National & Kapodistrian University of Athens

Master Thesis

The Yamabe problem

Author:
M. Paschalis

Supervisor:
P. Gianniotis

A thesis submitted in fulfillment of the requirements for the degree of M.Sc in Pure Mathematics

in the

Department of Mathematics

October 4, 2023

NATIONAL & KAPODISTRIAN UNIVERSITY OF ATHENS

Abstract

Department of Mathematics

M.Sc in Pure Mathematics

The Yamabe problem

by M. Paschalis

The main subject of this graduate thesis is the Yamabe problem, which is about conformal deformation of Riemannian metrics to ones of constant scalar curvature. In Chapter 1, we give an overview of the problem, as well as a solution for the 2-dimensional case using methods of Riemann surfaces. In Chapter 2, we give a review of the prerequisites, which are the classical theories of Riemannian manifolds and elliptic PDEs, up to the point that they are usually treated in graduate-level courses. Chapter 3, the main one in this thesis, systematically treats the Yamabe problem. Section 3.1 shifts the problem to the value of the Yamabe invariant, while Section 3.2 is concerned with the determination of this value, completing the solution to the problem. Finally, Chapter 4 is a survey on the spinorial Yamabe problem, which can be considered a first-order analogue of the Yamabe problem and shares a lot of similarities.

Βασικό αντικείμενο της παρούσας διπλωματικής εργασίας είναι το πρόβλημα του Yamabe, το οποίο αφορά τη σύμμορφη παραμόρφωση μιας μέτρικής Riemann σε κάποια με σταθερή βαθμωτή καμπυλότητα. Στο Κεφάλαιο 1 κάνουμε μια ανασκόπιση του προβλήματος, και επιπλέον παρουσιάζουμε μία λύση του προβλήματος για διάσταση 2 χρησιμοποιόντας μεθόδους επιφανειών Riemann. Στο Κεφάλαιο 2 παρουσιάζουμε εν τάχει τα προαπαιτούμενα, τα οποία είναι οι κλασικές θεωρίες των πολλαπλοτήτων Riemann και των ελλειπτικών διαφορικών εξισώσεων δεύτερης τάξης, μέχρι το σημείο που αυτές συνήθως καλύπτονται σε μαθήματα μεταπτυχιακού επιπέδου. Στο Κεφάλαιο 3, που είναι το κυριότερο σε αυτή την εργασία, ασχολούμαστε συστηματικά με τη λύση του προβλήματος του Yamabe. Στην Ενότητα 3.1 μετατοπίζουμε το πρόβλημα στην τιμή της αναλοίωτης Yamabe, ενώ στην Ενότητα 3.2 ασχολούμαστε με τον προσδιορισμό αυτής της τιμής, ολοκληρώνοντας τη λύση. Τέλος, το Κεφάλαιο 4 αποτελεί μια ανασκόπιση του "σπινοριακού" προβλήματος Yamabe, το οποίο μπορεί να θεωρηθεί ως ένα πρώτης τάξης ανάλογο του προβλήματος Yamabe και μοιράζεται με αυτό πολλά κοινά.

<u>Τριμελής εξεταστική επιτροπή:</u> Π. Γιαννιώτης, Α. Μελάς, Γ. Μπαρμπάτης

Acknowledgements

I am indebted to my supervisor, professor P. Gianniotis, for his guidance and willingness to supervise this thesis in a rather urgent circumstance, as well as proposing this beautiful topic. Also my Ph.D supervisor, professor G. Barbatis, who has been a constant source of guidance and support during my years as a student of the Depertment. This section of acknowledgments would be amiss not to include professor I. Androulidakis, whose lectures and guidance in topics related to index theory have been an inspiration during my involvement with differential geometry.

Finally, I would like to express my gratitude towards my parents. They have been of immense support, during times good and bad, and I probably wouldn't be were I am today without them.

Contents

Dedicated to my parents...

Chapter 1

The Yamabe problem

In this introductory chapter we offer an overview of the Yamabe problem and its historical development, as well as an outline of our presentation. We also present an elegant solution of the 2-dimensional case using the theory of Riemann surfaces.

1.1 Introduction

Let M be a smooth manifold and let $g, \tilde{g}$ be Riemannian metrics on M. The metric $\tilde{g}$ is said to be *conformal* to g if there exists a smooth positive function $u \in C_+^\infty(M)$ such that $\tilde{g} = ug$. This relation will be denoted by $\tilde{g} \simeq g$ and it is obviously an equivalence relation. Likewise, a map between Riemannian manifolds $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ is said to be conformal if $\phi^*\tilde{g} = ug$ for some $u \in C_+^\infty(M)$. If ϕ is a diffeomorphism, the manifolds are said to be *conformally equivalent*, denoted $(M, g) \simeq (\tilde{M}, \tilde{g})$. The term *conformal* is synonymous to *angle-preserving*, and thus being conformal is a geometrically profound property. The field of conformal geometry is concerned with the study of conformal transformations, as well as conformal invariants.

A conformal class $\mathbf{G}$ of M is an equivalence class of conformal metrics of M. The conformal class of a given metric g will be denoted by

$$\overline{g} := \{ug : u \in C_+^\infty(M)\}.$$

In the context of conformal geometry, manifolds are more appropriately equipped with conformal classes rather than specific Riemannian metrics. Such a structure is called a *conformal structure.* If M is a smooth manifold and $\mathbf{G}$ is a conformal class of M, the pair $(M, \mathbf{G})$ is called a *conformal manifold.* At the same time, given a conformal manifold $(M, \mathbf{G})$ we often choose to work with a specific representative $g \in \mathbf{G}$. In doing so, we are often interested in a choice of representative that simplifies the underlying analysis and geometry. The Yamabe problem is a problem of exactly that nature, in particular:

THE YAMABE PROBLEM: Given a compact conformal manifold $(M, \mathbf{G})$, is it possible to find a representative $g \in \mathbf{G}$ such that the scalar curvature S_g is constant?

Why scalar curvature? Why not pursue the bolder claim of finding a representative of constant sectional curvature, a significantly stronger assertion?

Such a quest readily leads to failure, as the problem becomes heavily overdetermined as dimension goes up. If $n = \dim M$, the space of curvature-like tensors has dimension

$$\frac{1}{12}n^2(n^2 - 1),$$

while the unknown function u offers only one degree of freedom. One can expect to solve this problem for $n = 2$ however, since the above expression becomes 1 in that case. An elegant solution using complex geometry will be given in Section 1.2. For $n = 0, 1$ the problem is trivial. As of Ricci curvature, one runs to similar problems. There are compact manifolds of dimension $n = 3, 4$ that are known to be unable admit any Einstein metrics (i.e of constant Ricci curvature) at all. This leaves us placing all our hopes on scalar curvature.

Thus the Yamabe problem becomes truly relevant in the case $n \geq 3$, which we assume to be the case from now on, with the single exception of Section 1.2. The answer to its question is *yes*, and it was given through the collective work of several authors over the span of roughly 25 years. In 1960, Yamabe proposed a solution using techniques of elliptic PDEs and calculus of variations. Unfortunately, his proof contained a serious error, which was discovered by Trudinger in 1968. Trudinger salveged what he could of Yamabe's original proof, at the cost of a rather restrictive assumption. In 1976, Aubin complemented Trudinger's result, giving a positive answer to the problem in all cases where a suitable conformal invariant $\lambda(M, \mathbf{G})$ (called the Yamabe invariant, to be explained later) is less than that of the standard sphere $(\mathbb{S}^n, g_\circ)$, where $g_\circ$ denotes the round metric. In particular:

Theorem (Yamabe, Trudinger, Aubin)**.** *The Yamabe problem possesses a solution for any compact conformal manifold $(M, \mathbf{G})$ provided that $\lambda(M, \mathbf{G}) < \lambda(\mathbb{S}^n, g_\circ)$.*

The standard sphere certainly has a metric of constant (sectional) curvature, namely the standard one, and it is also relatively easy to show that in general $\lambda(M, \mathbf{G}) \leq \lambda(\mathbb{S}^n, g_\circ)$. This shifts the problem to determining the value of Yamabe invariants, and in particular whether they assume the critical value $\lambda(\mathbb{S}^n, g_\circ)$ or not. Aubin was also able to prove:

Theorem (Aubin)**.** *If $(M, \mathbf{G})$ has dimension ≥ 6 and is not locally conformally flat, then $\lambda(M, \mathbf{G}) < \lambda(\mathbb{S}^n, g_\circ)$.*

The remaining cases where resolved by Schoen in 1984, by introducing methods involving Green functions of the conformal Laplacian, as well as the positive mass theorem, a result from the theory of general relativity that was found to be unexpectedly relevant. He proved:

Theorem (Schoen)**.** *If $(M, \mathbf{G})$ has dimension 3, 4 or 5 or if it is locally conformally flat and not conformal to $(\mathbb{S}^n, g_\circ)$, then $\lambda(M, \mathbf{G}) < \lambda(\mathbb{S}^n, g_\circ)$.*

This completes the solution of the Yamabe problem, as well as the historical introduction. The aim of this thesis is to explain these steps in detail, as well as some additional developments. The main references are the article of Lee & Parker [20], which is the first place to present a unified solution to the problem, as well as the book of Schoen & Yau [23]. Our presentation is aimed towards a more clear, less coordinate dependent exposition of the source material. Nevertheless, coordinates cannot be avoided entirely; the existence of conformal normal coordinates is still essential for the proof. In this regard, we have simplified the presentation by introducing conformal normal coordinates earlier on, relying on more recent results by Günther [13, 14] that exclude unnecessary remainder terms that complicate the original source material. Proofs are almost always given - some are merely outlined if judged to be too long and technical to be of educational value - with the single exception of the positive mass theorem, which is too involved to include here and far beyond our scope.

The prerequisites for following this presentation is familiarity with the theory of Riemannian manifolds as well as the standard theory of elliptic PDEs, both classical subjects that are usually treated in most curricula of graduate mathematics. We review the required notions in Chapter 2, which we encourage the reader to at least skim through, so as to become familiar with our notation. The last section on the spinorial Yamabe problem would also make good use of any prior knowledge of spin geometry, although an effort has been made to keep it self-contained.

Chapter 3, the main one in this thesis, systematically treats the Yamabe problem. Section 3.1 shifts the problem to the value of the Yamabe invariant, while Section 3.2 is concerned with the determination of this value, completing the solution to the problem.

Finally, Chapter 4 is a survey on the spinorial Yamabe problem, which can be considered a first-order analogue of the Yamabe problem and shares a lot of similarities. It is mostly based in the works of Ammann [4, 2, 3].

1.2 The case $n = 2$

The 2-dimensional case admits a special, elegant treatment using results from the theory of Riemann surfaces. These are 2-dimensional manifolds whose charts are additionally required to have transition maps that satisfy the Cauchy-Riemann equations. More precisely:

Definition 1.2.1. A *Riemann surface* $(\Sigma, \mathcal{C})$ is a second countable Hausdorff topological space Σ that is locally homeomorhic to $\mathbb{C}$, together with a set $\mathcal{C}$ of such local homeomorphisms that satisfy the following conditions:

1. The domains of the elements of $\mathcal{C}$ form an open cover of Σ,
2. Whenever $z, w \in \mathcal{C}$ and $z : U \to \mathbb{C}$, $w : V \to \mathbb{C}$, either $U \cap V = \varnothing$ or the transition maps $z \circ w^{-1}$ and $w \circ z^{-1}$ are both holomorphic in their respective domains of definition.

If $\mathcal{C}$ is maximal with respect to inclusion, it is called a *complex structure*.

Of course, not every 2-dimensional real manifold is a Riemann surface, since a holomorphic map is a special case of a smooth map. But given a surface with a smooth real structure, can we expect to find a complex substructure? If the manifold is orientable, the answer is yes, a result that is a consequence of the existence of a special kind of coordinates.

Definition 1.2.2. Let (M, g) be a 2-dimensional Riemannian manifold, and let (x, y) be coordinates on an open set U. If there exists a $u \in C^\infty_+(U)$ such that

$$g = u(dx \otimes dx + dy \otimes dy),$$

the set of coordinates (x, y) is called *isothermal*.

In particular, if for any $\mathbf{p} \in M$ there exist isothermal coordinates centered at $\mathbf{p}$, the manifold is locally conformally flat (see Section 3.1.4). Isothermal coordinates always exist, for a classical proof see Chern [8]; note however that the existence of conformal normal coordinates, proven here in Section 3.1.4, is also equivalent to this for $n = 2$. It is easy to see that the transition maps between isothermal charts are conformal. Moreover, on an orientable manifold they can also be taken to be orientation-preserving by possibly changing a sign. Holomorhic maps between open subsets of the complex plane are precicely the orientation-preserving conformal maps, so isothermal coordinates readily define a complex structure on any orientable surface.

In particular, simply connected surfaces are orientable, and thus admit the structure of a Riemann surface. Then the Uniformisation Theorem applies:

Theorem 1.2.1 (Uniformisation Theorem). *Every simply connected Riemann surface is conformally equivalent to one of the three:*

1. *the complex plane* $\mathbb{C}$,
2. *the unit disc* $\mathbb{D}$,
3. *the Riemann sphere* $\hat{\mathbb{C}}$.

These are equipped with the standard metrics of constant sectional curvature $K = 0, -1$ and 1 respectively. This solves the Yamabe problem for simply connected surfaces, and in fact the stronger version of sectional curvature.

The non-simply connected case easily reduces to the simply connected one: just consider the universal cover $\pi : \tilde{\Sigma} \to \Sigma$. If we equip $\tilde{\Sigma}$ with the pull-back metric $\pi^* g$, then $\pi : (\tilde{\Sigma}, \pi^* g) \to (\Sigma, g)$ becomes a local isometry. Local isometries preserve curvature, and the Uniformisation Theorem applies for $(\tilde{\Sigma}, \pi^* g)$. This concludes the proof of the 2-dimensional Yamabe problem.

The Uniformisation Theorem was proven in 1907 by Poincaré [22] and indipendently by Koebe, who later gave several more proofs and generalisations. For a modern proof, see Ahlfors [1].

Chapter 2

Preliminaries

In this chapter we review the notions and main results of Riemannian Geometry and Elliptic PDEs that will be needed for the solution of the Yamabe problem. Most of the results stated here are considered prerequisites for our presentation. The confident reader may skip this chapter and/or refer later to it when needed.

2.1 Review of Riemannian geometry

All material reviewed here can be found in many excellent differential geometry textbooks, see [19, 11, 16] to name a few.

2.1.1 Review of tensor algebra

First, we give a quick review of tensor algebra. Let V be a real vector space of finite dimension n. A tensor of valence (k,l) over V is an element of the tensor product

$$\otimes_l^k V = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ times}}.$$

This space can be identified with the space of multilinear functions

$$T : \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ times}} \to \mathbb{R}.$$

If $\{e_1, \ldots, e_n\}$ is a basis of V and $\{\varepsilon^1, \ldots, \varepsilon^n\}$ is the dual basis of V^*, a basis of $\otimes_l^k V$ is $\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}\}$, so given such a choice of basis, any (k,l)-tensor over V can be decomposed uniquely as[1]

$$T = T^{i_1 \cdots i_k}_{j_1 \cdots j_l} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}.$$

If $k = 0$ then $\otimes_l^0 V = \otimes^l V^*$. We recall now a few important subspaces of $\otimes^k V^*$ that occur frequently in practice. First is the space $\Lambda^m V^*$ of antisymmetric $(0,m)$-tensors (or m-forms); its dimension is $(n|m)$. We can multiply a k-form and an l-form to obtain a $(k+l)$-form using the exterior product

$$\eta \wedge \theta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\eta \otimes \theta),$$

[1] We freely use the Einsten summation convention throughout the text, summing over matching upper and lower indices.

where Alt is the projection of $\otimes^m V^*$ onto $\Lambda^m V^*$,

$$\mathrm{Alt}(\omega)(v_1,\dots,v_m)=\frac{1}{m!}\sum_{\pi\in S_m}\mathrm{sgn}(\pi)\omega(\pi(v_1),\dots,\pi(v_m)).$$

Likewise we have $S^m V^*$, the space of symmetric $(0,m)$-tensors, with the symmetrisation operation

$$\mathrm{Sym}(\omega)(v_1,\dots,v_m)=\frac{1}{m!}\sum_{\pi\in S_m}\omega(\pi(v_1),\dots,\pi(v_m)),$$

which is a projection of $\otimes^m V^*$ onto $S^m V^*$. When working with indices, we have the following notation:

$$\omega_{[i_1\cdots i_m]}:=\mathrm{Alt}(\omega)_{i_1\cdots i_m},\quad \omega_{(i_1\cdots i_m)}:=\mathrm{Sym}(\omega)_{i_1\cdots i_m}.$$

Note that it is also possible to (anti-)symmetrise only some of the components, as in $T_{i(jk)}$ etc. The symmetrisation and antisymmetrisation operations can also be considered for upper indices (the arguments permuted in this case will be covectors).

Another subspace, which is very central in the theory of curvature, is the subspace $S^2\Lambda^2V^*$ of $\otimes^4V^*$. This is the space of $(0,4)$-tensors that are antisymmetric with respect to the permutations $(1)\leftrightarrow(2)$ and $(3)\leftrightarrow(4)$ and symmetric with respect to $(1,2)\leftrightarrow(3,4)$; its dimension is

$$\frac{n(n-1)(n^2-n+2)}{8}.$$

It is not hard to construct elements of $S^2\Lambda^2V^*$ from 2-forms: if $\alpha,\beta\in\Lambda^2V^*$, it suffices to consider the symmetric product

$$\alpha\odot\beta=\alpha\otimes\beta+\beta\otimes\alpha.$$

Conversely, every element of $S^2\Lambda^2V^*$ is a linear combination of such symmetric products. Given a $T\in S^2\Lambda^2V^*$, its Bianchi Symmetrisation is the average of cyclic permutations of its first three arguments:

$$b(T)(X,Y,Z,W)=\frac{1}{3}(T(X,Y,Z,W)+T(Y,Z,X,W)+T(Z,X,Y,W)).$$

It can be shown that b is a projection of $S^2\Lambda^2V^*$ onto the subspace Λ^4V^*. It follows that $S^2\Lambda^2V^*=\ker(b)\oplus\Lambda^4V^*$, and we call $\mathrm{Curv}(V):=\ker(b)$ the space of curvature-like tensors over V. One way to construct curvature-like tensors is by taking the Kulkarni-Nomizu product

$$A\owedge B(X,Y,Z,W)=A(X,Z)B(Y,W)-A(Y,Z)B(X,W)$$
$$-A(X,W)B(Y,Z)+A(Y,W)B(X,Z)$$

of two symmetric $(0,2)$-tensors A, B. The space of curvature-like tensors has dimension
$$\frac{1}{12}n^2(n^2-1).$$

2.1.2 Review of Riemannian metrics

In what follows we denote by M a differentiable manifold of dimension n. Its tangent and cotangent space at point $\mathbf{p} \in M$ will be denoted by $T_{\mathbf{p}}M$ and $T^*_{\mathbf{p}}M$ respectively. Then we have as usual the tensor bundles of valence (k,l),

$$\otimes^k_l TM = \bigsqcup_{\mathbf{p}\in M} \otimes^k_l T_{\mathbf{p}}M.$$

Tensor fields are defined to be the smooth sections of these tensor bundles, so a (k,l)-tensor field is an element of $\Gamma(\otimes^k_l TM)$.

