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Weyl Geometries and Holographic Renormalization in AlAdS manifolds

Master Thesis

by

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Abstract

Weyl geometry is a generalization of Riemannian geometry such that the geometric description of our space is covariant not just under general coordinate transformations but also local scale transformations. In this context we apply the method of Holographic Renormalization in asymptotically locally Anti-de Sitter (AlAdS) manifolds. Our motivation is that Weyl geometry, by construction, is ideal for the study of AlAdS manifolds and Conformal Field Theories (CFTs) which are connected through AdS/CFT correspondence. Additionally, we investigate the impact of Weyl geometry on cosmology. This exploration is conducted in such a manner that our findings do not contradict Standard Cosmology but rather present the latter as a specific gauge of Weyl cosmology.

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Chapter 1

Introduction

Our journey commences in 1918 with H. Weyl's pioneering introduction of an additional symmetry into Riemannian geometry, aiming to forge a comprehensive geometric model unifying electromagnetism with gravity. This innovation involved the variation of both vector orientation and length under parallel transport, diverging from the established norm of solely altering orientation in Riemannian geometry. These resultant structures, termed Weyl geometries, represent a coherent and comprehensive extension of Riemannian geometries [1].

While this novel geometry fell short of unifying electromagnetism with gravity, it nonetheless paved the way for a profound generalization of Riemannian geometry and, consequently, General Relativity as well. At the heart of General Relativity lies the principle of invariance under general coordinate transformations. By examining this theory within the framework of Weyl geometry, we arrive at a theory that exhibits invariance under local scale transformations as well.

In this thesis, we delve into the inherent suitability of Weyl geometry for investigating asymptotically locally Anti-de Sitter (AlAdS) manifolds [2–4]. These manifolds serve as the foundational building blocks of the AdS/CFT correspondence [5], which stands as one of the most promising yet unproven theories bridging gravitational theories with quantum field theories. While our focus does not center on exploring this correspondence within the confines of this thesis, in essence, it posits a profound connection: the geometric properties within a $(d+1)$-dimensional AdS manifold can be correlated with quantities of a d-dimensional conformal field theory (CFT) residing on its boundary. The precise equivalence between the two formulations implies that, in principle, comprehensive information on one facet of the duality can be attained through computations carried out on the opposite facet. This is very useful since computations in quantum field theories can be quite difficult.

We begin, in Chapter 2 by examining the method of Holographic Renormalization in the usual Riemannian geometry [6–13]. This is a method of solving difficult differencial equations, such as Einstein equations, asymptotically. To do that we start from the AlAdS metric prescribed by Fefferman and Graham in [14], and construct Einstein equations of the $(d+1)$-dimensional AlAdS manifold. Then we asymptotically expand the metric tensor of the $(d+1)$-dimensional space that appears in these equations. Solving them order by order with respect to this expansion yields very interesting results. Notably, the leading term of this expansion—viewed as the induced metric on the (conformal) boundary of the $(d+1)$-dimensional space—remains entirely unconstrained by the Einstein equations. However, given this term, we can determine all the other coefficients of the expansion up to some order. In the case where the number of our dimensions d is even, this procedure breaks when the order becomes equal to the number of dimensions. To overcome this obstacle we introduce a logarithm term in this order of the expansion. Solving Einstein equations for this order enables us to determine the coefficient of the logarithmic term, called the obstruction tensor. This method can also be applied in the presence of matter. For this purpose we consider the action of a free scalar field and solve asymptotically for this field the Klein-Gordon equation. Where we see that the results are very similar to those of the pure gravity situation.

In Chapter 3 we translate the method of Holographic Renormalization in a Weyl geometry background [15–20]. At first we examine why it is quite natural to consider Weyl transformations on the AlAdS metric. Then we redefine our connection, and thus our covariant derivative, in order for it to be covariant under those transformations. This means that we are not working with the usual Levi-Civita (LC) connection as we do in Riemannian geoemtry, but with a new one which induces different geometric quantities. We find the curvature of our space imposed by this new connection, called Weyl connection, and then construct Einstein equations. What we gain from this procedure is that now these equations are not just covariant under general coordinate transformations, but also under Weyl transformations (i.e. local scale covariant). This comes from the fact that our new curvature tensors appearing in the left hand side of Einstein equations, are now Weyl covariant. Furthermore we solve these new (Weyl-) Einstein equations again asymptotically using the method of Holographic Renormalization. The main difference from the Riemannian case is that we asymptotically expand not just the induced metric on the radial slice, but also the Weyl gauge field we introduced in the construction of the Weyl connection. The result is that now the coefficients of the expansion of the metric will not be determined solely from the leading term (i.e. the induced metric) but will also contain terms of the expansion of the Weyl gauge field.

But does this exploration of Weyl geometry merely constitute a mathematical indulgence, or does it hold relevance for our observable universe? Motivated by this inquiry, Chapter 4 embarks on an investigation starting from the metric of our Universe, namely the Friedmann–Robertson–Walker (FRW) metric, analyzed within the framework of Weyl geometry. The inherent homogeneity and isotropy of our Universe simplify our analysis, as they impose constraints on the form of the Weyl parameter. We derive the Friedmann equations, now incorporating this Weyl term. Notably, the symmetry of the Einstein tensor remains intact.

Furthermore, we derive the geodesic equations induced by Weyl connection. Given our focus on Cosmology, it would be remiss not to address the concept of Inflation. Recognized as the most elegant and effective solution to the horizon problem to date, we are keenly interested in the conditions facilitating this process within our Weyl framework. While analogous work has been conducted from a modified gravity perspective [21, 22], our approach does not entertain modifications to the Einstein-Hilbert action. Consequently, the sole modification to the corresponding Einstein equations arises from the Weyl connection.

Chapter 2

AlAdS manifolds

2.1 Asymptotically locally Anti-de Sitter manifolds

Anti-de Sitter (AdS) spacetime is a solution to Einstein's field equations for an empty universe with a negative cosmological constant. It was first introduced by the Dutch mathematician and physicist Willem de Sitter in 1917 and it describes a maximally symmetric Lorentzian manifold with constant negative scalar curvature.

From a geometric point of view, in the absence of matter or energy, the cosmological constant models the intrinsic curvature of spacetime. And from a physics perspective this corresponds to the vacuum having energy density and pressure.

We can embed $(d+1)$-dimensional AdS space in $(d+2)$-dimensional Minkowski space having two time directions and d spacial directions

$$ds^2 = -(dx^0)^2 - (dx^1)^2 + \sum_{i=1}^{d}(dx^i)^2 \tag{2.1}$$

as the hyperboloid

$$(x^0)^2 + (x^1)^2 - \sum_{i=1}^{d}(x^i)^2 = L^2 \tag{2.2}$$

where L is a positive constant, called AdS radius.

Introducing the parametrization

$$\begin{aligned} x^0 &= L \sin \frac{t}{L} \cosh \frac{r}{L} \\ x^1 &= L \cos \frac{t}{L} \cosh \frac{r}{L} \\ \vec{x} &= L \sinh \frac{r}{L} \hat{n} \end{aligned} \tag{2.3}$$

where n is the unit normal in d-dimensions and $t \in [0, 2\pi L]$, we get the line element

$$ds^2 = L^2 \left(-\cosh^2 \frac{r}{L} dt^2 + dr^2 + \sinh^2 \frac{r}{L} d\Omega_{d-1}^2 \right) \tag{2.4}$$

This is mathematically correct but if we look closely we see that periodicity of time leads to (physically) not desirable results like closed time-like curves. To avoid situations like this, we extend the time coordinate so $t \in \mathbb{R}$, which is the so-called universal covering.

As AdS is a maximally symmetric solution of Einstein's equations, it has the following properties for the Riemann tensor, the Ricci tensor and the Ricci scalar respectively

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad R_{\mu\nu} = -\frac{d}{L^2}g_{\mu\nu}, \quad R = -\frac{d(d+1)}{L^2} \tag{2.5}$$

which can be derived by brute-force computation starting from the metric 2.4. And thus AdS solves the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.6}$$

with cosmological constant $\Lambda = -\frac{d(d-1)}{2L^2}$.

So one can interpret AdS space as the analog of Minkowski flat space in the case of a negative cosmological constant, or in other words a generalization of the flat space to $\Lambda < 0$.

AdS is best known for its role in the AdS/CFT correspondence that was first proposed by Juan Maldacena in late 1997 [13, 23–26] which suggests an equivalence between gravity in AdS spacetime and a conformal field theory (CFT) living on the boundary of that spacetime. And how computing geometric quantities like tensors in AdS can result in expectation values of operators in the CFT, something that is quite difficult to do in general. For example equations of motion in the bulk correspond to the beta functions in QFT.

Now we are ready to discuss asymptotically locally AdS (AlAdS) manifolds which are a type of spacetime that is asymptotically similar to AdS space near infinity but may have local deviations from the exact AdS geometry in some regions. It is a generalization of asymptotically AdS spaces that incorporates localized effects or sources within the spacetime.

We define a non compact (pseudo) Riemannian manifold to be an asymptotically locally AdS (AlAdS) manifold iff it is a conformally compact Einstein manifold [6].

Let M be a (d+1)-dimensional closed manifold (often called the bulk) with boundary ∂M and (pseudo) Riemannian metric g on M^o (the interior of M). M^o is said to be conformally compact Einstein manifold iff there exists a smooth non-negative function ρ on M such that

- $\rho(\partial M) = 0$
- $d\rho(\partial M) \neq 0$
- $\tilde{g} = \rho^2 g$ extends smoothly to a non-degenerate metric on M (i.e. g has a second order pole at the boundary)

If it exists, the defining function ρ is not unique and hence the conformal compactification is not unique.

Let's see an example

Performing the change of variables $\sinh \frac{r}{L} = \tan \frac{\theta}{L}$ in 2.4 we get the metric

$$ds^2 = \frac{L^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right) \tag{2.7}$$

which is a certain form of the AdS metric in global coordinates with Lorentzian signature, $d\Omega_{d-1}^2$ being the metric of the (d-1)-sphere and $0 \leq \theta < \pi/2$. Apparently it has a second order pole at $\theta = \pi/2$, this is where the boundary is located. A nice to visualize it is imagine sending a light ray from the center of the "cylinder" parametrized by 2.7 towards the boundary $\theta = \pi/2$. From the metric 2.7 we see that it reaches the boundary in finite proper time. So we can indeed interpret it as a boundary. And to be more precise as a conformal boundary, in the sence that the metric diverges for $\theta \to \pi/2$ but if we guess a defining function like in the previous definition then the remaining metric has indeed a boundary there. Now we will guess a defining function ρ, as we discussed above, that could erase this problem of divergence.

The obvious choice is to take $\rho = \cos\theta$ and it is also straightforward to see that this choice satisfies all three conditions of the above definition. So this $\rho(\theta)$ is indeed our defining function. In particular, the metric $\tilde{g} \equiv \frac{\rho^2}{L^2} g$ where g is the metric 2.7 is that of the flat cylinder $\mathbb{R} \times S^d$ and it is called Einstein Static Universe (ESU). We call the metric

$$\lim_{\rho \to 0} \rho^2 g \tag{2.8}$$

as boundary metric. But is this defining function unique?

The answer is no. For instance one could also take

$$\rho' = e^w \cos\theta \tag{2.9}$$

where w is an arbitrary smooth function on M with no zeros or poles on the boundary. And we see it also does the job. The difference now is that the induced metric on ∂M is just conformal to $\mathbb{R} \times S^d$. The choice of a particular rescaling factor 2.9 defines a representative of the corresponding conformal class of boundary metrics. Since now the limit gives

$$\lim_{\rho \to 0} \rho'^2 g = e^{2w} \lim_{\rho \to 0} \rho^2 g \tag{2.10}$$

This representative is often called just boundary metric for simplicity but what we mean by this is that a particular choice has been made, sometimes this is called choice of conformal frame. Therefore the AdS metric yields a conformal structure at the boundary, i.e. a metric up to conformal transformations (conformal class of metrics).

A straightforward computation [6] shows that near the boundary we have

$$R_{\mu\nu\rho\sigma} = |d\rho|^2_{\tilde{g}} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) + \mathcal{O}(\rho^{-3}) \tag{2.11}$$

where we defined the quantity[1]

$$|d\rho|^2_{\tilde{g}} \equiv \tilde{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho \tag{2.12}$$

Combining this with Einstein equations 2.6 we get that on the boundary holds

$$|d\rho|^2_{\tilde{g}} = \frac{1}{L^2} \tag{2.13}$$

[1] This quantity extends smoothly on ∂M.

and so the Riemann tensor of an AlAdS space on the boundary looks like that of an AdS space. It is a known theorem that a choice for defining function like 2.13 can always be made [27].

In order to see that AlAdS manifolds indeed approach AdS we should first choose a suitable set of coordinates and then try a power-series solution to the Einstein equations, by the so-called Fefferman-Graham expansion [14]. We begin by considering the Fefferman-Graham coordinates on a finite neighborhood U of ∂M. What we mean by it is that we choose the arbitrary function w in 2.9 so that the defining function r (which is not unique) obeys 2.13 on U, where of course $\tilde{g} = r^2 g$. This choice can always be made according to a known theorem [28].

Now we consider the defining function to be a coordinate near the boundary and choose the other d-coordinates x^i to be orthogonal (as for $\tilde{g}$) to r in U. So, according to [14] the metric 2.7 takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{r^2}(dr^2 + \gamma_{ij}dx^i dx^j) \tag{2.14}$$

where $\gamma_{ij} = \gamma_{ij}(x, r)$.

In these coordinates the conformal boundary is located at $r = 0$.

From now on we set $L = 1$ for simplicity and if someone wants to recover it at some point it is easy by just applying simple dimensional analysis. So we can interpret this space as the vacuum solution for that dynamical system where the Riemann and Ricci tensors are given respectively by:

$$R_{\mu\nu\rho\sigma} = (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}), \qquad R_{\mu\nu} = -d g_{\mu\nu} \tag{2.15}$$

Its Weyl tensor vanishes (therefore is conformally flat) and thus we can see that AdS is the analogous of Minkowski space if we choose not $\Lambda = 0$ (as in Minkowski) but $\Lambda = -d(d-1)/2$.

What we want now is expand asymptotically our induced metric $r^{-2}\gamma$ and if we can tell something interesting about the coefficients and the terms of the expansion. Such asymptotic analysis was done in detail in [14] for pure gravity by Fefferman and Graham. But their analysis extends straightforwardly to include matter with soft enough behavior at infinity (see [7, 8, 29, 30]).

By construction, γ_{ij} can be extended to ∂M (it has a smooth limit at $r \to 0$) and thus we can expand as follows

$$\gamma_{ij}(x,r) = \gamma_{ij}^{(0)}(x) + r\gamma_{ij}^{(1)}(x) + r^2\gamma_{ij}^{(2)}(x) + ... \tag{2.16}$$

where the upper-index indicates the number of derivatives involved in that term.

Since Einstein equations are second order PDEs, if we plug in them the expansion 2.16 we will end up with algebraic equations for $\gamma^{(n)}$. By brute-force computation one can show that in pure gravity situation all coefficients multiplying odd poewrs of r vanish up to order of d. In the case where d is odd these equations admit solutions for all $\gamma^{(n)}$. After we specify $\gamma^{(0)}$ we can uniquely determine $\gamma^{(n)}$ for all $n < d$. When d is even though, the situation is a bit more complicated [19]. Without further ado let's dive into this procedure.

2.2 Holographic Renormalization

We set $\rho = r^2$ in 2.7 for simplicity in our calculations and so we get [2] (in so-called Gaussian coordinates)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4\rho^2}d\rho^2 + \frac{1}{\rho}\gamma_{ij}dx^i dx^j \tag{2.17}$$

Our goal is to derive the Einstein equations that stem from this metric and subsequently solve them asymptotically. The significance lies in the transformation of potentially complex differential equations into more manageable algebraic ones. To initiate this process, we calculate the Christoffel symbols as defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{2.18}$$

Straightforward computation shows that the non-vanishing Christoffel symbols are[3]

$$\begin{aligned}
\Gamma^\rho_{\rho\rho} &= -\rho^{-1} \\
\Gamma^\rho_{ij} &= 2\gamma_{ij} - 2\rho\gamma'_{ij} \\
\Gamma^i_{\rho j} &= \frac{1}{2}\gamma^{ik}\gamma'_{kj} - \frac{1}{2\rho}\delta^i{}_j \\
\Gamma^i_{jk} &= \Gamma^i_{jk}[\gamma]
\end{aligned} \tag{2.19}$$

where

[2] Note that Greek indices go from 1 to $d+1$ and latin ones from 1 to d.

[3] We denoted by prime the derivative with respect to ρ, i.e. $\gamma'_{ij} = \partial_\rho\gamma_{ij}$

$$\Gamma^i_{jk}[\gamma] = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right) \tag{2.20}$$

What we want to do now is express the geometric quantities of the big $(d+1)$-dimensional metric regarding the corresponding ones for small d-dimensional metric.

