Black Hole Uniqueness Theorems in the Theory of General Relativity

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Abstract

In this thesis we provide a detailed proof of Israel's uniqueness theorem of the Schwarzschild solution [28] in the more general setting proven by Bunting and Masood-ul-Alam [8]. The methods used in the proof of the generalized uniqueness theorem has had an important impact in many later proofs such as higher dimensional analogs of uniqueness theorems for the Schwarzschild solution [26], [21] and the Riemannian Penrose inequality by Bray [7]. General relativity is the best theory so far describing gravity together with Einstein's equation which relates the spacetime geometry to the matter distribution. One of the most important exact solutions of Einstein's equation is the Schwarzschild solution. It describes the exterior gravitational field of a static, spherically symmetric body, it predicts several phenomena of general relativity in our solar system and for a massive, spherical body that has gravitationally collapsed it describes the spacetime in vacuum which contains a singularity within a black hole.

After showing some facts for Lorentzian geometry and special relativity, we prove Birkhoff's theorem, that the Schwarzschild metric is the unique spherical solution in vacuum, and describe the Kruskal coordinates which extends the Schwarzschild metric to the whole spacetime with a singularity. Afterwards we show what are the initial data for the well-posedness of the Einstein's equation in vacuum and their constraint equations. At last we prove that the Schwarzschild metric is the unique static, asymptotically flat, vacuum spacetime with regular event horizon without assuming that the event horizon is connected.

Contents

Introduction

Before the formulation of general relativity, Newton's model of the solar system could predict precicely the trajectory of most planets. But the perihelion of Mercury was falling behind its observed position at the rate of about 43 seconds of arc per century. In 1915, Einstein completed his famous equation that binds together gravity and the curvature of spacetime. Before a relativistic model of the solar system, he eliminated the 43 seconds lag of the perihelion of Mercury. Some weeks later Karl Schwarzschild had discovered the relativistic model outside an isolated, spherically symmetric star and the first non-trivial exact solution of the Einstein equations [49]. Later it was called Schwarzschild metric. At the time, Schwarzschild, was serving in the German army and was hospitalized by an illness that soon proved mortal ([37, p. xv]). Later, it was discovered that massive stars which gravitationally collapse to themselves, produces a black hole that can be described by the Schwarzschild model with an event horizon at $r = 2M$. In 1923 Birkhoff proved that the only spherically symmetric solution that can exist in a vacuum is the Schwarzschild metric without assuming the metric being static [6]. This is called Birkhoff's theorem. According to Johansen and Ravndal in [29, p. 2], the theorem was discovered and published in 1921 with a different method by Jørg Tofte Jebsen. At the time he was ill and died in 1922. In 1969 Yvonne Choquet-Bruhat together with Robert Geroch showed the well-posedness of the initial value problem for the Einstein equations globally. They showed that for a given initial data, which satisfies the constraint equations, there exists a unique maximal globally hyperbolic development.

In 1967, Werner Israel anounced the first black hole uniqueness theorem. He proved that a certain class of static, asymptotically flat solutions of the Einstein equations in a vacuum can only be the Schwarzschild solution. This result initiated research on black hole uniqueness theorems which continues today ([42, p. 2]). After a year he extended his result for static, electrovac spacetimes [27]. Meaning that the only static spacetime satisfing his assumptions was the spherically symmetic Reissner-Nordström solution. In the next years some of the assumptions of Israels first proof was shown that they were not neccesary [23].

A new approach was introduced in 1986 for proving uniqueness of black holes by Gary L. Bunting and A. K. M. Masood-ul-Alam in [8]. By using results from the positive mass theorem, proved in 1979 [46] by Schoen and Yau, and Bartnik's n-dimensional positive mass theorem for spin manifolds, proved in 1986 [1], they showed a generalization of Israel's theorem in which the assumption of connected event horizon was not necessary. This means that there cannot be more than one black hole in a static, assymptotically flat, vacuum spacetime. This was done by first constructing an appropriate conformal manifold to satisfy the conditions of the positive mass theorem. Then by using the positive mass theorem from Bartnik, they show that the starting manifold is conformally flat. Here, it is important to be noted that the metric needed for the positive mass theorem of Bartnik is required to be of lower regularity than that of Shoen and Yau's, this is essential for the proof of Bunting and Masood-ul-Alam. After that, it can be shown that the conformally flat metric of the manifold is spherically symmetric and thus the metric of the spacetime is the Schwarzshild metric. The first geometric approach of constructing the conformal manifold had the benefit that it didn't need any assumptions on the dimension of the manifold. So it has been used for higher dimensional proofs of the same kind. The next part of the proof which uses conformal flatness to prove spherical symmetry doesnt generalize that simply to higher dimensions because it uses the Cotton tensor.

For a very insightfull presentation of the Schwarzschild metric, Israel's proof, Bunting and Masood-ul-Alam's proof and black hole uniqueness theorems in higher dimensions we refer the video lectures from ICTP School on Geometry and Gravity [11], [12], [13], [14]. Also, for an extended description of the history of black hole uniqueness theorems we refer the reader to [42].

The purpose of this thesis is to make an introduction to special and general relativity with some of their important implications such as the Birkhoffs theorem, the Kruskal diagram, which is used to better understand the singularities of the Schwarzschild metric, and the Cauchy hypersurfaces. In the end we provide a detailed proof of the generalized uniqueness theorem from Bunting and Masood-ul-Alam using many important results from [1]. For that we first need to understand some basic concepts of semi-Riemannian manifolds and the causal character of Lorentzian geometry so that we can distinguish the possibilities that are presented to us when using Lorentzian or Riemannian geometry.

Chapter 1

Semi-Riemannian Geometry

In Riemannian geometry we define the Riemannian metric as a symmetric, positive definite, (0,2) tensor. In general relativity we use a generalization of the Riemannian metric which is called **semi-Riemannian metric** (or pseudo-Riemannian metric). In this section we will show some basic properties.

1.1 Scalar Product Spaces

Definition 1.1.1. *Suppose V is a finite dimensional vector space and $b : V \times V \to \mathbb{R}$ is a symmetric $\mathbb{R}$-bilinear function.*

1. *If $b(v, v) > 0$ for $v \neq 0$ then it is called* ***positive definite***.
2. *If $b(v, v) \geq 0$ for all $v \in V$ then it is called* ***positive semidefinite***.
3. *If $b(v, v) < 0$ for $v \neq 0$ then it is called* ***negative definite***.
4. *If $b(v, v) \leq 0$ for all $v \in V$ then it is called* ***negative semidefinite***.
5. *If $b(v, w) = 0$ for all $w \in V$ implies $v = 0$ then it is called* ***nondegenerate***.

Lemma 1.1.2. *Suppose b is a symmetric covariant 2-tensor on a finite dimensional vector space V. The following are equivalent:*

(a) b is nondegenerate.

(b) For every nonzero $v \in V$, there is some $w \in V$ such that $b(v, w) \neq 0$.

(c) The linear map $\tilde{b} : V \to V^*$ defined by $\tilde{b}(v)(w) = b(v, w)$ for all $v, w \in V$ is an isomorphism.

Proof. (a) $\iff$ (b) is immediate.
(b) $\iff$ (c) because
Assume

$$\ker \tilde{b} = \{v \in V : \tilde{b}(v)(w) = b(v, w) = 0\ \forall w \in V\}$$
$$\operatorname{im} \tilde{b} = \{k \in V^* : \tilde{b}(v)(w) = k(w)\}$$

So we have (b) equivalent with Im$\tilde{b} = V^*$ and $b(v)(w) = 0$ only for $v = 0$ which is equivalent to Ker$\tilde{b} = \{0\}$. So $\tilde{b}$ is an isomorphism. □

Remark 1.1.3. *If b is a symmetric bilinear form on V then for every W subspace of V the restriction $b|(W \times W)$ is again symmetric and bilinear. We will denote the restriction as $b|W$*

Lemma 1.1.4 (Polarization Identity)**.** *Suppose symmetric bilinear form. Then*

$$\begin{aligned} b(v,w) &= \frac{1}{4}(b(v+w,v+w) - b(v-w,v-w)) \\ &= \frac{1}{2}(b(v+w,v+w) - b(v,v) - b(w,w)) \end{aligned}$$

For a base $e_1, \dots, e_n$ of V we denote $b_{ij} = b(e_i, e_j)$ which is the elements of the metrix of b relative to the base $e_1, \dots, e_n$. And since b is symetric we can write

$$b(\sum v^i e_i, \sum w^j e_j) = \sum b_{ij} v_i w_j$$

Lemma 1.1.5. *[38, p. 47] A symmetric bilinear form is nondegenerate if and only if its matrix relative to one basis is invertible.*

Proof. Let $e_1, \dots, e_n$ a basis of V and $v \in V$. Then we have for all $w \in V$

$$\begin{aligned} b(v,w) = 0 \quad &\iff \quad b(v, \sum w^i e_i) = 0 \\ &\iff \quad \sum w^i b(v, e_i) \\ &\iff \quad b(v, e_i) = 0 \end{aligned}$$

for $i = 1, \dots, n$. the matrix (b_{ij}) is symmetric and so

$$b(v, e_i) = b(\sum v^j e_j, e_i) = \sum b_{ij} v_j$$

Thus b is degenerate if and only if there exist numbers $v^1, \dots, v^n$ not all zero such that

$$\sum b_{ij} v^j$$

for $i = 1, \dots, n$. But this is equivalent to the linear dependence of the columns of (b_{ij}) which is equivalent to (b_{ij}) being singular. So b is nondegenerate if and only if (b_{ij}) is invertible. $\square$

Definition 1.1.6. *Suppose a vector space V and a bilinear form $b : V \times V \to \mathbb{R}$. If b is nondegererate and symmetric then we call b **scalar product** and V **scalar product space**.*

One of the new phenomena that happens when we have a scalar product in contrast to the inner product (positive definite) is that there exists vectors $v \neq 0$ which have $b(v,v) = 0$. These vectors are called **null**.

Example 1.1.7. *[38, p. 48] Define the symmetric bilinear form $b : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$b(v,w) = v_1 w_1 - v_2 w_2$$

The null vectors fill the lines 45 degrees of the axes. For vectors $v, w \in V$ and $c \neq 0$ the quadratic forms $b(v,v) = c$, $b(w,w) = -c$ are hyperbolas asymptotic to the null lines.

Two vectors $v, w \in V$ are still defined orthogonal when $b(v,w) = 0$. But when b is a scalar product we cannot imagine orthogonal vectors as vectors at 90 degrees with each other.

- *If $v \in \mathbb{R}^2$ is a null vector then $v \perp v$.*
- *If $(x,0),(0,y) \in \mathbb{R}^2$ then $(x,0) \perp (0,y)$.*
- *If $(x,y),(y,x) \in \mathbb{R}^2$ then $(x,y) \perp (y,x)$*

Assume (V,b) is a scalar product space and W is a subspace of V, then

$$W^{\perp} = \{v \in V : b(v,w) = 0,\ \forall w \in W\}$$

Another difference in scalar product spaces is that in general for a subspace $W \subset V$ we can have $W + W^{\perp} \neq V$. For example let $W = \operatorname{span}\{(1,1)\}$, then $W^{\perp} = W$.

Lemma 1.1.8. *[31, p. 41] If W is a subspace of a scalar product space V, then*

1. $\dim W + \dim W^{\perp} = \dim V$
2. $(W^{\perp})^{\perp} = W$

Proof. (1.) We define a linear map

$$\Phi : V \to W^* \text{ such that } \Phi(v) = \tilde{b}(v)|_W$$

We have that

$$v \in \ker \Phi \quad \Longleftrightarrow \quad b(v,w) = \tilde{b}(v)(w) = 0,\ \text{for all } w \in W$$

So $\ker \Phi = W^{\perp}$. If $\phi \in W^*$ then there exists an extension $\tilde{\phi} \in V$ such that $\tilde{\phi}|_{W^*} = \phi$. Since $\tilde{b}$ is isomorphic (see Lemma 1.1.2) there exists a $v \in V$ such that $\tilde{b}(v) = \tilde{\phi}$ and from the restriction we have $\Phi(v) = \phi$ which implies that Φ is surjective.

From the rank-nullity theorem we have that

$$\dim V - \dim \ker \Phi = \dim \operatorname{im} \Phi$$

but we have shown that $\ker \Phi = W^{\perp},\ \operatorname{im} \Phi = W^*$, so

$$\dim V - \dim W^{\perp} = \dim W^* = \dim W \quad \Longrightarrow \quad \dim W + \dim W^{\perp} = \dim V$$

(2.) We note that every $w \in W$ is orthogonal to every $z \in W^{\perp}$ by definition, so

$$W \subset (W^{\perp})^{\perp} \tag{1.1}$$

From the previous assertion we have

$$\left.\begin{aligned} \dim W + \dim W^{\perp} = \dim V \\ \dim W^{\perp} + \dim (W^{\perp})^{\perp} = \dim V \end{aligned}\right\} \quad \Longrightarrow \quad \dim W = \dim (W^{\perp})^{\perp}$$

So this together with (1.1) we have $W = (W^{\perp})^{\perp}$. □

We will call a subspace $W \subset V$ nondegenerate if $b|W$ is nondegenerate. Subspaces of scalar product spaces are not necessarilly scalar product spaces. Some subspaces are degenerate, for example a null vector $z \in V$ has $\operatorname{span}(z)$ degenerate since if non zero $w_1, w_2 \in \operatorname{span}(z)$ then

$$b(w_1, w_2) = b(\alpha z, \beta z) = 0 \quad \forall w_2 \in \operatorname{span}(z)$$

Lemma 1.1.9. *Suppose $W \subset V$ is a subspace of a scalar product space. The following are equivalent:*

(a) W is nondegenerate

(b) $W \cap W^\perp = \{0\}$

(c) $V = W \oplus W^\perp$

Proof. (a) $\iff$ (b): We write

$$W \cap W^\perp = \{w \in W : b(w,k) = 0 \ \forall k \in W\}$$

and by definition

$$W \text{ nondegenerate} \quad \iff \quad (b(w,k) = 0, \ \forall k \in W \Rightarrow w = 0)$$

So it is immediate

$$W \text{ nondegenerate} \quad \iff \quad W \cap W^\perp = 0$$

(b) $\iff$ (c): Its known that

$$\dim(W + W^\perp) + \dim(W \cap W^\perp) = \dim W + \dim W^\perp$$

so $W \cap W^\perp = 0$ if and only if

$$\dim(W + W^\perp) = \dim W + \dim W^\perp = \dim V,$$

where the last equality is from Lemma 1.1.8, and this is equivalent with

$$W \oplus W^\perp = V$$

□

Lemma 1.1.10. *Suppose $W \subset V$ of a scalar product space. Then W is non degenerate if and only if $W^\perp$ is nondegenerate.*

Proof. We use the previous Lemma and the fact that $(W^\perp)^\perp = W$

$$\begin{aligned} W^\perp \text{ nondegenetate} \quad &\iff \quad V = W^\perp \oplus (W^\perp)^\perp \\ &\iff \quad V = W^\perp \oplus W \\ &\iff \quad W \text{ nondegenerate} \end{aligned}$$

□

A nondegenerate subspace can always be expanded to the nondegenerate space.

Lemma 1.1.11 (Completion of Nondegenerate Bases)**.** *[31, p. 41] Suppose V a scalar product space and $v_1, \ldots, v_k$ span a nondegenerate k-dimensional subspace in V with $0 \leq k < n$. Then there exist vectors $v_{k+1}, \ldots, v_n$ such that $v_1, \ldots, v_n$ is a nondegenerate basis for V.*

Proof. Let $W = \text{span}(v_1, \dots, v_k) \subset V$. Since $k < n$ we have $W^\perp \neq 0$ nondegenerate and $W^\perp \oplus W = V$ from Lemma 1.1.9. We have $W^\perp \neq 0$ because if $W^\perp = 0$ nondegenerate then

$$\{0\} \oplus W = V \implies W = V$$

which is a contradiction from hypothesis. So because $W^\perp$ is nondegenerate there exists a vector $v_{k+1} \in W^\perp$ such that $b(v_{k+1}, v_{k+1}) \neq 0$ and then $(v_1, \dots, v_{k+1})$ span a nondegenerate subspace. We repeat this to get $v_1, \dots, v_n$. □

We define the norm of the vectors in a scalar product space (V, b) as

$$|u| = |b(u,u)|^{1/2}$$

since $b(u,u)$ can be negative. We call $u \in V$ a **unit vector** when $|u| = 1$ meaning $b(u,u) = \pm 1$. We can always find an orthonormal base to a scalar product space similarly to a vector space with inner product.

Proposition 1.1.12. *(Gram-Schmidt Algorithm for Scalar Products)[31, p. 42] Suppose V an n-dimensional scalar product space. If $(v_1, \dots, v_n)$ is a basis of V, then there is an orthonormal basis $(w_1, \dots, w_n)$ with the property that* $\text{span}(w_1, \dots, w_k) = \text{span}(v_1, \dots, v_k)$ *for each $k = 1, \dots, n$.*

Proof. We will prove it by induction.

Let $w_1 = v_1/|v_1|$. Since $\text{span}(v_1)$ is a nondegenerate subspace then $|v_1| \neq 0$.

Assume that we have an orthonormal base $(w_1, \dots, w_k)$. We write

$$z = v_{k+1} - \sum_{i=1}^{k} \frac{b(v_{k+1}, w_i)}{b(w_i, w_i)} w_i$$

We notice that $b(w_i, w_i) = \pm 1$ and so $z \in V$ is nonzero, $z \perp w_1, \dots z \perp w_k$ and $\text{span}(w_1, \dots w_k, z) = \text{span}(v_1, \dots, v_{k+1})$.

- $b(w_i, w_i) = \pm 1$ since $w_1, \dots, w_k$ is an orthonormal base.
- $z \perp w_1, \dots, z \perp w_k$ because

$$\begin{aligned} b(z, w_1) &= b\left(v_{k+1} - \sum_{i=1}^{k} \frac{b(v_{k+1}, w_i)}{b(w_i, w_i)} w_i, w_1 \right) \\ &= b(v_{k+1}, w_1) - \sum_{i=1}^{k} \frac{b(v_{k+1} w_i)}{b(w_i, w_i)} b(b_i, b_1) \\ &= b(v_{k+1}, w_1) - b(v_{k+1}, w_1) \\ &= 0 \end{aligned}$$

It is the same for every w_i.

We need to exclude the possibility that z is a null vector. If $b(z,z) = 0$ then $z \perp z$, but $z \perp w_1, \dots, z \perp w_k$. So $z \perp \text{span}(w_1, \dots, w_k)$ and since $z \perp z$ we have that $z \perp \text{span}(w_1, \dots, w_k, z)$. But $\text{span}(w_1, \dots, w_k, z) = \text{span}(v_1, \dots, v_{k+1})$ which implies $z \perp \text{span}(v_1, \dots, v_{k+1})$. This means that

$$b(z, v) = 0, \quad \forall v \in \text{span}(v_1, \dots, v_{k+1})$$

then by nondegeneracy of $\operatorname{span}(v_1, \ldots, v_{k+1})$ we have that $z = 0$ which is a contradiction.

We complete the step of induction by writing

$$w_{k+1} = \frac{z}{|z|}$$

□

Suppose $e_1, \ldots e_n$ is an orthonormal base for V. Then the matrix of b relative to the base is diagonal and

$$b(e_i, e_j) = \delta_{ij}\varepsilon_j$$

where $\varepsilon_j = b(e_j, e_j) = \pm 1$. After reordering the orthonormal base such that the negative signs come first in the **signature** $(\varepsilon_1, \ldots, \varepsilon_n)$.

Like in inner product spaces we can still write vectors by their **orthonormal expansion**.

Lemma 1.1.13. *Let (V, b) be an n-dimensional scalar product space. If $e_1, \ldots, e_n$ is an orthonormal base of V, then each $v \in V$ has a unique expression*

$$v = \sum \varepsilon_i b(v, e_i) e_i$$

such that $\varepsilon_i = b(e_i, e_i)$.

Proof. We will show that

$$b(v - \sum \varepsilon_i b(v, e_i) e_i, w) = 0 \quad \forall w \in V$$

then from nondegeneracy of V we have the Lemma. It suffices to prove the above for each element of the base $e_1, \ldots, e_n$.

$$\begin{aligned} b\left(v - \sum \varepsilon_i b(v, e_i) e_i, e_j\right) &= b(v, e_j) - b\left(\sum \varepsilon_i b(v, e_i) e_i, e_j\right) \\ &= b(v, e_j) - \sum \varepsilon_i b(v, e_i) b(e_i, e_j) \\ &= b(v, e_j) - \varepsilon_j b(v, e_j) b(e_j, e_j) \\ &= b(v, e_j) - \varepsilon_j^2 b(v, e_j) \\ &= b(v, e_j) - b(v, e_j) \\ &= 0 \end{aligned}$$

□

Definition 1.1.14. *Let V be a scalar product space and W a nondegenerate subspace of V. We the **orthogonal projection** $\pi : V \to W$ to be the surjective linear map such that $\pi|_{W^\perp} = 0$ and $\pi|_W = \mathrm{id}_W$. An orthonormal base $(e_1, \ldots, e_k)$ of W can always be expanded to a basis for V, and so*

$$\pi(v) = \sum_{i=1}^{k} \varepsilon_i b(v, e_i) e_i$$

Definition 1.1.15. *[38, p. 47] The **index** ν of a symmetric bilinear form b on V is the largest integer that is the dimension of a subspace $W \subset V$ on which $b|W$ is negative definite.*

Sometimes we will refer to the index ν of the scalar product b of V as the index of V and we will write $\nu = \text{ind}V$. The following Lemma tells us that the number of negative signs in an orthonormal base and the largest dimension of a subspace in which b is negative definite are equal and that they are independent from the choice of orthonormal base. This property is sometimes called *Sylvester's Law of Inertia.*

Lemma 1.1.16. *[38, p. 51] Let V be an n-dimensional scalar product space with b scalar product. For any orthonormal basis $e_1, \ldots, e_n$ of V the number of negative signs in the signature $(\varepsilon_1, \ldots, \varepsilon_n)$ is the index ν of V.*

Proof. Suppose the first d numbers of the signature $(\varepsilon_1, \ldots, \varepsilon_n)$ have negative signs. If b is definite then the proof is trivial.

Suppose $0 < d < n$. Then the index ν of b satisfies $\nu \geq d$ by definition of the index. Now we wish to show the opposite inequality. Suppose W is a subspace of V such that b is negative definite on W and $\dim W = \nu$. Let N be the subspace of V such that $N = \text{span}(e_1, \ldots, e_d)$, so b is negative definite. Then we define the orthogonal projection $\pi : W \to N$

$$\pi(w) = -\sum_{i=1}^{d} b(w, e_i)e_i$$

If we show that π is injective then $\dim W \leq \dim N$ and so we have $\nu = d$. To show this, first we write w by its orthonormal expansion

$$w = -\sum_{i=1}^{d} b(w, e_i)e_i + \sum_{i=d+1}^{n} b(w, e_i)e_i$$

To show π is injective, suppose $\pi(w) = 0$. So we can write

$$w = \sum_{i=d+1}^{n} b(w, e_i)e_i$$

Then we compute

$$\begin{aligned} b(w, w) &= b(w, \sum_{i=d+1}^{n} b(w, e_i)e_i) \\ &= \sum_{i=d+1}^{n} b(w, e_i)b(w, e_i) \\ &= \sum_{i=d+1}^{n} b(w, e_i)^2 \end{aligned}$$

So $b(w, w) \geq 0$. But $w \in W$ and g is negative definite in W. So we have

$$b(w, e_i) = 0, \quad \text{for } j > d$$

and since b is nondegenerate $w = 0$. Thus π is injective. □

These are important because for nondegenerate subspaces W we can find an orthonormal base of V such that $V = W + W^{\perp}$ and from the previous lemma

Lemma 1.1.17. *Let V be an n-dimensional scalar product space and W a nondegenerate subspace. then*

$$\mathrm{ind}V = \mathrm{ind}W + \mathrm{ind}W^{\perp}$$

Proof. Since W is a nondegenerate subspace we have shown that $W^{\perp}$ is nondegenerate. Since $W, W^{\perp}$ are scalar product spaces, they have orthonormal bases $(e_1, \ldots, e_k)$ for W and $(e_{k+1}, \ldots, e_n)$ for $W^{\perp}$ with indexes $\mathrm{ind}W, \mathrm{ind}W^{\perp}$. But $(e_1, \ldots, e_k, \ldots, e_n)$ is an orthonormal base for V. Assume that the number of negative signs in the signature of W is d, of $W^{\perp}$ is l and of V is r. Then we have

$$\mathrm{ind}V = r = d + l = \mathrm{ind}W + \mathrm{ind}W^{\perp}$$

□

The kind of maps which preserve the index of scalar product spaces are linear isometries.

Definition 1.1.18. *Assume scalar product spaces V and $\overline{V}$ with scalar products b and $\bar{b}$. A linear map $T : V \to \overline{V}$ is called* ***linear isometry*** *when:*

- *T preserves scalar products such that*

$$\bar{b}(Tv, Tw) = b(v, w) \text{ for all } v, w \in V$$

Remark 1.1.19. *We notice that a linear isometry is necessarily injective, because*

$$Tv = 0 \implies b(v, w) = 0 \text{ for all } w \implies v = 0$$

Lemma 1.1.20. *[38, p. 52] Suppose (V, b) and $(\overline{V}, \bar{b})$ scalar product spaces. Then $\dim V = \dim \overline{V}$ and $\mathrm{ind}\, V = \mathrm{ind}\, \overline{V}$ if and only if there exists a linear isometry $T : V \to \overline{V}$.*

Proof. ($\Longrightarrow$) Suppose orthonormal bases $e_1, \ldots, e_n$ for V and $\bar{e}_1, \ldots, \bar{e}_n$ for $\bar{V}$. Since $\mathrm{ind}\, V = \mathrm{ind}\, \bar{V}$, we have

$$b(e_i, e_i) = \bar{b}(\bar{e}_i, \bar{e}_i) \text{ for all } i \in \{1, \ldots, n\}$$

Let T be a linear transformation such that $Te_i = \bar{e}_i$. So this means $\bar{b}(Te_i, Te_j) = b(e_i, e_j)$, which is a linear isometry. So for $v, w \in V$ we have

$$\left.\begin{aligned} v &= \sum v^i e_i \\ w &= \sum w^i e_i \end{aligned}\right\} \implies \bar{b}(Tv, Tw) = b(v, w)$$

($\Longleftarrow$) Since T is a linear isometry, if $(e_1, \ldots, e_n)$ is an orthonormal base of V then $(Te_1, \ldots, Te_n)$ is an orthonormal base of $\overline{V}$. Hence

$$\dim V = \dim \overline{V}$$

and because from the linear isometry

$$\bar{b}(Te_i, Te_i) = b(e_i, e_i) = \varepsilon_i$$

we have equal number of negative signs of the signatures of V and $\overline{V}$. Hence

$$\mathrm{ind}\, V = \mathrm{ind}\, \overline{V}.$$

□

1.2 Semi-Riemannian Metric

Riemannian metrics are symmetric tensor fields $g \in \mathcal{T}_2^0(M)$ such that on every point p of M they are positive definite. We define the semi-Riemannian metric accordingly for scalar product spaces.

Definition 1.2.1. *Assume M a smooth manifold. If $g \in T_2^0(M)$ is a symmetric nondegenerate tensor field of constant index, then it is called* ***metric tensor*** *when for each $p \in M$, g_p is a scalar product of T_pM and has the same index for all p.*

When a smooth manifold M has a metric tensor g it will be called semi-Riemannian manifold. Since by definition the index ν of the scalar product g_p is the same for all p we can sometimes simply say the index of M.

- If $\nu = 0$ then M is a Riemannian manifold.
- If $\nu = 1,\ n \geq 2$ then M will be called **Lorentz manifold**.

For the metric tensor g sometimes we will write equivalently $g(v, w) = \langle v, w\rangle$.

Example 1.2.2. *In the Riemannian case one of the simplest examples of a flat Riemannian manifold was the euclidean space $\mathbb{R}^n$. For vectors $v_p, w_p \in T_p\mathbb{R}^n$ it can be defined a positive definite metric*

$$\langle v_p, w_p\rangle = \sum_{i=1}^{n} v^i w^i$$

In the semi-Riemannian case one of the simplest examples of a flat semi-Riemannian manifold is the ***Minkowski space*** *$\mathbb{R}^n_\nu$. This is the Euclidean space with a metric tensor g of index ν such that:*

$$\langle v_p, w_p\rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j$$

Remark 1.2.3. *It is not true that if we have an n-dimensional semi-Riemannian manifold M with index ν then every base of T_pM containts ν timelike vectors and $n-\nu$ spacelike vectors. We can see that in the Lorentzian case, the Minkowski space $\mathbb{R}^4_1$ has a base of vectors $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$. They are one timelike vector, one null vector and two spacelike vectors.*

We notice that, as was the case in scalar product spaces, there exists non-zero vectors with positive, negative and zero length.

Definition 1.2.4. *Suppose M is a semi-Riemannian manifold. Then for a vector $v_p \in T_pM$*

- *if $\langle v_p, v_p\rangle > 0$ for $v_p \neq 0$ or if $v_p = 0$ then it is called* ***spacelike****,*
- *if $\langle v_p, v_p\rangle = 0$ for $v_p \neq 0$ then it is called* ***null****,*
- *if $\langle v_p, v_p\rangle < 0$ for $v_p \neq 0$ then it is called* ***timelike****.*

Remark 1.2.5. *[38, p. 56] The set of all null vectors in T_pM is called the **nullcone** at $p \in M$. The category into which a given tangent vector falls is called its **causal character**. This terminology derives from relativity theory, and particularly in the Lorentz case, null vectors are also said to be **lightlike**.*

So on a semi-Riemannian manifold M we have a decomposition of the tangent spaces T_pM, for every point $p \in M$, in spacelike, null and timelike vectors. Lets see again the Example 1.1.7 the Minkowski space $\mathbb{R}^2_1$ with the metric tensor $\langle v, w\rangle = -v^1w^1 + v^2w^2$. Assume $a \neq 0$ and a timelike vector v, such that $\langle v, v\rangle = -a^2$. So

$$\langle v, v\rangle = -a^2 \quad \Longrightarrow \quad -(v^1)^2 + (v^2)^2 = -a^2 \quad \Longrightarrow \quad \frac{(v^1)^2}{a^2} - \frac{(v^2)^2}{a^2} = 1$$

We conclude that the timelike vectors with $g(v, v) = -a^2$ form rectangular hyperbolas with major semi-axis y-coordinate axis. Null vectors form their asymptotes since $g(w, w) = 0$. And spacelike vectors u form rectangular hyperbolas with major semi-axis the x-coordinate axis since for $b \neq 0$

$$\langle u, u\rangle = b^2 \quad \Longrightarrow \quad \frac{(u^2)^2}{b^2} - \frac{(u^1)^2}{b^2} = 1$$

The above are in contrast the Euclidean case since for $\langle v, v\rangle = a^2$ the set of vectors formed a circle. This may give us a hint that also the angles will be measured differently. In a later section we will show that for timelike vectors in Lorentz vector space the inverse Cauchy-Schwarz inequality holds.

A difference between Riemannian metric and semi-Riemannian metrics is that we cant always induce a metric tensor to a submanifold from the ambient manifold. Let M be a Riemannian manifold with metric g and N a submanifold. For each subspace T_pN of T_pM we can induce the metric $i^*(g)$, from the inclusion $i : N \hookrightarrow M$, to N which makes it a Riemannian manifold. But in the case of a nondegenerate metric tensor g of M, $i^*(g)$ is not always a metric tensor because T_pN may not be nondegenerate for some $p \in M$ or the index may not be the same for all p.

Definition 1.2.6. *Suppose (M, g) is a semi-Riemannian manifold and $N \subset M$ is a submanifold such that we have the inclusion $i : N \hookrightarrow M$. If $i^*(g)$ is a metric tensor on N, then N is a semi-Riemannian submanifold of M.*

1.3 Parallel Translation

Suppose (M, g) is a semi-Riemannian manifold and $\gamma : I \to M$ a smooth curve. Then a smooth map $X : I \to TM$, where TM is the tangent bundle, is said to be an element of $\mathfrak{X}(\gamma)$. X assigns to each $t \in I$ a tangent vector to M at $a(t)$. The velocity γ' is a vector field on γ, as is the restriction Y_γ of any $Y \in \mathfrak{X}(M)$.

Proposition 1.3.1. *[41, p. 18] Let (M, g) be a semi-Riemannian manifold, $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. Then there is a unique function*

$$X \mapsto X' = \frac{DX}{\mathrm{d}t}$$

such that $X, X' \in \mathfrak{X}(\gamma)$. This map satisfies the following properties:

$$(aX_1 + bX_2)' = aX_1' + bX_2' \qquad (a, b \in \mathbb{R}) \tag{1.2}$$

$$(fX)' = (\mathrm{d}f/\mathrm{d}t)X + fX' \qquad (h \in C^\infty(I)) \tag{1.3}$$

$$(Y_\gamma)' = D_{\gamma'(t)}(Y) \qquad (t \in I, Y \in \mathfrak{X}(M)) \tag{1.4}$$

$$(\mathrm{d}/\mathrm{d}t)\langle X_1, X_2\rangle = \langle X_1', X_2\rangle + \langle X_1, X_2'\rangle \tag{1.5}$$

For a vector field X on γ when X is tangent to γ, meaning $X = \gamma'$, we can write $X' = \gamma''$ and we call it the acceleration of γ. We could try to define $X' = D_{\gamma'}X$ and $\gamma'' = D_{\gamma'}\gamma'$ for X tangent to γ, but this would not be entirely correct since X is not a vector field of M but a vector field of γ. But it is correct to write $X' = D_{\gamma'(t_0)}X$ only at points $\gamma(t_0)$ where $\gamma'(t_0) \neq 0$ and some neighborhood of t_0.

In coordinates we write

$$X' = \sum_k \left\{ \frac{\mathrm{d}X^k}{\mathrm{d}t} + \sum_{ij} \Gamma^k_{ij} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}t} X^j \right\} \partial_k \tag{1.6}$$

Definition 1.3.2. *[41, p. 19] Let (M,g) be a semi-Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $\gamma : I \to M$ a smooth curve. Then $X \in \mathfrak{X}(\gamma)$ is* ***parallel along*** *γ if and only if $X' = 0$.*

Proposition 1.3.3. *Let (M, g) be a semi-Riemannian manifold, $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. If $t_0 \in I$ and $\xi \in T_{\gamma(t_0)}M$, then there is a unique $X \in \mathfrak{X}(\gamma)$ such that $X' = 0$ and $X(t_0) = \xi$.*

Due to the previous proposition we can define a map called **parallel translation**.

Definition 1.3.4. *Let (M, g) be a semi-Riemannian manifold. $I \subset \mathbb{R}$ be an open interval with $t_0, t_1 \in I$ and $\gamma : I \to M$ be a smooth curve. Suppose $\xi \in T_{\gamma(t_0)}M$ and $X \in \mathfrak{X}(\gamma)$ such that $X' = 0$ and $X(t_0) = \xi$. Then the map*

$$P : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$

such that $P(\xi) = X(t_1)$ is called parallel translation along γ from $\gamma(t_0)$ to $\gamma(t_1)$

Proposition 1.3.5. *Let (M, g) be a semi-Riemannian, $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. Suppose $t_0, t_1 \in I$ and $p_i = \gamma(t_i)$, $i = 0, 1$. then parallel translation along γ from $\gamma(t_0)$ to $\gamma(t_1)$ is a linear isometry from $T_{p_0}M$ to $T_{p_1}M$.*

Since parallel translation is a linear isometry of the tangent spaces then the causal character of tangent vectors, curves, and submanifolds are preserved from Lemma 1.1.20. Another map that preserves the causal character is the conformal maps with positive conformal factor.

Definition 1.3.6. *Suppose $X \in \mathfrak{X}(\gamma)$ is parallel along γ such that $X = X^i\partial_{x_i}$ and $\gamma(t_0) = p, \gamma(t_1) = q$. If the parallel translation from point p to point q along any curve is just the canonical isomorphism*

$$P(X^i\partial_{x_i}|_p) = X^i\partial_{x_i}|_q$$

then we call it ***distant parallelism****.*

Example 1.3.7. *Assume we have the Minkowski space $\mathbb{R}^n_\nu$, a smooth curve γ and a vector field $X \in \mathfrak{X}(\gamma)$ parallel along γ. Since Minkowski space has $\Gamma^k_{ij} = 0$, with respect to the natural coordinates, from equation* (1.6) *we have*

$$X' = 0 \quad \Longrightarrow \quad \sum_k \frac{\mathrm{d}X^k}{\mathrm{d}t}\partial_k = 0$$

It follows that X is parallel along γ if and only if the X^k are constants on γ with respect to the standard basis. So the parallel translations are distant parallelisms since the vectors are constant along arbitrary smooth curves of the Minkowski space. This means that the result does not depend on the curve. This is the case for Euclidean and Minkowski space because the natural coodrinate vector fields are parallel and hence so are their restrictions on any curve. This is not true in general for other spaces.

We can use parallel transport in a very convinient way. Assume an orthonormal basis $e_1, \ldots, e_n$ for $T_{\gamma(t_0)}M$ and then parallel transport the vectors e_i along the curve γ. By that we get the parallel vector fields $E_1, \ldots, E_n$ along γ and because parallel transport is a linear isometry we have that the parallel vector field is an orthonomal basis for each $\gamma(t)$. For a vector field $X(t) = X^i(t)E_i(t)$ by taking its covariant derivative along a curve we have

$$\begin{aligned} X' &= \frac{\mathrm{d}X^i(t)}{\mathrm{d}t}E_i(t) + X^i(t)(E_i(t))' \\ &= \frac{\mathrm{d}X^i(t)}{\mathrm{d}t}E_i(t) \end{aligned}$$

In conclusion we have that a vector field is parallel along γ if and only if its components are constant with respect to the frame (E_i).

1.4 Geodesics

Definition 1.4.1. *[41, p. 20] Let M, g be a semi-Riemannian manifold, $I \subset \mathbb{R}$ be an open interval and $\gamma : I \to M$ be a smooth curve. Then γ is said to be a* ***geodesic*** *if $\gamma'' = 0$ or equivalently if $\gamma' \in \mathfrak{X}(\gamma)$ is parallel.*

Corollary 1.4.2. *[38, p. 67] Let $x^1, \ldots, x^n$ be a coordinate system on $U \subset M$. A curve γ in U is a geodesic of M if and only if its coordinate functions $x^k \circ \gamma$ satisfy*

$$\frac{\mathrm{d}^2(x^k \circ \gamma)}{\mathrm{d}t^2} + \sum_{i,j} \Gamma^k_{ij}(\gamma)\frac{\mathrm{d}^2(x^i \circ \gamma)}{\mathrm{d}t}\frac{\mathrm{d}^2(x^j \circ \gamma)}{\mathrm{d}t} = 0$$

for $1 \leq k \leq n$. The above is sometimes called ***geodesic equation****.*

When the context is obvious we will use the abbreviation of the geodesic equation

$$\frac{\mathrm{d}^2(x^k)}{\mathrm{d}t^2} + \sum_{i,j} \Gamma^k_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = 0$$

for $1 \leq k \leq n$ such that

$$x^k = x^k \circ \gamma, \quad \Gamma^k_{ij} = \Gamma^k_{ij}(\gamma)$$

Proposition 1.4.3. *Let (M, g) be a semi-Riemannian manifold, $p \in M$ and $v \in T_pM$. Then there is a unique geodesic $\gamma : I \to M$ with the properties that*

- *$I \subset \mathbb{R}$ is an open interval such that $0 \in I$,*
- *$\gamma'(0) = 0$,*
- *I is maximal in the sense that if $\alpha : J \to M$ is a geodesic (with J an open interval, $0 \in J$ and $\alpha'(0) = 0$), then $J \subset I$ and $\alpha = \gamma|_J$.*

The geodesics in the previous proposition are called **maximal geodesics** or **geodesically inextendible**. For a maximal geodesic γ with initial velocity $\gamma'(0) = v$ we will frequently use the notation γ_v.

Definition 1.4.4. *Let (M, g) be a semi-Riemannian manifold and $\gamma : I \to M$ be a maximal geodesic of M. If $I = \mathbb{R}$ then γ is called **complete geodesic**. If all maximal geodesics the of semi-Riemannian manifold M are complete geodesics then M is called **complete**.*

So a complete geodesic is a curve $\gamma : I \to M$ that if given an inicial velocity γ' then it doesnt stop. If we remove a point from a complete geodesic then it loses its completeness since given an initial velocity it will stop for some $t \in I$.

Example 1.4.5. *Suppose the Minkowski space $\mathbb{R}^n_\nu$. Since its Christofell symbols vanish it easy to see that the geodesic equation becomes*

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t} = 0$$

such that $1 \leq i \leq n$. So the components of the geodesics in the Minkowski space are

$$x^i(t) = tv^i + p^i$$

for all t and $p^i, v_i \in \mathbb{R}$. Meaning that the geodesics are straight lines

$$\gamma(t) = tv + p$$

and $\mathbb{R}^n_\nu$ is a geodesically complete.

Note that if we remove a point from a comples space then it becomes not complete. For example $\mathbb{R}^n_\nu \setminus \{0\}$ is not complete.

Remark 1.4.6. *If γ is a geodesic then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \gamma', \gamma' \rangle = \langle \gamma'', \gamma' \rangle + \langle \gamma', \gamma'' \rangle = 0$$

So $\langle \gamma', \gamma' \rangle$ is constant.

Definition 1.4.7. *Let M be a semi-Riemannian manifold, $I \subset \mathbb{R}$ be an open intervan and α be a curve in M. Then*

- *if $\langle \alpha', \alpha' \rangle > 0$ and $\alpha' \neq 0$ or $\alpha' = 0$ then α is called **spacelike curve**,*
- *if $\langle \alpha', \alpha' \rangle < 0$ and $\alpha' \neq 0$ then α is called **timelike curve**,*

- *if* $\langle \alpha', \alpha' \rangle = 0$ *and* $\alpha' \neq 0$ *then* α *is called* ***null curve****,*
- *if* α *is timelike or null then it is called* ***causal curve****.*

An arbitrary curve α is not necessarilly any of the above, so it doesnt have a causal character. This means that an arbitrary curve may have some spacelike velocity vectors, some timelike velocity vectors and some null vectors for $t \in I$. But a geodesic γ has one causal character of the above since γ' is parallel and parallel translation preserves the causal character of vectors.

1.5 Exponential Map

In the next Lemma we see that it is possible by increasing the velocity of the geodesic to decrease the length of the interval that the geodesic is defined. Conversely, we can decrease the interval that the geodesic is defined to increase the velocity of the geodesic.

Lemma 1.5.1 (Rescaling Lemma)**.** *[31, p. 127] Suppose M is a semi-Riemannian manifold and $\gamma : (-\delta, \delta) \to M$ a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then the geodesic $\gamma_c v$ for $c \in \mathbb{R}, c > 0$ is defined on the interval $(-\delta/c, \delta/c)$ and*

$$\gamma_{cv}(t) = \gamma_v(ct)$$

This Leema makes it possible to define a very importan map for semi-Riemannian and Riemannian geometry.

Definition 1.5.2. *[31, p. 128] Suppose M a semi-Riemannian manifold and a geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Let the subset $\mathcal{D} \subset TM$ be*

$$\mathcal{D} = \{v \in TM : \gamma_v \text{ is defined on an interval containing } [0,1]\}$$

The we define the exponential map $\exp : \mathcal{D} \to M$ *by*

$$\exp(v) = \gamma_v(1)$$

For each $p \in M$ we can define the exponential map $\exp_p$ *as the restriction of* $\exp$ *to the set $\mathcal{D}_p = \mathcal{D} \cap T_pM$.*

We notice that the exponential map $\exp_p$ carries lines through the origin of T_pM to geodesics of M through p since

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$$

Proposition 1.5.3. *For each $p \in M$ there exists a neighborhood $\tilde{U}$ of $0 \in T_pM$ such that the exponential map exp_p is a diffeomorphism onto a neghborhood U of p in M.*

- Suppose S is a subset of a vector space. If $v \in S$ implies $tv \in S$ for all $0 \leq t \leq 1$ then S is called **starshaped** about 0.
- Suppose U and $\tilde{U}$ are as in the proposition 1.5.3. Then if $\tilde{U}$ is starshaped about 0 (the zero section), then U is called **normal neighborhood** of p.

Proposition 1.5.4. *[38, p. 72] If U is a normal neighborhood of $p \in M$, then for each point $q \in U$ there is a unique geodesic $\sigma : [0,1] \to U$ from p to q in U. Furthermore $\sigma'(0) = exp_p^{-1}(p) \in \tilde{U}$.*

Definition 1.5.5. *Suppose σ is the geodesic of a semi-Riemannian manifold M, where $\sigma(0) = p, \sigma(1) = q$ and U a normal neighborhood of $p \in M$. Let $v = \exp_p^{-1}(q) \in \tilde{U}$ and $\rho(t) = tv \in \tilde{U}$ then the geodesic $\sigma(t) = \exp_p \circ \rho$, which lies in U, is called **radial geodesic**.*

Example 1.5.6. *[38, p. 73] In the Example 1.4.5 we showed that the geodesics of the Minkowski space with initial velocity $v_p \in T_p\mathbb{R}^n_\nu$ are the straight lines $\gamma(t) = tv + p$. So from the definition map we have that*

$$\exp_p(v_p) = \gamma_v(1) = v + p$$

We can express the exponential map $\exp_p : T_p\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ as the composition of the canonical isomorphism $T_p\mathbb{R}^n_\nu \cong \mathbb{R}^n_\nu$ and the translation $x \mapsto p + x$. This maps are both diffeomorphisms and so we have that $\exp_p : T_p\mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is a diffeomorphism. If $T_p\mathbb{R}^n_\nu$ has the usual metric tensor then the canonical isomorphism and the translation are isometries, so $\exp_p$ is also an isometry.

