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The Fourier and Fourier-Stieltjes algebras of a locally compact group

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Contents

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Abstract

In this thesis, we present the Fourier and Fourier-Stieltjes algebras of a locally compact group and discuss some of their properties.

Let G be a locally compact group. We denote by $\mathbf{\Sigma}(G)$ the family of equivalence classes of unitary representations of G up to unitary equivalence. Then, the Fourier-Stieltjes algebra of G, denoted by $B(G)$, is the set of coefficient functions of representations of G. That is,

$$B(G) := \{\langle \pi(\cdot)\xi, \eta\rangle : \pi \in \mathbf{\Sigma}(G) \,\text{and}\, \xi, \eta \in H_\pi\}.$$

We endow $B(G)$ with pointwise addition and multiplication (the sum of functions corresponds to the direct sum of representations and the product of functions corresponds to the tensor product of representations) and the norm

$$\|u\| = \inf\{\|\xi\|\,\|\eta\| : u(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle,\, (\pi, H_\pi) \in \mathbf{\Sigma}(G), \xi, \eta \in H_\pi\}$$

for every $u \in B(G)$. With those operations and norm, $B(G)$ becomes a Banach algebra (2.0.30) and it is isometrically isomorphic to the dual of the group C^*-algebra (2.0.11). The duality is given by $\langle u \big| f\rangle = \int f(x)u(x)\,dx$ for $u \in B(G)$ and $f \in \mathbf{L}^1(G)$. The Fourier algebra, $A(G)$, is equal to the set of coefficient functions of the left regular representation of G, that is

$$A(G) = \{\langle \lambda(\cdot)f, g\rangle : f, g \in \mathbf{L}^2(G)\},$$

where λ is the left regular representation of G. Clearly $A(G)$ is a subset of $B(G)$. Endowed with the norm of $B(G)$, it is actually a closed ideal in $B(G)$ (3.1.5) and therefore, $A(G)$ is a Banach algebra.

In the first chapter, we give the necessary definitions and results from the theory of C^*-algebras, von Neumann algebras and locally compact groups, that will be needed for the rest of this thesis. Most of the results in this chapter, are stated without proof and the reader is referred to the literature for proofs. An exception to this, is the section 0.3, where we construct the universal enveloping von Neumann algebra of a C^*-algebra A in detail and show that it is isometrically isomorphic to A^{**} (0.3.2).

In the second chapter, we define the unitary representations of a locally compact group G and we show that there are bijective correspondences between unitary representations of G, non-degenerate representations of $\mathbf{L}^1(G)$ and representations of the measure algebra $M(G)$, whose restrictions to $\mathbf{L}^1(G)$ are non-degenerate (1.1.8 and 1.1.11). We also introduce the functions of positive type (1.2.1) and give a short overview of some of their properties. The first definition of $B(G)$ that we provide in the next chapter, will be based on the functions of positive type.

In the third chapter, we introduce $B(G)$, along with its subspaces, $B_{\mathcal{S}}(G)$ (2.0.12), where S is a class of unitary representations of G and we study some of its properties.

In the fourth chapter, we introduce $A(G)$ and show that it is a closed ideal in $B(G)$ (3.1.5). We also specify its spectrum and its dual. More specifically, we show that the spectrum of $A(G)$ is homeomorphic to the group G (3.2.1) and we show that the dual of $A(G)$ is isometrically isomorphic to the group von Neumann algebra, vN(G) (3.3.5).

Περίληψη

Σκοπός αυτής της εργασίας, είναι να μελετήσουμε τις άλγεβρες Fourier και Fourier-Stieltjes μιας τοπικά συμπαγούς ομάδας G και να περιγράψουμε ορισμένες τους ιδιότητες.

Έστω G μια τοπικά συμπαγής ομάδα και έστω $\mathbf{\Sigma}(G)$ η οικογένεια των κλάσεων ισοδυναμίας των unitary αναπαραστάσεων της G ως προς unitary equivalence. Τότε, η άλγεβρα Fourier-Stieltjes, $B(G)$, της G, αποτελείται από τις συναρτήσεις συντελεστών των αναπαραστάσεων της G, δηλαδή,

$$B(G) = \{\langle \pi(\cdot)\xi, \eta\rangle : (\pi, H_\pi) \in \mathbf{\Sigma}(G), \xi, \eta \in H_\pi\}.$$

Εφοδιάζουμε τη $B(G)$ με κατά σημείο πρόσθεση και πολλαπλασιασμό (το άθροισμα συναρτήσεων αντιστοιχεί σε ευθύ άθροισμα αναπαραστάσεων και το γινόμενο συναρτήσεων σε τανυστικό γινόμενο αναπαραστάσεων) και ορίζουμε νόρμα στην $B(G)$ με

$$\|u\| = \inf\{\|\xi\|\,\|\eta\| : u(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle, (\pi, H_\pi) \in \mathbf{\Sigma}(G), \xi, \eta \in H_\pi\}$$

για u στη $B(G)$. Με αυτές τις πράξεις και τη νόρμα, η $B(G)$ είναι άλγεβρα Banach (2.0.30) και είναι ισομετρικά ισόμορφη με τον δυϊκό της C^*-άλγεβρας της ομάδας, με τον δυϊσμό να δίνεται από $\langle u|f\rangle = \int f(x)u(x)\,dx$ για $f \in \mathbf{L}^1(G)$ και $u \in B(G)$ (2.0.11).

Η άλγεβρα Fourier, $A(G)$, της G, ορίζεται από

$$A(G) = \{\langle \lambda(\cdot)f, g\rangle : f, g \in \mathbf{L}^2(G)\},$$

όπου λ η αριστερή κανονική αναπαράσταση της G. Με αυτόν τον ορισμό, σαφώς η $A(G)$ είναι υπόχωρος της $B(G)$. Εφοδιασμένη με τη νόρμα που κληρονομεί από τη $B(G)$, η $A(G)$ είναι κλειστό ιδεώδες στη $B(G)$ (3.1.5) και έτσι η ίδια είναι άλγεβρα Banach.

Στο πρώτο κεφάλαιο δίνουμε τους απαραίτητους ορισμούς και αποτελέσματα που χρειαζόμαστε από τη θεωρία των C^*-αλγεβρών, των αλγεβρών von Neumann και των τοπικά συμπαγών ομάδων. Τα περισσότερα αποτελέσματα σε αυτό το κεφάλαιο δίνονται χωρίς απόδειξη και παραπέμπουμε στη βιβλιογραφία για αποδείξεις. Εξαίρεση αποτελεί η παράγραφος 0.3, στην οποία αναπτύσσουμε με λεπτομέρεια την κατασκευή της universal enveloping άλγεβρας von Neumann μιας C^*-άλγεβρας A και δείχνουμε ότι αυτή είναι ισομετρικά ισόμορφη με τον δεύτερο δυϊκό της A.

Στο δεύτερο κεφάλαιο, ορίζουμε τις unitary αναπαραστάσεις μιας τοπικά συμπαγούς ομάδας και δείχνουμε την αντιστοιχία ανάμεσα σε unitary αναπαραστάσεις της G, σε μη εκφυλισμένες αναπραστάσεις του $\mathbf{L}^1(G)$ και σε αναπαραστάσεις της άλγεβρας των μέτρων, $M(G)$, οι οποίες περιορισμένες στον $\mathbf{L}^1(G)$ είναι μη εκφυλισμένες (1.1.8,1.1.11). Ακόμα εισάγουμε τις συναρτήσεις θετικού τύπου (1.2.1) και κάνουμε μια σύντομη επισκόπηση κάποιων ιδιοτήτων τους. Βάσει αυτών θα ορίσουμε για πρώτη φορά στο επόμενο κεφάλαιο την $B(G)$.

Στο τρίτο κεφάλαιο, εισάγουμε την $B(G)$ μαζί με τους υποχώρους $B_{\mathcal{S}}(G)$ (2.0.12), όπου $\mathcal{S}$ είναι κλάση unitary αναπαραστάσεων της G και μελετάμε κάποιες από τις ιδιότητές της.

Στο τέταρτο κεφάλαιο εισάγουμε την $A(G)$ και δείχνουμε ότι είναι κλειστό ιδεώδες στην $B(G)$ (3.1.5). Ακόμα, προσδιορίζουμε το φάσμα και τον δυϊκό της. Πιο συγκεκριμένα, δείχνουμε ότι το φάσμα της $A(G)$ είναι ομοιομορφικό με την G (3.2.1) και δείχνουμε ότι ο δυϊκός της $A(G)$ είναι ισομετρικά ισόμορφος με την άλγεβρα von Neumann της ομάδας, vN(G) (3.3.5).

Preliminaries

0.1 C^*-algebras

In this section, we give some basic results and definitions related to C^*-algebras, that will be needed for the rest of this thesis.

Definition 0.1.1. *A Banach algebra is an algebra A, equipped with a norm $\|\cdot\|$, such that $(A, \|\cdot\|)$ is a Banach space and for every $a, b \in A$, we have $\|ab\| \leqslant \|a\| \, \|b\|$.*

Definition 0.1.2. *Let A be a complex algebra. An involution on A is a map $* : A \to A$ such that:*

- $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$
- $(a^*)^* = a$
- $(ab)^* = b^* a^*$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$. The algebra A equipped with an involution is called a $*$-algebra.

Definition 0.1.3. *A Banach-$*$ algebra is a Banach algebra A, equipped with an isometric involution.*

Definition 0.1.4. *Let A be a $*$-algebra and let $a \in A$. Then a is said to be selfadjoint if $a = a^*$. The set of selfadjoint elements of A is denoted by A_{sa}.*

Definition 0.1.5. *A C^*-algebra is a Banach $*$-algebra A that satisfies the C^*- property, that is*

$$\|a^*a\| = \|a\|^2$$

for every $a \in A$.

The prototypical example of a C^*-algebra is the algebra of bounded operators on a Hilbert space.

Example 0.1.6. *Let H be a Hilbert space and let $B(H)$ be the algebra of bounded operators on H. We equip $B(H)$ with the operator norm and with the involution that maps any bounded operator to its adjoint. Then $B(H)$ is a C^*- algebra. In fact, every closed $*$-subalgebra of $B(H)$ is a C^*-algebra.*

Definition 0.1.7. *Let A, B be C^*- algebras and let ϕ be a map from A to B. Then, ϕ is called a $*$-homomorphism if it is linear, it respects the multiplication and the involution, that is :*

- $\phi(a + \lambda b) = \phi(a) + \lambda\phi(b)$
- $\phi(ab) = \phi(a)\phi(b)$
- $\phi(a^*) = (\phi(a))^*$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$.

It is remarkable, that any such map is automatically continuous.

Theorem 0.1.8. *Let A, B be C^*-algebras and let $\phi : A \to B$ be a $*$-homomorphism. Then ϕ is continuous. Moreover, if ϕ is injective, then ϕ is isometric.*

The proof of this theorem can be found in [9](proposition 2.2.5. and corollary 2.2.6.).

Proposition 0.1.9. *Let A be a C^*-algebra and let I be a closed two sided ideal in A. Then we can define an involution on the algebra A/I, by defining $(a + I)^* := a^* + I$ for every $a \in A$. This involution is well defined on A/I and equipped with the quotient norm, A/I becomes a C^*-algebra.*

Proof. See [2](proposition 1.8.2.) □

We have the following immediate corollary to proposition 0.1.9.

