Hardy - Sobolev Inequalities

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Abstract

In the present work we study two types of the Hardy-Sobolev inequality, the one involving distance to the origin and the other involving distance to the boundary. For the Hardy-Sobolev inequality involving distance to the origin, we also obtain the sharp constant. We relate this inequality to a limiting Caffarelli-Kohn-Nirenberg inequality and we prove that they are equivalent. Particularly in three dimensions the sharp constant coincides with the best Sobolev constant. Similarly, for the Hardy-Sobolev inequality involving distance to the boundary we prove that the sharp constant of the inequality on the upper half space $\mathbb{R}^3_+$ is given by the Sobolev constant.
In both cases we added a Sobolev term with the best constant on the Hardy inequality which has already a best constant.

Περίληψη

Στην παρούσα εργασία θα μελετηθούν δύο ανισότητες Hardy-Sobolev, μια που αφορά απόσταση από σημείο και μια που αφορά απόσταση από σύνορο. Για την ανισότητα Hardy-Sobolev που αφόρα απόσταση απο σημείο βρίσκουμε βέλτιστη σταθερά. Την συσχετίζουμε με μια οριακή περίπτωση της ανισότητας Caffarelli-Kohn-Nirenberg inequality και αποδεικνύουμε ότι είναι ισοδύναμες. Συγκεκριμένα στις τρείς διαστάσεις η βέλτιστη σταθερά ταυτίζεται με την βέλτιστη σταθερά Sobolev. Όμοια, για την ανισότητα Hardy-Sobolev που αφορά απόσταση από σύνορο, αποδεικνύουμε οτι η βέλτιστη σταθερά στον θετικό ημίχωρο $\mathbb{R}^3_+$ δίνεται από την σταθερά Sobolev.
Και στις δύο περιπτώσεις προστίθεται ένας όρος Sobolev με βέλτιστη σταθερά σε μια ανισότητα Hardy που έχει ήδη βέλτιστη σταθερά.

Contents

1 Introduction

1.1 Weak derivatives

Let $\Omega \in \mathbb{R}^n$ be open $f : \Omega \to \mathbb{R}$.Then we start with the following notation.
Notation. Let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable and compactly supported functions $\phi : \Omega \to \mathbb{R}$. We will call a function ϕ belonging to $C_c^\infty(\Omega)$ a test function.
We start with the motivation for definition of weak derivatives. Let $\Omega \in \mathbb{R}^n$ be open, $u \in C^1(\Omega)$ and $\phi \in C_c^\infty(\Omega)$. Integration by parts gives,

$$\int_\Omega u \frac{\partial \phi}{\partial x_j}\, dx = -\int_\Omega \frac{\partial u}{\partial x_j} \phi \, dx$$

There is no boundary term, since ϕ has a compact support in Ω and thus vanishes near $\partial\Omega$.
Let then $u \in C^k(\Omega) \quad k = 1, 2, \ldots$ and let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \cap \{0\}$ be a multi-index such that the order of multi-index $|a| = a_1 + \cdots + a_n$ is at most k. We denote,

$$D^a u = \frac{\partial^{|a|} u}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}} = \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} u$$

The order of a multi-index tells the total number of differentiation. Integration by parts gives,

$$\int_\Omega u D^a \phi \, dx = (-1)^{|a|} \int_\Omega D^a u \phi \, dx$$

We notice that the left-hand side makes sense even under the assumption $u \in L^1_{\text{loc}}(\Omega)$.

Definition 1.1. *Assume that $u \in L^1_{\text{loc}}(\Omega)$ and let $a \in \mathbb{N}^n \cap \{0\}$ be a multi-index.Then $v \in L^1_{\text{loc}}(\Omega)$ is the a-th weak partial derivative u, written $D^a u = v$, if*

$$\int_\Omega u D^a \phi \, dx = (-1)^{|a|} \int_\Omega v \phi \, dx \tag{1}$$

for every test function $\phi \in C_c^\infty(\Omega)$. We denote, $D^0 u = D^{(0,\ldots,0)} = u$. If $|a| = 1$, then

$$Du = (D_1 u, D_2 u, \ldots, D_n u)$$

is the weak gradient of u. Here,

$$D_j u = \frac{\partial u}{\partial x_j} = D^{(0,\ldots,1\ldots,0)} u, \quad j = 1, \ldots, n$$

(the j-th component is 1)

In other words, if we are given u and if there happens to exist a function v which verifies (1) for all ϕ, we say that $D^a u = v$ in the weak sense. If there does not exist such a function v, then u does not have a weak a-th partial derivative. We observe that changing the function on set of measure zero does not affect its weak derivatives.

Lemma 1.1. *If $f \in L^1_{\text{loc}}(\Omega)$ satisfies*

$$\int_\Omega f\phi \, dx = 0$$

for every $\phi \in C_c^\infty(\Omega)$ then $f = 0$ almost everywhere in Ω

Corollary 1. *(Uniqueness of weak derivatives). A weak a-th partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{\text{loc}}(\Omega)$ satisfy

$$\int_\Omega u D^a \phi \, dx = (-1)^a \int_\Omega v\phi \, dx = (-1)^a \int_\Omega \tilde{v}\phi \, dx$$

for every $\phi \in C_c^\infty(\Omega)$. This implies that,

$$\int_\Omega (v - \tilde{v})\phi \, dx = 0$$

for every $\phi \in C_c^\infty(\Omega)$.

□

1.2 Sobolev Spaces

Fix $1 \leq p \leq \infty$ and let k be an non-negative integer. We define certain function spaces, whose elements have weak derivatives of various orders lying in various L^p spaces.

Definition 1.2. *Assume that Ω is an open subset of $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index a with $|a| \leq k$ the weak derivative $D^a u$ exists and $D^a u \in L^p(\Omega)$. Thus,*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^a u \in L^p(\Omega), |a| \leq k\}$$

If $u \in W^{k,p}(\Omega)$, we define its norm

$$\|u\|_{W^{k,p}(\Omega)} = (\sum_{|a| \leq k} \int_{\Omega} |D^a u|^p \, dx)^{\frac{1}{p}} \quad 1 \leq p < \infty$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|a| \leq k} \operatorname{ess\,sup}_{\Omega} |D^a u|$$

Notice that $D^0 u = D^{(0,\ldots,0)} u = u$. Assume that Ω' is an open subset of Ω. We say that Ω' is compactly contained in Ω, denoted $\Omega' \subset\subset \Omega$, if $\overline{\Omega'}$ is a compact subset of Ω. A function $u \in W^{k,p}_{\text{loc}}(\Omega)$, if $u \in W^{k,p}(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Remark 1. *If $p = 2$, we usually write*

$$H^k(\Omega) = W^{k,2}(\Omega), \quad k = 0, 1 \ldots$$

The letter H is used, since $H^k(\Omega)$ is a Hilbert space. Note that, $H^0(\Omega) = L^2(\Omega)$.

Definition 1.3. *1. Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(\Omega)$. We say that u_m converges to u in $W^{k,p}(\Omega)$, written*

$$u_m \to u \quad \textit{in} \quad W^{k,p}(\Omega)$$

provided

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

2. We write

$$u_m \to u \quad \textit{in} \quad W^{k,p}_{\text{loc}}(\Omega)$$

to mean

$$u_m \to u \quad \textit{in} \quad W^{k,p}(U)$$

for each $U \subset\subset \Omega$

Theorem 1.2. *The Sobolev space $W^{k,p}(\Omega)$ $\quad 1 \leq p \leq \infty \quad k = 1, 2, \ldots$ is a Banach space.*

Proof. 1. Let us first of all check that $\|u\|_{W^{k,p}(\Omega)}$ is an norm. Clearly, $\|u\|_{W^{k,p}(\Omega)} \leftrightarrow u = 0$ almost everywhere. It is easy to see that $\|u\|_{W^{k,p}(\Omega)} =$

0 implies that $\|u\|_{L^p(\Omega)} = 0$, which implies that $u = 0$ almost everywhere. Conversely, $u = 0$ almost everywhere in Ω implies,

$$\int_\Omega D^a u\phi \, dx = (-1)^{|a|} \int_\Omega u D^a \phi \, dx = 0$$

for all $\phi \in C_c^\infty(\Omega)$. This implies that $u = 0$ almost everywhere in Ω for all $a, |a| \leq k$.
It is obvious that, $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \|u\|_{W^{k,p}(\Omega)}, \quad \lambda \in \mathbb{R}$ Next assume $u, v \in W^{k,p}(\Omega)$. Then if $1 \leq p \leq \infty$ Minkowski's inequality implies,

$$\begin{aligned} \|u+v\|_{W^{k,p}(\Omega)} &= \Big(\sum_{|a|\leq k} \|D^a u + D^a v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{|a|\leq k} (\|D^a u\|_{L^p(\Omega)} + \|D^a v\|_{L^p(\Omega)})^p \Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{|a|\leq k} \|D^a u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}} + \Big(\sum_{|a|\leq k} \|D^a v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}} \\ &= \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)} \end{aligned}$$

2. It remains to show that $W^{k,p}(\Omega)$ is complete. So assume $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(\Omega)$. Then for each $|a| \leq k, \quad \{D^a u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exist functions $u_a \in L^p(\Omega)$ such that,

$$D^a u_m \to u_a \quad \text{in} \quad L^p(\Omega)$$

for each $|a| \leq k$. In particular,

$$u_m \to u_{(0,\ldots,0)} = u \quad \text{in} \quad L^p(\Omega)$$

3. We now claim,

$$u \in W^{k,p}(\Omega), \quad D^a u = u_a \quad |a| \leq k \tag{2}$$

To verify this assertion, fix $\phi \in C_c^\infty(\Omega)$. Then,

$$\begin{aligned}\int_\Omega u D^a \phi \, dx &= \lim_{m\to\infty} \int_\Omega u_m D^a \phi \, dx \\ &= \lim_{m\to\infty} (-1)^{|a|} \int_\Omega D^a u_m \phi \, dx \\ &= (-1)^{|a|} \int_\Omega u_a \phi \, dx\end{aligned}$$

On the second line we used the definition of the weak derivative. Next we show how to conclude the first and last equalities above.
For $1 < p < \infty$. Let $\phi \in C_c^\infty(\Omega)$. By Hölder's inequality we have,

$$\begin{aligned}&| \int_\Omega u_m D^a \phi \, dx - \int_\Omega u D^a \phi \, dx| = | \int_\Omega (u_m - u) D^a \phi \, dx| \\ &\leq \|u_m - u\|_{L^p(\Omega)} \|D^a \phi\|_{L^{p'}(\Omega)} \\ &\to 0\end{aligned}$$

and consequently we obtain the first inequality above. The last inequality follows in the same way since,

$$| \int_\Omega D^a u_m \phi \, dx - \int_\Omega u_a \phi \, dx| \leq \|D^a u_m - u_a\|_{L^p(\Omega)} \|\phi\|_{L^{p'}(\Omega)} \to 0$$

For $p = 1, p = \infty$ we argue in a similar way as above. This means that the weak derivative $D^a u$ exist and $D^a u = u_a, \quad |a| \leq k$. As we also know that, $D^a u_m \to u_a = D^a u, \quad |a| \leq k$, we conclude that $\|u_m - u\|_{W^{k,p}(\Omega)} \to 0$. Thus $u_m \to u$ in $W^{k,p}(\Omega)$.

□

Smooth functions are dense in Sobolev spaces. Thus, every Sobolev function can be approximated with a smooth function in the Sobolev norm. The next result shows that.

Theorem 1.3. *(Meyers-Serrin Theorem)*
Assume that Ω is bounded, and suppose as well that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$.
Then there exist functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that

$$u_m \to \quad \textit{in} \quad W^{k,p}(\Omega)$$

The Meyers-Serrin theorem gives the following characterization for the Sobolev spaces $W^{k,p}(\Omega), 1 \leq p < \infty : u \in W^{k,p}(\Omega)$ if and only if there exist functions $u_i \in C^\infty \cap W^{k,p}(\Omega), i = 1, 2, \dots,$ such that $u_i \to u$ in $W^{k,p}(\Omega) \quad \text{as} \quad i \to \infty$. In other words, $W^{k,p}(\Omega)$ is the completion of $C^\infty(\Omega)$ in the Sobolev norm.

Definition 1.4. *Let $1 \leq p < \infty$. The Sobolev space with zero boundary values $W_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm. Thus $u \in W_0^{1,p}(\Omega)$ if and only if there exist functions $u_i \in C_c^\infty(\Omega), i = 1, 2, \dots,$ such that $u_i \to u \quad in \quad W^{1,p}(\Omega) \quad as \quad i \to \infty$. The space $W_0^{1,p}(\Omega)$ is endowed with the norm of $W^{1,p}(\Omega)$.*

The difference compared to $W^{1,p}(\Omega)$ is that functions in $W_0^{1,p}(\Omega)$ can be approximated by $C_c^\infty(\Omega)$ functions instead of C^∞ functions, that is

$$W^{1,p}(\Omega) = \overline{C^\infty(\Omega)} \quad \text{and} \quad W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}$$

where the completions are taken with respect to the Sobolev norm. A function in $W_0^{1,p}(\Omega)$ has zero boundary values in Sobolev's sense. We may say that $u, v \in W^{1,p}(\Omega)$ have the same boundary values in Sobolev's sense, if $u - v \in W_0^{1,p}(\Omega)$.
Notation. We write,

$$H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

Remark 2. $W_0^{1,p}(\Omega)$ *is a closed subspace of* $W^{1,p}(\Omega)$ *and thus complete.*

Lemma 1.4.

$$W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n) \quad \textit{with} \quad 1 \leq p < \infty$$

1.3 The Classical Hardy Inequality

The standard Hardy inequality involving the distance to the origin asserts that if $n \geq 3$ and $u \in C_c^\infty(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \, dx \tag{3}$$

The constant $(\frac{n-2}{2})^2$ is the best possible constant. So we would present the theorem for the Hardy Inequality and its proof.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$. For all $u \in C_c^\infty(\Omega)$, the Hardy Inequality holds,*

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx \tag{4}$$

Proof. In order to prove the Hardy inequality it is enough to to show that,

$$\int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx \geq 0 \tag{5}$$

1. At first we consider functions $u \in C_c^\infty(\Omega \backslash \{0\})$ which vanish near zero. We define,

$$v(x) = u(x)|x|^{\frac{n-2}{2}} \Leftrightarrow u(x) = \frac{v(x)}{|x|^{\frac{n-2}{2}}}$$

$$\begin{aligned} \nabla u &= \frac{\nabla v|x|^{\frac{n-2}{2}} - \frac{n-2}{2} v|x|^{\frac{n-2}{2}-2} x}{|x|^{n-2}} \\ &= \frac{\nabla v|x|^{\frac{n-2}{2}} - \frac{n-2}{2} v|x|^{\frac{n-6}{2}} x}{|x|^{n-2}} \end{aligned}$$

So,

$$\begin{aligned} |\nabla u|^2 &= \frac{(\nabla v|x|^{\frac{n-2}{2}} - \frac{n-2}{2} v|x|^{\frac{n-6}{2}} x)^2}{|x|^{2n-4}} \\ &= \frac{|\nabla v|^2 |x|^{n-2}}{|x|^{n-2}|x|^{n-2}} - \frac{(n-2)\nabla v \cdot x v |x|^{\frac{2(n-4)}{2}}}{|x|^n |x|^{n-4}} + \frac{(n-2)^2}{4} \frac{v^2 |x|^{n-6}|x|^2}{|x|^{2n-4}} \\ &= \frac{1}{|x|^{n-2}}((\nabla v)^2) - \frac{(n-2)\nabla v \cdot x}{|x|^n} + \frac{(n-2)^2}{4} \frac{v^2}{|x|^n} \end{aligned}$$

It is easily observed,

$$\frac{u^2}{|x|^2} = \frac{v^2}{(|x|^{\frac{n-2}{2}})^2|x|^2} = \frac{v^2}{|x|^{n-2}|x|^2} = \frac{v^2}{|x|^n}$$

Returning to (5) we have,

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\,dx - \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx \\
=\;& \int_\Omega \Big[\frac{(\nabla v)^2}{|x|^{n-2}} - \frac{(n-2)\nabla v\cdot xv}{|x|^n} + \frac{(n-2)^2}{4}\frac{v^2}{|x|^n} - \frac{(n-2)^2}{4}\frac{v^2}{|x|^n}\Big]\,dx \\
=\;& \int_\Omega \frac{(\nabla v)^2}{|x|^{n-2}}\,dx - \int_\Omega \frac{(n-2)\nabla v\cdot xv}{|x|^n}\,dx
\end{aligned}$$