From that we also notice that since $T_p\mathbb{R}^n_\nu$ is starshaped, then $\mathbb{R}^n_\nu$ is a normal neighborhood around all of its points.

Assume a semi-Riemannian manifold (M, g).

- For an orthonormal basis (b_i) of T_pM we define the **basis isomorphism**

  $$B : \mathbb{R}^n \to T_pM$$

  such that $B(x^1, \ldots, x^n) = x^i b_i$.

- Around p we define $U = \exp_p(V)$ the normal neighborhood of p

Then by combining the above we get a smooth coordinate map

$$\phi = B^{-1} \circ (\exp_p|_V)^{-1} : U \to \mathbb{R}^n$$

$$\begin{tikzcd}
T_pM \arrow[r, "B^{-1}"] & \mathbb{R}^n \\
U \arrow[u, "(\exp_p|_V)^{-1}"] \arrow[ur, "\phi"] &
\end{tikzcd}$$

This coordinate map is called **normal coordinate centered at** p.

Proposition 1.5.7. *(Uniqueness of Normal Coordinates)[31, p. 132] Let (M, g) be a Riemannian or semi-Riemannian n-manifold, $p \in M$ and U be a normal neighborhood of p. Then*

1. *For every normal coordinate chart on U centered at p, the coordinate basis is orthonormal at p.*

2. *For every orthonormal basis (b_i) of T_pM, there is a unique normal coordinate chart (x^i) on U such that*
$$\partial_i|_p = b_i$$
for $i = 1, \ldots, n$.

Proof. For (1). Suppose normal coordinate chart (U, ϕ) centered at p such that $\phi = (x^1, \ldots, x^n)$ and $\phi(p) = 0$. Then from the definition we have that
$$\phi = B^{-1} \circ \exp_p^{-1}$$
such that $B : \mathbb{R}^n \to T_pM$ is the basis isomorphism $B(v^1, \ldots, v^n) = v^i b_i$ where (b_i) is some orthonormal basis of T_pM. We notice that
$$\mathrm{d}\phi_p^{-1} = \mathrm{d}(\exp_p)_0 \circ \mathrm{d}B_0$$
Because $\mathrm{d}(\exp_p)_0 = \mathrm{id}_{T_pM}$ and B is a linear map we have
$$\mathrm{d}\phi_p^{-1} = B$$
So for the coordinate basis of T_pM we have
$$\partial_{x_i}|_p = \mathrm{d}\phi_p^{-1}(\partial_{x_i}|_{\phi(p)=0}) = B(\partial_{x_i}|_0) = b_i$$
which shows that the coordinate basis is orthonormal at p.

For (2). Suppose orthonormal basis (b_i) on T_pM. We define the basis isomorphism B determined on the orthonormal basis (b_i), then we can define a normal coordinate map $\phi = B^{-1} \circ \exp_p^{-1}$ which satisfies
$$\partial_{x_i}|_p = b_i$$
as it was shown in (1).

Now to show the uniqueness, assume another normal coordinate chart
$$\tilde{\phi} = \tilde{B}^{-1} \circ \exp_p^{-1}$$
Then we notice that
$$\tilde{\phi} \circ \phi^{-1} = \tilde{B}^{-1} \circ \exp_p^{-1} \circ \exp_p \circ B = \tilde{B}^{-1} \circ B$$
So $\tilde{\phi} \circ \phi^{-1}$ is a linear map and so $\mathrm{d}(\tilde{\phi} \circ \phi^{-1}) = \tilde{\phi} \circ \phi^{-1}$. We know that two the coordinate charts $\phi, \tilde{\phi}$ are the same if and only if $\mathrm{d}(\tilde{\phi} \circ \phi^{-1}) = \mathrm{id}_{\mathbb{R}^n}$. So from this we have that
$$\tilde{\phi} = \phi$$
and so the normal coordinate chart centered at p is unique. □

Definition 1.5.8. *[38, p. 129] An open set $\mathcal{V}$ in a semi-Riemannian manifold is **convex** when $\mathcal{V}$ is a normal neighborhood of each of its points.*

In particular for any tow points $p, q \in \mathcal{V}$ there is a unique geodesic segment $\sigma_{pq} : [0, 1] \to M$ from p to q that lies entirely in $\mathcal{V}$.

Definition 1.5.9. *If p and q are points of a convex open set $\mathcal{V}$ and σ_{pq} is the geodesic in $\mathcal{V}$ from $p = \sigma_{pq}(0)$ to $q = \sigma_{pq}(1)$, the **displacement vector** $\vec{pq}$ is $\sigma'_{pq}(0) \in T_pM$.*

An important fact is that local isometries of connected manifolds are completely determined by their values and differentials at a single point.

Proposition 1.5.10. *Let M and N be semi-Riemannian manifolds and M connected. Suppose $\phi, \psi : M \to N$ are local isometries. If for $p \in M$ we have $\phi(p) = \psi(p)$ and $\mathrm{d}\phi_p = \mathrm{d}\psi_p$, then $\phi = \psi$ at every point in M.*

Chapter 2

Lorentzian Geometry

2.1 Causality of Lorentzian Geometry

Definition 2.1.1. *A scalar product space of index* 1 *and dimension* $n \geq 2$ *is called* ***Lorentz vector space****.*

Definition 2.1.2. *[38, p. 141] Suppose* V *is a Lorentz vector space with scalar product* g *and* W *its a subspace.*

- *If* $g|_W$ *is positive definite, then* W *is said to be spacelike.*
- *If* $g|_W$ *is nondegenerate of index* 1*, then* W *is timelike.*
- *If* $g|_W$ *is degenerate, then* W *is lightlike.*

Similar to vectors in the tangent space, the category in which the subspaces fall is called the causal character.

In Lemma 1.1.9 we showed that a subspace W of a scalar product space V is nondegenerate if and only if its $W^{\perp}$ is nondegenerate if and only if $V = W \oplus W^{\perp}$. Here we make a similar argument by taking into account its causal character also.

Lemma 2.1.3. *[38, p. 141] If* z *is a timelike vector in a Lorentz vector space* V*, then the subspace* $z^{\perp}$ *is spacelike and* $V = span(z) \oplus z^{\perp}$*.*

From Lemma 1.1.8 we have that $(W^{\perp})^{\perp} = W$, so from the above we have that

- W is timelike if and only if $W^{\perp}$ is spacelike,
- W is lightlike if and only if $W^{\perp}$ is lightlike.

For the spacelike subspaces all the properties of vector subspaces with inner product are the same. For the timelike and lightlike we have some characteristic properties.

Lemma 2.1.4. *[38, p. 141] Let* W *be a subspace of dimension* $n \geq 2$ *in a Lorentz vector space. Then the following are equivalent:*

1. W *is timelike, hence is itself a Lorentz vector space.*
2. W *contains two linearly independent null vector.*

3. *W contains a timelike vector.*

Lemma 2.1.5. *[38, p. 142] For a subspace W of a Lorentz vector space the following are equivalent:*

1. *W is lightlike, that is, degenerate.*

2. *W contains a null vector but not a timelike vector.*

3. *$W \cap \Lambda = L \setminus 0$, where L is a one-dimensional subspace and Λ is the nullcone of V.*

2.2 Timecones

We can now examine if there are analogue properties of timelike vector with the spacelike vectors such as triangle inequality or the Cauchy-Schwarz inequality. First we need to be able to distinguish two different sets of timelike vectors.

Definition 2.2.1. *[38, p. 143] Let $\mathcal{T}$ be the set of all timelike vectors in a Lorentz vector space V. For $u \in \mathcal{T}$*

- $$C(u) = \{v \in \mathcal{T} : \langle u, v\rangle < 0\}$$
  *is the **timecone** of V containing u.*

- *The **opposite timecone** is*
  $$C(-u) = -C(u) = \{v \in \mathcal{T} : \langle u, v\rangle > 0\}$$

We have shown that $u^{\perp}$ is spacelike, so $\mathcal{T} = C(u) \sqcup C(-u)$.

We can distinguish when two timelike vectors are in the same timecone with the following Lemma.

Lemma 2.2.2. *[38, p. 143] Assume V is a Lorentzian vector space and the timelike vectors $v, w, u \in V$. Then $v, w \in C(u)$ or $v, w \in -C(u)$ if and only if $\langle v, w\rangle < 0$.*

Proof. We can assume that u is a unit timelike vector since $C(u/|u|) = C(u)$.

From Lemma 1.1.12 there is an orthonormal base $e_0, e_1, \dots, e_n$ of V such that $e_0 = u$. So $e_1, \dots, e_n \in u^{\perp}$ and by Lemma 2.1.3 $e_1, \dots, e_n$ are spacelike vectors. Now we can write $v = \sum v^i e_i$ for $i \in \{0, 1, \dots, n\}$ and

$$\begin{aligned}
\langle v, u\rangle &= \langle v^0 u + v^1 e_1 + \cdots + v^n e_n, u\rangle \\
&= \langle v^0 u, u\rangle + \langle v^1 e_1, u\rangle + \cdots + \langle v^n e_n, u\rangle \\
&= v^0 \langle u, u\rangle \\
&= -v^0
\end{aligned}$$

If $v \in C(u)$, this implies with the above that $v^0 > 0$. We showed

- $v \in C(u)$ if and only if $v^0 > 0$,

- $v \in -C(u)$ if and only if $v^0 < 0$.

Let $x = \sum x^i e_i$ where $\bar{x} = x^1 e_1 + \cdots x^n e_n$.

$$\begin{aligned}
\langle x, x \rangle &= \langle \sum x^i e_i, \sum x^i e_i \rangle \\
&= (x^0)^2 \langle u, u \rangle + (x^1)^2 \langle e_1, e_1 \rangle + \cdots + (x^n)^2 \langle e_n, e_n \rangle \\
&= -(x^0)^2 + (x^1)^2 + \cdots + (x^1)^2 \\
&= -(x^0)^2 + \|\bar{x}\|^2_{\mathbb{R}^n}
\end{aligned}$$

If x is timelike then

$$\begin{aligned}
\langle x, x \rangle < 0 \quad &\iff \quad -(x^0)^2 + \|\bar{x}\|^2_{\mathbb{R}^n} < 0 \\
&\iff \quad (x^0)^2 > \|\bar{x}\|^2_{\mathbb{R}^n} \\
&\iff \quad |x^0| > \|\bar{x}\|_{\mathbb{R}^n} \qquad (1)
\end{aligned}$$

Since we used equivalences, we showed

- x is timelike if and only if $|x^0| > \|\bar{x}\|_{\mathbb{R}^n}$

For $\bar{v} = v^1 e_1 + \cdots v^n e_n$ and $\bar{w} = w^1 e_1 + \cdots w^n e_n$ we define the timelike vectors v, w

$$v = v^0 u + \bar{v}, \qquad w = w^0 u + \bar{w}$$

Then we see that

$$\begin{aligned}
\langle v, w \rangle &= \langle v^0 u + \bar{v}, w^0 u + \bar{w} \rangle_{\mathbb{R}^n} \\
&= v^0 w^0 \langle u, u \rangle + \langle \bar{v}, \bar{w} \rangle_{\mathbb{R}^n} \\
&= -v^0 w^0 + \langle \bar{v}, \bar{w} \rangle_{\mathbb{R}^n}
\end{aligned}$$

We have shown that v, w timelike vectors if and only if $|v^0| > \|\bar{v}\|_{\mathbb{R}^n}, |w^0| > \|\bar{w}\|_{\mathbb{R}^n}$. This together with Cauchy-Schwarz gives us

$$|\langle \bar{v}, \bar{w} \rangle|_{\mathbb{R}^n} \leq \|\bar{v}\|_{\mathbb{R}^n} \|\bar{w}\|_{\mathbb{R}^n} < |v^0 w^0|$$

This means

$$\langle \bar{v}, \bar{w} \rangle_{\mathbb{R}^n} < v^0 w^0 \quad \text{or} \quad \langle \bar{v}, \bar{w} \rangle_{\mathbb{R}^n} < -v^0 w^0$$

This shows that

- $\langle v, w \rangle < 0$ if and only if $\operatorname{sign}(v^0) = \operatorname{sign}(w^0)$.

Assume that $v, w \in C(u)$, then $v^0, w^0 > 0$. So we have $\langle v, w \rangle < 0$. Similar for $v, w \in -C(u)$.

Assume that $\langle v, w \rangle < 0$. Then v^0, w^0 have the same sign and so $v, w \in C(u)$ or $v, w \in -C(u)$. $\square$

Also the timecones are convex sets. This is easy to see since for $v, w \in C(u)$ and if $a \geq 0, b \geq 0$ not both zero then $av + bw \in C(u)$ because

$$\langle av + bw, av + bw \rangle = a^2 \langle v, v \rangle + b^2 \langle w, w \rangle + 2ab \langle v, w \rangle$$

From definition of timecone v, w are timelike and $\langle v, w\rangle < 0$. This implies that

$$\langle av + bw, av + bw\rangle < 0 \quad \Longleftrightarrow \quad av + bw \in C(u)$$

From Lemma 2.2.2 we have a way to characterize vectors in a timecone. From that we showed that the timecones are convex so linear combination of timelike vectors cant escape from them.

Now we show that for timelike vectors in a Lorentz vector space has some analogous properties with the inner product case.

Proposition 2.2.3. *[38, p. 144] Let V be a Lorentz vector space and $v, w \in V$ are timelike vectors. Then*

1. *the inverse Cauchy-Schwarz inequality holds*

$$|\langle v, w\rangle| \geq |v||w|,$$

with equality if and only if v, w are collinear.

2. *If v, w are in the same timecone of V,. there is a unique number $\phi \geq 0$, called the **hyperbolic angle** between v, w, such that*

$$\langle v, w\rangle = -|v||w| \cosh\phi$$

Proof. (1.) We write $w = av + \bar{w}$, with $\bar{w} \in v^{\perp}$. So we have

$$\begin{aligned}\langle w, w\rangle &= a^2\langle av + \bar{w}, av + \bar{w}\rangle \\ &= a^2\langle v, v\rangle + \langle \bar{w}, \bar{w}\rangle + 2\langle v, \bar{w}\rangle \\ &= a^2\langle v, v\rangle + \langle \bar{w}, \bar{w}\rangle \end{aligned} \tag{2.1}$$

So

$$\begin{aligned}\langle v, w\rangle^2 &= \langle v, av + \bar{w}\rangle^2 \\ &= (a\langle v, v\rangle + \langle v, \bar{w}\rangle)^2 \\ &= a^2\langle v, v\rangle^2 \\ &= (\langle w, w\rangle - \langle \bar{w}, \bar{w}\rangle)\langle v, v\rangle \\ &= \langle w, w\rangle\langle v, v\rangle - \langle \bar{w}, \bar{w}\rangle\langle v, v\rangle \end{aligned} \tag{2.2}$$

where in the second from last equality we use (2.1). We know that w, v are timelike vectors and $\bar{w}$ is a spacelike vector, so

$$\left.\begin{aligned}\langle w, w\rangle\langle v, v\rangle > 0 \\ \langle \bar{w}, \bar{w}\rangle\langle v, v\rangle < 0\end{aligned}\right\} \quad \Longrightarrow \quad \langle v, w\rangle^2 \geq \langle w, w\rangle\langle v, v\rangle = |v|^2|w|^2$$

where the implication comes from (2.2).

If $v, w \in V$ are collinear it holds that

$$\begin{aligned} w, v \text{ collinear} \quad &\Longleftrightarrow \quad w = av \\ &\Longleftrightarrow \quad \bar{w} = 0 \\ &\Longleftrightarrow \quad \langle \bar{w}, \bar{w}\rangle \\ &\Longleftrightarrow \quad \langle v, w\rangle^2 = \langle w, w\rangle\langle v, v\rangle \end{aligned}$$

(2.) From Lemma 2.2.2 if $v, w \in C(u)$ then $\langle v, w\rangle < 0$ and so from (1.)

$$-\frac{\langle v, w\rangle}{|v||w|} \geq 1$$

From the equation

$$\cosh^2\phi - \sinh^2\phi = 1,$$

we have

$$\cosh\phi \geq 1$$

So for a unique $\phi \geq 0$ we have

$$\cosh\phi = -\frac{\langle v, w\rangle}{|v||w|}$$

□

Because the Cauchy-Schwarz inequality is backwards so is the triangle inequality.

Corollary 2.2.4. *[38, p. 144] If v, w are timelike vectors in the same timecone then*

$$|v| + |w| \leq |v + w|,$$

with equality if and only if v, w are collinear.

Proof. v, w are in the same timecone if and only if $\langle v, w\rangle < 0$. From the inverse Cauchy-Schwarz we have

$$|v||w| \leq -\langle v, w\rangle \tag{2.3}$$

hence

$$(|v| + |w|)^2 = |v|^2 + 2|v||w| + |w|^2 \leq |v|^2 - 2\langle v, w\rangle + |w|^2 \tag{2.4}$$

where the inequality holds from (2.3),

$$\begin{aligned}
|v|^2 - 2\langle v, w\rangle + |w|^2 &= |\langle v, v\rangle| - 2\langle v, w\rangle + \langle w, w\rangle \\
&= -\langle v, v\rangle - \langle w, w\rangle - 2\langle v, w\rangle \\
&= -(\langle v, v\rangle + \langle w, w\rangle + \langle v, w\rangle + \langle v, w\rangle) \\
&= -(\langle v, w + v\rangle + \langle v + w, w\rangle) \\
&= -\langle v + w, v + w\rangle \\
&= |\langle v + w, v + w\rangle| \qquad (2.5) \\
&= |v + w|^2 \qquad (2.6)
\end{aligned}$$

where the Equation 2.5 holds since if $v, w \in C(u)$ then $v + w \in C(u)$.

So from (2.4) and (2.6) we have

$$(|v| + |w|)^2 \leq |v + w|^2 \quad \Longrightarrow \quad |v| + |w| \leq |v + w|$$

From the inverse Cauchy-Schwarz we have that v, w are collinear if and only if $|v||w| = -\langle v, w\rangle$. So the inequality in (2.4) becomes an equality and we have from the same calculations

$$|u| + |w| = |v + w|$$

and the inverse holds since we only used equalities. □

The above Corollary tells us that when we are in a timecone the shortest path between two points is not a straight line anymore. In fact it will be longer to follow a straight line than going around it.

Definition 2.2.5. *Assume V a Lorentz vector space.*

- *If $v \in V$ is a null or timelike vector then we call it* ***causal***.
- *If v is a timelike vector then $\bar{C}(v)$ is the set of all causal vectors w such that $\langle v, w\rangle < 0$ and we call it the* ***causal cone*** *containing v.*

Since we have shown that timecones are convex we consider if timelike curves can change timecones. First some definitions that we will find useful

- A map $\alpha : [a, b] \to M$ is a **curve segment** provided that it has a smooth extension to an open interval containing a, b.
- A map $\beta : [a, b] \to M$ is a **piecewise smooth curve** segment provided there is a partition $a = t_0 < t_1 < \cdots < t_{k+1} = b$ of $[a, b]$ such that each $\beta|_{[t_i,t_{i+1}]}$ is a curve segment.
- A map $\beta : I \to M$, with I being an open interval, is piecewise smooth provided that for all $a < b$ in I the restriction $\beta|_{[a,b]}$ is piecewise smooth.

Another possibility arises for piecewise smooth curves, that is if they can change timecones on a break t_i.

- A piecewise smooth curve $\alpha : I \to M$ is timelike if

  $$\langle \alpha'(t), \alpha'(t)\rangle < 0 \quad \forall t \in I$$

  and for each break t_i

  $$\langle \alpha'(t_i^-), \alpha'(t_i^+)\rangle < 0$$

  where t_i^- is from $\alpha|_{[t_{i-1},t_i]}$ and t_i^- is from $\alpha|_{t_i}, t_{i+1}$.

So if a piecewise smooth is timelike then it cant change timecones at a break t_i. The next Lemma tells us if a curve is initially timelike then it stays in a single timecone.

Lemma 2.2.6. *[38, p. 146] Let p be a point of a Lorentz manifold M. Suppose that $\beta : [0, b] \to T_pM$ is a piecewise smooth curve starting at 0 such that $\alpha = \exp_p \circ \beta$ is timelike. Then β remains in a single timecone of T_pM.*

The analogous statement holds for causal curve and causal cone.

2.3 Time Orientation

From the definition of timecones we notice that for each tangent space we divide the timelike vectors into two components. Also there is no intrinsic way to distinguish them form each other. So we have to choose one of them. This is called choosing a **time orientation** for a Lorentz scalar product space. When a Lorentz scalar product space has a time orientation it is called **time oriented Lorentz scalar**

product space. If we have chosen a time orientation then the timelike vectors that belong to the specific timecone are called **future oriented**.

For example on a Lorentz scalar space (V, g) we can choose a time orientation if we choose the timecone $C(u)$. Then for timelike vectors v, if $g(v, u) < 0$ then v is future oriented and if $g(v, u) > 0$ then v is past oriented.

Definition 2.3.1. *[41, p. 10] Let (M, g) be a Lorentz manifold.*

- *A **time orientation** of (M, g) is a choice of time orientation of each scalar product space $(T_pM, g_p), p \in M$, such that the following holds. For each $p \in M$, there is an open neighborhood U of p and a smooth vector field X on U such that X_q is future oriented for all $q \in U$.*
- *A Lorentz metric g on a manifold M is said to be **time orientable** if (M, g) has a time orientation.*
- *A Lorentz manifold (M, g) is said to be **time orientable** if (M, g) has a time orientation.*
- *A Lorentz manifold with a time orientation is called a **time oriented Lorentz manifold**.*

Not all Lorentzian manifolds are time orientable. The Lorentizan manifolds that admit a time orientation are called **time orientable**.

Example 2.3.2. *Minkowski space $\mathbb{R}^n_1$ is time-orientable by choosing as time-orientation to be the one that containts the coordinate vector field ∂_{u_0} of the natural coordinates $u^0, \ldots, u^n$.*

We have the following Lemma to characterise time-orientability.

Lemma 2.3.3. *[38, p. 145] A Lorentz manifold M is time orientable if and only if there exists a timelike vector field $X \in \mathcal{X}(M)$*

For Lorentizan manifolds there is no relation between orientability and time orientability since we can find examples that admits either ones.

2.4 Riemannian and Lorentzian Geometry

An important difference between Riemannian and Lorentizan Geometry is that of existence of metrics in a smooth manifold. In Riemannian geometry it is known that every smooth manifold admits a Riemannian metric [31, p. 11]. This is not the case in Lorentzian geometry. Not every smooth manifold can be made into a Lorentzian manifold.

Proposition 2.4.1. *[38, p. 149] For a smooth manifold M the following are equivalent:*

1. *There exists a Lorentz metric on M.*
2. *There exists a time-rientable Lorentz metric on M.*
3. *there is a nonvanishing vector field on M.*

4. *Either M is noncompact, or M is compact and has Euler number $\chi(M) = 0$*

Next we will see a local property of geodesics that changes from Riemannian manifold to Lorentzian manifolds. First we define the arc length of a curve.

Definition 2.4.2. *Suppose a semi-Riemannian manifold M and a piecewise smooth curve segment $\sigma : [a, b] \to M$. Then the* ***arc length*** *of σ is*

$$L(\sigma) = \int_a^b |\sigma'(s)| \mathrm{d}s$$

In general we restrict our attention to curves which are either causal or spacelike such that

- if a curve σ is causal then the length is

$$L(\sigma) = \int \sqrt{-g(\sigma', \sigma')\mathrm{d}s}$$

- and if a curve σ is spacelike then the length is

$$L(\sigma) = \int \sqrt{g(\sigma', \sigma')\mathrm{d}s}$$

Remark 2.4.3. *The length of curves which change from timelike to spacelike is not defined.*

In Riemannian geometry it is known that locally the shortest length of a curve between two points is on a radial geodesic.

Lemma 2.4.4. *Suppose M be a Riemannian manifold. Let U be a normal neighborhood of p. If $q \in U$ then the radial geodesic segment (up to reparametrization) $\sigma : [0, 1] \to U$ from p to q is the unique shortest curve in U from p to q.*

This is reasonable as we see in the case of the Euclidean space where the radial geodesics are straight lines and intuitively they are the shortest curves between two points. In Lorentzian geometry this is not always the case since in Corollary 2.2.4 we showed that for timelike vectors inside a timecone the inverse of the triangle inequality holds, meaning that the straight path is no longer the shortest path between two points in a timecone. In fact locally inside a timecone the timelike radial geodesics have the longest length between two points compared to the rest.

Proposition 2.4.5. *Suppose M a Lorentzian manifold. Let U be a normal neighborhood of p. If $q \in U$ and there exists a timelike curve in U from p to q, then the radial geodesic segment (up to reparametrization) σ from p to q is the unique longest timelike curve in U from p to q.*

For complete Riemannian manifolds we have the Hof-Rinow theorem

Theorem 2.4.6 (Hopf-Rinow)**.** *[38, p. 138] For a connected Riemannian manifold M the following conditions are equivalent:*

1. *As a metric space under Riemannian distance d, M is complete, which means that every Cauchy sequence converges.*

2. *There exists a point $p \in M$ from which M is geodesically complete, meaning that* $\exp_p$ *is defined on the entire tangent space T_M.*

3. *M is geodesically complete.*

4. *Every closed bounded subset of M is compact.*

But unfortunately for the semi-Riemannian case there isnt such a generalization of the Hopf-Rinow theorem. In a semi-Riemannian manifold we can decompose its completeness by looking at the causal character. Meaning that a semi-Riemannian manifold can be **spacelike complete** (every maximal spacelike curve is complete), **null complete** and **timelike complete**. If it is complete in all three categories we say that the semi-Riemannian manifold is complete. There is an example of a Lorentz surface which is null and spacelike complete but not timelike complete (sf [38, p. 154] Example 5.43).

Chapter 3

Special Relativity

In this section we will see the consequences of the Lorentzian geometry in the flat spacetime known as Minkowski spacetime. The differences between Minkowski and Euclidean space have great impact on the physical explanations of several phenomena. In the next sections we will try to showcase these differences. For this purpose we briefly review some basic features of Newtonian space.

3.1 Newtonian Space and Time

Definition 3.1.1. *[38, p. 159]* ***Newtonian space*** *is a Euclidean 3-dimensional space E, meaning its a Riemannian manifold isometric to $(\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$.*

We dont define Newtonian space as simply $\mathbb{R}^3$ because in nature there are no coordinate axes and by choosing a coordinate system we can change where we place the axes to take measurements. With the same reasoning we will define the Minkowski spacetime.

Definition 3.1.2. *A* ***Newtonian particle*** *is a curve $\alpha : I \to E$ in Newtonian space, with I an interval in Newtonian time.*

Definition 3.1.3. *A* ***Euclidean coordinate system*** *for E is an isometry $\xi : E \to \mathbb{R}^3$.*

In Euclidean coordinates geodesics have affine coordinates $x^i(\gamma(t)) = a^i t + b^i$, tangent vectors from parallel translations have the same components and the distance from p to q is given by the usual Pythagorean formula

$$d(p,q) = \left(\sum (x^i(q) - x^i(p))^2\right)^{1/2}$$

3.1.1 Newtonian Space-Time

To define particles a propagating in time we can draw the graph $\{(t, a(t)) | t \in I\}$. For that reason we can think the particle moving in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The next definition doesnt have any physical meaning, we use it to show later the differences in properties of the Minkowski spacetime.

Definition 3.1.4. ***Newtonian space-time*** *is the Riemannian product manifold $\mathbb{R} \times E$ of Newtonian time and Newtonian space.*

- A point $(t, x) \in \mathbb{R} \times E$ is called an **event**.
- The natural projection $T : \mathbb{R} \times E \to \mathbb{R}$ is the universal Newtonian clock which measures the time interval between two events.

Instead of representing a particle by its equations of motions, we can represent it by its worldline in $\mathbb{R} \times E$.

Definition 3.1.5. *A **wordline** in Newtonian space-time is a one-dimensional submanifold W such that $T|_W$ is a diffeomorphism onto an interval.*

Wordlines and particles are equivalent. If we have a particle $a : I \to E$, then its graph $\{(t, a(t)) : t \in I\}$ is a wordline. Conversely, if we have a worldline W, then we have the particle $a = S \circ (T|_W)^{-1}$.

In Newtonian mechanics

1. The speed of an object can attain arbitrarily high speeds.
2. The speed of light changes relative to the observer.
3. A particle is either at rest or not.

There have been experiments showing that the above are not necessarily true. For example it has been known for 300 years that light travels in vacuum at very high but finite speed. The first must be treated such that no material particle can travel faster than the speed of light. We assume that the tangent directions of light are constant of speed c and so they determine a cone the tangent space of every point in $\mathbb{R} \times E$. The material particles are required to have their tangent lines inside this cone, and so their is speed is bellow c.

To get such cones we can change the sign of the time coordinate in the metric tensor of $\mathbb{R} \times E$. Thus the cones become the *nullcones* and the Newtonian space-time becomes the Minkowski spacetime.

3.2 Minkowski Spacetime

Definition 3.2.1. *A connected, time-oriented, four-dimensional Lorentz manifold is called **spacetime**.*

Definition 3.2.2. *A **Minkowski spacetime** M is a spacetime that is isometric to Minkowski $\mathbb{R}^4_1$.*

- The time-orientation of the Minkowski spacetime is called the **future**, and its negative is the **past**.
- A tangent vector in a future causal cone is called **future-pointing** or **future-directed**.
- A causal curve is **future-pointing** if all its velocity vectors are future-pointing.

The point is Minkowski spacetime are still called events and particles are called worldlines. But in contrast to $\mathbb{R} \times E$ and $\mathbb{R}^4_1$ there doesnt exist a canonical time function. Meaning there is not an absolute time, but several.

Definition 3.2.3. *[38, p. 163] A* ***material particle*** *in M is a timelike future-pointing curve $a : I \to M$ such that $|a'(\tau)| = 1$ for all $\tau \in I$. The parameter τ is called the* ***proper time*** *of the particle.*

So the material particle a has arc length parametrization. Which means we define the proper time as the arc length of a timelike future-pointing curve $\gamma : [a, b] \to M$ with arbitrary velocity

$$\tau(t) = \int_0^t |\gamma'(u)| \mathrm{d}u$$

Proper time is the elapsed time that each particle experiences from the event $\gamma(a)$ to $\gamma(b)$. Its like each particle comes equipped with a clock measuring its proper time.

Definition 3.2.4. *[38, p. 163] A* ***lightlike particle*** *is a future-pointing null geodesic $\gamma : I \to M$.*

Any particle $\beta : I \to M$ is a regular curve, and its $\beta(I)$ is a one dimensional submanifold of M called the *wordlline* of β.

Particles in M have *mass*, positive for material particles and necessarily zero for lightlike particles.

A fundamental hypothesis in relativity is that light moves in geodesics and since $\langle \gamma', \gamma' \rangle = 0$ for lightlike particle we cannot parametrize by proper time.

A particle that is a geodesic is said to be **freely falling**. By saying free falling we mean that something moves under the influence of gravity alone. Since Minkowski spacetime is flat it means that special relativity is limited to situations where gravitation is negligible.

Definition 3.2.5. *[38, p. 164] A* ***Lorentz*** *coordinate system in M is a time-orientation-preserving isometry $\xi : M \to \mathbb{R}^4_1$.*

Lemma 3.2.6. *Suppose (M, g) the Minkowski spacetime. Then a coordinate system $\xi : M \to \mathbb{R}^4$ is Lorentz if and only if $g_{ij} = \delta_{ij}\varepsilon_j$, where $\varepsilon = (-1, 1, 1, 1)$, and ∂_{x_0} is future-pointing.*

Proof. Suppose the coordinate system $(M, \xi = (x^1, \dots, x^n))$ has the coordinate basis (∂_{x_i}) and suppose the coordinate basis ∂_{u_i} of $\mathbb{R}^4_1$. We have that

$$\partial_{x_i} = \mathrm{d}\xi^{-1}(\partial_{u_i}) \quad \Longleftrightarrow \quad \mathrm{d}\xi(\partial_{x_i}) = \partial_{u_i}$$

So from isometry

$$\begin{aligned} \langle \mathrm{d}\xi(\partial_{x_i}), \mathrm{d}\xi(\partial_{x^j}) \rangle_{\mathbb{R}^4_1} = \langle \partial_{x_i}, \partial_{x_j} \rangle_M \quad &\Longleftrightarrow \quad \langle \partial_{u_i}, \partial_{u_j} \rangle_{\mathbb{R}^4_1} = \langle \partial_{x_i}, \partial_{x_j} \rangle_M \\ &\Longleftrightarrow \quad \delta_{ij}\varepsilon_j = g_{ij} \end{aligned}$$

Let X is a future oriented timelike vector field, then $\mathrm{d}\xi(X)$ is also timelike vector field since

$$\langle \mathrm{d}\xi(X), \mathrm{d}\xi(X), \rangle = \langle X, X \rangle < 0$$

So we have

$$\langle \mathrm{d}\xi(\partial_{x_0}), \mathrm{d}\xi(X) \rangle_{\mathbb{R}^4_1} = \langle \partial_{x_0}, X \rangle_M \quad \Longleftrightarrow \quad \langle \partial_{u_0}, \mathrm{d}\xi(X) \rangle_{\mathbb{R}^4_1} = \langle \partial_{x_0}, X \rangle_M$$

From Example 2.3.2 we know that ∂_{u_0} is future oriented in $\mathbb{R}^4_1$ and from Lemma 2.2.2 that two timelike vectors remain in the same timecone if and only if their scalar product is negative. So

$$\langle \partial_{u_0}, \mathrm{d}\xi(X)\rangle_{\mathbb{R}^4_1} < 0 \quad \iff \quad \langle \partial_{x_0}, X\rangle_M < 0$$

if and only if ∂_{x_0} is in the same timecone with X if and only if ∂_{x_0} is future oriented.

Since we used only equivalences, the converse is also true. □

Lemma 3.2.7. *Suppose (M, g) the Minkowski spacetime and (e_i) is an orthonormal basis of T_pM such that e_0 is future pointing. Then there is a unique Lorentz coordinate system ξ such that*

$$\partial_{x_i}|_p = e_i$$

for $0 \le i \le 3$.

Proof. This is immediate from Proposition 1.5.7. □

3.3 Minkowski Geometry

Since for Minkowski spacetime M we have an isometry $\phi : M \to \mathbb{R}^4_1$, we have the following properties:

1. For any points $p, q \in M$ there is a unique geodesic σ such that $\sigma(0) = p$ and $\sigma(1) = q$. This is because we know that $\mathbb{R}^4_1$ is a complete space, then from the isometry we have that the Minkowski spacetime is also complete.

2. There is a natural isometry $T_pM \cong T_qM$ which is distant parallelism (see Definition 1.3.6). Suppose that γ is a smooth curve on M such that $\gamma(t_0) = p, \gamma(t_1) = q$, $X \in \mathfrak{X}(\gamma)$ is parallel along γ such that $X = X^i\partial_{x_i}$. Let $P : T_pM \to T_qM$ be the parallel translation of γ on M and $\overline{P} : T_{\phi(p)} \to T_{\phi(q)}$ be the distant parallelism of $\phi(\gamma)$ on $\mathbb{R}^4_1$. Because we have an isometry ϕ, we know that (see [38, p. 91])

$$\overline{P} \circ \mathrm{d}\phi = \mathrm{d}\phi \circ P \quad \iff \quad P = (\mathrm{d}\phi)^{-1} \circ \overline{P} \circ \mathrm{d}\phi$$

$$\begin{tikzcd}
T_{\phi(p)}\mathbb{R}^4_1 \arrow[r, "\overline{P}"] & T_{\phi(q)}\mathbb{R}^4_1 \arrow[d, "\mathrm{d}\phi_q"] \\
T_pM \arrow[u, "\mathrm{d}\phi_p"] \arrow[r, "P"] & T_qM
\end{tikzcd}$$

So we have

$$\begin{aligned}
\mathrm{d}\phi^{-1} \circ \overline{P} \circ \mathrm{d}\phi(X) &= \mathrm{d}\phi^{-1}(\overline{P}(\mathrm{d}\phi(X))) \\
&= \mathrm{d}\phi^{-1}(\overline{P}(\mathrm{d}\phi(X^i\partial_{x_i}|_p))) \\
&= X^i\mathrm{d}\phi^{-1}(\overline{P}(\mathrm{d}\phi(\partial_{x_i}|_p))) \\
&= X^i\mathrm{d}\phi^{-1}(\mathrm{d}\phi(\partial_{x_i}|_q)) \\
&= X^i\partial_{x_i}|_q
\end{aligned}$$

This means that
$$P(X^i\partial_{x_i}|_p) = X^i\partial_{x_i}|_q$$
which tells us that P is a distant parallelism.

3. Each exponential map $\exp_p : T_pM \to M$ is an isometry. This is because from the naturality of the exponential map (see [31, p. 130]), if $\phi : M \to \mathbb{R}^n_\nu$ is an isometry then

$$\phi \circ \exp_p = \exp_{\phi(p)} \circ \mathrm{d}\phi_p \quad \Longrightarrow \quad \exp_p = \phi^{-1} \circ \exp_{\phi(p)} \circ \mathrm{d}\phi_p$$

$$\begin{tikzcd} T_{\phi(p)}\mathbb{R}^n_\nu \arrow[r, "\exp_{\phi(p)}"] & \mathbb{R}^n_\nu \arrow[d, "\phi^{-1}"] \\ T_pM \arrow[u, "\mathrm{d}\phi_p"] \arrow[r, "\exp_p"] & M \end{tikzcd}$$

From the above we conclude that M viewed from p is geometrically the same as T_pM viewed from 0. This tells us that instead of looking the causality of T_pM now we can look at the causality M itself. Since M is complete, for all $p, q \in M$ the displacement vector $\vec{pq} = \sigma'(0)$ (see Definition 1.5.9) is well defined. We want to describe the displacement vector in terms of a Lorentz coordinate system. To do that we notice that
$$\exp_p(\vec{pq}) = q$$
We have that $\exp^{-1}_{\phi(p)}(\phi(q)) = \phi(q) - \phi(p)$. Also $\sigma(0) = p, \sigma(1) = q$, this implies $\phi \circ \sigma(0) = \phi(p), \phi \circ \sigma(1) = \phi(q)$. Since ϕ is an isometry and σ is a geodesic of M, we know that $\phi \circ \sigma$ is also a geodesic of $\mathbb{R}^4_1$. Suppose $\phi(q) = X^i e_i|_{\phi(q)}$ and $\phi(p) = X^i e_i|_{\phi(p)}$, then

$$\begin{aligned} \vec{pq} &= \mathrm{d}\phi^{-1} \circ \exp_{\phi(p)} \circ \phi(q) \\ &= \mathrm{d}\phi^{-1}(\phi(q) - \phi(p)) \\ &= \mathrm{d}\phi^{-1}(\phi(q) - \mathrm{d}\phi^{-1}(\phi(p)) \\ &= \sum X^i(\mathrm{d}\phi^{-1}(e_i|_{\phi(q)}) - \mathrm{d}\phi^{-1}(e_i|_{\phi(p)})) \\ &= \sum X^i\partial_{x_i}|_q - X^i\partial_{x_i}|_p \\ &= \sum X^i(q)\partial_{x_i}(q) - X^i(p)\partial_{x_i}(p) \end{aligned}$$

since we have shown in (2.) that there is a natural isometry $T_pM \cong T_qM$ which is distant parallelism, we denote $\partial_{x_i}(q) = \partial_{x_i}(p)$, and so

$$\vec{pq} = (\sum X^i(q) - X^i(p))\partial_{x_i}$$

We imagine $\vec{pq}$ as the vector with its start on p and its end on q.

Definition 3.3.1. *[38, p. 165] Suppose M is a Minkowski spacetime.*

- *For an event $p \in M$ we define*

$$\mathcal{I}_p^+ = \{q \in M : \vec{pq} \text{ is timelike an future-pointing}\}$$

*to be the **future timecone** of p.*

- *The boundary of the future timecone of p except p itself is*

$$\mathcal{N}_p^+ = \partial\mathcal{I}_p^+ = \{q \in M : \vec{pq} \text{ is null and future-pointing}\}$$

*and we call it the **future lightcone** of p.*

- *The union of the future timecone of p and the future lightcone of p is called the **future causal cone** of p*

$$\mathcal{J}_p^+ = \mathcal{I}_p^+ \cup \mathcal{N}_p^+$$

- *Similarly, we define the **past timecone** of p as $\mathcal{I}_p^-$, the **past lightcone** of p as $\mathcal{N}_p^-$ and the **past causal cone** of p as $\mathcal{J}_p^-$.*

- *The union of the past and future lightcones of p is called **lightcone***

$$\mathcal{N}_p = \mathcal{N}_p^+ \cup \mathcal{N}_p^-$$

- *If a point q is not an element of neither future or causal cone of p, then it is called **spacelike relative** to p.*

- *An event p can **influence** an event q if and only if there is a particle (material or lightlike) from p to q.*

From Lemma 2.2.6 and the the definition of influence it becomes evident why the word causal has been used till now.

1. If there is an event $p \in M$ then the only events that can be *influenced* by p are $q \in \mathcal{J}_p^+$.

2. If there is an event $p \in M$ then the only events that can *influence* the event p are $q \in \mathcal{J}_p^-$.

Thus if there is an event $p \in M$ then it cannot be influenced by events spacelike relative to p and it cannot influence events spacelike relative to p. This is in contrast to Newtonian space-time since for an event $(t_0, x_0) \in \mathbb{R} \times E$ the past and the future fill the whole space-time except the hyperplane $t = t_0$.

Definition 3.3.2. *[38, p. 166] For $p, q \in M$ the number $pq = |\vec{pq}| \geq 0$ is called the **separation** between p and q.*

In terms of a Lorentz coordinate system the separation is

$$pq = \left| -(x^0(q) - x^0(p))^2 + \sum_1^3 (x^j(q) - x^j(p))^2 \right|$$

Remark 3.3.3 (Physical Significance of Separation)**.** *[38, p. 166] Suppose M is a Minkowski spacetime and $p, q \in M$*

1. *If $\vec{pq}$ is timelike future-pointing, then pq is the elapsed proper time $L(\sigma)$ of the unique freely falling material particle from p to q.*
2. *$\vec{pq}$ is lightlike if and only if $pq = 0$ if and only if there is a lightlike particle through p and q.*
3. *If $\vec{pq}$ is spacelike, then $pq \geq 0$ is the distance from p to q as measured by any freely falling observer orthogonal to $\vec{pq}$ (we will soon define what observer means).*

We can now prove some trigonometry facts in Minkowski spacetime.

Lemma 3.3.4. *[38, p. 166] Suppose $\vec{op}$ is spacelike and $\vec{oq}$ is timelike, then*

1. *If $\vec{pq}$ is lightlike and $\vec{op} \perp \vec{oq}$ then $op = oq$.*
2. *If $\vec{op} \perp \vec{oq}$ and $op = oq$ then $\vec{pq}$ is lightlike.*
3. *If $\vec{pq}$ is lightlike and $op = oq$ then $\vec{op} \perp \vec{oq}$.*

Proof. By moving the $\vec{pq}$ to the point o we have

$$\vec{pq} = (X(q) - X(p))\partial_{x_i} = (X(q) - X(o) - X(p) + X(o))\partial_{x_i} = \vec{oq} - \vec{op}$$

and

$$\begin{aligned} \langle \vec{pq}, \vec{pq} \rangle &= \langle \vec{oq} - \vec{op}, \vec{oq} - \vec{op} \rangle \\ &= \langle \vec{oq}, \vec{oq} \rangle + \langle \vec{op}, \vec{op} \rangle - 2\langle \vec{oq}, \vec{op} \rangle \end{aligned}$$

From which we get

$$\pm pq^2 = -oq^2 + op^2 - \langle \vec{oq}, \vec{op} \rangle \tag{3.1}$$

(1) We have $\langle \vec{pq}, \vec{pq} \rangle = 0$ and $\langle \vec{op}, \vec{oq} \rangle = 0$. From 3.1 we get

$$op^2 = oq^2$$

(2) We have $\langle \vec{op}, \vec{oq} \rangle = 0$ and $op^2 = oq^2$ so from 3.1

$$\pm pq^2 = 0$$

(3) We have $\langle \vec{pq}, \vec{pq} \rangle = 0$ and $op = oq$ then from 3.1

$$\langle \vec{oq}, \vec{op} \rangle = 0$$

□

Remark 3.3.5. *The convention of writing the null cone at angle of 45 degrees comes from the previous lemma, since for $\vec{op} = \partial_{x_0}$ and $\vec{oq} = \partial_{x_1}$ we have from (2) that $\vec{pq}$ is lightlike.*

We prove the corresponding Pythagorean formula and the corresponding orthogonal projections of an orthogonal triangle in the Euclidean space for the Minkowski spacetime.

