Tripartite Entanglement and its transfer

Master Thesis

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June 13, 2022

Abstract

In this work we study the quantification of entanglement for a tripartite quantum system and the transfer of maximally entangled tripartite states. This problem is part of a broader active research field concerned with the study and manipulation of entanglement of multipartite systems with many applications in quantum information.

Specifically, we mention some important properties for the classification of tripartite systems in terms of their entanglement. Subsequently, we consider a newly proposed entanglement measure for tripartite systems, called Concurrence Fill, and discuss how it compares with previously proposed entanglement measures in the literature and highlight its advantages.

Finally, we focus on the transfer of tripartite states with maximal Concurrence Fill in a chain of $\frac{1}{2}$-spins described by the Hamiltonian of the Heisenberg model. We find that such a transfer is possible for the chain lengths we considered and we present the characteristics of the solutions.

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1 Introduction

Entanglement has played a crucial role in the development of quantum physics. Initially it was mainly perceived as the qualitative feature of quantum theory that most strikingly distinguishes it from classical mechanics.

The development of Bell's inequalities has made this distinction quantitative, and thus the non-local features of quantum theory were made accessible to experimental verification. The technological advances of the last few decades have made possible to prepare, manipulate, and measure individual quantum systems, as well as create controllable quantum correlations.

In addition, quantum correlations have come to be recognized as a novel resource that may be used to perform tasks that are either impossible or very inefficient in the classical realm. Such developments have helped in the development of modern quantum information science.

In the past twenty years many efforts have been made to study tripartite entangled states, and in general multipartite states, regarding their usefulness in quantum information or quantum computation tasks.

For example, in [16] the authors have studied the teleportation of a quantum state using three particle entanglement to either one of two receivers in such a way that, generally, either one of the two, but only one, can fully reconstruct the quantum state conditioned on the measurement outcome of the other. This teleportation is facilitated using the maximally entangled GHZ state 3.29.

Another interesting application for tripartite entangled states was studied in [13], namely the authors study how a secret sharing procedure can be implemented using GHZ states. It is shown that GHZ states can be used to split information in such a way that if one party is in possession of all of the parts, the information can be recovered, but if one party has some of the parts, it cannot. As a result, an eavesdropper will introduce errors and therefore can be detected.

Furthermore, in [11] dense coding with a three particle entangled state was investigated. In this scenario, one party (Alice) can send information to the second party (Bob), while the local measurement of the third party (Cliff) serves as quantum erasure. Again, the three particle state considered is the maximally entangled GHZ state 3.29.

In [33] a protocol to securely encode classical information among three users via entangled GHZ states is investigated. This is implemented in cavity QED with atomic qubits, where the GHZ state is created in terms of excited and ground states of the atoms. The authors show that it is not possible to get complete information about the state without cooperation of the third party; hence the protocol is secure.

In a recent work, [18], GHZ states were used in error-detection. In this article, a protocol for error detection in entanglement-based tasks was presented, in which EPR pairs are replaced by GHZ states. A general task under the influence of a weak bit-flip noise is considered, and it is shown that it is always possible to increase the efficiency of the task, using the protocol with the GHZ states.

From the developments, briefly mentioned above, it is evident that it is very important to better understand the concept of entanglement as a resource and to develop its theoretical description. This description aims to characterise, manipulate, and quantify entanglement [26]. The last part is broadly understood as the theory of entanglement measures.

Entanglement measures, especially for bipartite quantum systems, have been thoroughly studied in the literature with detailed reviews, for example, in [26] and [14].

Entanglement measures are constructed by defining a real valued function, which satisfies the basic properties of entanglement, and using those functions to quantify the amount of entanglement in a given quantum state. Various such measures can be found in literature, for example in the review articles cited above.

For the purposes of this Master Thesis, we want to study the entanglement in tripartite systems. In general, stepping from the bipartite case to the multipartite case in the description of entanglement is a difficult, and still unresolved, problem [29].

In [1] it was found that tripartite quantum systems can be classified in four distinct classes, which are the product states, the biseparable states, the GHZ states, and the W states. This classification should be considered in the proposal of a tripartite quantum measure.

We review the newly proposed entanglement measure, called Concurrence Fill proposed in [32]. This measure uses the Concurrence, defined by Wootters in [31]. The measure successfully classifies the GHZ state as more entangled than the W state, and produces appropriate results for product and biseparable states. It is also compared with other tripartite entanglement measures and found superior. Concurrence Fill is also generalized to the case of mixed states via the convex roof construction, but such a calculation is computationally very difficult.

Next, we study the problem of perfect state transfer along a chain of spins of a GHZ state, which is successfully termed as maximally entangled by Concurrence Fill.

State transfer was firstly studied by Bose in [5] for an unmodulated chain. The system is characterized by the Heisenberg Hamiltonian with equal coupling strengths. The purpose is to transfer a state placed at one end of the chain to the other by letting the system evolve in time. It is found, that this can be achieved only up to N = 4 spin chains.

Later in [30] this idea was extended to engineered chains. As before, the basic idea is to transfer a particular state along the chain by letting the state evolve under the Heisenberg Hamiltonian. In contrast to having equal coupling strengths, the idea of optimizing them to achieve perfect state transfer is explored.

We use this method of optimizing the coupling strengths of the time-independent Hamiltonian to solve the problem of transferring a GHZ state, placed at the first three qubits of the chain to the last three qubits. We find that this is possible for chains of length N=4,5,6,7,8,9. As this is an open problem, more research is required to explore the theoretical framework and extend numerical calculations to larger chains.

2 Composite Systems

In this chapter we provide a brief overview of quantum composite systems. The mathematical tools describing those systems are necessary to approach the concept of entanglement which follows in the next chapter. We start by introducing an alternative approach to describing quantum states, namely the density operator or density matrix. This tool proves very important when describing quantum states of composite systems.

Then we introduce the mathematical framework of composite systems and the construction of the Hilbert space for such systems, as well as the states describing them. For simplicity we confine this analysis in the case of composite systems of two subsystems, since the generalization to larger systems is straightforward.

Next, the definition of operators acting on the system and operators acting on subsystems only is presented. Finally, the trace operation is introduced. In the example of bipartite systems, taking the partial trace over one subsystem results in the density matrix of the other subsystem. This so called reduced density matrix plays an important role in the next chapter when calculating various measures of entanglement.

2.1 Density Operators

In addition to the state vector formulation of quantum mechanics, an alternate formulation is possible using the density operator [22]. This alternate formulation is mathematically equivalent to the state vector approach, but as we will see it is more convenient when studying multipartite systems.

To a pure state described by the normalized state vector $|\psi\rangle$, we can assign the density operator [3]:

$$\rho = |\psi\rangle \langle\psi| \tag{2.1}$$

From its definition, the density operator has the following properties:

$$\begin{aligned} \rho > 0 \quad & (Hermitian : \rho^\dagger = \rho) \\ \operatorname{Tr} \rho = 1 & \\ \rho^2 = \rho & \end{aligned}$$

From the first two properties, the last property is equivalent to $\operatorname{Tr} \rho^2 = 1$, which holds only for pure states.

The expectation value of an observable $\langle A \rangle$ is given by the following equation:

$$\langle A \rangle = \operatorname{Tr} \rho A \tag{2.2}$$

The density operator formalism is convenient for describing quantum systems whose state is not completely known. Suppose that a quantum system is in one of a number of states $|\psi_i\rangle$ with respective probabilities p_i. We call $\{p_i, |\psi_i\rangle\}$ an ensemble of pure states. The density operator for the system is defined by the equation:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| = \sum_i p_i \rho_i \tag{2.3}$$

A system described by 2.3 is called a statistical mixture, and it can be a pure state as a special case. The p_i are classical probabilities and are normalized:

$$\sum_i p_i = 1$$

The expectation value of an observable A in the statistical mixture, is calculated as follows:

$$\langle A \rangle = \sum_i p_i \operatorname{Tr} \rho_i A = \operatorname{Tr} \rho A \tag{2.4}$$

For the generalized density operator in 2.3 the properties of being positive and having unit trace are easily understood by its definition.

For the last property we consider the spectral decomposition of ρ:

$$\rho = \sum_n \lambda_n |n\rangle \langle n| \tag{2.5}$$

where $\{|n\rangle\}$ is an orthonormal basis. From the first properties we find that $\lambda_n = \lambda_n^*$, $\lambda_n >= 0$ and $\sum_n \lambda_n = 1$. It follows then that:

$$0 \leq \lambda_n \leq 1$$

and therefore:

$$\mathrm{Tr}\,\rho^2 = \sum_n \lambda_n^2 \leq 1 \tag{2.6}$$

The equal sign in 2.6 unambiguously characterises the occurrence of a pure state, while the inequality holds only for mixtures.

2.2 Bipartite Systems

From classical physics we are accustomed to the fact that composite systems can be decomposed into their subsystems and that conversely, individual systems can be combined to give overall composite systems. The classical total system is completely describable in terms of the states of its subsystems and their mutual dynamic interactions [3].

In quantum physics, however, composite systems can have in addition completely different and surprisingly unified properties. These come to light when the composite quantum systems are in entangled states.

Composite systems are particular quantum systems which exhibit an internal structure, meaning that one can distinguish in them two or more subsystems which can be accessed separately. Subsystems can be experimentally identified and individual interventions can be carried out on them. Such operations are referred to as local operations.

