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Operator Systems and Correlation Sets

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Abstract

An operator system can be described as a self-adjoint subspace of a unital C^*-algebra containing the unit of this C^*-algebra. A celebrated result of Choi and Effros shows that equivalently we can consider an operator system as an Archimedean matrix ordered *-vector space. The tensor product of two operator systems can also be equipped with suitable matrix orderings, making it an operator system. In the first part of the present paper we examine three of these matrix orderings. In the second part we study the connection between tensor products of operator systems and several classes of non-signalling correlations.

We will now briefly describe the contents of each Chapter.

In the first Chapter we give the definition of a matrix ordering and review some basic results regarding matrix ordered spaces such as order units, positivity, duality and the Archimedeanization process.

In Chapter 2 we introduce the notion of an operator system structure on the tensor product of two operator systems. A tensor product of operator systems equipped with such a structure is once again an operator system. The main focus of this chapter will be the study of the minimal, maximal and commuting operator system tensor products. We will see that in order to determine the states on the minimal tensor product we require the maximal tensor product and vice-versa.

In the following two Chapters (3 and 4) we define and examine the quotient operator system and we describe the co-product of operator systems using this concept.

In Chapter 5 we define some classes consisting of non-signalling correlations with the use of Positive Operator Valued Measures (POVM's), namely the local, quantum, approximately quantum and quantum commuting classes. The geometrical properties of the sets of these correlation classes are studied and it is shown that they satisfy a chain of inclusions. Moreover, we establish bijective correspondences between the correlations belonging to each of these classes and the states on the tensor products of certain operator systems. More specifically these operator systems will be co-products of copies of the operator system $l_k^\infty := \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{k-times}$ while the tensor products in question are the aforementioned ones (Chapter 2).

In Chapter 6 some distinctions between the various correlation sets are proven. This is achieved with the help of non-local game theory and through methods of operator system theory. To be more precise we will see that the set of all local correlations differs from that of all quantum correlations and that the set of all quantum commuting correlations differs from that of all non-signalling correlations. We should also note that there was a long standing debate on whether the set of all approximately quantum correlations coincides with that of all quantum commuting correlations, referred to as the weak Tsirelson's problem. This was answered in the negative by Slofstra in [21]. Another long-standing question, known as the strong Tsirelson's problem, was whether or not the set of all approximately quantum correlations coincides with that of all quantum correlations. In a recent paper [10] it was shown that the answer to the strong Tsirelson's problem is also negative. We will not examine these two separations as their proofs, found in the papers given above, use techniques beyond the scope of this paper.

In the literature, the various correlation classes we discussed are sometimes defined using Projection Valued Measures (PVM's) instead of Positive Operator

Valued Measures. In the last Chapter it is shown that in both cases the same correlation classes are obtained.

Περίληψη

Ένα σύστημα τελεστών μπορεί να περιγραφεί ως ένας αυτοσυζυγής υπόχωρος μιας μοναδιαίας C^*-άλγεβρας ο οποίος περιέχει την μονάδα αυτής της C^*-άλγεβρας. Σύμφωνα με ένα θεμελιώδες αποτέλεσμα των Choi και Effros μπορούμε ισοδύναμα να θεωρήσουμε ένα σύστημα τελεστών σαν έναν *-διανυσματικό χώρο με Αρχιμήδεια διάταξη πινάκων. Το τανυστικό γινόμενο δύο συστημάτων τελεστών δύναται επίσης να εφοδιαστεί με μία κατάλληλη διάταξη πινάκων ούτως ώστε να γίνει και αυτό ένα σύστημα τελεστών. Στο πρώτο μέρος της παρούσας εργασίας εξετάζουμε τρεις από αυτές τις διατάξεις πινάκων. Στο δεύτερο μέρος μελετούμε την σχέση μεταξύ ορισμένων non-signalling κλάσεων συσχετίσεων και κάποιων τανυστικών γινομένων συστημάτων τελεστών.

Ακολουθεί μια σύντομη περιγραφή των περιεχομένων κάθε κεφαλαίου.

Στο πρώτο κεφάλαιο δίνεται ο ορισμός της διάταξης πινάκων καθώς και μία ανάλυση των βασικών αποτελεσμάτων που αφορούν τους χώρους με διάταξη πινάκων όπως είναι οι μονάδες διάταξης, η θετικότητα, ο δυϊσμός και η Αρχιμηδοποίηση.

Στο δεύτερο κεφάλαιο εισάγουμε την έννοια της δομής συστήματος τελεστών στο τανυστικό γινόμενο δύο συστημάτων τελεστών. Ένα τανυστικό γινόμενο δύο συστημάτων τελεστών εφοδιασμένο με τέτοια δομή είναι και αυτό ένα σύστημα τελεστών. Εδώ επικεντρωνόμαστε στην μελέτη του ελαχιστικού (minimal), του μεγιστικού (maximal) και του commuting τανυστικών γινομένων συστημάτων τελεστών. Θα δούμε πως προκειμένου να περιγράψουμε τις καταστάσεις (states) του ελαχιστικού τανυστικού γινομένου χρειαζόμαστε το μεγιστικό τανυστικό γινομενο και αντιστρόφως.

Στα δύο επόμενα κεφάλαια (3 και 4) ορίζουμε και μελετούμε το σύστημα τελεστών πηλίκο και βασιζόμενοι στην έννοια αυτή περιγράφουμε το co-product συστημάτων τελεστών. Θα δούμε πως το co-product συστημάτων τελεστών μπορεί να κατασκευαστεί ως σύστημα τελεστών πηλίκο.

Στο πέμπτο Κεφάλαιο θα ορίσουμε διάφορες κλάσεις αποτελούμενες από non-signalling συσχετίσεις χρησιμοποιώντας μέτρα με τιμές θετικούς τελεστές (POVM's), πιο συγκεκριμένα τις κλάσεις των τοπικών (local), κβαντικών (quantum), προσσεγγιστικά κβαντικών (approximately quantum) και quantum commuting συσχτίσεων. Ακολούθως εξετάζονται οι γεωμετρικές ιδιότητες των συνόλων αυτών των συσχετίσεων και αποδεικνύεται ότι τα σύνολα αυτά ικανοποιούν μία σειρά από εγκλεισμούς.
Επιπροσθέτως, θα αποδείξουμε την ύπαρξη αμφιμονοσήμαντων αντιστοιχιών ανάμεσα στις συσχετίσεις που ανήκουν στις παραπάνω κλάσεις και στις καταστάσεις στα τανυστικά γινόμενα ορισμένων συστημάτων τελεστών.
Ειδικότερα, τα ζητούμενα συστήματα τελεστών είναι co-products αντιγράφων του συστήματος τελεστών $l_k^\infty := \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{k-times}$ ενώ τα εν λόγω τανυστικά γινόμενα είναι τα προαναφερθέντα (Κεφάλαιο 2). Στην ουσία θα δούμε ότι:

1. Υπάρχει μία ένα προς ένα και επί αντιστοιχία μεταξύ του συνόλου όλων των non-signalling συσχετίσεων και του συνόλου των καταστάσεων (states) του χώρου $\mathcal{S}(n,k) \otimes_{max} \mathcal{S}(m,l)$, $n,k,m,l \in \mathbb{N}$.
2. Υπάρχει μία ένα προς ένα και επί αντιστοιχία μεταξύ του συνόλου όλων των quantum commuting συσχετίσεων και του συνόλου των καταστάσεων (states) του χώρου $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$, $n,k,m,l \in \mathbb{N}$.

3. Υπάρχει μία ένα προς ένα και επί αντιστοιχία μεταξύ του συνόλου όλων των approximately quantum συσχετίσεων και του συνόλου των καταστάσεων (states) του χώρου $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l), n,k,m,l \in \mathbb{N}$.

οπου $\mathcal{S}(n,k)$ εικαι το co-product n αντιγράφων του χώρου l_k^∞ και με $\otimes_{max}, \otimes_c$ και $\otimes_{min}$ συμβολίζουμε το μεγιστικό, το commuting και το ελαχιστικό τανυστικό γινόμενο συστημάτων τελεστών αντίστοιχα.

Στο Κεφάλαιο 6 δείχνουμε πως οι εγκλεισμοί μεταξύ των συνόλων των συσχετίσεων που ορίσαμε προηγουμένως είναι γνήσιοι. Αυτό επιτυγχάνεται με την βοήθεια της μη-τοπικής θεωρίας παιγνίων αλλά και με την χρήση μεθόδων της θεωρίας τανυστικών γινομένων συστημάτων τελεστών. Θα εστιάσουμε τις προσπάθειες μας στην απόδειξη των δυο ακόλουθων ισχυρισμών:

1. Το σύνολο όλων των τοπικών (local) συσχετίσεων είναι γνήσιο υποσύνολο του συνόλου όλων των κβαντικών (quantum) συσχετίσεων.

2. Το σύνολο όλων των quantum commuting συσχετίσεων είναι γνήσιο υποσύνολο του συνόλου όλων των non-signalling συσχετίσεων.

Στο σημείο αυτό ωφείλουμε να αναφέρουμε πως δεν θα μελετήσουμε το αν το σύνολο όλων προσσεγγιστικά κβαντικών (approximately quantum) συσχετίσεων ταυτίζεται με το σύνολο όλων των κβαντικών (quantum) συσχετίσεων. Αυτο το ερώτημα, γνωστό και ως ισχυρό πρόβλημα του Tsirelson, απαντήθηκε αρνητικά στην εργασία [10]. Επισημαίνουμε ακόμα πως ούτε η απάντηση στο αθενές πρόβλημα του Tsirelson, αν δηλαδή το σύνολο όλων προσσεγγιστικά κβαντικών (approximately quantum) συσχετίσεων ταυτίζεται με το σύνολο όλων των quantum commuting συσχετίσεων, αναλύεται στην παρούσα εργασία. Η απάντηση στο ασθενές πρόπλημα είναι και αυτή αρνητική όπως έδειξε ο Slofstra στο [21]. Επιλέξαμε να μην ασχολειθούμε (παρά μόνον επιδερμικά) με τα δύο αυτά ζητήματα καθώς για την αποσειξή τους απαιτούνται τεχνικές οι οποίες δεν παρουσιάζονται στην παρούσα εργασία.

Στην βιβλιογραφία είναι σύνηθες οι διάφορες κλάσεις συσχετίσεων που περιγράψαμε να ορίζονται με την χρήση μέτρων με τιμές προβολές (PVM's) αντί των μέτρων με τιμές θετικούς τελεστές (POVM's). Στο τελευταίο κεφάλαιο θα δούμε πως και στις δυο περιπτώσεις παίρνουμε τις ίδιες κλάσεις συσχετίσεων.

Ευχαριστίες

Η αποπεράτωση της παρούσας διπλωματικής εργασίας - η οποία εκπονήθηκε με τριμελή επιτροπή τους κ.κ. Αριστείδη Κατάβολο, Μιχάλη Ανούση και Ιβάν Τοντορώφ - σηματοδοτεί την ολοκλήρωση των σπουδών μου για την απόκτηση του Διπλώματος Μεταπτυχιακών Σπουδών με ειδίκευση στα Θεωρητικά Μαθηματικά που απονέμεται από το Τμήμα Μαθηματικών του Πανεπιστημίου Αθηνών.

Θα ήθελα να ευχαριστήσω την τριμελή επιτροπή στο σύνολο της για την συμμετοχή της σε αυτήν την προσπάθεια. Ιδιαιτέρως, ευχαριστώ τον επιβλέποντα της διπλωματικής κ. Αριστείδη Κατάβολο για το ενδιαφέρον και την βοήθεια που μου παρείχε σε κάθε βήμα της διαδικασίας αυτής που συνέβαλαν καθοριστικά στην συγγραφή και ολοκλήρωση της εργασίας.

Τέλος επιθυμώ να ευχαριστήσω τους γονείς μου και τον αδερφό μου για την στηριξή και τις συμβουλές τους κατά την διάρκεια των σπουδών μου. Επίσης, ευχαριστώ τους φίλους μου για την συμπαράστασή τους.

1 Introduction

We will assume that the reader has some familiarity with basic C^*-algebraic theory such as the Gelfand-Naimark Theorem and the tensor product construction in the C^*-algebra category.
For a thorough review of these topics the reader is advised to see [13].

1.1 Matrix Ordered Spaces

If V is a complex vector space, we denote the space of $n \times m$ matrices whose entries are elements of V by $M_{n,m}(V)$, which is also a vector space in a natural way. We set $M_{n,n}(V) := M_n(V)$ and $M_{n,m} := M_{n,m}(\mathbb{C})$.
The space $M_{n,m}$ has the canonical basis $\{E_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ where $E_{i,j}$ is the $n \times m$ matrix with 1 in the (i, j) entry and 0 everywhere else. If $m = n$ we will write $M_{n,n} = M_n$.

A ***-vector space** is a complex vector space together with a conjugate linear map $^* : V \to V$ which is involutive (that is $(v^*)^* = v$). We say that an element v is hermitian (self-adjoint) if $v^* = v$ and we let V_h denote the real subspace of V containing all such elements. Note that if we have a $v \in V$ then there exists a decomposition of v into self adjoint elements $v = x + iy$ where $x = \frac{1}{2}(v + v^*)$ and $y = \frac{1}{2i}(v - v^*)$ so:

$$V = V_h + iV_h, \qquad V_h \cap iV_h = \{0\}$$

If V is a *-vector space, then we define a *-operation on $M_n(V)$ by letting $[v_{ij}]^* = [(v_{ji})^*]$, with this operation $M_n(V)$ becomes a *-vector space. We let $M_n(V)_h$ denote the set of all hermitian elements of $M_n(V)$.

We call a subset K of a real vector-space V a cone if it satisfies the following properties:

1. $\lambda v \in K$, for every $\lambda \in \mathbb{R}^+ = [0, \infty)$ and $v \in K$
2. $v_1 + v_2 \in K$, for every $v_1, v_2 \in K$

An ordered *-vector space is a pair (V, V^+) satisfying:

1. V^+ is a cone in V_h
2. $V^+ \cap (-V)^+ = 0$ (i.e., V^+ is proper)

If V is an ordered *-vector space we may define a partial order $\leq$ on V_h by declaring $w \leq v$ if and only if $v - w \in V^+$. Then $v \in V^+$ if and only if $0 \leq v$, for this reason V^+ is called the cone of positive elements of V. Note that $w \leq v$ implies that:

1. $w + x \leq v + x$ for any $x \in V$
2. $\lambda w \leq \lambda v$ for any $\lambda \in [0, \infty)$

Remark: We used the cone V^+ to define a partial ordering, however we could have done the opposite: If $\leq$ is a partial order on V_h satisfying (1) and (2) as above and we set V^+={$v \in V : 0 \leq v$} then one can easily check that the pair (V, V^+) is an ordered vector space.

Let H be a Hilbert space we let $B(H)$ denote the space of all bounded linear operators on H and H^n the direct sum of n-copies of H. We will denote an element $h \in H^n$ by,

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = (h_1, \ldots, h_n)^t$$

Throughout this Chapter t will denote the transpose.
The inner product on H^n is defined by

$$\left(\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \right)_{H^n} = \sum_{i=1}^{n} (h_i, k_i)_H$$

with this inner product H^n is a Hilbert space and for an element of H^n its norm is given by

$$\left\| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|_{H^n} = \sqrt{\|h_1\|_H^2 + \|h_2\|_H^2 + \cdots + \|h_n\|_H^2}$$

Moreover $B(H)$ with the usual *-operation and positive cone that of the positive operators is an ordered vector space. (An operator T is called positive if $(Th, h) \geq 0$ for every $h \in H$).

We also have the identification $M_n(B(H)) = B(H^n)$.
Let $[\alpha_{ij}] \in M_n(B(H))$ we define an operator $A : H^n \to H^n$ via the rule:

$$A \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = [\alpha_{ij}] \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \alpha_{1j} h_j \\ \vdots \\ \sum_{j=1}^{n} \alpha_{nj} h_j \end{pmatrix}$$

Then $[a_{ij}] \in B(H^n)$. Indeed, A is clearly well-defined, linear and for
$h = (h_1, \ldots, h_n)^t \in H^n$ we have that

$$\begin{aligned} \|Ah\|^2 = \|[\alpha_{ij}]h\|^2 &= \|\sum_{j}^{n} \alpha_{1j} h_j\|^2 + \cdots + \|\sum_{j}^{n} \alpha_{nj} h_j\|^2 \\ &\leq (\sum_{j}^{n} \|\alpha_{1j}\|^2)(\sum_{j}^{n} \|h_j\|^2) + \cdots + (\sum_{j}^{n} \|\alpha_{nj}\|^2)(\sum_{j}^{n} \|h_j\|^2) \\ &= (\sum_{i,j}^{n} \|\alpha_{ij}\|^2)\|h\|^2 \end{aligned}$$

so $\|A\| \leq (\sum_{i,j=1}^{n} \|\alpha_{ij}\|^2)^{\frac{1}{2}} < \infty$ because for each i, j we have that α_{ij} is a bounded operator.

On the other hand, every operator $A \in B(H^n)$ can be written in the above form. To see this, for every $j = 1, \dots, n$, let $V_j : H \to H^n$ be the map that sends an $h \in H$ to the element $(\xi_1, \dots, \xi_n)^t$ of H^n where $\xi_i = 0$ for $i \neq j$ and $\xi_j = h$. The adjoint of this map $V_j^* : H \to H^n$ is the projection on the j-th coordinate, i.e.,

$$V_j^* \left((h_1, \dots, h_n)^t \right) = h_j$$

Set $\alpha_{ij} = V_i^* A V_j$ then for $h, k \in H^n$ we have that

$$\begin{aligned}(Ah, k) &= (A(V_1 h_1 + \dots + V_n h_n), (V_1 k_1 + \dots + V_n k_n)) \\ &= \sum_{i,j=1}^{n} (AV_j h_j, V_i k_i) = \sum_{i,j=1}^{n} (V_i^* A V_j h_j, k_i) \\ &= ([\alpha_{ij}] h, k)\end{aligned}$$

Thus, $A = [\alpha_{ij}]$.

Now we claim that the map $\Phi : M_n(B(H)) \to B(H^n) : [\alpha_{ij}] \to A$ is an $*$-isomorphism.

Indeed, let $[\alpha_{ij}], [\beta_{ij}] \in M_n(B(H))$ then

$$\begin{aligned}[\alpha_{ij}][\beta_{ij}](h_1, \dots, h_n)^t &= [\alpha_{ij}](\sum_{j=1}^{n} \beta_{1j} h_j, \dots, \sum_{j=1}^{n} \beta_{nj} h_j)^t \\ &= (\sum_{l=1}^{n} \alpha_{1l} \sum_{j=1}^{n} \beta_{lj} h_j, \dots, \sum_{l=1}^{n} \alpha_{nl} \sum_{j=1}^{n} \beta_{lj} h_j)^t \\ &= (\sum_{l,j=1}^{n} \alpha_{1l} \beta_{lj} h_j, \dots, \sum_{l,j=1}^{n} \alpha_{nl} \beta_{lj} h_j)^t \\ &= [\sum_{l} \alpha_{il} \beta_{lj}](h_1, \dots, h_n)^t\end{aligned}$$

so $\Phi([\alpha_{ij}])\Phi([\beta_{ij}]) = \Phi([\alpha_{ij}][\beta_{ij}])$, in addition we see that for every $h, k \in H^n$

$$(\Phi([\alpha_{ij}])^* h, k) = (h, \Phi([\alpha_{ij}])k) = \sum_{i=1}^{n} \left(h_i, \sum_{j=1}^{n} \alpha_{ij} h_j \right) = \sum_{i,j=1}^{n} \left(\alpha_{ij}^* h_i, k_j \right) =$$

$$\sum_{i,j=1}^{n} \left(V_j^* A^* V_i h_i, k_j \right) = \sum_{i,j=1}^{n} (A^* V_i h_i, V_j k_j) = (A^* h, k) = (\Phi([\alpha_{ij}]^*) h, k)$$

hence our claim was true.

Now we know that $B(H^n)$ with the operator norm is a C^*-algebra so if we transfer this norm to $M_n(B(H))$ by setting

$$\|[\alpha_{ij}]\| = \|\Phi([\alpha_{ij}])\|_{B(H^n)}$$

then $M_n(B(H))$ becomes a C^*-algebra.

Example 1: In this way we may identify $M_n = M_n(\mathbb{C})$ with $B(\mathbb{C}^n)$.

Example 2: Let $\mathcal{A}$ be a C^*-algebra and take the universal representation $\rho : \mathcal{A} \to B(H)$. We use it to define the following injective *-homomorphism $\Psi : M_n(\mathcal{A}) \to B(H^n) : [\alpha_{ij}] \to \Psi([\alpha_{ij}])$ where

$$\Psi([\alpha_{ij}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \rho(\alpha_{1j})h_j \\ \vdots \\ \sum_{j=1}^n \rho(\alpha_{nj})h_j \end{pmatrix}$$

Then we can define a C^*-norm on $M_n(\mathcal{A})$ via the rule

$$\|[\alpha_{ij}]\| = \|\Psi([\alpha_{ij}])\|_{B(H^n)}$$

Some other identifications we will use frequently include :

$$(\mathbb{C}^n)^m = \mathbb{C}^{nm} \text{ and } M_n(M_m(V)) = M_{nm}(V), \text{ for any } V \text{ *-vector space}$$

Note that the map:

$$((x_1, \ldots, x_n), \ldots, (y_1, \ldots, y_n)) \to (x_1, \ldots, x_n, \ldots, y_1, \ldots, y_n)$$

is an isometry from $(\mathbb{C}^n)^m$ onto $\mathbb{C}^{nm}$, where C^{nm} has the euclidean norm $\|\cdot\|_2$ and

$$\|((x_1, \ldots, x_n), \ldots, (y_1, \ldots, y_n))\|_{(C^n)^m} =$$

$$\sqrt{\|((x_1, \ldots, x_n)\|_2^2 + \cdots + \|(y_1, \ldots, y_n))\|_2^2}$$

Similarly for any *-vector space V we can identify $M_n(M_m(V))$ with $M_{nm}(V)$ via the map:

$$\begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix} \to \begin{bmatrix} \alpha_{11}^{11} & \ldots & \alpha_{1m}^{11} & \ldots & \ldots & \alpha_{11}^{1n} & \ldots & \alpha_{1m}^{1n} \\ \vdots & \ddots & \vdots & \ldots & \ldots & \vdots & \ddots & \vdots \\ \alpha_{m1}^{11} & \ldots & \alpha_{mm}^{11} & \ldots & \ldots & \alpha_{m1}^{1n} & \ldots & \alpha_{mm}^{1n} \\ \vdots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \vdots \\ \alpha_{11}^{n1} & \ldots & \alpha_{1m}^{n1} & \ldots & \ldots & \alpha_{11}^{nn} & \ldots & \alpha_{1m}^{nn} \\ \vdots & \ddots & \vdots & \ldots & \ldots & \vdots & \ddots & \vdots \\ \alpha_{m1}^{n1} & \ldots & \alpha_{mm}^{n1} & \ldots & \ldots & \alpha_{m1}^{nn} & \ldots & \alpha_{mm}^{nn} \end{bmatrix}$$

Where each $A_{kl} = [\alpha_{ij}^{kl}]_{i,j} \in M_m(V)$, for every $k, l \in \{1, \ldots, n\}$.

If V is a (unital) C^*-algebra it is easy to see that the above identification preserves the multiplication and the *-operation, i.e., it defines a *-isomorphism.
It follows that by removing the inner brackets we obtain the following isomorphisms:

$$M_n(M_m(\mathcal{V})) \cong M_{nm}(\mathcal{V})) \cong M_m(M_n(\mathcal{V}))$$

Another way to see the latter identification with the use of tensor products:

$M_{n,m}(V) = M_{n,m} \otimes V$ and $M_{n,m}(V) = V \otimes M_{n,m}$ via the maps,

$$[v_{ij}] \to \sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij} \qquad \text{and} \qquad [v_{ij}] \to \sum_{i,j=1}^{n,m} v_{ij} \otimes E_{i,j}$$

Indeed, define $\pi : M_n(V) \to M_n \otimes V$ to be the map given by

$$\pi([v_{ij}]) = \sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij}$$

then it is clearly linear. We will show that π is a *-isomorphism.

π is injective : Assume that $[v_{ij}] \in \ker \pi$ then $\sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij} = \pi([v_{ij}]) = 0$. Since the elements E_{ij} are linearly independent for all $1 \leq i \leq n$ and $1 \leq j \leq m$, we have that $v_{ij} = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus $[v_{ij}] = 0$.

π is surjective : Let $v \in V \otimes M_n$ then v can be written as $v = \sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij} = \pi([v_{ij}])$, for some $v_{ij} \in V$.

π is $*$-preserving : Let $[v_{ij}] \in M_{n,m}(V)$ then

$$\pi([v_{ij}]^*) = \pi([v_{ji}^*]) = \sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ji}^* = \sum_{i,j=1}^{n,m} E_{j,i}^* \otimes v_{ji}^* = (\sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij})^* = \pi([v_{ij}])^*$$

In the case in which V is a C^*-algebra we have seen that there exists a norm on $M_n(V)$ making it a C^*-algebra. Moreover, π as defined above is a *-homomorphism between $M_n(V)$ and $M_n \otimes V$. Thus $M_n \otimes V$ is a C^*-algebra with respect to the norm it inherits from $M_n(V)$.

To see this, let $[v_{ij}]$, $[w_{ij}] \in M_n(V)$ then:

$$\begin{aligned}
\pi([v_{ij}][w_{ij}]) = \pi([\sum_k v_{ik}w_{ki}]) &= \sum_{i,j=1}^{n} E_{i,j} \otimes (\sum_k v_{ik}w_{ki}) \\
&= \sum_{i,j=1}^{n} E_{i,k}E_{k,j} \otimes (\sum_k v_{ik}w_{ki}) = \sum_{i,j,k,l} E_{i,k}E_{s,j} \otimes (v_{ik}w_{lj}) \\
&= (\sum_{i,j=1}^{n} E_{i,j} \otimes v_{ji})(\sum_{i,j=1}^{n} E_{i,j} \otimes w_{ij}) \\
&= \pi([v_{ij}])\pi([w_{ij}])
\end{aligned}$$

Since every C^*-algebra admits a unique complete C^*-norm we have proved the following proposition :

Proposition 1.1 *For every C^*-algebra $\mathcal{A}$ and for every $n \in \mathbb{N}$ there exists a unique C^*-norm on the algebraic tensor product $M_n \otimes \mathcal{A}$, i.e., M_n is a nuclear C^*-algebra.*

If V and W are vector spaces then $V \otimes W$ is the linear span of the set $\{v \otimes w : v \in V, w \in W\}$. Thus, $M_n(V \otimes W) = M_n \otimes (V \otimes W)$ is the linear span of the set $\{E_{ij} \otimes (v \otimes w) : 1 \leq i, j \leq n, v \in V, w \in W\}$. Hence, the map

$$E_{i,j} \otimes (v \otimes w) \to (E_{i,j} \otimes v) \otimes w$$

extends to a linear isomorphism between the spaces $M_n \otimes (V \otimes W)$ and $(M_n \otimes V) \otimes W$, so $M_n(V \otimes W)$ is linearly isomorphic to $M_n(V) \otimes W$.

For $v \in M_{n,m}(V)$ and $u \in M_{k,l}(V)$ we use the notation

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+k,m+l}(V)$$

for the direct sum of v and w.

Definition 1.2 *If (V, V^+) is an ordered *-vector space, an element $e \in V_h$ is called an order unit for V if for all $v \in V_h$ there exists a real positive number r such that $-re \leq v \leq re$. Equivalently such an e is called an order unit if and only if $\cup[-\lambda e, \lambda e] = V_h$ for all real $\lambda > 0$.*

Lemma 1.3 ([16]) *If (V, V^+) is an ordered *-vector space with order unit e, then:*

1. *$e \in V^+$*
2. *If $v \in V$ and a real number $r \geq 0$ is chosen so that $re \geq v$, then $se \geq v$, for all $s \geq r$.*
3. *$V_h = V^+ - V^+$ (V^+ is a full cone of V_h)*

4. *If* $v_1, \dots, v_n \in V^+$ *and* $v_1 + \dots + v_n = 0$, *then* $v_1 = \dots = v_n = 0$

5. *If* $v_1, \dots, v_n \in V^+$ *and there are real numbers* $0 \leq \alpha_i$, *for* $0 \leq i \leq n$ *such that* $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ *then either* $v_i = 0$ *or* $\alpha_i = 0$, *for every* $0 \leq i \leq n$.

Definition 1.4 *If* (V, V^+) *is an ordered *-vector space with order unit* e, *we say that* e *is an Archimedean order unit if whenever* $v \in V$ *with* $0 \leq re + v$, *for every* $r > 0$ *then* $v \in V^+$. *Equivalently, if* $\cap[-\lambda e, \lambda e] = \{0\}$.
In this case the triple (V, V^+, e) *is called Archimedean ordered *-vector space, AOU for short.*

Definition 1.5 *Let* V *be a *-vector space.*
The family $\{C_n\}_{n=1}^{\infty}$ *where* $C_n \subseteq M_n(V)_h$ *for every* n, *is called a matrix ordering on* V *if:*

(1) C_n *is cone in* $M_n(V)_h$, *for every* n.

(2) $C_n \cap (-C_n) = \{0\}$, *for every* n *(*C_n *is a proper cone,* $\forall n$*)*

(3) for every $n, m \in \mathbb{N}$ *and for every* $X \in M_{n,m} : X^* C_n X \subseteq C_m$

Condition (3) is often referred to as the compatibility of the family $\{C_n\}_{n=1}^{\infty}$.

We call the pair $(V, \{C_n\}_{n=1}^{\infty})$ *a* ***matrix ordered *-vector space****.*

Note: It follows from the properties (1) and (2) of the above definition that $(M_n(V), C_n)$ is an ordered *-vector space for every $n \in \mathbb{N}$. If $A, B \in M_n(V)_h$, we write $A \leq B$ when $B - A \in C_n$.

Definition 1.6 *Let* $(V, \{C_n\}_{n=1}^{\infty})$ *be a matrix ordered *-vector space. For an element* $e \in V_h$ *we set* e_n *to be the corresponding diagonal matrix in* $M_n(V)$, *with entries:* e *in the main diagonal and 0 everywhere else:*

$$e_n := \begin{bmatrix} e & & \\ & \ddots & \\ & & e \end{bmatrix} = I_n \otimes e$$

where I_n *denotes the identity matrix of* M_n.

We say that e *is a matrix order unit for* (V, V^+) *if for every* $n \in \mathbb{N}$ *we have that* e_n *is an order unit for* $(M_n(V), C_n)$. *Furthermore,* e *is called an Archimedean matrix order unit when* e_n *is an Archimedean order unit for* $(M_n(V), C_n)$.

1.2 Positive maps on Matrix ordered spaces

In this subsection we will briefly examine positive and completely positive maps on matrix ordered spaces. Positive and completely positive maps are essential to the study of operator systems.

Definition 1.7 *Let (V, V^+) and (W, W^+) be ordered *-vector spaces with order units e_1 and e_2 respectively, then a linear map $\varphi : V \to W$ is called:*

1. *unital if $\varphi(e_1) = e_2$.*
2. *positive if $\varphi(V^+) \subseteq W^+$.*
3. *order isomorphism if it is an isomorphism of vector spaces and both φ, φ^{-1} are positive, in this case we have: $v \in V^+ \iff \varphi(v) \in W^+$.*

Let V and W be vector spaces and suppose that $\phi : V \to W$ is a linear map then for every $n \in \mathbb{N}$ the map ϕ induces a linear map $\phi^n : M_n(V) \to M_n(W)$ given by

$$\phi^n([v_{ij}]_{i,j}) := [\phi(v_{ij})]_{i,j}$$

Definition 1.8 *If $(V, \{C_n\}_{n=1}^{\infty})$ and $(W, \{D_n\}_{n=1}^{\infty})$ are matrix ordered *-vector spaces, a linear map $\varphi : V \to W$ is called:*

(i) ***completely positive*** *(c.p. or CP for short) if $\varphi^n(C_n) \subseteq D_n$, for all $n \in \mathbb{N}$, that is to say the induced map $\varphi^n : M_n(V) \to M_n(W)$ is positive.*

(ii) ***order isomorphism*** *if it is bijective and both φ, φ^{-1} are positive.*

(iii) ***complete order isomorphism*** *if it is bijective and both φ, φ^{-1} are completely positive.*

(iv) ***complete order embedding*** *if it is an injective completely positive map and whenever $\phi^n([v_{ij}]_{i,j}) \in D_n$ then $[v_{ij}]_{i,j} \in C_n$.*

Given matrix ordered spaces V and W we let $\mathcal{L}(V, W)$ denote the space of all linear maps from V to W. The cone of completely positive maps provides a partial ordering of $\mathcal{L}(V, W)$.

Definition 1.9 *Let (V, V^+) be an ordered *-vector space with order unit e, a linear map $\varphi : V \to \mathbb{C}$ that is positive and unital is called a state. We denote the set of all states on V by $\mathcal{S}(\mathcal{V})$.*

Let V be a *-vector space and $\varphi : V \to M_n$ a linear map then we associate to φ a linear functional $s_\varphi : M_n(V) \to \mathbb{C}$ via the formula:

$$s_\varphi(A) = \frac{1}{n} \sum_{i,j=1}^{n} (\varphi(\alpha_{ij})e_j, e_i)\,,\ A = [\alpha_{ij}] \in M_n(V)$$

where $\{e_k\}_{k=1}^n$ denotes the standard basis of $\mathbb{C}^n$.

Alternatively, if we let $x_0 = \dfrac{1}{\sqrt{n}} e_1 \oplus \cdots \oplus e_n \in \mathbb{C}^{n^2} = \mathbb{C}^n \oplus \ldots . \oplus \mathbb{C}^n$ we have that for each $A = [\alpha_{ij}]$ in $M_n(V)$:

$$s_\varphi(A) = (\varphi^n(A)x_0, x_0) = (\varphi([\alpha_{ij}])x_0, x_0)$$

where the inner product is that of $\mathbb{C}^{n^2}$.

If φ is unital then so is s_φ and the map $\Phi : \mathcal{L}(V, M_n(\mathbb{C})) \to \mathcal{L}(M_n(V), \mathbb{C})$ which takes φ to s_φ is linear.

Conversely, if $s : M_n(V) \to \mathbb{C}$ is a linear functional, we define the map $\varphi_s : V \to M_n$ via:

$$(\varphi_s(\alpha)e_j, e_i) = ns(\alpha \otimes E_{ij}),\ \alpha \in V$$

where $\alpha \otimes E_{ij}$ is the element of $M_n(V) = V \otimes M_n$ which has α in the (i, j) - entry and 0 everywhere else and $(\varphi_s(\alpha)e_j, e_i) = [\varphi_s(\alpha)]_{(i,j)}$ is the (i, j)-entry of the complex matrix $[\varphi_s(\alpha)]_{ij}$.

Let $\varphi \in \mathcal{L}(V, M_n)$, $s \in \mathcal{L}(M_n(V), \mathbb{C})$ and $A = [a_{ij}] \in M_n(V)$. Then,

$$s_{\phi_s}(A) = \frac{1}{n} \sum_{i,j=1}^{n} (\phi_s(a_{ij})e_j, e_i) = \frac{1}{n} \sum_{i,j=1}^{n} ns(a_{ij} \otimes E_{ij}) = s(A)$$

Now, let $v \in V$ then

$$\big(\phi_{s_\phi}(v)e_j, e_i\big) = ns_\phi(v \otimes E_{ij}) = (\phi(v)e_j, e_i)$$

so we have that $\varphi = \varphi_{s_\varphi}$ and $s = s_{\varphi_s}$, hence the maps $s \to \varphi_s$ and $\varphi \to s_\varphi$ are mutual inverses.

Theorem 1.10 *Let $(V, \{C_n\}_{n=1}^\infty)$ be a matrix ordered *-vector space. If $s : M_n(V) \to \mathbb{C}$ is a linear functional and $\varphi = \varphi_s : V \to M_n$ is the associated linear map. Then the following are equivalent:*

1. $s(C_n) \subseteq M_n^+$
2. $\varphi : V \to M_n$ *is n-positive.*
3. $\varphi : V \to M_n$ *is completely positive.*

Proof: Obviously (3) implies (2).

(2) ⇒ (1): Fix a $n \in \mathbb{N}$ and assume that φ is n-positive, so for the map $\varphi^n : M_n(V) \to M_n(M_n) = M_{n^2}$ we have that $\varphi^n(C_n) \subseteq M_{n^2}^+$. Let $A = [\alpha_{ij}]$ be in C_n, we will show that $s(A) \in M_{n^2}^+$. We have that

$$\varphi^n(A) = [\varphi(\alpha_{ij}] \in M_{n^2}^+, \quad \text{therefore} \quad (\varphi^n(A)h, h) \geq 0 \ \text{ for every } \ h \in \mathbb{C}^{n^2}$$

Applying this to the vector $x_0 = \dfrac{1}{\sqrt{n}} e_1 \oplus \cdots \oplus e_n$ which is in $\mathbb{C}^{n^2}$ and we obtain: (using the fact that $s = s_\varphi$ and $\varphi = \varphi_s$)

$$s(A) = s_\varphi(A) := (\varphi^n(A)x_0, x_0) = (\varphi([\alpha_{ij}])x_0, x_0) \geq 0,$$

as required.

(1) ⇒ (3): Assume that $s(A) \geq 0$ for all A in C_n. We want to show that φ is completely positive or equivalently that for every m the map $\varphi^m : M_m(V) \to M_m(M_n)$ is positive. Let $X = [v_{ij}] \in C_m := (M_m(V))^+$, we want:

$$\varphi^m([v_{ij}]) = [\varphi(v_{ij})] \in M_m(M_n)^+$$

Since $[\varphi(v_{ij})]$ acts on $\mathbb{C}^{mn}$ we need to show that $([\varphi(v_{ij})]h, h) \geq 0$, for every $h \in \mathbb{C}^{mn}$.

We write h as m-column vector $h = \begin{bmatrix} h_1^t \\ h_2^t \\ \vdots \\ h_m^t \end{bmatrix}$ where for every $i \in \{1, \ldots, m\}$ each h_i is a row vector in $\mathbb{C}^n$ and the superscript t denotes the transpose. Now $[\varphi(v_{ij})]$ is a $m \times m$ block matrix (with the blocks being $n \times n$ matrices) so:

$$([\varphi(v_{ij})]h, h) = \sum_{i,j=1}^{m} \left(\varphi(v_{ij})h_j^t, h_i^t\right)$$

(the first inner product is on $\mathbb{C}^{mn}$ the second on $\mathbb{C}^n$)

Remark: Given row vectors $h = [h_1 \ldots h_n]$ and $k = [k_1 \ldots k_n]$ of scalars since $\varphi(v) \in B(\mathbb{C}^n)$ for any $v \in V$, we have that:

$$\left(\varphi(v)h^t, k^t\right) = \sum_{i,j=1}^{n} h_j \overline{k_i} \left(\varphi(v)e_j, e_i\right)_{\mathbb{C}^n} =$$

$$\sum_{i,j=1}^{n} \left(\varphi(\overline{k_i} v h_j)e_j, e_i\right)_{\mathbb{C}^n} = ns([\overline{k_i} v h_j]) = ns(k^* v h)$$

where $k^* = \begin{bmatrix} \bar{k_1} \\ \bar{k_2} \\ \vdots \\ \bar{k_n} \end{bmatrix}$ (Remember that $\varphi = \varphi_s$)

so $k^* v h \in M_n(V)$ is the matrix product whose (i, j) entry is $[\bar{k_i} v h_j]$.

Using the above remark we have that:

$$\begin{aligned}([\varphi(v_{ij})]h, h)_{\mathbb{C}^{n^2}} &= \sum_{i,j=1}^{m} \left([\varphi(v_{ij})]h_j^t, h_i^t\right)_{\mathbb{C}^n} \\ &= \sum_{i,j=1}^{m} s(h_i^* v_{ij} h_j) \\ &= s(\sum_{i,j=1}^{m} h_i^* v_{ij} h_j)\end{aligned}$$

Let $A \in M_{mn}$ denote the matrix $A = \begin{bmatrix} h_1 & \dots & h_1 \\ \vdots & \dots & \vdots \\ h_m & \dots & h_m \end{bmatrix}$

whose m - rows are the vectors $h_1, \dots, h_m \in \mathbb{C}^n$ then $\sum_{i,j=1}^{m} h_i^* v_{ij} h_j$ is just the matrix product A^*XA. We assumed that $X \in C_m$ and the family $\{C_n\}_{n=1}^{\infty}$ is compatible so we have that $A^*XA \in C_n$, and therefore $s(A^*XA)$ is positive.

From the above theorem we obtain that the maps $s \to \varphi_s$ and its inverse are positive (they take completely positive maps to completely positive maps).

1.3 Dual of matrix ordered space

We will say that the real or complex vector spaces V and V_1 are in duality if there exists a bi-linear map $(v, f) \to v \cdot f$ from $V \times V_1$ to the scalars such that

(a) $v \in V$ is 0 if and only if $v \cdot f = 0$ for all $f \in V_1$

(b) $f \in V_1$ is 0 if and only if $v \cdot f = 0$ for all $v \in V$.

If V and V_1 are in duality, each defines the weak topology on the other. We refer to the weak topology on V as the $\sigma(V, V_1)$ topology (i.e., $v_i \to 0$ means that $v_i \cdot f \to 0$ in C for all $f \in V_1$).

Theorem 1.11 *The map $G : V_1 \to V^\delta : f \to \phi_f$, where $\phi_f(v) = v \cdot f$, $(v \in V)$ is an isomorphism.*

Proof: Notice that for each $f \in V_1$ the functional $\phi_f : V \to C$ is weakly continuous by definition of the weak topology $\sigma(V, V_1)$, so $\phi_f \in V^\delta$.

G is linear and 1-1: Let $f_1, f_2 \in V_1$ and $v \in V$ then

$$G(f_1 + \lambda f_2)[v] = \phi_{f_1+\lambda f_2}(v) = v \cdot (f_1 + \lambda f_2) =$$

$$v \cdot f_1 + v \cdot \lambda f_2 = \phi_{f_1}(v) + \lambda\phi_{f_2}(v) = G(f_1) + \lambda G(f_2)$$

Let $f \in V_1$ such that $G(f) = 0$, that is $\phi_f(v) = 0$ for every $v \in V$ which means that $v \cdot f = 0$ for every $v \in V$. Thus from the properties of duality we have that f must be the zero mapping, so $\ker(G) = \{0\}$ hence G is injective.

G is onto: Let $\phi \in V^\delta$, so ϕ is weakly continuous. We will show that $\phi = G(f) = \phi_f$ for some $f \in V_1$. Since $\sigma(V, V_1)$ is the weakest topology on V making each ϕ_f continuous, given ϵ there exist a $\delta > 0$ and a finite set $\{f_1, \dots, f_n\} \subseteq V_1$ such that:

$$|f_i(v)| < \delta, \ \forall i = 1, \dots, n \text{ which implies that } |\phi(v)| < \epsilon$$

In particular, if $|f_i(v)| = 0, \ \forall\, i = 1, \dots, n$, then for all $m \in N$ we have that:

$$|f_i(mv)| = 0, \ \forall\, i = 1, \dots, n \text{ and, hence } |\phi(mv)| < \epsilon$$

Thus $|\phi(v)| < \frac{1}{m}\epsilon$, for all m in $\mathbb{N}$ and so $\phi(v) = 0$. This shows that if $|f_i(mv)| = 0, \ \forall\, i = 1, \dots, n$ then $\phi(v) = 0$. It follows that:

$$\bigcap_{i=1}^{n} \ker \phi_{f_i} \subseteq \ker \phi$$

This condition implies that the linear map ϕ is a linear combination of the maps ϕ_{f_i}, so there exist scalars c_i such that $\phi = \sum_{i=1}^{n} c_i \phi_{f_i}$.
Letting $f := \sum_{i=1}^{n} c_i f_i \in V_1$ we have $\phi = \phi_f$ as claimed.

Applying the above theorem to the space V^δ we have that: $(V^\delta)^\delta = V$

If V and V^δ are complex spaces in duality then $M_m(V)$ and $M_m(V^\delta)$ are in duality under the bi-linear function

$$[v_{ij}] \cdot [f_{ij}] := \sum_{i,j} f_{ij}(v_{ij}).$$

Let $[v_{ij}] \in M_n(V)$ such that:

$$[v_{ij}] \cdot [f_{ij}] = 0, \text{ for every } [f_{ij}] \in M_n(V^\delta) \qquad (*)$$

Condition $(*)$ is equivalent to saying that: $\sum_{i,j} f_{ij}(v_{ij}) = 0$ for every $f_{ij} \in V^\delta$, $i, j \in \{1, \ldots, n\}$.
Condition $(*)$ is satisfied by every $[f_{ij}] \in M_n(V^\delta)$ so all the matrices that have only the (i,j) - entry non zero satisfy $(*)$ and from this we obtain that: $f_{ij}(v_{ij}) = 0$ for every $i, j \in \{1, \ldots, n\}$. Since $(*)$ is valid for any $[f_{ij}] \in V^\delta$ we have that for all i and j, f_{ij} can be any element of V^δ so from the duality of V and V^δ we obtain that $v_{ij} = 0$, for every $i, j \in \{1, \ldots, n\}$ therefore $[v_{ij}]$ is the zero matrix of $M_n(V)$.

Conversely,

if $\quad [v_{ij}] = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \in M_n(V)$ then obviously:

$[v_{ij}] \cdot [f_{ij}] := \sum_{i,j} f_{ij}(v_{ij}) = 0$, for every $[f_{ij}] \in M_n(V^\delta)$.

The fact that $[f_{ij}] = 0$ if and only if $[v_{ij}] \cdot [f_{ij}] = 0$ for all $[v_{ij}]$ in $M_n(V)$ is proven in a similar fashion.

So, $M_n(V)^\delta = M_n(V^\delta)$ and we can consider an element $[f_{ij}] \in M_n(V^\delta)$ as a weakly continuous linear map $F : M_n(V) \to \mathbb{C}$.

Note: Let $\mathcal{S} \subseteq B(H)$ be an operator system and $s = [s_{ij}] \in M_n(S)^+$ then for any arbitrary $n \times m$ matrix $A = [\alpha_{ik}]$ of complex numbers we have that:

$$A^* s A = [\sum_{i,j=1}^{n} \overline{\alpha_{ik}} r_{ij} \alpha_{jl}]_{kl} \in M_m(S)^+$$

.
Indeed, we have that s is positive in $M_n(\mathcal{S})$, so it can be written as $s = b^*b$, $b \in M_n(\mathcal{S})$. Therefore, $A^*sA = (bA)^*bA$ which is positive.

Lemma 1.12 *Let $(V, \{C_n\}_{n=1}^\infty)$ be a matrix ordered space, and $v = [v_{ij}] \in M_n(V)$ and $A = [\alpha_{ij}] \in M_n$. We define $\Phi(v) : M_n \to V$ by : $\Phi(v)(A) = \sum_{i,j=1}^n \alpha_{ij} v_{ij}$, then the map $\Phi : M_n(V) \to \mathcal{L}(M_n, V)$ is an order isomorphism.*

Proof: It is clear that Φ is linear and it is easy to see that it is injective.

Φ is surjective: Assume that $f \in \mathcal{L}(M_n, V)$ with $f(E_{i,j}) = v_{ij}$, $1 \leq i, j \leq n$. Letting $v = [v_{ij}] \in M_n(V)$ we have that

$$\Phi(v)(E_{i,j}) = \sum_{k,l=1}^{n} (\delta_{ki}\delta_{lj})v_{kl} = v_{ij} = f(E_{i,j})$$

where $E_{i,j} = [\delta_{ki}\delta_{lj}]_{k,l}$. It follows that $f = \Phi(v)$ ergo Φ is surjective.

Now we check the positivity of Φ and its inverse.
Let $v = [v_{ij}] \in M_n(V)$ be such that $\Phi(v)$ is completely positive, that means: $\Phi(v)^n : M_n(M_n) \to M_n(V)$ is positive, for every $n \in \mathbb{N}$.
Consider the element

$$E = [E_{i,j}] \in M_n(M_n) \text{ then } E = \begin{bmatrix} E_{1,1} & \dots & E_{1,n} \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} E_{1,1} & \dots & E_{1,n} \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{bmatrix}^*$$

so $E \in M_n(M_n)^+$.Thus,

$$v = [v_{ij}] = [\Phi(v)(E_{i,j})]_{i,j} := \Phi(v)^n(E) \geq 0$$

($E_{i,j} = [e_{kl}]$ is then $n \times n$ matrix with (i, j) - entry 1 and 0 everywhere else and $\Phi(s)(E_{i,j}) := \sum_{k,l=1}^{n} e_{kl}v_{kl} = v_{ij}$).