Proposition 3.3.6. *[38, p. 167] Let p, q be events in the same timecone of o and such that $\vec{op} \perp \vec{pq}$. Then*

1. $oq^2 = op^2 - pq^2$.
2. *If ϕ is the hyperbolic angle between $\vec{op}$ and $\vec{oq}$, then*

$$op = oq \cosh \phi, \qquad pq = oq \sinh \phi$$

Proof. We parallel translate $\vec{pq}$ to o and so we have

$$\vec{oq} = \vec{op} + \vec{pq}$$

Then

$$\begin{aligned}\langle \vec{oq}, \vec{oq} \rangle &= \langle \vec{op} + \vec{pq}, \vec{op} + \vec{pq} \rangle \\ &= \langle \vec{op}, \vec{op} \rangle + \langle \vec{pq}, \vec{pq} \rangle + 2\langle \vec{op}, \vec{pq} \rangle\end{aligned}$$

But $\vec{op} \perp \vec{pq}$, which means

$$oq^2 = op^2 - pq^2$$

(2) Suppose

$$u = \frac{\vec{op}}{op}, \quad v = \frac{\vec{oq}}{oq}$$

So

$$\begin{aligned}\langle \vec{op}, \vec{oq} \rangle &= \langle u \cdot op, v \cdot oq \rangle \\ &= op \cdot oq \langle u, v \rangle \\ &= -op \cdot oq \cosh \phi\end{aligned}$$

and

$$\begin{aligned}\langle \vec{op}, \vec{oq} \rangle &= \langle \vec{op}, \vec{op} + \vec{pq} \rangle \\ &= \langle \vec{op}, \vec{op} \rangle + \langle \vec{op}, \vec{pq} \rangle \\ &= -op^2\end{aligned}$$

By combining the previous two equations we have

$$op = oq \cosh \phi$$

From (1)

$$\begin{aligned}oq^2 &= op^2 - pq^2 \\ &= oq^2 \cosh^2 q + pq^2\end{aligned}$$

and so

$$\begin{aligned}pq^2 &= oq^2(1 - \cosh^2 \phi) \\ &= oq^2 \sinh^2 \phi\end{aligned}$$

And from the fact that $\phi \geq 0$ we have $\sinh \phi \geq 0$ and so

$$pq = oq \sinh \phi$$

$\square$

We have seen the properties of the Minkowski spacetime and its geometry. Now we want to be able to make measurements of phenomenas relative to ourselves, the observer. By comparing such measurements from one observer to the other we will be able to see the differences that each observer experiences and on what they can agree on.

When we say **observer** we mean a material particle of the Minkowski spacetime M. For a Lorentz coordinate system ξ the x^0 axis of ξ will be the worldline of a freely falling observer ω such that $x^0(\omega(t)) = t$, meaning that the proper time of ω is t. The Lorentz coordinate system produces the measurements taken by an observer ω.

From Lemma 3.2.7 we see that for every freely falling observer $\omega : I \to M$ by choosing a different orthonormal base on a point $\omega(t_0)$ there exists a unique Lorentz coordinate system ξ. So for every freely falling observer there are many Lorentz coordinate systems. When choosing one we will call it the **associated Lorentz coordinate system**.

For a Minkowski spacetime M we cannot distinguish between the time and the space with a natural way. What we can do is use the Lorentz coordinates to make some observations.

Definition 3.3.7. *Suppose M be a Minkowski spacetime with Lorentz coordinate system ξ.*

- *We denote the coordinate slice $x^0 = 0$ by*
$$E_0 = \{q \in M : (0, x^1(q), x^2(q), x^3(q)) \in \mathbb{R}^4_1\}$$
which is a Euclidean space identified with $\mathbb{R}^3$.

- *For each event $p \in M$ we call $x^0(p)$ the ξ-**time** of p.*

- *For each event $p \in M$ we call $\vec{p} = (x^1(p), x^2(p), x^3(p)) \in \mathbb{R}^3$ the ξ-**position** of p.*

Suppose we have $a : I \to M$ be a particle that is either material or lightlike. We told before that each particle has an internal clock which measures the time that it experiences it passes. By using the ξ-time $t = x^0(a(s))$ of an event $a(s)$ we can paramitrize the ξ-position $(x^1(a(s)), x^2(a(s)), x^3(a(s)))$ by the ξ-time. So from the measurements of a we get what the observer ω observes of a. To do that we need to show that $x^0 \circ a$ is a diffeomorphism.

$$\frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}s} = \mathrm{d}x^0(a') = \langle \operatorname{grad} x^0, a' \rangle$$

We see that

$$\operatorname{grad} x^0 = \sum_{i,j} g^{ij} \frac{\partial x^0}{\partial x^i} \partial_{x_j} = \sum_j \varepsilon_j \delta_{ij} \partial_{x_j} = -\partial_{x_0}$$

and so

$$\frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}s} = -\langle \partial_{x_0}, a' \rangle$$

but we know that a is causal curve and future-pointing which means $\langle \partial_{x_0}, a' \rangle < 0$. This gives us

$$\frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}s} > 0$$

Hence $x^0 \circ a$ is a diffeomorphism of I onto an interval $J \subset \mathbb{R}$. Let $u : J \to I$ be the inverse function. As we wanted we have at ξ-time the ξ-position of a, which is

$$\vec{a}(t) = (x^1 au(t), x^2 au(t), x^3 au(t)).$$

The curve $\vec{a} : J \to \mathbb{R}^3$ is called the **ξ-associated Newtonian particle** of a.

Remark 3.3.8. *In general, unless said otherwise, we will denote the parameters t and s by $t = x^0 a(s)$ and $s = u(t)$. We will write*

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}s} > 0$$

and from the chain rule

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{a}/\mathrm{d}s}{\mathrm{d}t/\mathrm{d}s}$$

We can now check how the speed of lightlike and material particles change relative to free falling observers. In the next Lemma we see that light has a constant speed when measuring it relative to every free falling observer.

Lemma 3.3.9. *Let M be a Minkowski spacetime and $\gamma : I \to M$ be a lightlike particle of M. Suppose ξ is the Lorentz coordinate system of M, then the associated Newtonian particle $\vec{\gamma}$ of γ is a straight line in $\mathbb{R}^3$ and it has constant speed*

$$v = \left| \frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}t} \right| = 1$$

Proof. From definition of lightlike particle we have that γ is a geodesic in M. So $\xi \circ \gamma$ is also a geodesic in $\mathbb{R}^4_1$, since ξ is an isometry. Thus γ can be expressed as

$$x^i(\gamma(s)) = a_i s + b_i$$

for $i = 0, \ldots, 3$. Hence its ξ-position

$$\vec{\gamma}(s) = (x^1(\gamma(s)), x^2(\gamma(s)), x^3(\gamma(s)))$$

is a straight line in $\mathbb{R}^3$ and its reparametrization $\vec{\gamma}(t)$, the associated Newtonian particle of γ, is a straight line in $\mathbb{R}^3$. We write

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}s} &= \frac{\mathrm{d}(x^0 \circ \gamma)}{\mathrm{d}s}\partial_{x_0} + \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i} \\ &= \frac{\mathrm{d}t}{\mathrm{d}s}\partial_{x_0} + \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i}\end{aligned}$$

We take the scalar product

$$\begin{aligned}\langle \frac{\mathrm{d}\gamma}{\mathrm{d}s}, \frac{\mathrm{d}\gamma}{\mathrm{d}s} \rangle &= \langle \frac{\mathrm{d}\gamma}{\mathrm{d}s}\partial_{x_0} + \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i}, \frac{\mathrm{d}\gamma}{\mathrm{d}s}\partial_{x_0} + \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i} \rangle \\ &= \langle \frac{\mathrm{d}t}{\mathrm{d}s}\partial_{x_0}, \frac{\mathrm{d}t}{\mathrm{d}s}\partial_{x_0} \rangle + \langle \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i}, \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i} \rangle + 2\langle \frac{\mathrm{d}t}{\mathrm{d}s}\partial_{x_0}, \sum_{i=1}^{3} \frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\partial_{x_i} \rangle \\ &= -\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \sum_{i=1}^{3} \left(\frac{\mathrm{d}(x^i \circ \gamma)}{\mathrm{d}s}\right)^2\end{aligned}$$

Since $\mathrm{d}\gamma/\mathrm{d}s$ is a null vector and $\mathrm{d}t/\mathrm{d}s > 0$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left|\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}s}\right|$$

so we have that the speed v of the associated Newtonian particle $\vec{\gamma}$ is

$$v = \left|\frac{\mathrm{d}\vec{\gamma}}{\mathrm{d}t}\right| = \frac{|\mathrm{d}\vec{\gamma}/\mathrm{d}s|}{\mathrm{d}t/\mathrm{d}s} = 1$$

□

In the next Proposition we show the formula of the speed v that the free falling observer ω measures of the material particle a and that it cant reach or surpass the speed of light. Also we show the formula for the relation of the proper time of the material particle and the time of the free falling observer.

Proposition 3.3.10. *Let M be a Minkowski spacetime and $a : I \to M$ be a material particle. Suppose ξ is the Lonrentz coordinate system of M. If $\vec{a}$ is the associated Newtonian particle of a, then*

1. *The speed of $\vec{a}$ is*

$$v = \left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}t}\right| = \tanh\phi$$

where $\phi \geq 0$ is the hyperbolic angle between $a' = \mathrm{d}a/\mathrm{d}\tau$ and the coordinate vector ∂_{x_0} of ξ. Since $v = \tanh\phi$ we have $0 \leq v < 1$.

2. *The time τ of a and its ξ-time are related by*

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}\tau} = \cosh\phi = \frac{1}{\sqrt{1-v^2}} \geq 1$$

The v and ϕ depend on τ.

Proof. First we prove (2.). From hypothesis we have that a' and ∂_{x_0} are timelike and future-pointing such that $|a'| = |\partial_{x_0}| = 1$. From Proposition 2.2.3 we have a unique hyperbolic angle $\phi \geq 0$ such that

$$\begin{aligned}\langle a', \partial_{x_0}\rangle &= -|a'||\partial_{x_0}|\cosh\phi \\ &= -\cosh\phi\end{aligned}$$

meaning $\langle a', \partial_{x_0}\rangle = \cosh\phi \geq 1$ for $\phi \geq 0$. Since

$$a'(\tau) = \sum_i \left(\frac{\mathrm{d}(x^i \circ a)}{\mathrm{d}\tau}\right)\partial_{x_i}$$

we have

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}\tau} = -\langle a', \partial_{x_0}\rangle = \cosh\phi \tag{3.2}$$

It is known that $\cosh^2\phi - \sinh^2\phi = 1$, so its easy to see

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = -\langle a', \partial_{x_0}\rangle = \cosh\phi = \frac{1}{\sqrt{1-\tanh^2\phi}}$$

and since we will show $\tanh\phi = v$ this proves (2.).

(1.) We compute the scalar product for a'

$$\begin{aligned}
\langle a', a'\rangle &= \langle \sum_i \frac{\mathrm{d}(x^i \circ a)}{\mathrm{d}\tau}\partial_{x_i}, \sum_j \frac{\mathrm{d}(x^j \circ a)}{\mathrm{d}\tau}\partial_{x_j}\rangle \\
&= \sum_{i,j} \frac{\mathrm{d}(x^i \circ a)}{\mathrm{d}\tau}\frac{\mathrm{d}(x^j \circ a)}{\mathrm{d}\tau}\langle \partial_{x_i}\partial_{x_j}\rangle \\
&= \sum_{i,j} \frac{\mathrm{d}(x^i \circ a)}{\mathrm{d}\tau}\frac{\mathrm{d}(x^j \circ a)}{\mathrm{d}\tau}\delta_{ij}\varepsilon_i \\
&= -\left(\frac{\mathrm{d}(x^0 \circ a)}{\mathrm{d}\tau}\right)^2 + \sum_{i=1}^{3}\left(\frac{\mathrm{d}(x^i \circ a)}{\mathrm{d}\tau}\right)^2 \\
&= -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau}\right|^2
\end{aligned}$$

Because $\langle a', a'\rangle = -1$ we have

$$-\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau}\right|^2 = -1 \quad\Longrightarrow\quad \left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau}\right|^2 = -1 + \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

From equation (3.2)

$$\left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau}\right| = \sqrt{-1+\cosh^2\phi} = \sinh\phi \geq 0$$

Thus the speed of the associated Newtonian particle $\vec{a}$ is

$$v = \left|\frac{\mathrm{d}\vec{a}}{\mathrm{d}t}\right| = \frac{|\mathrm{d}\vec{a}/\mathrm{d}\tau|}{\mathrm{d}t/\mathrm{d}\tau} = \frac{\sinh\phi}{\cosh\phi} = \tanh\phi$$

□

With the free falling observer we can make the following remarks.

Remark 3.3.11. *Let M be a Minkowski spacetime and ω a free falling observer.*

- *For a associated Lorentz coordinate system ξ of ω we have that the coordinate hyperplane*
$$E_t = \{q \in M : (t, x^1(q), x^2(q), x^3(q)) \in \mathbb{R}^4_1\}$$
is perpendicular to ω on the point $\omega(t) \in M$ because $x^0(\omega(t)) = t$, they are perpendicular in $\mathbb{R}^4_1$ and the isometry ξ. And so for all choices of ξ, x^0 is the same and the events that ω sees as simultaneous with $\omega(t)$ are the events in E_t.

- *From the definition of the observer, the associated Newtonian particle $\vec{\omega}$ is contant. We will call E_0 the **restspace** of ω. Every E_t can serve as a restspace of ω since we can project an event $p \in E_t$ to and event $q \in E_s$ such that $\vec{pq}$ is parallel to ω.*

- *We notice that since ω is the x^0 coordinate axis and we have shown that distant parallelism in $\mathbb{R}^4_1$ implies distant parallelism in M, we have that ω' is distant parallel to ∂_{x_0}. In proposition 3.3.10 we found the speed of a by using its hyperbolic angle with ∂_{x_0}. Since parallel translation preserves angles we see that $\phi(\tau)$ is the hyperbolic angle between ω' and a'. We will say the funtion $v = |\mathrm{d}\vec{a}/\mathrm{d}t|$ is the speed of a relative to ω and the function $\phi = \tanh^{-1} v$ is the velocity parameter of a relative to ω.*

- *In proposition 3.3.10 we showed that*

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}}$$

  for $0 \leq v < 1$. From the above we notive that the faster the particle is moving relative to the observer, the slower the particles clock τ runs relative to the observer's clock t.

- *If $\vec{pq}$ is orthogonal to a freely falling observer ω then p, q lie in a hyperplane E_t and so we have $x^0(p) = x^0(q)$. This means that their separation is*

$$pq = \left(\sum_{j=1}^{3}(x^j(p) - x^j(q))^2\right)^{1/2}$$

  This means that the distance between events makes sense when observers consider the events simultaneous.

From the above we see that for two different obervers ω_1, ω_2 which are nonparallel they have different restspaces. This tells us that events $p, q \in M$ can be simultaneous for ω_1 but not for ω_2 and vice verca. This happens because $\vec{pq}$ can be orthogonal to ω_1 but not to ω_2 and vice verca. So the concept of simultaneity is relative to the observer.

We have seen the notion of speed relative to ∂_{x_0} and relative to an observer. We can generalize it for material particles a and b by defining the instanteneous velocity parameter as their hyperbolic angle ϕ of their velocities $a'(\sigma), b'(\tau)$ and the instanteneous relative speed $v = \tanh\phi$.

Supose b is a free falling particle, then the free falling observers that are parallel to b consider him to be at rest. Other free falling observers can see him habing constant speeds $0 \leq v < 1$. But if $b'' \neq 0$ then no freely falling observer considers b to be at rest.

Example 3.3.12 (Relativistic Addition of Velocities)**.** *[38, p. 172] Assume we have a space station α free falling in space where inside it there is a rocketship and inside the rocketship there is a spaceman.*

At the event p the rocketship β leaves the space station by free falling with speed $v_1 > 0$ relative to the space station α. At the event q the spaceman γ ejects himself into space while free falling with speed v_2 relative to the rocketship β. We assume γ' is in the same plane with α and β. For $v_2 > 0$ we mean forward, away from α and $v_2 < 0$ means backward, towards α. We want to find the speed v of spaceman γ relative to α.

If ϕ_1 is the hyperbolic angle between α', β' at the event p and ϕ_2 is the hyperbolic angle between β', γ' at the event q, then $v_1 = \tanh\phi_1$ and $v_2 = \tanh\phi_2$. By distant *parallelism we parallel translate γ' from q to q, this tells us that b' is between α' and γ'. So from the additivity of hyperbolic angles $\phi = \phi_1 + \phi_2$.*

This implies

$$\begin{aligned} v &= \tanh\phi \\ &= \tanh(\phi_1 + \phi_2) \\ &= \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} \end{aligned}$$

where the last equality is a well known hyperbolic equation. And so

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

The same holds if $v_2 < 0$.

So instead of the addition of speed in the Newtonian spacetime we have addition of velocity parameters in the Minkowski spacetime.

Example 3.3.13 (The Twin Paradox)**.** *[38, p. 173] Assume there exists two twin brothers Peter and Paul living in a spaceship free falling in space. On their 21st birthday Peter decides to leave Paul to go on a journey. When Peter leaves he is free falling with constant speed $v = 24/25$ relative to Peter, this is event o. After free falling for 7 years of his proper time Peter turns and starts coming back symmetrically, this is event p. After free falling for another 7 years of his proper time he arrives back to Paul, this is event q. We will see that upon arriving Peter is 35 years old, but Paul is 71 years old.*

We draw a perpendicular from p which crosses the worldline of the spaceship at x. From propositions 3.3.6 and 3.3.10

$$\begin{aligned} ox &= op\cosh\phi \\ &= \frac{7}{(1 - (24/25)^2)^{1/2}} \\ &= 25 \end{aligned}$$

Since Peter turned symmetrically we have $xq = 25$. So the proper time for Paul that has passed till Peter returned is

$$oq = ox + xq = 2ox = 50$$

And so Paul now is 71 years old.

Corollary 3.3.14. *[38, p. 174] Let M be a Minkowski spacetime and $\sigma : [a, b] \to M$ be a material particle such that $\sigma(a) = p$ and $\sigma(b) = q$. Then we have for its elapsed time*

$$\Delta\tau = b - a \leq pq$$

with equality if and only if σ is free falling.

Proof. This comes from the fact that M is a normal neighborhood of every point $p \in M$ and from proposition 2.4.5. $\square$

So free fall is the unique slowest way to go from one event to another in the Minkowski spacetime.

From this chapter we see that the Minkowski spacetime has the possibility to solve some problems with simple trigonometry, like we would in Newtonian space. But from the hyperbolic nature of the geometry in the timecones of the Minkowski spacetime some physical effects change. What allows us to put physics in this geometric framework is what is called **general covariance principle**.

- All physical laws are independent of the choice of a particular coordinate system. Equally this is expressed as, every equation of physical laws must be written in terms of tensors.

3.4 Isometry Group of Minkowski Space

Suppose M a Minkowski spacetime and two Lorentz coordinate systems ξ, η. We would like to determine what kind of maps are the coodrinate changes $\xi^{-1} \circ \eta$. This maps will be the change of measurements between different observers in M. Since ξ, η are isometries then the coordinate change is also an isometry and will preserve the Minkowski metric. So to find this maps we need to describe the isometry group of M.

Definition 3.4.1. *Suppose M is a semi-Riemannian manifold. Its isometry group $I(M)$ with composition as its operation is*

$$I(M) = \{\phi : M \to M | \phi \text{ isometry}\}$$

In general we know that the tangent space of a semi-Riemannian manifold with index ν is isometric to $\mathbb{R}^n_\nu$. So the isometry group $I(\mathbb{R}^n_\nu)$ is important.

Lemma 3.4.2. *Suppose M is a Minkowski spacetime and $\xi : M \to \mathbb{R}^4_1$ a Lorentz coordinate system of M. Then the isometry group $I(M)$ of M is isomorphic to the isometry group $I(\mathbb{R}^4_1)$ of $\mathbb{R}^4_1$.*

Proof. We define a map $F : I(M) \to I(\mathbb{R}^4_1)$ such that $F(\phi) = \xi \circ \phi \circ \xi^{-1}$.

- Composition of isometries is an isometry, so $F(\phi) \in I(\mathbb{R}^4_1)$
- It is a homomorphism since

$$\begin{aligned} F(\phi \circ \psi) &= \xi \circ \psi \circ \xi^{-1} \\ &= \xi \circ \phi \circ \xi^{-1} \circ \xi \circ \psi \circ \xi^{-1} \\ &= F(\phi) \circ F(\psi) \end{aligned}$$

- It is injective since

$$\begin{aligned} F(\phi) = \mathrm{id}_{\mathbb{R}^4_1} \quad &\implies \quad \xi \circ \phi \circ \xi^{-1} = \mathrm{id}_{\mathbb{R}^4_1} \\ &\implies \quad \phi = \xi^{-1} \circ \mathrm{id}_{\mathbb{R}^4_1} \circ \xi = \mathrm{id}_M \end{aligned}$$

And obviously is surjective. □

From the previous Lemma we see that the isometry group of $\mathbb{R}^4_1$ describes the isometry group of the Minkowski spacetime M.

First we will turn our attention on the linear isometries of $I(\mathbb{R}^n_\nu)$.

For $0 \leq \nu \leq n$ and $v, w \in \mathbb{R}^n_\nu$,

- the **signature matrix** is the diagonal matrix $\varepsilon = \delta_{ij}\varepsilon_j$ such that

$$\varepsilon_1 = \cdots = \varepsilon_\nu = -1 \quad \text{and} \quad \varepsilon_{\nu+1} = \cdots = \varepsilon_n = 1,$$

  Since it is diagonal with units in it, we have $\varepsilon^{-1} = \varepsilon = \varepsilon^t$

- the scalar product of $\mathbb{R}^n_\nu$ is written equally as

$$\langle v, w\rangle = \varepsilon v \cdot w$$

- the set of all matrices $g \in \mathrm{GL}(n, \mathbb{R})$ such that

$$\langle gv, gw\rangle = \langle v, w\rangle$$

  is denoted by $\mathrm{O}_\nu(n)$ and it is called **semiorthogonal group**. This is the same as the set of all linear isometries of $\mathbb{R}^n_\nu$. Since $\mathrm{O}_\nu(n)$ is a closed set and a subgroup of $\mathrm{GL}(n, \mathbb{R})$ then it is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ and Lie group by itself.

Lemma 3.4.3. *[38, p. 234] Suppose $g \in M_n(\mathbb{R})$. The following are equivalent:*

1. *$g \in O_\nu(n)$.*
2. *$g^t = \varepsilon g^{-1}\varepsilon$.*
3. *The columns (rows) of g form an orthonormal base for $\mathbb{R}^n_\nu$ on which the first ν vectors are timelike.*
4. *g sends an orthonormal base of $\mathbb{R}^n_\nu$ to an orthonormal base.*

Proof. (1.) $\iff$ (2.) For g relative to an orthonormal base we have that the adjoint relative to the dot product is g^t. For all $v, w \in \mathbb{R}^n_\nu$

$$\begin{aligned}
\langle gv, gw\rangle = \langle v, w\rangle \quad &\iff \quad \varepsilon gv \cdot gv = \varepsilon \cdot w & (3.3)\\
&\iff \quad g^t\varepsilon gv \cdot gv = \varepsilon v \cdot w & (3.4)\\
&\iff \quad g^t\varepsilon gv = \varepsilon v & (3.5)\\
&\iff \quad g^t\varepsilon g = \varepsilon & (3.6)\\
&\iff \quad g^t = \varepsilon g^{-1}\varepsilon & (3.7)
\end{aligned}$$

(1.) $\iff$ (4.) We know that linear isometries send orthonormal bases to orthonormal bases and from Lemma 1.1.20 we know that it also preserves the index of the scalar product space.

(4.) $\iff$ (3.)

- (For columns)

  Let $u_1, \ldots, u_n \in \mathbb{R}^n$ is the natural base and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator. Then $gu_1, \ldots, gu_n$ are the columns of g since

  $$g(u_k) = gu_k = \sum_j g_{ij}(u_k)_j$$

  The columns $gu_1, \ldots, gu_n$ are orthonormal relative to $\mathbb{R}^n_\nu$ if and only if $u_1, \ldots, u_n$ is are orthonormal relative to $\mathbb{R}^n_\nu$ and g sends them to an orthonormal base.

- (For rows)

  From (2.) we have

  $$g \in \mathrm{O}_\nu(n) \quad \Longleftrightarrow \quad g^t \in \mathrm{O}_\nu(n)$$

  and so it holds for the rows if and only if it holds for the columns.

□

From the previous Lemma we notice that it is necessary for the timelike vectors in the matrix $g \in \mathrm{O}_\nu(n)$ to appear first or else g wont be an element of the semiorthogonal group.

Example 3.4.4. *Let the orthogonal unit vectors* $(0,1), (1,0) \in \mathbb{R}^2_1$ *and the matrix*

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we notice that g is not an element of $O_\nu(n)$ even though $(0,1)$ *orthonormal spacelike and* $(1,0)$ *orthonormal timelike. This happens because of the order of the collumns.*

- If $\nu = 0$ or $\nu = n$ then $\mathrm{O}_0(n) = \mathrm{O}_n(n) = \mathrm{O}(n)$ which is the **orthogonal group** of all linear isometries of the Euclidean space $\mathbb{R}^n$.

- If $n \geq 2$, then $\mathrm{O}_1(n)$ is called the **Lorentz group** of all linear isometries of the Minkowski space $\mathbb{R}^n_1$.

Lemma 3.4.5. *For $0 \leq \nu \leq n$ the Lie groups $O_\nu(n)$ and $O_{n-\nu}(n)$ are isomorphic.*

Proof. For $\mathrm{GL}(n, \mathbb{R})$ we have the Lie group automorphism $C_\sigma : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ such that

$$C_\sigma(g) = \sigma g \sigma^{-1}$$

where, for I_q the identity matrix with q number of units in its diagonal,

$$\sigma = \begin{pmatrix} 0 & I_q \\ I_p & 0 \end{pmatrix}$$

By taking the restriction to $\mathrm{O}_\nu(n)$ it suffices to show that elements of $\mathrm{O}_\nu(n)$ are sent to elements of $\mathrm{O}_{n-\nu}(n)$ by C_σ.

Let $g \in O_\nu(n)$. From Lemma 3.4.3 we have that the first ν columns of g are timelike orthonormal vectors and the rest $n - \nu$ are spacelike orthonormal vectors.

For example we notice that for $1 \leq j \leq \nu$ columns relative to an orthonormal base $e_i \in \mathbb{R}^n_\nu$

$$\begin{aligned}
\langle g_{ij}e_i, g_{ij}e_i\rangle = -1 \quad &\iff \quad -\sum_{i=1}^{\nu} g_{ij}^2 + \sum_{i=\nu+1}^{n} g_{ij}^2 = -1 \\
&\iff \quad -\sum_{i=\nu+1}^{n} g_{ij}^2 + \sum_{i=1}^{\nu} g_{ij}^2 = 1
\end{aligned}$$

Similarly, for the $\nu + 1 \leq j \leq n$ columns

$$\begin{aligned}
\langle g_{ij}e_i, g_{ij}e_i\rangle = 1 \quad &\iff \quad -\sum_{i=1}^{\nu} g_{ij}^2 + \sum_{i=\nu+1}^{n} g_{ij}^2 = 1 \\
&\iff \quad -\sum_{i=\nu+1}^{n} g_{ij}^2 + \sum_{i=1}^{\nu} g_{ij}^2 = -1
\end{aligned}$$

From the above is evident that by interchanging the first ν rows with the last $n-\nu$ rows we will get the spacelike and timelike columns interchanged, but that leaves them in the wrong position since we need the timelike vectors to be the first $n-\nu$ for it to be an element of $\mathrm{O}_{n-\nu}(n)$. And so after the permutation of the rows by interchanging the ν columns with the last $n-\nu$ columns we will have what we want. More speciffically

$$\sigma = \begin{pmatrix} 0 & I_{n-\nu} \\ I_\nu & 0 \end{pmatrix}$$

This is a permutation matrix and so its inverse equals its transpose

$$\sigma^{-1} = \begin{pmatrix} 0 & I_\nu \\ I_{n-\nu} & 0 \end{pmatrix}$$

For permutation matrix P we have that Pg permutates the rows and gP permutates the columns. And so we have that

$$\begin{aligned}
\sigma g \sigma^{-1} &= \begin{pmatrix} 0 & I_{n-\nu} \\ I_\nu & 0 \end{pmatrix} \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} 0 & I_\nu \\ I_{n-\nu} & 0 \end{pmatrix} \\
&= \begin{pmatrix}
g_{(\nu+1)(\nu+1)} & g_{(\nu+1)(\nu+2)} & \cdots & g_{(\nu+1)n} & g_{(\nu+1)1} & \cdots & g_{(\nu+1)\nu} \\
g_{(\nu+2)(\nu+1)} & g_{(\nu+2)(\nu+2)} & \cdots & g_{(\nu+2)n} & g_{(\nu+2)1} & \cdots & g_{(\nu+2)\nu} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
g_{\nu(\nu+1)} & g_{n(\nu+2)} & \cdots & g_{nn} & g_{n1} & \cdots & g_{n\nu} \\
g_{1(\nu+1)} & g_{1(\nu+2)} & \cdots & g_{1n} & g_{11} & \cdots & g_{1\nu} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
g_{\nu(\nu+1)} & g_{\nu(\nu+2)} & \cdots & g_{\nu n} & g_{\nu 1} & \cdots & g_{\nu\nu}
\end{pmatrix}
\end{aligned}$$

and so $C_\sigma(g) \in \mathrm{O}_{n-\nu}(n)$. □

Lemma 3.4.6. *[38, p. 235] The Lie algebra $\mathfrak{o}_\nu(n)$ of $O_\nu(n)$ is the subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ consisting of all S for which $S^t = -\varepsilon S \varepsilon$. Such S have the form*

$$\begin{pmatrix} a & x \\ x^t & b \end{pmatrix},$$

where $a \in \mathfrak{o}(\nu), b \in \mathfrak{o}(n-\nu)$, and x in an arbitrary $\nu \times (n-\nu)$ matrix.

Since from the orthogonal group $\dim \mathfrak{o}(\nu) = \nu(\nu-1)/2$, then for the semiorthogonal group we have

$$\dim \mathrm{O}_\nu(n) = \dim \mathfrak{o}_\nu(n) = \frac{n(n-1)}{2}$$

From the previous Lemma for $S \in \mathfrak{o}_\nu(n)$, we have $S^t = -\varepsilon S \varepsilon$ which implies

$$\begin{aligned}
\langle Sv, w\rangle &= \varepsilon Sv \cdot w = (\varepsilon Sv)^t w \\
&= v^t S^t \varepsilon w = v^t(-\varepsilon S\varepsilon)\varepsilon w \\
&= -v^t \varepsilon S w = -(\varepsilon v)^t S w \\
&= -\varepsilon v \cdot Sw = -\langle v, Sw\rangle
\end{aligned}$$

This is equivalent with $\langle Sv, w\rangle = -\langle v, Sw\rangle$ for all $v, w \in \mathbb{R}^n_\nu$. So all the elements of $\mathfrak{o}_\nu(n)$ are the skew adjoint linear operators on $\mathbb{R}^n_\nu$.

Example 3.4.7. *The orthogonal group $O(2)$ describes the rotations and the reflections of two vectors in a circle in $\mathbb{R}^2$. We could see that by taking the polar coordinates of two vectors $u_1, u_2 \in \mathbb{R}^2$ and then rotating them by an angle θ. This was the linear map with the matrix*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

then from the determinant map, since it is continouous, we have that $O(n)$ is disjoint. Its components are $O^+(2)$ which is the set of rotations in $\mathbb{R}^2$ and $O^-(2)$ which is the set of a reflaction and rotations in $\mathbb{R}^2$.

The corresponding group is the semiorthogonal group $O_1(2)$. We can similarly take the hyperbolic coordinates of two vectors vectors similarly and then turn them by an angle θ. This will produce a linear map with matrix

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

This kind of matrices are called ***boosts*** *of $\mathbb{R}^2_1$. Here each $a \in O_1(2)$ will send each hyperbola $\langle v, v\rangle = 1$ and $\langle v, v\rangle = -1$ to itself but in the process it may reverse one of its branches or even both. This choices give us the decomposition of $O_1(2)$ into 4 disjoint open subsets. The set preserving all branches is the set of all boosts.*

So the orthogonal group $\mathrm{O}(n)$ and the semiorthogonal group $\mathrm{O}_\nu(n)$ have two main differences

1. $\mathrm{O}(n)$ is compact since it is closed and bounded in $\mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ but $\mathrm{O}_\nu(n)$ is not compact since it is unbounded in $\mathbb{R}^{n^2}$. For example the elements of the form
$$\begin{pmatrix} \cosh\phi & \sinh\phi & 0 \\ \sinh\phi & \cosh\phi & 0 \\ 0 & I & 0 \end{pmatrix}$$
constitutes an unbounded set on $\mathbb{R}^{n^2}$.

2. $\mathrm{O}(n)$ has two components, but $\mathrm{O}_\nu(n)$ has four components.

For $g \in \mathrm{O}_\nu(n)$ we observe that by keeping the timelike components of a vector $v \in \mathbb{R}^{n-\nu}$ unchanged we can rotate the spacelike part of it just as we would in $\mathrm{O}(n)$. For example by the element

$$\begin{pmatrix} I_\nu & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & I_{n-(\nu+2)} \end{pmatrix}$$

To define the components of the semiorthogonal group first we will check the matrix $g \in \mathrm{O}_\nu(n)$ by its block matrices. Let $0 < \nu < n$ such that $\mathbb{R}^n_\nu = \mathbb{R}^\nu_\nu \times \mathbb{R}^{n-\nu}$. Then for $g \in \mathrm{O}_\nu(n)$ we have

$$\begin{pmatrix} g_T & a \\ b & g_S \end{pmatrix}$$

such that g_T is $\nu \times \nu$ matrix and g_S is $(n-\nu) \times (n-\nu)$ matrix. We will call $g_T : \mathbb{R}^\nu_\nu \to \mathbb{R}^\nu_\nu$ the **timelike part** and $g_S : \mathbb{R}^{n-\nu} \to \mathbb{R}^n - \nu$ the **spacelike part**.

Definition 3.4.8. *For $0 < \nu < n$ and $g \in O_\nu(n)$ we will say that*

- *it **preserves time-orientation** when* $\det g_T > 0$*,*
- *it **reverses time-orientation** when* $\det g_T < 0$*,*
- *it **preserves space-orientation** when* $\det g_S > 0$*,*
- *it **reverses space-orientation** when* $\det g_S < 0$*.*

And so the decomposition of $\mathrm{O}_\nu(n)$ to its components is the following

Definition 3.4.9. *For $g \in O_\nu(n)$*

- *$g \in O^{++}_\nu(n)$ if and only if it preserves time and space orientation,*
- *$g \in O^{+-}_\nu(n)$ if and only if it preserves time-orientation and reverses space-orientation,*
- *$g \in O^{-+}_\nu(n)$ if and only if it reverses time-orientation and preserves space-orientation,*
- *$g \in O^{--}_\nu(n)$ if and only if it reverses time and space orientation.*

From Lemma 3.4.3 we have

$$g^t = \varepsilon g^{-1} \varepsilon \quad \Longleftrightarrow \quad g \varepsilon g^t = \varepsilon$$

and from that $(\det g)^2 = 1$. As in the case of orthogonal matrices, every semiorthogonal matrix has determinant ± 1. It can be shown that $\mathrm{O}^{++}_\nu \cup \mathrm{O}^{--}_\nu$, $\mathrm{O}^{++}_\nu \cup \mathrm{O}^{+-}_\nu$, and $\mathrm{O}^{++}_\nu \cup O^{-+}_\nu$ are subgroups of $\mathrm{O}_\nu(n)$ (see Corollary 9.7 from [38]). We have the special semiorthogonal group

$$\mathrm{SO}_\nu(n) = O^{++}_\nu(n) \cup \mathrm{O}^{--}_\nu(n) = \{g \in \mathrm{O}_\nu(n) : \det g = 1\}$$

And so

- $g \in \mathrm{O}_\nu^{++} \cup \mathrm{O}_\nu^{--}$ is the linear isometry that preserves orientation,
- $g \in \mathrm{O}_\nu^{++} \cup \mathrm{O}_\nu^{+-}$ is the linear isometry that preserves time-orientation,
- $g \in \mathrm{O}_\nu^{++} \cup \mathrm{O}_\nu^{-+}$ is the linear isometry that preserves space-orientation.

We have seen the linear isometries of $\mathbb{R}^n_\nu$ form the subgroup $\mathrm{O}_\nu(n)$ of the isometry group $I(\mathbb{R}^n_\nu)$. Another isometry of $\mathbb{R}^n_\nu$ are the translations $T_x : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ such that $x \in \mathbb{R}^n_\nu$ and $T_x(v) = x + v$. We have

- $T_x \circ T_y = T_{x+y} = T_y \circ T_x$,
- $T_0 = \mathrm{id}_{\mathbb{R}^n_\nu}$,
- $(T_x)^{-1} = T_{-x}$.

The set $T(\mathbb{R}^n_\nu)$ of all translations of $\mathbb{R}^n_\nu$ is an abelian subgroup of $I(\mathbb{R}^n_\nu)$ and it is isomorphic to $\mathbb{R}^n$ via the $F : T(\mathbb{R}^n_\nu) \to \mathbb{R}^n$ such that $F(T_x) = x$. Now we prove that the semiorthogonal isometries and the translations are the only isometries of the semi-euclidean space.

Proposition 3.4.10. *[38, p. 240] Each isometry of $\mathbb{R}^n_\nu$ has a unique expression as $T_x \circ g$, with $x \in \mathbb{R}^n_\nu$ and $g \in O_\nu(n)$. Furthermore, $T_x \circ g \circ T_y \circ h = T_{x+g(y)} \circ g \circ h$.*

Proof. **Claim:** If $\phi : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ is an isometry such that $\phi(0) = 0$, then $\phi \in \mathrm{O}_\nu(n)$.

Proof of Claim. Since ϕ is an isometry, then $\mathrm{d}\phi_0$ is a linear isometry. We have the canonical linear isometry $T_v\mathbb{R}^n_\nu \cong \mathbb{R}^n_\nu$. So for canonical isometry $F : T_0\mathbb{R}^n_\nu \to \mathbb{R}^n\nu$ we can find a linear isometry $g : \mathbb{R}^n_\nu \to \mathbb{R}^n_\nu$ by writing $g = F \circ \mathrm{d}\phi_0 \circ F^{-1}$. Then $\mathrm{d}g_0 = \mathrm{d}\phi_0$, so from Proposition 1.5.10 we have that $\phi = g$. □

Suppose $\phi \in I(\mathbb{R}^n_\nu)$ and $x = \phi(0) \in \mathbb{R}^n_\nu$. We see that $(T_{-x} \circ \phi)(0) = 0$, so from the previous claim we have that $T_{-x} \circ \phi = g$ for some $\phi \in \mathrm{O}_\nu(n)$. And so $\phi = T_x \circ g$.

For uniqueness suppose $T_x \circ g = T_y \circ h$, then

$$x = (T_x \circ g)(0) = (T_y \circ h)(0) = y$$

and so

$$T_x \circ g = T_x \circ h \quad \Longleftrightarrow \quad g = h$$

For the last assertion, for all $v \in \mathbb{R}^n_\nu$ we have

$$(g \circ T_y)(v) = g(y + v) = g(y) + g(v) = T_{g(y)}g(v)$$

and so $g \circ T_y = T_{g(y)} \circ g$. From that

$$T_x \circ g \circ T_y \circ h = T_x \circ T_{g(y)} \circ g \circ h = T_{x+g(y)} \circ g \circ h$$

□

The multiplication rule that we proved shows that the translation subgroup $T(\mathbb{R}^n_\nu)$ is normal in $I(\mathbb{R}^n_\nu)$ since for all $g \in \mathrm{O}_\nu(n)$ and $T_x \in T(\mathbb{R}^n_\nu)$

$$T_x \circ g \circ T_y = T_{x+g(y)} \circ g \quad \Longleftrightarrow \quad g^{-1} \circ T_x \circ g = T_{x+g(y)} \circ T_{-y} \in T(\mathbb{R}^n_\nu)$$

By making $I(\mathbb{R}^n_\nu)$ into a smooth manifold we can define a diffeomorphism $f : \mathbb{R}^n \times \mathrm{O}_\nu(n) \to I(\mathbb{R}^n_\nu)$ such that $f(x, g) = T_x \circ g$, from that $I(\mathbb{R}^n_\nu)$ is a Lie group. The dimension of the set of isometries of the Minkowski space is

$$\dim I(\mathbb{R}^n_\nu) = \dim \mathbb{R}^n + \dim \mathrm{O}_\nu(n) = \frac{n(n+1)}{2}$$

$I(\mathbb{R}^n_\nu)$ for $0 < \nu < n$ has four components and for $\nu = 0, n$ it has two components.

Definition 3.4.11. *[38, p. 240]*

- $I(\mathbb{R}^n_\nu)$ *is called* ***semi-Euclidean group***.
- $I(\mathbb{R}^n)$ *is called* ***Euclidean group***.
- $I(\mathbb{R}^n_1)$ *is called* ***Poincare group*** *or* ***inhomogeneous Lorentz group***.

3.5 Poincare Group of Minkowski Spacetime

Now that we told some of the properties of the semi-Euclidean group $I(\mathbb{R}^n_\nu)$ we can see what kind of map are the coordinate changes of the Minkowski spacetime M. From Lemma 3.4.2 we have $I(M) \cong I(\mathbb{R}^4_1)$. The Poincare group $I(\mathbb{R}^4_1)$ has dimension

$$\dim I(\mathbb{R}^4_1) = \frac{4(4+1)}{2} = 10$$

and the Lorentz group $\mathrm{O}_1(4)$ has dimension

$$\dim \mathrm{O}_1(4) = \frac{4(4-1)}{2} = 6$$

From Proposition 3.4.10 we know that isometries of the Poincare group $g \in I(\mathbb{R}^4_1)$ are the combination of

- translations, for example $T_x(v) = x + v$
- rotations in the spacelike directions, for example
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- and boosts, for example
$$\begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
  which can be thought as rotations between space and time directions.

Elements of the Lorentz group $g \in \mathrm{O}_1(4)$ have $\det g = \pm 1$ and can preserve or reverse the time and space orientations of vectors. For example if we reverse the time-orietation we will change the timecone which we define the future directed vectors.

Chapter 4

General Relativity

4.1 Einstein Equations

General relativity is the extension of special relativity. It gives gravity the meaning as the curvature in the spacetime. General relativity follows these principles:

- **Equivalence principle:** One cannot distinguish locally between constant acceleration and constant gravitational field. This implies the inertial mass and the gravitational mass are equal.

- Freely falling particles follow timelike geodesics and photons follow null geodesics in the spacetime.

- **General covariance:** All physical laws written as equations must be independent from a choice of a coordinate system. This means that all physical laws must be written in terms of tensors.

Now we need equations that will tell us how the metric of the spacetime depends from the distribution of matter. For a spacetime (M, g) this equation is the **Einstein equation:**

$$\mathrm{Ric} - \frac{1}{2}Rg = 8\pi T \tag{4.1}$$

where Ric is the Ricci tensor, R is the Ricci scalar and T is the **energy-momentum tensor**. The energy-momentum tensor is a 2-covariant symmetric tensor which is divergence free and depends on the matter model. This means that in vacuum, in the absence of mass, we have $T = 0$.

We notice that in vacuum, by taking the trace of (4.1) we have:

$$\left.\begin{aligned} \mathrm{Ric} - \frac{1}{2}Rg = 0 \\ R = 0 \end{aligned}\right\} \quad \Longrightarrow \quad \mathrm{Ric} = 0$$

We say that the **Einstein vacuum equations** are:

$$\mathrm{Ric} = 0 \tag{4.2}$$

Minkowski spacetime is the trivial solution of the Einstein vacuum equations.

Remark 4.1.1. *From now on we will use the* ***Einstein summation notation*** *to denote sums in indices. This is done for simplicity. The summation notation tells us that when the same index is on top and on bottom then is is a sum. For example in a 3-dimensional Riemannian manifold (M, g) the vector field $X \in \mathfrak{X}(M)$ will be written as*

$$X = \sum_{i=1}^{3} X^i \partial_i = X^i \partial_i$$

or the Ricci scalar

$$R = \sum_{i,j=1}^{3} g^{ij} R_{ij} = g^{ij} R_{ij}$$

4.2 The Schwarzschild Metric

One of the most important exact solutions of the Einstein equations is the Schwarzschild metric

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \tag{4.3}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2$$

is the metric of the 2-round sphere and M is the mass of the body. Schwarzschild spacetime is a model of a universe containing a single star, where the star is spherically symmetric and static.

Definition 4.2.1. *A spacetime is called* ***spherically symmetric*** *if its isometry group has a subgroup G which is isomorphic to $SO(3)$ and the orbits of G are two-spheres.*

A spherically symmetric spacetime is a spacetime where its metric is invariant under rotations.

Definition 4.2.2. *A spacetime is called* ***static*** *if it admits a timelike Killing vector field which is orthogonal to a spacelike hypersuface of M.*

Killing vector fields are vector fields such that their flow is a family of isometries. Which means that their flow leaves the metric invariant. Since in the static case the Killing vector field is orthogonal to a spacelike hypersuface, the spacelike hypersurfaces are the level sets of $\{t = c\}$ which propagate in the flow of the Killing vector field. Since the metric is invariant under the flow, the components do not depend on the t-parameter and since they are orthogonal to Σ_t they dont have cross terms $\mathrm{d}t \, \mathrm{d}x^i$.