Corollary 0.1.10. *Let A, B be two C^*-algebras and let $\phi : A \to B$ be a $*$-homomorphism. Then, the map $\tilde{\phi} : A/\operatorname{Ker}(\phi) \to B$ is a $*$-homomorphism from the C^*- algebra A/I to B and it is injective, therefore, by proposition 0.1.9 it is isometric. Moreover, the image of ϕ is clearly a closed $*$-subalgebra of B and therefore it is a C^*-subalgebra of B.*

Definition 0.1.11. *Let A be a Banach algebra. Then, A is called unital if it contains a multiplicative unit, that is, there is an element $1_A \in A$ such that $a1_A = 1_A a = a$ for every $a \in A$.*

Definition 0.1.12. *Let A be a unital Banach algebra and $a \in A$. Then a is called invertible if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1_A$, where 1_A is the unit of A.*

Definition 0.1.13. *Let A be a unital Banach algebra and let $a \in A$. We define the spectrum of a by*

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1_A \textit{ is invertible}\}$$

Proposition 0.1.14. *Let A be a unital Banach algebra and let $a \in A$. Then $\sigma(a) \neq \varnothing$.*

Proof. See [9] (theorem 1.2.8.). □

Definition 0.1.15. *Let A be a $*$-algebra and let $a \in A$. Then a is called selfadjoint if $a^* = a$.*

Definition 0.1.16. *Let A be a unital Banach algebra and let $a \in A$. Then a is said to be positive if a is selfadjoint and $\sigma(a) \subset [0, \infty)$.*

Now, let A be a non-unital Banach $*$-algebra and let $a \in A$. There is a natural way to define the spectrum of a in A and it is true that $\sigma(a) \neq \varnothing$, so the definition 0.1.16 makes sense and therefore, we say that a is positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For the details of this discussion see [9](1.2.12 and 1.2.1.13.)

If A is a C^*-algebra, there exists a very interesting characterization of the positive elements.

Theorem 0.1.17. *Let A be a C^*-algebra and let $a \in A$. Then a is positive if and only if there is some $x \in A$ such that $a = x^*x$.*

Proof. See [9] (theorem 3.1.10.) □

Definition 0.1.18. *Let A be a C^*-algebra and let $\phi : A \to \mathbb{C}$ be a linear functional on A. Then, ϕ is called positive if $\phi(a) \geqslant 0$ for every $a \in A$ that is positive. We denote the set of positive functionals on A by A^*_+.*

A very interesting fact about positive functionals is that they are automatically continuous. For a proof, see [9](theorem 4.1.5).

Definition 0.1.19. *A state on a C^*-algebra A is a positive functional of norm 1. We denote the set of states on A by $S(A)$.*

Definition 0.1.20. *Let H be a Hilbert space and $A \subset B(H)$ be a subalgebra. A vector $\xi \in H$ is said to be cyclic for A if the set $\{a(\xi) : a \in A\}$ is dense in H.*

Proposition 0.1.21. *Let A be a C^*-algebra and $\phi \in S(A)$. Then there exists a Hilbert space H_ϕ, a $*$-homomorphism $\pi_\phi : A \to B(H_\phi)$ and a vector $\xi_\phi \in H_\phi$ that is cyclic for $\pi(A)$ such that $\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi \rangle$ for every $a \in A$.*

Proof. See [9](4.2.1. and theorem 4.3.1.). □

Definition 0.1.22. *We define the universal representation of a C^*-algebra A, to be the map $\pi : A \to B(H)$, where $H = \bigoplus_{\phi \in S(A)} H_\phi$ and $\pi(a) = (\pi_\phi(a))_{\phi \in S(A)}$.*

Theorem 0.1.23 (Gelfand-Naimark)**.** *Let A be a C^*-algebra and $\pi : A \to B(H)$ be its universal representation. Then π is an injective $*$-homomorphism and therefore A is isometrically $*$-isomorphic to a C^*-subalgebra of $B(H)$.*

Proof. See [10](theorem 9.18.) □

Definition 0.1.24. *Let A be a* $\mathbf{C}^*$*- algebra and ϕ a bounded linear functional on A. Then we define $\phi^* : A \to \mathbb{C}, \quad a \mapsto \overline{\phi(a^*)}$. The functional ϕ^* is called the adjoint of ϕ. If $\phi^* = \phi$, we call ϕ selfadjoint. The set of selfadjoint functionals on A is denoted by A^*_{sa}.*

A simple calculation shows that ϕ^* is bounded and linear, with $\|\phi^*\| = \|\phi\|$. Moreover, $(\phi^*)^* = \phi$ for every $\phi \in A^*$ and $(\lambda \cdot \phi)^* = \overline{\lambda} \cdot \phi^*$ for every $\phi \in A^*$ and every $\lambda \in \mathbb{C}$, so the $*$- operation is an involution on A^*.

Proposition 0.1.25 (Jordan decomposition)**.** *Let A be a C^*-algebra and let $\phi \in A^*_{sa}$. Then there exist unique positive functionals ϕ_+, ϕ_- such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$.*

Proof. See [10] (proposition III.2.1.). □

Corollary 0.1.26. *Let $\phi \in A^*$. Then, $a = \frac{1}{2}(a + a^*) - \frac{1}{2i}(ia^* - ia)$ and $a + a^*, ia^* - ia \in A^*_{sa}$ and therefore $A^* = \operatorname{span} A^*_+$.*

0.2 von Neumann algebras

In this section we provide the basic definitions and results related to von Neumann algebras, that will be needed for the rest of this thesis.

Definition 0.2.1. *Let H be a Hilbert space and $B(H)$ be the bounded operators on H. We define the weak operator topology (WOT) on $B(H)$ to be the locally convex topology generated by the seminorms*

$$a \mapsto |\langle a\xi, \eta\rangle|$$

for $a \in B(H)$ and $\xi, \eta \in H$.

Definition 0.2.2. *Let H be a Hilbert space, we define the strong operator topology (SOT) on $B(H)$, to be the locally convex topology generated by the seminorms*

$$a \mapsto \|a\xi\|$$

for $a \in B(H)$ and $\xi \in H$.

Definition 0.2.3. *Let H be a Hilbert space and let $B(H)$ be the bounded operators on H. We define the σ-weak or ultraweak topology on $B(H)$ to be the locally convex topology generated by the family of seminorms*

$$a \mapsto \left| \sum_{j=1}^{\infty} \langle a\xi_j, \eta_j \rangle \right|,$$

where $(\xi_j)_j, (\eta_j)_j$ are sequences of vectors in H such that $\sum_{j=1}^{\infty} \|\xi_j\|^2 < \infty$ and $\sum_{j=1}^{\infty} \|\eta_j\|^2 < \infty$.

Definition 0.2.4 (von Neumann algebra)**.** *Let H be a Hilbert space and let A be a unital $*$-subalgebra of $B(H)$. If A is closed in the weak operator topology, then A is called a von Neumann algebra.*

Remark 0.2.5. *Let $(a_n)_n$ be a sequence in a von Neumann algebra A such that $(a_n)_n$ converges in the operator norm to some $a \in B(H)$. Then, for any $\xi, \eta \in H$, we have $|\langle a\xi, \eta\rangle - \langle a_n\xi, \eta\rangle| = |\langle (a-a_n)\xi, \eta\rangle| \leqslant \|a - a_n\| \|\xi\| \|\eta\|$ which converges to 0 and therefore $a_n \to a$ in the weak operator topology. Since A is WOT-closed, we get that $a \in A$ and therefore A is norm-closed. Since A is a norm-closed $*$-subalgebra of $B(H)$, we conclude that A is a C^*-subalgebra of $B(H)$. So every von Neumann algebra is also a C^*-algebra.*

Definition 0.2.6. *Let H be a Hilbert space and let $A \subset B(H)$. We define the commutant of A to be the set of elements of $B(H)$ that commute with the elements of A and we denote it by A'. That is,*

$$A' = \{b \in B(H) : ab = ba \,\forall\, a \in A\}.$$

Proposition 0.2.7. *Let $A \subset B(H)$ be closed under taking adjoints. Then A' is a $*$-subalgebra of $B(H)$ and it is closed in the weak operator topology. Moreover, A' is unital and therefore it is a von Neumann algebra.*

Proof. A' is clearly a subalgebra of $B(H)$ and it contains the identity operator, so it is unital.

Let $x \in A'$ and $a \in A$. Then, $x^*a = (a^*x)^* = (xa^*)^*$, since $x \in A'$ and $a^* \in A$, so $x^*a = ax^*$ and therefore, $x^* \in A'$, so A' is a $*$-subalgebra of $B(H)$.

To see that A' is WOT-closed, take a net (x_i) in A', such that x_i converges to some $x \in B(H)$ in the weak operator topology and let $a \in A$. Let $\xi, \eta \in H$. Then, $\langle x_i(a\xi), \eta\rangle \to \langle x(a\xi), \eta\rangle$, since $x_i \to x$ in the weak operator topology.

At the same time, $\langle a(x_i\xi), \eta\rangle = \langle x_i\xi, a^*\eta\rangle \to \langle x\xi, a^*\eta\rangle = \langle (ax)\xi, \eta\rangle$. Therefore, $ax = xa$ and $x \in A'$, so A' is WOT-closed and therefore it is a von Neumann algebra. □

Theorem 0.2.8 (von Neumann's double commutant theorem)**.** *Let H be a Hilbert space and let $A \subset B(H)$ be a $*$-subalgebra of $B(H)$ that acts non-degenerately on H, that is, there is no $\xi \in H$ non-zero such that $a(\xi) = 0$ for every $a \in A$. Then $\overline{A}^{WOT} = A''$.*

Proof. See [9] (theorem 5.2.7.). □

Proposition 0.2.9. *Let H be a Hilbert space and let C be a convex subset of $B(H)$. Then C is WOT-closed if and only if C is SOT-closed.*

Proof. See [12] (corollary 2.7.5.)

□

Corollary 0.2.10. *Let A be a $*$-subalgebra of $B(H)$. Then A is convex and therefore $\overline{A}^{WOT} = \overline{A}^{SOT}$.*

Proposition 0.2.11. *Let $X \subset B(H)$ be bounded, then the weak and ultra-weak topologies coincide on X.*

Proof. See [12] (proposition 2.7.19.). □

Theorem 0.2.12 (Kaplansky's density theorem)**.** *Let H be a Hilbert space and let A be a $*$-subalgebra of $B(H)$, with SOT-closure B. Then:*

1. *A_{sa} is SOT-dense in B_{sa}.*
2. *The norm-closed unit ball of A is SOT-dense in the norm-closed unit ball of B.*

Proof. See [7] (theorem 4.3.3.) □

Let A be a $*$-subalgebra of $B(H)$ with SOT closure B. Clearly the closed unit ball of A is convex and therefore, by 0.2.9, we get that the closure of the closed ball of A in the weak operator topology is the same as the closure in the strong operator topology and therefore, the closed unit ball of A is WOT-dense in the closed unit ball of B. Moreover, since the closed unit ball of A is bounded, by 0.2.11, we get that the closed unit ball of A is ultraweakly dense in the closed unit ball of B.