On the other hand, suppose that $v \in C_n$. Let $m \geq 1$, we will show that if $A \in M_m(M_n)^+$, then $\Phi(v)^m(A) \geq 0$. If a matrix $A \in M_m(M_n) = M_{mn}$ is positive semi-definite, i.e., $A \in M_{mn}^+$, then from the spectral theorem it admits a representation $\sum_{k=1}^{N} \beta_k\beta_k^*$ for $\beta_k \in \mathbb{C}^{mn}$, $k \in \{1, \dots, N\}$ where N denotes the rank of the matrix.
Let $\beta_k = (\beta_{k_1}, \dots, \beta_{k_m})$ where for each $1 \leq j \leq m$: $\beta_{k_j} \in \mathbb{C}^n$.
Then $\beta_k\beta_k^* = [\beta_{k_i}\beta_{k_j}^*]_{i,j=1}^{m}$ where every $\beta_{k_i}\beta_{k_j}^*$ is a $n \times n$ complex positive matrix and for each $1 \leq i \leq m$ we have that $\beta_{k_i} = (\beta_{k_{i1}}, \dots, \beta_{k_{in}})$.
We let $B = [\beta_{k_{it}}], 1 \leq i \leq m, 1 \leq t \leq n$ then B is a $m \times n$ complex matrix. Moreover,

$$\beta_k\beta_k^* = [\beta_{k_i}\beta_{k_j}^*]_{i,j=1}^{m} = [[\beta_{k_{it}}\beta_{k_{jp}}^*]_{t,p=1}^{n}]_{i,j=1}^{m}$$

and

$$\Phi(v)^m(\beta_k\beta_k^*) = [\Phi(v)[\beta_{k_{it}}\beta_{k_{jp}}^*]_{t,p=1}^{n}]_{i,j=1}^{m} = [\sum_{t,p=1}^{n} \beta_{k_{it}}v_{tp}\beta_{k_{jp}}^*]_{i,j=1}^{m} = BvB^* \in C_m$$

from the compatibility of the family $\{C_n\}_{n=1}^{\infty}$. Since the sum of positive elements remains positive it follows that

$$\Phi(v)^m(A) \geq 0$$

Since every operator system is a matrix ordered space we also have that if $\mathcal{S}$ is an operator system then the map Φ of Lemma 1.12 is an order isomorphism from $M_n(\mathcal{S})$ to $\mathcal{L}(M_n, \mathcal{S})$.

Dual cone: Let $(V, \{C_n := M_n(V)^+\})$ be a matrix ordered space and let V^δ be the dual of V, we partially order V^δ by the dual cone: $(V^\delta)^+ = V^\delta \bigcap (V^d)^+$ where $V^d := \mathcal{L}(V, \mathbb{C})$ = {linear functions from V to $\mathbb{C}$}. We regard $M_n(V^\delta)$ as the dual of $M_n(V)$, i.e., $M_n(V^\delta) := M_n(V)^\delta$ and we partially order it for each $n \in \mathbb{N}$ with the cone $(M_n(V)^\delta)^+ = M_n(V)^\delta \bigcap (M_n(V)^d)^+$.

Lemma 1.13 *If V is a matrix ordered space* ***with matrix order unit e*** *and with dual V^δ, then the above structure is a matrix ordering on V^δ.*

Proof: We need to check that the conditions (1),(2),(3) of Definition 1.5 are satisfied. Let $F \in (M_n(V)^\delta)^+ \bigcap -(M_n(V)^\delta)^+$ and $v \in M_n(V)^+$ then

$$F(v) = F_1(v) = -F_2(v) \;\text{ for some } F_1, F_2 \in M_n(V)^{d+}$$

therefore $F(v)$ must be equal to zero.
Since V has an order unit we have that $V_h = V^+ - V^+$ so $V = (V^+ - V^+) + i(V^+ - V^+)$ hence we have that $F(v) = 0$ for every v in $M_n(V)$.
Thus F is the zero function on $M_n(V)$.

For (3): Assume that $F = [f_{ij}] \in M_n(V^\delta)^+$ and X is a $n \times m$ complex matrix, and let $v = [v_{rs}] \in M_m(V)^+$ then:

$$v \cdot X^* F X = [v_{rs}] \cdot [\sum_{i,j=1}^{n} \bar{x}_{ir} x_{js} f_{ij}]_{r,s=1}^{m} = \sum_{r,s=1}^{m} \sum_{i,j=1}^{n} \bar{x}_{ir} x_{js} f_{ij}(v_{ij}) =$$

$$\sum_{i,j=1}^{n} \sum_{r,s=1}^{m} \bar{x}_{ir} x_{js} f_{ij}(v_{ij}) = [\sum_{r,s=1}^{m} \bar{x}_{ir} x_{js} v_{rs}]_{i,j} \cdot [f_{ij}] = (X^t)^* v X^t \cdot F$$

where X^t denotes the transpose of X. Since V is matrix ordered we have that: $A^*(M_n(V)^+)A \subseteq (M_m(V)^+)$ for any complex $n \times m$ matrix A so in our case: $(X^t)^* v X^t \in M_n(V)^+$. Since $F \in M_n(V^\delta)^+$ it follows that $((X^t)^* v X^t) \cdot F \geq 0$.

We proved that when $F \in M_n(V^\delta)^+$, then $X^* F X$ takes any positive element of $M_m(V)$ to $[0, +\infty)$. Consequently $X^* F X \in M_m(V^\delta)^+$, so

$$X^* M_n(V^\delta)^+ X \subseteq M_m(V^\delta)^+.$$

which proves that condition (3) is satisfied.

If V is a matrix ordered space with a matrix order unit, then we will call the matrix ordered space $(V^\delta, \{(M_n(V)^\delta)^+\}_{n=1}^{\infty}$ the **matrix ordered dual** of V.

If V and W are vector spaces with duals V^δ and W^δ respectively, then each weakly continuous linear map $\phi : V \to W$ induces the adjoint map $\phi^\delta : W^\delta \to V^\delta$ via the formula:

$$\phi^\delta(f)[v] := f(\phi(v)), \;\text{ for every }\; f \in W^\delta \;\text{ and }\; v \in V$$

Lemma 1.14 *The map* $\delta : B_\sigma(V,W) \to B_\sigma(W^\delta, V^\delta) : \phi \to \phi^\delta$ *is a linear surjective isomorphism.*

Proof: It is clearly linear and injective.

Surjective: Let $\psi \in B_\sigma(W^\delta, V^\delta)$, that is $\psi : W^\delta \to V^\delta$ linear and weakly continuous. For $v \in V$ consider the functional $\alpha_v : W^\delta \to \mathbb{C}$ with $\alpha_v(f) = \psi(f)[v]$. Then α_v is weakly continuous, so $\alpha_v \in B_\sigma(W^\delta, \mathbb{C}) := W^{\delta\delta}$ (=W).

From this fact we obtain that there exists $w \in W$ such that $\alpha_v(f) = f(w)$, for every $f \in W^\delta$. We set $\phi(v) = w$ then $\phi : V \to W$ is well defined (since the dual separates the points of the space). Now $f(\phi(v)) = f(w)$ for every $f \in W^\delta$, therefore $\phi^\delta(f)[v]$ = $\psi(f)[v]$. Hence $\phi^\delta = \psi$.

Remark: For every $\phi \in B_\sigma(V,W)$ we have that:

$$(\phi^n)^\delta = (\phi^\delta)^n \text{ and } (\phi^*)^\delta = (\phi^\delta)^*.$$

If the above Remark is true, then it follows that in the case in which V, W are matrix ordered *-vector spaces we have that a linear map $\phi : V \to W$ is completely positive if and only if $\phi^\delta : W^\delta \to V^\delta$ is completely positive.

Indeed, let $n \geq 1$, $[v_{ij}] \in M_n(V)$ and $[f_{ij}] \in M_n(W^\delta)^+$. Then

$$(\phi^\delta)^n([f_{ij}]) \cdot [v_{ij}] = (\phi^n)^\delta([f_{ij}]) \cdot [v_{ij}] = [f_{ij}] \cdot \phi^n[(v_{ij})] \geq 0$$

whenever $[v_{ij}]$ is positive, since ϕ is completely positive.
Therefore, we conclude that indeed ϕ^δ is CP.

Conversely, if $\phi^\delta : V^\delta \to W^\delta$ is CP since $(V^\delta)^\delta = V$ and $(W^\delta)^\delta = W$ we have that $(\phi^\delta)^\delta = \phi$ and using the above argument we have that $\phi = (\phi^\delta)^\delta$ is completely positive.

We conclude that the map δ in Lemma 1.14 is an order isomorphism, where the positive cone of $B_\sigma(V,W)$ is the space of all completely positive weakly continuous linear maps from V to W.

Proof of the Remark: If $\phi \in B_\sigma(V,W)$ then $(\phi^\delta)^n$ is a map from $M_n(W^\delta)$ to $M_n(V^\delta)$. Let $[f_{ij}] \in M_n(W^\delta)$ and $[v_{ij}] \in M_n(V)$. Then,

$$\begin{aligned}\left((\phi^\delta)^n([f_{ij}])\right)([v_{ij}]) = [\phi^\delta(f_{ij})] \cdot [v_{ij}] = \sum_{i,j=1}^{n} f_{ij}(\phi(v_{ij}))\\ = [f_{ij}] \cdot [\phi(v_{ij})] = [f_{ij}] \cdot (\phi^n([v_{ij}]))\\ = \left((\phi^n)^\delta([f_{ij}])\right)([v_{ij}])\end{aligned}$$

The other part follows from the way we defined the dual map and the *-operation.

Let V, V^δ be matrix ordered *-vector spaces in duality. Given $v = [v_{ij}] \in M_n(V)$ we define $\Psi(v) : V^\delta \to M_n$ by: $\Psi(v)[f] = [f(v_{ij})]$.

We will show that $\Psi(v) = \Phi(v)^\delta$ where Φ is the map defined in Lemma 1.12.

Indeed, let $f \in V^\delta$ and $A = [\alpha_{ij}] \in M_n$ then,

$$\begin{aligned} f(\Phi(v)[A]) &= f(\sum_{i,j=1}^{n} \alpha_{ij} v_{ij}) \\ &= \sum_{i,j=1}^{n} \alpha_{ij} f(v_{ij}) \\ &= [\alpha_{ij}] \cdot [f(v_{ij})] \\ &= A \cdot \Psi(v)[f] \end{aligned}$$

Now the following hold true:

1. $\Phi : M_n(V) \to B_\sigma(M_n, V)$ and $\delta : B_\sigma(M_n, V) \to B_\sigma(V^\delta, M_n)$, both of which are order isomorphisms.
2. $\Psi : M_n(V) \to B_\sigma(V^\delta, M_n)$, and $\Psi = \delta \circ \Phi$. Since it is a composition of surjective linear order isomorphisms Ψ too is an order isomorphism.

So we obtain:

Lemma 1.15 *If V is a matrix ordered *- vector space with an order unit and V^δ is its matrix ordered dual, then the map $\Psi : M_n(V) \to B_\sigma(V^\delta, M_n)$ is an order isomorphism.*

1.4 Archimedeanization of a matrix ordered *-vector space with a matrix order unit

Given an ordered *-vector space (V, V^+, e) there exists a process, introduced in [16], called Archimedeanization that allows us to obtain an Archimedean ordered *-vector space. In [18] it was shown that if $(V, \{C_n\}_{n=1}^\infty, e)$ is a matrix ordered *- vector space then by applying the Archimedeanization process to each level $(M_n(V), C_n, e_n)$ we obtain an Archimedean matrix ordered *-vector. space. In this subsection we will review the procedures described above. For more information on this subject the reader is instructed to see [18].

Firstly, we will consider real vector spaces. Suppose that (V, V^+) is a matrix ordered real vector space with an order unit e. Let

$$D := \{v \in V : re + v \in V^+, \ \forall r > 0\}$$

and set

$$N := D \cap (-D)$$

It is easy to see that D is a cone with $V^+ \subseteq D$ and that N is a real subspace of V. The following proposition proven in [16] gives us another useful characterization of N.

Proposition 1.16 *[16, Proposition 2.34.] Let (V, V^+) be an ordered real vector space with an order unit e and define N as above. Then,*

$$N = \bigcap_{f \in \mathcal{S}(V)} \ker f$$

Theorem 1.17 *Let (V, V^+) be an ordered real vector space with an order unit e, and define N and D as in the paragraph at the start of this subsection. Set*

$$(V/N)^+ := D + N = \{d + N : d \in D\}$$

Then $(V/N, (V/N)^+)$ is an ordered vector space with $e + N$ as an Archimedean order unit.

Proof: The fact that $(V/N)^+$ is a cone follows readily from the fact that D is a cone. Next we show that it is a proper cone, let $v + N \in (V/N)^+ \bigcap -(V/N)^+$ we shall show that $v + N = 0 + N$.
We have that $v + N = d + N$ and $v + N = -d' + N$ for some $d, d' \in D$. Thus, we obtain that $v - d \in N \subseteq D$ and $v + d' \in N \subseteq -D$. However, D is a cone and $d, d' \in D$ it follows that $v \in D$ and $v \in -D$, i.e., $v \in D \bigcap (-D) := N$ so $v + N = 0 + N$.

$e + N$ is an order unit: Since e is an order unit for (V, V^+) there exists $r > 0$ such that $re + v \in V^+$ for any $v \in V$. Let $v + N \in V/N$ then there exists $r > 0$ such that $r(e + N) + (v + N) = (re + v) + N \in V^+ + N \subseteq D + N = (V/N)^+$.

$e+N$ is an Archimedean order unit: Assume that $v+N \in V/N$ with $r(e+N)+(v+N) \in (V/N)^+,\ \forall r > 0$. Then, $(re + v) + N \in D + N$ and $re + v \in D$ for all $r > 0$. Choose $r' > 0$, then $\frac{r'}{2}e + v \in D$. By the definition of D we have that $\frac{r'}{2}e + v \in V^+$ so $\frac{r'}{2}e + (\frac{r'}{2}e + v) \in V^+$. It follows that $r'e + v \in V^+$ for all $r' > 0$. Once again by the definition of D we have that $v \in D$. Therefore, $v + N \in D + N = (V/N)^+$.
The proof is now complete.

Definition 1.18 *Let (V, V^+) be an ordered real vector space with an order unit e. Let*

$$D := \{v \in V : re + v \in V^+,\ \forall r > 0\}$$

and set

$$N := D \cap (-D)$$

We define V_{Arch} to be the Archimedean ordered vector space $(V/N, (V/N)^+, e + N)$. We call V_{Arch} the Archimedeanization of V.

We now turn our attention towards *-vector spaces.
Let (V, V^+) be an ordered *-vector space with order unit $e \in V_h$.
For $u, v \in V_h$ we define: $[u, v] = \{x \in V_h : u \leq x \leq v\}$. Consider the set

$$E := [-e, e] = \{v \in V_h : -e \leq v \leq e\}$$

and the Minkowski functional of that set which is:

$$p_E : E \to \mathbb{R}, \quad p_E(x) := \inf\{\lambda > 0 : x \in \lambda E\}, \quad x \in V_h.$$

This defines a semi norm on V_h.

Moreover, notice that e is an Archimedean order unit if and only if p_E is a norm on V_h.

Remark: Let (V, V^+) be an ordered *-vector space with order unit e and (W, W^+) be an ordered *-vector space. If $\varphi : V \to W$ is a positive linear map, then $\varphi(v^*) = \varphi(v)^*$ for every $v \in V$.

Proof: Since e is an order unit we have that $V_h = V^+ - V^+$ and φ is positive so $\varphi(V_h) \subseteq W_h$. Let $v \in V$ then v can be written as $v = x + iy$ where x, y are in V_h, thus

$$\varphi(v^*) = \varphi(x - iy) = \varphi(x) - i\varphi(y) = (\varphi(x) + i\varphi(y))^* = \varphi(x + iy)^* = \varphi(v)^*$$

Using the above remark one can see that if (V, V^+) is an ordered *-vector space with an order unit and $f : V \to \mathbb{C}$ is a positive $\mathbb{C}$-linear functional then $f(v)^* = \overline{f(v)}$ for every $v \in V$.

Definition 1.19 *Let (V, V^+) be an ordered *-vector space and $f : V_h \to \mathbb{R}$ linear map, then we define $\tilde{f} : V \to \mathbb{C}$ by $\tilde{f}(v) := f(\mathrm{Re}(v)) + if(\mathrm{Im}(v))$.*

The proofs of the following propositions can be found in [16]

Proposition 1.20 *Let (V, V^+) be an ordered *-vector space. If $f : V_h \to \mathbb{R}$ is $\mathbb{R}$-linear, then $\tilde{f} : V \to \mathbb{C}$ is $\mathbb{C}$-linear. Moreover, f is positive if and only if $\tilde{f}$ is positive and f is state if and only if $\tilde{f}$ is a state.*

Proposition 1.21 *Let (V, V^+) be an ordered *-vector space with an order unit e.
If $f : V \to \mathbb{C}$ is $\mathbb{C}$ linear then f is positive if and only if: $f = \tilde{g}$ for some linear and positive map $g : V_h \to \mathbb{R}$.*

Lemma 1.22 *Let (V, V^+) be a ordered *-vector space with order unit e.
Given $u \in V_h$, let*

$$\alpha := \sup\{r \in \mathbb{R} : re \leq u\} \quad \textit{and} \quad \beta := \inf\{s \in \mathbb{R} : u \leq se\}$$

Then,

(a) $[\alpha, \beta] = \{f(u) : f \in \mathcal{S}(V)\}$.

(b) $p_E(u) = \max\{|\alpha|, |\beta|\} = \sup\{|f(u)| : f \in \mathcal{S}(V)\}$.

Proof: We shall show that $f(u) \in [\alpha, \beta]$ for every state $f : V \to \mathbb{C}$. If $re \leq u \leq se$ then $\alpha \leq r \leq s \leq \beta$, hence

$$\alpha \leq r = rf(e) \leq f(u) \leq sf(e) = s \leq \beta$$

Therefore, $\{f(u) : f \in \mathcal{S}(V)\} \subseteq [\alpha, \beta]$.

For the reverse containment, we will show that for every $\gamma \in [\alpha, \beta]$ there exist a state $f_\gamma : V \to \mathbb{C}$ such that $f_\gamma(u) = \gamma$. Consider the following $\mathbb{R}$-linear subspace of V_h:

$$W := \{re + tu : r, t \in \mathbb{R}\}$$

and the $\mathbb{R}$-linear functional

$$g_\gamma : W \to \mathbb{R} : re + tu \to r + t\gamma$$

Notice that $g_\gamma(u) = \gamma$ and $g_\gamma(e) = 1$. We claim that g_γ is positive.

Indeed, suppose that $re + tu \in V^+$. It is obvious that $g_\gamma(re + tu) \geq 0$ for $t = 0$. For $t > 0$ the relation $re + tu \geq 0$ gives $-\frac{r}{t}e \leq u$ and so $-\frac{r}{t} \leq \alpha \leq \gamma$ which means that $r + t\gamma \geq 0$. Similarly when $t < 0$ we have that $-\frac{r}{t} \geq u$ and hence $-\frac{r}{t} \geq \beta \geq \gamma$ so once again $r + t\gamma \geq 0$.
Now we use the usual Zorn's Lemma argument for the family of all pairs of $(\tilde{W}, \tilde{g})$ where $\tilde{W}$ are sub-spaces of V_h containing W and $\tilde{g}$ are positive linear functionals form $\tilde{W}$ to $\mathbb{R}$ extending g, this allows us to extend g_γ to a positive linear form on the whole of V_h, which we will call g'. This extension will satisfy $g'(e) = 1$ and $g'(u) = \gamma$. Now define $f_\gamma : V \to \mathbb{C}$ by $f_\gamma = \tilde{g'}$ this proves part (a).

For part (b) notice that if $-re \leq u \leq re$ then r must be non-negative.
Furthermore, in this case we have that:

$$-r \geq \alpha \text{ and } r \geq \beta \text{ and so } r > \max\{|\alpha|, |\beta|\}$$

Consequently, $p_E(u) \geq \max\{|\alpha|, |\beta|\}$.

On the other hand, if $t > \max\{|\alpha|, |\beta|\}$ then $-t < \alpha$ and $t > \beta$. Thus, $-te \leq u \leq te$ and therefore $p_E(u) \leq t$. Since t was arbitrary we obtain that $p_E(u) \leq \max\{|\alpha|, |\beta|\}$. We conclude that

$$p_E(u) = \max\{|\alpha|, |\beta|\} \overset{(a)}{=} \sup\{|f(u)| : f \in \mathcal{S}(V)\}$$

and the proof is complete.

Proposition 1.23 *Let Let (V, V^+) be an ordered *-vector space with an order unit e. Then e is an Archimedean order unit if and only if for every $v \in V$ the following holds:*

$$f(v) = 0, \text{ for every } f \in \mathcal{S}(\mathcal{V}) \iff v = 0$$

Proof: Let $E := [-e, e]$ and $p_E : V_h \to \mathbb{R}$ be the Minkowski functional of E. The states separate the points of V so they separate the points of V_h. Thus from the equality: $p_E(v) = \max\{|\alpha|, |\beta|\} = \sup\{|f(v)| : f \in \mathcal{S}(V)\}$, for every v in V_h, we have that p_E is a norm on V_h and so e is an Archimedean order unit.

Conversely, if e is an Archimedean order unit and for a $v \in V$ we have that $f(v) = 0, \forall f \in \mathcal{S}(\mathcal{V})$. Then $f(Re(v)) = f(Im(v)) = 0$, for all $f \in \mathcal{S}(\mathcal{V})$. Hence from Lemma 1.22 : $p_E(Re(v)) = p_E(Im(v)) = 0$ which implies that $Im(v) = Re(v) = 0$.
Thus, $v = 0$.

Proposition 1.24 *Let (V, V^+, e) be an Archimedean ordered $*$-vector space. Then for an element $v \in V$ we have that:*

$$v \in V^+ \iff f(v) \geq 0, \text{ for every state } f : V \to \mathbb{C}$$

Proof: If $v \in V^+$ then clearly $f(v) \geq 0$.

Conversely, if $f(v) \geq 0$ for every state f then $f(v) \in \mathbb{R}$ so $f(v^*) = f(v)^* = f(v)$. Hence, since $v - v^* \in V_h$ this element is annihilated by every state and thus by Proposition 1.23 we have that $v - v^* = 0$, i.e., $v \in V_h$. Using Lemma 2.3 we see that there exists a state f_α such that $f_\alpha(v) = \sup\{r \in \mathbb{R} : re \leq v\}$. Now from our hypothesis $f_\alpha(v) \geq 0$, and so $\sup\{r \in \mathbb{R} : re \leq v\} \geq 0$. It follows that for every $r < 0$: $re \leq v$ or equivalently that $v + (-r)e \in V^+$. Since e is Archimedean and $-r > 0$ we have that $v \in V^+$.

Let (V, V^+) be an ordered *-vector space with order unit e, then we can produce an Archimedean ordered *-vector space in the following way: We define the sets

$$D := \{v \in V_h : re + v \geq 0 \text{ for every } r \in \mathbb{R}^+\} \text{ and } N_{\mathbb{R}} = D \cap (-D)$$

Then $N_{\mathbb{R}}$ is a real subspace of V_h and by Proposition 1.16 we have that:
$N_{\mathbb{R}} = \bigcap_{f:V_h \to \mathbb{R}, f state} \ker(f)$. Now we define

$$N := \bigcap\{\ker(f) : f \in \mathcal{S}(\mathcal{V})\}$$

It follows from Proposition 1.21 that $N = N_{\mathbb{R}} \oplus iN_{\mathbb{R}}$.
Moreover, N is a complex subspace of V closed under the *-operation (of V), so the quotient V/N with the well defined *-operation: $(v^* + N) = v^* + N$ is a *-vector space and

$$(V/N)_h = \{v + N : v \in V_h\}$$

(if $v + N \in (V/N)_h$ then $v + N = v^* + N$ and so $v = \frac{v+v^*}{2} + N \in V_h/N$).
We define $(V/N)^+ = \{v + N : v \in D\}$ and $V_{Arch} := (V/N, (V/N)^+, e + N)$.
We claim that the spaces $((V/N)_h, (V/N)^+)$ and $(V_h/N_{\mathbb{R}}, D + N_{\mathbb{R}})$ are order isomorphic via the map $v + N \to v + N_{\mathbb{R}}$.

Indeed, it is straightforward to see that this map is an isomorphism between vector spaces. Moreover, if $v + N \in (V/N)^+$ then $v \in D$ and so $v + N_{\mathbb{R}} \in D + N_{\mathbb{R}}$. Conversely, if $v + N_{\mathbb{R}} \in D + N_{\mathbb{R}}$ then $v \in D$ and so $v + N \in (V/N)^+$.

Now by Theorem 1.17 we have that the space V_{Arch} is an Archimedean ordered *-vector space. We call V_{Arch} the **Archimedeanization** of the *-vector space V.

Lemma 1.25 *Let $(V, \{C_n\}_{n=1}^\infty)$ be a matrix ordered *-vector space with matrix order unit e and N as described above. For each $n \in \mathbb{N}$, we define*

$$N_n := \bigcap\{\ker(f) : f \in \mathcal{S}(M_n(V))\}$$

Then $N_n = M_n(N)$, for every $n \in \mathbb{N}$.

Proof: Let $n \in \mathbb{N}$ and $A = [\alpha_{kl}] \in N_n$ then $f(A) = 0$ for every $f \in \mathcal{S}(M_n(V))$ by the definition of N_n, which implies that $g(A) = 0$ for every positive linear functional $g : M_n(V) \to \mathbb{C}$. If $s \in \mathcal{S}(\mathcal{V})$ and $P = [p_{ij}] \in M_n^+$ then the map $s_P : M_n(V) \to \mathbb{C}$ given by $s_P([x_{ij}]) := \sum_{i,j=1}^n s(p_{ij}x_{ij})$ is a linear functional on $M_n(V)$. We will show that s_p is positive.

Let P be a rank one positive matrix in M_n then $P = u^*u$ for some $u \in M_{1,n}$ (P has only one non zero eigenvalue λ, take an eigenvector u with $\|u\| = \sqrt{\lambda}$). Since $\{C_n\}$ is a matrix ordering we have that $u^*Xu \in C_1$ for any $X = [x_{ij}] \in C_n$. Therefore,

$$s_P([x_{ij}]) = \sum_{i,j=1}^n s(p_{ij}x_{ij}) = \sum_{i,j=1}^n u_i x_{ij} \bar{u_j} = s(u^*Xu) \geq 0 \tag{$*$}$$

Every positive matrix has a decomposition into a sum of rank-one matrices so if we take P to be any arbitrary positive $n \times n$-complex matrix of rank r, then we have that $P = \sum_i^r u_i^* u_i$, so from $(*)$ and the linearity of s_P it follows that $s_P(X) \geq 0$, for all $P \in M_n^+$, $X \in C_n$.

It follows that $s_P(A) = \sum_{i,j=1} s(\alpha_{ij}p_{ij}) = 0$ for any $s \in \mathcal{S}(\mathcal{V})$ and any matrix $P \in M_n^+$. If we choose $1 \leq$ k $\leq n$ and let D be the diagonal $n \times n$ matrix with 1 in the (k, k) entry and zeroes elsewhere, then $D \in M_n^+$ so we have that $s_D(A) = 0$ for every state $s \in \mathcal{S}(\mathcal{V})$. Hence,

$$s(\alpha_{kk}) = 0, \text{ for every } s \in \mathcal{S}(\mathcal{V}) \tag{I}$$

Now choose $1 \leq k, l \leq n$ and let $u \in M_{1,n}$ be the row vector with entry 1 in the k-th and l-th positions and 0 elsewhere and set $P := u^*$u then $P \in M_n^+$. Since P has entries: 1 in the (k, k), (k, l), (l, k), and (l, l) positions and 0 elsewhere, we have that $s_P(A) = s(\alpha_{kk}) + s(\alpha_{Kl}) + s(\alpha_{lk}) + s(\alpha_{ll}) = 0$. Combining this with (I) we see that

$$s(\alpha_{kl}) + s(\alpha_{lk}) = 0 \tag{II}$$

Similarly if let $b \in M_{1,n}$ be the vector with 1 in the k-th position, and i in the l-th position, and zeroes elsewhere. Then $Q := b^*b \in M_n^+$ and has 1 in (k, k) and (l, l) entries, i in the (k, l) entry, $-i$ in the (l, k) entry, and zeroes elsewhere. Then, $s_Q(A) = 0$ so $s(\alpha_{kk}) + s(\alpha_{ll}) + is(\alpha_{kl}) - is(\alpha_{kl}) = 0$. Thus by (II) we have that:

$$is(\alpha_{kl}) - is(\alpha_{kl}) = 0 \Rightarrow -s(\alpha_{kl}) + s(\alpha_{kl}) = 0 \tag{III}$$

It follows from (II) and (III) that for any $l, k \in \{1, \ldots, n\}$ and any $s \in \mathcal{S}(\mathcal{V}) : s(\alpha_{k,l}) = 0$. Therefore $\alpha_{kl} \in N$ i.e., $A \in M_n(N)$.

Conversely, we assume that $A = [\alpha_{kl}] \in M_n(N)$ and $s : M_n(V) \to \mathbb{C}$ is a state on $M_n(V)$. For $1 \leq k, l \leq n$ we define $s_{kl} : V \to \mathbb{C}$ via: $s_{kl}(v) := s(E_{k,l} \otimes v)$.

The $s_{k,l}$ are linear functionals and for every $s \in \mathcal{S}(\mathcal{V})$, $s(A) = \sum_{k,l=1}^n s_{kl}(\alpha_{kl})$. Choose k in {1,...,n}, then for any $v \in C_1 = V^+$ the diagonal matrix D_v with v in the (k, k) - entry and 0 elsewhere is positive because:

$$diag(0, \ldots, 0, v, 0, \ldots, 0) = diag(0, \ldots, 0, 1, 0, \ldots, 0)^* \cdot v \cdot diag(0, \ldots, 0, 1, 0, \ldots, 0)$$

which is in C_n. Hence, $s_{kk}(v) = s(D_v) \geq 0$. Thus $s_{kk} : V \to \mathbb{C}$ is a positive linear functional, therefore

$$s_{kk}(x) = 0, \ \forall x \in N \tag{a}$$

Let $v \in V^+ = C_1$ and $1 \leq k, l \leq n$. We consider the matrix $P \in M_n(V)$ which has v in the $(k,k), (k,l), (l,k), (l,l)$ - entries and 0 elsewhere, then $P \in C_n$ and we have that $s_{kk}(v) + s_{kl}(V) + s_{lk}(v) + s_{ll}(v) = s(P) \geq 0$.
Thus $s_{kk} + s_{kl} + s_{lk} + s_{ll} : V \to \mathbb{C}$ is a positive linear functional, so $s_{kk}(x) + s_{kl}(x) + s_{lk}(x) + s_{ll}(x) = 0$ for all x in N. Using (a) we obtain:

$$s_{kl}(x) + s_{lk}(x) = 0, \ \forall x \in N \tag{b}$$

In a similar fashion if we set $Q \in M_n(V)$ to be the matrix with v in the $(k,k), (l,l)$ - entries, iv in the (k,l) - entry and $-iv$ in the (l,k) - entry then $Q \in C_n$. We have that $s_{kk}(v) + is_{kl}(V) - is_{lk}(v) + s_{ll}(v) = s(P) \geq 0$. Thus $s_{kk} + is_{kl} - is_{lk} + s_{ll} : V \to \mathbb{C}$ is a positive linear functional. As a result, $s_{kk}(x) + is_{kl}(x) - is_{lk}(x) + s_{ll}(x) = 0$ for all x in N, by (a) we have:

$$is_{kl}(x) - is_{lk}(x) = 0, \ \forall x \in N \Rightarrow -s_{kl}(x) + s_{lk}(x) = 0, \ \forall x \in N \tag{c}$$

It follows from (b) and (c) that $s_{kl}(x) = 0$ for all x in N. Consequently, since $A = [\alpha_{ij}] \in M_n(N)$ we have that $s(A) = \sum_{k,l=1}^{n} s_{kl}(\alpha_{kl}) = 0$, so $A \in N_n$.

Let V be a matrix ordered *-vector space with an order unit e and N the *-subspace of V we defined before. Identifying $M_n(V/N) = M_n(V)/M_n(N)$ we see that $(A + M_n(N))^* = A^* + M_n(N)$ and $(M_n(V)/M_n(N))_h = \{A + M_n(N) : A^* = A, \ A \in M_n(V)\}$. Moreover, $(e + N)_n = e_n + M_n(N)$.

Definition 1.26 *Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be a matrix ordered *-vector space with matrix order unit e. We set:*

$$C_n^{Arch} := \{A + M_n(N) \in M_n(V)/M_n(N) : (re_n + A) + M_n(N) \in C_n + M_n(N), \ \forall r > 0\}$$

and let,

$$V_n^{Arch} := (V/N, \{C_n^{Arch}\}_{n=1}^{\infty}, e + N) .$$

Proposition 1.27 *Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be a matrix ordered *-vector space with matrix order unit e. Then $V_{Arch} := (V/N, \{C_n^{Arch}\}_{n=1}^{\infty}, e + N)$ is an Archimedean matrix ordered *-vector space with $e + N$ being the Archimedean matrix order unit.*

Proof: Under the identification $M_n(V/N) = M_n(V)/M_n(N)$ and using Lemma 1.25 we see that for any $n \in \mathbb{N}$:

$$(M_n(V/N), C_n^{Arch}, e + M_n(N)) = (M_n(V)/N_n, C_n^{Arch}, e + N_n)$$

Hence, $(M_n(V/N), C_n^{Arch}, e + M_n(N))$ is the Archimedeanization of $(M_n(V), C_n, e_n)$.
The Archimedeanization is always an AOU space so C_n^{Arch} is a proper cone and $e_n + M_n(N)$ is an Archimedean order unit.

It remains to show the compatibility of the family $\{C_n^{Arch}\}_{n=1}^{\infty}$. Let $A \in C_n^{Arch}$ and $X \in M_{n,m}$. Then $X^* e_n X \in M_m(V)$ and e is a matrix order unit, so there exists some $r_0 > 0$ such that $r_0 e_m - X^* e_n X \in C_m$. Since $A \in C_n^{Arch}$ we have that $(re_n + A) + M_n(N) \in C_n + M_n(N)$, for all $r > 0$. Thus for all $r > 0$:

$$(\frac{r}{r_0} e_n + A) + M_n(N) \in C_n + M_n(N)$$

We also have that $X^* C_n X \subseteq C_m$ and $X^* M_n(N) X \subseteq M_m(N)$.
Combining the above facts we obtain:

$$X^*(\frac{r}{r_0} e_n + A)X + M_m(N) \in C_m + M_m(N)$$

or equivalently

$$(\frac{r}{r_0} X^* e_n X + X^* A X) + M_m(N) \in C_m + M_m(N)$$

Now the element $B := re_m - \frac{r}{r_0} X^* e_n X = \frac{r}{r_0}(r_0 e_m - X^* e_n X) \in C_m$

So $B + (\frac{r}{r_0} X^* e_n X + X^* A X) + M_m(N) \in C_m + M_m(N)$, i.e.,

$$(re_m + X^* A X) + M_m(N) \in C_m + M_m(N)$$

The above relation holds for all $r > 0$ so we have that $X^* A X + M_m(N) \in C_m^{Arch}$. Consequently $X^* C_n^{Arch} X \subseteq C_m^{Arch}$, thus $\{C_n^{Arch}\}_{n=1}^{\infty}$ is indeed a compatible family.

Remark: In particular we are interested in the case when $(V, \{C_n\}_{n=1}^{\infty}, e)$ is a matrix ordered *-vector space with matrix order unit e and (V, C_1, e) is an Archimedean ordered *-vector space. Since e is an Archimedean order unit for $(V, C_1 := V^+)$ we have from the above proposition that,

$$N := \bigcap \{\ker(f) : f \in \mathcal{S}(\mathcal{V})\} = \{0\}$$

Thus, in this case: $V/N = V$ and $C_1^{Arch} = C_1$.

In addition, since $N = \{0\}$, for $n \geq 2$ we have that:

$$C_n^{Arch} = \{A \in M_n(V) : re_n + A \in C_n, \forall r > 0\}$$

We conclude that in this case C_n^{Arch} is obtained by enlarging C_n.

1.5 Operator Systems

An abstract operator system is a triple $(V, \{C_n\}_{n=1}^{\infty}, e)$ where V is a complex *-vector space, $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V and $e \in V_h$ is an Archimedean matrix order unit.

Definition 1.28 *A (concrete) operator system $\mathcal{S}$ is a subspace of $B(H)$ such that the identity operator $I \in \mathcal{S}$ and if $s \in \mathcal{S}$, then $s^* \in \mathcal{S}$.*

If $\mathcal{S} \subseteq B(H)$ is a concrete operator system then it is a *-vector space with respect to the adjoint operation of $B(H)$ and it inherits an order structure form $B(H)$ that is,

$$\mathcal{S}_h = \mathcal{S} \cap B(H)_h \text{ and } \mathcal{S}^+ = \mathcal{S} \cap B(H)^+$$

Furthermore, $\mathcal{S} \subseteq B(H)$, so $M_n(\mathcal{S}) \subseteq M_n(B(H)) = B(H^n)$, hence $M_n(\mathcal{S})$ inherits an involution and order structure from $B(H^n)$ and has the diagonal $n \times n$ matrix $diag(I, .., I)$ as an Archimedean order unit.
Thus we may regard $(\mathcal{S}, M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap B(H^n)^+, e)$ as an abstract operator system.

The converse is also true as shown by the following theorem of Choi and Efrros (see [4]):

Theorem 1.29 *If $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an Archimedean matrix ordered *-vector space, then there exist a Hilbert space H, an operator system $\mathcal{S} \subseteq B(H)$ and a unital complete order isomorphism $\Phi : V \to \mathcal{S}$.*

Using the above theorem we may identify abstract and concrete operator systems and refer to them as operator systems.
We will denote the order unit of an operator system $\mathcal{S}$ as e and will use the symbol $M_n(\mathcal{S})^+$ for the cone of positive elements of $M_n(\mathcal{S})$, $n \in \mathbb{N}$. Notice that any unital C^*-algebra is also an operator system in a canonical way.

If $\mathcal{S}$ is an operator system then any unital and self-adjoint subspace $\mathcal{S}_0$ of $\mathcal{S}$ with the induced matrix order structure is again an operator system. We say that $\mathcal{S}_0$ is an operator subsystem of $\mathcal{S}$. Observe that in this case the inclusion $\mathcal{S}_0 \hookrightarrow \mathcal{S}$ is a unital complete order embedding.

Every matrix ordered space with an Archimedean order unit may be equipped with a norm:

Proposition 1.30 *[15, Proposition 13.3] Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be an Archimedean matrix ordered space, for every $v \in M_n(V)$ set*

$$\|v\|_n = \inf\{r \in \mathbb{R} : \begin{bmatrix} re_n & v \\ v^* & re_n \end{bmatrix} \in C_{2n}\}$$

Then $\|\cdot\|_n$ is a norm on $M_n(V)$. Moreover, with respect to this norm C_n is a closed subspace of $M_n(V)$, for every $n \in \mathbb{N}$.

Proof: We will prove the case $n = 1$. The other cases can be proven in a similar fashion.

Positive definiteness: Let $\begin{bmatrix} re & v \\ v^* & re \end{bmatrix} \in C_2$ and set $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in M_2$. Then,

$$\begin{bmatrix} re & -v \\ -v^* & re \end{bmatrix} = X^* \begin{bmatrix} re & v \\ v^* & re \end{bmatrix} X \in C_2 \quad (\{C_n\}_{n=1}^{\infty} \text{ is compatible})$$

Therefore,

$$2r diag(e, e) = \begin{bmatrix} re & v \\ v^* & re \end{bmatrix} + \begin{bmatrix} re & - \\ -v^* & re \end{bmatrix} \in C_2 \quad (C_2 \text{ is a cone})$$

Since C_2 is a proper cone and $diag(e, e) \in C_2$ it follows that $r \geq 0$.
Thus, $\|v\|_1 \geq 0$ for every $v \in V$.

Furthermore, if $\|v\|_1 = 0$, then from the compatibility of the family $\{C_n\}_{n=1}^{\infty}$ we have that for every $t \in \mathbb{C}$:

$$\begin{aligned} C_1 \ni \begin{pmatrix} 1 & \bar{t} \end{pmatrix} \begin{bmatrix} re & v \\ v^* & re \end{bmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix} &= re + \bar{t}v^* + tv + \bar{t}tre \\ &= r(1 + |t^2|)e + \underbrace{(tv)^* + tv}_{V_h} \end{aligned}$$

Since e is an Archimedean order unit we obtain that $(tv)^* + tv \in C_1, \ \forall t \in \mathbb{C}$. Now setting $t = 1, -1$ gives $v + v^* = 0$ while setting $t = i, -i$ gives $(iv)^* + iv = 0$. Thus, $v = 0$. It is straightforward to see that when $v = 0$ then $\|v\|_1 = 0$.

Homogeneity: Let $\lambda \neq 0$ and notice that

$$\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{(\bar{\lambda})} \end{bmatrix} \begin{bmatrix} re & v \\ v^* & re \end{bmatrix} \begin{bmatrix} \sqrt{(\bar{\lambda})} & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} |\lambda| re & \lambda v \\ \bar{\lambda} v^* & |\lambda| re \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} re & v \\ v^* & re \end{bmatrix} \in C_2 \text{ if and only if } \begin{bmatrix} |\lambda| re & \lambda v \\ \bar{\lambda} v^* & |\lambda| re \end{bmatrix} \in C_2 \tag{I}$$

(because $\{C_n\}_{n=1}^{\infty}$ is compatible).

Thus,

$$\begin{aligned} \|\lambda v\|_1 &= \inf\{r \in \mathbb{R} : \begin{bmatrix} re & \lambda v \\ (\lambda v)^* & re \end{bmatrix} \in C_2\} \\ &= \inf\{r \in \mathbb{R} : \begin{bmatrix} |\lambda|^{-1} re & v \\ v^* & |\lambda|^{-1} re \end{bmatrix} \in C_2\} \qquad \text{from (I)} \\ &= \inf\{|\lambda| r' \in \mathbb{R} : \begin{bmatrix} r'e & v \\ v^* & r'e \end{bmatrix} \in C_2\} \qquad (r' = |\lambda|^{-1} r) \\ &= |\lambda| \|v\|_1 \end{aligned}$$

$\|v\|_1 = \|v^*\|_1$: It follows from the compatibility of C_2 and the fact that for

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}: \quad A^* \begin{bmatrix} re & v \\ v^* & re \end{bmatrix} A = \begin{bmatrix} re & v^* \\ v & re \end{bmatrix}$$

Triangle inequality: Let $v_1, v_2 \in V$ and consider $r_1 \in \{r \in \mathbb{R} : \begin{bmatrix} re & v_1 \\ v_1^* & re \end{bmatrix} \in C_2\}$ and $r_2 \in \{r \in \mathbb{R} : \begin{bmatrix} re & v_2 \\ v_2^* & re \end{bmatrix} \in C_2\}$.

Then, $r_1 + r_2 \in \{r \in \mathbb{R} : \begin{bmatrix} re & v_1 + v_2 \\ (v_1 + v_2)^* & re \end{bmatrix} \in C_2\}$ and so

$$\inf\{r \in \mathbb{R} : \begin{bmatrix} re & v_1 + v_2 \\ (v_1 + v_2)^* & re \end{bmatrix} \in C_2\} \leq r_1 + r_2, \quad \text{for all such } r_1, r_2$$

It follows that:

$$\begin{aligned} \|v_1 + v_2\|_1 &= \inf\{r \in \mathbb{R} : \begin{bmatrix} re & \lambda v_1 + v_2 \\ (v_1 + v_2)^* & re \end{bmatrix} \in C_2\} \\ &\leq \inf\{r \in \mathbb{R} : \begin{bmatrix} re & v_1 \\ v_1^* & re \end{bmatrix} \in C_2\} + \inf\{r \in \mathbb{R} : \begin{bmatrix} re & v_2 \\ v_2^* & re \end{bmatrix} \in C_2\} \\ &= \|v_1\|_1 + \|v_2\|_1 \end{aligned}$$

For the second part, let $(v_n)_n$ be a sequence of elements of C_1 with $v_n \xrightarrow{\|\cdot\|_1} v$. We shall show that $v \in C_1$. Since $C_1 \subseteq V_h$ we have that $v_n = v_n^*, \ \forall n \in \mathbb{N}$ which implies that $v = v^*$. Given any $r > 0$ we can find some $n_1 \in \mathbb{N}$ such that $\|x_{n_1} - x\|_1 < r$. Now from the definition of the norm $\|\cdot\|_1$ we have that

$$\begin{bmatrix} re & v - v_{n_1} \\ v - v_{n_1} & re \end{bmatrix} \in C_2$$

Set $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then

$$2re + 2v - 2v_{n_1} = X^* \begin{bmatrix} re & v - v_{n_1} \\ v - v_{n_1} & re \end{bmatrix} X \in C_1$$

Since $v_{n_1} \in C_1$ and C_1 is a cone, it follows that $re + v \in C_1$ and in turn since e is Archimedean it follows that $v \in C_1$. The proof is now complete.

Remark: Using Lemma 1.3 we can see that if $\mathcal{S}$ is an operator system then $\mathcal{S}_h = \mathcal{S}^+ - \mathcal{S}^+$. For another way to see this observe that any $s \in \mathcal{S}_h$ can be written as

$$s = \frac{e_{\mathcal{S}}\|s\|_1 + s}{2} - \frac{e_{\mathcal{S}}\|s\|_1 - s}{2} \qquad (e_{\mathcal{S}} \text{ denotes the unit of } \mathcal{S})$$

It is straightforward to generalize this result for $M_n(\mathcal{S})$.

Given operator systems $\mathcal{S}$ and $\mathcal{T}$ we will use the notations CP($\mathcal{S}, \mathcal{T}$) and UCP($\mathcal{S}, \mathcal{T}$) for the cones of all completely positive maps and unital and completely positive maps from $\mathcal{S}$ to $\mathcal{T}$ respectively.

The following theorems will be used frequently throughout this paper, their proofs can be found in ([15, Theorem's 7.5., 3.9., 3.11.]):

Theorem 1.31 ***(Arveson's extension theorem).****Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{S} \subseteq \mathcal{A}$ an operator system and $\varphi : \mathcal{S} \to \mathbb{C}$ a a completely positive map. Then there exists a completely positive map $\tilde{\varphi} : \mathcal{A} \to \mathbb{C}$ which extends φ.*

Theorem 1.32 *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map between unital C^*-algebras. If $\mathcal{A}$ or $\mathcal{B}$ is commutative then φ is positive if and only if it is completely positive. This statement remains true in the case in which $\mathcal{A}$ is an operator system and $\mathcal{B}$ is a commutative unital C^*-algebra.*

Lemma 1.33 *Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be a matrix ordered *-vector space with matrix order unit e and such that (V, C_1, e) is an AOU space. Suppose that $\mathcal{T}$ is an operator system and $\varphi : V \to \mathcal{T}$ is a linear map. Then,*

$$\varphi^n(C_n^{Arch}) \subseteq M_n(\mathcal{T})^+ \ \text{ if and only if } \ \varphi^n(C_n) \subseteq M_n(\mathcal{T})^+, \ \text{ for each } \ n \in \mathbb{N}$$

Proof: Since $C_n \subseteq C_n^{Arch}$ for each $n \in \mathbb{N}$ when $\varphi^n(C_n^{Arch}) \subseteq M_n(\mathcal{T})^+$ we have that $\varphi^n(C_n) \subseteq M_n(\mathcal{T})^+$.

On the other hand, let $D \in C_n^{Arch}$ then $D + re_n \in C_n, \forall r > 0$ and we have that $\varphi^n(D + re_n) \in \varphi^n(C_n) \subseteq M_n(\mathcal{T})^+, \forall r > 0$. Since φ is linear and $M_n(\mathcal{T})^+$ is closed:

$$\varphi^n(D) + r\varphi^n(e_n) \in M_n(\mathcal{T})^+, \forall r > 0$$

and by letting $r \to 0$ we obtain, $\varphi^n(D) \in M_n(\mathcal{T})^+$.

Lemma 1.34 *Let $\mathcal{S}, \mathcal{T}$ be operator systems with underlying vector space V. If UCP($\mathcal{S}, B(H)$) = UCP($\mathcal{T}, B(H)$), for every Hilbert space H, then $\mathcal{S}$ is completely order isomorphic to $\mathcal{T}$.*

Proof: Suppose that $\mathcal{S} \subseteq B(H_1)$ for some Hilbert space H_1 as a concrete operator system. The identity map $id_1 : \mathcal{S} \to B(H_1)$ is a unital completely positive map so by our hypothesis it will be unital and completely positive when consider as a map from $\mathcal{T}$ to $B(H_1)$ and thus $M_n(\mathcal{T})^+ \subseteq M_n(\mathcal{S})^+$.

Reversing the roles of $\mathcal{S}$ and $\mathcal{T}$ in the above argument we can see that $M_n(\mathcal{S})^+ \subseteq M_n(\mathcal{T})^+$. Consequently, we have that $M_n(\mathcal{S})^+ = M_n(\mathcal{T})^+$. The requested complete order isomorphism will be the identity map on V.

The following Lemma will be instrumental in proving many a result in the chapters that follow.

Lemma 1.35 *Let $\mathcal{S}$ be an operator system. Then for a $P \in M_n(\mathcal{S})$ we have that:*

$$P \in M_n(S)^+ \iff \phi^n(P) \in M_{nk}^+, \ \forall \phi \in \text{UCP}(\mathcal{S}, M_k) \ , \forall \ k \geq 1$$

Proof: Assume that $\mathcal{S} \subseteq B(H)$ for some Hilbert space H and that for $P = [p_{ij}] \in M_n(S)^+$, $\phi^n(P) \in M_{nk}^+$, for all $\phi \in \text{UCP}(\mathcal{S}, M_k), k \in \mathbb{N}$. Let $h = (h_1, \ldots, h_n)^t \in H^n$ (where t denotes the transpose and H^n the direct sum of n - copies of H) and let $\psi : \mathcal{S} \to M_n$ be the map given by $\psi(s) = [(sh_j, h_i)]_{i,j}, s \in \mathcal{S}$.