We notice that for $r \to \infty$ the metric components approach the components of the Minkowski spacetime in spherical coordinates. We say that this kind of metric is **asymptotically flat**. This definition is made precise in Definition 6.0.5.

By computing the geodesics of the Schwarzschild metric we can find many interesting results such as the perihelion precession and the bending of light. For more details we refer to [50, p. 136] and [38, p. 372].

In the next sections we will present two important facts about the Schwarzschild metric. One is that in vacuum it is the only possible spherically symmetric metric and for $r = 0, 2M$ it has singularities.

4.3 Birkhoff Theorem

This presentation for the proof of Birkhoff's theorem is based on [10, p. 197] which we refer to for more details.

Theorem 4.3.1. *For a spacetime (M, g) in vacuum, $Ric = 0$, the unique spherically symmetric metric is the Schwarzschild metric.*

Proof. First we will show that from the geometry of the spherically symmetric spacetime, the metric takes the form:

$$g = \mathrm{d}\tau^2(u, v) + r^2(u, v)\,\mathrm{d}\Omega^2(\theta, \phi)$$

and after that by plugging in the Einstein's vacuum equations we will get the Schwarzschild spacetime.

An equivalent definition for the spherically symmetric spacetime is that it has three Killing vector fields that are the same as those on the 2-round sphere S^2. This Killing vector fields are:

$$\begin{aligned} X &= \partial_\phi \\ Y &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi \\ Z &= -\sin\phi\partial_\phi - \cot\cos\phi\partial_\phi \end{aligned}$$

It can be shown that:

$$\begin{aligned} [X, Y] &= Z \\ [Y, Z] &= X \\ [Z, X] &= Y \end{aligned}$$

Hence the above Killing vector fields describe an involutive distribution D on the spacetime. We have the following theorem from [32, p. 502]

Theorem 4.3.2 (Global Frobenius Theorem)**.** *Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

Since the integral manifolds of D are the 2-spheres and we have a folliation of the spacetime by 2-spheres S^2 except the origin.

Now we want to give the spacetime coordinates (u, v, θ, ϕ) such that for $u = c_1, v = c_2$ constants each sphere can be specified by (c_1, c_2, θ, ϕ) and the metric g takes the form

$$g(c_1, c_2, \theta, \phi) = f(c_1, c_2)\,\mathrm{d}\Omega^2 \tag{4.4}$$

and for $\theta = c_3, \phi = c_4$

$$g(u, v, c_3, c_4) = \mathrm{d}\tau^2(u, v) \tag{4.5}$$

Assume a point $p \in S_p$ where S_p is a sphere. Assume S_p has the coordinates (θ, ϕ). At each point $q \in S_p$ let the set O_q be the set of geodesics which pass throught q and their tanget vectors at q are orthogonal to S_p. O_q is a two-dimensional subspace which is orthogonal to S_p. Let the one-dimensional subgroup I_q which leave the point q fixed. Then I_q leaves any vector perpendicular to S_p at q fixed and O_q is left invariant by I_q.

Next let $m \in S_m \cap O_q$, then we notice that O_q is orthogonal to both S_m and S_p. By connecting the points q, m with a unique geodesic we assign the same coordinates (θ, ϕ) of S_p to S_m.

Assume $X, Y \in T_qM$ such that $\text{span}(X, Y) = T_qO_q$. Then any other sphere will be connected with a unique orthogonal geodesic such that its tangent vector is $uX + vY \in T_qM$. By doing the same for every point $p \in S_q$ we assign the components of $uX + vY$ to be the coordinates (u, v).

So we have for the spacetime M the coordinates (u, v, θ, ϕ) and the metric satisfies (4.4), (4.4) with no cross terms between (u, v) and (θ, ϕ), for example $\mathrm{d}\theta\,\mathrm{d}u$.

We concluded that spherical symmetry gives us the metric

$$g = g_{uu}(u, v)\,\mathrm{d}u^2 + g_{uv}(u, v)(\mathrm{d}u\,\mathrm{d}v + \mathrm{d}v\,\mathrm{d}u) + g_{vv}(u, v)\,\mathrm{d}v^2 + r^2(u, v)\,\mathrm{d}\Omega^2$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

We change coordinates to (u, r). If r was a function of v alone, then we would change to (v, r). The metric becomes:

$$g = g_{uu}(u, r)\,\mathrm{d}u^2 + g_{ur}(u, r)(\mathrm{d}u\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}u) + g_{rr}(u, r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

Now we want to eliminate the cross terms $\mathrm{d}u\,\mathrm{d}r$ by changing to a suitable coordinate system (t, r) and give to the metric the form

$$m(t, r)\,\mathrm{d}t^2 + n(t, r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

If $t(u, r)$ was such a function then

$$\mathrm{d}t = \frac{\partial t}{\partial u}\,\mathrm{d}u + \frac{\partial t}{\partial r}\,\mathrm{d}r$$

and

$$\mathrm{d}t^2 = \left(\frac{\partial t}{\partial u}\right)^2 \mathrm{d}u^2 + \left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}u\,\mathrm{d}r + \mathrm{d}r\,\mathrm{d}u) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2$$

So we would need

$$m\left(\frac{\partial t}{\partial u}\right)^2 \mathrm{d}u^2 + 2m\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial r}\right)\mathrm{d}u\,\mathrm{d}r + \left[\left(\frac{\partial t}{\partial r}\right)^2 m + n\right]\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

hence the following equations must be satisfied

$$\begin{aligned} m\left(\frac{\partial t}{\partial u}\right)^2 &= g_{uu} \\ m\left(\frac{\partial t}{\partial u}\right)\left(\frac{\partial t}{\partial r}\right) &= g_{ur} \\ \left(\frac{\partial t}{\partial r}\right)^2 m + n &= g_{rr} \end{aligned}$$

Now the metric can be written as:

$$m(t,r)\,\mathrm{d}t^2 + n(t,r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

By comparing to the simplest spherically symmetric spacetime, the Minkowski, we take the first term to be negative. Then we assume $f(t,r), g(t,r)$ such that:

$$g = -e^{2f(t,r)}\,\mathrm{d}t^2 + e^{2w(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{4.6}$$

Now we need to compute the Christoffel symbols of g. For simplicity we denote

$$\partial_t = \partial_0, \quad \partial_r = \partial_1 \quad , \partial_\theta = \partial_2, \quad \partial_\phi = \partial_3$$

Lemma 4.3.3. *The non-zero Christoffel symbols of the metric g in equation* (4.6) *are*

$$\begin{aligned}
&\Gamma^0_{00} = \partial_0 f && \Gamma^1_{01} = \partial_0 w && \Gamma^2_{12} = \frac{1}{r}\\
&\Gamma^0_{01} = \partial_1 f && \Gamma^1_{11} = \partial_1 w && \Gamma^2_{33} = -\sin\theta\cos\theta\\
&\Gamma^0_{11} = e^{2(w-f)}\partial_0 w && \Gamma^1_{22} = -re^{-2w} && \Gamma^3_{13} = \frac{1}{r}\\
&\Gamma^1_{00} = e^{2(f-w)}\partial_1 f && \Gamma^1_{33} = -re^{-2w}\sin^2\theta && \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

Proof.

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2}g^{0l}\left(\partial_0 g_{l0} + \partial_0 g_{l0} - \partial_l g_{00}\right)\\
&= \frac{1}{2}g^{00}(2\partial_0 g_{00} - \partial_0 g_{00})\\
&= \frac{1}{2}g^{00}\partial_0 g_{00}\\
&= \frac{1}{2}(-e^{-2f}\partial_0(-e^{2f}))\\
&= \frac{1}{2}e^{-2f}2(\partial_0 f)e^{2f}\\
&= \partial_0 f
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{01} &= \frac{1}{2}g^{0l}\left(\partial_0 g_{1l} + \partial_1 g_{0l} - \partial_l g_{01}\right)\\
&= \frac{1}{2}g^{00}(\partial_0 g_{10} + \partial_1 g_{00} - \partial_0 g_{01})\\
&= \frac{1}{2}(-e^{-2f})(\partial_1(-e^{2f}))\\
&= \frac{1}{2}e^{-2f}2e^{2f}\partial_1 f\\
&= \partial_1 f
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2}g^{00}(\partial_1 g_{10} + \partial_1 g_{10} - \partial_0 g_{11}) \\
&= \frac{1}{2}(-e^{-2f})(-\partial_0 e^{2w}) \\
&= \frac{1}{2}(-e^{-2f})(-2\partial_0 w e^{2w}) \\
&= e^{2(w-f)}\partial_0 w
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}(\partial_0 g_{10} + \partial_0 g_{10} - \partial_1 g_{00}) \\
&= \frac{1}{2}(e^{-2w})(-\partial_1(-e^{2f})) \\
&= -\frac{1}{2}e^{-2w}(-2\partial_1 f e^{2f}) \\
&= e^{2(f-w)}\partial_1 f
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{01} &= \frac{1}{2}g^{11}(\partial_0 g_{11} + \partial_1 g_{01} - \partial_1 g_{01}) \\
&= \frac{1}{2}e^{-2w}(\partial_0 e^{2w}) \\
&= \frac{1}{2}e^{-2w}2\partial_0 w e^{2w} \\
&= \partial_0 w
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}(\partial_1 g_{11}) \\
&= \frac{1}{2}e^{-2w}(\partial_1 e^{2w}) \\
&= \frac{1}{2}e^{-2w}2\partial_1 w e^{2w} \\
&= \partial_1 w
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}(2\partial_2 g_{12} - \partial_1 g_{22}) \\
&= \frac{1}{2}e^{-2w}(-\partial_1(r^2)) \\
&= \frac{1}{2}e^{-2w}(-2r) \\
&= -re^{-2w}
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2}g^{11}(2\partial_3 g_{13} - \partial_1 g_{33}) \\
&= \frac{1}{2}e^{-2w}(-\partial_1(r^2\sin^2\theta)) \\
&= \frac{1}{2}e^{-2w}(-2r\sin^2\theta) \\
&= -r\sin^2\theta e^{-2w}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^2_{12} &= \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) \\
&= \frac{1}{2} r^{-2} \partial_1 r^2 \\
&= \frac{1}{2} r^{-2} 2r \\
&= \frac{1}{r}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{33} &= \frac{1}{2} g^{22}(2\partial_3 g_{23} - \partial_2 g_{33}) \\
&= \frac{1}{2} r^{-2}(-\partial_2(r^2 \sin^2\theta)) \\
&= \frac{1}{2} r^{-2}(-2\cos\theta\sin\theta r^2) \\
&= -\sin\theta\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{13} &= \frac{1}{2} g^{33}(\partial_1 g_{33} + \partial_3 g_{13} - \partial_3 g_{13}) \\
&= \frac{1}{2} r^{-2} \sin^{-2}\theta(\partial_1(r^2 \sin^2\theta)) \\
&= \frac{1}{2} r^{-2} \sin^{-2}\theta 2r \sin^2\theta \\
&= r^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^3_{23} &= \frac{1}{2} g^{33}(\partial_2 g_{33} + \partial_3 g_{23} - \partial_3 g_{23}) \\
&= \frac{1}{2} r^{-2} \sin^{-2}\theta(\partial_2 r^2 \sin^2\theta) \\
&= \frac{1}{2} r^{-2} \sin^{-2}\theta 2r^2 \sin\theta\cos\theta \\
&= \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

□

Next we need to compute the Riemman tensor.

Lemma 4.3.4. *The non-zero coefficients of the Riemann tensor for the metric g in equation* (4.6) *are:*

$$
\begin{aligned}
R_{011}{}^0 &= e^{2(w-f)}\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f \partial_0 w\right] + \left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w \partial_1 f\right] \\
R_{022}{}^0 &= -re^{-2w}\partial_1 f \\
R_{033}{}^0 &= -re^{-2w}\sin^2\theta\partial_1 f \\
R_{122}{}^0 &= -re^{-2f}\partial_0 w \\
R_{133}{}^0 &= -re^{-2f}\sin^2\theta\partial_0 w \\
R_{122}{}^1 &= re^{-2w}\partial_1 w \\
R_{133}{}^1 &= re^{-2w}\sin^2\theta\partial_1 w \\
R_{233}{}^2 &= \sin^2\theta(1 - e^{-2w})
\end{aligned}
$$

Proof. We use the fact that the coefficients of the Riemann tensor in local coordinates are:

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}$$

We begin the computations.

$$\begin{aligned}
R_{011}{}^{0} &= \partial_0 \Gamma^0_{11} - \partial_1 \Gamma^0_{01} + \Gamma^m_{11}\Gamma^0_{0m} - \Gamma^m_{01}\Gamma^0_{1m} \\
&= \partial_0(e^{2(w-f)}\partial_0 w) - \partial_1\partial_1 f + \Gamma^0_{11}\Gamma^0_{00} + \Gamma^1_{11}\Gamma^0_{01} - \Gamma^0_{01}\Gamma^0_{10} - \Gamma^1_{01}\Gamma^0_{11} \\
&= 2(\partial_0 w - \partial_0 f)e^{2(w-f)}\partial_0 w - e^{2(w-f)}\partial_0^2 w + \partial_1^2 f - e^{2(w-f)}\partial_0 w \partial_0 f \\
&\qquad - \partial_1 w \partial_1 f + (\partial_1 f)^2 + \partial_1 w e^{2(w-f)}\partial_0 w \\
&= e^{2(w-f)}\left[-2(\partial_0 w)^2 + 2\partial_0 f \partial_0 w - \partial_0^2 w - \partial_0 w \partial_0 f + \partial_1 w \partial_0 w\right] + \left[\partial_1^2 f - \partial_1 w \partial_1 w \partial_1 f + (\partial_1 f)^2\right] \\
&= e^{2(w-f)}\left[-(\partial_0 w)^2 + \partial_0 f \partial_0 w - \partial_0^2 w\right] + \left[\partial_1^2 f - \partial_1 w \partial_1 f + (\partial_1 f)^2\right] \\
&= -e^{2(w-f)}\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f \partial_0 w\right] + \left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w \partial_1 f\right]
\end{aligned}$$

$$\begin{aligned}
R_{022}{}^{0} &= \partial_0 \Gamma^0_{22} - \partial_2 \Gamma^0_{02} + \Gamma^m_{22}\Gamma^0_{0m} - \Gamma^m_{02}\Gamma^0_{2m} \\
&= \Gamma^1_{22}\Gamma^0_{01} \\
&= -re^{-2w}\partial_1 f
\end{aligned}$$

$$\begin{aligned}
R_{033}{}^{0} &= \partial_0 \Gamma^0_{33} - \partial_3 \Gamma^0_{03} + \Gamma^m_{33}\Gamma^0_{0m} - \Gamma^m_{03}\Gamma^0_{3m} \\
&= \Gamma^1_{33}\Gamma^0_{01} + \Gamma^2_{33}\Gamma^0_{02} \\
&= -re^{-2w}\sin^2\theta \partial_1 f
\end{aligned}$$

$$\begin{aligned}
R_{122}{}^{0} &= \partial_1 \Gamma^0_{22} - \partial_2 \Gamma^0_{12} + \Gamma^m_{22}\Gamma^0_{1m} - \Gamma^m_{12}\Gamma^0_{2m} \\
&= \Gamma^1_{22}\Gamma^0_{11} - \Gamma^2_{12}\Gamma^0_{22} \\
&= -re^{-2w}e^{2(w-f)}\partial_0 w \\
&= -re^{-2f}\partial_0 w
\end{aligned}$$

$$\begin{aligned}
R_{133}{}^{0} &= \partial_1 \Gamma^0_{33} - \partial_3 \Gamma^0_{13} + \Gamma^m_{33}\Gamma^0_{1m} - \Gamma^m_{13}\Gamma^0_{3m} \\
&= \Gamma^1_{33}\Gamma^0_{11} + \Gamma^2_{33}\Gamma^0_{12} - \Gamma^3_{13}\Gamma^0_{33} \\
&= -re^{-2w}\sin^2\theta e^{2(w-f)}\partial_0 w \\
&= -re^{-2f}\sin^2\theta \partial_0 w
\end{aligned}$$

$$\begin{aligned}
R_{122}{}^{1} &= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^1_{12} + \Gamma^m_{22}\Gamma^1_{1m} - \Gamma^m_{12}\Gamma^1_{2m} \\
&= \partial_1(-re^{-2w}) + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^1_{22} \\
&= -e^{-2w} + 2r\partial_1 w e^{-2w} + (-re^{-2w}\partial_1 w) - \frac{1}{r}(-re^{-2w}) \\
&= re^{-2w}\partial_1 w
\end{aligned}$$

$$\begin{aligned}
R_{133}{}^{1} &= \partial_1 \Gamma^1_{33} - \partial_3 \Gamma^1_{13} + \Gamma^m_{33}\Gamma^1_{1m} - \Gamma^m_{13}\Gamma^1_{3m} \\
&= \partial_1(-re^{-2w}\sin^2\theta) + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33}\gamma^1_{12} - \Gamma^3_{13}\Gamma^1_{33} \\
&= -e^{-2w}\sin^2\theta + 2r\partial_1 w e^{-2w}\sin^2\theta - re^{-2w}\sin^2\theta\partial_1 w + \frac{1}{r}re^{-2w}\sin^2\theta \\
&= -e^{-2w}\sin^2\theta + r\partial_1 w e^{-2w}\sin^2\theta + e^{-2w}\sin^2\theta \\
&= re^{-2w}\sin^2\theta\partial_1 w
\end{aligned}$$

$$\begin{aligned}
{R_{233}}^2 &= \partial_2\Gamma^2_{33} - \partial_3\Gamma^2_{23} + \Gamma^m_{33}\Gamma^2_{2m} - \Gamma^m_{23}\Gamma^2_{3m} \\
&= \partial_2(-\sin\theta\cos\theta) + \Gamma^1_{33}\Gamma^2_{21} + \Gamma^2_{33}\Gamma^2_{22} - \Gamma^3_{23}\Gamma^2_{33} \\
&= -\cos^2\theta + \sin^2\theta + (-re^{-2w}\sin^2\theta)\frac{1}{r} - \frac{\cos\theta}{\sin\theta}(-\sin\theta\cos\theta) \\
&= \sin^2\theta - e^{-2w}\sin^2\theta \\
&= \sin^2\theta(1-e^{-2w})
\end{aligned}$$

□

Lemma 4.3.5. *The non-zero coefficients of the Ricci tensor of the metric g in the equation* (4.6) *are:*

$$\begin{aligned}
R_{00} &= -\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f\partial_0 w\right] - e^{2(f-w)}\left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w\partial_1 f - \frac{2}{r}\partial_1 f\right] \\
R_{11} &= e^{2(w-f)}\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f\partial_0 w\right] + \left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w\partial_1 f + \frac{2}{r}\partial_1 w\right] \\
R_{01} &= \frac{2}{r}\partial_0 w \\
R_{22} &= e^{-2w}\left[r(\partial_1 w - \partial_1 f) - 1\right] + 1 \\
R_{33} &= \sin^2\theta R_{22}
\end{aligned}$$

Proof. The Ricci tensor is defined

$$R_{ij} = {R_{kij}}^k = g^{km}R_{kijm}$$

We begin the computations.

$$R_{00} = {R_{k00}}^k = {R_{100}}^1 + {R_{200}}^2 + {R_{300}}^3$$

$$\begin{aligned}
{R_{100}}^1 &= g^{1l}R_{100l} \\
&= g^{11}R_{1001} \\
&= g^{11}R_{0110} \\
&= g^{11}g_{0l}{R_{011}}^l \\
&= g^{11}g_{00}{R_{011}}^0
\end{aligned}$$

Similarly

$${R_{200}}^2 = g^{22}g_{00}{R_{022}}^0, \qquad {R_{300}}^3 = g^{33}g_{00}{R_{033}}^0$$

So

$$\begin{aligned}
R_{00} &= g_{00}\left[g^{11}{R_{011}}^0 + g^{22}{R_{022}}^0 + g^{33}{R_{033}}^0\right] \\
&= -e^{2f}\Big[e^{-2w}\left(e^{2(w-f)}((\partial_0 2)^2 + \partial_0^2 w - \partial_0 f\partial_0 w) + (-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w\partial_1 f)\right) \\
&\qquad + r^{-2}(-re^{-2w}\partial_1 f) + r^{-2}\sin^{-2}\theta(-re^{-2w}\sin^2\theta\partial_1 f)\Big] \\
&= -e^{-2(w-f)}e^{2(w-f)}\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f\partial_0 w\right] - e^{2(f-2)}\left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w\partial_{1f}\right] \\
&\qquad + e^{2f}r^{-1}e^{-2w}\partial_1 f + r^{-1}e^{2f}e^{-2w}\partial_1 f \\
&= -\left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f\partial_0 w\right] - e^{2(f-w)}\left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w\partial_1 f - \frac{2}{r}\partial_1 f\right]
\end{aligned}$$

$$R_{11} = R_{k11}{}^{k} = R_{011}{}^{0} + R_{211}{}^{2} + R_{311}{}^{3}$$

$$\begin{aligned}
R_{211}{}^{2} &= g^{2l} R_{211l} \\
&= g^{22} R_{2112} \\
&= g^{22} R_{1221} \\
&= g^{22} g_{1l} R_{122}{}^{l} \\
&= g^{22} g_{11} R_{122}{}^{1}
\end{aligned}$$

Similarly

$$R_{311}{}^{3} = g^{33} R_{3113} = g^{33} R_{1331} = g^{33} g_{11} R_{133}{}^{1}$$

which gives us

$$\begin{aligned}
R_{11} &= R_{011}{}^{0} + g^{22} g_{11} R_{122}{}^{1} + g^{33} g_{11} R_{133}{}^{1} \\
&= e^{2(w-f)} \left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f \partial_0 w \right] + \left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w \partial_1 f \right] \\
&\qquad + r^{-2} e^{2w} r e^{-2w} \partial_1 w + r^{-2} \sin^{-2}\theta e^{2w} r e^{-2w} \sin^2\theta \partial_1 w \\
&= e^{2(w-f)} \left[(\partial_0 w)^2 + \partial_0^2 w - \partial_0 f \partial_0 w \right] + \left[-\partial_1^2 f - (\partial_1 f)^2 + \partial_1 w \partial_1 f 2 r^{-1} \partial_1 w \right]
\end{aligned}$$

$$R_{01} = R_{k01}{}^{k} = R_{201}{}^{2} + R_{301}{}^{3}$$

And also

$$\begin{aligned}
R_{201}{}^{2} &= g^{22} R_{2012} = g^{22} R_{1220} = g^{22} g_{00} R_{122}{}^{0} \\
R_{301}{}^{3} &= g^{33} R_{3013} = g^{33} R_{1330} = g^{33} g_{00} R_{133}{}^{0}
\end{aligned}$$

Hence

$$\begin{aligned}
R_{01} &= g_{00} \left[g^{22} R_{122}{}^{0} + g^{33} R_{133}{}^{0} \right] \\
&= -e^{2f} \left[r^{-2} (-r e^{-2f} \partial_0 w) + r^{-2} \sin^{-2}\theta (-r e^{-2f} \sin^2\theta \partial_0 w) \right] \\
&= -e^{2f} \left[-r^{-1} e^{-2f} \partial_0 w - r^{-1} e^{-2f} \partial_0 w \right] \\
&= \frac{2}{r} \partial_0 w
\end{aligned}$$

$$R_{22} = R_{k22}{}^{k} = R_{022}{}^{2} + R_{122}{}^{1} + R_{322}{}^{3}$$

and

$$R_{322}{}^{3} = g^{33} R_{3223} = g^{33} R_{2332} = g^{33} g_{22} R_{233}{}^{2}$$

Hence

$$\begin{aligned}
R_{22} &= R_{022}{}^{0} + R_{122}{}^{1} + g^{33} g_{22} R_{233}{}^{2} \\
&= -r e^{-2w} \partial_1 f + r e^{-2w} \partial_1 w + r^{-2} \sin^{-2}\theta r^2 \sin^2\theta (1 - e^{-2w}) \\
&= e^{-2w} \left[r(\partial_1 w - \partial_1 f) - 1 \right] + 1
\end{aligned}$$

$$\begin{aligned}
R_{33} &= R_{k33}{}^{k} \\
&= R_{033}{}^{0} + R_{133}{}^{1} + R_{233}{}^{2} \\
&= -re^{-2w}\sin^2\theta\partial_1 f + re^{-2w}\sin^2\theta\partial_1 w + \sin^2\theta(1-e^{-2w}) \\
&= \sin^2\theta\left[re^{-2w}(\partial_1 w - \partial_1 f) + 1 - e^{-2w}\right] \\
&= \sin^2\theta\left[e^{-2w}\left((\partial_1 w - \partial_1 f)r - 1\right) + 1\right] \\
&= \sin^2\theta R_{22}
\end{aligned}$$

□

No we use the Einstein equations in vacuum $R_{ij} = 0$. First we have

$$R_{01} = 0 \quad \Longrightarrow \quad \partial_0 w = 0$$

Hence

$$\begin{aligned}
\partial_0 R_{22} = 0 \quad &\Longrightarrow \quad \partial_0\left[e^{-2w}(r(\partial_1 w - \partial_1 f) - 1) + 1\right] = 0 \\
&\Longrightarrow \quad -2e^{-2w}\partial_0 w\left[r((\partial_1 w - \partial_1 f) - 1)\right] + e^{-2w}\partial_0(r(\partial_1 w - \partial_1 f)) = 0 \\
&\Longrightarrow \quad e^{-2w}r(\partial_0\partial_1 w - \partial_0\partial_1 f) = 0 \\
&\Longrightarrow \quad \partial_0\partial_1 f = 0
\end{aligned}$$

So we have that

$$\begin{aligned}
w &= w(r) \\
f &= k(r) + h(t)
\end{aligned}$$

So the metric

$$g = -e^{2f(t,r)}\,\mathrm{d}t^2 + e^{2w(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

becomes

$$g = -e^{2k(r)}e^{2h(t)}\,\mathrm{d}t^2 + e^{2w(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

by redifining the time coordinate such that $\mathrm{d}t \to e^{-h(t)}\,\mathrm{d}t$, we have the static metric

$$g = -e^{2k(r)}\,\mathrm{d}t^2 + e^{2g(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{4.7}$$

The metric is the same but now w and k depend only on r. So from the previous lemmas and by the same calculations we have that for the g static metric in equation (4.7) the non-zero Christoffel symbols are:

$$\begin{aligned}
&\Gamma^0_{01} = \partial_1 k && \Gamma^1_{22} = -re^{-2w} && \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^1_{00} = e^{2(k-w)}\partial_1 k && \Gamma^1_{33} = -re^{-2w}\sin^2\theta && \Gamma^3_{13} = \frac{1}{r} \\
&\Gamma^1_{11} = \partial_1 w && \Gamma^2_{12} = \frac{1}{r} && \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The non-zero coefficients of the Riemann tensor are:

$$\begin{aligned}
R_{011}{}^{0} &= -\partial_1^2 k - (\partial_1 k)^2 + \partial_1 w \partial_1 k \\
R_{022}{}^{0} &= -re^{-2w}\partial_1 k \\
R_{033}{}^{0} &= -re^{-2w}\sin^2\theta\partial_1 k \\
R_{122}{}^{1} &= re^{-2w}\partial_1 w \\
R_{133}{}^{1} &= re^{-2w}\sin^2\theta\partial_1 w \\
R_{233}{}^{2} &= \sin^2\theta(1-e^{-2w})
\end{aligned}$$

And the non-zero coefficients of the Ricci tensor are:

$$\begin{aligned}
R_{00} &= e^{2(k-w)}\left[\partial_1^2 k + (\partial_1 k)^2 - \partial_1 w\partial_1 k + \frac{2}{r}\partial_1 k\right] \\
R_{11} &= -\partial_1^2 k - (\partial_1 k)^2 + \partial_1 w \partial_1 k + \frac{2}{r}\partial_1 w \\
R_{22} &= e^{-2w}\left[r(\partial_1 w - \partial_1 k) - 1\right] + 1 \\
R_{33} &= \sin^2\theta R_{22}
\end{aligned}$$

Again using the Einstein equations in vacuum we write:

$$\begin{aligned}
e^{2(w-k)}R_{00} + R_{11} = 0 \quad &\implies \quad \frac{2}{r}(\partial_1 k + \partial_1 w) = 0 \\
&\implies \quad k + w = C
\end{aligned}$$

Set the constant zero by rescaling $t \to e^{-C}t$. So

$$k = -w \tag{4.8}$$

Also

$$\begin{aligned}
R_{22} = 0 \quad &\implies \quad e^{-2w}\left[r(\partial_1 w - \partial_1 k)\right] - e^{-2w} + 1 = 0 \\
&\implies \quad e^{2k}\left[r(-\partial_1 k - \partial_1 k)\right] - e^{2k} + 1 = 0 \\
&\implies \quad -2e^{2k} r\partial_1 k - e^{2k} + 1 = 0 \\
&\implies \quad e^{2k}(2r\partial_1 k + 1) = 1 \\
&\implies \quad \partial_1(re^{2k}) = 1 \\
&\implies \quad re^{2k} = r - C \\
&\implies \quad e^{2k} = 1 - \frac{C}{r}
\end{aligned} \tag{4.9}$$

Hence from equations (4.8), (4.9) and

$$g = -e^{2k}\,\mathrm{d}t^2 + e^{-2k}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

we have:

$$g = -\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

where R_S is called the **Schwarzschild radius** and we set $R_S = 2M$.

□

4.4 Kruskal Extension

In the Schwarzschild metric

$$g = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2,$$

according to [50, p. 148], for any static equilibrium configuration the region $r \le 2M$ will be within the matter-filled star. So examining what is happening for $r = 2M$ or $r = 0$ in the Schwarzschild metric is irrelevant to the study of the gravitational field of a static star. But stars with massive bodies will undergo gravitational collapse and the study of the region $r \le 2M$ becomes relevant.

We notice that for $r = 0,\ r = 2M$ the metric diverges to infinity. Because the metric coefficients are coordinate dependent it is possible that we can fix the divergent terms by a coordinate change. For example polar coordinates in a plane

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$$

becomes degenerate on $g^{\theta\theta} = r^{-2}$. The definition of a singularity in a spacetime is not an obvious one and may be defined differently (we refer to chapter 9 of [50]). To check when something is wrong we can see when does the curvature becomes infinite. But since its components are coordinate dependent we check one of the various scalar quantities of the curvature. Some examples are

$$R = g^{\mu\nu}R_{\mu\nu}, \quad R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \quad R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu}$$

If any of them diverges to infinity, then on that point on the manifold we will say that we have a singularity.

By direct calculations it can be shown that:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}$$

So $r = 0$ has a singularity but not $r = 2M$.

In our study of the singularites we will use only the part of the Schwarzschild

$$g = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2$$

because of the spherical symmetry.

We check the null geodesics of the Schwarzschild for θ, ψ =constant

$$-\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 = 0 \quad \Longrightarrow \quad \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 = \left(1 - \frac{2M}{r}^{-2}\right)^{-1}\mathrm{d}r^2$$

Assume geodesic $\gamma(\lambda) = (t(\lambda), r(\lambda), c_1, c_2)$, then

$$\begin{aligned}
g(\dot\gamma(\lambda), \dot\gamma(\lambda)) = 0 \quad &\implies \quad \frac{\mathrm{d}\gamma^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^j}{\mathrm{d}\lambda} g_{ij} = 0 \\
&\implies \quad -\dot t^2 \left(1 - \frac{2M}{r}\right) + \dot r^2 \left(1 - \frac{2M}{r}\right)^{-1} = 0 \\
&\implies \quad \dot t^2 \left(1 - \frac{2M}{r}\right) = \dot r^2 \left(1 - \frac{2M}{r}\right)^{-1} \\
&\implies \quad \dot t^2 = \dot r^2 \left(1 - \frac{2M}{r}\right)^{-2} \\
&\implies \quad \dot t = \pm \dot r \left(1 - \frac{2M}{r}^{-1}\right) \\
&\implies \quad \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \pm \frac{\mathrm{d}r}{\mathrm{d}\lambda}\left(1 - \frac{2M}{r}\right)^{-1} \\
&\implies \quad \frac{\mathrm{d}t}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^{-1} = \pm \left(1 - \frac{2M}{r}\right)^{-1}
\end{aligned}$$

which gives us

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm \left(1 - \frac{2M}{r}\right)^{-1} \quad \implies \quad \left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \left(\frac{r}{r - 2M}\right)^2$$

hence

$$t = \pm \int \frac{r}{r - 2M}\,\mathrm{d}r + C$$

Suppose $r' = r - 2M$ and $\mathrm{d}r' = \mathrm{d}r$, which implies

$$\begin{aligned}
t &= \pm \int \frac{r'^{+2M}}{r'}\,\mathrm{d}r' \\
&= \pm \int 1 + \frac{2M}{r'}\,\mathrm{d}r' \\
&= \pm \left[r - 2M + 2M \log(r - 2M) + C_1\right] + C
\end{aligned}$$

Also

$$\begin{aligned}
\log(r - 2M) &= \log\left(2M\left(\frac{r}{2M} - 1\right)\right) \\
&= \log(2M) + \log\left(\frac{r}{2M} - 1\right)
\end{aligned}$$

So we have

$$t = \pm\left[r + 2M\log\left(\frac{r}{2M} - 1\right)\right] + C_2$$

where $C_2 = C \pm (\log(2M) + C_1)$. We also write it as

$$t = \pm r^* + C_2$$

This is called Regge-Wheeler tortoise coordinate r^* and is defined by

$$r^* = r + 2M\log\left(\frac{r}{2M} - 1\right)$$

which satisfies:

$$\begin{aligned}
\frac{\mathrm{d}r^*}{\mathrm{d}r} &= \left(1 + 2M\frac{1}{\frac{r}{2M} - 1}\left(\frac{1}{2M}\right)\right) \\
&= \left(1 + \frac{1}{\frac{r}{2M} - 1}\right) \\
&= \left(\frac{1 - \frac{r}{2M} - 1}{1 - \frac{r}{2M}}\right) \\
&= \left(\frac{1 - \frac{r}{2M}}{1 - \frac{r}{2M} - 1}\right)^{-1} \\
&= \left(\frac{1 - \frac{r}{2M}}{-\frac{r}{2M}}\right)^{-1} \\
&= \left(\frac{2M - r}{-r}\right)^{-1} \\
&= \left(1 - \frac{2M}{r}\right)^{-1}
\end{aligned}$$

So if we changed coordinates with respect to r^* we would have:

$$g = \left(1 - \frac{2M}{r}\right)\left(-\,\mathrm{d}t^2 + \mathrm{d}r^*\right) + r^2\,\mathrm{d}\Omega^2$$

which eliminates the singularity at $r = 2M$ but pushes the surface to infinity since $r^* = -\infty$.

We define the null coordinates u, v such that:

$$\begin{aligned}
u &= t - r^* \\
v &= t + r^*
\end{aligned}$$

We write the metric as

$$\begin{aligned}
g &= -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 \\
&= -\left(1 - \frac{2M}{r}\right)\left(\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-2}\mathrm{d}r^2\right)
\end{aligned}$$

Also

$$\mathrm{d}u = \mathrm{d}t - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r, \qquad \mathrm{d}v = \mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r$$

and

$$\begin{aligned}
\mathrm{d}u^2 &= \left(\mathrm{d}t - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r\right)^2 \\
&= \mathrm{d}t^2 - 2\left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}t\,\mathrm{d}r + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}v^2 &= \left(\mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r\right)^2 \\
&= \mathrm{d}t^2 + 2\left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}t\,\mathrm{d}r + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2
\end{aligned}$$

hence

$$\begin{aligned}
\mathrm{d}u\,\mathrm{d}v &= \left(\mathrm{d}t - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r\right)\left(\mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r\right) \\
&= \mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-2} \mathrm{d}r^2 - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r\,\mathrm{d}t + \left(1 - \frac{2M}{r}\right) \mathrm{d}r\,\mathrm{d}t \\
&= \mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-2} \mathrm{d}r^2
\end{aligned}$$

So the metric g becomes:

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}u\,\mathrm{d}v$$

We notice that

$$r^* = t - u = \frac{v-u}{2}$$

and we see that $r = r(u,v)$, defined by

$$r + 2M\log\left(\frac{r}{2M} - 1\right) = r^* = \frac{v-u}{2}$$

From the above we will again change the metric by:

$$\begin{aligned}
r + 2M\log\left(\frac{r}{2M} - 1\right) = r^* = \frac{v-u}{2} \quad &\implies \quad \frac{r}{2M} + \log\left(\frac{r}{2M} - 1\right) = \frac{v-u}{4M} \\
&\implies \quad e^{r/2M}\left(\frac{r}{2M} - 1\right) = e^{(v-u)/4M} \\
&\implies \quad -\frac{2M}{r} + 1 = e^{(v-u)/4M} e^{-r/2M} \frac{2M}{r} \\
&\implies \quad -\left(1 - \frac{2M}{r}\right) = -e^{(v-u)4M} e^{-r/2M} \frac{2M}{r}
\end{aligned}$$

Hence we have

$$g = -e^{(v-u)/4M} e^{-r/2M} \frac{2M}{r} \mathrm{d}u\,\mathrm{d}v$$

Suppose

$$\begin{aligned}
U &= -e^{-u/4M} \\
V &= e^{v/4M}
\end{aligned}$$

then

$$\left.\begin{aligned}
\mathrm{d}U &= \frac{1}{4M} e^{-u/4M}\,\mathrm{d}u \\
\mathrm{d}V &= \frac{1}{4M} e^{v/4M}\,\mathrm{d}v
\end{aligned}\right\} \quad \implies \quad \begin{aligned}
4M\,\mathrm{d}U &= e^{-u/4M}\,\mathrm{d}u \\
4M\,\mathrm{d}V &= e^{v/4M}\,\mathrm{d}v
\end{aligned}$$

So the metric is

$$g = -\frac{32M^3 e^{-r/2M}}{r}\,\mathrm{d}U\,\mathrm{d}V$$

and there is no more a singularity at $r = 2M$, meaning in $U = 0, V = 0$. Now we can extend the region of the Schwarzschild metric in U, V coordinates such that $r > 0$. The signularity at $r = 0$ doesn't disappear because of

$$R_{abcd}R^{abcd}(0) = \infty$$

With the next transformation we will have the desired Kruskal coordinates. Suppose

$$\begin{aligned} T &= \frac{(U+V)}{2} \\ X &= \frac{(V-U)}{2} \end{aligned}$$

where

$$\left.\begin{aligned} \mathrm{d}T &= \frac{\mathrm{d}U + \mathrm{d}V}{2} \\ \mathrm{d}X &= \frac{\mathrm{d}V - \mathrm{d}U}{2} \end{aligned}\right\} \quad \Longrightarrow \quad \begin{aligned} \mathrm{d}T^2 &= \frac{(\mathrm{d}U + \mathrm{d}V)^2}{4} \\ \mathrm{d}X^2 &= \frac{(\mathrm{d}V - \mathrm{d}U)^2}{4} \end{aligned}$$

Hence

$$\begin{aligned} -\mathrm{d}T^2 + \mathrm{d}X^2 &= \frac{-\mathrm{d}U^2 - \mathrm{d}V^2 + \mathrm{d}V^2 + \mathrm{d}U^2}{4} - \frac{2\,\mathrm{d}U\,\mathrm{d}V}{4} - \frac{2\,\mathrm{d}U\,\mathrm{d}V}{4} \\ &= -\mathrm{d}U\,\mathrm{d}V \end{aligned}$$

and so the full metric takes the form:

$$g = 32M^3\frac{e^{-r/2M}}{r}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

such that

$$\begin{aligned} X^2 - T^2 &= \frac{(U-V)^2}{4} - \frac{(U+V)^2}{4} \\ &= \frac{U^2 + V^2 - U^2 - V^2 - 2UV - 2UV}{4} \\ &= -UV \\ &= e^{-u/4M}e^{v/4M} \\ &= e^{(v-u)4M} \\ &= e^{r^*/2M} \\ &= e^{r/2M}e^{\log\left(\frac{r}{2M}-1\right)} \\ &= \left(\frac{r}{2M} - 1\right)e^{r/2M} \end{aligned}$$

It is known that

$$\operatorname{arctanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

hence

$$
\begin{aligned}
2\text{arctanh}\left(\frac{T}{X}\right) &= \log\left(\frac{1+\frac{T}{X}}{1-\frac{T}{X}}\right) \\
&= \log\left(\frac{X+T}{X-T}\right) \\
&= \log\left(\frac{\frac{V-U}{2}+\frac{U+V}{2}}{\frac{V-U}{2}-\frac{U+V}{2}}\right) \\
&= \log\left(\frac{-V}{U}\right) \\
&= \log\left(\frac{e^{v/4M}}{e^{-u/4M}}\right) \\
&= \log\left(e^{(v+u)/4M}\right) \\
&= \frac{v+u}{4M} \\
&= \frac{t}{2M}
\end{aligned}
$$

We have shown that for the Kruskal metric

$$
g = 32M^3\frac{e^{-r/2M}}{r}(-\,\mathrm{d}T^2+\mathrm{d}X^2)+r^2\,\mathrm{d}\Omega^2 \tag{4.10}
$$

The following equations hold

$$
\left(\frac{r}{2M}-1\right)e^{r/2M} = X^2-T^2 \tag{4.11}
$$

$$
\frac{t}{2M} = \log\left(\frac{T+X}{X-T}\right) = 2\tanh^{-1}\left(\frac{T}{X}\right) \tag{4.12}
$$

where r is defined by equation (4.11). If $r>0$ then $X^2-T^2>-1$. For points $X=C_1$, $T=C_2$ we have spheres and for $r=0$ we have singularities

$$
X^2 = T^2-1 \quad\implies\quad X=\pm\sqrt{T^2-1}
$$

To better understand the causal character that the Kruskal diagram has, we can look at the figure 4.1.

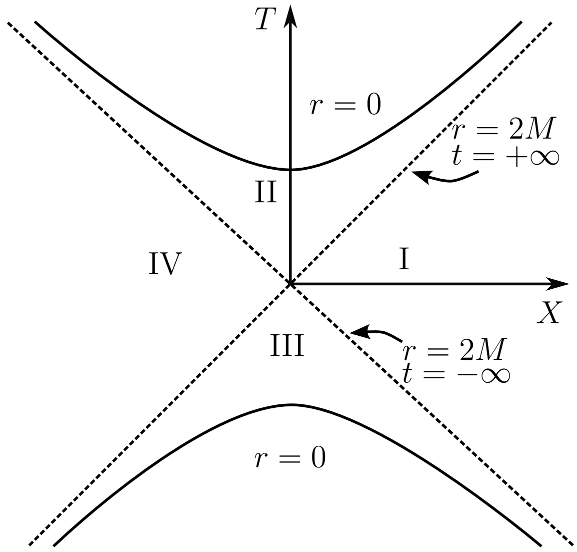

Figure 4.1: The Kruskal diagram in coordinates (T, X) is divided in the regions I, II, III, IV. The singularities for $r = 0$ are in the regions II, III. The horizon is for $r = 2M,\ t = +\infty$ and $r = 2M,\ t = -\infty$.

From [50, p. 155] we have the following remarks:

- the singularities for $r = 0$ where pushed to the regions II and III.
- The region for $r > 2M$ was pushed to the region I.
- An observer falling radially from region I once he crosses $X = T$ to region II he will stay there and in finite proper time he will fall to the $X = \sqrt{T^2 - 1}$ singularity. All the light signals sent from him will remain in region II. For the reason described, region II is called a **black hole**.
- An observer in region III must have originated from the singularity $X = -\sqrt{T^2 - 1}$ and within finite time must leave region I. This region represents the time reversal of region II and it is called a **white hole**.
- Light signals from region I cannot cross to region IV because they will fall to region II, the black hole.

Chapter 5

Cauchy Hypersurfaces

For the well-posedness of the Cauchy problem, the solution for the Einstein equations, we need well defined initial data. For that we need the following.

Definition 5.0.1 (Cauchy Hypersurfaces)**.** *[17, p. 6] Assume (M, g) is a spacetime manifold. Then a Cauchy hypersurface is a complete spacelike hypersurface Σ in M such that every causal curve through any point $p \in M$ intersects Σ at exactly one point.*

Definition 5.0.2. *[17, p. 6] A spacetime admitting a Cauchy hypersurface is called **globally hyperbolic**.*

An important result about the Cauchy hypersurfaces is the following Lemma.

Lemma 5.0.3. *[4, p. 4] Let M be a (C^k-)spacetime which admits a C^r-Cauchy hypersurface S, $r \in \{0, 1, \dots, k\}$. Then M is C^r-diffeomorphic to $\mathbb{R} \times S$ and all the C^r-Cauchy hypersurfaces are C^r diffeomorphic.*

According to [43], Geroch in [20] proved that if a Cauchy hypersurface exists then the spacetime is globally hyperbolic and, conversely:

Theorem 5.0.4. *If M is globally hyperbolic, there exists a continuous function $t: M \to \mathbb{R}$ such that:*

1. *t is strictly increasing on any future-directed causal curve*
2. *Each level set $S_a := t^{-1}(a)$ is a Cauchy hypersurface $\forall a \in \mathbb{R}$.*

Geroch proved this theorem by considering the time function t as

$$t(z) = \ln\left(\frac{\mathrm{vol}(J^-(z))}{\mathrm{vol}(J^+(z))}\right)$$

for a (suitable) finite measure on M.

From this result we have the existence of a time function t which gives as topological Cauchy hypersurfaces. What wasnt known was the existence of a smooth function that its level sets are smooth spacelike Cauchy hypersurfaces.