Corollary 0.2.13. *Let A be a $*$-subalgebra of $B(H)$ with SOT closure B. Then the unit ball of A is WOT dense in the closed unit ball of B.*

Definition 0.2.14. *Let H be a Hilbert space and let $\mathcal{M} \subset B(H)$ be a von Neumann subalgebra of $B(H)$. Then, we define $\mathcal{M}_*$ to be the subspace of the dual of $\mathcal{M}$, consisting of the ultraweakly continuous functionals in $\mathcal{M}$. That is,*

$$\mathcal{M}_* = \{\phi \in \mathcal{M}^* : \phi \text{ ultraweakly continuous}\}.$$

Proposition 0.2.15. *Let $\mathcal{M}$ be a von Neumann subalgebra of $B(H)$. Then, $\mathcal{M}_*$ is a closed subspace of $\mathcal{M}^*$ and $\mathcal{M}$ is isometrically isomorphic to $(\mathcal{M}_*)^*$ via the pairing*

$$\langle a | \phi \rangle = \phi(a)$$

for $a \in \mathcal{M}$ and $\phi \in \mathcal{M}_*$.

Proof. See [10] (theorem II.2.6.). □

Remark 0.2.16. *Notice that with this duality, the w^*-topology on $\mathcal{M}$ is exactly the ultraweak topology.*

Proposition 0.2.15 shows that every von Neumann algebra is a dual space. By a famous theorem of Sakai, von Neumann algebras have unique preduals.

Theorem 0.2.17 (Sakai)**.** *Let $\mathcal{M}$ be a von Neumann algebra. If E is a Banach space whose dual is isometrically isomorphic to $\mathcal{M}$, then E is isometrically isomorphic to $\mathcal{M}_*$.*

Proof. See [10] (corollary 3.9). □

0.3 The universal enveloping von Neumann algebra of a C^*-algebra

Every von Neumann algebra is also a $\mathbf{C}^*$- algebra. While the converse is not true, we do have a natural way to associate to any $\mathbf{C}^*$- algebra A a unique von Neumann algebra, called the universal enveloping von Neumann algebra of A. In fact, this von Neumann algebra is isometrically isomorphic to the double dual of A.

Proposition 0.3.1. *Let A be a $\mathbf{C}^*$- algebra, H a Hilbert space and $\pi : A \to B(H)$ a representation of A. Let $\mathcal{M} = (\pi(A))''$ be the von Neumann algebra generated by the image of π in $B(H)$. Then, if $i : A \to A^{**}$ is the natural embedding of A into A^{**}, there exists a unique $\tilde{\pi} : A^{**} \to \mathcal{M}$ such that $\tilde{\pi} \circ i = \pi$. The map $\tilde{\pi}$ is onto and it is continuous with respect to the w^* and the ultraweak topology on A^{**} and $\mathcal{M}$ respectively.*

Proof. First of all, notice that $i(A)$ is w^*- dense in A^{**}. Then, if $\tilde{\pi}$ is w^*-continuous, $\tilde{\pi}$ is determined by its values on the dense $i(A)$ and is therefore unique. So we only need to check that such a map exists.

Let $\mathcal{M}_*$ denote the predual of $\mathcal{M}$. Since $\pi : A \to \mathcal{M}$, we have $\pi^* : \mathcal{M}^* \to A^*$. Let $r = \pi^*|_{\mathcal{M}_*}$, then its transpose, r^* is a map from A^{**} to $(\mathcal{M}_*)^* = \mathcal{M}$, so : $A^{**} \to \mathcal{M}$. As we will see, r^* is exactly the map we are looking for.

Let $a \in A$ and $\phi \in \mathcal{M}_*$, then $\langle r^*(i(a)) \big| \phi \rangle = \langle i(a) \big| r\phi \rangle = \langle \pi^*\phi \big| a \rangle = \langle \phi \big| \pi(a) \rangle$ and so $r^*(i(a)) = \pi(a)$ for every $a \in A$ and $r^* \circ i = \pi$. Also, r^* is a dual map and therefore it is continuous with respect to the w^* and the ultraweak topologies. We only need to show that r^* is onto.

Assume first that π is isometric. By Goldstine's theorem, $i(B_A)$ is w^*-dense in $B_{A^{**}}$ and by the Banach-Alaoglou theorem, $B_{A^{**}}$ is w^*- compact. Since r^* is w^*- continuous, $r^*(B_{A^{**}})$ is ultraweakly compact in $\mathcal{M}$ and $r^*(i(B_A)) = \pi(B_A)$ is ultraweakly dense in $r^*(B_{A^{**}})$. Now, π is isometric, so $\pi(B_A) = B_{\mathcal{M}} \cap \pi(A)$, where $B_{\mathcal{M}}$ denotes the closed unit ball of $\mathcal{M}$. The ultraweak topology on $\mathcal{M}$ is the w^*- topology induced by $\mathcal{M}_*$, so by the

Banach-Alaoglou theorem, $B_{\mathcal{M}}$ is ultraweakly compact and therefore closed in $\mathcal{M}$. Moreover, by Kaplansky's density theorem, $\pi(A) \cap B_{\mathcal{M}}$ is ultraweakly dense in $B_{\mathcal{M}}$ and so $\overline{\pi(B_A)}^{w^*} \subset r^*(B_{A^{**}}) \subset B_{\mathcal{M}}$, since π is isometric and therefore r^* is contractive. But $\overline{\pi(B_A)}^{w^*} = B_{\mathcal{M}}$, so $r^*(B_A^{**}) = B_{\mathcal{M}}$ and r^* is onto.

Now for the general case, let $q : A \to A/\operatorname{Ker}(\pi)$ be the quotient map. Then there exists a unique $\tilde{\pi} : A/\operatorname{Ker}(\pi) \to \mathcal{M}$ such that $\tilde{\pi} \circ q = \pi$. In that case, $\tilde{\pi}$ is isometric, since it is an injective $*$-homomorphism between C^*-algebras (0.1.8). We only need to show that $\pi(B_A)$ is ultraweakly dense in $B_{\mathcal{M}}$, then the conclusion follows in exactly the same way as before. By the isometric case, we know that $\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)})$ is ultraweakly dense in $B_{\mathcal{M}}$. Let B_A° and $B_{A/\operatorname{Ker}(\pi)}^\circ$ denote the open unit balls of A and $A/\operatorname{Ker}(\pi)$ respectively. We claim that $\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)}^\circ) = \pi(B_A^\circ)$.

Clearly, $\pi(B_A^\circ) \subset \tilde{\pi}(B_{A/\operatorname{Ker}(\pi)}^\circ)$, since $\|a + \operatorname{Ker}(\pi)\| \leqslant \|a\|$ for every $a \in A$. Let $a \in A$ such that $a + \operatorname{Ker}(\pi) \in B_{A/\operatorname{Ker}(\pi)}^\circ$. Then $\|a + \operatorname{Ker}(\pi)\| < 1$ and we can find a sequence $(x_n)_n$ in $\operatorname{Ker}(\pi)$ such $\|a + x_n\| \to \|a + \operatorname{Ker}(\pi)\|$ and so for large enough n, we have $\|a + x_n\| < 1$ and $a + x_n \in B_A^\circ$. Moreover, $\pi(a + x_n) = \pi(a) = \tilde{\pi}(a + \operatorname{Ker}(\pi))$ for every n, so $\tilde{\pi}(a + \operatorname{Ker}(\pi)) \in \pi(B_A^\circ)$ and therefore $\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)}^\circ) = \pi(B_A^\circ)$. $\tilde{\pi}$ is an isometry, so

$\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)}^\circ) = B_{\mathcal{M}}^\circ \cap \tilde{\pi}(A/\operatorname{Ker}(\pi)) = B_{\mathcal{M}}^\circ \cap \pi(A)$ and then $\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)}) = B_{\mathcal{M}} \cap \pi(A)$. By the isometric case, $\tilde{\pi}(B_{A/\operatorname{Ker}(\pi)})$ is ultraweakly dense in $B_{\mathcal{M}}$ and so $B_{\mathcal{M}} \cap \pi(A)$ is ultraweakly dense in $B_{\mathcal{M}}$ and we are done. □

Among all representations of A, there is one making the map $\tilde{\pi}$ of the previous theorem isometric and since this map is always surjective, A^{**} is isometrically isomorphic to the von Neumann algebra generated by the particular representation of A, through $\tilde{\pi}$.

Theorem 0.3.2. *Let A be a* $\mathbf{C}^*$*- algebra and $\pi : A \to B(H)$ its universal representation. Then $\tilde{\pi} : A^{**} \to \mathcal{M} = (\pi(A))''$ is an isometric isomorphism. Moreover, the following also hold:*

1. *$\tilde{\pi}$ is a homeomorphism with respect to the w^* and the ultraweak topolo-*

gies on A and $\mathcal{M}$ respectively.

2. *If $f \in A^*$, there exist $\xi, \eta \in H$ such that $f(a) = \langle \pi(a)\xi, \eta \rangle$ for every $a \in A$.*

3. *If $\mathcal{N}$ is a von Neumann algebra and $\rho_0 : A \to \mathcal{N}$ is a $*-$ homomorphism, there exists an ultraweakly continuous $*-$ homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ such that $\rho_0 = \rho \circ \pi$ and ρ maps $\mathcal{M}$ onto $(\rho_0(A))''$.*

$\mathcal{M}$ is called the universal enveloping von Neumann algebra of A.

Proof. With the notation of proposition 0.3.1, we have that $\tilde{\pi} = r^*$ and $\mathrm{Ker(r^*)} = (\mathrm{Im(r)})^\perp$, where $(\mathrm{Im(r)})^\perp$ denotes the orthogonal complement of $\mathrm{Im(r)}$ in H. Let $f \in A^*$ be a positive functional of norm 1. Using the GNS construction [10] (theorem 9.14), we can find a Hilbert space H_f, a representation π_f of A on H_f and $\xi_f \in H_f$ a cyclic vector for A such that $f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle$ for every $a \in A$. Recall that $H = \bigoplus_\sigma H_\sigma$, where σ runs over all of the positive functionals on A of norm 1, so we can define $\xi \in H$ such that the f coordinate of ξ is ξ_f and all others are 0. Then, if we denote the σ coordinate of ξ by ξ_σ, we have $\langle \pi(a)\xi, \xi \rangle = \sum_\sigma \langle \pi_\sigma(a)\xi_\sigma, \xi_\sigma \rangle = \langle \pi_f(a)\xi_f, \xi_f \rangle = f(a)$ and so every positive functional of norm 1 is of the form $\langle \pi(\cdot)\xi, \xi \rangle$ for some $\xi \in H$ and by scaling, the same holds for any positive functional on A. Since the positive functionals span the whole A^*, by polarization we conclude that every $f \in A^*$ is of the form $\langle \pi(\cdot)\xi, \eta \rangle$ for some $\xi, \eta \in H$, which proves 2. For convenience, we will denote the functional on $\mathcal{M}$ defined by $T \mapsto \langle T\xi, \eta \rangle$ by $\omega_{\xi,\eta}$. Then, $\langle f | a \rangle = \langle \omega_{\xi,\eta} | \pi(a) \rangle = \langle \pi^* \omega_{\xi,\eta} | a \rangle$ for every $f \in A^*$ and $a \in A$, so $\pi^*(\{\omega_{\xi,\eta} : \xi, \eta \in H\}) = A^*$ and so $\pi^*(\mathcal{M}^*) = A^*$. Moreover, clearly $\omega_{\xi,\eta} \in \mathcal{M}_*$, so $r(\mathcal{M}_*) = A^*$ and r is surjective. Then, $\mathrm{Ker(r^*)} = (\mathrm{Im(r)})^\perp = \{0\}$ and r^* is injective. From the proof of the previous proposition, we know that $r^*(B_A^{**}) = B_{\mathcal{M}}$. Assume that there exists a $a^{**} \in B_A^{**}$ with $\|a^{**}\| = 1$ such that $\|r^*(a^{**})\| = \lambda \neq 1$. Recall that r^* is a contraction, so $\lambda < 1$ and r^* is injective, so λ cannot be 0. Now, $r^*(a^{**}) \in B_{\mathcal{M}} = r^*(B_{A^{**}})$, so there exists $x \in B_{A^{**}}$ such that $r^*(x) = \frac{r^*(a^{**})}{\lambda}$ and therefore $r^*(\lambda \cdot x) = r^*(a^{**})$. Since r^* is injective, we have $\lambda \cdot x = a^{**}$ and $x = \frac{a^{**}}{\lambda}$, but $\frac{a^{**}}{\lambda} \notin B_{A^{**}}$ and therefore $x \notin B_{A^{**}}$, which is absurd and thus r^* is isometric. We showed in the previous proposition that it is surjective, so $r^* : A^{**} \to \mathcal{M}$ is an isometric isomorphism.

