas. Such an assessment adds to the ambiguity of the scientific method by leaving open the possibility of a different match to reality. Additionally, an alternative model may explain the world based on different principles, further obscuring the matter. Giere's main argument aims at showing that perspectivism is as much realism as science can provide, and that is why scientific practice is impossible to support objectivist realism.

To deal with the contingency of scientific practice, Giere suggests methodological naturalism to be the general framework on which perspectival interpretations of the scientific method are based (2006, p. 12). When science encounters obstacles, philosophy does not have to resort to supernatural assumptions in order to interpret the scientific practice. Instead, perspectivism should assume that the causes of these deadlocks are empirical or naturalistic in order to be measured, quantified, and studied. This position rejects claims to absolute, supernatural or a priori truth/hypotheses and advocates for the replacement of metaphysical doctrines with methodologies such as a posteriori investigations, which result in synthetic knowledge about the world. The only accepted hypotheses for methodological naturalism are those that produce useful methods, which is why absolute truth is rejected as unattainable. In contrast to objective realists, who, as Weinberg suggests, support an absolute notion of truth for our scientific knowledge, perspectivists seek a method to improve our current knowledge.

We can now summarize the argument in a few points:

- Instruments
  - are used to gain a better understanding of the world.

  - are open in some stimuli but not in others, indicating that they are not transparent.
  - yield partial and perspectival knowledge, implying that empirical claims based on them are perspectival.

- Scientific practice is subject to contingency.
- Philosophy should articulate methodologies to face science's contingency and obstacles (methodological naturalism).

3.3 A scheme of scientific theorizing

In this section, we will develop Giere's scheme of scientific theorizing, which supports the perspectival nature of knowledge. Giere's perspectivism conceives scientific practice as the creation of models based on certain principles and specific conditions. Scientists assert claims about the fitness of these models by applying them to the actual world. As knowledge grows, hypotheses are stated and compared to data models generated by data analysis employing logical and mathematical techniques to cluster actual observations. Following that, scientists can generalize and improve the fitted hypotheses by applying them to previously unconsidered classes of objects (Giere, 2006, p. 60). As a result, we can safely assume that scientific perspectivism supports the co-determination of hypotheses and data models created by experiments. Giere's scheme seems to follow Bachelard's description of an "evolutive relation of co-determination" between noumenology (including mathematical hypotheses) and phenomenotechique (including physical theories and technological means of experimentation) (Fabry, 2019, p. 2).

Giere defines two processes for analyzing the aforementioned method of scientific theorizing. The first is a top-down process that starts with the principles and specific conditions that scientists used to create representational models. They compare these models to a specific class of real-world objects and draw specific conclusions about their compatibility. The second process is a bottom-up approach that starts with undifferentiated raw observations, which are then stacked into smaller sets and combined into data models using data analysis techniques. As a result, specific hypotheses extracted from top-down processes are compared to the data models inferred from the bottom-up approach, and if they agree, they are extended to more objects.

The key to understanding perspectival knowledge is to examine each component of Giere's scientific theorising scheme, beginning with the issue of general principles. A more precise illustration of this concept can be found in Newton's three laws of motion, Maxwell's equations of electrodynamics, and the thermodynamic axioms. General principles have been wrongly interpreted as empirical laws, namely "generalizations that are both universal and true" (ibid, p. 61). Giere (1988; 1999), Cartwright (1983; 1999), and Teller (2001; 2004) observe that when scientific principles are interpreted as universal statements capable of making true empirical claims, they turn out to be vaguely true or false when tested experimentally. The reason for this is that general principles cannot be both universal and refer to specific empirical objects. For example, Newton's three laws of motion use terms such as force, mass, and acceleration, yet they do not relate these concepts to actual things in the world that act as forces or as masses. A general principle does not pertain to specific objects, but rather to ideal entities, like points and vectors. Thus, the question of how theories bridge the gap between principles and reality must be addressed.

One way to bridge the discrepancy between the principles and reality is by seeing science as a linguistic system (Giere, 2006, p. 61). This approach considers principles to be genuine statements, namely capable of declarative strength. In other words, principles can make normative positions regarding objects of the world. Scientists are then obliged to find an abstract object for which the aforementioned principles are valid, and their task is to find relations among the world's elements and the abstract model to satisfy the normative character of principles. The object created by such a process is subject to a generalized model. The simple gravity pendulum, for example, is the generalized model for all real pendulums that exist in a gravity field. Giere identifies models with abstract objects, suggesting that science studies strictly defined objects that retain their basic properties in time and distinguish their existence from their surroundings.

However, an abstract model is still far from an empirical claim about reality, which is why the linguistic approach to science cannot provide additional insights. One way to close the gap between theoretical and empirical is to assign precise mathematical quantities to the model (ibid, p. 62). For example, by defining a mass of 5 kg swinging from a pendulum of 20 centimetres of length at a local strength of gravity, we can empirically ascertain whether a real mass movement corresponds to the calculated motion of the abstract model. Therefore, Giere's "specific conditions" shape and limit further the structure of an abstract model to a specific example with direct empirical claims. For this reason, scientists need to add specific conditions to general principles in order to produce more precise empirical statements.

Additionally, Giere acknowledges two ways to restrict an abstract model. The first is a suitable "interpretation" of the theoretical terms involved in general principles. The Newtonian principles, for example, incorporate the terms of force and mass into a context by determining relationships with other terms such as position, velocity and acceleration. By correctly interpreting the terms mass and weight through their relations with other terms used in Newtonian principles, we can better restrict the pendulum model and apply it more effectively in real-world situations. The second activity involves the identification of elements of a model with specific things in the world. For example, the bottom point of a massless rod in the pendulum model has to be identified with the massive bob of the real pendulum. If we identify all relevant elements with the appropriate real objects, the model is as specific as possible.

Here, it may be helpful to consider what Giere means by the term "model" (2006, p. 62). Particularly, scientists use models to represent world aspects in order to achieve the greatest degree of identification with reality. This is achieved by forcing the model to have certain similarities with the part of the world it represents. To do so, scientists select certain model characteristics that are similar to the characteristics of the designed "actual" system. For example, there are some similarities between Watson and Crick's models and the actual structure of DNA, but they have nothing to do with the metal of the plates used to create a double helix effigy.

The DNA example demonstrates that determining which features of the model apply to the actual world is part of scientists' work. In particular, the angles of Watson and Crick's first model did not reflect the actual DNA bonding angles, but the model's angles were close enough to imply that DNA has a helical shape. To address this issue, scientists define a measure of model-to-real-world similarity that determines the deviation limits that are assumed to be sufficiently similar. Though such a measure of similarity is extremely useful in applications, what is sufficiently similar depends on the purposes for which the model is implemented, implying that similarity is a function of context rather than simply a relationship between the model and the designed real system (Giere, 2006, p. 65-66). Giere suggests replacing the phrase 'a model is true of' with 'similar to', which implies "that one does not expect a perfect fit between models and the world" (ibid., p. 66). That is why he investigates the relationship between similarity and truth-value.

Giere argues that ascribing truth-value to a model is irrational because models are closer to predicates than statements. Predicates apply to a specific region of the universe, and thus they are only "true" for that particular. Similarly, models can be "true of" or "apply to" a thing. Given that a system is experimentally isolated aspects of the world, a general position on the model's truth-value can include several system-specific statements. The claim that a model fits a specific system implies that the values of the model's specific parameters are consistent with the measured parameters of the actual system. In turn, experiments test the validity of such statements.

However, the claims concerning the values of the specific parameters for an actual system cannot exhaust the fitness of the model. We need also to define dynamic relationships between variables that describe the behaviour of the system. By making statements about the relationship between variables within the model we exhaust the content of the general statement that the model "fits", "applies to", and "is applicable to" the actual system (Giere, 2006, p. 65). Thus, Giere prefers to discuss fitting rather than truth since it is more useful for studying the similarities between models and systems in terms of the representational relationships between the world and the models used to represent it. With this move, he replaces the metaphysical doctrine of truth with a method of studying how we represent objects of knowledge.

The second concept we need to discuss in Giere's scheme of scientific theorizing is generalizations (2006, p. 67). These can be certain equations that have been wrongly interpreted as laws of nature instead of extensively used equations. Giere demarcates the supposed laws of nature between those that are more like lower-level equations and those that are grand principles. As a consequence, statements based on the former case are not universal. They should be regarded as ordinary statements that are precisely true only in certain abstract (representative) models "and thus being strictly true of the model" (ibid.). Abstract models must be constrained by actual systems or groups of systems that fit better or worse.

According to Giere, the concept of law has been misunderstood, and for this reason, it is critical to clarify the types of equations included in such a concept. Laws of nature, for Giere (2006, p. 70), are general principles that define highly abstract models such as Newton's laws of motion. In addition, there are other equations called laws that define more specific models that embody a set of general principles. Also, there are "laws" that define models that do not include any general principles. These models, like statistical laws governing medical knowledge, are limited empirical generalizations. In order to avoid confusion, Giere supposes as "laws" only the most general principles that exist in every physical theory.

Data models, first mentioned by Suppes (1962) and Woodward (1989), are the next component of Giere's scientific theorizing scheme. Consider a model that integrates several variables, two of which are connected by the equation $y = ax$, which is a linear relationship. To put this model to the test, we must create an actual system that displays two measurable quantities, which are then identified with the model's x and y. One could argue that the actual system's design is based on a reconstruction of certain conditions that we believe suffice to reproduce a phenomenon governed by the linear regression $y = ax$. Each measurement is a pair of data points that can be fitted by a linear regression calculated by a software (e.g., least-square algorithm). It creates a linear data model based on a normal distribution of the difference between calculated and measured values. This choice supposes the profound metaphysical assumption that a measurement error results from random fluctuations whose effect follows a specific distribution. The difference between a and a_0 must be small enough to say that the measured value matches the value provided by the model (Giere, 2006, p. 68).

The fitting of the experimental phenomenon to certain distributions indicates a new conceptualization of experiment, namely the physical implementation of a theoretical model rather than the passive observation of nature with no pre-conceptualization. The analysis of the Kochen-Specker theorem is inclined towards the same conclusion since it demonstrates that every measurement is pre-conditioned by the choice of a context. In Giere's view, the experimental fitness of a model is determined by comparing it to data models, not by comparing it to unconceptualized reality (2006, p. 68). Using various statistical techniques, scientists manipulate the data to get a model of data. Then, a second comparison of the data model with abstract representative models suffices to establish a general fit between the model and the actual world. It is a fact that different fields of study require different ways to determine what is a good fit. Such conventions have a pragmatic rationale to do with each field's special needs and there is no way to make these inferences universal across all sciences.

Giere's scheme of scientific theorizing concludes with the concept of scientific theories. He claims that there are no single elements that correspond to 'the theory' (2006, p. 69). This is because we use these terms to describe broad meanings in practice and in meta-level discussions about sciences. He argues that when attempting to understand scientific practice, the use of these terms does not allow for the distinction of important aspects that should be distinguished. The term ‘evolutionary theory’, for example, is frequently used to refer to the principles of this theory. These principles describe an abstract object rather than a specific object in the world, whereas almost everyone would rely on the empirical arguments of evolutionary theory for specific species. This trust is built by scientists who create evolutionary models that are based on real evolutionary populations. Giere defines ‘theory’ as a collection of concepts such as principles, models, specific conditions, empirical claims, and so on.

The following diagram illustrates Giere’s scheme of scientific theorization:

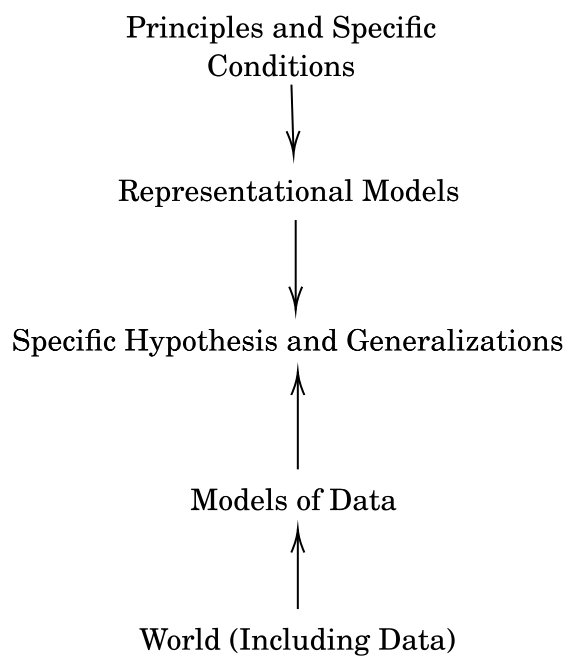

Figure 1: Giere’s scheme of scientific theorizing (2006, p. 61).