To describe composite systems we introduce the tensor product $\mathcal{H}^{AB}$ of two Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ as:

$$\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B \tag{2.7}$$

which is itself a Hilbert space. The above expression can be generalised for systems with more than two subsystems.

For each pair of vectors in the subsystems $|\phi^A\rangle \in \mathcal{H}^A$ and $|\chi^A\rangle \in \mathcal{H}^B$, there is a product vector in $\mathcal{H}^{AB}$ which can be written as:

$$|\phi^A\rangle \otimes |\chi^B\rangle = |\phi, \chi\rangle$$

where in the last expression we dropped the superscript referring to the subsystem for simplicity. In the rest, we assume that the first element in the ket notation refers to the first subsystem, the second element refers to the second subsystem, and so on.

If $\{|n\rangle\}$ is a basis of the Hilbert space $\mathcal{H}^A$ and $\{|i\rangle\}$ is a basis of the Hilbert space $\mathcal{H}^B$, then $\{|n\rangle \otimes |i\rangle\}$ is a basis for the Hilbert space $\mathcal{H}^{AB}$. The dimension of $\mathcal{H}^{AB}$ is $dim\mathcal{H}^{AB} = (dim\mathcal{H}^B) \cdot (dim\mathcal{H}^B)$.

We can expand every vector $|\psi^{AB}\rangle$ in the composite system in terms of the basis as:

$$|\psi^{AB}\rangle = \sum_{n,i} \alpha_{n,i} |n, i\rangle \tag{2.8}$$

This definition can be directly applied to the product of a finite number of Hilbert spaces. Vectors in $\mathcal{H}^{AB}$ which are not product vectors are called entangled. They can be written only as a superposition of product vectors.

An example of a composite system is a system of 2-qubits. The Hilbert space of this system is $\mathcal{H}^{AB} = \mathcal{H}_2^A \otimes \mathcal{H}_2^B$. A particular base for this space is given by the following vectors:

$$|\Phi_\pm^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \quad |\Psi_\pm^{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{2.9}$$

which are the so called Bell states.

As a next point we have to consider operators on the composite system. If C^A is a linear operator acting on the space $\mathcal{H}^A$ and D^A is a linear operator on the space $\mathcal{H}^B$, then the tensor product

$$C^A \otimes D^B = C^A D^B \tag{2.10}$$

refers to a product operator, which acts "space by space" as:

$$C^A \otimes D^B |\phi, \chi\rangle = |C^A\phi, D^B\chi\rangle \tag{2.11}$$

and the product operator is a linear operator on $\mathcal{H}^{AB}$.

Using the identity operator of a subsystem, we can construct product operators acting on a subsystem, which are particularly important for physical applications. Those operators are defined on the composite system space $\mathcal{H}^{AB}$, but they act only in the individual factor Hilbert spaces. They are called local operators, and they are of the form:

$$\hat{C}^A = C^A \otimes \hat{1}^B \quad \hat{D}^B = \hat{1}^A \otimes D^B \tag{2.12}$$

By definition, those operators commute with each other within $\mathcal{H}^{AB}$.

Another useful tool in the study of composite systems is the trace and partial trace of an operator. Using an orthonormal basis of the space $\mathcal{H}^{AB}$ the trace is defined as:

$$\operatorname{Tr} Z^{AB} = \operatorname{Tr}_{AB} Z^{AB} = \sum_{n,i} \langle n, i| \, Z^{AB} \, |n, i\rangle \tag{2.13}$$

For product operators we have:

$$\operatorname{Tr} C^A \otimes D^B = \sum_{n,i} C^A_{nn} D^B_{ii} = \operatorname{Tr}_A C^A \operatorname{Tr}_B D^B \tag{2.14}$$

Of particular interest for physical applications, and especially for the study of entanglement, is the partial trace over the space of one of the subsystems. For example, the partial trace over the space $\mathcal{H}^A$ is defined by:

$$\operatorname{Tr}_A Z^{AB} = \sum_{n} \langle n^A| \, Z^{AB} \, |n^A\rangle \tag{2.15}$$

The partial trace over $\mathcal{H}^A$ generates an operator on $\mathcal{H}^B$. For example, the partial trace over product operators is:

$$\operatorname{Tr}_A C^A \otimes D^B = D^B \operatorname{Tr}_A C^A \tag{2.16}$$

For composite systems a measurement of an observable on a subsystem is associated with an operator which acts only in that subsystem, for example C^A acting on $\mathcal{H}^A$. This local measurement corresponds in $\mathcal{H}^{AB}$ to a local observable:

$$\hat{C}^{AB} = C^A \otimes \hat{1}^B \tag{2.17}$$

We want to use the density operator formalism of quantum mechanics to describe measurements. For that reason, it is necessary to introduce the reduced density operator. We associate to each subsystem a reduced density operator by taking the partial trace over the other subsystem:

$$\rho^A = \operatorname{Tr}_B \rho^{AB} \quad \rho^B = \operatorname{Tr}_A \rho^{AB} \tag{2.18}$$

The expectation value of this observable is found to be:

$$\langle \hat{C}^A \rangle = \operatorname{Tr}_{AB} \rho^{AB} \hat{C}^A = \operatorname{Tr}_A \rho^A C^A \tag{2.19}$$

We observe that all probability statements about local measurements on a subsystem A are obtained by associating the reduced density operator ρ^A to subsystem A and applying the rules postulated for the density operators of isolated systems.

Since all probability statements for measurements on subsystem A are unambiguously determined by the reduced density operator ρ^A, we may say that the subsystem A is in the state ρ^A.

The composite system may be in a product state, such as $|\alpha_k, \beta_k\rangle$. In that case, subsystem A is in the pure state $|\alpha_k\rangle$, and similarly for subsystem B. The composite system may be in a statistical mixture of product states:

$$\rho^{AB} = \sum_s p_s \, |\alpha_s, \beta_s\rangle \, \langle \alpha_s, \beta_s| = \sum_s p_s \, |\alpha_s\rangle \, \langle \alpha_s| \otimes |\beta_s\rangle \, \langle \beta_s| \tag{2.20}$$

where the normalization $\sum_s p_s = 1$ condition holds. In that case, the states of the subsystems are likewise statistical mixtures:

$$\rho^A = \mathrm{Tr}_B \, \rho^{AB} = \sum_s p_s \, |\alpha_s\rangle \, \langle \alpha_s| \quad \sum_s p_s \, |\beta_s\rangle \, \langle \beta_s| \tag{2.21}$$

of the states $|\alpha_k\rangle$ or $|\beta_k\rangle$.

As an example, we consider a bipartite system in one of the Bell states. In this case, the states of the subsystems are maximally mixed as a result of the entanglement. We find:

$$\rho^A = \mathrm{Tr}_B \, \Phi_\pm^{AB} = \frac{1}{2} \hat{1}^A \quad \rho^A = \mathrm{Tr}_B \, \Psi_\pm^{AB} = \frac{1}{2} \hat{1}^A \tag{2.22}$$

with a similar result for ρ^B. We observe that the subsystems are in entangled states, even though the state of the composite system is pure.

3 Entanglement

In this chapter we will discuss the main characteristics of entangled states. First, we describe how quantum systems might be correlated in a classical manner. Such correlations arise by performing Local Operations assisted by Classical Communication to a product state.

Then we briefly touch on the concept of separability of quantum states of multipartite systems. Intuitively this definition helps define a system as entangled, that is a system which has quantum correlations.

Then we move on to an important tool in identifying quantum states of bipartite systems, namely the Schmidt decomposition. The Schmidt number, defined as the number of non zero elements in this decomposition, may be used to quantify entanglement. Specifically, we call a state of a bipartite system entangled when the Schmidt number is larger than one.

Another useful tool in quantifying entanglement is the von Neumann Entropy. Its definition is an extension of the classical entropy to the case of quantum systems. The von Neumann entropy may be used as an entanglement measure for pure states.

Using the von Neumann entropy and expanding a mixed state as a decomposition of pure states, entanglement of formation is defined as an entanglement measure for mixed states. We describe the work of Wootters in associating the entanglement of formation with a new measure for entanglement of bipartite systems, called concurrence.

The definition of concurrence and its calculation are essential in the development of the newly proposed entanglement measure for tripartite systems, called Concurrence Fill. We review the definition and the characteristics of this measure, noting the strengths over previously defined tripartite measures.

3.1 Correlations

We continue our description using bipartite systems as they are easier to understand. A bipartite system can be in a product state, as $|00\rangle$ and $|11\rangle$, or it can be in a superposition of such states, as $\alpha\,|00\rangle + \beta\,|11\rangle$, which is an example of an entangled state [3].

Composite systems in entangled states are correlated. For the state shown before, if we measure the observable σ_z on each of the subsystems, then we find $(-1,-1)$ or $(+1,+1)$ as the combination of measured values. The correlations found in entangled states are of different nature compared with classical correlations. That can be understood when measuring another observable, for example σ_x, in that above state.

Consider a bipartite system AB, with subsystems A and B. If the composite system is in a product state $\rho^{AB} = \rho^A \otimes \rho^B$ the subsystems are not correlated. We call states ρ^{AB} of composite systems correlated, if they are not product states:

$$\rho^{AB} \neq \rho^A \otimes \rho^B \tag{3.1}$$

We start by describing quantum states with classical correlations. The composite system AB is supposed to be prepared starting from product states and using Local Operations on the subsystems. Such operations are for example unitary operations or measurements. In addition, Classical Communication may be exchanged between the two subsystems. From now on we will use the abbreviation LOCC for Local Operations and Classical Communication.