We have already seen that $\tilde{\pi}$ is w^*-ultraweakly continuous. Let T_i be a net in $\mathcal{M}$ such that $T_i \to T$ ultraweakly for some $T \in \mathcal{M}$. Then, there exist unique $f_i, f \in A^{**}$ such that $r^*(f_i) = T_i$ for every i and $r^*(f) = T$. Then, $r^* f_i \to r^* f$ ultraweakly $\iff$ $\langle r^*(f_i)\big|\psi\rangle \to \langle r^*(f)\big|\psi\rangle$ for every $\psi \in \mathcal{M}_*$ $\iff$ $\langle f_i\big|r\psi\rangle \to \langle f\big|r\psi\rangle$ for every $\psi \in \mathcal{M}_*$ and as we saw $r(\mathcal{M}_*) = A^*$ and therefore, $\langle f_i\big|r\psi\rangle \to \langle f\big|r\psi\rangle$ $\iff$ $\langle f_i\big|a^*\rangle \to \langle f\big|a^*\rangle$ for every $a^* \in A^*$ which means exactly that $f_i \xrightarrow{w^*} f$ and therefore, $\tilde{\pi}$ is a homeomorphism with respect to the w^* and the ultraweak topology respectively, which proves 1.

For 3, let $\mathcal{N}$ be a von Neumann algebra and $\rho_0 : A \to \mathcal{M}$ a $*-$ homomorphism. Then, consider $\tilde{\rho}_0 : A^{**} \to (\rho(A))'' \subset \mathcal{N}$ the map defined in proposition 0.3.1 and define $\rho : \mathcal{M} \to \mathcal{N}$ by $\rho = \tilde{\rho}_0 \circ (\tilde{\pi})^{-1}$. Then, $\rho \circ \pi = \rho_0$ and ρ is an ultraweakly continuous $*-$ homomorphism onto $(\rho_0(A))''$.

□

In fact, the enveloping von Neumann algebra of A is the unique up to ultraweakly continuous $*-$ isomorphism von Neumann algebra satisfying (3). To be more specific, we will need a definition.

Definition 0.3.3. *Let A be a $\mathbf{C}^*$- algebra and $\pi : A \to B(H)$ a representation of A on some Hilbert space H. The representation (π, H) is called universal if for any other representation (ρ, K) of A there exists an ultraweakly continuous $*$- homomorphism $\tilde{\rho} : (\pi(A))'' \to (\rho(A))''$ that is onto and such that $\tilde{\rho} \circ \pi = \rho$.*

By the previous theorem, it is clear that the universal representation of A is universal in the sense of definition 0.3.3. As the next proposition shows it is the only universal representation in this sense, thus justifying its name.

Proposition 0.3.4. *Let A be a $\mathbf{C}^*$- algebra and $(\pi, H), (\rho, K)$ be two universal representations of A. Then there exists an ultraweakly continuous $*$-isomorphism $T : (\rho(A))'' \to (\pi(A))''$ such that $T\tilde{\rho} = \tilde{\pi}$*

Proof. Since (π, H) is universal, there exists a $\phi_1 : (\pi(A))'' \to (\rho(A))''$ such that $\phi_1 \circ \pi = \rho$. In the same way, there is $\phi_2 : (\rho(A))'' \to (\pi(A))''$ such that $\phi_2 \circ \rho = \pi$ and so $\phi_1 \circ \phi_2 \circ \rho = \rho$. Now, $\rho(A)$ is ultraweakly dense in $(\rho(A))''$ and $\phi_1 \circ \phi_2$ is ultraweakly continuous, so $\phi_1 \circ \phi_2 = Id_{(\pi(A))''}$. In exactly the same way, $\phi_2 \circ \phi_1 = Id_{(\rho(A))''}$ and ϕ_1 is an ultraweakly continuous $*$- isomorphism between $(\pi(A))''$ and $(\rho(A))''$ and $\phi_1 \circ \pi = \rho$. □

0.4 Locally compact groups

Definition 0.4.1. *(Topological group) Let G be a group endowed with some topology. We call G a topological group if the multiplication $m : G \times G \to G, \quad (x, y) \mapsto xy$ and inversion $i : G \to G, \quad x \mapsto x^{-1}$ maps are continuous.*

Definition 0.4.2. *A locally compact group is a topological group G such that the topology of G is Hausdorff and locally compact.*

Proposition 0.4.3. *Let G be a locally compact group and $x, y \in G$. Let $C_c(G)$ denote the set of compactly supported continuous functions on G. For every $f \in C_c(G)$ we define $L_x f : G \to \mathbb{C}$ by $(L_x f)(z) = f(x^{-1}z)$ and $R_y f : G \to \mathbb{C}$ by $(R_y f)(z) = f(zy)$ for every $z \in G$. Then, $L_x f$ and $R_y f$ lie in $C_c(G)$ and for every $f \in C_c(G)$, we have that $\|L_x f - f\|_\infty \to 0$ as x tends to the identity element e and $\|R_y f - f\|_\infty \to 0$ as y tends to e.*

Proof. See [4](proposition 2.6.) □

Proposition 0.4.4. *Let K, C be two compact subsets of G. Then, the sets KC and K^{-1} are compact.*

Proof. Let $m : G \times G \to G$ be the multiplication map. Then, $KC = m(K \times C)$ and therefore, KC is compact, since $K \times C$ is compact and m is continuous. In the same way, $K^{-1} = i(K)$ and therefore K^{-1} is a compact subset of G. □

Locally compact groups have the unique property that they can be equipped with a very special measure, the so called Haar measure. Before introducing this measure, we will need some additional terminology.

Definition 0.4.5. *Let μ be a Borel measure on a topological group G. Then μ is called left invariant if $\mu(xE) = \mu(E)$ for every $x \in G$ and every Borel subset $E \subset G$.*

Definition 0.4.6. *Let X be a locally compact topological space and let μ be a positive Borel measure on X. Then, μ is called a Radon measure if the following hold:*

1. *$\mu(K) < \infty$ for every compact $K \subset G$.*
2. *For every open set $U \subset G$, we have that*
$$\mu(U) = \sup\{\mu(K) : K \subset U \, compact\}.$$
3. *For every E Borel subset of G, we have that*
$$\mu(E) = \inf\{\mu(U) : E \subset U \, open\}.$$

Theorem 0.4.7. *[Haar measure] Let G be a locally compact group. Then there exists a non-zero left invariant Radon measure on G. This measure is unique up to a multiplicative constant and we denote it by λ_G. If it is clear that it is a measure on G, we simply denote it by λ.*

Proof. See [4](theorem 2.10). □

Proposition 0.4.8. *Let μ be a non-zero Radon measure on the locally group G. Then μ is a Haar measure if and only if $\int (L_x f)(y) \, d\mu(y) = \int f(y) \, d\mu(y)$ for every $f \in C_c(G)$ and every $x \in G$.*

Proof. See [4](proposition 2.9.) □

Proposition 0.4.9. *Let G be a locally compact group and let U be a nonempty open subset of G. Then $\lambda(U) > 0$.*

Proof. Assume that there is an open $U \subset G$ such that $\lambda(U) = 0$. Then, for every compact subset G that we denote by K, we have that K can covered by finitely many translates of U and therefore $\lambda(K) = 0$ for every compact

$K \subset G$. Now, λ is a Radon measure and therefore, $\lambda(V) = \sup\{\lambda(K) : K \subset U \text{ compact}\}$ for every $V \subset G$ open. The same holds for $V = G$ and therefore $\lambda(G) = 0$ and $\lambda = 0$, which is a contradiction. $\square$

After choosing a Haar measure λ for G, for every $p \in [1, \infty)$, we can define the corresponding Lebesgue space $\mathbf{L}^p(\lambda)$ and we denote it simply by $\mathbf{L}^p(G)$.

In the case $p = \infty$, we will need to define $\mathbf{L}^\infty(G)$ in a slightly different way. Of course, the classical Lebesgue space $\mathbf{L}^\infty(\lambda)$ makes sense for the Haar measure, but since in general the Haar measure is not σ-finite, it is generally not the case that $\mathbf{L}^\infty(G)$ is isometrically isomorphic to the dual of $\mathbf{L}^1(G)$. To fix this, we will define $\mathbf{L}^\infty(G)$ as follows.

Definition 0.4.10. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \subset X$. Then, A is said to be locally in A, if $A \cap B \in \mathcal{A}$ for every $B \in \mathcal{A}$ such that $\mu(B) < \infty$.*

Definition 0.4.11. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \subset X$ be a locally measurable set. Then A is said to be locally null if $\mu(A \cap B) = 0$ for every $B \in \mathcal{A}$.*

Definition 0.4.12. *Let $(X, \mathcal{A}, \mu)$ be a measure space. A property is said to hold locally almost everywhere on X, if it holds everywhere, except possibly on a locally null set.*

Definition 0.4.13. *Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$. Then f is said to be locally measurable if $f^{-1}(B)$ is locally measurable for every $B \subset \mathbb{C}$ Borel.*

Definition 0.4.14. *Let $(X, \mathcal{A}, \mu)$ be a measure space. We define $\mathbf{L}^\infty(\mu)$ to be the space of locally measurable functions on X, that are bounded except on a locally null set, equipped with the norm $\|\cdot\|_\infty$, defined by*

$$\|f\|_\infty = \inf\{c > 0 : |f(x)| \leqslant c \textit{ locally almost everywhere}\}.$$

When X is a locally compact Hausdorff space and μ is a Radon measure on X, then $\mathbf{L}^\infty(\mu)$ as defined above, is isometrically isomorphic to the dual of $\mathbf{L}^1(G)$, via the pairing

$$\langle \phi | f \rangle = \int f(x)\phi(x)\, d\mu(x)$$

for $\phi \in \mathbf{L}^\infty(\mu)$ and $f \in \mathbf{L}^1(\mu)$. For a proof of this result and a discussion about how to overcome the problems created by the fact that the Haar measure is generally not σ-finite, see [4] (section 2.3.). Therefore, we define $\mathbf{L}^\infty(G)$ to be $\mathbf{L}^\infty(\lambda)$, where $\mathbf{L}^\infty(\lambda)$ is as defined in 0.4.14.