A Riemannian metric on M is a $(0,2)$-tensor field g which defines pointwise an inner product on TM (symmetric, positive definite). Then there is a canonical way to extend this inner product to the other tensor bundles, by pairing indices using the metric; for example the inner product on T^*M is

$$g(\eta,\theta) = g^{ij}\eta_i\theta_j,$$

where g^{ij} denotes the inverse of g_{ij}. Likewise, it is possible to convert vectors to covectors and vice versa via the musical isomorphism $\flat : TM \to T^*M$,

$$\flat(X)(Y) = g(X,Y),$$

and its inverse (denoted $\sharp$). This process is known as "raising" or "lowering" indices, owing to the fact that

$$\flat(X)_i = g_{ij}X^j, \quad \sharp(\omega)^i = g^{ij}\omega_j.$$

A connection (or covariant derivative) on M is a directional derivative for vector fields $\nabla : TM \times \Gamma(TM) \to TM$. The covariant derivative of the vector field X along the tangent vector ξ is denoted $\nabla_\xi X$, and is locally of the form

$$(\nabla_\xi X)^k = \xi X^k + \Gamma^k_{ij}\xi^i X^j,$$

where $\Gamma^k_{ij} = (\nabla_{\partial_i}\partial_j)^k$ are the Christoffel symbols of ∇. The torsion tensor of ∇ is defined to be the $(1,2)$-tensor field

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

with local expression $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. Then ∇ is called torsion-free if $T = 0$, or equivalently if $\Gamma^k_{ij} = \Gamma^k_{ji}$. Moreover, ∇ is said to be compatible with the metric if it satisfies the Leibniz rule

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

To each Riemannian metric g corresponds a unique connection, called the Levi-Civita connection, which is torsion-free and compatible with the metric. It satisfies the Koszul formula

$$\begin{aligned}2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &\quad - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X).\end{aligned}$$

and its Christoffel symbols are

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right\}.$$

It is also possible to uniquely extend the connection for all tensor fields in a manner such that $\nabla u = du$ for $u \in C^\infty(M)$, ∇ commutes with traces, and

$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$$

for any tensor fields T, S. For example, the metric compatibility condition can be rewritten as $\nabla g = 0$. In this context, a connection can be seen as a map

$$\nabla : \Gamma(\otimes^k_l TM) \to \Gamma(\otimes^k_{l+1} TM).$$

In index notation, we set $\nabla_a := \nabla_{\partial_a}$ and if $T \in \Gamma(\otimes^k_l TM)$, denote

$$T^{i_1 \cdots i_k}_{j_1 \cdots j_l; a} := (\nabla_a T)^{i_1 \cdots i_k}_{j_1 \cdots j_l}.$$

Note that covariant differentiation can be performed multiple times, so we get a map $\nabla^m : \Gamma(\otimes^k_l TM) \to \Gamma(\otimes^k_{l+m} TM)$ and likewise we encounter expressions of the form

$$T^{i_1 \cdots i_k}_{j_1 \cdots j_l; a_1 \cdots a_m} := (\nabla_{a_m} \cdots \nabla_{a_1} T)^{i_1 \cdots i_k}_{j_1 \cdots j_l}.$$

Each Riemannian manifold (M, g) is naturally equipped with the Riemannian volume form, expressed locally as

$$\omega_g = \Omega_g(x) dx,$$

where $\Omega_g(x) = \sqrt{\det g_{xx}}$ and $dx = dx^1 \wedge \cdots \wedge dx^n$ (here g_{xx} is the matrix obtained by expressing g in the x-coordinates). The associated Borel measure on M is

$$\mu_g(B) = \int_B \omega_g.$$

The classical differential operators are now in order. Given $u \in C^\infty(M)$, the gradient of u is the vector field

$$\operatorname{grad}(u) = \sharp(du) = g^{ij}(\partial_i u)\partial_j.$$

The divergence of a 1-form $\theta \in \Gamma(T^*M)$ is

$$\operatorname{div}(\theta) := \operatorname{tr}(\nabla \theta) = g^{ab} \theta_{a;b},$$

and the Laplacian of a function $u \in C^\infty(M)$ is

$$\Delta u := \operatorname{div}(du) = \frac{1}{\Omega_g(x)} \frac{\partial}{\partial x^i} \Omega_g(x) g^{ij} \frac{\partial u}{\partial x^j}.$$

These satisfy the divergence theorem

$$\int_M \operatorname{div}(\theta)\, d\mu_g = \int_{\partial M} \theta(\nu)\, d\sigma_g,$$

where ∂M is the boundary of M (if any), σ_g is the induced surface measure on ∂M that is obtained from the restriction of g on $T\partial M$ and ν is the outward unit normal vector on ∂M, as well as the integration by parts formula

$$\int_M \langle du, dv\rangle\, d\mu_g = -\int_M u\Delta v\, d\mu_g,$$

where M is a compact manifold without boundary. More generally, in a compact manifold M without a boundary, the following integration by parts formula holds for all appropriate tensor fields T, S:

$$\int_M \langle \nabla T, S\rangle\, d\mu_g = -\int_M \langle T, \operatorname{tr}(\nabla S)\rangle\, d\mu_g,$$

where the trace is with respect to the last two arguments.

2.1.3 Review of curvature

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The curvature of (M, g) is then defined to be the $(1,3)$-tensor field

$$R(X,Y)Z = [\nabla_Y, \nabla_X]Z - \nabla_{[Y,X]}Z.$$

Locally, its components are given by

$$R^l_{ijk} = -\Gamma^a_{jk}\Gamma^l_{ia} + \Gamma^a_{ik}\Gamma^l_{ja} - \partial_i\Gamma^l_{jk} + \partial_j\Gamma^l_{ik}.$$

It satisfies the Bianchi identities

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

$$\nabla_X R(Y,Z,W) + \nabla_Y R(Z,X,W) + \nabla_Z R(X,Y,W) = 0,$$

as well as the Ricci identity

$$R(X,Y)Z = \nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z,$$

where $\nabla^2_{Y,X} = \nabla_Y\nabla_X - \nabla_{\nabla_Y X}$ is the second covariant derivative. Using the metric, it is possible to turn the curvature into a $(0,4)$-tensor field

$$R(X,Y,Z,W) = g(R(X,Y)Z, W),$$

called the Riemann curvature, which enjoys, unsurprisingly, all the symmetries of a curvature-like tensor, see Section 2.1.1.

The Ricci tensor Ric is obtained by taking the Ricci contraction c of the Riemann curvature, i.e by pairing arguments (2) and (4). Its components are

$$R_{ij} = c(R)_{ij} = g^{ab} R_{iajb}.$$

Then the scalar curvature is obtained by further tracing over the remaining arguments:

$$S := \mathrm{tr}(\mathrm{Ric}) = g^{ab} R_{ab}.$$

The Weyl tensor is defined to be

$$W = R - A \owedge g,$$

where A is the Schouten tensor

$$A = \frac{1}{n-2}\left\{\mathrm{Ric} - \frac{S}{2(n-1)} g\right\},$$

while the traceless Ricci tensor (or Einstein tensor) is

$$E = \mathrm{Ric} - \frac{S}{n} g.$$

Then we have the orthogonal decomposition

$$R = W \oplus \frac{1}{n-2} E \owedge g \oplus \frac{S}{2n(n-2)} g \owedge g.$$

Given a 2-dimensional tangent plane $\Pi = \mathrm{span}\{X, Y\}$ of $T_{\mathbf{p}}M$, we define its sectional curvature

$$K(\Pi) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

which does not depend upon the choice of basis $\{X, Y\}$. Given a tangent line $L = \mathrm{span}\{X\}$ of $T_{\mathbf{p}}M$, we define its Ricci curvature

$$\mathrm{RIC}(L) = \frac{\mathrm{Ric}(X,X)}{g(X,X)},$$

which is again independent of the choice of the generating vector. It follows that the manifold has constant sectional curvature κ if $R = \frac{\kappa}{2} g \owedge g$, and constant Ricci curvature ρ if $\mathrm{Ric} = \rho g$. Metrics of constant Ricci curvature are also called Einstein metrics, and satisfy $E = 0$.

2.2 Review of elliptic PDEs

The material presented here is classical and is contained in standard references such as [9, 12].

2.2.1 Function spaces and embeddings

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. We denote by $\mathcal{C}^{k,p}(M)$ the space of all k times differentiable functions u on the smooth manifold M with finite Sobolev norm

$$\|u\|_{k,p} = \sum_{i=0}^{k} \|\nabla^i u\|_p,$$

where $\|\cdot\|_p$ stands for the standard L^p norm. The Sobolev space $W^{k,p}(M)$ is then defined to be the completion of $\mathcal{C}^{k,p}(M)$ with respect to the $\|\cdot\|_{k,p}$ norm. On a compact manifold, the elements of $W^{k,p}(M)$ are precisely the ones admitting weak derivatives up to order k with finite L^p norm. Specifically, this means that for every smooth differential operator D of order at most k and $u \in W^{k,p}(M)$, there exists a function $Du \in L^p(M)$ such that

$$\int_M vDu\, d\mu_g = \int_M uD^*v\, d\mu_g \quad \forall v \in C^\infty(M),$$

where D^* is the formal adjoint of D. Moreover, on compact manifolds, $C^\infty(M)$ is dense in $W^{k,p}(M)$, and $W^{k,p}(M)$ is independent of g.

Next we define the Hölder space $C^{k,\alpha}(M)$ for $0 \leq \alpha \leq 1$. This is the collection of all functions $u \in C^k(M)$ that have finite Hölder norm

$$\|u\|_{C^{k,\alpha}} = \sum_{i=1}^{k} \sup_M |\nabla^i u| + \sup_{x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha},$$

where the last supremum is taken over all x, y such that y is contained in a normal coordinate neighbourhood of x and $\nabla^k u(y)$ is parallelly transported from y to x along a radial geodesic in this neighbourhood.

These spaces are related via the Sobolev Embedding Theorem:

Theorem 2.2.1 (Sobolev Embedding Theorem)**.** *Let M be a compact Riemannian manifold of dimension n (possibly with C^1 boundary). Then:*

1. *If*
$$\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n},$$
then $W^{k,q}(M) \hookrightarrow L^r(M)$.

2. (Rellich Lemma). *If the previous inequality for the exponents is strict, the embedding is compact.*

3. *If $0 < \alpha < 1$ and*
$$\frac{1}{q} \leq \frac{k-\alpha}{n},$$
then $W^{k,q}(M) \hookrightarrow C^{0,\alpha}(M)$.

In particular, $W^{1,p}(M) \hookrightarrow L^r(M)$ for $r \leq p^*$, where $p^* = np/(n-p)$ is the critical Sobolev exponent, while the embedding is compact for $r < p^*$.

2.2.2 Important results for elliptic PDEs

Now we give a review of some of the most powerful tools that the theory of elliptic PDEs has to offer. First is local elliptic regularity, which roughly states that a weak solution of $-\Delta u = f$ must also be a solution in the strong sense. The same result can be extended globally in the case of a compact Riemannian manifold.

Theorem 2.2.2 (Local Elliptic Regularity). *Let $\Omega \subset \mathbb{R}^n$ be open and consider the Laplacian Δ with respect to any metric on Ω, and suppose that $u \in L^1_{loc}(\Omega)$ is a weak solution of $-\Delta u = f$. Then the following statements are true:*

1. *If $f \in W^{k,q}(\Omega)$, then $u \in W^{k+2,q}(K)$ for any $K \subset\subset \Omega$, and if $u \in L^q(\Omega)$, then*
$$\|u\|_{W^{k+2,q}(K)} \leq C(\|f\|_{W^{k,q}(\Omega)} + \|u\|_{L^q(\Omega)}).$$
2. *If $f \in C^{k,\alpha}(\Omega)$, then $u \in C^{k+2,\alpha}(K)$ for any $K \subset\subset \Omega$, and if $u \in C^{0,\alpha}(\Omega)$, then*
$$\|u\|_{C^{k+2,\alpha}(K)} \leq C(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^{0,\alpha}(\Omega)}).$$

Theorem 2.2.3 (Global Elliptic Regularity). *Let M be a compact Riemannian manifold with Laplacian Δ, and suppose that $u \in L^1_{loc}(M)$ is a weak solution of $-\Delta u = f$. Then the following statements are true:*

1. *If $f \in W^{k,q}(M)$, then $u \in W^{k+2,q}(M)$, and*
$$\|u\|_{W^{k+2,q}(M)} \leq C(\|f\|_{W^{k,q}(M)} + \|u\|_{L^q(M)}).$$
2. *If $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$, and*
$$\|u\|_{C^{k+2,\alpha}(M)} \leq C(\|f\|_{C^{k,\alpha}(M)} + \|u\|_{C^{0,\alpha}(M)}).$$

Next we recall the strong maximum principle for compact Riemannian manifolds.

Theorem 2.2.4 (Strong Maximum Principle). *Let M be a connected Riemannian manifold with Laplacian Δ, and suppose that h is a non-negative smooth function of M. If $u \in C^2(M)$ satisfies $(-\Delta + h)u \geq 0$ and attains a minimum ≤ 0, then u is constant in M.*

Last but not least, we state a result concerning removable singularities, whose proof can be found in Lee & Parker [20]. For the sake of completeness we include it here as well.

Theorem 2.2.5 (Weak Removable Singularities). *Let U be an open subset of the compact Riemannian manifold (M, g) and let $\mathbf{p} \in U$. If $u \in L^q(U)$ for some $q > 2^*/2$ is a weak solution of $(-\Delta + h)u = 0$ in $U \setminus \mathbf{p}$ for $h \in L^{n/2}(U)$, then $(-\Delta + h)u = 0$ weakly in all of U.*

Proof. We need to show that $-\Delta u + hu = 0$ holds in the weak sense in all of U, i.e

$$\int_U (-u\Delta v + huv)\, d\mu_g = 0 \quad \forall v \in C_c^\infty(U).$$

Let $B \subset U$ be a ball of small enough radius centered at $\mathbf{p}$, and denote by ϵB the ball centered at $\mathbf{p}$ with radius ϵ-times that of B. Let $\eta \in C_c^\infty(U)$ be a cut-off function such that $\operatorname{supp} \eta \subset B$ and $\eta|_{B/2} = 1$. Define $\eta_\epsilon = \eta \circ \delta_\epsilon$, where $\delta_\epsilon(x) = x/\epsilon$ in normal coordinates centered at $\mathbf{p}$ denotes the dilation by ϵ about $\mathbf{p}$. Then $\operatorname{supp} \eta_\epsilon \subset \epsilon B$, and $(1-\eta_\epsilon)v \in C_c^\infty(U \setminus \mathbf{p})$ for every $v \in C_c^\infty(U)$. Since $-\Delta u + hu = 0$ away from $\mathbf{p}$, it follows that

$$\int_U (-u\Delta v + huv)\, d\mu_g = \int_{\epsilon B} (-u\Delta(\eta_\epsilon v) + hu\eta_\epsilon v)\, d\mu_g,$$

and our task is to show that the RHS converges to zero as $\epsilon \to 0$.

By Hölder's inequality, hu is integrable and hence the second term goes to zero as $\epsilon \to 0$ by the dominated convergence theorem. For the first term note that

$$\Delta(\eta_\epsilon v) = v\Delta\eta_\epsilon + 2\langle d\eta_\epsilon, dv\rangle + \eta_\epsilon \Delta v,$$

and it is straightforward that $|d\eta_\epsilon| \leq C/\epsilon$ and $|\Delta\eta_\epsilon| \leq C/\epsilon^2$. Therefore, if $1/p + 1/q = 1$, i.e p and q are Hölder-conjugate exponents, we have

$$\begin{aligned}
\Big| \int_{\epsilon B} u\Delta(\eta_\epsilon v)\, d\mu_g \Big| &\leq \frac{C}{\epsilon^2} \int_{\epsilon B} |u|\, d\mu_g \\
&\leq C\epsilon^{-2} \|u\|_q \mu_g(\epsilon B)^{\frac{1}{p}} \\
&\leq C\epsilon^{n/p-2} \|u\|_q,
\end{aligned}$$

so it follows that this goes to zero as $\epsilon \to 0$ when $q > 2^*/2$, since $n/p > 2$ in that case. This completes the proof. $\square$

Chapter 3

Solution of the Yamabe problem

In this section we solve the Yamabe problem. This is done in two steps, and we devote a separate section to each. Section 3.1 shifts the problem to the value of the Yamabe invariant, while Section 3.2 is concerned with the determination of this value, completing the solution to the problem.

3.1 Solution in terms of the Yamabe invariant

3.1.1 Yamabe's approach

Given a conformal manifold $(M, \mathbf{G})$, we will often choose to work with a Riemannian manifold (M, g) where $g \in \mathbf{G}$. This has the advantage of reducing the problem to the solution of a geometric partial differential equation, associated to which is a conformally invariant functional. This was Yamabe's original approach, the core of which remains intact until today despite the error that occurred at a later step, which we will point out when the time comes.

The first step is to specify the way curvature changes under conformal transformation of the metric. To this end, suppose that $\tilde{g}$ is of the (convenient) form $\tilde{g} = e^{2v}g$. It is straightforward to verify that in this case the connection transforms as

$$\widetilde{\nabla}_X Y = \nabla_X Y + (Xv)Y + (Yv)X - g(X,Y)\,\mathrm{grad}\,v,$$

and using this we get the transformation law for curvature, which reads

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z &+ g(\nabla_X \,\mathrm{grad}\,v, Z)Y - g(\nabla_Y \,\mathrm{grad}\,v, Z)X \\ &+ g(X,Z)\nabla_Y \,\mathrm{grad}\,v - g(Y,Z)\nabla_X \,\mathrm{grad}\,v + (Yv)(Zv)X \\ &- (Xv)(Zv)Y - g(\mathrm{grad}\,v, \mathrm{grad}\,v)(g(Y,Z)X - g(X,Z)Y) \\ &+ ((Xv)g(Y,Z) - (Yv)g(X,Z))\,\mathrm{grad}\,v,\end{aligned}$$

see for example Kühnel [17] for details. From this we can obtain the transformation laws for Riemann, Ricci and scalar curvature:

$$\widetilde{R} = e^{2v}R - e^{2v}g \owedge \left(\nabla^2 v - dv \otimes dv + \frac{1}{2}|dv|^2 g\right), \tag{3.1}$$

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (n-2)(\nabla^2 v - dv \otimes dv) - (\Delta v + (n-2)|dv|^2)g, \tag{3.2}$$

$$\widetilde{S} = e^{-2v}S - 2(n-1)e^{-2v}\Delta v - (n-2)(n-1)e^{-2v}|dv|^2. \tag{3.3}$$

To bring formula (3.3) in a form which is more manageable, we substitute $e^{2v} = u^{2^*-2}$, where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Straightforward calculation then yields

$$\widetilde{S} = u^{1-2^*}\big(-4\frac{n-1}{n-2}\Delta u + Su\big),$$

so if $\widetilde{S}$ is to assume the constant value λ, it follows that the following PDE, called the *Yamabe equation*, must be satisfied:

$$\mathbb{D}\, u = \lambda u^{2^*-1}, \tag{3.4}$$

where $\mathbb{D} = -\rho\Delta + S$, $\rho = 4(n-1)/(n-2)$ is the *conformal Laplacian*. To solve the Yamabe problem λ could assume any value, so this can be thought of as a non-linear eigenvalue problem.