For the first term, it is obvious that

$$\int_\Omega \frac{(\nabla v)^2}{|x|^{n-2}} \geq 0$$

Now we shall prove that the second term is zero. Indeed, we know that $v = 0$ in an area near zero, so the function that we integrate is a C^∞ function and we can use Green's identity. Here $\hat{n}$ is the outward pointing unit normal vector on $\partial\Omega$

$$\begin{aligned}
\int_\Omega \frac{\nabla v\cdot xv}{|x|^n}\,dx &= \frac{1}{2}\int_\Omega \frac{x}{|x|^n}\nabla v^2\,dx \\
&= \frac{1}{2}\int_{\partial\Omega} v^2\frac{x}{|x|^n}\hat{n}\,dS - \frac{1}{2}\int_\Omega v^2 \text{div}\Big(\frac{x}{|x|^n}\Big)\,dx
\end{aligned}$$

We notice that the the first term is equal to zero because $v \in C_c^\infty(\Omega\backslash\{0\})$. For the second term,

$$\begin{aligned}
\text{div}\Big(\tfrac{x}{|x|^n}\Big) &= \text{div}x\frac{1}{|x|^n} + x\nabla\Big(\frac{1}{|x|^n}\Big) \\
&= n\frac{1}{|x|^n} + x(-n|x|^{-(n-2)}x) \\
&= \frac{n}{|x|^n} - n\frac{|x|^2}{|x|^{n+2}} = \frac{n}{|x|^n} - \frac{n}{|x|^n} = 0
\end{aligned}$$

Thus we proved the Hardy inequality when $u \in C_c^\infty(\Omega\backslash\{0\})$

2. Now we continue by proving the general case of the Hardy inequality, where we consider functions in $C_c^\infty(\Omega)$. We suppose that $\phi(x) \in C^\infty(\mathbb{R}^n)$ is a function with the following property,

$$\phi(x) = \begin{cases} 0, & |x| < 1 \\ \\ 1, & |x| > 2. \end{cases}$$

Thus we have that $|\nabla\phi(x)| \leq c$. Using $\phi(x)$ we consider the sequence of functions ϕ_m where $\phi_m(x) = \phi(xm)$. Hence,

$$\phi_m(x) = \begin{cases} 0, & |x| < \frac{1}{m} \\ \\ 1, & |x| > \frac{2}{m}. \end{cases}$$

Consequently, $|\nabla\phi_m(x)| = m|\nabla\phi(mx)| \leq cm$.
We now define the sequence of functions $u_m(x) = u(x)\phi_m(x)$.
We then have,

$$u_m(x) = u(x)\phi_m(x) = \begin{cases} 0, & |x| < \frac{1}{m} \\ \\ u(x), & |x| > \frac{2}{m}. \end{cases}$$

Therefore, we have that $u_m \in C_c^\infty(\Omega\backslash\{0\})$. From (i) for every $m \geq 1$ we have that

$$\int_\Omega |\nabla u_m|^2\,dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u_m|^2}{|x|^2}\,dx$$

In order to prove the Hardy inequality in this case, it is enough to prove that as $m \to \infty$

$$\int_\Omega |\nabla u_m|^2\,dx \to \int_\Omega |\nabla u|^2\,dx$$

and

$$\left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u_m^2}{|x|^2}\,dx \to \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx$$

Thus, we have to prove that, $\|\nabla u_m - \nabla u\|_2 \to 0$ and

$$\int_\Omega \frac{|u_m - u|^2}{|x|^2}\,dx \to 0$$

First we have that,

$$\int_\Omega |\nabla u_m - \nabla u|^2\,dx = \int_{|x|<\frac{2}{m}} |\nabla u_m - \nabla u|^2\,dx$$
$$\leq 2\int_{|x|<\frac{2}{m}} |\nabla u_m|^2\,dx + 2\int_{|x|<\frac{2}{m}} |\nabla u|^2\,dx$$

From Lebesgue's Dominated Convergence Theorem, the second term tends to zero. Moreover,

$$\int_{|x|<\frac{2}{m}} |\nabla u_m|^2\,dx = \int_{|x|<\frac{2}{m}} |\phi_m \nabla u + u\nabla\phi_m|^2\,dx$$
$$\leq 2\int_{|x|<\frac{2}{m}} |\nabla u|^2\,dx + 2\int_{|x|<\frac{2}{m}} |u|^2|\nabla\phi_m|^2\,dx$$

The first term tends to zero, as $m \to \infty$. Then we prove that the second term tends to zero. Let $M \geq 0$ such that, $|u| \leq M$ in Ω. Finally, we have that,

$$\int_{|x|<\frac{2}{m}} |u|^2|\nabla\phi_m|^2\,dx \leq M^2c^2m^2\int_{|x|<\frac{2}{m}} dx \leq M^2m^2\frac{c}{m^n} \to 0 \quad \text{as } m \to \infty$$

Now the proof is complete.

□

Remark 3. *Brezis and Vazquez improved the classical Hardy inequality on bounded domains by establishing that for* $u \in C_c^\infty(B_1)$*,*

$$\int_{B_1} |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2\int_{B_1}\frac{u^2}{|x|^2}\,dx + \mu_1\int_{B_1} u^2\,dx \tag{6}$$

where the constant μ_1 *is the first eigenvalue of the Laplacian of the unit disk in* $\mathbb{R}^2$*. We note that* μ_1 *is the best constant in the inequality independent of the dimension* $n \geq 3$*.*
When taking distance to the boundary , the following Hardy inequality where the constant $\frac{1}{4}$ *is optimal, is also well known for* $n \geq 2$ *and* $u \in C_c^\infty(B_1)$*,*

$$\int_{B_1} |\nabla u|^2\,dx \geq \frac{1}{4}\int_{B_1}\frac{u^2}{(1-|x|)^2}\,dx \tag{7}$$

Similarly, Brezis and Marcus established an improved Hardy inequality for a convex bounded domain in $\mathbb{R}^n$,

$$\int_{B_1} |\nabla u|^2\, dx \geq \frac{1}{4}\int_{B_1} \frac{u^2}{(1-|x|)^2}\, dx + b_n \int_{B_1} u^2\, dx \tag{8}$$

for some positive constant b_n. This time the best constant b_n depends on the space dimension with $b_n \geq \mu_1$ when $n \geq 4$ but in the $n = 3$ case, one has that $b_n = \mu_1$.

1.4 Sobolev Inequality

We shall prove the Sobolev inequality without the sharp constant.

Definition 1.5. *If $1 \leq p < n$ the Sobolev conjugate of p is*

$$p^* = \frac{np}{n-p}$$

Note that,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

The foregoing scaling analysis show the estimate,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{9}$$

for certain constants $c > 0$, $1 \leq q < \infty$ and all functions $u \in C_c^\infty(\mathbb{R}^n)$. This can only be true for $q = p^*$

Remark 4. *1. We consider an inequality of the form,*

$$\Big(\int_{\mathbb{R}^n} |u|^q\, dx\Big)^{\frac{1}{q}} \leq c\Big(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\Big)^{\frac{1}{p}}$$

for every $u \in C_c^\infty(\mathbb{R}^n)$, where constant $0 < c < \infty$ and exponent $1 \leq q < \infty$ are independent of u. Let $u \neq 0$, $1 \leq p < n$ and consider $u_\lambda(x) = u(\lambda x)$ with $\lambda > 0$. Since $u \in C_c^\infty(\mathbb{R}^n)$ it follows that the inequality above holds for every u_λ with c and q independent of λ. Thus,

$$\Big(\int_{\mathbb{R}^n} |u_\lambda|^q\, dx\Big)^{\frac{1}{q}} \leq c\Big(\int_{\mathbb{R}^n} |\nabla u_\lambda|^p\Big)^{\frac{1}{p}}$$

for every $\lambda > 0$. We change the variables $y = \lambda x$, $dx = \frac{1}{\lambda^n} dy$, we can see that,

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q \, dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q \, dx = \int_{\mathbb{R}^n} |u(y)|^q \frac{1}{\lambda^n} \, dy$$
$$= \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(x)|^q \, dx$$

and

$$\int_{\mathbb{R}^n} |\nabla u_\lambda(x)|^p \, dx = \int_{\mathbb{R}^n} \lambda^p |\nabla u(\lambda x)|^p \, dx$$
$$= \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u(y)|^p \, dy = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx$$

So,

$$\frac{1}{\lambda^{\frac{n}{q}}} \Big(\int_{\mathbb{R}^n} |u|^q \, dx \Big)^{\frac{1}{q}} \leq \frac{\lambda}{\lambda^{\frac{n}{p}}} (|\nabla u|^p \, dx)^{\frac{1}{p}}$$

for every $\lambda > 0$ and equivalently

$$\|u\|_{L^q(\mathbb{R}^n)} \leq c \lambda^{1 - \frac{n}{p} + \frac{n}{q}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Since, the inequality has to hold for every $\lambda > 0$ we have,

$$1 - \frac{n}{p} + \frac{n}{q} = 0 \Leftrightarrow q = \frac{np}{n-p}$$

This is the only possible exponent for which the inequality may hold true.

2. *The classical Sobolev inequality*

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq S_n \Big(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \Big)^{\frac{n-2}{n}} \tag{10}$$

is valid for any $u \in C_c^\infty(\mathbb{R}^n)$ where $S_n = \pi n(n-2) (\frac{\Gamma(\frac{n}{2})}{\Gamma(n)})^{\frac{2}{n}}$ is the best constant.

The generalized Hölder's inequality will be useful in order to prove the Sobolev inequality.

Lemma 1.6. *Let* $1 \leq p_1 \dots p_k < \infty$ *with* $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$ *and assume* $f_i \in L^{p_i}(\Omega)$*,* $i = 1, \dots k$*. Then,*

$$\int_\Omega |f_1 \dots f_k| \, dx \leq \int_\Omega \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\Omega)} \tag{11}$$

Proof. To prove the generalized Hölder's inequality we will use induction and Hölder's inequality.
The classical Hölder's inequality states that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $u \in L^p(\Omega), \quad v \in L^q(\Omega)$, then $uv \in L^1$ and

$$\int_\Omega |u(x)v(x)| \, dx \leq \|u\|_p \|v\|_p$$

When $k = 2$ we are given $p_1 p_2 > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$. In particular we have, $p_1 p_2 > 1$ and so (11) is reduced to the classical Hölder's inequality. Now, we suppose that (11) holds for some $k \geq 2$. We claim that it holds for $k+1$. So let $p_1 \dots p_{k+1} > 0$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_{k+1}} = 1$ ad let $f_i \in L^{p_i}$, $i = 1, \dots, k+1$. Note that, $p_i > 1$ for $i = 1, \dots, k+1$. In particular, we have

$$p_1 > 0 \quad \frac{1}{p_1 - 1} > 0, \quad \frac{1}{p_1} + \frac{1}{\frac{p_1}{p_1 - 1}} = 1$$

By the classical Hölder's inequality we have.

$$\begin{aligned} \int_\Omega \prod_{i=1}^{k+1} |f_i| \, dx &= \int_\Omega |f_1| \prod_{i=2}^{m+1} |f_i| \, dx \\ &= \|f_1\|_{p_1} \left[\int_\Omega \Big(\prod_{i=2}^{k+1} |f_i| \Big)^{\frac{p_1}{p_1 - 1}} \, dx \right]^{\frac{p_1 - 1}{p_1}} \\ &= \|f_1\|_{p_1} \left[\int_\Omega \prod_{i=2}^{k+1} |f_i|^{\frac{p_1}{p_1 - 1}} \, dx \right]^{\frac{p_1 - 1}{p_1}} \end{aligned}$$

Furthermore, since

$$\frac{p_i(p_1 - 1)}{p_1} > 0 \quad \text{for} \quad i =, \dots, k+1$$

$$\sum_{i=2}^{k+1} \frac{1}{\frac{p_i(p_1 - 1)}{p_1}} = \frac{p_1}{p_1 - 1} \sum_{i=2}^{k+1} \frac{1}{p_i} = \frac{p_1}{p_1 - 1} \Big(1 - \frac{1}{p_1} \Big) = 1$$

By the induction hypothesis we have,

$$\int_{\Omega}\prod_{i=2}^{k+1}|f_i|\,dx \le \|f_1\|_{p_1}\left[\prod_{i=2}^{k+1}\Big(\int_{\Omega}|f_i|^{\frac{p_1}{p_1-1}\cdot\frac{p_i(p_1-1)}{p_1}}\,dx\Big)^{\frac{p_1}{p_i(p_1-1)}}\right]^{\frac{p_1-1}{p_1}}$$

$$= \|f_1\|_{p_1}\prod_{i=2}^{k+1}\Big(\int_{\Omega}|f_i|^{p_i}\,dx\Big)^{\frac{1}{p_i}}$$

and so the assertion follows □

Sobolev proved the following theorem in the case $p > 1$ and Nirenberg and Gagliardo in the case $p = 1$.

Theorem 1.7. *Assume that* $1 \le p < n$. *There exists a constant* c, *depending only on* p *an* n, *such that,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le c\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{12}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$

We really do need u to have compact support for (12) to hold, as the example $u \equiv 1$ shows. But remarkably the constant here does not depend at all upon the size of the support of u.

Proof. We start by proving the estimate for $u \in C_c^\infty(\mathbb{R}^n)$
First assume $p = 1$.
Since u has compact support, for each $i = 1, \dots n$ and $x \in \mathbb{R}^n$ we have,

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, y_i, x_{i+1}, \dots, x_n)\,dy_i$$

and so,

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, y_i, \dots, x_n)|\,dy_i \quad i = 1, \dots, n$$

Consequently,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n}\Big(\int_{-\infty}^{\infty}|\nabla u(x_1, \dots, y_i, \dots, x_n)|\,dy_i\Big)^{\frac{1}{n-1}}$$

We integrate the inequality with respect to x_1

$$\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1 \le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} |Du|\,dy_i\Big)^{\frac{1}{n-1}}\,dx_1$$

$$= \Big(\int_{-\infty}^{\infty} |\nabla u|\,dy_1\Big)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{\infty} |\nabla u|\,dy_i\Big)^{\frac{1}{n-1}}\,dx_1$$

$$\le \Big(\int_{-\infty}^{\infty} |\nabla u|\,dy_1\Big)^{\frac{1}{n-1}} \Big(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,dx_1\,dy_i\Big)^{\frac{1}{n-1}}$$

the last inequality resulting from the general Hölder's inequality. We now integrate with respect to x_2, for for,

$$I_1 = \int_{-\infty}^{\infty} |\nabla u|\,dy_1, \quad I_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,dx_1\,dy_i \quad i = 3, \dots, n$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1\,dx_2 \le \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,dx_1\,dx_2\Big)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{\substack{i=1\\ i\neq 2}}^{n} I_i^{\frac{1}{n-1}}\,dx_2$$

Applying once more the extended Hölder's inequality, we find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u|^{\frac{n}{n-1}}\,dx_1\,dx_2 \le \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,dx_1\,dy_2\Big)^{\frac{1}{n-1}} \prod_{i=3}^{n} \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\nabla u|\,dx_1\,dx_2\,dy_i\Big)^{\frac{1}{n-1}}$$

We continue by integrating with respect to $x_3 \dots, x_n$ to finally find that,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx \le \prod_{i=1}^{n} \Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\nabla u|\,dx_1 \dots dy_i \dots dx_n\Big)^{\frac{1}{n-1}} = \Big(\int_{\mathbb{R}^n} |\nabla u|\,dx\Big)^{\frac{n}{n-1}} \tag{13}$$

This is estimate (22) for $p = 1$

2. Consider now the case that $1 < p < n$. We apply estimate (13) to $v = |u|^\gamma$, where $\gamma > 1$ is to be selected. Then,

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-}}\, dx\Big)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla |u|^\gamma|\, dx \\
&= \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u|\, dx \\
&\leq \gamma \Big(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\, dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\Big)^{\frac{1}{p}}
\end{aligned}
$$

We choose γ so that,

$$\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1}$$

That is we set

$$\gamma = \frac{p(n-1)}{n-p} > 1$$

In which case,

$$\frac{\gamma n}{n-1} = (\gamma - 1)\frac{p}{p-1} = \frac{np}{n-p} = p^*$$

in view of (9) the estimate above becomes,

$$\Big(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\Big)^{\frac{1}{p^*}} \leq \Big(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\Big)^{\frac{1}{p}}$$

Assume that then $u \in W^{1,p}(\mathbb{R}^n)$. We have that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$. Thus there exist $u_i \in C_c^\infty(\mathbb{R}^n), i = 1, 2, \dots$ such that $\|u_i - u\|_{W^{1,p}}(\mathbb{R}^n) \to 0 \quad \text{as} \quad i \to \infty$. In particular $\|u_i - u\|_{L^p}(\mathbb{R}^n) \quad \text{as} \quad i \to \infty$. Thus there exists a subsequence (u_i) such that $u_i \to u$ almost everywhere in $\mathbb{R}^n$ and $u_i \to u$ in $L^p(\mathbb{R}^n)$.