Proposition 0.4.15. *Let G be a locally compact compact group and let λ be a Haar measure on G. Then $C_c(G)$ is dense in $\mathbf{L}^p(G)$ for every $p \in [1, \infty)$.*

Proof. See [5] (proposition 7.9.). $\square$

Let $x \in G$. For every $E \subset G$ Borel, we define $\lambda_x(E) := \lambda(Ex)$. Then λ_x is a left invariant Radon measure on G and by theorem 0.4.7, there exists a $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$. It is proved in [4](proposition 2.24) that the function $\Delta : G \to \mathbb{R}_\times$, is a continuous group homomorphism, where $\mathbb{R}_\times$ denotes the multiplicative group of positive numbers and that for every $f \in C_c(G)$, we have that

$$\int R_y f \, d\lambda = \Delta(y^{-1}) \int f \, d\lambda.$$

Proposition 0.4.16. *Let G be a locally compact group and let $p \in [1, \infty)$. If $f \in \mathbf{L}^p(G)$, then the maps $G \to \mathbf{L}^p(G), \quad x \mapsto L_x f$ and $G \to \mathbf{L}^p(G), \quad x \mapsto R_x f$ are continuous.*

Proof. See [4] (proposition 2.41.). $\square$

Definition 0.4.17. *Let X be a locally compact topological space and let μ be a complex Borel measure on X. Then μ is called a complex Radon measure if it is the complex linear combination of positive Radon measures on X.*

Proposition 0.4.18 (Riesz's representation theorem)**.** *Let X be a locally compact Hausdorff space and let $M(X)$ be the Banach space of the complex Radon measures on X, equipped with the norm of total variation. Then*

$M(X)$ is isometrically isomorphic to the dual of $C_0(X)$, where $C_0(X)$ denotes the space of continuous functions on X vanishing at infinity, equipped with the supremum norm. The duality is given by

$$\langle \mu | f \rangle := \int f(x)\, d\mu(x)$$

for $f \in C_0(G)$ and $\mu \in M(X)$.

Proof. See [5](theorem 7.17) $\square$

Let G be a locally compact group and $\mu, \nu \in M(G)$. We define $\phi : C_0(G) \to \mathbb{C}$ by

$$f \mapsto \iint f(xy)\, d\mu(x) d\nu(y).$$

Then,

$$|\phi(f)| \leqslant \iint |f(xy)|\, d|\mu|(x) d|\nu|(y) \leqslant \|f\|_\infty \|\mu\| \|\nu\|, \tag{*}$$

so ϕ is well defined, clearly linear and bounded. Thus $\phi \in (C_0(G))^*$ and by proposition 0.4.18, there exists a unique measure in $M(G)$, which we denote by $\mu \star \nu$, such that $\phi(f) = \int f(x)\, d(\mu \star \nu)(x)$ for every $f \in C_0(G)$ and therefore

$$\int f(x)\, d(\mu \star \nu)(x) = \iint f(xy)\, d\mu(x) d\nu(y)$$

for every $f \in C_0(G)$. We call the measure $\mu \star \nu$ the convolution of μ and ν. Moreover, by (*), we get that $\|\mu \star \nu\| \leqslant \|\mu\| \|\nu\|$ and therefore, $M(G)$ equipped with the convolution product is a Banach algebra.

Proposition 0.4.19. *Let G be a locally compact group and let $\mu \in M(G)$. Then, for every $E \subset G$ Borel, we define $\mu^*(E) := \overline{\mu(E^{-1})}$. Then $\mu^* \in M(G)$ and the map $* : M(G) \to M(G), \quad \mu \mapsto \mu^*$ is an involution on $M(G)$.*

Proof. See [4](2.35.) $\square$

When there is no risk of confusion, we write dx instead of $d\lambda(x)$ for the Haar measure on the locally compact group G.

Definition 0.4.20 (Measure algebra)**.** *Let G be a locally compact group. We define the measure algebra of the group to be the Banach $*$-algebra $M(G)$, where $M(G)$ is equipped with the convolution product and the involution defined in 0.4.19.*

Let $f \in \mathbf{L}^1(G)$. By identifying f with the measure $fd\lambda$, where λ is the Haar measure on G, we can consider $\mathbf{L}^1(G)$ as a subspace of $M(G)$. Then $\|fd\lambda\| = \|f\|_1$, therefore $\mathbf{L}^1(G)$ can be isometrically embedded into $M(G)$. For $f, g \in \mathbf{L}^1(G)$, the convolution of the measures $fd\lambda$ and $gd\lambda$ is given by the measure $(f \star g)d\lambda$, where $f \star g \in \mathbf{L}^1(G)$ is defined by $(f \star g)(x) = \int f(y)g(y^{-1}x)\, dy$ for $x \in G$. This function is called the convolution of f, g and it agrees with the usual convolution of functions in $\mathbf{L}^1(\mathbb{R})$, for $G = \mathbb{R}$. Moreover, $(fd\lambda)^* = f^*d\lambda$, where $f^* \in \mathbf{L}^1(G)$ and $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ for $x \in G$. Therefore, $\mathbf{L}^1(G)$, equipped with the convolution, the involution and the norm it inherits from $M(G)$ is a Banach $*$-algebra. Moreover, for $\mu \in \mathbf{L}^1(G)$ and $\mu \in M(G)$, we have that $\mu \star f$ and $f \star \mu$ both lie in $\mathbf{L}^1(G)$. We have

$$(\mu \star f)(x) = \int f(y^{-1}x)\, d\mu(y)$$

and

$$(f \star \mu)(x) = \int f(y)\, d\mu(y^{-1}x).$$

Therefore, $\mathbf{L}^1(G)$ is a closed two sided ideal in $M(G)$. For the details of this discussion, see [4](section 2.5.).

Unitary representations of a locally compact group

1.1 Unitary representations

Representation theory of locally compact groups made its first appearance at the beginning of the the previous century and since then

Definition 1.1.1. *Let G be a locally compact group and H be a Hilbert space. A unitary representation of G on H is a group homomorphism $\pi : G \to U(H)$, where $U(H)$ is the set of unitary operators on H, that is continuous with respect to the strong operator topology on $B(H)$.*

Example 1.1.2. *Let H be a Hilbert space and G be a locally compact group. We define $\pi : G \to B(H)$ by $\pi(x) = Id_H$ for every $x \in G$, where Id_H is the identity operator on H. Clearly $\pi(x)$ is a unitary operator for every $x \in G$, so π is indeed a map from G to $\mathcal{U}(H)$ and it is clearly a group homomorphism. Moreover, π is a constant map, so it is continuous with respect to any topology on $B(H)$ and in particular, it is continuous with respect to the strong operator topology and therefore π is a unitary representation of G on H. We call π the trivial representation of G on H.*

There is another representation that we can define on any group and it is one that is going to play a major role in the rest of this thesis

Example 1.1.3 (Left regular representation)**.** *Let G be a locally compact group. For every $x \in G$, we define a map $\lambda(x) : C_c(G) \to C_c(G)$ as follows: If $f \in C_c(G)$ and $y \in G$, we define*

$$(\lambda(x)f)(y) := f(x^{-1}y).$$

Clearly, $\lambda(x)f \in C_c(G)$. Now,

$$\|\lambda(x)f\|_2^2 = \int |(\lambda(x)f)(y)|^2 \, dy = \int |f(x^{-1}y)|^2 \, dy = \int |f(y)|^2 \, dy = \|f\|_2^2 \, ,$$

since the Haar measure is left invariant. Therefore, $\lambda(x)$ is an isometry on $C_c(G)$ with respect to $\|\cdot\|_2$ and therefore it has a unique extension to an isometry from $\mathbf{L}^2(G)$ to $\mathbf{L}^2(G)$, which we denote again by $\lambda(x)$. Moreover, if $f \in C_c(G)$, then the function $g : G \to \mathbf{C}, \quad y \mapsto f(xy)$ lies in $C_c(G)$ and $(\lambda(x)g)(y) = f(y)$ for every $y \in G$. Therefore, $\lambda(x)$ maps $C_c(G)$ onto $C_c(G)$ and $\lambda(x) : \mathbf{L}^2(G) \to \mathbf{L}^2(G)$ is onto and isometric, so it is unitary.

It is easy to check that λ is indeed a group homomorphism from G to $\mathcal{U}(H)$ and the fact that λ is SOT-continuous follows from [4](proposition 2.41). Therefore λ is a unitary representation of G.

In a similar way, we define the right regular representation of G.

Example 1.1.4. *Let G be a locally compact group and $x \in G$. We define $\rho(x) \in B(L^2(G))$ by $(\rho(x)f)(y) = \Delta(x)^{\frac{1}{2}} f(yx)$ for $y \in G$ and $f \in \mathbf{L}^2(G)$. One can check that $\rho : G \to B(\mathbf{L}^2(G))$ is indeed a unitary representation.*

Definition 1.1.5. *Let G be a locally compact group and $(\pi_1, H_{\pi_1}), (\pi_2, H_{\pi_2})$ two unitary representations of G. We say that the two representations are unitarily equivalent if there exists a unitary operator $T : H_{\pi_1} \to H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for every $x \in G$.*

Definition 1.1.6. *Let A be a Banach algebra and H be a Hilbert space. A representation of A on H is a Banach algebra homomorphism $\pi : A \to B(H)$, that is, π is linear, bounded and $\pi(ab) = \pi(a)\pi(b)$ for every $a, b \in A$. If in addition A is a $*$-algebra and $\pi(a^*) = (\pi(a))^*$ for every $a \in A$, π is called* *a $*$-representation. If there is a non-zero $\xi \in H$ such that $\pi(a)\xi = 0$ for every $a \in A$, we call π degenerate and if there is no such $\xi \in H$, we call π non-degenerate.*

Let π be a unitary representation of G on the Hilbert space H. Then there is a natural way to define a representation of $\mathbf{L}^1(G)$ on H. To see this, let $f \in \mathbf{L}^1(G)$ and define $\psi : H \times H \to \mathbb{C}$ by

$$\psi : (\xi, \eta) \mapsto \int f(x)\langle \pi(x)\xi, \eta\rangle \, dx.$$

Then ψ is clearly sesquilinear and if $\xi, \eta \in H$, we have

$$|\psi(\xi, \eta)| = \left| \int f(x)\langle \pi(x)\xi, \eta\rangle \, dx \right| \leqslant \|f\|_1 \cdot \|\xi\| \cdot \|\eta\| \, .$$

So ψ is a bounded sesquilinear form on H with $\|\psi\| \leqslant \|f\|_1$ and therefore there exists a unique $\tilde{\pi}(f) \in B(H)$ such that

$$\psi(\xi, \eta) = \langle \tilde{\pi}(f)\xi, \eta\rangle$$

for every $\xi, \eta \in H$. Moreover, $\|\tilde{\pi}(f)\| = \|\psi\| \leqslant \|f\|_1$.