We assume that two experimenters, Alice and Bob, perform those actions on their subsystems. Alice prepares subsystem A in the state ρ_r^A and then informs Bob. He prepares the subsystem B in the state ρ_r^B. After repeating this sequence many times and with relative frequencies p_r, the composite system is by construction a convex combination or a statistical mixture of product states:

$$\rho^{AB} = \sum_r^m p_r \rho_r^A \otimes \rho_r^B \quad p_r \geq 0 \quad \sum_r p_r = 1 \tag{3.2}$$

If the sum cannot be reduced to a single term, we have a correlated state. Since the correlations are produced by LOCC in a classical manner with the probabilities p_r, we say that ρ^{AB} is classically-correlated.

In general, we say that a composite system is in a classically-correlated state if its statistical properties can be reproduced by a LOCC mechanism.

We use the following classically-correlated quantum state to explore the analogy with classical states:

$$\rho^{AB} = p_1 \left|0^A\right\rangle \left\langle 0^A\right| \otimes \left|0^B\right\rangle \left\langle 0^B\right| + p_2 \left|1^A\right\rangle \left\langle 1^A\right| \otimes \left|1^B\right\rangle \left\langle 1^B\right| \tag{3.3}$$

Measuring the states $|0\rangle$ and $|1\rangle$ has a similarity to having a set of many pairs of boxes which both contain either red or blue balls with probabilities p_1, p_2, respectively. Opening the boxes we can observe the correlation of the colors.

For our quantum state 3.3, if we measure in the computational basis on the subsystem A, then with probability p_1 we find the state $|0^A\rangle$, and subsystem B is found in the state $|0^B\rangle$ which is correlated with $|0^A\rangle$.

However, we may also carry out a measurement in a "rotated" basis. Such a measurement has no classical analogue. For example, if in our state 3.3 we set $p_1 = p_2 = \frac{1}{2}$ and carry out a measurement in the basis $\{|0_x^A\rangle, |1_x^A\rangle\}$, the composite state after the measurement, for example after the measurement outcome $|0_x^A\rangle$, is:

$$\rho^{AB} \to \rho^{AB'} = \left|0_x^A\right\rangle \left\langle 0_x^A\right| \otimes \frac{1}{2}\hat{1}^B \tag{3.4}$$

We find that the state of subsystem B is maximally mixed and in this sense it is completely undetermined. The results of the measurements of σ_x on both subsystems are uncorrelated.

3.2 Separability

A useful concept for the study of multipartite systems is that of separable states. We call the state ρ^{AB} of the bipartite system AB separable, when we can write it in the form of equation 3.2. A pure or mixed quantum state which is not separable is called entangled. An entangled quantum state contains non-classical correlations.

The preparation procedure with LOCC leads to separable states. Entangled states cannot be produced from product states via LOCC. This characterisation can also be considered to be an equivalent definition of entanglement, [3].

It can be shown that the density operator for a pure state $|\psi^{AB}\rangle$ cannot be decomposed in terms of a convex sum, it cannot be classically-correlated.

Therefore, a pure state is either not correlated (it is a product state), or it is entangled.

3.3 Schmidt Decomposition

A useful tool to study composite systems and entanglement is the Schmidt decomposition, which we will shortly review here. If $|\psi^{AB}\rangle$ is a pure state of a composite system AB, then there exist orthonormal states $|i^A\rangle$ for subsystem A, and orthonormal states $|i^B\rangle$ of subsystem B, such that:

$$|\psi^{AB}\rangle = \sum_i \lambda_i |i^A\rangle |i^B\rangle \tag{3.5}$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i = 1$ and are known as Schmidt coefficients, [22].

From the Schmidt decomposition we find that reduced density matrices for subsystems A and B, of a pure state, are:

$$\rho^A = \sum_i \lambda_i^2 |i^A\rangle \langle i^A| \quad \rho^B = \sum_i \lambda_i^2 |i^B\rangle \langle i^B| \tag{3.6}$$

which means that the eigenvalues of ρ^A and ρ^B are identical and equal to λ_i^2.

Many important properties of quantum systems are completely determined by the eigenvalues of the reduced density operator of the system, for example the von Neumann Entropy we will study later. From the Schmidt decomposition, we know that for a pure state of a composite system such properties will be the same for both subsystems.

The bases $|i^A\rangle$ and $|i^B\rangle$ are called the Schmidt bases for the subsystems A and B, respectively. The number of non-zero values λ_i is called the Schmidt number for the state $|\psi^{AB}\rangle$. The Schmidt number is an important property of a composite system and is preserved under unitary transformations on the subsystem A or subsystem B alone.

The Schmidt number can be used as a measure of the entanglement of pure states, [3]. The state $|\psi^{AB}\rangle$ is a product state (not entangled) if and only if the Schmidt number is equal to one. Furthermore, when the composite system AB is in a pure state, it is impossible that one of the subsystems be in a pure state and the other in a genuine mixture.

3.4 von Neumann Entropy

A useful tool for studying the entanglement of quantum systems is the von Neumann entropy, which is a generalization of the definition of Shannon entropy for classical systems.

The Shannon entropy measures the uncertainty associated with a classical probability distribution. In contrast, in the definition of the von Neumann entropy density operators replace the probability distributions [22].

The von Neumann entropy of a quantum state ρ is defined by the formula:

$$\mathcal{S}(\rho) = -\operatorname{Tr} \rho \log \rho \tag{3.7}$$

where the logarithm is taken with base two. If we know the decomposition of ρ and λ_i are its eigenvalues, then the formula for the von Neumann entropy is expressed as:

$$\mathcal{S}(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{3.8}$$

The last formula is usually preferred for calculations. The properties of the von Neumann entropy are:

1. A pure state $\rho = |\psi\rangle \langle\psi|$ has the minimal value of the entropy: $\mathcal{S}(\rho) = 0$.

2. For a density operator with d non-vanishing eigenvalues, the following inequality holds:

$$0 \leq \mathcal{S}(\rho) \leq \log d \tag{3.9}$$

3. The von Neumann entropy is concave:

$$\mathcal{S}(p_1\rho_1 + ... + p_k\rho_k) \geq p_1\mathcal{S}(\rho_1) + ... + p_k\mathcal{S}(\rho_k) \tag{3.10}$$

where $\sum_i p_1 = 1$.

As we will now see the von Neumann entropy of the subsystems may be used as a measure of entanglement.

As mentioned earlier, pure states of bipartite system AB are either in a product state or they are entangled. Since we consider only pure states, we always have the maximum information about the composite state [3]:

$$\mathcal{S}(\rho^{AB}) = 0 \tag{3.11}$$

If the system AB is in a product state, the maximum information about the state of the subsystems is also present:

$$\mathcal{S}(\rho^A) = \mathcal{S}(\rho^B) = 0 \tag{3.12}$$

On the other hand, for one of the Bell states, which are maximally entangled, we have:

$$\rho^A = \rho^B = \frac{1}{2}\hat{1} \tag{3.13}$$

which means that the von Neumann entropy for the subsystems is maximal:

$$\mathcal{S}(\rho^A) = \mathcal{S}(\rho^B) = 1 \tag{3.14}$$

The entropy of the subsystems is a measure of the missing information compared with the pure state of the composite system. If we consider only the subsystems, we lose information about the composite system, the more information is contained in the correlations between the subsystems. The greater the entropy of the subsystems, the more strongly is the pure state of the composite system entangled.

For a pure state $|\psi^{AB}\rangle$ we take the value $E(\psi)$ of the entropy of the subsystems:

$$0 \leq E(\psi) = \mathcal{S}(\rho^A) = \mathcal{S}(\rho^B) \leq 1 \tag{3.15}$$

as a measure of the entanglement of the state. $E(\psi)$ is also called the entropy of entanglement and holds only for pure qubit states. We note that the entropy of entanglement depends only on the Schmidt coefficients, it is independent of the basis and does not change under local unitary transformations.

3.5 Concurrence

In this section, we briefly review the work of Wootters in [31] and the introduction of concurrence as a measure of entanglement. The starting point is the entanglement of formation, which will be defined shortly.

Given a density matrix ρ of a pair of quantum systems A and B, we may consider all possible pure-state decompositions of ρ and write:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \tag{3.16}$$

where $|\psi_i\rangle$ are the ensembles of states occurring with probability p_i.

For each pure state, the entanglement E is defined as the entropy of either of the two subsystems A and B, as mentioned in 3.15.

The entanglement of formation of the mixed state ρ is defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of ρ:

$$E(\rho) = min \sum_i p_i E(\psi_i) \tag{3.17}$$

The significance in the work of Wootters was proving that for a pair of qubits the minimum value in equation 3.17 can be expressed as a function of ρ.