Claim: (u_i) is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$.

Reason: Since $u_i - u_j \in C_c^\infty(\mathbb{R}^n)$, we use the Sobolev inequality for compactly supported smooth functions and Minkowski's inequality to conclude

$$
\begin{aligned}
&\|u_i - u_j\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|\nabla u_i - \nabla u_j\|_{L^p(\mathbb{R}^n)} \\
&\leq c(\|\nabla u_i - \nabla u_j\|_{L^p(\mathbb{R}^n)} + \|\nabla u - \nabla u_j\|_{L^p(\mathbb{R}^n)})
\end{aligned}
$$

Since $L^{p^*}(\mathbb{R}^n)$ is complete there exists $v \in L^{p^*}(\mathbb{R}^n)$ such that $u_i \to v \in L^{p^*}(\mathbb{R}^n) \quad \text{as} \quad i \to \infty$.

Since $u_i \to u$ almost everywhere in $\mathbb{R}^n$ and $u_i \to v$ in $L^{p^*}(\mathbb{R}^n)$ we have $u = v$ almost everywhere in $\mathbb{R}^n$. This implies that $u_i \to u$ in $L^{p^*}(\mathbb{R}^n)$ and that $u \in L^{p^*}(\mathbb{R}^n)$.
Now we can apply Minkowski's inequality and the Sobolev inequality for compactly supported smooth functions to conclude that,

$$\begin{aligned} \|u\|_{L^{p^*}(\mathbb{R}^n)} &\leq \|u - u_i\|_{L^{p^*}(\mathbb{R}^n)} + \|u_i\|_{L^{p^*}(\mathbb{R}^n)} \\ &\leq \|u - u_i\|_{L^{p^*}(\mathbb{R}^n)} + c\|\nabla u_i\|_{L^p(\mathbb{R}^n)} \\ &= \|u - u_i\|_{L^{p^*}(\mathbb{R}^n)} + c(\|\nabla u_i - \nabla u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}) \\ &\to c\|\nabla u\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

since $u_i \to u$ in $L^{p^*}(\mathbb{R}^n)$ and $\nabla u_i \to \nabla u$ in $L^p(\mathbb{R}^n)$ □

1.5 Elements of Operator Theory

Let V, W be Hilbert spaces. A linear operator $A : \mathrm{Dom(A)} \subset \mathrm{V} \to \mathrm{W}$ is called bounded, if there exists some $C > 0$ such that for all $u \in \mathrm{Dom(A)}$

$$\|Au\|_W \leq C\|u\|_V$$

A densely defined operator is a linear operator that is defined on a dense linear subspace Dom(A) of V and takes values in W,$\overline{\mathrm{Dom(A)}} = V$.
We now define an operator as closed, if the graph

$$G(A) = \Big\{(u, Au) : u \in \mathrm{Dom}(A)\Big\} \subset V \times W$$

is closed as a subspace of $V \times W$.
Let V be a Hilbert space and $A : \mathrm{Dom(A)} \subset \mathrm{V} \to \mathrm{V}$ a densely defined operator. A is called symmetric if for every $u, v \in \mathrm{Dom(A)}$

$$\langle\, Au, v\rangle = \langle\, v, Au\rangle$$

$A : \mathrm{Dom(A)} \subset \mathrm{V} \to \mathrm{W}$ is a densely defined operator if we define a linear operator $A^* : \mathrm{Dom(A^*)} \subset \mathrm{W^*} \to \mathrm{V^*}$

$$\mathrm{Dom}(A^*) = \{f \in W| \quad g \mapsto \langle\, Ag, f\rangle \quad \text{is bounded}\}$$

and

$$\langle\, Ag, f\rangle = \langle\, g, A^* f\rangle$$

this holds for every $g \in \mathrm{Dom}(A)$, $f \in \mathrm{Dom}(A^*)$.
The operator A^* is called adjoint of A. If $A = A^*$, (so $V = W$), then A is called self-adjoint.

1.6 Functional Calculus

Theorem 1.8. *Let A be a self adjoint operator on a Hilbert space $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the continuous functions on $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ so that*

1. *$\hat{\phi}$ is an algebraic $*$-homomorphism.*
2. *If h is bounded, $\hat{\phi}(h)$ is a norm continuous, that is $\|\phi\|_{\mathcal{L}(H)} \leq \|h\|_\infty$*
3. *Let $h_n(x)$ be a sequence of a bounded continuous functions with $h_n(x) \to_{n\to\infty} x$ for each x and $|h_n(x)| \leq |x|$ for all x and n. Then for any $\psi \in D(A)$, $\lim_{n\to\infty} \hat{\phi}(h_n)\psi = A\psi$.*
4. *If $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\hat{\phi}(h_n) \to \phi(\hat{h})$ strongly.*
   In addition
5. *If $A\psi = \lambda\psi$, $\hat{\phi}(h)\psi = h(\lambda)\psi$*
6. *If $h \geq 0$, then $\hat{\phi}(h) \geq 0$*

It is often convenient to allow our functions to take the values $\pm\infty$ on small sets in which case we require $f^{-1}[\pm\infty]$ to be continuous. The functional calculus is very useful in order to define the exponential e^{itA} and prove easily many of its properties as a function. In the case where A is bounded we do not need the functional calculus to define the exponential since we can define e^{itA} by the power series which converges norm.

1.7 Quadratic forms

Quadratic forms will help us define the Laplace operator and more generally Schrödinger operators. Let a linear subspace, Dom(Q), of a real Hilbert space $\mathcal{H}$. A bilinear form on Dom(Q), is a mapping Q : Dom(Q) $\times$ Dom(Q) $\to \mathbb{R}$ such that,

1. $Q(au + \beta v, w) = aQ(u, w) + \beta Q(v, w)$
2. $Q(w, au + \beta v) = aQ(w, u) + \beta Q(w, v)$
3. $Q(u, v) = Q(v, u)$

for every $u, w, w \in$ Dom(Q) and $a, \beta \in \mathbb{R}$. We note that (2) is a consequence of (1) and (3). Using Q we define the following quadratic form,

$$Q(u) = \begin{cases} Q(u,u) & \text{if} \quad f \in \text{Dom}(Q) \\ +\infty & \text{else} \end{cases}$$

A non-negative quadratic form Q is closed if and only if

$$\begin{cases} (u_n) \subseteq \text{Dom}(Q) & \text{and} \\ u_n \to u \quad \in \mathcal{H} & \text{and} \\ Q(u_n - u_m) \to 0 \end{cases}$$

it implies that $u \in \text{Dom}(Q)$ and $Q(u_m - u) \to 0$.

Theorem 1.9. *Let Q a non-negative quadratic form, then there exists a unique non-negative self-adjoint operator H such that Dom$(H^{\frac{1}{2}})$ = Dom(Q) and*

$$\langle Hu, v\rangle = Q(u, v), \quad \forall u \in Dom(H), v \in Dom(Q)$$

1.8 The Laplace Operator

We define the Laplace operator(with Dirichlet boundary conditions) $H : L^2(\Omega) \to L^2(\Omega)$ as the non-negative self-adjoint operator and according to the previous theorem is equivalent to the quadratic form Q with Dom(Q) = $\mathrm{H}_0^1(\square)$ and,

$$Q(u) = \int_\Omega |\nabla u|^2 \, dx, \quad u \in H_0^1(\Omega)$$

So,

$$\text{Dom}(H) = \Big\{ u \in H_0^1(\Omega) : \exists f \in L^2(\Omega) \quad \text{such that} \quad Q(u, \phi) = \int_\Omega f\phi \, dx, \forall \phi \in C_c^\infty(\Omega) \Big\}$$

and if $u \in$ Dom(H), then f is unique and we define $Hu = f$.
A direct consequence of the definition and Green's identity is the following remark.

Remark 5. *If $\partial\Omega \in \mathcal{H}$ then for every $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0 \in$* Dom(H) *and $Hu = -\Delta u$*

1.9 Heat Kernel and Green Function

By using Fourier transforms one sees that

$$e^{-Ht}f = K_t * f$$

for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ where

$$K_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}}\, dt$$

Using the formula

$$(H + \lambda)^{-1} = \int_0^\infty e^{-Ht} e^{-\lambda t}\, dt$$

one deduces that if $Re\lambda > 0$ one has

$$(H + \lambda)^{-1} = G_\lambda * f$$

where

$$G_\lambda(x) = \int_0^\infty (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}} e^{-\lambda t}\, dt$$

The kernel G_λ is strictly positive and becomes infinite as $x \rightarrow$. It is dominated pointwise by the kernel G_0 of the unbounded operator Q^{-1} which is given by

$$\begin{aligned} G_0(x) &= \int_0^\infty (4\pi t)^{-\frac{n}{2}} e^{-\frac{x^2}{4t}}\, dt \\ &= C_n |x|^{-(n-2)} \end{aligned}$$

provided $n > 2$.

2 Hardy-Sobolev Inequality Involving Distance to the origin

Maz'ya combined both the Hardy and the Sobolev term in one inequality, valid in the upper half space. After a conformal transformation, it leads to the following Hardy-Sobolev-Maz'ya inequality,

$$\int_{B_1} |\nabla u|^2\,dx \geq \frac{1}{4}\int_{B_1} \frac{u^2}{(1-|x|)^2}\,dx + B_n\Big(\int_{B_1} |u|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}} \tag{14}$$

valid for any $u \in C_c^\infty(B_1)$. Clearly $B_n \leq S_n$ and it was shown that $B_n < S_n$ where $n \geq 4$. Again the case $n = 3$ it turns out to be special. It has been established that $B_3 = S_3$. To state the result we first define

$$X_1(a,s) = (a - \ln s)^{-1}, \quad a > 0, \quad 0 < s \leq 1$$

Our main concern is to prove the following theorem

Theorem 2.1. *Let $n \geq 3$. The best constant in $C_n(a)$ in*

$$\int_{B_1} |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2\int_{B_1} \frac{u^2}{|x|^2}\,dx + C_n(a)\Big(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \tag{15}$$

is given by:

$$C_n(a) = \begin{cases} (n-2)^{\frac{-2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

when restricted to radial functions, the best constant in (15) *is given by*

$$C_{n,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n \quad \textit{for all} \quad a \geq 0$$

In all cases there is no $H_0^1(B_1)$ minimizer.

Remark 6. *One easily checks that $C_n(a) < S_n$ when $n \geq 4$. We observe that in the $n = 3$ case one has that $C_3(a) = S_3 = 3(\frac{\pi}{2})^{\frac{4}{3}} = B_3$ for all $a \geq 1$, that is the classical Sobolev inequality, the Hardy-Sobolev-Maz'ya and* (15) *share the same best constant.*

We define the following space $W_0^1(B_1; |x|^{-(n-2)})$ as the completion of $C_c^\infty(B_1)$ under the norm $\Big(\int_{B_1}|x|^{-(n-2)}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}$.

Theorem 2.2. *Let $n\geq 3$. The best constant $C_n(a)$ in the limiting Caffarelli-Kohn-Nirenberg inequality*

$$\int_{B_1}|x|^{-(n-2)}|\nabla v|^2\,dx \geq C_n(a)\Big(\int_{B_1}|x|^{-n}X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}, v\in C_c^\infty(B_1) \tag{16}$$

is given

$$C_n(a) = \begin{cases} (n-2)^{\frac{-2(n-1)}{n}}S_n, & a\geq \frac{1}{n-2} \\ a^{\frac{2(n-1)}{n}}S_n, & 0<a<\frac{1}{n-2}. \end{cases}$$

When restricted to radial functions the best constant in (16) *is given by*

$$C_{n,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}}S_n \quad \textit{for all} \quad a\geq 0$$

In all cases, there is no $W_0^{1,2}(B_1,|x|^{-(n-2)})$ minimizer

Remark 7. *Estimate* (16) *is a limiting case of a Caffarelli-Kohn-Nirenberg inequality. Indeed, for any $-\frac{n-2}{2}<b<\infty$, the following inequality holds:*

$$\int_{\mathbb{R}^n}|x|^{2b}|\nabla v|^2\,dx \geq S(b,n)\Big(\int_{\mathbb{R}^n}|x|^{\frac{2bn}{n-2}}|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{2}}, v\in C_c^\infty(\mathbb{R}^n) \tag{17}$$

Moreover, for $b=-\frac{n-2}{2}$ estimate (17) fails. Clearly, estimate (16) is the limiting case of (17) for $b=-\frac{n-2}{2}$. Thus we have:

We note that that the nonexistence of a $W_0^{1,2}(B_1,|x|^{-(n-2)})$ minimizer of Theorem 2.2 is stronger than the nonexistence of an $H_0^1(B_1)$ minimizer of Theorem 2.1.This is due to the fact that the existence of an $H_0^1(B_1)$ minimizer for (15) would imply the existence of a $W_0^{1,2}(B_1,|x|^{-(n-2)})$ for (16).

Lemma 2.3.

1. *If $u\in H_0^1(\Omega)$ then $|x|^{\frac{n-2}{2}}u\in W_0^{1,2}(\Omega,|x|^{-(n-2)})$.*

2. *If* $w \in W_0^{1,2}(\Omega), |x|^{-(n-2)})$ *then* $|x|^{-a}w \in H_0^1(\Omega)$ *for all* $a < \frac{n-2}{2}$.

3. $\left(\int_\Omega |x|^{-(n-2)}|\nabla w|^2\,dx\right)^{\frac{1}{2}}$ *is an equivalent norm for the space* $W_0^{1,2}(\Omega, |x|^{-(n-2)})$.

Proof. 1. Let $u \in H_0^1(\Omega)$. A simple calculation shows that.

$$\int_\Omega |x|^{-(n-2)}|\nabla(|x|^{\frac{n-2}{2}}u)|^2 dx = \int_\Omega |x|^{-(n-2)}|\frac{n-2}{2}|x|^{\frac{n-6}{2}}ux + |x|^{\frac{n-2}{2}}\nabla u|^2 dx$$
$$\leq 2\Big(\frac{n-2}{2}\Big)^2\int_\Omega \frac{u^2}{|x|^2}, dx + 2\int_\Omega |\nabla u|^2 dx \leq c\|u\|_{H_0^1(\Omega)} < +\infty$$

where in the last line we used the classical Hardy inequality.

2. Concerning the second statement let $w \in C_c^\infty(\Omega)$. If $v = |x|^{-a}w$ then,

$$\int_\Omega |\nabla v|^2\,dx \leq 2a^2\int_\Omega |x|^{-2a-2}w^2\,dx + 2\int_\Omega |x|^{-2a}|\nabla w|^2\,dx \qquad (18)$$

The classical Hardy inequality, when applied to $v = |x|^{-a}w$ yields,

$$\nabla w = \frac{\nabla w|x|^a - aw|x|^{a-1}x}{|x|^{2a}}$$

$$\begin{aligned} |\nabla v|^2 &= \frac{(\nabla w|x|^a - aw|x|^{a-1}x)^2}{|x|^{4a}} \\ &= \frac{|\nabla w|^2|x|^{2a} - 2a\nabla w\cdot xw|x|^{2a-1} + a^2w^2|x|^{2a-2}|x|^2}{|x|^{4a}} \\ &= \frac{|\nabla w|^2}{|x|^{2a}} - \frac{2a\nabla w\cdot xw}{|x|^{2a+1}} + \frac{a^2w^2}{|x|^{2a}} \end{aligned}$$

Furthermore,

$$\frac{v^2}{|x|^2} = \frac{w^2|x|^{-2a}}{|x|^2} = |x|^{-a-2}w^2$$

Returning to the classical Hardy inequality we have,

$$\int_\Omega \left(\frac{|\nabla w|^2}{|x|^{2a}} - \frac{2a\nabla w\cdot xw}{|x|^{2a+1}} + \frac{a^2w^2}{|x|^{2a}} - \frac{(n-2)^2}{|x|^{-a-2}}w^2\right)dx$$

As we proved earlier, the second term is zero. So we conclude that,

$$\Big(a - \frac{n-2}{2}\Big)^2 \int_\Omega |x|^{-2a-2} w^2 \, dx \le \int_\Omega |x|^{-2a} |\nabla w|^2 \, dx \tag{19}$$

from (18) and (19) we get for some constant C_a depending only on a:

$$\|v\|^2_{H^1_0(\Omega)} \le C_a \int_\Omega |x|^{-2a} |\nabla w|^2 \, dx \le C_a \int_\Omega |x|^{-(n-2)} |\nabla w|^2 \, dx < +\infty$$

The result then follows by a standard density argument.