We can deduce from Giere’s scheme of scientific theorizing that scientific reasoning does not occur from an objective God’s eye view, but is embedded in several perspectives. According to Giere’s scheme, an empirical claim about the period of an oscillation, for example, depends on several perspectives: the assumptions of the data model, which is a low-level perspective of the real oscillation, the similarities that many particular oscillatory systems may have (generalizations), the representational models of oscillation, and the general principles of the gravitational field in which the oscillation occurs. A general perspective of gravitation based on relativistic principles, for example, will give us richer empirical conclusions about oscillations than Newtonian principles, which are only a limit of the first (Babusci et al., 2013).

Although Giere's scheme is important for the demonstration of the perspectival nature of knowledge, it does not explain the cognitive process by which these various perspectives are linked to create the success of contemporary physical theories. In the following section, we will look at a very elaborate analogy that will help us conceptualize how the various perspectives on a topic could possibly be connected to create the result of scientific thought.

In this respect, knowledge has a perspectival character not limited to the empirical. As we will see in the next section, perspectivity extends as far as the postulates and primitive notions of theories.

3.4 Similarities between Giere's scheme and Cartography

An intriguing aspect of Giere's perspectivism is that it shares many essential features with cartography (2006, pp. 72-80). Maps, like scientific theories, are representational in the sense that they depict morphological data about a geographical area. When creating a map, a cartographer must decide what world stuff to depict, just as a scientist must decide what kind of stuff to represent with any model. The relationship between models and maps is often very clear in some sciences, while it is more metaphorical in others.

Giere uses several examples to back up the similarities between cartography and scientific practice. The most instructive is the problem of flattening the Earth, or mapping the Earth's surface to a flat surface. In a way, this is similar to the practice of building models in science, which has both benefits and drawbacks. The benefit is that maps serve as an example of a purposeful construction of a representation. Navigation and general geography, for example, are two very different purposes for creating a map. These two goals result in different maps that depict the same area differently. In other words, every perspective reveals and emphasizes different aspects of the earth's surface, depending on the cartographer's goal.

Concerning the drawbacks, in cartography, the manipulation of coordinates and scale distinguishes maps of the same area as non-comparable. In this sense, perspectives are incompatible with one another and cannot exchange any information regarding the same issue. While this may be true, perspectivism does not overly restrict communication between different perspectives, allowing them in principle to be compatible (2006, p. 80). The key concept underlying the similarity between cartography and perspectivism is spatiality, as defined by Riemann, which we believe is closely related to perspectivism. In Riemannian geometry we can define the overall space of a manifold as patchwork-like assemblies of local spaces (Plotnitsky, 2009, p. 192). Each assembly describes a different region of the total space and manages the metric uniquely, just as each map manipulates the coordinates in its own unique way. With this move, Riemann eliminates the need for the entire space to have the same type of structure, or the same coordinate manipulation.

Notably, we can give the global space a global determination that is distinct from the local one. For instance, we could define an overall metric structure, which is a formula for calculating the distance between two points. If these points are close to each other, they are likely to belong to the same local space, which we can assume is a Euclidean space without losing generality. The non-Euclidean global metric is then specialized in a formula for calculating distances in the local Euclidean space. In other words, the global structure is contextualized or localized into the local space and acquires constant curvature metric characteristics. As a result, we can see how knowledge can be contextualized to meet the local needs of a specific science, experiment, or formula.

By using cartography as an example, Giere's perspectivism seeks to capture the various ways in which scientists can organize data, empirical claims, principles, and knowledge in general through perspectives (assemblages) that determine the product of scientific theorizing. In our interpretation of Giere's perspectivism, we identify local space congregations (assemblages) as perspectives, scientific knowledge as the global space of manifolds, and theorizing as the patchwork. Perspectivism, by referring to the various ways in which these assemblies congregate, allows for the quality of heterogeneity in scientific knowledge. Thus, the global manifold of knowledge does not impose any characteristics on the local spaces of sciences, but rather allows the epistemologist to study and criticize each formula, theorem, principle, law, model, and so on in its own right. In this way, perspectivism takes seriously the pluralism of methods in contemporary scientific practice.

Another aspect of Riemannian geometry, which is closely related to perspectivism, is the practice of combining different assemblages/perspectives when approaching problems in a single field. For example, the concept of a manifold is created by combining algebra, topology, functional analysis, and geometry, thereby combining divergent fields of mathematical practice to address a single concept or problem. Each field enables us to approach the same problem from a different angle, expressing a different quality. With this goal in mind, we use geometry's perspective to measure distances due to the metric structure that space provides. Alternatively, Riemannian geometry allows us to examine a space from a topological standpoint, ignoring distance measurement and scale issues in favor of focusing on the morphology of space. In this case, we look at the shape of a space, whether it has holes, and determine the shape of the equivalent space based on the number of holes.

Perspectivism, in this sense, allows us to analyze conceptual structures through perspectives that examine the structure based on a single quality (e.g., geometrical, topological, algebraic, logical) and as subjects in their own right, rather than a hierarchical approach that assumes a primary structure that pre-defines their being. By acknowledging that each perspective addresses a concept or a problem in its own right, we can examine the various paths that reasoning can take when they are combined. Non-Euclidean geometry, for example, is a mathematical field of new geometries rather than a single theory that simply negates Euclid's fifth axiom. Perspectivism defies the hierarchical nature of knowledge by studying each perspective in its own right while keeping all other perspectives ready for investigation without elevating a unique perspective to the status of fundamental. We strongly believe that, as Riemannian geometry classifies spaces by their different qualities, similarly perspectivism can create a space for knowledge that classifies different ways of thinking while treating them equally. These heterogeneous spaces created by the synthesis of different perspectives might give rise to new ways of describing the world.

The analogy of perspectivism with cartography can play an important role in motivating the development of a perspectivist methodology that can address the problem of managing the various perspectives involved in scientific thought. But, before we get into a potential method for manipulating perspectives, let's look at the relationship between Kuhn's concept of paradigm and Giere's perspective.

3.5 Similarities and differences between Giere's perspective and Kuhn's paradigm

In Giere's approach, there are many similarities between the concepts of perspective and paradigm, which should be explored so as to understand the relationship of perspectivism to the history of science. Giere (2006, p. 82) compares the notion of perspectives with the paradigms of Thomas Kuhn as defined in his book "The Structure of Scientific Revolutions" (1967). To begin with the similarities, paradigms and perspectives both express scientific statements about the fitness of models. To determine the validity of a claim within a perspective or paradigm, it is compared to data models or lower-level observational perspectives. In this regard, paradigms and perspectives serve the same purpose: they provide a means of validating scientific claims.

Another similarity between paradigms and perspectives is that they are gradually being replaced by new perspectives that "provide resources for new research leading to new verified claims or successful models" (Giere, 2006, p. 82). It is true that a dominant paradigm or perspective is never confirmed definitively, and is always subject to new experiments and observations. As a result, a dominant paradigm that fails to produce verifiable claims can be replaced by new rival paradigms that provide successful empirical statements and new research directions. Therefore, perspectivism preserves a Kuhnian view of scientific change since perspectives, like paradigms, are subject to change.

In terms of differences, Giere (2006, p. 82) agrees with Masterman (1970) and Shapere (1964) that Kuhn's paradigm is ambiguous because it has at least two meanings. According to the first one, the paradigm is an "entire constellation of beliefs, values, techniques, and so on shared by the members of a given community" (Kuhn, 1970, p. 237). The second definition of paradigm is only one element of the previous constellation that is the "concrete puzzle-solutions which, employed as models or examples, can replace explicit rules as a basis for the solution of the remaining puzzles of normal science". The first definition is referred to by Kuhn as a "disciplinary matrix," while the second is referred to as "exemplars". According to Giere, neither definition can be compared to the notion of perspective because a disciplinary matrix is a broader concept than a perspective, whereas the exemplar is more specific and close to the idea of the representational model used by his scheme of theorizing.

Furthermore, Kuhn's disciplinary matrices differ from perspectives in that the former exhibit incommensurability or asymmetry, while the latter are not as rigid (Giere, 2006, p. 82). Incommensurability can be interpreted in two ways: linguistically and methodologically. According to the first, the meanings of the terms in the various successive disciplinary matrices are so dissimilar that they cannot be translated from one to the other. A well-known example is the definition of the term 'mass', which in Newtonian perspective is conserved, as opposed to relativistic mass, which increases with velocity and is convertible to energy. In general, the meaning of a term is determined by its relationships with the other elements of the disciplinary matrix, and when these elements change, so does the meaning of the given term. With the methodological interpretation of incommensurability, the criteria for determining the validity of the argument depend on a specific paradigm and for this reason we cannot compare arguments belonging to different paradigms. Giere is only concerned with the linguistic understanding of incommensurability.

Specifically, Giere contends that the historical background in which historians and philosophers of science were at the time of Kuhn influenced how asymmetrical the paradigms were perceived to be (ibid., p. 83). For example, Kuhn was struck by the incommensurability between modern science and Aristotelian physics, which may have influenced his emphasis on paradigm asymmetry in "The Structure of Scientific Revolutions" (1967). Alternatively, in "Copernican Revolution" (1957), Kuhn advocates for a more gradual transition between the Ptolemaic and Copernican paradigms, arguing that incommensurability was not always an issue. The first case essentially discusses science as a linguistic system, which overemphasizes incommensurability. In this case, translating from one paradigm to another is an extremely challenging task, since all inferential relationships need to be preserved in the new paradigm, which is not possible.

In the following section, we will look at perspectivism's position on a variety of issues, including its relationship with objectivist realism and constructivism, knowledge justification, scientific change, and the concept of truth.

3.6 Giere's perspectivism

Giere's perspectivism is a framework developed in response to the debate between scientific realism and various philosophical approaches inspired by sociology and history of science (e.g., social constructivism, relativism, constructive empiricism) (2006, p. 88). The disagreement, according to Giere, between these two schools of thought is based on a characterization of scientific realism as strictly objectivist realism, which supports the idea that our various claims must be true or false; otherwise, it is synonymous with relativism. In addition, objectivist realism stresses the permanence of empirically supported knowledge, in contrast to constructivism, which emphasizes paradigm change.

An objectivist understanding of science attributes the success of scientific theories in their ability to describe objectively how things are, meaning that our knowledge should be absolutely true or false. This dualistic notion of empirical success supposes that any description that does not encompass all phenomena will fail. In this case, the argument of the pessimistic meta-induction (Laudan, 1981) prevails, since there is always a part of reality that alludes to our supposedly 'universal' theories.

We should, however, always begin our analysis with scientific practice itself and ask how scientists justify their statements based on the relative alignment of their models with the world. Any recognized scientific paper discussing the results of a successful experiment concludes with the need for further independent examination of the results and methods used, as well as their limited application to other areas. Accordingly, a perspectival understanding of scientific realism would describe how scientific methods convert perspectives' hypotheses into alignment with experimental 'reality' (data models). One might suggest that philosophy of science should comprehend how scientific perspectives construct true claims by converting, applying, and specializing the general principles upon which they are founded.

As we saw in Giere's scheme of theorizing, general principles need to be accompanied by specific conditions and many other lower-level perspectives in order to be applied to real circumstances. When the old hypotheses (principles and specific hypotheses) do not produce new representational models that comply with the experimental data, they must be rejected or modified as inapplicable to the physical domain under study. The problem of justifying scientific knowledge lessens if philosophers and scientists accept perspectival knowledge claims that apply to certain aspects of the world rather than to all (localness of knowledge), as with Riemannian geometry. Giere argues that the perspectival conclusions are not relativistic, but they sought to be the most precise conclusions (2006, p. 92).