Firstly, we introduce the "spin flip" transformation which is used throughout the calculations. This transformation is a function applicable to both state vectors and density matrices of arbitrary number of qubits. For a pure state of a single qubit, it is defined as:

$$|\tilde{\psi}\rangle = \sigma_y |\psi^*\rangle \tag{3.18}$$

where $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ expressed in a fixed basis, as for example $\{|0\rangle, |0\rangle\}$ and σ_y is the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. To perform the spin flip on n qubits, we apply the above transformation to each individual qubit. Similarly, we find that for a general state ρ of two qubits, the spin flipped density matrix is:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{3.19}$$

where the complex conjugate is taken in the standard basis, which for a pair of spin-1/2 particles is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

The spin flip transformation can be used for pure states as well. For example, we consider a general state for a bipartite system, which can be expanded as [3]:

$$|\psi^{AB}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{3.20}$$

Given $|\psi^{AB}\rangle$ we can find the corresponding density matrix ρ^{AB} and then calculate the entanglement as the von Neumann entropy of each of the subsystems. It is straightforward to show that this calculation leads to the following equality [31]:

$$E(\psi^{AB}) = \mathcal{E}(C(\psi^{AB})) \tag{3.21}$$

where the concurrence C is defined as:

$$C(\psi^{AB}) = |\langle\psi^{AB}||\tilde{\psi}^{AB}\rangle| \tag{3.22}$$

The function $\mathcal{E}$ is the binary entropy function [3]:

$$\mathcal{E}(x) = -x\log_2 x - (1-x)\log_2(1-x) \tag{3.23}$$

which is monotonically increasing and ranges from 0 to 1 as C goes from 0 to 1, so it is suggested in [31] that concurrence can be taken as an entanglement measure.

It is proved in [31] that the entanglement of formation of a mixed state ρ of two qubits can be calculated by the formula:

$$E(\rho) = \mathcal{E}(C(\rho)) \tag{3.24}$$

where

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{3.25}$$

with λ_i being the eigenvalues, in decreasing order, of the Hermitian matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$. Alternatively, the λ_i can be found from the square roots of the eigenvalues of the non-Hermitian matrix $R = \rho\tilde{\rho}$.

In [31] it is shown that the equation 3.24 is correct for arbitrary states of a two qubit system.

In [28] the authors offer a generalization for the calculation of concurrence. For a pure state ψ the concurrence is calculated as:

$$C(\psi) = \sqrt{2(1 - \mathrm{Tr}\,\rho^2)} \tag{3.26}$$

in that case ρ is the reduced density matrix referring to the part of the system we want to calculate the concurrence for. If the system under consideration is bipartite, then ρ would be either ρ^A or ρ^B. The usefulness of this formula is found when we have larger systems.

For example, in a tripartite system ABC we could calculate the concurrence between qubit A and the remaining two qubits BC. In that case, we find $C_{A(BC)} = \sqrt{2(1 - \mathrm{Tr}\,(\rho^{BC})^2)}$.

These calculations are important for the definition of Concurrence Fill, which we discuss in detail later.

3.6 Three Qubit Entanglement Classes

In this section we review an important result for the entanglement of three qubits, namely that they can be entangled in inequivalent ways, studied in [10].

This result should be considered when defining a tripartite entanglement measure, as the measure should be able to distinguish between states with different entanglement.

In [10] the entanglement of two states $|\psi\rangle$ and $|\phi\rangle$ of a multipartite system is called equivalent if and only if there exist local protocols that allow to convert each of the two states into the other one with some a priori probability of success. This definition is motivated by the observation that if the entanglement of $|\psi\rangle$ and $|\phi\rangle$ is equivalent, then the two states can be used to perform the same tasks.

In their work Dür, Vidal, and Cirac [10] show that for tripartite systems there are four distinct classes of entanglement. The first class is that of product states. Those cases are not entangled. In this case, the state of the system is a tensor product of the states of the subsystems:

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle \tag{3.27}$$

The second class is that of biseparable states, which are of the form A-BC, AB-C, and C-AB. Those states contain only bipartite entanglement between two of the qubits. Therefore, they are not termed as genuinely tripartite entangled. For example, the states in A-BC have entanglement between subsystems B and C, while they are product with subsystem A. The state:

$$\frac{1}{\sqrt{2}} |0\rangle (|0\rangle |0\rangle + |1\rangle |1\rangle) \tag{3.28}$$

is maximally entangled in the bipartite system BC, and may be used as a representative of this class. Any state within this class can be derived from this one with certainty by LOCC.

The other two classes of states which are genuinely entangled are the GHZ class and the W class. The representative for the GHZ class is the state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{3.29}$$

and the representative of the W class is the state:

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{3.30}$$

It is shown in [10] that, any tripartite entangled state can be converted, by means of SLOCC, into one of those two standard forms.

In addition, if $|\psi\rangle$ can be converted into the state $|GHZ\rangle$ and $|\phi\rangle$ can be converted into the state $|W\rangle$, then it is not possible to transform, not even with a small probability of success, $|\psi\rangle$ into $|\phi\rangle$ or vice-versa.

The GHZ state is considered as more entangled than the W state, and we will see that this condition would be important in testing the validity of Concurrence Fill. One result supporting the above statement was found in [15], where it was shown that a perfect teleportation using the GHZ state is possible, while this is not possible with a W state.

The classification provided in this section will be useful in characterizing tripartite entanglement measures in the following.

3.7 Residual Entanglement

In the previous sections we discussed some basic concepts relating to the entanglement of bipartite systems. It is still an open problem to define entanglement measures for larger systems. In this section we will review some aspects concerning the entanglement of tripartite systems, as well as a first measured proposed for such systems.

In [9], the authors set out to answer the question: To what extent can an object be simultaneously entangled with two other objects?

It is pointed out that quantum entanglement cannot be freely shared among many objects. For example, if we have a system of three 1/2-spin particles A, B, C, if particle A is fully entangled with particle B (for example in a Bell state), then particle A cannot be simultaneously entangled with particle C. If that was the case, then the pair AB would be entangled with C and would then be described by a mixed state, but that cannot be the case since the Bell state is pure. It is expected that if A is partly entangled with B, then A can have only a limited entanglement with C.

The question posed before is reformulated in terms of the concurrence between pairs in the tripartite system. Concurrence was defined in the previous section in Eq.3.25. We mention that for a pure state, as a special case, the

concurrence of a bipartite system AB is given by: $C_{AB} = 2\sqrt{\det \rho_A}$, where ρ_A is the density matrix of qubit A, calculated from ρ_{AB} after tracing out qubit B.

The reformulated question states: Given a pure state of three qubits A, B, and C, how is the concurrence between A and B related to the concurrence between A and C?

For a pure state the formula for concurrence simplifies. Each pair of qubits, being entangled with only one other qubit in a joint pure state, is described by a density matrix having at most two nonzero eigenvalues. As a consequence, the product $\rho_{AB}\tilde{\rho}_{AB}$ has only two nonzero eigenvalues. With these observations the concurrence between A and B is:

$$C_{AB}^2 = (\lambda_1 - \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \tag{3.31}$$

$$= \operatorname{Tr} \rho_{AB}\tilde{\rho}_{AB} - 2\lambda_1\lambda_2 \leq \operatorname{Tr} \rho_{AB}\tilde{\rho}_{AB} \tag{3.32}$$

In this expression, ρ_{AB} is density matrix of the pair AB and is obtained by tracing out qubit C from the tripartite state. In a similar manner C_{AC}^2 may be calculated. We then arrive at the following bound:

$$C_{AB}^2 + C_{AC}^2 \leq \operatorname{Tr} \rho_{AB}\tilde{\rho}_{AB} + \operatorname{Tr} \rho_{AC}\tilde{\rho}_{AC} \tag{3.33}$$

Evaluation of the right hand side, derived in [9], finally leads to

$$C_{AB}^2 + C_{AC}^2 \leq 4 \det \rho_A \tag{3.34}$$

We may think of the object BC as a single object and speak about the concurrence between qubit A and the pair BC, $C_{A(BC)}$. That is because even though the state space of BC is four dimensional, only two of those dimensions are necessary to describe the state of ABC, which follows from the two nonzero eigenvalues of ρ_{BC}. There are only two nonzero eigenvalues since A is a qubit and the whole state of the system is pure.

Therefore A and BC are treated as qubits in a pure state. As it was mentioned earlier, this leads to concurrence $2\sqrt{\det \rho_A}$. With that, we arrive at the following inequality:

$$C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2 \tag{3.35}$$

In [9] the authors point out that the difference between the two sides of the above inequality can be thought of as the amount of entanglement

between A and BC that cannot be accounted for by the entanglement of A with B and C separately. This quantity is called "residual entanglement" and is given by:

$$\tau_{ABC} = C^2_{A(BC)} - C^2_{AB} - C^2_{AC} \tag{3.36}$$

and an explicit calculation shows that it represents a property of the three qubits that is unchanged under permutations.

From the definition of residual entanglement we may identify a drawback in its use. Even though $\tau_{ABC} = 1$ for the GHZ state 3.29, it is $\tau_{ABC} = 0$ for the W state. We know that the W state is genuinely tripartite entangled and therefore a good measure should not be zero for this state.

3.8 Concurrence Fill

In this section we will study a measure for tripartite entangled systems, proposed recently by Xie and Eberly in [32].

The authors point out that, given the known result of classification of entangled states for tripartite systems (discussed previously) it is understood that a good multipartite entanglement measure, has to satisfy the following two conditions to be called a genuine multipartite entanglement measure (GME): (a) the measure must be zero for product and biseperable states, (b) the measure must be positive for all non-biseperable states.