We define the Riemann tensor

$$R^\lambda{}_{\mu\sigma\nu} \equiv \partial_\sigma\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\sigma} + \Gamma^\kappa_{\mu\nu}\Gamma^\lambda_{\kappa\sigma} - \Gamma^\kappa_{\mu\sigma}\Gamma^\lambda_{\kappa\nu} \tag{2.21}$$

In order to simplify things we use the more compact notation $\partial_{[\mu}\Gamma^\kappa_{\nu]\lambda} \equiv \partial_\mu\Gamma^\kappa_{\nu\lambda} - \partial_\nu\Gamma^\kappa_{\mu\lambda}$, so the Riemann tensor can be written as $R^\lambda{}_{\mu\sigma\nu} = \partial_{[\sigma}\Gamma^\lambda_{\nu]\mu} + \Gamma^\kappa_{\mu[\nu}\Gamma^\lambda_{\sigma]\kappa}$.

Therefore the Ricci tensor will be

$$R_{\mu\nu} \equiv \partial_{[\lambda}\Gamma^\lambda_{\nu]\mu} + \Gamma^\kappa_{\mu[\nu}\Gamma^\lambda_{\lambda]\kappa} \tag{2.22}$$

Calculating its ij-component one finds

$$R_{ij} = R_{ij}[\gamma] + \Gamma^{\rho}_{ij}{}' + \Gamma^\rho_{ij}\Gamma^\rho_{\rho\rho} + \Gamma^\rho_{ij}\Gamma^l_{\rho l} - \Gamma^\rho_{il}\Gamma^l_{\rho j} - \Gamma^k_{i\rho}\Gamma^\rho_{kj} \tag{2.23}$$

where $R_{ij}[\gamma]$ is the Ricci tensor of the small metric defined by[4]

$$R_{ij}[\gamma] \equiv \partial_{[k}\Gamma^k_{j]i} + \Gamma^l_{i[j}\Gamma^k_{k]l} \tag{2.24}$$

Combining this with the ij-component of the Einstein equations 2.6 yields

$$R_{ij}[\gamma] - 2\rho\gamma''_{ij} + 2\rho\gamma'_{ik}\gamma^{kl}\gamma'_{lj} - \rho\gamma^{kl}\gamma'_{kl}\gamma'_{ij} + (d-2)\gamma'_{ij} + \gamma_{ij}\gamma^{kl}\gamma'_{kl} = 0 \tag{2.25}$$

Similarly the $i\rho$-component gives [5]

$$\nabla_i\left(\gamma^{kl}\gamma'_{kl}\right) - \nabla^k\gamma'_{ki} = 0 \tag{2.26}$$

And finally the $\rho\rho$-component

[4]In general it is not necessarily true that $\Gamma^i_{jk}[g] = \Gamma^i_{jk}[\gamma]$. This is a nice property of this perticular metric. In this definition we mean the latter, but since both are the same we use the same notation Γ^i_{jk}.

[5]The covariant derivative ∇_i is constructed from the metric γ.

$$\gamma^{kl}\gamma''_{kl} - \frac{1}{2}\gamma^{ij}\gamma'_{jk}\gamma^{kl}\gamma'_{li} = 0 \tag{2.27}$$

So, in total, the equations of motion read [8, 18]

$$\begin{aligned} R_{ij}[\gamma] - 2\rho\gamma''_{ij} + 2\rho\gamma'_{ik}\gamma^{kl}\gamma'_{lj} - \rho\gamma^{kl}\gamma'_{kl}\gamma'_{ij} + (d-2)\gamma'_{ij} + \gamma_{ij}\gamma^{kl}\gamma'_{kl} &= 0 \\ \nabla_i\left(\gamma^{kl}\gamma'_{kl}\right) - \nabla^k\gamma'_{ki} &= 0 \\ \gamma^{kl}\gamma''_{kl} - \frac{1}{2}\gamma^{ij}\gamma'_{jk}\gamma^{kl}\gamma'_{li} &= 0 \end{aligned} \tag{2.28}$$

To demonstrate how Holographic Renormalization works we expand γ_{ij} as

$$\gamma_{ij}(x,\rho) = \gamma^{(0)}_{ij}(x) + \rho\gamma^{(2)}_{ij}(x) + ... + \rho^{d/2}\gamma^{(d)}_{ij}(x) + \rho^{d/2}h^{(d)}_{ij}(x)\ln\rho + \mathcal{O}(\rho^{\frac{d}{2}+1}) \tag{2.29}$$

where $h^{(d)}$ is called the obstruction tensor and it appears only when d is even. This follows from solving Einstein equations order by order in ρ. Let us examine this process.

First we expand the inverse metric

$$\gamma^{ij}(x,\rho) = \gamma^{ij}_{(0)}(x) - \rho\left(\gamma^{-1}_{(0)}(x)\gamma^{(2)}(x)\gamma^{-1}_{(0)}(x)\right)^{ij} + ... \tag{2.30}$$

and then insert the two expansions into the first equation of 2.28, which gives

$$\begin{aligned} &R_{ij} - 2\rho\left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)'' \\ &+ 2\rho\left(\gamma^{(0)}_{ik} + \rho\gamma^{(2)}_{ik} + ...\right)'\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(0)}_{lj} + \rho\gamma^{(2)}_{lj} + ...\right)' \\ &- \rho\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(0)}_{kl} + \rho\gamma^{(2)}_{kl} + ...\right)'\left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)' \\ &+ (d-2)\left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)' \\ &+ \left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(0)}_{kl} + \rho\gamma^{(2)}_{kl} + ...\right)' = 0 \end{aligned} \tag{2.31}$$

Performing the differentiation with respect to ρ yields

$$
\begin{aligned}
&R_{ij} - 2\rho\left(2\gamma^{(4)} + 6\rho\gamma^{(6)} + ...\right) \\
&+ 2\rho\left(\gamma^{(2)}_{ik} + 2\rho\gamma^{(4)}_{ik} + ...\right)'\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(2)}_{lj} + 2\rho\gamma^{(4)}_{lj} + ...\right) \\
&- \rho\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(0)}_{kl} + \rho\gamma^{(2)}_{kl} + ...\right)'\left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)' \\
&+ (d-2)\left(\gamma^{(2)}_{ij} + 2\rho\gamma^{(4)}_{ij} + ...\right) \\
&+ \left(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...\right)\left(\gamma^{kl}_{(0)} - \rho\left(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)}\right)^{kl} + ...\right)\left(\gamma^{(2)}_{kl} + 2\rho\gamma^{(4)}_{kl} + ...\right) = 0
\end{aligned} \tag{2.32}
$$

At the order of ρ^0 we get

$$
R_{ij}[\gamma^{(0)}] + (d-2)\gamma^{(2)}_{ij} + \gamma^{(2)}_{kl}\gamma^{(0)}{}^{kl}\gamma^{(0)}_{ij} = 0 \tag{2.33}
$$

Where we defined

$$
\Gamma^{i}{}^{(0)}_{jk} \equiv \frac{1}{2}\gamma^{im}_{(0)}\left(\partial_j\gamma^{(0)}_{km} + \partial_k\gamma^{(0)}_{jm} - \partial_m\gamma^{(0)}_{jk}\right) \tag{2.34}
$$

and the Ricci tensor of the induced metric

$$
R_{ij}[\gamma^{(0)}] \equiv \partial_m\Gamma^{m}{}^{(0)}_{ij} - \partial_j\Gamma^{m}{}^{(0)}_{im} + \Gamma^{m}{}^{(0)}_{ij}\Gamma^{n}{}^{(0)}_{mn} - \Gamma^{m}{}^{(0)}_{in}\Gamma^{n}{}^{(0)}_{mj} \tag{2.35}
$$

So, if we take the trace of 2.33 we find the Ricci scalar of the induced metric

$$
R[\gamma^{(0)}] = -2(d-1)\gamma^{(2)}_{ij}\gamma^{(0)}{}^{ij} \tag{2.36}
$$

And thus

$$
\boxed{\gamma^{(2)}_{ij} = -\frac{1}{d-2}\left(R_{ij}[\gamma^{(0)}] - \frac{R[\gamma^{(0)}]}{2(d-1)}\gamma^{(0)}_{ij}\right)} \tag{2.37}
$$

The right hand side, without the minus sign, is also known as the Schouten tensor.

Similarly from the ρ^1-order we get the $\gamma^{(4)}_{ij}$ coefficient in terms only of $\gamma^{(0)}_{ij}$, from the ρ^2-order the $\gamma^{(6)}_{ij}$ and so on. If we continue this process we can determine all coefficients[6] $\gamma^{(n)}_{ij}$ in the expansion of the metric up to the order of $n < d$ and the obstruction tensor $h^{(d)}_{ij}$ in terms of $\gamma^{(0)}_{ij}$ using the first equation in 2.28. The first coefficient $\gamma^{(0)}_{ij}$ ends up to be completely arbitrary and is considered to be the induced metric on the boundary ($\rho = 0$) since $\rho g_{ij} = \gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ... \xrightarrow{\rho\to 0} \gamma^{(0)}_{ij}(x)$ (more precisely a representative of the conformal class of metrics). As for the term $\gamma^{(d)}_{ij}$ one we can determine, again by solving Einstein equations (in particular the last two equations in 2.28), its trace and covariant divergence[7] [8].

2.2.1 Pure gravity in 5d

To see this procedure more clearly let us suppose that $d = 4$. We expect to determine $\gamma^{(0)}_{ij}$ and $\gamma^{(2)}_{ij}$ from the first equation of 2.28, the divergence and trace of $\gamma^{(4)}_{ij}$ from the last two equations and the obstruction tensor $h^{(4)}_{ij}$ again from the first equation.

We already proved the expression for $\gamma^{(2)}_{ij}$ in 2.37 so in our case we just have to put $d = 4$ in the expression. For the rest of the terms we first expand the Christoffel Symbols of the small metric up to the order of ρ.

$$\begin{aligned}\Gamma^k_{ij} =& \frac{1}{2}\gamma^{km}\left(\partial_i\gamma_{mj} + \partial_j\gamma_{mi} - \partial_m\gamma_{ij}\right)\\ =& \frac{1}{2}\left(\gamma^{km}_{(0)} - \rho(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)})^{km} + ...\right)\\ &\times\left(\partial_i(\gamma^{(0)}_{mj} + \rho\gamma^{(2)}_{mj} + ...) + \partial_j(\gamma^{(0)}_{mi} + \rho\gamma^{(2)}_{mi} + ...) - \partial_m(\gamma^{(0)}_{ij} + \rho\gamma^{(2)}_{ij} + ...)\right)\end{aligned} \tag{2.38}$$

So in the order of $\mathcal{O}(\rho)$ we get

$$\begin{aligned}\Gamma^{(2)k}_{ij} =& \frac{1}{2}\gamma^{km}_{(0)}\left(\partial_i\gamma^{(2)}_{mj} + \partial_j\gamma^{(2)}_{mi} - \partial_m\gamma^{(2)}_{ij}\right)\\ &- \frac{1}{2}(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)})^{km}\left(\partial_i\gamma^{(0)}_{mj} + \partial_j\gamma^{(0)}_{mi} - \partial_m\gamma^{(0)}_{ij}\right)\end{aligned} \tag{2.39}$$

where we denoted the coefficients $\Gamma^k_{ij} = \Gamma^{(0)k}_{ij} + \rho\Gamma^{(2)k}_{ij} + ...$

Furthermore

[6] In general the corresponding expressions for $\gamma^{(n)}_{ij}$ are singular for $n = d$.
[7] In odd dimensions they turn out to be zero.

$$\begin{aligned}
\nabla_i^{(0)}\gamma_{mj}^{(2)} &= \partial_i\gamma_{mj}^{(2)} - \Gamma_{im}^{(0)n}\gamma_{nj}^{(2)} - \Gamma_{ij}^{(0)n}\gamma_{mn}^{(2)} \\
\nabla_j^{(0)}\gamma_{mi}^{(2)} &= \partial_j\gamma_{mi}^{(2)} - \Gamma_{jm}^{(0)n}\gamma_{ni}^{(2)} - \Gamma_{ij}^{(0)n}\gamma_{mn}^{(2)} \\
\nabla_m^{(0)}\gamma_{ij}^{(2)} &= \partial_m\gamma_{ij}^{(2)} - \Gamma_{im}^{(0)n}\gamma_{nj}^{(2)} - \Gamma_{mj}^{(0)n}\gamma_{in}^{(2)}
\end{aligned} \tag{2.40}$$

So adding the first two rows and subtracking the third we get

$$\partial_i\gamma_{mj}^{(2)} + \partial_j\gamma_{mi}^{(2)} - \partial_m\gamma_{ij}^{(2)} = \nabla_i^{(0)}\gamma_{mj}^{(2)} + \nabla_j^{(0)}\gamma_{mi}^{(2)} - \nabla_m^{(0)}\gamma_{ij}^{(2)} + 2\Gamma_{ij}^{(0)n}\gamma_{mn}^{(2)} \tag{2.41}$$

Substituting into 2.39 we get

$$\begin{aligned}
\Gamma_{ij}^{(2)k} =& \frac{1}{2}\gamma_{(0)}^{km}\left(\nabla_i^{(0)}\gamma_{mj}^{(2)} + \nabla_j^{(0)}\gamma_{mi}^{(2)} - \nabla_m^{(0)}\gamma_{ij}^{(2)}\right) + \gamma_{(0)}^{km}\gamma_{mn}^{(2)}\Gamma_{ij}^{(0)n} \\
& - \frac{1}{2}(\gamma_{(0)}^{-1}\gamma^{(2)}\gamma_{(0)}^{-1})^{km}\left(\partial_i\gamma_{mj}^{(0)} + \partial_j\gamma_{mi}^{(0)} - \partial_m\gamma_{ij}^{(0)}\right)
\end{aligned} \tag{2.42}$$

where the last two terms cancel its other out and thus yields

$$\Gamma_{ij}^{(2)k} = \frac{1}{2}\gamma_{(0)}^{km}\left(\nabla_i^{(0)}\gamma_{mj}^{(2)} + \nabla_j^{(0)}\gamma_{mi}^{(2)} - \nabla_m^{(0)}\gamma_{ij}^{(2)}\right) \tag{2.43}$$

Now that we've got the expression for the Christoffel Symbols we can compute the ρ-order of the Ricci tensor which also suppose expands as

$$R_{ij} = R_{ij}^{(0)} + \rho R_{ij}^{(2)} + ... \tag{2.44}$$

So from its definition we get

$$\begin{aligned}
R_{ij} =& \partial_m\Gamma_{ij}^m - \partial_j\Gamma_{im}^m + \Gamma_{ij}^m\Gamma_{mn}^n - \Gamma_{in}^m\Gamma_{mj}^n \Rightarrow \\
R_{ij}^{(2)} =& \partial_m\Gamma_{ij}^{(2)m} - \partial_j\Gamma_{im}^{(2)m} + \Gamma_{ij}^{(0)m}\Gamma_{mn}^{(2)n} - \Gamma_{in}^{(0)m}\Gamma_{mj}^{(2)n} \\
& + \Gamma_{ij}^{(2)m}\Gamma_{mn}^{(0)n} - \Gamma_{in}^{(2)m}\Gamma_{mj}^{(0)n} \\
=& \nabla_m^{(0)}\Gamma_{ij}^{(2)m} \Rightarrow \\
R_{ij} =& R_{ij}^{(0)} + \rho\nabla_m^{(0)}\Gamma_{ij}^{(2)m} + ...
\end{aligned} \tag{2.45}$$

Now it's time to return to the Einstein equations 2.28. In these equations we substitute the expression we found for the Ricci tensor and for the metric tensor

$$\begin{aligned}
\gamma_{ij}(x,\rho) =& \gamma^{(0)}_{ij}(x) + \rho\gamma^{(2)}_{ij}(x) + \rho^2(\gamma^{(4)}_{ij}(x) + h^{(4)}_{ij}(x)\ln\rho) + ... \Rightarrow \\
\gamma'_{ij}(x,\rho) =& \gamma^{(2)}_{ij} + 2\rho\gamma^{(4)}_{ij} + \rho h^{(4)}_{ij} + 2\rho\ln\rho h^{(4)}_{ij} + ... \Rightarrow \\
\gamma''_{ij}(x,\rho) =& 2\gamma^{(4)}_{ij} + 3h^{(4)}_{ij} + 2h^{(4)}_{ij}\ln\rho + ...
\end{aligned} \tag{2.46}$$

Substituting into the first equation of 2.28 and taking the order $\mathcal{O}(\rho)$ gives

$$\begin{aligned}
0 =& \nabla^{(0)}_m\Gamma^{(2)m}_{ij} - 2(2\gamma^{(4)}_{ij} + 3h^{(4)}_{ij}) + 2\gamma^{(2)}_{ik}\gamma^{kl}_{(0)}\gamma^{(2)}_{lj} - \gamma^{kl}_{(0)}\gamma^{(2)}_{kl}\gamma^{(2)}_{ij} \\
&+ (d-2)(2\gamma^{(4)}_{ij} + h^{(4)}_{ij}) + \gamma^{(0)}_{ij}\gamma^{kl}_{(0)}(2\gamma^{(4)}_{kl} + h^{(4)}_{kl}) \\
&- \gamma^{(0)}_{ij}(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)})^{kl}\gamma^{(2)}_{kl} + \gamma^{(2)}_{ij}\gamma^{kl}_{(0)}\gamma^{(2)}_{kl}
\end{aligned} \tag{2.47}$$

where, from the second term in the first row and the first term in the second row, we can see that the coefficient of $\gamma^{(4)}_{ij}$ is proportional to $d-4$. But since $d=4$ there is no $\gamma^{(4)}_{ij}$ in the above expression and this is why we cannot compute it as we did for $\gamma^{(2)}_{ij}$.