Moreover, the most realistic argument that we can make has the general form: provided the presumed theoretical and observational perspective, model M shows a good fit for the particular subject of interest. There are no more objective, less perspectival arguments as to how the world should be. Possible alternatives to model M would all be constructed within the existing observational and theoretical perspectives, and the overall situation would not change much. If several other perspectives appear to have the same conclusion, that does not mean that the conclusion is more objective than a simple claim stated from a single perspective (ibid.). But, it is good evidence that there is something out there. For example, the speed of light shows up in several theoretical perspectives such as special relativity, electromagnetism and quantum field theory which exhibit a good fit with real physical systems. This conclusion, however, does not make the speed of light independent of all theoretical perspectives. According to perspectivism, an appropriate conclusion is that the speed of light is a fundamental property of our most experimentally successful theoretical perspectives, allowing us to assume it as a fundamental part of nature as confirmed by our instruments. Such a combination of perspectives is still perspectival and not objective in a stronger sense.

Perspectivism should also address the issue of how a perspective forms its phenomena of interest. Since knowledge claims refer to certain isolated aspects of the world, which are empirically accessible only through technical means, perspectivism needs to describe how these aspects are isolated in order to form the object to be observed, either theoretical or experimental. Most sciences today agree that phenomena are not simply found in nature but are constituted through a process of purification and exacerbation of certain aspects. Also, it became apparent from Giere's scheme that every choice of perspective (data models, representational models, general principles, specific hypotheses) has an irreversible impact on its final fitness to reality. For this reason, perspectivism should address how the various perspectives involved in scientific practice influence the investigation.

3.7 Scientific change

As we said earlier, scientific perspectivism inspires to preserve a Kuhnian view for scientific change. According to Giere (2006, p. 95), historians of science describe a situation in which most previous theories proved false and were replaced by newer theories that fit better with experimental data. By induction, it is very probable for our current theories to be false and better theories to take their place when new experimental data accumulate. This is essentially the pessimistic meta-induction argument, which has been a major argument against scientific realism (Laudan, 1981). For Giere's perspectivism, theories cannot be considered merely sets of propositions that have universal truth or falsity, as pessimistic meta-induction suggests, but rather interconnected models that have good or bad fits with reality. An assertion about the fitness of a general perspective, as we discussed earlier, involves a number of specific propositions about many empirical tests. It is therefore not sufficient just to characterize a general perspective, which is to some degree resistant to refutation in a number of experiments, as true or false, but it is necessary to describe weaknesses and strengths in specific cases.

Even though the notion of fitness explains how perspectives become established or refuted, it does not explain how perspectives evolve and why certain aspects of previous perspectives may be retained in new ones while overall knowledge is extended. For example, while the relativistic perspective fits the new astronomical data of the twentieth century better than the Newtonian perspective, fitness does not explain the embedding of the Newtonian gravity potential in the relativistic perspective. Indeed, the Newtonian perspective assumes 'an action at a distance' that is rejected by general relativity due to its inability to fit the empirical data related to the finiteness of the speed of light. Nevertheless, the mathematical form of the relativistic gravitational potential near a planet maintained Newton's law of universal gravitation as the first term in an infinite series of terms. In general, the recovery of Newtonian true empirical claims within general relativity is based on a completely different and extended worldview about the universe. In the event of a scientific change, fitness is not the only criterion by which perspectives are formed.

As a whole, Giere's approach to perspectivism is a good way forward because it acknowledges a Riemannian landscape of different perspectives as we discussed in the section of cartography while adopting a naturalistic approach to the philosophy of science. Furthermore, Giere correctly motivates the role of context/perspective in scientific practice by developing its scheme and highlighting the various perspectives-models involved in scientific theorizing. Despite the intention of combining constructivism and scientific realism, the history of science is only used sporadically, leaving the issue of perspective development unresolved. In the following session, we will look at how Massimi (2018) attempts to answer the question of how perspectives track true empirical claims based on examples from the history of science.

3.8 Massimi's perspectival realism

One of the most important questions raised by Giere's perspectivism is that knowledge claims are perspective dependent. Such a position involves a tension because, on the one hand, perspectives can make true claims, but on the other they are also subject to change. As a result, Massimi's (2018) article "Four Kinds of Perspectival Truth" delves into the concept of perspectival truth, which is the central notion to her interpretation of perspectivism as a type of realism. This effort is part of a larger effort to understand science "from a human point of view", moving beyond hard realist interpretations and sociologists' constructivism.

Massimi specifically acknowledges that realism and constructivism are complementary ideas, rather than diametrically opposed, and suggests perspectival realism as a possible synthesis. This is accomplished in two steps: first, Massimi's version of perspectivism maintains the metaphysical view of scientific realism[4], namely that the state of affairs is independent of the perspective, and second, perspectivism specializes the epistemological commitment of realism by arguing that our knowledge of a state of affairs is based on a specific perspective while describing the world as it is (Massimi, 2018, p. 342). In order to remain realistic about scientific knowledge and account for scientific change, the second step requires refinement of the concept of truth. At the end, the question becomes how we can re-establish the concept of truth so that knowledge claims can be perspective-dependent while remaining normative.

[4]Scientific realism, according to Psillos (2000), consists of three major theses: metaphysical, epistemological, and semantical. Massimi's article (2018) discusses the epistemological thesis that interprets knowledge claims as true, that is, describing the world as it is, while adopting the metaphysical (i.e., mind-independent state of affairs) and semantic components of scientific realism (i.e., scientific language must be read literally).

The normative aspect of knowledge derives from science's ability to tell us how things should be. Giere's concept of perspectival truth, or "truth relative to perspective", is incapable of normatively interpreting science and making judgments about empirical facts as a consequence of its connection with a perspective subject to change (Massimi, 2018, pp. 343-344). To put things right, Massimi contends that three criteria for perspectival truth must be met in order to interpret science as capable of making normative claims. First, perspectival true claims need to be true across multiple perspectives; second, a perspective cannot assert its own truth; and, third, a true claim must meet certain performance standards. These include accuracy, empirical adequacy, and projectability (ibid., p. 355). Massimi considers cross-perspectival standards, or standards adopted from different perspectives, due to the risk of perspectivism being reduced to relativism. These standards will be discussed in greater depth later on.

In order to provide evidence of these criteria, Massimi uses examples from the history of science. Her argument consists of citing experimental results that do indeed track objective states of affairs, such as the mass-charge ratio of electrons, to show that they cannot be presented as raw data without interpretative perspective since they are "so intertwined with the scientific perspective of the time" (2018, p. 344). Particularly, scientific discoveries are embedded in a context from their inception, and they are not created ex nihilo; rather, they are interpreted either concurrently as they are discovered through rival synchronic scientific perspectives, or later through future theories (diachronic Kuhnian-like scientific perspectives). Therefore, states of affairs cannot be presented without the specification of any perspective and there is a constant shift of interpretative perspectives on science that forces us to reconsider how we are realists about science (Massimi, 2018, p. 345).

Given the unavoidable pluralism of interpretive perspectives on scientific results, it is necessary to consider what all perspectives that describe something real have in common. To put it in another way, Massimi wishes to establish the minimum requirement for a perspective to be true. The minimum requirement is a commitment to a theory or perspective that can withstand challenges from the history of science or from opposing viewpoints. Massimi interprets resistance to any experimental challenge as the ability of a perspective to track something true about reality.

However, at first glance, perspectivism appears to place constraints on meeting the previously mentioned minimum commitment. This deficiency stems from the desire to comprehend how one can know that their point of view is true. According to Massimi, a realist's default position is that determining whether a theory is true is based on the cumulative success of its predictions in more and more experiments. This is what Massimi refers to as "success from nowhere" (ibid.).

At the same time, a position at a different point in history may mean something entirely different, and any model considered successful today may be rejected in the future. This indicates that performance standards have shifted over time, and a hypothesis that was highly successful for the ancient Greeks has now been completely replaced (Massimi, 2018, p. 346). Furthermore, this type of success evaluates candidate models after the experiment has confirmed or refuted them, which is ineffective. Consequently, "success from nowhere" does not appear to be consistent with the history of science and pluralism.

In a similar position is the "success from everywhere," that is a type of success that is decided by the scientific community in which the issue of truth-value is raised. Here, the concept of truth becomes relativized because each community establishes its own standards and may manipulate the results to suit its own purposes. This type of truth would remove the realistic basis of perspectival truth, and thus true knowledge claims would lose their ability to refer to objective reality. For these reasons, Massimi rejects "success from nowhere" and "success from everywhere" as standards characterizing the scientific practice and continues to study four types of perspectival truth to find the right one (2018, p. 347).

As such, Massimi explores how historical and/or intellectual scientific perspectives affect knowledge, which are introduced during the epistemic process and do not involve "a metaphysical view of reality" (2018, p. 347). One could say, however, that introducing a perspective during the scientific process is not necessarily the result of a rational choice and almost always involves implicit metaphysical views about reality. For example, the Newtonian perspective implicitly accepts a mysterious "action at a distance," which is an instantaneous non-local interaction between separated objects, as well as an absolute space and time. In any case, Massimi investigates how perspectivism is compatible with realism by identifying four types of dependency on perspective, three of which she rejects and adopts the fourth. In this study, we will develop all of them in order to achieve completeness.

The first type of perspective-dependence is identified when the propositional content (subject-matter) of a knowledge claim depends on the perspective in which we express it (Massimi, 2018, p. 348). In this kind of perspectival truth, different states of affairs correspond to different propositional content, which provides meaning to sentences that appear to reflect the same comprehension. For example, the propositions "Earth orbits the sun on September 8, 1638" and "Earth orbits the sun on March 18, 2016" present two different states of affairs (i.e., truth-makers)[5], which provide different propositional content for apparently similar claims. We can think of the perspectival truth of this kind as one that is based on the propositional content we used to formulate it and applies to a specific state of affairs. In this way, Massimi interprets the various perspectives as if they were distinct truth-makers, or different states of affairs that provide the proposition with its truth-value.

The second type of perspective-dependence assumes that a knowledge claim is perspective-dependent when the truth-value of the claim depends on the perspective from which it is asserted (ibid., p. 348). This type of dependence and Giere's perspective on perspectival truth have a lot in common in that "truth claims are always relative to a perspective" (Giere, 2006, p. 81). In contrast to the first type of dependency, which captures the concept of perspective-indexicality, this dependency captures the concept of perspective-relativity. The significance of this change is that, while the propositional content of a given scientific claim is the same in all perspectives, the truth-value is context-dependent, which means that perspectives provide the scientific proposition's truth-value. For example, the proposition "The earth revolves around the sun" is true in the context of Galileo or Copernicus, but has a different truth-value in the case of Ptolemy. As a result of this type of dependence, the truth is relativized according to a perspective that assigns different truth-values to the same content.

The third type of dependence captures a form of contextualism. In this kind of perspectival truth, knowledge depends on the perspective when the truth-conditions[6] of our claims depend on the context (Massimi, 2018, p.

[5]The identification of truth-makers with the state of affairs is an assumption made by Armstrong's truth-maker theory (1997).

[6]According to MacFarlane (2005, p. 236, ft 7), "there are at least six different things that might be meant by "truth conditions": (i) function (in the mathematician's extensional sense) from contexts to truth values, (ii) rule for determining truth values based on features of context, (iii) function from circumstances of evaluation to truth values, (iv) rule for determining truth values based on features of circumstances of evaluation,(v) function from possible worlds (and perhaps times) to truth values, (vi) rule for determining truth

349). Perspectives, for Massimi, establish the contexts of use[7] that determine the truth conditions of knowledge claims. She uses the following example: the truth conditions of the sentence “The earth revolves around the sun” may vary depending on the context of use; that is, the truth conditions depend on whether this sentence is used by us or by Ptolemy in which case it would be false. Accordingly, if one formulates the proposition in a different context of use, the truth-conditions might change. For Massimi, truth-conditions are rules that determine the truth-values based on the characteristics of the context (ibid., p. 350).

The three types of perspective-dependence can be summarized in a few points to highlight their differences:

- In the first type, the propositional content of scientific claims is determined by the perspective.
- In the second type, the truth-value is determined by the perspective.
- In the third type, perspectives determine the context of use (circumstances) that defines the truth-conditions for scientific claims.