Arguably introducing a measure for genuine tripartite entanglement is a difficult problem and many proposed measures are shortly reviewed in [32] (in addition to the residual entanglement), but the authors find that these measures are not GME, based on the two conditions discussed above. Their motivation is to introduce a new triangle measure specifically for three qubit systems which has a simple form and an elegant geometric interpretation, and satisfies the two conditions for a GME mentioned earlier.

The concurrence of Wootters is used as the basic ingredient in the definition of Concurrence Fill, but in this case we consider the entanglement between one qubit and the remaining two taken together as an "other" single party. Thus, we have three bipartite entanglements to consider, namely $C_{1(23)}$, $C_{2(31)}$, $C_{3(12)}$.

It was found in [27] that these bipartite entanglements are not completely independent. In that work, the entanglement polygon inequality states that

one entanglement cannot exceed the sum of the other two:

$$C_{i(jk)} \leq C_{j(ki)} + C_{k(ij)} \tag{3.37}$$

A stronger version of this inequality was found in [34], where all three concurrences are replaced by their squared forms:

$$C^2_{i(jk)} \leq C^2_{j(ki)} + C^2_{k(ij)} \tag{3.38}$$

A geometric interpretation for these inequalities is that the three squared one-to-"other" concurrences can represent the lengths of the three edges of a triangle, as seen in figure 3.1.

It is mentioned in [32] that there is a physical meaning for the perimeter of the concurrence triangle as well. The perimeter was considered as a tripartite entanglement measure and was called global entanglement. In [32], the authors argue that global entanglement is zero only for product states, but is positive for biseparable states, therefore it violates condition (a) for a GME measure.

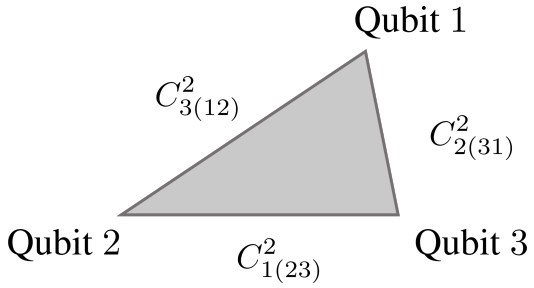

Figure 3.1: Concurrence Triangle for a three qubit system, [32]

Next, the area of the concurrence triangle is considered. This is zero for product and biseparable states, but there exists one class of concurrence triangle with zero area corresponding to non-biseperable states. In that respect, the area of the concurrence triangle seems to violate condition (b) for a GME measure.

It is proved in [32] that such a class of concurrence triangle does not exist. The authors prove the following theorem: The area of the concurrence triangle is zero if and only if it has at least one edge with zero length.

In general, a triangle has zero area when its three vertices are collinear. The non-biseperable state, which has concurrence triangle area equal to zero

is formed by three collinear vertices with no two vertices coinciding. This situation is excluded from the above theorem. Therefore, the area of the concurrence triangle satisfies condition (b) of a GME measure.

With the previous result in mind, the authors in [32] the new genuine tripartite entanglement measure, called Concurrence Fill, is defined as the square root of the area of the concurrence triangle. Using Heron's formula for the triangle area, Concurrence Fill is calculated from the following expression:

$$F_{123} = \Big[\frac{16}{3}Q(Q - C^2_{1(23)})(Q - C^2_{2(13)})(Q - C^2_{3(12)})\Big]^{1/4} \tag{3.39}$$

where $Q = \frac{1}{2}(C^2_{1(23)} + C^2_{2(13)} + C^2_{3(12)})$.

Q is the half-perimeter and it is equivalent to the global entanglement measure. The factor $\frac{16}{3}$ is introduced to ensure that Concurrence Fill is normalized: $0 \leq F_{123} \leq 1$. Taking the square root of the triangle area guarantees local monotonicity under LOCC.

The authors in [32] make use of the result found in [15] regarding the ability of the GHZ to faithfully teleport an arbitrary single-qubit state, compared with the W state which is less efficient. They argue that this result should be incorporated in the criteria for a GME measure. Therefore a new condition is added, namely: (c) A GME measure ranks the GHZ state as more entangled than the W state. A measure satisfying all conditions can be called a "proper" GME measure.

Calculating the Concurrence Fill for the GHZ state 3.29 we find that it takes the maximum value, that is $F_{123} = 1$, while for the W state 3.30 we find $F_{123} = 8/9 \approx 0.889$. Therefore, Concurrence Fill considers the GHZ state as more entangled than the W state and can be regarded as a proper GME measure.

An additional step considered by the authors in [32] to ensure that Concurrence Fill is a good measure for tripartite entanglement was to compare it with other GME measures in the literature. The other measures considered are equivalent or dependent with each other, hence the comparison was restricted only with the so called genuinely multipartite concurrence (GMC), denoted as C_{GME}.

For a three qubit system, C_{GME} is the square root of the length of the shortest edge of the concurrence triangle. For simplicity, the square root is ignored and C_{GME} is considered as the length of the shortest edge of the concurrence triangle.

In [21], Nielsen pointed out that a pair of states in one class, although stochastically equivalent, can still be incomparable. That means that the ranking of their entanglement cannot be judged simply by local monotonicity. In [32], it is argued that two GME measures can be inequivalent, meaning that they provide different opinions on the ranking of one specific pair of states. We will see that Concurrence Fill and C_{GME} are two inequivalent measures.

To understand this observation, we use the following states to calculate their concurrence triangle and Concurrence Fill:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \sin\frac{\pi}{5} |000\rangle + \frac{1}{\sqrt{2}} \cos\frac{\pi}{5} |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \tag{3.40}$$

$$|\psi_2\rangle = \cos\frac{\pi}{8} |000\rangle + \sin\frac{\pi}{\sqrt{8}} |111\rangle \tag{3.41}$$

$$|\psi_3\rangle = \frac{1}{2} |000\rangle + \frac{1}{2} |100\rangle + \frac{1}{\sqrt{2}} |111\rangle \tag{3.42}$$

From the following figure we see that GMC considers state $|\psi_2\rangle$ as more entangled than $|\psi_1\rangle$, because $C_{GME}(\psi_2) > C_{GME}(\psi_1$ as can be seen from the shortest edge of the respective triangles. However, Concurrence Fill considers the opposite: $F_{123}(\psi_2) < F_{123}(\psi_1)$.

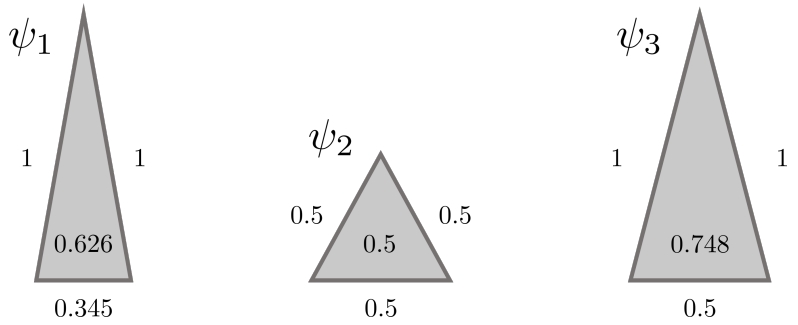

Figure 3.2: Concurrence triangles for the states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ for comparison of F_{123} and C_{GME}. [32]

From their definitions, it is natural to assume that Concurrence Fill contains more information than GMC because it is calculated using all edges of the concurrence triangle, as opposed to the shortest one.

The state $|\psi_3\rangle$ presented above is offered as an example to illustrate the previous comment. GMC cannot differentiate the entanglements of the states

$|\psi_2\rangle$ and $|\psi_3\rangle$, since the shortest edge of the respective concurrence triangles is the same. However, the overall triangles are different since the other two edges are different. This information is not detected by GMC. Concurrence Fill incorporates this information and shows that $|\psi_3\rangle$ is more entangled than $|\psi_2\rangle$. In that way, it is thought that Concurrence Fill has an advantage.

Another important comment made about Concurrence Fill in relation to GMC is that the former is by definition "smooth" for pure state, while GMC might have "sharp peaks" due to the minimization required for its calculation.

Finally, Concurrence Fill can be generalized to the case of mixed states via the convex roof construction:

$$F_{123}(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i F_{123}(\psi_i) \tag{3.43}$$

where the minimum is taken over all ensembles $\{p_i, |\psi_i\rangle\}$.

In conclusion, Concurrence Fill is a useful measure for tripartite quantum systems as it accurately classifies the GHZ state as more entangled than the W state, but it has some limitations due to its difficult calculation for mixed states.

Now that we have presented the definition and the main characteristics of Concurrence Fill, we want to test if it is possible for states other than the GHZ to have Concurrence Fill unity. If we construct the general state for a three qubit system, that is as a superposition of the allowed states in that space $|000\rangle$, $|100\rangle$, $|010\rangle$, $|001\rangle$, $|110\rangle$, $|101\rangle$, $|011\rangle$, $|111\rangle$, and then optimize the coefficients in the superposition to achieve maximum Concurrence Fill, the states we find are not easy to verify that they belong to the GHZ class. By that we mean, that it is not easy to find the LOCC operations which could transform those states to the form given in 3.29.

Fortunately we can overcome this problem by making use of the generalized Schmidt decomposition for three qubit states, developed in [1] by Acín *et. al.* The authors showed that a three qubit state can be written in the following, minimal form:

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle \tag{3.44}$$

with $\lambda_i \geq 0$, $0 \leq \phi \leq \pi$, $\sum_i \lambda_i^2 = 1$. Therefore five free parameters are needed to describe a generic three qubit system.