Then ψ is completely positive:

Let $[s_{pq}]_{p,q=1}^l \in M_l(S)^+$ and consider the following elements of M_n,

$$Y_{pq} := [(s_{pq}h_j, h_i)]_{i,j}$$

If we show that the matrix $Y := [Y_{pq}] = [\psi([s_{pq}])] \in M_l(M_n)$ is positive we are done.

Let $\lambda_r = \begin{bmatrix} \lambda_{r1} \\ \lambda_{r2} \\ \vdots \\ \lambda_{rn} \end{bmatrix} \in \mathbb{C}^n$, for $r \in \{1, \ldots, l\}$, then setting $\tilde{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_l \end{bmatrix} \in \mathbb{C}^{ln}$,

$\tilde{h_r} = \sum_{i=1}^n \lambda_{ri} h_i$ and $\tilde{h} = \begin{bmatrix} \tilde{h_1} \\ \tilde{h_2} \\ \vdots \\ \tilde{h_l} \end{bmatrix}$ we have that:

$$\begin{aligned}
\left(Y\tilde{\lambda}, \tilde{\lambda}\right) = \sum_{p,q=1}^{l} (Y_{pq}\lambda_q, \lambda_p) &= \sum_{p,q=1}^{l} \sum_{i,j}^{n} \left((s_{pq}h_j, h_i)\, \lambda_{qj}, \lambda_{pi}\right) \\
&= \sum_{p,q=1}^{l} \sum_{i,j}^{n} (s_{pq}h_j, h_i)\, \lambda_{qj}\overline{\lambda_{pi}} \\
&= \sum_{p,q=1}^{l} \left(s_{pq} \sum_{j=1}^{n} \lambda_{qj}h_j, \sum_{i=1}^{n} \lambda_{pi}h_i \right) \\
&= \sum_{p,q=1}^{l} \left(s_{pq}\tilde{h_q}, \tilde{h_p} \right) \\
&= \left([s_{pq}]\tilde{h}, \tilde{h} \right)
\end{aligned}$$

which is ≥ 0 because $[s_{pq}]_{p,q=1}^l \in M_l(S)^+$.
This proves that ψ is CP. Hence $\psi^n(P) \geq 0$.

Now let $\theta = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \in \mathbb{C}^{n^2}$ where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$, then

$$(Ph, h) = \sum_{i,j}^{n} (p_{ij}h_j, h_i) = \sum_{i,j=1}^{n} (\psi^n(p_{ij})e_j, e_i) = (\psi^n(P)\theta, \theta) \geq 0$$

so $P \in M_n(B(H))^+$. Thus $P \in M_n(B(H))^+ \cap M_n(\mathcal{S}) = M_n(\mathcal{S})^+$.

The other direction is straightforward.

Lemma 1.36 *Let V be an operator system (matrix ordered space with Archimedean order unit) then both $V \otimes M_n$ and $M_n(V)$ are Archimedean ordered $*$-vector spaces. Furthermore, the $*$-isomorphism $\pi : M_n(V) \to M_n \otimes V$ we defined in section 1.1 is now an order isomorphism.*

Proof: Remember that for a $[v_{ij}] \in M_n(V)$, $\pi([v_{ij}]) = \sum_{i,j}^{n} E_{ij} \otimes v_{ij}$.

Suppose that $\pi([v_{ij}]) \in (M_n \otimes V)^+$ we will show that $[v_{ij}] \in M_n(V)^+$, in order to achieve this we will appeal to Lemma 1.35.

Let $k \in \mathbb{N}$ and let $\Phi : V \to M_k$ be a unital completely positive map, then $(id_n \otimes \Phi) \in UCP(M_n \otimes V, M_{nk})$ and

$$\Phi^n([v_{ij}]) = (id_n \otimes \Phi)(\sum_{i,j=1}^{n} E_{ij} \otimes v_{ij}) \geq 0$$

so by Lemma 1.35 we have that $[v_{ij}] \in M_n(V)^+$.

For the other part, assume that $V \subseteq B(H)$ for some Hilbert space H and consider π as a map whose domain is $B(H)$, notice that in this case π is a $*$-homomorphism. Let $[v_{ij}] \in M_n(V)^+ = M_n(V) \bigcap M_n(B(H))^+$ then $[v_{ij}] = [w_{ij}][w_{ij}]^*$ for some $[w_{ij}] \in M_n(B(H))$. Thus,

$$\pi([v_{ij}]) = \pi([w_{ij}][w_{ij}]^*) = \pi([w_{ij}])\pi([w_{ij}]^*) = \pi([w_{ij}])\pi([w_{ij}])^* \geq 0$$

so the proof is complete.

Let $\mathcal{S}$ be an operator system and $\mathcal{S}^*$ denote its Banach space dual. We define a *-operation on $\mathcal{S}^*$ by: $f^*(s) = \overline{f(s^*)}$ for every $f \in \mathcal{S}^*$. This operation turns $\mathcal{S}^*$ into a *-vector space and the cone of positive linear functionals defines a partial order on S^* (because their image is in $\mathbb{C}$: positive $\iff$ completely positive). We declare an element $[f_{ij}] \in M_n(S^*)$ to be positive if and only if the map $F : \mathcal{S} \to M_n$ given by $F(s) := [f_{ij}(s)]$ is completely positive. From Lemmas 1.15 and 1.14, and the fact that every operator system is a matrix ordered *- vector space with a unit we have the following:

The family $\{C_n\}_{n=1}^{\infty}$ where $C_n = \{[f_{ij}] \in M_n(\mathcal{S}^*) \mid F : \mathcal{S} \to M_n$ is CP$\}$is a matrix ordering on $\mathcal{S}^*$. We will write $\mathcal{S}^d$ for the arising matrix ordered $*$-vector space. That is,

$$\mathcal{S}^d := (\mathcal{S}^*, \{C_n\}_{n=1}^{\infty})$$

In the case where $\mathcal{S}$ is a finite dimensional operator systems we have a stronger result, the matrix ordered space $\mathcal{S}^d$ is in fact an operator system as shown in Corollary 4.5 of [4].

2 Tensor Products Of Operator Systems

In this Chapter we review the theory of tensor products in the category of operator systems as established in [12].

Consider two operator systems $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$, we wish to endow the vector space tensor product $\mathcal{S} \otimes \mathcal{T}$ with a matrix ordering (see Definition 1.5) $\{C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T}) : n \in \mathbb{N}\}$ such that $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty}, e_1 \otimes e_2)$ will be an operator system.

Definition 2.1 *Suppose that $(\mathcal{S}, \{P_n\}_{n=1}^{\infty}, e_1)$ and $(\mathcal{T}, \{Q_n\}_{n=1}^{\infty}, e_2)$ are operator systems, then an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ is a family of cones $\tau = \{C_n\}_{n=1}^{\infty}$, where $C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, $\forall n$, such that:*

1. $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty}, e_1 \otimes e_2)$ *is an operator system, denoted* $\mathcal{S} \otimes_\tau \mathcal{T}$
2. $P_n \otimes Q_m \subseteq C_{nm}, \ \forall n, m \in \mathbb{N}$, *i.e., if $P = [p_{ij}] \in P_n$ and $Q = [q_{kl}] \in Q_m$ then* $P \otimes Q := [p_{ij} \otimes q_{kl}] \in C_{nm}$
3. *If $\phi \in UCP(\mathcal{S}, M_n)$ and $\psi \in UCP(\mathcal{T}, M_m)$ then $\phi \otimes \psi \in UCP(\mathcal{S} \otimes \mathcal{T}, M_{nm})$*

We may write $C_n := M_n(\mathcal{S} \otimes \mathcal{T})^+$. Suppose that τ_1 and τ_2 are two operator system structures we say that τ_1 is greater than τ_2 if the identity map on $\mathcal{S} \otimes \mathcal{T}$ from $\mathcal{S} \otimes_{\tau 1} \mathcal{T}$ to $\mathcal{S} \otimes_{\tau_2} \mathcal{T}$ is completely positive that means $M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$. In other words the operator system structure with the smaller cones is the bigger one, this is parallel to the fact that for two norms on a complex vector space the bigger one is the one with the smaller unit ball. Looking at Proposition 1.30, one can also see that smaller cones give bigger norms.

Let $\mathcal{O}$ denote the category which has operator systems as objects and unital CP maps as morphisms. By an operator system tensor product we mean a map $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ for every pair of operator systems $\mathcal{S}, \mathcal{T}$. We denote it by $\mathcal{S} \otimes_\tau \mathcal{T}$.

Definition 2.2 *We call an operator system tensor product τ:*

1. ***functorial*** *if for any $\phi \in UCP(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in UCP(\mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{S}_i, \ i = 1, 2$, and $\mathcal{T}_j, \ j = 1, 2$, are operator systems we have that $\phi \otimes \psi \in UCP(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1, \mathcal{S}_2 \otimes_\tau \mathcal{T}_2)$*
2. ***associative*** *if for any three operator systems $\mathcal{S}_i, i = 1, 2, 3$ the operator system tensor products $(\mathcal{S}_1 \otimes_\tau \mathcal{S}_2) \otimes_\tau \mathcal{S}_3$ and $\mathcal{S}_1 \otimes_\tau (\mathcal{S}_2 \otimes_\tau \mathcal{S}_3)$ are canonically completely order isomorphic.*
3. ***symmetric*** *if for any two operator systems $\mathcal{S}, \mathcal{T}$ the flip map $\theta : \mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S}$ extends to a unital complete order isomorphism from $\mathcal{S} \otimes_\tau \mathcal{T}$ to $\mathcal{T} \otimes_\tau \mathcal{S}$.*

We say that a functorial operator system product $\otimes_\tau$ is **injective** if for all operator systems $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ the embedding $\mathcal{S}_1 \otimes_\tau \mathcal{T}_1 \subseteq \mathcal{S}_2 \otimes_\tau \mathcal{T}_2$ is a complete order isomorphism onto its range, i.e.,

$$M_n(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1) \cap M_n(\mathcal{S}_2 \otimes_\tau \mathcal{T}_2)^+ = M_n(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1)^+, \ \forall n \in \mathbb{N}.$$

2.1 Minimal Tensor Product

Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems. For each $n \in \mathbb{N}$, we set

$$C_n^{min} = C_n^{min}(\mathcal{S}, \mathcal{T}) = \{[p_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T}) : [(\phi \otimes \psi)(p_{ij})] \in M_{nkm}^+,$$

$$\forall \phi \in \text{UCP}(\mathcal{S}, M_k),\ \psi \in \text{UCP}(\mathcal{T}, M_m), \forall k, m \in \mathbb{N}\}$$

Lemma 2.3 *Let $\mathcal{S}, \mathcal{T}$ be operator systems and $P \in M_n(\mathcal{S}) \otimes \mathcal{T}$. If*

$$(\phi^n \otimes \psi)(P) \geq 0,\ \forall \phi \in \bigcup_{m=1}^{\infty} \{f : \mathcal{S} \to M_m : f\ UCP\}, \psi \in \bigcup_{m=1}^{\infty} \{f : \mathcal{T} \to M_m : f\ UCP\},$$

then $(\Phi \otimes \psi)(P) \geq 0, \forall \Phi \in \bigcup_{m=1}^{\infty} \{f : M_n(\mathcal{S}) \to M_m : f$ UCP$\}$.

Proof: Fix $m \in \mathbb{N}$ and $\psi \in \{f : \mathcal{T} \to M_m : f$ UCP$\}$.
For every functional $\omega : M_m \to \mathbb{C}$ let $g_\omega : M_n(\mathcal{S}) \otimes \mathcal{T} \to M_n(\mathcal{S})$ be the map given by $g_\omega(X \otimes y) := \omega(\psi(y))X$.

If $v_1, v_2 \in \mathbb{C}^m$ we let $\omega_{v_1,v_2} : M_m \to \mathbb{C}$ be the functional given by $\omega_{v_1,v_2}(x) = (xv_1, v_2)$. Let $v_1, \ldots, v_r \in \mathbb{C}^m$ and $k \in \mathbb{N}$, we define the following map:

$$[L_{\omega_{v_t,v_s}}]_{s,t} : M_{nkm} = M_{nk} \otimes M_m \to M_{nkr} : A \to [L_{\omega_{v_t,v_s}}(A)]_{s,t}$$

with $[L_{\omega_{v_t,v_s}}(A_1 \otimes A_2)]_{s,t} = [A_1 \omega_{v_s,v_t}(A_2)]_{s,t}$, and we extend it linearly.

Claim: The map $[L_{\omega_{v_t,v_s}}]_{s,t}$ is positive.
Indeed, let $A \in M_{nkm}^+ = (M_{nk} \otimes M_m)^+$ then there exist $B = \sum_i N_i \otimes M_i \in M_{nk} \otimes M_m$ such that $A = B^*B$. Thus for any $\xi = (\xi_1, \ldots, \xi_r) \in (\mathbb{C}^{nk})^r$:

$$\begin{aligned}
\big([L_{\omega_{v_t,v_s}}(A)]_{s,t}\xi, \xi\big) &= \big([L_{\omega_{v_t,v_s}}(B^*B)]_{s,t}\xi, \xi\big) \\
&= \left([L_{\omega_{v_t,v_s}}(\sum_{i,j} N_i^* N_j \otimes M_i^* M_j)]_{s,t})\xi, \xi\right) \\
&= \sum_{s,t=1}^{r} \left(L_{\omega_{v_t,v_s}}(\sum_{i,j} N_i^* N_j \otimes M_i^* M_j)\xi_t, \xi_s\right) \\
&= \sum_{s,t=1}^{r} \sum_{i,j} (\omega_{v_t,v_s}(M_i^* M_j) N_i^* N_j \xi_t, \xi_s) \\
&= \sum_{s,t=1}^{r} \sum_{i,j} ((M_i^* M_j v_t, v_s)\, N_i^* N_j \xi_t, \xi_s) \sum_{s,t=1}^{r} \sum_{i,j} (M_i^* M_j v_t, v_s)\,(N_i^* N_j \xi_t, \xi_s) \\
&= \sum_{s,t=1}^{r} \sum_{i,j} ((N_i^* N_j \otimes M_i^* M_j)\xi_t \otimes v_t, \xi_s \otimes v_s) \\
&= \sum_{s,t=1}^{r} ((B^*B)\xi_t \otimes v_t, \xi_s \otimes v_s) \geq 0
\end{aligned}$$

and so the Claim is proved.

Now suppose that $(\phi^n \otimes \psi)(P) \in M_{nkm}^+$ for any UCP map $\phi : \mathcal{S} \to M_k$, $k \in \mathbb{N}$. We know that the map $[L_{\omega_{v_t,v_s}}]_{s,t} : M_{nkm} \to M_{nkr} : A \to [L_{\omega_{v_t,v_s}}(A)]_{s,t}$, $1 \leq s,t \leq r$, is positive so

$$[L_{\omega_{v_t,v_s}}((\phi^n \otimes \psi)(P))]_{s,t} \in M_{nkr}^+$$

Now we shall show that $\phi^{nr}([g_{\omega_{v_t,v_s}}(P)]_{s,t}) \geq 0,\ \forall \phi \in \text{UCP}(\mathcal{S}, M_k)$.
It suffices to verify this on elementary tensors of the form $P = X \otimes y$.
For all $\phi \in \text{UCP}(\mathcal{S}, M_k)$,

$$\begin{aligned}\phi^{nr}([g_{\omega_{v_t,v_s}}(P)]_{s,t}) &= \phi^{nr}([g_{\omega_{v_t,v_s}}(X \otimes y)]_{s,t}) = ([\phi^n(\omega_{v_t,v_s}(\psi(y))X)]_{s,t}) \\ &= [\omega_{v_t,v_s}(\psi(y))\phi^n(X)]_{s,t} = [L_{\omega_{v_t,v_s}}(\phi^n(X) \otimes \psi(y))]_{s,t} \geq 0\end{aligned}$$

Applying Lemma 1.35 we obtain $[g_{\omega_{v_t,v_s}}(P)] \in M_{nr}(\mathcal{S})^+$.

Hence, $\Phi^r([g_{\omega_{v_t,v_s}}(P)]) \geq 0$, for all $\Phi \in \text{CP}(M_n(\mathcal{S}), M_k)$, for every $k \in \mathbb{N}$.

Now fix such a Φ then, $[L_{\omega_{v_t,v_s}}((\Phi \otimes \psi)(P))]_{s,t} \geq 0$. Thus if $h_1, \ldots, h_r \in \mathbb{C}^k$ then,

$$\left((\Phi \otimes \psi)(P)(\sum_{t=1}^{r} h_t \otimes v_t), \sum_{s=1}^{r} h_s \otimes v_s\right) =$$

$$\sum_{t,s=1}^{r} (\Phi(X)h_t \otimes \psi(y)v_t, h_s \otimes v_s) = \sum_{t,s=1}^{r} (\Phi(X)h_t, h_s)\,(\psi(y)v_t, v_s) =$$

$$\sum_{t,s=1}^{r} (\omega_{v_t,v_s}(\psi(y))\Phi(X)h_t, h_s) = \sum_{t,s=1}^{r} \left(L_{\omega_{v_t,v_s}}((\Phi \otimes \psi)(P))h_t, h_s\right) =$$

$$\left([L_{\omega_{v_t,v_s}}((\Phi \otimes \psi)(P))]_{s,t}[h_1 \ldots h_r]^t, [h_1 \ldots h_r]^t\right) \geq 0$$

It follows that $(\Phi \otimes \psi)(P)$ is indeed positive.

Lemma 2.4 *If $\psi \in$ UCP$(\mathcal{S}, M_k)$ and $\psi \in$ UCP$(\mathcal{T}, M_m)$, for operator systems $\mathcal{S}, \mathcal{T}$. Then $(\phi \otimes \psi)^n = \phi^n \otimes \psi$.*

Proof: We will prove the result for elementary tensors of the form $P = X \otimes y$, where $X = [x_{ij}] \in M_n(\mathcal{S})$ and $y \in \mathcal{T}$, then the general case follows by linearity.
Let P be as described above then,

$$(\phi^n \otimes \psi)(P) = \phi^n([x_{ij}]) \otimes \psi(y) = \phi^n([x_{ij}]) \otimes \psi(y)$$

and

$$(\phi \otimes \psi)^n(P) = [(\phi \otimes \psi)(x_{ij} \otimes y)]_{i,j} = [\phi(x_{ij}) \otimes \psi(y)]_{i,j} = [\phi(x_{ij})]_{i,j} \otimes \psi(y)$$

The result follows.

Theorem 2.5 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems, and let $i_{\mathcal{S}} : \mathcal{S} \to B(H)$ and $i_{\mathcal{T}} : \mathcal{T} \to B(K)$ be embeddings that are complete order isomorphisms onto their ranges. The family $\{C_n^{min}\}_{n=1}^{\infty} := \{C_n^{min}(\mathcal{S}, \mathcal{T})\}$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the embedding $i_{\mathcal{S}} \otimes i_{\mathcal{T}} : \mathcal{S} \otimes \mathcal{T} \to B(H \otimes K)$.*

Proof: Let $P \in C_n^{min}$ and set $Q := (i_{\mathcal{S}} \otimes i_{\mathcal{T}})^n(P)$.
We will show that $Q \in B((H \otimes K)^n)^+$. Assume that $Q = \sum_{r=1}^{m} X_r \otimes y_r$, for $X_r \in M_n(i_{\mathcal{S}}(\mathcal{S}))$ and $y_r \in i_{\mathcal{T}}(\mathcal{T})$, $1 \leq r \leq m$. Let $\xi_s \in H^n$ and $\eta_s \in K$ for $1 \leq s \leq k$ and set $\zeta = \sum_{s=1}^{k} \xi_s \otimes \eta_s$. We define the mappings $\Phi : M_n(i_{\mathcal{S}}(\mathcal{S})) \to M_k$ by $\Phi(X) = [(X\xi_t, \xi_s)]_{s,t}$ and $\psi : i_{\mathcal{T}}(\mathcal{T}) \to M_k$ by $\psi(y) = [(y\eta_t, \eta_s)]_{s,t}$. In a similar way as in the Lemma 1.35 it can be shown that Φ and ψ are completely positive. Since $Q \in C_n^{min}(\mathcal{S}, \mathcal{T})$ we have from the definition of C_n^{min} and Lemma 2.4 that $(\phi_0^n \otimes \psi_0)(Q) = (\phi_0 \otimes \psi_0)^n(Q) \in M_{nk^2}^+$, $\forall \phi_o \in \mathrm{UCP}(i_{\mathcal{S}}(\mathcal{S}), M_k)$ ($\phi_0^n : M_n(i_{\mathcal{S}}(\mathcal{S})) \to M_k$ is positive) and $\forall \psi_0 \in \mathrm{UCP}(i_{\mathcal{T}}(\mathcal{T}), M_k)$. Now, from Lemma 2.3, $(\Phi \otimes \psi)(Q) \in M_{nk^2}^+$.

Let $\theta = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_k \end{bmatrix} \in \mathbb{C}^{k^2}$ where $\{e_1, \ldots, e_k\}$ is the standard basis of $\mathbb{C}^k$.Then,

$$(Q\zeta, \zeta) = \sum_{r=1}^{m} \sum_{s,t=1}^{k} (X_r\xi_t, \xi_s)(y\eta_t, \eta_s) = \sum_{r=1}^{m} ((\Phi(X_r) \otimes \psi(y_r))\theta, \theta) =$$

$$((\Phi \otimes \psi)(Q)\theta, \theta) \geq 0, \text{ because } (\Phi \otimes \psi)(Q) \in M_{nk^2}^+$$

Consequently, $Q \in B((H \otimes K)^n)^+$. Thus if $D_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$ is the cone arising from the inclusion of $i_{\mathcal{S}}(\mathcal{S}) \otimes i_{\mathcal{T}}(\mathcal{T})$ into $B(H \otimes K)$, in other words

$$D_n = \{A \in M_n(\mathcal{S} \otimes \mathcal{T}) : (i_{\mathcal{S}} \otimes i_{\mathcal{T}})^n(A) \in B((H \otimes K)^n)^+\} =$$

$$[(i_{\mathcal{S}} \otimes i_{\mathcal{T}})^n]^{-1}(B((H \otimes K)^n)^+)$$

then $C_n^{min}(\mathcal{S}, \mathcal{T}) \subseteq D_n$ (Remember that $M_n(\mathcal{S} \otimes \mathcal{T})$ is identified with $M_n(\mathcal{S}) \otimes \mathcal{T}$ and $(i_{\mathcal{S}} \otimes i_{\mathcal{T}})^n = i_{\mathcal{S}}^n \otimes i_{\mathcal{T}}$).

For the inverse inclusion, let $\phi \in \mathrm{UCP}(\mathcal{S}, M_m)$ and $\psi \in \mathrm{UCP}(\mathcal{T}, M_k)$. We identify $\mathcal{S}$ with $i_{\mathcal{S}}(\mathcal{S}) \subseteq B(H)$ and by applying Arveson's extension Theorem, we find a UCP map $\tilde{\phi} : B(H) \to M_m$ such that $\tilde{\phi} \restriction_{\mathcal{S}} = \phi$. In the same way we find a UCP map $\tilde{\psi} : B(K) \to M_k$ that extends ψ. Now, the minimal C^* tensor product $\otimes_{C^*min}$ of the C^* algebras $B(H), B(K)$ satisfies the following ([15, Chapter 12])

$$B(H) \otimes_{C^*min} B(K) \subseteq B(H \otimes K)$$

Furthermore, there exists a UCP map $\tilde{\phi} \otimes \tilde{\psi} : B(H) \otimes_{C^*min} B(K) \to M_{mk}$. We once again apply Arveson's extension theorem and obtain a UCP map $\Psi : B(H \otimes K) \to M_{mk}$, with $\Psi \restriction_{B(H) \otimes_{C^*min} B(K)} = \tilde{\phi} \otimes \tilde{\psi}$.
Therefore, if $A = [\alpha_{ij}] \in D_n \subseteq B((H \otimes K)^n)^+ = M_n(B(H \otimes K))^+$, then

$$[(\phi \otimes \psi)(\alpha_{ij})] = [(\tilde{\phi} \otimes \tilde{\psi})(\alpha_{ij})] = [\Psi(\alpha_{ij})] \in M_{nmk}^+$$

Hence, $D_n = C_n^{min}(\mathcal{S}, \mathcal{T})$.

It follows that $C_n^{min}(\mathcal{S}, \mathcal{T})$ is an operator system structure on the vector space $\mathcal{S} \otimes \mathcal{T}$ (see Definition 2.1) with $e_1 \otimes e_2$ as Archimedean matrix order unit.

Definition 2.6 *We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{C_n^{min}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}, e_1 \otimes e_2)$ the minimal tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{min} \mathcal{T}$.*

The next corollary is immediate:

Corollary 2.7 *For $\mathcal{A}$ and $\mathcal{B}$ unital C^*-algebras denote their C^*-algebraic minimal tensor product as $\mathcal{A} \otimes_{C^* min} \mathcal{B}$. Then $\mathcal{A} \otimes_{C^*-min} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside $\mathcal{A} \otimes_{C^* min} \mathcal{B}$.*

Theorem 2.8 *The operator system tensor product $\otimes_{min}$ is functorial, injective , associative and symmetric. Moreover, if $\mathcal{S}$ and $\mathcal{T}$ are operator systems then $\otimes_{min}$ is the smallest operator system structure on $\mathcal{S} \otimes \mathcal{T}$. This means that, if τ is any other operator system structure on $\mathcal{S} \otimes \mathcal{T}$ then $C_n^{\tau} \subseteq C_n^{min}$, $n \geq 1$.*

Proof: The fact that $\otimes_{min}$ is functorial and injective follows from Theorem 2.5. Next, suppose that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are operator system and $i_j : \mathcal{S}_j \to B(H_j)$ is a complete order embedding which is a complete order isomorphism onto its range, $j = 1, 2, 3$. From the associativity of the Hilbert space tensor product we have that $(H_1 \otimes H_2) \otimes H_3 = H_1 \otimes (H_2 \otimes H_3)$.

It follows that $(\mathcal{S}_1 \otimes_{min} \mathcal{S}_2) \otimes_{min} \mathcal{S}_3$ and $\mathcal{S}_1 \otimes_{min} (\mathcal{S}_2 \otimes_{min} \mathcal{S}_3)$ are completely order isomorphic.
The symmetry follows in a similar way.

Lastly, if τ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ then by property 3 of the Definition 2.1 we indeed have that $C_n^{\tau} \subseteq C_n^{min}$, $n \geq 1$.

Lemma 2.9 *Let $\mathcal{S}$ be an operator system, then*

$$M_n \otimes_{min} \mathcal{S} \cong_{c.o.i.} M_n(\mathcal{S})$$

Proof: Consider $\mathcal{S}$ as an operator subsystem of a unital C^*-algebra $\mathcal{A}$. Since the minimal operator system tensor product is injective we have that

$$(M_n \otimes_{min} \mathcal{S})^+ = M_n \otimes_{min} \mathcal{S} \cap (M_n \otimes_{min} \mathcal{A})^+$$

For the C^*-algebra $\mathcal{A}$ we also have that

$$M_n \otimes_{min} \mathcal{A} \cong_{c.o.i.} M_n \otimes_{C^*-min} \mathcal{A} \text{ and } M_n \otimes_{C^*-min} \mathcal{A} = M_n(\mathcal{A})$$

It follows that

$$M_n \otimes_{min} \mathcal{S} \cong_{c.o.i.} M_n(\mathcal{S})$$

Let V and W be vector spaces (V finite dimensional) then we can identify each element of the space $V \otimes W$ with a linear function from V^d to W via the map $L_u : V^d \to W$ given by $L_u(f) = \sum_{i=1}^{n} f(s_i)t_i$, for $u = \sum_i s_i \otimes t_i \in V \otimes W$.

Proposition 2.10 *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let $u = [u_{ij}] \in M_n(\mathcal{S} \otimes \mathcal{T})$. The following are equivalent:*

1. $u = [u_{ij}] \in M_n(\mathcal{S} \otimes_{min} \mathcal{T})^+$
2. *The map $L_u : \mathcal{S}^d \to M_n(\mathcal{T}) : f \to [L_{u_{ij}}(f)]_{i,j}$, where*

$$[L_{u_{ij}}(f)] = [L_{\sum_\beta s_\beta^{ij} \otimes t_\beta^{ij}}(f)]_{i,j} = \left[\sum_\beta f(s_\beta^{ij}) t_\beta^{ij} \right]_{i,j}$$

   is completely positive.

Proof: Using the identification $M_n(\mathcal{S} \otimes_{min} \mathcal{T}) = S \otimes_{min} M_n(\mathcal{T})$ it suffices to show this for $n = 1$.

Let $u = \sum_i s_i \otimes t_i \in (\mathcal{S} \otimes_{min} \mathcal{T})^+$ and let $[f_{rs}] \in M_k(\mathcal{S}^d)^+$ for some $k \in \mathbb{N}$, then map $F : \mathcal{S} \to M_k$ given by $F(v) = [f_{rs}(v)], \forall v \in \mathcal{S}$ will be CP.

We will show that $(L_u)^k([f_{rs}]) \in M_k(\mathcal{T})^+$, to do this we will appeal to Lemma 1.35.

Let $\phi \in$ UCP$(\mathcal{T}, M_m)$. Then for each $1 \leq p, q \leq m$ there exists a (unique) $\phi_{pq} \in \mathcal{T}^d$ such that $\phi(t) = [\phi_{pq}(t)]$, for every $t \in \mathcal{T}$. Hence,

$$\phi^k((L_u)^k([f_{rs}])) = [\phi \circ L_u(f_{rs})]_{r,s} = [[\phi_{pq} \circ L_u(f_{rs})]_{p,q}]_{r,s} =$$

$$[[\phi_{pq}(\sum_i f_{rs}(s_i)t_i)]_{p,q}]_{r,s} = [[f_{rs}(\sum_i s_i \phi_{pq}(t_i))]_{p,q}]_{r,s} =$$

$$[(f_{rs})^m[\sum_i s_i \phi_{pq}(t_i)]_{p,q}]_{r,s} = \tau(F^m([\sum_i s_i \phi_{pq}(t_i)]_{p,q}))$$

where $\tau : M_m \otimes M_k \to M_k \otimes M_m$ is the canonical flip isomorphism.
Now since $\otimes_{min}$ is functorial (see Definition 2.2) and $id \in$ UCP$(\mathcal{S}, \mathcal{S})$ we have that $id \otimes \phi : \mathcal{S} \otimes_{min} T \to M_m(\mathcal{S})$ is UCP, and

$$0 \leq (id \otimes \phi)(u) = \sum_i id(s_i) \otimes \phi(t_i) = [\sum s_i \phi_{pq}(t_i)]_{p,q}$$

Thus, $\phi^k((L_u)^k([f_{rs}]) \geq 0$ and our objective follows from Lemma 1.35.
For the opposite direction, suppose that for $u = \sum_\zeta s_\zeta \otimes t_\zeta \in S \otimes_{min} \mathcal{T}$ the map $L_u : \mathcal{S}^d \to \mathcal{T}$ is completely positive. Let $\phi \in$ UCP$(\mathcal{S}, M_k)$ and $\psi \in$ UCP$(\mathcal{T}, M_m)$, for $k, m \geq 1$. We will show that $(\phi \otimes \psi)(u) \in M_{km}^+$. For $1 \leq i, j \leq k$ and $1 \leq p, q \leq m$ there exist $\phi_{ij} \in \mathcal{S}^d$ and $\psi_{pq} \in \mathcal{T}^d$ such that $\phi(s) = [\phi_{ij}(s)]$ and $\psi(t) = [\psi_{pq}(t)]$, $\forall s \in \mathcal{S}, t \in \mathcal{T}$. Since, the maps ϕ and ψ are UCP we have that $[\phi_{ij}] \in M_k(\mathcal{T}^d)^+$ and

$[\psi_{pq}] \in M_m(\mathcal{S}^d)^+$. Thus,

$$\begin{aligned}(\phi \otimes \psi)(u) &= \sum_{\zeta} \phi(s_\zeta) \otimes \psi(t_\zeta) = [[\sum_{\zeta} \phi_{ij}(s_\zeta)\psi_{pq}(t_\zeta)]_{p,q}]_{i,j} \\ &= [[\psi_{pq}(\sum_{\zeta} \phi_{ij}(s_\zeta)t_\zeta)]_{p,q}]_{i,j} = [[\psi_{pq}(L_u(\phi_{ij}))]_{p,q}]_{ij} \\ &= [\psi((L_u(\phi_{ij}))]_{i,j} = \psi^k([(L_u(\phi_{ij})]_{i,j}) \\ &= \psi^k \circ (L_u)^k([\phi_{ij}]_{i,j})\end{aligned}$$

which is in M_{km}^+ since both ψ and L_u are completely positive and $[\phi_{ij}] \in M_k(\mathcal{S}^d)^+$. Hence, $u \in (\mathcal{S} \otimes_{min} \mathcal{T})^+$.

2.2 Maximal Tensor Product

Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems. For each $n \in \mathbb{N}$, we set

$D_n^{max} = D_n^{max}(\mathcal{S}, \mathcal{T}) =$

$\{X(P \otimes Q)X^* : P \in M_k(\mathcal{S})^+, Q \in M_m(\mathcal{T})^+, X \in M_{n,km}, k, m \in \mathbb{N}\}$

Lemma 2.11 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems and $\{D_n\}_{n=1}^{\infty}$ be a compatible family of cones, with $D_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying property 2 of Definition 2.1. Then $D_n^{max} \subseteq D_n$ for every $n \in \mathbb{N}$.*

Proof: Let $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$ then by property 2, $P \otimes Q \in D_{km}$. Since $\{D_n\}_n$ is compatible it follows that $X(P \otimes Q)X^* \in D_n, \forall X \in M_{n,km}$. Thus $D_n^{max} \subseteq D_n$.

Lemma 2.12 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems, $P = [P_{ij}]_{i,j} \in M_k(M_n(\mathcal{S}))^+$ and $Q = [q_{ij}]_{i,j} \in M_k(\mathcal{T})^+$. Then $\sum_{i,j=1} P_{ij} \otimes q_{ij} \in D_n^{max}$.*

Proof: Let I_n be the identity matrix in M_n, and $X = [X_1 \dots X_{k^2}] \in M_{n,nk^2}$, where $X_m \in M_n$, $1 \leq m \leq k^2$, such that

$$X_1 = X_{k+2} = X_{2k+3} = \cdots = X_{k^2} = I_n$$

and $X_m = 0, \forall m \notin \{1, k+2, 2k+3, \dots, k^2\}$. Then,

$$\textstyle\sum_{i,j=1}^{k} P_{ij} \otimes q_{ij} = X(P \otimes Q)X^* \in D_n^{max}.$$

Proposition 2.13 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems. Then $\mathcal{S} \otimes \mathcal{T}$ together with the family $\{D_n^{max}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty} = \{D_n^{max}\}_{n=1}^{\infty}$ is a matrix ordered space with matrix order unit $e_1 \otimes e_2$.*

Proof: Let $n \in \mathbb{N}$, $X_i \in M_{n,k_i m_i}$, $P_i \in M_{k_i}(\mathcal{S})^+$ and $Q_i \in M_{m_i}(\mathcal{T})^+$, for $i = 1, 2$. Then $X_1(P_1 \otimes Q_1)X_1^*$, $X_2(P_2 \otimes Q_2)X_2^* \in D_n^{max}$. Moreover,

$$X_1(P_1 \otimes Q_1)X_1^* + X_2(P_2 \otimes Q_2)X_2^* =$$

$$\begin{bmatrix} X_1 & 0 & 0 & X_2 \end{bmatrix} \cdot \begin{bmatrix} P_1 \otimes Q_1 & 0 & 0 & 0 \\ 0 & P_1 \otimes Q_2 & 0 & 0 \\ 0 & 0 & P_2 \otimes Q_1 & 0 \\ 0 & 0 & 0 & P_2 \otimes Q_2 \end{bmatrix} \cdot \begin{bmatrix} X_1^* \\ 0 \\ 0 \\ X_2^* \end{bmatrix} =$$

$$\begin{bmatrix} X_1 & 0 & 0 & X_2 \end{bmatrix} ((P_1 \oplus P_2) \otimes (Q_1 \oplus Q_2)) \begin{bmatrix} X_1 & 0 & 0 & X_2 \end{bmatrix}^* \in D_n^{max}$$

where $\begin{bmatrix} X_1 & 0 & 0 & X_2 \end{bmatrix} \in M_{n,k_1m_1+k_1m_2+k_2m_1+k_2m_2}$ and

$$(P_1 \oplus P_2) \otimes (Q_1 \oplus Q_2) = (P_1 \otimes Q_1) \oplus (P_1 \otimes Q_2) \oplus (P_2 \otimes Q_1) \oplus (P_2 \otimes Q_2)$$

Clearly D_n^{max} is closed under scalar multiplication, so from the above we have that $\{D_n^{max}\}_{n=1}^{\infty}$ is a family of cones. The fact that this family is compatible is obvious from the way we defined its elements. Now we know that $\{C_n^{min}\}_{n=1}^{\infty}$ is a compatible family of proper cones and by Lemma 2.11, $D_n^{max} \subseteq C_n^{min}$, hence $D_n^{max} \bigcap (-D_n^{max}) \subseteq C_n^{min} \bigcap (-C_n^{min}) = \{0\}$, so D_n^{max} is a proper cone for every $n \in \mathbb{N}$. Furthermore since $e_1 \otimes e_2$ is a matrix order unit for $\{C_n^{min}\}$, it follows that it will be a matrix order unit for $\{D_n^{max}\}$. We conclude from all the above that $\{D_n^{max}\}$ is a matrix ordering with matrix order unit $e_1 \otimes e_2$.

Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems. Then $(\mathcal{S} \otimes \mathcal{T}, \{D_n^{max}\}_{n=1}^{\infty}, e_1 \otimes e_2)$ is a matrix ordered space. However there exist examples where $e_1 \otimes e_2$ fails to be Archimedean. For this reason we consider the Archimedeanization of $\{D_n^{max}\}$, which we denote by $\{C_n^{max}(\mathcal{S}, \mathcal{T})\} := \{C_n^{max}\}$.
In general for a matrix ordered *-vector space $(V, \{C_n\}_n, e)$ we have seen that for each $n \in \mathbb{N}$, C_n^{Arch} is the set:

$$\{A + M_n(N) \in M_n(V)/M_n(N) : (re_n + A) + M_n(N) \in C_n + M_n(N),\ \forall r > 0\}$$

where $N = \bigcap\{\ker f : f \in \mathcal{S}(V)\}$.
In our case we have that $V = \mathcal{S} \otimes \mathcal{T}$, $C_n = D_n^{max},\ n \in \mathbb{N}$ and the matrix order unit is $e_1 \otimes e_2$.
Notice that (this will be proven in Theorem 2.15): if we endow the space $\mathcal{S} \otimes \mathcal{T}$ with the cones D_n^{max} then whenever $\varphi_1 \in \text{UCP}(\mathcal{S}, \mathbb{C})$ and $\varphi_2 \in \text{UCP}(\mathcal{T}, \mathbb{C})$ we have that $\varphi_1 \otimes \varphi_2 \in \text{UCP}(\mathcal{S} \otimes \mathcal{T}, \mathbb{C})$. Thus if $\varphi_1,\ \varphi_2$ are states on $\mathcal{S}$ and $\mathcal{T}$ respectively then $\varphi_1 \otimes \varphi_2$ will be a state on $(\mathcal{S} \otimes \mathcal{T}, \{D_n^{max}\})$.
Now suppose that $N \neq \{0\}$, this would imply that we could find an non-zero element of $\mathcal{S} \otimes \mathcal{T}$ which would be annihilated by every state on $(\mathcal{S} \otimes \mathcal{T}, \{D_n^{max}\})$. We call that element v and we write v as $v = \sum_i s_i \otimes t_i$ where the t_i are chosen to be linearly

independent. Now our hypothesis implies that for every φ_1 state on $\mathcal{S}$ and every φ_2 state on $\mathcal{T}$,

$$(\varphi_1 \otimes \varphi_2)(v) = 0$$

,i.e.,

$$(\varphi_1 \otimes \varphi_2)(\sum_i s_i \otimes t_i) = 0$$

hence

$$\sum_i \varphi_1(s_i) \otimes \varphi_2(t_i) = 0$$

since for each i, $\varphi_1(s_i),\ \varphi_2(t_i) \in \mathbb{C}$ the above is equivalent to

$$\sum_i \varphi_1(s_i)\varphi_2(t_i) = 0$$

and φ_2 is $\mathbb{C}$-linear so the above gives

$$\varphi_2(\sum_i \varphi_1(s_i)t_i) = 0$$

Since $\mathcal{T}$ is an operator system it has an Archimedean order unit, it follows from Proposition 1.23 that

$$\sum_i \varphi_1(s_i)t_i = 0$$

and from the fact that the t_i's are linearly independent we have that

$$\varphi_1(s_i) = 0 \text{ for every } i$$

,i.e, for every state φ_1 on $\mathcal{S}$ and for every i: $\varphi_1(s_i) = 0$ which, again by Proposition 1.23, would give that

$$s_i = 0, \text{ for every } i$$

which leads to a contradiction ($v = 0$).

Definition 2.14 *We call the operator system $(\mathcal{S}\otimes\mathcal{T}, \{C_n^{max}\}_{n=1}^{\infty}, e_1\otimes e_2)$ the maximal operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{max} \mathcal{T}$.*

Theorem 2.15 *The mapping $max : \mathcal{O}\times\mathcal{O} \to \mathcal{O} : (\mathcal{S}, \mathcal{T}) \to \mathcal{S}\otimes_{max}\mathcal{T}$ is a symmetric, associative and functorial operator system tensor product. Moreover, $\otimes_{max}$ is the largest operator system tensor structure on $\mathcal{S}\otimes\mathcal{T}$ in the sense that if τ is another operator system structure on $\mathcal{S} \otimes \mathcal{T}$ with cones $\{C_n^{\tau}\}_{n\in\mathbb{N}}$ then $C_n^{max} \subseteq C_n^{\tau}, \forall n \in \mathbb{N}$.*

Proof: Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We need to check if the family $\{C_n^{max}\}_n$ satisfies the properties 1, 2 and 3 of Definition 2.1. We have shown 1, and 2 follows from the definition of $\{C_n^{max}\}_n$.Furthermore, since $C_n^{max} \subseteq C_n^{min}$ it follows that it satisfies property 3, because $\{C_n^{min}\}_n$ does.

Assume that $\phi \in \text{UCP}(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in \text{UCP}(\mathcal{T}_1, \mathcal{T}_2)$, and let $P \in M_k(S_1)^+$, $Q \in M_m(\mathcal{T}_1)^+$ and $X \in M_{n,km}$. Then $\phi^k(P) \in M_k(\mathcal{S}_2)^+$ and $\psi^m(Q) \in M_m(\mathcal{T}_2)^+$. Thus,

$$(\phi \otimes \psi)^n(X(P \otimes Q)X^*) = X(\phi^k(P) \otimes \psi^m(Q))X^* \in D_n^{max}(\mathcal{S}_2, \mathcal{T}_2)$$

So $(\phi \otimes \psi)^n(D_n^{max}(\mathcal{S}_1, \mathcal{T}_1)) \subseteq D_n^{max}(\mathcal{S}_2, \mathcal{T}_2)$, and by Lemma 1.33 we have that $\otimes_{max}$ is functorial.

Now, suppose that $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$. Consider the map $\theta : \mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S} : s \otimes t \to t \otimes s$. Then, after conjugation with a permutation matrix U: $\theta^{(km)}(P \otimes Q) = U(Q \otimes P)U^*$. Thus, for all $X \in M_{n,km}$

$$\theta^n(\underbrace{X(P \otimes Q)X^*}_{\in D_n^{max}(\mathcal{S},\mathcal{T})}) = X\theta^{(km)}(P \otimes Q)X^* = XU(Q \otimes P)U^*X^* = \underbrace{(XU)(Q \otimes P)(XU)^*}_{\in D_n^{max}(\mathcal{T},\mathcal{S})}$$

Hence, $\theta : \mathcal{S} \otimes_{max} \mathcal{T} \to \mathcal{T} \otimes_{max} \mathcal{S}$ is a complete order isomorphism, i.e., max is symmetric. Lemma 2.11 implies that max is the largest operator system tensor product (it has the smallest cones). In particular let $\{C_n\}$ be any matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ for which $e_1 \otimes e_2$ is an Archimedean matrix order unit. If $P \in C_n^{max}$ then for every $r > 0$, $P + r(e_1 \otimes e_2)_n \in D_n^{max} \subseteq_{2.11} C_n$. Since $e_1 \otimes e_2$ is Archimedean matrix order unit for C_n we have that $P \in C_n$.
We omit the proof of the associativity.

Let $(\mathcal{S}, e)$ be an operator system we call an element $s \in \mathcal{S}$ **strictly positive** if there exists a real number $\delta > 0$ such that $s \geq \delta e$.

Lemma 2.16 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems. If $u \in \mathcal{S} \otimes_{max} \mathcal{T}$ is strictly positive, then there exists $n \in \mathbb{N}$, $A = [a_{ij}] \in M_n(\mathcal{S})^+$ and $B = [b_{ij}] \in M_n(\mathcal{T})^+$ such that*

$$u = \sum_{i,j=1}^{n} a_{ij} \otimes b_{ij}$$

.

Proof: Since u is strictly positive we have that there exists some $\delta > 0$ such that $u - \delta(e_1 \otimes e_2) \in (\mathcal{S} \otimes_{max} \mathcal{T})^+ = C_1^{max}(\mathcal{S}, \mathcal{T})$. By the definition of C_1^{max} and D_1^{max} there exist $P = [p_{ij}] \in M_n(\mathcal{S})^+$ and $Q = [q_{kl}] \in M_m(\mathcal{T})^+$ and $X = \begin{bmatrix} x_{11} & \cdots & x_{1m} & x_{21} & \cdots & x_{2m} & \cdots & x_{n1} & \cdots & x_{nm} \end{bmatrix} \in M_{1,nm}$ for $n, m \in \mathbb{N}$ such that

$$u = (u - \delta(e_1 \otimes e_2)) + \delta(e_1 \otimes e_2) = X(P \otimes Q)X^* = \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} \bar{x}_{ik} p_{ij} \otimes q_{kl} x_{jl}$$

For each pair (i, j) we set $a_{ij} = p_{ij}$, thus $A = P \in M_n(\mathcal{S})^+$ and $b_{ij} = \sum_{k,l=1}^{m} \bar{x_{ik}} q_{kl} x_{jl}$. Then $B = [b_{ij}] = (X^t)^* Q(X^t)$ where t denotes the transpose, so $B \in M_n(\mathcal{T})^+$. The result follows.

If V, W and U are vector spaces and $\phi : V \times W \to U$ is a bi-linear map, then for $n, m \in \mathbb{N}$ we let $\phi^{(n,m)} : M_n(V) \times M_m(W) \to M_n(U)$ to be the bi-linear map given by $\phi^{(n,m)}([v_{ij}]_{i,j}, [w_{kl}]_{k,l}) := [\phi(v_{ij}, w_{kl})]_{(i,k),(j,l)}$

Definition 2.17 *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We call a bi-linear map $\phi : \mathcal{S} \times \mathcal{T} \to B(H)$ jointly completely positive if $\phi^{(n,m)}(P,Q) \in M_{nm}(B(H))^+$, for every $P \in M_n(\mathcal{S})^+$ and every $Q \in M_m(\mathcal{T})^+$.*

Theorem 2.18 *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Then*

1. *If $\phi : \mathcal{S} \times \mathcal{T} \to B(H)$ is jointly c.p. map, then its linearization $\phi_L : \mathcal{S} \otimes \mathcal{T} \to B(H)$, which is given by $\phi_L(s \otimes t) = \phi(s,t)$, is completely positive on $\mathcal{S} \otimes_{max} \mathcal{T}$.*
2. *If $\psi : \mathcal{S} \otimes_{max} \mathcal{T} \to B(H)$ is completely positive, then the map $\phi : \mathcal{S} \times \mathcal{T} \to B(H)$ given by $\phi(x,y) = \psi(x \otimes y)$, $x \in \mathcal{S}$ and $y \in \mathcal{T}$, is jointly completely positive.*
3. *Let τ be an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ such that the linearization of every UCP map $\phi : \mathcal{S} \times \mathcal{T} \to B(H)$ is completely positive on $\mathcal{S} \otimes_\tau \mathcal{T}$, then $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{max} \mathcal{T}$.*
4. *For every $n \in \mathbb{N}$, set*

$$K_n := \{P \in M_n(\mathcal{S} \otimes \mathcal{T}) : \phi_L^n(P) \geq 0 \text{ for every jointly completely positive}$$
$$\phi : \mathcal{S} \times \mathcal{T} \to B(H) \text{ and every } H : \text{Hilbert space}\}$$

   Then, the following holds:

$$C_n^{max}(\mathcal{S}, \mathcal{T}) = K_n, \ \forall\, n \in \mathbb{N}$$

Proof: Fix $\mathcal{S}$ and $\mathcal{T}$ operator systems, $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$.