We see that the above equation can be written as

$$\begin{aligned}
0 =& \nabla^{(0)}_m\Gamma^{(2)m}_{ij} - 4h^{(4)}_{ij} + 2(\gamma^{(2)}\gamma^{-1}_{(0)}\gamma^{(2)})_{ij} - \cancel{Tr(\gamma^{(2)})\gamma^{(2)}_{ij}} \\
&+ 2\gamma^{(0)}_{ij}Tr(\gamma^{(4)}) + \gamma^{(0)}_{ij}Tr(h^{(4)}) - \gamma^{(0)}_{ij}Tr(\gamma^2_{(2)}) + \cancel{Tr(\gamma^{(2)})\gamma^{(2)}_{ij}}
\end{aligned} \tag{2.48}$$

To find the trace of $\gamma^{(4)}$ we take the trace of this equation, which gives

$$0 = \cancelto{0}{\nabla^{(0)}_m\Gamma^{(2)m}_{ij}\gamma^{(0)}_{ij}} + 8Tr(\gamma^{(4)}) - 2Tr(\gamma^2_{(2)}) \Rightarrow Tr(\gamma^{(4)}) = \frac{1}{4}Tr(\gamma^2_{(2)}) \tag{2.49}$$

Notice that we couldn't find the trace of the obstruction tensor from the above equation since it vanished from the expression. In order to find we must take the ρ^0-order third equation of 2.28. If we do so we deduce that it is zero.

$$
\begin{aligned}
\gamma^{kl}_{(0)}(2\gamma^{(4)}_{kl} + 3h^{(4)}_{kl}) =& \frac{1}{2}\gamma^{ij}_{(0)}\gamma^{(2)}_{jk}\gamma^{kl}_{(0)}\gamma^{(2)}_{li} \Rightarrow \\
2Tr(\gamma^{(4)}) + 3Tr(h^{(4)}) =& \frac{1}{2}Tr(\gamma^2_{(2)}) \Rightarrow \\
Tr(\gamma^{(4)}) + \frac{3}{2}Tr(h^{(4)}) =& \frac{1}{4}Tr(\gamma^2_{(2)}) \Rightarrow \\
Tr(h^{(4)}) = 0 &
\end{aligned}
\tag{2.50}
$$

Replacing this into the expression 2.47 and solving for the obstruction tensor we find

$$
\begin{aligned}
0 =& \nabla^{(0)}_m \Gamma^{(2)m}_{ij} - 4h^{(4)}_{ij} + 2(\gamma^{(2)}\gamma^{-1}_{(0)}\gamma^{(2)})_{ij} - \cancel{Tr(\gamma^{(2)})\gamma^{(2)}_{ij}} \\
&+ \frac{1}{2}\gamma^{(0)}_{ij}Tr(\gamma^2_{(2)}) + \cancelto{0}{\gamma^{(0)}Tr(h^{(4)})} - \gamma^{(0)}_{ij}(\gamma^{-1}_{(0)}\gamma^{(2)}\gamma^{-1}_{(0)})^{kl}\gamma^{(2)}_{kl} + \cancel{Tr(\gamma^{(2)})\gamma^{(2)}_{ij}}
\end{aligned}
\tag{2.51}
$$

Therefore[8] [8]

$$
\boxed{
\begin{aligned}
h^{(4)}_{ij} =& \frac{1}{2}(\gamma_{(2)})^2_{ij} - \frac{1}{8}\gamma^{(0)}_{ij}Tr(\gamma^2_{(2)}) \\
&+ \frac{1}{8}\left(\nabla^k_{(0)}\nabla^{(0)}_i\gamma^{(2)}_{jk} + \nabla^k_{(0)}\nabla^{(0)}_j\gamma^{(2)}_{ik} - \nabla^2_{(0)}\gamma^{(2)}_{ij} - \nabla^{(0)}_i\nabla^{(0)}_j Tr(\gamma^{(2)})\right)
\end{aligned}
}
\tag{2.52}
$$

is the obstruction tensor in 4 dimensions.

Now let's see what the second equation in 2.28 has to tell us. We expect to find the the divergence of $\gamma^{(4)}$. To do so we first take the ρ^0-order of this equation and find the divergence of $\gamma^{(2)}$. Indeed at $\mathcal{O}(\rho^0)$ we get

$$
\begin{aligned}
\nabla^{(0)k}\gamma^{(2)}_{ki} &= \nabla^{(0)}_i(\gamma^{kl}_{(0)}\gamma^{(2)}_{kl}) \Rightarrow \\
&= \nabla^{(0)k}(\gamma^{(0)}_{ki}\gamma^{mn}_{(0)}\gamma^{(2)}_{mn}) \Rightarrow \\
\nabla^{(0)k}\gamma^{(2)}_{ki} &= \nabla^{(0)k}\left(\gamma^{(0)}_{ki}Tr(\gamma^{(2)})\right)
\end{aligned}
\tag{2.53}
$$

Finally, in order to find the divergence of $\gamma^{(4)}$ we first take the divergence of the expression 2.48. This results in $h^{(4)}$ being divergence-free. Using this we take the ρ-order of the second equation of motion and find [8]

[8] Where we denoted $(\gamma^{(2)}\gamma^{-1}_{(0)}\gamma^{(2)})_{ij} \equiv (\gamma^2_{(2)})_{ij}$

$$\nabla^{(0)i}\gamma^{(4)}_{ij} = \nabla^{(0)i}\left(-\frac{1}{8}\gamma^{(0)}_{ij}(Tr(\gamma^2_{(2)}) - Tr^2(\gamma^{(2)}) + \frac{1}{2}\gamma^2_{(2)ij} - \frac{1}{4}\gamma^{(2)}_{ij}Tr(\gamma^{(2)})\right) \tag{2.54}$$

To summarize, we saw that by solving asymptotically Einstein equations in $d = 4$ dimensions we were able to determine the subleading term $\gamma^{(2)}$ of the metric expansion in terms of the induced metric $\gamma^{(0)}$. For the latter there was no constraint added by the equations of motion and thus it is completely arbitrary in pure gravity. Furthermore we saw that the number of dimensions didn't allow us to determine the coefficient $\gamma^{(4)}$ but we found its divergence and trace from the last two equations of motion. The introduction, however, of the logarithmic term in the expansion of the metric gave us the opportunity to determine the exact form of the obstruction tensor, since due to this term the coefficient of $h^{(4)}$ was not zero as the one of $\gamma^{(4)}$ and we were thus able to solve for it.

2.2.2 Massive scalar in $(d+1)$-dimensions

If we are interested in the case of matter, these results can be extended. In the presence of matter we suppose we have a free massive scalar field coupled to gravity. Therefore its action will have the form [6]

$$S = \frac{1}{2}\int d^dx\sqrt{g}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi\right). \tag{2.55}$$

where we consider it to be in the Euclidean version of AdS spacetime (hyperbolic space). What we want to do is to see how can we perform the same procedure we did earlier but now for the massive scalar. The equations of motion now are given by the Klein-Gordon equation:

$$-\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi\right) + m^2\Phi = 0. \tag{2.56}$$

We are looking for solutions in the form $\Phi(x,\rho) = \rho^{(d-\Delta)/2}\phi(x,\rho)$.

Plugging this into 2.56 gives[9]

$$\left(m^2 - \Delta(\Delta - d)\right)\phi - \rho\left(\partial_\mu\partial^\mu\phi + 2\phi'(d - 2\Delta + 2) + 4\rho\phi''\right) = 0 \tag{2.57}$$

[9] Again prime denotes derivative with respect to ρ.

Taking the limit $\rho = 0$ we see that

$$m^2 - \Delta(\Delta - d) = 0 \tag{2.58}$$

and thus

$$\partial_\mu \partial^\mu \phi + 2\phi'(d - 2\Delta + 2) + 4\rho\phi'' = 0 \tag{2.59}$$

We perform the asymptotic expansion

$$\phi(x,\rho) = \phi^{(0)}(x) + \rho\phi^{(2)}(x) + \rho^2\phi^{(4)} + ... \tag{2.60}$$

and solving order by order in 2.59 yields

$$\boxed{\phi^{(2)} = \frac{1}{2(2\Delta - d - 2)}\partial_\mu \partial^\mu \phi^{(0)}} \tag{2.61}$$

Continuing this procedure we find all coefficients in the expansion 2.60. However, this process stops when $2\Delta - d - 2 = 0$ (the analogue for $n = d$ in the even dimension case for the pure gravity situation we analyzed earlier).

$$\phi^{(2n)} = \frac{1}{2(2\Delta - d - 2n)}\partial_\mu \partial^\mu \phi^{(2n-2)} \tag{2.62}$$

And this is the very reason why we need once more to introduce a logarithmic term in order to obtain a solution. And thus the asymptotic expansion becomes

$$\phi(x,\rho) = \phi^{(0)}(x) + \rho\left(\phi^{(2)} + \psi^{(2)}\ln\rho\right) + ... \tag{2.63}$$

If we insert this into 2.57 we find

$$\psi^{(2)} = -\frac{1}{4}\partial_\mu \partial^\mu \phi^{(0)} \tag{2.64}$$

One very interesting feature in both situations is that we didn't have to solve any differential equation, as we are used to when equations of motion are needed. Instead, Einstein equations on pure gravity and Klein-Gordon in the presence of matter ended up amounted to algebraic equations. We solved them and observed two things. First the leading order equation leaves $\gamma^{(0)}$ and $\phi^{(0)}$ undetermined. Second we can determine the subleading terms in the asymptotic expansion up to some order where field equations cannot help us anymore.

In the next section we will explore how all those terms in the expansion can enjoy a particular kind of symmetry. The motivation for this symmetry comes not only from the structure of AlAdS but also from electromagnetism.

Chapter 3

Weyl Geometries

"It appears as one of the fundamental principles of nature that the equations expressing the basic laws of physics should be invariant under the widest possible group of transformations." (P.A.M.Dirac)

In this chapter we are going to explore a generalization of the Riemannian geometry, introduced by Weyl, and then see why this geometry is more suitable for the analysis of AlAdS manifolds and CFTs through Holographic Renormalization.

3.1 Weyl connection

In 1918, H. Weyl conceived a simple, but important as we will see, generalization of Riemaniann geometry [1]. His idea was to allow both the orientation and the length of vectors to vary under parallel transport instead of just the orientation as in Riemaniann geometry. Mathematically this means that instead of the familiar metricity condition

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{3.1}$$

we introduce the 1-form σ with components σ_α to a local coordinate basis $\{\partial_\alpha\}$ and we assume the more general condition

$$\nabla_\alpha g_{\mu\nu} = \sigma_\alpha g_{\mu\nu} \tag{3.2}$$

which is also known as Weyl metricity condition.

In this condition if we perform the so called Weyl transformations [1]

$$g \to \mathcal{B}^{-2}(x)g \tag{3.3}$$

we see that

$$\begin{aligned}\nabla g \to \nabla(\mathcal{B}^{-2}g) =& \mathcal{B}^{-2}\nabla g + gd(\mathcal{B}^{-2}) \\ =& \mathcal{B}^{-2}\nabla g - 2g\mathcal{B}^{-3}d\mathcal{B} \\ =& \left(\nabla g - 2gd\ln\mathcal{B}(x)\right)\mathcal{B}^{-2}(x)\end{aligned} \tag{3.4}$$

So we are motivated to introduce the 1-form a that transforms as

$$a \to a - d\ln\mathcal{B}(x) \tag{3.5}$$

and now it is easy to check that this new connection (i.e. the Weyl connection) satisfies

$$\nabla g - 2ag \to (\nabla g - 2ag)\,\mathcal{B}^{-2}(x) \tag{3.6}$$

Which means that this connection is actually covariant under Weyl transformations. Here the factor -2 appears because we supposed that the metric transforms as 3.3. When this happens we say that the tensor (in our case the metric) has Weyl weight of -2 and we denote it by w_g. So in general, a tensor A has Weyl weight w_A iff $A \to \mathcal{B}(x)^{w_A}A$.

Now, suppose we live in a AlAdS manifold. Fefferman and Graham taught us [14] that our metric can always be brought in the form [2]

$$ds^2 = \frac{dz^2}{z^2} + h_{ij}(x,z)dx^idx^j \tag{3.7}$$

which is called Fefferman-Graham (FG) gauge. The reason we are interested in Weyl transformations and their link to the AlAdS metric is that the latter induces a conformal

[1] The basic difference between Weyl geometry and conformal geometry is that in the latter we have the conformal transformations

$$g \to \omega(x)^{-2}g$$

where $\omega(x)$ is a specific function associated with a diffeomorphism that is a conformal symmetry of the theory, whereas in Weyl geometry $\mathcal{B}(x)$ is completely arbitrary.

[2] Here we use the letter h for the induced metric on the radial slice and not for the obstruction tensor as we did in Chapter 2.

structure on its boundary. By this we mean that starting from 3.7 and taking the leading term of $h_{ij}(x,z) \approx \frac{1}{z^2} h^{(0)}_{ij}(x)$ we get that on the (conformal) boundary

$$z^2 ds^2 \xrightarrow{z\to 0} h^{(0)}_{ij} dx^i dx^j \equiv ds^2_{\text{boundary}} \tag{3.8}$$

We see that this does not define an induced metric on the boundary in a unique way. It defines it up to the transformations $z \to \mathcal{B}(x)^{-1} z$ and $x^i \to x^i$. This is what we mean by a conformal class of metrics.

Unfortunately in FG gauge the form of the AlAdS metric is not preserved under Weyl diffeomorphisms.

$$z \to \mathcal{B}(x)^{-1} z, \qquad x^i \to x^i, \qquad a_i \to a_i - \partial_i \ln \mathcal{B}(x) \tag{3.9}$$

In order to preserve the form of the metric we consider instead the Weyl-Fefferman-Graham (WFG) gauge

$$ds^2 = \left(\frac{dz}{z} - a_i(x,z)dx^i\right)^2 + h_{ij}(x,z)dx^i dx^j \tag{3.10}$$

and then indeed if we perform Weyl transformations 3.9 we get

$$ds^2 = \left(\frac{dz}{z} - a_i(x,z)dx^i\right)^2 + \tilde{h}_{ij}(x,\mathcal{B}^{-1}z)dx^i dx^j \tag{3.11}$$

where $\tilde{h}_{ij}(\mathcal{B}^{-1}z, x) = h_{ij}(z,x)$ since $z \to \mathcal{B}^{-1}z$.

It is worth noticing that since, from FG theorem, every AlAdS metric can be written as 3.7 this means that a_i is a pure gauge in the bulk. In another perspective, if we set $a_i = 0$ in the WFG gauge we simply get the FG gauge. In that sense the WFG gauge is a generalization of the FG gauge.

Rewriting the metric 3.10 we get

$$ds^2 = \left(\frac{dz}{z}\right)^2 - 2a_i \frac{dz}{z} dx^i + \gamma_{ij}(x,z)dx^i dx^j \tag{3.12}$$

where $\gamma_{ij} = h_{ij} + a_i a_j$[3] and under Weyl transformations $\gamma_{ij} \to \mathcal{B}^{-2}\gamma_{ij}$.