Using the decomposition 3.44 with

$$\begin{aligned}
\lambda_0 &= \cos\theta_0 \\
\lambda_1 &= \sin\theta_0 \cos\theta_1 \\
\lambda_2 &= \sin\theta_0 \sin\theta_1 \cos\theta_2 \\
\lambda_3 &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
\lambda_4 &= \sin\theta_0 \sin\theta_1 \sin\theta_2 \sin\theta_3
\end{aligned}$$

we can calculate the respective density matrix for the general three qubit state. Given this density matrix we calculate Concurrence Fill and attempt to maximize it by optimizing the coefficients λ_i. The optimization we perform is based on the Nelder-Mead method, which will be briefly discussed in the next chapter.

Our numerical results show that Concurrence Fill reaches its maximum value, $F_{123} = 1$, only for the GHZ state.

An interesting problem is to study how the entanglement between three qubits evolves with the time evolution of a spin chain with more than three qubits. In that case, we consider the tripartite system as a subsystem of the whole chain, meaning we have to trace out the rest of the qubits to find the density matrix for the tripartite system. In that case, we generally have a mixed quantum state which means that for such an application Concurrence Fill would not be useful.

4 State Transfer

In this chapter we discuss the concept of state transfer along a spin chain. We will review some important results in the development of this field and apply those ideas in the problem of transferring a GHZ state along a spin chain.

Firstly, we provide a basic overview and the motivations behind this study which are linked to the fields of quantum information and quantum computation.

We continue with the description of the problem as identified by Bose in his work with unmodulated spin chains. Bose worked on the problem of transferring a one-excitation state from the first qubit of the chain to the last by allowing the system to evolve under the Heisenberg Hamiltonian, without imposing any control to the coupling strengths.

This problem was considered again later by Christandl, but he expanded the problem by searching for the optimal coupling strengths that would facilitate perfect state transfer along the chain under a predetermined time and for arbitrary length of spin chain.

In the last section, we employ this optimization scheme to study the case of transferring a GHZ state along the spin chain. More specifically, we consider the possibility of transferring a maximally entangled GHZ state, namely with Concurrence Fill unity. This requirement gives us some flexibility in defining the target state by introducing a relative phase, which may be thought of as a LOCC operation on the GHZ state.

We present our results for different chain lengths and discuss their characteristics. Admittedly this is still an open problem, therefore many additional research possibilities need to be pursued for a more detailed analysis and understanding.

4.1 Overview

Quantum communication is the act of transferring a quantum state from one place to another. A well-known application is quantum key distribution, where a quantum state prepared by one party needs to be measured by another party at a distance. For this purpose, photons are well suited as they easily travel long distances through optical fibres or empty space and can be readily measured by a receiving party [6]. In addition, quantum communication is important for quantum computation, specifically in connecting distinct quantum processors or registers to make a powerful quantum computer.

For such an application, it is also important to map a quantum state from/to the elements of the quantum register sending/receiving it. Exchange of quantum information between the elements of a quantum computer and the entities carrying the information between the computers is necessary.

The transfer is needed only over short distances separating distinct registers and alternative to photons is very useful. One alternative, which will be described based on [6] is where the quantum state transfer is accomplished purely through the natural dynamical evolution of a permanently coupled chain of quantum systems.

For classical computers Moore's law, which states that computer power will double for constant cost roughly once every two years, is expected to end in the next couple of decades. Conventional approaches to the fabrication of computer technology are beginning to run up against fundamental difficulties of size, which means that quantum effects are beginning to interfere in the functioning of electronic devices at smaller scales.

An alternative computing paradigm is based on the idea of using quantum mechanics to perform computations instead of classical physics. While an ordinary classical computer can be used to simulate a quantum computer, it appears impossible to perform the simulation in an efficient fashion. In that way, quantum computers offer an essential speed advantage over classical computers.

A problem with profound scientific and technological implications is the promise of using quantum computers to simulate quantum mechanical systems which are too difficult to simulate using classical computers [22].

We may think of a quantum computer as a collection of quantum two state systems, qubits, on which arbitrary unitary operations can be performed. A

quantum computer becomes more powerful with an increase in the number of qubits [6]. An obstacle to increasing the number of qubits in a quantum computer is that the quantum computer may be based on a common bus through which qubits interact. In that case there is a physical limitation on the number of qubits which can be linked by the same bus. Another obstacle is if the quantum computer is based on direct interactions between qubits, then either the qubits have to be moved close enough to interact, or their states have to be transferred to qubits which are already within the range of each other's interaction.

Considering a quantum computer as composed of a number of quantum registers connected to each other by quantum communication channels offers a solution to the above mentioned problems. Then most operations would take place between qubits of the same register and occasionally the quantum channels would be used to transfer qubits from one register to another, and enable quantum gates between qubits of different registers.

Quantum communication channels would be useful even if quantum computers are scaled up and do not require internal communication channels. In that scenario, those channels would be necessary to connect distinct quantum computers.

4.2 Unmodulated Chain

In this section, we briefly consider the work of Bose in [5], where a scheme for using an unmodulated and umeasured spin chain is proposed as a channel for short distance quantum communications.

The communication is achieved by placing a spin encoding the state at one end of the chain and waiting for a specific amount of time to let this state propagate to the other end, visualized in the following Fig.4.1.

The spin chain does not require the ability to switch "on" and "off" the interactions between the spins, except for the start and the end of the protocol. Furthermore, the channel does not require any modulation by external fields. This simplicity in comparison to a quantum computer makes it an ideal connector between quantum computers and realizable before a quantum computer.

Bose [5] considered a graph of spins with ferromagnetic Heisenberg interactions. Assuming that we have N spins in the graph, the Hamiltonian is

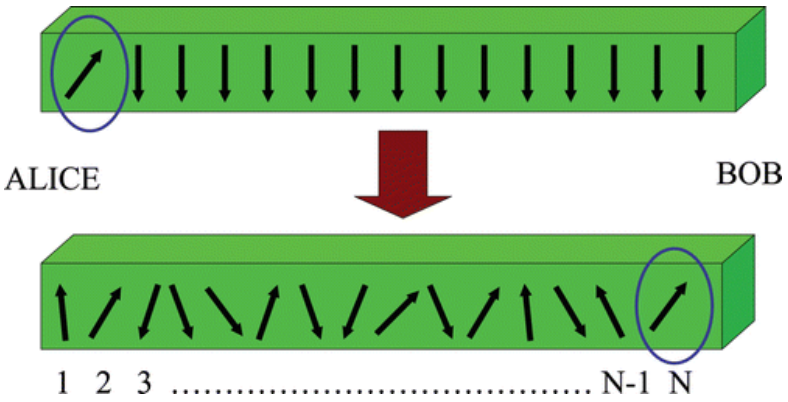

Figure 4.1: Spin chain communication protocol. A spin chain with all spins facing down is the quantum channel. Alice places a quantum state at one end of the chain and Bob picks up a close approximation of this state from his end of the chain after some time, [6]

given by:

$$\mathcal{H} = -\sum_{<i,j>} J_{ij}\vec{\sigma}^i\vec{\sigma}^j - \sum_{i=1}^{N} B_i\sigma_z^i \tag{4.1}$$

where $\vec{\sigma}^i = (\sigma_x^i, \sigma_y^i, \sigma_z^i)$ are the Pauli matrices corresponding to the ith spin. B_i is the static magnetic field applied to the ith spin, $J_{ij} > 0$ are the coupling strengths, and the Hamiltonian describes an arbitrary ferromagnet with isotropic Heisenberg interactions.

It is assumed that Alice is close to the sender spin (sth spin) and Bob is close to the receiver spin (rth spin). The sending and receiver spins are assumed to be detachable from the chain. Alice replaces the existing sender spin with a spin encoding the state to be transferred. After some time has passed, the unknown state placed by Alice travels to the receiver spin with some fidelity. Bob then obtains the receiver spin, which is in a state close to the state Alice wanted to transfer.

The system is initially cooled to the ground state $|\mathbf{0}\rangle = |000...0\rangle$, with $|0\rangle$ denoting the spin in the down state of a spin(along -z direction). It is useful to introduce the set of states $|\mathbf{j}\rangle = |00...010...0\rangle$, where the spin at the jth site is in the up state $|1\rangle$.

The state Alice places at the sender spin in the spin chain can in general

be of the form:

$$|\psi_{in}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \tag{4.2}$$

The whole chain at time t=0 can be written as:

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\mathbf{s}\rangle \tag{4.3}$$

Bob has to wait for a specific time until the initial state evolves to a final state which is as close as possible to the state:

$$|\psi\rangle = \cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\mathbf{r}\rangle \tag{4.4}$$

At this point it is useful to mention an important property of the Heisenberg Hamiltonian. It is straightforward to see that

$$\big[\mathcal{H}, \sum_{i=1}^{N}\sigma_z^i\big] = 0 \tag{4.5}$$

This means that the total spin in the z-direction is conserved. This result shows that the total number of spin in the up state $|1\rangle$ is constant throughout the time evolution of the chain. This effectively divides the Hamiltonian $\mathcal{H}$ into subspaces with specific number of qubits in the state $|1\rangle$, referred to as the number of excitations.