Proposition 1.1.7. *Let G be a locally compact group, H a Hilbert space and π a unitary representation of G on H. Define $\tilde{\pi} : \mathbf{L}^1(G) \to B(H)$ by $f \mapsto \tilde{\pi}(f)$. Then $\tilde{\pi}$ is a non-degenerate $*$- representation of $\mathbf{L}^1(G)$ on H and $\|\tilde{\pi}(f)\| \leqslant \|f\|_1$ for every $f \in \mathbf{L}^1(G)$.*

Proof. It is clear that $\tilde{\pi}$ is linear. Now let $f, g \in \mathbf{L}^1(G)$ and $\xi, \eta \in H$. Then,

$$
\begin{aligned}
\langle \tilde{\pi}(f \star g)\xi, \eta \rangle = & \int (f \star g)(x) \langle \pi(x)\xi, \eta \rangle \, dx \\
& \int \int f(y) g(y^{-1}x) \langle \pi(x)\xi, \eta \rangle \, dy dx = \\
& \int \int f(y) g(x) \langle \pi(yx)\xi, \eta \rangle \, dx dy = \\
& \int \int f(y) g(x) \langle \pi(x)\xi, \pi(y^{-1}\eta \rangle \, dx dy = \\
& \int f(y) \langle \tilde{\pi}(g)\xi, \pi(y^{-1})\eta \rangle \, dy = \\
& \int f(y) \langle \pi(y)\tilde{\pi}(g)\xi, \eta \rangle \, dy = \\
& \langle \tilde{\pi}(f)\tilde{\pi}(g)\xi, \eta \rangle
\end{aligned}
$$

and therefore,

$$\tilde{\pi}(f \star g) = \tilde{\pi}(f)\tilde{\pi}(g).$$

Let $f \in \mathbf{L}^1(G)$ and $\xi, \eta \in H$, then,

$$
\begin{aligned}
\langle \xi, (\tilde{\pi}(f))^* \eta \rangle = & \langle \tilde{\pi}(f)\xi, \eta \rangle = \int f(x) \langle \pi(x)\xi, \eta \rangle \, dx = \\
& \int \Delta(x)\Delta(x^{-1}) f(x) \langle \pi(x)\xi, \eta \rangle \, dx = \\
& \int \Delta(x^{-1}) f(x^{-1}) \langle \pi(x^{-1})\xi, \eta \rangle \, dx = \\
& \int \overline{f^*(x) \langle \pi(x)\eta, \xi \rangle} \, dx = \overline{f^*(x) \langle \pi(x)\eta, \xi \rangle} \, dx = \\
& \overline{\langle \tilde{\pi}(f^*)\eta, \xi \rangle} = \langle \eta, \tilde{\pi}(f^*)\xi \rangle
\end{aligned}
$$

and therefore,

$$\tilde{\pi}(f^*) = (\tilde{\pi}(f))^*.$$

Moreover, $\|\tilde{\pi}(f)\| \leqslant \|f\|_1$, so $\tilde{\pi}$ is bounded and it is a $*$- representation of $\mathbf{L}^1(G)$ on H. We still need to show that $\tilde{\pi}$ is non-degenerate.

To see this, let $\xi \in H$ be non-zero. Then, since $\pi : G \to B(H)$ is SOT- continuous and $\pi(e)\xi = \xi$, there exists an open relatively compact neighborhood of the identity V such that

$$\|\pi(x)\xi - \xi\| < \frac{\|\xi\|}{2}$$

for every $x \in V$.

Let $f = \frac{1}{\lambda(V)}\mathbf{1}_V$. Then f is well defined and lies in $\mathbf{L}^1(G)$, since V is open and non empty and so $\lambda(V) > 0$ and $\overline{V}$ is compact and therefore $\lambda(V) \leqslant \lambda(\overline{V}) < \infty$.

Then, notice that

$$\|\tilde{\pi}(f)\xi - \xi\| = \sup\{|\langle\tilde{\pi}(f)\xi - \xi, \eta\rangle| : \eta \in H, \|\eta\| \leqslant 1\}$$

and for every $\eta \in H$ with $\|\eta\| \leqslant 1$ we have,

$$\begin{aligned}
|\langle\tilde{\pi}(f)\xi - \xi, \eta\rangle| =& |\langle\tilde{\pi}(f)\xi, \eta\rangle - \langle\xi, \eta\rangle| = \\
& \left|\int_V \frac{1}{\lambda(V)}\langle\pi(x)\xi, \eta\rangle\, dx - \langle\xi, \eta\rangle\right| = \\
& \left|\frac{1}{\lambda(V)}\int_V (\langle\pi(x)\xi, \eta\rangle - \langle\xi, \eta\rangle)\, dx\right| = \\
& \left|\frac{1}{\lambda(V)}\int_V \langle\pi(x)\xi - \xi, \eta\rangle\, dx\right| < \\
& \frac{1}{\lambda(V)}\int_V \|\pi(x)\xi - \xi\|\, \|\eta\|\; dx < \\
& \frac{1}{\lambda(V)} \sup\{\|\pi(x)\xi - \xi\| : x \in V\}\, \|\eta\|\, \lambda(V) < \\
& \frac{\|\xi\|}{2}
\end{aligned}$$

and therefore, $\tilde{\pi}(f)\xi$ cannot be 0. So for each $0 \neq \xi \in H$ we have found a function $f \in \mathbf{L}^1(G)$ such that $\tilde{\pi}(f)\xi \neq 0$ and so $\tilde{\pi}$ is non-degenerate.

□

In fact, all the non-degenerate $*$- representations of $\mathbf{L}^1(G)$ arise in this way. To see this, let $\pi : \mathbf{L}^1(G) \to B(H)$ be a non-degenerate $*$- representation of $\mathbf{L}^1(G)$ on H. We want to define an operator $\pi(x)$ for every $x \in G$.

Let $(\psi_V)_{V\in\mathcal{V}}$ be an approximate unit for $\mathbf{L}^1(G)$ such that $\|\psi_V\|_1 = 1$ for every $V \in \mathcal{V}$ and $x \in G$. Such an approximate unit exists ([4] 2.42.). A simple calculation shows that $(L_x\psi_V) \star f = L_x(\psi_V \star f)$ for every $f \in \mathbf{L}^1(G)$ and so, for $f \in \mathbf{L}^1(G)$, we have

$$(L_x\psi_V) \star f = L_x(\psi_V \star f) \to L_x f$$

and therefore,

$$\pi(L_x\psi_V)\pi(f) = \pi((L_x\psi_V) \star f) \to \pi(L_x f).$$

Let $D = \mathrm{span}\{\pi(f)\xi : f \in \mathbf{L}^1(G), \xi \in H\} = \{\pi(f)\xi : f \in \mathbf{L}^1(G), \xi \in H\}$ and let $\eta \in D^\perp$. Then,

$$\langle \pi(f)\xi, \eta \rangle = 0$$

for every $f \in \mathbf{L}^1(G)$ and every $\xi \in H$ and so,

$$\langle \xi, \pi(f^*)\eta \rangle = 0$$

for every $\xi \in H$ and every $f \in \mathbf{L}^1(G)$ and so,

$$\pi(f^*)\eta = 0$$

for every $f \in \mathbf{L}^1(G)$. Now, $(\pi(\mathbf{L}^1(G)))^* = \mathbf{L}^1(G)$ and therefore,

$$\pi(f)\eta = 0$$

for every $f \in \mathbf{L}^1(G)$, but π is non-degenerate, so $\eta = 0$ and therefore,

$$D^\perp = \{0\}$$

so D is dense in H.

Let $u \in D$, then $u = \sum_{j=1}^n \pi(f_j)\xi_j$ for some $f_j \in \mathbf{L}^1(G)$ and $\xi_j \in H$. Then, for every $V \in \mathcal{V}$, we have

$$\pi(L_x\psi_V)u = \sum_{j=1}^n \pi(L_x\psi_V)\pi(f_j)\xi_j \to \sum_{j=1}^n \pi(L_x f_j)\xi_j.$$

And we define

$$T_x : D \to H, \quad \sum_{j=1}^n \pi(f_j)\xi_j \mapsto \sum_{j=1}^n \pi(L_x f_j)\xi_j$$

To see that T_x is well defined, we need to check that if $\sum_{j=1}^{n} \pi(f_j)\xi_j = 0$, then $\sum_{j=1}^{n} \pi(L_x f_j)\xi_j = 0$.

Indeed, $\sum_{j=1}^{n} \pi(L_x f_j)\xi_j = \lim \pi(L_x\psi_V)(\sum_{j=1}^{n} \pi(f_j)\xi_j) = 0$ and T is well defined and clearly it is linear. Moreover, for every $V \in \mathcal{V}$, we have

$$\|\pi(L_x)\psi_V\| \leqslant \|L_x\psi_V\|_1 = \|\psi_V\|_1 = 1$$

so for every $u \in D$, we have

$$\|\pi(L_x\psi_V)(u)\| \leqslant \|u\|$$

for every $V \in \mathcal{V}$ and so

$$\|T_x u\| \leqslant \|u\| .$$

Therefore, T_x is bounded and since D is dense in H, we can extend T_x to H and we denote the extension again by T_x.

We define $\tilde{\pi} : G \to B(H)$, by $\tilde{\pi}(x) = T_x$. We will show that $\tilde{\pi}$ is a unitary representation of G.

At first, notice that for every $f \in \mathbf{L}^1(G), \xi \in H$ and $x \in G$, we have

$$\tilde{\pi}(x)\pi(f)\xi = \pi(L_x f)\xi$$

and therefore, for every $x, y \in G$, we have

$$\tilde{\pi}(xy)(\pi(f)\xi) = \pi(L_{xy}f)\xi = \pi(L_x(L_y f))\xi = \tilde{\pi}(x)\tilde{\pi}(y)(f)\xi$$

so $\tilde{\pi}(xy)u = \tilde{\pi}(x)\tilde{\pi}(y)u$ for every $u \in D$ and since D is dense in H, we conclude that $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ and $\tilde{\pi}$ is a group homomorphism.

Let $f \in \mathbf{L}^1(G)$ and $\xi \in H$, then $\tilde{\pi}(e)(\pi(f)\xi) = \pi(L_e f)\xi = \pi(f)\xi$ and therefore $\tilde{\pi}(e)(u) = u$ for every $u \in D$ and since D is dense in H, we conclude that $\tilde{\pi}(e) = Id_H$. Now, for every $x \in G$, we have

$$Id_H = \tilde{\pi}(xx^{-1}) = \tilde{\pi}(x)\tilde{\pi}(x^{-1})$$

and therefore $\tilde{\pi}(x)$ is invertible for every $x \in G$ and in particular it maps H onto H.

Let $x \in G$, $f \in \mathbf{L}^1(G)$ and $\xi \in H$, then

$$\|\tilde{\pi}(x)(\pi(f)\xi)\| \leqslant \|\pi(f)\xi\| = \left\|\tilde{\pi}(x^{-1})\tilde{\pi}(x)\pi(f)\xi\right\| \leqslant \|\tilde{\pi}(x)\pi(f)\xi\|$$

and so,

$$\|\tilde{\pi}(x)u\| = \|u\|$$

for every $u \in D$ and thus $\pi(x)$ is an isometry onto, that is, $\tilde{\pi}(x)$ is a unitary.

To see that $\tilde{\pi}$ is strongly continuous, let $(x_i)_{i\in I}$ be a net in G, converging to some $x \in G$ and let $f \in \mathbf{L}^1(G)$ and $\xi \in H$. Then, for every $i \in I$, we have

$$\tilde{\pi}(x_i)(\pi(f)\xi) = \pi(L_{x_i}f)\xi \to \pi(L_x f)\xi = \tilde{\pi}(x)(\pi(f)\xi)$$

and therefore,

$$\tilde{\pi}(x_i)u \to \tilde{\pi}(x)u$$

for every $u \in D$. Now, let $\xi \in H$ and $\epsilon > 0$. Since D is dense in H, there is some $u \in D$ such that $\|\xi - u\| < \frac{\epsilon}{3}$.