Now, let us examine how these three types of dependency are combined with a minimally realistic commitment. In terms of the first type of dependency, Massimi asserts that knowledge is represented by perspectives in such a way that truth-makers rely on those perspectives. Assuming that truth-makers are states of affairs that determine the ontological or metaphysical ground on which a proposition is true, then, first, truth is ontologically grounded in the state of affairs, and second, the state of affairs is perspectival, that is, intrinsic to the perspective. When truth-makers are

values based on features of possible worlds (and perhaps times)”. Massimi (2018) defines truth-conditions as “rules for determining truth values based on features of the context” understood by the notion of context of use (i.e. the context in which the scientific claim is made and employed). The context of use dictates these rules. In this sense, Massimi views "truth-conditions" as standards of justification rather than conditions for propositional content’s correspondence to states of affairs.

[7]However, this is not what is normally meant when one talks about truth-conditions that vary with the context of use. The concept of a "context of use" is drawn from semantics. The sentence "the earth now revolves around the sun" has truth-conditions that can vary depending on the context of use. The reason is the word "now": whenever the sentence is used, it is used at some point, and it is true at that point. Similarly, the sentence "I am woman" has truth-conditions that vary with the context of use and, more specifically, with who is the speaker.

linked with perspectives, there is a need to find a state of affairs that serves as an ontological foundation for them, and vice versa, the state of affairs is inherent in the perspective. Thus, we need to find the state of affairs that confirms each sentence.

According to Massimi (2018, p. 351), this situation creates a problem in the first type of dependency. The problem arises when a proposition is false, in which case we must either assume that a hypothesis, in a given context that is true by a community of scientists, is incorrect because we have not found a state of affairs to confirm it, or the obligation to find a state of affairs to confirm the claims burdens those who consider it true. In the first case, such a view would be disastrous for past or opposing views because it would prove them incorrect simply because we have not found a situation to confirm their views thus far, and would collapse perspectival truth to a correspondence theory of truth. While in the second case, there is a risk of constructing the state of affairs in such a way that it confirms the community's position (i.e., fact-constructivism).

However, regarding the second case of the first type of dependency, we should observe that contemporary sciences attempt to recreate the phenomena that they are interested in examining and the states of affairs that verify their claims. Scientists at the Large Hadron Collider, for example, are attempting to recreate the state of affairs of the universe in its early stages in order to be more probable to measure particles such as the Higgs particle. Though, the same states of affairs need to be recreated by other communities (e.g. ATLAS, CMS) in order to establish the same results. Given these considerations, we can conclude that first, the possibility of fact-constructivism is almost nil if there are enough communities recreating the same state of affairs, and second, states of affairs can be contextual without losing their objectivity. When it is impossible to recreate the states of affairs that verify the claim by several communities, a limit is established within which the claim has a very low probability of being true.

Furthermore, there are properties in quantum mechanics that depend on the state of the system, such as spin. It is important for Massimi to clarify whether she is discussing physical quantities like mass and electric charge or state-dependent observables like spin. In the second case, states of affairs that confirm our claims are generally contextual, because this type of property has different values (e.g. spin projection) depending on the system's state and the observable that we will choose to contextualise the system. Due to this reasoning, the claim that "electron has half-integer spin", which

Massimi examines in (2018), is of no special significance unless we are interested in the projection of spin that can be either positive or negative ($\pm\hbar/2$) depending on the context.

The second type of perspective-dependency appears to describe Kuhn's picture of scientists who, having different frameworks of thought, assign different truth-values to the same propositions (ibid., p. 352). However, it is possible to have both true and contradictory propositions based on this type of context-dependent knowledge. For example, the sentences "Earth revolves around the sun" and "Earth does not revolve around the sun" are both correct in their own context, geocentric and heliocentric respectively. According to Goodman (1978), we can eliminate such a contradiction by extending the proposals to refer to the context in which they are true. For example, "Under Ptolemy, the earth does not rotate around the sun" and "Under Copernicus, the earth rotates around the sun." This, however, does not help us understand what is going on in reality because both arguments are valid within their own perspective. This kind of perspectival truth does not provide us with new information and does not meet the standards of realism.

The third type, according to Massimi (ibid., pp. 352-353), captures the sensitivity of a claim to the perspective (i.e., use-sensitivity; see MacFarlane, 2005), which provides the conditions or context of use that determine the truthfulness of the sentences. In this sense, perspectival truth accepts Kuhn's view that there is no such thing as a God's eye view. For this reason, Massimi expresses the intuitive view that perspectives should establish the conditions of truth by cross-validating claims and thus arrive at the minimal commitment of realism. To accomplish this, Massimi discusses the third type of perspectival truth in order to improve it by defining the fourth and final type of perspectival truth.

Particularly, the third type realizes the concept of context of use, or context-sensitivity, adopted by various forms of contextuality. Essentially, in this type of truth, the context provides the circumstances defining the truth-conditions of the knowledge claims (Massimi, 2018, p. 352). The main problem with Massimi, however, is that the context of use implies that the correspondence between the claim and the state of affairs depends on the context. This is problematic because Massimi is committed from the start to the perspective-independence of state of affairs in order to preserve traditional scientific realism as part of perspectivism, creating in this way a confusion on whether Massimi's perspectivism adopts perspective-independent states of affairs or not. Also, we have seen that quantum theory supports the perspectivity of correspondence between the states of affairs and the relevant propositional content as projected within a context, determined by the physical magnitude to be measured. Nevertheless, for Massimi, the third type is problematic because it fails to preserve the meaning of a physical concept outside of the context of use. Therefore, Massimi (2018, p. 353) defines a fourth type of perspectival truth while retaining the context-sensitivity of the third type.

A useful observation is the fact that we can refer to a property considered as a derivative concept in one context while being fundamental in another. For example, viscosity is a fundamental macroscopic property of fluid mechanics that can also be approached from the perspective of statistical mechanics based on the movement of individual atoms, which is a more fundamental explanation of viscosity (Massimi, 2018, p. 354). Provided that, Massimi argues that the more fundamental perspective can function as a framework that could validate knowledge in a cross-perspectival way. A perspective that can be used in such a way is called context-of-assessment[8]. As a result, in order to accept propositions as true, we must validate their perspectival conditions of truth from multiple perspectives.

In order to define the forth type of perspectival truth, Massimi identifies the truth-conditions provided by the context as standards of performance-adequacy (2018, p. 354). These are defined as standards that need to be met for scientific claims across perspectives. To be more specific, claims of a given perspective must satisfy certain conditions such as obeying specific physical laws of other perspectives, passing calibration tests, and displaying features that can be generalized to more than one perspective in order to be considered true in a stronger way than claims considered true by only one perspective. Standards of performance-adequacy can be summed up by the following notions; accuracy, empirical testability, projectibility and heuristic fruitfulness (ibid, p. 355). Clearly, Massimi, in her attempt to avoid relativism, rejects any perspectival feature of scientific notions by demanding universal standards of truth.

Therefore, Massimi proposes that every perspective can work in two ways. On the one hand, perspectives function as context of use that determines the conditions for a sentence to be true (i.e., third type of perspective-dependence). On the other hand, perspectives play the role of a context of

[8]This kind of context is discussed from MacFarlane (2005), however Massimi (2018) rejects the notion of relativized truth of MacFarlane.

assessment from which we can assess the truth-conditions of propositions from other perspectives. Thus, Massimi defines her fourth kind of perspectival truth as follows:

"Perspective-dependence: Knowledge claims in science are perspective-dependent when their truth-conditions (understood as rules for determining truth-values based on features of the context of use) depend on the scientific perspective in which such claims are made. Yet such knowledge claims must also be assessable from the point of view of other (subsequent or rival) scientific perspectives" (Massimi, 2018, p. 354).

Massimi argues that in order to preserve a realist account of knowledge, that is to make normative claims about reality, the truth-conditions of those claims should be accessible from the point of view of perspectives other than the original ones. In this way, the mostly assessed claims can be normative claims of how the world should be. As long as the properties described by these claims perform adequately across scientific perspectives, we can safely say that they accurately reflect some aspect of reality. Even so, one may argue that claims and conditions that perform adequately across several perspectives cannot be considered perspectival in the first place. In other words, Massimi, by requiring a cross-perspectival confirmation of conditions and claims in order to preserve traditional scientific realism, removes the perspectival nature of those notions.

Moreover, our conclusions from the Kochen-Specker theorem may not be comparable to this kind of perspectival truth. Specifically, the context in quantum theory is constitutive of the empirical reality involving the states of affairs that define the conditions for the truth-values of claims. In other words, states of affairs are not predetermined but manifest themselves to us in relation to the context. This becomes apparent from the fact that the measurement process in quantum theory changes the state of the system under investigation always in accordance with the specification of the measurement context, i.e., in accordance with the selected physical magnitude to be measured. The context, according to its definition, allows the manifestation of certain properties characterising the states of affairs, which is why they are not pre-given to us. As a result, the context as constitutive of the cognitive process cannot act passively, as Massimi argues, namely as a stage for the verification of empirical claims, but as a pre-condition for the manifestation of quantum states of affairs (Karakostas, 2014, p. 14). Such a function, also, contrasts with Massimi's view that the states of affairs should be independent from perspective, which is expressed at the start of her essay

(2018). As a consequence, our analysis of the Kochen-Specker theorem and Massimi's perspectival truth tend to diverge.

Besides, the analysis of the Kochen-Specker theorem clarifies that the truth-conditions are defined by the context as a necessary part of any attempt to comprehend a quantum system (Karakostas, 2014, p. 11-12) and they are in stark contrast with Massimi's conditions expressed as standards of performance adequacy (2018, p. 355). We should observe that conditions in quantum mechanics are not necessarily material, namely experimental arrangements and operations made on quantum systems, but they are also conditions for understanding the quantum systems. In other words, quantum existence is not found ready to be investigated by our measuring apparatuses, but has to conform to the human conditions of cognizability. It is not at all a given that quantum phenomena should fall in with our measuring devices. In fact only a small part of reality is accessible by non-technological means and, furthermore, quantum measurement changes the state of the measured system.

By contrast, Massimi examines only the material aspects of conditions expressed mainly as empirical performance of perspectives (accuracy, empirical testability, and projectibility). An argument of this kind does not account for the change of perspectives, but only for their establishment. For example, Newtonian mechanics is not comparable with quantum mechanics in terms of accuracy because their experiments differ completely. For many purposes (e.g., astronomical, engineering), Newtonian mechanics is more appropriate than quantum mechanics, even if the latter is considered to be a much more fundamental perspective. Therefore, we realize that progress in science cannot be justified purely in terms of empirical performance alone. At the same time, we should point out that when Massimi's conditions are fulfilled by every true perspective and detached from their context of use, then they are transformed into universal values of truth and lose their perspectival character.

In the following section, we will go over to Karakostas and Zafiris' (2021) perspectivist methodological framework to discuss a more systematic way of managing perspectives based on quantum mechanics and to help us better understand the problem of truth-conditions.

3.9 Karakostas and Zafiris' perspectivist methodology

Until this point, we have seen that the Kochen-Specker theorem demonstrated the critical role of contexts in the truth-value assignment of propositions describing quantum properties. In this line of thought, the chapter of scientific perspectivism, more precisely, Giere, acknowledges both the historical and epistemic function of perspective and attempts to analyze scientific practice in terms of various types of models/perspectives (e.g., principles and specific conditions, representational models, specific hypotheses and generalizations, data models). Additionally, we discussed how Massimi emphasizes the historical significance of perspectives that interpret scientific events as part of a historical stage. As a result, we reach two important points: first, although the concept of context in quantum theory and Massimi's perspective both provide the conditions for the truth-values of empirical claims, the former has a constitutive character compared to the latter, and second, Massimi's context of assessment needs to take into consideration the existence of incompatible perspectives.