This was proven by Bernal and Sanchez, speciffically, according to [4, p. 2] they showed

1. Any globally hyperbolic spacetime admits a smooth spacelike Cauchy hypersurface S.

2. Besides the existence of a time function, there exists a "temporal" function, i.e. smooth with timelike gradient.

3. Any globally hyperbolic spacetime admits a smooth splitting $M = \mathbb{R} \times S$ with Cauchy hypersurfaces slices $\{t_0\} \times S$ orthogonal to ∇t and that M is isometric to $\mathbb{R} \times S$.

One can look for the detailed proof in [3] and [5].

5.1 3+1 Splitting

Let (M, g) be a globally hyperbolic spacetime, we define a smooth time function $t : M \to \mathbb{R}$ and the diffeomorphism $\psi : M \to \mathbb{R} \times \Sigma_0$ where each level set Σ_t is diffeomorphic to Σ_0 and Σ_0 is a Cauchy hypersurface. So we have a folliation of M with leaves Σ_t.

We define the lapse function

$$V = \frac{1}{\sqrt{-g(\nabla t, \nabla t)}}$$

We notice that for $X \in T_p\Sigma_\tau$

$$g(\nabla t, X) = \mathrm{d}t(X) = X(t)|_{\Sigma_\tau} = 0$$

and so $\nabla t \perp \Sigma_\tau$. So V measures the normal separation between the leaves Σ_τ. We define the vector field

$$T = -V^2 \cdot \nabla t$$

which is orthogonal to Σ_τ. From that we have

$$T = -V^2 \cdot \nabla t = -\frac{1}{-g(\nabla t, \nabla t)}\nabla t = \frac{\nabla t}{g(\nabla t, \nabla t)}$$

and

$$g(T, \nabla t) = g\left(\frac{\nabla t}{g(\nabla t, \nabla t)}, \nabla t\right) = 1$$

The vector fields $T, \nabla t$ are on opposite cones. We choose T to be the future directed timelike vector field and ∇t to be the past directed. Also

$$\begin{aligned}
T(t) &= -V^2 \nabla t(t) \\
&= \frac{\nabla t(t)}{g(\nabla t, \nabla t)} \\
&= \frac{1}{g(\nabla t, \nabla t)} g^{\mu\nu}\partial_\mu(t)\partial_\nu(t) \\
&= \frac{1}{g(\nabla t, \nabla t)} g(\nabla t, \nabla t) \\
&= 1
\end{aligned}$$

We assume the integrals curves of T are parametrized by λ.

$$\begin{aligned} T(t) = 1 \quad &\Longrightarrow \quad T^{\mu}\frac{\partial t}{\partial x^{\mu}} = 1 \\ &\Longrightarrow \quad \frac{\partial x^{m}}{\partial \lambda}\frac{\partial t}{\partial x^{\mu}} = 1 \\ &\Longrightarrow \quad \frac{\partial t}{\partial \lambda} = 1 \\ &\Longrightarrow \quad t = \lambda + c \end{aligned}$$

where c is constant. So the integral curves of T are orthogonal curves to the level sets Σ_t and they are parametrized by t. The flow ϕ_τ of T takes the leaves of the foliation to another leave by $\phi_\tau(\Sigma_t) = \Sigma_{t+\tau}$.

We assume the unit normal

$$N = V^{-1}T.$$

Its a unit normal since

$$\begin{aligned} g(N,N) &= V^2 g(T,T) \\ &= V^2 V^4 g(\nabla t, \nabla t) \\ &= V^2 g(\nabla t, \nabla t) \\ &= -\frac{1}{g(\nabla t, \nabla t)} g(\nabla t, \nabla t) \\ &= -1 \end{aligned}$$

Also for $N^{\mu} = \frac{\partial x^{\mu}}{\partial s}$, where s is arc length, we have

$$\frac{\partial x^{\mu}}{\partial t} = \frac{\partial x^{\mu}}{\partial s}\frac{\partial s}{\partial t} = N^{\mu}\frac{\partial s}{\partial t}$$

which implies

$$T^{\mu} = N^{\mu}\frac{\partial s}{\partial t} \quad \Longrightarrow \quad VN^{\mu} = N^{\mu}\frac{\partial s}{\partial t} \quad \Longrightarrow \quad V = \frac{\partial s}{\partial t}$$

And so the integral curves of N are the same with the integral curves of T but instead they are parametrized by arc length.

So for $\partial_t = T$ we have that

$$\begin{aligned} g_{00} &= g(\partial_t, \partial_t) \\ &= V^4 g(\nabla t, \nabla t) \\ &= V^2 \left(-\frac{1}{g(\nabla t, \nabla t)} \right) g(\nabla t, \nabla t) \\ &= -V^2 \end{aligned}$$

and

$$g_{i0} = -V^2 g(\nabla t, \partial_i) = 0$$

where the last equation holds since $\partial_i \in T_p\Sigma_t$ for some $p \in M$, $t \in \mathbb{R}$. For induced metric $\bar{g}(t)$ of Σ_t, the metric g is

$$g = -V^2\,\mathrm{d}t^2 + \bar{g}$$

Notice that $V, \bar{g}$ both depend from t.

5.2 Initial Value Problem

The Einstein equations can be written as

$$\mathrm{Ric} - \frac{1}{2}Rg = 8\pi T$$

where T is the energy-momentum tensor. The solution of the Einstein equation is the metric g of a spacetime M.

For the well-posedness of the Einstein equation we need a set of initial data. According to Christodoulou [17, p. 22]

"initial data for the Einstein equations consist of a pair $(\bar{g}_{ij}, k_{ij})$ where $\bar{g}_{ij}$ is a Riemannian metric and k_{ij} is a 2-covariant symmetric tensor field on the 3-manifold $\bar{M}$, which is to be identified with the initial hypersurface Σ_0. Once we have a solution (M, g) with $M = [0,T] \times \Sigma_0$ and $\Sigma_0 = \bar{M}$, then $\bar{g}_{ij}$, k_{ij} shall be, respectively, the 1st and 2nd fundamental form of $\Sigma_0 = \{0\} \times \Sigma_0$ in (M, g)."

From Choquet-Bruhat and Geroch [15] we have the following result

Theorem 5.2.1. *[43, p. 8] Let $(\Sigma, \bar{g})$ be a (connected) Riemannian 3-manifold, and k a symmetric two covariant tensor which satisfies the compatibility conditions of a second fundamental form (Gauss and Codazzi equations). Then there exist a unique spacetime (M, g) satisfying the following conditions:*

1. *$\Sigma \hookrightarrow M$, consistenly with g, k (i.e. $\bar{g} = g|_\Sigma$ etc.).*
2. *Vacuum: Ric $\equiv 0$ (this can be extened to more general T).*
3. *Σ is a Cauchy hypersurface of (M, g).*
4. *Maximality: if (M', g') satisfies (1)-(3), then it is isometric to an open subset of (M, g).*

As the previous theorem stated, for it to hold, the initial data needs to satisfy some constraint equations which are derived from the Gauss and Codazzi equations. Also the initial data have the evolution equations of g. Next we will find the constraint and evolution equations that hold for the initial data of a spacetime in a vacuum.

5.3 Constraint Equations of the Einstein Equations in Vacuum

In general, unless otherwise specified, when writing latin indices we will mean the spatial indices (i.e. $i, j, k \in \{1, 2, 3\}$) and the greek indices will mean the spacetime indices (i.e. $\mu, \nu, \lambda \in \{0, 1, 2, 3\}$).

5.3.1 First and Second Variation of the Metric

Proposition 5.3.1. *Let (M, g) be a spacetime such that $M \equiv [0, \tau] \times \Sigma_0$ where*

$$g = -V^2 \, \mathrm{d}t^2 + \bar{g},$$

*$\bar{g}(t)$ is the induced metric on Σ_t and V is the lapse function. Then the **first variation equation** is*

$$\frac{\partial \bar{g}_{ij}}{\partial t} = 2V k_{ij}$$

where k_{ij} is the second fundamental form.

Proof. Suppose

$$T = \frac{\partial}{\partial t}, \quad E_i = \frac{\partial}{\partial x^i}$$

The unit normal to Σ_t is

$$N = V^{-1} T,$$

and the second fundamental form is

$$k_{ij} = \langle \nabla_{E_i} N, E_j \rangle .$$

So we have

$$T(\bar{g}_{ij}) = VN \langle E_i, E_j \rangle = V \left(\langle \nabla_N E_i, E_j \rangle + \langle E_i, \nabla_N E_j \rangle \right)$$

First we compute

$$\begin{aligned}
\nabla_N E_i &= \nabla_{E_i} N + [N, E_i] \\
&= \nabla_{E_i} N + \left[V^{-1} T, E_i \right] \\
&= \nabla_{E_i} N + V^{-1} [T, E_i] + \left[V^{-1}, E_i \right] T \\
&= \nabla_{E_i} N - E_i(V^{-1}) T
\end{aligned}$$

And so

$$\begin{aligned}
\langle \nabla_N E_i, E_j \rangle &= \left\langle \nabla_{E_i} N - E_i(V^{-1}) T, E_j \right\rangle \\
&= \langle \nabla_{E_i} N, E_j \rangle - E_i(V^{-1}) \langle T, E_j \rangle \\
&= \langle \nabla_{E_i} N, E_j \rangle
\end{aligned}$$

With the above

$$\frac{\partial}{\partial t} \bar{g}_{ij} = 2V k_{ij}$$

□

Proposition 5.3.2. *Let (M, g) be a spacetime such that $M \equiv [0, \tau] \times \Sigma_0$ where*

$$g = -V^2 \, \mathrm{d}t^2 + \bar{g},$$

*$\bar{g}(t)$ is the induced metric on Σ_t and V is the lapse function. Then the **second variation equation** is*

$$\frac{\partial k_{ij}}{\partial t} = \overline{\nabla}_i \overline{\nabla}_j V + V k_j^m k_{im} + V R_{0i0j}$$

where k_{ij} is the second fundamental form, $\overline{\nabla}$ is the covariant derivative instrinsic over Σ_t and $R_{0i0j} = R(E_i, E_0, E_j, E_0)$ on the coordinate frame field (E_1, E_2, E_3) and $E_0 = V^{-1} \partial_0$ being the future directed unit normal on Σ_t.

Proof. Suppose
$$T = \frac{\partial}{\partial t}, \quad E_i = \frac{\partial}{\partial x^i}$$
The unit normal to Σ_t is
$$N = V^{-1}T$$
and the second fundamental form is
$$k_{ij} = \langle \nabla_{E_i} N, E_j \rangle .$$

$$\begin{aligned}
T(k_{ij}) &= T \langle \nabla_{E_i} N, E_j \rangle \\
&= T \left\langle \nabla_{E_i}(V^{-1}T), E_j \right\rangle \\
&= T \left(V^{-2} E_i(V) \langle T, E_j \rangle + V^{-1} \langle \nabla_{E_i} T, E_j \rangle \right) \\
&= T \left(V^{-1} \langle \nabla_{E_i} T, E_j \rangle \right) \\
&= T(V^{-1}) \langle \nabla_{E_i} T, E_j \rangle + V^{-1} \langle \nabla_T \nabla_{E_i} T, E_j \rangle + V^{-1} \langle \nabla_{E_i} T, \nabla_T E_j \rangle \\
&= -V^{-2} T(V) \langle \nabla_{E_i} T, E_j \rangle + V^{-1} \left[\langle \nabla_T \nabla_{E_i} T, E_j \rangle + \langle \nabla_{E_i} T, \nabla_T E_j \rangle \right] \\
&= -V^{-1} N(V) \langle \nabla_{E_i}(VN), E_j \rangle + V^{-1} \left[\langle \nabla_T \nabla_{E_i} T, E_j \rangle + \langle \nabla_{E_i} T, \nabla_T E_j \rangle \right] \\
&= -V^{-1} N(V) [\langle \nabla_{E_i}(V) N, E_j \rangle + \langle V \nabla_{E_i} N, E_j \rangle] + V^{-1} \left[\langle \nabla_T \nabla_{E_i} T, E_j \rangle + \langle \nabla_{E_i} T, \nabla_T E_j \rangle \right] \\
&= -N(V) k_{ij} + V^{-1} \left[\langle \nabla_T \nabla_{E_i} T, E_j \rangle + \langle \nabla_{E_i} T, \nabla_T E_j \rangle \right]
\end{aligned}$$

We have shown that
$$T(k_{ij}) = -N(V) k_{ij} + V^{-1} \left[\langle \nabla_T \nabla_{E_i} T, E_j \rangle + \langle \nabla_{E_i} T, \nabla_T E_j \rangle \right] \tag{5.1}$$

Denote
$$A = \langle \nabla_T \nabla_{E_i} T, E_j \rangle , \quad B = \langle \nabla_{E_i} T, \nabla_T E_j \rangle$$
so we can write
$$T(k_{ij}) = -N(V) k_{ij} + V^{-1} [A + B] \tag{5.2}$$

First we will compute B. We know that
$$[E_i, T] = \nabla_{E_i} T - \nabla_T E_i$$
and
$$[E_i, T] = 0$$
So we have
$$\nabla_{E_i} T = \nabla_T E_i$$

$$\begin{aligned}
B &= \langle \nabla_{E_i} T, \nabla_T E_j \rangle \\
&= \left\langle \nabla_{E_i} T, \nabla_{E_j} T \right\rangle \\
&= \left\langle \nabla_{E_i}(VN), \nabla_{E_j}(VN) \right\rangle \\
&= E_i(V) \left\langle N, \nabla_{E_j}(VN) \right\rangle + V \left\langle \nabla_{E_i} N, \nabla_{E_j}(VN) \right\rangle \\
&= E_i(V) \left[E_j(V) \langle N, N \rangle + V \left\langle N, \nabla_{E_j} N \right\rangle \right] + V \left[E_j(V) \langle N, \nabla_{E_i} N \rangle + V \left\langle \nabla_{E_j} N, \nabla_{E_i} N \right\rangle \right] \\
&= -E_i(V) E_j(V) + E_i(V) V \left\langle N, \nabla_{E_j} N \right\rangle + V E_j(V) \langle N, \nabla_{E_i} N \rangle + V^2 \left\langle \nabla_{E_j} N, \nabla_{E_i} N \right\rangle \\
&= -E_i(V) E_j(V) + V^2 \left\langle \nabla_{E_i} N, \nabla_{E_j} N \right\rangle \\
&= -E_i(V) E_j(V) + V^2 \left\langle \nabla_{E_i} N, g^{mi} k_{ij} E_m \right\rangle
\end{aligned}$$

The last equation comes from the fact that

$$\nabla_{E_j} N = \sharp k(E_i, E_j)$$

And so we have

$$B = -E_i(V)E_j(V) + V^2 k_j^m k_{im} \tag{5.3}$$

Now we compute A.

$$\begin{aligned}
A &= \langle \nabla_T \nabla_{E_i} T, E_j \rangle \\
&= \left\langle R(T, E_i)T + \nabla_{E_i} \nabla_T T + \nabla_{[T,E_i]} T \right\rangle \\
&= \langle R(T, E_i)T, E_j \rangle + \langle \nabla_{E_i} \nabla_T T, E_j \rangle \\
&= V^2 \langle R(N, E_i)N, E_j \rangle + \left\langle \nabla_{E_i} \nabla_{(VN)}(VN), E_j \right\rangle
\end{aligned}$$

Denote

$$C = \left\langle \nabla_{E_i} \nabla_{(VN)}(VN), E_j \right\rangle$$

So we can write

$$A = V^2 R_{0i0j} + C \tag{5.4}$$

Now we compute C.

$$\begin{aligned}
C &= \langle \nabla_{E_i} \nabla_{VN}(VN), E_j \rangle \\
&= \langle \nabla_{E_i} (V \nabla_N (VN)), E_j \rangle \\
&= \left\langle \nabla_{E_i} [VN(V)N + V^2 \nabla_N N], E_j \right\rangle \\
&= \left\langle \nabla_{E_i} (VN(V)N) + \nabla_{E_i} (V^2 \nabla_N N), E_j \right\rangle \\
&= \langle E_i(V)N(V)N + VE_i(N(V))N + (\nabla_{E_i} N)VN(V), E_j \rangle \\
&\qquad + \left\langle E_i(V^2) \nabla_N N + V^2 \nabla_{E_i} \nabla_N N, E_j \right\rangle \\
&= VN(V) \langle \nabla_{E_i} N, E_j \rangle + 2VE_i(V) \langle \nabla_N N, E_j \rangle + V^2 \langle \nabla_{E_i} \nabla_N N, E_j \rangle \\
&= VN(V) k_{ij} + 2VE_i(V) \langle \nabla_N N, E_j \rangle + V^2 \langle \nabla_{E_i} \nabla_N N, E_j \rangle
\end{aligned}$$

And so we have

$$C = VN(V) k_{ij} + 2VE_i(V) \langle \nabla_N N, E_j \rangle + V^2 \langle \nabla_{E_i} \nabla_N N, E_j \rangle \tag{5.5}$$

Next we have to compute $\nabla_N N$.

$$\begin{aligned}
\nabla_N N &= \nabla_{V^{-1}T}(V^{-1}T) \\
&= V^{-1} \nabla_T (V^{-1}T) \\
&= V^{-1}(T(V^{-1})T + V^{-1} \nabla_T T) \\
&= V^{-1}(-V^{-2})T(V)T + V^{-2} \nabla_T T \\
&= -V^{-3} T(V)T + V^{-2} \Gamma_{00}^{\alpha} E_a
\end{aligned}$$

$$\Gamma_{00}^{0} = \frac{1}{2} g^{0\beta} \left(E_0 g_{0\beta} + E_0 g_{\beta 0} - E_\beta g_{00} \right)$$

We know that $g^{0\beta} = 0$ for $\beta \neq 0$, so

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{00} \left(E_0 g_{00} + E_0 g_{00} - E_0 g_{00} \right) \\
&= \frac{1}{2} (-V^{-2}) E_0 (-V^2) \\
&= \frac{1}{2} V^{-2} 2 V E_0(V) \\
&= V^{-1} E_0(V)
\end{aligned}$$

For $k = 1, 2, 3$ and $\beta = 0, \dots, 3$

$$\Gamma^k_{00} = \frac{1}{2} g^{k\beta} \left(E_0 g_{0\beta} + E_0 g_{\beta 0} - E_\beta g_{00} \right)$$

We know $g^{k0} = 0$, so we must have $\beta \neq 0$. We write equivalently

$$\Gamma^k_{00} = \frac{1}{2} g^{kl} \left(E_0 g_{0l} + E_0 g_{l0} - E_l g_{00} \right)$$

But $g_{0l} = 0$

$$\begin{aligned}
\Gamma^k_{00} &= \frac{1}{2} g^{kl} (-E_l g_{00}) \\
&= -\frac{1}{2} g^{kl} E_l (-V^2) \\
&= \frac{1}{2} 2 V g^{kl} E_l(V) \\
&= V g^{kl} E_l(V)
\end{aligned}$$

We have shown that

$$\Gamma^0_{00} = V^{-1} E_0(V), \quad \Gamma^k_{00} = V g^{kl} E_l(V)$$

together with the previous computation of $\nabla_N N$ we have

$$\begin{aligned}
\nabla_N N &= -V^{-3} T(V) T + V^{-2} \Gamma^\alpha_{00} E_a \\
&= -V^{-3} E_0(V) E_0 + V^{-2} V^{-1} E_0(V) E_0 + V^{-2} V g^{kl} E_l(V) E_k \\
&= V^{-1} g^{kl} E_l(V) E_k \\
&= V^{-1} \overline{\nabla} V
\end{aligned}$$

We have shown

$$\nabla_N N = V^{-1} \overline{\nabla} V \tag{5.6}$$

Now we substitute (5.6) to (5.5)

$$\begin{aligned}
C &= VN(V)k_{ij} + 2VE_i(V)V^{-1}g^{kl}E_l(V)g_{kj} + V^2\left\langle \nabla_{E_i}(V^{-1}\overline{\nabla V}), E_j\right\rangle \\
&= VN(V)k_{ij} + 2E_i(V)E_l(V)\delta^l_j + V^2\left[E_i(V^{-1})\left\langle \overline{\nabla}V, E_j\right\rangle + V^{-1}\left\langle \nabla_{E_i}(\overline{\nabla}V), E_j\right\rangle\right] \\
&= VN(V)k_{ij} + 2E_i(V)E_j(V) - V^2V^{-2}E_i(V)g^{kl}E_l(V)g_{kj} + V\left\langle \nabla_{E_i}(\overline{\nabla}V), E_j\right\rangle \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left\langle \nabla_{E_i}(\sharp\, \mathrm{d}V), E_j\right\rangle \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left[\left\langle \sharp\left(\nabla_{E_i}(\mathrm{d}V)\right), E_j\right\rangle\right] \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left\langle \sharp\left[\nabla_{E_i}\left(E_k(V)\,\mathrm{d}x^k\right)\right], E_j\right\rangle \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left\langle \sharp\left[(E_iE_k(V) - E_l(V)\Gamma^l_{ik})\,\mathrm{d}x^k\right], E_j\right\rangle \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left\langle g^{kp}(E_iE_p(V) - E_l(V)\Gamma^l_{ip})E_k\right\rangle \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\left[g^{kp}E_iE_p(V)g_{kj} - g^{kp}E_l(V)\Gamma^l_{ip}g_{kj}\right] \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + VE_iE_jV - E_l(V)\Gamma^l_{ij}V \\
&= VN(V)k_{ij} + E_i(V)E_j(V) + V\overline{\nabla}_i\overline{\nabla}_jV
\end{aligned}$$

Now we substitute C to (5.4).

$$A = V^2R_{0i0j} + VN(V)k_{ij} + E_i(V)E_j(V) + V\overline{\nabla}_i\overline{\nabla}_jV \tag{5.7}$$

Now we substitute both (5.7) and (5.3) to (5.2).

$$\begin{aligned}
T(k_{ij}) &= -N(V)k_{ij} + V^{-1}\left[V^2R_{0i0j} + VN(V)k_{ij} + E_i(V)E_j(V)\right] \\
&\qquad + \left[V\overline{\nabla}_i\overline{\nabla}_jV + -E_i(V)E_j(V) + V^2k^m_jk_{im}\right] \\
&= \overline{\nabla}_i\overline{\nabla}_jV - N(V)k_{ij} + VR_{0i0j} + N(V)k_{ij} + Vk^m_jk_{im} \\
&= \overline{\nabla}_i\overline{\nabla}_jV + VR_{0i0j} + Vk^m_jk_{im}
\end{aligned}$$

□

5.3.2 Gauss and Codazzi Equations

Theorem 5.3.3 (Gauss Equation)**.** *[38, p. 100] Let (M, g) be a semi-Riemannian manifold and $(\overline{M}, \overline{g})$ be a semi-Riemannian submanifold of M. Then for all $W, X, Y, Z \in \mathfrak{X}(\overline{M})$, the following equation holds:*

$$R(W,X,Y,Z) = \overline{R}(W,X,Y,Z) - \langle \mathrm{I\!I}(W,Z), \mathrm{I\!I}(X,Y)\rangle + \langle \mathrm{I\!I}(W,Y), \mathrm{I\!I}(X,Z)\rangle$$

Assume $W, X, Y, Z \in T_p\overline{M}$ and $N \in T_pM$ is a timelike unit vector normal to $\overline{M}$. Then from theorem 5.3.3

$$\begin{aligned}
R(W,X,Y,Z) &= \overline{R}(W,X,Y,Z) - \langle \mathrm{I\!I}(W,Z), \mathrm{I\!I}(X,Y)\rangle + \langle \mathrm{I\!I}(W,Y), \mathrm{I\!I}(X,Z)\rangle \\
&= \overline{R}(W,X,Y,Z) - \langle k(W,Z)N, k(X,Y)N\rangle + \langle k(W,Y)N, k(X,Z)N\rangle \\
&= \overline{R}(W,X,Y,Z) + k(W,Z)k(X,Y) - k(W,Y)k(X,Z)
\end{aligned}$$

And so in coordinates the **Gauss equation** is:

$$R_{imjl} = \overline{R}_{imjl} + k_{il}k_{mj} - k_{ij}k_{ml} \tag{5.8}$$

Theorem 5.3.4 (Codazzi equation)**.** *[38, p. 115] Let (M, g) be a semi-Riemannian manifold and $(\overline{M}, \overline{g})$ be a semi-Riemannian submanifold of M. Then for all $W, X, Y \in \mathfrak{X}(M)$, the following equation holds:*

$$\begin{aligned}(R(W,X)Y)^{\perp} = &-\nabla^{\perp}_{W}(\mathrm{I\!I}(X,Y)) + \mathrm{I\!I}(\nabla_W X, Y) + \mathrm{I\!I}(X, \nabla_W Y)\\ &+ \nabla^{\perp}_{X}(\mathrm{I\!I}(W,Y)) - \mathrm{I\!I}(\nabla_X W, Y) - \mathrm{I\!I}(W, \nabla_X Y)\end{aligned}$$

Suppose $W, X, Y, N \in \mathfrak{X}(M)$, then from theorem 5.3.4

$$\begin{aligned}R(W,X,Y,N) &= \langle -\nabla_W(\mathrm{I\!I}(X,Y)) + \mathrm{I\!I}(\nabla_W X, Y) + \mathrm{I\!I}(X, \nabla_W Y), N\rangle\\ &\quad + \langle \nabla_X(\mathrm{I\!I}(W,Y)) - \mathrm{I\!I}(\nabla_X W, Y) - \mathrm{I\!I}(\nabla_X Y, W), N\rangle\\ &= \langle -\nabla_W(k(X,Y)N) + k(\nabla_W X, Y)N + k(X, \nabla_W Y)N, N\rangle\\ &\quad + \langle \nabla_X(k(W,Y)N) - k(\nabla_X W, Y)N - k(W, \nabla_X Y)N, N\rangle\\ &= -\langle \nabla_W(k)(X,Y)N, N\rangle + \nabla_X(k)(W,Y)\langle N, N\rangle\\ &= -\langle N, N\rangle [\nabla_W(k)(X,Y) - \nabla_X(k)(W,Y)]\end{aligned}$$

So for $W = E_i,\ X = E_j,\ Y = E_m,\ N = V^{-1}\partial_0$ we have the **Codazzi equation** in coordinates:

$$R_{ijm0} = k_{jm;i} - k_{im;j} \tag{5.9}$$

We remind that $g^{0k} = 0$ for $k \in \{1, 2, 3\}$ and $g^{00} = -1$ on the frame field (E_1, E_2, E_3, N).

We take the trace of the Gauss equation (5.8) in respect to the hypersurface Σ_t

$$g^{ml} R_{imjl} = -\overline{R}_{ij} + k_{il}k_{mj}g^{ml} - k_{ij}k_m{}^m \tag{5.10}$$

We notice that

$$\begin{aligned}g^{ml} R_{imjl} &= -g^{ml} R_{imlj}\\ &= -g^{\mu\nu} R_{i\mu\nu j} + g^{0l} R_{i0lj} + g^{m0} R_{im0j} + g^{00} R_{i00j}\\ &= -R_{ij} + g^{00} R_{i00j}\\ &= -R_{ij} - R_{i00j}\end{aligned}$$

Together with equation (5.10) the trace of the Gauss equation becomes

$$R_{ij} + R_{i00j} = +\overline{R}_{ij} - k_{il}k^l{}_j + k_{ij}k_m{}^m$$

Taking the second trace on the hypersurface we have

$$g^{ij}R_{ij} + g^{ij}R_{0ij0} = \overline{R} - k^j{}_l k^l{}_j + k^i{}_i k_m{}^m \quad \Longrightarrow \quad R - g^{00}R_{00} + g^{ij}R_{0ij0} = \overline{R} - k^j{}_l k^l{}_j + k^i{}_i k_m{}^m$$

but

$$g^{00}R_{00} = -g^{ij}R_{i00j} - g^{0j}R_{000j} - g^{i0}R_{i000} = -g^{ij}R_{i00j}$$

because of the antisymmetry of the first two and the last two indices of the Riemannian metric. So we get in coordinates

$$R + 2R_{00} = \overline{R} - k^j{}_l k^l{}_j + k^i{}_i k_m{}^m$$

and in invariant form

$$R + 2Ric(N,N) = \overline{R} - |k|^2 + (\mathrm{tr}k)^2$$

where $|k|^2 = k^j{}_l k^l{}_j$.

For the Codazzi equation (5.9) we take the trace on the hypersurface

$$g^{jm} R_{ijm0} = g^{jm} k_{jm;i} - g^{jm} k_{im;j}$$

but

$$R_{i0} = g^{jm} R_{ijm0} + g^{00} R_{i000} = g^{jm} R_{ijm0}$$

So in coordinates

$$R_{0i} = k_{ij}{}^{;j} - \partial_j(\mathrm{k})$$

and in invariant form

$$Ric(N, \cdot) = \overline{\mathrm{div}} k - \bar{\mathrm{d}} k$$

We have proven the equations

$$R + 2Ric(N, N) = \overline{R} - |k|^2 + (\mathrm{tr} k)^2$$
$$Ric(N, \cdot) = \overline{\mathrm{div}} k - \bar{\mathrm{d}} k$$

By imposing the Einstein equations in a vacuum we have the **constraint equations**

$$\overline{R} - |k|^2 + (\mathrm{tr} k)^2 = 0 \tag{5.11}$$
$$\overline{\mathrm{div}} k - \bar{\mathrm{d}} k = 0 \tag{5.12}$$

So the initial data $(k, \overline{g})$ on a spacetime (M, g), which has the decomposion from the spatial hypersurfaces $(\Sigma_t, \overline{g}(t))$, need to satisfy the above constraint equations and the time evolution equations from propositions 5.3.1, 5.3.2:

$$\frac{\partial}{\partial t} \bar{g}_{ij} = 2V k_{ij} \tag{5.13}$$
$$\frac{\partial}{\partial t} k_{ij} = V_{;ji} + V k_j^m k_{im} + V R_{0i0j} \tag{5.14}$$

5.4 Static Spacetime

Assume a static spacetime (M, g) where

$$g = -V^2 \, \mathrm{d}t^2 + \overline{g}$$

static means that V and $\overline{g}$ dont depend from t. Because of this independence we observe that spatial hypersurfaces Σ_t of the static spacetimes are totally geodesic. This happens because the first variation equation (5.13) and the second variation equation (5.14) become

$$k_{ij} = 0$$
$$V_{;ji} + V R_{0i0j} = 0$$

and the Gauss equation (5.8) similarly become

$$R_{imjl} = \overline{R}_{imjl}$$

Proposition 5.4.1. *Let* (M, g) *be a static spacetime in vacuum and*

$$g = -V^2 \,\mathrm{d}t^2 + \overline{g}$$

Then the following equations hold:

$$\begin{aligned} \overline{R}_{ij} &= V^{-1} V_{;ji} \\ \overline{\Delta} V &= 0 \end{aligned}$$

where the bar above the operators are the induced on the hypersurface Σ_t*.*

Proof. We observe that

$$\begin{aligned} R_{\mu\nu} &= g^{\alpha\beta} R_{\alpha\mu\nu\beta} \\ &= g^{0\beta} R_{0\mu\nu\beta} + g^{\alpha 0} R_{\alpha\mu\nu 0} + g^{kl} R_{k\mu\nu l} + g^{00} R_{0\mu\nu 0} \\ &= g^{kl} R_{k\mu\nu l} + g^{00} R_{0\mu\nu 0} \end{aligned}$$

So we have the equations

$$\begin{aligned} R_{ij} &= g^{kl} R_{kijl} - R_{0ij0} \\ R_{00} &= g^{kl} R_{k00l} \end{aligned}$$

From the second variation equation we have that

$$R_{k00l} = V^{-1} V_{;ji}$$

By taking the trace we get

$$R_{00} = V^{-1} \overline{\Delta} V$$

and together with the Einstein equation in vacuum we have

$$\overline{\Delta} V = 0$$

We showed before that

$$R_{ij} = g^{kl} R_{kijl} - R_{0ij0}$$

together with the Gauss equation in static spacetime we have

$$\begin{aligned} R_{ij} = g^{kl} \overline{R}_{kijl} - R_{0ij0} \quad &\Longrightarrow \quad R_{ij} = \overline{R}_{ij} - R_{0ij0} \\ &\Longrightarrow \quad \overline{R}_{ij} = R_{ij} + V^{-1} V_{;ij} \\ &\Longrightarrow \quad \overline{R}_{ij} = V^{-1} V_{;ij} \end{aligned}$$

□

Chapter 6

Uniqueness of Asymptotically Euclidean Static Vacuum Spacetime

In this chapter we will prove the following theorem:

Theorem 6.0.1. *The exterior Schwarzschild solution is the only maximally extended static, vacuum, asymptotically Euclidean spacetime with regular, compact black-hole boundary.*

In 1967, this theorem, with more assumptions, was first proved by Israel in [28]. In 1986, Gary L. Bunting and A. K. M. Masood-ul-Alam proved the above generalization of Israel's theorem in [8]. According to [42, p. 18] they introduced a new approach by using results from the positive mass theorem [46] which was proved by Schoen and Yau in 1979. In 1986 Bartnik, prior to Bunting and Masood-ul-Alam's paper, generalized the positive mass theorem in n dimensions with the hypothesis of spin manifolds in [1]. Bunting and Masood-ul-Alam didn't use the assumption that the intersection of the event horizon with the closure of a t =constant hypersurface is connected, which proves that there doesn't exist multiple black holes in an asymptotically Euclidean, static, vacuum space-time.

The proof can be decomposed in three parts. First part proves a suitable asymptotic expansion for the metric g of the three manifold and the lapse function V. The second part proves some of the facts needed to use the positive mass theorem and the third part constructs the suitable manifold using all the previous parts and finally uses the positive mass theorem to prove the theorem.

We assume a static space-time with metric

$$^4g = -V^2(x^\tau)\,\mathrm{d}t^2 + g_{ab}(x^\tau)\,\mathrm{d}x^a\,\mathrm{d}x^b$$

for $a = 1, 2, 3$, where V is the lapse function and g is the Riemannian metric of the t =constant hypersurfaces.

Let Σ be the $t = 0$ slice. We assume the following

- $V > 0$ on Σ,
- $V = 0$ on $\partial\Sigma = \overline{\Sigma} \setminus \Sigma$,
- $\overline{\Sigma}$ is a spacelike oriented manifold,

- the boundary $\partial\Sigma$ is compact C^3,
- g and V are smooth in Σ and C^2 on $\overline{\Sigma}$.

We will denote the connected components of the boundary $\partial\Sigma$ with $(\partial\Sigma)_i$. Since the boundary is compact, we have that the number of $(\partial\Sigma)_i$ is finite.

We know that the extrinsic curvature of $\overline{\Sigma}$ is zero in M, but it follows that:

- $\partial\Sigma$ has extrinsic curvature zero in $\overline{\Sigma}$
- $|\nabla V|^2$ are positive constants on each connected component of $\partial\Sigma$.

We have shown in proposition 5.4.1 that the Einstein field equations in static vacuum take the form

$$\Delta_g V = 0 \tag{6.1}$$
$$Ric(g)_{ab} = V^{-1}V_{;ab} \tag{6.2}$$

for $a, b \in \{1, 2, 3\}$.

Remark 6.0.2. *To show that the second fundamental form of $(\partial\Sigma)_i$ in $\overline{\Sigma}$ is zero and that $|\nabla V|^2$ is a positive constant on each $\partial\Sigma$ we do the following:*

The static equation (6.2) *can be equally be written as:*

$$V Ric(g) = \mathrm{Hess}(V)$$

Since $V = 0$ in the boundary, we have that $\mathrm{Hess}(V) = 0$ *in the boundary. We also have*

$$\begin{aligned}\nabla_X(|\nabla V|^2) &= 2\,\langle \nabla_X \nabla V, \nabla V\rangle \\ &= 2\mathrm{Hess}V(X, \nabla V)\end{aligned}$$

Hence on $(\partial\Sigma)_i$ we have

$$\nabla_X(|\nabla V|^2) = 0$$

which implies that $|\nabla V|$ is a positive constant on $(\partial\Sigma)_i$.

From [40, p. 91] we have that

$$k(X, Y) = \frac{1}{|\nabla V|}\mathrm{Hess}V(X, Y) \tag{6.3}$$

where $X, Y \in T(\partial\Sigma)_i$ and k is the second fundamental form on $(\partial\Sigma)_i$. Then from equation (6.3) *and the static equation* (6.2) *we have*

$$V Ric(g) = |\nabla V| k \tag{6.4}$$

So for k to be zero we need $|\nabla V| \neq 0$ in $(\partial\Sigma)_i$. Assume the orthogonal vector field

$$N = \frac{\nabla V}{|\nabla V|}$$

from which we have

$$N(V) = |\nabla V|$$

on the boundary $(\partial\Sigma)_i$. We have just shown that $N(V)$ =constant in the boundary, hence it is sufficient to show that $N(V) \neq 0$ in a point $p \in (\partial\Sigma)_i$. We have that $V \in C^2(\overline{\Sigma})$, $-V$ has a maximum in $(\partial\Sigma)_i$ and from the static equation (6.1) $\Delta V = 0$, *then from Hopf's maximum principle [19, p. 347] we have that*

$$N(V)|_p = N(-V)|_p > 0$$

where $p \in (\partial\Sigma)_i$, but $N(V)$ is also a positive constant on the whole boundary. Hence $N(V) \neq 0$ on the whole boundary and so from equation (6.4) *we have that the second fundamental form $k = 0$ in the boundary.*

We assume that the spacetime $(M, {}^4g)$ is **asymptotically Euclidean** ([8, p. 2]).

Definition 6.0.3 (Asymptotically Euclidean Manifold)**.** *There exists a compact set $K \subset \Sigma$ such that $\Sigma \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B}_1(0)$ where $\overline{B}_1(0)$ is the closed unit ball centered at the origin. With respect to the standard coordinate system (y^a) in $\mathbb{R}^3$ we have on $\Sigma \setminus K$*

$$g_{ab} = \delta_{ab} + h_{ab} \tag{6.5}$$

$$V = 1 - \frac{m}{|y|} + v \tag{6.6}$$

such that for $|y|^2 \to \infty$

$$h_{ab} = O(|y|^{-1}), \quad \frac{\partial h_{ab}}{\partial y^k} = O(|y|^{-2}), \quad v = O(|y|^{-2}), \quad \frac{\partial v}{\partial y^k} = O(|y|^{-3})$$

For some $\lambda > 0$ and $4 < q < +\infty$

$$\frac{\partial^2 h_{ab}}{\partial y^k \partial y^l} \in L^q_{-\lambda-2}(\mathbb{R}^3 \setminus B_2(0)), \quad \frac{\partial^2 v}{\partial y^k \partial y^l} \in L^q_{-\lambda-3}(\mathbb{R}^3 \setminus B_2(0))$$

*The constant m is positive and is called the **mass** of (Σ, g).*

Definition 6.0.4. *[1, p. 663] The **weighted Lebesgue space** L^q_δ, for $1 \leq q \leq \infty$, and weight $\delta \in \mathbb{R}$ are the spaces of measureable functions in $L^q_{loc}(\mathbb{R}^n \setminus \{0\})$, such that the norm is defined by*

$$\|u\|_{q,\delta} = \begin{cases} \left(\int_{\mathbb{R}^n \setminus \{0\}} |u|^q r^{-\delta q - n} \, \mathrm{d}x \right)^{\frac{1}{q}}, & p < \infty \\ \text{ess sup}_{\mathbb{R}^n \setminus \{0\}} (r^{-\delta} |u|), & p = \infty \end{cases}$$

is finite.

*The **weighted Sobolev space** $W^{k,q}_\delta$ is defined by*

$$\|u\|_{k,q,\delta} = \sum_{n=0}^{k} \|D^j u\|_{q,\delta-j}$$

From [1, p. 675] we have a more general definition which is called **asymptotically flat**. This definition is the following

Definition 6.0.5 (Asymptotically Flat Manifold). *A smooth n-dimensional manifold (M, g) with Riemannian metric $g \in W^{1,q}_{loc}(M)$ for some $n < q < +\infty$ is said to be **asymptotically flat** if there is a compact $K \subset\subset M$ such that $M \setminus K$ has a **structure of infinity**:*

There is $R \geq 1$ and for $E_R = \mathbb{R}^n \setminus \overline{B}_R(0)$ a C^∞ diffeomorphism $\Phi : M \setminus K \to E_R$ which satisfies:

1. *$(\Phi_* g)_{ij}$ is uniformly equivalent to the flat metric δ_{ij} on E_R, so that there is a $\lambda \geq 1$ such that*
$$\lambda^{-1}|\xi|^2 \leq (\Phi_* g)_{ij}(x)\xi^i\xi^j \leq \lambda|\xi|^2$$
for all $x \in E_R, \xi^i \in \mathbb{R}^n$,

2. $$(\Phi_* g)_{ij} - \delta_{ij} \in W^{1,q}_{-\tau}(E_R)$$
for some decay rate $\tau > 0$.

We will now prove that the conditions of asymptotically Euclidean implies the conditions of asymptotically flat manifold.

1. Asymptotic Euclidean implies first condition of asymptotic flatness: g_{ab} is uniformly equivalent to the flat metric δ_{ab}.

Proof. For $y \in K$, g_{ab} is continuous and positive definite on a compact set

$$g_{ab}(y)\xi^a\xi^b > 0 \quad \text{for } 0 \neq \xi \in \mathbb{R}^3$$

If $|n| = 1$, let $F : K \times S^2$ such that $F(y, n) = g_{ab}(y)n^a n^b > 0$. Then $F > 0$ in the compact set and it has a minimum

$$\begin{aligned} F(y,n) \geq F_0 > 0 \quad &\implies \quad g_{ab}(y)n^a n^b \geq F_0 \text{ for } n^a = \frac{\xi^a}{|\xi|} \\ &\implies \quad g_{ab}(y)\xi^a\xi^b \geq F_0|\xi|^2 \quad \text{for } \xi \neq 0 \end{aligned}$$

and it a maximum F_1

$$g_{ab}(y)\xi^a\xi^b \leq F_1|\xi|^2$$

If $y \in M \setminus K$ then $y \in \mathbb{R}^3\setminus_R$ for suitable R

$$g_{ab} = \delta_{ab} + h_{ab}, \quad h_{ab} = O(|y|^{-1})$$

which implies

$$g_{ab}\xi^a\xi^b = |\xi|^2 + h_{ab}\xi^a\xi^b$$

We have the inner product of matrices $\text{tr}(B^\intercal A) = (B, A)$, so for symmetric matrices $(B, A) = \text{tr}(BA)$ which gives us the norm

$$\|B\| = \text{tr}(B^2)^{1/2}$$

Assume the matrix with components $P_{ab} = \xi^a \xi^b$ and $\mathrm{tr}(P^2) = \sum_a (P^2)$, then

$$\begin{aligned}(P^2)_{aa} &= \sum_b P_{ab} P_{ba} \\ &= \sum_b \xi^a \xi^b \xi^b \xi^a \\ &= (\xi^a)^2 \sum_b (\xi^b)^2 \\ &= (\xi^a)^2 |\xi|^2\end{aligned}$$

From that we have

$$\mathrm{tr}(P)^2 = \sum_a |\xi|^2 \xi_a^2 = |\xi|^4 \quad \Longrightarrow \quad \|P\| = \mathrm{tr}(P^2)^{1/2} = |\xi|^2$$

so from Cauchy-Schwarz

$$|h_{ab} \xi^a \xi^b| \leq \|h_{ab}\| \cdot |\xi|^2$$

It follows

$$\begin{aligned}|g_{ab} \xi^a \xi^b| &= ||\xi|^2 + h_{ab} \xi^a \xi^b| \\ &\leq |\xi|^2 + |h_{ab}||\xi|^2\end{aligned}$$

From the definition of asymptotically Euclidean manifold we have

$$|h_{ab} \xi^a \xi^b| \leq \frac{C}{|y|} |\xi|^2$$

which implies

$$g_{ab} \xi^a \xi^b \leq |\xi|^2 + \frac{C}{|y|} |\xi|^2$$

Similarly

$$\begin{aligned}g_{ab} \xi^a \xi^b &\geq |\xi|^2 - |h||\xi|^2 \\ &\geq |\xi|^2 - \frac{C}{|y|} |\xi|^2\end{aligned}$$

for $|y| > R$. We choose $|y| > R' > R$ such that $C/R' \leq 1/2$. From that

$$\begin{aligned}g_{ab} \xi^a \xi^b &\geq \frac{1}{2} |\xi|^2 \\ g_{ab} \xi^a \xi^b &\leq \left(1 + \frac{C}{R'}\right) |\xi|^2 \leq \frac{3}{2} |\xi|^2 \leq 2|\xi|^2\end{aligned}$$

And so we get

$$\frac{1}{2} |\xi|^2 \leq g_{ab} \xi^a \xi^b \leq 2|\xi|^2$$

for suitable R' where $|y| > R'$.