3. This easily follows from (19) with $a = \frac{n-2}{2} - 1$.

□

Proof. : At first we will show that,

$$C_n(a) = (n-2)^{\frac{2(n-1)}{n}} S_n \quad \text{when} \quad a \ge \frac{1}{n-2}$$

We have that,

$$C_n(a) = \inf_{v \in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-n-2} |\nabla v|^2 \, dx}{\big(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |v|^{\frac{2n}{n-2}} \, dx\big)^{\frac{n-2}{n}}} \tag{20}$$

We change variables by ($r = |x|$)

$$v(x) = y(\tau, \theta), \quad \tau = \frac{1}{X_1(a,r)} = a - \ln r, \quad \theta = \frac{x}{|x|}$$

This change of variables maps the unit ball $B_1 = \{x : |x| < 1\}$ to the complement of the ball of radius a, that is

$$B_a^c = \{(\tau, \theta) : a < \tau < +\infty, \theta \in S^{n-1}\}.$$

Noticing that $X_1'(a,r) = \frac{X_1^2(a,r)}{r} = -\frac{dr}{r}$ we also have,

$$
\begin{aligned}
|\nabla v|^2 &= v_r^2 + \frac{1}{r^2}|\nabla_\theta v|^2 \\
&= y_\tau^2\Big(\frac{d\tau}{dr}\Big)^2 + \frac{1}{r^2}|\nabla_\theta y|^2 \\
&= y_\tau^2\Big(-\frac{dr}{rdr}\Big)^2 + \frac{1}{r^2}|\nabla_\theta y|^2 \\
&= \frac{1}{r^2}y_\tau^2 + \frac{1}{r^2}|\nabla_\theta y|^2 \\
&= \frac{1}{r^2}(y_\tau^2 + |\nabla_\theta y|)^2 \\
&= e^{2(\tau-a)}(y_\tau^2 + |\nabla_\theta y|)^2.
\end{aligned}
$$

A straightforward calculation shows that for $y \in C^\infty([a,\infty) \times S^{n-1})$ under the Dirichlet boundary condition on $\tau = a$ we have,

$$
\begin{aligned}
&r = -e^{(\tau-a)} \\
&dr = (-e^{(\tau-a)})d\tau \rightarrow r^{n-1}dr = e^{n(\tau-a)}d\tau
\end{aligned}
$$

Therefore

$$
\begin{aligned}
C_n(a) &= \inf_{v\in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-n-2}|\nabla v|^2\, dx}{\Big(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\, dx\Big)^{\frac{n-2}{n}}} \\
&= \inf_{y(a,\theta)=0} \frac{\int_a^\infty \int_{S^{n-1}} e^{-(n-2)(\tau-a)}(e^{2(\tau-a)}(y_\tau^2 + |\nabla_\theta y|^2))e^{n(\tau-a)}\, dt\, dS}{\Big(\int_a^\infty \int_{S^{n-1}} e^{-n(\tau-a)}\tau^{-\frac{2(n-1)}{n-2}}|y|^{\frac{2n}{n-2}}e^{n(\tau-a)}\, d\tau\, dS\Big)^{\frac{n-2}{n}}}
\end{aligned}
$$

We conclude that,

$$
C_n(a) = \inf_{y(a,\theta)=0} \frac{\int_a^\infty \int_{S^{n-1}}(y_\tau^2 + |\nabla_\theta y|^2)\, dSd\tau}{\Big(\int_a^\infty \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}}|y|^{\frac{2n}{n-2}}\, dS\, d\tau\Big)^{\frac{n-2}{n}}} \tag{21}
$$

In the sequel we will relate $C_n(a)$ with the best Sobolev constant S_n. It is well known that for any R with $0 < R \leq \infty$

$$S_n = \inf_{u \in C_c^\infty(B_R)} \frac{\int_{B_R} |\nabla u|^2 \, dx}{\left(\int_{B_R} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}} \tag{22}$$

We also know that $S_n = S_{n,radial}$ the latter being the infimum when taken over radial functions. Changing variables by:

$$u(x) = z(t, \theta), \quad t = |x|^{-(n-2)}, \quad \theta = \frac{x}{|x|}$$

We compute the following,

$$t = r^{2-n} \to r = t^{\frac{1}{2-n}}$$

$$dr = \frac{1}{2-n} t^{\frac{1}{2-n}-1} dt = \frac{1}{2-n} t^{\frac{1-(2-n)}{2-n}} dt = \frac{1}{2-n} t^{\frac{n-1}{n-2}} dt$$

$$r^{n-1} dr = \frac{1}{2-n} t^{\frac{n-1}{2-n}} t^{\frac{n-1}{n-2}} dt = \frac{1}{2-n} t^{\frac{2(n-1)}{2-n}} dt$$

$$\begin{aligned} |\nabla u|^2 &= u_r^2 + \frac{1}{r^2} |\nabla_\theta u|^2 = z_t^2 (\frac{dt}{dr})^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \\ &= (2-n)^2 r^{2(1-n)} z_t^2 + t^{\frac{2}{n-2}} |\nabla_\theta z|^2 \\ &= (n-2)^2 t^{\frac{2(n-1)}{n-2}} z_t^2 + t^{\frac{2}{n-2}} |\nabla_\theta z|^2 \end{aligned}$$

Taking (22) and applying the change of variables we finally get that:

$$
\begin{aligned}
S_n &= \inf_{u\in C_c^\infty(B_R)} \frac{\int_{B_R} |\nabla u|^2\,dx}{\left(\int_{B_R}|u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}}}\\
&= \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}[(n-2)^2 t^{\frac{2(n-1)}{n-2}} z_\theta^2 + t^{\frac{2}{n-2}}|\nabla_\theta z|^2]\frac{1}{n-2}t^{\frac{-2(n-1)}{n-2}}\,dt\,dS}{\left(\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}\frac{1}{n-2}t^{\frac{-2(n-1)}{n-2}}\,dt\,dS\right)^{\frac{n-2}{n}}}\\
&= \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}[(n-2)^2 z_t^2 + t^{\frac{4-2n}{n-2}}|\nabla_\theta z|^2]\frac{1}{n-2}\,dt\,dS}{(n-2)^{-\frac{n-2}{n}}\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}t^{\frac{-2(n-1)}{n-2}}\,dt\,dS)^{\frac{n-2}{n}}}\\
&= \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}[(n-2)^2 z_t^2 + \frac{1}{n-2}\frac{1}{t^2}|\nabla_\theta z|^2]\,dt\,dS}{(n-2)^{-\frac{n-2}{n}}\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}t^{\frac{-2(n-1)}{n-2}}\,dt\,dS)^{\frac{n-2}{n}}}\\
&= \inf_{z(R^{-(n-2)},\theta)=0} \frac{n-2}{(n-2)^{-\frac{n-2}{n}}} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2)^2}\frac{1}{t^2}|\nabla_\theta z|^2)}{\left(\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{n-2}}\,dt\,dS\right)^{\frac{n-2}{n}}}\\
&= (n-2)^{\frac{2(n-1)}{n}} \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2)^2}\frac{1}{t^2}|\nabla_\theta z|^2)}{\left(\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{n-2}}\,dt\,dS\right)^{\frac{n-2}{n}}}
\end{aligned}
$$

It follows that for any $R \in (0, \infty]$,

$$
(n-2)^{\frac{-2(n-1)}{n}} S_n = \inf_{z(R^{-(n-2)},\theta)=0} \frac{\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2)^2}\frac{1}{t^2}|\nabla_\theta z|^2)}{\left(\int_{R^{-(n-2)}}^\infty \int_{S^{n-1}} |z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{n-2}}\,dt\,dS\right)^{\frac{n-2}{n}}} \quad (23)
$$

We note that a function u is radial in x if and only if the function z is a function of t only. Comparing (21) and (23) we have that,

$$
C_n(a) \le C_{n,radial}(a) = (n-2)^{\frac{-2(n-1)}{n}} S_{n,radial} \le (n-2)^{\frac{-2(n-1)}{n}} S_n \qquad (24)
$$

On the other hand, assuming that, $a \ge \frac{1}{n-2}$ and observing (23),(24) let us take

$R = a^{-\frac{1}{n-2}}$ so that $a = R^{-(n-2)}$. Then, $\frac{1}{(n-2)^2}\frac{1}{t^2} \leq 1$ since $t \geq a \geq \frac{1}{n-2}$ and therefore $C_n(a) \geq (\frac{1}{n-2})^{\frac{2(n-1)}{n}} S_n$.

Combining this with (24) we conclude our claim that,

$$C_n(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n \text{ when } a \geq \frac{1}{n-2}.$$

Our next step is to prove the following. For any $a > 0$ we have that,

$$C_n(a) \leq a^{\frac{2(n-1)}{n}} S_n.$$

To this end let $0 \neq x_0 \in B_1$ and consider the minimizing sequence of functions,

$$U_\epsilon(x) = (\epsilon + |x - x_0|^2)^{\frac{n-2}{n}} \phi_\delta(|x - x_0|)$$

where $\phi_\delta(t)$ is a C_c^∞ cutoff function which is zero for $t > \delta$ and equal to one for $t < \frac{\delta}{2}$ is small enough so that $|x_0| + \delta < 1$ and therefore, $U_\epsilon \in C_c^\infty(B_\delta(x_0)) \subset C_c^\infty(B_1)$

Then it is well known that,

$$S_n = \lim_{\epsilon \to 0} \frac{\int_{B_1} |\nabla U_\epsilon|^2 \, dx}{\left(\int_{B_1} |U_\epsilon|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}}}$$

From

$$C_n(a) = \inf_{v \in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 \, dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |v|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}}}$$

we have that for any $\epsilon > 0$ small enough,

$$
\begin{aligned}
C_n(a) &= \inf_{v\in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)|\nabla v|^2}\, dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}} \\
&\le \frac{\int_{B_\delta(x_0)} |x|^{-(n-2)}|\nabla U_\epsilon|^2\, dx}{\left(\int_{B_\delta(x_0)} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|U_\epsilon|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{n}}} \\
&\le \left(\frac{|x_0|+\delta}{|x_0|-\delta}\right)^{n-2} \frac{1}{X_1^{\frac{2(n-1)}{n}}(a,|x_0|-\delta)} \frac{\int_{B_\delta(x_0)} |\nabla U_\epsilon|^2\, dx}{\left(\int_{B_\delta(x_0)} |U_\epsilon|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}}
\end{aligned}
$$

where we used the fact that $X_1(a,s)$ is an increasing function of s. Taking the limit $\epsilon \to 0$ we conclude:

$$
C_n(a) \le \left(\frac{|x_0|+\delta}{|x_0|-\delta}\right)^{n-2} \frac{S_n}{X_1^{\frac{2(n-1)}{n}}(a,|x_0|-\delta)}
$$

This is true for any $\delta > 0$ small enough therefore,

$$
C_n(a) \le X_1^{-\frac{2(n-1)}{n}}(a,|x_0|)S_n
$$

Since x_0 is arbitary and $X_1(a,s)$ is an increasing function of s, we end up with

$$
C_n(a) \le X_1^{-\frac{2(n-1)}{n}}(a,1)S_n = a^{\frac{2(n-1)}{n}} S_n \tag{25}
$$

and this proves our claim that,

$$
C_n(a) \ge \left(\frac{1}{n-2}\right)^{\frac{2(n-1)}{n}} S_n
$$

To complete the calculation of $C_n(a)$ we will finally show that,

$$
C_n(a) \ge a^{\frac{2(n-1)}{n}} S_n \quad \text{when} \quad 0 < a < \frac{1}{n-2}
$$

To prove this we will relate the infimum $C_n(a)$ to a Caffarelli-Kohn-Nirenberg inequality. We will need the following result.

Proposition 1. *Let $b > 0$ and,*

$$S_n(b) = \inf_{v \in C_c^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |x|^{2b} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |x|^{\frac{2bn}{n-2}} |u|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}} \tag{26}$$

Then $S_n(b) = S_n$ and this constant is not achieved in the appropriate function space. This is proved in Theorem 1.1 of F. Catrina, Z.-Q Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence) and symmetry of extremal functions.

Caffarelli, Kohn and Nirenberg established the following inequalities: For all $u \in C_c^\infty(\mathbb{R}^N)$:

$$\left(\int_{\mathbb{R}^N} |x|^{-bp} |u|^p \, dx \right)^{\frac{2}{p}} \leq C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx, \quad N \geq 3$$

$$, -\infty < a < \frac{N-2}{2}, a \leq b \leq a+1, p = \frac{2n}{N-2+2(b-a)}$$

We change variables in (26)

$$u(x) = z(t, \theta), \quad t = |x|^{-(n-2)-2b}, \quad \theta = \frac{x}{|x|}$$

As before we compute,

$$t = r^{-(n-2)-2b} \to r = t^{\frac{1}{2-n-2b}}$$

$$\begin{aligned} dr &= \frac{1}{2-n-2b} t^{\frac{1}{2-n-2b}} dt \\ &= \frac{1}{2-n-2b} t^{\frac{1}{2-n-2b} - \frac{2-n-2b}{2-n-2b}} dt = \frac{1}{2-n-2b} t^{\frac{1-(2-n-2b)}{2-n-2b}} \\ &= \frac{1}{2-n-2b} t^{\frac{n-1+2b}{2-n-2b}} \\ r^{n-1} &= t^{\frac{n-1}{2-n-2b}} \end{aligned}$$

$$
\begin{aligned}
r^{n-1}dr &= \frac{1}{2-n-2b} t^{\frac{n-1+2b}{2-n-2b}} t^{\frac{n-1}{2-n-2b}} dt = \frac{1}{2-n-2b} t^{\frac{n-1+2b+n-1}{2-n-2b}} dt = \\
&= \frac{1}{2-n-2b} t^{\frac{2n-2+2b}{2-n-2b}} dt
\end{aligned}
$$

$$
\begin{aligned}
|\nabla u|^2 &= u_r^2 + \frac{1}{r^2}|\nabla_\theta u|^2 = z_t^2 (\frac{dt}{dr})^2 + \frac{1}{r^2}|\nabla_\theta u|^2 \\
&= z_t^2((-(n-2)-2b)r^{-(n-2)-2b-1})^2 + \frac{1}{r^2}|\nabla_\theta u|^2 \\
&= z_t^2(2-n-2b)^2 r^{2(1-n-2b)} + \frac{1}{r^2}|\nabla_\theta u|^2 \\
&= z_t^2(2-n-2b)^2 t^{\frac{2(1-n-2b)}{2-n-2b}} + \frac{1}{t^{\frac{2}{2-n-2b}}}|\nabla_\theta z|^2
\end{aligned}
$$