Although the Kochen-Specker theorem and perspectivism support the contextuality of quantum systems and scientific inquiry respectively, they do not provide a method for managing contexts and perspectives systematically. The scientific perspectivism of Giere and Massimi illustrates the perspectival nature of knowledge, but without articulating how perspectives interact, exchange information, and come to their conclusions. To accomplish this, Karakostas and Zafiris' "perspectivist methodology" employs the mathematics of category theory (2021, p. 4). This technical aspect is motivated by the fact that category theory can provide a method for meaningfully comparing and combining perspectives into higher theoretical structures. Karakostas and Zafiris' method, based on quantum mechanics, introduces a new type of perspective that differs from Massimi and Giere's in that it is "endo-theoretic/interactive" in nature (ibid, p. 2). This kind of perspective is conceived as a "vehicle of tracing and investigating the world" or a "probe" that can carve out a particular route of access to the world (ibid.). It is important to distinguish this concept of perspective from the visual metaphor of projection, which regards the act of knowing as passive. As in quantum theory, this perspective targets a particular by defining a set of compatible variables that describe it. For this reason, it moves away from a passive role and actively shapes the observed object by detaching a particular aspect of it that may now be amenable for further investigation within a suitable context.

Furthermore, in Karakostas and Zafiris' perspectivist methodology, the act of probing is internalized in the relations between the investigated object's parts or with other objects (2021, pp. 13-14). The information extracted from the observed object, i.e. the content of a perspective, can be completely "resolved" in the relationships of the object with its environment. This position is critical for scientific perspectivism because it reinforces the fact that an object's properties—at least those that are state-dependent—are determined by the object's relationships with the context (ibid., p. 10). Given that perspectivism supports the perspectival nature of knowledge, Karakostas and Zafiris articulate a method of perspectivism that can be applied to scientific practice and thus fulfil the aim of perspectivism to encourage scientific research to move forward. But regarding our study, let's look at how this framework answers the question of what exactly qualifies as a perspective.

According to Karakostas and Zafiris' framework, a proper probe belongs to the same category (level of inquiry) as the object-system being observed (ibid., p. 14). A category of objects, particularly, consists of all the objects that share the same being and the relations between them. The probes must belong to the same category and be able to internalize the mode of being of the object. It is very important that the function of the probe preconditions the notion of perspective. In Karakostas and Zafiris' method, the main function of perspective is to relate an object to its environment. Now, if the relationships of the object with the environment can be internalized in another object related to the object of inquiry, then the probe created from the related object can be qualified as a perspective. In categorical terms, such a situation is described as a structure-respecting morphism. Given that any relation between the object-system and another object can be thought of as a potential perspective if it encompasses all the relations between the object and its environment as well as it is structurally-respecting, then we understand that a potential perspective should conceive an invariant context for the object. This conclusion is very important since it emphasises a similar function of perspective to the frame of reference in special relativity (ibid., p. 30).

At a second level, we can compare perspectives, namely how we can transit from one to the other and to what degree they probe equivalent aspects of an object. At this level, we stumble upon "the horizon of perspectives", whereby applying reasoning derived from the mathematics of category theory, we can have a view on the perspectives targeting the system and combine them to create an overall structure that stands for the entire system (ibid., p. 7). We can then overcome the limitations of the Kochen-Specker theorem by using the overall structure created by the category theory and drawing conclusions about the entire system. This is achieved by the overall structure created based on the combination of perspectives, which results in the categorical notion of colimit. This notion expresses the mathematical fact that the interconnection of partial and local perspectives generates an object as the result of a limit process. One way of visualizing such an object, but only at the limit, is as a multi-layered surface of concatenated and stacked sieves covering the object-system where their joint coverage and concatenation approximate the targeted system structurally (ibid.). In this way, the colimit object is not an a priori object but is a process of combining perspectives under appropriate and faithful conditions.

Yet, perspective combinations must be made in such a way as to produce a synthesized unity (ibid., p. 3). In other words, unification must not be a chimeric approach, in which perspectives are merged as separate parts of another perspective, but rather as a process designed to transcend the locality of knowledge. The categorical concept of colimit (or inductive limit) succeeds in connecting all possible perspectives of the same object-system while ensuring coherent unity (Karakostas, Zafiris, 2021, p. 16) by fulfilling certain binding factors. In order for colimit to exist, there needs to be a correlation between the non-Boolean global quantum algebra of projections referring to the system's properties and the colimit object of stitched Boolean perspectives. This correlation is "a bi-directional functorial correlation" or, in category theory's terms, an adjunction (ibid., pp. 9, 25).

As demonstrated by the Kochen-Specker theorem, the overall algebra of projections of quantum systems is non-Boolean and, for this reason, the physical content of quantum systems can not be reduced to a single Boolean context. The categorical notion of adjunction allows us to correlate the non-directly accessible non-Boolean global quantum algebra of projections to all possible partial assemblages of perspectives constituted by families of Boolean sub-algebras of projections. Based on invariant aspects of the Boolean perspectives, this correlation (adjunction) is able to encode or decode the global content of the quantum algebra of events to locally interconnected assemblages of Boolean perspectives (ibid., pp. 18, 30). Since there is no unique way to cover the entire system with perspectives, each possible combination (colimit object) emphasizes different invariant characteristics. As a result, another level of abstraction is created where this methodology allows us to compare different combinations and, by extension, include multiple levels of relations. Additionally, the overall structure of the colimit created by the synthesis of perspectives can vary between different unifications of perspectives while covering the entire system, taking into consideration the evolution of those structures.

Nevertheless, the synthesis of perspectives, that leads to even higher multilevel structures, cannot reach an absolute point from which we can know everything or attain a "God's eye view" (ibid., p. 31). Despite the fact that a combination of perspectives can assert even more encompassing knowledge claims, it has its own set of physical limitations involving requirements under which the combination of perspectives is possible, referred to previously as binding factors.

We can summarize the novel features of Karakostas and Zafiris' framework in a few points:

- This new type of perspective
  - actively objectifies aspects of the world into targeted systems.
  - considers an object (a quantum system) to be constituted of its relations with other objects, implying that a system is entirely determined by its interactions with a permissible multiplicity of contexts.
  - should be of the same being as the observed object.
  - is a probe relation that additionally preserves the structure of the object.
- This conceptual and mathematical framework allows us to examine relationships between perspectives and combine them as a result of a limit process into a new overall structure that can cover the entire system.
- The overall structure of combined perspectives (colimit object) can vary across the system, encompassing all possible unifications of perspectives and account for their evolution.
- The possibility of the colimit object is ensured by the existence of a correlation between the non-Boolean quantum algebra of projections and the colimit object, that is called adjunction. This notion is able

to encode the information of the non-Boolean structure to families of Boolean perspectives and decode back.

4 Conclusion

In summary, the current study investigated whether knowledge claims can be perspective-dependent and still provide a realistic account of science. As a result of our analysis, it became clear that we can be realists about scientific knowledge if empirical claims refer to a restricted, probed aspect of reality conditioned by experimental means (the context). Also, we can be realists about theoretical claims referring to the entirety of quantum theory, such as the non-separability of quantum systems. We arrived at these conclusions through an analysis of the Kochen-Specker theorem, which we saw as an epistemic foundation for the philosophical framework of scientific perspectivism. In this way, the perspectival nature of knowledge is supported by the fact that a context assigns truth-values to propositions referring to an observable's properties through the supply of truth-conditions.

In quantum mechanics, it is impossible to assign determined properties to systems prior to measurement, and the propositions describing them do not have determined truth-values. Our analysis of the Kochen-Specker theorem showed that quantum properties are contextual in the sense that they can only be determined after defining the context of co-measurable observables. It is the specification of a context that provides the truth conditions, i.e., the conditions that allow the properties associated with the quantum mechanical observable under consideration to manifest themselves to us and the corresponding propositions to be assigned truth-values. Based on the contextual nature of truth-values, we see, therefore, that empirical claims about the properties of quantum systems cannot be propositions about microphysical reality independently of the epistemic process; rather, they can only be asserted within a context and assessed by compatible contexts.

In addition, we explored the philosophical approach of scientific perspectivism, based on the assumptions of methodological naturalism and the contingency thesis. Giere, in particular, emphasizes the similarity between cartography and how perspectivism views scientific practice. We note, in this respect, that such a similarity promotes the Riemannian quality of knowledge. To be more specific, Giere articulates its scheme of scientific theorizing, which attempts to analyze scientific practice in terms of models, in order to demonstrate the perspectival nature of knowledge. At the same time, Giere, in order to take into consideration the historical and social situatedness of knowledge, compares the notion of perspective with Kuhn's paradigm. He concludes that the main two definitions of paradigm, even though they have many elements in common with perspective, do not exactly match.

Another view of perspectivism is that of Massimi, whose concept of perspectival truth is not comparable to the analysis of the Kochen-Specker theorem. As a way of illustrating the perspectivity of knowledge, Massimi examines how perspectives can track true states of affairs while they are likely to change by citing examples from the history of science. To that end, she proposes a concept of perspectival truth, which requires, first, the perspective-independence of the states of affairs, second that the truth-conditions of knowledge claims are provided by the context in which they are used, and third, that there are contexts capable of assessing the truth-conditions of other perspectives in order to be considered as true.

We have pointed out that Massimi's realist's requirement for perspective-independent states of affairs does not align with the kind of context (context of use) that she ends up proposing for her kind of perspectival truth. More precisely, Massimi begins by imposing perspective-independent states of affairs on her perspectivism (2018, p. 342), probably inspired by the metaphysical thesis of traditional scientific realism, and in the end she concludes that perspectives provide the "circumstances" (i.e., states of affairs) under which claims are assigned their truth-values and equivalently that perspectives function as contexts of use (2018, p. 354). If we accept that perspectives provide the circumstances for the verification of our claims, then circumstances cannot be found independently of their perspective and, as a result, they are perspectival.

Concomitantly, in the example of the electron's spin on which Massimi grounds the perspective-independence of states of affairs, she overlooks that spin is a state-dependent property of quantum mechanics. This means that we are unable to assign properties ('up' or 'down' value for the projection of spin) to the states of affairs before measurement and thus the measurement does not just look to find a perspective-independent state of affairs corresponding to the half-integer spin. In any entangled spin state, as in a typical EPR-state, the states of affairs are in a superposition of possible spin states that are mediated by the context of co-measurable observables in order to provide the value for the projection of spin. As such, it becomes clear that the notion of context in quantum mechanics plays a constitutive role both in the states of affairs and in our experience expressed by the empirical propositions. Therefore, Massimi supports a rather passive notion of perspective, one that illuminates the world without affecting it in any way, which contrasts with contemporary science and especially the analysis of the

Kochen-Specker theorem.

The conclusions drawn from perspectivism and the absence of a method to manage the multiplicity of contexts in quantum mechanics led to the study of Karakostas and Zafiris' perspectivist methodology. Based on the mathematical framework of category theory, the perspectivist methodology of Karakostas and Zafiris sheds light on several issues related to the nature of perspective, the object-system itself, and the management of perspectives. Our results from this study can be summarized into a few points. First, in this methodology, the perspective objectifies aspects of reality into systems and so it moves away from the passive role of viewing. Second, to be able to probe a system, a perspective must be of the same being as the observed object, that is to say, it belongs to the same category as that object, and preserves its structure. Third, the object-system is defined by its relationship with other objects or with the context. And, forth, the extension of knowledge about an object-system is achieved by the combination of perspectives that probe the system.

Karakostas and Zafiris' perspectivist methodology, through the categorical notion of colimit, is able to cover the object-system by interconnecting all possible perspectives and thus describe an object-system in its entirety. As a consequence, the locality of the perspectives probing an object can be overcome with the appropriate application of category theory as interpreted by the scientific perspectivist approach of Karakostas and Zafiris. Last but not least, if we accept that perspectivism describes the implicit way of how contemporary science evolves, then we could make it explicit through the application of Karakostas and Zafiris' methodology to scientific problems. In other words, given that quantum theory can be analyzed through the interconnection of perspectives, the consistent application of a perspectivist methodology in physics would save us from a lot of trouble.

The study of those matters led to several challenges concerning the truth-conditions provided by context and the role they play in perspective evolution. To begin, Giere's scheme of scientific theorizing does not clarify how the various models/perspectives are linked in order to result in the established discoveries that perspectivism investigates to support the perspectival nature of knowledge. Subsequently, Massimi accepts the perspectivity of the truth-conditions but afterwards she essentially denies it by upgrading them into cross-perspectival requirements of true perspectives. Thereby, in Massimi's approach the question of truth-conditions remains unresolved. Further, although the analysis of the Kochen-Specker theorem supports that conditions are not necessarily material and are related to our cognitive capabilities, it is a shortcoming that we do not specify in greater detail what they are and how they may be established in a given perspective. We believe that truth-conditions are directly linked to how perspectives are combined, and we intend to examine this through Karakostas and Zafiris' perspectivist methodology.