□

A usefull Lemma that we will use in general is the following:

Lemma 6.0.6. *Suppose a function $f : E_R \to \mathbb{R}$ where $E_R = \mathbb{R}^3 \setminus B_R$ and B_R is an open ball of radius R. If $f(x) = O(|x|^{-k})$ as $|x| \to \infty$ for $k > 0$, then $f \in L^q_{-\lambda}(E_R)$, for $0 < \lambda < k$ and $4 < q < +\infty$.*

Proof.

$$\begin{aligned}\int_{E_R} |f|^q |x|^{\lambda q-3}\,\mathrm{d}x &\le C_1 \int_{E_r} |x|^{\lambda q-3-kq}\,\mathrm{d}x \\ &= C_1 \int_{E_R} |x|^{q(\lambda-k)-3}\,\mathrm{d}x \\ &= C_2 \int_R^{+\infty} r^{(\lambda-k)q-1}\,\mathrm{d}r \\ &= C_3 r^{(\lambda-k)q}\Big|_R^{+\infty}\end{aligned}$$

where in the 3rd line we substituted spherical coordinates and denoted $C_2 = C_1 \int \mathrm{d}\Omega$ for the angle coordinates. The above converges for $0 < \lambda < k$. So we have proven that

$$\|f\|_{q,-\lambda} < +\infty$$

when $\lambda \in (0, k)$. □

2. Asymptotic Euclidean implies second condition of asymptotic flatness:

Proof. From the asymptotically Euclidean hypothesis and Lemma 6.0.6 we have for $0 < \tau < 1$

$$\left.\begin{aligned} h_{ab} = O(|y|^{-1}) \quad &\Longrightarrow \quad \|h_{ab}\|_{q,-\tau} < +\infty \\ \partial_k h_{ab} = O(|y|^{-2}) \quad &\Longrightarrow \quad \|\partial_k h_{ab}\|_{q,-\tau-1} < +\infty \end{aligned}\right\} \quad \Longrightarrow \quad \|h_{ab}\|_{1,q,-\tau} < +\infty$$

This means that $h_{ab} \in W^{1,q}_{-\tau}(E_R)$ for $0 < \tau < 1$. □

Remark 6.0.7. *Together with $D^2 h_{ab} \in L^q_{-\lambda-2}(E_R)$ we have*

$$h_{ab} \in W^{2,q}_{-\tau}(E_R)$$

for $\tau \in (0, 1)$.

From [46] we have the following corollary of the positive mass theorem.

Theorem 6.0.8. *[8, p. 2] Let (N, γ) be a complete oriented three dimensional Riemannian manifold which is asymptotically euclidean in the sense that N is topologically Euclidean outside a compact set and the metric γ satisfies the decay condition*

$$\gamma_{ab} = \left(1 + \frac{2c}{|y|}\right)\delta_{ab} + a_{ab}$$

where

$$a_{ab} = O(|y|^{-2}), \quad \frac{\partial a_{ab}}{\partial y^k} = O(|y|^{-3})$$

as $|y| \to \infty$ *and for some* $\lambda > 0$ *and* $4 < q < +\infty$

$$\frac{\partial^2 a_{ab}}{\partial y^k \partial y^l} \in L^q_{-\lambda-3}(\mathbb{R}^3 \setminus B_1(0))$$

If the scalar curvature of γ *is nonnegative and the mass* $c = 0$*, then* (N, γ) *is isometric to* $\mathbb{R}^3$ *with the standard euclidean metric.*

One of the important features of the positive mass theorem is that it gives as a precise meaning of mass as the divergence from the Euclidean space.

6.1 Asymptotic Form of the Metric and the Lapse Function

In this section we prove a series of Lemmas with the goal to gain a suitable form for the asymptotic expansion of the metric g and the lapse function V which is in proposition 6.1.11. Most of the following Lemmas uses the previous ones for their proofs.

First we need to change the coordinate system to **harmonic coordinates**.

Definition 6.1.1. *[16, p. 90] Local coordinates* $\{x^i\}$ *are called* ***harmonic coordinates*** *if each coordinate function* x^i *is harmonic:*

$$\Delta x^i = 0$$

We notice that

$$\begin{aligned}\Delta x^i &= g^{jk}\left(\partial_j \partial_k x^i - \Gamma^l_{kj} \partial_l x^i\right) \\ &= g^{jk}\left(\partial_j \delta^i_k - \Gamma^l_{jk} \delta^i_l\right) \\ &= -g^{jk} \Gamma^i_{jk}\end{aligned}$$

and so for harmonic coordinates it holds that

$$g^{jk} \Gamma^i_{jk} = 0 \tag{6.7}$$

Now we will choose an asymptotic coordinate system (x^a) harmonic relative to the metric $\Lambda = V^2 g$ such that the components of g and the function V have better decay properties than before. This is achieved by the following proposition of Bartnik:

Proposition 6.1.2. *[1, p. 19] Suppose that* (M, g) *has a structure of infinity* Φ *with decay rate* $\eta > 0$*, so* $(\Phi_* g - \delta) \in W^{2,q}_{-\eta}(E_R)$ *for some* $q > n, R \geq 1$*, and that the Ricci tensor of* (M, g) *satisfies*

$$Ric(g) \in L^q_{-2-\tau}(M) \quad \textit{for some nonexceptional } \tau > \eta$$

Then there is a structure of infinity Θ *defined by coordinates harmonic near infinity which satisfies* $(\Theta_* g - \delta) \in W^{2,q}_{-\tau}(E_{R_1})$*, for some* $R_1 \geq R$*.*

We have already showed that $h_{ab} \in W^{2,q}_{-\eta}(E_R)$ for $0 < \eta < 1$ and that the manifold has a structure of infinity. We need to show the condition for the Ricci tensor. First we conformally change the metric to give the conditions needed.

Lemma 6.1.3. *Let g be the metric and V the lapse function of the static, asymptotically Euclidean spacetime . If $\Lambda = V^2 g$ and $U = \log V$. Then*

$$\Delta_\Lambda U = 0 \tag{6.8}$$
$$Ric(\Lambda)_{ab} = 2U_{;a}U_{;b} \tag{6.9}$$

Proof. First we will calculate equation 6.8

$$\Delta_\Lambda U = \frac{1}{\sqrt{|\Lambda|}}\partial_i\left(\Lambda^{ij}\sqrt{|\Lambda|}U_{;j}\right)$$

Where $\Lambda^{ij} = V^{-2}g^{ij}$, for $|\Lambda| = \det \Lambda$

$$|\Lambda| = |V^2 g| = V^6|g| \quad \Longrightarrow \quad \sqrt{|\Lambda|} = V^3\sqrt{|g|}$$

and $\partial_j U = V^{-1}\partial_j V$. Then

$$\begin{aligned}
\Delta_\Lambda U &= \frac{1}{V^3\sqrt{|g|}}\partial_i\left(V^{-2}g^{ij}\sqrt{|g|}V^3V^{-1}\partial_j V\right)\\
&= \frac{1}{V^3\sqrt{|g|}}\partial_i(g^{ij}\sqrt{|g|}\partial_j V)\\
&= \frac{1}{V^3}\Delta_g V\\
&= 0
\end{aligned}$$

The change of metric is conformal so the Ricci curvature becomes (from [45, p. 184])

$$\begin{aligned}
Ric(\Lambda)_{ab} &= R(g)_{ab} - \frac{1}{2}(\log V^2)_{;ab} + \frac{1}{4}(\log V^2)_{;a}(\log V^2)_{;b} - \left[\frac{\Delta(\log V^2)}{2} + \frac{1}{4}|\nabla \log V^2|^2\right] g_{ab}\\
&= \frac{V_{;ab}}{V} - (\log V)_{;ab} + (\log V)_{;a}(\log V)_{;b} - \left[\Delta(\log V) + |\nabla \log V|^2\right] g_{ab}
\end{aligned}$$

We have

$$\begin{aligned}
(\log V)_{;ab} &= \frac{V_{;ab}}{V} - \frac{V_{;a}V_{;b}}{V^2}\\
\Delta(\log V) &= \frac{\Delta V}{V} - \frac{|\nabla V|^2}{V^2}\\
|\nabla \log V|^2 &= \frac{|\nabla V|^2}{V^2}
\end{aligned}$$

and so

$$\begin{aligned}
Ric(\Lambda)_{;ab} &= \frac{V_{;ab}}{V} - \frac{V_{;ab}}{V} + \frac{V_{;a}V_{;b}}{V^2} + \frac{V_{;a}V_{;b}}{V^2} - \frac{\Delta V}{V}g_{ab}\\
&= 2U_{;a}U_{;b}
\end{aligned}$$

□

Now we can show the following lemma:

Lemma 6.1.4. *$Ric(\Lambda)_{ab} \in L^q_{-2-\tau}(M)$ for some $\tau > \eta$.*

Proof.

$$Ric(\Lambda)_{ab} = 2U_{;a}U_{;b} = 2V^{-2}V_{;a}V_{;b}$$

where

$$V = 1 - \frac{m}{|y|} + O(|y|^{-2}) \quad \Longrightarrow \quad V_a = -\frac{m}{|y|^3}y^a + O(|y|^{-3})$$

$$\begin{aligned}
Ric(\Lambda)_{ab} &= 2\frac{1}{\left(1-\frac{m}{|y|}+O(|y|^{-2})\right)^2}\left(-\frac{m}{|y|^3}y^a + O(|y|^{-3})\right)\left(-\frac{m}{|y|^3}y^b + O(|y|^{-3})\right) \\
&= 2\left(1+\frac{m}{|y|}+O(|y|^{-2})\right)^2\left[\frac{m^2}{|y|^6}y^a y^b + O(|y|^{-6})\right] \\
&= 2\left[\frac{m^2}{|y|^6}y^a y^b + \frac{m^3}{|y|^7 y^a y^b} + O(|y|^{-6})\right] \\
&= 2\frac{m^2}{|y|^6}y^a y^b + O(|y|^{-5}) \\
&= O(|y|^{-4})
\end{aligned}$$

where in the second equality we use the Taylors expansion of $1/(1-x)$. So from lemma 6.0.6 we have that for $\lambda < 4$ and $\eta < \tau < 2$

$$Ric(\Lambda)_{ab} \in L^q_{-\lambda}(M) \quad \Longrightarrow \quad Ric(\Lambda)_{ab} \in L^q_{-2-\tau}(M)$$

□

Now from proposition 6.1.2 we have that there exists an asymptotic coordinate system (x^a) harmonic relative to Λ such that $\tilde{\Lambda}_{ab}$ components of the metric Λ satisfy

$$\tilde{\Lambda}_{ab} = \delta_{ab} + \Pi_{ab} \tag{6.10}$$

where

$$\Pi_{ab} = O(|x|^{-1}), \quad \partial_k \Pi_{ab} = O(|x|^{-2})$$

and for $\tau \in (1,2)$

$$\partial_k\partial_l\Pi_{ab} \in L^q_{-\tau-2}(\mathbb{R}^3 \setminus B_3(0))$$

From equation (6.7) and lemma 6.1.3 we have

$$\begin{aligned}
\Delta_\Lambda U = 0 \quad &\Longrightarrow \quad \tilde{\Lambda}^{ab}\left(\frac{\partial^2 U}{\partial x^a \partial x^b} - \Gamma^k_{ab}\partial_k U\right) = 0 \\
&\Longrightarrow \quad \tilde{\Lambda}^{ab}\left(\frac{\partial^2 U}{\partial x^a \partial x^b}\right) = 0
\end{aligned}$$

Lemma 6.1.5. *[16, p. 92] In harmonic coordinates, the Ricci tensor is given by*

$$-2R_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g)$$

where $\Delta(g_{ij})$ denotes the Laplacian of the component g_{ij} and Q denotes a sum of terms which are quadratic in the metric inverse g^{-1} and its first derivatives ∂g.

Proof.

$$\begin{aligned}R_{jk} &= R_{qjk}{}^{q}\\ &= \partial_q\Gamma^q_{jk} - \partial_j\Gamma^q_{qk} + \Gamma^p_{jk}\Gamma^q_{qp} - \Gamma^p_{qk}\Gamma^q_{jp}\end{aligned}$$

The last two terms will be absorbed by Q since they will be quadratic in g^{-1} and ∂g. For the first two terms we have

$$\begin{aligned}\partial_q\Gamma^q_{jk} &= \frac{1}{2}\left[\partial_q(g^{qr})\left(\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk}\right)\right] + \frac{1}{2}g^{qr}\left[\left(\partial_q\partial_j g_{kr} + \partial_q\partial_k g_{jr} - \partial_q\partial_r g_{jk}\right)\right]\\ \partial_j\Gamma^q_{qk} &= \frac{1}{2}\left[\partial_j(g^{qr})\left(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{qk}\right)\right] + \frac{1}{2}g^{qr}\left[\left(\partial_j\partial_q g_{kr} + \partial_j\partial_k g_{qr} - \partial_j\partial_r g_{qk}\right)\right]\end{aligned}$$

where again the first terms in the above equations are absorbed in Q. So we have

$$-2R_{jk} = g^{qr}\left(-\partial_q\partial_k g_{jr} + \partial_q\partial_r g_{jk} + \partial_j\partial_k g_{qr} - \partial_j\partial_r g_{qk}\right) + Q \tag{6.11}$$

We want to show that

$$-2R_{jk} = \Delta(g_{jk}) - g^{qr}\partial_k(\Gamma^s_{qr}g_{sj}) - g^{qr}\partial_j(\Gamma^s g_{sk}) + Q \tag{6.12}$$

because if the above holds then

$$\begin{aligned}-2R_{jk} &= \Delta(g_{jk}) - g^{qr}\partial_k(\Gamma^s_{qr}g_{sj}) - g^{qr}\partial_j(\Gamma^s g_{sk}) + Q\\ &= \Delta(g_{jk}) - \partial_k(g^{qr}\Gamma^s_{qr}g_{sj}) - \partial_j(g^{qr}\Gamma^s_{qr}g_{sk}) + \partial(g^{qr})\Gamma^s_{qr}g_{sj} + \partial_j(g^{qr})\Gamma^s_{qr}g_{sk} + Q\end{aligned}$$

where the last terms are absorbed from Q, and the second and third terms are zero from the necessary condition of the harmonic coordinates (6.7). And so the equation holds

$$-2R_{jk} = \Delta(g_{jk}) + Q(g^{-1}, \partial g)$$

Now we will open the first three terms in equation (6.12) to show that it is the same equation as (6.11). We will do the computation in normal coordinates.

$$\begin{aligned}\Delta(g_{jk}) - g^{qr}\partial_k(\Gamma^s_{qr}g_{sj}) - g^{qr}\partial_j(\Gamma^s_{qr}g_{sk}) &= g^{qr}\left[\partial_q\partial_r g_{jk} - g_{sj}\partial_k(\Gamma^s_{qr}) - g_{sk}\partial_j(\Gamma^s_{qr})\right]\\ &= g^{qr}\Big[\partial_q\partial_r g_{jk} - \frac{1}{2}g_{sj}\partial_k\left(g^{sl}(\partial_q g_{rl} + \partial_r g_{ql} - \partial_l)\right)\\ &\qquad - \frac{1}{2}g_{sk}\partial_j\left(g^{sl}\left(\partial_q g_{rl} + \partial_r g_{ql} - \partial_l g_{qr}\right)\right)\Big]\end{aligned}$$

We will ignore the first term $g^{qr}\partial_q\partial_r g_{jk}$ since it is one of the terms that we need in equation (6.11)

$$\begin{aligned}&-\frac{1}{2}g^{qr}\left(g_{sj}g^{sl}\left[\partial_k\partial_q g_{rl} + \partial_k\partial_r g_{ql} - \partial_k\partial_l g_{qr}\right] + g_{sk}g^{sl}\left[\partial_j\partial_q g_{rl} + \partial_j\partial_r g_{ql} - \partial_j\partial_l g_{qr}\right]\right)\\ &= -\frac{1}{2}g^{qr}\left(\left[\partial_k\partial_q g_{rj} + \partial_k\partial_r g_{qj} - \partial_k\partial_j g_{qr}\right] - g^{qr}\left[\partial_j\partial_q g_{rk} + \partial_j\partial_r g_{qk} - \partial_j\partial_k g_{qr}\right]\right)\end{aligned}$$

we reverse the indices q, r since its in a sum

$$= -\frac{1}{2}g^{qr}\left[2\partial_k\partial_r g_{qj} + 2\partial_j\partial_r g_{qk} - 2\partial_k\partial_j g_{qr}\right] = -g^{qr}\left(\partial_q\partial_k g_{rj} + \partial_j\partial_r g_{qk} - \partial_j\partial_k g_{qr}\right)$$

which is rest of the terms in equation (6.11). ∎

From lemma 6.1.5 we have

$$
\begin{aligned}
Ric(\tilde{\Lambda})_{ij} &= -\frac{1}{2}\tilde{\Lambda}^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}\tilde{\Lambda}_{ij} + Q_{ij}(\tilde{\Lambda}, \partial\tilde{\Lambda}) \\
\implies \quad 2\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} &= -\frac{1}{2}\tilde{\Lambda}^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^l}\tilde{\Lambda}_{ij} + Q_{ij}(\tilde{\Lambda}, \partial\tilde{\Lambda}) \\
\implies \quad \tilde{\Lambda}^{ab}\left(\frac{\partial^2\tilde{\Lambda}_{ij}}{\partial x^a \partial x^b}\right) &= -4\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} + Q_{ij}(\tilde{\Lambda}, \partial\tilde{\Lambda})
\end{aligned}
$$

In the proof of Theorem 4.3 in [1, p. 22], says that from an observation of in [48] we have the expansion in harmonic coordinates

$$
\tilde{\Lambda}_{ij} = \delta_{ij} + A_{ij}|x|^{-1} + O(|x|^{-\tau}) \tag{6.13}
$$

for $\tau > 1$ and A_{ij} constant matrix.

We will prove that $A_{ij} = 0$, by using the harmonicity condition

Lemma 6.1.6. *If we have an asymptotic expansion*

$$
\tilde{\Lambda}_{ij} = \delta_{ij} + A_{ij}|x|^{-1} + O(|x|^{-\tau})
$$

in harmonic coordinates, then $A_{ij} = 0$.

Proof. From harmonic coordinates we have

$$
\Delta x^k = 0 \quad \implies \quad \frac{1}{\sqrt{|\tilde{\Lambda}|}}\partial_i\left(\sqrt{|\tilde{\Lambda}|}\tilde{\Lambda}^{ij}\partial_j x^k\right) = 0
$$

but first we need to calculate $|\tilde{\Lambda}|$ and $\tilde{\Lambda}^{ij}$. We have

$$
\begin{aligned}
\tilde{\Lambda}_{ij} &= \delta_{ij} + \varepsilon A_{ij} + O(|x|^{-\tau}) \\
&= \delta_{ij} + \varepsilon\mu_{ij}
\end{aligned}
$$

where $\varepsilon = |x|^{-1}$, $\mu_{ij} = A_{ij} + O(|x|^{-\tau+1})$. We write the determinant as

$$
|\tilde{\Lambda}| = \varepsilon^{ijk}\tilde{\Lambda}_{1i}\tilde{\Lambda}_{2j}\tilde{\Lambda}_{3k}
$$

where ε is the levi-civita symbol [10, p. 24]

$$
\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 123 \\ -1, & \text{if } ijk \text{ is an off permutation of } 123 \\ 0, & \text{otherwise} \end{cases}
$$

We have

$$
\begin{aligned}
|\tilde{\Lambda}| &= \varepsilon^{ijk}\tilde{\Lambda}_{1i}\tilde{\Lambda}_{2j}\tilde{\Lambda}_{3k} \\
&= \varepsilon^{ijk}\left(\delta_{1i} + \varepsilon\mu_{1i}\right)\left(\delta_{2j} + \varepsilon\mu_{2j}\right)\left(\delta_{3k} + \varepsilon\mu_{3k}\right) \\
&= \varepsilon^{ijk}\left(\delta_{1i}\delta_{2j}\delta_{3k} + \varepsilon(\mu_{1i}\delta_{2j}\delta_{3k} + \mu_{2j}\delta_{1i}\delta_{3k} + \mu_{3k}\delta_{1i}\delta_{2j})\right) + O(\varepsilon^3) \\
&= \varepsilon^{123} + \varepsilon(\varepsilon^{i23}\mu_{1i} + \varepsilon^{1j3}\mu_{2j} + \varepsilon^{12k}\mu_{3k}) + O(\varepsilon^2) \\
&= 1 + \varepsilon(\mu_{11} + \mu_{22} + \mu_{33}) + O(\varepsilon^2) \\
&= 1 + \varepsilon\,\mathrm{tr}(\mu) + O(\varepsilon^2)
\end{aligned}
$$

But $\mu = A + O(|x|^{-\tau+1}),\ \varepsilon = |x|^{-1}$,

$$\begin{aligned}|\tilde{\Lambda}| &= 1 + \varepsilon \mathrm{tr}\left(A + O(|x|^{-\tau+1}) + O(\varepsilon^2)\right) \\ &= 1 + \frac{\mathrm{tr}A}{|x|} + O(|x|^{-\tau}) + O(|x|^{-2})\end{aligned}$$

But $\tau > 1$, and so

$$|\tilde{\Lambda}| = 1 + \frac{\mathrm{tr}A}{|x|} + O(|x|^{-\tau}) \tag{6.14}$$

for $\tau \in (1,2)$. We also have

$$\tilde{\Lambda}^{ij} = \delta_{ij} - \varepsilon A_{ij} + O(\varepsilon^2) \tag{6.15}$$

From the harmonicity condition

$$\begin{aligned}\Delta x^k = 0 \quad &\Longrightarrow \quad \frac{1}{\sqrt{|\tilde{\Lambda}|}}\partial_i\left(\sqrt{|\tilde{\Lambda}|}\tilde{\Lambda}^{ij}\partial_j x^k\right) = 0 \\ &\Longrightarrow \quad \partial_i\left(\sqrt{|\tilde{\Lambda}|}\tilde{\Lambda}^{ij}\delta_{jk}\right) = 0 \\ &\Longrightarrow \quad \partial_i\left(\sqrt{\tilde{\Lambda}}\tilde{\Lambda}^{ij}\right) = 0 \end{aligned} \tag{6.16}$$

From 6.14

$$\sqrt{\tilde{\Lambda}} = \sqrt{1 + \frac{\mathrm{tr}A}{|x|} + O(|x|^{-\tau})}$$

From Taylor expansion

$$\sqrt{1+t} = 1 + \frac{t}{2} + O(t^2), \quad \text{for } |t| < 1$$

assume

$$t = \frac{\mathrm{tr}A}{|x|} + O(|x|^{-\tau})$$

So we have from the above

$$\begin{aligned}\sqrt{|\tilde{\Lambda}|} &= 1 + \frac{\mathrm{tr}A}{2|x|} + O(|x|^{-\tau}) \\ \tilde{\Lambda}^{ij} &= \delta^{ij} - \frac{A_{ij}}{|x|} + O(|x|^{-\tau})\end{aligned}$$

Now we continue equation (6.16)

$$\begin{aligned}\partial_i\left(\sqrt{|\tilde{\Lambda}|}\tilde{\Lambda}^{ij}\right) = 0 \quad &\Longrightarrow \quad \partial_i\left[\left(1 + \frac{\mathrm{tr}A}{2|x|} + O(|x|^{-\tau})\right)\left(\delta^{ij} - \frac{A_{ij}}{|x|} + O(|x|^{-\tau})\right)\right] = 0 \\ &\Longrightarrow \quad \partial_i\left[\delta^{ij}\left(\delta^{ij}\frac{\mathrm{tr}A}{2|x|} - \frac{A_{ij}}{|x|}\right) + O(|x|^{-\tau})\right] = 0 \\ &\Longrightarrow \quad \partial_i\left(\frac{\delta_{ij}\mathrm{tr}A}{2|x|} - \frac{A_{ij}}{|x|}\right) + \partial_i\left(O(|x|^{-\tau})\right) = 0\end{aligned}$$

We observe that

$$\partial_i(O(|x|^{-\tau})) = O(|x|^{-(\tau+1)}), \quad \partial_i\left(\delta_{ij}\frac{\mathrm{tr}A}{2|x|} - \frac{A_{ij}}{|x|}\right) = O(|x|^{-2})$$

But $\tau + 1 > 2$ for $\tau \in (1,2)$, so both terms must be zero.

$$\begin{aligned}
\partial_i\left(\delta^{ij}\frac{\mathrm{tr}A}{2|x|} - \frac{A_{ij}}{|x|}\right) = 0 \quad &\Longrightarrow \quad \delta^{ij}\partial_i\left(\frac{\mathrm{tr}A}{2|x|}\right) - \partial_i\left(A_{ij}|x|\right) = 0 \\
&\Longrightarrow \quad -\delta^{ij}\frac{\mathrm{tr}A}{2|x|^2}2x^i + A_{ij}\frac{2x^i}{|x|^2} = 0 \\
&\Longrightarrow \quad \frac{\mathrm{tr}A}{2}x^i\delta_{ij} = A_{ij}x^i
\end{aligned}$$

for all $x \in \mathbb{R}^3$.

Suppose $x = e^1 = (1,0,0)$.

$$\left(A_{ij} - \frac{\mathrm{tr}A}{2}\delta_{ij}\right)x^i = 0 \quad \Longrightarrow \quad A_{1j} - \frac{\mathrm{tr}A}{\delta_{1j}} = 0 \quad \Longrightarrow \quad \begin{cases} A_{1j} = 0, & j \neq 1 \\ A_{11} = \dfrac{\mathrm{tr}A}{2}, & j = 1 \end{cases}$$

Similarly,

$$A_{22} = \frac{\mathrm{tr}A}{2}, \quad A_{33} = \frac{\mathrm{tr}A}{2}$$

From the above

$$A_{11} + A_{22} + A_{33} = 3\frac{\mathrm{tr}A}{2} \quad \Longrightarrow \quad \mathrm{tr}A = 3\frac{\mathrm{tr}A}{2} \quad \Longrightarrow \quad \mathrm{tr}A = 0$$

and together with the fact that $A_{1j} = A_{2j} = A_{3j} = 0$ for $j \neq 1$ we have that

$$A = 0$$

$\square$

So equation (6.13) becomes

$$\tilde{\Lambda}_{ab} = \delta_{ab} + O(|x|^{-\tau}) \tag{6.17}$$

Remark 6.1.7. *A usefull calculation that we will use is the following*

$$\begin{aligned}
\partial_a\left(\frac{1}{|y|^n}\right) &= \partial_a\frac{1}{\left(\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}\right)^n} \\
&= \partial_a\left((y^1)^2 + (y^2)^2 + (y^3)^2\right)^{-n/2} \\
&= -\frac{n}{2}\frac{1}{\left((y^1)^2 + (y^2)^2 + (y^3)^2\right)^{n+2}}\left(2y^1\delta^1_a + 2y^2\delta^2_a + 2y^3\delta^3_a\right) \\
&= -\frac{n}{2}\frac{1}{|y|^{n+2}}2y^a \\
&= -n\frac{y^a}{|y|^{n+2}}
\end{aligned}$$

Lemma 6.1.8. *The $U = \log V$ can be written as*

$$U = -\frac{m}{|x|} + \frac{c_a x^a}{|x|^3} + u \tag{6.18}$$

where c_a are constants and

$$u = O(|x|^{-\tau-1}), \quad \partial_a u = O(|x|^{-\tau-2}), \quad \partial_a\partial_b u \in L^q_{-\tau-3}L^q(\mathbb{R}^3 \setminus B_R(0))$$

Proof. From equations (6.8) and (6.17), for $\lambda \in (1,2)$ we have:

$$\begin{aligned}\tilde{\Lambda}^{ab}U_{;ab} = 0 \quad &\Longrightarrow \quad \delta^{ab}U_{;ab} + O(|x|^{-\lambda})U_{;ab} = 0 \\ &\Longrightarrow \quad \Delta_E U = -O(|x|^{-\lambda})U_{;ab}\end{aligned}$$

Where the operator Δ_E is the Euclidean Laplacian. We denote $M_{ab} = O(|x|^{-\lambda})$ so we can write

$$\Delta_E U = -M_{ab}U_{;ab}$$

First we write

$$\begin{aligned}\frac{\partial^2 U}{\partial x^a \partial x^b} &= \frac{\partial}{\partial x^a}\left(\frac{1}{V}\frac{\partial V}{\partial x^b}\right) \\ &= -\frac{1}{V^2}\partial_a V \partial_b V + \frac{1}{V}\frac{\partial^2 V}{\partial x^a \partial x^b}\end{aligned}$$

where

$$V = O(1), \quad \partial_a V = O(|x|^{-2}), \quad \frac{\partial^2 V}{\partial x^a \partial x^b} = O(|x|^{-3}) + \frac{\partial^2 v}{\partial x^a \partial x^b}$$

and

$$\frac{\partial^2 v}{\partial x^a \partial x^b} \in L^q_{\lambda-3}$$

$$v = O(|x|^{-2}), \quad \partial_a v = O(|x|^{-3}), \quad \partial_a\partial_b v \in L^q_{\lambda-3}(\mathbb{R}^3 \setminus B_3(0))$$

Hence we have

$$\begin{aligned}|\Delta_E U| &= |M_{ab}D^2 U| \\ &\leq \frac{C}{|x|^{4+\lambda}} + \frac{C}{|x|^{3+\lambda}} + \frac{C}{|x|^{\lambda}}|D^2 v|\end{aligned}$$

From lemma 6.0.6 we have

$$\frac{C}{|x|^{3+\lambda}} \in L^q_{-k}, \quad \frac{C}{|x|^{4+\lambda}} \in L^q_l$$

for $l < 4+\lambda$ and $k < 3+\lambda$. Substituting for $l = 4+\tau$ and $k = 3+\tau$ we have that the above holds for $\tau < \lambda$. For the third term we check for what exponents is it in $L^q_{\tau-3}$

$$\int_{E_R} |x|^{-\lambda q}|D^2 v|^q |x|^{-\omega}\,\mathrm{d}x = \int_{E_R} |D^2 v|^q |x|^{-\omega-\lambda q}$$

Suppose $-\omega - \lambda q = (\tau+3)q - 3$

$$\begin{aligned}\omega = -\tau q - 3q + 3 - \lambda q \quad &\Longrightarrow \quad \tau q = -\omega - 3q - \lambda + 3 \\ &\Longrightarrow \quad q < -\omega - 3q - \lambda q + 3 < 2q \\ &\Longrightarrow \quad -q(5+\lambda) + 3 < \omega < -q(4+\lambda) + 3\end{aligned}$$

so for suitable choice of ω we have:

$$\int_{E_R} |D^2 v|^q |x|^{-\omega-\lambda q} < +\infty \quad \Longrightarrow \quad |x|^{-\lambda} D^2 v \in L^q_{-\tau-3}$$

Hence

$$\Delta_E U \in L^q_{-\tau-3}$$

for $\tau < \lambda$.

We now prove that the first two terms in equation (6.18) are harmonic:

First term:

$$\begin{aligned}\Delta_E\left(\frac{m}{|x|}\right) &= m\partial^2_{\lambda\lambda}\frac{1}{|x|} \\ &= -m\partial_\lambda \frac{x^\lambda}{|x|^3} \\ &= 3m\frac{x^\lambda x^\lambda}{|x|^5} - m\frac{3}{|x|^3} \\ &= 0\end{aligned}$$

Second term:

$$\Delta_E\left(\frac{c_a x^a}{|x|^3}\right) = \frac{1}{|x|^3}\Delta_E(c_a x^a) + 2\nabla\left(\frac{1}{|x|^3}\right)\cdot\nabla(c_a x^a) + c_a x^a \Delta_E\left(\frac{1}{|x|^3}\right)$$

where

- $\Delta_E(c_a x^a) = 0$
- $\partial_\lambda(c_a x^a) = c_a \partial_\lambda x^a = c_a \delta^a_\lambda = c_\lambda$
-
$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}\frac{1}{|x|^3} &= \frac{\partial}{\partial\lambda}\left(-\frac{3x^\lambda}{|x|^5}\right) \\ &= \frac{15x^\lambda x^\lambda}{|x|^7} - \frac{9}{|x|^5} \\ &= \frac{15}{|x|^5} - \frac{9}{|x|^5} \\ &= \frac{6}{|x|^5}\end{aligned}$$

hence

$$\begin{aligned}\Delta_E\left(\frac{c_a x^a}{|x|^3}\right) &= -2c_\lambda\frac{3x^\lambda}{|x|^5} + c_a x^a \frac{6}{|x|^5} \\ &= 0\end{aligned}$$

From the above we have that

$$\begin{aligned}U = -\frac{m}{|x|} + \frac{c_a x^a}{|x|^3} + u \quad &\Longrightarrow \quad \Delta_E U = -\Delta_E\left(\frac{m}{|x|}\right) + \Delta_E\left(\frac{c_a x^a}{|x|^3}\right) + \Delta_E u \\ &\Longrightarrow \quad \Delta_E U = \Delta_E u\end{aligned}$$

And so

$$\Delta_E U \in L^q_{-\tau-3} \quad \Longrightarrow \quad \Delta_E u \in L^q_{-\tau-3} \quad \Longrightarrow \quad \|\Delta_E u\|_{0,q,-\tau-3} < +\infty$$

From [1, p. 667] we have the following Theorem

Theorem 6.1.9. *Suppose that δ is* ***nonexceptional****, $1 < p < \infty$, and s is a non-negative integer. Then the map*

$$\Delta : W^{s+2,p}_{\delta} \to W^{s,p}_{\delta-2}$$

is an isomorphism and there is a constant $C = C(n, p, \delta, s)$ such that

$$\|u\|_{s+2,p,\delta} \leq C\|\Delta u\|_{s,p,\delta-2}$$

We call $\delta \in \mathbb{R}$ nonexceptional if $\delta \in \mathbb{R} \setminus \{k \in \mathbb{Z} : k \neq -1, -2, \ldots, 3-n\}$.
From the above Theorem, for $s = 0$ and $\delta = -\tau - 1$, we have

$$\|u\|_{2,q,-\tau-1} \leq C\|\Delta u\|_{0,q,-\tau-3} < +\infty$$

From that we have our first estimate needed for the Lemma:

$$D^2 u \in L^q_{-\tau-3}$$

For the first derivative we will need from [1, p. 664] the following results

Theorem 6.1.10. *If $u \in W^{k,p}_{\delta}$, such that $0 < a \leq k - n/p \leq 1$, then*

$$\|u\|_{C^{0,a}_{\delta}} \leq C\|u\|_{k,p,\delta} \tag{6.19}$$

where the weighted Holder norm is defined by

$$\|u\|_{C^{0,a}_{\delta}} = \sup_{x\in\mathbb{R}^n} \left((\sqrt{1+|x|^2})^{-\delta+a}(x) \sup_{4|x-y|\leq\sqrt{1+|x|^2}} \frac{|u(x)-u(y)|}{|x-y|^a} \right) + \sup_{x\in\mathbb{R}^n} \{(\sqrt{1+|x|^2})^{-\delta}|u(x)|\} \tag{6.20}$$

For E_R we have

$$u \in W^{2,q}_{-\tau-1}(E_R) \quad \Longrightarrow \quad Du \in W^{1,q}_{-\tau-2}(E_R)$$

and from the above theorem for $0 < a \leq 1 - 3/q \leq 1$ and $q > 4$ we have

$$\|Du\|_{C^{0,a}_{-\tau-2}} \leq C\|Du\|_{1,q,-\tau-2} < +\infty$$

which gives us

$$|Du(x)| \leq \frac{C}{|x|^{\tau+2}}$$

hence we have the desired estimate for the first derivative of u

$$|Du(x)| = O(|x|^{-\tau-2}) \tag{6.21}$$

We also have

$$|Du(x) - Du(y)| \leq \frac{C|x-y|^a}{|x|^{\tau+2+a}} \tag{6.22}$$

We cant use the theorem for $k = 0$ because of the limitation for choosing a.
From the continuity of Du we can use the mean value theorem for

$$\xi = tx + (1-t)x_0$$

such that $t \in (0, 1)$. So we have

$$|u(x) - u(x_0)| = |Du(\xi)(x - x_0)| \leq |Du(\xi)||x - x_0|$$

Now from equations (6.21), and (6.22)

$$|Du(\xi)| = |Du(\xi) - Du(x) + Du(x)| \leq \frac{C|\xi - x|^a}{|x|^{\tau+2+a}} + \frac{C}{|x|^{\tau+2}}$$

But

$$|\xi - x| = |tx + (1-t)x_0 - x| = |(1-t)x_0 - (1-t)x| \leq |x - x_0|$$

Therefore

$$|Du(\xi)| = |Du(\xi) - Du(x) + Du(x)| \leq \frac{C|x - x_0|^a}{|x|^{\tau+2+a}} + \frac{C}{|x|^{\tau+2}}$$

But we have

$$\frac{C|x - x_0|^a}{|x|^{\tau+2+a}} \leq \frac{C(|x| + |x_0|)^a}{|x|^{\tau+2+a}} \leq \frac{C}{|x|^{\tau+2}} \left(1 + \frac{|x_0|}{|x|}\right)^a$$

We notice for $|x| \to +\infty$

$$\left(1 + \frac{|x_0|}{|x|}\right)^a \to 1$$

Then for $|x| > R_1$

$$\left(1 + \frac{|x_0|}{|x|}\right)^a \leq \frac{3}{2}$$

and so

$$|Du(\xi)| \leq \frac{5C}{2|x|^{\tau+2}}$$

Hence

$$|u(x) - u(x_0)| \leq \frac{5C}{2|x|^{\tau+2}}|x - x_0|$$

And so

$$\begin{aligned} |u(x)| &= |u(x) - u(x_0) + u(x_0)| \\ &\leq |u(x) - u(x_0)| + |u(x_0)| \\ &\leq \frac{5C}{2|x|^{\tau+2}}|x - x_0| + |u(x_0)| \\ &\leq \frac{5C}{2|x|^{\tau+1}} + \frac{5C|x_0|}{2|x|^{\tau+2}} + |u(x_0)| \end{aligned}$$

Since as $|x| \to +\infty$

$$\frac{5C|x_0|}{2|x|^{\tau+2}} + |u(x_0)| \to |u(x_0)|$$

then for $|x| > R_2$

$$\frac{5C|x_0|}{2|x|^{\tau+2}} + u(x_0) \leq \frac{3}{2}|u(x_0)|$$

hence for $R = \max(R_1, R_2),\ |x| > R$

$$|u(x)| \leq \frac{5C}{2|x|^{\tau+1}} + \frac{3}{2}|u(x_0)|$$

The later holds for $|x_0|, |x| > R$, then

$$\lim_{|x_0| \to +\infty} |u(x_0)| = 0$$

because u is uniform continuous (bounded derivative) and integrable we have that for $|x_0| \to +\infty$ we get the desired estimate for $|x| > R$

$$|u(x)| \leq \frac{5C}{2|x|^{\tau+1}}$$

or differently

$$u(x) = O(|x|^{-\tau-1})$$

□

Now we will give the metric g and the lapse function V the desirable form and decay conditions using the previous results.

Proposition 6.1.11. *Let $(M, {}^4g)$ be a static, vacuum, asymptotically Euclidean spacetime with a metric*

$${}^4g = -V^2\,\mathrm{d}t^2 + g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$$

Then the spatial part of the metric g_{ab} and the lapse function V can be written, in harmonic coordinates $\{x^a\}$ while $|x| \to +\infty$, as

$$g_{ab} = \left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}$$
$$V = 1 - \frac{m}{|x|} + \frac{m^2}{2|x|^2} + \frac{c_a x^a}{|x|^3} + u$$

where, for $\tau \in (1,2)$,

$$\mathcal{H}_{ab} = O(|x|^{-\tau}), \quad D\mathcal{H}_{ab} = O(|x|^{-\tau-1}), \quad D^2\mathcal{H}_{ab} \in L^q_{\tau-2}(E_{R_1})$$

and

$$u = O(|x|^{-\tau-1}), Du = O(|x|^{-\tau-2}), \quad D^2u = L^q_{-\tau-3}(E_{R_2})$$

Proof. In the previous steps we have used the transformation $\Lambda = V^2 g$, we proved for harmonic coordinates the equation (6.17) and (6.18).

$$\Lambda = V^2 g \quad \Longrightarrow \quad g = \frac{1}{V^2}\Lambda$$

From $U = \log V$ we have

$$V = e^U = 1 + U + \frac{U^2}{2} + \dots$$

Assume

$$\hat{V} = U + \frac{U^2}{2}$$

where we ignore terms $O(U^3)$. From equation (6.18) we have

$$U = O(|x|^{-1}) \quad \Longrightarrow \quad \hat{V} = O(|x|^{-1})$$

Thus

$$g = \frac{1}{(1+\hat{V})^2}\Lambda = \left(1 - 2\hat{V} + 3\hat{V}^2 + O(\hat{V}^3)\right)\Lambda$$

where the second equality comes from

$$\frac{1}{(1+x)^2} = -\frac{\partial}{\partial x}\left(\frac{1}{1-(-x)}\right) = -\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)$$

from (6.17) we have

$$\Lambda = 1 + O(|x|^{-\tau})$$

where $\tau \in (1,2)$. Lets denote $\pi = O(|x|^{-\tau})$, then

$$\begin{aligned} g &= \left(1 - 2\hat{V} + 3\hat{V}^2 + O(|x|^{-3})\right)(1+\pi) \\ &= \left(1 - \hat{V} + 3\hat{V}^2\right)(1+\pi) \\ &= 1 - 2\hat{V} + 3\hat{V}^2 + \pi - 2\pi\hat{V} + 3\hat{V}^2\pi \end{aligned}$$

We notice that

$$\pi = O(|x|^{-\tau}), \quad \pi\hat{V} = O(|x|^{-\tau-1}), \quad \pi\hat{V}^2 = O(|x|^{-\tau-2})$$

so

$$g = 1 - 2\hat{V} + \pi - 2\hat{V}\pi + 3\hat{V}^2 + O(|x|^{-\tau-2})$$

which means

$$\begin{aligned} g_{ab} &= \left(1 - 2\hat{V} + 3\hat{V}^2\right)\delta_{ab} + \pi_{ab} - 2\hat{V}\pi_{ab} \\ &= \left(1 - 2\hat{V} + 3\hat{V}^2\right)\delta_{ab} + \left(1 - 2\hat{V}\right)\pi_{ab} \end{aligned}$$

Denote

$$M = \left(1 - 2\hat{V} + 3\hat{V}^2\right)$$

Substituting $\hat{V} = U + U^2/2$ we have

$$\begin{aligned}
M &= \left(1 - 2U - U^2 + 3\left(U + \frac{U^2}{2}\right)^2\right) \\
&= 1 - 2U - U^2 + 3U^2\left(\frac{U}{2} + 1\right)^2 \\
&= 1 - 2U - U^2 + 3U^2\left(1 + U + \frac{U^2}{4}\right) \\
&= 1 - 2U - U^2 + 3U^2 \\
&= 1 - 2U + 2U^2
\end{aligned}$$

where we discarded terms $O(|x|^{-3})$. Substituting (6.18)

$$\begin{aligned}
M &= 1 + \frac{2m}{|x|} - 2\frac{c_a x^a}{|x|^3} - 2u + \frac{1}{2}\left(\frac{-m}{|x|} - \frac{c_a x^a}{|x|^3} + u\right)^2 \\
&= 1 + \frac{2m}{|x|} - 2\frac{c_a x^a}{|x|^3} - 2u + 2\left(-\frac{m}{|x|} + \frac{c_a x^a}{|x|^3} + u\right)^2 \\
&= 1 + \frac{2m}{|x|} - 2\frac{c_a x^a}{|x|^3} - 2u + 2\left(\frac{m^2}{|x|^2} - \frac{c_a x^a}{|x|^4}m - \frac{m}{|x|}u - \frac{c_a x^a}{|x|^4}m + \right. \\
&\qquad \left. + \frac{(c_a x^a)^2}{|x|^6} + \frac{c_a x^a}{|x|^3}u - \frac{m}{|x|}u + \frac{c_a x^a}{|x|^3}u + u^2\right)
\end{aligned}$$

where $u = O(|x|^{-\tau-1})$, for $\tau \in (1, 2)$. So from the above after checking the decaying terms we have

$$M = 1 + \frac{2m}{|x|} - \frac{2c_a x^a}{|x|^3} - 2u + 2\frac{m^2}{|x|^2} + O(|x|^{-\tau-2})$$

Substituting M back to g_{ab} we have

$$\begin{aligned}
g_{ab} &= \left(1 + \frac{2m}{|x|} - \frac{2c_a x^a}{|x|^3} + \frac{2m^2}{|x|^2} - 2u\right)\delta_{ab} + (1 - 2\hat{V})\pi_{ab} \\
&= \left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \pi_{ab} + \left(-\frac{2c_a x^a}{|x|^3} + \frac{2m^2}{|x|^2}\right)\delta_{ab} + u\delta_{ab} - 2\hat{V}\pi_{ab}
\end{aligned}$$

where

$$\left(-2\frac{c_a x^a}{|x|^3} + 2\frac{m^2}{|x|^2}\right)\delta_{ab} = O(|x|^{-2}), \quad u\delta_{ab} - 2\hat{V}\pi_{ab} = O(|x|^{-\tau-1})$$

and so

$$g_{ab} = \left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}$$

where $\mathcal{H}_{ab} = O(|x|^{-\tau})$. Working similarly as before from

$$V = 1 + U + \frac{U^2}{2}$$

and (6.18)

$$\begin{aligned} V &= 1 - \frac{m}{|x|} + \frac{c_a x^a}{|x|^3} + u + \frac{1}{2}\left(-\frac{m}{|x|} + \frac{c_a x^a}{|x|^3} + u\right)^2 \\ &= 1 - \frac{m}{|x|} + \frac{m^2}{2|x|^2} + \frac{c_a x^a}{|x|^3} + u \end{aligned}$$

where

$$u = O(|x|^{-\tau-1})$$

□

We observe that the decay conditions of Lemma 6.1.11 does not match the decay conditions for the positive mass theorem 6.0.8. From [1, p. 680] we have the following Theorem which tells us that the mass of a manifold is independent of the structure of infinity.