Since $u \in D$, we have that $\tilde{\pi}(x_i)u \to \tilde{\pi}(x)u$ and therefore there is some $i_0 \in I$ such that $\|\tilde{\pi}(x_i)u - \tilde{\pi}(x)u\| < \frac{\epsilon}{3}$ for every $i \geqslant i_0$.

Then, for $i \geqslant i_0$ we have,

$$\begin{aligned}\|\tilde{\pi}(x_i)\xi - \tilde{\pi}(x)\xi\| \leqslant &\|\tilde{\pi}(x_i)\xi - \tilde{\pi}(x_i)u\| + \|\tilde{\pi}(x_i)u - \tilde{\pi}(x)\xi\| \leqslant \\ &\|\tilde{\pi}(x_i)\|\,\|\xi - u\| + \|\tilde{\pi}(x_i)u - \tilde{\pi}(x)u\| + \|\tilde{\pi}(x)u - \tilde{\pi}(x)\xi\| \leqslant \\ &\frac{\epsilon}{3} + \frac{\epsilon}{3} + \|\tilde{\pi}(x)\|\,\|u - \xi\| \leqslant \epsilon\end{aligned}$$

and therefore,

$$\tilde{\pi}(x_i)\xi \to \tilde{\pi}(x)\xi$$

and $\tilde{\pi}$ is strongly continuous, so it is a unitary representation of G.

Now, it is proved in [4](theorem 3.11) that the representation of $\mathbf{L}^1(G)$ associated to $\tilde{\pi}$ as in proposition 1.1.7 is exactly π.

We have proven the following:

Proposition 1.1.8. *Let G be a locally compact group, then there is a one to one and onto correspondence between unitary representations of G and* *non-degenerate $*$- representations of $\mathbf{L}^1(G)$. For (π, H_π) a unitary representation of G, for any $f \in \mathbf{L}^1(G)$ we define $\tilde{\pi}(f) \in B(H)$ by $\langle \pi(f)\xi, \eta\rangle = \int f(x)\langle \pi(x)\xi, \eta\rangle\, dx$ for every $\xi, \eta \in H_\pi$ and for (π, H_π) a non-degenerate $*$- representation of $\mathbf{L}^1(G)$ on H_π and $x \in G$, we define $\tilde{\pi}(x) \in B(H)$ by $\tilde{\pi}(x) = \lim_V \pi(L_x\psi_V)$ with respect to the strong operator topology, where $(\psi_V)_{V\in\mathcal{V}}$ is an approximate unit for $\mathbf{L}^1(G)$.*

Remark 1.1.9. *While $\tilde{\pi}(\mathbf{L}^1(G))$ and $\pi(G)$ can be quite different in general, they generate the same von Neumann algebra in $B(H)$, that is*

$$(\pi(G))'' = (\tilde{\pi}(\mathbf{L}^1(G)))''.$$

For a proof, see [4] (theorem 3.12.).

Remark 1.1.10. *We will need to know explicitly the representations of $\mathbf{L}^1(G)$ corresponding to the left and the right regular representation of G. It is proved in [4](example 3.8) that in the case of the left regular representation λ, for $f \in \mathbf{L}^1(G)$ and $g \in \mathbf{L}^2(G)$, we get $(\tilde{\lambda}(f))(g) = f \star g$. For the right regular representation we get that $(\tilde{\rho}(f))(g) = g \star f$, for $f \in \mathbf{L}^1(G)$ and $g \in C_c(G)$.*

We are going to extend the previous proposition to include representations of the measure algebra $M(G)$.

Let (π, H_π) be a unitary representation of G and $\mu \in M(G)$. We are going to define an operator $\tilde{\pi}(\mu) \in B(H)$ in exactly the same way as we previously did for $f \in \mathbf{L}^1(G)$. More specifically, we define

$$\psi : H_\pi \times H_\pi \to \mathbb{C}, \quad (\xi, \eta) \mapsto \int \langle \pi(x)\xi, \eta\rangle\, d\mu(x).$$

Then, ψ is clearly sesquilinear and for every $\xi, \eta \in H_\pi$, we have

$$|\psi(\xi, \eta)| = \left| \int \langle \pi(x)\xi, \eta\rangle\, d\mu(x) \right| \leqslant \|\xi\|\, \|\eta\|\, \|\mu\|$$

and therefore ψ is bounded with $\|\psi\| \leqslant \|\mu\|$. Therefore, there exists a unique operator $\tilde{\pi}(\mu) \in B(H)$ such that $\langle \tilde{\pi}(\mu)\xi, \eta\rangle = \psi(\xi, \eta)$ for every $\xi, \eta \in H_\pi$.

Similar calculations to the case of $\mathbf{L}^1(G)$ show that $\tilde{\pi} : M(G) \to B(H_\pi), \quad \mu \mapsto \tilde{\pi}(\mu)$ is a $*$- representation of $M(G)$ and the restriction of $\tilde{\pi}$ on $\mathbf{L}^1(G)$ is the representation of $\mathbf{L}^1(G)$ associated to π as in proposition 1.1.8 and so, by the same proposition, we get that the restriction of $\tilde{\pi}$ on $\mathbf{L}^1(G)$ is non-degenerate.

In the opposite direction, let $\pi : M(G) \to B(H)$ be a $*$- representation of $M(G)$ on a Hilbert space H, such that the restriction of π on $\mathbf{L}^1(G)$ is non-degenerate. Then consider $\tilde{\pi} : G \to B(H)$ as in proposition 1.1.8. Since $\tilde{\pi}$ restricted to $\mathbf{L}^1(G)$ is non-degenerate, proposition 1.1.8 implies that $\tilde{\pi}$ is a unitary representation of G. Now, notice that $\tilde{\pi}$ is determined by the restriction of $\tilde{\pi}$ on $\mathbf{L}^1(G)$ and therefore, two representations of $M(G)$ that are non-degenerate when restricted on $\mathbf{L}^1(G)$ and that agree on $\mathbf{L}^1(G)$, induce the same representation of G. We have therefore proven the following proposition:

Proposition 1.1.11. *There is a one to one and onto correspondence between unitary representations of G and $*$-representations of $M(G)$, whose restriction on $\mathbf{L}^1(G)$ is non-degenerate.*

Remark 1.1.12. *It is important to note that if (H, π) is a unitary representation of G, then for every $x \in G$, we have*

$$\tilde{\pi}(\delta_x) = \pi(x)$$

where $\tilde{\pi}$ is the corresponding representation of $M(G)$ on H.

To see this, let $\mathcal{V}$ be a neighborhood basis of the identity, consisting of relatively compact open sets. Then, it is proved in [4](proposition 2.42) that there exists an approximate identity of $(g_V)_{V \in \mathcal{V}}$ for $\mathbf{L}^1(G)$ such that $\operatorname{supp} g_V \subset V$, $g_V \geqslant 0$ and $\int g_V(x)\,dx = 1$ for every $V \in \mathcal{V}$. Now, let $\mu \in M(G)$ and define $f_V = \mu \star g_V$ for every $V \in \mathcal{V}$.

In light of propositions 1.1.8 and 1.1.11, we will use the same symbol for the corresponding representations of G, of $\mathbf{L}^1(G)$ and $M(G)$, without necessarily mentioning this every time.

1.2 Functions of positive type

Definition 1.2.1. *Let $u \in \mathbf{L}^\infty(G)$. We call u a function of positive type if it defines a positive functional on $\mathbf{L}^1(G)$, that is*

$$\int (f^* \star f)(x)u(x)\,dx \geqslant 0 \tag{*}$$

for every $f \in \mathbf{L}^1(G)$. We denote the set of functions of positive type by $P(G)$.

It is clear from the definition that $P(G)$ is closed under addition and multiplication with positive scalars and it is therefore a cone in $\mathbf{L}^\infty(G)$.

Remark 1.2.2. *Expanding the integral in (*) we get that a function $\phi \in \mathbf{L}^\infty(G)$ is in $P(G)$ if and only if*

$$\iint f(x)\overline{f(y)}u(y^{-1}x)\,dxdy \geqslant 0$$

for every $f \in \mathbf{L}^1(G)$. Clearly, if this holds for ϕ, it also holds for $\overline{\phi}$ and therefore $\overline{\phi} \in P(G)$ for every $\phi \in P(G)$.

Example 1.2.3. *Let π be a unitary representation and $\xi \in H_\pi$. We define $u : G \to \mathbb{C}$, by $u(x) := \langle \pi(x)\xi, \xi\rangle$. Then, for every $f \in \mathbf{L}^1(G)$, we have*

$$\int (f \star f^*)(x)v(x)\,dx = \int (f \star f^*)(x)\langle \pi(x)\xi, \xi\rangle\,dx =$$
$$\langle \pi(f \star f^*)\xi, \xi\rangle = \langle \pi(f^*)\xi, \pi(f^*)\xi\rangle = \|\pi(f^*)\|^2 \geqslant 0$$

and therefore, v is of positive type.

In fact, as the next proposition shows, every function of positive type occurs in this way.

Proposition 1.2.4. *Let $u : G \to \mathbb{C}$ be a function of positive type. Then there exists a triple (π_u, H_u, ξ_u) where π_u is a unitary representation of G on the Hilbert space H_u and ξ_u is a cyclic vector for $\pi_u(\mathbf{L}^1(G))$ in H_u, such that $u(x) = \langle \pi_u(x)\xi_u, \xi_u\rangle$ for locally almost every $x \in G$, where by the abuse of notation we have previously mentioned, we denote by π_u the corresponding unitary representation of $\mathbf{L}^1(G)$ on H_u.*

Proof. See [4] (theorem 3.20.). We define $\langle\cdot,\cdot\rangle_u$ on $\mathbf{L}^1(G)$ by

$$\langle f,g\rangle_u = \int (g^*\star f)(x)u(x)\,dx \tag{*}$$

for every $f,g\in\mathbf{L}^1(G)$. Then $\langle\cdot,\cdot\rangle_u$ is clearly linear in the first variable and antilinear in the second one and

$$\langle f,f\rangle_u = \int (f^*\star f)(x)u(x)\,dx \geqslant 0$$

for every $f\in\mathbf{L}^1(G)$, since $u\in P(G)$.

Notice that

$$|\langle f,g\rangle_u| \leqslant \|g^*\star f\|_1\,\|u\|_\infty \leqslant \|g\|_1\,\|f\|_1\,\|u\|_\infty\,. \tag{**}$$

We define $\omega:\mathbf{L}^1(G)\to\mathbb{C},\quad f\mapsto\int f(x)u(x)\,dx$. Then ω is clearly bounded and linear and $\omega(g^*\star f)=\langle f,g\rangle_u$ for every $f,g\in\mathbf{L}^1(G)$. Moreover, for every $f\in\mathbf{L}^1(G)$, we have $\omega(f^*\star f)=\langle f,f\rangle_u\geqslant 0$ and therefore ω is positive, therefore by [10](chapter 1 lemma 9.11), we see that $\omega((g^*\star f)^*)=\overline{\omega(f^*\star g)}$ for every $f,g\in\mathbf{L}^1(G)$ and therefore, $\langle g,f\rangle_u=\overline{\langle f,g\rangle_u}$ for every $f,g\in\mathbf{L}^1(G)$ and therefore, $\langle\cdot,\cdot\rangle_u$ is a semi-inner product on $\mathbf{L}^1(G)$.