Another issue raised by our study of Giere's "fitness" and Massimi's "performance-adequacy standards" is that they fail to account for the change of perspectives. It is a fact that the knowledge of successive perspectives involves a broader understanding of the world. This is a two-fold position involving what is already known and what could be learned, suggesting successive perspectives can unveil both explanations for known phenomena as well as predictions for unknown phenomena through a wider scope of application. In other words, successive perspectives transcend the locality of knowledge. Given that, perspectivism should investigate scientific change by examining how and why certain conditions are rejected or preserved by successive perspectives. Empirical success can account for the establishment and the specialization of perspectives in a specific area, but not for the extension of our knowledge, such as the prediction of new phenomena, or the statement of more fundamental reasons and explanations of known phenomena.

Furthermore, the history of science is replete with examples where the unification of existing perspectives is the driving force for scientific progress. Historically speaking, one of the reasons for the establishment of perspectives as true was that they could combine the truth-conditions of past perspectives coherently into one. As a result, perspectivism should provide an answer to the question of what happens to the truth-conditions in a scientific change in order to include the broadening of understanding of successive perspectives as well as the unification of existing perspectives. Indeed, Massimi is right to argue that successive perspectives can assess a previous perspective's truth-conditions. For example, relativity theory can assess the conditions of Newtonian mechanics. But Massimi's view is not universally applicable since the classical limit of quantum mechanics cannot evaluate the conditions of classical physics. Thus, Massimi's prespectivism does not provide a satisfying answer regarding truth-conditions with respect to scientific change. In order to address this question, we should develop an endo-theoretical framework that analyses the way theories change using Karakostas and Zafiris' perspectivist methodology.

It was also revealed from the analysis of the Kochen-Specker theorem that truth conditions should not only relate to standards of performance, but also to those that permit the cognizance of physical systems. It is important to realize that truth-conditions are not necessarily associated with a material counterpart, such as our measuring devices or criteria of empirical success, but also with cognitive abilities enabling our experimental perception and theoretical conceptualization of quantum reality. The conditions are primarily related to the type of observables we use to examine the system, as eigenstates play an important role in determining the truth-values of propositions. It follows that we can only investigate quantum phenomena through the specification of observables pertaining to the system under investigation. In this respect, the notion of perspective may shed light on the relationship that exists between observables and conditions in cognitive processes.

Ultimately, we could ask what truth-conditions are and how they can be identified. Also, the question of truth-conditions can be formulated at the level of physical theories in order to provide a systematic account, through category theory, of what happens with truth-conditions of perspectives in a scientific change. By utilising Karakostas and Zafiris' methodology and, more precisely, the notion of colimit, we could combine partially compatible perspectives, at the level of physical theories, and in the direction of developing objects-systems in which previous held perspectives (eg., general relativity, quantum theory) coexist. These questions, however, should not be answered from an Archimedean point of view of a final theory but as an immanent way (a mathematical method) of constant investigation and further development.

Appendix A Quantum Theory

The aim of this section is to facilitate the mathematical structures of quantum theory that are essential for the discussion of the Kochen-Specker theorem. In order to set an early context for quantum theory, we present some basic concepts such as the quantum state, vector spaces, and linear operators, followed by the axioms of quantum theory.

In light of the fact that a physical theory is intrinsically linked with reality, it is difficult to present it in a bare form, without any interpretation and without making any assumptions about reality. Because of this, we adopt an interpretation that is as minimal as possible.

A.1 Quantum States

Let's say a few things first on quantum systems before we go into the mathematical formalism of quantum theory. In quantum mechanics, a *system* is defined by an equivalence class of preparations. In other words, in experimental physics a system is defined by all the equivalent instructions used to construct systems. Note that a *preparation* is a set of instructions followed by an experimenter. For example, there are several macroscopic procedures that are equivalent for producing what we call a photon or a hydrogen atom, etc. We then perform adequate tests to check the equivalence of the various preparation techniques (Peres, 2002, p. 24).

Unlike quantum systems, which do not usually have a clear definition, quantum states are easily described with repeated measurements. Take, for example, a set of tests that are mutually incompatible. By performing several tests on the same preparations, we can determine the statistical distribution of the results. On the basis of such a distribution, we can see that each outcome tends toward a limit, which is the probability that it will occur. Accordingly, the quantum state is described as follows: A state is a mathematical object that expresses the probability of different outcomes to any test, represented as a function from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$ and provides all the information about the system. Among the characteristics of quantum states is their linear combination, which means we can combine two quantum states linearly and create a new one, or multiply a quantum state to produce another. The appropriate definitions are:

$$(\lambda\psi)(x) := \lambda\psi(x), \tag{4}$$

$$(\psi+\phi)(x) := \psi(x)+\phi(x), \quad (5)$$

for all $x \in \mathbb{R}$ (see Isham, 1995, p. 18). The function ψ is also called *wave function* and the most general form of a wave function can be $\psi(x) = \psi_1 u_1(x)+\psi_2 u_2(x)$ where ψ_1 and ψ_2 are any pair of complex numbers since ψ is a function from real to complex numbers. The values ψ_1 and ψ_2 determine completely the state ψ and can be represented by the column matrix $\binom{\psi_1}{\psi_2}$.

The discussion above can be generalised to include any finite set of eigenfunctions $u_1, u_2, ..., u_M$ and in that case the wave functions can be expanded in the form $\psi(x) = \sum_{i=1}^{N} \psi_i u_i(x)$. In fact, we can use an infinite set of eigenfunctions after extending the definition of the infinite sum to include the eigenfunctions of the self-adjoint operator, and the states of any system can always be represented by an infinite column matrix. The combination laws in equations (4) and (5) cannot be justified by appeal to an underlying wave theory but are instead imposed *ab initio*. Ultimately, the empirical success of the resulting theory proves or disproves such assumptions in physics. In the following sections, we will provide the necessary definitions in order to introduce the notion of vector space.

A.2 (Complex) Vector Spaces

The crucial role of vector spaces in physics is demonstrated by the fact that they allow us to construct structures that represent probabilities. In quantum mechanics, physicists can only predict the probability that some observable A, which is an element of the spectrum $\sigma(A)$, falls within the Borel-measurable set $E \in R$[9]. Even if we know the state of a quantum system, we cannot predict with certainty the outcome of an isolated measurement. This is due to the fact that the measurement process in quantum theory affects the state of our system and gives a (potentially) new state. For example, we can predict with what probability we get the possible final states before we measure, and once we make the measurement, we get a final state that depends on the measurement result.

There is also another form of probability in quantum mechanics concerned with the 'ignorance' of the experimenter. The states of these systems

[9]A Borel set is any set in a topological space that can be constructed from open sets (or, equivalently, closed sets) through countable union operation, countable intersection, and relative complement.

shall be referred to as mixed states and shall be examined in the following sections. But let us first see how the notion of group is defined.

A.2.1 The Concept of a Group

The concept of vector space can be defined in many ways, but we are going to focus mainly on vectors and numbers. This manipulation of vectors and numbers is possible due to combination laws, which play a central role in mathematics and, specifically, in defining group theory. There is a particular importance in group theory to most areas of physics because it reveals the invariant properties of a physical system. Using conservation laws, such as energy conservation, angular momentum, and spin, we express the relationship between invariant properties. That is why the notion of a group is central to quantum physics.

Definition A.1. A *group* is a set G equipped with a 'combination law' that associates with each pair of elements $a, b \in G$ another element, written ab, that satisfies the following three axioms:

- The combination law is *associative*. That is, for all elements $a, b, c \in G$ we have $a(bc) = (ab)c$.
- There exists a *unit* element $e \in G$ with the property that, for all $g \in G$, $ge = eg = g$.
- To each element $g \in G$ there exists an *inverse* element, written g^{-1}, with the property that $gg^{-1} = g^{-1}g = e$.

The group is said to be *abelian* (or *commutative*) if, for all $a, b \in G$ we have $ab = ba$.

A.2.2 The Notion of a Vector Space

We are now ready to define the notion of vector space.

Definition A.2. A (complex) *vector space* is a set V equipped with a law of combination that associates each pair of vectors $\overrightarrow{u}, \overrightarrow{v} \in V$ with a third vector written $\overrightarrow{u} + \overrightarrow{v}$. Another combination law called *scalar multiplication* associates with each $\overrightarrow{v} \in V$ and $\lambda \in V$ a vector, written $\lambda\overrightarrow{v}$. These laws satisfy the following axioms:

- The '+' law makes V into an Abelian group for the following reasons:
  - For all $\vec{u}, \vec{v}, \vec{w} \in V$ holds the associative property: $\vec{u} + (\vec{v} + \vec{w})$.
  - There is a *null* vector, that is the unit element for the abelian group and satisfies $\vec{v} + \vec{0} = \vec{0} + \vec{v}$ for all $\vec{v} \in V$.
  - For each vector $\vec{v} \in V$ there exists an inverse element, denoted $-\vec{v}$, such that $\vec{v} + (-\vec{v}) = \vec{0}$.
  - For all $\vec{u}, \vec{v} \in V$ we have $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
- The '+' law can be combined with scalar multiplication in the sense that

$$\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v},$$
$$(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v},$$
$$\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v},$$
$$1\vec{v} = \vec{v},$$
$$0\vec{v} = \vec{0},$$

  for all $\vec{u}, \vec{v} \in V$ and $\alpha, \beta \in \mathbb{C}$.

Another important concept for vector spaces is that of *morphism*. This is a map between two structures of the same type, e.g. between two groups or two vector spaces. If the structure of the target space is the same as that of the source space, we are saying that morphisms preserve the underlying structure, or that they are structured-preserving maps. For instance, a morphism between groups G and H is called *homomorphism* if exists a map $\phi : G \to H$ such that $\phi(a * b) = \phi(a)\phi(b)$ for all $a, b \in G$. If ϕ is also a *bijection* between the groups G, H, that is a *one-to-one* (i.e., $f(x_1) = f(x_2)$ implies $x_1 = x_2$) and *onto* (i.e., for any $y \in Y$ there exists an $x \in X$ such that $y = f(x)$), then the groups are called *isomorphic* to each other and they are different manifestations of the same abstract group (Isham, 1995, p. 26).

A morphism between vector spaces is called a linear map:

Definition A.3. A *linear map* between two vector spaces V_1 and V_2 is a map $L : V_1 \to V_2$ which is compatible with the vector space structure if:

$$L(\alpha\vec{u} + \beta\vec{v}) = \alpha L(\vec{u}) + \beta L(\vec{v}),$$

for all $\alpha, \beta \in \mathbb{C}$ and $\vec{u}, \vec{v} \in V_1$

Definition A.4. The map is called *anti-linear* if

$$L(\alpha\vec{u}+\beta\vec{v}) = \alpha^* L(\vec{u}) + \beta^* L(\vec{v}),$$

for all complex numbers α, β and vectors $\vec{u}, \vec{v} \in V_1$

Definition A.5. If a linear map $L: V_1 \to V_2$ is a bijection, then L is an isomorphism between V_1 and V_2 and symbolised as $V_1 \simeq V_2$

Isomorphism is a key concept in algebra because isomorphic spaces are regarded as identical, even though the underlying sets may be different.

A.3 Basis Vectors

Definition A.6.

A set of vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_N$, $N < \infty$ is *linearly dependent* if there is a set of numbers $\alpha_1, \alpha_2, ..., \alpha_N$ (not all zero) such that

$$\sum_{i=1}^{N} \alpha_i \vec{u}_i = \vec{0}.$$

If the set of numbers $\{\alpha_i\}$ does not exist or, equivalently, all $\{\alpha_i\}$ are zero then the set of vectors $\{\vec{u}_i\}$ is *linearly independent*. Linear independence means that none of them are linear combinations of the other.

Definition A.7. The linear independence of an infinite set of vectors is guaranteed if and only if every finite subset of vectors is linearly independent.