[3]Here the metric γ_{ij} is not related to γ_{ij} in Chapter 2

In matrix form

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{z^2} & -\frac{a_i}{z} \\ -\frac{a_j}{z} & \gamma_{ij} \end{pmatrix} \quad \Rightarrow \quad g^{\mu\nu} = \begin{pmatrix} \frac{z^2}{a_i a^i} & \frac{z a_i}{a_j a^j} \\ \frac{z a_i}{a_j a^j} & \gamma^{ij} + \frac{a^i a^j}{1 - a_k a^k} \end{pmatrix} \tag{3.13}$$

Now we define the Weyl connection so it would be invariant under 3.9 and so [4]

$$\mathcal{G}^k_{ij} \equiv \Gamma^k_{ij}[\gamma] - \left(a_{(i}\delta^k{}_{j)} - a^k \gamma_{ij} \right) \tag{3.14}$$

Since there are 3 different "Christoffel symbols" we can define 3 different covariant derivatives with them and one more through the definition of the Weyl weight we gave above. Given a tensor A[5] we define

$$\begin{aligned}
&\nabla_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda, \qquad \text{for} \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu (g_{\rho\nu}) + \partial_\nu (g_{\mu\rho}) - \partial_\rho (g_{\mu\nu}) \right) \\
&\mathcal{D}_i A_j \equiv \partial_i A_j - \mathcal{G}^k_{ij} A_k = D_i A_j + \left(a_{(i} A_{j)} - a^k A_k \gamma_{ij} \right), \qquad \text{for} \quad \mathcal{G}^k_{ij} \equiv \Gamma^k_{ij}[\gamma] - \left(a_{(i}\delta^k{}_{j)} - a^k \gamma_{ij} \right) \\
&D_i A_j \equiv \partial_i A_j - \Gamma^k_{ij}[\gamma] A_k, \qquad \text{for} \quad \Gamma^k_{ij}[\gamma] = \frac{1}{2} \gamma^{km} \left(\partial_i \gamma_{mj} + \partial_j \gamma_{mi} - \partial_m \gamma_{ij} \right) \\
&\mathscr{D}_i A_j \equiv \mathcal{D}_i A_j + w_A a_i A_j
\end{aligned} \tag{3.15}$$

The last connection is constructed so it would be covariant under Weyl transformations 3.9, indeed it has the property

$$\mathscr{D}_i A \to \mathcal{B}(x)^{w_A} \mathscr{D}_i A \tag{3.16}$$

In particular, for the metric tensor one gets

$$\mathscr{D}_i \gamma_{jk} = 0 \tag{3.17}$$

[4]Here we denoted $a_{(i}\delta^k{}_{j)} \equiv a_i \delta^k{}_j + a_j \delta^k{}_i$. In other authors you will see a normalization factor of $\frac{1}{2}$. Also we raise or lower indices in a_i by γ_{ij}, i.e. $a^k = \gamma^{ik} a_i$

[5]This tensor can be of any kind

which the new metricity condition, called Weyl metricity condition. Notice that our motivation behind this construction 3.14 was to get a connection that satisfies

$$\mathcal{D}_i \gamma_{jk} = 2a_i \gamma_{jk} \tag{3.18}$$

since our initial idea was that of 3.2. The difference between the two is that $\mathscr{D}$ is Weyl-covariant (it satisfies 3.16) whereas $\mathcal{D}$ is not. An also important fact and crucial for our calculations is that the introduction of a_i happened in such a way that it didn't break the torsion-free property of our connection, since $\mathcal{G}^i_{jk}$ is still symmetric in jk.

What we want to explore now is how this Weyl connection can lead us into defining geometric quantities that are covariant under transformations 3.9. This Weyl covariance property however comes with a great cost, these geometric quantities will have less symmetries than the corresponding ones from the usual Levi-Civita connection ∇ due to the introduction of a_i.

We begin from the most important one which is not other than Riemann tensor 2.21. Since we want to construct Weyl-covariant geometric tensors we should start from the derivative that is covariant under Weyl transformations, i.e. $\mathcal{D}$. We know that for an arbitrary contravariant tensor A^i the usual Riemann tensor can be defined as the part of the commutator of covariant derivatives which is proportional to A^i

$$R^k{}_{imn} A^i \equiv [D_m, D_n] A^k \tag{3.19}$$

so now we define a Weyl-invariant Riemann tensor as

$$\mathcal{R}^k{}_{imn} A^i \equiv [\mathcal{D}_m, \mathcal{D}_n] A^k \tag{3.20}$$

After quite a bit of calculation we get the tensor in the familiar

$$\boxed{\mathcal{R}^l{}_{knm} = \partial_{[n} \mathcal{G}^l_{m]k} + \mathcal{G}^l_{i[n} \mathcal{G}^i_{m]k}} \tag{3.21}$$

and what this beautiful form of the tensor shows us is that it is in fact Weyl-invariant since $\mathcal{G}^i_{jk}$ are Weyl-invariant. But how is it related to our usual (LC) Riemann tensor?

To answer that we have no other option than to start from 3.21 (or equivalently 3.20) and by straightforward calculations show the explicit form of the tensor. So after we do this we find that the Weyl-invariant Riemann tensor in its full glory can be written as[6]

$$\mathcal{R}^l{}_{knm} = R^l{}_{knm} + \delta^l{}_k D_{[m} a_{n]} + \delta^l{}_{[n} D_{m]} a_k + \gamma_{k[m} D_{n]} a^l + (a_k a_{[m} - a^2 \gamma_{k[m}) \delta^l{}_{n]} + a^l a_{[n} \gamma_{m]k} \tag{3.22}$$

The full derivation will be found in Appendix A. From there we can see that the Weyl-Riemann tensor is actually antisymmetric in the last two indices (as the usual one) but it lacks the antisymmetry of the first two indices and the interchange symmetry of the index pairs. So the introduction of a_i did come with a cost. Now, contracting the two indices to get the Weyl-invariant Ricci tensor yields

$$\mathcal{R}_{km} = \delta^n{}_l \mathcal{R}^l{}_{knm} \tag{3.23}$$

and thus get

$$\boxed{\mathcal{R}_{km} = R_{km} + D_{[m} a_{k]} + (d-2) D_m a_k + \gamma_{km} D \cdot a + (d-2)(a_k a_m - a^2 \gamma_{km})} \tag{3.24}$$

Before going to the Ricci scalar lets take a moment and think about what can we deduce from the last two expressions. First from 3.23 we see that since the tensor we defined is actually Weyl-invariant so $\mathcal{R}_{ij}$ will be also Weyl-invariant (as expected). The second observation, coming from 3.24, is that the Weyl-invariant Ricci tensor is not symmetric. In fact we can measure exactly how it fails to be symmetric and if we do so we find

$$\mathcal{R}_{[km]} = d D_{[m} a_{k]} \equiv -d F_{km} \tag{3.25}$$

where we defined the field strength tensor to be $F_{ij} \equiv D_i a_j - D_j a_i$. So our tensor in not symmetric but it is not symmetric in a nice way. A useful observation here regarding field strength tensor is that it is also Weyl-invariant. This can be easily seen due to its form, since instead of the covariant derivative D we can use either $\mathcal{D}$ or $\mathscr{D}$ (or even the usual partial derivative ∂).

$$F_{ij} \equiv D_{[i} a_{j]} = \mathcal{D}_{i[} a_{j]} = \mathscr{D}_{i[} a_{j]} = \partial_{i[} a_{j]} \tag{3.26}$$

[6] Where $R^l{}_{knm}$ is the Riemann tensor of γ_{ij} using Levi-Civita connection.

which of course remains the same after performing 3.9.

As for the Weyl-covariant Ricci scalar, from the trace of 3.24 we get

$$\boxed{\mathcal{R} = R + 2(d-1)D \cdot a - (d-1)(d-2)a^2} \tag{3.27}$$

and since

$$\begin{aligned} &\mathcal{R} = \gamma^{km}\mathcal{R}_{km} \Rightarrow \\ &\mathcal{R} \to \mathcal{B}(x)^2\gamma^{km}\mathcal{R}_{km} = \mathcal{B}(x)^2\mathcal{R} \end{aligned} \tag{3.28}$$

it indeed transforms covariantly (in particular it is a tensor with Weyl-weight 2).

Now that we warmed up lets define our final Weyl tensors[7]. Following the same thinking as before, we start from the Weyl connection $\mathscr{D}$ and construct the Weyl-Riemann tensor as

$$\mathscr{R}^k{}_{imn}A^i \equiv [\mathscr{D}_m, \mathscr{D}_n]A^k \tag{3.29}$$

and after some calculations[8] we find

$$\boxed{\mathscr{R}^l{}_{kmn} = \mathcal{R}^l{}_{kmn} + \delta^l{}_k F_{mn}} \tag{3.30}$$

Therefore, by contracting the two indices, we get the Weyl-Ricci tensor

$$\boxed{\mathscr{R}_{ij} = \mathcal{R}_{ij} + F_{ij}} \tag{3.31}$$

and finally the Weyl-Ricci scalar

$$\boxed{\mathscr{R} = \gamma^{ij}\mathscr{R}_{ij} = \mathcal{R}} \tag{3.32}$$

where we see that $\mathscr{R}^l{}_{kmn}$ and $\mathscr{R}_{ij}$ have no Weyl-weight (i.e. they are Weyl-invariant) and $\mathscr{R}$ has weight 2.

[7]Some authors call Weyl-tensors those that are built from $\mathcal{D}$ and others those from $\mathscr{D}$. Here we choose the second terminology.

[8]Here we chose A_i to be a basis vector of the tangent space, which has Weyl-weight 1. In general this tensor depends on the weight of the vector field we choose to act on, check A. This is not a problem however since it is a well-defined tensor on each vector field. Here we define it so it acts on the basis of the tangent space.

This property of those geometric quantities is very important. Since it indicates that one can construct a Weyl-covariant (gravitational) Lagrangian from which yields a Weyl-invariant (gravitational) action. This action combined with a (not necessarily Weyl-invariant) matter action can give very interesting cosmological results as we are going to see in the next chapter [21, 22].

It also worth noticing that, although Weyl-Riemann tensor possesses less symmetries than the LC Riemann tensor, it satisfies the Bianchi identity[9]

$$\boxed{\mathscr{D}_i \mathscr{R}^l{}_{mjk} + \mathscr{D}_j \mathscr{R}^l{}_{mki} + \mathscr{D}_k \mathscr{R}^l{}_{mij} = 0} \tag{3.33}$$

Now one can define also a Weyl-Einstein tensor

$$\mathscr{G}_{ij} \equiv \mathscr{R}_{ij} - \frac{1}{2}\mathscr{R}\gamma_{ij} \tag{3.34}$$

which will also be Weyl-invariant. And the Weyl-Schouten tensor as

$$\mathscr{P}_{ij} \equiv \frac{1}{d-2}\left(\mathscr{R}_{ij} - \frac{1}{2(d-1)}\mathscr{R}g_{ij}\right) \tag{3.35}$$

From these two expressions we see that both Weyl-Einstein and Weyl-Schouten are Weyl-invariant. This invariance however did not come to us for free. In the usual LC connection it is known that both of these tensors are symmetric, i.e. $G_{ij} = G_{ji}$ and $P_{ij} = P_{ji}$. This symmetry now is lost. Since the antisymmetric part of $\mathscr{R}_{ij}$ is $\mathscr{R}_{[ij]} = -(d-2)F_{ij}$ and thus $\mathscr{G}_{[ij]} = \mathscr{P}_{[ij]} = -F_{ij}$. Finally, the Weyl-Cotton tensor can be defined as

$$\mathscr{C}_{ijk} \equiv \mathscr{D}_k \mathscr{P}_{ij} - \mathscr{D}_j \mathscr{P}_{ik} \tag{3.36}$$

which is also Weyl-invariant.

If we would like to construct the Weyl-Weyl tensor $\mathscr{W}$[10], its quite easy because we know it is the part of the Weyl-Riemann tensor 3.30 that is trace-free. So it is just the part of the Riemann tensor that is trace-free, which is exactly the Weyl tensor W (in the LC connection). Thus, the Weyl tensor is the same in the LC and the Weyl connection.

[9]For the proof check A

[10]Weyl tensor measures how the shape of a body changes due to the tidal force when moving along a geodesic, it does not contain the information on how the volume of the body changes. This information is contained in the Ricci tensor

We saw how we can define all the known tensors we have to describe the geometry of our universe. If one would like to search for how the geodesic equations change can read [16]. In the next section we are going to explore how we can apply the method of Holographic Renormalization in this geometric background.

3.2 Weyl geometry in Holography

As we mentioned earlier we supposed that we live in a (d+1)-dimensional AlAdS manifold and thus the metric of our space can always be brought in the form 3.7, where the conformal boundary is located at $z = 0$. According to [17] near $z = 0$ we expand $h(x, z)$ as follows

$$h_{ij}(x,z) = \frac{1}{z^2}\left(h_{ij}^{(0)}(x) + z^2 h_{ij}^{(2)}(x) + z^4 h_{ij}^{(4)}(x) + ...\right) \tag{3.37}$$

We are interested where Weyl geometry come into play? As we have seen before Weyl transformations change the form of the FG gauge but they preserve the form of WFG gauge. So we consider the AlAdS metric in the WFG gauge, where now near the boundary not just $h_{ij}(x, z)$ can be expanded as 3.37 but also the Weyl connection [11]

$$a_i(x,z) = a_i^{(0)}(x) + z^2 a_i^{(2)}(x) + z^4 a_i^{(4)}(x) + ... \tag{3.38}$$

When a_i is zero all the subleading terms $h_{ij}^{(2k)}$ are determined by $h_{ij}^{(0)}$ as we show in 2. Now, in the presence of a_i, they will also depend on $a_i^{(0)}$, $a_i^{(2)}$, etc.

From 3.9 and the expansions 3.38 3.37 we can see that

$$\begin{aligned} h_{ij}^{(k)}(x) &\to \mathcal{B}(x)^{2k-2} h_{ij}^{(k)}(x), \qquad \text{for} \quad k \geq 0 \\ a_i^{(k)}(x) &\to \mathcal{B}(x)^{2k} a_i^{(k)}(x), \qquad \text{for} \quad k \geq 1 \\ a_i^{(0)}(x) &\to a_i^{(0)}(x) - \partial_i \ln \mathcal{B}(x) \end{aligned} \tag{3.39}$$

Where one sees that all terms, except $a_i^{(0)}$, transform covariantly under Weyl transformations. Therefore this term does not have a Weyl weight, whereas the weight of the other terms can be read off from the power of $\mathcal{B}(x)$.

Now lets expand the Weyl connection 3.14 near $z = 0$ and take the leading order. We then find

[11] The asymptotic expansion of a_i will also contain a second part series as γ_{ij} did.

$$\mathcal{G}^{k(0)}_{\ \ ij} = \frac{1}{2} h^{km}_{(0)} \left(\partial_i h^{(0)}_{mj} + \partial_j h^{(0)}_{mi} - \partial_m h^{(0)}_{ij} \right) - \left(a^{(0)}_{(i} \delta^k{}_{j)} - a^{(0)}_m h^{mk}_{(0)} h^{(0)}_{ij} \right) \tag{3.40}$$

which means that on the boundary we can define a connection $\mathcal{D}^{(0)}$ built by 3.40 that is torsion-free and satisfies (by construction) the metricity condition

$$\mathcal{D}^{(0)}_i h^{(0)}_{kj} = 2 a^{(0)}_i h^{(0)}_{kj} \tag{3.41}$$

This shows that the leading term $a^{(0)}_i$ of a_i can be interpreted as a (Weyl) connection at the boundary and the term $h^{(0)}_{ij}$ as an induced metric. Together they form a Weyl geometry at the boundary.

Since $h^{(0)}_{ij}$ and $a^{(0)}_i$ transform as 3.39 following the logic of the previous section we can define the Weyl-covariant connection[12] denoted by $\mathscr{D}^{(0)}$ as

$$\mathscr{D}^{(0)}_i A \equiv \mathcal{D}^{(0)}_i A + w_A a^{(0)}_i A \tag{3.42}$$

which is indeed Weyl-covariant since

$$\mathscr{D}^{(0)}_i A \to \mathcal{B}(x)^{w_A} \mathscr{D}^{(0)}_i A \tag{3.43}$$

We happily obtained a geometric structure on the boundary that has the beatiful property of Weyl-covariance so we are now interested in developing the corresponding geometric tensors as we did before. The idea is that for every geometric quantity we built from $h^{(0)}$ and the usual LC connection, now we have a Weyl-covariant one constructed by $h^{(0)}_{ij}$, $a^{(0)}_i$ and the connection $\mathcal{D}^{(0)}_i$. So given these, we first construct the Weyl-Riemann tensor on the boundary as

$$\boxed{\mathcal{R}^{l(0)}_{\ \ ikj} \equiv \partial_{[k} \mathcal{G}^{l(0)}_{\ \ ij]} + \mathcal{G}^{m(0)}_{\ \ i[j} \mathcal{G}^{l(0)}_{\ \ mk]}} \tag{3.44}$$

thus the Ricci tensor and Ricci scalar are given by

$$\mathcal{R}^{(0)}_{ij} \equiv \partial_{[k} \mathcal{G}^{k(0)}_{\ \ ij]} + \mathcal{G}^{m(0)}_{\ \ i[j} \mathcal{G}^{k(0)}_{\ \ mk]} \qquad\qquad \mathcal{R}^{(0)} = h^{ij}_{(0)} \mathcal{R}^{(0)}_{ij} \tag{3.45}$$

And as we did before we have[13]

[12] Some authors call Weyl connection the Weyl-covariant connection $\mathscr{D}$ whereas some others the connection $\mathcal{D}$ that satisfies the Weyl-metricity condition.