In terms of calculus of variations, Yamabe noted that equation (3.4) is essentially the Euler-Lagrange equation associated to the functional $\mathcal{Q} : \mathbf{G} \to \mathbb{R}$,

$$\mathcal{Q}(g) = \frac{\int_M S\, d\mu_g}{\mu_g(M)^{2/2^*}}, \tag{3.5}$$

To see this, note that there is an equivalent way to express $\mathcal{Q}$ as follows. Fixing $g \in \mathbf{G}$, then if $\tilde{g} = u^{2^*-2}g$ it is straightforward to check that $d\mu_{\tilde{g}} = u^{2^*} d\mu_g$ and

$$\begin{aligned}\mathcal{Q}(\tilde{g}) = \frac{\int_M \widetilde{S}\, d\mu_{\tilde{g}}}{\big(\int_M d\mu_{\tilde{g}}\big)^{2/2^*}} &= \frac{\int_M u^{1-2^*}(-\rho\Delta u + Su)u^{2^*}\, d\mu_g}{\big(\int_M u^{2^*}\, d\mu_g\big)^{2/2^*}} \\ &= \frac{\int_M(-\rho u\Delta u + Su^2)\, d\mu_g}{\|u\|_{2^*}^2},\end{aligned}$$

so $\mathcal{Q}(\tilde{g}) = \mathcal{Q}_g(u)$ where $\mathcal{Q}_g : C^\infty_+(M) \to \mathbb{R}$,

$$\mathcal{Q}_g(u) = \frac{\mathcal{E}_g(u)}{\|u\|_{2^*}^2}, \quad \mathcal{E}_g(u) = \int_M (\rho|du|^2 + Su^2)\, d\mu_g. \tag{3.6}$$

$\mathcal{Q}$ and $\mathcal{Q}_g$ are obviously different expressions of the same functional defined on a conformal class, the first one being independent of a representative and the second one depending on such a choice, but instead offering a more functional analytic formulation.

The critical points of $\mathcal{Q}_g$ are all functions $u \in C^\infty_+(M)$ for which the Gateaux derivative

$$\frac{d}{dt}\Big|_{t=0} \mathcal{Q}_g(u+tw)$$

vanishes for all $w \in C^\infty(M) \cong T_u C^\infty_+(M)$. Explicitly,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{Q}_g(u+tw) = \frac{\|u\|_{2^*}^2 \frac{d}{dt}\big|_{t=0}\mathcal{E}_g(u+tw) + \mathcal{E}_g(u)\frac{d}{dt}\big|_{t=0}\|u+tw\|_{2^*}^2}{\|u\|_{2^*}^4}, \tag{3.7}$$

and it is a matter of straightforward calculations to show that

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}_g(u+tw) = 2\int_M (-\rho\Delta u + Su)w\,d\mu_g, \tag{3.8}$$

$$\left.\frac{d}{dt}\right|_{t=0} \|u+tw\|_{2^*}^2 = 2\|u\|_{2^*}^{2-2^*}\int_M u^{2^*-1}w\,d\mu_g. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we get the final formula for the Gateaux derivative

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{Q}_g(u+tw) = \frac{2}{\|u\|_{2^*}^2}\int_M \left(\mathbb{D}\,u - \frac{\mathcal{E}_g(u)}{\|u\|_{2^*}^{2^*}}u^{2^*-1}\right) w\,d\mu_g, \tag{3.10}$$

so it follows that u is a critical point of $\mathcal{Q}_g$ if and only if it is a solution of the Yamabe equation

$$\mathbb{D}\,u = \lambda u^{2^*-1}, \quad \lambda = \mathcal{E}_g(u)/\|u\|_{2^*}^{2^*}. \tag{3.11}$$

Now note that by Hölder's inequality we have $\left|\int_M Su^2\,d\mu_g\right| \le C\|u\|_{2^*}^2$ for some constant $C > 0$, thus

$$\mathcal{Q}_g(u) = \frac{\int_M (\rho|du|^2 + Su^2)\,d\mu_g}{\|u\|_{2^*}^2} \ge \frac{\int_M Su^2\,d\mu_g}{\|u\|_{2^*}^2} \ge -C,$$

so the Yamabe functional is bounded from below. We call its infimum the *Yamabe invariant*, denoted by

$$\lambda(M,g) = \inf\{\mathcal{Q}_g(u) : u \in C_+^\infty(M)\} = \inf\{\mathcal{Q}(\tilde g) : \tilde g \simeq g\}. \tag{3.12}$$

It is obvious that the Yamabe invariant is a conformal invariant, i.e. $\lambda(M,\tilde g) = \lambda(M,g)$ whenever $\tilde g \simeq g$, so it makes sense to talk about the Yamabe invariant

$$\lambda(M,\mathbf{G}) = \inf\{\mathcal{Q}(g) : g \in \mathbf{G}\} \tag{3.13}$$

of the entire conformal class.

3.1.2 The subcritical problem

In view of the previous section, it has become apparent that if we can prove the existence of a minimizer for the Yamabe functional $\mathcal{Q}_g$, then this would also be a solution to the Yamabe equation and would therefore yield a solution to the Yamabe problem. The standard approach would be to consider a minimising sequence of $\mathcal{Q}_g$ and then hope that, up to a subsequence, it converges to an actual minimizer.

This direct approach fails due to the criticality of the exponent 2^* than appears in $\mathcal{Q}_g$. This can be mended by considering the subcritical problem and then passing to the limit, but let us take a moment to see where the problem is. Suppose $\{u_k\} \subset C_+^\infty(M)$ is a minimising sequence of $\mathcal{Q}_g$, i.e. $\mathcal{Q}_g(u_k) \to \lambda(M,g)$ as $k \to \infty$. We may assume that $\|u_k\|_{2^*} = 1$ for all $k \in \mathbb{N}$ by rescaling, since $\mathcal{Q}_g(\kappa u) = \mathcal{Q}_g(u)$ for any $\kappa > 0$. Then

$$
\begin{aligned}
\|u_k\|_{1,2}^2 &= \int_M (|du_k|^2 + u_k^2)\, d\mu_g \\
&= \frac{1}{\rho}\mathcal{Q}_g(u_k) + \int_M \left(1 - \frac{S}{\rho}\right) u_k^2\, d\mu_g \\
&\leq \frac{1}{\rho}\mathcal{Q}_g(u_k) + C\|u_k\|_{2^*}^2,
\end{aligned}
$$

the last inequality being true for some $C > 0$ due to Hölder's inequality. It follows that $\{u_k\}$ is bounded in $W^{1,2}(M)$, which is a Hilbert space. A well known consequence of the Banach-Alaoglu theorem is that bounded subsets of a Hilbert space are weakly precompact, see for example Folland [10], so in particular there is a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ that converges weakly to an element $u \in W^{1,2}(M)$. It remains to be shown that $u \in C_+^\infty(M)$. To show that u is smooth would be a matter of elliptic regularity, but we face a more fundamental problem: since the embedding $W^{1,2}(M) \hookrightarrow L^{2^*}(M)$ is not compact (see the Rellich lemma in Section 2.2.1), we cannot conclude that the subsequence can be chosen so that $\|u_{k_l}\|_{2^*} \to 1$, as in general there is no strong convergence in $L^{2^*}(M)$. In particular, we could have $u = 0$.

So it is clear that the problem is the critical exponent 2^*. We try to get around this difficulty by considering the subcritical problem first. Consider the perturbed functional $\mathcal{Q}_g^s : C_+^\infty(M) \to \mathbb{R}$,

$$
\mathcal{Q}_g^s(u) = \frac{\mathcal{E}_g(u)}{\|u\|_s^2} \tag{3.14}
$$

for $2 \leq s < 2^*$ and the perturbed Yamabe constant

$$
\lambda_s(M,g) = \inf\{\mathcal{Q}_g^s(u) : u \in C_+^\infty(M)\}. \tag{3.15}
$$

Likewise, one can show that the minimizers of $\mathcal{Q}_g^s$ that are normalised by $\|u\|_s = 1$ must satisfy the Euler-Lagrange equation

$$
\mathbb{D}\, u = \lambda_s u^{s-1}, \tag{3.16}
$$

where we write $\lambda_s = \lambda_s(M,g)$ for brevity. In that case we have the following regularity theorem.

Theorem 3.1.1. *Suppose that $u \in W^{1,2}(M)$ is a weak solution of (3.16) for some fixed $2 \leq s \leq 2^*$. Suppose in addition that $u \in L^r(M)$ for some $r > (s-2)n/2$. Then either $u = 0$ or $u \in C_+^\infty(M)$ and for any $0 \leq \alpha \leq 1$, $\|u\|_{C^{2,\alpha}} < C$ for some $C = C(M, g, |\lambda_s|, \|u\|_r)$.*

Proof. Rewriting Equation (3.16) we have that

$$
-\rho\Delta u = \lambda_s u - S u^{s-1}, \tag{3.17}
$$

and since $u \in L^r(M)$ it readily follows that $\Delta u \in L^q(M)$ for $q = r/(s-1)$. Using global elliptic regularity (see Section 2.2.2), it follows that $u \in W^{2,q}(M)$. Now applying the critical case of the Sobolev embedding theorem (see Section 2.2.1), we get that $u \in L^{r'}(M)$ for $r' = nr/(ns - n - 2r)$, while our hypothesis ensures that $r' > r$. Recursive application of this process then yields that $u \in W^{2,q}(M)$ for all $q > 1$.

Considering the Hölder space version of the Sobolev embedding theorem with $k = 2$ and any $0 \leq \alpha \leq 1$, it follows that $u \in C^{0,\alpha}(M)$, and then the Hölder space version of global elliptic regularity yields $u \in C^{2,\alpha}(M)$, as well as the desired estimate. Moreover, recursive application of global elliptic regularity yields $u \in C^\infty(M)$.

Finally, equation (3.17) implies that $(-\Delta + \kappa)u \geq 0$ for

$$\kappa = \max\{0, (S - \lambda_s u^{s-2})/\rho\}$$

Then one can apply the strong maximum principle to verify the positivity claim. □

Since the hypothesis on the exponent r may seem somewhat technical or *ad hoc*, let us mention two special cases that we are interested in. The first one arises when we consider the subcritical problem and $r = s < 2^*$, while the second one arises when we consider the critical problem and $s = 2^* < r$. We will need to consider both of these cases in the sequel.

We are now in a position to prove the existence of smooth solutions in the subcritical case. Note that since $C^\infty(M)$ is dense in $W^{1,2}(M)$ and $\mathcal{Q}_g^s(u)$ is independent of the sign of u, we may continuously and symmetrically extend $\mathcal{Q}_g^s$ from $C_+^\infty(M)$ to $W^{1,2}(M)$ in a unique manner, which we will do from now on without further mention.

Theorem 3.1.2 (Yamabe). *For $2 \leq s < 2^*$, there exists a solution $u_s \in C_+^\infty(M)$ of $\mathbb{D}\, u = \lambda_s u^{s-1}$, for which $\mathcal{Q}_g^s(u_s) = \lambda_s$ and $\|u_s\|_s = 1$.*

Proof. Consider a minimising sequence $\{u_k\} \subset C_+^\infty(M)$ of $\mathcal{Q}_g^s$ such that $\|u_k\|_s = 1$ (such a choice can again be made due to homogeneity). Similar to the critical case, it is straightforward to show that $\{u_k\}$ is bounded in $W^{1,2}(M)$, and thus, possibly up to a subsequence, it converges weakly in $W^{1,2}(M)$ and strongly in $L^s(M)$ to some $u_s \in W^{1,2}(M)$.

Since $L^s(M) \hookrightarrow L^2(M)$, it follows that

$$\lim_{k\to\infty} \int_M S u_k^2 \, d\mu_g = \int_M S u_s^2 \, d\mu_g,$$

while weak convergence in $W^{1,2}(M)$ and the Cauchy-Swartz inequality imply

$$\begin{aligned}\int_M |\operatorname{grad}_g u_s|^2 \, d\mu_g &= \lim_{k\to\infty} \int g(\operatorname{grad}_g u_k, \operatorname{grad}_g u_s)\, d\mu_g \\ &\leq \liminf_{k\to\infty} \left(\int_M |\operatorname{grad}_g u_k|^2 \, d\mu_g \right)^{1/2} \left(\int_M |\operatorname{grad}_g u_s|^2 \, d\mu_g \right)^{1/2}\end{aligned}$$

and so

$$
\begin{aligned}
\mathcal{Q}_g^s(u_s) &= \int_M (|\operatorname{grad}_g u_s|^2 + S u_s^2)\, d\mu_g \\
&\le \liminf_{k\to\infty} \int_M (|\operatorname{grad}_g u_k|^2 + S u_k^2)\, d\mu_g \\
&= \lim_{k\to\infty} \mathcal{Q}_g^s(u_k) = \lambda_s.
\end{aligned}
$$

But then λ_s is defined so that $\mathcal{Q}_g^s(u_s) \ge \lambda_s$, therefore $\mathcal{Q}_g^s(u_s) = \lambda_s$ and u_s is a non-zero minimizer, since $\|u_s\|_s = 1$. Since (3.16) is the Euler-Lagrange equation of $\mathcal{Q}_g^s$, it follows that u_s is a weak solution that is in $L^s(M)$, so by Theorem 3.1.1 we also have $u_s \in C_+^\infty(M)$. □

Back to the Yamabe problem, we want to consider the limit $s \to 2^*$. In particular, our goal is to investigate under what assumptions, if any, does u_s converge to an actual smooth, positive solution of the Yamabe equation (3.4) with $\lambda = \lambda(M,g)$. The error in Yamabe's proof was to assume that the sequence u_s is uniformly bounded as $s \to 2^*$, which may be false in general. The subtlety of this question will be further explored in the subsequent sections. The following useful lemma regarding the limit behaviour of λ_s is a good start and an indication that such expectations are plausible.

Lemma 3.1.3 (Aubin)**.** *If $\mu_g(M) = 1$, then $s \mapsto |\lambda_s|$ is a non-increasing function of $s \in [2, 2^*]$. If, in addition, we have that $\lambda(M,g) \ge 0$, then $s \mapsto \lambda_s$ is continuous from the left.*

Proof. First, we show that $s \mapsto |\lambda_s|$ is non-increasing. Due to Hölder's inequality and the fact that $\mu_g(M) = 1$, for every $u \in C^\infty(M)$ we have that $\|u\|_s \le \|u\|_{s'}$ whenever $s \le s'$. For any s, s', the functionals $\mathcal{Q}_g^s$ and $\mathcal{Q}_g^{s'}$ are related by

$$
\mathcal{Q}_g^{s'}(u) = \frac{\|u\|_s}{\|u\|_{s'}} \mathcal{Q}_g^s(u). \tag{3.18}
$$

Consequently, we have that $|\lambda_{s'}| \le |\lambda_s|$ for $s \le s'$, which proves the claim.

If $\lambda_s < 0$ for some $s \in [2, 2^*]$, we can choose a function $u \in C^\infty(M)$ such that $\mathcal{Q}_g^s(u) < 0$. It follows from (3.18) that $\lambda_{s'} \le \mathcal{Q}_g^{s'}(u) < 0$ for any $s' \in [2, 2^*]$. Therefore, if $\lambda(M,g) \ge 0$, then $\lambda_s \ge 0$ for all $s \in [2, 2^*]$ as well.

Now we prove the continuity claim. Given $\epsilon > 0$, by the definition of λ_s there is a function $u \in C^\infty(M)$ such that $\mathcal{Q}_g^s(u) < \lambda_s + \epsilon/2$. Since $s \mapsto \|u\|_s$ is continuous, it follows from (3.18) that for all $s' \le s$ that are sufficiently close to s, $\mathcal{Q}_g^{s'}(u) < \mathcal{Q}_g^s(u) + \epsilon/2$, and thus

$$
\lambda_s \le \lambda_{s'} \le \mathcal{Q}_g^{s'}(u) < \lambda_s + \epsilon,
$$

hence $\lambda_{s'} \to \lambda_s$ as $s' \to s^-$. □

Note that in the context of conformal geometry we can always choose g so that $\mu_g(M) = 1$, possibly by multiplying with a positive constant.

3.1.3 Sharp Sobolev inequality and $\lambda(\mathbb{S}^n, g_\circ)$

In this section we explore the unexpected connection between two seemingly very different things, namely the sharp Sobolev inequality of $\mathbb{R}^n$ and the Yamabe problem of the standard sphere $(\mathbb{S}^n, g_\circ)$. As stated in the introduction, the Yamabe invariant

$$\lambda(\mathbb{S}^n, g_\circ) = \inf_{u \in C_+^\infty(\mathbb{S}^n)} \frac{\int_{\mathbb{S}^n} (\rho |\operatorname{grad}_\circ u|^2 + u^2)\, d\mu_\circ}{\left(\int_{\mathbb{S}^n} u^{2^*}\, d\mu_\circ\right)^{2/2^*}}$$

turns out to be the key for the solution of the general case.

Let $\mathbf{n} = (0, \ldots, 0, 1)$ denote the north pole of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. It is well known that $\mathbb{S}^n \setminus \mathbf{n}$ can be covered by a single chart using the stereographic projection $\sigma : \mathbb{S}^n \setminus \mathbf{n} \to \mathbb{R}^n$, whose components are given by

$$\sigma^i(x_1, \ldots, x_n, x_{n+1}) = \frac{x_i}{1 - x_{n+1}}.$$

Its inverse, i.e the parametrisation $\chi : \mathbb{R}^n \to \mathbb{S}^n \setminus \mathbf{n}$, is given by

$$\chi_i(x^1, \ldots, x^n) = \frac{2x^i}{1 + |x|^2}, \quad 1 \le i \le n, \quad \chi_{n+1}(x^1, \ldots, x^n) = \frac{-1 + |x|^2}{1 + |x|^2}.$$

Let $g_\circ$ and g_0 denote the standard metrics of $\mathbb{S}^n$ and $\mathbb{R}^n$ respectively. Then it follows by straightforward calculations that

$$\chi^* g_\circ(x) = \frac{4}{(1 + |x|^2)^2} g_0(x), \quad x \in \mathbb{R}^n.$$

A similar argument for the antipodal stereographic projection excluding the south pole $\mathbf{n}'$ proves that $(\mathbb{S}^n, g_\circ)$ is *locally conformally flat*, which means that its metric is locally conformal to a flat metric. Since the Weyl tensor as a $(1,3)$-tensor is conformally invariant[1], it follows that $W_\circ = 0$.

Since $(\mathbb{S}^n, g_\circ)$ is locally conformally flat, the structure of its conformal diffeomorphisms is locally determined by the conformal diffeomorphisms of $(\mathbb{R}^n, g_0)$. There is a well known rigidity theorem, originally dew to Liouville, which gives a precise description of these transformations; they are the Möbius transformations.

Theorem 3.1.4 (Conformal Liouville Theorem)**.** *Let $n \ge 3$ and $\phi : U \to \phi(U)$ be a conformal diffeomorphism from the open set $U \subset \mathbb{R}^n$ to its image. Then ϕ is a composition of similarities (translations, rotations, reflections and dilations) and inversions.*

For a proof of this beautiful result, see Blair [7]; also Schoen & Yau [23]. Now, the round sphere is invariant under rotations and reflections, so these conformal maps are not interesting. Neither is inversion, since it amounts to the transition maps between the two charts obtained by stereographic projection

[1] This follows directly from the definition of the Weyl tensor (see Section 2.1.3) and the transformation formulas of Section 3.1.1.

from either pole. So what we are left with are the translations $\tau_b : \mathbb{R}^n \to \mathbb{R}^n$,

$$\tau_b(x) = x - b,$$

as well as the dilations $\delta_a : \mathbb{R}^n \to \mathbb{R}^n$,

$$\delta_a(x) = \frac{x}{a},$$

where $b \in \mathbb{R}^n$ and $a > 0$. For reasons that will become apparent in the sequel, putting

$$u_a(x) = \left(\frac{a}{|x|^2 + a^2}\right)^{\frac{n-2}{2}},$$

we have

$$\chi^* g_\circ(x) = u_1^{2^*-2}(x) g_0(x), \quad \delta_a^* \chi^* g_\circ = u_a^{2^*-2}(x) g_0(x/a). \tag{3.19}$$

An obvious choice of constant scalar (and sectional) curvature of the sphere is of course the round metric $g_\circ$. We now prove the following rigidity theorem regarding metrics of constant scalar curvature within the conformal class of $g_\circ$.