A straightforward calculation shows that for any R':

$$
\begin{aligned}
S_n &= \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}} t^{\frac{2b}{2-n-2b}}[z_t^2(2-n-2b)^2 t^{\frac{2(1-n-2b)}{2-n-2b}} + \frac{1}{t^{\frac{2}{2-n-2b}}}|\nabla_\theta z|^2]\frac{1}{2-n-2b} t^{\frac{2n-2+2b}{2-n-2b}}\,dt\,dS}{\left[\int_{R'}^{\infty}\int_{S^{n-1}} t^{\frac{2bn}{(n-2)(2-n-2b)}}|z|^{\frac{2n}{n-2}}\frac{1}{(2-n-2b)} t^{\frac{2n-2+2b}{2-n-2b}}\,dt\,dS\right]^{\frac{n-2}{n}}} \\
&= \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}[z_t^2(n-2+2b)^2 t^{\frac{2-2n-4b}{2-n-2b}+\frac{2b}{2-n-2b}} + \frac{t^{\frac{2b}{2-n-2b}}}{t^{\frac{2}{2-n-2b}}}|\nabla_\theta z|^2]\frac{1}{(n-2+2b)} t^{\frac{2n-2+2b}{2-n-2b}}\,dt\,dS}{\left[\int_{R'}^{\infty}\int_{S^{n-1}} t^{\frac{2bn}{(n-2)(2-n-2b)}+\frac{2n-2+2b}{2-2b-n}}\frac{1}{(n-2+2b)^2}|z|^{\frac{2n}{n-2}}\,dt\,dS\right]^{\frac{n-2}{n}}} \\
&= \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}[(n-2+2b)^2 z_t^2 t^{\frac{2-2n-2b}{2-n-2b}} + t^{\frac{2b-2}{n-2+2b}}|\nabla_\theta z|^2]\frac{1}{(n-2+2b)} t^{\frac{2n-2-2b}{2-n-2b}}\,dt\,dS}{(n-2+2b)^{-\frac{n-2}{n}}\left[\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}} t^{\frac{2bn+(n-2)(2n-2+2b)}{(n-2)(2-n-2b)}}\,dt\,dS\right]^{\frac{n-2}{n}}}
\end{aligned}
$$

$$= \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}[(n-2+2b)^2 z_t^2 t^{\frac{2-2n-2b+2n-2+2b}{2-n-2b}} + t^{\frac{2b-2+2n-2+2b}{2-n-2b}}|\nabla_\theta z|^2]\frac{1}{(n-2+2b)}\,dt\,dS}{(n-2+2b)^{-\frac{n-2}{n}}\left[\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{-\frac{2(n-1)}{n-2}}\,dt\,dS\right]^{\frac{n-2}{n}}}$$

$$= \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}[(n-2+2b)^2 z_t^2 + t-\frac{2(-2b-n+2)}{-2b-n+2}|\nabla_\theta z|^2]\frac{1}{(n-2+2b)}\,dt\,dS}{(n-2+2b)^{-\frac{n-2}{n}}\left[\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{(n-2)}}\,dt\,dS\right]^{\frac{n-2}{n}}}$$

$$= \inf_{z(R',\theta)=0} \frac{(n-2+2b)\int_{R'}^{\infty}\int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2+2b)^2}\frac{1}{t^2}|\nabla_\theta z|^2)\,dtdS}{(n-2+2b)^{-\frac{n-2}{n}}\left[\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{(n-2)}}\,dt\,dS\right]^{\frac{n-2}{n}}}$$

$$S_n = (n-2+2b)^{\frac{2(n-1)}{n}} \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2+2b)^2}\frac{1}{t^2}|\nabla_\theta z|^2)\,dtdS}{\left(\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{(n-2)}}\,dt\,dS\right)^{\frac{n-2}{n}}}$$

$$(n-2+2b)^{-\frac{2(n-1)}{n}}S_n = \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2+2b)^2}\frac{1}{t^2}|\nabla_\theta z|^2)\,dtdS}{\left(\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{(n-2)}}\,dt\,dS\right)^{\frac{n-2}{n}}}$$

Therefore we conclude that,

$$(n-2+2b)^{-\frac{2(n-1)}{n}}S_n = \inf_{z(R',\theta)=0} \frac{\int_{R'}^{\infty}\int_{S^{n-1}}(z_t^2 + \frac{1}{(n-2+2b)^2}\frac{1}{t^2}|\nabla_\theta z|^2)\,dtdS}{\left(\int_{R'}^{\infty}\int_{S^{n-1}}|z|^{\frac{2n}{n-2}}t^{-\frac{2(n-1)}{(n-2)}}\,dt\,dS\right)^{\frac{n-2}{n}}} \quad (27)$$

Condition $1 \geq \frac{1}{(n-2+2b)^2t^2}$ for $t \geq a$ is satisfied if we choose $b \in (0,\infty)$ such that,

$$\frac{1}{n-2} > a = (n-2+2b)^{-1} > 0$$

Taking $R' = a$ and comparing (27) and (21) we have that if,

$$1 \geq \frac{1}{(n-2+2b)^2t^2} \quad \text{for} \quad t \geq a$$

Then

$$C_n(a) \geq (n-2+2b)^{-\frac{2(n-1)}{n}}S_n$$

For such b it follows from $C_n(a) \geq (n-2+2b)^{-\frac{2(n-1)}{n}}S_n$ that,

$$C_n(a) \geq a^{\frac{2(n-1)}{n}}S_n$$

And this proves our claim that,

$$C_n(a) \geq a^{\frac{2(n-1)}{n}} S_n \quad \text{when} \quad 0 < a < \frac{1}{n-2}.$$

We finally establish the nonexistence of a minimizer. We will argue by contradiction. Suppose that $\tilde{v} \in W_0^{1,2}(B_1; |x|^{-(n-2)})$ is a minimizer of

$$C_n(a) = \inf_{v \in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} |\nabla v|^2 \, dx}{\left(\int_{B_1} |x|^{-n} X_1^{\frac{2(n-1)}{n-2}}(a, |x|) |v|^{\frac{2n}{n-2}} \, dx\right)^{\frac{n-2}{n}}}.$$

Through the change of variables we did the quotient in (21) also admits a minimizer $\tilde{y}$.

Consider first the case $a \geq \frac{1}{n-2}$. Comparing (21) with (23) with $R = a^{-\frac{1}{n-2}}$, we conclude that $\tilde{y}$ is a radial minimizer of (23) as well. It follows that (22) admits a radial $H_0^1(B_R)$ minimizer $\tilde{u}(r) = \tilde{y}(t)$, $t = r^{-(n-2)}$, which contradicts the fact that the Sobolev inequality (22) has no H_0^1 minimizers.

In case when $0 < a < \frac{1}{n-2}$ we use a similar argument comparing (21) and (27) to conclude the existence of a radial minimizer to (27) with b such that $\frac{1}{n-2} > a = (n-2+2b)^{-1} > 0$. This contradicts the nonexistence of minimizer for (26).The proof of Theorem 2.2 is now complete. $\square$

We can now prove Theorem2.1 by the change of variables in (15) and using the change of variables,

$$u(x) = |x|^{-\frac{n-2}{2}} v(x)$$

we have that,

$$\nabla u = \frac{\nabla v |x|^{\frac{n-2}{2}} - \frac{n-2}{2} v |x|^{\frac{n-6}{2}} x}{|x|^{n-2}}$$

So,

$$\begin{aligned}
|\nabla u|^2 &= \frac{(\nabla v |x|^{\frac{n-2}{2}} - \frac{n-2}{2} v |x|^{\frac{n-6}{2}} x)^2}{|x|^{2n-4}} \\
&= \frac{|\nabla v|^2 |x|^{n-2}}{|x|^{n-2}|x|^{n-2}} - \frac{(n-2)\nabla v \cdot x v |x|^{\frac{2(n-4)}{2}}}{|x|^n |x|^{n-4}} + \frac{(n-2)^2}{4} \frac{v^2 |x|^{n-6} |x|^2}{|x|^{2n-4}} \\
&= \frac{1}{|x|^{n-2}} (|\nabla v|^2) - \frac{(n-2)\nabla v \cdot x}{|x|^n} + \frac{(n-2)^2}{4} \frac{v^2}{|x|^n}
\end{aligned}$$

The second term, as in the proof of the classical Hardy inequality, is zero. Furthermore,

$$\frac{u^2}{|x|^2} = \frac{v^2}{(|x|^{\frac{n-2}{2}})^2|x|^2} = \frac{v^2}{|x|^{n-2}|x|^2} = \frac{v^2}{|x|^n}$$

It is easily observed that,

$$\begin{aligned}
&C_n(a)\Big(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\\
&= C_n(a)\Big(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|x^{\frac{2-n}{2}}v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\\
&= C_n(a)\Big(\int_{B_1} X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|x|^{\frac{2-n}{2}\frac{2n}{n-2}}|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\\
&= C_n(a)\Big(\int_{B_1} |x|^{-n}X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}
\end{aligned}$$

Returning to (15),

$$\begin{aligned}
&\int_{B_1}\Big(|x|^{-(n-2)}|\nabla v|^2 + \frac{(n-2)^2}{4}\frac{v^2}{|x|^n}\Big)\,dx \geq \frac{(n-2)^2}{4}\frac{v^2}{|x|^n}\\
&+C_n(a)\Big(\int_{B_1} |x|^{-n}X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}
\end{aligned}$$

So (15) is equivalent to

$$\int_{B_1} |x|^{-(n-2)}|\nabla v|^2\,dx \geq C_n(a)\Big(\int_{B_1} |x|^{-n}X_1^{\frac{2(n-1)}{n-2}}(a,|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}, v \in C_c^\infty(B_1) \tag{28}$$

Corollary 2. *Let $n \geq 3$. For any $u \in C_0^\infty(B_1^c)$, there holds,*

$$\int_{B_1^c} |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2\int_{B_1^c}\frac{u^2}{|x|^2}\,dx + C_n(a)\Big(\int_{B_1^c} X_1^{\frac{2(n-1}{(n-2)}}\Big(a,\frac{1}{|x|}\Big)|u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \tag{29}$$

where the best constant $C_n(a)$ is the same as in Theorem 2.1

We also cover the case of a general bounded domain Ω containing the origin.

Theorem 2.4. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain containing origin. Set $D = \sup_{x\in\Omega} |x|$. For any $u \in C_c^\infty(\Omega)$ there holds,*

$$\int_\Omega |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_\Omega \frac{u^2}{|x|^2}\,dx + C_n(a)\Big(\int_\Omega X_1^{\frac{2(n-1)}{n-2}}(a, \frac{|x|}{D})|u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \tag{30}$$

where the best constant $C_n(a)$ is independent of Ω and is given by

$$C_n(a) = \begin{cases} (n-2)^{\frac{-2(n-1)}{n}} S_n, & a \geq \frac{1}{n-2} \\ \\ a^{\frac{2(n-1)}{n}} S_n, & 0 < a < \frac{1}{n-2}. \end{cases}$$

It follows easily from Theorem 2.2 that there are no minimizers for (29) *and* (30) *in $H_0^1(\Omega)$.*

Proof. : The lower bound on the best constant follows from Theorem 2.1, the fact that if $\in C_c^\infty(\Omega)$ then $\in C_c^\infty(B_D)$, since $\Omega \subset B_D$ and a simple scaling argument. To establish the upper bound in the case where $0 < a < \frac{1}{n-2}$ we argue exactly as in the proof of Theorem 1.2 using test functions.
Let $0 \neq x_0 \in B_1$ and consider the sequence of functions,

$$U_\epsilon(x) = (\epsilon + |x - x_0|^2)^{\frac{n-2}{n}} \phi_\delta(|x - x_0|)$$

where $\phi_\delta(t)$ is a C_c^∞ cutoff function which is zero for $t > \delta$ and equal to one for $t < \frac{\delta}{2}$ is small enough so that $|x_0| + \delta < 1$ and therefore, $U_\epsilon \in C_c^\infty(B_\delta(x_0)) \subset C_c^\infty(B_1)$
The sequence U_ϵ concentrates near a point of the boundary Ω, that realizes the $\max_{x\in\Omega} |x|$. Let as now consider the case where $a \geq \frac{1}{n-2}$. For $a > 0$ and $0 < \rho < 1$, we set

$$\tilde{C}_n(a, \rho) = \inf_{u \in C_c^\infty(B_\rho)} \frac{\int_{B_\rho} |\nabla u|^2\,dx - (\frac{n-2}{n})^2 \int_{B_\rho} \frac{u^2}{|x|^2}\,dx}{\Big(\int_{B_\rho} X_1^{\frac{2(n-1)}{n-2}}(a, |x|)|u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}}$$

A simple scaling argument and Theorem 1.1 shows that,

$$\tilde{C}_n(a, \rho) = C_n(a - \ln \rho).$$

Thus, for ρ small enough we have that

$$\tilde{C}_n(a, \rho) = (n-2)^{-\frac{2(n-1)}{n}} S_n.$$

Since for ρ small, $B_\rho \subset \Omega$ the upper bound follows easily as well. □

2.1 The k-improved Hardy-Sobolev Inequality

We next consider the k-improved inequality. Let k be a fixed positive integer. For X_1 as in (2) we define for $s \in (0,1)$

$$X_{i+1}(a,s) = X_1(a, X_i(a,s)), i = 1,2,\dots,k$$

Noticing that $X_i(a,s)$ is a decreasing function of a we easily check that there exist unique positive constant $0 < a_k < \beta_{n,k} \leq 1$ such that:

1. The $X_i(a_k, s)$ are well defined for all $i = 1,2,\dots,k+1$ and all $s \in (0,1)$ and $X_{k+1}(a_k, 1) = \infty$. In other words, a_k is the minimum value of the constant a so that the X_i's, $i = 1,2,\dots,k+1$ are well defined in (0,1).

2. $X_1(\beta_{n,k},1)X_2(\beta_{n,k},1)\dots X_{k+1}(\beta_{n,k},1) = n-2$
   For $n \geq 3$, k a fixed positive integer and $u \in C_c^\infty(B_1)$ there holds

$$\int_{B_1} |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2}\,dx + \frac{1}{4}\sum_{i=1}^{k}\int_{B_1} \frac{X_1^2(a,|x|)\dots X_i^2(a,|x|)}{|x|^2} u^2\,dx$$

$$+C_{n,k}(a)\Big(\int_{B_1} (X_1(a,|x|)\dots X_{k+1}(a|,x|)^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \tag{31}$$

In our next result we calculate the best constant $C_{n,k}(a)$ in (31)

Theorem 2.5. *Let $n \geq 3$ and $k = 1,2,\dots$ be a fixed positive integer. The best constant $C_{n,k}(a)$ in*

$$\int_{B_1} |\nabla u|^2\,dx \geq \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{u^2}{|x|^2}\,dx + \frac{1}{4}\sum_{i=1}^{k}\int_{B_1} \frac{X_1^2(a,|x|)\dots X_i^2(a,|x|)}{|x|^2} u^2\,dx$$

$$+C_{n,k}(a)\Big(\int_{B_1} (X_1(a,|x|)\dots X_{k+1}(a|,x|)^{\frac{2(n-1)}{n-2}} |u|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \tag{32}$$

satisfies:

$$C_n(a) = \begin{cases} (n-2)^{\frac{-2(n-1)}{n}} S_n, & a \geq \beta_{n,k} \\ \\ (\prod_{i=1}^{k+1} X_i(a,1))^{-\frac{2(n-1)}{n}} S_n, & a_k < a < \beta_{n,k}. \end{cases}$$

When restricted to radial functions, the best constant of (31) *is given by,*

$$C_{n,k,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_n \quad \textit{for all} \quad a > a_k$$

Again we notice that $C_{n,k}(a) < S_n$ *for all* $a > \beta_{3,k}$.