Definition A.8. A vector space is *N-dimensional* (where $N < \infty$) if it contains a subset of N linearly independent vectors, but contains no subset of $N+1$ such vectors. A vector space is *infinite dimensional* if it contains N linearly-independent vectors for each positive integer N.

Definition A.9. A finite set of N linearly independent vectors in an N-dimensional vector space is called a *basis* set for the space.

A.4 Scalar Products

A mathematical theory of quantum mechanics requires finding the equivalent of the overlap function when the states of the system are defined in a general vector space. The overlap function, defined for any $\psi, \phi \in \mathscr{L}^2(\mathbb{R})$ as:

$$<\psi, \phi> := \int_{-\infty}^{\infty} \psi^*(x)\phi(x)dx.$$

The procedure of developing the framework of quantum probabilities requires to generalise the well-known dot product $\mathbf{u} \cdot \mathbf{v}$ between a pair of vectors $\mathbf{u}, \mathbf{v}$ into a dot product between two quantum states. The dot product is known to be proportional to the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ and can be used to determine quantum probabilities.

In order to determine the dot product between two quantum states we can follow a path analogous to wave mechanics but with a twist of using $\mathbb{C}^N$ as a state space. By doing so, we can focus on wave functions which can be written in the form $\psi(x) = \sum_{i=1}^{N} c_i u_i(x)$ where $\{u_1, u_2, ..., u_N\}$ is a set of non-degenerate eigenfunctions of some self-adjoint operator. Likewise when the second wave function is written as $\phi(x) = \sum_{j=1}^{N} d_j u_j(x)$, we have

$$<\psi, \phi> = \sum_{i,j=1}^{N} \int_{-\infty}^{\infty} c_i^* u_i^*(x) d_j u_j(x) dx. \tag{6}$$

Since for $i \neq j$ the eigenvalues α_i and α_j are different due to non-degeneracy of eigenfunctions, we claim

$$\int_{-\infty}^{\infty} u_i^* u_j(x) dx = \delta_{ij}.$$

Therefore eq. (6) becomes

$$<\psi, \phi> = \sum_{i=1}^{N} c_i^* d_i.$$

The above equation suggests that, in general, if quantum mechanical state space of some system is $\mathbb{C}^N$, an appropriate definition for the equivalent of the overlap function for quantum states $\overrightarrow{a} = (a_1, a_2, ..., a_N)^T$ and $\overrightarrow{b} = (b_1, b_2, ..., b_N)^T$ might be

$$<\overrightarrow{a}, \overrightarrow{b}> := \sum_{i=1}^{N} a_i^* b_i$$

and can be written in matrix form as

$$< \overrightarrow{a}, \overrightarrow{b} >= \left(a_1^*, a_2^*, ..., a_N^*\right) \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_N \end{pmatrix}.$$

Based on this similarity, we can make the following plausible assertions:

- The overlap function of wave mechanics can successfully be regarded as an analogue of the dot product of elementary vector calculus.
- This analogy can be used for any vector space that represents the quantum state space of a physical system.

An appropriate structure to make explicit the above analogy is the scalar product.

Definition A.10. A scalar product (or inner product) on a complex vector space V is an assignment to each pair of vectors $\psi, \phi \in V$ of a complex number $< \psi, \phi >$ satisfying the following conditions:

$$< \overrightarrow{\psi}, (\alpha_1 \overrightarrow{\phi_1} + \alpha_2 \overrightarrow{\phi_2}) >= \alpha_1 < \overrightarrow{\psi}, \overrightarrow{\phi}_1 > + \alpha_2 < \psi, \phi_2 >, \tag{7}$$

$$< \overrightarrow{\psi}, \overrightarrow{\phi} >^* =< \overrightarrow{\phi}, \overrightarrow{\psi} >, \tag{8}$$

$$< \overrightarrow{\psi}, \overrightarrow{\psi} > \geq 0 \text{ with } < \overrightarrow{\psi}, \overrightarrow{\psi} >= 0 \text{ only if } \overrightarrow{\psi} = \overrightarrow{0}. \tag{9}$$

Note that scalar product on Hilbert spaces is a map $< \cdot | \cdot >: H \times H \to \mathbb{C}$.

A.5 Linear Self-Adjoint Operators

In quantum mechanics, observables are represented by self-adjoint differential operators and the possible results of a measurement of an observable are the eigenvalues of the corresponding operators. For this reason in this section we will define the analogue of a self-adjoint differential operator for a general Hilbert space $\mathscr{H}$. First, we define a linear operator as a special case of a linear map in which both the domain or source space V_1 and the target space V_2 are the same space $\mathscr{H}$. Second, we define the *hermiticity* of a self-adjoint operator based on the inner product. The definition of linear operator is as follows:

Definition A.11. A *linear operator* (or just *operator*) $\hat{A}$ on Hilbert space $\mathscr{H}$ associates a vector, denoted by $\hat{A}\overrightarrow{\psi}$ with every vector $\overrightarrow{\psi}$ in $\mathscr{H}$ such that

$$\hat{A}(\alpha\overrightarrow{\psi}+\beta\overrightarrow{\phi}) = \alpha\hat{A}\overrightarrow{\psi}+\beta\hat{A}\overrightarrow{\phi},$$

for all $\alpha,\beta \in \mathbb{C}$ and $\overrightarrow{\psi},\overrightarrow{\phi} \in \mathscr{H}$.

Definition A.12. The *sum* of a pair of operators $\hat{A},\hat{B}$ is the operator $\hat{A}+\hat{B}$ defined by

$$(\hat{A}+\hat{B})\overrightarrow{\psi} := \hat{A}\overrightarrow{\psi}+\hat{B}\overrightarrow{\psi},$$

for all $\overrightarrow{\psi} \in \mathscr{H}$. The *product* of $\hat{A}$ and $\hat{B}$ is the operator $\hat{A}\hat{B}$ defined by

$$(\hat{A}\hat{B})\overrightarrow{\psi} := \hat{A}(\hat{B}\overrightarrow{\psi}),$$

for all $\overrightarrow{\psi} \in \mathscr{H}$.

Definition A.13. The product of an operator $\hat{A}$ with a complex number λ is the operator $\lambda\hat{A}$ defined by

$$(\lambda\hat{A})\overrightarrow{\psi} := \lambda(\hat{A}\overrightarrow{\psi}), \tag{10}$$

for all $\overrightarrow{\psi} \in \mathscr{H}$.

Definition A.14. A (non-zero) vector $\overrightarrow{u} \in \mathscr{H}$ is an *eigenvector* of $\hat{A}$ with *eigenvalue* α if

$$\hat{A}\overrightarrow{u} = \alpha\overrightarrow{u}.$$

Definition A.15. The set of *matrix elements* of an operator $\hat{A}$ on a Hilbert space $\mathscr{H}$ is the collection of all numbers $<\overrightarrow{\psi},\hat{A}\overrightarrow{\phi}>$ where $\overrightarrow{\psi},\overrightarrow{\phi} \in \mathscr{H}$.

Definition A.16. The *adjoint* (or *hermitian conjugate*) of an operator $\hat{A}$ is the operator $\hat{A}^\dagger$ defined by the condition of its matrix elements:

$$<\overrightarrow{\psi},\hat{A}^\dagger\phi> = <\hat{A}\overrightarrow{\psi},\overrightarrow{\phi}>, \tag{11}$$

for all $\overrightarrow{\psi},\overrightarrow{\phi} \in \mathscr{H}$.

Definition A.17. An operator $\mathscr{A}$ is *self-adjoint* (or *hermitian*) if $\hat{A} = \hat{A}^{\dagger}$. That is, for all $\vec{\psi}, \vec{\phi} \in \mathscr{H}$, the matrix elements of $\hat{A}$ satisfy the conditions

$$< \vec{\psi}, \hat{A}\phi >=< \hat{A}\vec{\psi}, \vec{\phi} >=< \vec{\phi}, \hat{A}\vec{\psi} >^* .$$

The above equation takes the following form for wave functions

$$\int_{-\infty}^{\infty} \psi^*(x)(\hat{A}\phi)(x)dx = \int_{-\infty}^{\infty} (\hat{A}\psi)^*(x)\phi(x)dx,$$

for all square-integrable functions ψ and ϕ.

Definition A.18. A self-adjoint operator $\hat{A}$ is *bounded* if its eigenvalues are contained in a finite subspace of the real line.

A.6 Projection Operators

We call an operator that projects a vector into a subspace of Hilbert space a projection operator. They are self-adjoint and have 0 and 1 as their own values. They thus are 'binary-valued' observables and can be interpreted as propositions for the properties of the quantum system. As such, they play a crucial role in the discussion of the conceptual basis of quantum theory (Isham, 1995, p. 53). Let us begin with some preliminary definitions:

Definition A.19. A linear subspace W of a Hilbert space $\mathscr{H}$ is *topologically closed* if for every strongly convergent sequence of vectors $\vec{v}_1, \vec{v}_2, \ldots$ lying in W, the limit vector $\vec{v} \in \mathscr{H}$ also belongs to W. In the case where the dimension of $\mathscr{H}$ is finite, the above definition is true for every linear subspace W. If the Hilbert space is infinite-dimensional, then we can find non-closed linear subspaces in $\mathscr{H}$[10].

Definition A.20. For any linear subspace W of $\mathscr{H}$ there exists a smallest linear subspace of $\mathscr{H}$ that is closed and contains W as a subspace. This is known as the *closure* of W and is denoted $\bar{W}$.

Definition A.21. Two linear subspaces W_1 and W_2 of $\mathscr{H}$ are *orthogonal* if every vector in W_1 is orthogonal to every vector in W_2. The *orthogonal*

[10]For example, the set of all finite linear combinations of an orthonormal basis set $\{\vec{e}_1, \{\vec{e}_2, \ldots\}$ of $\mathscr{H}$.

complement $W^{\perp}$ of a linear subspace W of $\mathscr{H}$ is the set of all vectors that are orthogonal to every vector in W[11]:

$$W^{\perp} := \{\overrightarrow{\psi} \in \mathscr{H} | \forall \overrightarrow{w} \in W, < \overrightarrow{w}, \overrightarrow{\psi} >= 0\}.$$

Apparently W and $W^{\perp}$ are orthogonal subspaces of $\mathscr{H}$.

It is straightforward that $W^{\perp}$ is linear subspace of $\mathscr{H}$ and topologically closed. In addition, if $\mathscr{H}$ has a finite-dimension then $(W^{\perp})^{\perp} = W$, whereas if $\mathscr{H}$ id infinite-dimensional then $(W^{\perp})^{\perp}$ is the closure $\bar{W}$ of W.

The motivation behind definitions of the present section is that, given a topologically closed subspace W of $\mathscr{H}$, any vector $\overrightarrow{\psi}$ can be decomposed as a unique sum

$$\overrightarrow{\psi} = \overrightarrow{\psi}_W + \overrightarrow{\psi}_{W^{\perp}} \tag{12}$$

of vectors $\overrightarrow{\psi}$ and $\overrightarrow{\psi}_{W^{\perp}}$ that lie in W and $W^{\perp}$ respectively. In order to show that such decomposition is unique, first we need to show that the limit of a strongly convergent sequence is unique. Let us recall the strongly convergent sequence $a_n \to a$:

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^{+} \text{ such that } |a_n - a| < \epsilon.$$

Now let us assume that the limit a of the sequence a_n is not unique and that there is a second limit b to which the sequence a_n converges, with $a \neq b$. Then:

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^{+} \text{ such that } |a_n - b| < \epsilon.$$

Since the two limits are different from each other, we may define their difference $\epsilon = |a - b| > 0$. Then we can say that

$$\text{since } a_n \to a, \exists N_1 \in \mathbb{Z}^{+} \text{ such that } |a_n - a| < \frac{\epsilon}{2}$$

and

$$\text{since } a_n \to b, \exists N_2 \in \mathbb{Z}^{+} \text{ such that } |a_n - b| < \frac{\epsilon}{2}.$$

We set $M = max\{N_1, N_2\}$ and then

$$\forall n > M \quad |a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon = |a - b|.$$

[11]The notation $\{x \in X | P(x)\}$ means that proposition $P(x)$ it true for all elements x of the set X.