[13] We defined $F^{(0)}_{ij} = \partial_{[i} a^{(0)}_{j]}$, which can be viewd as the curvature of $a^{(0)}_i$.

$$\boxed{\mathscr{R}^{l\,(0)}_{\ ikj} = \mathcal{R}^{(0)}_{ij} + F^{(0)}_{ij} \qquad \mathscr{R}^{(0)}_{ij} = \mathcal{R}^{(0)}_{ij} + F^{(0)}_{ij} \qquad \mathscr{R}^{(0)} = \mathcal{R}^{(0)}} \tag{3.46}$$

and of course the Weyl-Einstein, Weyl-Schouten and Weyl-Cotton tensors respectively

$$\begin{aligned} \mathscr{G}^{(0)}_{ij} &\equiv \mathscr{R}^{(0)}_{ij} - \frac{1}{2}\mathscr{R}^{(0)}\gamma^{(0)}_{ij} \\ \mathscr{P}^{(0)}_{ij} &\equiv \frac{1}{d-2}\left(\mathscr{R}^{(0)}_{ij} - \frac{1}{2(d-1)}\mathscr{R}^{(0)}\gamma^{(0)}_{ij}\right) \\ \mathscr{C}^{(0)}_{ijk} &\equiv \mathscr{D}^{(0)}_{k}\mathscr{P}^{(0)}_{ij} - \mathscr{D}^{(0)}_{j}\mathscr{P}^{(0)}_{ik} \end{aligned} \tag{3.47}$$

An important note on these definitions is that nothing hasv changed regarding the Weyl-weights of the tensors, they are the same as in the previous chapter. This as far as the good news go. As for the bad news, we see that the Weyl-quantities do not necessarily have the same symmetries as the corresponding ones in the LC connection. For instance there is once again an antisymmetric part in the Weyl-Ricci and Weyl-Schouten tensors (as we expected)

$$\mathscr{R}^{(0)}_{[ij]} = -(d-2)F^{(0)}_{ij} \Rightarrow \mathscr{G}^{(0)}_{[ij]} = \mathscr{P}^{(0)}_{[ij]} = -F^{(0)}_{ij} \tag{3.48}$$

Now that we saw we can define all known geometric quantities in the Weyl-connection background one can apply the same procedure as we did to find 2.37 to this background. In this process the first step was to form the Einstein equations. Of course someone could do it the Weyl-connection case also by brute-force, i.e. start from the metric 3.13, compute the Christoffel symbols, then the Ricci tensor components, insert them into Einstein equations and then solve them order by order. In practise, however, this can be quite difficult. So a better idea is to perform a compactification of the metric 3.10 in order to form Einstein equations. This compacitfication is made in such a way so that the metric becomes diagonal. This makes our computations much easier. On the other hand, the corresponding Christoffel symbols will have torsion. After that we expand all quantities inside Einstein's equations and solving order by order yields the same result for the leading term as we did in 2.37 (where only the symmetric part of the Weyl-Ricci tensort appears). The whole procedure is described in detail in [17]. Here we are going to remark the main points and the details can be found in Appendix B.

First we define the differential form $e \equiv \frac{dz}{z} - a_i dx^i$ and thus the metric 3.10 can be written as

$$g = e \otimes e + h_{ij}(x,z)dx^i \otimes dx^j \tag{3.49}$$

We can check that the 1-form e is Weyl-invariant

$$\begin{aligned} e \equiv \frac{dz}{z} - a_i dx^i \rightarrow & \frac{d(\mathcal{B}^{-1}z)}{\mathcal{B}^{-1}z} - (a_i - \partial_i \ln \mathcal{B})\, dx^i \\ &= \frac{dz}{z} - \frac{d\mathcal{B}}{\mathcal{B}} - a_i dx^i + \partial_i \ln \mathcal{B} dx^i \\ &= \frac{dz}{z} - a_i dx^i \end{aligned} \tag{3.50}$$

and by construction h_{ij} is Weyl-invariant since it expands as 3.37 and γ_{ij} transforms as 3.39. Therefore the metric 3.49 is Weyl-invariant.

Defining the dual vectors such that they form a basis on the tangent space T_pM

$$\underline{e} \equiv z\partial_z \qquad \underline{\partial}_i \equiv \partial_i + a_i z \partial_z \tag{3.51}$$

With their help we define the Riemann tensor and after quite a bit of work we find the Einstein field equations. The ij-component[14] of Einstein equations gives

$$\begin{aligned} 0 =& \hat{G}_{ij} + \Lambda g_{ij} \Rightarrow \\ 0 =& \underline{G}_{ij} - \underline{D}_j \phi_i - (\underline{e} + \theta) \left(\rho_{ij} + \frac{1}{2} f_{ij} \right) - \phi_i \phi_j + 2\rho^k{}_i \rho_{kj} \\ &+ \frac{1}{2} f_{jk} f^k{}_i + h_{ij} \left(\underline{e}(\theta) - \frac{1}{8} Tr(ff) + \frac{1}{2} Tr(\rho\rho) + \underline{D}_k \phi^k + \frac{1}{2}\theta^2 + \phi^2 + \Lambda \right) \end{aligned} \tag{3.52}$$

where we defined $\phi_i \equiv \underline{e}(a_i)$, $f_{ij} \equiv \underline{\partial}_i a_j - \underline{\partial}_j a_i$, $\rho^i{}_j \equiv \frac{1}{2} h^{ik} \underline{e}(h_{kj})$, $\psi^i{}_j \equiv \rho^i{}_j + \frac{1}{2} h^{ik} f_{kj}$ and $\theta \equiv \rho^i{}_i$. We denoted by $\underline{R}$ the new Ricci scalar in this coordinate system constructed by 3.49.

In Chapter 2 this is the step where we inserted the asymptotic expansion of the metric into the Einstein equations. And this is what we are going to do now but this time we will also expand the Weyl connection a_i.

$$\begin{aligned} h_{ij}(x,z) &= z^{-2} \left(h^{(0)}_{ij}(x) + z^2 h^{(2)}_{ij}(x) + z^4 h^{(4)}_{ij}(x) + ... \right) + z^{d-2} \left(\pi^{(0)}_{ij}(x) + z^2 \pi^{(2)}_{ij}(x) + ... \right) \\ a_i(x,z) &= \left(a^{(0)}_i(x) + z^2 a^{(2)}_{ij}(x) + z^4 a^{(4)}_i(x) + ... \right) + z^{d-2} \left(p^{(0)}_i(x) + z^2 p^{(2)}_i(x) + ... \right) \end{aligned} \tag{3.53}$$

[14]In this coordinate system the components of the metric are not ρ and i but e and i.

The expansion of h_{ij} in 3.53 holds for generic d. If d is an even integer additional logarithmic terms appear in the expansion [19].

We know how the first part of the series behaves under Weyl transformations. According to these expansions we can also deduce that

$$\pi_{ij}^{(k)} \to \mathcal{B}(x)^{d-2+k}\pi_{ij}^{(k)} \qquad p_i^{(k)} \to \mathcal{B}(x)^{d-2+k}p_i^{(k)} \tag{3.54}$$

Now according to these expansions we also expand our tensors in order to solve field equations order by order. Those expressions can be found in Appendix B. This work has also be done in [17].

We should note that in the FG gauge, where $a_i = 0$, after solving field equations the terms $h^{(2k)}$ were going to be determined by the induced metric $h^{(0)}$ and its derivatives. In our situation (i.e. WFG gauge) those subleading terms will also depend on $a^{(2k)}$.

The leading order of 3.52 gives

$$\boxed{h_{ij}^{(2)} = -\frac{1}{d-2}\left(\frac{1}{2}R_{(ij)}^{(0)} - \frac{1}{2(d-1)}R^{(0)}h_{ij}^{(0)}\right)} \tag{3.55}$$

where $R_{ij}^{(0)}$ is the leading term of the expansion of $\underline{R}_{ij}$ constructed by 3.49 and $R_{(ij)}^{(0)} = R_{ij}^{(0)} + R_{ji}^{(0)}$. For the explicit expansions and the exact way we obtain the result 3.55 we refer to the Appendix B.

Continuing this procedure we find that in WFG gauge all the subleading terms in the expansion of the metric are determined by the Weyl curvature exactly as happened in the FG gauge using the usual LC connection. The difference is that now (in WFG gauge using Weyl-connection) all those terms are Weyl-covariant.

But is this new geometry purely a mathematical heist of curiosity or does it have also a meaning in our physical world? In the next section we are going to see that it does give some very interesting results in cosmology. These results are not in contrast with our known cosmology, but they are rather a generalization of it so that the FRW cosmology appears as just a special gauge of the so called Weyl cosmology. This should not be a surprise since Weyl geometry is just a generalization of Riemaniann geometry.

Chapter 4

Weyl Cosmology

In this chapter, our exploration delves into the properties of the metric that defines our physical universe. Rather than confining ourselves to conventional Riemannian background geometry, our focus shifts towards the utilization of Weyl geometry and its implications for well-established cosmological phenomena, such as inflation. Through this analysis, we aim to unveil novel perspectives on cosmic evolution and structure, broadening our understanding beyond traditional frameworks.

4.1 Standard Cosmology

The Cosmological Principle asserts that on large scales[1], the universe exhibits homogeneity and isotropy. Homogeneity implies uniformity throughout the universe, suggesting that the metric describing it should remain consistent across all regions, as well as the curvature. Isotropy means that the universe should look the same across every direction. This suggests that there is no difference in what two different observers in different parts of the universe see. Furthermore it suggests that our own place in the universe is not special by any means. The isotropy of the universe is also heavily supported by astronomical observation, most famously the Cosmic Microwave Background radiation (CMB)

In the realm of theoretical physics, the Friedmann-Robertson-Walker (FRW) metric serves as a cornerstone in describing the large-scale structure and evolution of the universe. This metric, a fundamental component of Einstein's general theory of relativity, offers a mathematical framework to comprehend the dynamics of cosmic expansion, the distribution of matter, and the fabric of spacetime itself.

[1] Those scales are the galaxy filaments scales and are about 100 million lys.

At its core, the FRW metric embodies the principles of homogeneity and isotropy on cosmological scales. These principles assert that the universe, on average, appears the same to all observers regardless of their position or direction of observation. Homogeneity implies that the universe is uniform, exhibiting the same properties at every point in space, while isotropy suggests that it looks the same in all directions.

Mathematically, the FRW metric takes the form:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{4.1}$$

where $k = 0, \pm 1$ is a curvature parameter[2] and $a(t)$ is the scale factor. The values $0, +1, -1$ of k correspond to Euclidean, spherical and hyperbolic space respectively. The scale factor describes how distances between cosmological objects change over time. As the universe evolves, the scale factor determines the expansion or contraction of spatial distances. The dynamics of $a(t)$ are governed by the solutions to Einstein's equations, which relate the distribution of matter and energy in the universe to the curvature of spacetime. is considered to be 1 today and 0 at the Big Bang.

In this chapter we will denote all spacetime indices with Greek letters $\mu, \nu \in \{0, 1, 2, 3\}$ and spacial indices with Latin $i, j \in \{1, 2, 3\}$[3].

The FRW metric allows for the inclusion of various forms of energy and matter content in the universe, including radiation, matter (both ordinary and dark), and dark energy. These components influence the dynamics of cosmic expansion and the evolution of the universe over time. The relative densities of these components dictate the fate of the universe, determining whether it will continue expanding indefinitely, reach a maximum size and contract, or undergo an accelerated expansion due to dark energy.

It also serves as the foundation for numerous cosmological models that aim to explain observed phenomena such as the cosmic microwave background radiation, the large-scale distribution of galaxies, and the redshift-distance relationship of distant objects. The full derivation of the metric, due to the Cosmological principle, can be found in the beginning of Appendix C.

Starting from 4.1 we can compute its Christoffel symbolds and find that the non-vanishing ones are[4]

[2] It has dimension of length^{-2} and should not be confused with the curvature of spacetime which is captured by the Riemann tensor.

[3] Note that the analogue of γ_{ij} of chapter 3 in our cosmological case is $g_{\mu\nu}$ defined in 4.1.

[4] We have denoted with dot the derivative with respect to t.

$$
\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a} \\
&\Gamma^1_{11} = \frac{kr}{1-kr^2} \qquad \Gamma^1_{22} = r(1-kr^2) \qquad \Gamma^1_{33} = r(1-kr^2)\sin^2\theta \\
&\Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^2_{33} = \sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta
\end{aligned}
\tag{4.2}
$$

Using these symbolds we calculate the components of the Ricci tensor. Doing so yields

$$
\begin{aligned}
&R_{00} = \frac{-3\ddot{a}}{a} \qquad R_{11} = \frac{1}{1-kr^2}(\ddot{a}a + 2\dot{a}^2 + 2k) \\
&R_{22} = r^2(\ddot{a}a + 2\dot{a}^2 + 2k) \quad R_{33} = r^2\sin^2\theta(\ddot{a}a + 2\dot{a}^2 + 2k)
\end{aligned}
\tag{4.3}
$$

And of course the Ricci scalar

$$
R = -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \tag{4.4}
$$

Finally, the Einstein tensor is

$$
G^0{}_0 = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} \qquad G^i{}_j = \left(\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)\delta^i{}_j \tag{4.5}
$$

But what equations should it satisfy?

If we suppose our Universe was filled with a perfect fluid of energy density ρ and pressure P then the field equations, coming from the principle of least action, will be

$$
G^\mu{}_\nu = 8\pi G T^\mu{}_\nu \tag{4.6}
$$

where G is the gravitational constant and $T^\mu{}_\nu$ is the stress energy tensor.

These equations describe how matter and energy curve the spacetime around them, thereby influencing the paths of particles and the geometry of the universe and vice versa. They show how the geometry of our space dictates the physical laws.

Since our Universe is homogeneous and isotropic the stress-energy tensor of a perfect fluid by a comoving observer must have the form

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix} \tag{4.7}$$

This tensor represents the distribution of energy, momentum, and stress within a given spacetime. In this expression ρ and P are understood as the sum of all contributions to the energy density and pressure respectively in the Universe. Of course both of them are function of time.

Substituting 4.7 and 4.5 in 4.6 gives us the Friedmann equations

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \end{aligned} \tag{4.8}$$

The first equation relates the rate of expansion of the universe, given by the Hubble parameter

$$H \equiv \frac{\dot{a}}{a} \tag{4.9}$$

to the density of matter and energy (ρ) and the curvature of space (k). It essentially quantifies how the energy density and curvature influence the rate at which the universe expands. While the second equation describes how the rate of change of the expansion rate (given by the derivative of the Hubble parameter, $\dot{H}$) is influenced by the energy density and pressure of various components in the universe. It accounts for the acceleration or deceleration of the expansion due to the presence of matter, radiation, or other forms of energy.

In order to work on the cosmological dictionary we need to define also the Hubble radius $(aH)^{-1}$ which has the physical meaning of the maximal distance at which particles can communicate with each other at a given time[5].

And there lies a problem that tortured cosmologists for years. If we suppose that our Universe did not existed before $t = 0$ then there wouldn't be enough time for the photons of the CMB to communicate with each other and thus to make CMB almost perfectly

[5]Hubble radius should not be confused with particle horizon $x = \int_{t_i}^{t} \frac{dt}{a}$ which is the maximal distance at which particles could have ever been able to communicate with each other.

isotropic as it is observed today. This is called the horizon problem. In standard cosmology the most elegant and successful way to solve the horizon problem is through inflation. There are many ways to define inflation, but the most common is as the period of shrinking Hubble sphere (i.e. the Hubble radius decreases).

Another way some authors define inflation is as an accelerated expansion period since

$$\frac{d(aH)^{-1}}{dt} < 0 \Rightarrow -\frac{\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \ddot{a} > 0 \tag{4.10}$$

Now in order to measure whether inflation occurs or not and if so how much, we define two quantities that reflect the possibility and the amount of inflation. These quantities in Riemannian geometry are called inflation parameters and are respectively

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \qquad \eta \equiv \frac{d\ln\epsilon}{dN} \tag{4.11}$$

where N is the so called number of e-folds defined by $N \equiv \ln a$. Cosmological evidence have shown that in order to solve the horizon problem we want inflation to last for at least 40 to 60 e-folds.

Inflation response to $\epsilon < 1$ since

$$\frac{d(aH)^{-1}}{dt} = -\frac{1}{a}(1-\epsilon) \Rightarrow \epsilon < 1 \tag{4.12}$$

and the second condition regarding the parameter η is $|\eta| < 1$ which guaranties that the fractional change of ϵ per Hubble time is small.

The simplest model of inflation is that of a single scalar field $\phi(t)$, called the inflaton. The fact that our Universe is homogeneous and isotropic dectates that $\phi = \phi(t)$. This field carries potential energy density $V(\phi)$ and if it is dynamical it carries also kinetic energy density. We want to establish under which conditions of these two energy densities can give rise to inflation (i.e. accelerated expansion). So from now on whenever we write ρ and P we will mean the energy density and pressure of the inflaton scalar field respectively.

From Noether's theorem we know that the stress-energy of a scalar field tensor can be written as

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi - V(\phi)\right) \tag{4.13}$$

In order for this result to agree with the high amount of symmetries of FRW metric it must equals with 4.7 and so we get tow equations for the energy density and the pressure of the inflaton

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{4.14}$$

We saw earlier that if we want inflation to occur we must demand ϵ to be small. This, in Riemannian geometry leads to a period of domination of a fluid of negative pressure $P < -\frac{1}{3}\rho$.

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{P}{\rho}\right) < 1 \Rightarrow P < -\frac{1}{3}\rho \tag{4.15}$$

Finally, since inflaton is a scalar, its equations of motion will be given by the Klein-Gordon equation. To derive this equation we first prove the continuity equation. We want our stress-energy tensor to be conserved in our gravitational backgraound and thus

$$\nabla_\mu T^\mu{}_\nu = 0 \tag{4.16}$$

Taking the $\nu = 0$ component of this equation and using the Christoffel symbols we calculated earlier we get the evolution of the energy density (i.e. the continuity equation)

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{4.17}$$

Once we have this equation we simply substitute 4.14 into this expression and yields

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{4.18}$$

This equation represents the evolution of the inflaton. Here, the potential serves as a driving force, influencing the field's behavior, while the expansion of the universe introduces a damping effect, akin to friction in the system.