For 1: Let $\phi : \mathcal{S} \times \mathcal{T} \to B(H)$ be a jointly completely positive map, then

$$\phi_L^{(km)}(P \otimes Q) = [\phi_L(p_{ij} \otimes q_{rs})]_{(i,r),(j,s)} = [\phi(p_{ij}, q_{rs})]_{(i,r),(j,s)} = \phi^{(k,m)}(P,Q) \geq 0$$

Thus, if $X \in M_{n,km}$ then

$$\phi_L^n(X(P \otimes Q)X^*) = X(\phi_L^{(km)}(P \otimes Q))X^* \geq 0$$

so $\phi^n(D_n^{max}) \subseteq M_n(B(H))^+$.
Thus, from Lemma 1.33 ϕ_L is completely positive.

For 2: As above $\phi^{(k,m)}(P,Q) = \psi^{(km)}(P \otimes Q) \geq 0$, because ψ is completely positive.

For 3: We know that the cones of the maximal tensor product are the smallest possible, so every UCP map from $\mathcal{S} \otimes_\tau \mathcal{T}$ to $B(H)$ is a UCP map from $\mathcal{S} \otimes_{max} \mathcal{T}$ to $B(H)$. By the hypothesis of 3 combined with 1,2 we have that the converse is also true. Hence, UCP($\mathcal{S} \otimes_\tau \mathcal{T}, B(H)$) = UCP($\mathcal{S} \otimes_{max} \mathcal{T}, B(H)$). By Lemma 1.34 we have that $\mathcal{S} \otimes_\tau \mathcal{T}$ = $\mathcal{S} \otimes_{\text{max}} \mathcal{T}$.

For 4: One can check that $\{K_n\}_{n=1}^{\infty}$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ and denote it by τ. Then τ satisfies property 3 by definition, so we have the desired result.

Given a bounded bilinear map $\phi : \mathcal{S} \times \mathcal{T} \to \mathbb{C}$ we define $\mathcal{L}(\phi) : \mathcal{S} \to \mathcal{T}^d : s \to \mathcal{L}(\phi)(s)$ (resp, $\mathcal{R}(\phi) : \mathcal{T} \to \mathcal{S}^d$) by $\mathcal{L}(\phi)(s)(t) = \phi(s,t)$ (resp. $\mathcal{R}(\phi)(t)(s) = \phi(s,t)$).

Lemma 2.19 *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let $\phi : \mathcal{S} \times \mathcal{T} \to \mathbb{C}$ be a bilinear map. The following are equivalent*

1. *ϕ is jointly completely positive.*
2. *$\mathcal{L}(\phi) : \mathcal{S} \to \mathcal{T}^d$ is completely positive.*
3. *$\mathcal{R}(\phi) : \mathcal{T} \to \mathcal{S}^d$ is completely positive.*

Proof: We will show the equivalence 1 $\iff$ 2. With a similar argument one can show the equivalence 1 $\iff$ 3.

The map $\mathcal{L}(\phi)$ is CP if and only if: for every $v = [v_{ij}] \in M_k(S)^+$ we have that $\mathcal{L}(\phi)^k([v_{ij}]) = [\mathcal{L}(\phi)(v_{ij})] \in M_k(\mathcal{T}^d)^+$ or equivalently that the map $\mathcal{L}(\phi)\hat{(}v_{ij}) : \mathcal{T} \to M_k : t \to [\mathcal{L}(\phi)(v_{ij})(t)]$ is CP. That is to say, for all $w = [w_{rs}] \in M_m(T)^+$

$$0 \leq \mathcal{L}(\phi)\hat{(}v_{ij})^m([w_{rs}]) = [[\mathcal{L}(\phi)(v_{ij})(w_{rs})]_{i,j}]_{r,s} =$$

$$[\mathcal{L}(\phi)(v_{ij})(w_{rs})]_{(i,r),(j,s)} = [\phi(v_{ij}, w_{rs})]_{(i,r),(j,s)} = \phi^{(k,m)}(v,w)$$

Hence, we have the equivalence of 1 and 2.

Lemma 2.20 *Let $(\mathcal{S}, e_1)$ be a finite dimensional operator system. The canonical isomorphism $\hat{}: \mathcal{S} \to (\mathcal{S}^d)^d : x \to \hat{x}$, where $\hat{x}(f) = f(x), \forall f \in \mathcal{S}^d$, is a complete order isomorphism.*

Proof: It suffices to show the following:

$$M_n(S)^+ \ni [x_{ij}] \iff [\hat{x_{ij}}] \in M_n((S^d)^d)^+$$

Remember that for any operator system $\mathcal{R}$ an element $[f_{ij}] \in M_n(\mathcal{R}^d)^+$ if and only if the map $F : \mathcal{R} \to M_n$ given by $F(r) = [f_{ij}(r)]$ is CP. Assume that $[x_{ij}] \in M_n(S)^+$. We will show that the map $\Phi : \mathcal{S}^d \to M_n$ given by

$$\Phi(f) = [\hat{x_{ij}}(f)]_{i,j} = [f(x_{ij})]_{i,j},\ f \in \mathcal{S}^d$$

is completely positive. This implies that $[\hat{x_{ij}}] \in M_n((S^d)^d)^+$.
Suppose that $[g_{kp}] \in M_m(\mathcal{S}^d)^+$ or equivalently that the map $\mathcal{S} \ni s \to [g_{kp}(s)]_{k,p} \in M_m$ is CP. Then,

$$\Phi^m([g_{kp}]_{k,p}) = [\Phi(g_{kp})]_{k,p} =$$
$$[[g_{kp}(x_{ij})]_{i,j}]_{k,p} = [g_{kp}^n([x_{ij}]_{i,j})]_{k,p} \geq 0$$

because $[x_{ij}] \in M_n(S)^+$ and the mapping $\mathcal{S} \ni s \to [g_{kp}(s)]_{k,p} \in M_m$ is CP. It follows that Φ is completely positive.

For the opposite direction, we suppose that $[x_{ij}] \in M_n(\mathcal{S})$ is such that $[\hat{x_{ij}}] \in M_n((S^d)^d)^+$ we shall show that $[x_{ij}] \in M_n(\mathcal{S})^+$. Let $k \in \mathbb{N}$ and $\phi \in$ UCP$(\mathcal{S}, M_k)$ then for each $1 \leq p, q \leq k$ there exist a unique $\phi_{pq} \in \mathcal{S}^d$ such that $\phi(s) = [\phi_{pq}(s)]$ and since ϕ is UCP we have that $[\phi_{pq}] \in M_k(\mathcal{S}^d)^+$. Now, letting $\Phi : \mathcal{S}^d \to M_n$ be the map given as in the previous part then Φ is CP and

$$\phi^n([x_{ij}]_{i,j}) = [\phi(x_{ij})]_{i,j} = [[\phi_{pq}(x_{ij})]_{p,q}]_{i,j}$$

$$= [[\hat{x}_{ij}(\phi_{pq})]_{p,q}]_{i,j} = \tau(\Phi^k([\phi_{pq}])) \geq 0$$

where τ is the canonical *-isomorphism $M_n \otimes M_k \cong M_k \otimes M_n$. By Lemma 1.35 we obtain that $[x_{ij}] \in M_n(\mathcal{S})^+$.

Lemma 2.21 *Let $\mathcal{S}$ be an operator system, then*

$$M_n \otimes_{min} \mathcal{S} \cong_{c.o.i.} M_n(\mathcal{S}) \cong_{c.o.i.} M_n \otimes_{max} \mathcal{S}$$

Proof: The first identification is Lemma 2.9. For the other one, suppose that $P \in M_k(M_n(\mathcal{S}))^+$, then we could write:

$$P = X(I_n \otimes P)X^*, \ \text{ where } \ X := \begin{bmatrix} I_k & 0 & \cdots & 0 \end{bmatrix},$$

and $I_k \in M_k$ is the identity matrix.

Therefore we have that $X \in M_{k,nk}$, $I_n \in M_n^+$ and $P \in M_k(M_n(\mathcal{S}))^+$ so $P = X(I_n \otimes P)X^* \in D_k^{max}(M_n, \mathcal{S})$.

Thus, $M_k(M_n(\mathcal{S}))^+ \subseteq D_k^{max}(M_n, \mathcal{S}) \subseteq (M_k(M_n \otimes_{max} \mathcal{S}))^+ := C_k^{max}(M_n, \mathcal{S})$.

Now we know that $C_k^{max}(M_n, \mathcal{S}) \subseteq C_k^{min}(M_n, \mathcal{S}), \forall k \in \mathbb{N}$ and $C_k^{min}(M_n, \mathcal{S}) = M_k(M_n(\mathcal{S}))^+$ hence we conclude that

$$(M_k(M_n \otimes_{max} \mathcal{S}))^+ = M_k(M_n(\mathcal{S}))^+, \ k \in \mathbb{N}$$

Theorem 2.22 *Let $\mathcal{S}$ and $\mathcal{T}$ be finite dimensional operator systems. Then $(\mathcal{S} \otimes_{max} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{min} \mathcal{T}^d$ and $(\mathcal{S} \otimes_{min} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{max} \mathcal{T}^d$.*

Proof: From Proposition 2.10 we have that $(S \otimes_{min} T)^+ = CP(\mathcal{S}^d, \mathcal{T})$. Furthermore, from Theorem 2.18 and Lemma 2.19 we have that a map $f : \mathcal{S} \otimes_{max} \mathcal{T} \to \mathbb{C}$ is completely positive if and only if $\phi_f : \mathcal{S} \times \mathcal{T} \to \mathbb{C}$, given by $\phi_f(x, y) := f(x \otimes y)$ is jointly completely positive and this is equivalent with $\mathcal{L}(\phi_f) : \mathcal{S} \to \mathcal{T}^d$ being completely positive. Hence,

$$(\mathcal{S} \otimes_{max} \mathcal{T})^{d+} = CP(\mathcal{S}, \mathcal{T}^d)$$

for all operator systems $\mathcal{S}, \mathcal{T}$, so this is true if we put $\mathcal{S}^d$ and $\mathcal{T}^d$ in the place of $\mathcal{S}$ and $\mathcal{T}$ respectively. Thus, from the above and the fact that $\mathcal{T}$ and $(\mathcal{T}^d)^d$ are complete order isomorphic (Lemma 2.20) we obtain

$$((\mathcal{S}^d \otimes_{max} \mathcal{T}^d)^d)^+ = CP(\mathcal{S}^d, \mathcal{T}) = (\mathcal{S} \otimes_{min} \mathcal{T})^+ \tag{I}$$

Thus far we have shown that there is a bijective correspondence between positive linear functionals on $\mathcal{S}^d \otimes_{max} \mathcal{T}^d$ and positive elements of $\mathcal{S} \otimes_{min} \mathcal{T}$. Hence, there exists a bijective linear map from $\mathcal{S} \otimes_{min} \mathcal{T}$ to $(\mathcal{S}^d \otimes_{max} \mathcal{T}^d)^d$ which is an order isomorphism. We need to show that it is a complete order isomorphism.

We identify $M_n(\mathcal{S}) \otimes_{min} \mathcal{T}$ with $M_n(\mathcal{S} \otimes_{min} \mathcal{T})$. Since max is associative we have that $(M_n \otimes \mathcal{S}^d) \otimes_{max} \mathcal{T}^d = M_n \otimes (\mathcal{S}^d \otimes_{max} \mathcal{T}^d)$. Moreover, for any $\mathcal{R}$ operator system we have the identification $M_n(R^d) = M_n(R)^d$. In particular we have that $M_n(R^d) \ni [f_{ij}] \to \phi \in \mathcal{L}(\mathcal{R}, M_n)$ where $\phi(r) = [f_{ij}(r)]$ and then from the section about positive maps we have that

$$\mathcal{L}(\mathcal{R}, M_n) \ni \phi \to s_\phi \in \mathcal{L}(M_n(\mathcal{R}), \mathbb{C}) = M_n(\mathcal{R})^d \text{ where } s_\phi([r_{ij}]) = \sum_{i,j=1}^{n} f_{ij}(r_{ij})$$

Thus,

$$(M_n(\mathcal{S}^d) \otimes_{max} \mathcal{T}^d)^d = (M_n(\mathcal{S}^d \otimes_{max} \mathcal{T}^d))^d = M_n((\mathcal{S}^d \otimes_{max} \mathcal{T}^d)^d) \tag{II}$$

Now replacing $\mathcal{S}$ by $M_n(\mathcal{S})$, from the relation (I) we have that $(M_n(\mathcal{S}^d) \otimes_{max} \mathcal{T}^d)^{d+} = (M_n(\mathcal{S}) \otimes_{min} \mathcal{T})^+ = M_n(\mathcal{S} \otimes_{min} \mathcal{T})^+$ and combining this with (II) we obtain

$$(M_n((\mathcal{S}^d \otimes_{max} \mathcal{T}^d)^d))^+ = M_n(\mathcal{S} \otimes_{min} \mathcal{T})^+$$

Hence we conclude that

$$(\mathcal{S}^d \otimes_{max} \mathcal{T}^d)^d \text{ and } \mathcal{S} \otimes_{min} \mathcal{T} \text{ are completely order isomorphic} \tag{III}$$

Remember that for any operator system $\mathcal{R}$ we have that it is completely order isomorphic with $(\mathcal{R}^d)^d$. Now replacing $\mathcal{S} \to \mathcal{S}^d$ and $\mathcal{T} \to \mathcal{T}^d$ in (III) we have that $(\mathcal{S} \otimes_{max} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{min} \mathcal{T}^d$. By taking duals in (III) we have that $(\mathcal{S} \otimes_{min} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{max} \mathcal{T}^d$.

For the remainder of this subsection whenever $\mathcal{A}$, $\mathcal{B}$ are C^*-algebras we will use the following notation: $\mathcal{A} \otimes \mathcal{B}$ for their C^*-algebraic tensor product and $\mathcal{A} \otimes_{C^*max} \mathcal{B}$ for their maximal C^*-algebraic tensor product. We will see that $\otimes_{max}$ gives an extension of the maximal C^*-algebraic tensor product from the category of C^*-algebras to the category of operator systems.

We will need the following [2, theorem 3.5.3] :

Proposition 2.23 *Let $\mathcal{A}_1,\ \mathcal{A}_2,\ \mathcal{A}_3,\ \mathcal{A}_4$ be C^*-algebras and $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ and $\psi : \mathcal{A}_3 \to \mathcal{A}_4$ be completely positive maps then the algebraic tensor product map $\varphi \odot \psi : \mathcal{A}_1 \otimes \mathcal{A}_3 \to \mathcal{A}_2 \otimes \mathcal{A}_4$ extends to a completely positive map from $\mathcal{A}_1 \otimes_{C^*max} \mathcal{A}_3 \to \mathcal{A}_2 \otimes_{C^*max} \mathcal{A}_4$.*

Theorem 2.24 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Then the operator system $\mathcal{A} \otimes_{max} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside $\mathcal{A} \otimes_{C^*max} \mathcal{B}$.*

Proof: Let $\mathcal{C} := \mathcal{A} \otimes_{C^*max} \mathcal{B}$.
Claim: The faithful inclusion $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{A} \otimes_{C^*max} \mathcal{B}$ endows $\mathcal{A} \otimes \mathcal{B}$ with an operator system structure.
For this claim to be true we need to check whether the conditions of Definition 2.1 are satisfied. Indeed, 1 and 2 are clearly true as for 3 it follows from proposition 2.23. We denote the arising operator system by $\mathcal{A} \otimes_\tau \mathcal{B}$.
For every $n \in \mathbb{N}$, let $D_n = M_n(\mathcal{A} \otimes_\tau \mathcal{B})^+ = M_n(\mathcal{A} \otimes \mathcal{B}) \bigcap M_n(\mathcal{C})^+$. Since max is the largest operator system structure on $\mathcal{A} \otimes \mathcal{B}$, i.e. it has the smallest cones, we see that $C_n^{max}(\mathcal{A}, \mathcal{B}) \subseteq D_n$.

Now we will show that the Archimedean ordered $*$-vector spaces $(M_n(\mathcal{A} \otimes \mathcal{B}), C_n^{max}(\mathcal{A}, \mathcal{B}))$ and $(M_n(\mathcal{A} \otimes \mathcal{B}), D_n)$ have the same state space.
For the above to be true it suffices to show that whenever we have a linear map $f : M_n(\mathcal{A} \otimes \mathcal{B}) \to \mathbb{C}$ with $f(C_n^{max}(\mathcal{A}, \mathcal{B})) \subseteq \mathbb{R}^+$ then $f(D_n) \subseteq \mathbb{R}^+$, because the inverse follows form the inclusion $C_n^{max}(\mathcal{A}, \mathcal{B}) \subseteq D_n$.
To this end fix a linear map $f : M_n(\mathcal{A} \otimes \mathcal{B}) \to \mathbb{C}$ such that $f(C_n^{max}(\mathcal{A}, \mathcal{B})) \subseteq \mathbb{R}^+$. Suppose that $X = \sum_{i=1}^k \alpha_i \otimes \beta_i$, where $\alpha_i \in M_n(\mathcal{A})$ and $\beta_i \in \mathcal{B}$. Notice that $P = [\alpha_i \alpha_j^*]_{i,j=1}^k \in M_k(M_n(\mathcal{A}))^+$ and $Q = [\beta_i \beta_j^*]_{i,j=1}^k \in M_k(\mathcal{B})^+$. Thus, from Lemma 2.12 we have that

$$XX^* = \sum_{i,j=1}^k \alpha_i \alpha_j^* \otimes \beta_i \beta_j^* \in C_n^{max}(\mathcal{A}, \mathcal{B})$$

So $f(XX^*) \geq 0$.

On the other hand, we know from C^*-algebraic theory that the C^*-algebraic tensor product is associative and that M_n is a nuclear C^*-algebra. Hence we have the natural identification $M_n(\mathcal{C}) = M_n(\mathcal{A}) \otimes_{C^*max} \mathcal{B}$.
By the definition of states on the C^*-algebraic tensor product (see [7] p.7-9) we have that the state space of $M_n(\mathcal{C}) = M_n(\mathcal{A}) \otimes_{C^*max} \mathcal{B}$ denoted $\mathcal{S}(M_n(\mathcal{A}) \otimes_{C^*max} \mathcal{B})$ is the following set of linear functionals

$$\{g : M_n(\mathcal{A}) \otimes \mathcal{B} \to \mathbb{C} : g \text{ unital and } g(yy^*) \geq 0,\ \forall y \in M_n(\mathcal{A}) \otimes \mathcal{B}\}$$

Since $X \in M_n(\mathcal{A}) \otimes \mathcal{B}$ we observe that the linear map

$$f : M_n(\mathcal{A} \otimes \mathcal{B}) = M_n(\mathcal{A}) \otimes \mathcal{B} \to \mathbb{C}$$

is an element of the above set, i.e., $f \in \mathcal{S}(M_n(\mathcal{C}))$. Consequently, $f(M_n(\mathcal{C})^+) \subseteq \mathbb{R}^+$ and thus $f(D_n) \subseteq \mathbb{R}^+$.

Finally, let $A \in D_n$ and $f : M_n(\mathcal{A} \otimes_{max} \mathcal{B}) \to \mathbb{C}$ be a positive map, that is, $f(C_n^{max}(\mathcal{A}, \mathcal{B})) \subseteq \mathbb{R}^+$. By the above discussion : $f(A) \geq 0$ and by Proposition 1.24 $A \in C_n^{max}(\mathcal{A}, \mathcal{B})$ which completes the proof.

2.3 The Commuting Tensor Product

Let$(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems. We set,

$$ucp(\mathcal{S},\mathcal{T}) = \{(\phi,\psi) : H \text{ is a Hilbert space, } \phi : \mathcal{S} \to B(H) \text{ and } \psi : \mathcal{T} \to B(H)$$

$$\text{are unital completely positive maps with commuting ranges}\}$$

We call a pair (ϕ, ψ) as above *commuting* and we let $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to B(H)$ be the map given on elementary tensors by $(\phi \cdot \psi)(u \otimes v) = \phi(u)\psi(v) = \psi(v)\phi(u)$, $u \in \mathcal{S}$ and $v \in \mathcal{T}$.

For each $n \in \mathbb{N}$, let

$$C_n^{com} = C_n^{com}(\mathcal{S},\mathcal{T}) = \{u = [u_{ij}] \in M_n(\mathcal{S}\otimes\mathcal{T}) : (\phi\cdot\psi)^n(u) = [(\phi\cdot\psi)(u_{ij})] \in B(H^n)^+,$$

$$\text{for all } (\phi,\psi) \in ucp(\mathcal{S},\mathcal{T})\}$$

Proposition 2.25 *The collection $\{C_n^{com}\}_{n=1}^{\infty}$ is a matrix ordering on $\mathcal{S}\otimes\mathcal{T}$ with Archimedean matrix order unit $e_1 \otimes e_2$.*
In other words, $(\mathcal{S} \otimes \mathcal{T}, \{C_n^{com}\}_{n=1}^{\infty}, e_1 \otimes e_2)$ is an operator system.

Proof: The fact that C_n^{com} is a cone follows from the linearity of $(\phi \cdot \psi)$ which in turn follows from the linearity of ϕ and ψ.

In order to prove compatibility let $u = [u_{kl}] \in C_m^{com}$ and $X = [x_{ik}] \in M_{n,m}$ then,

$$\begin{aligned}(\phi \cdot \psi)^n(XuX^*) &= (\phi \cdot \psi)^n[\sum_{k,l=1}^{m} x_{ik}u_{kl}\overline{x_{jl}}]_{ij,=1}^{n} \\ &= [(\phi \cdot \psi)(\sum_{k,l=1}^{m} x_{ik}u_{kl}\overline{x_{jl}})]_{i,j=1}^{n} \\ &= [\sum_{k,l=1}^{m} x_{ik}(\phi \cdot \psi)(u_{kl})\overline{x_{jl}}]_{i,j=1}^{n} \\ &= X(\phi \cdot \psi)^m(u)X^*\end{aligned}$$

and $X(\phi \cdot \psi)^m(u)X^* \in B(H^m)^+$, so $XuX^* \in C_m^{com}$. Hence the family $\{C_n^{com}\}_{n=1}^{\infty}$ is compatible.

Let $\phi \in \mathrm{UCP}(\mathcal{S}, M_k)$ and $\psi \in \mathrm{UCP}(\mathcal{T}, M_m)$ we define $\tilde{\phi} : \mathcal{S} \to M_k \otimes I_m$ and $\tilde{\psi} : \mathcal{S} \to I_k \otimes M_m$ by $\tilde{\phi}(u) = \phi(u) \otimes I_m$ and $\tilde{\psi}(v) = I_k \otimes \psi(v)$. Notice that if $[v_{ij}] \in M_n(\mathcal{S})^+$ then

$$(\tilde{\phi})^n([v_{ij}]) = [\tilde{\phi}(v_{ij})] = [\phi(v_{ij}) \otimes I_m] = [\phi(v_{ij})] \otimes I_m = \phi^n([v_{ij}]) \otimes I_m$$

which is positive because ϕ is completely positive. In the same way we see that $(\tilde{\psi})^n([u_{ij}]) \geq 0$ for any $[u_{ij}] \in M_n(\mathcal{T})^+$. Hence $(\tilde{\phi}, \tilde{\psi}) \in ucp(\mathcal{S},\mathcal{T})$. Let $P = [p_{ij}] \in C_n^{com}$ where $p_{ij} = \sum_{\alpha,\beta} u_{ij}^{\alpha} \otimes v_{ij}^{\beta}$, $u_{ij}^{\alpha} \in \mathcal{S}$ and $v_{ij}^{\beta} \in \mathcal{T}$, then

$$(\phi\otimes\psi)^n(P)=[(\phi\otimes\psi)(p_{ij})]=[\sum_{\alpha,\beta}\phi(u_{ij}^\alpha)\otimes\psi(v_{ij}^\beta)]=$$

$$[\sum_{\alpha,\beta}(\phi(u_{ij}^\alpha)\otimes I_m)(I_k\otimes\psi(v_{ij}^\beta)]=[\sum_{\alpha,\beta}\tilde{\phi}(u_{ij}^\alpha)\tilde{\psi}(v_{ij}^\beta)]=$$

$$[(\tilde{\phi}\cdot\tilde{\psi})(\sum_{\alpha,\beta}u_{ij}^\alpha\otimes v_{ij}^\beta)]=[(\tilde{\phi}\cdot\tilde{\psi})(p_{ij})]=(\tilde{\phi}\cdot\tilde{\psi})^n(P)\geq 0$$

Now remembering the definition of the cones C_n^{min} (see subsection 2.1), we see that $P\in C_n^{min}$. Thus $C_n^{com}\subseteq C_n^{min}$ for every $n\in\mathbb{N}$.
Since $C_n^{min}\bigcap(-C_n^{min})=\{0\}$ and $e_1\otimes e_2$ is a matrix order unit for $\{C_n^{min}\}_{n=1}^\infty$ we have that: $C_n^{com}\bigcap(-C_n^{com})=\{0\}$ and $e_1\otimes e_2$ is a matrix order unit for $\{C_n^{com}\}_{n=1}^\infty$.

Finally, suppose that $u\in M_n(\mathcal{S}\otimes\mathcal{T})$ is such that $r(e_1\otimes e_2)_n+u\in C_n^{com}$ for all $r>0$. Then for all $(\phi,\psi)\in ucp(\mathcal{S},\mathcal{T})$,

$$0\leq(\phi\cdot\psi)^n(r(e_1\otimes e_2)_n+u)=r(I_H)_n+(\phi\cdot\psi)^n(u),\ \forall r>0$$

Since I_H is an Archimedean matrix order unit for $B(H)$ we have that
$(\phi\cdot\psi)^n(u)\geq 0$, for all $(\phi,\psi)\in ucp(\mathcal{S},\mathcal{T})$. It follows that $u\in C_n^{com}$. Thus, $e_1\otimes e_2$ is an Archimedean matrix order unit.

Definition 2.26 *We call the operator system $(\mathcal{S}\otimes\mathcal{T},\{C_n^{com}\}_{n=1}^\infty,e_1\otimes e_2)$ the commuting tensor product of $\mathcal{S}$ and $\mathcal{T}$ and we denote it by $\mathcal{S}\otimes_c\mathcal{T}$.*

Theorem 2.27 *The mapping $c:\mathcal{O}\times\mathcal{O}\to\mathcal{O}:(\mathcal{S},\mathcal{T})\to\mathcal{S}\otimes_c\mathcal{T}$ is a symmetric and functorial operator system tensor product.*

Proof: We need to check that it satisfies properties 1,2 and 3 of Definition 2.1. From the previous proposition we have that this is true for 1 and 3.

For 2: Suppose that $P=[p_{ij}]\in M_n(\mathcal{S})^+$ and $Q=[q_{kl}]\in M_m(\mathcal{T})^+$, and let $(\phi,\psi)\in ucp(\mathcal{S},\mathcal{T})$. We shall show that $P\otimes Q\in C_{nm}^{com}$. Indeed,

$$(\phi\cdot\psi)^{(nm)}(P\otimes Q)=(\phi\cdot\psi)^{(nm)}([p_{ij}\otimes q_{kl}])=$$

$$\begin{bmatrix}(\phi\cdot\psi)(p_{11}\otimes q_{11}) & \cdots & (\phi\cdot\psi)(p_{11}\otimes q_{1m}) & \cdots & (\phi\cdot\psi)(p_{1n}\otimes q_{11}) & \cdots & (\phi\cdot\psi)(p_{1n}\otimes q_{1m})\\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ (\phi\cdot\psi)(p_{11}\otimes q_{m1}) & \cdots & (\phi\cdot\psi)(p_{11}\otimes q_{mm}) & \cdots & (\phi\cdot\psi)(p_{1n}\otimes q_{m1}) & \cdots & (\phi\cdot\psi)(p_{1n}\otimes q_{mm})\\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ (\phi\cdot\psi)(p_{n1}\otimes q_{11}) & \cdots & (\phi\cdot\psi)(p_{n1}\otimes q_{1m}) & \cdots & (\phi\cdot\psi)(p_{nn}\otimes q_{11}) & \cdots & (\phi\cdot\psi)(p_{nn}\otimes q_{1m})\\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots\\ (\phi\cdot\psi)(p_{n1}\otimes q_{m1}) & \cdots & (\phi\cdot\psi)(p_{n1}\otimes q_{mm}) & \cdots & (\phi\cdot\psi)(p_{nn}\otimes q_{m1}) & \cdots & (\phi\cdot\psi)(p_{nn}\otimes q_{mm})\end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \phi(p_{11})\psi(q_{11}) & \cdots & \phi(p_{11})\psi(q_{1m}) & \cdots & \phi(p_{1n})\psi(q_{11}) & \cdots & \phi(p_{1n})\psi(q_{1m}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \phi(p_{11})\psi(q_{m1}) & \cdots & \phi(p_{11})\psi(q_{mm}) & \cdots & \phi(p_{1n})\psi(q_{m1}) & \cdots & \phi(p_{1n})\psi(q_{mm}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \phi(p_{n1})\psi(q_{11}) & \cdots & \phi(p_{n1})\psi(q_{1m}) & \cdots & \phi(p_{nn})\psi(q_{11}) & \cdots & \phi(p_{nn})\psi(q_{1m}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \phi(p_{n1})\psi(q_{m1}) & \cdots & \phi(p_{n1})\psi(q_{mm}) & \cdots & \phi(p_{nn})\psi(q_{m1}) & \cdots & \phi(p_{nn})\psi(q_{mm}) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \begin{bmatrix} \phi(p_{11}) & & \\ & \ddots & \\ & & \phi(p_{11}) \end{bmatrix} \psi^m(Q) & \cdots & \begin{bmatrix} \phi(p_{1n}) & & \\ & \ddots & \\ & & \phi(p_{1n}) \end{bmatrix} \psi^m(Q) \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} \phi(p_{n1}) & & \\ & \ddots & \\ & & \phi(p_{n1}) \end{bmatrix} \psi^m(Q) & \cdots & \begin{bmatrix} \phi(p_{nn}) & & \\ & \ddots & \\ & & \phi(p_{nn}) \end{bmatrix} \psi^m(Q) \end{bmatrix}$$

$$=$$

$$[(\phi(p_{ij}) \otimes I_m)\psi^m(Q)]_{i,j} = [(\phi(p_{ij}) \otimes I_m)]_{i,j}(I_n \otimes \psi^m(Q)) \geq 0$$

The last term is positive because $[(\phi(p_{ij}) \otimes I_m)]_{i,j} = \phi^n(P) \otimes I_m$ which is positive since ϕ is CP and $P \in M_n(\mathcal{S})^+$ and it commutes with $(I_n \otimes \psi^m(Q))$ which in turn is positive since ψ is completely positive and $Q \in M_m(\mathcal{T})^+$. Thus $P \otimes Q \in C_{nm}^{com}$ and property 3 is satisfied.

Functoriality: Let $\rho : \mathcal{S}_1 \to \mathcal{S}_2$ and $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ be unital completely positive maps, and let $v \in M_n(\mathcal{S}_1 \otimes_c \mathcal{T}_1)$ be positive. If $(\phi, \psi) \in ucp(\mathcal{S}_2, \mathcal{T}_2)$ we have that $(\phi \circ \rho, \psi \circ \eta) \in ucp(\mathcal{S}_1, \mathcal{T}_1)$. Moreover,

$$(\phi \cdot \psi)^n((\rho \otimes \eta)^n(v)) = ((\phi \circ \rho)(\psi \circ \eta))^n(v) \geq 0$$

Hence $(\rho \otimes \eta)^n(v) \in M_n(\mathcal{S}_2 \otimes_c \mathcal{T}_2)^+$ and the functoriality follows.

Symmetric: Consider the map $\theta : \mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S} : u \otimes v \to v \otimes u$, we shall show that it extends to a unital complete order isomorphism from $\mathcal{S} \otimes_c \mathcal{T}$ onto $\mathcal{T} \otimes_c \mathcal{S}$. Firstly notice that $(\phi, \psi) \in ucp(\mathcal{S}, \mathcal{T})$ if and only if $(\psi, \phi) \in ucp(\mathcal{T}, \mathcal{S})$.

Now let $v = \sum_p s_p \otimes t_p \in \mathcal{S} \otimes \mathcal{T}$ and $(\phi, \psi) \in ucp(\mathcal{S}, \mathcal{T})$. Then,

$$
\begin{aligned}
(\phi \cdot \psi)(v) &= \sum_p \phi(s_p)\psi(t_p) = \sum_p \psi(t_p)\phi(s_p) \\
&= (\psi \cdot \phi)(\sum_p t_p \otimes s_p) = (\psi \cdot \phi)(\theta(\sum_p s_p \otimes t_p)) \\
&= (\psi \cdot \phi)(\theta(v))
\end{aligned}
$$

Hence for a $v \in M_n(\mathcal{S} \otimes_c \mathcal{T})$ we have that $v \in M_n(\mathcal{S} \otimes_c \mathcal{T})^+$ if and only if $(\theta(v)) \in M_n(\mathcal{S} \otimes_c \mathcal{T})^+$. It follows that the commuting tensor product is symmetric.

Theorem 2.28 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, then $\mathcal{A} \otimes_c \mathcal{B} = \mathcal{A} \otimes_{max} \mathcal{B}$*

Proof: By theorem 2.15 we have that $C_n^{max}(\mathcal{A}, \mathcal{B}) \subseteq C_n^{com}(\mathcal{A}, \mathcal{B})$.

Conversely, suppose that $u \in C_n^{com}(\mathcal{A}, \mathcal{B})$. From theorem 2.24 we have that $\mathcal{A} \otimes_{max} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ inside the maximal C^*-algebraic tensor product $\mathcal{A} \otimes_{C^*max} \mathcal{B}$. Let $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes_{C^*max} \mathcal{B}$ given by $\iota_{\mathcal{A}}(a) = a \otimes 1_{\mathcal{B}}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \otimes_{C^*max} \mathcal{B}$ given by $\iota_{\mathcal{B}}(b) = 1_{\mathcal{A}} \otimes b$. Obviously, these maps are completely positive and their ranges commute. Moreover, Theorem 2.24 implies that $u \in C_n^{max}(\mathcal{A}, \mathcal{B})$ if and only if $(\iota_{\mathcal{A}} \cdot \iota_{\mathcal{B}})^n(u) \geq 0$. However the latter is true by the definition of the commuting tensor product. Thus the proof is complete.

We have shown that the cones of the maximal tensor product are the smallest possible and those of the minimal tensor product are the largest. Thus, we have the following inclusions

$$C_n^{max} \subseteq C_n^{com} \subseteq C_n^{min}$$

as well as the following completely positive maps

$$\mathcal{S} \otimes_{max} \mathcal{T} \overset{id}{\hookrightarrow} \mathcal{S} \otimes_c \mathcal{T} \overset{id}{\hookrightarrow} \mathcal{S} \otimes_{min} \mathcal{T}$$

It turns out that the above inclusions are in fact strict as we will see in later Chapters.

3 The Quotient

In this Chapter we recall some fundamental results regarding operator system quotients introduced in [19] are examined. In the following Chapter we will use the quotient theory in order to construct the coproduct of operator systems.

Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems, and $\varphi : \mathcal{S} \to \mathcal{T}$ be a non-zero (unital) completely positive map. Note that the kernel, $\ker\varphi$, of φ is a closed (because φ is continuous/bounded) *-subspace of $\mathcal{S}$ and does not contain e_1 (obviously). Furthermore, it is an **order ideal** of $\mathcal{S}$, that means:

$$\text{If } x \in \ker\varphi \text{ and } 0 \leq y \leq x \text{ then } y \in \ker\varphi$$

However the reverse of the above arguments, in general, is not true. (for example: span $\{E_{1,1}\} \subseteq M_n$).

Let $(\mathcal{S}, e_1)$ be an operator system and $\mathcal{J}$ a closed *-subspace of $\mathcal{S}$ which does not contain e_1. On the algebraic quotient $\mathcal{S}/\mathcal{J}$ we let $q : \mathcal{S} \to \mathcal{S}/\mathcal{J}$ be the canonical quotient map. The vector space $\mathcal{S}/\mathcal{J}$ has a natural involution induced by q, $(s+\mathcal{J})^* = q(s)^* = q(s^*) = s^* + \mathcal{J}$, which turns it into a *-vector space. For each $n \in \mathbb{N}$ we set,

$$D_n(\mathcal{S}/\mathcal{J}) = \{[s_{ij}+\mathcal{J}]_{i,j} \in M_n(\mathcal{S}/\mathcal{J}) : \exists\, k_{ij} \in \mathcal{J} \text{ such that } [s_{ij}+k_{ij}] \in M_n(\mathcal{S})^+\}$$

The family $\{D_n(\mathcal{S}/\mathcal{J})\}_{n=1}^{\infty}$ is a matrix ordering on $\mathcal{S}/\mathcal{J}$ with $e_1 + \mathcal{J}$ as a matrix order unit. Unfortunately, it is not Archimedean.

Definition 3.1 *We call a subspace $\mathcal{J}$ of an operator system $\mathcal{S}$ a kernel if there exist some operator system $\mathcal{T}$ and a (unital) completely positive map $\varphi : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{J} = \ker\varphi$.*

Remark: Let $\mathcal{S}$ be an operator system and $\mathcal{J}$ a kernel in $\mathcal{S}$. Then the following holds

$$\{[s_{ij} + \mathcal{J}] : [s_{ij}] \in M_n(\mathcal{S})^+\} = D_n(\mathcal{S}/\mathcal{J})$$

Proof: The proof for the general case is no different than that for the case in which $n = 1$.
Call the set on the left hand side B. If $s + \mathcal{J} \in B$, then $s \in \mathcal{S}^+$ and so for $k = 0 \in \mathcal{J}$ we have that $s + k = s + 0 = s \in \mathcal{S}^+$. It follows that $B \subseteq D_1(\mathcal{S}/\mathcal{J})$.

Conversely, if $s + \mathcal{J} \in D_1(\mathcal{S}/\mathcal{J})$ then there exist some $k \in \mathcal{J}$ such that $s + k \in \mathcal{S}^+$. Hence, $(s + k) + \mathcal{J} \in \{x + \mathcal{J} : x \in \mathcal{S}^+\}$. Hence, $D_1(\mathcal{S}/\mathcal{J}) \subseteq B$ and the result follows.

Let $(\mathcal{S}, e_1)$ be an operator system and $\mathcal{J} \subseteq \mathcal{S}$ a kernel. Consider the family of cones $\{C_n(\mathcal{S}/\mathcal{J})\}_{n=1}^{\infty}$, where for each $n \in \mathbb{N}$:

$$C_n(\mathcal{S}/\mathcal{J}) = \{[s_{ij} + \mathcal{J}] \in M_n(\mathcal{S}/\mathcal{J}) : \ \forall \epsilon > 0 \text{ there exist } k_{ij} \in \mathcal{J} \text{ such that } \epsilon(e_1)_n + [x_{ij} + k_{ij}] \in M_n(\mathcal{S})^+\}$$

$$= \{[s_{ij} + \mathcal{J}] \in M_n(\mathcal{S}/\mathcal{J}) : \ \forall \epsilon > 0, \ \epsilon(e_1 + \mathcal{J})_n + [s_{ij} + \mathcal{J}] \in D_n(\mathcal{S}/\mathcal{J})\}$$

It was shown in [19, Proposition 3.4] that if we endow $\mathcal{S}/\mathcal{J}$ with this family of cones, then the quotient $\mathcal{S}/\mathcal{J}$ becomes a matrix ordered *-vector space with Archimedean matrix order unit $e_1 + \mathcal{J}$ and the quotient map $q : \mathcal{S} \to \mathcal{S}/\mathcal{J}$ is completely positive.

Definition 3.2 *[19, Definition 3.5] The operator system* $(\mathcal{S}/\mathcal{J}, \{C_n(\mathcal{S}/\mathcal{J})\}_{n=1}^{\infty}, e_1+\mathcal{J})$ *arising from the above construction is called the* ***quotient operator system****.*

Definition 3.3 *Let $\mathcal{S}$ be an operator system and $\mathcal{J}$ a kernel in $\mathcal{S}$. We call the kernel $\mathcal{J}$ order proximinal if $D_1(\mathcal{S}/\mathcal{J}) = C_1(\mathcal{S}/\mathcal{J})$ and completely order proximinal if $D_n(\mathcal{S}/\mathcal{J}) = C_n(\mathcal{S}/\mathcal{J}), \ \forall n \in \mathbb{N}$.*

The quotient operator system satisfies an operator system version of the First Isomorphism Theorem (see [19, Proposition 3.6.]):

Proposition 3.4 *Let $\mathcal{S}$ be an operator system and $\mathcal{J}$ a kernel in $\mathcal{S}$.*

Whenever $\mathcal{R}$ is an operator system and $\phi : \mathcal{S} \to \mathcal{R}$ is a unital completely positive map with $\mathcal{J} \subseteq \ker\phi$. Then the induced map $\tilde{\phi} : \mathcal{S}/\mathcal{J} \to \mathcal{R}$ given by $\tilde{\phi}(s+\mathcal{J}) = \phi(s)$, that is $\tilde{\phi} \circ q = \phi$, is also unital and completely positive.

Conversely, if $\psi : \mathcal{S}/\mathcal{J} \to \mathcal{R}$ is a UCP map between operator systems then there exists a UCP map $\phi : \mathcal{S} \to \mathcal{R}$ with, necessarily $\mathcal{J} \subseteq \ker\phi$ such that $\phi = q \circ \psi$.

Note: The above proposition remains true if we drop the condition on the unitality of both sides.

A completely positive surjective linear map between operator systems $\phi : \mathcal{S} \to \mathcal{T}$ is called **complete quotient map** if the induced map $\tilde{\phi} : \mathcal{S}/\ker\phi \to \mathcal{T}$ is a complete order isomorphism.

Lemma 3.5 *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems, and $\phi : \mathcal{S} \to \mathcal{T}$ a complete quotient map. Then for every $n \in \mathbb{N}$ and every strictly positive $y \in M_n(\mathcal{T})$ there exists a strictly positive $x \in M_n(\mathcal{S})$ such that $\phi^n(x) = y$.*

Proof: We will prove it for $n = 1$ the proof for the general case is similar.
Suppose that $y \in \mathcal{T}$ is strictly positive. Then by definition, $\exists\delta > 0$ such that: $y \geq \delta e_2$. Hence, $y' = y - \delta e_2 \in T^+$ and consequently $z = y' + \frac{\delta}{2}e_2 \in T^+$. Notice that $y = y' + \delta e_2 = z + \frac{\delta}{2}e_2$. From the hypothesis we have that $\tilde{\phi} : \mathcal{S}/\ker\phi \to \mathcal{T}$ is a complete order isomorphism, in particular it is surjective, so there exists $\tilde{h} = h + \ker\phi \in \mathcal{S}/\ker\phi$ such that $z = \tilde{\phi}(\tilde{h})$. The positive elements of the quotient are those in the cone

$$C_1(\mathcal{S}/\ker\phi) = \{(s + \ker\phi) : \ \forall \ \epsilon > 0, \ \exists \ k \in \ker\phi \text{ such that } \epsilon e_1 + s + k \in \mathcal{S}^+\}$$

Take $\epsilon = \frac{\delta}{4}$ then $\exists \ k \in \ker\phi$ such that, $\frac{\delta}{4}e_1 + h + k \in \mathcal{S}^+$. Set $\beta = \frac{\delta}{2}e_1 + h + k = \frac{\delta}{4}e_1 + (\frac{\delta}{4}e_1 + h + k)$. Since $\frac{\delta}{4}e_1 + h + k \geq 0$ we have that $\beta \geq \frac{\delta}{4}e_1$, this means that $\beta \in \mathcal{S}$ is strictly positive. Moreover, $\phi(\beta) = \tilde{\phi}(\tilde{\beta}) = \frac{\delta}{4}e_2 + \frac{\delta}{4}e_1 + \tilde{\phi}(\tilde{h}) + 0 = \frac{\delta}{4}e_2 + z = y$ and the proof is complete.

Theorem 3.6 *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\phi : \mathcal{S} \to \mathcal{T}$ be a complete quotient map. Then the dual map $\phi^d : \mathcal{T}^d \to \mathcal{S}^d$ is a complete order embedding.*

Proof: ϕ^d is completely positive: Fix a $n \in \mathbb{N}$ then for any $G = [g_{ij}] \in M_n(\mathcal{T}^d)^+$, which we identify with the CP map $\hat{G} : \mathcal{T} \to M_n : t \to [g_{ij}(t)]$ we have that

$$(\phi^d)^n(G) = [\phi^d(g_{ij})] = [g_{ij} \circ \phi] \longleftrightarrow \hat{G} \circ \phi$$

and $\hat{G} \circ \phi$ is a composition of CP maps thus it is CP.

ϕ^d is injective: It suffices to show that $\ker \phi^d = \{0\}$. To that end let $f : \mathcal{T} \to \mathbb{C}$ be a linear map with $f \in ker\phi^d$ then $\phi^d(f)(s) = 0$ or equivalently $f(\phi(s)) = 0$ for every $s \in \mathcal{S}$. Since $\phi : \mathcal{S} \to \mathcal{T}$ is surjective $\forall\ t \in \mathcal{T},\ \exists\ s \in \mathcal{S}$ such that $\phi(s) = t$. Thus we have that $f(t) = 0$ for every $t \in \mathcal{T}$ and the desired result follows.

ϕ^d is a complete order embedding: We will show that if $G = [g_{ij}] \in M_n(\mathcal{T}^d)$ is such that $(\phi^d)^n(G) \in M_n(\mathcal{S}^d)^+$ then necessarily $G \in M_n(\mathcal{T}^d)^+$ or equivalently that the mapping $\hat{G} : \mathcal{T} \to M_n : t \to [g_{ij}(t)]$ is completely positive.

Let $k \in \mathbb{N}$ and $[t_{lm}] \in M_k(\mathcal{T})^+$. Then $\hat{G}^k([t_{lm}]) = [[g_{ij}(t_{lm})]_{i,j}]_{l,m}$. For any $\epsilon \geq 0$ we set $[t^\epsilon_{lm}] = [t_{lm}] + \epsilon(e_2)_k \in M_k(\mathcal{T})$, where e_2 denotes the unit of $\mathcal{T}$. This element is strictly positive so from Lemma 3.5 we have that there exists a (strictly) positive $[s^\epsilon_{lm}] \in M_k(\mathcal{S})$ such that $[t^\epsilon_{lm}] = \phi^k([s^\epsilon_{lm}]) = [\phi(s^\epsilon_{lm})]$. Thus,

$$[g_{ij}(t^\epsilon_{lm})]_{i,j}]_{l,m} = [[g_{ij}(\phi(s^\epsilon_{lm}))]_{i,j}]_{l,m} = [\hat{G}(\phi(s^\epsilon_{lm}))]_{l,m} = (\hat{G} \circ \phi)^k([s^\epsilon_{lm}]_{l,m})$$

Now the map $\hat{G} \circ \phi : \mathcal{S} \to M_n$ corresponds to the positive element $\phi^d(G)$ of $M_n(\mathcal{S}^d)^+$ so it is completely positive. Since $[s_{lm}]$ is a positive element of $M_k(\mathcal{S})$ we see that $[g_{ij}(t^\epsilon_{lm})]_{i,j}]_{l,m}$ is positive in $M_k(M_n)$. Considering that ϵ was arbitrary and $t^\epsilon_{lm} \xrightarrow[\epsilon \to 0]{} t_{lm}$ we conclude that $[[g_{ij}(t_{lm})]_{i,j}]_{l,m} \in M_k(M_n)^+$. This implies that $\hat{G}$ is indeed CP.

Lemma 3.7 *Let $(\mathcal{S}, e_1)$ be an operator system and y a self-adjoint element of $\mathcal{S}$ which is neither positive nor negative. Then the set* span $\{y\} = \{\lambda y : \lambda \in \mathbb{C}\}$ *is a proximinal kernel in $\mathcal{S}$.*

Proof: Firstly, we will show that it is a kernel. Assume that $\mathcal{S} = \mathcal{A}$ is a unital C^*-algebra and let $\mathcal{J} = \text{span}\{y\}$. We equip $\mathcal{A}/\mathcal{J}$ with the cone $D_1 = D_1(\mathcal{A}/\mathcal{J}) = \{\alpha + \mathcal{J} : \alpha \in \mathcal{A}^+\}$ and observe that $D_1 \bigcap (-D_1) = \{0\}$. Indeed, assume that $d \in D_1 \bigcap (-D_1)$ then $d = x_1 + \mathcal{J} = -x_2 + \mathcal{J}$, where $x_i \in \mathcal{A}^+,\ i = 1, 2$. Thus there exists some $j \in \mathcal{J}$ such that $x_1 = -x_2 + j$ which means that $j = x_1 + x_2 \geq 0$. However j is neither positive nor negative, hence $x_1 = x_2 = 0$ and $d = 0 + \mathcal{J} = 0_{\mathcal{A}/\mathcal{J}}$. Now we shall show that $e_1 + \mathcal{J}$ is an Archimedean order unit for $(\mathcal{A}/\mathcal{J}, D_1)$. Let $x + \mathcal{J} \in \mathcal{A}/\mathcal{J}$ be such that

$$\epsilon(e_1 + \mathcal{J}) + x + \mathcal{J} \in D_1,\ \forall \epsilon > 0 \tag{$*$}$$

We will prove that $x + \mathcal{J} \in D_1$ and it suffices do this for a self-adjoint x. Condition $(*)$ is equivalent to the following: $(\epsilon e_1 + x) + \mathcal{J} \in D_1,\ \forall \epsilon > 0$. Thus for every $\epsilon > 0$

there exists some $\alpha_\epsilon \in \mathbb{C}$ such that $\epsilon e_1 + x + \alpha_\epsilon y \in \mathcal{A}^+$. Since $\epsilon e_1 + x + \alpha_\epsilon y \in \mathcal{A}^+$ it is self-adjoint, which implies that $\alpha_\epsilon \in \mathbb{R}$.