Theorem 6.1.12. *[1, p. 682] Let $(\Phi, x), (\Psi, z)$ be two structures of infinity for (M, g) satisfying the* ***mass decay conditions*** *with decay rates τ_1, τ_2, respectively, so*

$$\tau = \min\{\tau_1, \tau_2\} \geq \frac{1}{2}(n-2)$$

Then the mass of M for both structures of infinity are well defined and equal.

Definition 6.1.13. *The* ***mass decay conditions*** *for a manifold M and asymptotic structure Φ are*

1.

$$(\Phi_* g - \delta) \in W^{2,q}_{-\tau}(E_{R_0})$$

for some $R_0 > 1,\ q > n$ and $\tau \geq 1/2(n-2)$,

2.

$$R(g) \in L^1(M)$$

In the last section we will show that the mass decay conditions hold.

Remark 6.1.14. *In section 6 of [1, p. 689] says that the positive mass theorem, written there, in n dimensions holds for complete, asymptotically flat n-dimensional spin manifolds satisfying the mass decay conditions and having non-negative scalar curvature. Since all* 3*-dimensional oriented manifolds are spin manifolds, the positive mass theorem holds also for the decay conditions of 6.1.11. We observe that the conditions here need the metric to be of lower regularity than in the positive mass theorem stated in Theorem 6.0.8 where the metric is smooth.*

We can prove that the m constant in the metric g_{ab} for $|x| \to +\infty$ is equal to the ADM mass.

Lemma 6.1.15. *If we have the metric of a 3-manifold*

$$g_{ab} = \left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}$$

being the same as in Lemma 6.1.11, then

$$M_{ADM} = m$$

Proof. We have the derivative of g_{ab} being

$$g_{ab;c} = -\frac{2mx^c}{|x|^3}\delta_{ab} + h_{ab;c}$$

Assume $n^a = x^a/|x|$

$$\begin{aligned}
M_{ADM} &= \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}(g_{ab;b} - g_{bb;a})n^a \,\mathrm{d}\Sigma \\
&= \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}\left(\frac{2mx^b}{|x|^3}\delta_{ab} + h_{ab;b} + \frac{2mx^a}{|x|^3}\delta_{bb} - h_{bb;a}\right)n^a \,\mathrm{d}\Sigma \\
&= \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}\frac{2m}{R^3}\left(-x^b n^b + 3x^a n^a\right)\mathrm{d}\Sigma + \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}\left(h_{ab;b} - h_{bb;a}\right)n^a \,\mathrm{d}\Sigma
\end{aligned}$$

Let $M_{ADM} = I_1 + I_2$ where

$$I_1 = \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}\frac{2m}{R^3}\left(-x^b n^b + 3x^a n^a\right)\mathrm{d}\Sigma$$

We have

$$n^a x^a = \frac{(x^a x^a)}{|x|} = \frac{|x|^2}{|x|} = |x| = R$$

and

$$c^b\delta_{ab}n^a = n_a n^a = R$$

These gives us

$$\begin{aligned}
I_1 &= \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}\frac{2m}{R^3}\left(-R + 3R\right)\mathrm{d}\Sigma \\
&= \frac{1}{16\pi}\lim_{R\to+\infty}(2R)\frac{2m}{R^3}\int_{S_R}\mathrm{d}\Sigma \\
&= \frac{1}{16\pi}\lim_{R\to+\infty}2R\frac{2m}{R^3}4\pi R^2 \\
&= m
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &= \frac{1}{16\pi}\lim_{R\to+\infty}\left|\int_{S_R}\left(h_{ab;b} - h_{bb;a}\right)n^a \,\mathrm{d}\Sigma\right| \\
&\le \frac{1}{16\pi}\lim_{R\to+\infty}\int_{S_R}|h_{ab;b}| - |h_{bb;a}||n^a|\,\mathrm{d}\Sigma \\
&\le \frac{1}{16\pi}\lim_{R\to+\infty}\frac{2C}{|x|^{\tau+1}}\,\mathrm{d}\Sigma \\
&= \frac{C}{8\pi}\lim_{R\to+\infty}\int_{S_1}\frac{R^2}{R^{\tau+1}}\,\mathrm{d}\Omega \\
&= \frac{C}{8\pi}\lim_{R\to+\infty}\frac{1}{R^{\tau-1}}4\pi \\
&= 0
\end{aligned}$$

for $\tau > 1$.

$\square$

6.2 Conditions Needed for the Positive Mass Theorem

In this section we prove some Propositions that are needed to use the positive mass theorem. Starting with the zero scalar curvature.

Proposition 6.2.1. *Let* ${}^{\pm}\gamma = b^2 (1 \pm V)^4 g$*, where* b^2 *is a positive constant and* V, g *are* C^2*. Then* ${}^{\pm}\gamma$ *has zero scalar curvature.*

Proof. Suppose the metric transformation $\hat{g} = \Omega^2 g$, we notice that for $f = \log \Omega$ then the transformation is the same as $\hat{g} = e^{2f} g$. From [31, p. 217] we have

$$\hat{R} = \Omega^{-2} \left(R - 4\Delta f - 2|\,\mathrm{d}f|_g^2 \right)$$

where $\hat{R}$ is the scalar curvature of $\hat{g}$ and R is the scalar curvature of g. From Einstein equations in vacuum we know that $R = 0$, so we need to calculate Δf and $|\,\mathrm{d}f|_g^2$.

$$\begin{aligned}
\Delta f &= \frac{1}{\sqrt{|g|}} \partial_a \left(g^{ab} \sqrt{|g|} \partial_b f \right) \\
&= \frac{1}{\sqrt{|g|}} \partial_a \left(g^{ab} \sqrt{|g|} \Omega^{-1} \partial_b \Omega \right) \\
&= \frac{1}{\sqrt{|g|}} g^{ab} \sqrt{|g|} \partial_b \Omega \partial_a (\Omega^{-1}) + \frac{1}{\sqrt{|g|}} \Omega^{-1} \partial_a \left(g^{ab} \sqrt{|g|} \partial_b \Omega \right) \\
&= -g^{ab} \Omega^{-2} \partial_a \Omega \partial_b \Omega + \Omega^{-1} \Delta \Omega
\end{aligned}$$

and

$$\begin{aligned}
|\,\mathrm{d}f|^2 &= \langle \mathrm{d}f, \mathrm{d}f \rangle \\
&= \langle \nabla f, \nabla f \rangle \\
&= g \left(g^{ab} \partial_a f \partial_b, g^{cd} \partial_c f \partial_d \right) \\
&= g^{ab} g^{cd} \partial_a f \partial_c f g(\partial_b, \partial_d) \\
&= g^{ab} \delta_b^c \partial_a f \partial_c f \\
&= g^{ab} \Omega^{-2} \partial_a \Omega \partial_b \Omega
\end{aligned}$$

From Δf, $|\,\mathrm{d}f|^2$ and the equation for $\hat{R}$.

$$\begin{aligned}
\hat{R} &= \Omega^{-2} R - 4\Omega^{-2} \left[-g^{ab} \Omega^{-2} \partial_a \Omega \partial_b \Omega + \Omega^{-1} \Delta \Omega \right] - 2\Omega^{-2} g^{ab} \Omega^{-2} \partial_a \Omega \partial_b \Omega \\
&= \Omega^{-2} R - 4\Omega^{-3} \Delta \Omega + 2 g^{ab} \Omega^{-4} \partial_a \Omega \partial_b \Omega \\
&= \Omega^{-2} R - 4\Omega^{-3} \Delta \Omega + 2\Omega^{-4} |\nabla \Omega|^2
\end{aligned} \tag{6.23}$$

Now we need to calculate $\Delta \Omega$ and $|\nabla \Omega|^2$ for $\Omega = \Omega(V)$.

$$\begin{aligned}
|\nabla \Omega|^2 &= g^{ab} \partial_a \Omega \partial_b \Omega \\
&= g^{ab} \frac{\mathrm{d}\Omega}{\mathrm{d}V} \partial_a V \frac{\mathrm{d}\Omega}{\mathrm{d}V} \partial_b \\
&= \left(\frac{\mathrm{d}\Omega}{\mathrm{d}V} \right)^2 |\nabla V|^2
\end{aligned}$$

and

$$
\begin{aligned}
\Delta\Omega &= g^{ab}\Omega_{;ab} \\
&= g^{ab}\left(\partial_a\partial_b\Omega - \Gamma^c_{ab}\partial_c\Omega\right) \\
&= g^{ab}\partial_a\left(\frac{\mathrm{d}\Omega}{\mathrm{d}V}\partial_b V\right) - \Gamma^c_{ab}g^{ab}\partial_c V\frac{\mathrm{d}\Omega}{\mathrm{d}V} \\
&= g^{ab}\frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2}\partial_b V\partial_a V + g^{ab}\frac{\mathrm{d}\Omega}{\mathrm{d}V}\partial_a\partial_b V - \Gamma^c_{ab}g^{ab}\partial_c V\frac{\mathrm{d}\Omega}{\mathrm{d}V} \\
&= \frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2}|\nabla V|^2 + \frac{\mathrm{d}\Omega}{\mathrm{d}V}g^{ab}\left[\partial_a\partial_b V - \Gamma^c_{ab}\partial_c V\right] \\
&= \frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2}|\nabla V|^2 + \frac{\mathrm{d}\Omega}{\mathrm{d}V}\Delta V \\
&= \frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2}|\nabla V|^2
\end{aligned}
$$

Now we substitute $|\nabla\Omega|^2$, $\Delta\Omega$ and $R = 0$ to equation (6.23)

$$
\begin{aligned}
\hat{R} &= \Omega^{-2}R - 4\Omega^{-3}\Delta\Omega + 2\Omega^{-4}|\nabla|^2 \\
&= -4\Omega^{-3}\frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2}|\nabla V|^2 + 2\Omega^{-4}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}V}\right)^2|\nabla V|^2
\end{aligned}
$$

which gives us

$$
\hat{R} = -2\Omega^{-4}|\nabla V|^2\left[2\Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2} - \left(\frac{\mathrm{d}\Omega}{\mathrm{d}V}\right)^2\right]
$$

Now suppose

$$
\Omega = b(1 \pm V)^2
$$

we notice that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}V}\frac{\mathrm{d}}{\mathrm{d}V}\left(b(1\pm V)^2\right) &= b\frac{\mathrm{d}}{\mathrm{d}V} \pm 2(1 \pm V) \\
&= 2b
\end{aligned}
$$

and

$$
\left(\frac{\mathrm{d}\Omega}{\mathrm{d}V}\right)^2 = (2b(1\pm V))^2
$$

So we have

$$
\begin{aligned}
\hat{R} &= -2\Omega^{-4}|\nabla V|^2\left[2\Omega\frac{\mathrm{d}^2\Omega}{\mathrm{d}V^2} - \left(\frac{\mathrm{d}\Omega}{\mathrm{d}V}\right)^2\right] \\
&= -2\Omega^{-4}|\nabla V|^2\left[(2b)^2(1\pm V)^2 - (2b(1\pm V))^2\right] \\
&= 0
\end{aligned}
$$

□

Assume that $b^2 = 1/16$.

Proposition 6.2.2. *${}^{+}\gamma = (1/16)(1+V)^4 g$ is asymptotically euclidean with mass zero.*

Proof.

$$(1+V)^4 = \left(2 - \frac{m}{|x|} + \frac{m^2}{2|x|^2} + \frac{c_a x^a}{|x|^3} + u\right)^4$$

We remind that $u = O(|x|^{-\tau-1})$ for $\tau \in (1,2)$. Assume $k = O(|x|^{-2})$ and $\zeta = 2+k$. Then

$$(1+V)^4 = \left(2 - \frac{m}{|x|} + k\right)^4 = \left(\zeta - \frac{m}{|x|}\right)^4$$

- $\zeta = 2 + k$
- $\zeta^2 = 4 + O(|x|^{-2})$
- $\zeta^3 = 8 + O(|x|^{-2})$
- $\zeta^4 = 16 + O(|x|^{-2})$

$$\begin{aligned}\left(\zeta - \frac{m}{|x|}\right)^4 &= \zeta^4 + 4\zeta^3\left(-\frac{m}{|x|}\right) + 6\zeta^2\frac{m^2}{|x|^2} - 4\zeta\frac{m^3}{|x|^3} + \frac{m^4}{|x|^4} \\ &= 16 - 32\frac{m}{|x|} + O(|x|^{-2})\end{aligned}$$

$$\begin{aligned}{}^{+}\gamma &= \frac{1}{16}(1+V)^4 g \\ &= \left(1 - \frac{2m}{|x|} + O(|x|^{-2})\right)\left(\left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}\right) \\ &= \left(1 - \frac{2m}{|x|}\right)\left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \left(1 - \frac{2m}{|x|}\right)\mathcal{H}_{ab} \\ &= \left(1 - \frac{4m^2}{|x|^2}\right)\delta_{ab} + \mathcal{H}_{ab} \\ &= \delta_{ab} + \mathcal{H}_{ab}\end{aligned}$$

where $\mathcal{H}_{ab} = O(|x|^{-\tau})$. So

$$|{}^{+}\gamma_{ab;b}| = O(|x|^{-\tau-1})$$

So from the definition of ADM mass

$$|\text{mass}| = |\frac{1}{16\pi}\int_{S_\infty}({}^{+}\gamma_{ab;b} - {}^{+}\gamma_{bb;a})n^a\,\mathrm{d}\Sigma|$$

we take the estimates

$$\begin{aligned}|\text{mass}| &\leq \frac{1}{16\pi} \lim_{R\to+\infty} \int_{S_R} |{}^{+}\gamma_{ab;b}| + |{}^{+}\gamma_{bb;a}| \,\mathrm{d}\Sigma \\ &\leq \frac{2C}{16\pi} \lim_{R\to\infty} \int_{S_R} |x|^{-\tau-1} \,\mathrm{d}\Sigma \\ &= \frac{2C}{16\pi} \lim_{R\to+\infty} R^{-\tau-1} R^2 \int_{S_1} \mathrm{d}\Omega \\ &= \frac{C}{2} \lim_{R\to+\infty} R^{-\tau+1}\end{aligned}$$

which is zero for $\tau > 1$ as needed. □

Proposition 6.2.3. *The second fundamental form of $(\partial\Sigma)_i$ in the ${}^{\pm}\gamma$ metrics are given by*

$$^{\pm}A_{jk} = \pm(-8)w_i^{1/2}\,{}^{\pm 2}\gamma_{jk}, \quad j,k = 1,2$$

${}^{\pm 2}\gamma$ metric on $(\partial\Sigma)_i$ induced from ${}^{\pm}\gamma$, and $w_i = |\nabla V|^2$ evaluated at $(\partial\Sigma)_i$ are positive constants and the second fundamental forms are with respect to the inward (or outward) pointing vectors of the manifold with metric ${}^{\pm}\gamma$.

Proof. We have $\hat{g} = \Omega^2 g,\ \Omega = \Omega(V)$ and

$$N = -\frac{\hat{\nabla} V}{\hat{w}_i^{1/2}}, \quad n = -\frac{\nabla V}{w_i^{1/2}}$$

are the unit normals of $(\partial\Sigma)_i$ in the $\hat{g}$ metric and the g metric respectively.

The second fundamental form on the $\hat{g}$ metric on $(\partial\Sigma)_i$ is by definition

$$[A(\hat{g})]_{jk} = \hat{g}(\hat{\nabla}_{E_j} N, E_k)$$

We notice that

$$\begin{aligned}N &= -\frac{\hat{\nabla} V}{\hat{w}_i^{1/2}} \\ &= -\frac{\hat{g}^{ij}\partial_i V \partial_j}{\sqrt{\hat{g}^{lk}\partial_l \partial_j V}} \\ &= -\frac{\Omega^{-2}}{\Omega^{-1}} \frac{\nabla V}{w_i^{1/2}} \\ &= \Omega^{-1} n\end{aligned}$$

and from that we have

$$\begin{aligned}\hat{g}(\hat{\nabla}_{E_j} N, E_k) &= \Omega^{-1}\hat{g}(\hat{\nabla}_{E_j} n, E_k) + \hat{g}(n\hat{\nabla}_{E_j}\Omega^{-1}, E_k) \\ &= \Omega^{-1}\hat{g}(\hat{\nabla}_{E_j} n, E_k) \\ &= \Omega g(\hat{\nabla}_{E_j} n, E_k)\end{aligned}$$

From [31, p. 217] we have for $\hat{g} = e^{2U} g$

$$\begin{aligned}\hat{\nabla}_X Y &= \nabla_X Y + X(U)Y + Y(U)X - g(X,Y)\nabla U \\ &= \nabla_X Y + X(\log\Omega)Y + Y(\log\Omega)X - g(X,Y)\nabla(\log\Omega)\end{aligned}$$

so we have

$$\begin{aligned}\hat{\nabla}_{E_j} n &= \nabla_{E_j} n + E_j(\log\Omega)n + n(\log\Omega)E_j - g(E_j,N)\nabla(\log\Omega) \\ &= \nabla_{E_j} n + E_j(\log\Omega)n + n(\log\Omega)E_j\end{aligned}$$

We calculate the terms seperately

$$\begin{aligned}\partial_j(\log\Omega) &= \frac{1}{\Omega}\partial_j\Omega \\ &= \frac{1}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}V}\partial_j V\end{aligned}$$

and

$$\begin{aligned}n(\log\Omega) &= -\frac{\nabla V}{w_i^{1/2}}(\log\Omega) \\ &= -\frac{g^{kl}\partial_k V\partial_l(\log\Omega)}{w_i^{1/2}} \\ &= -\frac{g^{kl}\partial_k V\partial_l V}{w_i^{1/2}}\frac{\mathrm{d}\Omega}{\mathrm{d}V}\Omega^{-1} \\ &= -\frac{\nabla V(V)}{w_i^{1/2}}\frac{\mathrm{d}\Omega}{\mathrm{d}V}\Omega^{-1} \\ &= n(V)\frac{\mathrm{d}\Omega}{\mathrm{d}V}\Omega^{-1} \\ &= \frac{\mathrm{d}\Omega}{\mathrm{d}V}\Omega^{-1}g(n,\nabla V)\end{aligned}$$

no we can calculate the second fundamental form

$$\begin{aligned}\Omega g(\hat{\nabla}_{E_j} n, E_k) &= \Omega\left[g(\nabla_{E_j} n, E_k) + \Omega^{-1}\frac{\mathrm{d}\Omega}{\mathrm{d}V}\partial_j V g(n,E_k) + \Omega^{-1}\frac{\mathrm{d}\Omega}{\mathrm{d}V}g(n,\nabla V)g(E_j,E_k)\right] \\ &= \Omega\left[A(g)\right]_{jk} + \Omega^{-2}\frac{\mathrm{d}\Omega}{\mathrm{d}V}g(n,\nabla V)\hat{g}_{jk}\end{aligned}$$

meaning

$$[A(\hat{g})]_{jk} = \Omega\left[A(g)\right]_{jk} + \Omega^{-3}\frac{\mathrm{d}\Omega}{\mathrm{d}V}g(n,\nabla V)\hat{g}_{jk}$$

We know the second fundamental form in the boundary $(\partial\Sigma)_i$ of the metric g is zero and the lapse function V is zero, so $\Omega = \frac{1}{4}$.

Also

$$\begin{aligned}\frac{\mathrm{d}\Omega}{\mathrm{d}V} &= \frac{\mathrm{d}}{\mathrm{d}V}\left(\frac{(1\pm V)^2}{4}\right)\\ &= \pm\frac{1}{2}(1\pm V)\\ &= \pm\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}g(n,\nabla V) &= n(V)\\ &= -\frac{\nabla V(V)}{|\nabla V|}\\ &= -\frac{g^{ij}\partial_i V\partial_j V}{|\nabla V|}\\ &= -\frac{|\nabla V|^2}{|\nabla V|}\\ &= -|\nabla V|\\ &= -w_i^{1/2}\end{aligned}$$

We conclude

$$\begin{aligned}[A(\hat{g})]_{jk} &= \pm\frac{4^2}{2}(-w_i^{1/2})^2\hat{g}_{jk}\\ &= \pm 8(-w_i^{1/2})^{\pm 2}\gamma_{jk}\end{aligned}$$

□

Proposition 6.2.4. *$^{-}\gamma$ compactifies the infinity: If P is the point at infinity, then there is a $W^{2,q}$ extension of $^{-}\gamma$ to $\Sigma\cup\{P\}$.*

Proof. From the equations in lemma 6.1.11 we have

$$\begin{aligned}{}^{-}\gamma_{ab} &= \left(\frac{1}{16}\right)(1-V)^4\, g_{ab}\\ &= \left(\frac{1}{16}\right)\left(\frac{m}{|x|}-\frac{m^2}{2|x|^2}-\frac{c_a x^a}{|x|^3}+v\right)^4\left[\left(1+\frac{2m}{|x|}\right)\delta_{ab}+\mathcal{H}_{ab}\right]\\ &= \left(\frac{m^4}{16|x|^4}\right)\left(1-\frac{m}{2|x|}-\frac{c_a x^a}{|x|^2 m}+\frac{v|x|}{m}\right)^4\left[\left(1+\frac{2m}{|x|}\right)\delta_{ab}+\mathcal{H}_{ab}\right]\end{aligned}$$

We remind that for $\tau\in(1,2)$

$$v = O(|x|^{-\tau-1}),\quad \mathcal{H}_{ab} = O(|x|^{-\tau})$$

which implies that

$$\frac{v|x|}{m} = O(|x|^{-\tau})$$

Assume that

$$\lambda = 1+k,\quad \pi = \frac{m}{2|x|}+\frac{c_a x^a}{|x|^2 m}$$

where $\lambda = O(1),\ \pi = O(|x|^{-1}), k = v|x|/m$.

$$(\lambda - \pi)^4 = \lambda^4 - 4\lambda^3\pi + 16\lambda^2\pi^2 - 4\lambda\pi^3 + \pi^4$$

where

$$\lambda^4 = O(1), \quad \lambda^3\pi = O(|x|^{-1}), \quad \lambda^2\pi^2 = O(|x|^{-2}), \quad \lambda\pi^3 = O(|x|^{-3}), \quad \pi^4 = O(|x|^{-4})$$

we ignore the terms $\lambda\pi^3,\ \pi^4$ and we write

$$(\lambda - \pi)^4 = \lambda^4 - 4\lambda^3\pi + 16\lambda^2\pi^2$$

we have the equations

$$\begin{aligned}
\lambda^2 &= 1 + 2k + k^2 \\
&= 1 + 2k + O(|x|^{-\tau}) \\
\lambda^3 &= 1 + 3k + 3k^2 + k^3 \\
&= 1 + 3k + O(|x|^{-2\tau}) \\
\lambda^4 &= 1 + 4k + 16k^2 + 4k^3 + k^4 \\
&= 1 + 4k + O(|x|^{-\tau})
\end{aligned}$$

So

$$\begin{aligned}
(\lambda - \pi)^4 &= 1 + 4k - 4(1 + 3k)\pi + 16(1 + 2k)\pi^2 \\
&= 1 + 4k - 4\pi - 12k\pi + 16\pi^2 + 16 \cdot 2k\pi^2 \\
&= 1 + 4k - 4\pi + 16\pi^2
\end{aligned}$$

Therefore

$$\begin{aligned}
(\lambda - \pi)^4 \left[\left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}\right] &= \left(1 + 4k - 4\pi + 16\pi^2\right)\left[\left(1 + \frac{2m}{|x|}\right)\delta_{ab} + \mathcal{H}_{ab}\right] \\
&= \left(1 + \frac{2m}{|x|} + 4k\left(1 + \frac{2m}{|x|}\right) - 4\pi\left(1 + \frac{2m}{|x|}\right) + 16\pi^2\left(1 + \frac{2m}{|x|}\right)\right)\delta_{ab} + \\
&\quad + \left(1 + 4k - 4\pi + 16\pi^2\right)\mathcal{H}_{ab} \\
&= \left(1 + \frac{2m}{|x|} + 4k + \frac{8km}{|x|} - 4\pi - \frac{8\pi m}{|x|} + 16\pi^2 + \frac{16\pi^2 \cdot 2m}{|x|}\right)\delta_{ab} + \\
&\quad + \left(1 + 4k - 4\pi 16\pi^2\right)\mathcal{H}_{ab} \\
&= \left(1 + \frac{2m}{|x|} + 4\frac{|x|v}{m} + 8\frac{|x|vm}{m|x|} - 4\frac{m}{2|x|} - 4\frac{c_a x^a}{|x|^2 m} - 8\frac{m^2}{2|x|^2}\right. \\
&\quad \left. -8\frac{c_a x^a m}{|x|^3 m} + 16\frac{m^2}{4|x|^2} + 16\frac{2mc_a x^a}{2|x|^3 m} + 16\frac{(c_a x^a)^2}{|x|^4 m^2}\right)\delta_{ab} + \left(1 + 4k - 4\pi + 16\pi^2\right)\mathcal{H}_{ab} \\
&= \left(1 + \frac{4|x|v}{m} + 8v - \frac{4c_a x^a}{|x|^2 m} + 8\frac{c_a x^a}{|x|^3}\right)\delta_{ab} + \left(1 + 4k - 4\pi + 16\pi^2\right)\mathcal{H}_{ab}
\end{aligned}$$

and so

$$^-\gamma_{ab} = \left(\frac{m^4}{16|x|^4}\right)\left[\delta_{ab} - 4\frac{c_a x^a}{m|x|^2}\delta_{ab} + \left(4\frac{|x|v}{m}\delta_{ab} + \mathcal{H}_{ab}\right) + 8\frac{c_a x^a}{|x^3|}\delta_{ab} + (8v\delta_{ab} - 4\pi\mathcal{H}_{ab})\right]$$

where we denote

$$\Psi_{ab} = \left(4\frac{|x|v}{m}\delta_{ab} + \mathcal{H}_{ab}\right), \quad \Phi_{ab} = (8v\delta_{ab} - 4\pi\mathcal{H}_{ab})$$

The above equation becomes

$$^{-}\gamma_{ab} = \left(\frac{m^4}{16|x|^4}\right)\left[\delta_{ab} - 4\frac{c_a x^a}{m|x|^2}\delta_{ab} + \Psi_{ab} + 8\frac{c_a x^a}{|x^3|}\delta_{ab} + \Phi_{ab}\right] \tag{6.24}$$

s.t. for $\tau \in (1,2)$

$$\Psi_{ab} = O(|x|^{-\tau}), \quad \Phi_{ab} = O(|x|^{-\tau-1}), \quad D\Phi_{ab} = O(|x|^{\tau-2})$$

and

$$D^2\Phi_{ab} \in L^q_{-\tau-3}(\mathbb{R}^3 \setminus B_3(0)) \tag{6.25}$$

Now we take the coordinate transformation

$$z^a = \frac{x^a}{|x|^2}$$

from which we have

$$|z| = \frac{1}{|x|} \quad \Longrightarrow \quad x^a = z^a|z|^{-2}$$

and

$$\begin{aligned}
\frac{\partial x^a}{\partial z^b} &= \delta|z|^{-2} - z^a\frac{2}{|z|^3}\frac{\partial|z|}{\partial z^b} \\
&= \delta_{ab}|z|^{-2} - \frac{z^a}{|z|^3}\frac{2}{2|z|}2z^b \\
&= \delta_{ab}|z|^{-2} - 2\frac{z^a z^b}{|z|^4} \\
&= \left(\delta_{ab} - 2\frac{z^a z^b}{|z|^2}\right)|z|^{-2}
\end{aligned}$$

This transformation is called inversion of the sphere because it reflects the points of the open ball inside the sphere to points outside of the sphere while keeping the points in the sphere the same. We use this because we want to check the neighborhood around zero, instead of checking the neighborhood around infinity.

For the coordinate change we will need

$$\begin{aligned}
\frac{\partial x^a}{\partial z^c}\frac{\partial x^b}{\partial z^d} &= \left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)|z|^{-2}\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)|z|^{-2} \\
&= \left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)|z|^{-4}
\end{aligned}$$

We denote for simplicity

$$\begin{aligned}
\Psi'_{ab} &= \left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)\Psi_{cd} \\
\Phi'_{ab} &= \left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)\Phi_{cd} \\
M &= 1 - 4\frac{c_a x^a}{m|x|^2} + 8\frac{c_a x^a}{|x^3|}
\end{aligned}$$

where $\Phi_{cd} = O(|x|^{-\tau-2})$ as $|x| \to +\infty$ and $\Phi_{cd} = O(|z|^{\tau+1})$ as $|z| \to 0$. This implies

$$\Phi'_{ab} = O(|z|^{\tau+1})$$

So the equation (6.24) becomes

$$^{-}\gamma_{ab} = \left(\frac{m^4}{16|x|^4}\right)[M\delta_{ab} + \Psi_{ab} + \Phi_{ab}]$$

which after the transformation we have

$$\begin{aligned} ^{-}\gamma &= \left(\frac{m^4|z|^4}{16}\right)[M\delta_{cd} + \Psi_{cd} + \Phi_{cd}]\left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)|z|^{-4} \\ &= \left(\frac{m^4}{16}\right)\left[M\left(\delta_{ac} - 2\frac{z^a z^c}{|z|^2}\right)\left(\delta_{bd} - 2\frac{z^b z^d}{|z|^2}\right)\delta_{cd} + \Psi'_{ab} + \Phi'_{ab}\right]\mathrm{d}z^a\,\mathrm{d}z^b \end{aligned}$$

but we notice

$$\begin{aligned} \left(\delta_{ac} - \frac{2z^a z^c}{|z|^2}\right)\left(\delta_{bd} - \frac{2z^b z^d}{|z|^2}\right)\delta_{cd} &= \left(\delta_{ad}\delta_{bd} + \frac{4z^a z^b z^d z^d}{|z|^4} - \frac{2z^a z^d}{|z|^2}\delta_{bd} - \frac{2z^b z^d}{|z|^2}\delta_{ad}\right) \\ &= \left(\delta_{ab} + \frac{4z^a z^b|z|^2}{|z|^4} - \frac{2z^a z^b}{|z|^2} - \frac{2z^b z^a}{|z|^2}\right) \\ &= \delta_{ab} \end{aligned}$$

and so the above equation becomes

$$^{-}\gamma = \left(\frac{m^4}{16}\right)\left[\delta_{ab} - 4\frac{c_a x^a}{m|x|^2}\delta_{ab} + 8\frac{c_a x^a}{|x^3|}\delta_{ab} + \Psi'_{ab} + \Phi'_{ab}\right]\mathrm{d}z^a\,\mathrm{d}z^b \tag{6.26}$$

We know that $D_x^2\Phi \in L^q_{-\tau-3}(\mathbb{R}^3 \setminus B_3(0))$, but we dont know about $D_z^2\Phi$. This is what we are going to find out.

We can see that

$$\begin{aligned} D_z\Phi' &= D_z x D_x\Phi \\ &= O(|z|^{-2})O(|z|^{\tau+2}) \\ &= O(|z|^{\tau}) \end{aligned}$$

and

$$\begin{aligned} D_z^2\Phi' &= D_z\left(D_z x D_x\Phi\right) \\ &= \left(D_z^2 x\right)D_x\Phi + D_z x D_z D_x\Phi \\ &= \left(D_z^2 x\right)D_x\Phi + D_z x D_z x D_{xx}\Phi \\ &= \left(D_z^2 x\right)D_x\Phi + \left(D_z x\right)^2 D_{xx}\Phi \end{aligned}$$

$$\left.\begin{aligned} D_z^2 x &= O(|z|^{-3}) \\ D_x\Phi &= O(|z|^{\tau+2}) \end{aligned}\right\} \implies D_z^2 x D_x\Phi = O(|z|^{\tau-1})$$

and $D_z x = O(|z|^{-2})$. From these we have

$$(D_z x)^2 \, D_{xx}\Phi = O(|z|^{-4}) D_{xx}\Phi$$

where $D_{xx}\Phi \in L^q_{-\tau-3}$.

$$\begin{aligned}
\int_{|z|<\varepsilon} |D_z^2\Phi'|^q &\leq \int_{|z|<\varepsilon} \left(|D_z^2 x||D_x\Phi| + |D_z x|^2 |D_{xx}\Phi|\right)^q \mathrm{d}z \\
&\leq C \int_{|z|<\varepsilon} \left(|z|^{\tau-1} + |z|^{-4}|D_{xx}\Phi|\right) \mathrm{d}z \\
&= C_1 \int_{|z|<\varepsilon} \left(|z|^{\tau-1} + |z|^{-4}|D_x^2\Phi|\right)^q |z|^2 \, \mathrm{d}|z| \\
&= C_2 \int_{|x|>1/\varepsilon} \left(|x|^{-\tau+1} + |x|^4 |D_x^2\Phi|\right)^q \frac{1}{|x|^4} \, \mathrm{d}|x| \\
&\leq C_3 \int_{1/\varepsilon}^{+\infty} |x|^{-q(\tau-1)-4} \, \mathrm{d}|x| + \int_{1/\varepsilon}^{+\infty} |x|^{4q-4} |D_x^2\Phi|^q \, \mathrm{d}|x|
\end{aligned}$$

where the last we changes to spherical coordinates and then we used

$$|z| = \frac{1}{|x|} \quad \Longrightarrow \quad |z|^2 \, \mathrm{d}|z| = \frac{1}{|x|^2}\left(-\frac{\mathrm{d}|x|}{|x|^2}\right) = -\frac{\mathrm{d}|x|}{|x|^4}$$

We have that

$$|D_x^2\Phi| \in L^q_{-\tau-3}(\mathbb{R}^3 \setminus B_3(0)) \quad \Longrightarrow \quad \int_{\mathbb{R}^3 \setminus B_2(0)} |D_x^2\Phi|^q |x|^{(\tau+3)q-3} \, \mathrm{d}x < +\infty$$

We take our attention on the two terms of the last inequality.

- First term:

  $$\int_{1/\varepsilon}^{+\infty} r^{-q(\tau-1)-4} \, \mathrm{d}r = C r^{-q(\tau-1)-3}|_{1/\varepsilon}^{+\infty}$$

  which is finite for $q > 4$

  $$-q(\tau-1) - 3 < 0 \quad \Longrightarrow \quad -q(\tau-1) < 3 \quad \Longrightarrow \quad \tau > 1$$

- Second term:

  $$\begin{aligned}
  \int_{1/\varepsilon}^{+\infty} |x|^{4q-4} |D_x^2\Phi|^q \, \mathrm{d}|x| &= \int_{1/\varepsilon}^{+\infty} |x|^{4q-6} |D_x^2\Phi|^q \, \mathrm{d}x \\
  &= \int_{|x|>1/\varepsilon} |x|^{4q-6-\tau q-3q+3} |x|^{(\tau+3)q-3} |D_x^2\Phi|^q \, \mathrm{d}x \\
  &= \int_{|x|>1/\varepsilon} |x|^{q(1-\tau)-3} |x|^{(\tau+3)q-3} |D_x^2\Phi|^q \, \mathrm{d}x
  \end{aligned}$$

  We denote

  $$-k = q(1-\tau) - 3 < 0 \quad \Longrightarrow \quad |x| > 1/\varepsilon \quad \Longrightarrow \quad \frac{1}{|x|^k} \leq \frac{1}{(1/\varepsilon)^k} = \varepsilon^k$$

and so

$$\begin{aligned}\int_{1/\varepsilon}^{+\infty} |x|^{4q-4}|D_x^2\Phi|^q \,\mathrm{d}|x| &= \int_{|x|>1/\varepsilon} |x|^{q(\tau+3)-3}|D_x^2\Phi|^q \,\mathrm{d}x \\ &\leq \varepsilon^k \int_{|x|>1/\varepsilon} |x|^{q(\tau+3)-3}|D_x^2\Phi|^q \,\mathrm{d}x \\ &< +\infty\end{aligned}$$

Using Lemma 6.0.6 for Φ', $D_z\Phi'$ and $\tau \in (1,2)$ we have

$$\|\Phi'\|_{2,q} = \|\Phi'\|_q + \|D\Phi'\|_q + \|D^2\Phi'\|_q < +\infty$$

we work similarly for the other terms of ${}^{-}\gamma_{ab}$. We define the metric ${}^{-}\gamma^*$ such that

$$\begin{aligned}{}^{-}\gamma^* &= {}^{-}\gamma(z), \quad \text{ on } \{z : 0 < |z| < \varepsilon\} \\ &= \frac{m^4}{16}\delta_{ab}\,\mathrm{d}z^a\,\mathrm{d}z^b, \quad \text{ at } P\end{aligned}$$

and from the previous results we have that

$${}^{-}\gamma^* \in W^{2,q}(B_\varepsilon(P))$$

where $B_\varepsilon(P) = \{z : 0 < |z| < \varepsilon\} \cup P$. □

6.3 Doubling of the 3-manifold and Proof of the Main Theorem

We begin with the 3-dimensional spatial manifold (Σ, g) with the assumptions in the start of the current section and the metric in the asymptotic form of Proposition 6.1.11.

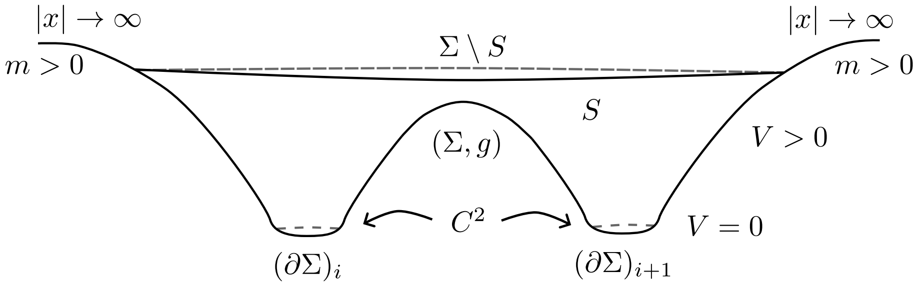

Figure 6.1: The 3-dimensional spacelike slice (Σ, g) for $\{t = 0\}$ such that: if S is compact then its asymptotically Euclidean on the set $\Sigma \setminus S$ with constant positive mass, it has compact boundary $(\partial\Sigma)_i$, the metric g and the lapse function V are smooth on Σ and C^2 on $\overline{\Sigma}$, V is positive in Σ and zero on $(\partial\Sigma)_i$.

We have the following Theorem from [32, p. 224]:

Theorem 6.3.1. *Let M and N be smooth n-manifolds with nonempty boundaries, and suppose $h : \partial N \to \partial M$ is a diffeomorphism. Let $M \cup_h N$ is a topological* *manifold (without boundary), and has a smooth structure such that there are regular domains $M', N' \subseteq M \cup_h N$ diffeomorphic to M and N, respectively, and satisfying*

$$M' \cup N' = M \cup_h N, \qquad M' \cap N' = \partial M' = \partial N'$$

If M and N are both compact, then $M \cup_h N$ is compact, and if they are both connected, then $M \cup_h N$ is connected.

We use Theorem 6.3.1 to glue Σ along its boundaries $(\partial\Sigma)_i$ with itself. We denote for simplicity $\tilde{\Sigma}$ the copy of Σ which we glue together. In $\Sigma \sqcup \tilde{\Sigma}$ we glue the points in the boundaries using the identity maps $\mathrm{id}_i : (\partial\Sigma)_i \to (\partial\tilde{\Sigma})_i$, we denote the adjunction space as $\check{\Sigma} = \Sigma \cup_{\mathrm{id}} \tilde{\Sigma}$, $(\partial\Sigma)_i \cup_{\mathrm{id}} (\partial\tilde{\Sigma})_i = (\partial\Sigma^{\pm})_i$ and the quotient map $\pi : \Sigma \sqcup \tilde{\Sigma} \to \check{\Sigma}$. Suppose the collar neighborhoods of $(\partial\Sigma)_i$ and $(\partial\tilde{\Sigma})_i$, are the sets $U_i^+ \subset \Sigma$ and $U_i^- \subset \tilde{\Sigma}$ respectively. From that we have the diffeomorphisms

$$f_i^+ : (\partial\Sigma)_i \times [0,1) \to U_i^+, \qquad f_i^- : (\partial\tilde{\Sigma})_i \times [0,1) \to U_i^-$$

Let $U_i = U_i^+ \cup_{\mathrm{id}} U_i^-$ and $\Phi_i : U_i^+ \sqcup U_i^- \to (\partial\tilde{\Sigma})_i \times (-1,1)$ be

$$\Phi_i(x) = \begin{cases} (p, x_3), & x = f_i^+(p, x_3) \in U_i^+ \\ (p, -x_3), & x = f_i^-(p, x_3) \in U_i^- \end{cases}$$

From that we can define the homeomorphism $\tilde{\Phi} : \pi(U_i^+ \sqcup U_i^-) \to (\partial\tilde{\Sigma})_i \times (-1,1)$. It can be shown that $\pi(U_i^+ \sqcup U_i^-)$ and $\pi(\mathrm{int}\Sigma \sqcup \mathrm{int}\tilde{\Sigma})$ are topological manifolds. From the above, the smooth charts of $\check{\Sigma}$ outside the glued boundaries are

$$\left(\pi(W), \psi \circ \pi^{-1}|_{\pi(W)}\right)$$

where (W, ψ) are smooth charts of $\mathrm{int}\Sigma$ or $\mathrm{int}\tilde{\Sigma}$. The charts around the glued collar neighborhoods are

$$(\tilde{\Phi}_i^{-1}(U_i), \phi \circ \tilde{\Phi}_i|_{\tilde{\Phi}_i^{-1}(U_i)})$$

where ϕ are smooth charts of $(\partial\tilde{\Sigma})_i \times (-1,1)$.

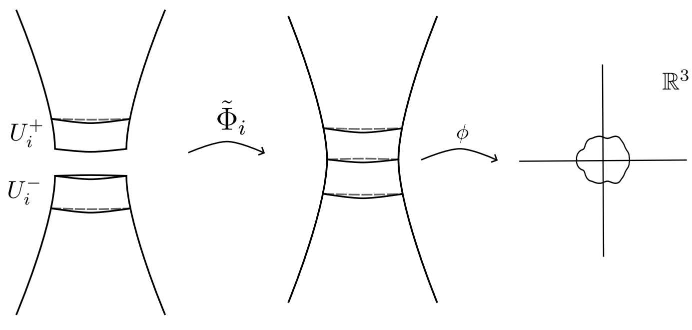

Figure 6.2: The chart of the collar neighborhood around the boundaries $(\partial\Sigma)_i$ and $(\partial\tilde{\Sigma})_i$

This produces a smooth **doubled manifold** $(\check{\Sigma}, \check{g})$. For more details about the above statements we refer to the proof of theorem 9.28 in [32]. The manifold produced by attaching manifolds along their boundaries is smooth, but this isnt true for the functions attached to it e.g. the metric tensor and the lapse function. In fact we will see that they may lose some degree of smoothness.

In $\tilde{\Sigma}$ we define $\tilde{V} = -V < 0$ on $\tilde{\Sigma}$ and $\tilde{V} = 0$ on $(\partial\tilde{\Sigma})_i$. Because $(\partial\Sigma)_i$ is totally geodesic in $\overline{\Sigma}$ and $N(V)$ =constant for each i we will show that the metric is $C^{1,1}$ on $(\partial\Sigma)_i$. For V because $V = \tilde{V} = 0$ on $(\partial\Sigma)_i$ it is continuous on the doubled manifold and also $C^{1,1}$ on $(\partial\Sigma)_i$.

So we have the doubled manifold $(\check{\Sigma}, \check{g})$ such that

$$\check{\Sigma} = \Sigma \cup \tilde{\Sigma}, \qquad \check{g} = \begin{cases} g, & \text{in } \Sigma \\ \tilde{g}, & \text{in } \tilde{\Sigma} \end{cases}, \qquad \check{V} = \begin{cases} V, & \text{in } \Sigma \\ \tilde{V}, & \text{in } \tilde{\Sigma} \end{cases}$$

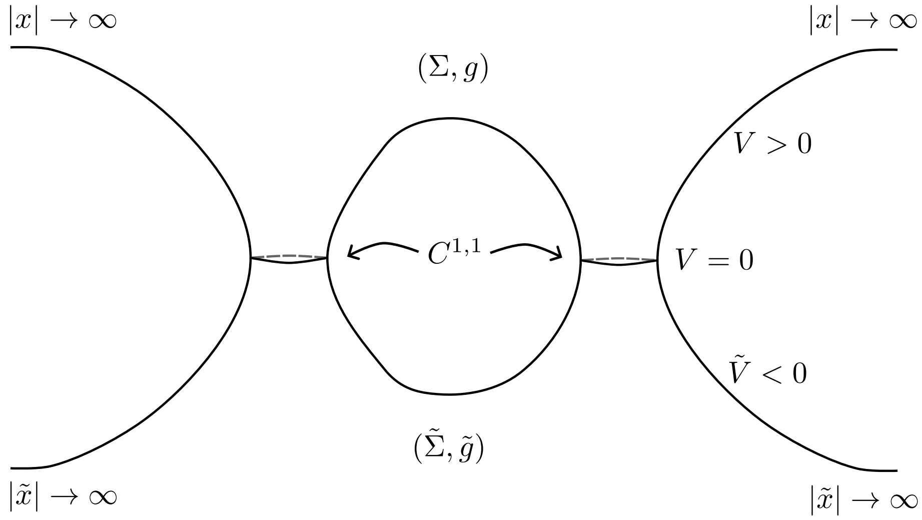

Figure 6.3: The doubled manifold $(\check{\Sigma}, \check{g})$ such that: $\check{g}, \check{V}$ are $C^{1,1}$ in the the glued boundaries.