All the aforementioned findings were obtained within the framework of Riemannian geometry. While this model stands as the most promising explanation for our physical universe, recent evidence, such as the Hubble tension problem, prompts us to consider the possibility of its refinement. The introduction of the Weyl term into the connection is intriguing; not only does it render the new connection Weyl-invariant, but it also endows the Weyl term with units of time^{-1}, akin to the Hubble parameter. This raises suspicions regarding its potential cosmological implications on the expansion of our Universe.

4.2 Weyl connection in Cosmology

We know that the gravitational part of equations 4.6 is obtained by varying the Einstein-Hilbert action

$$S_g = \frac{1}{16\pi G} \int R\sqrt{g} d^4x \tag{4.19}$$

with respect to the metric.

An interesting idea would be to study gravitational theories in Cosmology using Weyl connection. And then see what kind of Inflation those gravitational theories imply.

So we define once more the Weyl-invariant connection

$$\mathcal{G}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \left(A_{(\mu}\delta^\lambda{}_{\nu)} - A^\lambda g_{\mu\nu}\right) \tag{4.20}$$

where we denoted with A_μ the Weyl gauge field in order not to be confused with the scale factor a.

According to [21, 22] one can construct a Weyl-invariant action so that the theory as a whole enjoys Weyl symmetry. So the main difference with Chapter 3 is that now Weyl symmetry is not the result of diffeomorphisms in the bulk, since there is no bulk in our case. Weyl symmetry is considered a priori in our theory through the gravitational action

$$S_w = \int d^4x\sqrt{g}\left(a_1\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + a_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + a_3\mathcal{R}^2 + a_4F_{\mu\nu}F^{\mu\nu}\right) \tag{4.21}$$

for some constants (a_1, a_2, a_3, a_4) and $F_{\mu\nu}$ being the field strength tensor of A_λ.

This action is indeed Weyl-invariant, since under Weyl transformations

$$\begin{gathered}\mathcal{R}^\mu{}_{\nu\rho\sigma} \to \mathcal{R}^\mu{}_{\nu\rho\sigma} \Rightarrow \\ \mathcal{R}_{\mu\nu\rho\sigma} = g_{\mu\lambda}\mathcal{R}^\lambda{}_{\nu\rho\sigma} \to \mathcal{B}^{-2}g_{\mu\lambda}\mathcal{R}^\lambda{}_{\nu\rho\sigma} = \mathcal{B}^{-2}\mathcal{R}_{\mu\nu\rho\sigma}\end{gathered} \tag{4.22}$$

In the same way, the Riemann with upper indices

$$\begin{aligned}\mathcal{R}^{\mu\nu\rho\sigma} &= g^{\nu\alpha}g^{\rho\beta}g^{\sigma\gamma}\mathcal{R}^{\mu}{}_{\alpha\beta\gamma} \\ &\to \mathcal{B}^6 g^{\nu\alpha}g^{\rho\beta}g^{\sigma\gamma}\mathcal{R}^{\mu}{}_{\alpha\beta\gamma}\end{aligned} \tag{4.23}$$

Therefore the product transforms as

$$\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} \to \mathcal{B}^4\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} \tag{4.24}$$

Similarly we get that

$$\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} \to \mathcal{B}^4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}, \qquad \mathcal{R}^2 \to \mathcal{B}^4\mathcal{R}^2, \qquad F_{\mu\nu}F^{\mu\nu} \to \mathcal{B}^4 F_{\mu\nu}F^{\mu\nu} \tag{4.25}$$

And since the metric transforms as $g_{\mu\nu} \to \mathcal{B}^{-2}g_{\mu\nu}$ its determinant will transform as $g \to \mathcal{B}^{-8}g$, which means that $\sqrt{g} \to \mathcal{B}^{-4}\sqrt{g}$.

Therefore, the whole gravitational action

$$\begin{aligned} S_w = &\int d^4x\sqrt{g}\left(a_1\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + a_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + a_3\mathcal{R}^2 + a_4F_{\mu\nu}F^{\mu\nu}\right) \\ &\to \int d^4x\mathcal{B}^{-4}\sqrt{g}\mathcal{B}^4\left(a_1\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + a_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + a_3\mathcal{R}^2 + a_4F_{\mu\nu}F^{\mu\nu}\right) \\ &= S_w \end{aligned} \tag{4.26}$$

But it is not really necessary to create an action of order $\mathcal{R}^2$ in order to obtain Weyl symmetry. For example one can start from the closest action to the Einstein-Hilbert one, which is

$$\int \mathcal{R}\sqrt{g}d^4x \tag{4.27}$$

the problem is that this action is not Weyl-invariant but rather Weyl-covariant, since $\mathcal{R}\sqrt{g} \to \mathcal{B}^{-2}\mathcal{R}\sqrt{g}$. We can easily, however, overcome this problem by adding a suitable scalar field ϕ in such a way that would cancel the $\mathcal{B}^{-2}$ term. So we take our scalar field ϕ to transform as

$$\phi \to \phi + \ln\mathcal{B} \tag{4.28}$$

we can define the Weyl-invariant action

$$S_w = \int \mathcal{R}\sqrt{g}e^{2\phi}d^4x \tag{4.29}$$

If we do so we can check that is indeed invariant

$$\begin{aligned} S_w &= \int \mathcal{R}\sqrt{g}e^{2\phi}d^4x \\ &\rightarrow \int \mathcal{B}^{-2}\mathcal{R}\sqrt{g}e^{2\phi+2\ln\mathcal{B}}d^4x \\ &= \int \mathcal{B}^{-2}\mathcal{R}\sqrt{g}\mathcal{B}^2e^{2\phi}d^4x \\ &= S_w \end{aligned} \tag{4.30}$$

It would be an interesting future research direction to obtain the Weyl-invariant field equations from this action by varying with respect to the metric. Also, varying with respect to the scalar field ϕ will give us its equations of motion. The difference from the work done in [21, 22] is that the modified action we proposed in 4.29 has a GR limit.

In conclusion, although Weyl's original intention for this particular geometry may not have been fulfilled, it has nonetheless provided us with a valuable tool. Not only in the realm of AdS/CFT correspondence and Holography but also in Cosmology. There are many ways to insert Weyl-invariance in a cosmological theory but we proposed one that has a GR limit. We hope that Weyl connection in Cosmology will be able in the future to explain observational results in the cosmological regime that our Standard Cosmology right now cannot. If so, it would signify that this geometry aids us in gaining a deeper understanding of the world around us.

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Appendix A

Connections

First let's prove that $\mathcal{G}^i_{jk}$ are invariant under Weyl transformations

$$\begin{aligned}
\mathcal{G}^i_{jk} \equiv& \frac{1}{2}\gamma^{im}\left(\partial_j\gamma_{mk} + \partial_k\gamma_{mj} - \partial_m\gamma_{jk}\right) - \left(a_{(j}\delta^i{}_{k)} - a^i\gamma_{jk}\right) \\
\to& \frac{1}{2}\mathcal{B}(x)^2\gamma^{im}(\partial_j(\mathcal{B}(x)^{-2}\gamma_{mk}) + \partial_k(\mathcal{B}(x)^{-2}\gamma_{mj}) - \partial_m(\mathcal{B}(x)^{-2}\gamma_{jk})) \\
& - \big(a_{(j} - \partial_{(j}\ln\mathcal{B}(x)\big)\,\delta^i{}_{k)} + \left(a_l - \partial_l\ln\mathcal{B}(x)\right)\mathcal{B}(x)^2\gamma^{il}\mathcal{B}(x)^{-2}\gamma_{jk} \\
=& \Gamma^i_{jk}[\gamma] - \mathcal{B}(x)^{-1}\gamma^{im}(\gamma_{mk}\partial_j\mathcal{B}(x) + \gamma_{mj}\partial_k\mathcal{B}(x) - \gamma_{jk}\partial_m\mathcal{B}(x)) \\
& - a_{(j}\delta^i{}_{k)} + \partial_{(j}\ln\mathcal{B}(x)\delta^i{}_{k)} + a^i\gamma_{jk} - \gamma_{jk}\partial^i\ln\mathcal{B}(x) \\
=& \Gamma^i_{jk}[\gamma] - \left(a_{(j}\delta^i{}_{k)} - a^i\gamma_{jk}\right)
\end{aligned} \tag{A.1}$$

Now, if we take the covariant derivative of the metric

$$\begin{aligned}
\mathcal{D}_i\gamma_{jk} =& \partial_i\gamma_{jk} - \mathcal{G}^l_{i(j}\gamma_{k)l} \\
=& \partial_i\gamma_{jk} - \frac{1}{2}\gamma_{l(k}\gamma^{lm}\left(\partial_{j)}\gamma_{im} + \partial_i\gamma_{j)m} - \partial_m\gamma_{j)i}\right) \\
& + \gamma_{l(k}\left(a_i\delta^l{}_{j)} + a_{j)}\delta^l{}_i - a^l\gamma_{j)i}\right) \\
=& \partial_i\gamma_{jk} - \frac{1}{2}\delta^m{}_{(k}\left(\partial_{j)}\gamma_{im} + \partial_i\gamma_{j)m} - \partial_m\gamma_{j)i}\right) \\
& + a_i\gamma_{(jk)} + \cancel{\gamma_{i(k}a_{j)}} - \cancel{a_{(k}\gamma_{j)i}} \\
=& 2a_i\gamma_{jk}
\end{aligned} \tag{A.2}$$

Therefore

$$\mathscr{D}_i\gamma_{jk} = \mathcal{D}_i\gamma_{jk} + wa_i\gamma_{jk} = 2a_i\gamma_{jk} - 2a_i\gamma_{jk} = 0 \tag{A.3}$$

since $\gamma_{jk} \to \mathcal{B}(x)^{-2}\gamma_{jk}$ (in other words γ_{jk} has Weyl weight -2)

We derive a Weyl-invariant Riemann tensor

$$\begin{aligned}
\mathcal{D}_n A_k =& D_n A_k + (a_n\delta^i{}_k + a_k\delta^i{}_n - a^i\gamma_{nk})A_i \Rightarrow \\
\mathcal{D}_m\mathcal{D}_n A_k =& \partial_m\mathcal{D}_n A_k - \mathcal{G}^l_{nm}\mathcal{D}_l A_k - \mathcal{G}^l_{mk}\mathcal{D}_n A_l \\
=& \partial_m D_n A_k + \partial_m(a_n A_k + a_k A_n - a\cdot A\gamma_{nk}) - \Gamma^l_{nm}D_l A_k - \Gamma^l_{nm}(a_l A_k + a_k A_l - a\cdot A\gamma_{lk}) \\
&+ (a_n\delta^l{}_m + a_m\delta^l{}_n - a^l\gamma_{mn})(D_l A_k + a_l A_k + a_k A_l - a\cdot A\gamma_{kl}) - \Gamma^l_{mk}D_n A_l \\
&- \Gamma^l_{mk}(a_n A_l + a_l A_n - a\cdot A\gamma_{mk}) + (a_m\delta^l{}_k + a_k\delta^l{}_m - a^l\gamma_{mk})(D_n A_l + a_{(n}A_{l)} - a\cdot A\gamma_{nl}) \\
=& D_m D_n A_k + \partial_m\left(a_{(n}A_{k)} - a\cdot A\gamma_{nk}\right) - \Gamma^l_{mn}\left(a_{(l}A_{k)} - a\cdot A\gamma_{lk}\right) \\
&+ \left(a_{(n}\delta^l{}_{m)} - a^l\gamma_{mn}\right)(D_l A_k + a_{(l}A_{k)} - a\cdot A\gamma_{kl}) - \Gamma^l_{mk}(a_n A_l + a_l A_n - a\cdot A\gamma_{nl}) \\
&+ (a_m\delta^l{}_k + a_k\delta^l{}_m - a^l\gamma_{mk})(D_n A_l + a_{(n}A_{l)} - a\cdot A\gamma_{nl}) \Rightarrow \\
\mathcal{D}_{[m}\mathcal{D}_{n]}A_k =& D_{[m}D_{n]}A_k + \partial_{[m}(a_{n]}A_k + a_k A_{n]} - a\cdot A\gamma_{n]k}) - \Gamma^l_{k[m}(a_{n]}A_l + a_l A_{n]} - a\cdot A\gamma_{n]l}) \\
&+ (a_{[m}\delta^l{}_k + a_k\delta^l{}_{[m} - a^l\gamma_{k[m]})(D_{n]}A_l + a_{n]}A_l + a_l A_{n]} - a\cdot A\gamma_{n]l}) \\
=& D_{[m}D_{n]}A_k + \partial_{[m}(a_{n]}A_k + a_k A_{n]} - a\cdot A\gamma_{n]k}) - \Gamma^l_{k[m}(a_{n]}A_l + a_l A_{n]} - a\cdot A\gamma_{n]l}) \\
&+ a_{[m}D_{n]}A_k + a_k a_{[m}A_{n]} - \cancel{a\cdot A a_{[m}\gamma_{n]k}} + a_k D_{[n}A_{m]} + \cancelto{0}{a_k a_{[n}A_{m]} + a_k a_{[m}A_{n]}} \\
&- \cancelto{0}{a\cdot A a_k\gamma_{[mn]}} - a^l\gamma_{k[m}D_{n]}A_l - \cancel{a\cdot A\gamma_{k[m}a_{n]}} - a^2\gamma_{k[m}A_{n]} + a\cdot A a_{[n}\gamma_{km]}
\end{aligned} \tag{A.4}$$

The terms proportional to A^l give

$$
\begin{aligned}
\mathcal{R}^l{}_{knm}A_l =& R^l{}_{knm}A_l + A_k\partial_{[m}a_{n]} + A_{[n}\partial_{m]}a_k - A_l a^l \partial_{[m}\gamma_{n]k} - A_l\gamma_{k[n}\partial_{m]}a^l \\
&- A_l\Gamma^l_{k[m}a_{n]} - \Gamma^i_{k[m}A_{n]}a_i + a^l A_l \gamma_{i[n}\Gamma^i_{m]k} \\
&- a_{[m}\Gamma^l_{n]k}A_l + a_k a_{[m}A_{n]} - \cancelto{0}{a_k\Gamma^l_{[nm]}A_l} + a^l\gamma_{k[m}\Gamma^i_{n]l}A_i - a^2\gamma_{k[m}A_{n]} + a\cdot A a_{[n}\gamma_{km]} \\
=& A_l R^l{}_{knm} + A_l\delta^l{}_k\partial_{[m}a_{n]} + A_l\delta^l{}_{[n}\partial_{m]}a_k - A_l a^l\partial_{[m}\gamma_{n]k} - A_l\gamma_{k[n}\partial_{m]}a^l \\
&- A_l\Gamma^l_{k[m}a_{n]} - A_l\Gamma^i_{k[m}\delta^l{}_{n]}a_i + A_l a^l\gamma_{i[n}\Gamma^i_{m]k} \\
&- A_l a_{[m}\Gamma^l_{n]k} + A_l a_k a_{[m}\delta^l{}_{n]} + A_l a^j\gamma_{k[m}\Gamma^l_{n]j} - A_l a^2\gamma_{k[m}\delta^l{}_{n]} + A_l a^l a_{[n}\gamma_{km]} \Rightarrow
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
\mathcal{R}^l{}_{knm} =& R^l{}_{knm} + \delta^l{}_k\partial_{[m}a_{n]} + \delta^l{}_{[n}\partial_{m]}a_k - a^l\partial_{[m}\gamma_{n]k} - \gamma_{k[n}\partial_{m]}a^l - \cancel{\Gamma^l_{k[m}a_{n]}} - \Gamma^i_{k[m}\delta^l{}_{n]}a_i + a^l\gamma_{i[n}\Gamma^i_{m]k} \\
& \cancel{-a_{[m}\Gamma^l_{n]k}} + a_k a_{[m}\delta^l{}_{n]} + a^j\gamma_{k[m}\Gamma^l_{n]j} - a^2\gamma_{k[m}\delta^l{}_{n]} + a^l a_{[n}\gamma_{km]} \\
=& R^l{}_{knm} + \delta^l{}_k D_{[m}a_{n]} + \delta^l{}_{[n}D_{m]}a_k + \gamma_{k[m}D_{n]}a^l + (a_k a_{[m} - a^2\gamma_{k[m})\delta^l{}_{n]} + a^l a_{[n}\gamma_{m]k}
\end{aligned}
\tag{A.6}
$$