Proof. To simplify the representation we will write $X_i(|x|)$ instead of $X_i(a,|x|)$. Let k be a fixed positive integer.
We first consider the case $a \geq \beta_{n,k}$.We change variables in (31) by,

$$u(x) = |x|^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(|x|) \dots X_k^{-\frac{1}{2}}(|x|) v(x)$$

We know that

$$X_1 = \frac{1}{1 - \ln r} \quad \text{and} \quad X_1' = \frac{X_1^2(r)}{r}$$

also

$$X_{k+1} = X_1(X_k(r))$$

so,

$$X_k'(r) = \frac{1}{r} X_1 \dots X_{k-1}(r) X_k^2(r)$$

$$X_{i+1}(a,|x|) = X_1(a, X_i(a,|x|)) = X_1(a, X_i(a,r))$$

In general,it is easily proven by induction that

$$\frac{d}{dr} X_k^a = \frac{a}{r} X_1(r) \dots X_{k-1}(r) X_k^{a+1}(r)$$

We observe that, $u(x) = \phi(r)v(x)$, where,

$$\phi(x) = |x|^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(|x|) \dots X_k^{-\frac{1}{2}}(|x|)$$

Furthermore,

$$
\begin{aligned}
&\int_{B_1} |\nabla u|^2\,dx \\
&= \int_{B_1} |\phi\nabla v + v\nabla\phi|^2\,dx \\
&= \int_{B_1} \phi^2|\nabla v|^2\,dx + \int_{B_1} |\nabla\phi|^2 v^2\,dx + 2\int_{B_1} \phi v\nabla v\nabla v\,dx \\
&= \int_{B_1} \phi^2|\nabla v|^2\,dx + \int_{B_1} |\nabla\phi|^2 v^2\,dx + \int_{B_1} \phi\nabla\phi(\nabla v)^2 \\
&= \int_{B_1} \phi^2|\nabla v|^2\,dx + \int_{B_1} |\nabla\phi|^2 v^2\,dx - \int_{B_1} v^2\text{div}(\phi\nabla\phi)\,dx \\
&= \int_{B_1} \phi^2|\nabla v|^2\,dx + \int_{B_1} |\nabla\phi|^2 v^2\,dx - \int_{B_1} v^2|\nabla\phi|^2 - \int_{B_1} v^2\phi\Delta\phi\,dx \\
&= \int_{B_1} \phi^2|\nabla v|^2\,dx - \int_{B_1} v^2\frac{\Delta\phi}{\phi}\,dx
\end{aligned}
$$

Lemma 2.6. *For every $m \in \mathbb{N}$ we have that,*

$$\Delta\phi_m + V_m\phi_m = 0$$

where,

$$\phi_m(r) = r^{-\frac{n-2}{2}} X_1^{-\frac{1}{2}}(r)\dots X_k^{-\frac{1}{2}}(r)$$
$$V_m(r) = \Big(\frac{n-2}{2}\Big)^2\frac{1}{r^2} + \frac{1}{4r^2}(X_1^2 + X_1^2X_2^2 + \cdots + X_1^2X_2^2\dots X_m^2)$$
$$n_m(r) = X_1 + X_1X_2 + \cdots + X_1X_2\dots X_m$$

Proof. We can prove this lemma using induction. It is easy to see that for $m = 1$ the equality holds. We assume that it is true for $m = k$ and we will prove it for $m = k + 1$. So,

$$
\begin{aligned}
\Delta\phi_{k+1} &= \Delta\Big(\phi_k X_{k+1}^{-\frac{1}{2}}\Big) \\
&= (\Delta\phi_k)X_{k+1}^{-\frac{1}{2}} + 2\nabla\phi_k\nabla X_{k+1}^{-\frac{1}{2}} + \phi_k\Big(\Delta X_{k+1}^{-\frac{1}{2}}\Big)
\end{aligned}
$$

First we compute,

$$\begin{aligned}
\phi_k'(r) &= -\frac{n-2}{2} r^{-\frac{n}{2}} \Big(X_1^{-\frac{1}{2}} X_2^{-\frac{1}{2}} \dots X_k^{-\frac{1}{2}} \Big) + \\
& r^{-\frac{n-2}{2}} \Big[-\frac{1}{2r} X_1^{\frac{1}{2}} X_2^{-\frac{1}{2}} \dots X_k^{-\frac{1}{2}} - \dots - \frac{1}{2r} X_1^{\frac{1}{2}} X_2^{\frac{1}{2}} \dots X_k^{\frac{1}{2}} \Big] \\
&= -\frac{n-2}{2} r^{-\frac{n}{2}} \Big(X_1^{-\frac{1}{2}} \Big) \Big(X_1^{-\frac{1}{2}} X_2^{-\frac{1}{2}} \dots X_k^{-\frac{1}{2}} \Big) + \\
& r^{-\frac{n}{2}} \Big[X_1^{\frac{1}{2}} X_2^{-\frac{1}{2}} \dots X_k^{-\frac{1}{2}} - \dots X_1^{\frac{1}{2}} X_2^{\frac{1}{2}} \dots X_k^{\frac{1}{2}} \Big] \\
&= r^{-\frac{n}{2}} \Big[-\frac{n-2}{2} X_1^{-\frac{1}{2}} \dots X_k - \frac{1}{2} - \frac{1}{2} X_1^{\frac{1}{2}} \dots X_k^{-\frac{1}{2}} - \dots - \frac{1}{2} X_1^{\frac{1}{2}} \dots X_k^{\frac{1}{2}} \Big] \\
&= -\frac{n-2}{2r} \phi_k(r) - \frac{1}{2r} \phi_k(r) \eta_k(r)
\end{aligned}$$

So,

$$\begin{aligned}
& \nabla \phi_k(r) \nabla X_{k+1}^{-\frac{1}{2}} \\
&= \phi_k(r) (X_{k+1}^{-\frac{1}{2}})' \\
&= \Big[-\frac{n-2}{2} \phi_k(r) - \frac{1}{2r} \phi_k(r) \eta_k(r) \Big] \Big[-\frac{1}{2r} X_1 X_2 \dots X_k X_{k+1}^{\frac{1}{2}} \Big] \\
&= \frac{n-2}{4r^2} r^{-\frac{n-2}{2}} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} + \frac{1}{4r^2} r^{-\frac{n-2}{2}} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} \eta_k(r)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Big(X_{k+1}^{-\frac{1}{2}} \Big)'' &= \Big[-\frac{1}{2r} X_1 X_2 \dots X_k X_{k+1}^{\frac{1}{2}} \Big]' \\
&= \frac{1}{2r^2} X_1 X_2 \dots X_{k+1}^{\frac{1}{2}} - \frac{1}{2r^2} X_1^2 X_2 \dots X_{k+1}^{\frac{1}{2}} - \frac{1}{2r^2} X_1^2 X_2^2 \dots X_{k+1}^{\frac{1}{2}} \\
& - \dots - \frac{1}{2r^2} X_1^2 \dots X_k^2 X_{k+1}^{\frac{1}{2}} - \frac{1}{4r^2} X_1 \dots X_k X_{k+1}^{\frac{3}{2}} \\
&= \frac{1}{2r^2} X_1 \dots X_k X_{k+1}^{\frac{1}{2}} (1 - \eta_k(r)) - \frac{1}{4r^2} X_1^2 \dots X_k^2 X_{k+1}^{\frac{3}{2}}
\end{aligned}$$

Finally, we have

$$
\begin{aligned}
&\Delta\phi_{k+1} + V_{k+1}\phi_{k+1} \\
&= \Delta\phi_{k+1} + \Big(V_k(r) + \frac{X_1^2 \dots X_{k+1}^2}{4r^2}\Big)\phi_k X_{k+1}^{-\frac{1}{2}} \\
&= (\Delta\phi_k)X_{k+1}^{-\frac{1}{2}} + \frac{n-2}{2r^2} r^{-\frac{n-2}{2}} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} + \frac{1}{2r^2} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} \eta_k(r) \\
&- \frac{n-2}{2r^2} r^{-\frac{n-2}{2}} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} - \frac{1}{2r^2} X_1^{\frac{1}{2}} \dots X_{k+1}^{\frac{1}{2}} \eta_k(r) \\
&- \frac{1}{4r^2} X_1^2 \dots X_{k+1}^2 \phi_k(r) + V_k\phi_k X_{k+1}^{-\frac{1}{2}} + \frac{1}{4r^2} X_1^2 \dots X_{k+1}^2 \phi_k X_{k+1}^{-\frac{1}{2}} \\
&= 0
\end{aligned}
$$

Hence, it is true for $m = k+1$ and so it holds for $m = k$. $\square$

Using Lemma 2.6 we observe that,

$$
\begin{aligned}
&- \int_{B_1} v^2 \frac{\Delta\phi}{\phi}\, dx = \int_{B_1} v^2 V_m(r)\, dx \\
&= \Big(\frac{n-2}{2}\Big)^2 \int_{B_1} \frac{v^2}{|x|^2}\, dx + \frac{1}{4}\sum_{i=1}^{k} \int_{B_1} \frac{X_1^2(a,|x|) \dots X_i^2(a,|x|)}{|x|^2} v^2\, dx
\end{aligned}
$$

Thus, continuing with the change of variables

$$C_{n,k}(a)\Big(\int_{B_1}(|x|^{-\frac{n-2}{2}}X_1^{-\frac{1}{2}}(|x|)\dots X_k^{-\frac{1}{2}}(|x|)|v|)^{\frac{2n}{n-2}}(X_1(|x|)\dots X_{k+1})^{\frac{2(n-1)}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

$$= C_{n,k}(a)\Big(\int_{B_1}|x|^{-\frac{n-2}{2}\frac{2n}{n-2}}X_1^{-\frac{1}{2}\frac{2n}{n-2}}(|x|)\dots X_k^{-\frac{1}{2}\frac{2n}{n-2}}(|x|)(X_1(|x|)\dots X_{k+1})^{\frac{2(n-1)}{n-2}}|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

$$= C_{n,k}(a)\Big(\int_{B_1}|x|^{-n}X_1^{-\frac{n}{n-2}}(|x|)\dots X_k^{-\frac{n}{n-2}}(|x|)(X_1(|x|)\dots X_{k+1}(|x|))^{\frac{2(n-1)}{n-2}}|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

$$= C_{n,k}\Big(\int_{B_1}|x|^{-n}X_1^{-\frac{n}{n-2}+\frac{2n-2}{n-2}}(|x|)\dots X_k^{-\frac{n}{n-2}+\frac{2n-2}{n-2}}(|x|)\dots X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

$$= C_{n,k}(a)\Big(\int_{B_1}|x|^{-n}X_1^{\frac{n-2}{n-2}}(|x|)\dots X_k^{\frac{n-2}{n-2}}(|x|)X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

$$= C_{n,k}(a)\Big(\int_{B_1}X_1(|x|)\dots X_k(|x|)X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}$$

Finally we obtain,

$$\int_{B_1}|x|^{-(n-2)}X_1^{-1}(|x|)\dots X_k^{-1}(|x|)|\nabla v|^2\,dx$$

$$\geq C_{n,k}(a)\Big(\int_{B_1}|x|^{-n}X_1(|x|)\dots X_k(|x|)X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}, v\in C_c^\infty(B_1)$$

We further change variables by,

$$v(x)=y(\tau,\theta),\quad \tau=\frac{1}{X_{k+1}(r)},\quad \theta=\frac{x}{|x|},\quad (r=|x|)$$

This change of variables maps the unit ball $B_1=\{x:|x|<1\}$ to the complement of the ball of radius $r_a=X_{k+1}^{-1}(1)$ that is,

$$B_{r_a}^c=\{(\tau,\theta):X_{k+1}^{-1}(r)<\tau<\infty,\theta\in S^{n-1}\}.$$

Note that,

$$d\tau=-\frac{X'_{k+1}(r)}{X_{k+1}^2(r)}dr=-\frac{X_1(r)\dots X_k(r)}{r}dr$$

Let us denote by $f_1(t)$ the inverse function of $X_1(t)$.We also set

$$f_{i+1}(t)=f_1(f_i(t)),\quad i=1,2,\dots k$$

Consequently $r = f_{k+1}(\tau^{-1})$. Also, $X_1(r) = f_k(\tau^{-1}), X_2(r) = f_{k-1}(\tau^{-1}), \dots X_k(r) = f_1(\tau^{-1})$.
We then find,

$$C_{n.k}(a) = \inf_{v \in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} X_1^{-1}(|x|) \dots X_k^{-1}(|x|) |\nabla v|^2 \, dx}{\left(\int_{B_1} |x|^{-n} X_1(|x|) \dots X_k(|x|) X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|) |v|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}}} \tag{33}$$

We then compute,

$$\begin{aligned} |\nabla v|^2 &= (v_r)^2 + \frac{1}{r^2} |\nabla_\theta v|^2 = y_\tau^2 (\frac{d\tau}{dr})^2 + \frac{1}{r^2} |\nabla_\theta y|^2 \\ &= y_\tau^2 (-\frac{X_1 \dots X_k(r)}{r})^2 + \frac{1}{r^2} |\nabla_\theta y|^2 \\ &= y_\tau^2 \frac{(X_1(r) \dots X_k(r))^2}{r^2} + \frac{1}{r^2} |\nabla_\theta y|^2 \\ &= \frac{1}{r^2} (y_\tau^2 (X_1(r) \dots X_k(r))^2 + |\nabla_\theta y|^2) \\ &= \frac{1}{f_{k+1}(\tau^{-1})^2} (y_\tau^2 (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^2 + |\nabla_\theta y|^2) \end{aligned}$$

$$X_{k+1}(|x|) = X_{k+1}(r) = \tau^{-\frac{2(n-1)}{n-2}}$$

$$X_1^{-1}(|x|) \dots X_k^{-1}(|x|) = X_1^{-1}(r) \dots X_k^{-1}(r) = (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^{-1}$$

$$\begin{aligned} r^{n-1} dr &= \frac{-r \cdot r^{n-1}}{X_1(r) \dots X_k(r)} d\tau \\ &= -\frac{r^n}{X_1(r) \dots X_k(r)} d\tau \\ &= -\frac{(f_{k+1}(\tau^{-1}))^n}{f_k(\tau^{-1}) \dots f_1(\tau^{-1})} \end{aligned}$$

Finally, we apply the change of variables in (33),

$$
\begin{aligned}
C_{n,k}(a) &= \inf_{y(r_a,\theta=0)} \int_{r_a}^{\infty} \int_{S^{n-1}} \left[(f_{k+1}(\tau)^{-1})^{-(n-2)} (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^{-1} \frac{1}{(f_{k+1}(\tau^{-1}))^2} \right] \\
&\cdot \frac{(y_\tau^2 (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^2 + |\nabla_\theta y|^2) \frac{(f_{k+1}(\tau^{-1}))^n}{(f_k(\tau^{-1}) \dots f_1(\tau^{-1}))} \, d\tau \, dS}{\left(\int_{r_a}^{\infty} \int_{S^{n-1}} (f_{k+1}(\tau^{-1}))^{-n} (f_k(\tau^{-1}) \dots f_1(\tau^{-1})) \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} \frac{(f_{k+1}(\tau^{-1}))^n}{f_k(\tau^{-1}) \dots f_1(\tau^{-1})} \, d\tau \, dS \right)^{\frac{n-2}{n}}} \\
&= \int_{r_a}^{\infty} \int_{S^{n-1}} \frac{(f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^{-2} (y_\tau^2 (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^2 + |\nabla_\theta y|^2) \, d\tau \, dS}{\left(\int_{r_a}^{\infty} \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} \, d\tau \, ds \right)^{\frac{n-2}{n}}} \\
&= \int_{r_a}^{\infty} \int_{S^{n-1}} \frac{(y_\tau^2 + (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^{-2} |\nabla_\theta y|^2) \, d\tau \, dS}{\left(\int_{r_a}^{\infty} \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} \, d\tau \, ds \right)^{\frac{n-2}{n}}}
\end{aligned}
$$

We conclude that,

$$
C_{n,k}(a) = \inf_{y(r_a,\theta)=0} \frac{\int_{r_a}^{\infty} \int_{S^{n-1}} (y_\tau^2 + (f_k(\tau^{-1}) \dots f_1(\tau^{-1}))^{-2} |\nabla_\theta y|^2) \, d\tau \, dS}{\left(\int_{r_a}^{\infty} \int_{S^{n-1}} \tau^{-\frac{2(n-1)}{n-2}} |y|^{\frac{2n}{n-2}} \, d\tau \, ds \right)^{\frac{n-2}{n}}} \tag{34}
$$

Again we will relate this with the best constant S_n. From (23) we have that,

$$
(n-2)^{-\frac{2(n-1)}{n}} S_n = \inf_{z(r_a,\theta)=0} \frac{\int_{r_a}^{\infty} \int_{s^{n-1}} (z_t^2 + \frac{1}{(n-2)^2 t^2} |\nabla_\theta z|) \, dt \, dS}{(\int_{r_a}^{\infty} \int_{S^{n-1}} t^{-\frac{2(n-1)}{n-2}} |z|^{\frac{2n}{n-2}} \, dt \, dS)^{\frac{n-2}{n}}}
$$

Comparing this with (34) we have that,

$$
C_{n,k}(a) \le C_{n,k,radial}(a) = (n-2)^{-\frac{2(n-1)}{n}} S_{n,radial} = (n-2)^{-\frac{2(n-1)}{n}} S_n \tag{35}
$$