Thus

$$|a-b| < |a-b|,$$

which is a contradiction. So now that we showed that the limit of a strongly convergent sequence is unique, and since a topologically closed subspace W (or $W_{\perp}$) contains the limits of every strongly convergent sequence, we can infer that $\vec{\psi}$ and $\vec{\psi}_{W^{\perp}}$ are limits of strongly convergent sequences. Then, as limits of strongly convergent sequence are unique and their sum—the decomposition of $\vec{\psi} = \vec{\psi}_W + \vec{\psi}_{W^{\perp}}$— is also unique.

For the exact definition of $\vec{\psi}$ and $\vec{\psi}_{W^{\perp}}$, let $\{\vec{f}_1, \vec{f}_2, ...\}$ be any orthonormal basis for the subspace W. Then we define

$$\vec{\psi}_W := \sum_i < f_i, \psi > \vec{f}_i$$

[12] and

$$\vec{\psi}_{W^{\perp}} := \vec{\psi} - \vec{\psi}_W$$

At this point we need to show that vectors $\vec{\psi}_W$ are independent of the choice of orthonormal basis for the subspace W. We recall that a vector $\vec{\psi}$ can be expanded on an orthonormal basis as follows

$$\vec{\psi} = \sum_i < \psi_i, f_i > .$$

By substituting in the definition of $\vec{\psi}_W$, we have

$$\vec{\psi}_W = \sum_i < \vec{f}_i, \vec{\psi} > \vec{f}_i \tag{13}$$

$$= \sum_i < \vec{f}_i, \sum_i \vec{f}_i < \psi_i, f_i > \vec{f}_i \tag{14}$$

$$= \sum_i \psi_i < \vec{f}_i, \vec{f}_i >< \vec{f}_i, \vec{f}_i > \tag{15}$$

$$= \sum_i \psi_i \tag{16}$$

where ψ_i are complex numbers, independent of the choice of basis. It can be shown similarly that $\vec{\psi}_{W^{\perp}}$ does not depend on the choice of orthonormal basis.

[12]If $\mathscr{H}$ is infinite-dimensional it is necessary to show that this sum converges strongly.

The map $\vec{\psi} \to \vec{\psi}_W$ is clearly linear and hence can be regarded as the action of some operator P_W:

$$\hat{P}_W \vec{\psi} := \vec{\psi}_W.$$

Thus, we define the *projection operator* onto the subspace W. The projector to the orthogonal complement $W^\perp$ is $\hat{P}^{W^\perp}$ and is equal to $\mathbb{1} - \hat{P}_W$.

A.7 The Axioms of Quantum Theory

Axioms of quantum theory will be introduced in this section with references to mathematical concepts from the previous sections. The five axioms that follow establish a mathematical framework from which quantum mechanical systems can be described.

<u>*Axiom 1:*</u> A quantum system's *state space* corresponds to a complex Hilbert space $\mathscr{H}$. A normalized unit vector in $\mathscr{H}$ can be considered as the mathematical representative of the physical concept of *pure state* of the system. The vector includes all information that is available for the system. Therefore, if the maximum amount of information is available for the system, this vector predicts probabilistically the measurements' results to the greatest degree of accuracy.

Definition A.22. A complex Hilbert space $(\mathscr{H}, +, *, <\cdot,\cdot>)$ is a set satisfying the following properties:
$+ : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$
$* : \mathbb{C} \times \mathscr{H} \to \mathscr{H}$
$\mathrm{C} : x + y = y + x, \quad x * y = y * x$
$\mathrm{A} : (x+y)+z = x+(y+z), \quad x*(y*z) = (x*y)*z$
$\mathrm{N} : 0 + x = x + 0 = x, \quad x * 1 = x$
$\mathrm{I} : (-x)+x = x+(-x) = 0, \quad x * x^{-1} = 1$

A norm on a vector space V over K is a function from V to $\mathbb{R}$ denoted $||\cdot||$, satisfying for all $v, w \in V$ and $k \in K$:

1. $||v|| = 0$ iff $v = 0$
2. $||kv|| = |k|||v||$
3. $||v + w|| \leq ||v|| + ||w||$

It follows that $||v|| \geq 0$ for all $v \in V$. As we said in earlier section, Hilbert space is an inner product space that is complete with respect to the norm topology, meaning that the limit of any sequence of vectors is itself contained in the space.

Assuming the state vector is $|\psi>$, we may now define the probability that measurement of the observable A results in the eigenvalue a_i:

$$Prob(A = a_i; |\psi>) =< \psi|P_i|\psi> \tag{17}$$

where $P_i := \sum_{j=1}^{d(i)} |a_i, j>< a_i, j|$ is the projector onto the eigenspace of vectors with eigenvalue a_i, for non-degenerate eigenvalues $j = 1$.

<u>*Axiom 2:*</u> The observables of a quantum system are represented mathematically by self-adjoint operators $A : \mathscr{D}_A \to \mathscr{H}$ that act on the Hilbert space $\mathscr{H}$.

Definition A.23. A *linear map* $A : \mathscr{D}_A \to \mathscr{H}$ is called self-adjoint if it coincides with its adjoint map $A^* : \mathscr{D}_{A^*} \to \mathscr{H}$. Coincide means that $\mathscr{D}_{A^*} = \mathscr{D}_{\mathscr{A}}$ and $A^*\psi = A\psi, \quad \forall \psi \in \mathscr{D}_{\mathscr{A}}$.

Definition A.24. The *adjoint map* $A^* : \mathscr{D}_{A^*} \to \mathscr{H}$ of a linear map $A : \mathscr{D}_A \to \mathscr{H}$ is defined by

i) $\mathscr{D}_A := \{\psi \in \mathscr{H} | \forall \alpha \in \mathscr{D}_A, \exists \eta \in \mathscr{H} :< \psi, A\alpha >=< \eta, \alpha >\}$

ii) $A^*(\psi) := \eta$.

An adjoint map is well-defined iff for each $\alpha \in \mathscr{D}_A$ and $\psi \in \mathscr{H}$ there exists at most one $\eta \in \mathscr{H}$ such that $< \psi|A\alpha >=< \eta|\alpha >$.

<u>*Axiom 3:*</u> Given that an observable quantity A and a state of the system are represented by a self-adjoint and the normalized[13] vector $\overrightarrow{\psi} \in \mathscr{H}$, then the expected value $< A >_\psi$ of measuring A is

$$< A >_\psi =< \overrightarrow{\psi}, A\overrightarrow{\psi} > \tag{18}$$

or in Dirac notation

$$< A >_\psi =< \overrightarrow{\psi}|\hat{A}|\overrightarrow{\psi} > .$$

[13]A vector $\overrightarrow{\psi}$ is *normalized* if $< \overrightarrow{\psi}, \overrightarrow{\psi} >= 1$.

Projection Axiom 4: This axiom was proposed by John von Neumann (1932/2018) and essentially determines the state of the system following the measurement. The postulate is expressed as follows:

$$|\psi> \rightarrow \frac{P_{|a_m>}|\psi>}{<\psi|P_{|a_m>}|\psi>^{1/2}} \tag{19}$$

where $P_{|a_m} = \sum_{j=1}^{d(m)} |a_m, j>< a_m, j|$. When a measurement is performed on the system in state $|\psi>$, its wave function changes according to the projection postulate; it collapses to its normalized projection onto the subspace of its Hilbert space associated with the result of the measurement. The associated change of quantum state is usually referred to as the wave function collapse or as the reduction of the state vector. The projection axiom governs, not any measurement, but only the *ideal measurement*, one which changes the system's state as little as possible while obtaining the relevant result a_m.

Note that, if the eigenvalues of observable A are non-degenerate, then $P_{|a_m>} = |a_m>< a_m|$. In this case, the right-hand side of the projection axiom becomes $\frac{<a_m|\psi>}{|<\psi|a_m>|}|a_m>$ and since the factor of $|a_m>$ is a complex number of modulus 1, we essentially get:

$$|\psi> \rightarrow |a_m>. \tag{20}$$

Axiom 5: In quantum mechanics, the evolution of a system is identified with the evolution of the state vector. In the absence of any external influence (i.e., in a *closed* system), the time evolution of the state vector of the quantum system denoted by $|\psi(t)>$ changes smoothly in time t according to the time-dependent Schrödinger equation:

$$i\hbar \frac{d|\psi(t)>}{dt} = \hat{H}|\psi(t)>$$

where $\hat{H}$ is a special operator known as the *Hamiltonian* and is related with the total energy of the system. The Hamiltonian of the system is written as:

$$\hat{H} = \hat{T} + \hat{V} = \frac{p^2}{2m} + V(x) \tag{21}$$

where $\hat{x}|\psi(x)>= x|\psi(x)>$ is the operator of the position and $\hat{p}|\psi(x)>= -i\hbar\frac{\partial|\psi(x)>}{\partial x}$ is the operator of momentum. Thus, the Hamiltonian can be written:

$$\hat{H}|\psi(x)>=\left(\frac{p^2}{2m}+V(x)\right)|\psi(x)>=\frac{-\hbar^2}{2m}\frac{\partial|\psi(x)>}{\partial x}+V(x)|\psi(x)>. \tag{22}$$

Thus, the Schrödinger equation takes the form:

$$i\hbar\frac{\partial|\psi(x,t)>}{\partial t}=\left(\frac{-\hbar^2}{2m}\frac{\partial}{\partial x}+V(x)\right)|\psi(x,t)>. \tag{23}$$

We see, therefore, that the Schrödinger equation is a deterministic equation that describes the evolution of the system's state, which is a catalogue of probability amplitudes concerning all physical quantities pertaining to the system. Indeed, the deterministic evolution of quantum probabilistic amplitudes should not be considered as equivalent with the deterministic evolution of classical systems.

Depending on the problem, if we define the appropriate Hamiltonian of the system and the initial state $|\psi(t_1)>$ at a time t_1, we can know the system's state at a future time t_2 in the absence of external influence. This is achievable by writing the state $|\psi(t_2)>$ as follows:

$$|\psi(t_2)>=e^{-\frac{i}{\hbar}(t_2-t_1)\hat{H}}|\psi(t_1)>=\left(1-\frac{i}{\hbar}(t_2-t_1)\hat{H}\right)|\psi(t_1)>:=\hat{U}(t_2,t_1)|\psi(t_1)> \tag{24}$$

and the quantity $\hat{U}(t_2,t_1)$ is defined as the unitary operator.

A.8 Mixed States and Density Matrices

In this section, we will examine how the formalism of quantum theory defines mixed states, namely states that do not provide all the information relative to the system. Mixed states allow us to study many systems simultaneously, which are not in the same state, or a system for which we know only the possibility of being in different pure states $\{|\psi_i>\}$. Also, mixed states cannot be described by kets as pure states, but only by a density matrix (or density operator) ρ. We define the density matrix or mixed-state operator ρ as

$$\hat{\rho}:=\sum_{d=1}^{D}w_d P_{|\psi_d>}=\sum_{d=1}^{D}w_d|\psi_d><\psi_d|. \tag{25}$$

Where w_d are the purely classical probabilities $w_1, w_2, ..., w_D$ for each state $|\psi_d>$, with $0 < w_D \leq 1$ and $\sum_{d=1}^{D} w_D = 1$.

The first axiom for a pure state $|\psi>$ is $Prob(A = a_n; |\psi>) =< \psi|P_n|\psi>$, and so the corresponding rule for a mixed state is:

$$Prob(A = a_n; \rho) = \sum_{d=1}^{D} w_d Prob(A = a_n; |\psi>) = \sum_{d=1}^{D} w_d < \psi_d|P_n|\psi_d> \tag{26}$$

while presupposing that quantum and classical probabilities are independent of each other.

Based on the notion of the trace of an operator $tr(O) = \sum_{d=1}^{D} < \psi_d|O|\psi_d>$, we can prove that

$$Prob(A = a_n; \rho) = tr(\rho P_n) \tag{27}$$

and the corresponding expression for the expected value is

$$< A >_\rho = tr(\rho A). \tag{28}$$

All these expressions can be reduced to the usual results for a pure state $|\psi>$, namely for density matrix $\rho = P_{|\psi>} = |\psi><\psi|$.

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