To prove the Bianchi identity for $\mathcal{R}^l{}_{mjk}$ begin from the Jacobi identity for the (Weyl-) covariant derivative

$$[\mathcal{D}_i, [\mathcal{D}_j, \mathcal{D}_k]] + [\mathcal{D}_j, [\mathcal{D}_k, \mathcal{D}_i]] + [\mathcal{D}_k, [\mathcal{D}_i, \mathcal{D}_j]] = 0 \tag{A.7}$$

and act on an arbitrary vector field A^l

$$\begin{aligned}
[\mathcal{D}_i, [\mathcal{D}_j, \mathcal{D}_k]]A^l &= \mathcal{D}_i[\mathcal{D}_j, \mathcal{D}_k]A^l - [\mathcal{D}_j, \mathcal{D}_k]\mathcal{D}_i A^l \\
&= \mathcal{D}_i\left(\mathcal{R}^l{}_{mjk}A^m\right) - \mathcal{R}^l{}_{mjk}\mathcal{D}_i A^m + \mathcal{R}^m{}_{ijk}\mathcal{D}_m A^l \\
&= \mathcal{D}_i\mathcal{R}^l{}_{mjk}A^m + \mathcal{R}^l{}_{mjk}\mathcal{D}_i A^m - \mathcal{R}^l{}_{mjk}\mathcal{D}_i A^m + \mathcal{R}^m{}_{ijk}\mathcal{D}_m A^l \\
&= \mathcal{D}_i\mathcal{R}^l{}_{mjk}A^m + \mathcal{R}^m{}_{ijk}\mathcal{D}_m A^l
\end{aligned} \tag{A.8}$$

Inserting this into the Jacobi identity we get

$$\left(\mathcal{D}_i\mathcal{R}^l{}_{mjk} + \mathcal{D}_j\mathcal{R}^l{}_{mki} + \mathcal{D}_k\mathcal{R}^l{}_{mij}\right)A^m + \left(\mathcal{R}^l{}_{knm} + \mathcal{R}^l{}_{nmk} + \mathcal{R}^l{}_{mkn}\right)\mathcal{D}_m A^l = 0 \tag{A.9}$$

The second parenthesis vanishes identically and thus

$$\boxed{\mathcal{D}_i\mathcal{R}^l{}_{mjk} + \mathcal{D}_j\mathcal{R}^l{}_{mki} + \mathcal{D}_k\mathcal{R}^l{}_{mij} = 0} \tag{A.10}$$

Since $\mathcal{R}^l{}_{mjk}$ is Weyl-invariant we can replace $\mathcal{D}$ with $\mathscr{D}$.

$$\mathscr{D}_i\mathcal{R}^l{}_{mjk} + \mathscr{D}_j\mathcal{R}^l{}_{mki} + \mathscr{D}_k\mathcal{R}^l{}_{mij} = 0 \tag{A.11}$$

And from the definition of $\mathscr{R}^l{}_{mjk}$ we have

$$\mathscr{D}_i\mathscr{R}^l{}_{mjk} = \mathcal{D}_i\mathcal{R}^l{}_{mjk} + \delta^l{}_m\mathcal{D}_i F_{jk} \tag{A.12}$$

We see that

$$\begin{aligned}
&\mathcal{D}_i F_{jk} + \mathcal{D}_j F_{ki} + \mathcal{D}_k F_{ij} = \\
&\mathcal{D}_i \left(\mathcal{D}_j a_k - \mathcal{D}_k a_j\right) + \mathcal{D}_j \left(\mathcal{D}_k a_i - \mathcal{D}_i a_k\right) + \mathcal{D}_k \left(\mathcal{D}_i a_j - \mathcal{D}_j a_i\right) = \\
&\left(\mathcal{R}^l{}_{kij} + \mathcal{R}^l{}_{ijk} + \mathcal{R}^l{}_{jki}\right) a_l = 0
\end{aligned} \tag{A.13}$$

Therefore

$$\boxed{\mathscr{D}_i \mathscr{R}^l{}_{mjk} + \mathscr{D}_j \mathscr{R}^l{}_{mki} + \mathscr{D}_k \mathscr{R}^l{}_{mij} = 0} \tag{A.14}$$

We claim that the Weyl weight of an arbitrary tensor A_k is equal to the weight of its Weyl-covariant derivative. Of course this is something that occurs by construction of the Weyl-covariant derivative, but in order to check that our definitions are valid we show it explicitly

$$\begin{aligned}
\mathcal{D}_n A_k =& \partial_n A_k - \mathcal{G}^l_{nk} A_l \to \partial_n \left(\mathcal{B}(x)^w A_k\right) - \mathcal{G}^l_{nk}\mathcal{B}(x)^w A_l \\
=& \mathcal{B}(x)^w \mathcal{D}_n A_k + A_k \partial_n \mathcal{B}(x)^w \\
=& \mathcal{B}(x)^w \mathcal{D}_n A_k + w\mathcal{B}(x)^w A_k \partial_n \ln \mathcal{B}(x) \Rightarrow \\
\mathscr{D}_n A_k =& \mathcal{D}_n A_k + w a_n A_k \to \\
& \mathcal{B}(x)^w \mathcal{D}_n A_k + w\mathcal{B}(x)^w A_k \partial_n \ln \mathcal{B}(x) + w(a_n - \partial_n \ln \mathcal{B}(x)) A_k \mathcal{B}(x)^w \\
=& \mathcal{B}(x)^w \mathcal{D}_n A_k + w a_n A_k \mathcal{B}(x)^w \\
=& \mathcal{B}(x)^w \mathscr{D}_n A_k
\end{aligned} \tag{A.15}$$

Now we compute the quantity

$$\begin{aligned}
\mathscr{D}_n A_k =& \mathcal{D}_n A_k + w a_n A_k = \partial_n A_k - \mathcal{G}^l_{nk} A_l + w a_n A_k \Rightarrow \\
\mathscr{D}_m \mathscr{D}_n A_k =& \mathcal{D}_m \left(\mathcal{D}_n A_k + w a_n A_k\right) + w a_m \mathscr{D}_n A_k \\
=& \mathcal{D}_m \mathcal{D}_n A_k + w a_n \mathcal{D}_m A_k + w A_k \mathcal{D}_m a_n + w a_m \left(\mathcal{D}_n A_k + w a_n A_k\right) \\
=& \mathcal{D}_m \mathcal{D}_n A_k + w a_n \partial_m A_k - w a_n \mathcal{G}^l_{mk} A_l + w A_k \mathcal{D}_m a_n \\
& + w a_m \partial_n A_k - w a_m \mathcal{G}^l_{nk} A_l + w^2 a_m a_n A_k \Rightarrow \\
\mathscr{D}_{[m} \mathscr{D}_{n]} A_k =& \mathcal{D}_{[m} \mathcal{D}_{n]} A_k + w a_{[n} \partial_{m]} A_k - w a_{[n} \mathcal{G}^l_{m]k} A_l \\
& + w A_k \mathcal{D}_{[m} a_{n]} + w a_{[m} \partial_{n]} A_k - w a_{[m} \mathcal{G}^l_{n]k} A_l \Rightarrow
\end{aligned} \tag{A.16}$$

$$\boxed{\mathscr{R}^l{}_{kmn} = \mathcal{R}^l{}_{kmn} + w\delta^l{}_k \mathcal{D}_{[m} a_{n]}} \tag{A.17}$$

And so

$$\boxed{\mathscr{R}_{ij} = \mathcal{R}_{ij} + w f_{ij}} \tag{A.18}$$

Appendix B

Einstein equations

In order to work on the Holographic dictionary it is useful to write the metric as

$$g = e \otimes e + h_{ij}(x,z)dx^i \otimes dx^j \tag{B.1}$$

where $e \equiv \frac{dz}{z} - a_i dx^i$.

The advantage of this notation is that now our metric is diagonal

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & h_{ij} \end{pmatrix} \Rightarrow g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & h^{ij} \end{pmatrix} \tag{B.2}$$

The dual vectors are given by

$$\underline{e} \equiv z\partial_z \qquad \underline{\partial}_i \equiv \partial_i + a_i z \partial_z$$

and they indeed form an orthonormal basis:

$$e(\underline{e}) = 1, \qquad e\left(\underline{\partial}_i\right) = 0, \qquad dx^i\left(\underline{\partial}_j\right) = \delta^i{}_j, \qquad dx^i(\underline{e}) = 0 \tag{B.3}$$

where $\{\mu,\nu\} \in \{1,...,d+1\}$ and $\{i,j\} \in \{1,...,d\}$.

If we set

$$\phi_i \equiv \underline{e}(a_i), \qquad f_{ij} \equiv \underline{\partial}_i a_j - \underline{\partial}_j a_i$$

where ϕ can be thought as the acceleration of the radial congruence $\underline{e}$ and f as the field strength tensor of a_i, we get

$$[\underline{e}, \underline{\partial}_i] = \phi_i \underline{e}, \qquad [\underline{\partial}_i, \underline{\partial}_j] = f_{ij}\underline{e} \tag{B.4}$$

.

By definition the LC coefficients are $[\underline{e}_\mu, \underline{e}_\nu] = C_{\mu\nu}{}^\rho \underline{e}_\rho$ and since in our case $\mu \in \{e, i\}$, $\underline{e}_\mu = \{\underline{e}, \underline{\partial}_i\}$ we compute the commutators and find

$$C_{ei}{}^e = \phi_i, \qquad C_{ij}{}^e = f_{ij}, \qquad C_{ij}{}^k = 0 \tag{B.5}$$

.

With these coefficients we define the most general non-coordinated LC connection as

$$\widehat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\underline{e}_\mu(g_{\rho\nu}) + \underline{e}_\nu(g_{\mu\rho}) - \underline{e}_\rho(g_{\mu\nu})\right) - \frac{1}{2} g^{\lambda\rho} \left(C_{\mu\rho}{}^\sigma g_{\sigma\nu} + C_{\nu\mu}{}^\sigma g_{\sigma\rho} - C_{\rho\nu}{}^\sigma g_{\sigma\mu}\right) \tag{B.6}$$

For our future calculations we also define the quantities

$$\rho^i{}_j \equiv \frac{1}{2} h^{ik} \underline{e}(h_{kj}), \qquad \psi^i{}_j \equiv \rho^i{}_j + \frac{1}{2} h^{ik} f_{kj}, \qquad \theta \equiv \rho^i{}_i \tag{B.7}$$

Then, by direct computation we get

$$\begin{aligned} &\widehat{\Gamma}^e_{ee} = 0, \quad \widehat{\Gamma}^e_{ei} = \phi_i, \quad \widehat{\Gamma}^e_{ie} = 0, \quad \widehat{\Gamma}^e_{ij} = -\frac{1}{2}\underline{e}(h_{ij}) + \frac{1}{2} f_{ij} \quad \widehat{\Gamma}^i_{ee} = -h^{ij}\phi_j, \\ &\widehat{\Gamma}^i_{ej} = \widehat{\Gamma}^i_{je} = \rho^i{}_j + \frac{1}{2} f^i{}_j, \quad \widehat{\Gamma}^i_{jk} = \underline{\Gamma}^i_{jk} \equiv \frac{1}{2} h^{im} \left(\underline{\partial}_j h_{mk} + \underline{\partial}_k h_{jm} - \underline{\partial}_m h_{jk}\right) \end{aligned} \tag{B.8}$$

Those Christoffel symbols have now torsion. This is the cost we have to pay for the coordinate transformation we did to bring the metric in a nice diagonal form.

In order not to create confusion according the connections we are going to have, given a tensor A_μ we define

$$
\begin{aligned}
\widehat{\nabla}_\mu A_\nu &\equiv \underline{e}_\mu A_\nu - \widehat{\Gamma}^\lambda_{\mu\nu} A_\lambda, && \text{for} \quad \widehat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\underline{e}_\mu(g_{\rho\nu}) + \underline{e}_\nu(g_{\mu\rho}) - \underline{e}_\rho(g_{\mu\nu}) \right) \\
& && \qquad\qquad - \frac{1}{2} g^{\lambda\rho} \left(C_{\mu\rho}{}^\sigma g_{\sigma\nu} + C_{\nu\mu}{}^\sigma g_{\sigma\rho} - C_{\rho\nu}{}^\sigma g_{\sigma\mu} \right) \\
\overline{\nabla}_i A_j &\equiv \underline{\partial}_i A_j - \overline{\Gamma}^k_{ij} A_k, && \text{for} \quad \overline{\Gamma}^k_{ij} \equiv \underline{\Gamma}^k_{ij} - \left(a_{(i}\delta^k{}_{j)} - a^k h_{ij} \right) \\
\underline{D}_i A_j &\equiv \underline{\partial}_i A_j - \underline{\Gamma}^k_{ij} A_k, && \text{for} \quad \underline{\Gamma}^k_{ij} = \frac{1}{2} h^{km} \left(\underline{\partial}_i h_{mj} + \underline{\partial}_j h_{mi} - \underline{\partial}_m h_{ij} \right) \\
\overline{\mathscr{D}}_i A_j &\equiv \overline{\nabla}_i A_j + w_A a_i A_j && \text{such that} \qquad A \xrightarrow{\text{Weyl}} \mathcal{B}(x)^{w_A} A
\end{aligned}
\tag{B.9}
$$

We see that they obey the usual rules of covariant derivatives, such as

$$
\underline{D}_i h_{jk} = 0, \qquad \overline{\nabla}_i h_{jk} = 2a_i h_{jk}, \qquad \overline{\mathscr{D}}_i h_{jk} = 0 \tag{B.10}
$$

and taking the leading order like we did in 3.40 we can construct the Weyl-covariant derivative on the boundary in our coordinate system exactly as happened in 3.42. This derivative will appear soon in our computations.

The corresponding Riemann tensor in these coordinates will be

$$
\widehat{R}^\lambda{}_{\mu\rho\nu} = \underline{e}_{[\rho}(\widehat{\Gamma}^\lambda_{\nu]\mu}) + \widehat{\Gamma}^\sigma_{[\nu\mu}\widehat{\Gamma}^\lambda_{\rho]\sigma} - C_{\rho\nu}{}^\sigma \widehat{\Gamma}^\lambda_{\sigma\mu} \tag{B.11}
$$

Contracting the two indices we get the Ricci tensor

$$
\widehat{R}_{\mu\nu} = \underline{e}_{[\lambda}(\widehat{\Gamma}^\lambda_{\nu]\mu}) + \widehat{\Gamma}^\sigma_{[\nu\mu}\widehat{\Gamma}^\lambda_{\lambda]\sigma} - C_{\lambda\nu}{}^\sigma \widehat{\Gamma}^\lambda_{\sigma\mu} \tag{B.12}
$$

with components

$$
\begin{aligned}
\widehat{R}_{ee} &= -\underline{D}_i \phi^i - \phi^2 - \underline{e}(\theta) - Tr(\rho\rho) - \frac{1}{4} Tr(ff) \\
\widehat{R}_{ei} = \widehat{R}_{ie} &= \underline{D}_j \left(\rho^j{}_i + \frac{1}{2} f^j{}_i \right) - \underline{\partial}_i \theta + \phi^j f_{ji} \\
\widehat{R}_{ij} &= \underline{R}_{ij} - \underline{D}_j \phi_i - (\underline{e} + \theta) \left(\rho_{ij} + \frac{1}{2} f_{ij} \right) - \phi_i \phi_j + 2\rho^k{}_i \rho_{kj} + \frac{1}{2} f_{jk} f^k{}_i
\end{aligned}
\tag{B.13}
$$

where

$$\underline{R}^i{}_{jkm} = \underline{\partial}_{[k}\underline{\Gamma}^i_{m]j} + \underline{\Gamma}^n_{[mj}\underline{\Gamma}^i_{k]n} \tag{B.14}$$

and the two Ricci scalars are related by

$$\boxed{\widehat{R}[g] = \underline{R}[h] - 2\underline{e}(\theta) - \theta^2 - 2\phi^2 - 2\underline{D}_i\phi^i + \frac{1}{4}Tr(ff) - Tr(\rho\rho)} \tag{B.15}$$

We see that although the bulk Christoffel symbols were not symmetric (i.e. there is torsion) the Ricci tensor is symmetric. Therefore the Einstein tensor will also be symmetric.

The Einstein tensor is defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{B.16}$$

and equations of motion read

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{B.17}$$

But since this is a tensor equation, it is invariant under coordinate transformations and thus it will has the same form for the hated tensors:

$$\widehat{G}_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{B.18}$$

Of course the last two equations do not mean that the components of the Einstein (or Riemann) tensor are the same in the two bases, since the hated μ, ν indices take values in $\{e, i\}$ in the Einstein tensor and the metric according to its basis while the unhated ones take values in $\{z, i\}$ (again according to its basis)[1].

So, by direct computation, starting from the Ricci tensor components and the Ricci scalar, we get the components of the Einstein tensor. In the third equation we expressed the Einstein tensor of the bulk metric $g_{\mu\nu}$ with the corresponding one for the small metric h_{ij} which is denoted by $\underline{G}_{ij}$ and is constructed from B.14.