Theorem 3.1.5. *Let $g \in \bar{g}_\circ$ and suppose that g has constant scalar curvature. Then g has constant sectional curvature.*

Proof. First we show that g is an Einstein metric. Working with g as the background metric and since $g_\circ \simeq g$, it follows that $g_\circ = e^{2f} g$ for some $f \in C^\infty(\mathbb{S}^n)$. Substituting $e^{2f} = u^{-2}$ and using the conformal transformation formula (3.2) for Ricci curvature, we obtain

$$\mathrm{Ric}_\circ = \mathrm{Ric} + \frac{1}{u}\left((n-2)\nabla^2 u - (n-1)\frac{|\operatorname{grad} u|^2}{u} g - \Delta u\, g\right).$$

Since $g_\circ$ is of constant curvature, it is Einstein and hence

$$E + \frac{n-2}{u}(\nabla^2 u + \frac{1}{n}\Delta u\, g) = E_\circ = 0.$$

Now, the Einstein tensor is traceless, meaning that $\operatorname{tr}_g E = \langle E, g\rangle = 0$ in $\Gamma(\oplus_2^0 T\mathbb{S}^n)$, so integration by parts (see Section 2.1.2) yields

$$\begin{aligned}
\int_{\mathbb{S}^n} u|E|^2\, d\mu_g &= \int_{\mathbb{S}^n} u\langle E, E\rangle\, d\mu_g \\
&= -(n-2)\int_{\mathbb{S}^n} \langle E, \nabla^2 u + \frac{1}{n}\Delta u\, g\rangle\, d\mu_g \\
&= -(n-2)\int_{\mathbb{S}^n} \langle E, \nabla^2 u\rangle\, d\mu_g \\
&= (n-2)\int_{\mathbb{S}^n} \langle \operatorname{tr}_g \nabla E, \nabla u\rangle\, d\mu_g = 0.
\end{aligned}$$

Since $u > 0$, this implies that $E = 0$ and g is indeed an Einstein metric.

In addition, $W = 0$ since the round sphere is locally conformally flat and $g \simeq g_\circ$. Since the scalar curvature S is a constant by assumption, using the orthogonal decomposition formula for curvature (see Section 2.1.3) we conclude that g is of constant curvature. □

As a consequence we get the following corollary regarding the structure of minimizers.

Corollary 3.1.6. *If the Yamabe functional of* $(\mathbb{S}^n, g_\circ)$ *attains its infimum, then the infimum is attained by every metric of the form* $\phi^*\kappa g_\circ$ *where* $\kappa > 0$ *and* $\phi : (\mathbb{S}^n, g_\circ) \to (\mathbb{S}^n, g_\circ)$ *is a conformal diffeomorphism, and those are the only metrics that attain it.*

Proof. Since minimizers have constant scalar curvature, it follows from the theorem that they must have constant sectional curvature. Hence the minimizers are isometric to constant multiples of the round metric $g_\circ$. This means that if $g \in \bar{g}_\circ$ is a minimizer, there exists $\kappa > 0$ and a diffeomorphism $\phi : \mathbb{S}^n \to \mathbb{S}^n$ such that $g = \phi^*\kappa g_\circ$. In particular $\phi : (\mathbb{S}^n, g_\circ) \to (\mathbb{S}^n, g_\circ)$ is a conformal diffeomorphism. Since the Yamabe invariant is a conformal invariant, the claim follows. □

Proving the existence of a minimizer is a more delicate issue. It requires a renormalisation argument to treat the potential blow-up of the functions $\{u_s\}$ constructed in the proof of Theorem 3.1.2 as $s \to 2^*$. Here we offer a schetch of the proof, the complete version of which can be found in Lee & Parker [20].

Theorem 3.1.7. *The Yamabe functional of* $(\mathbb{S}^n, g_\circ)$ *attains its infimum.*

Sketch of proof. We summarise the proof in steps.

1. Consider the family of functions $\{u_s\} \subset C^\infty_+(\mathbb{S}^n)$ as in Theorem 3.1.2. Possibly composing with a rotation, we may assume that $\max_{\mathbb{S}^n} u_s = u_s(\mathbf{n}')$ for every s. If the sequence is unifomly bounded, elliptic regularity and the Arzella-Ascoli Theorem imply that u_s converges to an actual minimizer as $s \to 2^*$.

2. If $\max_{\mathbb{S}^n} u_s \to \infty$ as $s \to 2^*$, we may renormalise the sequence using the dilations δ_a so that the resulting sequence v_s is such that $\max_{\mathbb{S}^n} v_s = v_s(\mathbf{n}') = 1$. Using the transformation properties of the conformal Laplacian $\mathbb{D}$, one can show that $\|v_s\|_{1,2} \leq C\|u_s\|_{1,2}$. Hence $\{v_s\}$ is bounded in $W^{1,2}(\mathbb{S}^n)$ and consequently in $L^{2^*}(\mathbb{S}^n)$, and $v_s \rightharpoonup v$ in $W^{1,2}(\mathbb{S}^n)$ for some $v \in W^{1,2}(\mathbb{S}^n)$, possibly up to a subsequence.

3. It is easy to show that $\{v_s\}$ is bounded in $L^r_{loc}(\mathbb{S}^n \setminus \mathbf{n})$ for any $r > 1$ and consequently, using local elliptic regularity (see Section 2.2.2), we conclude that $\{v_s\}$ is bounded in $C^{2,\alpha}_{loc}(\mathbb{S}^n \setminus \mathbf{n})$. By considering an exhaustion of $\mathbb{S}^n \setminus \mathbf{n}$ by compact subsets and appealing to the Arzella-Ascoli Theorem once more, a diagonal argument shows that $v \in C^2(\mathbb{S}^n \setminus \mathbf{n})$, while the possibility of a singularity at $\mathbf{n}$ remains.

4. The Yamabe invariant $\Lambda := \lambda(\mathbb{S}^n, g_\circ)$ is certainly non-negative by its definition, and thus $\lambda_s \to \Lambda$ as $s \to 2^*$ in view of Lemma 3.1.3. Then one can show that $\mathbb{D}\, v = f v^{2^*-1}$ for some $f \in C^2(\mathbb{S}^n \setminus \mathbf{n})$ such that $0 \le f \le \Lambda$. As a consequence, the singularity at N is removable (see Section 2.2.2) and the same equation holds weakly in all of $\mathbb{S}^n$. Then direct calculations imply that $\mathcal{Q}_{g_\circ}(v) = \Lambda$.

5. The proof is finished by showing that v is positive and smooth. In view of Theorem 3.1.1, it suffices to show that $u \in L^r(\mathbb{S}^n)$ for some $r > 2^*$. This is done by considering a perturbation of the conformal Laplacian of the form $\mathbb{D}_\eta = \mathbb{D} - \eta \Lambda v^{2^*-2}$, where η is a cut-off function supported in a sufficiently small neighbourhood of $\mathbf{n}$. The operator $\mathbb{D} : W^{2,q} \to L^q$ is bijective, and η controls the operator norm of the perturbation, so it can be chosen so that $\mathbb{D}_\eta$ remains bijective. In this way we can prove that $v \in W^{2,q}(\mathbb{S}^n) \subset L^r(\mathbb{S}^n)$ for some suitable q and $r > 2^*$.

This completes the proof. □

Now we turn to the relationship between the sharp Sobolev inequality and the spherical Yamabe invariant. In what follows, for $u \in C^\infty(\mathbb{S}^n)$ denote by $\overline{u} \in C^\infty(\mathbb{R}^n) \cap W_0^{1,2}(\mathbb{R}^n)$ the weighted pull-back $\overline{u} = 4u_1\chi^* u$.

Theorem 3.1.8 (Sharp Sobolev Inequality)**.** *The inequality*

$$\int_{\mathbb{R}^n} |\operatorname{grad} u|^2 \, dx \ge \frac{\Lambda}{\rho} \left(\int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{2/2^*} \tag{3.20}$$

holds for all $u \in W_0^{1,2}(\mathbb{R}^n)$, where $\Lambda = \lambda(\mathbb{S}^n, g_\circ)$. The constant is sharp and attained only by constant multiples and translations of the functions u_a defined previously.

Proof. By definition, we have

$$\Lambda = \lambda(\mathbb{S}^n, g_\circ) = \inf_{u \in C^\infty(\mathbb{S}^n)} \frac{\int_{\mathbb{S}^n} (\rho |\operatorname{grad}_\circ u|^2 + u^2) \, d\mu_\circ}{\left(\int_{\mathbb{S}^n} |u|^{2^*} \, d\mu_\circ \right)^{2/2^*}}.$$

The integrals are unaffected if we remove the north pole, and since $\mathbb{S}^n \setminus \mathbf{n}$ is conformally flat, we get

$$\Lambda = \inf_{u \in C^\infty(\mathbb{S}^n)} \rho \frac{\int_{\mathbb{R}^n} |\operatorname{grad} \overline{u}|^2 \, dx}{\left(\int_{\mathbb{R}^n} |\overline{u}|^{2^*} \, dx \right)^{2/2^*}}.$$

Since $C^\infty(\mathbb{R}^n) \cap W_0^{1,2}(\mathbb{R}^n)$ is dense in $W_0^{1,2}(\mathbb{R}^n)$, it follows that Λ/ρ is indeed the best constant in (3.20). The exact form of the minimizers is a direct consequence of Corollary 3.1.6. □

The exact value of the best Sobolev constant is actually known to be

$$\sigma_n = \frac{n(n-2)\mu_\circ(\mathbb{S}^{n-1})^{2/n}}{4},$$

see for example Schoen & Yau [23] and also Talenti [24], where the best constant is determined for the general L^p version of the Sobolev inequality. In view of Theorem 3.1.8, we obtain the precise value of the spherical Yamabe invariant, which is

$$\lambda(\mathbb{S}^n, g_\circ) = n(n-1)\mu_\circ(\mathbb{S}^{n-1})^{2/n}.$$

To conclude this section, we provide the following lemma regarding the assymptotic L^2_{loc} behaviour of the minimizers as $a \to 0$, which will be useful on several occasions in the sequel.

Lemma 3.1.9. *Let $k > -n$, and for $\epsilon > 0$ define*

$$I(a) := \int_{B_\epsilon(0)} |x|^k u_a^2(x)\, dx, \quad a > 0.$$

Then as $a \to 0$, the following statements hold:

1. *If $k < n-4$, then $I(a) \sim a^{k+2}$,*
2. *If $k = n-4$, then $I(a) \sim a^{n-2} \log 1/a$,*
3. *If $k > n-4$, then $I(a) \sim a^{n-2}$.*

Proof. Passing to polar coordinates and then changing variables $r = a\xi$ we have

$$I(a) = \int_0^\epsilon \left(\frac{a}{a^2+r^2}\right)^{n-2} r^{n+k-1}\, dr = a^{k+2} \int_0^{\epsilon/a} \frac{\xi^{n+k-1}}{(1+\xi^2)^{n-2}}\, d\xi.$$

Now note that for $\xi \geq 1$ we have $\xi^2 \leq 1+\xi^2 \leq 2\xi^2$. This implies that, for $a \leq \epsilon$,

$$I(a) \sim a^{k+2}\left(C(n,k) + \int_1^{\epsilon/a} \xi^{-n+k+3}\, d\xi\right),$$

and finishing the proof is a matter of straightforward calculations. $\square$

3.1.4 Some aspects of conformal geometry

As we saw in the previous section, the standard sphere is locally conformally flat, and this has been the key for proving the relation between the best Sobolev constant σ_n of $\mathbb{R}^n$ and $\lambda(\mathbb{S}^n, g_\circ)$, as stated in Theorem 3.1.8. In general, a Riemannian manifold (M, g) is *locally conformally flat* if for every point $\mathbf{p} \in M$ there is a chart (U, x) such that $\mathbf{p} \in U$ and there is a positive function $u \in C^\infty_+(U)$ so that

$$g = u(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n).$$

Note that for $n = 2$ these are the isothermal coordinates, see Definition 1.2.2. While such coordinates always exist for surfaces, this is certainly not the case for $n \geq 3$. Since the Weyl tensor - as a $(1,3)$-tensor field - is a conformal invariant, every locally conformally flat manifold must have $W = 0$, which is not always the case. In fact we have the following characterisation theorem, generally known as the Weyl-Schouten Theorem.

Theorem 3.1.10 (Weyl-Schouten). *Let (M, g) be a Riemannian manifold of dimension n.*

1. *If $n = 3$, (M, g) is locally conformally flat if and only if*

$$\nabla A([X; Y], Z) = 0$$

for all $X, Y, Z \in \Gamma(TM)$, with A being the Schouten tensor[2].

2. *If $n \geq 4$, (M, g) is locally conformally flat if and only if $W = 0$.*

Local conformal flatness is a very useful property in the context of conformal geometry, as it allows us to locally carry out calculations exactly as in the Euclidean space $\mathbb{R}^n$. There is a weaker notion than local conformal flatness that retains much of this functionality:

Definition 3.1.1. The Riemannian manifold (M, g) is said to be *locally conformally volume preserving* if every $\mathbf{p} \in M$ belongs to a normal coordinate chart $(U, \tilde{x})$ of a metric $\tilde{g} \in \overline{g}$ so that $d\mu_{\tilde{g}} = d\tilde{x}$. Such coordinates are called *conformal normal coordinates.*

Intrinsically, the definition means that the Jacobian of the exponential map $\exp_{\mathbf{p}}^{\tilde{g}}$ is 1 for some $\tilde{g} \in \overline{g}$. While local conformal flatness is a quite rigid requirement, the locally conformally volume preserving condition is not rigid at all. In fact, every Riemannian manifold is locally conformally volume preserving as the following theorem shows.

Theorem 3.1.11 (Conformal Normal Coordinates). *Let $(M, \mathbf{G})$ be a conformal manifold and let $\mathbf{p} \in M$. Then there are a metric $g \in \mathbf{G}$ and normal coordinates (U, x) of g so that $x(\mathbf{p}) = 0$ and*

$$d\mu_g = dx, \quad S = O(|x|^2), \quad -\Delta S(P) = \frac{1}{6}|W(P)|^2.$$

A weaker version of this that is nonetheless sufficient for solving the Yamabe problem is given in Lee and Parker [20], where $d\mu_g = (1+O(|x|^m))dx$ and $m \in \mathbb{N}$ is arbitrary. The strong version stated above is due to Günther [13, 14], who proves this in an analytic as well as a $C^{k,\alpha}$ setting. Below we give an outline of Günther's proof for the smooth/analytic case.

Sketch of proof. Since the issue is local in nature, we may work in a ball $B := B_R(0) \subset \mathbb{R}^n$ (with the correspondence $\mathbf{p} \equiv 0$) equipped with a background metric g, and then look for a conformal metric $\tilde{g} = ug$ accompanied by a set of normal coordinates $\tilde{x}$ for that metric with the desired properties. As polynomials are dense in $C^\infty(B)$[3], it suffices to work with analytic functions. The proof is given in steps.

[2] For the definition of the Schouten tensor, see Section 2.1.3

[3] Density is with respect to the weak C^∞-topology, which is generated by the finite rank C^k-norms evaluated in relatively compact subsets; it is a consequence of the Stone-Weierstrass Theorem

1. Suppose that $\tilde{g} = ug$ and $\tilde{x}$ are as desired. Then for $v = |\tilde{x}|^2$,

$$|dv|^2_{\tilde{g}} = 4v, \quad \Delta_{\tilde{g}} v = 2n.$$

These two relations characterise v as the square distance function with respect to $\tilde{g} = ug$. Since u is still unknown, we will instead work in normal coordinates x of the background metric g and opt to solve the resulting system of partial differential equations for u and v. These are

$$D_1(u,v) := g^{ij}\frac{\partial v}{\partial x^i}\frac{\partial v}{\partial x^j} - 4uv = 0 \quad (3.21)$$

$$D_2(u,v) := \frac{u}{\Omega_g(x)}\frac{\partial}{\partial x^i}\Omega_g(x)g^{ij}\frac{\partial v}{\partial x^j} + \frac{n-2}{2}g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} - 2nu^2 = 0, \quad (3.22)$$

together with the initial conditions

$$u(x) = 1 + O(|x|), \quad v(x) = |x|^2 + O(|x|^3). \quad (3.23)$$

2. The partial differential operators D_1 and D_2 are both non-linear and therefore we cannot apply standard elliptic theory to the system. Equations (3.21) and (3.22) can be rewritten as

$$L_1(u,v) = L_1(u,v) - D_1(u,v) =: P_1(u,v) \quad (3.24)$$
$$L_2(u,v) = L_2(u,v) - D_2(u,v) =: P_2(u,v), \quad (3.25)$$

where $L_i(u,v)$ is the linearisation of $D_i(u,v)$ with respect to the triple $(g^{ij}, u, v) = (\delta^{ij}, 1, r^2)$ for $i = 1,2$:

$$L_1(u,v) := 4x^i\partial v/\partial x^i - 4v - 4r^2 u \quad (3.26)$$
$$L_2(u,v) := \Delta_0 v + (n-2)x^i\partial u/\partial x^i - 2nu. \quad (3.27)$$

3. Since we assume analyticity, we will work with polynomials. Let $\mathcal{P}_m[x]$ denote the set of homogeneous polynomials of order m in x. We consider the non-homogeneous problem

$$L_1(u,v) = f_1, \quad L_2(u,v) = f_2, \quad (3.28)$$

where f_1 and f_2 are homogeneous polynomials that will later be determined by iteration. To this end we need the following "analytic regularity" lemma.

Lemma 3.1.12 ([14], Satz 1)**.** *Suppose that $f_1 \in \mathcal{P}_{m+2}[x]$ and $f_2 \in \mathcal{P}_m[x]$ with coefficients*

$$f_1 = \sum_{|\alpha|=m+2} f^1_\alpha x^\alpha, \quad f_2 = \sum_{|\alpha|=m} f^2_\alpha x^\alpha.$$

Then, for $m \geq 4$, there exist homogeneous polynomials $v \in \mathcal{P}_{m+2}[x]$ and $u \in \mathcal{P}_m[x]$ with coefficients

$$v = \sum_{|\alpha|=m+2} v_\alpha x^\alpha, \quad u = \sum_{|\alpha|=m} u_\alpha x^\alpha$$

that satisfy the equations (3.28), as well as the estimates

$$\sum_{|\alpha|=m} |v_\alpha| \leq \frac{C}{m^2}\Big(\sum_{|\alpha|=m-2} |f^2_\alpha| + m \sum_{|\alpha|=m} |f^1_\alpha|\Big) \tag{3.29}$$

$$\sum_{|\alpha|=m} |u_\alpha| \leq \frac{C}{m}\Big(\sum_{|\alpha|=m-2} |f^2_\alpha| + m \sum_{|\alpha|=m} |f^1_\alpha|\Big). \tag{3.30}$$

If $m = 3$ and $u = u_i x^i$, v also satisfies $\Delta_0 v = 8 f_2$, as well as the estimate

$$\sum_{|\alpha|=3} |v_\alpha| \leq C\Big(\sum_{i=1}^{n} |u_1| + \sum_{|\alpha|=3} |f^1_\alpha|\Big). \tag{3.31}$$

The constant in all estimates is uniform and independent of m.