Consider the set $P_\epsilon = \{\alpha \in \mathbb{R} : \epsilon e_1 + x + \alpha y \in \mathcal{A}^+\}$ this is a closed subset of $\mathbb{R}$ (because $\mathcal{A}^+$ is closed in $\mathcal{A}$) and for every $\delta \geq \epsilon$, $P_\epsilon \subseteq P_\delta$. Take the Jordan decomposition, $y = y_1 - y_2,\ y_i \in \mathcal{A}^+$ with $y_1 y_2 = y_2 y_1 = 0$, of y (remember that $\mathcal{A}$ is a $C^* -$ algebra). For $\epsilon = 1$ we have that $e_1 + x + \alpha y \geq 0$ and by multiplying left and right by y_1 we obtain:

$$y_1^2 + y_1 x y_1 + \alpha y_1 y y_1 \geq 0$$

$$y_1^2 + y_1 x y_1 + \alpha (y_1 y_1 - y_1 y_2) y_1 \geq 0$$

$$\alpha y_1^3 \geq -y_1^2 - y_1 x y_1 \tag{I}$$

Since y_1 is non-zero, (I) gives us a lower bound for α. In particular, consider $\mathcal{A}$ as a C^*-subalgebra of $B(H)$ for some Hilbert space H. Then since y_1 is non-zero and positive there exists some $h \in H$ such that $(y_1 h, h)_H > 0$. Therefore from (I): $\alpha \geq \frac{\left((-y_1^2 - y_1 x y_1)h, h\right)_H}{\left(y_1^3 h, h\right)_H} = \beta \in \mathbb{R}$. Correspondingly multiplying both sides by y_2 we obtain an upper bound for α. We conclude that P_1 is bounded. Hence, $(P_\epsilon)_{0<\epsilon\leq 1}$ is a decreasing ϵ-net of closed and bounded subsets of $\mathbb{R}$, i.e., compact, thus they have a non-empty intersection. It follows that there exists some $\alpha_0 \in \bigcap P_\epsilon$, then $\epsilon e_1 + x + \alpha_0 y \in \mathcal{A}^+,\ \forall\ 0 < \epsilon \leq 1$ and letting $\epsilon \to 0$, we have $x + \alpha_0 y \geq 0$. This implies that $x + \mathcal{J} \in D_1$.

It is clear from the above points that $(\mathcal{S}/\mathcal{J}, D_1, e_1 + \mathcal{J})$ is an Archimedean ordered *-vector space, so we can equip it with the minimal operator system structure $OMIN(\mathcal{A}/\mathcal{J})$ (for more details we refer the reader to [18, definition 3.1]). The quotient map $q : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is UCP and has $\mathcal{J}$ as a kernel, so from [18, Theorem 3.4] it will be UCP from $\mathcal{A}$ to $OMIN(\mathcal{A}/\mathcal{J})$, and $\mathcal{J}$ remains its kernel. This proves that $\mathcal{J}$ is a kernel of a UCP map from $\mathcal{A}$ to an operator system and thus it is a kernel in $\mathcal{A}$.

For the general case, suppose that $\mathcal{A}$ is a unital C^*-algebra which contains $\mathcal{S}$. We have shown that there exist an operator system $\mathcal{R}$ and a UCP map $\phi : \mathcal{A} \to \mathcal{R}$ with kernel span $\{y\}$. Consider the restriction of ϕ on $\mathcal{S}$, this remains a UCP map between operator systems with kernel span $\{y\}$. This completes the first part of the proof.

Now we will work towards proving the proximinality of span $\{y\}$ which we will once more denote by $\mathcal{J}$. As before we start by examining the case in which $\mathcal{S} = \mathcal{A}$ is a unital C^*-algebra.

Let $x + \mathcal{J}$ be element in $(\mathcal{A}/\mathcal{J})^+ = C_1(\mathcal{A}/\mathcal{J}) = C_1$. We can assume that x is self-adjoint. By the definition of C_1 for every $\epsilon > 0$ there exists $\alpha_\epsilon y \in \mathcal{J}$ such that $x + \epsilon e_1 + \alpha_\epsilon y \in \mathcal{A}^+$. As in the previous case we have that $\alpha_\epsilon \in \mathbb{R}$. Set $\Pi_\epsilon = \{\alpha \in \mathbb{R} : x + \alpha y + \epsilon e_1 \in \mathcal{A}^+\}$, then $(\Pi_\epsilon)_{0<\epsilon\leq 1}$ is a decreasing ϵ-net of compact subsets of $\mathbb{R}$. Thus, $\bigcap \Pi_\epsilon$ is a non-empty set, which means that there exists $\alpha_0 \in \mathbb{R}$ such that $x + \alpha_0 y + \epsilon e_1 \in \mathcal{A}^+,\ \forall\ 0 < \epsilon \leq 1$. Hence, $x + \alpha_0 y \in \mathcal{A}^+$, i.e., $\exists\ j \in \mathcal{J}$ such that $x + j \in \mathcal{A}^+$ so $x + J \in D_1(\mathcal{A}/\mathcal{J})$. This shows that $C_1(\mathcal{A}/\mathcal{J}) \subseteq D_1(\mathcal{A}/\mathcal{J})$ and since the other inclusion is always true we conclude that $C_1(\mathcal{A}/\mathcal{J}) = D_1(\mathcal{A}/\mathcal{J})$. Now assume that y is an element in an operator system $\mathcal{S}$. Consider $\mathcal{S}$ as an operator subsystem of a unital C^*-algebra $\mathcal{A}$. Then $y \in \mathcal{A}$ so from the above we have that $\mathcal{J}$ is a proximinal kernel in $\mathcal{A}$. Let $q : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ be the quotient map, this map is UCP with kernel $\mathcal{J}$. If $q_0 : \mathcal{S} \to \mathcal{A}/\mathcal{J}$ is the restriction of q on $\mathcal{S}$, then q_0 is UCP with kernel $\mathcal{J}$. Therefore from theorem 3.4 we have that the induced map $\tilde{q_0} : \mathcal{S}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ is UCP. Let $s+\mathcal{J} \in (\mathcal{S}/\mathcal{J})^+ = C_1(\mathcal{S}/\mathcal{J})$, then $s+\mathcal{J} \in (\mathcal{A}/\mathcal{J})^+ = C_1(\mathcal{A}/\mathcal{J}) = D_1(\mathcal{A}/\mathcal{J})$ so there exists an element $\alpha \in \mathcal{A}^+$ such that $s + \mathcal{J} = \alpha + \mathcal{J}$ and since $\mathcal{J} \subseteq \mathcal{S}$, α must be in $\mathcal{S}$. Hence, $\alpha \in \mathcal{S} \bigcap \mathcal{A}^+ = \mathcal{S}^+$ and $s + \mathcal{J} = \alpha + \mathcal{J} \in D_1(\mathcal{S}/\mathcal{J})$. The result follows.

Let $\mathcal{S}$ be an operator system. A finite dimensional *-closed subspace $\mathcal{J}$ of $\mathcal{S}$, which contains no other positive element of $\mathcal{S}$ except from 0 is called a **null subspace** of $\mathcal{S}$.

An example of a one dimensional null subspace of an operator system is the set span $\{y\}$ of the previous proposition.

Lemma 3.8 *Let V be a vector space and $v_1, \ldots, v_n \in V$.*
Set $\mathcal{J} = \text{span}\{v_1, \ldots, v_n\}$ and let $\mathcal{J}_0 = \text{span}\{v_1, \ldots, v_k\} \subset \mathcal{J}$. Then for $\mathcal{J}_1 = \text{span}\{\sum_{j=k+1}^n v_j + \mathcal{J}_0\} \subseteq V/\mathcal{J}_0$,

$$V/\mathcal{J} \cong (V/\mathcal{J}_0)/\mathcal{J}_1$$

In the case in which V ***is an operator system*** *and $\mathcal{J}_0$, $\mathcal{J}_1$ are proximinal kernels in V and $V/\mathcal{J}_0$ respectively then we have that,*

1. *There exists an order isomorphism between the matrix ordered spaces*

$$(V, D_1(V/\mathcal{J}_0)) \quad \textit{and} \quad ((V/\mathcal{J}_0)/\mathcal{J}_1, D_1((V/\mathcal{J}_0)/\mathcal{J}_1))$$

2. *There exists an order isomorphism between the Archimedean matrix ordered spaces*

$$(V, C_1(V/\mathcal{J}_0)) \quad \textit{and} \quad ((V/\mathcal{J}_0)/\mathcal{J}_1, C_1((V/\mathcal{J}_0)/\mathcal{J}_1))$$

Proof: Consider the map $T : V/\mathcal{J}_0 \to V/\mathcal{J}$ defined by $T(v + \mathcal{J}_0) = v + \mathcal{J},\ v \in V$. It is easy to see that T is well-defined, linear and surjective. We will show that $\ker T = \mathcal{J}_1$.

$$\begin{aligned}\ker T &= \{v + \mathcal{J}_0 \in V/\mathcal{J}_0 : T(v + \mathcal{J}_0) = 0_{V/\mathcal{J}}\} \\ &= \{v + \mathcal{J}_0 \in V/\mathcal{J}_0 : v + \mathcal{J} = 0 + \mathcal{J}\} \\ &= \{v + \mathcal{J}_0 \in V/\mathcal{J}_0 : v \in \mathcal{J}\}\end{aligned}$$

Let $v \in \mathcal{J}$ then there exist $\lambda_i,\ 1 \leq i \leq n+1$, such that $v = \sum_{i=1}^n \lambda_i v_i$. Hence,

$$v + \mathcal{J}_0 = \sum_{i=1}^n \lambda_i v_i + \mathcal{J}_0 = \sum_{i=1}^k \lambda_i v_i + \sum_{i=k+1}^n \lambda_i v_i + \mathcal{J}_0 = \sum_{i=k+1}^n \lambda_i v_i + \mathcal{J}_0 \in \mathcal{J}_1$$

Thus, $\ker T \subseteq \mathcal{J}_1$ and the other inclusion is trivial. So we have that $T : V/\mathcal{J}_0 \to V/\mathcal{J}$ is a surjective linear map with $\ker T = \mathcal{J}_1$. Hence it induces a well-defined isomorphism $G : (V/\mathcal{J}_0)/\mathcal{J}_1 \to V/\mathcal{J}$ given by $G((v + \mathcal{J}_0) + \mathcal{J}_1) = T(v + \mathcal{J}_0)$.

Now for the case in which V **is an operator system**.

Firstly we recall the following:

$C_1(V/\mathcal{J}) = \{v+\mathcal{J} \in V/\mathcal{J} : \ \forall\epsilon > 0$ there exist $k_\epsilon \in \mathcal{J}$ such that $\epsilon e_1+v+k \in V^+\}$

and

$$D_1(V/\mathcal{J}) = \{v + \mathcal{J} \in V/\mathcal{J} : v \in V^+\}$$

Furthermore,

$$C_1(V/\mathcal{J}_0) = \{(v + \mathcal{J}_0) \in V/\mathcal{J}_0 : \forall\epsilon > 0, \ \exists\ \tau_\epsilon \in \mathcal{J}_0 \text{ s.t. } \epsilon e_1 + v + \tau_\epsilon \in V^+\}$$

this is the positive cone of the operator system quotient $V/\mathcal{J}_0$ and because $\mathcal{J}_0$ is proximinal it is equal to

$$D_1(V/\mathcal{J}_0) = \{v + \mathcal{J}_0 \in V/\mathcal{J}_0 : v \in V^+\}$$

Moreover,

$$C_1((V/\mathcal{J}_0)/\mathcal{J}_1)) = \{(v + \mathcal{J}_0) + \mathcal{J}_1 \in (V/\mathcal{J}_0)/\mathcal{J}_1 :$$
$$\forall\epsilon > 0, \ \exists k'_\epsilon \in \mathcal{J}_1 \text{ s.t. } \epsilon(e_1 + \mathcal{J}_0) + ((v + \mathcal{J}_0) + k'_\epsilon) \in C_1(V/\mathcal{J}_0)\}$$

Now we prove 1: Let $(v + \mathcal{J}_o) + \mathcal{J}_1 \in D_1((V/\mathcal{J}_0)/\mathcal{J}_1)$ then $v + \mathcal{J}_0 \in C_1(V/\mathcal{J}_0)$. However, since $\mathcal{J}_0$ is proximinal in V, $C_1(V/\mathcal{J}_0) = D_1(V/\mathcal{J}_0)$ and so we have that that $v \in V^+$. Thus,

$$G((v + \mathcal{J}_o) + \mathcal{J}_1) = T(v + \mathcal{J}_o) = v + \mathcal{J}, \text{ with } v \in V^+$$

so $G((v + \mathcal{J}_o) + \mathcal{J}_1) \in D_1(V/\mathcal{J})$.

Conversely, we will show that whenever $G((v + \mathcal{J}_o) + \mathcal{J}_1) \in D_1(V/\mathcal{J})$ then necessarily $(v + \mathcal{J}_o) + \mathcal{J}_1 \in D_1((V/\mathcal{J}_0)/\mathcal{J}_1)$.
Indeed, $G((v + \mathcal{J}_o) + \mathcal{J}_1) \in D_1(V/\mathcal{J})$ means that $T(v + \mathcal{J}_0) \in D_1(V/\mathcal{J})$ but $T(v + \mathcal{J}_0) = v + \mathcal{J}$ which implies that $v \in V^+$. Thus,

$$v + \mathcal{J}_0 \in D_1(V/\mathcal{J}_0)$$

so $(v + \mathcal{J}_0) + \mathcal{J}_1 \in D_1((V/\mathcal{J}_0)/\mathcal{J}_1))$.

For 2: Let $(v + \mathcal{J}_o) + \mathcal{J}_1 \in C_1((V/\mathcal{J}_0)/\mathcal{J}_1)$, since $\mathcal{J}_1$ is a proximinal kernel in $V/\mathcal{J}_0$ this is equivalent to $(v + \mathcal{J}_o) + \mathcal{J}_1 \in D_1((V/\mathcal{J}_0)/\mathcal{J}_1)$ which from 1 means that $G((v + \mathcal{J}_o) + \mathcal{J}_1) \in D_1(V/\mathcal{J})$, however $G((v + \mathcal{J}_o) + \mathcal{J}_1) = T(v + \mathcal{J}_0) = v + \mathcal{J}$ so we have that $v \in V^+$. Hence, for every $\epsilon > 0$: $\epsilon e_1 + v \in V^+$. Thus, letting $k_\epsilon = 0 \in \mathcal{J}$ for every $\epsilon > 0$, we have that

$$\epsilon e_1 + k_\epsilon + v \in V^+$$

which implies that $G((v+\mathcal{J}_0)+\mathcal{J}) \in C_1(V/\mathcal{J})$.

On the other hand, suppose that $G((v+\mathcal{J}_o)+\mathcal{J}_1) \in C_1(V/\mathcal{J})$ we shall show that then necessarily $(v+\mathcal{J}_o)+\mathcal{J}_1 \in C_1((V/\mathcal{J}_0)/\mathcal{J}_1)$.
Indeed, $G((v+\mathcal{J}_o)+\mathcal{J}_1) \in C_1(V/\mathcal{J})$ means that $T(v+\mathcal{J}_0) \in C_1(V/\mathcal{J})$ however $T(v+\mathcal{J}_0) = v+\mathcal{J}$.
It follows from the above that $\forall\epsilon$ there exists $k_\epsilon \in \mathcal{J}$ such that $\epsilon e_1+v+k_\epsilon := x \in V^+$.
Hence, $\forall\epsilon$ there exist $\lambda_{i,\epsilon}$ such that $(\epsilon e_1+v+\sum_{i=1}^n \lambda_{i,\epsilon}v_i)+\mathcal{J}_0 = x+\mathcal{J}_0 \in D_1(V/\mathcal{J}_0)$ so

$$(\epsilon e_1 + v + \underbrace{\sum_{i=1}^{k} \lambda_{i,\epsilon} v_i}_{\in \mathcal{J}_0} + \sum_{i=k+1}^{n} \lambda_{i,\epsilon} v_i) + \mathcal{J}_0 = x + \mathcal{J}_0,\ x \in V^+,\ \forall\epsilon > 0$$

or equivalently,

$$\epsilon(e_1+\mathcal{J}_0) + (v+\mathcal{J}_0) + \underbrace{(\sum_{i=k+1}^{n} \lambda_{i,\epsilon} v_i + \mathcal{J}_0)}_{\in \mathcal{J}_1} = x + \mathcal{J}_0 \in D_1(V/\mathcal{J}_0),\ \forall\epsilon > 0$$

meaning that for every $\epsilon > 0$ there exist $k'_\epsilon = \sum_{i=k+1}^n \lambda_{i,\epsilon}v_i + \mathcal{J}_0 \in \mathcal{J}_1$ such that

$$\epsilon(e_1+\mathcal{J}_0) + (v+\mathcal{J}_0) + k'_\epsilon \in D_1(V/\mathcal{J}_0)$$

This is equivalent to $(v+\mathcal{J}_o)+\mathcal{J}_1 \in C_1((V/\mathcal{J}_0)/\mathcal{J}_1)$.

Proposition 3.9 *Let $\mathcal{S}$ be an operator system and $\mathcal{J}$ a null subspace of $\mathcal{S}$. Then $\mathcal{J}$ is a completely proximinal kernel.*

Proof: Firstly we will show that $\mathcal{J}$ is a proximinal kernel in $\mathcal{S}$. This will be done by induction. If $\mathcal{J} = \text{span}\{y\}$, where y is a self-adjoint element of $\mathcal{S}$ then Lemma 3.7 proves the point. Suppose that the statement holds for every null-subspace of $\mathcal{S}$ generated by n self-adjoint elements of $\mathcal{S}$ and let $\mathcal{J}$ be a null-subspace of $\mathcal{S}$ generated by $n+1$ self-adjoint elements. Then $\mathcal{J} = \text{span}\{y_1,\dots,y_n,y_{n+1}\}$ where every $y_i,\ 1 \le i \le n+1$, is self-adjoint. Set $\mathcal{J}_0 = \text{span}\{y_1,\dots,y_n\}$ then $\mathcal{J}_0$ is a n-dimensional null-subspace of $\mathcal{S}$ so from the induction hypothesis it is a proximinal kernel in $\mathcal{S}$.

Claim 1: The element $y_{n+1}+\mathcal{J}_0$ of $\mathcal{S}/\mathcal{J}_0$ is self-adjoint and is neither positive nor negative.
Proof of Claim 1: It is obviously self-adjoint (from the way we defined the involution on the quotient). Now assume that it is positive, i.e., $y_{n+1}+\mathcal{J}_0 \in C_1(\mathcal{S}/\mathcal{J}_0) = D_1(\mathcal{S}/\mathcal{J}_0) = \{s+\mathcal{J}_0 : s \in \mathcal{S}^+\}$ ($\mathcal{J}_0$ is proximinal). Thus there exists $x \in \mathcal{S}^+$ such that $y_{n+1}+\mathcal{J}_0 = x+\mathcal{J}_0$. Hence, $x - y_{n+1} \in \mathcal{J}_0 = \text{span}\{y_1,\dots,y_n\}$, so $\exists\lambda_i \in \mathbb{C},\ 1 \le i \le n$ such that $x - y_{n+1} = \sum_{i=1}^n \lambda_i y_i$. It follows that $x \in \mathcal{J}$, which means that we have found a positive element in $\mathcal{J}$, however $\mathcal{J}$ contains no other positive

element except from zero. Consequently, $x = 0$. Therefore $y_{n+1} + \mathcal{J}_0 = 0 + \mathcal{J}_0$, so $y_{n+1} \in \mathcal{J}_0$ which is a contradiction. The fact that y_{n+1} cannot be negative is proven in a similar way.
It follows from the above claim that span $\{y_{n+1} + \mathcal{J}_0\}$ is the linear span of a self-adjoint element of $\mathcal{S}/\mathcal{J}_0$ which in neither positive nor negative and thus from Lemma 3.7 we have that it is a proximinal kernel in $\mathcal{S}/\mathcal{J}_0$.

We set $K :=$ span $\{y_{n+1} + \mathcal{J}_0\}$ and consider the following quotient maps:

$$\mathcal{S} \xrightarrow{q_0} \mathcal{S}/\mathcal{J}_0 \xrightarrow{q_1} (\mathcal{S}/\mathcal{J}_0)/K$$

For the map $q := q_1 \circ q_0$ we have that:

$$\begin{aligned}\ker q &= \{s \in \mathcal{S} : (q_1 \circ q_0)(s) = 0_{(\mathcal{S}/\mathcal{J}_0)/K}\} = \{s \in \mathcal{S} : q_1(s + \mathcal{J}_o) = 0_{\mathcal{S}/\mathcal{J}_0} + K\} \\ &= \{s \in \mathcal{S} : (s + \mathcal{J}_0) + K = 0 + \mathcal{J}_0 + K\} = \{s \in \mathcal{S} : s + \mathcal{J}_0 \in K\} = \mathcal{J}\end{aligned}$$

Since $\mathcal{J}_0$ is a proximinal kernel in $\mathcal{S}$ and K is a proximinal kernel in $\mathcal{S}/\mathcal{J}_0$, using Lemma 3.8 we have that

$$D_1(\mathcal{S}/\mathcal{J}) = D_1((\mathcal{S}/\mathcal{J}_0)/K) = C_1((\mathcal{S}/\mathcal{J}_0)/K) = C_1(\mathcal{S}/\mathcal{J})$$

We conclude that $\mathcal{J}$ is indeed a proximinal kernel in $\mathcal{S}$.
In order to show that $\mathcal{J}$ is completely order proximinal we will use the identification $M_n(\mathcal{S}/\mathcal{J}) = M_n(\mathcal{S})/M_n(\mathcal{J})$.

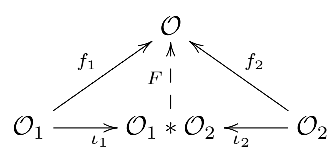
Claim 2: $M_n(\mathcal{J})$ is a null-subspace of $M_n(\mathcal{S})$.

Proof of the Claim 2: $M_n(\mathcal{J})$ is clearly a $*$-closed subspace of $M_n(\mathcal{S})$. Suppose that there exist non-zero positive elements in $M_n(\mathcal{J})$and let $[j_{kl}]$ be one of them. Then for every unital (completely) positive map $\phi : \mathcal{S} \to \mathbb{C}$ we would have that $\phi^n([j_{kl}]) = [\phi(j_{kl})] \in M_n^+$, i.e., the matrix $[\phi(j_{kl})]$ would be positive semi-definite. Thus, all of its diagonal entries would be positive or zero. In the scenario where all the diagonal entries are zero then the matrix would be the zero-matrix which in turn would imply that $[j_{kl}] = 0$ in $M_n(\mathcal{J})$ which contradicts our hypothesis. We conclude that for every state ϕ on $\mathcal{S}$ the matrix $[\phi(j_{kl})]$ must have some positive (non-zero) diagonal entry. More specifically, there would be some $k_1 \in \{1, \ldots, n\}$ such that for every $\phi : \mathcal{S} \to \mathbb{C}$ completely positive, $\phi(j_{k_1k_1}) \geq 0$ and it would be non-zero for at least one of these ϕ. Since $\mathcal{S}$ is an operator system (it has an Arch. order unit) this would mean that $j_{k_1k_1}$ is a non-zero positive element of $\mathcal{J}$ which is a contradiction.

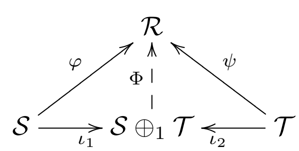
Finally, from the first part of the proof and claim 2 it is immediate that $\mathcal{J}$ is a completely order proximinal kernel in $\mathcal{S}$.

4 The Coproduct

In category theory the coproduct of two objects $\mathcal{O}_1$ and $\mathcal{O}_2$ in a category is: another object (in the same category) denoted $\mathcal{O}_1 * \mathcal{O}_2$ together with morphisms of this category $\iota_1 : \mathcal{O}_1 \to \mathcal{O}_1 * \mathcal{O}_2$ and $\iota_2 : \mathcal{O}_2 \to \mathcal{O}_1 * \mathcal{O}_2$, satisfying the following universal property:
If $f_1 : \mathcal{O}_1 \to \mathcal{O}$ and $f_2 : \mathcal{O}_2 \to \mathcal{O}$ are morphisms then there exists a unique morphism $F : \mathcal{O}_1 * \mathcal{O}_2 \to \mathcal{O}$ such that $F \circ \iota_1 = f_1$ and $F \circ \iota_2 = f_2$. In other words we have the following commuting diagram,



In the category of operator systems the morhpisms are the UCP maps.
Given two operator systems $\mathcal{S}$ and $\mathcal{T}$ their coproduct denoted by $\mathcal{S} \oplus_1 \mathcal{T}$ is an operator system together with UCP maps $\iota_1 : \mathcal{S} \to \mathcal{S} \oplus_1 \mathcal{T}$ and $\iota_2 : \mathcal{S} \to \mathcal{S} \oplus_1 \mathcal{T}$ satisfying the following: If $\mathcal{R}$ is an operator system and, $\varphi : \mathcal{S} \to \mathcal{R}$ and $\psi : \mathcal{T} \to \mathcal{R}$ are UCP maps then there exists a unique UCP map $\Phi : \mathcal{S} \oplus_1 \mathcal{T} \to \mathcal{R}$ such that $\Phi \circ \iota_1 = \varphi$ and $\Phi \circ \iota_2 = \psi$, i.e.,



is a commuting diagram.

We will construct this object with the help of operator system quotients.
The construction shown below is presented in Section 8 of [11], for a different construction see Section 3 of [9].

Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be two operator systems. Consider their direct sum $\mathcal{S} \oplus \mathcal{T} \subset B(H_1) \oplus B(H_2)$, for some Hilbert spaces $H_1,\ H_2$, this is an operator system in a canonical way with unit $e_1 \oplus e_2$. The element $(e_1, -e_2) := e_1 \oplus (-e_2)$ is self-adjoint and neither positive nor negative. It follows from Lemma 3.7 that $\mathcal{J} = \text{span}\{(e_1, -e_2)\} = \{\lambda(e_1 \oplus (-e_2)) : \lambda \in \mathbb{C}\}$ is a proximinal kernel in $\mathcal{S} \oplus \mathcal{T}$. In particular it is a null-subspace of $\mathcal{S}$, so using Proposition 3.9 we see that it is a completely order proximinal kernel. Thus we obtain the quotient operator system $\mathcal{S} \oplus \mathcal{T}/\mathcal{J}$. We shall show that this quotient equipped with the maps $\iota_1 : \mathcal{S} \to \mathcal{S} \oplus \mathcal{T}/\mathcal{J} : s \to (2s, 0) + \mathcal{J}$ and $\iota_2 : \mathcal{T} \to \mathcal{S} \oplus \mathcal{T}/\mathcal{J} : t \to (0, 2t) + \mathcal{J}$ satisfies the universal property of the coproduct.

Note: If $H_1,\ H_2$ are Hilbert spaces and $T_1 \oplus T_2 \in (B(H_1) \oplus B(H_2))^+$, then $T_1 \oplus T_2 = (A \oplus B)(A \oplus B)^*$, for some $A \in B(H_1)$ and $B \in B(H_2)$. Thus, $T_1 \oplus T_2 = AA^* \oplus BB^*$ and so $T_1 = AA^* \in B(H_1)^+$ and $T_2 = BB^* \in B(H_2)^+$.

Firstly, we will show that ι_1 and ι_2 are complete order isomorphisms.

Indeed, ι_1 is unital since $\iota_1(e_1) = (2e_1, 0) + \mathcal{J} = (e_1, e_2) + \mathcal{J}$ where the last equality stems from the fact that $(-e_1, e_2) \in \mathcal{J}$. Furthermore, it is completely positive because it can be written as a composition of CP maps, specifically $\iota_1 = q \circ f$ where $q : \mathcal{S} \oplus \mathcal{T} \to \mathcal{S} \oplus \mathcal{T}/\mathcal{J}$ denotes the quotient map and $f : \mathcal{S} \to \mathcal{S} \oplus \mathcal{T}$ is the map given by $f(s) = (2s, 0)$. It remains to show that ι_1^{-1} is CP. For this it suffices to show that whenever we have $[s_{ij}] \in M_n(\mathcal{S})$ such that $\iota_1^n([s_{ij}]) = [(2s_{ij}, 0) + \mathcal{J}]$ is a positive element of $M_n(\mathcal{S} \oplus \mathcal{T}/\mathcal{J})$ then necessarily $[s_{ij}]$ is positive.

If $[(2s_{ij}, 0) + \mathcal{J}] \in M_n(\mathcal{S} \oplus \mathcal{T}/\mathcal{J})^+$ from the way the positivity in the quotient is defined we have that there exist scalars $\alpha_{ij} \in \mathbb{C}$ such that

$$[(2s_{ij}, 0)+\alpha_{ij}(e_1, -e_2)]_{ij} \in M_n(\mathcal{S}\oplus\mathcal{T})^+, \text{ i.e., } [(2s_{ij}+\alpha_{ij}e_1, -\alpha_{ij}e_2)]_{ij} \in M_n(\mathcal{S}\oplus\mathcal{T})^+$$

From this and the previous Note we obtain that $[-\alpha_{ij}e_2] \in M_n(\mathcal{T})^+$ and consequently $[-\alpha_{ij}e_1] \in M_n(\mathcal{S})^+$ (α_{ij} are scalars). Hence, since $M_n(\mathcal{S})^+$ is a cone

$$[s_{ij}] = [s_{ij} + \alpha_{ij}e_1] + [-\alpha_{ij}e_1] \in M_n(\mathcal{S})^+$$

It follows that ι_1 is a complete order isomorphism and in a similar way one can show that this is also true for ι_2.

We conclude from the above that both $\iota_1 : \mathcal{S} \to \mathcal{S}\oplus\mathcal{T}/\mathcal{J}$ and $\iota_2 : \mathcal{T} \to \mathcal{S}\oplus\mathcal{T}/\mathcal{J}$ are complete order isomorphisms.

Now assume that $(\mathcal{R}, e_{\mathcal{R}})$ is an operator system and $\varphi : \mathcal{S} \to \mathcal{R}$, $\psi : \mathcal{T} \to \mathcal{R}$ are UCP maps. Consider the map $\Phi : \mathcal{S} \oplus \mathcal{T}/\mathcal{J} \to \mathcal{R}$ given by the formula $\Phi((s, t) + \mathcal{J}) = \frac{\varphi(s)+\psi(t)}{2}$, it will be CP because φ and ψ are CP. Moreover, notice that

(i) $\Phi((e_1, e_2) + \mathcal{J}) = \frac{\varphi(e_1)+\psi(e_2)}{2} = \frac{e_{\mathcal{R}}+e_{\mathcal{R}}}{2} = e_{\mathcal{R}}$

(ii) $(\Phi \circ \iota_1)(s) = \Phi((2s, 0) + \mathcal{J}) = \frac{\varphi(2s)+\psi(0)}{2} = \frac{2\varphi(s)+0}{2} = \varphi(s),\ \forall s \in \mathcal{S}$

(iii) as above $(\Phi \circ \iota_2)(t) = \psi(t),\ \forall t \in \mathcal{T}$

Thus, Φ is a UCP map with $\Phi \circ \iota_1 = \varphi$ and $\Phi \circ \iota_2 = \psi$ which implies that $\mathcal{S} \oplus \mathcal{T}/\operatorname{span}\{(e_1, -e_2)\}$ satisfies the universal property of the coproduct.

Subsequently, we have that

$$\mathcal{S} \oplus \mathcal{T}/\operatorname{span}\{(e_1, -e_2)\} = \mathcal{S} \oplus_1 \mathcal{T}.$$

Note: For operator systems $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ we have that in the coproduct $\mathcal{S} \oplus_1 \mathcal{T}$ their units coincide. Indeed,

$$(2e_1, 0) + \mathcal{J} = (e_1, e_2) + (e_1, -e_2) + \mathcal{J} = (e_1, e_2) + \mathcal{J} =$$

$$(e_1, e_2) + (-e_1, e_2) + \mathcal{J} = (0, 2e_2) + \mathcal{J}$$

Remark (i): The category with objects unital C^*-algebras and morphisms the $*$-homomorphisms also admits a coproduct, the free product amalgamated over the unit. If $\mathcal{A}_1,\ \mathcal{A}_2$ are unital C^*-algebras their free product amalgamated over the unit, denoted $\mathcal{A}_1 *_1 \mathcal{A}_2$, is a C^*-algebra equipped with inclusions

$$\iota_j : \mathcal{A}_j \to \mathcal{A}_1 *_1 \mathcal{A}_2,\ j = 1,\ 2$$

satisfying the following: If $\pi_i : \mathcal{A}_i \to B(H),\ i = 1,\ 2$ are $*$-homomorphisms then there exists a (unique) $*$-homomorphism $\pi : \mathcal{A}_1 *_1 \mathcal{A}_2 \to B(H)$ with $\pi \circ \iota_j = \pi_j,\ j = 1,\ 2$.

Remark (ii): The coproduct of operator systems (respectively C^*-algebras) can be extended in the obvious way to the case in which we have more than two terms.
In that case we would take for kernel

$$\mathcal{J} = \{(e, -e, 0, \dots, 0), (e, 0, -e, 0, \dots, 0), \dots, (e, 0, \dots, 0, -e)\}$$

(here e denotes the unit of the corresponding operator system)

In order to prove some of our next results we will invoke the following theorem found in [1] (for another proof see [5]).

Theorem 4.1 *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be unital C^*-algebras and $\varphi_i : \mathcal{A}_i \to B(H)$ be unital completely positive maps, $1 \leq i \leq n$. Then there exists a unital completely positive map $\varphi : \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n \to B(H)$ whose restriction to each $\mathcal{A}_i$ is φ_i.*

Theorem 4.2 *Let $n \in \mathbb{N}$ and $\mathcal{S}_i$ be an operator subsystem of a C^*-algebra $\mathcal{A}_i$ for $1 \leq i \leq n$. Set*

$$\mathcal{S} = \text{span}\,\{s_1 + \cdots + s_n : s_i \in \mathcal{S}_i,\ 1 \leq i \leq n\} \subseteq \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n$$

Then the canonical map $\mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_n \hookrightarrow \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n$ arising from the inclusions $i_k : \mathcal{S}_k \to \mathcal{A}_k$, $k = 1, \dots, n$ is a unital complete order embedding with image $\mathcal{S}$.

Proof: We will show that $\mathcal{S}$ satisfies the universal property of the coproduct.

Suppose that $\mathcal{T} \subseteq B(H)$ is an operator system and $\varphi_m : \mathcal{S}_m \to \mathcal{T}$ is a UCP map, for $m = 1, \dots, n$. Let $\tilde{\varphi}_m : \mathcal{A}_m \to B(H)$ be a unital completely positive extension of φ_m (obtained by Arveson's theorem), using theorem 4.1 we obtain a unital completely positive map $\varphi : \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n \to B(H)$ such that $\varphi \restriction_{\mathcal{A}_m} = \tilde{\varphi}_m$. Let $s \in \mathcal{S}$ then, $s = \sum_{i=1}^n \lambda_i s_i$ for some $\lambda_i \in \mathbb{C}$ and $s_i \in \mathcal{S}_i$ so

$$\varphi(s) = \sum_{i=1}^n \lambda_i \varphi(s_i) = \sum_{i=1}^n \lambda_i \tilde{\varphi}_i(s_i) = \sum_{i=1}^n \lambda_i \varphi_i(s_i) \in \mathcal{T}$$

Now it follows that for the map $\Phi := \varphi \restriction_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$ we have that, $\Phi \restriction_{\mathcal{S}_m} = \varphi_m$, i.e., $\mathcal{S}$ satisfies the desired universal property.

Remark: Using theorem 4.2 we see that if $\mathcal{A}_i$, $1 \leq i \leq n$, are unital C^*-algebras then the operator subsystem

$$\mathcal{A}_1 + \cdots + \mathcal{A}_n \subseteq \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n$$

is complete order isomorphic to the free product $\mathcal{A}_1 \oplus_1 \cdots \oplus_1 \mathcal{A}_n$.

Proposition 4.3 *Let $(\mathcal{S}_1, e_1), (\mathcal{S}_2, e_2), \ldots, (\mathcal{S}_n, e_n)$ be finite dimensional operator systems. Then up to a (canonical) complete order isomorphism:*

$$(\mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n)^d = \{\varphi_1 \oplus \cdots \oplus \varphi_n \in \mathcal{S}_1^d \oplus \cdots \oplus \mathcal{S}_n^d : \varphi_1(e_1) = \cdots = \varphi_n(e_n)\}$$

Moreover, $(\mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n)^d$ is completely order isomorphic to a subspace of $(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n)^d$.

Proof: Firstly, notice that the quotient map

$$q : \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n \to \mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n$$

is clearly a complete quotient map between operator systems, i.e., the induced map $\tilde{q}$ is a complete order isomorphism. Thus from theorem 3.6 the dual map

$$q^d : (\mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n)^d \to (\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n)^d$$

is a complete order embedding. This proves the second part.

Moreover, since $(\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n)^d = \mathcal{S}_1^d \oplus \mathcal{S}_2^d \oplus \cdots \oplus \mathcal{S}_n^d$, by identifying $(\mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n)^d$ with its image under q^d we have that

$$(\mathcal{S}_1 \oplus_1 \mathcal{S}_2 \oplus_1 \cdots \oplus_1 \mathcal{S}_n)^d \subseteq (\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_n)^d = \mathcal{S}_1^d \oplus \mathcal{S}_2^d \oplus \cdots \oplus \mathcal{S}_n^d$$

For the proof of first part we will discuss the case in which we have two operator systems $(\mathcal{S}_1, e_1)$ and $(\mathcal{S}_2, e_2)$ the general case can be proved in a similar way.

Set $\mathcal{J} = \{\lambda(e_1 \oplus (-e_2)) : \lambda \in \mathbb{C}\}$, then $\mathcal{S}_1 \oplus_1 \mathcal{S}_2 = \mathcal{S}_1 \oplus \mathcal{S}_2 / \mathcal{J}$ and let

$$\mathcal{J}^0 = \{\varphi \in (\mathcal{S}_1 \oplus \mathcal{S}_2)^d : \varphi \restriction_{\mathcal{J}} = 0\}$$

Suppose that $\phi \in \mathcal{J}^0$. Since $(\mathcal{S}_1 \oplus \mathcal{S}_2)^d = \mathcal{S}_1^d \oplus \mathcal{S}_2^d$, ϕ can be written as $\phi = \phi_1 \oplus \phi_2$ where $\phi_i \in \mathcal{S}_i,\ i = 1, 2$.
Furthermore, $\phi \circ j_1 = \phi_1$ and $\phi \circ j_2 = \phi_2$ where $j_1 : \mathcal{S}_1 \to \mathcal{S}_1 \oplus \mathcal{S}_2 : s_1 \to (s_1, 0)$ and $j_2 : \mathcal{S}_2 \to \mathcal{S}_1 \oplus \mathcal{S}_2 : s_2 \to (0, s_2)$ are the natural embeddings. Observe that $\varphi \restriction_{\mathcal{J}} = 0$ means that:

$$\begin{aligned}
&\varphi(\lambda(e_1 \oplus (-e_2))) = 0 \in S_1 \oplus S_2,\ \forall \lambda \in \mathbb{C} \\
&\iff \varphi(e_1 \oplus (-e_2)) = 0 \\
&\iff \varphi(e_1 \oplus 0 + 0 \oplus (-e_2)) = 0 \\
&\iff \varphi(e_1 \oplus 0) + \varphi(0 \oplus (-e_2)) = 0 \\
&\iff \varphi(e_1 \oplus 0) = \varphi(0 \oplus e_2)
\end{aligned}$$

Thus,

$$\phi_1(e_1) = (\phi \circ j_1)(e_1) = \phi(e_1, 0) = \phi(0, e_2) = (\phi \circ j_2)(e_2) = \phi_2(e_2)$$

Consequently,

$$\mathcal{J}^0 = \{\varphi_1 \oplus \varphi_2 \in \mathcal{S}_1^d \oplus \mathcal{S}_2^d : \varphi_1(e_1) = \varphi_2(e_2)\}$$

Now we will show that the map $L : \mathcal{J}^0 \to (\mathcal{S}_1 \oplus \mathcal{S}_2/\mathcal{J})^d$ with $L(f) = \hat{f}$, where $\hat{f}((s_1, s_2) + \mathcal{J}) = f((s_1, s_2))$, is a complete order isomorphism.

Firstly, we see that $\hat{f}$ is well-defined because whenever $(s_1', s_2') + \mathcal{J} = (s_1, s_2) + \mathcal{J}$ then $(s_1', s_2') - (s_1, s_2) \in \mathcal{J}$ so $f((s_1', s_2') - (s_1, s_2)) = 0$, which implies that $\hat{f}((s_1', s_2') + \mathcal{J}) = \hat{f}((s_1, s_2) + \mathcal{J})$.
Moreover notice that for every $f \in \mathcal{J}^0$, $\ker f \subseteq \mathcal{J}$ and $\hat{f} \circ q = f$, so it follows from Proposition 3.4 that L is a complete order isomorphism.

L is injective: It can be easily checked that $\ker L = \{0\}$

L is surjective: Let $\psi \in (\mathcal{S}_1 \oplus_1 \mathcal{S}_2)^d = (\mathcal{S}_1 \oplus \mathcal{S}_2/\mathcal{J})^d$ then $q^d(\psi) \in (\mathcal{S}_1 \oplus \mathcal{S}_2)^d$ and $q^d(\psi) \restriction_{\mathcal{J}} = 0$. Thus $q^d(\psi) \in \mathcal{J}^0$ and for every $(s_1, s_2) + \mathcal{J} \in \mathcal{S}_1 \oplus \mathcal{S}_2/\mathcal{J}$ we have that

$$L(q^d(\psi))((s_1, s_2) + \mathcal{J}) = q^d(\psi)((s_1, s_2)) = \psi(q((s_1, s_2))) = \psi((s_1, s_2) + \mathcal{J})$$

Hence $L(q^d(\psi)) = \psi$.

We conclude that

$$(\mathcal{S}_1 \oplus_1 \mathcal{S}_2)^d = \{\varphi_1 \oplus \varphi_2 \in \mathcal{S}_1^d \oplus \mathcal{S}_2^d : \varphi_1(e_1) = \varphi_2(e_2)\}$$

Remark: Suppose that $\mathcal{S}$ and $\mathcal{T}$ are operator systems and $\mathcal{S}_0 \subseteq \mathcal{S}$ and $\mathcal{T}_0 \subseteq \mathcal{T}$ are operator subsystems, then the identity map

$$id : \mathcal{S}_0 \otimes_c \mathcal{T}_0 \to \mathcal{S} \otimes_c \mathcal{T}$$

is a completely positive.

Indeed, let $(\phi, \psi) \in ucp(\mathcal{S}, \mathcal{T})$ and $v \in M_n(\mathcal{S}_0 \otimes_c \mathcal{T}_0)^+$. If $\phi_0 = \phi \restriction_{\mathcal{S}_0}$ and $\psi_0 = \psi \restriction_{\mathcal{T}_0}$ then $(\phi_0, \psi_0) \in ucp(\mathcal{S}_o, \mathcal{T}_0)$ and

$$(\phi \cdot \psi)^n(v) = (\phi_0 \cdot \psi_0)^n(v) \in M_n((B(H))^+.$$

This map can sometimes be a complete order embedding as shown in Lemma 2.6 of [17]:

Theorem 4.4 *[17] Let $\mathcal{A}_i$, $i = 1, \ldots, n$, and $\mathcal{B}_j$, $j = 1, \ldots, m$ be unital C^*-algebras. Set $\mathcal{S} = \mathcal{A}_1 \oplus_1 \cdots \oplus_1 \mathcal{A}_n$, $\mathcal{T} = \mathcal{B}_1 \oplus_1 \cdots \oplus_1 \mathcal{B}_m$, $\mathcal{A} = \mathcal{A}_1 *_1 \cdots *_1 \mathcal{A}_n$ and $\mathcal{B} = \mathcal{B}_1 *_1 \cdots *_1 \mathcal{B}_m$. Then, the inclusion of $\mathcal{S} \otimes_c \mathcal{T}$ into $\mathcal{A} \otimes_{max} \mathcal{B}$ is a complete order isomorphism onto its range, i.e.:*

$$\mathcal{S} \otimes_c \mathcal{T} \subseteq_{c.o.i.} \mathcal{A} \otimes_{max} \mathcal{B}$$

5 Quantum Correlations

Quantum mechanics is a mathematical framework used for the development of physical theories that attempt to describe the universe in a subatomic scale. In what follows we state some of the basic postulates of quantum mechanics. For more information on the topic of quantum mechanics the reader is advised to see [14].

Postulate I : To each isolated physical system, there corresponds a (complex) Hilbert space H, called the **state space**. Every unit vector in H represents a possible state, called **state vector** or **pure state**. The system is completely described by its state vector.

The first Postulate tells us that the state space of a quantum system is described by a (complex) Hilbert space. However, it neither tells us which Hilbert space corresponds to a given physical system nor what the state vector of the system is. More often than not figuring out these facts is quite difficult.
We will focus on the study of quantum systems which are not closed (they interact with the environment). In particular in our scenarios there will be ”observers” conducting measurements on the systems. The next Postulate tells us in what way these measurements affect the system.

Postulate II : Quantum measurements are always described by a class of operators $\{M_i\}_{i\in J}$, where J is the set of all possible outcomes.
The probability that we observe outcome i, when the system is in a state y is given by $p_i = \|M_i y\|^2$ and if we observe outcome i then the system changes to the state $\frac{M_i y}{\|M_i y\|}$.
The measurement operators satisfy the so called completeness equation :
$\sum_{i\in J} M_i^* M_i = I$, where I denotes the identity operator of the state space.
The completeness equation expresses the fact that the sum of all the probabilities of all possible outcomes must be 1 ($\sum_{i\in J} p_i = 1$).

Remark : The completeness equation need not necessarily be included in Postulate II as it can be derived from the fact that the sum of all the probabilities of all possible outcomes must be 1.
Indeed, let $m \in \mathbb{N}$ and consider a quantum experiment with at most m possible outcomes and let H denote the state space of the system. Now let $\{M_i\}_{i=1}^m$ be a family of operators where, $M_i \in B(H), \forall i \in \{1, \ldots, m\}$. Suppose that before the measurement the system is at a state y, where y is a unit vector in H. Then,

$$1 = \sum_{i=1}^m p_i = \sum_{i=1}^m \|M_i y\|^2 = \sum_{i=1}^m (M_i y, M_i y) = \sum_{i=1}^m (M_i^* M_i y, y) = \left(\sum_{i=1}^m M_i^* M_i y, y\right)$$

Since the above equality holds for every unit vector $y \in H_1$, it follows that:

$$\sum_{i=1}^m M_i^* M_i = I$$

The following Definition stems form the above Postulate.

Definition 5.1 *Let H_1 and H_2 be finite dimensional Hilbert spaces. A finite family of operators $\{M_i : 1 \leq i \leq k\}$, with $M_i : H_1 \to H_2$ is called a **measurement system** if $\sum_{i=1}^{k} M_i^* M_i = I$.*
If $H_1 = H_2$ then we call $\{M_i\}_i$ a measurement system on H_1.

In one hand, Postulate II gives us a rule which determines the respective probabilities of the different possible measurement outcomes. On the other hand, it also describes the state of the system after the measurement. However, here we are mostly interested in the former, i.e. in the probabilities of the respective outcomes. A mathematical tool which is extremely useful in such instances is the Positive Operator Valued Measures (POVM's for short).

Definition 5.2 *Let H be a Hilbert space and $k \in \mathbb{N}$, a family $\{P_i\}_{i=1}^{K}$ of operators on H is called a (K-outcome) **positive operator-valued measure or POVM** for short if:*

1. *for each i, P_i is a positive operator $((P_i h, h) \geq 0, \ \forall h \in H)$*
2. *$\sum_{i=1}^{K} P_i = I_H$, where I_H is the identity operator on H.*

Remark : Whenever we have a measurement system $\{M_i\}_i$ on some Hilbert space H which is a state space of some system, then there exists a POVM $\{P_i\}_i$ on H such that

$$p_i = (P_i y, y)$$

where as before p_i is the probability to observe outcome i when the system is in a state y.
To see this, set $P_i = M_i^* M_i$ for every i.

When $H = \mathbb{C}^n$ is finite dimensional, we identify the operators acting on H with the elements of the algebra M_n of $n \times n$-matrices, via the following process:
Let $\{e_i\}_{i=1}^{n}$ be an orthogonal basis for H we define for each $T \in \mathcal{L}(H)$ a $n \times n$-matrix A given by: $A = [(Te_j, e_i)]_{i,j}$.
Thus we can consider the POVM's $\{P_i\}_{i=1}^{n}$ acting on $H = \mathbb{C}^n$ as a subset of M_n.

Definition 5.3 *Let H be a Hilbert space, a family $\{R_i\}_{i=1}^{K}$ of orthogonal projections (i.e., $R_i = R_i^2 = R_i^*, \ \forall i$) on H is called a (K-outcome) **projection-valued measure or PVM** for short if: $\sum_{i=1}^{K} R_i = I_H$.*

Clearly every PVM is a POVM. As we will see in the discussion that follows it is also true that every POVM dilates to a PVM.