As a result, the state $|\mathbf{s}\rangle$ only evolves to states $|\mathbf{j}\rangle$, which leads to the time evolved state:

$$|\psi(t)\rangle = \cos\frac{\theta}{2}|\mathbf{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}\sum_{j=1}^{N}\langle\mathbf{j}|\,e^{-i\mathcal{H}t}\,|\mathbf{s}\rangle\,|\mathbf{j}\rangle \tag{4.6}$$

The state of the rth spin is in general a mixed state which can be found by tracing out the rest of the qubits in the chain. In [5] the time evolution of the state at the rth spin is used to find a formula for the fidelity of quantum communication, for a predetermined time at which Bob will pick up the rth qubit. In addition, the special case of a linear spin chain is considered.

For the linear spin chain it is assumed that the interactions are only between nearest neighbors with equal strength, and that the magnetic field is uniform for all spins in the chain. With these assumptions an analytical solution for the eigenstates of the Hamiltonian is possible, and then the

transition amplitude for the excitation to travel from the sender qubit to the receiver qubit is calculated.

In [5] various chain lengths from N=2 to N=80 spins were considered, for which the maximum transition amplitude was calculated. It was found that the waiting time for optimal quantum communication varies with N. In that scheme, a spin chain with N=4 allows perfect state transfer (F = 1.000), while for N=8 there is near perfect state transfer (F=0.994). As the chain length is increased, fidelity is reduced. Finally, it was found that a chain of N as high as 80 exceeds the highest fidelity for classical transmission of a state, which is 2/3.

4.3 Engineered Chain

The problem of perfect state transfer was further studied in [8], where it was found that a spin chain with engineered interactions can be used to transfer a quantum state in fixed time t_0.

The basic idea is to find the optimal coupling strengths in the Hamiltonian to achieve perfect state transfer, measured by the fidelity between the time evolved state and the target state at the end of the chain.

We consider the Hamiltonian, for an open end chain:

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N-1} J_i(\sigma_x^i\sigma_x^{i+1} + \sigma_y^i\sigma_y^{i+1}) \tag{4.7}$$

where the coupling strengths J_i are different for each site of the chain, and N is the length of the chain.

The protocol we consider is to start with a spin chain where all spins are in the $|0\rangle$ state. Then an excitation $|1\rangle$ is introduced in the chain, for example on the first site of the chain. The time evolution governed by the above Hamiltonian; hence the time evolved state can be a superposition of states with one-excitation only.

As a next step we introduce the spin ladder operators:

$$\sigma_+ = \sigma_x + i\sigma_y \quad \sigma_- = \sigma_x - i\sigma_y \tag{4.8}$$

and express the Hamiltonian in the following way:

$$\mathcal{H} = \sum_{i=1}^{N-1} J_i(\sigma_+^i\sigma_+^{i+1} + \sigma_-^i\sigma_-^{i+1}) \tag{4.9}$$

where a factor of $\frac{1}{4}$ is incorporated in the coupling strengths for simplicity. With this form it is easier to calculate the matrix elements of the Hamiltonian, given by $\langle n| \mathcal{H} |m\rangle$, where n,m are the possible states in the chain.

For example, in a chain with 4 qubits the allowed states are $|0000\rangle$, $|1000\rangle$, $|0100\rangle$, $|0010\rangle$, $|0001\rangle$, $|1100\rangle$, $|1010\rangle$, $|1001\rangle$, $|0110\rangle$, $|0101\rangle$, $|0011\rangle$, $|1110\rangle$, $|1101\rangle$, $|1011\rangle$, $|0111\rangle$, $|1111\rangle$. A straightforward calculation leads to:

$$\mathcal{H} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & J_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & J_1 & 0 & J_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & J_2 & 0 & J_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & J_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & J_2 & 0 & J_3 & J_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & J_3 & 0 & 0 & J_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & J_1 & 0 & 0 & J_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & J_1 & J_3 & 0 & J_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_3 & 0 & J_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_2 & 0 & J_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{4.10}$$

If we study state transfer where only one-excitation is allowed, then the relevant Hamiltonian for that problem is the respective block from the above matrix:

$$\mathcal{H}_1 = \begin{pmatrix}
0 & J_1 & 0 & 0 \\
J_1 & 0 & J_2 & 0 \\
0 & J_2 & 0 & J_3 \\
0 & 0 & J_3 & 0
\end{pmatrix} \tag{4.11}$$

After some time t, the state $|\psi(0)\rangle$ of the spin chain will evolve to the state:

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle = exp(-i\mathcal{H}t) |\psi(0)\rangle \tag{4.12}$$

In the one excitation example, we introduce the state $|1\rangle$ in the first qubit of the chain and we want to transfer it to the end of the chain. Therefore our target state is $|00...01\rangle$. We measure our success with the fidelity, defined as:

$$F = |\langle \psi(t)|\psi_{target}\rangle|^2 \tag{4.13}$$

Our goal is then to optimize the coupling strengths in our Hamiltonian with the condition that F=1. In practice, we calculate the fidelity and try to minimize the loss function:

$$F' = 1 - F \tag{4.14}$$

since fidelity takes values in the interval $[0, 1]$.

The optimization algorithm we use throughout all our calculations is based on the Nelder-Mead method [19]. This method was proposed in 1965 for the minimization of a function with n variables. The method depends on the comparison of function values at the (n+1) vertices of a polytope, referred to as simplex.

The Nelder-Mead method starts initially with a randomly-generated simplex, meaning that we provide the algorithm with an initial guess for the parameters. At every iteration this simplex is reshaped or moved, one vertex at a time, towards an optimal region in the search space. At each step, the method tries some modifications on the current simplex, and chooses the one that shifts it towards a "better" region of the domain. In the end, the vertex of the simplex that yields the most optimal function value is returned. More details and a short animations may be found in [20].

4.4 GHZ State Transfer

In this section we will use the concepts described in the previous section to study the problem of transferring a GHZ state along a spin chain. Namely, we let the system evolve in time under the Heisenberg Hamiltonian 4.9 and optimize the coupling strengths J_i to achieve unit Fidelity, between the time evolved state and the target state. We select the time to achieve this perfect state transfer as $t_0 = \pi$.

We will use as an example the chain of N=4 spins to describe the transfer of a GHZ state along the chain. The initial state of the chain is described by a product state in which the first three qubits are in the GHZ state, while the fourth qubit is in the $|0\rangle$, spin down, state. Then this state is given by:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes |0\rangle \tag{4.15}$$

What we want to achieve is to have the GHZ state in the last three qubits, while the first qubit would be in the down spin state. In that case, our final

state would be again a product state and when tracing out the first qubit we would have a pure state with Concurrence Fill unity, that is maximally entangled.

If we set as a target state the following:

$$|\psi_{target}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{4.16}$$

it turns out that we restrict our idea of having transferred a maximally entangled state along the chain.

To better understand this point we note, as described earlier, that the Heisenberg Hamiltonian preserves the total spin under time evolution. The $|000\rangle$ part of the GHZ state stays fixed through time, since it has zero energy. Due to the second term, $|111\rangle$, the spin chain will evolve to a superposition of states in the three-excitation subspace.

Now we can generalize Eq.4.6 to find the time evolution of our initial state, in the case of N = 4:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|0000\rangle + \frac{1}{\sqrt{2}}\sum_{k=1}^{M}\langle \mathbf{k}|\, e^{-i\mathcal{H}t}\,|1110\rangle\,|\mathbf{k}\rangle \tag{4.17}$$

where $\mathbf{k}$ are the states with three excitations, and M is the number of those states in a given spin chain with N total spins, which is given by: $M = \binom{N}{3} = \frac{N!}{3!(N-3)!}$.

From the previous equation, we see that even if we succeed in transferring the three up spins of the GHZ state at the end of the chain, they would likely have a different phase due to the time evolution. For that reason, we would consider as a target state the following:

$$|\psi_{target}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|000\rangle + e^{i\phi}\,|111\rangle) \tag{4.18}$$

This is a product state where $|0\rangle$ is the state of the first qubit, while the last three qubits are in the following GHZ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{i\phi}\,|111\rangle) \tag{4.19}$$

We may think of the relative phase term as a unitary local operation in one of the $|1\rangle$ qubits, in comparison with the usual GHZ state 3.29. Since this

state comes from the usual GHZ state after implementing a LOCC it is still classified as GHZ and it is easy to show that it has $F_{123} = 1$.

We use this generalized version of the GHZ state to expand our understanding of transferring a maximally entangled state. It then follows that the relative phase introduced in the GHZ state would be part of our optimization scheme, and in addition to searching for the optimized coupling strengths of the Hamiltonian J_i, we want to find the relative phase to achieve perfect state transfer.

Considering the above, we proceed with our results for spin chains of length N = 4, N = 5, N = 6, N = 7, N = 8, and N = 9. Overall, we find that a perfect state transfer of a maximally entangled GHZ state is possible for all those chains and show in detail the optimized coupling strengths J_i, as well as the optimal relative phase ϕ, below.

For N = 4 chain the coupling strengths are J = [0.866, 1., 0.866] and the relative phase in the GHZ state is $\phi = \frac{\pi}{2}$. For the N = 5 chain we find J = [3.000, 1.871, 1.871, 3.000] and $\phi = \pi$. For the N = 6 chain, J = [2.291, 1.414, 2.5, 1.414, 2.291] and $\phi = \frac{\pi}{2}$.