4. Given u, v satisfying the required initial conditions, we denote the asymptotic expansions of $P_i(u, v)$, $i = 1, 2$ by

$$P_1(u, v) = \sum_{|\alpha|\geq 4} p^1_\alpha x^\alpha, \quad P_2(u, v) = \sum_{|\alpha|\geq 2} p^2_\alpha x^\alpha$$

and attempt to solve the problem using the power series method (it is easy to check that lower order terms vanish). The condition

$$\Delta_0 \sum_{|\alpha|=3} p^2_\alpha x^\alpha = 8 \sum_{|\alpha|=1} p^1_\alpha x^\alpha$$

holds trivially, therefore all estimates of Lemma 3.1.12 apply recursively. What remains is to show that the power series of u and v converge in a neighborhood of zero.

5. To this end define

$$N_m := \max\Big\{ (m+2) \sum_{|\alpha|=m+2} |v_\alpha|, \sum_{|\alpha|=1} |u_\alpha| \Big\}.$$

By induction one can show that

$$N_m \leq C_0\Big(\sum_{i+j\leq m+1} N_i N_j + \sum_{\substack{i+j=m \\ i,j\neq 0}} N_i N_j \Big).$$

Next we define the complex power series $\psi(z) = \sum_{m=0}^{\infty} \tilde{N}_m z^m$ via the equation

$$N_0 + C_0((\psi(z) - N_0) + (z + z^2 + \cdots)\psi^2(z)) = \psi(z),$$

and one can show that $N_m \leq \tilde{N}_m$ and that the inverse of $\psi - N_0$ is analytic with non-vanishing derivative in a neighborhood of zero. It follows that the same is true for ψ, and the power series of u and v are absolutely convergent.

This proves the existence of solutions u, v for the system (3.21)-(3.22) together with the initial conditions (3.23). The desired coordinates can then be obtained by exponential mapping.

6. Properties $S = O(|\tilde{x}|^2)$ and $-\Delta S(\mathbf{p}) = |W(\mathbf{p})|^2/6$ are a consequence of $\Omega_{\tilde{g}}(\tilde{x}) = 1$, for details see Lee & Parker [20].

This completes the proof. □

Note that the construction of the conformal factor in the proof is subject to the initial conditions $u = 1 + u_i x^i + O(|x|^2)$. This means that the resulting locally conformally volume preserving metric is not unique; one can get the required properties and still maintain a lot of freedom. This fact is nevertheless irrelevant in the sequel.

3.1.5 Resolution in terms of $\lambda(\mathbb{S}^n, g_\circ)$

It is a fact of crucial importance that $\lambda(\mathbb{S}^n, g_\circ)$ is actually an upper bound for the Yamabe invariant of any other conformal manifold - compact or non-compact. This is a consequence of the fact that the minimizers u_a of the Sobolev inequality are concentrated at zero as $a \to 0$ and can be approximated by compactly supported functions. Then one can use conformal normal coordinates to obtain a test function whose Yamabe quotient approximates $\lambda(\mathbb{S}^n, g_\circ)$ as close as we wish as. Geometrically, note that this corresponds to blowing up a small neighborhood to a sphere.

Theorem 3.1.13 (Aubin)**.** *Let $(M, \mathbf{G})$ be a conformal manifold of dimension $n \geq 3$. Then $\lambda(M, \mathbf{G}) \leq \lambda(\mathbb{S}^n, g_\circ)$.*

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a radially symmetric cut-off function such that $\operatorname{supp}\eta \subset B_2(0)$ and $\eta|_{B_1(0)} = 1$, and denote by $\eta_\epsilon = \eta \circ \delta_\epsilon$ the dilation of η by a factor $\epsilon > 0$. Then $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n)$ satisfies $\operatorname{supp}\eta_\epsilon \subset B_{2\epsilon}(0)$, $\eta_\epsilon|_{B_\epsilon(0)} = 1$ and $\operatorname{grad}\eta_\epsilon = \epsilon^{-1}\operatorname{grad}\eta \circ \delta_\epsilon$. If u_a is a minimizer of the Sobolev quotient as in Theorem 3.1.8, consider the compactly supported approximation $u_{a,\epsilon} = \eta_\epsilon u_a$. Recall that u_a satisfies the equality $\rho\|\operatorname{grad} u_a\|_2^2 = \Lambda\|u_a\|_{2^*}^2$ and

$$u_a(x) = \left(\frac{a}{|x|^2 + a^2}\right)^{\frac{n-2}{2}}, \quad \operatorname{grad} u_a(x) = -(n-2)\left(\frac{a}{|x|^2 + a^2}\right)^{\frac{n}{2}} \frac{x}{a},$$

the gradient being obtained by straightforward calculation.

Our goal is to use $u_{a,\epsilon}$ as a test function whose Yamabe quotient can get arbitrarily close to Λ. The role of ϵ is to make the support of $u_{a,\epsilon}$ small enough to fit into a conformal normal coordinate chart of $(M, \mathbf{G})$, and taking $a \to 0$ should take care of the rest. We estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \rho |\operatorname{grad} u_{a,\epsilon}|^2 dx &= \int_{\mathbb{R}^n} \rho(\eta_\epsilon^2 |\operatorname{grad} u_a|^2 + 2\eta_\epsilon u_a \langle \operatorname{grad} \eta_\epsilon, \operatorname{grad} u_a\rangle \\ &\quad + u_a^2 |\operatorname{grad} \eta_\epsilon|^2)\, dx \\ &\leq \int_{\mathbb{R}^n} \rho |\operatorname{grad} u_a|^2\, dx + C(\epsilon) a^{n-2},\end{aligned}$$

and in addition, by the sharp Sobolev inequality (3.20) and Taylor expanding,

$$\begin{aligned}\int_{\mathbb{R}^n} \rho |\operatorname{grad} u_a|^2\, dx &= \Lambda \|u_a\|_{2^*}^2 \\ &\leq \Lambda \left(\int_{B_\epsilon(0)} u_a^{2^*}\, dx + \int_{\mathbb{R}^n \setminus B_\epsilon(0)} u_a^{2^*}\, dx \right)^{2/2^*} \\ &\leq \Lambda \|v_{a,\epsilon}\|_{2^*}^2 + O_\epsilon(a^n).\end{aligned}$$

Moreover, for $0 < a \leq \epsilon$ we have the lower estimate

$$\|u_{a,\epsilon}\|_{2^*} \geq \|u_a \chi_{B_a(0)}\|_{2^*} \geq C.$$

Taking ϵ sufficiently small, we may regard $u_{a,\epsilon}(x)$ as a function of some conformal normal coordinates (U, x) of $(M, \mathbf{G})$ as in Theorem 3.1.11. Note that in normal coordinates, for $r = |x|$ one has $g(\partial_r, \partial_r) = 1$, thus the gradient of $u_{a,\epsilon}$ is unaffected due to radial symmetry. Then by the previous estimates the Yamabe quotient of $u_{a,\epsilon}$ is

$$\begin{aligned}\mathcal{Q}_g(u_{a,\epsilon}) &\leq \Lambda + C(\epsilon) a^{n-2} + C \int_{B_{2\epsilon}(0)} S u_{a,\epsilon}^2\, dx \\ &\leq \Lambda + C(\epsilon) a^{n-2}.\end{aligned} \tag{3.32}$$

for $0 < a \leq \epsilon < 1$. Taking $a \to 0$ yields the conclusion. □

In order to proceed we need the following variant of the sharp Sobolev inequality for compact Riemannian manifolds. In this case the best value σ_n of the Sobolev constant of $\mathbb{R}^n$ can be approximated as close as we wish, at the cost of an additional L^2 term.

Theorem 3.1.14 (Aubin). *Let (M, g) be a compact Riemannian manifold. Then for every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that the inequality*

$$\sigma_n \|u\|_{2^*}^2 \leq (1+\epsilon) \int_M |\operatorname{grad}_g u|^2\, d\mu_g + C(\epsilon) \int_M u^2\, d\mu_g \tag{3.33}$$

holds for all $u \in C^\infty(M)$.

Proof. Let $\epsilon > 0$ and assume an open cover of M consisting of normal coordinate charts $\{(U_\mathbf{p}, x_\mathbf{p})\}_{\mathbf{p} \in M}$ centered around each $\mathbf{p}$, respectively, such that the metric and local volume density satisfy $|g_{ij}(x) - \delta_{ij}(x)| < \epsilon$ and $|\Omega_g(x) - 1| < \epsilon$ in U_P for all P. Then we may pass to a finite subcover $\{(U_k, x_k)\}_{k=1}^m$, and assume a partition of unity subordinate to that cover, which we write in the form $\{\varphi_k^2\}_{k=1}^m$ and $\sum_{k=1}^m \varphi_k^2 = 1$.

Then, for $u \in C^\infty(M)$,

$$\begin{aligned}\|u\|_{2^*}^2 = \|u^2\|_{2^*/2} &= \left\| \sum_{k=1}^m \varphi_k^2 u^2 \right\|_{2^*/2} \le \sum_{k=1}^m \|\varphi_k^2 u^2\|_{2^*/2} \\ &= \sum_{k=1}^m \|\varphi_k u\|_{2^*}^2 = \sum_{k=1}^m \left(\int_{U_k} |\varphi_k u|^{2^*} \, d\mu_g \right)^{2/2^*} \\ &\le (1+\epsilon)^{2/2^*} \sum_{k=1}^m \left(\int_{x_k(U_k)} |\varphi_k u|^{2^*} \, dx \right)^{2/2^*}.\end{aligned}$$

Applying the sharp Sobolev inequality of $\mathbb{R}^n$, we obtain

$$\begin{aligned}\left(\int_{x_k(U_k)} |\varphi_k u|^{2^*} \, dx \right)^{2/2^*} &\le \frac{1}{\sigma_n} \int_{x_k(U_k)} |\operatorname{grad} \varphi_k u|^2 \, dx \\ &\le \frac{(1+\epsilon)^2}{\sigma_n} \int_{U_k} |\operatorname{grad}_g \varphi_k u|^2 \, d\mu_g,\end{aligned}$$

and subsequently

$$\begin{aligned}|\operatorname{grad}_g \varphi_k u|^2 &= \varphi_k^2 |\operatorname{grad}_g u|^2 + 2\varphi_k u \langle \operatorname{grad}_g \varphi_k, \operatorname{grad}_g u \rangle + u^2 |\operatorname{grad}_g \varphi_k|^2 \\ &\le (1+\epsilon)\varphi_k^2 |\operatorname{grad}_g u|^2 + (1 + 1/\epsilon) u^2 |\operatorname{grad}_g \varphi_k|^2,\end{aligned}$$

where in the last step we have used the Cauchy-Schwartz inequality as well as the inequality $2ab \le \epsilon a^2 + b^2/\epsilon$. Summing up, we obtain the conclusion. Note here that the constant $C(\epsilon)$ depends also on the dimension as well as the chosen partition of unity. $\square$

It is worth noting that $C(\epsilon) \to \infty$ as $\epsilon \to 0$, so as we get closer to the sharp Sobolev constant the collateral L^2 term blows up. Nevertheless, this fact will be of no consequence to the sequel, and we are in fact in a position to state and prove the main result of this section, which is the existence of a solution of the Yamabe problem provided that the Yamabe invariant does not attain the critical value $\lambda(\mathbb{S}^n, g_\circ)$.

Theorem 3.1.15 (Yamabe, Trudinger, Aubin). *Let (M, g) be a compact Riemannian manifold. Let $\{u_s\}$ be the sequence of normalised subcritical solutions constructed in Theorem 3.1.2. Then a subsequence converges uniformly to a minimizer $u \in C_+^\infty(M)$ of the Yamabe functional, i.e*

$$\mathcal{Q}_g(u) = \lambda(M, g), \quad \mathbb{D}\, u = \lambda(M, g) u^{2^*-1},$$

provided that $\lambda(M, g) < \lambda(\mathbb{S}^n, g_\circ)$. In particular, in this case the metric $u^{2^*-2} g \in \bar{g}$ has constant scalar curvature.

Proof. Without loss of generality we may assume that $\mu_g(M) = 1$. Let $\lambda = \lambda(M, g)$, $\Lambda = \lambda(\mathbb{S}^n, g_\circ)$ for brevity. The proof consists of two parts. In the first part we prove that the condition $\lambda < \Lambda$ implies that the sequence is uniformly bounded in $L^r(M)$ as $s \to 2^*$ for some $r > 2^*$, in view of Theorem 3.1.1. In the second part we use the first part in conjunction with the Arzela-Ascoli theorem and Lemma 3.1.3 to complete the proof.

1. Let $\delta > 0$. Multiplying the equation $\mathbb{D}\, u_s = \lambda_s u_s^{s-1}$ by $u_s^{1+2\delta}$ and integrating by parts we obtain

$$\int_M \rho \langle \operatorname{grad}_g u_s, (1+2\delta) u_s^{2\delta} \operatorname{grad}_g u_s \rangle \, d\mu_g + \int_M S u_s^{2(1+\delta)} \, d\mu_g$$
$$= \lambda_s \int_M u_s^{s+2\delta} \, d\mu_g.$$

Setting $v_s := u_s^{1+\delta}$, it follows that

$$\frac{1+2\delta}{(1+\delta^2)} \int_M \rho |\operatorname{grad}_g v_s|^2 \, d\mu_g = \int_M (\lambda_s v_s^2 u_s^{s-2} - S v_s^2) \, d\mu_g.$$

Then the Riemannian version of the Sobolev inequality (3.33) implies

$$\begin{aligned} \sigma_n \|v_s\|_{2^*}^2 &\le (1+\epsilon) \int_M |\operatorname{grad}_g v_s|^2 \, d\mu_g + C(\epsilon) \int_M v_s^2 \, d\mu_g \\ &\le (1+\epsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{\lambda_s}{\rho} \int_M v_s^2 u_s^{s-2} \, d\mu_g + C(\epsilon, \delta) \int_M v_s^2 \, d\mu_g, \end{aligned}$$

so by Hölder's inequality and the fact that $\Lambda = \rho\sigma_n$, it follows that

$$\|v_s\|_{2^*}^2 \le (1+\epsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{\lambda_s}{\Lambda} \|v_s\|_{2^*}^2 \|u_s\|_{(s-2)n/2}^{s-2} + C(\epsilon, \delta) \|v_s\|_2^2.$$

Since $\mu_g(M) = 1$ and $2 \le s < 2^*$, it follows that $(s-2)n/2 < s$ and $\|u_s\|_{(s-2)n/2} \le \|u_s\|_s = 1$, and consequently

$$\left(1 - (1+\epsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{\lambda_s}{\Lambda}\right) \|v_s\|_{2^*}^2 \le C(\epsilon, \delta) \|v_s\|_2^2.$$

For δ sufficiently small we have $2(1+\delta) < 2^*$, so

$$\|v_s\|_2^2 = \|u_s\|_{2(1+\delta)}^{2(1+\delta)} \le \|u_s\|_s^{2(1+\delta)} = 1$$

as $s \to 2^*$. In addition, we observe that $\|v_s\|_{2^*}^2 = \|u_s\|_{(1+\delta)2^*}^{2(1+\delta)}$, so if we can prove that the constant of the LHS can be manipulated to be positive by adjusting δ and ϵ, we are done.

We distinguish two cases. If $\lambda < 0$, Lemma 3.1.3 implies that $\lambda_s < 0$ for all s and the conclusion follows. If, on the other hand, we have $\lambda \ge 0$, $s \mapsto \lambda_s$ is non-increasing and continuous from the left, so there is an $s_0 < 2^*$ such that $0 \le \lambda_s \le \lambda_{s_0} < \Lambda$ whenever $s_0 \le s < 2^*$. Hence by choosing δ and ϵ small the constant remains positive.

2. Since the sequence $\{u_s\}$ is uniformly bounded in $L^r(M)$ for some $r > 2^*$ as $s \to 2^*$, by Theorem 3.1.1 it is also uniformly bounded in $C^{2,\alpha}(M)$ for $0 < \alpha < 1$. Then the Arzela-Ascoli theorem implies that a subsequence converges uniformly in $C^2(M)$ to a function $u \in C^2(M)$. Therefore u must satisfy
$$\mathbb{D}\, u = \lambda^* u^{2^*-1}, \quad \mathcal{Q}_g(u) = \lambda^*,$$
where $\lambda^* = \lim_{s\to 2^*} \lambda_s$ and the equation holds in the strong sense. If $\lambda \ge 0$, Lemma 3.1.3 implies that $\lambda^* = \lambda$. If $\lambda < 0$, by the same theorem $s \to \lambda_s$ is non-decreasing and hence $\lambda^* \le \lambda$. But since $\lambda = \inf \mathcal{Q}_g$, it follows that $\lambda^* = \lambda$ in that case as well.

   Finally, Hölder's inequality and Fatou's Lemma imply
$$\|u\|_{2^*} \ge \lim_{s\to 2^*} \|u_s\|_s = 1,$$
so in particular we have $u \neq 0$. Then we apply Theorem 3.1.1 one last time to conclude that $u \in C^\infty_+(M)$.

This completes the proof. □

It is an interesting fact that, among compact manifolds, the critical value $\lambda(\mathbb{S}^n, g_\circ)$ is actually only ever attained by the standard sphere, arguably the most perfectly symmetric of all geometric objects, and so the Yamabe problem possesses a solution for every compact manifold. This fact is anything but straightforward to prove, and will be explored in Section 3.2, using the complementary methods of Aubin and Schoen.

3.2 Chasing Yamabe invariants

3.2.1 The case $n \ge 6$ and $W \neq 0$

Following the same reasoning as in Theorem 3.1.13, Aubin was also able to prove that $\lambda(M, \mathbf{G}) < \lambda(\mathbb{S}^n, g_\circ)$ for all compact conformal manifolds $(M, \mathbf{G})$ that are not locally conformally flat and have dimension ≥ 6. This is, in essence, the easy case which can be obtained without extra effort directly from the framework we have developed so far. We have deliberately chosen to introduce conformal normal coordinates earlier in our presentation than Lee & Parker [20], as this simplifies several of the proofs in Section 3.1.5, and allows us to present this result without additional modifications.

Theorem 3.2.1. *Let $(M, \mathbf{G})$ be a compact conformal manifold of dimension ≥ 6, and suppose that the Weyl tensor is not identically zero. Then $\lambda(M, \mathbf{G}) < \lambda(\mathbb{S}^n, \mathbf{G}_\circ)$. In particular, there is a metric $g \in \mathbf{G}$ of constant scalar curvature.*

Proof. Let $\mathbf{p} \in M$ be such that $W(\mathbf{p}) \neq 0$, and $g \in \mathbf{G}$, (U, x) conformal normal coordinates with respect to g centered at $\mathbf{p}$ as in Theorem 3.1.11. Let $u_{a,\epsilon}$ be the test function constructed in the proof of Theorem 3.1.13. Then all we need to do is to examine the estimate (3.32) more carefully, and in particular the last term

$$\begin{aligned}\int_{B_{2\epsilon}(0)} Su_{a,\epsilon}^2\,dx &\leq \int_{B_\epsilon(0)} Su_a^2\,dx + C(\epsilon)\int_{R_{2\epsilon}(0)} u_a^2\,dx\\ &\leq \int_{B_\epsilon(0)} Su_a^2\,dx + C(\epsilon)a^{n-2}.\end{aligned}$$

The second term is of the same order as estimate (3.32), so it offers no additional advantage. But we are going to show that the first term is negative and of lower order as $a \to 0$, which is just what we need to dominate the positive contributions of higher order and bring the Yamabe quotient below the critical value.