Theorem 5.4 *Let $\{P_i\}_{i=1}^{K}$ be a POVM on a Hilbert space H. Then there exist a PVM $\{R_i\}_{i=1}^{K}$ on $H \otimes \mathbb{C}^K$ and an isometry $V : H \to H \otimes \mathbb{C}^K$ such that $P_i = V^* R_i V$, for all $1 \leq i \leq K$.*

Proof: We identify $H \otimes \mathbb{C}^K$ with the direct sum of K-copies of H, $H \oplus \cdots \oplus H = H^K$ (via the identification $\sum_{i=1}^K h_i \otimes e_i \longleftrightarrow (h_1, \ldots, h_n)$, where $\{e_i\}$ is the standard basis of $\mathbb{C}^K$).
Let $E_{i,j}$ be the $K \times K$ matrix with 1 in the (i,j) position and zeroes elsewhere and set $R_i = I_H \otimes E_{ii}$, this will be a $K \times K$ matrix with I_H on the (i,i)-position and zeroes elsewhere. Thus,

$$R_i \left(\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ h_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

That is, each $R_i : H \otimes \mathbb{C}^K \to H \otimes \mathbb{C}^K$ is the projection on the i-th copy of H in $H \otimes \mathbb{C}^K = H^K$. Obviously, $R_i = R_i^* = R_i^2$ and $\sum_{i=1}^K R_i = I_H \otimes I_K = I_{H \otimes \mathbb{C}^K}$. Hence, $\{R_i\}_{i=1}^K$ is a PVM on $H \otimes \mathbb{C}^K$. Now, define a map $V : H \to H \otimes \mathbb{C}^K$ by

$$V(h) = \sum_{i=1}^K P_i^{\frac{1}{2}}(h) \otimes e_i = \begin{bmatrix} P_1^{\frac{1}{2}}(h) \\ \vdots \\ P_K^{\frac{1}{2}}(h) \end{bmatrix}$$

This map is linear and an isometry. Indeed,

$$\|V(h)\|^2 = (V(h), V(h))_{H \otimes \mathbb{C}^K} =$$

$$\sum_{i=1}^K \left(P_i^{\frac{1}{2}}(h), P_i^{\frac{1}{2}}(h) \right)_H (e_i, e_i)_{\mathbb{C}^K} \overset{P_i = P_i^*}{=} \sum_{i=1}^K (h, P_i h) = (h, h) = \|h\|^2$$

Finally, notice that for every $1 \leq j \leq K$:

$$\begin{aligned}(V^* R_j V h, h)_H = \left(R_j \left(\begin{bmatrix} P_1^{\frac{1}{2}} h \\ \vdots \\ P_K^{\frac{1}{2}} h \end{bmatrix} \right), Vh \right)_{H \otimes \mathbb{C}^K} &= \left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_j^{\frac{1}{2}} h \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} P_1^{\frac{1}{2}}(h) \\ \vdots \\ P_j^{\frac{1}{2}} h \\ \vdots \\ P_K^{\frac{1}{2}}(h) \end{bmatrix} \right) \\ &= \left(P_j^{\frac{1}{2}} h, P_j^{\frac{1}{2}} h \right) \\ &= (P_j h, h)_H \end{aligned}$$

which implies that $P_j = V^* R_j V$.

Remark: We have shown that for $h \in H$ if we set $\tilde{h} = Vh$ then,

$$(P_i h, h)_H = (V^* R_i V h, h)_H = (R_i(Vh), Vh)_{H \otimes \mathbb{C}^K} = \left(R_i \tilde{h}, \tilde{h}\right)$$

and since V is an isometry $\|h\| = \|\tilde{h}\|$.
So, to sum up, whenever we are given a POVM $\{P_i\}_{i=1}^n$ on a Hilbert space H and $h \in H$, we can always dilate the POVM to a PVM $\{R_i\}_i$ via an isometry $V : H \to H \otimes \mathbb{C}^n$.

The elements of a POVM are not necessarily orthogonal projections, so the number of operators in a POVM can be larger than the dimension of the Hilbert space they act on. However, since a PVM consist of projections summing up to the identity their ranges are pairwise orthogonal, that is, $R_i R_j = 0 = R_j R_i$, for $i \neq j$.
(actually for the above claim to be true it suffices to have that their sum is less than the identity)
Indeed, suppose that $\sum_{m=1}^k R_m \leq I$ and fix i, j such that $i \neq j$ then

$$\sum_{m \neq i}^{k} R_m \leq I - R_i$$

Therefore,

$$0 \leq R_i R_j R_i \leq R_i (\sum_{m \neq i}^{k} R_m) R_i$$

$$R_i (I - R_i) R_i = R_i - R_i^2 = 0$$

Thus, $R_i R_j R_i = 0$. But, then

$$0 = R_i R_j^2 R_i \overset{R_i^* = R_i}{=} R_i^* R_j^* R_j R_i = (R_j R_i)^* R_j R_i$$

which implies that $R_j R_i = 0$.

Theorem 5.4 gives a way to obtain a PVM from a POVM in the case where we have one measurement system when we have several measurement systems an analogous result is given by the following.

Theorem 5.5 *Let $\{P_{t,i}\}_{i=1}^m$ be a family of POVM's on a Hilbert H, indexed by $t \in T$ where $|T| = n < \infty$. Then there exist a Hilbert space K and a family of PVM's $\{R_{t,i}\}_{i=1}^m$ acting on K for $t \in T$ and an isometry $V : H \to K$ such that $V^* R_{t,i} V = P_{t,i}, \ \forall t, i$. Moreover, if H is finite dimensional so is K.*

Proof: We will prove it by induction to number of elements of T. The case for $n = 1$ is Theorem 5.2. Assume that it is true for $|T| = n$. Now suppose that $|T| = n + 1$. By the induction hypothesis we know that there exist a Hilbert space K_1, an isometry $V_1 : H \to K_1$ and a family of PVM's $\{R_{t,i}\}_{i=1}^m$ for $1 \leq t \leq n$ such that $V_1^* P_{t,i} V_1 =$

$R_{t,i}$, $\forall i \in \{1,\ldots,m\}$ and $\forall t \in \{1,\ldots,n\}$. At first, let $\tilde{P}_{n+1,i} = V_1 P_{n+1,i} V_1^*$. Then $\tilde{P}_{n+1,i} \geq 0$ and

$$\sum_{i=1}^{m} \tilde{P}_{n+1,i} = V_1(\sum_{i=1}^{m} P_{n+1,i})V_1^* = V_1 V_1^* \tag{$*$}$$

which is a projection (V_1 is an isometry). We want $\{\tilde{P}_{n+1,i}\}_{i=1}^m$ to be a POVM on K_1 (the elements must sum up to the identity), so we adjust $\tilde{P}_{n+1,1}$ by setting $\tilde{P}_{n+1,1} = V_1 P_{n+1,1} V_1^* + (I - V_1^* V_1)$.

Now on K_1, we have PVM's $\{R_{t,i}\}_{i=1}^m$ and a POVM $\{\tilde{P}_{n+1,i}\}_{i=1}^m = (K_1)^m$. Let $K = K_1 \otimes \mathbb{C}^m$, and define a map $V_2 : K_1 \to K$ by:

$$V_2 k = \begin{bmatrix} (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \\ \vdots \\ (\tilde{P}_{n+1,m})^{\frac{1}{2}} k \end{bmatrix}$$

Notice that:

$$\begin{aligned}
\|V_2 k\|^2 &= \left(\begin{bmatrix} (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \\ \vdots \\ (\tilde{P}_{n+1,m})^{\frac{1}{2}} k \end{bmatrix}, \begin{bmatrix} (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \\ \vdots \\ (\tilde{P}_{n+1,m})^{\frac{1}{2}} k \end{bmatrix} \right) \\
&= \sum_{i=1}^{m} \left((\tilde{P}_{n+1,i})^{\frac{1}{2}} k, (\tilde{P}_{n+1,i})^{\frac{1}{2}} k \right) \\
&= \sum_{i=1}^{m} \left(\tilde{P}_{n+1,i} k, k \right) \\
&= \sum_{i=1}^{m} (Ik, k) = \|k\|^2
\end{aligned}$$

Set $R_{n+1,i} = I_{K_1} \otimes E_{ii}$, $1 \leq i \leq m$, where E_{ii} is the $m \times m$ matrix with 1 in the (i,i)-entry and 0 elsewhere. Then $\{R_{n+1,i}\}_{i=1}^m$ is a PVM and,

$$\begin{aligned}
(V_2^* R_{n+1,i} V_2 k, k) = (R_{n+1,i} V_2 k, V_2 k) &= \left(\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & I_{K_1} & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \\ \vdots \\ (\tilde{P}_{n+1,m})^{\frac{1}{2}} k \end{bmatrix}, \begin{bmatrix} (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \\ \vdots \\ (\tilde{P}_{n+1,m})^{\frac{1}{2}} k \end{bmatrix} \right) \\
&= \left((\tilde{P}_{n+1,1})^{\frac{1}{2}} k, (\tilde{P}_{n+1,1})^{\frac{1}{2}} k \right) \\
&= \left(\tilde{P}_{n+1,1} k, k \right)
\end{aligned}$$

which implies that $V_2^* R_{n+1,i} V_2 = \tilde{P}_{n+1,1}$.
For all $1 \leq t \leq n$ we set:

$$Q_{t,j} = V_2 R_{t,j} V_2^*,\ 2 \leq j \leq m \text{ and } Q_{t,1} = V_2 R_{t,1} V_2^* + (I - V_2 V_2^*)$$

Notice that $V_2^* Q_{t,j} V_2 = R_{t,j}$, thus $Q_{t,j} = V_2 R_{t,j} V_2^*$. For $1 \leq t \leq n$, $\{Q_{t,j}\}_{j=1}^m$ are POVM's, we need to show that they are PVM's. We have that for $t = n+1$, $\{R_{n+1,i}\}_{i=1}^m$ is a PVM. For the other ones, note that for $j \geq 2$

$$Q_{t,j}^2 = (V_2 R_{t,j} V_2^*)(V_2 R_{t,j} V_2^*) \overset{V_2^* V_2 = I}{=} V_2 R_{t,j} R_{t,j} V_2^* \overset{R_{t,j}^2 = R_{t,j}}{=} V_2 R_{t,j} V_2^* = Q_{t,j}$$

Finally,

$$\begin{aligned} Q_{t,1}^2 &= (V_2 R_{t,1} V_2^* + (I - V_2 V_2^*))(V_2 R_{t,1} V_2^* + (I - V_2 V_2^*)) \\ &= V_2 R_{t,1} V_2^* + (I - V_2 V_2^*) \\ &= Q_{t,1} \end{aligned}$$

It follows that $Q_{t,j}$, $j \geq 1$ are projections. Hence $\{Q_{t,j}\}_{j=1}^m$ are PVM's, for all $1 \leq t \leq n$ and there exists an isometry $V : H \to K$ such that $V^* Q_{t,j} V = P_{t,i}$, where $V = V_2 V_1$.

A system which can be thought of as being comprised by different parts is called a composite system. We will study composite systems that are made out of two distinct physical systems. The following Postulate gives us an axiomatic mathematical description of such a system.

Postulate III : The state space of a composite physical system is the tensor product of the respective state spaces of the components of the total system. Furthermore, if we have physical systems numbered 1 through n and each of them is in a state y_i then the joint state of the total system is : $y_1 \otimes \cdots \otimes y_n$.

For example : If we have two systems modelled by Hilbert spaces H_1 and H_2 which are in states h_1 and h_2 respectively. Then the total system is modelled by the space $H_1 \otimes H_2$ and is in a state $h_1 \otimes h_2$.

Non-local Games: Consider a two-person game which is played between two players, Alice and Bob, and arbitrated by a referee R. Let I_A, I_B, $\mathcal{O}_A$, $\mathcal{O}_B$ be finite non-empty sets. The sets I_A, I_B are the sets of questions (or inputs) and the sets $\mathcal{O}_A$, $\mathcal{O}_B$ are the sets of answers (or outputs) for Alice and Bob respectively. Since this is a game it will have some rules, which are described by a function $\lambda : I_A \times I_B \times \mathcal{O}_A \times \mathcal{O}_B \to \{0, 1\}$ where,

$$\lambda(x, y, a, b) = \begin{cases} 1, & \text{means that this answer is correct} \\ 0, & \text{means that this answer is wrong} \end{cases}$$

the players and the referee are all aware of the rules (the function λ).
Alice and Bob are playing the game cooperatively against the referee and they are not allowed to communicate during the game. However, they may agree on a strategy beforehand. We denote a game with input sets I, J, output sets A, B and rule function λ, by $\mathcal{G} = (I, J, A, B, \lambda)$.

For a **one-round** game the referee gives Alice a question $x \in I_A$ and Bob a question $y \in I_B$ and the players do not know what question the other was given. Then each of the players independently (without communicating) produces outputs (answers) $a \in \mathcal{O}_A$ and $b \in \mathcal{O}_B$. They win the game if $\lambda(x, y, a, b) = 1$ and lose if $\lambda(x, y, a, b) = 0$.

A two-person **non-local game** is a game $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ together with a probability distribution $\pi : I_A \times I_B \to [0, 1]$. In a single round of the game the referee selects a pair of questions $(x, y) \in I_a \times I_B$ according to the probability distribution π and communicates x to Alice and y to Bob, then they return answers a and b respectively. The tandem Alice-Bob wins the round whenever $\lambda(x, y, a, b) = 1$ and loses the round otherwise. Moreover, each of them knows neither the question the other was given nor his/her answer. We will concern ourselves only with non-local games that involve two players, so we will drop the term "two person" and will refer to them as "non-local games" or "games".

Obviously, the probability distribution π acquires significance only when the game is played multiple times. When we are concerned with winning every round of the game independently of the chosen pair (x, y) of questions then π can be omitted all together, in these cases by game we mean the quintuple $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$.

We mentioned that Alice and Bob, although not allowed to communicate during the game, can come up with some strategy beforehand in order to win the game. A strategy can be either deterministic or probabilistic (they win the game with a certain probability).

Definition 5.6 *A* ***deterministic strategy*** *for a non-local game* $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ *is a pair of functions* (f, g) *with* $f : I_A \to \mathcal{O}_A$ *and* $g : I_B \to \mathcal{O}_B$.

We interpret a deterministic strategy as follows, we view the pair $(f(x), g(y))$ as the answers that are given by Alice and Bob to the questions (x, y).
A deterministic strategy is called perfect if it yields a win for the players independently of the choice of an input pair, that is, if

$$\lambda(x, y, f(x), g(y)) = 1,\ \forall x \in I_A \text{ and } \forall y \in I_B$$

Notice that when following deterministic strategies, given a fixed $(x, y) \in I_A \times I_B$ that appears as an input pair in two different rounds of the game, the players have to respond with the same output pair, namely $(f(x), g(y))$.

As we will see probabilistic strategies offer a significant advantage to the players compared to deterministic ones.

We say that Alice and Bob follow a probabilistic strategy when they generate outputs according to some probability distribution. Formally, a probabilistic strategy is defined as follows:

Definition 5.7 *A probabilistic strategy for a non-local game* $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ *is a family of probability distributions:*

$$p = \{(p(a, b|x, y))_{(a,b) \in \mathcal{O}_A \times \mathcal{O}_B} : (x, y) \in I_A \times I_B\}$$

If $p = \{(p(a,b \mid x,y))_{(a,b)\in\mathcal{O}_A\times\mathcal{O}_B} : (x,y) \in I_A \times I_B\}$ is a probabilistic strategy for a non-local game $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$, we interpret the value $p(a,b \mid x,y)$ as the joint conditional probability that the players will answer with the pair (a,b), if they are given an input/questions (x,y).
Therefore it is clear that for any probabilistic strategy p:

$$p(a,b \mid x,y) \geq 0 \qquad \text{and} \quad \sum_{(a,b)\in\mathcal{O}_A\times\mathcal{O}_B} p(a,b \mid x,y) = 1,\ \forall x \in I_A,\ y \in I_B$$

Henceforth any tuple $((p(a,b \mid x,y))_{a\in\mathcal{O}_A, b\in\mathcal{O}_B, x\in I_A y\in I_B}$ satisfying these conditions will be called a correlation and we shall use the terms correlation and strategy interchangeably.
Notice that any correlation $((p(a,b \mid x,y))_{a\in\mathcal{O}_A, b\in\mathcal{O}_B, x\in I_A y\in I_B}$ can be viewed as a vector in $\mathbb{R}^N$ with non-negative coordinates, where $N = |I_A||I_B||\mathcal{O}_A||\mathcal{O}_B|$.

In the scenario in which the players follow a probabilistic strategy they are in a position to vary their answers for the same pair of questions, this flexibility is what gives them the aforementioned advantage (compared to a deterministic strategy).

Definition 5.8 *A probabilistic strategy*

$$p = \{(p(a,b \mid x,y))_{(a,b)\in\mathcal{O}_A\times\mathcal{O}_B} : (x,y) \in I_A \times I_B\}$$

for a non-local game $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ is called a non-signalling (or NS) correlation if for all $x \in I_A$ and $y \in I_B$ it satisfies the following:

1. $\sum_{b'\in\mathcal{O}_B} p(a,b' \mid x,y') = \sum_{b'\in\mathcal{O}_B} p(a,b' \mid x,y''),\ y',y'' \in I_B, a \in \mathcal{O}_A$

2. $\sum_{a'\in\mathcal{O}_A} p(a',b \mid x',y) = \sum_{a'\in\mathcal{O}_A} p(a',b \mid x'',y),\ x',x'' \in I_A,\ b \in \mathcal{O}_B.$

If we have a non-signalling correlation p we let $p(a \mid x)$ (resp. $(p(b \mid y))$ denote the values obtained from the sums in 1 (resp. 2) of definition 5.8. The conditions required for a probabilistic strategy to be non-signalling are formal incarnations of the requirement that Alice and Bob do not communicate during the game. Indeed, they ensure that the conditional probability distributions $p(*,* \mid x,y)$ have well-defined (the sums in 1 and 2 do not depend on a and b, respectively) marginal distributions, namely $p(* \mid x)$ and $p(* \mid y)$. Thus, there is a well-defined probability $p(a \mid x)$ that Alice responds with an answer a given a question x, independently of what Bob's question and answer is (and similarly for Bob $\exists\ p(b \mid y)$).

We denote the set of all non-signalling correlations by $\mathcal{C}_{ns}$.

Remark: $\mathcal{C}_{ns}$ is a convex set.

Indeed, let $p_1, p_2 \in \mathcal{C}_{ns}$ and $\lambda \in [0,1]$. Then for $p = \lambda p_1 + (1-\lambda)p_2$ we have that for all $x \in I_A,\ y', y'' \in I_B, a \in \mathcal{O}_A$:

$$\begin{aligned}
\sum_{b' \in \mathcal{O}_B} p(a, b' \mid x, y') &= \sum_{b' \in \mathcal{O}_B} (\lambda p_1(a, b' \mid x, y') + (1-\lambda) p_2(a, b' \mid x, y')) \\
&= \lambda \sum_{b' \in \mathcal{O}_B} p_1(a, b' \mid x, y') + (1-\lambda) \sum_{b' \in \mathcal{O}_B} p_2(a, b' \mid x, y') \\
&= \lambda \sum_{b' \in \mathcal{O}_B} p_1(a, b' \mid x, y'') + (1-\lambda) \sum_{b' \in \mathcal{O}_B} p_2(a, b' \mid x, y'') \\
&= \sum_{b' \in \mathcal{O}_B} (\lambda p_1(a, b' \mid x, y'') + (1-\lambda) p_2(a, b' \mid x, y'')) \\
&= \sum_{b' \in \mathcal{O}_B} p(a, b' \mid x, y'')
\end{aligned}$$

where the third equality stems from the fact that $p_1, p_2 \in \mathcal{C}_{ns}$.
Similarly $\sum_{a' \in \mathcal{O}_A} p(a', b \mid x', y) = \sum_{a' \in \mathcal{O}_A} p(a', b \mid x'', y),\ x', x'' \in I_A,\ j \in I_B, b \in \mathcal{O}_B$.

Definition 5.9 *The probabilistic strategy of definition 5.8 is called perfect if*

$$\textit{whenever } \lambda(x, y, a, b) = 0, \ \textit{then } p(a, b \mid x, y) = 0$$

Clearly, if the players follow a perfect strategy they win every round of the game, in that case we say they win the game with probability one.

If (f, g) is a deterministic strategy (see 5.6) then it gives rise to the probabilistic strategy $p_{f,g}$ defined by,

$$p_{f,g}(a, b \mid x, y) = \begin{cases} 1, & \text{if } a = f(x) \text{ and } b = g(y) \\ 0, & \text{otherwise} \end{cases}$$

Another way to see this is that $p_{f,g}(a, b \mid x, y) = p_f(a \mid x) p_g(b \mid y)$, where

$$p_f(a \mid x) = \begin{cases} 1, & \text{if } a = f(x) \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad p_g(b \mid y) = \begin{cases} 1, & \text{if } b = g(y) \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that such a strategy is non-signalling.

A more general class of probabilistic strategies consists of the convex combinations of strategies of the form $p_{f,g}$.

Definition 5.10 *A non-signalling correlation*

$$p = \{(p(a, b \mid x, y))_{(a,b)\in\mathcal{O}_A\times\mathcal{O}_B} : (x, y) \in I_A \times I_B\}$$

*is called **local** if there exists families of probability distributions*

$$p_k^{(1)} = \{(p_k^{(1)}(a \mid x))_{a\in\mathcal{O}_A} : x \in I_A\} \text{ and } p_k^{(2)} = \{(p_k^{(2)}(b \mid y))_{b\in\mathcal{O}_B} : y \in I_B\}$$

as well as non-negative scalars $\lambda_k,\ k = 1, \dots, m$ such that

$$p(a, b \mid x, y) = \sum_{k=1}^{m} \lambda_k p_k^{(1)}(a \mid x) p_k^{(2)}(b \mid y),\ x \in I_A,\ y \in I_B,\ a \in \mathcal{O}_A,\ b \in \mathcal{O}_B$$

as a convex combination.

We denote the set of all local correlations by $\mathcal{C}_{loc}$.

Remark (i): Viewing a non-signalling correlation p as a vector in $\mathbb{R}^N$, where $N = |I_A||I_B||\mathcal{O}_A||\mathcal{O}_B|$, and by appealing to Caratheodory's Theorem, one can show that the set $\mathcal{C}_{loc}$ is a closed subset of $\mathbb{R}^N$ (that is, it is closed in the product topology).

Indeed, let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_{loc}$ such that $p_n \to p$ in the product topology of $\mathbb{R}^N$. We have to prove that p lies in the convex set $\mathcal{C}_{loc}$.

Since $p_n \in \mathcal{C}_{loc},\ \forall n \in \mathbb{N}$, every p_n can be written as $\sum_{i=1}^{m} \lambda_{i,n} p_{i,n}^1 \otimes p_{i,n}^2$ where $p_{i,n}^1 : I_A \times \mathcal{O}_A \to [0, 1]$ and $p_{i,n}^2 : I_B \times \mathcal{O}_B \to [0, 1]$ are functions, that is,

$$p_n(a, b \mid x, y) = \sum_{i=1}^{m} \lambda_{i,n} p_{i,n}^{(1)}(a \mid x) p_{i,n}^{(2)}(b \mid y)$$

Now, by Caratheodory's Theorem each p_n is a convex combination of at most $N + 1$ elements of the form $p^1 \otimes p^2$, i.e., for every $n \in \mathbb{N}$

$$p_n = \sum_{t=1}^{N+1} \lambda_{t,n} p_{t,n}^{(1)} \otimes p_{t,n}^{(2)}$$

Notice that these sequences live in bounded sets of finite dimensional spaces, and that there are finitely many of them ($N + 1$ at most which is a fixed number), so we may pass to subsequences in order to ensure that there are scalars $\lambda_t \in [0, 1]$ as well as functions $p_t^1 \in [0, 1]^{I_A\times\mathcal{O}_A}$ and $p_t^2 \in [0, 1]^{I_B\times\mathcal{O}_B}$, $1 \le t \le N + 1$ such that

$$\lim_n \lambda_{t,n} = \lambda_t,\ \lim_n p_{t,n}^1(a \mid x) = p_t^1(a \mid x) \text{ and } \lim_n p_{t,n}^1(b \mid y) = p_t^2(b \mid y)$$

for all $(x, y, a, b) \in I_A \times I_B \times \mathcal{O}_A \times \mathcal{O}_B$ and $t = 1, \dots, N + 1$. It follows that

$$p = \lim_n p_n = \sum_{t=1}^{N+1} \lambda_t p_t^{(1)} \otimes p_t^{(2)}$$

which lies in the convex set $\mathcal{C}_{loc}$.

Remark (ii): The extreme points of $\mathcal{C}_{loc}$ are the strategies of the form $p_{f,g}$.

Proof: Let $p \in \mathcal{C}_{loc}$ and suppose that p is not an extreme point then there would be $p_1, p_2 \in \mathcal{C}_{loc}$ with $p_1 \neq p_2$ and $\lambda \in (0,1)$ such that $p = \lambda p_1 + (1-\lambda)p_2$. Thus, we would have that for all $x \in I_A, y \in I_B, a \in \mathcal{O}_A, b \in \mathcal{O}_B$:

$$p(a,b \mid x,y) = \lambda p_1(a,b \mid x,y) + (1-\lambda)p_2(a,b \mid x,y)$$

Fix $(x,y) \in I_A \times I_B$, since $p_1 \neq p_2$ we would also have that for some $(a_1,b_1) \in \mathcal{O}_A \times \mathcal{O}_B$:

$$p_1(a_1,b_1 \mid x,y) \neq p_2(a_1,b_1 \mid x,y)$$

Assume that $p_1(a_1,b_1 \mid x,y) > p_2(a_1,b_1 \mid x,y)$ then,

$$\begin{aligned} p(a_1,b_1 \mid x,y) &= \lambda p_1(a_1,b_1 \mid x,y) + (1-\lambda)p_2(a_1,b_1 \mid x,y) \\ &> \lambda p_2(a_1,b_1 \mid x,y) + (1-\lambda)p_2(a_1,b_1 \mid x,y) \\ &= p_2(a_1,b_1 \mid x,y) \geq 0 \end{aligned}$$

and

$$\begin{aligned} p(a_1,b_1 \mid x,y) &= \lambda p_1(a_1,b_1 \mid x,y) + (1-\lambda)p_2(a_1,b_1 \mid x,y) \\ &< \lambda p_1(a_1,b_1 \mid x,y) + (1-\lambda)p_1(a_1,b_1 \mid x,y) \\ &= p_1(a_1,b_1 \mid x,y) \leq 1 \end{aligned}$$

Thus we see that $0 < p(a_1,b_1 \mid x,y) < 1$ so p does not arise from a deterministic strategy.

We will now show that if $p \in \mathcal{C}_{loc}$ arises from a deterministic strategy then it is an extreme point. Consider a correlation p in $\mathcal{C}_{loc}$ arising from a deterministic strategy. Then p is of the form $p_{f,g}$ for some functions $f : I_A \to \mathcal{O}_A$ and $f : I_B \to \mathcal{O}_B$. Suppose that p was not an extreme point then there would exist $p_1, p_2 \in \mathcal{C}_{loc}$ with $p_1 \neq p_2$ and $\lambda \in (0,1)$ such that $p = \lambda p_1 + (1-\lambda)p_2$. Therefore, we would have that for all $x \in I_A, y \in I_B, a \in \mathcal{O}_A, b \in \mathcal{O}_B$:

$$p(a,b \mid x,y) = \lambda p_1(a,b \mid x,y) + (1-\lambda)p_2(a,b \mid x,y)$$

Since $p_1 \neq p_2$ we also have that $p_1(a,b \mid x,y) \neq p_2(a,b \mid x,y)$ for (at least) one quadruple $(x,y,a,b) \in I_A \times I_B \times \mathcal{O}_A \times \mathcal{O}_B$. Let (x_1,y_1,a_1,b_1) be that quadruple. Then,

$$p(a_1,b_1 \mid x_1,y_1) = \lambda p_1(a_1,b_1 \mid x_1,y_1) + (1-\lambda)p_2(a_1,b_1 \mid x_1,y_1)$$

If $p(a_1,b_1 \mid x_1,y_1) = 0$ then $p_1(a_1,b_1 \mid x_1,y_1), p_2(a_1,b_1 \mid x_1,y_1)$ must both be equal to zero which is a contradiction. On the other hand if $p(a_1,b_1 \mid x_1,y_1) = 1$ then since $p_1(a_1,b_1 \mid x,y) \neq p_2(a_1,b_1 \mid x,y)$ they cannot both be 1 which means that one of them is strictly less than 1, let $p_2(a_1, b_1 \mid x, y)$ be that one. However, in this case we would have that

$$1 = p(a_1, b_1 \mid x, y) = \lambda p_1(a_1, b_1 \mid x, y) + (1-\lambda) p_2(a_1, b_1 \mid x, y)$$
$$< \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$$

which once again is a contradiction.

We will now define some additional classes of NS-correlations, this will be done using POVM's.

Definition 5.11 *A non-signalling correlation*

$$p = \{(p(a, b \mid x, y))_{(a,b)\in\mathcal{O}_A\times\mathcal{O}_B} : (x, y) \in I_A \times I_B\}$$

is called

1. ***quantum**: if there exist a finite dimensional Hilbert spaces H_A (Alice's state space) and H_B (Bob's state space), a unit vector $\xi \in H_A \otimes H_B$ and for each $x \in I_A$ a POVM $\{E_{x,a}\}_{a\in\mathcal{O}_A}$ on H_A and for each $y \in I_B$ $\{F_{y,b}\}_{b\in\mathcal{O}_B}$ on H_B such that*

   $$p(a, b \mid x, y) = ((E_{x,a} \otimes F_{y,b}\xi, \xi), \text{ for all } x \in I_A,\ y \in I_B,\ a \in \mathcal{O}_A,\ b \in \mathcal{O}_B$$

2. ***approximately quantum**: if there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of quantum correlations such that $p_n \to_{n\to\infty} p$*

3. ***quantum commuting**: if there exist a (possibly infinite-dimensional) Hilbert space H (shared state space), a unit vector $\xi \in H$ as well as POVM's on H $\{E_{x,a}\}_{a\in\mathcal{O}_A}$ for each $x \in I_A$ and $\{F_{y,b}\}_{b\in\mathcal{O}_B}$ for each $y \in I_B$, such that: $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$, for all x, y, a, b and*

   $$p(a, b \mid x, y) = ((E_{x,a}F_{y,b})\xi, \xi), \text{ for all } x \in I_A,\ y \in I_B,\ a \in \mathcal{O}_A,\ b \in \mathcal{O}_B$$

The set of all correlations $(p(a, b \mid x, y))$ as in 1 arising from all choices of finite-dimensional Hilbert spaces H_A, H_B, all POVM's on H_A, H_B and all unit vectors in $H_A \otimes H_B$ is called the set of quantum correlations and is denoted by $\mathcal{C}_q$.
Similarly the set of all correlations $(p(a, b \mid x, y))$ as in 3 arising from all choices of the Hilbert space H, all POVM's on H and all unit vectors is called the set of quantum commuting correlations and will be denoted by $\mathcal{C}_{qc}$.
The set of all approximately quantum correlations will be denoted by $\mathcal{C}_{qa}$.

Proposition 5.12 *For the correlation sets defined, we have the following inclusions:*

$$\mathcal{C}_{loc} \subseteq \mathcal{C}_q \subseteq \mathcal{C}_{qa} \subseteq \mathcal{C}_{qc} \subseteq \mathcal{C}_{ns}$$

Proof: For the first inclusion: Suppose that $p = \sum_{i=1}^{m} \lambda_i p_i^{(1)} \otimes p_i^{(2)}$, i.e., $p(a, b \mid x, y) = \sum_{i=1}^{m} \lambda_i p_i^{(1)}(a \mid x) p_i^{(2)}(b \mid y)$ is a convex combination, where

$$p^1 : \mathcal{O}_A \times I_A \to [0,1] \text{ and } p^2 : \mathcal{O}_B \times I_B \to [0,1]$$

are probability distributions. Thus p is an element of $\mathcal{C}_{loc}$.
Now, for each $(a, x) \in \mathcal{O}_A \times I_A$ letting $e_{(a,x)}(i) = \sqrt{\lambda_i} p_i^1(a \mid x)$ we obtain a function $e_{(a,x)} : \{1, \ldots, m\} \to [0, 1]$, that is, an element of the Abelian finite dimensional von-Neumann algebra $l^\infty([m])$. Similarly letting $f_{(b,y)}(i) = \sqrt{\lambda_i} p_i^2(b \mid y)$ we obtain a function $f_{(b,y)} \in l^\infty([m])$. Moreover, if we set $E_{(x,a)}$ and $F_{(y,b)}$ to be the diagonal operators $diag(e_{(a,x)}(i))$ and $diag(f_{(b,y)}(i))$ respectively, and denote the standard basis of $l^2([m])$ by $\{\delta_i\}$ then

$$p(a, b \mid x, y) = \sum_{i=1}^{m} e_{(a,x)}(i) f_{(b,y)}(i) = \sum_{i=1}^{m} \big((E_{(x,a)} F_{(y,b)})\delta_i, \delta_i\big) = Tr(E_{(x,a)} F_{(y,b)})$$

Now, write $\tilde{E}_{(x,a)} = E_{(x,a)} \otimes I_m$ and $\tilde{F}_{(y,b)} = I_m \otimes F_{(y,b)}$ these are operators acting on $l^2([m]) \otimes l^2([m])$, and let $\xi = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \delta_i \otimes \delta_i$ (this is a unit vector on $l^2([m]) \otimes l^2([m])$) then:

$$p(a, b \mid x, y) = \Big((\tilde{E}_{(x,a)} \otimes \tilde{F}_{(y,b)})\xi, \xi\Big)$$

it follows that $p \in \mathcal{C}_q$.
In particular, we have shown that every local correlation can be written in the form of a quantum correlation as in Definition 5.11 with the added condition that the families of POVM's commute.

The second inclusion is obvious.

For the third inclusion: We will make use of a fact that will be proven later on, which is that the set $\mathcal{C}_{qc}$ is closed. Since it is closed it will contain the closure of any of its subsets. Thus, it suffices to show that $\mathcal{C}_q \subseteq \mathcal{C}_{qc}$.

To this end, suppose that we have a correlation p such that for all $x \in I_A,\ y \in I_B,\ a \in \mathcal{O}_A,\ b \in \mathcal{O}_B$:

$$p(a, b \mid x, y) = ((E_{x,a} \otimes F_{y,b})\xi, \xi)$$

where for each $x \in I_A$, $\{E_{x,a}\}_{a \in \mathcal{O}_A}$ are POVM's on a finite dimensional Hilbert space H_A and for each $y \in I_B$, $\{F_{y,b}\}_{b \in \mathcal{O}_B}$ are POVM's on a finite dimensional Hilbert space H_B, and $\xi \in H_A \otimes H_B$ is a unit vector. Then by letting $\tilde{E}_{(x,a)} = E_{(x,a)} \otimes I$ and $\tilde{F}_{(y,b)} = I \otimes F_{(y,b)}$ we see that for $x \in I_A$, $y \in I_B$: $\{\tilde{E}_{(x,a)}\}_a$ and $\{\tilde{F}_{(y,b)}\}_b$ are commuting families of POVM's acting on the Hilbert space $H := H_A \otimes H_B$. Furthermore,

$$p(a, b \mid x, y) = \Big((\tilde{E}_{x,a} \tilde{F}_{y,b} \xi, \xi\Big)$$

which proves the desired inclusion.

For the fourth inclusion we work as follows, assume that $p \in \mathcal{C}_{qc}$, then by Definition 5.11 there exist a Hilbert space H, a unit vector $\xi \in H$ and POVM's $\{E_{x,a}\}_{a\in\mathcal{O}_A}$ and $\{F_{y,b}\}_{b\in\mathcal{O}_B}$ acting on H whose elements commute, such that

$$p(a,b \mid x,y) = ((E_{x,a}F_{y,b})\xi, \xi)\,, \text{ for all } x \in I_A,\ y \in I_B,\ a \in \mathcal{O}_A,\ b \in \mathcal{O}_B$$

The aforementioned commutativity implies that $E_{x,a}^{\frac{1}{2}}F_{y,b}^{\frac{1}{2}} = F_{y,b}^{\frac{1}{2}}E_{x,a}^{\frac{1}{2}}$
(If the positive operators A, B commute then B commutes with with $f(A)$ for all polynomials f, and since $A^{\frac{1}{2}}$ is the limit of $g(A)$ for some polynomials g it follows that B and $A^{\frac{1}{2}}$ commute. The same reasoning gives us the commutativity between $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$)
Now using the above it follows that

$$p(a,b \mid x,y) = \left((E_{x,a}^{\frac{1}{2}}F_{y,b}^{\frac{1}{2}})\xi, (E_{x,a}^{\frac{1}{2}}F_{y,b}^{\frac{1}{2}})\xi\right) \geq 0$$

Since, $\{E_{x,a}\}_{a\in\mathcal{O}_A}$ and $\{F_{y,b}\}_{b\in\mathcal{O}_B}$ are POVM's and ξ is a unit vector we have that

$$\sum_{a\in\mathcal{O}_A, b\in\mathcal{O}_B} p(a,b \mid x,y) = \sum_{a\in\mathcal{O}_A, b\in\mathcal{O}_B} ((E_{x,a}F_{y,b})\xi, \xi) =$$

$$\left(\sum_{b\in\mathcal{O}_B} F_{y,b}\xi, \sum_{a\in\mathcal{O}_A} E_{x,a}\xi\right) = (\xi, \xi) = 1$$

and

$$\sum_{b\in\mathcal{O}_B} p(a,b \mid x,y) = \sum_{b\in\mathcal{O}_B} ((E_{x,a}F_{y,b})\xi, \xi) = \left(E_{x,a}\xi, \sum_{b\in\mathcal{O}_B} F_{y,b}\xi\right) = (E_{x,a}\xi, \xi) \geq 0$$

Similarly it is shown that $\sum_{a\in\mathcal{O}_A} p(a,b \mid x,y) = (F_{y,b}\xi, \xi) \geq 0$. Notice that $(E_{x,a}\xi, \xi)$ and $(F_{y,b}\xi, \xi)$ are independent of y and x respectively.
Thus, p satisfies the non-signalling condition, i.e., $p \in \mathcal{C}_{ns}$.

Proposition 5.13 *The set $\mathcal{C}_q$ of all quantum correlations is convex.*

Proof: Let $H_{A,j}$ and $H_{B,j}$ be Hilbert spaces, $j = 1, 2$. Suppose that ξ_j are unit vectors on $H_{A,j} \otimes H_{B,j}$, and for each x, y: $\{E_{x,a}^j\}_a$, $\{F_{b,y}\}_b$ are POVM's acting on $H_{A,j}$ and $H_{B,j}$ respectively. Then

$$p_1(a,b \mid x,y) = \left((E_{x,a}^1 \otimes F_{y,b}^1)\xi_1, \xi_1\right) \text{ and } p_2(a,b \mid x,y) = \left((E_{x,a}^2 \otimes F_{y,b}^2)\xi_2, \xi_2\right)$$

are elements of $\mathcal{C}_q$. Let $\lambda \in [0,1]$, we shall show that the probability distribution $\lambda p_1 + (1-\lambda)p_2$ also belongs to $\mathcal{C}_q$, i.e., it can be written as in 3 of definition 5.11.

Consider the following operators

$$E_{x,a} = E_{x,a}^1 \oplus E_{x,a}^2 \text{ and } F_{y,b} = F_{y,b}^1 \oplus F_{y,b}^2$$

acting on the direct sums $H_{A,1} \oplus H_{A,2}$ and $H_{B,1} \oplus H_{B,2}$.
Now,

$$(H_{A,1} \oplus H_{A,2}) \otimes (H_{B,1} \oplus H_{B,2}) = (H_{A,1} \otimes H_{B,1}) \oplus (H_{A,1} \otimes H_{B,2}) \oplus (H_{A,2} \otimes H_{B,1}) \oplus (H_{A,2} \otimes H_{B,2})$$

Thus in this Hilbert space the vector $\xi = \sqrt{\lambda}\xi_1 \oplus 0 \oplus 0 \oplus \sqrt{1-\lambda}\xi_2$ is a unit vector.
Moreover,

$$((E_{x,a} \otimes F_{y,b})\xi, \xi) =$$

$$\left(((E^1_{x,a} \otimes F^1_{y,b}) \oplus (E^1_{x,a} \otimes F^2_{y,b}) \oplus (E^2_{x,a} \otimes F^1_{y,b}) \oplus (E^2_{x,a} \oplus F^2_{y,b}))\xi, \xi\right) =$$

$$\left((E^1_{x,a} \otimes F^1_{y,b})\sqrt{\lambda}\xi_1 \oplus 0 \oplus 0 \oplus (E^2_{x,a} \otimes F^2_{y,b})\sqrt{1-\lambda}\xi_2, \sqrt{\lambda}\xi_1 \oplus 0 \oplus 0 \oplus \sqrt{1-\lambda}\xi_2\right) =$$

$$\left((E^1_{x,a} \otimes F^1_{y,b})\sqrt{\lambda}\xi_1, \sqrt{\lambda}\xi_1\right) + \left((E^2_{x,a} \otimes F^2_{y,b})\sqrt{1-\lambda}\xi_2, \sqrt{1-\lambda}\xi_2\right) =$$

$$\lambda\left((E^1_{x,a} \otimes F^1_{y,b})\xi_1, \xi_1\right) + (1-\lambda)\left((E^2_{x,a} \otimes F^2_{y,b})\xi_2, \xi_2\right) =$$

$$\lambda p_1(a, b \mid x, y) + (1-\lambda)p_2(a, b \mid x, y)$$

and the result follows.

Proposition 5.14 *The set $\mathcal{C}_{qa}$ of all approximately quantum correlations is convex.*

Proof: Suppose that $p, q \in \mathcal{C}_{qa}$ and let $\lambda \in [0, 1]$. There exist sequences $(p_n)_n, (q_n)_n$ of quantum correlations such that $p_n \longrightarrow_{n\to\infty} p$ and $q_n \longrightarrow_{n\to\infty} p$. Since $\mathcal{C}_{loc}$ we have that for every $n \in \mathbb{N}$, $\lambda p_n + (1-\lambda)q_n \in \mathcal{C}_{loc}$. Moreover,

$$\lambda p_n + (1-\lambda)q_n \longrightarrow_{n\to\infty} \lambda p + (1-\lambda)q$$

The result is immediate.

It is also true that $\mathcal{C}_{qc}$ is convex, a fact that will be shown in the sequel (see Subsection 5.1.3). It was shown in [6] that the set $\mathcal{C}_q$ is not closed.

5.1 Characterizations of the sets of correlations

We will try to interpret the definitions of the various correlations classes we have seen in terms of the theory of tensor products of operator systems.

Throughout this chapter we will make use of the notation $[n] = \{1, \ldots, n\}$ for a $n \in \mathbb{N}$.

Let $k \in \mathbb{N}$, then $\mathbb{C}^k$ with component-wise multiplication and component-wise complex conjugation is an Abelian C^*-algebra. The canonical basis of $\mathbb{C}^k$ will be denoted by $\{e_j\}_{j=1}^k$.

Notice that $\mathbb{C}^k$ is (canonically) $*$-isomorphic to the Abelian C^* algebra Δ_k of the $k \times k$ diagonal matrices with complex entries which in turn is (canonically) $*$-isomorphic to the C^* algebra of functions on k isolated points. To see this check that the map defined by sending a matrix $diag\{v_1, \dots, v_n\}$ to the function $f : [k] \to \mathbb{C} : i \to v_i$ is indeed a $*$-isomorphism. Furthermore, observe that the set $\{\delta_i\}_{i=1}^n$ where for each i, $\delta_i : [k] \to \mathbb{C}$ is given by

$$\delta_i(j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

is a set of orthogonal projections and it spans the aforementioned C^*-algebra.

We will denote this Abelian C^*-algebra by ℓ_k^∞, i.e.,

$$\ell_k^\infty = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\text{k- times}} = \{(\lambda_i)_{i=1}^k : \lambda_i \in \mathbb{C}, 1 \leq i \leq k\} = \mathbb{C}^k$$

If $\mathcal{S}$ is an operator system we write

$$\ell^\infty(k, \mathcal{S}) = \underbrace{\mathcal{S} \oplus \cdots \oplus \mathcal{S}}_{\text{k- times}}$$

5.1.1 Local Correlations

Let

$$\mathcal{D} = \underbrace{\ell_k^\infty \otimes \cdots \otimes \ell_k^\infty}_{\text{n-times}}$$

notice that we have the following isomorphisms

$$\mathcal{D} = \underbrace{\mathbb{C}^k \otimes \cdots \otimes \mathbb{C}^k}_{\text{n-times}} = \mathbb{C}^{k^n}$$

so $\mathcal{D}$ is $*$-isomorphic to the space of all (continuous) functions on k^n-points. In addition, for $1 \leq v \leq n$ we write $e'_{v,i}$ for the i-th standard basis vector e_i of ℓ_k^∞ occurring in the v-th term of the tensor product, that means

$$e'_{v,i} = \underbrace{1 \otimes \cdots \otimes 1 \otimes e_i \otimes 1 \otimes \cdots \otimes 1}_{\text{n-terms}}$$

Definition 5.15 *Let $\mathcal{A}$ be a C^*-algebra, a state of $\mathcal{A}$ is called pure if it is an extreme point of the state space of $\mathcal{A}$.*

Proposition 5.16 *A correlation $(p(i, j \mid v, w))$ is in $\mathcal{C}_{loc}$ if and only if there exists a state s on the tensor product $\mathcal{D} \otimes \mathcal{D}$ such that*

$$p(i, j \mid v, w) = s(e_{v,i} \otimes e_{w,j}) \tag{$*$}$$

Proof: Formula $(*)$ is equivalent to saying that $p_k^1(i \mid v) = s_k^1(e_{v,i})$ for some state $s_k^1 : \mathcal{D} \to \mathbb{C}$ and $p_k^2(j \mid w) = s_k^2(e_{w,j})$ for a state $s_k^2 : \mathcal{D} \to \mathbb{C}$. As we have seen an element of $\mathcal{C}_{loc}$ can be written as:

$$p(i, j \mid v, w) = \sum_{k=1}^{m} \lambda_k p_k^1(i \mid v) p_k^2(j \mid w)$$

so it becomes

$$p(i, j \mid v, w) = \sum_{k=1}^{m} \lambda_k s_k^1(e_{v,i}) s_k^2(e_{w,j}) = \sum_{k=1}^{m} \lambda_k s_k^1 \otimes s_k^2(e_{v,i} \otimes e_{w,j})$$

so it is a convex combination of product states. We shall show that such combinations yield all states on $\mathcal{D} \otimes \mathcal{D}$.
Recall that :

$$\mathcal{D} \otimes \mathcal{D} = C(\{1, \ldots, k^n\}) \otimes C(\{1, \ldots, k^n\}) = C(\{1, \ldots, k^n\} \times \{1, \ldots, k^n\})$$

For the last equality in particular we have that :

$$C([k^n]) \otimes C([k^n]) = \text{span}\{f \otimes g : f \in C([k^n]), g \in C([k^n])\} = C([k^n] \times [k^n])$$

where for $f \in C([k^n])$ and $g \in C([k^n])$, $f \otimes g \in C([k^n] \times [k^n])$ is the function given by $(f \otimes g)(i, j) = f(i)g(j),\ \forall i, j \in [k^n]$.
In general, given a compact (Hausdorff) space K it is known that the pure states of $C(K)$ are the evaluations at the points of K, so the pure states of $\mathcal{D} \otimes \mathcal{D}$ are the evaluations at the points of $\{1, \ldots, k^n\} \times \{1, \ldots, k^n\}$.
Let $(i, j) \in \{1, \ldots, k^n\} \times \{1, \ldots, k^n\}$, we will denote the evaluation at the point (i, j) by $\widehat{(i, j)}$.
Let ρ be a pure state of $\mathcal{D} \otimes \mathcal{D}$ then $\rho = \widehat{(i, j)}$ for some $i \in [k^n], j \in [k^n]$. Moreover, if $f = \sum_k f_k \otimes g_k \in \mathcal{D} \otimes \mathcal{D}$ then ,

$$\begin{aligned} \widehat{(i, j)}(f) = \widehat{(i, j)}(\sum_k f_k \otimes g_k) = (\sum_k f_k \otimes g_k)(i, j) = \sum_k f_k(i) g_k(j) \\ = \sum_k \hat{i}(f_k) \hat{j}(g_k) = (\hat{i} \otimes \hat{j})(\sum_k f_k \otimes g_k) \end{aligned}$$

Thus, every pure state ρ of $\mathcal{D} \otimes \mathcal{D}$ can be written as $\rho = \hat{i} \otimes \hat{j}$.

Since $\mathcal{D} \otimes \mathcal{D}$ is a unital C^*-algebra its state space is the closed convex hull of the pure states ([13, Corollary 5.1.9]).
Let ω be a state on $\mathcal{D} \otimes \mathcal{D}$ then there exists a sequence ω_n, whose every term is a convex combination of pure states, such that $\omega_n \to \omega$. Thus, for every $n \in \mathbb{N}$:

$$\omega_n = \sum_{i=1}^{n} \lambda_{i,n} s_{i,n}^1 \otimes s_{i,n}^2$$

now using Caratheodory's theorem we obtain the desired result.