For the N = 7 chain, J = [1.225, 1.581, 1.732, 1.732, 1.581, 1.225] and $\phi = 0$. For the N = 8 chain, J = [1.323, 1.732, 1.936, 2., 1.936, 1.732, 1.323] and $\phi = \frac{\pi}{2}$.

Finally, for the N = 9 chain, J = [1.414, 1.871, 2.121, 2.236, 2.236, 2.121, 1.871, 1.414] and $\phi = \pi$.

We see that the optimal J_i for all chain lengths we considered have a mirror symmetry around the center. The mirror symmetry of the coupling strengths was also found in [2]. There the authors show how to implement a mirror inversion of the state of the chain with respect to the center of the chain. In our scheme, this is the situation only when we have N = 7 spins. Then the relative phase is $\phi = 0$, meaning that the final state is a mirror inversion of our initial GHZ state.

Observing the relative phase outcome for the perfect state transfer, we see that for N even the relative phase is $\phi = \frac{\pi}{2}$, while for N odd we find $\phi = 0$ or $\phi = \pi$. This outcome is observed for one-excitation state transfer, as discussed also in [23], [24], where this relative phase was associated with a permutation symmetry connecting the initial with the target state.

In addition, it has been argued that with a suitable choice of basis one can map the one-excitation behavior to that observed in the multipartite excitation case [7]. However, a rigorous proof of the latter is not available and the subject needs further investigation.

In this context, we also mention that in the PhD Thesis [25] the transfer of a bipartite entangled state along a spin chain was studied. Specifically, the transfer of the state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is studied. There it is demonstrated that the relative phase between the two target sites considered can take the values $\phi = \nu\pi$ or $\phi = (2\nu + 1)\frac{\pi}{2}$, for odd and even length chains respectively.

After we have found the optimized coupling strengths that facilitate a perfect state transfer of the GHZ chain along the chain, we use the engineered Hamiltonian to calculate the time evolution of the spin chain.

In 4.2 we plotted the time evolution of fidelity with the optimized Hamiltonian. In addition, we plotted two correlation measures which we discuss below.

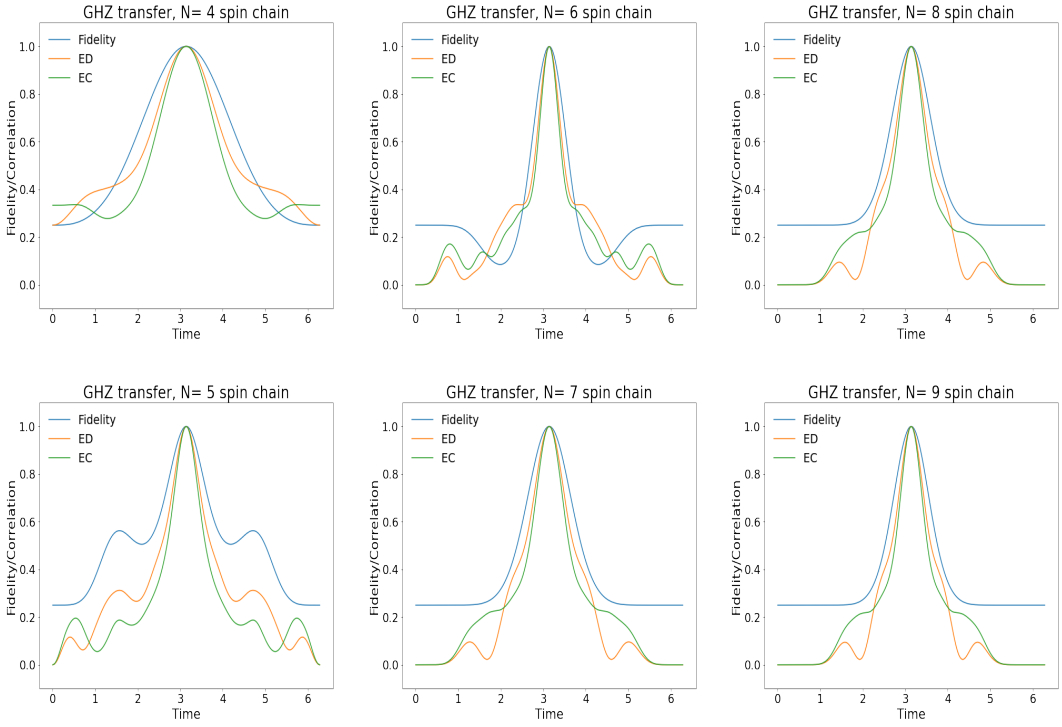

Figure 4.2: GHZ state transfer along spin chains with N = 4, 5, 6, 7, 8, 9 qubits. The graphs depict the time evolution for the fidelity, the tripartite correlation measure $\mathcal{E}_D$, and the bipartite correlation measure $\mathcal{E}_C$, after the optimized coupling strengths J_i and the relative phase ϕ are found. Perfect state transfer is achieved at $t = \pi$.

As we have discussed in the previous chapter, for the time evolved states it is difficult to calculate the Concurrence Fill for the tripartite system of the last three qubits of the chain, as those states are in general mixed. We always consider the last three qubits of the chain as a tripartite system, and its state is found by tracing out the rest of the qubits. An alternative is to calculate the correlations of the tripartite system under consideration.

In [4] a tensor specifying the tripartite correlations is given as:

$$D_{ijk} = \langle \sigma_i^A \sigma_j^B \sigma_k^C \rangle - \langle \sigma_i^A \rangle \langle \sigma_j^B \rangle \langle \sigma_k^C \rangle \tag{4.20}$$

where i,j,k = x,y,z for the respective Pauli matrices, and A,B,C are the subsystems of our tripartite system. To calculate the average values $\langle \sigma_i^A \rangle$, we trace out the necessary qubits from our tripartite system.

In [4] a tripartite correlation measure for qubits is defined as:

$$\mathcal{E}_D = \frac{1}{4} \sum_{i,j,k} D_{i,j,k}^2 \tag{4.21}$$

It is also pointed out that a tripartite system may have bipartite correlations as well. The bipartite correlation of a tripartite system is the sum of the correlation for the three bipartite pairs:

$$\mathcal{E}_C = \frac{n^2}{4(n^2-1)} \sum_{I,J(I \neq J)} \mathrm{Tr}\, \mathbf{C}^{\mathbf{I,J}} (\mathbf{C}^{\mathbf{I,J}})^T \tag{4.22}$$

where n is the level of the system, which for qubits is n=2.

The tensor $\mathbf{C}$ for two correlated qubits is constructed as:

$$C_{ij}^{AB} = \langle \sigma_i^A \sigma_j^B \rangle - \langle \sigma_i^A \rangle \langle \sigma_j^B \rangle \tag{4.23}$$

We see in Fig.4.2 that the correlations increase as we approach the maximum fidelity. Even though these measures provide as with an intuition of how much correlated are the subsystems in the tripartite system under consideration, they do not specify if the correlations are classical or quantum. For this distinction additional criteria have to be formulated and implemented. Nevertheless, it is clear that the tripartite correlation reaches its maximum value of unity, when the perfect state transfer is completed.

5 Conclusion

The study of multipartite quantum systems is of great importance for the fields of quantum information processing and quantum computation. The entanglement of those systems is a useful resource in performing tasks using those systems; hence it is essential to develop a deep understanding of entanglement. In addition, a way to quantify entanglement is of necessary for applications.

We have seen in this work that the task of quantifying entanglement for multipartite systems is very complex due to the richness of those systems and their properties. It is still an open problem and a lot of effort is still required. Even for tripartite systems that we mainly considered here the complexity is much higher than bipartite systems.

We have described in more detail the newly proposed entanglement measure for tripartite systems called Concurrence Fill. This measure has some advantages over other previously proposed measures. It can be used to correctly categorize tripartite states in the entanglement classes of product states, biseparable states, the W state, and the GHZ state.

Unfortunately it can be efficiently used only for pure states. The expansion to mixed states is very difficult to implement computationally. This point requires more efforts to overcome, but it is generally an issue found in entanglement measures.

Since Concurrence Fill is successful in identifying the GHZ state as maximally entangled, we used this observation to study the transfer of a maximally entangled state along a spin chain.

To transfer a GHZ state along the chain we have considered an engineered spin chain. The process we followed was to introduce the GHZ state in the first three qubits of the chain and then aim to have a GHZ state in the last three qubits of the chain, by optimizing the coupling strengths of the Hamiltonian.

We have used an observation about the GHZ state, namely that a LOCC operation transforms the state to another GHZ state, to increase our flexibility in the target state under time evolution. More specifically, we have introduced a relative phase in the $|111\rangle$ part of the GHZ state, which evolves with time. Therefore, our optimization had to consider this phase, in addition to the coupling strengths.

Through numerical calculations we found that this transfer was possible for the spin chains under consideration (N=4,5,6,7,8,9). An interesting characteristic of our solutions was that the coupling strengths exhibit a mirror symmetry. Further research on this topic is required to develop deeper understanding of the conditions, or symmetries, which allow the transfer of the GHZ state.

We note also, that the relative phase takes values 0 or π for odd chain lengths, while its value for even chain lengths is $\frac{\pi}{2}$. This is verified in several numerical simulations and it is also compatible with conjectures existing in the literature [7]. However, a rigorous proof is still missing.

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