Taylor expanding near $\mathbf{p}$ we see that

$$S = \frac{1}{2}\frac{\partial^2 S}{\partial x^i \partial x^j}(\mathbf{p})x^i x^j + O(|x|^3),$$

and furthermore[4]

$$\begin{aligned}\int_{B_\epsilon(0)} \frac{\partial^2 S}{\partial x^i \partial x^j}(\mathbf{p})x^i x^j u_a^2\,dx &= \int_0^\epsilon \int_{\mathbb{S}^{n-1}(r)} \frac{\partial^2 S}{\partial x^i \partial x^j}(\mathbf{p})x^i x^j u_a^2\,d\sigma_0 dr\\ &= C\int_{B_\epsilon(0)} \Delta S(\mathbf{p})|x|^2 u_a^2\,dx\\ &= -C|W(\mathbf{p})|^2\int_{B_\epsilon(0)} |x|^2 u_a^2\,dx,\end{aligned}$$

so in view of Lemma 3.1.9,

$$\mathcal{Q}_g(u_{a,\epsilon}) = \begin{cases} \Lambda - C(\epsilon)|W(\mathbf{p})|^2 a^4 + O_\epsilon(a^5), & n > 6\\ \Lambda - C(\epsilon)|W(\mathbf{p})|^2 a^4 \log 1/a + O_\epsilon(a^4), & n = 6\end{cases}$$

as $a \to 0$. This completes the proof. □

3.2.2 The case $n \in \{3, 4, 5\}$, or $n \geq 6$ and $W = 0$

The resolution of the remaining cases involve the construction of a global test function using the Green function of the conformal Laplacian. The resulting test metric has zero scalar curvature away from the blow-up area. First we recall the notion of a Green function.

Definition 3.2.1. Let (M, g) be a compact Riemannian manifold and D be a smooth linear differential operator in M. A *Green function* of D at $\mathbf{p} \in M$ is a

[4] Here we use the following fact: if A is a symmetric $n \times n$ matrix, then

$$\int_{\mathbb{S}^{n-1}} \langle Ax, x\rangle\,dx = C(n)\,\mathrm{tr}\,A.$$

function $\Gamma_{\mathbf{p}} \in C^{\infty}(M \setminus \mathbf{p})$ satisfying $D\Gamma_{\mathbf{p}} = \delta_{\mathbf{p}}$ in the sense of distributions, i.e

$$\int_M \Gamma_{\mathbf{p}} D^* u \, d\mu_g = u(\mathbf{p}) \quad \forall u \in C^{\infty}(M),$$

where D^* is the formal adjoint of D. Likewise, a *Green kernel* is a symmetric mapping $(\mathbf{p}, \mathbf{q}) \mapsto \Gamma(\mathbf{p}, \mathbf{q})$ that is C^{∞} off the diagonal and $\Gamma(\mathbf{p}, \cdot)$ is a Green kernel at $\mathbf{p}$.

The existence of Green functions and kernels for an operator D is invaluable in the study of non-homogeneous problems involving D. For example, it is straightforward to verify that the convolution

$$\Gamma * f(\mathbf{q}) = \int_M \Gamma(\mathbf{p}, \mathbf{q}) f(\mathbf{p}) d\mu_g(\mathbf{p}), \quad P \in M$$

is a solution of $Du = f$ in the sense of distributions, provided that the integral makes sense. We are particularly interested in the Green functions of the conformal Laplacian, so a natural starting point is the Laplace-Beltrami operator. In particular we have the following classical result, for a proof see Aubin [5].

Theorem 3.2.2 (Green kernel for $-\Delta$)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a Green kernel of $-\Delta$, such that for all $u \in C^2(M)$*

$$u(\mathbf{p}) = \frac{1}{\mu_g(M)} \int_M u(\mathbf{q}) \, d\mu_g(\mathbf{q}) - \int_M \Gamma(\mathbf{q}, \mathbf{p}) \Delta u(\mathbf{p}) \, d\mu_g(\mathbf{q}). \tag{3.34}$$

Moreover, Γ is bounded from below and satisfies the estimates

$$|\Gamma(\mathbf{p}, \mathbf{q})| \leq C d(\mathbf{p}, \mathbf{q})^{-(n-2)}, \quad |\nabla_{\mathbf{p}} \Gamma(\mathbf{p}, \mathbf{q})| \leq C d(\mathbf{p}, \mathbf{q})^{-(n-1)},$$

$$|\nabla^2_{\mathbf{p}} \Gamma(\mathbf{p}, \mathbf{q})| \leq C d(\mathbf{p}, \mathbf{q})^{-n}.$$

Since Γ is bounded from below, $\Gamma(\mathbf{p}, \mathbf{q}) > c$ for some $c \in \mathbb{R}$. It follows that $\Gamma - c$ is a positive Green kernel for $-\Delta$. The existence of a positive Green kernel is of central importance in problems involving the Laplacian. In fact, we can extend this result for the conformal Laplacian, provided that $S > 0$. Theorem 3.1.15 covers the case $\lambda(M, g) \leq 0$, so this is sufficient for our purposes.

In Section 3.1.3 we used stereographic projection to pull back the round metric $g_\circ$ of $\mathbb{S}^n$ to a metric of $\mathbb{R}^n$ that is conformal to the standard metric - namely $\chi^* g_\circ = u_1^{2^*-2} g_0$ - effectively transferring the spherical problem to Euclidean space, which has zero scalar curvature. It follows that

$$g_0 = \sigma^* u_1^{2-2^*} g_\circ,$$

so in view of the Yamabe equation the conformal factor $u_1^{2-2^*}$ must satisfy $\mathbb{D}\, u_1^{-1} = 0$. Moreover, note that the new metric is singular at the north pole $\mathbf{n}$. In fact, closer investigation reveals that u_1^{-1} is just a multiple of the Green function of $\mathbb{D}$ at $\mathbf{n}$.

This procedure is not restricted to $\mathbb{S}^n$. Given any compact Riemannian manifold (M, g) with $\lambda(M, g) > 0$ and a point $\mathbf{p} \in M$, constant multiples of the metric $\Gamma_{\mathbf{p}}^{2^*-2}g$ - with Γ_P being the Green function of $\mathbb{D}$ at $\mathbf{p}$ - all have zero scalar curvature in the non-compact manifold $M \setminus \mathbf{p}$. This motivates the following definition.

Definition 3.2.2 (Stereographic projection)**.** Let (M, g) be a compact Riemannian manifold such that $\lambda(M, g) > 0$. Moreover, let $\mathbf{p} \in M$ and $\Gamma_{\mathbf{p}}$ denote the Green function of $\mathbb{D}$ at $\mathbf{p}$. The *stereographic projection* of (M, g) from $\mathbf{p}$ is the natural map $\sigma : (M \setminus \mathbf{p}, g) \to (\hat{M}, \hat{g})$, where

$$\hat{M} = M \setminus \mathbf{p}, \quad \hat{g} = G_{\mathbf{p}}^{2^*-2}g, \quad G_P = \rho(n-2)\mu_\circ(\mathbb{S}^{n-1})\Gamma_{\mathbf{p}}. \tag{3.35}$$

So the image of a stereographic projection is always a non-compact manifold with zero scalar curvature. In fact, more is true: it is asymptotically flat. Before we give the definition of asymptotic flatness, let us fix some notation on asymptotic behaviour. For a tensor field T, we write $T = O^m(r^\tau)$ whenever

$$T = O(r^\tau), \quad \nabla^k T = O(r^{\tau-k}) \quad \forall k \in \{1, \ldots, m\}.$$

In particular, we write O' for O^1, O'' for O^2 and so forth.

Definition 3.2.3. A non-compact Riemannian manifold (M, g) is said to be *asymptotically flat* of order $\tau > 0$ if it admits a decomposition $M = M_0 \cup M_\infty$ where M_0 is compact and M_∞ is a neighbourhood of ∞ with the property that there exist $R > 0$ and coordinates $z : M_\infty \to \mathbb{R}^n \setminus B_R(0)$ such that

$$g = (1 + O''(|z|^{-\tau}))z^* g_0.$$

In that case, we call the coordinates z *asymptotic coordinates* of M of order τ.

To prove the asymptotic flatness of the stereographic projection, we need the following estimates for the Green function of the conformal Laplacian.

Theorem 3.2.3. *Let $G_{\mathbf{p}} = \rho(n-2)\mu_\circ(\mathbb{S}^{n-1})\Gamma_{\mathbf{p}}$ as in Definition 3.2.2, and let (U, x) be a conformal normal coordinate chart centered at $\mathbf{p}$. Then $G_{\mathbf{p}}$ possesses the asymptotic expansion*

$$G_{\mathbf{p}} = \frac{1}{|x|^{n-2}}\left(1 + \sum_{k=4}^{n} \varphi_k(x)\right) + c \log|x| + O''(1), \tag{3.36}$$

where $\varphi_k \in \mathcal{P}_k[x]$ for $k = 4, \ldots, n$ and c is a constant that can be taken to be 0 if n is even.

In particular, if $n = 3, 4,$ or 5 or if (M, g) is locally conformally flat at $\mathbf{p}$, the asymptotic expansion is

$$G_{\mathbf{p}} = \frac{1}{|x|^{n-2}} + \mathfrak{m} + O''(|x|), \tag{3.37}$$

where $\mathfrak{m}$ is a constant.

Sketch of proof. Switching to conformal normal coordinates (U, x) centered at $\mathbf{p}$ and then passing to polar coordinates (r, ξ) with $r = |x|$, we have

$$g = dr \otimes dr + \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta} d\xi^\alpha \otimes d\xi^\beta,$$

where $g_{\alpha\beta} := g(\partial_{\xi^\alpha}, \partial_{\xi^\beta})$. In these coordinates the Riemannian volume form is $\omega_g = dx = r^{n-1} dr \wedge d\xi$, thus the Laplacian has the form

$$\Delta_g = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \sum_{\alpha,\beta=1}^{n-1} \frac{\partial}{\partial \xi^\alpha} g^{\alpha\beta} \frac{\partial}{\partial \xi^\beta}.$$

In view of this formula, it is evident that the Laplacian Δ_g acts on radial functions centered at $\mathbf{p}$ the same way as the Euclidean Laplacian Δ_0. Moreover, since $g_{\alpha\beta} = O''(r^2)$, it follows that $g^{\alpha\beta} = O''(r^{-2})$.

Recalling the fact that $\Delta_0 r^{2-n} = (n-2)\mu_\circ(\mathbb{S}^{n-1})\delta_{\mathbf{p}}$ and since $S = O''(r^2)$, straightforward calculation reveals that

$$\mathbb{D}(G_{\mathbf{p}} - r^{2-n}) = O''(r^{4-n}), \tag{3.38}$$

so the problem now is to examine the remainder term. If $n \in \{3, 4, 5\}$ the remainder is $O(r^{-1})$. Since $r^{-1} \in L^q(U)$ for $q < n$, by local elliptic regularity we have that $G_{\mathbf{p}} - r^{2-n} \in W^{2,q}(U)$, and then the Sobolev embedding theorem implies that $G_{\mathbf{p}} - r^{2-n} \in C^{0,\alpha}(U)$ for $0 < \alpha < 6 - n$, and the expansion (3.37) follows. If (M, g) is locally conformally flat at $\mathbf{p}$, things are even simpler since we can choose $g_{ij} = \delta_{ij}$, and consequently $\mathbb{D}(G_{\mathbf{p}} - r^{2-n}) = 0$ and $G_{\mathbf{p}} - r^{2-n} \in C^\infty(U)$, as a consequence of local elliptic regularity.

The general expansion (3.36) requires a closer investigation of the remainder, and we omit it as it will not be needed in the sequel, for more details see Lee & Parker [20] or Schoen & Yau [23]. □

A fact of key importance for the resolution of the remaining cases of the Yamabe problem is that the constant $\mathfrak{m}$ in the above expansion is non-negative and becomes zero if and only if (M, g) is conformally equivalent to the standard sphere $(\mathbb{S}^n, g_\circ)$. We have chosen the symbol $\mathfrak{m}$ since this quantity turns out to be closely related to the concept of mass in general relativity, and the positivity of $\mathfrak{m}$ is then a consequence of the positive mass theorem, for a summary see Lee & Parker [20].

For the time being let us investigate the asymptotic behavior of the stereographic projection $(\hat{M}, \hat{g})$. To this end, we introduce *inverted conformal normal coordinates* as follows. Let $\mathbf{p} \in M$ and let (U, x) be conformal normal coordinates centered at $\mathbf{p}$. Putting $z = x/|x|^2$ on $U \setminus \mathbf{p}$, the induced vector fields are

$$\frac{\partial}{\partial z^i} = \frac{1}{|z|^2} \left(\delta_{ij} - \frac{2 z^i z^j}{|z|^2} \right) \frac{\partial}{\partial x^j}.$$

$U \setminus \mathbf{p}$ will play the role of the neighborhood of infinity of $\hat{M}$ that we are looking for. Setting $\gamma_{\mathbf{p}} = |x|^{n-2}G_{\mathbf{p}} = 1 + O(|x|)$, in the above coordinates we evaluate

$$\begin{aligned}
\hat{g} &= G_{\mathbf{p}}^{2^*-2} g \\
&= \gamma_{\mathbf{p}}^{2^*-2} |z|^4 g(\partial_{z^i}, \partial_{z^j}) dz^i \otimes dz^j \\
&= \gamma_{\mathbf{p}}^{2^*-2} (\delta_{ik} - \frac{2}{|z|^2} z^i z^k)(\delta_{jl} - \frac{2}{|z|^2} z^j z^l) g(\partial_{x^k}, \partial_{x^l}) dz^i \otimes dz^j \\
&= \gamma_{\mathbf{p}}^{2^*-2} (1 + O''(|z|^2))(dz^1 \otimes dz^1 + \cdots + dz^n \otimes dz^n),
\end{aligned}$$

where we have used the fact that $g(\partial_{x^k}, \partial_{x^l}) = \delta_{kl} + O''(|x|^2)$ in normal coordinates. In addition, if (M, g) is locally conformally flat at $\mathbf{p}$, the we may take $g(\partial_{x^k}, \partial_{x^l}) = \delta_{kl}$ and the $O''(|z|^{-2})$ term is redundant. Combining this with the assymptotic expansion of $\gamma_{\mathbf{p}}$ in view of Theorem 3.2.3, we conclude that:

Corollary 3.2.4. *Let $(\hat{M}, \hat{g})$ be the stereographic projection of the Riemannian manifold (M, g) from $\mathbf{p} \in M$. Then, in inverted conformal normal coordinates $z = x/|x|^2$ in a punctured neighbourhood of $\mathbf{p}$, $\hat{g}$ has the assymptotic expansion*

$$\hat{g} = \gamma_{\mathbf{p}}^{2^*-2} (1 + O''(|z|^{-2}))(dz^1 \otimes dz^1 + \cdots + dz^n \otimes dz^n), \tag{3.39}$$

where $\gamma_{\mathbf{p}} = |x|^{n-2} G_{\mathbf{p}}$.

In particular, if $n = 3, 4$ or 5, or (M, g) is locally conformally flat at $\mathbf{p}$, $\hat{g}$ has the assymptotic expansion

$$\hat{g} = (1 + O''(|z|^{2-n}))(dz^1 \otimes dz^1 + \cdots + dz^n \otimes dz^n),$$

and is thus aymptotically flat of order $n - 2$.

We are now in a position to give the proof of the solubility of the Yamabe problem in the remaining cases.

Theorem 3.2.5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, not conformally equivalent to $(\mathbb{S}^n, g_\circ)$. If $n = 3, 4$ or 5 or if (M, g) is locally conformally flat, then $\lambda(M, g) < \lambda(\mathbb{S}^n, g_\circ)$. In particular, the Yamabe problem possesses a solution in those cases.*

Proof. The main idea of the proof is to construct a global test function with zero scalar curvature away from the reference point $\mathbf{p}$ using the stereographic projection. We examine the case where $n \geq 6$ and (M, g) is locally conformally flat at $\mathbf{p}$. In this case, we may choose flat conformal normal coordinates (U, x) in a neighbourhood of $\mathbf{p}$, i.e $g_{ij} = \delta_{ij}$, while the Green function with pole at $\mathbf{p}$ has the expansion

$$G_{\mathbf{p}} = \frac{1}{|x|^{n-2}} + \mathfrak{m} + \alpha(x),$$

where $\alpha(x) = O''(|x|)$ as $x \to 0$ and $\mathfrak{m} > 0$ by the positive mass theorem.

Recall the minimizers of the Sobolev quotient u_a which we introduced in Section 3.1.3, as well as the test functions $u_{a,\epsilon} = \eta_\epsilon u_a$ used in the proof of Theorem 3.2.1. We modify these local test functions into global ones, in the spirit outlined above, to be

$$\tilde{u}_{a,\epsilon} = \begin{cases} u_a(x), & |x| \leq \epsilon \\ a_0(G_{\mathbf{p}}(x) - \eta_\epsilon \alpha(x)), & \epsilon \leq |x| \leq 2\epsilon \\ a_0 G_{\mathbf{p}}(x), & |x| \geq 2\epsilon \end{cases},$$

where $a_0 = a_0(a, \epsilon)$ is chosen so that $\tilde{u}_{a,\epsilon}$ is a Lipschitz continuous function in M, and therefore $\tilde{u}_{a,\epsilon} \in W^{1,2}(M)$. In particular, a_0 is given by

$$a_0 = (\epsilon^{2-n} + \mathfrak{m})^{-1} \left(\frac{a}{a^2 + \epsilon^2} \right)^{\frac{n-2}{2}} = O_\epsilon(a^{\frac{n-2}{2}}).$$

We proceed with estimating the energy of $\tilde{u}_{a,\epsilon}$. We break this estimate into two parts, the internal one in $B_\epsilon = \{|x| < \epsilon\}$ and the external one in the exterior of B_ϵ.