On the other hand for all $a \ge \beta_{k,n}$ and $\tau \ge r_a$ we have that,

$$
\begin{aligned}
(\tau^{-1} f_1(\tau^{-1}) \dots f_k(\tau^{-1}))^{-2} &\ge (r_a^{-1} f_1(r_a^{-1}) \dots f_k(r_a^{-1}))^{-2} \\
&= (X_1(a,1) \dots X_k(a,1) X_{k+1}(a,1))^{-2} \\
&\ge \frac{1}{(n-2)^2}
\end{aligned}
$$

Therefore,

$$(f_1(\tau^{-1})\dots f_k(\tau^{-1}))^{-2} \geq \frac{1}{(n-2)^2\tau^2}, \quad \tau \geq r_a$$

and consequently,

$$C_{n,k}(a) \geq (n-2)^{-\frac{2(n-1)}{n}} S_n$$

from this and (35) it follows that

$$C_{n,k}(a) = (n-2)^{-\frac{2(n-1)}{n}} \quad \text{when} \quad a \geq \beta_{n,k}$$

The case where $a_k < a < \beta_{n,k}$, is quite similar to the case $0 < a < \frac{1}{n-2}$ in the proof of Theorem 2.2.That is testing in (33) the sequence

$$U_\epsilon(x) = (\epsilon + |x - x_0|^2)^{-\frac{n-2}{2}} \phi_\delta(|x - x_0|)$$

Let, $0 \neq x_0 \in B_1$ and ϕ_δ a cutoff function which is zero for $t > \delta$ and equal to one for $t < \frac{\delta}{2}$, δ small enough so that $|x_0 + \delta| < 1$ and therefore $U_\epsilon \in C_c^\infty(B_\delta(x_0)) \subset C_c^\infty(B_1)$

$$\begin{aligned}
C_{n,k}(a) &= \inf_{v \in C_c^\infty(B_1)} \frac{\int_{B_1} |x|^{-(n-2)} X_1^{-1}(|x|)\dots X_k^{-1}(|x|)|\nabla v|^2\,dx}{(\int_{B_1} |x|^{-n} X_1(|x|)\dots X_k(|x|) X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|v|^{\frac{2n}{n-2}}\,dx)^{\frac{n-2}{n}}} \\
&\leq \frac{\int_{B_\delta(x_0)} |x|^{-(n-2)} X_1^{-1}(|x|)\dots X_k^{-1}(|x|)|\nabla U_\epsilon|^2\,dx}{(\int_{B_1} |x|^{-n} X_1(|x|)\dots X_k(|x|) X_{k+1}^{\frac{2(n-1)}{n-2}}(|x|)|U_\epsilon^{\frac{2n}{n-2}}|\,dx)^{\frac{n-2}{n}}} \\
&\leq \left(\frac{|x_0+\delta|}{|x_0-\delta|}\right)^{n-2} \frac{(X_1(a,|x_0|-\delta)\dots X_k(a,|x_0|-\delta))^{-\frac{2(n-1)}{n}}}{(X_{k+1}(a,|x_0|-\delta))^{\frac{2(n-1)}{n}}} \frac{\int_{B_{\delta(c_0)}} |\nabla U_\epsilon|^2\,dx}{(\int_{B_{\delta(x_0)}} |U_\epsilon|^{\frac{2n}{n-2}}\,dx)^{\frac{n-2}{n}}} \\
&\leq \left(\frac{|x_0+\delta|}{|x_0-\delta|}\right)^{n-2} ((X_1(a,|x_0|-\delta)\dots X_k(a,|x_0|-\delta) X_{k+1}(a,|x_0|-\delta))^{-\frac{2(n-1)}{n}} \\
&\cdot \frac{\int_{B_{\delta(c_0)}} |\nabla U_\epsilon|^2\,dx}{(\int_{B_{\delta(x_0)}} |U_\epsilon|^{\frac{2n}{n-2}}\,dx)^{\frac{n-2}{n}}}
\end{aligned}$$

Taking the limit $\epsilon \to 0$

$$C_{n,k}(a) \leq \left(\frac{|x_0+\delta|}{|x_0-\delta|}\right)^{n-2} \Big(\prod_{i=1}^{k+1} X_i(a,|x_0|-\delta)\Big)^{-\frac{2(n-1)}{n}} S_n$$

This is for any $\delta > 0$ small enough, therefore

$$C_{n,k}(a) \leq \Big(\prod_{i=1}^{k+1} X_i(a, |x_0| - \delta)\Big)^{-\frac{2(n-1)}{n}} S_n$$

Since $|x_0| < 1$ is arbitary and X_1 is an increasing function of s, we end up with

$$C_{n,k}(a) \leq \Big(\prod_{i=1}^{k+1} X_i(a, 1)\Big)^{-\frac{2(n-1)}{n}} S_n$$

Finally, in the case $a_k < a < \beta_{n,k}$ we obtain the opposite inequality by compraring the infimum in (35) with the infimum in (27). This time we take $R' = r_a$ and $b > 0$ is chosen so that

$$\prod_{i=1}^{k+1} X_i(a, 1) = n - 2 + 2b$$

□

3 Hardy-Sobolev Inequality Involving Distance From the Boundary in the Three Dimensional Upper Half-Space

The main purpose of this section is to prove the following theorem

Theorem 3.1. *For every $f \in C_c^\infty(\mathbb{R}^3_+)$ the inequality,*

$$\int_{\mathbb{R}^3_+} |\nabla f(x)|^2\, dx \geq \frac{1}{4}\int_{\mathbb{R}^3_+} \frac{|f(x)|}{x_3^2}\, dx + S_3\Big(\int_{\mathbb{R}^3_+} |f(x)|^6\, dx\Big)^{\frac{1}{3}} \tag{36}$$

holds where S_3 is the sharp Sobolev constant in three dimensions, i.e,

$$S_3 = 3\Big(\frac{\pi}{3}\Big)^{\frac{4}{3}}$$

At first sight (36) seems to contradict the well known fact that Hardy's inequality

$$\int_{\mathbb{R}^3_+} |\nabla f(x)|^2\, dx \geq \int_{\mathbb{R}^3_+} \frac{1}{4x_n^2}|f(x)|^2\, dx$$

as well as Sobolev's inequality

$$\int_{\mathbb{R}^3_+} |\nabla f(x)|^2\, dx \geq S_3\Big(\int_{\mathbb{R}^3_+} |f(x)|^6\Big)^{\frac{1}{3}}$$

are sharp in the sense that in each the constant on the right side cannot be replaced by a larger one. None of them, however, has a non-zero optimizer and the optimizing sequence in Hardy's inequality are far from optimal for Sobolev's inequality and vice versa.

We denote

$$\mathbb{R}^n_+ = \{\mathbf{x} = (x, y) : x \in \mathbb{R}^{n-1}, y > 0, \}$$

3.1 The Hyperbolic space $\mathbb{H}^n$

In this section we study the hyperbolic space $\mathbb{H}^n$. There are two standard models for $\mathbb{H}^n$, the first one is the half space model, $\mathbb{R}^n_+$, equipped with the Riemannian metric

$$ds^2 = \frac{dx^2}{x_n^2}$$

Under this model, we have

$$|\nabla_{\mathbb{H}^n} u|^2 = x_n^2 |\nabla u|^2, \quad dV = x_n^{-n} dx$$

and the hyperbolic Laplacian is given by

$$\Delta_{\mathbb{H}^n} w = x_n^2 \Delta w - (n-2) x_n w_{x_n}$$

where ∇ and Δ denote the Euclidean gradient and Laplacian. The Riemannian distance between two points $x = (x', x_n), y = (y', y_n)$ is given by,

$$\rho(x, y) = 2 \ln \Big(\frac{|x-y| + |x - \overline{y}|}{2\sqrt{x_n y_n}} \Big)$$

where $\overline{y} = (y', -y_n)$.
The second one is the unit ball model, where the unit ball B_1 is equipped with the Riemannian metric

$$ds^2 = \Big(\frac{1-|x|^2}{2} \Big)^{-2} dx^2$$

Under this model we have,

$$|\nabla_{\mathbb{H}^n} u|^2 = \Big(\frac{1-|x|^2}{2} \Big)^2 |\nabla w|^2, \quad dV = \Big(\frac{1-|x|^2}{2} \Big)^{-n} dx$$

and the distance of a point $x \in B_1$ to the origin is

$$\rho = \ln \Big(\frac{1+|x|}{1-|x|} \Big)$$

Proposition 2. *(Poincaré inequality in $\mathbb{H}^n$) Let $n \geq 2$. For any $w \in C_c^\infty(\mathbb{H}^n)$ there holds,*

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} w|^2 \, dV \geq \Big(\frac{n-1}{2} \Big)^2 \int_{\mathbb{H}^n} w^2 \, dV \tag{37}$$

Moreover, $\left(\frac{n-1}{2}\right)^2$ *is the best possible constant.*

Proof. Using the half-space model,

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} w|^2 \, dV - \Big(\frac{n-1}{2} \Big)^2 \int_{\mathbb{H}^n} w^2 \, dV \geq 0$$

$$\int_{\mathbb{R}^n_+} x_n^2 |\nabla w|^2 x_n^{-n} \, dx - \frac{(n-1)^2}{4} \int_{\mathbb{R}^n_+} x_n^{-n} w^2 \, dx \geq 0$$

$$\int_{\mathbb{R}^n_+} |\nabla w|^2 x_n^{2-n} \, dx - \frac{(n-1)^2}{4} \int_{\mathbb{R}^n_+} x_n^{-n} w^2 \, dx \geq 0$$

We now change the variables by, $w = ux_n^{\frac{n-2}{2}}$, we obtain that

$$\nabla w = x_n^{\frac{n-2}{2}} \nabla u + \frac{(n-2)}{2} u x_n^{\frac{n-4}{2}} e_n$$

Then,

$$\begin{aligned}|\nabla w|^2 &= \left(x_n^{\frac{n-2}{2}} \nabla u + \frac{(n-2)}{2} u x_n^{\frac{n-4}{2}} e_n\right)^2 \\ &= x_n^{n-2}|\nabla u|^2 + (n-2)x_n^{n-3} u \nabla u \cdot e_n + \frac{(n-2)^2}{4} u^2 x_n^{n-4}\end{aligned}$$

□

We also have that,

$$\int_{\mathbb{R}^n_+} u^2 x_n^{n-2} x_n^{-n}\, dx = \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\, dx$$

We now substitute,

$$\begin{aligned}&\int_{\mathbb{R}^n_+} |\nabla w|^2 x_n^{2-n} \\ &= \int_{\mathbb{R}^n_+} |\nabla u|^2 x_n^{2-n} x_n^{n-2}\, dx + (n-2)\int_{\mathbb{R}^n_+} x_n^{2-n} x_n^{n-3} u \nabla u \cdot e_n\, dx \\ &+ \frac{(n-2)^2}{4} \int_{\mathbb{R}^n_+} x_n^{2-n} u^2 x_n^{n-4}\, dx\end{aligned}$$

So,

$$\begin{aligned}&\int_{\mathbb{R}^n_+} |\nabla w|^2\, dx - \frac{(n-1)^2}{4}\int_{\mathbb{R}^n_+} x_n^{-n} w^2\, dx \\ &= \int_{\mathbb{R}^n_+} |\nabla u|^2\, dx + \Big(\frac{(n-2)^2}{4} - \frac{(n-1)^2}{4}\Big)\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\, dx + (n-2)\int_{\mathbb{R}^n_+} \frac{u}{x_n}\nabla u \cdot e_n\, dx \\ &= \int_{\mathbb{R}^n_+} |\nabla u|^2\, dx + \frac{2n-3}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\, dx + (n-2)\int_{\mathbb{R}^n_+} \frac{u}{x_n}\nabla u \cdot e_n\, dx\end{aligned}$$

Since $u \in C_c^\infty(\mathbb{R}^n_+)$ we can use Green's identity on the third integral,

$$\int_{\mathbb{R}^n_+} \frac{u}{x_n} \nabla u \cdot e_n \, dx = \frac{1}{2} \int_{\mathbb{R}^n_+} \frac{1}{x_n} \nabla u^2 \cdot e_n \, dx$$
$$= \frac{1}{2} \int_{\mathbb{R}^n_+} \frac{1}{x_n} (u^2) x_n \, dx = \frac{1}{2} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}$$

Finally we conclude that,

$$\int_{\mathbb{H}^n} |\nabla w|^2 \, dV - \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} w^2 \, dV$$
$$= \int_{\mathbb{R}^n_+} |\nabla w|^2 x_n^{2-n} \, dx - \frac{(n-1)^2}{4} \int_{\mathbb{R}^n_+} x_n^{-n} w^2 \, dx$$
$$= \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx + \frac{n-2}{2} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} \, dx + \frac{(n-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} \, dx - \frac{(n-1)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}$$
$$= \int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}$$

The sharpness of the constant $\frac{n-1}{4}$ follows easily from the sharpness of the Hardy inequality

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2} \, dx$$

3.2 The Green Function

From now on we shall use the following notation,

$$\mathbb{R}^n_+ = \{ \mathbf{x} = (x, y) : x \in \mathbb{R}^{n-1}, y > 0, \}$$

Similarly we shall write

$$\mathbf{x}' = (x', y') : x' \in \mathbb{R}^{n-1}, y' > 0$$

We start with the following heat type equation on the upper half space $\mathbb{R}^n_+$.

$$\begin{cases} u_t = \Delta u + \frac{1}{4y^2} u \\ u(x, y, 0) = u_0(x, y) \end{cases} \tag{38}$$

Using the change of variables, $u = \sqrt{y}g$, in (38) we have

$$u_t = \frac{\partial}{\partial t}(\sqrt{y}g(x,y,t)) = \sqrt{y}g_t(x,y,t)$$

and

$$\frac{1}{4y^2}u = \frac{1}{4y^2}(\sqrt{y}g(x,y,t)) = \frac{g}{4y^{\frac{3}{2}}}$$

Also,

$$\begin{aligned}\Delta u &= \Delta(\sqrt{y}g(x,y,t)) \\ &= \frac{4y(y(\Delta_x g + g_{yy}) + g_y) - g}{4y^{\frac{3}{2}}} \\ &= \frac{4y^2}{4y^{\frac{3}{2}}}(\Delta_x g + g_{yy}) + \frac{4yg_y}{4y^{\frac{3}{2}}} - \frac{g}{4y^{\frac{3}{2}}} \\ &= y^{\frac{1}{2}}(\Delta_x g + g_{yy}) + y^{-\frac{1}{2}}g_y - \frac{g}{4y^{\frac{3}{2}}}\end{aligned}$$

So we obtain that,

$$\begin{aligned} y^{\frac{1}{2}}g_t &= y^{\frac{1}{2}}(\Delta_x g + g_{yy}) + y^{-\frac{1}{2}}g_y - \frac{g}{4y^{\frac{3}{2}}} + \frac{g}{4y^{\frac{3}{2}}} \\ g_t &= \Delta_x g + g_{yy} + \frac{1}{y}g_y\end{aligned}$$

We conclude that the function g satisfies,

$$\begin{cases} g_t = \Delta_x g + g_{yy} + \frac{1}{y}g_y \\ g(x,y,0) = \frac{u_0(x,y)}{\sqrt{y}} = g_0(x,y)\end{cases} \tag{39}$$

We observe that $g_{yy} + \frac{1}{y}g_y$ is the Laplacian of a radial function in two dimensions so the right hand side of the equation is the $n+1$ dimensional Laplacian. That is,

$$\Delta_z g = g_{yy} + \frac{1}{y}g_y$$

where $y = |z|$, so $g = g(y)$ is a radially symmetric function.
Therefore, we can also write (39) as,

$$\begin{cases} g_t = \Delta_{\mathbb{R}^{n+1}} g \\ g(x,z,0) = g_0(x,z) = g_0(x,y) \quad y = |z|\end{cases} \tag{40}$$

Using the fundamental solution of the heat equation we get,

$$g(x,y,t) = (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n+1}} e^{-\frac{|x-\tilde{x}|^2+|z-\tilde{z}|^2}{4t}} g_0(\tilde{x},\tilde{z})\, d\tilde{x}$$

Using polar coordinates

$$\tilde{z} = (\tilde{r},\tilde{\phi}) = (\tilde{r}\cos\tilde{\phi}, \tilde{r}\sin\tilde{\phi}), \quad r = y, \quad \tilde{r} = \tilde{y}$$

and

$$\begin{aligned} |z-\tilde{z}| &= (r\cos\phi - \tilde{r}\cos\tilde{\phi})^2 + (r\sin\phi - \tilde{r}\sin\tilde{\phi})^2 \\ &= r^2 + \tilde{r}^2 - 2r\tilde{r}(\cos\phi\cos\tilde{\phi} + \sin\phi\sin\tilde{\phi}) \\ &= r^2 + \tilde{r}^2 - 2r\tilde{r}\cos(\phi - \tilde{\phi}) \end{aligned}$$