Now we transform conformally the metric such that:

$$\gamma = \phi_{\pm}^2 \check{g}$$

where the conformal factor is:

$$\phi_{\pm} = \frac{(1 \pm V)^2}{4}$$

This gives us manifold (N, γ^*) where

$$N = N^+ \cup N^-, \qquad \gamma^* = \begin{cases} {}^+\gamma = \phi_+^2 \check{g}, & \text{in } N^+ \\ {}^-\gamma = \phi_-^2 \check{g}, & \text{in } N^- \end{cases}$$

We have shown in Lemma 6.2.4 that we can compactify N^- in N by adding a point of infinity P. From that we get a manifold $N \cup \{P\}$. Thus we extend the metric ${}^*\gamma$ to a metric γ that includes the point P as we did in Lemma 6.2.4 where on a neighborhood around P, γ is $W^{2,q}$.

$$N = N^+ \cup N^-, \qquad \gamma = \begin{cases} {}^+\gamma = \phi_+^2 \check{g}, & \text{in } N^+ \\ {}^-\gamma^* = \phi_-^2 \check{g}, & \text{in } N^- \cup \{P\} \end{cases}$$

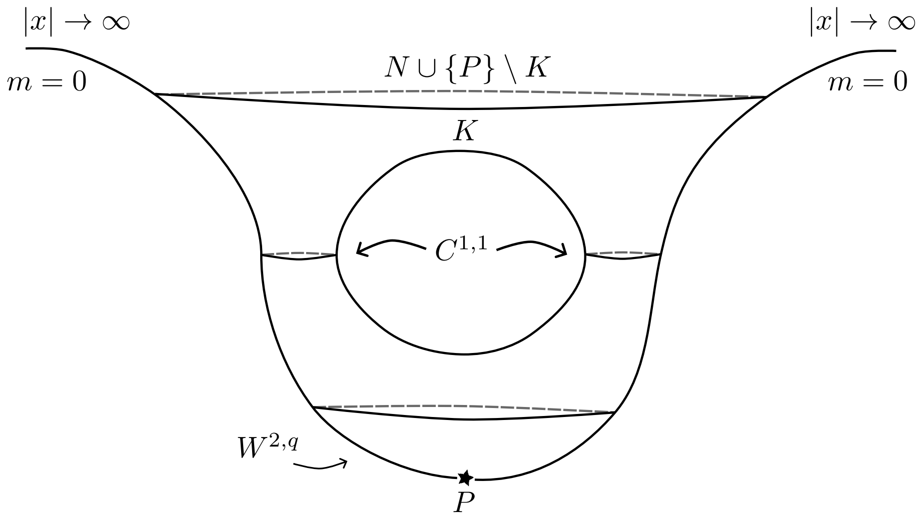

Figure 6.4: The manifold $(N \cup \{P\}, \gamma)$ such that: N^+ is asymptotically Euclidean on a set $N \cup \{P\}$ with zero mass, the metric γ is $C^{1,1}$ on the glued $(\partial\Sigma)_i$, $(\partial\tilde{\Sigma})_i$, N^- is compactified with a point of infinity P and γ is $W^{2,q}$ in a neigbourhood of P.

In Proposition 6.2.2 we proved that the metric ${}^+\gamma$ has asymptotically Euclidean structure and mass zero in $|x| \to \infty$ outside a compact set. Now we can prove that in fact $(N \cup \{P\}, \gamma)$ is a complete manifold in contrast to the original (Σ, g).

Proposition 6.3.2. *The Riemannian manifold $(N \cup \{P\}, \gamma)$ is complete.*

Proof. First we will need the following Theorem:

Theorem 6.3.3. *[22, p. 222] Let M be a Riemannian manifold which is C^3. Then M is complete with respect to g if and only it supports a proper C^3 function f such that:*

$$|\nabla f| \leq constant$$

The idea of the proof is that we will construct proper functions as the theorem states and then by gluing them we will have the desired outcome.

First we look on the upper half $(N^+, {}^+\gamma)$. Assume a function $f : N^+ \setminus K \to \mathbb{R}$ such that

$$f(x) = \log(r(x))$$

where $r(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, K is compact and $\phi(N^+ \setminus K) = E_R$ from the structure of infinity that we proved in Proposition 6.2.2. We remind that the metric on $N^+ \setminus K$ takes the form

$${}^+\gamma_{ij} = \delta_{ij} + \mathcal{H}_{ij}$$

where $\mathcal{H}_{ij} = O(|x|^{-\tau})$ for $\tau \in (1, 2)$. Then

$$\begin{aligned}
|\nabla f(x)|^2 &= {}^+\gamma^{ij} \partial_i f(x) \partial_j f(x) \\
&= {}^+\gamma^{ij} \partial_i (\log(r(x))) \partial_j (\log(r(x)))
\end{aligned}$$

but

$$\begin{aligned}
\partial_i \log(r(x)) &= \partial_i \log\left[((x^1)^2 + (x^2)^2 + (x^3)^2)^2\right] \\
&= \frac{1}{2} \frac{1}{r(x)} \frac{1}{r(x)} 2\left(x^1 \delta_i^1 + x^2 \delta_i^2 + \delta_i^3\right) \\
&= \frac{1}{|r(x)|^2} x^i
\end{aligned}$$

hence

$$\begin{aligned}|\nabla f(x)|^2 &= {}^+\gamma^{ij}\frac{1}{|x|^4}x^i x^j \\ &= \frac{1}{|x|^4}\delta^{ij}x^i x^j + \frac{1}{|x|^4}\mathcal{H}^{ij}x^i x^j\end{aligned}$$

and because

$$|\delta^{ij}x^i x^j| = |x|^2, \qquad |\mathcal{H}^{ij}x^i x^j| = O(|x|^{-\tau-2})$$

we write

$$|\nabla f(x)|^2 = |x|^{-2} + O(|x|^{-\tau-2})$$

which for $|x| \to \infty$ it goes to zero and so

$$|\nabla f(x)| \leq C$$

We also have the following Theorem [31, p. 175]:

Theorem 6.3.4. *Suppose (M, g) is a connected Riemannian manifold, $S \subseteq M$ is arbitrary, and $f : M \to [0, \infty)$ is the distance to S, that is,*

$$f(x) = d_g(x, S) = \inf\{d_g(x, p) : p \in S\}$$

for all $x \in M$. If f is smooth on some open subset $U \subseteq M \setminus S$, then

$$|\nabla f| \equiv 1$$

on U.

Suppose the function

$$r(x) = \mathrm{d}(x, \cup_i(\partial\Sigma^{\pm})_i)$$

we notice that $f(N^+ \setminus K) = [\log R, +\infty)$ and $r(K) = [0, R]$. We change the domain of r with the compact subset S such that $f(N^+ \setminus S_2) = [2R, +\infty)$, $r(S_1) = [0, R]$ and $R = \mathrm{d}(P, \cup_i(\partial\Sigma^{\pm})_i)$. We want to find a function

$$k(x) = \begin{cases} x, & x \in [0, R] \\ p(x), & x \in [R, e^{2R}] \\ \log x, & \in [e^{2R}, +\infty] \end{cases}$$

such that p is suitable to make $k \in C^3$. So we want p to satisfy the conditions:

$$\begin{aligned} p(R) &= R & p(e^{2R}) &= 2R \\ p'(R) &= 1 & p'(e^{2R}) &= e^{-2R} \\ p''(R) &= 0 & p''(e^{2R}) &= -e^{-4R} \\ p'''(R) &= 0 & p'''(e^{2R}) &= 2e^{-6R} \end{aligned}$$

We choose a polynomial $p(x) = a_7x^7 + \cdots + a_0$ to satisfy the above conditions. To see that the coefficients exist such that the polynomial satisfies the above conditions we need the determinant of the system to be nonzero. The determinant is an analytic function and so it has at most countable roots. If R is a root of the determinant then we can choose a different R. The gradient of the polynomial is bounded and $|x|$ is a proper function as the rest. So we have that

$$k(x) = \begin{cases} r(x), & x \in S \\ p(|x|), & x \in \S_2 \setminus S_1 \\ \log|x|, & x \in N^+ \setminus S_2 \end{cases}$$

which is proper C^3 on $N \cup \{P\}$ and has bounded gradient.

□

After gluing the manifold (Σ, g) along its boundaries $(\partial\Sigma)_i$ we made the double manifold $(\check{\Sigma}, \check{g})$, where we claimed that the metric $\check{g}$ and the lapse function $\check{V}$ are $C^{1,1}$ in a neighborhood of $(\partial\Sigma^{\pm})_i$. We prove that claim in the following proposition.

Proposition 6.3.5. *The metric γ of $N \cup \{P\}$ and the lapse function $\check{V}$ are $C^{1,1}$ in a neighborhood U_i of $(\partial\Sigma^{\pm})_i$.*

Proof. First we want to use a suitable metric representation. In a neighborhood of the attached boudaries we use a coordinate system where the coordinate curves (x^1, x^2, t) are unit geodesics such that they cross orthogonally the level sets of x^3. This coordinates are called **semigeodesic coordinates** (or else Gaussian coordinates). From [31, p. 183] we have the following:

Proposition 6.3.6. *Let (M, g) be a Riemannian n-manifold and (x^i) are the semigeodesic coordinates on an open set $U \subset M$. Then the metric can be written as:*

$$g = (\mathrm{d}x^n)^2 + g_{ab}(x^1, \dots, x^n)\,\mathrm{d}x^a\,\mathrm{d}x^b$$

As can be seen from the corollary, from semigeodesic coordinates we have an orthogonal decomposition of the metric tensor. We use this on a neighborhood of $(\partial\Sigma^{\pm})_i$ to get:

$$\gamma = (\mathrm{d}x^3)^2 + \gamma_{ab}(x^a, x^b, x^3)\,\mathrm{d}x^a\,\mathrm{d}x^b$$

where $a, b = 1, 2$ and γ_{ab} is the metric of the attached boundaries. Because of the conformal tranformation $\gamma = \phi_{\pm}^2 \check{g}$ and $\gamma_{3a} = 0$ for $a = 1, 2$ we notice that:

$$\begin{aligned} \gamma = \phi_{\pm}^2 \check{g} \quad &\implies \quad \check{g}_{3a} = 0 \\ &\implies \quad \check{g} = \rho(x^1, x^2, x^3)(\mathrm{d}x^3)^2 + \check{g}_{ab}(x^a, x^b, x^3)\,\mathrm{d}x^a\,\mathrm{d}x^b \end{aligned}$$

where $a, b = 1, 2$, ρ is some unknown function and $\check{g}_{ab}$ is the metric tensor on the attached boundaries. Following the same computations as in proposition 5.3.1, with small changes because of the spatial metric, we have the first evolution equation of the metric $\check{g}_{ab}$

$$\frac{\partial}{\partial x^3}\check{g}_{ab} = 2\rho[A(\check{g})]$$

where $[A(\check{g})]$ is the second fundamental form in the boundary. But as we have shown in Remark 6.0.2, we have that the second fundamental form on the boundary inside $\overline{\Sigma}$ is zero. And so

$$\frac{\partial}{\partial x^3}\check{g}_{ab} = 0 \tag{6.27}$$

In $\check{\Sigma}$ the metric $\check{g}$ in a coordinate chart of U_i can be written as:

$$\breve{g}_{ab} \circ \tilde{\Phi}_i^{-1} \circ \phi^{-1} = \breve{g}_{ab} \circ \tilde{\Phi}^{-1}(x)$$

where $\phi \circ \tilde{\Phi}_i$ is the chart of the collar neighborhood. Hence

$$\breve{g}_{ab} \circ \tilde{\Phi}_i = \begin{cases} \breve{g}_{ab} \circ \tilde{\Phi}^{-1}(x_1, x_2, x_3), & \text{for } x \in U^+ \\ \breve{g}_{ab} \circ \tilde{\Phi}^{-1}(x_1, x_2, -x_3), & \text{for } x \in U^- \end{cases}$$

so $\breve{g}_{ab}(x_1, x_2, x_3)$ on U^- is even in x_3. Then for $q \in \check{\Sigma}$ and $\psi = \phi \circ \tilde{\Phi}$

$$\begin{aligned} \frac{\partial}{\partial x_3} \breve{g}_{ab}\bigg|_q &= \frac{\partial}{\partial x_3}\left(\breve{g}_{ab} \circ \tilde{\Phi}^{-1} \circ \phi^{-1}\right)\bigg|_{\psi(q)} \\ &= \frac{\partial}{\partial x_3} \breve{g}_{ab}\left(\tilde{\Phi}^{-1}(x_1, x_2, x_3)\right)\bigg|_{\psi(q)} \end{aligned}$$

Assume $h_{ab} = \breve{g}_{ab} \circ \tilde{\Phi}^{-1}$. For U^+, U^- we denote the partial derivative on the boundary $\partial_{x_3} h_+(0)$, $\partial_{x_3} h_-(0)$ respectively. Hence

$$\begin{aligned} \partial_{x_3} h_+(0) &= \lim_{x_3 \to 0^+} \frac{h_{ab}(x_1, x_2, x_3) - h_{ab}(x_1, x_2, 0)}{x_3} \\ \partial_{x_3} h_-(0) &= \lim_{x_3 \to 0^-} \frac{h_{ab}(x_1, x_2, -x_3) - h_{ab}(x_1, x_2, 0)}{x_3} \\ &= \lim_{0^+} \frac{h_{ab}(x_1, x_2, x_3) - h_{ab}(x_1, x_2, 0)}{-x_3} \\ &= -h'_+(0) \end{aligned}$$

This holds when $\partial_{x_3} h_+(0) = \partial_{x_3} h_-(0) = 0$ which we have from (6.27). So $h \in C^1$ meaning that $\breve{g} \in C^1$.

For the second derivative we notice that

$$\partial_{x_i} \partial_{x_3} h_+(0) = -\partial_{x_i} \partial_{x_3} h_-(0)$$

for $i = 1, 2$. So $h \in C^2$ if $\partial_{x_i} \partial_{x_3} h(x_1, x_2, 0) = 0$, which is not necessary true.

We have the following Theorem:

Theorem 6.3.7. *[19, p. 294] Let U be open and bounded, with ∂U being C^1. Then $f : U \to \mathbb{R}$ is Lipschitz continuous if and only if $f \in W^{1,\infty}(U)$.*

Because $\breve{g}$ outside of $(\partial\Sigma^{\pm})_i$ is smooth, we have that $\partial_a \partial_b \breve{g} \in L^\infty(U_i)$ and so

$$\partial_a \breve{g} \in W^{1,\infty}_{\text{loc}} \implies \partial_a \breve{g} \in C^{0,1} \implies \breve{g} \in C^{1,1}$$

Similarly for $\check{V}$ we have

$$\check{V} \circ \tilde{\Phi}^{-1}(x) = \begin{cases} V \circ \tilde{\Phi}^{-1}(x_1, x_2, x_3), & x \in U^+ \\ -V \circ \tilde{\Phi}^{-1}(x_1, x_2, -x_3), & x \in U^- \end{cases}$$

Suppose $L = \check{V} \circ \tilde{\Phi}^{-1}$. We have that $L(x_1, x_2, x_3)$ on U^- is odd in x_3. By defining similarly the partial derivatives $\partial_{x_3} L_+(0), \partial_{x_3 L_-(0)}$ we have that

$$\begin{aligned} \partial_{x_3} L_+(0) &= \partial_{x_3} L_-(0) \\ \partial_{x_i} \partial_{x_3} L_+(0) &= -\partial_{x_i} \partial_{x_3} L_-(0) \end{aligned}$$

but again $\partial_{x_i}\partial_{x_3}\check{V}(x_1, x_2, 0)$ is not necessary zero. So $\check{V} \in C^1$ and

$$\partial_a\partial_b\check{V} \in L^\infty \quad \Longrightarrow \quad \partial_a\check{V} \in W^{1,\infty}_{\text{loc}} \quad \Longrightarrow \quad \partial_a\check{V} \in C^{0,1} \quad \Longrightarrow \quad \check{V} \in C^{1,1}$$

Hence the conformal transformation

$$\gamma = \frac{(1 \pm V)^4}{16}\check{g}$$

tells us that γ is also $C^{1,1}$.

□

Corollary 6.3.8. $\gamma \in W^{2,q}_{loc}(N \cup \{P\})$.

Proof. From proposition 6.3.5 we have that $\gamma \in C^{1,1}$ in a neighborhood of the glued boundaries $(\partial\Sigma^\pm)_i$. Then from the definition of the Holder space we have

$$\gamma \in C^{1,1} \quad \Longrightarrow \quad \gamma \in C^1, \quad \gamma' \in C^1, \quad f' \text{ is Lipschitz}$$

So we have

$$\gamma' \text{ Lipschitz} \quad \Longrightarrow \quad \gamma' \in W^{1,\infty}$$

hence

$$\|\gamma'\|_{1,\infty} = \|\gamma'\|_\infty + \|\gamma''\|_\infty$$

So we have

$$\gamma'' \in L^\infty \quad \Longrightarrow \quad \gamma'' \in L^q_{\text{loc}}$$

Hence we have $\gamma \in W^{2,q}_{\text{loc}}$. □

In remark 6.1.14 we said that the positive mass theorem for n-manifolds requires lower regularity for the metric than smoothness in all derivatives because of the mass decay conditions (Definition 6.1.13).

The first condition of the mass decay conditions is that there exists an asymptotical flat structure Φ such that $\gamma \in W^{2,q}_{-\tau}(E_R)$. We have shown that the metric γ is asymptotically Euclidean which satisfies the condition.

The second condition is that Ricci scalar is integrable on the manifold. In Proposition 6.2.1 we showed that the Ricci scalar is zero for a metric $\gamma^\pm = b^2(1 \pm V)^2 g$. But the proof required γ to be C^2 and $\Delta_g V = 0$ which requires V to be C^2. We have shown that γ, V in a neighborhood of the compactified point P and a neighborhood around $(\partial\Sigma^\pm)_i$ are $C^{1,1}$. Also in $N \cup \{P\} \setminus K$ where K is a compact set, γ is not smooth. So we need to prove that the Ricci scalar at those neighborhoods is L^1 so that the second condition of the mass decay conditions is satisfied.

Proposition 6.3.9. $R \in L^1(N \cup \{P\})$ *where* R *is the Ricci scalar in the metric* γ.

Proof. We have $\mathrm{d}\mu_\gamma$ is the Riemannian density. Suppose the covering of $N \cup \{P\}$ by W_i open sets with the smooth charts (W_i, ϕ_i). Let a partition of unity ψ_i in W_i such that:

$$\begin{aligned}\int_{N\cup\{P\}} |R|\,\mathrm{d}\mu_\gamma &= \sum_i \int_{\phi_i(W_i)} (\phi_i^{-1})^*|\psi_i R|\,\mathrm{d}\mu_\gamma \\ &= \sum_i \int_{\phi_i(W_i)} (\psi_i \circ \phi_i^{-1})|R \circ \phi_i^{-1}| \cdot |\sqrt{\det \gamma_{ij}}|\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\end{aligned}$$

because $\mathrm{d}\mu_\gamma = |\omega_\gamma|$, where ω_γ is the Riemannian volume form.

Suppose the sets

- $U_i^{\pm}$ is a neighborhood of $(\partial\Sigma^{\pm})$,
- $B_\varepsilon(P)$ is a neighborhood of the compactified point P,
- $N \cup \{P\} \setminus K$ is a neighborhood of the infinity such that:

$$\Phi : (N \cup \{P\} \setminus K) \to E_R$$

  is a diffeomorphism for $R > 0$.

From Proposition 6.2.1 we have that $R = 0$ except in the neighborhoods above. If $P \in W_1 \cap W_2$ then we choose $\varepsilon > 0$ such that $B_\varepsilon(P) \subset \phi_i(W_i)$ or $B_\varepsilon(P) \subset \phi_{i+1}(W_{i+1})$. Since $R = 0$ in $\phi_j(W_j) \setminus B_\varepsilon(P)$ for $j = i$ or $i+1$, we have:

$$\begin{aligned}\int_{N\cup\{P\}} |R|\,\mathrm{d}\mu_\gamma = &\int_{\Phi(N\cup\{P\})} (\psi_1 \circ \Phi^{-1}) \cdot |R(x)| \cdot |\sqrt{\det\gamma_{ij}}|\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \\ &+ \int_{B_\varepsilon(P)} (\psi_2 \circ \phi_2^{-1}) \cdot |R(x)| \cdot |\sqrt{\det\gamma_{ij}}|\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 \\ &+ \int_{\phi_3(U^{\pm})} (\psi_3 \circ \phi_3^{-1}) \cdot |R(x)| \cdot |\sqrt{\det\gamma_{ij}}|\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\end{aligned}$$

We have $\psi \circ \phi_i^{-1} \leq 1$. Also $\sqrt{\det\gamma_{ij}}$ in $B_\varepsilon(P)$, $\phi_3(U^{\pm})$ is bounded

$$\gamma_{ij} \in C^{1,1} \quad \Longrightarrow \quad \gamma_{ij} \in C^0 \quad \Longrightarrow \quad \det\gamma_{ij} \in C^0$$

hence it is bounded in $\phi_3(\overline{U^{\pm}})$, $\overline{B}_\varepsilon(P)$.

We will prove that $\det\gamma_{ij}$ is bounded in E_R.

We have that

$$|\gamma_{ij} - \delta_{ij}| \leq \frac{C_1}{|x|^\tau}, \qquad |\partial\gamma_{ij}| \leq \frac{C_2}{|x|^{\tau+1}}$$

hence $\gamma_{ij} = \delta_{ij} + A_{ij}$ where

$$|A_{ij}| \leq \frac{C_1}{|x|^\tau}$$

Then suppose $\gamma_{ij} = \delta_{ij} + |x|^{-\tau}\hat{A}_{ij}$ where

$$|\hat{A}_{ij}| = ||x^\tau|A_{ij}| \leq |x|^\tau \frac{C_1}{|x|^\tau} = C_1$$

For $|x| \geq \rho$, where $r = (2C_1)^{1/\tau}$, suppose

$$\frac{C_1}{|x|^\tau} \leq \frac{1}{2}$$

then

$$\|\gamma^{-1}\| = \| \left(1 - \left(-\frac{\hat{A}}{|x|^\tau}\right)\right)^{-1} \| \leq \frac{1}{1 - \|\frac{\hat{A}}{|x|^\tau}\|} \leq \frac{1}{1 - \frac{C_1}{|x|^\tau}}$$

From geometric-arithmetic inequality, for $x_1, \ldots, x_n > 0$

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

where in 3-dimensions

$$x_1 x_2 x_3 \leq \left(\frac{x_1 + x_2 + x_3}{3} \right)^3$$

Because γ is symmetric, positive definite matrix we have using its eigenvalues

$$\begin{aligned}(\det \gamma) \leq \left(\frac{\operatorname{tr}(\gamma)}{3} \right)^3 \quad \Longrightarrow \quad \det \left(1 + \frac{\hat{A}}{|x|^\tau} \right) &\leq \left(\frac{\operatorname{tr}\left(1 + \frac{\hat{A}}{|x|^\tau}\right)}{3} \right)^3 \\ &= \left(\frac{3 + \frac{\operatorname{tr}\hat{A}}{|x|^\tau}}{3} \right)^3\end{aligned}$$

defining the inner product of matrices $(A, B) = \operatorname{tr}(A \cdot B)$ we have $\|A\| = (\operatorname{tr}(A^2))^{1/2}$, and so from Cauchy-Schwarz:

$$\operatorname{tr}\hat{A} = (1, \hat{A}) \leq \|1\| \cdot \|\hat{A}\| = \sqrt{3}\operatorname{tr}(A^2)^{1/2} \leq \sqrt{3} C_3$$

Hence for $|x| \geq \rho$ we have

$$\det(\gamma) \leq \left(1 + \frac{\sqrt{3} C_3}{3|x|^\tau} \right)^3 \leq C_4$$

Because $\det(\gamma)$ is bounded in the required sets, we have

$$\begin{aligned}\int_{N \cup \{P\}} |R| \, \mathrm{d}\mu_\gamma \leq & C_4 \int_{E_\rho} |R(x)| \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 \\ & + C_5 \int_{B_\varepsilon(P)} |R(x)| \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3 \\ & + C_6 \int_{\phi_3(U^\pm)} |R(x)| \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3\end{aligned}$$

- In $\phi_3(U^\pm)$ we know that R is not continuous, but it is continuous in $\phi_3(U^+)$, $\phi_3(U^-)$, which means

$$\int_{\phi_3(U^+)} |R| \leq C_7$$
$$\int_{\phi_3(U^-)} |R| \leq C_8$$

hence

$$\int_{\phi_3(U^\pm)} |R| \leq C_7 + C_8$$

- In $B_\varepsilon(P)$ we have shown that $\gamma_{ij} \in W^{2,q}$, which means for $a \leq 1 - 3/q \leq 1$

$$D\gamma_{ij} \in W^{1,q} \quad \Longrightarrow \quad D\gamma_{ij} \in C^{0,a} \quad \Longrightarrow \quad \gamma_{ij} \in C^{1,a}$$

Hence γ_{ij}, $D\gamma_{ij} \in C^1$ are bounded in $B_\varepsilon(P)$. From equation (6.11) in lemma 6.1.5 we have:

$$\begin{aligned} -2R_{jk} &= \gamma^{qr}\left(-\partial_q\partial_k\gamma_{jr} + \partial_q\partial_r\gamma_{jk} + \partial_j\partial_k\gamma_{qr} - \partial_j\partial_r\gamma_{qk}\right) + Q(\gamma^{-1}, \partial\gamma) \\ &= \gamma^{-1} * \partial^2\gamma + Q(\gamma^{-1}, \partial\gamma) \end{aligned} \tag{6.28}$$

where $\gamma^{-1} * \partial^2\gamma$ are contractions of γ^{-1}, $\partial^2\gamma$ and $Q(\gamma^{-1}, \partial\gamma)$ are quadratic terms in $\gamma^{-1}, \partial\gamma$. Because $\gamma, \partial\gamma$ are bounded we have:

$$|R| \leq |\gamma^{-1} * \partial^2\gamma| + C_9 \leq C_{10}|\partial^2\gamma| + C_9$$

Hence

$$\int_{B_\varepsilon(P)} |R| \leq C_{10} \int_{B_\varepsilon(P)} |\partial^2\gamma| + C_9|B_\varepsilon(P)|$$

From Holder inequality

$$\int_{B_\varepsilon(P)} |\partial^2\gamma| \leq \left(\int_{B_\varepsilon(P)} 1\right)^{1/q^*} \cdot \left(\int_{B_\varepsilon(P)} |\partial^2\gamma|^q\right)^{1/q}$$

But we have

$$\gamma \in W^{2,q}(B_\varepsilon(P)) \quad \Longrightarrow \quad \|\partial^2\gamma\|_{L^q(B_\varepsilon(P))} < +\infty$$

and so

$$\int_{B_\varepsilon(P)} |R| \leq C_{11}$$

- In E_ρ we have that $\gamma_{ij} \in W^{2,q}_{-\tau}(E_\rho)$ for $\tau \in (1,2)$. And from the Ricci scalar equation (6.28) we notice that we have shown that the terms are bounded except the terms with $\partial^2\gamma$ and $(\partial\gamma)^2$. So

$$\int_{E_\rho} |R| \leq C_{12}\left(\int_{E_\rho} |\partial^2\gamma| + |\partial\gamma|^2\right)$$

To show that these terms are bounded in E_ρ we will use that:

$$\gamma \in W^{2,q}_{-\tau} \quad \Longrightarrow \quad \begin{cases} \displaystyle\int_{|x|\geq\rho} |\partial^2\gamma|^q|x|^{(\tau+2)q-3} < +\infty \\ \displaystyle\int_{|x|\geq\rho} |\partial\gamma|^q|x|^{(\tau+1)q-3} < +\infty \end{cases}$$

For the first the term $\partial^2\gamma$:

$$\begin{aligned} \int_{|x|\geq\rho} |\partial^2\gamma| &= \int_{|x|\geq\rho} w^{-1}w|\partial^2\gamma| \\ &\leq \left(\int_{|x|\geq\rho} w^{-q^*}\right)^{\frac{1}{q^*}} \cdot \left(\int_{|x|\geq\rho} w^q|\partial^2\gamma|^q\right)^{\frac{1}{q}} \end{aligned}$$

where $q^* = q/(q-1)$ and we want

$$w^q = |x|^{(\tau+2)q-3} \quad \Longrightarrow \quad w = |x|^{\tau+2-3/q}$$

hence

$$\left(\int_{|x|\geq\rho} w^q|\partial^2\gamma|^q\right)^{\frac{1}{q}} < +\infty$$

For w^{-q^*}

$$\begin{aligned} w^{-q^*} &= w^{-\frac{q}{q-1}} \\ &= |x|^{-\left[(\tau+2)-\frac{3}{q}\right]\frac{q}{q-1}} \\ &= |x|^{-(\tau+2)\frac{q}{q-1}+\frac{3}{q-1}} \end{aligned}$$

So we have

$$\begin{aligned} \int_{|x|\geq\rho} w^{-q^*} &= \int_{|x|\geq\rho} |x|^{-(\tau+2)\frac{q}{q-1}+\frac{3}{q-1}} \\ &= 4\pi \int_\rho^\infty |x|^{-(\tau+2)\frac{q}{q-1}+\frac{3}{q-1}+2} \end{aligned}$$

We need

$$\begin{aligned} (\tau+2)\frac{q}{q-1} - \frac{3}{q-1} - 2 > 1 \quad &\Longrightarrow \quad \tau+2)\frac{q}{q-1} > 3 + \frac{3}{q-1} \\ &\Longrightarrow \quad \tau+2 > 3\frac{q-1}{q} + \frac{3}{q} = 3 - \frac{3}{q} + \frac{3}{q} \\ &\Longrightarrow \quad \tau > 1 \end{aligned}$$

For the second term $(\partial\gamma)^2$:

$$\begin{aligned} \int_{|x|\geq\rho} |\partial\gamma|^2 &= \int_{|x|\geq\rho} w^{-1}w|\partial\gamma|^2 \\ &\leq \left(\int_{|x|\geq\rho} w^{-q^*}\right)^{\frac{1}{q^*}} \cdot \left(\int_{|x|\geq\rho} w^{\frac{q}{2}}|\partial\gamma|^q\right)^{\frac{1}{q}} \end{aligned}$$

where $q^* = (q/2)^*$ and we want:

$$w^{\frac{q}{2}} = |x|^{(\tau+1)q-3} \quad \Longrightarrow \quad w = |x|^{2(\tau+1)-\frac{6}{q}}$$

hence

$$\left(\int_{|x|\geq\rho} w^{\frac{q}{2}}|\partial\gamma|^q\right)^{\frac{1}{q}} < +\infty$$

For w^{-q^*}:

$$\begin{aligned} \int_{|x|\geq\rho} w^{-q^*} &= \int_{|x|\geq\rho} |x|^{-\frac{2q}{q-2}\left[\tau+1-\frac{3}{q}\right]} \\ &= 4\pi \int_\rho^\infty |x|^{-\frac{2q}{q-2}\left[\tau+1-\frac{3}{q}\right]+2} \end{aligned}$$

So we need

$$\begin{aligned}\frac{2q}{q-2}\left[(\tau+1)-\frac{3}{q}\right]-2>1 \quad &\Longrightarrow \quad \frac{2q}{q-2}\left[\tau+1-\frac{3}{q}\right]>3\\ &\Longrightarrow \quad \tau+1-\frac{3}{q}>\frac{3}{2}\left(\frac{q-2}{q}\right)\\ &\Longrightarrow \quad \tau+1>\frac{3}{2}\left(1-\frac{2}{q}\right)+\frac{3}{q}=\frac{3}{2}\\ &\Longrightarrow \quad \tau>\frac{1}{2}\end{aligned}$$

Hence

$$\int_{E_\rho}|R|\leq C_{13}$$

We have shown that

$$\int_{\phi_3(U^\pm)}|R(x)|\leq C_{14}, \qquad \int_{B_\varepsilon(P)}|R(x)|\leq C_{11}, \qquad \int_{E_\rho}|R(x)|\leq C_{13}$$

hence together with the previous results we have for the Ricci scalar that:

$$\int_{N\cup\{P\}}|R|\,\mathrm{d}\mu_\gamma\leq C_{14} \quad \Longrightarrow \quad R\in L^1(N\cup\{P\})$$

□

Proof of the main theorem 6.0.1. We have constructed the smooth 3-manifold $(N\cup\{P\},\gamma)$ such that the metric is conformally related to $\breve{g}$ metric by

$$\gamma=\frac{(1\pm V)^4}{16}\breve{g}$$

We remind that the metric $\breve{g}$ is the metric of the double manifold $\check{\Sigma}$ such that

$$\breve{g}=\begin{cases} g, & \text{in } \Sigma\\ \tilde{g}, & \text{in } \tilde{\Sigma}\end{cases}$$

We have proven the following facts:

- $N\cup\{P\}$ is complete by Proposition 6.3.2.
- The metric γ is asymptotically Euclidean and has zero mass by Proposition 6.2.2.
- The metric γ is $W^{2,q}_{\text{loc}}$ by Corollary 6.3.8.
- The Ricci scalar is zero for $\gamma,\check{V}\in C^2$ by Proposition 6.2.1.
- The mass decay conditions, from Definition 6.1.13, are satisfied by Proposition 6.3.9.

With the above, the positive mass theorem 6.3 from [1, p. 690] tells us that there exists an isometry between $(N \cup \{P\}, \gamma)$ and $(\mathbb{R}^3, \delta)$, where δ is the standard Euclidean metric. Hence the 3-manifold (Σ, g) is conformally flat with the metric

$$g = \frac{16}{(1+V)^4}\delta$$

We orthogonally decompose (Σ, g) by taking the level sets of the lapse function V and then extending it by following orthogonally the flow of $\nabla V/|\nabla V|^2$. So in the coordinates (V, x^1, x^2) the metric becomes

$$g = g_{33}\,\mathrm{d}V^2 + \overline{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

for $i, j = 1, 2$ and $\bar{g}_{ij}$ is the induced metric on the level sets of V.

We notice, for $a, b = 1, 2, 3$ that

$$\begin{aligned} |\nabla V|^2 &= g^{ab}\partial_a V \partial_b V \\ &= g^{33} + \sum_{j=1}^{2} g^{aj}\partial_a V \partial_j V \end{aligned}$$

But in the level set with coordinates (x^1, x^2) we have V =constant. So

$$|\nabla V|^2 = g^{33} \quad \Longrightarrow \quad g_{33} = \frac{1}{|\nabla V|^2}$$

Hence by denoting $W^2 = |\nabla V|^2$ and for $i, j = 1, 2$, we get:

$$g = W^{-2}\,\mathrm{d}V^2 + \overline{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{6.29}$$

For dimension $n \geq 4$ we know that the the **Weyl tensor** is identically zero if and only if the metric metric is conformally flat. But for dimension $n = 3$ the Weyl tensor is zero for every manifold ([31, p. 218]). Thus it doesnt give us a suitable condition for conformal flatness in 3-dimensions. For a 3-dimensional manifolds we have that the metric is conformally flat if and only if the **Cotton tensor** is zero ([31, p. 220]). The Cotton tensor is:

$$C_{abc} = R_{ab;c} - R_{ac;b} + \frac{1}{4}(g_{ac}R_{;b} - g_{ab}R_{;c})$$

Lemma 6.3.10. *Using the static equations of the spacetime (M, g) in vacuum*

$$\begin{aligned} \Delta_g V &= 0 \\ R_{ab} &= V^{-1}V_{;ab} \end{aligned}$$

The Cotton tensor can take the form:

$$C_{abc} = V^{-2}\left[2V_{;ca}V_{;b} - 2V_{;ab}V_{;c} + V_{;bm}V^{;m}g_{ca} - V_{;cm}V^{;m}g_{ba}\right]$$

Proof. We remind that $V_{;ab} = V_{;ba}$. First we compute the terms of the Cotton tensor

$$\begin{aligned} R_{ab;c} &= (V^{-1}V_{;ba})_{;c} = -V^{-2}V_{;c}V_{;ba} + V^{-1}V_{;bac} \\ R_{ac;b} &= -V^{-2}V_{;b}V_{;ca} + V^{-1}V_{;cab} \end{aligned}$$

Also

$$R = g^{ab}R_{ab} = g^{ab}V^{-1}V_{;ab} = V^{-1}\Delta V = 0$$

So

$$C_{abc} = V^{-1}(V_{;bac - V_{;cab}}) - V^{-2}(V_{;c}V_{;ba} - V_{;b}V_{;ca})$$

But we have the Ricci identity [31, p. 206]

$$V_{;bac} - V_{;cab} = R_{bcam}V^{;m}$$

and that the Riemann tensor in three dimensions is defined by ([23, p. 56]):

$$\begin{aligned} R_{bcam} &= -R_{ba}g_{cm} + R_{bm}g_{ca} - R_{cm}g_{ba} + R_{ca}g_{bm} - \frac{1}{2}R(g_{bm}g_{ca} - g_{ba}g_{cm}) \\ &= V^{-1}(-V_{;ab}g_{cm} + V_{;bm}g_{ca} - V_{;cm}g_{ba} + V_{;ca}g_{bm}) \end{aligned}$$

since the Ricci scalar is zero. So

$$\begin{aligned} C_{abc} &= V^{-2}(-V_{;ab}V_{;c} + V_{;bm}V^{;m}g_{ca} - V_{;cm}V^{;m}g_{ba} + V_{;ca}V_{;b} - V_{;c}V_{;ba} + V_{;b}V_{;ca}) \\ &= V^{-2}(2V_{;ca}V_{;b} - 2V_{;ab}V_{;c} + V_{;bm}V^{;m}g_{ca} - V_{;cm}V^{;m}g_{ba}) \end{aligned}$$

□

Using Lemma (6.3.10), the following equation can be derived ([23, p. 58]):

$$C_{abc}C^{abc} = 4W^6V^{-4}(\psi_{ij}\psi^{ij} + \frac{1}{4}W^{-6}(W^2)_{;i}(W^2)^{;i}) \tag{6.30}$$

where $a, b, c = 1, 2, 3$ and $i, j = 1, 2$

$$\psi_{ij} = W(H_{ij} - \frac{1}{2}\overline{g}_{ij}H),$$

such that H_{ij} is the second fundamental form on the level sets, H is the trace of H_{ij}, ψ_{ij}/W is the trace-free part of H_{ij} and $\overline{g}$ is the induced metric on the level sets of V. For the rest of the proof whenever we write tensors with indices i, j we will mean that $i, j = 1, 2$, which means that the tensors are on the level sets.

Since we have shown that g is conformally flat, we have from (6.30)

$$\psi_{ij} = 0, \qquad (W^2)_{;i} = 0$$

meaning

$$(H_{ij} - \frac{1}{2}\bar{g}_{ij}H) = 0, \qquad W_{;i} = 0$$

in coordinates (V, x^1, x^2). Hence W is a function of V only. Now we change coordinates such that $U = \log V$. For equation (6.29) we have:

$$\left.\begin{aligned} W &= e^U|\nabla U| \\ \mathrm{d}V^2 &= e^{2U}\,\mathrm{d}U \end{aligned}\right\} \implies W^{-2}\,\mathrm{d}V^2 = |\nabla U|^{-2}\,\mathrm{d}U^2$$

For the rest of the rest of the proof we denote $W = |\nabla U|$, and so we have

$$g = W^{-2}\,\mathrm{d}U^2 + \bar{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

such that $i, j = 1, 2$ and $\bar{g}_{ij}$ is the induced metric in the level sets of U.

The above are tensor equations, so we have that W is a function of U only and

$$H_{ij} = \frac{1}{2}\bar{g}_{ij}H \tag{6.31}$$

We need the equations from [30, p. 90]

$$\frac{\mathrm{d}\bar{g}_{ij}}{\mathrm{d}U} = 2W^{-1}H_{ij} \tag{6.32}$$

$$\overline{R} - H^2 + H_{ij}H^{ij} = 0 \tag{6.33}$$

$$W\frac{\mathrm{d}H}{\mathrm{d}U} + H_{ij}H^{ij} = 0 \tag{6.34}$$

$$\frac{\mathrm{d}W}{\mathrm{d}U} = -H \tag{6.35}$$

where $\bar{R}$ is the scalar curvature of the level sets of U.

- From equation (6.35) we have that H is a function of U only.
- From equation (6.34) we have that $H_{ij}H^{ij}$ is a function of U only.
- From equation (6.33) we have that $\overline{R}$ is a function of U only.

From the last statement we have that the level sets of U have constant scalar curvature.

From equations (6.31) and (6.32) we have that:

$$\frac{\mathrm{d}\overline{g}_{ij}}{\mathrm{d}U} = W^{-1}H\overline{g}_{ij} \tag{6.36}$$

where $W^{-1}H$ is a function of U only. We write $f(U) = W^{-1}H$, so

$$\frac{\mathrm{d}\overline{g}_{ij}}{\mathrm{d}U} = f(U)\overline{g}_{ij} \quad\implies\quad \overline{g}_{ij}(U) = \overline{g}_{ij}(U_0)e^{\int_{U_0}^{U} f(u)\,\mathrm{d}u}$$

Denote

$$r^2(U) = e^{\int_{U_0}^{U} f(u)\,\mathrm{d}u}$$

From theorem 1 in [30, p. 89] we have that

$$S_c = U^{-1}(c) \simeq S^2$$

for $|c|$ sufficiently small. In two dimensions it is known that the scalar curvature is two times the Gaussian curvature ([31, p. 250]). Since the level sets are diffeomorphic to spheres, we have that the scalar curvature is positive somewhere on the level set ([31, p. 279]). But we have shown that the level sets have constant scalar curvature, so they are isometric to two dimensional spheres of r radius. We assume that for a value c some S_c is the round sphere S^2. Let the level set of U_0 be from that value, so we have

$$\overline{g} = \overline{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

where

$$r^2 = \frac{2}{\overline{R}} = r^2(U)$$

From $H_{ij} = (1/2)H\overline{g}_{ij}$ we have

$$H_{ij}H^{ij} = \frac{1}{2}H^2 \tag{6.37}$$

So equations (6.37) and (6.33) gives us:

$$\begin{aligned} \overline{R} - H^2 + \frac{1}{2}H^2 = 0 \quad &\Longrightarrow \quad \overline{R} = \frac{1}{2}H^2 \\ &\Longrightarrow \quad \frac{2}{r^2} = \frac{1}{2}H^2 \\ &\Longrightarrow \quad H^2 = \frac{4}{r^2} \\ &\Longrightarrow \quad H = \frac{2}{r} \end{aligned} \tag{6.38}$$

Also

$$\begin{aligned} r^2(U) = e^{\int_{U_0}^{U}(W^{-1}H)\,\mathrm{d}u} \quad &\Longrightarrow \quad 2r\frac{\mathrm{d}r}{\mathrm{d}U} = W^{-1}Hr^2 \\ &\Longrightarrow \quad 2\frac{\mathrm{d}r}{\mathrm{d}U} = \frac{H}{W}r \\ &\Longrightarrow \quad H = 2W\frac{\mathrm{d}r}{\mathrm{d}U}\frac{1}{r} \end{aligned}$$

We denote

$$' = \frac{\mathrm{d}}{\mathrm{d}U}$$

So we have the equation:

$$H = 2W\frac{r'}{r} = 2W\mathcal{R}' \tag{6.39}$$

where $\mathcal{R} = \log r$. Using equation (6.39) together with equation (6.35) we have

$$\begin{aligned} W' + 2W\mathcal{R}' = 0 \quad &\Longrightarrow \quad 2WW' + 4W^2\mathcal{R}' = 0 \\ &\Longrightarrow \quad (W^2)' + 4\mathcal{R}'W^2 = 0 \end{aligned}$$

Now equation (6.39) gives us:

$$\begin{aligned} H = 2W\mathcal{R}' \quad &\Longrightarrow \quad \frac{H^2}{4} = W^2(\mathcal{R}')^2 \\ &\Longrightarrow \quad W^2(\mathcal{R}')^2 = \frac{1}{r^2} \end{aligned}$$

where the in the last equation we used that $H = 2/r$ from (6.38). And so W, r are determined from the equations:

$$(W^2)' + 4\mathcal{R}'W^2 = 0 \tag{6.40}$$

$$W^2(\mathcal{R}')^2 = \frac{1}{r^2} \tag{6.41}$$

From Morse theory [35, p. 15] we have that if there is no critical value of U in $[c_1, c_2]$, then the level sets S_{c_1} and S_{c_2} are diffeomorphic. So we have proven:

Proposition 6.3.11. *[30, p. 92] If all $c \in [c_1, c_2]$ are regular values of U and any S_c is a sphere, then $U^{-1}[(c_1, c_2)]$ is diffeomorphic to $S^2 \times \mathbb{R}$ and the metric in $U^{-1}[(c_1, c_2)]$ is*

$$W^{-2}\, \mathrm{d}U^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)$$

where $r(U)$ and $W(U)$ are determined up to constants from equations (6.40) *and* (6.41)*.*

This tells us that the metric g is spherically symmetric. It follows that the metric ${}^4g$ in the static spacetime M is also spherically symmetric. So from Birkhoffs theorem we have that the only spherically symmetric spacetime in vacuum is the Schwarzschild metric.

$\square$

Remark 6.3.12. *In the proof of this section we didnt use any arguments involving the dimension of the time slice Σ up until the point where we used the Cotton tensor. In fact the whole construction of doubling an asymptotically Euclidean time slice in static, vacuum spacetime can be generalized in higher dimensions. Again using the positive mass theorem for spin manifolds to prove that the metric of the time slice is conformally flat. After that it needs different arguments to proceed. We refer to the papers [26] and [21] for the proofs.*

In 2017 Schoen and Yau proved the positive mass theorem in n-dimensions without the assumption of spin manifolds ([47]).

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