For the exterior estimate we obtain

$$\begin{aligned}
\mathcal{E}_{\text{ext}}(\tilde{u}_{a,\epsilon}) &= \int_{M \setminus B_\epsilon} (\rho |\operatorname{grad} \tilde{u}_{a,\epsilon}|^2 + S \tilde{u}_{a,\epsilon}^2)\, d\mu_g \\
&= \int_{M \setminus B_{2\epsilon}} a_0^2 (\rho |\operatorname{grad} G_{\mathbf{p}}|^2 + S G_{\mathbf{p}}^2)\, d\mu_g \\
&\quad + \int_{B_\epsilon \setminus B_{2\epsilon}} a_0^2 (\rho |\operatorname{grad}(G_P - \eta_\epsilon \alpha)|^2 + S(G_{\mathbf{p}} - \eta_\epsilon \alpha))\, d\mu_g \\
&= \int_{M \setminus B_\epsilon} a_0^2 (\rho |\operatorname{grad} G_{\mathbf{p}}|^2 + S G_{\mathbf{p}}^2)\, d\mu_g \\
&\quad + \int_{B_\epsilon \setminus B_{2\epsilon}} a_0^2 \rho (|\operatorname{grad}(\eta_\epsilon \alpha)|^2 - 2 \langle \operatorname{grad} G_{\mathbf{p}}, \operatorname{grad}(\eta_\epsilon \alpha) \rangle)\, d\mu_g.
\end{aligned}$$

Since $\alpha = O''(|x|)$ and $|\operatorname{grad} G_{\mathbf{p}}| \leq C|x|^{1-n}$, it follows that the second term is $\leq C\epsilon a_0^2$. As for the first term, integrating by parts and taking into account that $\mathbb{D}\, G_{\mathbf{p}} = -\rho \Delta G_{\mathbf{p}} + S G_{\mathbf{p}} = 0$ away from P, we obtain

$$\int_{M \setminus B_\epsilon} a_0^2 (\rho |\operatorname{grad} G_P|^2 + S G_P^2)\, d\mu_g = a_0^2 \int_{\partial B_\epsilon} G_{\mathbf{p}} \frac{\partial G_{\mathbf{p}}}{\partial \nu}\, d\sigma_g.$$

Regarding the interior estimate, we proceed much like the case of the local test function in the proof of Theorem 3.2.1. Since $S = 0$ in B_ϵ, it follows that

$$\begin{aligned}
\mathcal{E}_{\text{int}}(\tilde{u}_{a,\epsilon}) &= \int_{B_\epsilon} (\rho |\operatorname{grad} \tilde{u}_{a\epsilon}|^2 + S \tilde{u}_{a,\epsilon}^2)\, d\mu_g \\
&= \int_{B_\epsilon} \rho |\operatorname{grad} u_a|^2\, dx \\
&= n(n-2)\rho \int_{B_\epsilon} u_a^{2^*}\, dx + \rho \int_{\partial B_\epsilon} u_a \frac{\partial u_a}{\partial \nu}\, d\sigma_0 \\
&\leq n(n-2)\rho \left(\int_{B_\epsilon} u_a^{2^*}\, dx \right)^{\frac{2}{n}} \left(\int_{B_\epsilon} u_a^{2^*}\, dx \right)^{\frac{2}{2^*}} + \rho \int_{\partial B_\epsilon} u_a \frac{\partial u_a}{\partial \nu}\, d\sigma_0
\end{aligned}$$

$$\leq \lambda(\mathbb{S}^n, g_\circ)\bigg(\int_M \tilde{u}_{a,\epsilon}^{2^*}\,dx\bigg)^{\frac{2}{2^*}} + \rho\int_{\partial B_\epsilon} u_a \frac{\partial u_a}{\partial \nu}\,d\sigma_0.$$

In addition, similar to the proof of Theorem 3.2.1, for $a \leq \epsilon \leq 1$ we have that

$$\int_M \tilde{u}_{a,\epsilon}^{2^*}\,d\mu_g \geq \int_{B_\epsilon} u_\epsilon^{2^*}\,dx \geq C(\epsilon).$$

We now turn our attention to the boundary terms on ∂B_ϵ. For $|x| = \epsilon$, by straightforward calculation we see that

$$\begin{aligned} G_{\mathbf{p}}\frac{\partial G_{\mathbf{p}}}{\partial \nu} &= -(n-2)a_0^2(\epsilon^{3-2n} + \mathfrak{m}\epsilon^{1-n}) + O(\epsilon^{2-n}),\\ u_a\frac{\partial u_a}{\partial \nu} &= -(n-2)a_0^2(\epsilon^{3-2n} + 2\mathfrak{m}\epsilon^{1-n}) + O(\epsilon^{2-n}). \end{aligned}$$

Finally, it follows that

$$\mathcal{E}(\tilde{u}_{a,\epsilon}) \leq \lambda(\mathbb{S}^n, g_\circ)\|\tilde{u}_{a,\epsilon}\|_{2^*}^2 + C\epsilon a_0^2 + \int_{\partial B_\epsilon}\bigg(u_a\frac{\partial u_a}{\partial \nu} - G_{\mathbf{p}}\frac{\partial G_{\mathbf{p}}}{\partial \nu}\bigg)\,d\sigma_0,$$

and consequently

$$\mathcal{Q}_g(\tilde{u}_{a,\epsilon}) \leq \lambda(\mathbb{S}^n, g_\circ) - C(\epsilon)(\mathfrak{m} - \epsilon)a^{n-2} + o_\epsilon(a^{n-2}).$$

Choosing first $\epsilon < \mathfrak{m}$ and then $a < \epsilon$ small enough yields the conclusion.

The case $n \in \{3, 4, 5\}$ is very similar. The expansion of the Green function is identical, and the main difference is that since no local conformal flatness is assumed, in conformal normal coordinates we instead have $g_{ij} = \delta_{ij} + O(|x|^2)$ and $S = O(|x|^2)$. These changes only contribute higher order terms which are of no consequence to the conclusion. □

This completes the solution of the Yamabe problem. ■

Chapter 4

The spinorial Yamabe problem

4.1 Introduction

Let (M, g, σ) be a compact Riemannian spin manifold of dimension $n \geq 2$. Then for (M, g, σ) there is an associated, canonically defined Dirac operator $\mathfrak{D}_g : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$, i.e a first-order, self-adjoint elliptic operator whose square is of Laplace type, acting on sections of the associated spinor bundle ΣM (spinor fields). $\mathfrak{D}_g$ possesses a discrete real spectrum $\{\lambda_n^{\pm}\}_{n=0}^{\infty}$ of the form

$$-\infty \leftarrow \lambda_n^- < \cdots \lambda_1^- < 0 = \cdots = 0 < \lambda_1^+ < \cdots < \lambda_n^+ \to +\infty,$$

while the multiplicity of the zero eigenvalue, i.e the dimension of $\ker \mathfrak{D}$, is known to be a conformal invariant. The eigenvalues are of course dependent on the metric g.

A problem which is directly related to the Yamabe problem is to minimize (resp. maximize) the value of the first positive eigenvalue $\lambda_1^+(\mathfrak{D}_g)$ (resp. the first negative eigenvalue $\lambda_1^-(\mathfrak{D}_g)$) whithin a given conformal class $\mathbf{G}$. To see the connection, note that in 1986 Hijazi was able to prove that for $n \geq 3$ the lower bound

$$|\lambda_1^{\pm}(\mathfrak{D}_g)|^2 \geq \frac{n}{4(n-1)} \lambda_1(\mathbb{D}_g)$$

holds, where $\lambda_1(\mathbb{D}_g)$ is the first eigenvalue of the conformal Laplacian $\mathbb{D}_g = -\rho\Delta_g + S_g$. On the other hand, the definition of the Yamabe invariant implies that

$$\lambda_1(\mathbb{D}_g)\mu_g(M)^{2/n} \geq \lambda(M, g),$$

so combining these we obtain the lower bound

$$|\lambda_1^{\pm}(\mathfrak{D}_g)|\mu_g(M)^{1/n} \geq \sqrt{\frac{n}{4(n-1)} \lambda(M, g)}$$

provided that the Yamabe invariant is positive. From our discussion so far it is obvious that the right hand side is conformally invariant, so this suggests that the quantity in the left hand side is the appropriate one to study in the setting of conformal geometry. Note that in this context one can choose $\mu_g(M) = 1$, so minimizing this quantity with that assumption also minimizes $|\lambda_1^{\pm}(\mathfrak{D}_g)|$. With this in mind, set

$$\lambda_{\min}^+(M, \mathbf{G}, \sigma) := \inf_{g \in \mathbf{G}} \lambda_1^+(\mathfrak{D}_g)\mu_g(M)^{1/n}. \tag{4.1}$$

Studying this functional, i.e the existence and regularity of minimizers, is known as the SPINORIAL YAMABE PROBLEM, and it can be though of as a first-order analogue of the classical Yamabe problem.

Indeed, the similarities are many. Note that by definition we have $\lambda^+_{\min} \geq 0$. It can actually be shown that

$$\lambda^+_{\min}(M, \mathbf{G}, \sigma) > 0,$$

and just as with the Yamabe problem,

$$\lambda^+_{\min}(M, \mathbf{G}, \sigma) \leq \lambda^+_{\min}(\mathbb{S}^n, g_\circ, \sigma_\circ).$$

Moreover, the infimum is attained within a generalised conformal class enlarged by some singular metrics provided that

$$\lambda^+_{\min}(M, \mathbf{G}, \sigma) < \lambda^+_{\min}(\mathbb{S}^n, g_\circ, \sigma_\circ).$$

These results were established by Ammann [4, 2, 3], on whom our presentation is based.

4.2 A rough outline of spin geometry

While a complete account of the classical theory of spinors is well beyond the scope and time restrictions of this thesis, we offer a review of the basic notions with the hope that the reader will feel comfortable with what will follow, or that they would perhaps feel motivated to study the material in further detail themselves. The lecture notes by Bär [6], by which this outline was inspired, is an excellent place to continue; see also the notes by Hijazi [15]. For a more comprehensive treatment, there is also the standard reference of Lawson & Michelsohn [18].

Clifford algebras and the spin group. Let V be a vector space of dimension n, equipped with a symmetric, non-degenerate bilinear form g. The *Clifford algebra* of (V, g) is the quotient

$$\mathrm{Cl}(V, g) := T^0V/I(V, g),$$

where

$$T^0V := \bigoplus_{k=0}^{\infty} T^0_k V, \quad I(V, g) := \langle v \otimes v + g(v, v)1 : v \in V\rangle.$$

Note that by the polarisation identity the quotient relation implies the relation

$$v \otimes w + w \otimes v = -2g(v, w)1$$

in $\mathrm{Cl}(V, g)$. The algebra multiplication $\otimes$ will instead be denoted by $\cdot$ within the Clifford algebra. Moreover, if $\{e_1, \ldots, e_n\}$ is a g-orthonormal basis, a basis of $\mathrm{Cl}(V,g)$ is given by

$$\{e_1^{a_1}\cdots e_n^{a_n} : a_1,\dots,a_n \in \{0,1\}\},$$

which implies that $\dim \mathrm{Cl}(V,g) = 2^n$. By lifting the antipodal map $-\mathrm{Id}_V$, we obtain a natural $\mathbb{Z}_2$-grading

$$\mathrm{Cl}(V,g) = \mathrm{Cl}^0(V,g) \oplus \mathrm{Cl}^1(V,g),$$

where the direct summands are the even and odd elements respectively. Note that $e_i^2 = -1$, so $\mathrm{Cl}(n)$ can be thought of a generalisation of the complex numbers in higher dimensions. In fact, $\mathrm{Cl}(1) \cong \mathbb{C}$ and $\mathrm{Cl}(2) \cong \mathbb{H}$.

The standard Clifford algebra in n elements is $\mathrm{Cl}(n) := \mathrm{Cl}(\mathbb{R}^n, g_0)$, which is generated by the standard orthonormal basis of $\mathbb{R}^n$, together with the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}.$$

Then for $v \in \mathbb{R}^n \setminus 0$, the Clifford relation implies $v^2 = -|v|^2 1$, so in particular $\mathbb{R}^n \setminus 0 \subset \mathrm{Cl}^\times(n)$. Moreover, if $v \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, then $v^{-1} = -v$. The *spin group* in n elements is the multiplicative subgroup of even spherical elements, i.e

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2m} : v_i \in \mathbb{S}^{n-1}, m \in \mathbb{N}_0\}.$$

Straightforward calculation reveals that if $v \in \mathbb{S}^{n-1}$, then $v \cdot w \cdot v^{-1} = -(w - 2\langle v, w\rangle v)$ is a reflection, hence the adjoint map $\mathrm{Ad}_v(w) = v \cdot w \cdot v^{-1}$ is the opposite of a reflection along the hyperplane $v^\perp$. As a matter of fact $\mathrm{Ad} : \mathrm{Spin}(n) \to SO(n)$ is a double covering, i.e we have the following short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\mathrm{Ad}} SO(n) \to 1.$$

In the words of Michael Atiyah, *spin geometry is the square root of geometry*, which is a reflection of just this fact. $\mathrm{Spin}(n)$, like $SO(n)$, is a Lie group of the same dimension, which is

$$\frac{1}{2}n(n-1).$$

For example, $\mathrm{Spin}(1) \cong \mathbb{Z}_2$ and $\mathrm{Spin}(2) \cong U(1)$.

Spinors and spin structures. Let $\mathbb{C}\mathrm{l}(n) := \mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the real Clifford algebra $\mathrm{Cl}(n)$. For even $n = 2m$, the standard basis of $\mathbb{C}^{2m}$ will be denoted $\{e_1, \tilde{e}_1, \dots, e_m, \tilde{e}_m\}$. We switch to the complex basis

$$z_i := \frac{1}{2}(e_i - i\tilde{e}_i), \quad \bar{z}_i := \frac{1}{2}(e_i + i\tilde{e}_i)$$

for $i = 1, \dots, m$. Putting $\bar{\omega} := \bar{z}_1 \cdots \bar{z}_m$, let

$$\Sigma_{2m} = \mathrm{span}\{z_1^{a_1} \cdots z_m^{a_m} \cdot \bar{\omega} : a_1, \dots a_m \in \{0,1\}\} \cong \mathbb{C}^{2^m}.$$

Then we have a representation $\mathbb{C}l(2m) \to \mathrm{End}(\Sigma_{2m})$ obtained by Clifford multiplication (which is actually a complex algebra isomorphism), as well as a decomposition of $\Sigma_{2m} = \Sigma_{2m}^+ \oplus \Sigma_{2m}^-$ into positive and negative *chirality* subspaces. The restriction $\sigma_{2m} : \mathrm{Spin}(2m) \to GL(\Sigma_{2m})$ is called the *spinor representation.* Note that σ_{2m} is chirality-preserving and hence not irreducible. Owing to this we obtain two sub-representations $\sigma_{2m}^\pm : \mathrm{Spin}(2m) \to GL(\Sigma_{2m}^\pm)$. The properties of the Clifford multiplication imply that it is skew-symmetric with respect to vector multiplication, i.e

$$\langle v \cdot \varphi, \psi \rangle = -\langle \varphi, v \cdot \psi \rangle, \quad v \in \mathbb{R}^{2m},\ \varphi, \psi \in \Sigma_{2m}.$$

Subsequently this implies that if $v \in \mathbb{S}^{2m-1}$, $\varphi \mapsto v \cdot \varphi$ is unitary, and hence σ_{2m} is a unitary representation.

For $n = 2m-1$, note first that the map $\mathbb{R}^n \to \mathrm{Cl}^0(2m)$, $v \mapsto v \cdot e_{2m}$ induces an algebra isomorphism $\mathrm{Cl}(2m-1) \cong \mathrm{Cl}^0(2m)$. We define the spinor space $\Sigma_{2m-1} := \Sigma_{2m}^+$, on which $\mathrm{Cl}(2m-1)$ acts with via the aforementioned identification. Restricting on the spin group we obtain the representation $\sigma_{2m-1} : \mathrm{Spin}(2m-1) \to GL(\Sigma_{2m-1})$, which is again unitary.

So, to summarise, we have constructed unitary representations $\sigma_n : \mathrm{Spin}(n) \to U(\Sigma_n)$ of dimension $\dim \Sigma_n = 2^{[n/2]}$, where $[\cdot]$ denotes the integer part. The elements of Σ_n are called *spinors*, and elements of the spin group may be identified with their unitary action on spinors.

Finally, a *spin structure* on a Riemannian manifold (M, g) of dimension n is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(M)$ which is a double cover $\sigma : \mathrm{Spin}(M) \to SO(M)$ of the orthonormal frame bundle $SO(M)$ that is equivariant with respect to the double covering $\mathrm{Ad} : \mathrm{Spin}(n) \to SO(n)$ outlined in the previous paragraph. A *Riemannian spin manifold* is then a triple (M, g, σ). The *spinor bundle* of (M, g, σ) is defined to be the associated vector bundle $\Sigma M := \mathrm{Spin}(n) \times_{\sigma_n} \Sigma_n$, on which the spin group acts via the unitary representation. It caries a natural Hermitian metric $\langle \cdot, \cdot \rangle_g$ inherited by g; the pair will be denoted by $(\Sigma M, g)$. Sections of ΣM are called *spinor fields.* It is worth noting here that not every manifold can admit a spin structure, and if it does, it need not be unique.

The classical Dirac operator. For each Riemannian spin manifold (M, g, σ) there is an associated canonically defined connection $\nabla^\Sigma : \Gamma(\Sigma M) \to \Gamma(T^*M \otimes \Sigma M)$ which is *metric* with respect to the standard spinor Hermitian form, i.e

$$X\langle \varphi, \psi \rangle = \langle \nabla_X^\Sigma \varphi, \psi \rangle + \langle \varphi, \nabla_X^\Sigma \psi \rangle,$$

and which is also compatible with the Levi-Civita connection ∇ of (M, g), i.e

$$\nabla_X^\Sigma (Y \cdot \psi) = \nabla_X Y \cdot \psi + Y \cdot \nabla_X^\Sigma \psi.$$

The superscript $\cdot^\Sigma$ will be dropped from now on when there is no risk of confusion. The *classical Dirac operator* of (M, g, σ), $\mathfrak{D}_g : \Gamma(\Sigma M) \to \Gamma(\Sigma M)$ is then given by the composition

$$\Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma M) \xrightarrow{\sharp} \Gamma(TM \otimes \Sigma M) \xrightarrow{\text{Cliff.}} \Gamma(\Sigma M),$$

where $\sharp = \sharp_g$ is the musical isomorphism and the last mapping is Clifford multiplication. Choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$, we have the local expression

$$\mathfrak{D}_g \psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \psi.$$

$\mathfrak{D}_g$ is a first-order elliptic self-adjoint differential operator, whose square is Laplace type (i.e has principal symbol $\sigma_2(\mathfrak{D}_g^2)(\xi) = -|\xi|^2$). It possesses a real, discrete, symmetric with respect to zero spectrum, converging to $\pm\infty$. By standard elliptic theory, each eigenvalue has finite multiplicity.

We now examine the regularity of solutions of the Dirac equation $\mathfrak{D}_g \varphi = \psi$.

Theorem 4.2.1 (Elliptic regularity for the Dirac operator)**.** *Let (M, g, σ) be a compact Riemannian spin manifold, and suppose that $\psi \in W^{k,q}(\Sigma M)$. If $\varphi \in W^{1,1}(\Sigma M)$ is a weak solution of $\mathfrak{D}_g \varphi = \psi$, then $\varphi \in W^{k+1,q}(\Sigma M)$ and there is a positive constant $C = C(M, g, \sigma)$ such that*

$$\|\varphi\|_{W^{k+1,q}(\Sigma M)} \le C(\|\psi\|_{W^{k,q}(\Sigma M)} + \|\pi_{\ker \mathfrak{D}_g}(\varphi)\|_{L^q(\Sigma M)}),$$

where $\pi_{\ker \mathfrak{D}_g}$ is the L^2-orthogonal projection onto the kernel of $\mathfrak{D}_g$.

The $C^{k,\alpha}$-version of this is entirely analogous.

4.3 The conformally invariant functional

Like the Yamabe problem, the spinorial Yamabe problem admits a variational formulation. Constructing the conformaly invariant functional is not straightforward; although the Dirac operator is, in a sense to be described, conformally invariant, its square is not, and this indicates that we should abandon the idea of a quadratic form, as was the case with the Yamabe problem which was second-order. Moreover, we are interested in a functional which is bounded, since we need to establish a relationship between extremals of the functional and minimizers of $\lambda^+_{\min}(M, g, \sigma)$. First we note the following.

Theorem 4.3.1 (Conformal transformation formula for the Dirac Operator)**.** *Let $\tilde{g} = u^2 g$ for some $u \in C^\infty_+(M)$. Then there is an isomorphism of vector bundles $F : (\Sigma M, g) \to (\Sigma M, \tilde{g})$ which is a fiber-wise isometry, such that*

$$\mathfrak{D}_{\tilde{g}} F(\psi) = F(u^{-(n+1)/2} \mathfrak{D}_g u^{(n-1)/2} \psi). \tag{4.2}$$

For a self-contained proof see Hijazi [15], note however that the result is originally due to Hitchin. It is convenient to define

$$\tilde{\psi} := F(u^{-(n-1)/2} \psi),$$

and implicitly make the identification $(\psi, g) \equiv (\tilde{\psi}, \tilde{g})$. Then it is a matter of straightforward calculations to show that

$$\mathfrak{D}_{\tilde{g}}\tilde{\psi} = u^{-1}\widetilde{\mathfrak{D}_g\psi},$$

and moreover

$$\langle\tilde{\psi}, \tilde{\varphi}\rangle_{\tilde{g}} = u^{1-n}\langle\psi, \varphi\rangle_g, \quad \omega_{\tilde{g}} = u^n\omega_g.$$

This implies that the top-form

$$\langle\mathfrak{D}_g\psi, \psi\rangle_g\,\omega_g$$

is conformally invariant (with the above identification in mind). It is also straightforward to verify that the quantities $\|\psi\|_{L^p(\Sigma M,g)}$ and $\|\mathfrak{D}_g\psi\|_{L^q(\Sigma M,g)}$ are conformally invariant if and only if $p = 2^\star := 2n/(n-1)$ and $q = 2^\dagger := 2n/(n+1)$ respectively[1].