[1]Here to be precise we should denote the new diagonal metric with $\widehat{g}$ to distinguish it form the old one g. But this would have made our notation very bad and difficult to read so we remark that since we made the coordinate transformation $\{z, i\} \rightarrow \{e, i\}$ whenever we write g we mean the new diagonal metric.

$$\begin{aligned}
\widehat{G}_{ee} =& -\frac{1}{2}Tr(\rho\rho) - \frac{3}{8}Tr(ff) - \frac{1}{2}\underline{R} + \frac{1}{2}\theta^2 \\
\widehat{G}_{ei} =& \widehat{G}_{ie} = \underline{D}_j\left(\rho^j{}_i + \frac{1}{2}f^j{}_i\right) - \underline{\partial}_i\theta + \phi^j f_{ji} \\
\widehat{G}_{ij} =& \underline{G}_{ij} - \underline{D}_j\phi_i - (\underline{e}+\theta)\left(\rho_{ij} + \frac{1}{2}f_{ij}\right) - \phi_i\phi_j + 2\rho^k{}_i\rho_{kj} \\
&+ \frac{1}{2}f_{jk}f^k{}_i + h_{ij}\left(\underline{e}(\theta) - \frac{1}{8}Tr(ff) + \frac{1}{2}Tr(\rho\rho) + \underline{D}_k\phi^k + \frac{1}{2}\theta^2 + \phi^2\right)
\end{aligned} \tag{B.19}$$

Combined with the Einstein equations yields

$$\begin{aligned}
0 =& -\frac{1}{2}Tr(\rho\rho) - \frac{3}{8}Tr(ff) - \frac{1}{2}\underline{R} + \frac{1}{2}\theta^2 + \Lambda \\
0 =& \underline{D}_j\left(\rho^j{}_i + \frac{1}{2}f^j{}_i\right) - \underline{\partial}_i\theta + \phi^j f_{ji} \\
0 =& \underline{G}_{ij} - \underline{D}_j\phi_i - (\underline{e}+\theta)\left(\rho_{ij} + \frac{1}{2}f_{ij}\right) - \phi_i\phi_j + 2\rho^k{}_i\rho_{kj} \\
&+ \frac{1}{2}f_{jk}f^k{}_i + h_{ij}\left(\underline{e}(\theta) - \frac{1}{8}Tr(ff) + \frac{1}{2}Tr(\rho\rho) + \underline{D}_k\phi^k + \frac{1}{2}\theta^2 + \phi^2 + \Lambda\right)
\end{aligned} \tag{B.20}$$

Before expanding the tensors we first define[2]

$$m^{(k)i}{}_j \equiv \left(h^{-1}_{(0)}h^{(k)}\right)^i{}_j \qquad n^{(k)i}{}_j \equiv \left(h^{-1}_{(0)}\pi^{(k)}\right)^i{}_j \tag{B.21}$$

and the scalars

$$\begin{aligned}
X^{(1)} &\equiv Tr(m^{(2)}) \\
X^{(2)} &\equiv Tr(m^{(4)}) - \frac{1}{2}Tr(m^2_{(2)}) + \frac{1}{4}\left(Tr(m^{(2)})\right)^2 \\
Y^{(1)} &\equiv Tr(n^{(0)})
\end{aligned} \tag{B.22}$$

From the expansions 3.53 we get the expressions

[2]Note that $h^{(0)}_{ij}$ acts as a metric only on the boundary terms. Which means that whenever we write $h^{ij}_{(2)}$ we simply mean $h^{ik}_{(0)}h^{(2)}_{kl}h^{lj}_{(0)}$, which is not the inverse of ${}^{(2)}_{ij}$.

$$h^{ij}(x,z) = z^2\left(h_{(0)}^{-1} - z^2 m^{(2)} h_{(0)}^{-1} - z^4(m^{(4)} - m_{(2)}^2)h_{(0)}^{-1} + ...\right)^{ij} - z^{d+2}\left(n^{(0)}h_{(0)}^{-1} + ...\right)^{ij}$$
$$\sqrt{-\det h(x,z)} = z^{-d}\sqrt{-\det h^{(0)}(x)}\left(1 + \frac{1}{2}z^2 X^{(1)} + \frac{1}{2}z^4 X^{(2)} + ... + \frac{1}{2}z^d Y^{(1)} + ...\right)$$
$$\rho^i{}_j(x,z) = -\delta^i{}_j + z^2 m^{(2)i}{}_j + z^4(2m^{(4)} - m_{(2)}^2)^i{}_j + ... + \frac{d}{2}z^d n^{(0)i}{}_j + ...$$
$$\theta(x,z) = -d + z^2 X^{(1)} + z^4 2(X^{(2)} - \frac{1}{4}(X^{(1)})^2) + ... + \frac{d}{2}z^d Y^{(1)} + ...$$
$$\phi_i(x,z) = z^2 2a_i^{(2)} + ... + z^{d-2}(d-2)p_i^{(0)} + ...$$
$$f_{ij}(x,z) = f_{ij}^{(0)}(x) + z^2\overline{\mathscr{D}}_{[i}^{(0)} a_{j]}^{(2)} + ... + z^{d-2}\overline{\mathscr{D}}_{[i}^{(0)} p_{j]}^{(0)} + ... \tag{B.23}$$

where the leading order of field strength tensor was defined by $f_{ij}^{(0)} \equiv \underline{\partial}_{[i}a_{j]}^{(0)}$. And the leading order of the Weyl-covariant derivative $\overline{\mathscr{D}}^{(0)}$ as we mentioned earlier.

We also expand the Christoffel symbols of the "small metric" as follows

$$\begin{aligned}\underline{\Gamma}^i_{jk} =& \underline{\Gamma}^{i(0)}_{jk} + z^2\left(\frac{1}{2}h_{(0)}^{im}\left(\overline{\mathscr{D}}_j^{(0)} h_{km}^{(2)} + \overline{\mathscr{D}}_k^{(0)} h_{jm}^{(2)} - \overline{\mathscr{D}}_m^{(0)} h_{jk}^{(2)}\right) - \left(a_{(j}^{(2)}\delta^i{}_{k)} - a_m^{(2)} h_{(0)}^{mi} h_{jk}^{(0)}\right)\right) \\ &+ ... - z^{d-2}\left(p_{(j}^{(0)}\delta^i{}_{k)} - p_m^{(0)} h_{(0)}^{mi} h_{jk}^{(0)}\right) + ...\end{aligned} \tag{B.24}$$

Therefore, by contracting B.14 and inserting the previous expansion, we get[3]

$$\begin{aligned}\overline{R}_{ij} =& R_{ij}^{(0)} + z^2\left(\frac{1}{2}\overline{\mathscr{D}}_n^{(0)}\left(h_{(0)}^{nm}\left(\overline{\mathscr{D}}_j^{(0)} h_{im}^{(2)} + \overline{\mathscr{D}}_i^{(0)} h_{jm}^{(2)} - \overline{\mathscr{D}}_m^{(0)} h_{ji}^{(2)}\right)\right)\right) \\ &+ z^2\left((d-1)\overline{\mathscr{D}}_j^{(0)} a_i^{(2)} - \overline{\mathscr{D}}_i^{(0)} a_j^{(2)} + h_{ij}^{(0)}\overline{\mathscr{D}}^{(0)}\cdot a^{(2)} - \frac{1}{2}\overline{\mathscr{D}}_j^{(0)}\overline{\mathscr{D}}_i^{(0)} X^{(1)}\right) \\ &+ ... + z^{d-2}\left((d-1)\overline{\mathscr{D}}_j^{(0)} p_i^{(0)} - \overline{\mathscr{D}}_i^{(0)} p_j^{(0)} + h_{ij}^{(0)}\overline{\mathscr{D}}^{(0)}\cdot p^{(0)}\right) + ...\end{aligned} \tag{B.25}$$

Similarly the Ricci scalar expands as

[3]We dropped the bar notation in the leading terms for simplicity.

$$\overline{R} = z^2 R^{(0)} + z^4 \left(h^{ij}_{(0)} \overline{\mathscr{D}}^{(0)}_i \overline{\mathscr{D}}^{(0)}_k \left(m^k_{(2)j} - Tr(m_{(2)}) \delta^k{}_j \right) + 2(d-1) \overline{\mathscr{D}}^{(0)} \cdot a^{(2)} - Tr(m_{(2)} h^{-1}_{(0)} R^{(0)}) \right)$$
$$+ ... + 2(d-1) z^d \overline{\mathscr{D}}^{(0)} \cdot p^{(0)} + ... \tag{B.26}$$

And finally we expand the Einstein tensor

$$\overline{G}_{ij} = G^{(0)}_{ij} + z^2 \left(\frac{1}{2} \overline{\mathscr{D}}^{(0)}_n \left(h^{nm}_{(0)} \left(\overline{\mathscr{D}}^{(0)}_j h^{(2)}_{km} + \overline{\mathscr{D}}^{(0)}_k h^{(2)}_{jm} - \overline{\mathscr{D}}^{(0)}_m h^{(2)}_{jk} \right) \right) \right)$$
$$+ z^2 \left((d-1) \overline{\mathscr{D}}^{(0)}_j a^{(2)}_i - \overline{\mathscr{D}}^{(0)}_i a^{(2)}_j + h^{(0)}_{ij} \overline{\mathscr{D}}^{(0)} \cdot a^{(2)} - \frac{1}{2} \overline{\mathscr{D}}^{(0)}_j \overline{\mathscr{D}}^{(0)}_i X^{(1)} \right)$$
$$+ z^2 \left(-\frac{1}{2} h^{(2)}_{ij} R^{(0)} - \frac{1}{2} h^{(0)}_{ij} \overline{\mathscr{D}}^{(0)}_k \overline{\mathscr{D}}^{(0)}_l \left((h^{-1}_{(0)} h^{(2)} h^{-1}_{(0)})^{kl} - X^{(1)} h^{kl}_{(0)} \right) + \frac{1}{2} h^{(0)}_{ij} Tr(m_{(2)} h^{-1}_{(0)} R^{(0)}) \right)$$
$$+ ... + z^{d-2} \left((d-1) \overline{\mathscr{D}}^{(0)}_j p^{(0)}_i - \overline{\mathscr{D}}^{(0)}_i p^{(0)}_j - (d-2) h^{(0)}_{ij} \overline{\mathscr{D}}^{(0)} \cdot p^{(0)} \right) + ... \tag{B.27}$$

First let us look on the zz-component of Einstein equations B.20. At the order $\mathcal{O}(z^0)$ we get the trivial equation $\Lambda = -\frac{d(d-1)}{2}$. The $\mathcal{O}(z^2)$ order gives

$$X^{(1)} = -\frac{1}{2(d-1)} R^{(0)} \tag{B.28}$$

And if we take the $\mathcal{O}(z^0)$ order of the ij-component and plug in B.28 we get 3.55.

Appendix C

Weyl cosmology

In order to derive the FRW metric we start from the most general form of a metric

$$ds^2 = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j \tag{C.1}$$

and consider an observer with spacetime coordinates (t, x^i). In the comoving frame its four-velocity will be

$$u^\mu = \begin{pmatrix} u^0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{C.2}$$

Since $u^\mu u_\mu = 1$ we get $u^0 = \frac{1}{\sqrt{g_{00}}}$.

Now, for a constant time slice of spacetime Σ_t, isotropy dectates that the projection of the four-velocity on this slice should be zero. This means that for ever $v \in \Sigma_t$ we have

$$u \cdot v = 0 \Rightarrow g_{0i}u^0 v^i = 0 \tag{C.3}$$

And since this should hold for every v on Σ_t, we get

$$g_{0i} = 0 \tag{C.4}$$

Furthermore, if we think about the acceleration of the observer, we know that it is given by

$$a^\mu = u^\nu \nabla_\nu u^\mu \tag{C.5}$$

Taking the i-component yields

$$a_i = \partial_i \ln \sqrt{g_{00}} \tag{C.6}$$

and isotropy demands that this should be zero. Therefore the coefficient g_{00} is only a function of time. This way we can define

$$t' = \int_0^t \sqrt{g_{00}(\lambda)} d\lambda \Rightarrow dt' = \sqrt{g_{00}} dt \tag{C.7}$$

Changing the notation from t' to t for simplicity gives us

$$ds^2 = dt^2 + g_{ij} dx^i dx^j \tag{C.8}$$

Also, considering a spatial vector $v \perp u$ and normalized (i.e. $v^\mu v_\mu = -1$ and $v^0 = 0$) we define

$$H = v^\mu v_\nu \nabla_\mu u^\nu \tag{C.9}$$

which is the spatial components of the gradient of the four-velocity. So it cannot depend on the direction of the spatial vector v^μ.

Doing a little bit of calculation we get

$$H = \frac{1}{2} v^i v^j \partial_t g_{ij} \tag{C.10}$$

But since $v^i v^j g_{ij} = -1$, and since we know that the result of $v^i v^j \partial_t g_{ij}$ cannot depend on the direction of v, we get that

$$\partial_t g_{ij} = A(t) g_{ij} \tag{C.11}$$

where A is scalar function of time. Inserting it back to the definition of H we get that $A = -2H$. Therefore

$$\partial_t g_{ij} = -2H g_{ij} \tag{C.12}$$

We also define

$$a(t) = e^{\int H(t)dt} \Rightarrow H = \frac{\dot{a}}{a} \tag{C.13}$$

and integrating $\partial_t g_{ij} = -2H g_{ij}$ gives

$$g_{ij} = -a^2 \tilde{g}_{ij} \Rightarrow \tag{C.14}$$

$$ds^2 = dt^2 - a^2(t)\tilde{g}_{ij}dx^i dx^j \tag{C.15}$$

Finally, we know that in a $(d+1)$-dimensional maximally symmetric space, like our Universe, the Riemann tensor can be written as

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}\left(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{C.16}$$

From this relation we get

$$R_{ij}[\tilde{g}] = \frac{R[\tilde{g}]}{3}\tilde{g}_{ij} \equiv 2k\tilde{g}_{ij} \tag{C.17}$$

Isotropy in the spatial metric $\tilde{g}$ implies that

$$\tilde{g}_{ij}dx^i dx^j = e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{C.18}$$

for an arbitrary function β which can be found from

$$R_{11}[\tilde{g}] = 2k\tilde{g}_{11} \Rightarrow \frac{2}{r}\beta'(r) = 2k\tilde{g}_{11} \Rightarrow \beta(r) = -\frac{1}{2}\ln\left(1 - kr^2\right) \tag{C.19}$$

and therefore we have the FRW metric in its full glory

$$ds^2 = dt^2 - a^2(t)\left(\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right) \tag{C.20}$$

To see the components more clearly we write it in a matrix form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{a^2(t)r^2\sin^2\theta} \end{pmatrix} \tag{C.21}$$

Homogeneity and isotropy implies that

$$A_\mu = (A(t), 0, 0, 0) \tag{C.22}$$

So we can compute the new "Christoffel symbols". The non-vanishing ones end up to be

$$\begin{gathered} \mathcal{G}^0_{00} = -A, \quad \mathcal{G}^0_{11} = \frac{a\dot{a} - Aa^2}{1-kr^2}, \quad \mathcal{G}^0_{22} = ar^2(\dot{a} - aA), \quad \mathcal{G}^0_{33} = ar^2\sin^2\theta(\dot{a} - aA) \\ \mathcal{G}^1_{01} = \frac{\dot{a}}{a} - A, \quad \mathcal{G}^1_{11} = \frac{kr^2}{1-kr^2}, \quad \mathcal{G}^1_{22} = r - kr^3, \quad \mathcal{G}^1_{33} = (r - kr^3)\sin^2\theta(\dot{a} - aA) \\ \mathcal{G}^2_{02} = \frac{\dot{a}}{a} - A, \quad \mathcal{G}^2_{12} = \frac{1}{r}, \quad \mathcal{G}^2_{33} = \sin\theta\cos\theta, \quad \mathcal{G}^3_{03} = \frac{\dot{a}}{a} - A, \quad \mathcal{G}^3_{13} = \frac{1}{r}, \quad \mathcal{G}^3_{23} = \cot\theta \end{gathered} \tag{C.23}$$

Next we define the Ricci tensor as

$$\mathcal{R}_{\mu\nu} = \partial_\lambda \mathcal{G}^\lambda_{\mu\nu} - \partial_\nu \mathcal{G}^\lambda_{\mu\lambda} + \mathcal{G}^\kappa_{\mu\nu}\mathcal{G}^\lambda_{\kappa\lambda} - \mathcal{G}^\kappa_{\mu\lambda}\mathcal{G}^\lambda_{\kappa\nu} \tag{C.24}$$

And with this definition we find that the non-vanishing components are

$$\begin{aligned} \mathcal{R}_{00} &= \frac{3(\dot{a}A + a\dot{A} - \ddot{a})}{a} \\ \mathcal{R}_{11} &= \frac{2k + 2\dot{a}^2 + a^2(2A^2 - \dot{A}) - 5a\dot{a}A + a\ddot{a}}{1-kr^2} \\ \mathcal{R}_{22} &= r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k + a^2(2A^2 - \dot{A}) - 5a\dot{a}A\right) \\ \mathcal{R}_{33} &= r^2\sin^2\theta\left(a\ddot{a} + 2\dot{a}^2 + 2k + a^2(2A^2 - \dot{A}) - 5a\dot{a}A\right) \end{aligned} \tag{C.25}$$

Therefore we can define the Ricci scalar to be $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, from which it yields

$$\mathcal{R} = -\frac{6\left(k + \dot{a}^2 + a\ddot{a} + a^2(A^2 - \dot{A}) - 3a\dot{a}A\right)}{a^2} \tag{C.26}$$