The remaining correlation classes will be described with the help of the following coproduct of operator systems

$$\mathcal{S}(n,k) = \underbrace{\ell_k^\infty \oplus_1 \cdots \oplus_1 \ell_k^\infty}_{\text{n-times}},\ n,k \in \mathbb{N}$$

In particular we will see that correlations belonging to the aforementioned classes can be realized as states on the tensor products of spaces of the above form (co-products of l_i^∞, $i \in \mathbb{N}$). We will see that each class of correlations corresponds to a different tensor product.

For the next subsections we will write $(e_{x,a})_{a=1}^k$ for the canonical basis of the x-th copy of ℓ_k^∞ in the coproduct $\mathcal{S}(n,k)$. Thus,

$$((e_{x,a})_{a=1}^k)_{x=1}^n = (e_{1,a})_{a=1}^k \oplus \cdots \oplus (e_{n,a})_{a=1}^k$$

Another space that we shall find to be useful is :

$$\mathcal{A}(n,k) = \ell_k^\infty *_1 \cdots *_1 \ell_k^\infty$$

to be more specific the usefulness of the last space lies in the fact that

$$\mathcal{S}(n,k) \subseteq_{c.o.i.} \mathcal{A}(n,k) \text{ (see Theorem 4.2)}$$

For a Hilbert space H, the following Lemma and Proposition reveal to us an interesting correspondence between unital completely positive maps from ℓ_k^∞ to $B(H)$ and POVM's on H, and respectively between unital completely positive maps from $\mathcal{S}(n,k)$ to $B(H)$ and families of POVM's on H.

Lemma 5.17 *Suppose that $\varphi : \ell_k^\infty \to B(H)$ is a unital completely positive map then $\{\varphi(e_j)\}_{1\le j\le k}$ is a POVM on H. Conversely, if $\{E_j\}_{1\le j\le k}$ is a POVM then the linear map $\varphi : \ell_k^\infty \to B(H)$ defined by $\varphi(e_j) = E_j$ is unital and completely positive.*

Proof: In order to prove the forward implication notice that e_j is a positive element of ℓ_k^∞, for all $j \in [k]$ and

$$\sum_{j\in[k]} e_j = (1,\dots,1) = 1_{\ell_k^\infty}$$

Thus $\varphi(e_j) \in B(H)^+,\ \forall j \in [k]$ and $\sum_{j\in[k]} \varphi(e_j) = I_H$ which proves the forward part.
For the converse, assume that $P = [p_{rs}]_{r,s} \in M_n(\ell_k^\infty)^+$ and define the map $\varphi : \ell_k^\infty \to B(H)$ via $\varphi(e_j) = E_j$. Since $\{e_j\}$ is a basis for the vector space ℓ_k^∞, φ extends uniquely to all of ℓ_k^∞. Moreover

$$\varphi(1_{\ell_k^\infty}) = \varphi(\sum_{j\in[k]} e_j) = \sum_{j\in[k]} E_j = I_H$$

so this extension is unital. Now taking into consideration that $\mathbb{C}$ is a commutative C^*-algebra, for the complete positivity of φ it suffices to show that it is a positive map. To this end let $\alpha \in (\ell_k^\infty)^+$, then $\alpha = \sum_{j\in[k]} \lambda_j e_j$ where all the coefficients λ_j are in $\mathbb{R}^+$. Consequently,

$$\varphi(\alpha) = \varphi(\sum_{j\in[k]} \lambda_j e_j) = \sum_{j\in[k]} \lambda_j \varphi(e_j) = \sum_{j\in[k]} \lambda_j E_j \in B(H)^+$$

which completes the proof.

Remark : The above theorem remains true if we replace unital completely positive maps with unital $*$-homomorphisms and POVM's with PVM's.

A sketch of the proof of this result using the terminology established in the proof of the Lemma:
In the case that $\varphi : \ell_k^\infty \to B(H)$ is a $*$-homomorphism we have that

$$(\varphi(e_j))^2 = \varphi(e_j^2) = \varphi(e_j)$$

and

$$(\varphi(e_j))^* = \varphi(e_j^*) = \varphi(e_j)$$

On the other hand when $\{E_j\}_j$ is a PVM on H we have that for an $a = \sum_{j\in[k]} \lambda_j e_j \in \ell_k^\infty$,

$$(\varphi(a))^* = \sum_{j\in[k]} \lambda_j^* E_j^* = \sum_{j\in[k]} \lambda_j^* E_j = \varphi(\sum_{j\in[k]} \lambda_j^* e_j) = \varphi(a)$$

.

Proposition 5.18 *If $\varphi : \mathcal{S}(n,k) \to B(H)$ is a unital completely positive map then $\{\varphi(e_{x,a})_{1\leq a\leq k}\}$ is a POVM on H for every $x \in \{1,\dots,n\}$. Conversely, if for every $x \in \{1,\dots,n\}$, $(E_{x,a})_{1\leq a\leq k}$ is a POVM acting on H then there exists a (unique) unital completely positive map $\varphi : \mathcal{S}(n,k) \to B(H)$ such that:*
$\varphi(e_{x,a}) = E_{x,a}$ for all $a \in \{1,\dots,k\}$.

Proof: Let $\ell_{k,x}^\infty$, $1 \leq x \leq n$ denote the x-th copy of ℓ_k^∞ in the coproduct $\mathcal{S}(n,k)$ then as operator systems $\ell_{k,x}^\infty \subseteq \mathcal{S}(n,k)$. Thus the restriction of φ on $\ell_{k,x}^\infty$ is a unital completely positive map and by applying Lemma 5.17 we have that $\{\varphi(e_{x,a})_{1\leq a\leq k}\}$ is a POVM on H for every $1 \leq x \leq n$.
For the converse, suppose that $(E_{x,a})_{1\leq a\leq k}$, $1 \leq x \leq n$ is a family of POVM's acting on H. Once again we apply Lemma 5.17 and for each $x \in [n]$ we find a UCP map $\varphi_x : \ell_{x,k}^\infty \to B(H)$ such that $\varphi_x(e_{x,a}) = E_{x,a}$, for all $a \in [k]$. By the universal property of the coproduct, there exists a (unique) unital completely positive map $\varphi : \mathcal{S}(n,k) \to B(H)$ such that $\varphi \restriction \ell_{k,x}^\infty = \varphi_x$ (remember that $\mathcal{S}(n,k) = \underbrace{\ell_k^\infty \oplus_1 \cdots \oplus_1 \ell_k^\infty}_{n-times}$).

The proof is now complete.

Consider the set

$$\mathcal{V}(n,k) =$$

$$\{((\lambda_{1,\alpha})_\alpha, \dots, (\lambda_{n,\alpha})_\alpha) : \lambda_{i,\alpha} \in \mathbb{C},\ \forall i \in [n], \alpha \in [k] \text{ and } \exists\, c \in \mathbb{C} \text{ s.t. } \sum_{\alpha=1}^{k} \lambda_{i,\alpha} = c, \forall i\}$$

$$= \{((\lambda_{1,\alpha})_\alpha, \dots, (\lambda_{n,\alpha})_\alpha) : (\lambda_{i,a})_\alpha \in \ell_k^\infty, \forall i \in [n] \text{ and } \sum_{\alpha=1}^{k} \lambda_{i,\alpha} = \sum_{\alpha=1}^{k} \lambda_{i',\alpha}, \forall i, i'\}$$

as an operator subsystem of ℓ_{nk}^∞, this means that $\mathcal{V}(n,k)$ is seen as an operator system with the operator system structure it inherits from ℓ_{nk}^∞.

Proposition 5.19 *The dual of $\mathcal{S}(n,k)$ is completely order isomorphic to $\mathcal{V}(n,k)$.*

Proof: We have that $\mathcal{S}(n,k)$ is the coproduct $\underbrace{\ell_k^\infty \oplus_1 \cdots \oplus_1 \ell_k^\infty}_{n-times}$ and the coproduct was defined as $\ell_k^\infty \oplus \cdots \oplus \ell_k^\infty / \mathcal{J} = \mathbb{C}^k \oplus \cdots \oplus \mathbb{C}^k / \mathcal{J}$, where

$$\mathcal{J} = \text{span}\{(e, -e, 0, \dots, 0), (e, 0, -e, 0, \dots, 0), \dots, (e, 0, \dots, 0, -e)\}$$

is a kernel/ null subspace in $\mathbb{C}^k \oplus \cdots \oplus \mathbb{C}^k = \mathbb{C}^{nk}$ (here e denotes the unit of $\mathbb{C}^k$). Moreover, by Proposition 4.3 we have the following complete order isomorphism

$$(\ell_k^\infty \oplus_1 \cdots \oplus_1 \ell_k^\infty)^d \cong_{c.o.i.} \{f_1 \oplus \cdots \oplus f_n \in (\ell_k^\infty)^d \oplus \cdots \oplus (\ell_k^\infty)^d : f_1(e) = \cdots = f_n(e)\}$$

$$= \{f_1 \oplus \cdots \oplus f_n \in (\ell_k^\infty)^d \oplus \cdots \oplus (\ell_k^\infty)^d : f_i(e) = f_{i'}(e),\ \forall i, i' \in [n]\}$$

Recall that we let $\{e_\alpha\}_{\alpha=1}^k$ denote the standard basis of $\ell_k^\infty = \mathbb{C}^k$. We will make the following identification between the spaces ℓ_k^∞ and $(\ell_k^\infty)^d$:
For the aforementioned basis let $\{e_\alpha^*\}_{\alpha=1}^k$ denote the dual basis ($e_j^*(e_i) = \delta_{ij}, i, j \in [k]$).
Then for linear functional $f : \ell_k^\infty \to \mathbb{C}$ we obtain the correspondence :

$$f = \sum_{\alpha=1}^{k} f(e_\alpha) e_\alpha^* \to \sum_{\alpha=1}^{k} f(e_\alpha) e_\alpha$$

$$\text{i.e., } (\ell_k^\infty)^d \ni f \longleftrightarrow (f(e_\alpha))_{\alpha=1}^k \in \ell_k^\infty, \text{ with respect to the basis } \{e_\alpha\}_{\alpha=1}^k$$

Note that f is completely positive if and only if $f(e_\alpha) \geq 0$ so the identification we used is a complete order isomorphism. To see this, let $\Psi : (\ell_k^\infty)^d \to \ell_k^\infty$ be the map given by $\Psi(f) = (f(e_\alpha))_{\alpha=1}^k$ then Ψ is unital and positive and since ℓ_k^∞ is a commutative C^*-algebra it will be completely positive. Now the inverse of Ψ is the positive map $\Psi^{-1} : \ell_k^\infty \to (\ell_k^\infty)^d : (\lambda_\alpha)_{\alpha=1}^k \to \sum_{\alpha=1}^k \lambda_\alpha e_\alpha^*$. Since ℓ_k^∞ is a finite dimensional operator system its dual is also an operator system and as such it will be a subspace of $B(H)$ for some Hilbert space H. Therefore we can consider the map Ψ^{-1} as a map from ℓ_k^∞ which is a commutative C^*-algebra to $B(H)$ which is a C^*-algebra in this case Ψ^{-1} remains positive. Now a theorem of Stinespring (see [15, Theorem 3.11]) allows us to obtain that Ψ^{-1} is completely positive.

Since $e = \sum_{\alpha=1}^k e_\alpha$, the condition

$$f_i(e) = f_{i'}(e),\ \forall i, i' \in [n]$$

takes the following form

$$\sum_{\alpha=1}^{k} f_i(e_\alpha)e^*_\alpha(e_\alpha) = \sum_{\alpha=1}^{k} f_{i'}(e_\alpha)e^*_\alpha(e_\alpha)$$

or equivalently

$$\sum_{\alpha=1}^{k} f_i(e_\alpha) = \sum_{\alpha=1}^{k} f_{i'}(e_\alpha)$$

Now for every $\alpha \in [k],\ i \in [n]$ set

$$f_i(e_\alpha) = \lambda^f_{i,\alpha}$$

then by combining the observations we made above we see that for a $f \in \mathcal{S}(n,k)^d$ we have the following identification (up to a complete order isomorphism) :

$$f \longleftrightarrow (\lambda^f_{1,\alpha})^k_{\alpha=1} \oplus \cdots \oplus (\lambda^f_{n,\alpha})^k_{\alpha=1}$$

where

$$(\lambda^f_{i,\alpha})^k_{\alpha=1} \in \ell^\infty_k \text{ and } \sum_{\alpha=1}^{k} \lambda^f_{i,\alpha} = \sum_{\alpha=1}^{k} \lambda^f_{i',\alpha},\ \forall i, i' \in [n]$$

so every $f \in \mathcal{S}(n,k)^d$ can be written as :

$$f = ((\lambda^f_{i,\alpha})^k_{\alpha=1})^n_{i=1},\ \text{where} \sum_{\alpha=1}^{k} \lambda^f_{i,\alpha} = \sum_{\alpha=1}^{k} \lambda^f_{i',\alpha},\ \forall i, i' \in [n]$$

Thus

$$\mathcal{S}(n,k)^d \text{ is order isomorphic to a subspace of } \mathcal{V}(n,k)$$

Conversely, let

$$\lambda = (\lambda_{1,\alpha})_\alpha \oplus \cdots \oplus (\lambda_{n,\alpha})_\alpha \in \mathcal{V}(n,k) \subseteq_{c.o.i.} \underbrace{\ell^\infty_k \oplus \cdots \oplus \ell^\infty_k}_{n-times}$$

for every $i \in \{1, \ldots, n\}$ we have that $(\lambda_{i,\alpha})_\alpha$ defines a linear functional f_i on $(\ell^\infty_k)^d$ via the rule

$$(\lambda_{i,\alpha})_\alpha = \sum_{\alpha=1}^{k} \lambda_{i,\alpha} e_\alpha \longleftrightarrow \sum_{\alpha=1}^{k} \lambda_{i,\alpha} e^*_\alpha = f_i$$

Furthermore for each i, it is easy to see that f_i is positive if and only if $(\lambda_{i,\alpha})_\alpha$ is a positive element of ℓ^∞_k so the identification above is an order isomorphism and since

$$\sum_{\alpha=1}^{k} \lambda_{i,\alpha} = \sum_{\alpha=1}^{k} \lambda_{i',\alpha}, \forall i, i' \in [n]$$

it follows that for every $1 \leq i, i' \leq n$

$$f_i(e) = \sum_{\alpha=1}^{k} \lambda_{i,\alpha} e_\alpha^* (\sum_{b=1}^{k} e_b) = \sum_{\alpha=1}^{k} \lambda_{i,\alpha} = \sum_{\alpha=1}^{k} \lambda_{i',\alpha} = \sum_{\alpha=1}^{k} \lambda_{i',\alpha} e_\alpha^* (\sum_{b=1}^{k} e_b) = f_{i'}(e)$$

Set $f = f_1 \oplus \cdots \oplus f_n$ then it is not hard to verify that $f \in \mathcal{S}(n,k)^d$ (and that f is positive if and only if each f_i is positive) which implies that

$$\mathcal{V}(n,k) \text{ is order isomorphic to a subspace of } \mathcal{S}(n,k)^d$$

Thus far we have shown that the spaces $\mathcal{S}(n,k)^d$ and $\mathcal{V}(n,k)$ are order isomorphic. Now we will show that they are completely order isomorphic.

Firstly, note that there exist an order isomorphism

$$\varphi : \mathcal{S}(n,k)^d \to \mathcal{V}(n,k) : f \to \lambda^f$$

Now if we consider the map φ as a map from $\mathcal{S}(n,k)^d$ to ℓ^∞_{nk} then it remains positive and its range is a subset of a commutative C^*-algebra, so it will be completely positive.

Moreover, the map

$$\psi : \mathcal{V}(n,k) \to \mathcal{S}(n,k)^d$$

$$\lambda = (\sum_{\alpha=1}^{k} \lambda_{1,\alpha} e_\alpha, \ldots, \sum_{\alpha=1}^{k} \lambda_{n,\alpha} e_\alpha) \to f_\lambda = (\sum_{\alpha=1}^{k} \lambda_{1,\alpha} e_\alpha^*, \ldots, \sum_{\alpha=1}^{k} \lambda_{n,\alpha} e_\alpha^*)$$

is the inverse of φ, which once again is a positive map. Consider ψ as a map from $\mathcal{V}(n,k)$ to $\underbrace{(\ell^\infty_k)^d \oplus \cdots \oplus (\ell^\infty_k)^d}_{n-times} \cong_{c.o.i.} \underbrace{\ell^\infty_k \oplus \cdots \oplus \ell^\infty_k}_{n-times} = \mathbb{C}^{nk}$ then as before ψ is a positive map whose range is a subset of a commutative C^*-algebra so it will be completely positive.

Indeed, let $\lambda = ((\lambda_{1,\alpha})_\alpha, \ldots, (\lambda_{n,\alpha})_\alpha) \in \mathcal{V}(n,k)^+$ then for every i we have that $(\lambda_{1,\alpha})_\alpha$ is positive in ℓ^∞_k and

$$\psi(\lambda) = f_1 \oplus \cdots \oplus f_n$$

where $f_i = \sum_{\alpha=1}^{k} \lambda_{i,\alpha} e_\alpha^*, \forall i \in \{1, \ldots, n\}$. Since every $\lambda_{i,\alpha}$ is positive it follows that every f_i is positive which implies that $f_1 \oplus \cdots \oplus f_n$ is positive.

Looking at the proof of the above theorem we see that the duality between the spaces $\mathcal{S}(n,k)$ and $\mathcal{V}(n,k)$ is given via the following formula :

$$\text{Let} \quad v = \sum_{i,\alpha} m_{i,\alpha} e_{i,\alpha} \in \mathcal{S}(n,k) \quad \text{and} \quad f = ((\lambda_{i,\alpha})_{\alpha=1}^k)_{i=1}^n \in \mathcal{V}(n,k)$$

$$\text{then} \quad f(v) = \sum_{i,\alpha} m_{i,\alpha} \lambda_{i,\alpha}$$

5.1.2 Non-Signalling Correlations

Let $n, k, m, l \in \mathbb{N}$ and A, B, X, Y be finite sets with $|A| = k, |B| = l, |X| = n, |Y| = m$. Recall that the canonical generators of $\mathcal{S}(n,k)$ are denoted $e_{x,a}$ for $x \in X,\ a \in A$. Similarly let $f_{y,b}$ denote the canonical generators of $\mathcal{S}(m,l)$.

If $s : \mathcal{S}(n,k) \otimes \mathcal{S}(m,l) \to \mathbb{C}$ is a linear functional we let $p_s : A \times B \times X \times Y$ be the map given by

$$p_s(a, b \mid x, y) = s(e_{x,a} \otimes f_{y,b}),\ \text{for } a \in A, b \in B, x \in X, y \in Y$$

Notice that the collection

$$\{p_s(a, b \mid x, y) : a \in A, b \in B, x \in X, y \in Y\}$$

is non-signalling.

Indeed, for every $x, x' \in X$ if we let $1_x, 1_{x'}$ denote the units of the x-th and x'-th copies of ℓ_k^∞ in the co-product $\mathcal{S}(n,k)$, then we have that:

$$\begin{aligned}
\sum_{a=1}^{k} p_s(a, b \mid x, y) &= \sum_{a=1}^{k} s(e_{x,a} \otimes f_{y,b}) = s(\sum_{a=1}^{k} e_{x,a} \otimes f_{y,b}) \\
&= s(1_x \otimes f_{y,b}) = s(1_{x'} \otimes f_{y,b}) \\
&= s(\sum_{a=1}^{k} e_{x',a} \otimes f_{y,b}) = \sum_{a=1}^{k} s(e_{x,a} \otimes f_{y,b}) \\
&= \sum_{a=1}^{k} p_s(a, b \mid x', y)
\end{aligned}$$

because in the co-product 1_x is identified with $1_{x'}$ for all $x, x' \in X$.
Similarly

$$\sum_{b=1}^{l} p_s(a, b \mid x, y) = \sum_{b=1}^{l} p_s(a, b \mid x, y'), \forall y, y' \in Y$$

On the other hand, the formula

$$s_p(e_{x,a} \otimes f_{y,b}) = p(a, b \mid x, y),\ a \in A, b \in B, x \in X, y \in Y$$

defines a unique linear functional on $\mathcal{S}(n,k) \otimes \mathcal{S}(m,l)$ when $\{p(a, b \mid x, y) : a \in A, b \in B, x \in X, y \in Y\}$ is a non-signalling correlation, because $(e_{x,a} \otimes f_{y,b})_{a,b,x,y}$ is a basis of $\mathcal{S}(n,k) \otimes \mathcal{S}(m,l)$.

The argument presented above shows that the map $s \to p_s$ is a bijective correspondence between the space $(\mathcal{S}(n,k) \otimes \mathcal{S}(m,l))^d$ and the set of non-signalling correlations on $A \times B \times X \times Y$ where $|A| = k, |B| = l, |X| = n, |Y| = m$.

In this chapter we will introduce another way to view the class $\mathcal{C}_{ns}$.

Theorem 5.20 *The map $s \to p_s$ is an (affine) isomorphism between the state space of $\mathcal{S}(n,k) \otimes_{max} \mathcal{S}(m,l)$ and $\mathcal{C}_{ns}$.*

Proof: Suppose that we have a linear functional $s : \mathcal{S}(n,k) \otimes_{max} \mathcal{S}(m,l) \to \mathbb{C}$ then for all $a \in A, b \in B, x \in X, y \in Y$

$$p_s(a,b \mid x,y) = s(e_{x,a} \otimes f_{y,b})$$

We shall show that if s is a state then $p_s \in \mathcal{C}_{ns}$ and vice-versa.

Since $e_{x,a}$, $f_{y,b}$ are positive in $\mathcal{S}(n,k)$ and $\mathcal{S}(m,l)$ respectively it follows that $e_{x,a} \otimes f_{y,b}$ is positive in $\mathcal{S}(n,k) \otimes_{max} \mathcal{S}(m,l)$. Consequently,

$$p_s(a,b \mid x,y) = s(e_{x,a} \otimes f_{y,b}) \geq 0, \ \forall a \in A, b \in B, x \in X, y \in Y$$

Furthermore,

$$\sum_{a,b=1}^{k,l} e_{x,a} \otimes f_{y,b} = (\sum_{a=1}^{k} e_{x,a}) \otimes (\sum_{b=1}^{l} f_{y,b}) = 1 \otimes 1$$

Thus

$$\sum_{a,b=1}^{k,l} s(e_{x,a} \otimes f_{y,b}) = s(\sum_{a,b=1}^{k,l} e_{x,a} \otimes f_{y,b}) = s(1 \otimes 1) = 1$$

so

$$\sum_{a,b=1}^{k,l} p_s(a,b \mid x,y) = 1$$

For the non-signalling conditions observe that in $\mathcal{S}(n,k)$ the following holds

$$\sum_{a=1}^{k} e_{x,a} = 1 = \sum_{a=1}^{k} e_{x',a}, \ \text{for all } x, x' \in X$$

Now tensoring with $f_{y,b}$ we obtain :

$$\sum_{a=1}^{k} e_{x,a} \otimes f_{y,b} = 1 \otimes f_{y,b} = \sum_{a=1}^{k} e_{x',a} \otimes f_{y,b}, \ \text{for all } x, x' \in X$$

and by applying s we get

$$\sum_{a=1}^{k} s(e_{x,a} \otimes f_{y,b}) = \underbrace{s(1 \otimes f_{y,b})}_{\in \mathbb{R}^+} = \sum_{a=1}^{k} s(e_{x',a} \otimes f_{y,b}), \ \text{for all } x, x' \in X$$

,i.e.,

$$\sum_{a=1}^{k} p_s(a,b \mid x,y) = \sum_{a=1}^{k} p_s(a,b \mid x',y), \ \text{for all } x, x' \in X$$

In a similar way (summing the $f_{y,b}$'s and tensoring with $e_{x,a}$) one can show that

$$\sum_{b=1}^{l} p_s(a, b \mid x, y) = \sum_{b=1}^{l} p_s(a, b \mid x, y'), \text{ for all } y, y' \in Y$$

Hence p_s satisfies conditions 1 and 2 of definition 5.8.

On the other hand, assume that $p_s = \{(p_s(a, b \mid x, y))_{a,b,x,y}\} \in \mathcal{C}_{ns}$. Notice that in this case

$$s(1 \otimes 1) = \sum_{a,b=1}^{k,l} s(e_{x,a} \otimes f_{y,b}) = \sum_{a,b=1}^{k,l} p_s(a, b \mid x, y) = 1$$

so s is unital.
So far we have that $s : \mathcal{S}(n, k) \otimes_{max} \mathcal{S}(m, l) \to \mathbb{C}$ is a unital linear functional, i.e., $s \in (\mathcal{S}(n, k) \otimes_{max} \mathcal{S}(m, l))^d$, by Theorem 2.22 we have the identification

$$(\mathcal{S}(n, k) \otimes_{max} \mathcal{S}(m, l))^d \cong_{c.o.i.} \mathcal{S}(n, k)^d \otimes_{min} \mathcal{S}(m, l)^d$$

identifying s with its image under the above complete order isomorphism it can be viewed as an element of

$$\mathcal{S}(n, k)^d \otimes_{min} \mathcal{S}(m, l)^d \cong_{c.o.i.} \mathcal{V}(n, k) \otimes_{min} \mathcal{V}(m, l) \subseteq_{c.o.i.} \ell^\infty_{nk} \otimes_{min} \ell^\infty_{ml}$$

(The complete order embedding above being the tensor product of identities.)
Remembering the way the duality between $\mathcal{S}(n, k)$ and $\mathcal{V}(n, k)$ is achieved, we see that there must be some elements $s_{k,1} \in \mathcal{S}(n, k)^d$ and $s_{k,2} \in \mathcal{S}(m, l)^d$ such that

$$s = \sum_{k=1}^{m} \beta_k s_{k,1} \otimes s_{k,2} \in \mathcal{V}(n, k) \otimes_{min} \mathcal{V}(m, l) \subseteq \ell^\infty_{nk} \otimes_{min} \ell^\infty_{ml}$$

where for every $k \in \{1, \ldots, m\}$

$$s_{k,1} = ((\lambda_{x,a})_a)_x = ((s_{k,1}(e_{x,a}))_a)_x \text{ and } s_{k,2} = ((\rho_{y,b})_b)_y = ((s_{k,2}(f_{y,b}))_b)_y$$

If we consider s as an element of $\ell^\infty_{nk} \otimes_{min} \ell^\infty_{ml}$ and recall that $(\ell^\infty_{nk} \otimes_{min} \ell^\infty_{ml})^+$ is the set:

$$\{z \in \ell^\infty_{nk} \otimes \ell^\infty_{ml} : (\varphi \otimes \psi)(z) \geq 0, \ \forall \varphi \in UCP(\ell^\infty_{nk}, \mathbb{C}), \ \forall \psi \in UCP(\ell^\infty_{ml}, \mathbb{C})\}$$

then is can be seen that s is positive if and only if every one of its coordinates in $\ell^\infty_{nkml} = \ell^\infty_{nk} \otimes \ell^\infty_{ml}$ is non-negative.

Now,

$$s = \sum_{k=1}^{m} \beta_k s_{k,1} \otimes s_{k,2} = \sum_{k=1}^{m} \beta_k ((s_{k,1}(e_{x,a}))_a)_x \otimes ((s_{k,2}(f_{y,b}))_b)_y$$

$$= \sum_{k=1}^{m} \beta_k \begin{pmatrix} (s_{k,1}(e_{1,a}))_a \\ \vdots \\ (s_{k,1}(e_{n,a}))_a \end{pmatrix} \otimes \begin{pmatrix} (s_{k,2}(f_{1,b}))_b \\ \vdots \\ (s_{k,2}(f_{m,b}))_b \end{pmatrix}$$

$$= \sum_{k=1}^{m} \beta_k \begin{pmatrix} \begin{pmatrix} s_{k,1}(e_{1,1}) \\ \vdots \\ s_{k,1}(e_{1,k}) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} s_{k,1}(e_{n,1}) \\ \vdots \\ s_{k,1}(e_{n,k}) \end{pmatrix} \end{pmatrix} \otimes \begin{pmatrix} \begin{pmatrix} s_{k,2}(f_{1,1}) \\ \vdots \\ s_{k,2}(f_{1,l}) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} s_{k,2}(f_{m,1}) \\ \vdots \\ s_{k,2}(f_{m,l}) \end{pmatrix} \end{pmatrix}$$

Deleting the inner brackets yields a complete order isomorphism, which implies that in order to obtain that s is a state we need to ensure that for all a, b, x, y

$$\sum_{k=1}^{m} \beta_k s_{k,1}(e_{x,a}) s_{k,2}(f_{y,b}) \geq 0$$

i.e.,

$$\sum_{k=1}^{m} (\beta_k s_{k,1} \otimes s_{k,2})(e_{x,a} \otimes f_{y,b}) \geq 0$$

or equivalently

$$s(e_{x,a} \otimes f_{y,b}) \geq 0$$

but for every a, b, x, y we have that $s(e_{x,a} \otimes f_{y,b}) = p_s(a, b \mid x, y)$ and the latter, being probabilities, are always positive.

5.1.3 Quantum Commuting Correlations

Remark: For the next Theorem we will require the following useful fact:
Given unital C^*-algebras $\mathcal{A}_1, \mathcal{A}_2$ then for any $*$-homomorphism $\pi : \mathcal{A}_1 \otimes \mathcal{A}_2 \to B(H)$ there exists a pair of (contractive) $*$-homomorphisms $\pi_j : \mathcal{A}_j \to B(H)$, $j = 1, 2$ with **commuting ranges** such that such that:

$$\pi(\alpha_1 \otimes \alpha_2) = \pi_1(\alpha_1)\pi_2(\alpha_2),\ \forall \alpha_j \in \mathcal{A}_j,\ j = 1, 2$$

Indeed, for every $\alpha_j \in \mathcal{A}_j,\ j = 1, 2$ just set $\pi_1(\alpha_1) = \pi(\alpha_1 \otimes 1_{\mathcal{A}_2})$ and $\pi_2(\alpha_2) = \pi(1_{\mathcal{A}_1} \otimes \alpha_2)$. Then

$$\pi(\alpha_1 \otimes 1_{\mathcal{A}_2}) = \pi((\alpha_1 \otimes 1_{\mathcal{A}_2})(1_{\mathcal{A}_1} \otimes \alpha_2)) = \pi(\alpha_1 \otimes 1_{\mathcal{A}_2})\pi(1_{\mathcal{A}_1} \otimes \alpha_2) = \pi_1(\alpha_1)\pi_2(\alpha_2)$$

On the other hand, every such pair of $*$-homomorphisms $\pi_j : \mathcal{A}_j \to B(H)$, $j = 1, 2$ with commuting ranges determines uniquely a $*$-homomorphism π on $\mathcal{A}_1 \otimes \mathcal{A}_2$ by setting $\pi(\alpha_1 \otimes \alpha_2) = \pi_1(\alpha_1)\pi_2(\alpha_2)$.

Theorem 5.21 *The following are equivalent for an element $p \in \mathcal{C}_{ns}$:*

1. *$p \in \mathcal{C}_{qc}$*

2. *There exists a state s of $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ such that $p = p_s$, where p_s is defined as in subsection 5.1.2.*

3. *There exist a Hilbert space H (possibly infinite-dimensional), a unit vector $\xi \in H$ as well as PVM's $\{E_{x,a}\}_{a \in A}$ and $\{F_{y,b}\}_{b \in B}$ on H, for $x \in X$ and $y \in Y$, such that: $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$, for all x, y, a, b and*

$$p(a, b \mid x, y) = ((E_{x,a}F_{y,b})\xi, \xi)\,, \text{ for all } x \in X,\ y \in Y,\ a \in A,\ b \in B$$

The map $s \to p_s$ is an (affine) isomorphism between the state space of $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ and $\mathcal{C}_{qc}$.

Proof: $1 \to 2$: Let H be a Hilbert space and $\{E_{x,a}\}_{a \in A}$ and $\{F_{y,b}\}_{b \in B}$ be commuting POVM's on H, for all $x \in X$ and all $y \in Y$ respectively, such that

$$p(a, b \mid x, y) = ((E_{x,a}F_{y,b})\xi, \xi)\,, \text{ for all } x \in X,\ y \in Y,\ a \in A,\ b \in B$$

Suppose that we have the following maps $\varphi : \mathcal{S}(n,k) \to B(H)$ and $\psi : \mathcal{S}(m,l) \to B(H)$ with $\varphi(e_{x,a}) = E_{x,a}$ and $\psi(f_{y,b}) = F_{y,b}$, for $x \in X,\ y \in Y,\ a \in A,\ b \in B$ by Proposition 5.18 these maps are unital and completely positive. Furthermore, since our POVM's commute these maps will have commuting ranges, i.e., $(\varphi, \psi) \in ucp(\mathcal{S}(n,k), \mathcal{S}(m,l))$. Now let $s : \mathcal{S}(n,k) \otimes \mathcal{S}(m,l) \to \mathbb{C}$ be the linear functional given by:

$$s(e_{x,a} \otimes f_{y,b}) = p(a, b \mid x, y), \text{ for } a \in A, b \in B, x \in X, y \in Y$$

so

$$s(e_{x,a} \otimes f_{y,b}) = ((E_{x,a}F_{y,b})\xi, \xi)\,, \text{ for } a \in A, b \in B, x \in X, y \in Y$$

hence

$$s(e_{x,a} \otimes f_{y,b}) = ((\varphi(e_{x,a})\psi(f_{y,b}))\xi, \xi)\,, \text{ for } a \in A, b \in B, x \in X, y \in Y$$

Thus for $u \in \mathcal{S}(n,k)$ and $v \in \mathcal{S}(m,l)$ we have that

$$s(u \otimes v) = (\varphi(u)\psi(v)\xi, \xi) = ((\varphi \cdot \psi)(u \otimes v)\xi, \xi)$$

(recall that $e_{x,a}$, $f_{y,b}$ are the generators of $\mathcal{S}(n,k)$, $\mathcal{S}(m,l)$ respectively and both s and the inner product are linear.) Notice that:

$$s(1 \otimes 1) = (\xi, \xi) = 1$$

and remembering the definition (see section 2.3) of the positive cones of the commuting tensor product of operator systems we see that

$$s(v_1 \otimes v_2) = \left(\underbrace{(\varphi \cdot \psi)(v_1 \otimes v_2)}_{\in B(H)^+} \xi, \xi \right) \geq 0, \ \forall v_1 \otimes v_2 \in (\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l))^+$$

(since s has range a subset of $\mathbb{C}$ positivity implies complete positivity).
We conclude that s is indeed a state on $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$.
Lastly, for $x \in X, \ y \in Y, \ a \in A, \ b \in B$:

$$p(a, b \mid x, y) = s(e_{x,a} \otimes f_{y,b}) := p_s(a, b \mid x, y)$$

so $p = p_s$.

$2 \to 3$: Let s be a state on $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ then by Theorem 4.4 s can be extended to a state $\tilde{s}$ on $\mathcal{A}(n,k) \otimes_{max} \mathcal{A}(m,l)$. Now the GNS representation of $\tilde{s}$ yields a Hilbert space H, a unit vector $\xi \in H$ and a $*$-representation $\pi : \mathcal{A}(n,k) \otimes_{max} \mathcal{A}(m,l) \to B(H)$ such that

$$\tilde{s}(u \otimes v) = (\pi(u \otimes v)\xi, \xi), \ u \in \mathcal{A}(n,k), v \in \mathcal{A}(m,l)$$

As in the *Remark* at the start of this subsection we may assume that $\pi = \pi_1 \pi_2$ for some $*$-homomorphisms $\pi_j : \mathcal{A}_j \to B(H), j = 1,2$ with commuting ranges. Hence, we have that

$$\tilde{s}(u \otimes v) = (\pi_1(u)\pi_2(v)\xi, \xi), \ u \in \mathcal{A}(n,k), v \in \mathcal{A}(m,l)$$

By Proposition 5.18 and from the fact that π_1, π_2 have commuting ranges we have that $(\pi_1(e_{x,a}))_{a \in A}$ and $(\pi_2(f_{y,b}))_{b \in B}$ are commuting families of POVM's. Since for $a \in A, b \in B, x \in X, y \in Y$

$$p(a, b \mid x, y) = p_s(a, b \mid x, y) = s(e_{x,a} \otimes f_{y,b}) = \tilde{s}(e_{x,a} \otimes f_{y,b}) = (\pi_1(e_{x,a})\pi_2(f_{y,b})\xi, \xi)$$

we conclude that $p \in \mathcal{C}_{qc}$.
In particular $(\pi_1(e_{x,a}))_{a \in A}$ and $(\pi_2(f_{y,b}))_{b \in B}$ are PVM's because $e_{x,a}$ and $f_{y,b}$ are projections in $\mathcal{A}(n,k)$ and $\mathcal{A}(m,l)$ respectively.

$3 \to 1$: Every PVM is a POVM, so this is straightforward.

Corollary 5.22 *The set $\mathcal{C}_{qc}$ is a closed convex set.*

Proof: Let T be the map $s \to p_s$ defined in Theorem 5.21. This is an affine map between the state space of $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ and $\mathcal{C}_{qc}$. To see this let $\lambda \in [0,1]$ and s_1, s_2 be states on $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$. Then, $(1-\lambda)s_1 + \lambda s_2$ is a state on $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ (the state space of an operator system is convex) and

$$\begin{aligned} T((1-\lambda)s_1 + \lambda s_2) = p_{(1-\lambda)s_1 + \lambda s_2} &:= ((1-\lambda)s_1 + \lambda s_2)(e_{x,a} \otimes f_{y,b}) \\ &= (1-\lambda)s_1(e_{x,a}) + \lambda s_2(f_{y,b}) \\ &= (1-\lambda)p_{s_1} + \lambda p_{s_2} \\ &= (1-\lambda)T(s_1) + \lambda T(s_2) \end{aligned}$$

The state space of $\mathcal{S}(n,k) \otimes_c \mathcal{S}(m,l)$ is convex and compact when equipped with the weak-* topology. In this case, it is easy to see that T is also a homeomorphism. Consequently $\mathcal{C}_{qc}$ **too is convex** and compact, and therefore **closed**.

5.1.4 Approximately Quantum Correlations

In this subsection we will give an alternative description of the correlations of the class $\mathcal{C}_{qa}$. To this end we will make use of the following Lemma.

Lemma 5.23 *Assume that H be a Hilbert space and $\mathcal{A} \subseteq B(H)$ is a unital C^*-algebra. Let $\mathcal{M} = \{\omega_\xi : \xi \in H, ||\xi|| = 1\}$ denote the vector state space of $\mathcal{A}$ (where $\omega_\xi : \mathcal{A} \to \mathbb{C} : \alpha \to (\alpha\xi, \xi)$) and $\mathcal{S}(\mathcal{A})$ denote the state space of $\mathcal{A}$. Then the convex hull of $\mathcal{M}$ is weak $*$-dense in $\mathcal{S}(\mathcal{A})$.*

Proof: Obviously, $\mathcal{M} \subseteq \mathcal{S}(\mathcal{A})$ and since $\mathcal{A}$ is unital the weak-$*$ limit of states is a state, so the weak-* convex hull of $\mathcal{M}$ denoted $\widetilde{\mathcal{M}}$ is contained in $\mathcal{S}(\mathcal{A})$. Suppose that there exists a state s in $\mathcal{S}(\mathcal{A})$ such that $s \notin \widetilde{\mathcal{M}}$. By the Hahn-Banach separation theorem there exist a weak-$*$ continuous functional $\varphi : \mathcal{A}^d \to \mathbb{C}$ and a real number $\beta \in \mathbb{R}$ such that:

$$Re(\varphi(\sigma)) < \beta < Re(\varphi(s)), \ \forall \sigma \in \widetilde{\mathcal{M}} \tag{I}$$

Now using the fact that weak-$*$ continuous functionals on $\mathcal{A}^d$ are given by evaluation at some element of $\mathcal{A}$ we have that: $\varphi = \hat{\alpha}$, for some $\alpha \in \mathcal{A}$ (where $\alpha(f) = f(\alpha), \ f \in \mathcal{A}^d$). Moreover, we know that α can be written as $\alpha = \alpha_1 + i\alpha_2$, for $\alpha_1, \alpha_2 \in \mathcal{A}_h$ hence for every $f \in (\mathcal{A}^d)^+$, $\varphi(f) = \hat{\alpha}(f) = f(\alpha) = f(\alpha_1 + i\alpha_2) = f(\alpha_1) + if(\alpha_2)$. Thus, $Re(\varphi(f) = f(\alpha_1), \forall f \in (\mathcal{A}^d)^+$ and relation (I) becomes:

$$\sigma(\alpha_1) < \beta < s(\alpha_1), \ \forall \sigma \in \widetilde{\mathcal{M}}$$

Subsequently, we have that

$$(\alpha_1\xi, \xi) < \beta = (\beta \cdot Id_H \xi, \xi)\,, \text{ for every unit vector } \xi \text{ in } H$$

Now, let η be an arbitrary vector in H, then $\frac{\eta}{||\eta||} \in H$ will be a unit vector so

$$\frac{1}{||\eta||^2}\,(\alpha_1\eta, \eta) < \frac{1}{||\eta||^2}\,(\beta Id_H \eta, \eta)$$

i.e,

$$(\alpha_1\eta, \eta) < (\beta \cdot Id_H \eta, \eta)$$

which means that in $\mathcal{A}$ we have that $\alpha_1 < \beta \cdot Id_H$ (inequality involving operators). However, since s is a state then $s(\alpha_1) < s(\beta \cdot Id_H) = \beta s(1_{\mathcal{A}}) = \beta$, which is a contradiction.

Theorem 5.24 *Let $p \in \mathcal{C}_{ns}$ the following are equivalent:*

1. $p \in \mathcal{C}_{qa}$
2. *There exists a state s of $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l)$ such that $p = p_s$*

The map $s \to p_s$ is an (affine) isomorphism between the state space of $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l)$ and $\mathcal{C}_{qa}$.

Proof: $2 \to 1$ We know that the minimal operator system tensor product is injective and that $\mathcal{S}(n,k) \subseteq \mathcal{A}(n,k)$, $\mathcal{S}(m,l) \subseteq \mathcal{A}(m,l)$ so the embedding $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l) \subseteq \mathcal{A}(n,k) \otimes_{min} \mathcal{A}(m,l)$ is a complete order isomorphism onto its range. Thus s which is a state of $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l)$ can be extended to a state $\tilde{s}$ of $\mathcal{A}(n,k) \otimes_{min} \mathcal{A}(m,l)$. Now let $\pi_1 : \mathcal{A}(n,k) \to B(H_1)$ and $\pi_2 : \mathcal{A}(m,l) \to B(H_2')$ be faithful (one to one) $*$-representations, then the representation

$$\pi_1 \otimes \pi_2 : \mathcal{A}(n,k) \otimes_{min} \mathcal{A}(m,l) \to B(H_1 \otimes H_2') : \alpha_1 \otimes \alpha_2 \to \pi_1(\alpha_1) \otimes \pi_2(\alpha_2)$$

is also faithful. Set $\mathcal{B} := (\pi_1 \otimes \pi_2)(\mathcal{A}(n,k) \otimes_{min} \mathcal{A}(m,l))$.
Let $E_{x,a} = \pi_1(e_{x,a}),\ x \in X, a \in A$ and $F'_{y,b} = \pi_2(f_{y,b}),\ y \in Y, b \in B$, then by Lemma 5.18 we have that for every $x \in X$, $\{E_{x,a}\}_{a \in A}$ is a PVM on H_1 and respectively for every $y \in Y$, $\{F_{y,b}\}_{b \in B}$ is a PVM on H_2'.

Consider a state s satisfying condition 2 and take $\epsilon > 0$ arbitrary. Call $\tilde{s}$ the state s, when considered as a state on $\mathcal{B}$. Lemma 5.23 tells us that $\tilde{s}$ belongs to the weak* closure of the convex hull of the set of all vector states of $\mathcal{B}$. Thus there exist unit vectors $\xi_j \in H_1 \otimes H_2',\ j = 1, \dots, r$ and positive scalars λ_j with $\sum_{j=1}^{r} \lambda_j = 1$ such that

$$|\tilde{s}(e_{x,a} \otimes f_{y,b}) - \sum_{j=1}^{r} \lambda_j \left((E_{x,a} \otimes F'_{y,b})\xi_j, \xi_j\right)| < \epsilon, \text{ for all } a \in A, b \in B, x \in X, y \in Y$$

(Notice that $\Big((E_{x,a} \otimes F'_{y,b})\xi_j, \xi_j\Big) = ((\pi_1 \otimes \pi_2)(e_{x,a} \otimes f_{y,b}))\xi_j, \xi_j)$ which is a vector state of $\mathcal{B}$)

Let $H_2 = H_2' \otimes \mathbb{C}^r$ and $\xi = \sum_{j=1}^{r} \sqrt{\lambda_j} \xi_j \otimes e_j$ (where $\{e_j\}$ is the standard basis of $\mathbb{C}^r$), clearly ξ is a unit vector in H_2. Furthermore, let $F_{y,b} = F'_{y,b} \otimes I_r,\ y \in Y, b \in B$ then $\{F_{y,b}\}_{b \in B}$ is a PVM on H_2 for every $y \in Y$. Indeed,

$$F_{y,b}^2 = (F'_{y,b} \otimes I_r)(F'_{y,b} \otimes I_r) = (F'_{y,b})^2 \otimes (I_r)^2 = F'_{y,b} \otimes I_r = F_{y,b}$$

and

$$F_{y,b}^* = (F'_{y,b})^* \otimes I_r^* = F'_{y,b} \otimes I_r = F_{y,b}$$

and

$$\sum_{b \in B} F_{y,b} = \sum_{b \in B} F'_{y,b} \otimes I_r = I_{H_2'} \otimes I_r = I_{H_2}.$$

Moreover, we now have

$$|\tilde{s}(e_{x,a} \otimes f_{y,b}) - ((E_{x,a} \otimes F_{y,b})\xi, \xi)| < \epsilon, \text{ for all } a \in A, b \in B, x \in X, y \in Y \quad \text{(I)}$$

Hence if we prove the following claim we will complete our proof:

CLAIM: Let H_1 and H_2 be two Hilbert spaces (not necessarily finite dimensional), $\xi \in H = H_1 \otimes H_2$ a unit vector and $\{E_{x,a}\}_{a\in A}$ (respectively $\{F_{y,b}\}_{b\in B}$) a POVM on H_1 (resp. H_2) for every $x \in X$ (resp. $y \in Y$). Then the correlation p defined by

$$p(a, b \mid x, y) = ((E_{x,a} \otimes F_{y,b}\xi, \xi)\,, \text{ for all } x \in X,\ y \in Y,\ a \in A,\ b \in B$$

is an element of $\mathcal{C}_{qa}$.

Proof of Claim: Assume that $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ be nets of finite dimensional projections that converge in the strong operator topology to the identity operators on H_1 and H_2, respectively (by relabelling the index sets of the nets, we can assume that these sets coincide). Now set $E^i_{x,a} = P_i E_{x,a} \restriction P_i H_1$ and $F^i_{y,b} = Q_i F_{y,b} \restriction Q_i H_2$, then and

$$E^i_{x,a} \text{ is a positive operator on } P_i H_1 \text{ for all } a \text{ in } A$$

and

$$\sum_{a\in A} E^i_{x,a} = \sum_{a\in A} P_i E_{x,a} \restriction P_i H_1 = P_i I_{H_1} \restriction P_i H_1 = I_{P_i H_1}$$

so $\{E^i_{x,a}\}_{a\in A}$ is a POVM on the Hilbert space $P_i H_1$ which is finite dimensional. Similarly $\{F^i_{y,b}\}_{b\in B}$ is a POVM on the finite dimensional Hilbert space $Q_i H_2$. In addition if (for each i) we call p_i the quantum non-signalling correlation arising from the above POVM's and the unit vector $\xi_i = \frac{P_i\otimes Q_i \xi}{||P_i\otimes Q_i\xi||}$ as in 2 of Definition 5.11 then

$$\begin{aligned}
p_i &= ((P_i E_{x,a} \otimes Q_i F_{y,b})\xi_i, \xi_i) \\
&= \frac{1}{\|(P_i \otimes Q_i)\xi\|^2}\,((P_i E_{x,a} \otimes Q_i F_{y,b})(P_i \otimes Q_i)\xi, (P_i \otimes Q_i)\xi) \\
&= \frac{1}{\|(P_i \otimes Q_i)\xi\|^2}\,((P_i E_{x,a} P_i \otimes Q_i F_{y,b} Q_i)\xi, (P_i \otimes Q_i)\xi) \\
&\xrightarrow{i\in I} ((I_{H_1} E_{x,a} I_{H_1} \otimes I_{H_2} F_{y,b} I_{H_2})\xi, (I_{H_1} \otimes I_{H_2})\xi) \\
&= ((E_{x,a} \otimes F_{y,b})\xi, \xi) = p
\end{aligned}$$

since the net $(P_i \otimes Q_i)$ converges strongly to the identity and is uniformly bounded.

Hence p is a limit of elements of $\mathcal{C}_q$, i.e., $p \in \mathcal{C}_{qa}$.