Hence.

$$g(x,z,t) = (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x-\tilde{x}|^2}{4t}} \int_{\tilde{y}=0}^{\infty} \int_{\tilde{\phi}=0}^{2\pi} e^{-\frac{y^2+\tilde{y}^2}{4t} + \frac{y\tilde{y}\cos(\phi-\tilde{\phi})}{2t}} \tilde{y} g_0(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}\, d\tilde{\phi}$$

Substituting $g = \frac{u}{\sqrt{y}}$ and $g_0(\tilde{x},\tilde{y}) = \frac{f(\tilde{x},\tilde{y})}{\sqrt{\tilde{y}}}$, we obtain

$$\begin{aligned} \frac{u(x,y,t)}{\sqrt{y}} &= (4\pi t)^{-\frac{n+1}{2}} \int_{\tilde{x}\in\mathbb{R}^{n-1}} \int_{\tilde{y=0}}^{\infty} \int_{\tilde{\phi}}^{2\pi} e^{-\frac{|x-\tilde{x}|^2}{4t}} e^{-\frac{y^2+\tilde{y}^2}{4t}} e^{\frac{y\tilde{y}\cos(\phi-\tilde{\phi})}{4t}} \frac{f(\tilde{x},\tilde{y})}{\sqrt{\tilde{y}}} \tilde{y}\, d\tilde{x}\, d\tilde{y}\, d\tilde{\phi} \\ u(x,y,t) &= (4\pi t)^{-\frac{n+1}{2}} \int_{\mathbb{R}^n_+} \left[\int_{\tilde{\phi}}^{2\pi} e^{-\frac{|x-\tilde{x}|^2}{4t}} e^{-\frac{y^2+\tilde{y}^2}{4t}} e^{\frac{y\tilde{y}\cos\phi}{2t}} \sqrt{y\tilde{y}}\, d\tilde{\phi} \right] f(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \end{aligned}$$

So, we conclude to the following formula for the solution of (38),

$$u(x,y,t) = \int_{\mathbb{H}^n} G(x-\tilde{x},y,\tilde{y};t) u_0(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \tag{41}$$

where

$$G(x-\tilde{x},y,\tilde{y};t) = \left(\frac{1}{4\pi t}\right)^{\frac{n+1}{2}} \sqrt{y\tilde{y}} e^{-\frac{(x-\tilde{x})^2+y^2+\tilde{y}^2}{4t}} \int_0^{2\pi} e^{\frac{y\tilde{y}}{2t}\cos\phi}\, d\phi \tag{42}$$

We can see that this is a heat kernel. As we previously defined the quadratic forms and the Laplacian operator, we notice that L is a self-adjoint operator and it is an extension of $-\Delta - \frac{1}{4y^2}$ originally defined on smooth functions with compact support in the three dimensional upper half space. We shall continue to use the symbol L to denote $-\Delta - \frac{1}{4y^2}$.

Definition 3.1. *We define the operator* L *using Theorem 1.8 by using the following quadratic form*

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{y^2} \, dx$$

and by the Hardy inequality

$$\int_{\mathbb{R}^3_+} |\nabla u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^3_+} \frac{u^2}{y^2} \, dx \geq 0$$

We define s a positive real valued number such that $s \to L$. The following relation holds for every $s \in \mathbb{R}$

$$s^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-\frac{\alpha-2}{2}} e^{-st} \, dt$$

Also e^{-Lt} a semigroup.
Since L is a self adjoint operator, we can define the fractional powers of L^α, $\alpha \in \mathbb{R}$.

$$L^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-\frac{\alpha-2}{2}} e^{-Lt} \, dt$$

By the heat kernel we find the kernel of the fractional powers.

$$L^{-\frac{\alpha}{2}}(\mathbf{x}; \tilde{\mathbf{x}}) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}} G(x - \tilde{x}, y, \tilde{y}; t) \frac{1}{t} \, dt$$

For all $0 < \alpha < n + 1$ we compute,

$$\begin{aligned} L^{-\frac{\alpha}{2}}(\mathbf{x}; \tilde{\mathbf{x}}) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \int_0^{2\pi} t^{\frac{\alpha}{2}} \sqrt{y\tilde{y}} e^{-\frac{(x-\tilde{x})^2+y^2+\tilde{y}^2}{4t}} e^{\frac{y\tilde{y}}{2t}\cos\phi} \frac{1}{t} \, dt \, d\phi \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})} \pi^{-\frac{n+1}{2}} \sqrt{y\tilde{y}} \int_0^\infty \int_0^{2\pi} \frac{t^{\frac{\alpha}{2}}}{t} \frac{1}{(4t)^{\frac{n+1}{2}}} e^{-\frac{(x-\tilde{x})^2+y^2+\tilde{y}^2+2y\tilde{y}\cos\phi}{4t}} \, d\phi \, dt \\ &= \frac{1}{\Gamma(\frac{\alpha}{2})} \pi^{-\frac{n+1}{2}} \sqrt{y\tilde{y}} \int_0^{2\pi} \int_0^\infty \frac{t^{-\frac{(n+1-\alpha)-1}{2}}}{4^{\frac{n+1}{2}}} e^{-\frac{(x-\tilde{x})^2+y^2+\tilde{y}+2y\tilde{y}\cos\phi}{4t}} \, d\phi \, dt \\ &= 2^{-\alpha} \frac{\Gamma(\frac{n+1-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \pi^{-\frac{n+1}{2}} \sqrt{y\tilde{y}} \int_0^{2\pi} ((x-\tilde{x})^2 + y^2 + \tilde{y}^2 - 2y\tilde{y}\cos\phi)^{-\frac{n+1-\alpha}{2}} \, d\phi \\ &=: \Phi_{n,\alpha}(\mathbf{x}; \tilde{\mathbf{x}}) \end{aligned}$$

For $0 < \alpha < n$ a similar expression holds for $(-\Delta)^{(-\frac{\alpha}{2})}$ on $\mathbb{R}^n$. We can compute the integral as

$$
\begin{aligned}
&(-\Delta)^{-\frac{\alpha}{2}}(\mathbf{x};\tilde{\mathbf{x}}) \\
&= 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}[(x-\tilde{x})^2+(y-\tilde{y})^2]^{-\frac{n-\alpha}{2}} \\
&=: \Gamma(\frac{n-\alpha}{2})\Psi_{n,\alpha}(\mathbf{x};\tilde{\mathbf{x}})
\end{aligned}
$$

Now we state some pointwise properties about the kernel $\Phi_{n,\alpha}$

Lemma 3.2. *Let $n \geq 2$ and $n-1 \leq \alpha < n$ we have that,*

$$\sup_l \Phi_{n,\alpha}(x, y+l; \tilde{x}, \tilde{y}+l) = \lim_{l\to\infty} \Phi_{n,\alpha}(x, y+l, \tilde{x}, \tilde{y}+l) = \Psi_{n,\alpha}(\mathbf{x};\tilde{\mathbf{x}})$$

In this case

$$\Phi_{n,\alpha}(\mathbf{x};\tilde{\mathbf{x}}) = \Psi_{n,\alpha}(\mathbf{x};\tilde{\mathbf{x}})F(A)$$

where

$$A = \frac{\sqrt{y\tilde{y}}}{|x-\tilde{x}|}$$

and $F(A)$ is strictly increasing towards $\Gamma(\frac{n-\alpha}{2})$

Proof. We will prove the above Lemma for $n = 3$ and $\alpha = 2$. F is defined as

$$F(A) = \frac{\Gamma(\frac{n+1-\alpha}{2})}{\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{A}{[1+2A^2(1-\cos\phi)]^{\frac{n+1-\alpha}{2}}}\, d\phi$$

All the statements are an immediate consequence of the following Lemma for $\beta = \frac{n+1-\alpha}{2}$ □

In the following Lemma we collect some facts about the function

$$F(A) = \frac{\Gamma(\beta)}{\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{A}{(1+2A^2(1-\cos\phi))^{\beta}}\, d\phi$$

where $\beta = \frac{n+1-\alpha}{2}$. All the statements are an immediate consequence of the following Lemma for $\beta = \frac{n+1-\alpha}{2}$

Lemma 3.3. *For $\frac{1}{2} < \beta \leq 1$, the function $F(A)$ has the following asymptotics as $A \to \infty$.*
Then $F(A)$ is strictly monotone increasing function and

$$\lim_{A\to\infty} F(A) = \Gamma(\beta - \frac{1}{2})$$

We will prove this Lemma for $\beta = 1$

Proof. Since,

$$F(A) = \frac{\Gamma(\beta)}{\sqrt{\pi}} G(A)$$

where

$$G(A) = \int_{-\pi A}^{\pi A} \frac{1}{(1 + 2A^2(1 - \cos(\frac{\phi}{A})))^{\beta}} \, d\phi$$

To see that the statement holds for $\beta = 1$ we have to perform the ϕ integration We change the variables,

$$\frac{\phi}{A} = u \quad \text{and} \quad du = \frac{1}{A} d\phi$$

So,

$$\begin{aligned} G(A) &= \int_{-\pi}^{\pi} \frac{A}{1 + 2A^2 - 2A^2 \cos u} \, du \\ &= A \int_{-\pi}^{\pi} \frac{1}{1 + 2A^2 - 2A^2 \cos u} \, du \end{aligned}$$

We change the variables again,

$$s = \tan\frac{u}{2} \quad \text{and} \quad ds = \frac{1}{2}\sec^2(\frac{u}{2})\, du$$

We substitute,

$$\sin u = \frac{2s}{s^2 + 1}, \quad \cos u = \frac{1 - s^2}{1 + s^2}, \quad \text{and} \quad du = \frac{2}{s^2 + 1} ds$$

Therefore,

$$\begin{aligned} G(A) &= A\int_{-\infty}^{\infty}\frac{1}{1+2A^2-2A^2(\frac{1-s^2}{1+s^2})}\frac{2}{s^2+1}\,ds \\ &= 2A\int_{-\infty}^{\infty}\frac{1}{(s^2+1)(1+2A^2-2A^2(\frac{1-s^2}{1+s^2}))}\,ds \\ &= 2A\int_{-\infty}^{\infty}\frac{1}{s^2+1+2A^2s^2+2A^2-2A^2+2A^2s^2}\,ds \\ &= 2A\int_{-\infty}^{\infty}\frac{1}{4A^2s^2+s^2+1}\,ds \\ &= 2A\int_{-\infty}^{\infty}\frac{1}{(1+4A^2)s^2+1}\,ds \end{aligned}$$

We finally change the variables

$$x=\sqrt{1+4A^2}s \quad \text{and} \quad dx=\sqrt{1+4A^2}ds$$

Since $f(x)=\frac{2A}{(1+4A^2)s^2+1}$ is an even function and the interval $(-\infty,\infty)$ is symmetric about 0. So,

$$\begin{aligned} G(A) &= 2A\int_{-\infty}^{\infty}\frac{1}{x^2+1}\frac{1}{\sqrt{1+4A^2}}\,dx \\ &= \frac{2A}{\sqrt{1+4A^2}}\int_{0}^{\infty}\frac{2}{x^2+1}\,dx \\ &= \frac{2\pi A}{\sqrt{1+4A^2}} \end{aligned}$$

Hence,

$$G(A)=\frac{2\pi A}{\sqrt{1+4A^2}}$$

which is obviously strictly increasing function with A. □

In the following section we shall only study the case for $n=3$ and $\alpha=2$ By the Lemmas above we conclude to the following estimate that

Corollary 3. *The integral kernel of the operator L^{-1} is less than or equal to the integral kernel of the operator $(-\Delta)^{-1}$*

$$\Phi_{n,\alpha}(\mathbf{x};\mathbf{x})\leq\Gamma(\frac{n-\alpha}{2})\Psi_{n,\alpha}(\mathbf{x};\mathbf{x})$$

3.3 L^p-estimates for fractional powers

Theorem 3.4. *Let $n = 3$ and $\alpha = 2$ then the operator,*

$$(-\Delta - \frac{1}{4y^2})^{-1}$$

is bounded from $L^{\frac{6}{5}}(\mathbb{R}^3_+)$ to $L^6(\mathbb{R}^3_+)$

Moreover,

$$\int_{\mathbb{R}^3_+} (|\nabla f|^2 - \frac{1}{4x_3^2}|f|^2)\,dx \geq C\Big(\int_{\mathbb{R}^3_+} |f|^6\,dx\Big)^{\frac{1}{3}} \tag{43}$$

and

$$C = \frac{1}{3\pi}\sqrt[3]{\frac{2}{\pi}}$$

is the sharp constant.

Proof. Let f, g be functions in $\mathbb{R}^3_+$. Then, for $f \in \mathrm{Dom(L)}$ and $g \in L^2(\mathbb{R}^3_+)$

$$\begin{aligned} \langle\, f, g\rangle^2 &= \langle\, L^{\frac{1}{2}}f, L^{-\frac{1}{2}}g\rangle^2 \\ &\leq \langle\, L^{\frac{1}{2}}f, L^{\frac{1}{2}}f\rangle\langle\, L^{-\frac{1}{2}}g, L^{-\frac{1}{2}}g\rangle \\ &= \langle\, Lf, f\rangle\langle\, L^{-1}g, g\rangle \end{aligned}$$

We extend the functions by zero in $\mathbb{R}^3$ and by Lemma 3.3 we have

$$\langle\, f, g\rangle^2 \leq \langle\, Lf, f\rangle_{L^2(\mathbb{R}^3_+)}\langle\,(-\Delta)^{-1}g, g\rangle_{L^2(\mathbb{R}^3)}$$

Then, we obtain a pointwise estimate of the inequality using the inequality of the Green functions

$$\langle\, Lg, g\rangle_{L^2(\mathbb{R}^3_+)} \leq \langle\,(-\Delta)^{-1}g, g\rangle_{L^2(\mathbb{R}^3)}$$

Moreover, we recall the Sobolev inequality in $\mathbb{R}^3$

$$\int_{\mathbb{R}^3} |\nabla w|^2\,dx \geq S_3\|w\|^2_{L^6(\mathbb{R}^3)}$$

Equivalently,

$$\langle\, -\Delta w, w\rangle_{L^2(\mathbb{R}^3)} \geq S_3\|w\|^2_{L^6(\mathbb{R}^3)}$$

Setting $-\Delta w = g$

$$\langle g, (-\Delta)^{-1} g\rangle \geq S_3 \|(-\Delta)^{-1} g\|^2_{L^6(\mathbb{R}^3)}$$

By Hölder's inequality with conjugate exponents 6 and $\frac{6}{5}$, we have,

$$\begin{aligned} \|(-\Delta)^{-1} g\|^2_{L^6(\mathbb{R}^3)} &\leq \frac{1}{S_3}\langle g, (-\Delta)^{-1} g\rangle \\ &\leq \frac{1}{S_3}\|(-\Delta)^{-1} g\|_{L^6(\mathbb{R}^3)}\|g\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \end{aligned}$$

Therefore

$$\|(-\Delta)^{-1} g\|_{L^6(\mathbb{R}^3)} \leq \frac{1}{S_3}\|g\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}$$

Hence,

$$\begin{aligned} \langle f, g\rangle^2 &\leq \langle Lf, f\rangle \|(-\Delta)^{-1} g\|_6 \|g\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\leq \frac{1}{S_3}\langle Lf, f\rangle \|g\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \quad g \in L^{\frac{6}{5}}(\mathbb{R}^3) \end{aligned}$$

$$\frac{\langle f, g\rangle^2}{\|g\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)}} \leq \frac{1}{S_3}\langle Lf, f\rangle$$

Finally, taking supremum on the left-hand side we have,

$$\sup_{g \in L^{\frac{6}{5}}(\mathbb{R}^3)} \frac{\langle f, g\rangle^2}{\|g\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)}} \leq \frac{1}{S_3}\langle Lf, f\rangle \Longleftrightarrow \|f\|^2_6 \leq \frac{1}{S_3}\langle Lf, f\rangle$$

Therefore

$$\langle Lf, f\rangle \geq S_3 \|f\|^2_{L^6(\mathbb{R}^3_+)}$$

□

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