While it might be tempting to define a functional using the simpler quantity $\|\cdot\|_{L^{2^\star}(\Sigma M,g)}$, it turns out that choosing the seminorm $\|\mathfrak{D}_g\cdot\|_{L^{2^\dagger}(\Sigma M,g)}$ offers more freedom and leads to Euler-Lagrange equations that can be more easily manipulated for our purposes. For reasons that are similar to the ones that occurred when treating the classical Yamabe problem, for $q \in [2^\dagger, 2]$ we define the family of functionals $\mathcal{Q}_g^q : W^{1,q}(\Sigma M) \setminus \ker\mathfrak{D}_g \to \mathbb{R}$ given by the quotient

$$\mathcal{Q}_g^q(\psi) := \frac{\int_M\langle\mathfrak{D}_g\psi, \psi\rangle_g d\mu_g}{\|\mathfrak{D}_g\psi\|^2_{L^q(\Sigma M,g)}}. \tag{4.3}$$

To check that this is well-defined, let p be the Hölder-conjugate of q, i.e $1/p + 1/q = 1$. Because of the embedding $W^{1,q} \hookrightarrow L^{q^*} \hookrightarrow L^p$, the nominator is integrable by Hölder's inequality, and the denominator is positive and finite by definition. Moreover, the self-adjointness of $\mathfrak{D}_g$ ensures that $\mathcal{Q}_g^q$ is real.

Now let $\psi \in W^{1,q}(\Sigma M) \setminus \ker\mathfrak{D}_g$. Since $\mathcal{Q}_g^q(\psi + \psi_{\text{ker}}) = \mathcal{Q}_g^q(\psi)$ for any $\psi_{\text{ker}} \in \ker\mathfrak{D}_g$, we may assume without loss of generality that $\psi \in (\ker\mathfrak{D}_g)^\perp$. Then Theorem 4.2.1 implies $\|\psi\|_{W^{1,q}(\Sigma M,g)} \leq C\|\mathfrak{D}_g\psi\|_{L^q(\Sigma M,g)}$, and therefore

$$|\mathcal{Q}_g^q(\psi)| \leq \frac{\|\mathfrak{D}_g\psi\|_{L^q(\Sigma M,g)}\|\psi\|_{L^p(\Sigma M,g)}}{\|\mathfrak{D}_g\psi\|^2_{L^q(\Sigma M,g)}} \leq C,$$

so $\mathcal{Q}_g^q$ is bounded. Let

$$\lambda_g^q := \sup\mathcal{Q}_g^q.$$

Chosing an eigenspinor ψ to a positive eigenvalue of $\mathfrak{D}_g$ as a test spinor, we readily see that $\lambda_g^q \geq \mathcal{Q}_g^q(\psi) > 0$.

Given a conformal class $\mathbf{G}$, there is a closely related functional outlined in Section 4.1, namely $\mathcal{J} : \mathbf{G} \to \mathbb{R}$,

$$\mathcal{J}(g) := \lambda_1^+(\mathfrak{D}_g)\mu_g(M)^{1/n}, \quad \lambda_{\min}^+(M, \mathbf{G}, \sigma) := \inf\mathcal{J}. \tag{4.4}$$

[1] Note that $2^\star$ is the *fractional* Sobolev exponent of order $s = 1/2$; in general it is $p^\star := np/(n-sp)$. $2^\dagger$ is then the Hölder-conjugate of $2^\star$

The relationship between quantities λ_g^q and $\lambda_{\min}^+(M,\bar{g},\sigma)$ will be given shortly; for the time being let us first give some properties of the values λ_g^q.

Lemma 4.3.2 (Properties of λ_g^q)**.** *The function $q \mapsto \lambda_g^q$, $q \in [2^\dagger, \infty)$ is continuous from the right and*

$$\lambda_g^2 = \frac{1}{\lambda_1^+(\mathfrak{D}_g)}.$$

If, moreover, $\mu_g(M) = 1$, then $q \mapsto \lambda_g^q$ is non-increasing.

Proof. Continuity and monotonicity are proven in a manner which is entirely analogous to Lemma 3.1.3. As for the claim $\lambda_g^2 = 1/\lambda_1^+(\mathfrak{D}_g)$, one has to simply use the spectral theorem: for $\psi = \sum_{k\in\mathbb{Z}} \alpha_k \psi_k$, direct computation yields

$$\mathcal{Q}_g^2(\psi) = \frac{\sum_{k\in\mathbb{Z}} \alpha_k^2 \lambda_k}{\sum_{k\in\mathbb{Z}} \alpha_k^2 \lambda_k^2}.$$

This is maximized in the direction of the first positive eigenspinor ψ_1^+, and the claim follows. □

Theorem 4.3.3. *There holds $\lambda_g^{2^\dagger} = 1/\lambda_{\min}^+(M,\bar{g},\sigma)$.*

Proof. Since $\lambda_g^{2^\dagger}$ is conformally invariant by construction, we have that for any $\tilde{g} \in \bar{g}$ with $\mu_{\tilde{g}}(M) = 1$,

$$\lambda_g^{2^\dagger} = \lambda_{\tilde{g}}^{2^\dagger} \geq \lambda_{\tilde{g}}^2 = 1/\lambda_1^+(\mathfrak{D}_{\tilde{g}}),$$

therefore $\lambda^{2^\dagger} \geq 1/\lambda_{\min}^+(M,\bar{g},\sigma)$.

For the opposite inequality, let $\psi_\epsilon \in \Gamma(\Sigma M)$ be such that $\mathcal{Q}_g^{2^\dagger}(\psi_\epsilon) \geq \lambda_g^{2^\dagger} - \epsilon$ and $\|\mathfrak{D}_g\psi_\epsilon\|_{L^{2^\dagger}(\Sigma M,g)} = 1$, and we may assume that $\mathfrak{D}_g\psi_\epsilon$ is non-vanishing (up to a small perturbation, which may be done without loss of generality). Setting $g_\epsilon := |\mathfrak{D}_g\psi_\epsilon|_g^{4/(n+1)} g$, we calculate $\mu_{g_\epsilon}(M) = 1$ and $|\mathfrak{D}_{g_\epsilon}\psi_\epsilon|_{g_\epsilon} = 1$. Therefore

$$\mathcal{Q}_g^{2^\dagger}(\psi_\epsilon) = \mathcal{Q}_{g_\epsilon}^{2^\dagger}(\psi_\epsilon) = \mathcal{Q}_{g_\epsilon}^2(\psi_\epsilon) \leq \lambda_{g_\epsilon}^2 \leq 1/\lambda_1^+(\mathfrak{D}_{g_\epsilon}) = 1/\lambda_{\min}^+(M,\bar{g},\sigma).$$

Taking $\epsilon \to 0$ yields the conclusion. □

With that settled let us turn our attention to the Euler-Lagrange equations of $\mathcal{Q}_g^q$. The critical points $\psi \in W^{1,q}(\Sigma M)$ of $\mathcal{Q}_g^q$ are the ones for which the Gateaux derivative

$$\frac{d}{dt}\Big|_{t=0} \mathcal{Q}_g^q(\psi + t\varphi) = \frac{2}{\|\mathfrak{D}_g\psi\|_{L^q(\Sigma M,g)}^2} \int_M \langle \psi - \frac{\mathcal{Q}_g^q(\psi)}{\|\mathfrak{D}_g\psi\|_{L^q(\Sigma M,g)}^{q-2}} |\mathfrak{D}_g\psi|^{q-2}\mathfrak{D}_g\psi, \mathfrak{D}_g\varphi\rangle$$

vanishes for all $\varphi \in W^{1,q}(\Sigma M)$. Hence a maximizer ψ of $\mathcal{Q}_g^q$ that is normalised by $\|\mathfrak{D}_g\psi\|_{q(\Sigma M,g)} = 1$ must satisfy

$$\mathfrak{D}_g(\lambda_g^q|\mathfrak{D}_g\psi|^{q-2}\mathfrak{D}_g\psi - \psi) = 0. \tag{4.5}$$

This is the Euler-Lagrange equation associated with $\mathcal{Q}_g^q$, which is fully non-linear, but we can actually do better. Since for any $\kappa \neq 0$ and $\psi_{\ker} \in \ker\mathfrak{D}_g$ we have $\mathcal{Q}^q_g(\kappa\psi+\psi_{\ker}) = \mathcal{Q}^q_g(\psi)$, there is a lot of room to manipulate the equation. Note that elliptic regularity implies that $\lambda^q_g|\mathfrak{D}_g\psi|^{q-2}\mathfrak{D}_g\psi - \psi$ is smooth. Setting $\psi_1 := \lambda^q_g|\mathfrak{D}_g\psi|^{q-2}\mathfrak{D}_g\psi$, we see that $\mathfrak{D}_g\psi_1 = \mathfrak{D}_g\psi$ and therefore $\psi_1 \in W^{1,q}(\Sigma M)$, $\|\mathfrak{D}_g\psi_1\|_{L^q(\Sigma M,g)} = 1$. Moreover, direct calculation yields

$$\mathfrak{D}_g\psi_1 = (\lambda^q_g)^{1-p}|\psi_1|^{p-2}\psi_1, \tag{4.6}$$

and further substituting $\varphi := \psi_1/\lambda^q_g$ yields the equation

$$\lambda^q_g\mathfrak{D}_g\varphi = |\varphi|^{p-2}\varphi, \tag{4.7}$$

which is equivalent to (4.5) and (4.6), while one can check by direct calculation that φ is still a maximizing spinor of $\mathcal{Q}^q_g$. Finally, note that for $q = 2^\dagger$, the equation becomes

$$\mathfrak{D}_g\varphi = \lambda|\varphi|^{2^\star-2}\varphi, \quad \lambda = \lambda^+_{\min}(M,g,\sigma). \tag{4.8}$$

Now if $\mathfrak{D}_g\psi$ is non-vanishing, φ is non-vanishing. setting $\tilde{g} := |\varphi|^{4/(n-1)}g$, we calculate $|\varphi|_{\tilde{g}} = 1$ and $\mu_{\tilde{g}}(M) = 1$, and moreover

$$1/\lambda^+_1(\mathfrak{D}_{\tilde{g}}) = \lambda^2_{\tilde{g}} \geq \mathcal{Q}^2_{\tilde{g}}(\varphi) \geq \mathcal{Q}^{2^\dagger}_{\tilde{g}}(\varphi) = \mathcal{Q}^{2^\dagger}_g(\varphi) = \lambda^{2^\dagger}_g = 1/\lambda^+_{\min}(M,\bar{g},\sigma).$$

Since by definition we have $\lambda^+_{\min}(M,\bar{g},\sigma) \leq \lambda^+_1(\mathfrak{D}_{\tilde{g}})$, it follows that $\tilde{g}$ is a minimizer of $\mathcal{J} : \bar{g} \to \mathbb{R}$. Note that by a simple rescaling argument the same is true for the metric $\tilde{g} := |\mathfrak{D}_g\psi|^{4/(n+1)}g$.

If, on the other hand, g is a smooth minimizer of $\mathcal{J} : \mathbf{G} \to \mathbb{R}$ and ψ is an eigenspinor to $\lambda^+_1(\mathfrak{D}_g)$, then ψ is a maximizer of $\mathcal{Q}^{2^\dagger}_g$. Thus we have established a correspondence between maximising solutions of (4.8) and minimizers of $\lambda^+_{\min}(M,\bar{g},\sigma)$.

4.4 Overview of main results

There are striking similarities between the spinorial and the classical Yamabe problem. One basic difference is the absence of a maximum principle for solutions of the Dirac equation, and therefore one does not get the positivity and nice regularity properties of solutions to the Yamabe equations. The situation can be amended if one looks for solutions in an enlarged conformal class modified to include some "almost smooth" metrics. In any case, one can show that, similar to the classical Yamabe problem, there is the following upper bound with respect to the standard round spin sphere $(\mathbb{S}^n, g_\circ, \sigma_\circ)$[2].

Theorem 4.4.1 (Ammann). *Let (M,g,σ) be a compact Riemannian spin manifold of dimension $n \geq 3$, or $n = 2$ and $\ker\mathfrak{D}_g \neq \{0\}$. Then*

$$\lambda^+_{\min}(M,\bar{g},\sigma) \leq \lambda^+_{\min}(\mathbb{S}^n,\bar{g}_\circ,\sigma_\circ).$$

[2] The sphere $\mathbb{S}^n$ is diffeomorphic to $\mathrm{Spin}(n+1)/\mathrm{Spin}(n)$, and carries a natural spin structure $\mathrm{Spin}(n+1) \to \mathbb{S}^n$, with $\sigma_\circ : \mathrm{Spin}(n+1) \to SO(\mathbb{S}^n, g_\circ)$ given in a more or less obvious manner (for a detailed construction, see the Appendix section of [21])

Sketch of proof. The proof is given by the construction of a suitable Aubin-type spinor, which is given as a linear combination of Killing fields and essentially corresponds to blow-ups by standard round spin spheres, as was the case with the classical Yamabe problem. □

To describe the enlarged conformal class within which one should look for solutions, set $C^{1,\alpha}_{\text{sing}}(M) := \{u \in C^{1,\alpha}(M) : u \in C^\infty(M \setminus u^{-1}(0)), \operatorname{supp} u = M\}$. These functions are smooth except from their zero-set, which is of measure zero, were they are at least $C^{1,\alpha}$. The enlarged conformal class of a metric g is then defined to be

$$\breve{g} = \{u^{2/(n-1)} g : u \geq 0,\, u \in C^{1,\alpha}_{\text{sing}}(M)\}.$$

An important fact is that enlarging the conformal class in such a manner does not change the value of $\lambda^+_{\min}$, i.e

$$\inf_{\breve{g}} \mathcal{J} = \inf_{\bar{g}} \mathcal{J}.$$

Then we have a result that resolves the problem in terms of the criticality of $\lambda^+_{\min}$, which is very similar to Theorem 3.1.15.

Theorem 4.4.2 (Ammann). *Let (M, g, σ) be a compact Riemannian spin manifold of dimension $n \geq 2$, and suppose that*

$$\lambda^+_{\min}(M, \bar{g}, \sigma) < \lambda^+_{\min}(\mathbb{S}^n, \bar{g}_\circ, \sigma_\circ).$$

In that case:

1. *There is a spinor field $\varphi \in C^{1,\alpha}_{\text{sing}}(\Sigma M)$ such that*

$$\mathfrak{D}_g \varphi = \lambda |\varphi|^{2^\star - 2} \varphi, \quad \lambda = \lambda^+_{\min}(M, \bar{g}, \sigma),$$

normalised by $\|\varphi\|_{2^\star} = 1$.

2. *There is a generalized conformal metric $\tilde{g} \in \breve{g}$ such that $\mu_{\tilde{g}}(M) = 1$ and*

$$\lambda^+_1(\mathfrak{D}_{\tilde{g}}) = \lambda^+_{\min}(M, \bar{g}, \sigma).$$

Sketch of proof. As with the classical Yamabe problem, we consider the family of functionals $\mathcal{Q}^q_g$ for $q > 2^\dagger$. In direct analogy, these are subcritical and possess maximizers $\varphi_q \in L^q(\Sigma M)$ satisfying (4.7) and $\|\varphi_q\|_{L^p(\Sigma M, g)} = 1$, and the question is to investigate the limit $q \to 2^\dagger$. A suitable regularity result then reveals that $\{\varphi_q\}$ is uniformly bounded in $C^{0,\alpha}(\Sigma M)$ as $q \to 2^\dagger$, and hence uniformly bounded in $C^{1,\alpha}(\Sigma M)$ in view of the $C^{1,\alpha}$-version of Theorem 4.2.1. The Arzela-Ascoli theorem and Theorem 4.3.2 then imply that φ_q converges to a solution φ satisfying (4.8) which is a maximizer of $\mathcal{Q}^{2^\dagger}_g$ with $\|\varphi\|_{2^\star} = 1$.

The second statement is a direct consequence of the first. Given such a spinor field φ, set $\tilde{g} = |\varphi|^{4/(n-1)} g$, and the discussion of the previous section implies the required properties. □

Bibliography

[1] Lars Ahlfors. *Conformal Invariants*. AMS Chelsea Publishing, vol. 371.H, 2010.

[2] Bernd Ammann. “A spin-conformal lower bound of the first positive Dirac eigenvalue”. In: *Differential Geometry and its Applications* 18.1 (2003), pp. 21–32.

[3] Bernd Ammann. *A Variational Problem in Conformal Spin Geometry (Habilitation)*. Universität Hamburg, 2003.

[4] Bernd Ammann. “The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions”. In: *Communications in Analysis and Geometry* 17 (Oct. 2003).

[5] Thiery Aubin. *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*. Springer New York, NY, 1982.

[6] Christian Bär. *Spin Geometry (Lecture notes)*. Universität Potsdam, 2011.

[7] David Blair. *Inversion Theory and Conformal Mapping*. American Mathematical Society, 2010.

[8] Shing-Shen Chern. “An Elementary Proof of the Existence of Isothermal Parameters on a Surface”. In: *Proceedings of the American Mathematical Society* 6.5 (1955), pp. 771–782.

[9] Lawrence Evans. *Partial Differential Equations (2nd Edition)*. American Mathematical Society, 2010.

[10] Gerald Folland. *Real Analysis: Modern Techniques and their Applications (2nd Edition)*. Wiley, 2007.

[11] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian Geometry*. Springer Berlin, Heidelberg, 2004.

[12] David Gilbarg and Niel Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin, Heidelberg, 2001.

[13] Matthias Günther. “Conformal normal coordinates”. In: *Annals of Global Analysis and Geometry* 11.2 (1993), pp. 173 –184.

[14] Matthias Günther. “Über konforme Abbildungen Riemannscher Räume auf einfach zentralharmonische Räume”. In: *Mathematische Nachrichten* 125.1 (1986), pp. 191–200.

[15] Oussama Hijazi. “Spectral Properties of the Dirac Operator and Geometrical Structures”. In: (Apr. 2001).

[16] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*. Wiley Classics Library, 2009.

[17] Wolfgang Kühnel. *Differential Geometry: Curves - Surfaces - Manifolds (3rd Edition)*. American Mathematical Society, 2015.

[18] Blaine Lawson and Marie Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1989.

[19] John Lee. *Introduction to Riemannian Manifolds (2nd Edition)*. Springer Cham, 2019.

[20] John Lee and Thomas Parker. "The Yamabe problem". In: *Bulletin (New Series) of the American Mathematical Society* 17.1 (1987), pp. 37 –91.

[21] Nikolai Nowaczyk. *Dirac Eigenvalues of higher Multiplicity (Dissertation)*. Universität Regensburg, 2015.

[22] Henri Poincaré. "Sur l'uniformisation des fonctions analytiques". In: *Acta Mathematica* 31.31 (1908), pp. 1 –63.

[23] Richard Schoen and Shing-Tung Yau. *Lectures on Differential Geometry*. International Press, Somerville, Massachusetts, 2010.

[24] Giorgio Talenti. "Best constant in Sobolev inequality". In: *Annali di Matematica Pura ed Applicata* 110 (1976), pp. 353 –372.