$1 \to 2$: Since the state space of an operator system is weak star compact (Banach-Alaoglu) we can assume that $p \in \mathcal{C}_q$ (then by taking limits we obtain the result for correlations in $\mathcal{C}_{qa}$).
Let H_1, H_2 be Hilbert spaces, $\xi \in H_1 \otimes H_2$ unit vector and POVM's $\{E_{x,a}\}_{a\in A}$ on H_1, $x \in X$ and $\{F_{y,b}\}_{b\in B}$ on H_2, $y \in Y$, such that

$$p(a, b \mid x, y) = ((E_{x,a} \otimes F_{y,b}\xi, \xi)\,, \text{ for all } x \in X,\ y \in Y,\ a \in A,\ b \in B$$

Using Proposition 5.18 we obtain unital completely positive maps

$$\varphi : \mathcal{S}(n,k) \to B(H_1) : e_{x,a} \to E_{x,a}$$

and

$$\psi : \mathcal{S}(m,l) \to B(H_2) : f_{y,b} \to F_{y,b}$$

The minimal tensor product of operator systems is functorial so the map $\varphi \otimes \psi : \mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l) \to B(H_1) \otimes_{min} B(H_2)$ is unital and completely positive. Thus, the linear functional $s : \mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l) \to \mathbb{C}$ given by

$$s(v_1 \otimes v_2) = ((\varphi(v_1) \otimes \psi(v_2))\xi, \xi)\,,\ v_1 \in \mathcal{S}(n,k), v_2 \in \mathcal{S}(m,l)$$

is positive (and its range is a subset of a commutative C^*-algebra consequently it is completely positive) and obviously unital, i.e., it is a state on the tensor product $\mathcal{S}(n,k) \otimes_{min} \mathcal{S}(m,l)$. Finally, it is clear that $p = p_s$

6 Distinguishing between correlation sets

In this Chapter we will demonstrate a number of separations between the correlation sets we defined previously. Remember that for each $k, n \in \mathbb{N}$ we have the following sequence of inclusions

$$\mathcal{C}_{loc}(n,k) \subseteq \mathcal{C}_q(n,k) \subseteq \mathcal{C}_{qa}(n,k) \subseteq \mathcal{C}_{qc}(n,k) \subseteq \mathcal{C}_{ns}(n,k)$$

our aim is to prove that the above inclusions are strict. These separations combined with the way we defined the various correlations sets via tensor products will in turn allow us to obtain the following strict inclusions

$$\mathcal{S} \otimes_{max} \mathcal{T} \subset \mathcal{S} \otimes_c \mathcal{T} \subset \mathcal{S} \otimes_{min} \mathcal{T}, \quad \text{for } \mathcal{S}, \mathcal{T} \text{ operator systems}$$

Since C_{qa} is the closure of C_q and we know that C_q is not closed we have that

$$C_q \neq C_{qa}$$

The other inequalities, as we will see, are not so easy to obtain.

The inequality $\mathcal{C}_{qa} \neq \mathcal{C}_{qc}$ was proved quite recently in ([10]). This inequality will not be studied here as it requires techniques not mentioned in this paper.

Suppose that we have a finite input-output game $\mathcal{G} = (X, Y, A, B, \lambda)$, let t denote one of $\{loc, q, qa, qc, ns\}$ and $\pi : X \times Y \to [0,1]$ be a probability density. We introduce the following quantity which will help us in our attempt to separate the correlation sets

$$\omega_t(\mathcal{G}, \pi) = \sup\{ \sum_{(x,y)\in X\times Y} \sum_{(a,b)\in A\times B} \pi(x,y)\lambda(x,y,a,b)p(a,b \mid x,y) : p \in \mathcal{C}_t\}$$

We call $\omega_t(\mathcal{G}, \pi)$ the t-value of the game and we set $\omega_t(\mathcal{G}) = \omega_t(\mathcal{G}, \pi_u)$, where π_u denotes the uniform distribution.
The idea is to find games whose value depends on the choice of t (i.e. they have different values for different t).

6.1 Separation of local and quantum correlations

We will move towards proving that $\mathcal{C}_{loc}(2,2) \neq \mathcal{C}_q(2,2)$ to do so we will consider the CHSH game which was introduced in 1969 by the physicists Clauser, Horne, Shimony, and Holt ([3]):

The CHSH game : Let $X = Y = A = B = 0, 1$ and

$$\lambda = \begin{cases} 1, & \text{if } a + b = xy(mod2) \\ 0, & \text{otherwise} \end{cases}$$

Proposition 6.1 *We have the following:*

1. $\omega_{loc}(CHSH) = \frac{3}{4}$
2. $\omega_q(CHSH) \geq \frac{1}{2} + \frac{\sqrt{2}}{4}$

Proof: Firstly notice that the players win the game in the following scenarios:

1. If $x = y = 0$ and they answer identically.
2. If $x = 1, y = 0$ and they answer identically.
3. If $x = 0, y = 1$ and they answer identically.
4. If $x = y = 1$ and their answers differ.

Since the extreme points of the convex set $\mathcal{C}_{loc}$ are precisely the strategies of the form $p_{f,g}$ where (f, g) is a deterministic strategy for the game and

$$p_{f,g}(a, b \mid x, y) = \begin{cases} 1, & \text{if } a = f(x) \text{ and } b = g(y) \\ 0, & \text{otherwise} \end{cases}$$

by enumerating the deterministic strategies for the CHSH game we can deduce its local value. Now a deterministic strategy for this particular game is a pair of functions $f : \{0, 1\} \to \{0, 1\}$ and $g : \{0, 1\} \to \{0, 1\}$ which determine Alice and Bob's responses respectively. Since for each player there are four functions of that form we see that the (maximum) number of all possible pairs of strategies for Alice and Bob is sixteen, while the set of all possible pairs of questions has four elements. It is not hard to verify that for each such strategy there exists a pair (x, y) of questions that will make the strategy fail (for example if they choose the strategy in which they always answer 1 then they win in every case except for the one in which $x = y = 1$). This shows that for every choice of (f, g) we always have $p_{f,g}(a, b \mid x, y) = 0$ for one pair of questions (x, y). Consequently, we deduce that

$$\omega_{loc}(CHSH) = \frac{1}{4}(1 + 1 + 1) = \frac{3}{4}$$

On the other hand, let $H_1 = H_2 = \mathbb{C}^2$ and let $\{e_0, e_1\}$ denote the standard basis of $\mathbb{C}^2$, we consider the following

$$e = \frac{1}{\sqrt{2}}(e_0 + e_1)\ ,\ f_0 = \cos(\frac{\pi}{8})e_0 + sin(\frac{\pi}{8})e_1\ ,\ f_1 = \cos(\frac{\pi}{8})e_0 - sin(\frac{\pi}{8})e_1$$

and the maximally entangled vector in $H_1 \otimes H_2$

$$\xi_{max} = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)$$

Moreover, for each $x \in X$ and for each $y \in Y$, let $\{E_{x,a}\}_{a \in A}$ and $\{F_{y,b}\}_{b \in B}$ be POVM's on $\mathbb{C}^2$ arising as follows:

$$E_{0,0} = e_0 e_0^* \qquad E_{1,0} = ee^* \qquad F_{0,0} = f_0 f_0^* \qquad F_{1,0} = f_1 f_1^*$$

$$E_{0,1} = I - e_0 e_0^* \qquad E_{1,1} = I - ee^* \qquad F_{0,1} = I - f_0 f_0^* \qquad F_{1,1} = I - f_1 f_1^*$$

Here we use the notation : $xy^*(z) = (z, y)\, x$.
Let p be the quantum correlation arising from this data, i.e.,

$$p(a, b \mid x, y) == ((E_{x,a} \otimes F_{y,b})\xi_{max}, \xi_{max}))\,,\ x \in X, y \in Y, a \in A, b \in B$$

We will compute the probabilities of winning for each pair of questions.

For $x = y = 0$:

$$\begin{aligned}
p(0, 0 \mid 0, 0) &= ((E_{0,0} \otimes F_{0,0})\xi_{max}, \xi_{max})) \\
&= \frac{1}{2}\,((E_{0,0} \otimes F_{0,0})(e_0 \otimes e_0) + (E_{0,0} \otimes F_{0,0})(e_1 \otimes e_1), e_0 \otimes e_0 + e_1 \otimes e_1) \\
&= \frac{1}{2}\,(e_0 e_0^*(e_0) \otimes f_0 f_0^*(e_0) + e_0 e_0^*(e_1) \otimes f_0 f_0^*(e_1), e_0 \otimes e_0 + e_1 \otimes e_1) \\
&= \frac{1}{2}\left(e_0 \otimes \cos(\frac{\pi}{8}) f_0 + 0, e_0 \otimes e_0 + e_1 \otimes e_1)\right) \\
&= \frac{1}{2}\cos(\frac{\pi}{8})\,(e_0 \otimes f_0, e_0 \otimes e_0) + \cos(\frac{\pi}{8})\,(e_0 \otimes f_0, e_1 \otimes e_1) \\
&= \cos(\frac{\pi}{8})\,(e_0, f_0) + 0 \\
&= \frac{1}{2}\cos(\frac{\pi}{8})\left(e_0, \cos(\frac{\pi}{8}) e_0 + sin(\frac{\pi}{8}) e_1\right) \\
&= \frac{1}{2}\cos^2(\frac{\pi}{8})
\end{aligned}$$

and

$$\begin{aligned}
p(1, 1 \mid 0, 0) &= ((E_{0,1} \otimes F_{0,1})\xi_{max}, \xi_{max}) \\
&= \frac{1}{2}\,(((I - e_0 e_0^*) \otimes (I - f_0 f_0^*)(e_0 \otimes e_0 + e_1 \otimes e_1), e_0 \otimes e_0 + e_1 \otimes e_1) \\
&= \frac{1}{2}(0 + 0 + 0 + \cos^2(\frac{\pi}{8})) \\
&= \frac{1}{2}\cos^2(\frac{\pi}{8})
\end{aligned}$$

Via similar calculations one can see that:

For $x = 1, y = 0$:

$$p(0, 0 \mid 1, 0) = \frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8})$$

and similarly

$$p(1, 1 \mid 1, 0) = \frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8})$$

For $x = 0, y = 1:$

$$p(0,0 \mid 0,1) = \frac{1}{2}\cos^2(\frac{\pi}{8})$$

and

$$p(1,1 \mid 0,1) = \frac{1}{2}\cos^2(\frac{\pi}{8})$$

For $x = y = 1:$

$$p(1,0 \mid 1,1) = \frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8}) = p(0,1 \mid 1,1)$$

We know set

$$p_1 := p(0,0 \mid 0,0) = p(1,1 \mid 0,0) = p(0,0 \mid 0,1) = p(1,1 \mid 0,1) = \frac{1}{2}\cos^2(\frac{\pi}{8})$$

and

$$p_2 := p(0,0 \mid 1,0) = p(1,1 \mid 1,0) = p(0,0 \mid 1,0) = p(1,1 \mid 0,1) = \frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8})$$

Then combining all of the above we obtain,

$$\begin{aligned}
\omega_q(CHSH) &\geq \frac{1}{4}(4p_1 + 4p_2) \\
&= \frac{1}{4}(2\cos^2(\frac{\pi}{8}) + 4(\frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8})) \\
&= \frac{1}{2}\cos^2(\frac{\pi}{8}) + \frac{1}{4} + \frac{1}{2}\cos(\frac{\pi}{8})\sin(\frac{\pi}{8}) \\
&= \frac{1}{4} + \frac{1}{2}\frac{\sqrt{2}}{4} + \frac{1}{4} + \frac{1}{2}\frac{\sqrt{2}}{4} \\
&= \frac{1}{2} + \frac{\sqrt{2}}{4}
\end{aligned}$$

Thus, we conclude that $\omega_q(CHSH) \geq \frac{1}{2} + \frac{\sqrt{2}}{4} > \frac{3}{4} = \omega_{loc}(CHSH)$.

It follows from Proposition 6.1 that when the players follow local strategies they win the CHSH game with probability 75% whereas if they follow a quantum one they win with probability (at least) 85%, so indeed $\mathcal{C}_{loc} \neq \mathcal{C}_q$.

The above example also concretely demonstrates that quantum strategies offer a significant advantage to the players in comparison to the deterministic ones. From a physics point of view the CHSH game shows that correlations arising from quantum entanglement cannot be explained by any non-quantum theory of physics (such as the local hidden variable theory).

6.2 Separation of quantum commuting and NS correlations

We will use the following result which is proved in [8, Theorem 6.3.]

Proposition 6.2 *For any operator system $\mathcal{R}$ the following holds*

$$\mathcal{R} \otimes_c \mathcal{S}(2,2) = \mathcal{R} \otimes_{min} \mathcal{S}(2,2)$$

Furthermore, recall that for every $k, n \in \mathbb{N}$ we defined

$$\mathcal{V}(n,k) =$$

$$\{((\lambda_{1,\alpha})_\alpha, \ldots, (\lambda_{n,\alpha})_\alpha) : (\lambda_{i,a})_\alpha \in \ell_k^\infty, \forall i \in [n] \text{ and } \sum_{\alpha=1}^k \lambda_{i,\alpha} = \sum_{\alpha=1}^k \lambda_{i',\alpha}, \forall i, i'\}$$

and we proved that $\mathcal{V}(n,k) \cong_{c.o.i.} \mathcal{S}(n,k)^d$.

The main objective of this chapter is to prove the following Theorem:

Theorem 6.3

$$\mathcal{V}(2,2) \otimes_{min} \mathcal{V}(2,2) \neq \mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2)$$

If the aforementioned theorem holds true then combining it with Proposition 5.19 we will obtain that

$$\mathcal{S}(2,2)^d \otimes_{min} \mathcal{S}(2,2)^d \neq \mathcal{S}(2,2)^d \otimes_{max} \mathcal{S}(2,2)^d$$

which by Theorem 2.22 and Proposition 6.2 is equivalent to

$$(\mathcal{S}(2,2) \otimes_{max} \mathcal{S}(2,2))^d \neq (\mathcal{S}(2,2) \otimes_{min} \mathcal{S}(2,2))^d = (\mathcal{S}(2,2) \otimes_c \mathcal{S}(2,2))^d$$

This inequality viewed in light of Propositions 5.20 and 5.21 implies that there exists a correlation p in $\mathcal{C}_{ns}$ which does not belong to $\mathcal{C}_{qc}$ and thus we obtain the desired separation.

For the proof of Theorem 6.3 several Lemma's will be required.

Firstly recall the following Lemma we proved in Section 2.1

Lemma [2.16] *Let $(\mathcal{S}, e_1)$ and $(\mathcal{T}, e_2)$ be operator systems. If $u \in \mathcal{S} \otimes_{max} \mathcal{T}$ is strictly positive, then there exists $n \in \mathbb{N}$, $A = [a_{ij}] \in M_n(\mathcal{S})^+$ and $B = [b_{ij}] \in M_n(\mathcal{T})^+$ such that*

$$u = \sum_{i,j=1}^n a_{ij} \otimes b_{ij}$$

.

Lemma 6.4 *Let $\mathcal{S}$ and $\mathcal{T}$ be finite dimensional vector spaces, in addition let $s_1, \ldots, s_m \in \mathcal{S}$ and $t_1, \ldots, t_n \in \mathcal{T}$. Moreover, let $[x_{ij}] \in M_p(\mathcal{S})$ and $[y_{ij}] \in M_p(\mathcal{T})$ and assume that for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ there exist $A_k, B_l \in M_p$ such that $[x_{ij}] = \sum_{k=1}^{m} A_k \otimes s_k$ and $[y_{ij}] = \sum_{l=1}^{n} B_l \otimes t_l$. Then, we have that*

$$\sum_{i,j}^{p} x_{ij} \otimes y_{ij} = \sum_{k=1}^{m}\sum_{l=1}^{n} Tr(A_k B_l^t) s_k \otimes t_l$$

Proof: For each $k = 1, \ldots, m$ we have that $A_k = [a_{ij}^{(k)}]_{i,j=1}^{p}$ where $a_{ij} \in \mathbb{C}, 1 \leq i, j \leq p$, and

$$A_k \otimes s_k = [a_{ij}^{(k)}] \otimes s_k = [a_{ij}^{(k)} s_k]$$

so

$$[x_{ij}] = \sum_{k=1}^{m} A_k \otimes s_k = \sum_{k=1}^{m} [a_{ij}^{(k)} s_k] = [\sum_{k=1}^{m} a_{ij}^{(k)} s_k]$$

and similarly

$$[y_{ij}] = \sum_{l=1}^{n} B_l \otimes t_l = \sum_{l=1}^{n} [b_{ij}^{(l)} t_l] = [\sum_{l=1}^{n} b_{ij}^{(l)} t_l]$$

for some $B_l = [b_{ij}^{(l)}] \in M_p$. Now,

$$\begin{aligned}
\sum_{i,j}^{p} x_{ij} \otimes y_{ij} &= \sum_{i,j}^{p} (\left(\sum_{k=1}^{m} a_{ij}^{(k)} s_k\right) \otimes \left(\sum_{l=1}^{n} b_{ij}^{(l)} t_l\right)) \\
&= \sum_{i,j}^{p} (\sum_{k=1}^{m}\sum_{l=1}^{n} a_{ij}^{(k)} s_k \otimes b_{ij}^{(l)} t_l) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n} (\sum_{i,j}^{p} a_{ij}^{(k)} b_{ij}^{(l)} s_k \otimes t_l) \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n} Tr(A_k B_l^t) s_k \otimes t_l
\end{aligned}$$

(the sums are all finite and the tensor product is bi-linear).

Now we will use both of the above in order to prove :

Lemma 6.5 *Let $\mathcal{S}$ and $\mathcal{T}$ be finite dimensional vector spaces, with linear bases $\{s_1, \ldots, s_m\} \subseteq \mathcal{S}$ and $\{t_1, \ldots, t_n\} \subseteq \mathcal{T}$ and let*

$$u = \sum_{k=1}^{m}\sum_{l=1}^{n} q_{k,l} s_k \otimes t_l \in \mathcal{S} \otimes \mathcal{T},\ q_{kl} \in \mathbb{C}$$

If u is strictly positive in $\mathcal{S} \otimes \mathcal{T}$ then there exist a $p \in \mathbb{N}$ as well as matrices $U_1 = \sum_{k=1}^{m} A_k \otimes s_k \in M_p(\mathcal{S})^+$ and $U_2 = \sum_{l=1}^{n} B_l \otimes t_l \in M_p(\mathcal{T})^+$, where for each $k = 1, \ldots, m$ and $l = 1, \ldots, n$: A_k, B_l are matrices in M_p such that $Tr(A_k B_l^t) = q_{k,l}$.

Proof: Suppose that $u = \sum_{k=1}^{m} \sum_{l=1}^{n} q_{k,l} s_k \otimes t_l \in \mathcal{S} \otimes \mathcal{T}$ is a strictly positive element of $M_p(\mathcal{S})$, then by Lemma 2.16 there exist a $p \in \mathbb{N}$, and positive elements $U_1 = [\alpha_{ij}]$ and $U_2 = [\beta_{ij}]$ in $M_p(\mathcal{S})$ and $M_p(\mathcal{T})$ respectively such that

$$u = \sum_{i,j}^{p} \alpha_{ij} \otimes \beta_{ij}$$

Since $M_p(\mathcal{S}) \cong_{c.o.i.} M_p \otimes \mathcal{S}$ and $M_p(\mathcal{T}) \cong_{c.o.i.} M_p \otimes \mathcal{T}$ we are able to write $U_1 = \sum_{k=1}^{m} A_k \otimes s_k$ and $U_2 = \sum_{l=1}^{n} B_l \otimes t_l$ where $A_k, B_l \in M_p$ for all $k = 1, \dots, m$ and all $l = 1, \dots, n$. Using Lemma 6.4 it is immediate that u can be written in the form we desire.

Observe that

$$\mathcal{V}(2,2) = \{((\lambda_{1,1}, \lambda_{1,2}), (\lambda_{2,1}, \lambda_{2,2})) : \lambda_{i,j} \in \mathbb{C} \text{ and } \sum_{j=1}^{2} \lambda_{i,j} = \sum_{j=1}^{2} \lambda_{i',j},\ \forall i, , i', j \in [2]\}$$

$$= \{(z_1, z_2, z_3, z_4) : z_i \in \mathbb{C},\ i \in [4] \text{ and } z_1 + z_2 = z_3 + z_4\}$$

Let $p \in \mathbb{N}$ then, using the identification $M_p(V(2,2)) = M_p \otimes V(2,2)$ we have that

$$M_p(\mathcal{V}(2,2)) = \{\sum_{i=1}^{4} X_i \otimes e_i : X_i \in M_p,\ X_1 + X_2 = X_3 + X_4,\ i \in [4]\}$$

Indeed, suppose that $p \in \mathbb{N}$ and $[\lambda_{mn}] \in M_p(\mathcal{V}(2,2))$ then

$$[\lambda_{mn}] = \sum_{m,n=1}^{p} E_{m,n} \otimes \lambda_{mn}$$

and for each $m, n \in [p]$ we have that $\lambda_{mn} = \sum_{i=1}^{4} b_i^{mn} e_i$ where $b_1^{mn} + b_2^{mn} = b_3^{mn} + b_4^{mn}$. Thus,

$$\sum_{m,n=1}^{p} E_{m,n} \otimes \lambda_{mn} = \sum_{m,n=1}^{p} E_{m,n} \otimes \sum_{i=1}^{4} b_i^{mn} e_i =$$

$$\sum_{m,n=1}^{p} \sum_{i=1}^{4} E_{m,n} \otimes b_i^{mn} e_i = \sum_{m,n=1}^{p} \sum_{i=1}^{4} b_i^{mn} E_{m,n} \otimes e_i =$$

$$\sum_{i=1}^{4} \left(\sum_{m,n=1}^{p} b_i^{mn} E_{m,n} \right) \otimes e_i$$

Now for $i = 1, 2, 3, 4$ set

$$X_i = \sum_{m,n=1}^{p} b_i^{mn} E_{m,n}$$

Then $[\lambda_{mn}] = \sum_{i=1}^4 X_i \otimes e_i$ and it is straightforward to see that $X_1 + X_2 = X_3 + X_4$.

Moreover, it follows from the above discussion that

$$M_p(\mathcal{V}(2,2))^+ = \{\sum_{i=1}^4 X_i \otimes e_i : X_1 + X_2 = X_3 + X_4,\ X_i \in M_p^+,\ \forall i \in [4]\}$$

To see this suppose that $(\sum_{i=1}^4 X_i \otimes e_i) \in (M_p \otimes l_4^\infty)^+$ and let $\xi \in \mathbb{C}^p$ then we have that for every $j = 1, \ldots, 4$:

$$\left(\sum_{i=1}^4 X_i \otimes e_i(\xi \otimes e_j), (\xi \otimes e_j) \right) \geq 0$$

and at the same time for every $j = 1, \ldots, 4$:

$$\left(\sum_{i=1}^4 X_i \otimes e_i(\xi \otimes e_j), (\xi \otimes e_j) \right) = \sum_{i=1}^4 (X_i \xi, \xi)\,(e_i e_j, e_j) = (X_j \xi, \xi)$$

Thus, $(X_j \xi, \xi) \geq 0$ for every $j = 1, \ldots, 4$ and every $\xi \in \mathbb{C}^p$ which proves our point.

The other direction is trivial.

The next proposition gives us a realization of the strictly positive elements of $\mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2)$.

Proposition 6.6 *Let $v = \sum_{i,j=1}^4 q_{ij} e_i \otimes e_j$ be a strictly positive element in $\mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2)$. Then there exist a $p \in \mathbb{N}$ and matrices $X_i, Y_j \in M_p^+$ satisfying: $X_1 + X_2 = X_3 + X_4$ and $Y_1 + Y_2 = Y_3 + Y_4 = I$ such that*

$$q_{ij} = Tr(X_i Y_j) \quad \textit{for}\ \ 1 \leq i, j \leq 4.$$

Proof: Using Lemma 6.5 we can find a $p \in \mathbb{N}$, $U_1 = \sum_{i=1}^4 X_i \otimes e_i$, $X_i \in M_p$ and $U_2 = \sum_{j=1}^4 B_j \otimes e_j$, $B_j \in M_p$ with $U_1, U_2 \in M_p(\mathcal{V}(2,2))^+$ such that $\sum_{i,j=1}^4 q_{ij} e_i \otimes e_j = \sum_{i,j=1}^4 Tr(X_i B_j^t) e_i \otimes e_j$. Furthermore, since $U_1, U_2 \in M_p(\mathcal{V}(2,2))^+$ we have that $X_i, B_j \in M_p^+$ and $X_1 + X_2 = X_3 + X_4$ and $B_1 + B_2 = B_3 + B_4$. Moreover, for each $i \in [4]$ we set $B_j = Y_j^t$ then the formulas above take the following form

$$q_{ij} = Tr(X_i (Y_j^t)^t) = Tr(X_i Y_j), \quad Y_1 + Y_2 = Y_3 + Y_4 \quad \text{and} \quad Y_j \in M_p^+$$

Let E be the projection onto $\ker(Y_1 + Y_2)^\perp$ where $^\perp$ denotes the orthogonal complement.

Claim 1: There exists a positive invertible matrix P such that:

$$\sum_{j=1}^4 P^{-1} Y_j P^{-1} = 2E$$

Proof of Claim 1: Firstly, notice that E is the orthogonal projection onto ker$(Y_1 + Y_2)^{\perp} = Im((Y_1+Y_2)^*) = Im(Y_1+Y_2)$= $Im\sqrt{Y_1+Y_2}$. We will make use of the following fact: If T is a positive semi-definite operator on a finite dimensional Hilbert space and π is the orthogonal projection onto $Im(T)$ then

$$\sqrt{T} = \sqrt{T}\pi = \pi\sqrt{T}$$

Proof : Let $x \in \mathbb{C}^p$ then $x = h_1 + h_2$ with $h_1 \in$ ker$(T)^{\perp} = Im(T^*) = Im(T)$ and $h_2 \in$ ker(T) so $\pi(x) = h_1$. Thus $\sqrt{T}\pi(x) = \sqrt{T}(h_1) = \sqrt{T}(x)$ (because ker$(\sqrt{T}) \subseteq$ ker(T)). On the other hand, since $Im(T)$= $Im\sqrt{T}$ we have that $\sqrt{T}(h_1) \in Im(T)$ therefore $\pi\sqrt{T}(x) = \pi\sqrt{T}(h_1) = \sqrt{T}(h_1) = \sqrt{T}\pi(x)$.

We apply the above for the operator $B := (Y_1 + Y_2)$ and for the projection E, so we obtain that

$$\sqrt{B} = \sqrt{B}E = E\sqrt{B}$$

which implies that

$$B = \sqrt{B}\sqrt{B} = \sqrt{B}\sqrt{B}E = \sqrt{B}E\sqrt{B}$$

Now consider the operator

$$Px = \begin{cases} \sqrt{B}x, & x \in \text{ker}(Y_1+Y_2)^{\perp} \\ x, & x \in \text{ker}(Y_1+Y_2) \end{cases}$$

then P is a positive and injective linear operator so it will be positive and invertible and the same will hold for its matrix (which we will denote by the same letter). Moreover, we have that $Y_1 + Y_2 = PEP$. Indeed, let let $x \in \mathbb{C}^p$ then it can be written as $x = h_1 + h_2$, for $h_1 \in$ ker$(Y_1 + Y_2)^{\perp}$ and $h_2 \in$ ker$(Y_1 + Y_2)$, hence

$$\begin{aligned}
(Y_1+Y_2)(x) &= (Y_1+Y_2)(h_1+h_2) \\
&= (Y_1+Y_2)(h_1) + (Y_1+Y_2)(h_2) \\
&= (Y_1+Y_2)(h_1) + 0 \\
&= B(h_1) + E(h_2) \\
&= \sqrt{B}E\sqrt{B}(h_1) + E(h_2) \\
&= PEP(h_1+h_2) \\
&= PEP(x)
\end{aligned}$$

(because $E(h_2) = 0$). Thus, $Y_1+Y_2 = PEP$ or equivalently $P^{-1}(Y_1+Y_2)P^{-1} = E$. Since $Y_1 + Y_2 = Y_3 + Y_4$ we also have that $P^{-1}(Y_3 + Y_4)P^{-1} = E$, the result now is immediate.

For each $i \in \{1, \ldots, 4\}$, set

$$\hat{Y}_i = P^{-1}Y_iP^{-1} \text{ and } \hat{X}_i = PX_iP$$

then for all $i, j \in \{1, \ldots, 4\}$ we have that $\hat{Y}_i$ and $\hat{X}_i$ are positive. Indeed, since P is positive and invertible its inverse is also positive and for A, B positive matrices we have that ABA is also positive ($(ABAx, x) = (B(Ax), (Ax)) \geq 0$).

In addition, notice that

$$\hat{Y}_1 + \hat{Y}_2 = \hat{Y}_3 + \hat{Y}_4 = E \quad \text{and} \quad \hat{X}_1 + \hat{X}_2 = \hat{X}_3 + \hat{X}_4 \tag{I}$$

(since: $Y_1 + Y_2 = PEP$ so $\hat{Y}_1 + \hat{Y}_2 := P^{-1}(Y_1 + Y_2)P^{-1} = P^{-1}PEPP^{-1} = E$)

and

$$\begin{aligned} Tr(X_iY_j) = Tr(PX_iY_jP^{-1}) = Tr(PX_iPP^{-1}Y_jP^{-1}) = \\ Tr(\hat{X}_i\hat{Y}_j) = Tr((E\hat{X}_iE)(E\hat{Y}_jE)) \end{aligned} \tag{II}$$

The last equality stems from the following claim :

Claim 2: $\hat{Y}_j = \hat{Y}_j E$.

Proof of Claim 2: E is the orthogonal projection onto $\ker(Y_1 + Y_2)^\perp = Im((Y_1 + Y_2)^*) = Im(Y_1 + Y_2)$. Let $x \in \mathbb{C}^p$ then it can be written as $x = h_1 + h_2$, for $h_1 \in Im(Y_1 + Y_2)$ and $h_2 \in \ker(Y_1 + Y_2)$ so

$$Y_1(x) = Y_1(h_1) + Y_1(h_2) = Y_1E(x) + Y_1(h_2)$$

It is also true that $\ker(Y_1 + Y_2) = \ker(Y_1) \bigcap \ker(Y_2)$ (because Y_1 and Y_2 are positive semi-definite) thus we have that $h_2 \in \ker(Y_1)$. Thus, $Y_1(h_2) = 0$ and so $Y_1(x) = Y_1E(x), \forall x \in \mathbb{C}^p$. In the same way it can be shown that $Y_j = Y_jE,\ j = 2, 3, 4$.

Now, once again let $x \in \mathbb{C}^p$, we can write $x = h_1 + h_2$ with h_1, h_2 defined as above. Then

$$PE(x) = PE(h_1 + h_2) = PE(h_1) + PE(h_2) = P(h_1) + 0 = \sqrt{B}(h_1)$$

and

$$EP(x) = EP(h_1) + EP(h_2) = E\sqrt{B}(h_1) = \sqrt{B}(h_1) = PE(x)$$

Thus, $EP = PE$. Multiplying with P^{-1} from left and right we see that $P^{-1}E = EP^{-1}$. Finally, we obtain that for every $x \in \mathbb{C}^p$ and for $j = 1, 2, 3, 4$:

$$\begin{aligned} \hat{Y}_jE(x) &:= P^{-1}Y_jP^{-1}E(x) \\ &= P^{-1}Y_jEP^{-1}(x) \\ &= P^{-1}Y_jP^{-1}(x) \\ &= \hat{Y}_j(x) \end{aligned}$$

It follows that $\hat{Y}_j = \hat{Y}_jE$. The proof of the Claim is now complete.

Furthermore, since $(\hat{Y}_j)^* = \hat{Y}_j,\ \forall j \in [4]$ and $E^* = E$ (E is an orthogonal projection onto a closed subspace) we also have that $E\hat{Y}_j = \hat{Y}_j = \hat{Y}_jE,\ j = 1, 2, 3, 4$. Thus,

$$E\hat{Y}_jE = E\hat{Y}_j = \hat{Y}_j$$

So for every $i,j \in \{1,\ldots,4\}$ we have that,

$$\hat{X}_i E \hat{Y}_j E = \hat{X}_i \hat{Y}_j$$

Notice that if we replace $\hat{X}_i$ and $\hat{Y}_j$ with $E\hat{X}_iE$ and $E\hat{Y}_jE$ respectively, the equations (I) and (II) still hold. Since E is a projection we can diagonalize it, in particular E can be written in the form: $diag\{1,\ldots,1,0,\ldots,0\}$ where the number of one's is equal to dim$(Im(E)) =$ dim(ker$(Y_1+Y_2)^{\perp}$) while the number of zeroes is equal to dim(ker E). This implies that we can regard the matrices $E\hat{X}_iE$ and $E\hat{Y}_iE$ as matrices of smaller size (by "cutting" the matrix at the point that all its rows and columns are zero) . Now, if we abuse the notation and denote these smaller matrices again by $E\hat{X}_iE$ and $E\hat{Y}_iE$, from (I) we have that

$$E\hat{Y_1}E + E\hat{Y_2}E = E\hat{Y_3}E + E\hat{Y_4}E = diag(1,\ldots,1)$$

Thus without loss of generality we may assume that $Y_1 + Y_2 = Y_3 + Y_4 = I$ or equivalently that $\sum_{i=1}^{4} Y_i = 2I$. The proof is now complete.
(If the original matrices do not satisfy our requirements we replace them with the matrices $E\hat{X}_iE$ and $E\hat{Y}_iE$ and regard them as matrices of smaller size in the way we described, for the smaller size matrices the Lemma is true in its entirety.)

Definition 6.7 *Let $A = [\alpha_{ij}]$ be a $n \times m$ matrix, the Frobenius norm of A is denoted by $\|\cdot\|_F$ and is given by*

$$\|A\|_F = \sqrt{\sum_{i,j} |\alpha_{ij}|^2} = \sqrt{Tr(A^*A)}$$

Note : In the space of $n \times m$ complex matrices we can define an inner product as follows, let $A = [\alpha_{ij}], B = [\beta_{ij}] \in M_{n,m}$ then

$$(A,B)_{M_n} = Tr(B^*A)$$

The induced norm of this inner product is the Frobenius norm. Since this is an inner product it will satisfy the Cauchy-Schwarz inequality, i.e.,

$$(A,B)_{M_n} \leq \|A\|_F \|B\|_F$$

Lemma 6.8 *Let $p \in \mathbb{N}$ and for $i,j \in \{1,\ldots,4\}$ suppose that $X_i, Y_j \in M_p^+$ with $X_1 + X_2 = X_3 + X_4$ and $Y_1 + Y_2 = Y_3 + Y_4 = I$. Moreover, for $i,j = 2$, set $q_{ij} = Tr(X_iY_j)$ and for $a,c \in \{0,2\}$, set*

$$S_{a,c}(j,k) = \min_{b\in\{0,2\}} \sum_{i=1}^{2} \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}}$$

Then,

$$Tr(X_1 + X_2) \leq \min_{a,c} \sum_{j,k=1}^{2} S_{a,c}(j,k)$$

Proof: We set $\beta = Tr(X_1 + X_2) = Tr(X_3 + X_4)$. Since $Y_1 + Y_2 = Y_3 + Y_4 = I$ and for all square matrices A, B we have that $Tr(A+B) = Tr(A) + Tr(B)$, we see that for all $a, b, c \in \{0, 2\}$:

$$\begin{aligned}\beta &= Tr((Y_{a+1} + Y_{a+2})(X_{b+1} + X_{b+2})(Y_{c+1} + Y_{c+2})) \\ &= \sum_{i,j,k=1}^{2} Tr(Y_{a+j} X_{b+i} Y_{c+k}) \\ &= \sum_{i,j,k=1}^{2} Tr((Y_{a+j} X_{b+i}^{\frac{1}{2}})(X_{b+i}^{\frac{1}{2}} Y_{c+k})) \\ &= \sum_{i,j,k=1}^{2} Tr((X_{b+i}^{\frac{1}{2}} Y_{a+j})^* (X_{b+i}^{\frac{1}{2}} Y_{c+k})) \\ &\leq \sum_{j,k=1}^{2} \sum_{i=1}^{2} \|X_{b+i}^{\frac{1}{2}} Y_{a+j}\|_F \|X_{b+i}^{\frac{1}{2}} Y_{c+k}\|_F\end{aligned}$$

(for all a, j, b, i, $Y_{a+j}, X_{b+i}^{\frac{1}{2}}$ are p.s.d. matrices so Hermitian)

Conversely, if X and Y are positive (semi-definite) matrices and $Y \leq I$, then

$$\|X^{\frac{1}{2}} Y\|_F^2 = Tr(X^{\frac{1}{2}} Y^2 X^{\frac{1}{2}}) \leq Tr(X^{\frac{1}{2}} Y X^{\frac{1}{2}}) = \|X^{\frac{1}{2}} Y^{\frac{1}{2}}\|_F^2$$

and

$$\|X^{\frac{1}{2}} Y^{\frac{1}{2}}\|_F^2 = Tr(X^{\frac{1}{2}} Y X^{\frac{1}{2}}) = Tr(XY)$$

Combining these results we have that

$$\begin{aligned}\beta &\leq \sum_{j,k=1}^{2} \sum_{i=1}^{2} \|X_{b+i}^{\frac{1}{2}} Y_{a+j}^{\frac{1}{2}}\|_F \|X_{b+i}^{\frac{1}{2}} Y_{c+k}^{\frac{1}{2}}\|_F \\ &\leq \sum_{j,k=1}^{2} \sum_{i=1}^{2} \sqrt{Tr(X_{b+i} Y_{a+j})} \sqrt{Tr(X_{b+i} Y_{c+k})} \\ &= \sum_{j,k=1}^{2} \sum_{i=1}^{2} \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}}\end{aligned}$$

and this holds for every $b \in \{0, 2\}$. Taking minimum over all $b \in \{0, 2\}$ we obtain the desired inequality.

Lemma 6.9 *Let* $u = \sum_{i,j=1}^{4} q_{ij} e_i \otimes e_j \in (\mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2))^+$. *For* $a, c \in \{0, 2\}$ *set*

$$S_{a,c}(j,k) = \min_{b \in \{0,2\}} \sum_{i=1}^{2} \sqrt{q_{b+i,a+j}} \sqrt{q_{b+i,c+k}}$$

Then, for all $d \in \{0,2\}$ we have that

$$\sum_{i,j=1}^{2} q_{d+i,d+j} \leq \min_{a,c} \sum_{i,j=1}^{2} S_{a,c}(j,k)$$

Proof: Given $\delta > 0$ the element

$$u + \delta 1 \otimes 1 = \sum_{i,j}^{2} (q_{ij} + \delta) e_i \otimes e_j$$

is strictly positive, so by Proposition 6.6 there exist a $p \in \mathbb{N}$ and matrices $X_i, Y_j \in M_p^+$ with $X_1 + X_2 = X_3 + X_4$ and $Y_1 + Y_2 = Y_3 + Y_4 = I$ such that

$$q_{ij} + \delta = Tr(X_i Y_j),\ 1 \leq i,j \leq 2$$

By Lemma 6.8 we obtain that,

$$Tr(X_1 + X_2) \leq \min_{a,c} \sum_{i,j=1}^{2} S_{a,c}(j,k) \tag{I}$$

Moreover, observe that for $d \in \{0,2\}$

$$Tr(X_1 + X_2) = Tr((X_{d+1} + X_{d+2})\underbrace{(Y_{d+1} + Y_{d+2})}_{I}) = \sum_{i,j=1}^{2} (q_{d+i,d+j} + \delta)$$

Letting $\delta \to 0$ and using relation (I) we obtain the desired result.

We are now in a position to prove Theorem 6.3,

Proof of Theorem 6.3 : We want to show that :

$$\mathcal{V}(2,2) \otimes_{min} \mathcal{V}(2,2) \neq \mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2)$$

We begin by identifying $\ell_4^\infty \otimes \ell_4^\infty$ with M_4 via the map

$$e_i \otimes e_j \to e_i e_j^* = E_{i,j},\ 1 \leq i,j \leq 4$$

Under this identification $\mathcal{V}(2,2) \otimes \mathcal{V}(2,2)$ coincides with the space of all 4×4-matrices such that :

1. The sum of the first two entries in each row is equal to the sum of the last two entries
2. The sum of the first two entries in each column is equal to the sum of the last two entries

To see this let $v \in \mathcal{V}(2,2) \otimes \mathcal{V}(2,2)$ then it can be written as

$$v = \sum_{i,j} \alpha_i e_i \otimes \beta_j e_j = \sum_{i,j} \alpha_i \beta_j e_i \otimes e_j$$

where $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ and $\beta_1 + \beta_2 = \beta_3 + \beta_4$. Using the aforementioned identification we that v corresponds to the following element of M_4 :

$$\sum_{i,j} \alpha_i \beta_j E_{i,j} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\beta_3 & \alpha_1\beta_4 \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\beta_3 & \alpha_2\beta_4 \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_3\beta_3 & \alpha_3\beta_4 \\ \alpha_4\beta_1 & \alpha_4\beta_2 & \alpha_4\beta_3 & \alpha_4\beta_4 \end{pmatrix}$$

Now it is straightforward to see that the above matrix satisfies both of the conditions given above. Moreover, we claim that this identification is a (complete) order isomorphism between the operator systems $(l_4^\infty \otimes_{min} l_4^\infty)^+$ and $\mathcal{M}_4$, where $\mathcal{M}_4$ is the space of the 4×4 complex matrices with involution given by

$$([v_{ij}]_{i,j=1}^4)^* = [v_{ij}^*]_{i,j=1}^4, \ \ [v_{ij}]_{i,j=1}^4 \in M_4$$

and is equipped with the cone of the real matrices with non-negative entries (the fact that $\mathcal{M}_4$ is indeed an operator system is very easy to prove).
Indeed, call the aforementioned identification Φ. The fact that Φ is unital is trivial. Moreover, notice that with respect to the usual involution on $\ell_4^\infty \otimes_{min} \ell_4^\infty$ and the involution we defined on $\mathcal{M}_4$, Φ is involution preserving.
Now set

$$K = \{[b_{ij}] \in M_4 : b_{ij} \geq 0, \ \forall i, j\}$$

We shall show that Φ and Φ^{-1} are positive maps with respect to K.
Suppose that $\Phi(\sum_{i,j} \alpha_i e_i \otimes \beta_j e_j) \in K$ this means that $[\alpha_i \beta_j] \in K$. Thus if $k, m \in \mathbb{N}$ and $\phi_1 \in UCP(\ell_4^\infty, M_k)$ and $\phi_2 \in UCP(\ell_4^\infty, M_m)$ then

$$(\phi_1 \otimes \phi_2)(\sum_{i,j} \alpha_i e_i \otimes \beta_j e_j) = \sum_{i,j} \underbrace{\alpha_i \beta_j}_{\geq 0} \phi_1(e_i) \otimes \phi_2(e_j) \in M_{km}^+$$

It follows that $\sum_{i,j} \alpha_i e_i \otimes \beta_j e_j \in (\ell_4^\infty \otimes_{min} \ell_4^\infty)^+$.

Conversely, let $\sum_{i,j} \alpha_i e_i \otimes \beta_j e_j \in (\ell_4^\infty \otimes_{min} \ell_4^\infty)^+$. Notice that for all $l, q \in \{1, \ldots, 4\}$ the following maps are unital completely positive from ℓ_4^∞ to $\mathbb{C}$

1. $\phi_l : \ell_4^\infty \to \mathbb{C}$ projection to the l-th coordinate

2. $\phi_q : \ell_4^\infty \to \mathbb{C}$ projection to the q-th coordinate

Thus

$$(\phi_l \otimes \phi_q)(\sum_{i,j} \alpha_i e_i \otimes \beta_j e_j) \geq 0, \ \forall l, q \in [4]$$

So for all $l, q \in \{1, \ldots, 4\}$ we have that

$$0 \leq \sum_{i,j} \alpha_i \phi_l(e_i) \otimes \beta_j \phi_q(e_j) = \sum_{i,j} \alpha_i \delta_{il} \otimes \beta_j \delta_{jq} = \alpha_l \beta_q$$

This implies that the matrix $[\alpha_i \beta_j]_{i,j} \in K$ and completes the proof of our claim.

Since $(\ell_4^\infty \otimes_{min} \ell_4^\infty)^+$ is the cone of all real matrices in M_4 with non-negative entries, we see that the matrix

$$Q = [q_{ij}] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is an element of $(\ell_4^\infty \otimes_{min} \ell_4^\infty)^+$ and it satisfies conditions 1 and 2 above, so

$$Q \in (\mathcal{V}(2,2) \otimes \mathcal{V}(2,2)) \cap (\ell_4^\infty \otimes_{min} \ell_4^\infty)^+ = (\mathcal{V}(2,2) \otimes_{min} \mathcal{V}(2,2))^+$$

Now we will work towards proving that $Q \notin (\mathcal{V}(2,2) \otimes_{max} \mathcal{V}(2,2))^+$, a fact that will complete the proof.
Let $d \in \{0, 2\}$ then $\sum_{i,j=1}^{2} q_{d+i,d+j} = 1 + 1 = 2$. Now set $a = 0, c = 2$, take $b = 0$ and consider the quantity $S_{a,c}(j,k)$ we defined in Lemma 6.9, we see that

$$0 \leq S_{0,2}(2,1) \leq \sum_{i=1}^{2} \sqrt{q_{i,2}} \sqrt{q_{i,3}} = 0 \cdot 1 + 1 \cdot 0 = 0$$

and

$$0 \leq S_{0,2}(1,2) \leq \sum_{i=1}^{2} \sqrt{q_{i,1}} \sqrt{q_{i,4}} = 1 \cdot 0 + 0 \cdot 1 = 0$$

On the other hand, if $b = 2$ then we have that

$$0 \leq S_{0,2}(1,1) \leq \sum_{i=1}^{2} \sqrt{q_{2+i,1}} \sqrt{q_{2+i,3}} = 0 \cdot 1 + 1 \cdot 0 = 0$$

and

$$0 \leq S_{0,2}(2,2) \leq \sum_{i=1}^{2} \sqrt{q_{2+i,2}} \sqrt{q_{2+i,4}} = 1 \cdot +0 \cdot 1 = 0$$

Thus in any case $\sum_{i,j=1}^{2} q_{d+i,d+j} = 2 > 0 = \min_{a,c} \sum_{i,j=1}^{2} S_{a,c}(j,k)$ which violates Lemma 6.9.

7 Disambiguation

The correlation classes in Definition 5.11 were defined via POVM's however some authors chose to use PVM's instead of POVM's while defining these classes of correlations. The main reason for this is that since the PVM's are pairwise orthogonal projections the computations with a PVM are much better than the computations with a POVM. In this chapter we shall see that in both cases we obtain the same correlation sets. The correlation classes defined using POVM's in Definition 5.11 were the quantum and quantum commuting and we denoted their respective sets by $\mathcal{C}_q$ and $\mathcal{C}_{qc}$. Hereafter we will let $\mathcal{C}'_q$ and $\mathcal{C}'_{qc}$ denote the aforementioned correlation sets when their correlations are defined via PVM's.
Notice that by Proposition 5.21 we already have that $\mathcal{C}_{qc} = \mathcal{C}'_{qc}$, so it remains to show that $\mathcal{C}_q = \mathcal{C}'_q$. The next proposition will give us this result.

Proposition 7.1

$$\mathcal{C}_q = \mathcal{C}'_q$$

Proof: Since every PVM is a POVM the inclusion $\mathcal{C}'_q \subseteq \mathcal{C}_q$ is immediate.
We will work towards proving the reverse inclusion. Let $p(a, b \mid x, y) \in \mathcal{C}_q$ then there exist finite dimensional Hilbert spaces H_A and H_B, a unit vector $\xi \in H_A \otimes H_B$ as well as families of POVM's $\{E_{x,a}\}_a, x \in X$ on H_A (for every $x \in X$, $E_{x,a} \geq 0, \forall a \in A$ and $\sum_a E_{x,a} = I_{H_A}$) and $\{F_{y,b}\}_b, y \in Y$ on H_B such that for all $x \in X, y \in Y, a \in A, b \in B$:

$$p(a, b \mid x, y) = ((E_{x,a} \otimes F_{y,b})\xi, \xi)$$

Using Theorem 5.5 for the space H_A and the family $\{E_{x,a}\}_a$ we can find a finite dimensional Hilbert space H'_A (H_A is finite dimensional), a family of PVM's $\{E'_{x,a}\}_a$ and an isometry $V_A : H_A \to H'_A$ such that for every $(x, a) \in X \times A$ we have that $E_{x,a} = V_A^* E'_{x,a} V_A$. Doing the same for the space H_B and the family $\{F_{y,b}\}_b$ we obtain a finite dimensional Hilbert space H'_B, a family of PVM's $\{F'_{y,b}\}_b$ and an isometry $V_B : H_B \to H'_B$ such that for every $(y, b) \in Y \times B$ we have that $F_{y,b} = V_B^* F'_{y,b} V_B$.

In the space $H'_A \otimes H'_B$, we define the following element $\xi' = (V_A \otimes V_B)(\xi)$ and notice that it is a unit vector. Moreover,

$$\begin{aligned}
\big((E'_{x,a} \otimes F'_{y,b})\xi', \xi'\big) &= \big((E'_{x,a} \otimes F'_{y,b})(V_A \otimes V_B)\xi, (V_A \otimes V_B)\xi\big) \\
&= \big((V_A^* \otimes V_B^*)(E'_{x,a} \otimes F'_{y,b})(V_A \otimes V_B)\xi, \xi\big) \\
&= \big((V_A^* E'_{x,a} V_A \otimes V_B^* F'_{y,b} V_B)\xi, \xi\big) \\
&= ((E_{x,a} \otimes F_{y,b})\xi, \xi) \\
&= p(a, b \mid x, y)
\end{aligned}$$

which shows that $\mathcal{C}'_q \subseteq \mathcal{C}_q$ and concludes the proof.

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