We define

$$\mathcal{N}_u=\{f\in\mathbf{L}^1(G):\langle f,f\rangle_u=0\}.$$

Let $f\in\mathcal{N}_u$ and $g\in\mathbf{L}^1(G)$, then, since $\langle\cdot,\cdot\rangle_u$ is a semi-inner product, the Cauchy-Schwarz inequality holds [1](1.4) and therefore,

$$|\langle f,g\rangle_u|^2\leqslant\langle f,f\rangle_u\langle g,g\rangle_u=0,$$

so,

$$\mathcal{N}_u=\{f\in\mathbf{L}^1(G):\langle f,g\rangle_u=0\,\forall\,g\in\mathbf{L}^1(G)\}$$

which is clearly a linear subspace of $\mathbf{L}^1(G)$. Now, notice that if f_1,f_2 and g_1,g_2 are in $\mathbf{L}^1(G)$ such that $f_1-f_2,g_1-g_2\in\mathcal{N}_u$, then we have that $\langle f_1,g_1\rangle_u=\langle f_2,g_2\rangle_u$ and therefore, $\langle\cdot,\cdot\rangle_u$ defines an inner product on $\mathbf{L}^1(G)/\mathcal{N}_u$. We then define $H_u:=\overline{\mathbf{L}^1(G)/\mathcal{N}_u}^{\langle\cdot,\cdot\rangle_u}$. Now, H_u is clearly a Hilbert space and we are going to define a unitary representation of G on H_u.

Expanding the integral in (*), we get that

$$\langle f,g\rangle_u=\int(g^*\star f)(x)u(x)\,dx=\int\int\Delta(y^{-1})\overline{g(y^{-1})}f(y^{-1}x)u(x)\,dydx=$$
$$\int\int\overline{g(y)}f(yx)u(x)\,dydx=\int\int\overline{g(y)}f(x)u(y^{-1}x)\,dydx.$$

Now, let $z\in G$ and let $f,g\in\mathbf{L}^1(G)$. Then,

$$\begin{aligned}\langle L_zf,L_zg\rangle_u=&\int\int\overline{(L_zg)(y)}(L_zf)(x)u(y^{-1}x)\,dydx=\\&\int\int\overline{g(z^{-1}y)}f(z^{-1}x)u(y^{-1}x)\,dydx\overset{s=z^{-1}y}{=\!=}\\&\int\int\overline{g(s)}f(z^{-1}x)u(s^{-1}z^{-1}x)\,dsdx\overset{t=z^{-1}x}{=\!=}\\&\int\int\overline{g(s)}f(t)u(s^{-1}t)\,dsdt=\\&\langle f,g\rangle_u.\end{aligned}$$

In particular, this shows that $L_z(\mathcal{N}_u)\subset\mathcal{N}_u$ for every $z\in G$ and therefore, we can define a map $\tilde{L}_z:\mathbf{L}^1(G)/\mathcal{N}_u\to\mathbf{L}^1(G)/\mathcal{N}_u$, by $\tilde{L}_z(f+\mathcal{N}_u)=L_zf+\mathcal{N}_u$ for every $f+\mathcal{N}_u\in\mathbf{L}^1(G)/\mathcal{N}_u$. Then clearly $\tilde{L}_z$ is linear.

Moreover, if $f+\mathcal{N}_u\in\mathbf{L}^1(G)/\mathcal{N}_u$, then we have that $\left\|(\tilde{L}_z)(f+\mathcal{N}_u)\right\|_u^2=\langle L_zf,L_zf\rangle_u=\langle f,f\rangle_u=\|f+\mathcal{N}_u\|_u^2$ and therefore $\tilde{L}_z$ is an isometry and $\mathbf{L}^1(G)/\mathcal{N}_u$ is dense in H_u, so $\tilde{L}_z$ extends to an isometry from H_u to H_u, which we denote again by $\tilde{L}_z$. The map $\tilde{L}_z$ maps $\mathbf{L}^1(G)/\mathcal{N}_u$ onto itself and therefore $\tilde{L}_z:H_u\to H_u$ is an isometry onto. That is, $\tilde{L}_z$ is a unitary.

We now define $\pi_u:G\to\mathcal{U}(H)$ by $\pi_u(x)=\tilde{L}_x$ for every $x\in G$. We will show that π_u is a unitary representation of G on H_u.

We will first show that π_u is a group homomorphism. To see this, notice that $\tilde{L}_e|_{\mathbf{L}^1(G)/\mathcal{N}_u}=Id|_{\mathbf{L}^1(G)/\mathcal{N}_u}$ and since $\mathbf{L}^1(G)/\mathcal{N}_u$ is dense in H_u, we get that $\tilde{L}_e=Id$. Likewise, for every $x,y\in G$, it is clear that $\tilde{L}_x\tilde{L}_y=\tilde{L}_{xy}$ in $\mathbf{L}^1(G)/\mathcal{N}_u$ and therefore $\tilde{L}_x\tilde{L}_y=\tilde{L}_{xy}$ and $\pi_u:G\to\mathcal{U}(H_u)$ is a group homomorphism.

We still need to show that π_u is SOT-continuous.

Let $(x_i)_{i\in I}$ be a net in G, converging to some $x\in G$ and let $\xi\in H_u$. We need to show that $\pi_u(x_i)\xi\to\pi_u(x)\xi$.

Let us first assume that $\xi = f + \mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$. Then, we have that

$$\|\pi_u(x_i)\xi - \pi_u(x)\xi\|_u^2 = \|\pi_u(x_i)(f + \mathcal{N}_u) - \pi_u(x)(f + \mathcal{N}_u)\|_u^2$$

and by (**) we have that

$$\|\pi_u(x_i)\xi - \pi_u(x)\xi\|_u^2 \leqslant \|L_{x_i}f - L_x f\|_1^2 \|u\|_\infty$$

which tends to zero by 0.4.16.

Now for the general case, let $\xi \in H_u$ and $\epsilon > 0$. Then, there is some $f + \mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$ such that $\|\xi - (f + \mathcal{N}_u)\|_u < \frac{\epsilon}{3}$.

Now, as we just showed, $\pi_u(x_i)f \to \pi_u(x)f$ and therefore, there is some $i_0 \in I$ such that $\|\pi_u(x_i)f - \pi_u(x)f\|_u < \frac{\epsilon}{3}$ for any $i \geqslant i_0$.

Then, for $i \geqslant i_0$, we have that

$$\begin{aligned}\|\pi_u(x_i)\xi - \pi_u(x)\xi\|_u \leqslant & \|\pi_u(x_i)\xi - \pi_u(x_i)(f + \mathcal{N}_u)\|_u \\ & + \|\pi_u(x)(f + \mathcal{N}_u) - \pi_u(x_i)\xi\|_u + \\ & \|\pi_u(x_i)(f + \mathcal{N}_u) - \pi_u(x_i)\xi\|_u < \epsilon.\end{aligned}$$

Therefore, π_u is SOT-continuous and therefore it is a unitary representation of G on H_u.

We still need to find a cyclic vector for $\pi_u(\mathbf{L}^1(G))$.

Let $(\psi_V)_{V\in\mathcal{V}}$ be an approximate unit for $\mathbf{L}^1(G)$ such that $\|\psi_V\|_1 = 1$ for every $V \in \mathcal{V}$. Then $(\psi_V^*)_{V\in\mathcal{V}}$ is also an approximate unit for $\mathbf{L}^1(G)$ and so, if $f + \mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$, we then have that

$$\langle f, \psi_V\rangle_u = \int (\psi_V^* \star f)(x)u(x)\,dx \to \int f(x)u(x)\,dx.$$

Define $\phi : \mathbf{L}^1(G)/\mathcal{N}_u \to \mathbb{C}$ by $f + \mathcal{N}_u \mapsto \int f(x)u(x)\,dx = \lim_V \langle f, \psi_V\rangle_u$.

Then ϕ is well defined and it is clearly linear. Moreover, for every $V \in \mathcal{V}$, we have that

$$\begin{aligned}|\langle f, \psi_V\rangle| \leqslant & \|f + \mathcal{N}_u\| \, \|\psi_V + \mathcal{N}_u\|_u \leqslant \\ & \|f + \mathcal{N}_u\| \, \|u\|_\infty^{\frac{1}{2}} \|\psi_V\|_1 = \\ & \|f + \mathcal{N}_u\|_u \, \|u\|_\infty^{\frac{1}{2}}\end{aligned}$$

and therefore,

$$|\phi(f+\mathcal{N}_u)| \leqslant \|f+\mathcal{N}_u\| \, \|u\|_\infty^{\frac{1}{2}}$$

and so ϕ is bounded. Now, ϕ can be uniquely extended to a bounded functional on H_u, which we denote again by ϕ. Then, by Riesz's representation theorem, there is a unique vector $\xi_u \in H_u$ such that $\phi(\xi) = \langle \xi, \xi_u \rangle$ for every $\xi \in H_u$.

First, notice that for $f+\mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$, we have that $\langle f+\mathcal{N}_u, \xi_u\rangle_u = \phi(f+\mathcal{N}_u) = \int f(x)u(x)\,dx$.

Now, let $f+\mathcal{N}_u, g+\mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$. Then,

$$\begin{aligned}\langle f+\mathcal{N}_u, g+\mathcal{N}_u\rangle = &\iint f(x)\overline{g(y)}u(y^{-1}x)\,dydx = \\ &\iint f(yx)\overline{g(y)}u(x)\,dxdy = \\ &\iint (L_{y^{-1}}f)(x)\overline{g(y)}u(x)\,dxdxy = \\ &\int \langle L_{y^{-1}}f+\mathcal{N}_u, \xi_u\rangle\overline{g(y)}\,dy = \\ &\int \overline{g(y)}\langle f+\mathcal{N}_u, \pi_u(y)\xi_u\rangle\,dy = \\ &\overline{\langle \pi_u(g)\xi_u, f+\mathcal{N}_u\rangle} = \\ &\langle f+\mathcal{N}_u, \pi_u(g)\xi_u\rangle\end{aligned}$$

and therefore,

$$\pi_u(g)\xi_u = g+\mathcal{N}_u \tag{***}$$

for every $g+\mathcal{N}_u \in \mathbf{L}^1(G)/\mathcal{N}_u$. Notice that this means that $\mathbf{L}^1(G)/\mathcal{N}_u \subset \pi_u(\mathbf{L}^1(G))\xi_u$ and $\mathbf{L}^1(G)/\mathcal{N}_u$ is dense in H_u, so $\pi_u(\mathbf{L}^1(G))\xi_u$ is dense in H_u and therefore ξ_u is cyclic for $\pi_u(\mathbf{L}^1(G))$.

Now, if $f \in \mathbf{L}^1(G)$, we have that,

$$\langle \pi_u(f)\xi_u, \xi_u\rangle_u = \int f(x)\langle \pi_u(x)\xi_u, \xi_u\rangle\,dx$$

and by (***), we get that

$$\langle f+\mathcal{N}_u, \xi_u\rangle_u = \int f(x)\langle \pi_u(x)\xi_u, \xi_u\rangle\,dx$$

and therefore,

$$\int f(x)u(x)\,dx = \int f(x)\langle \pi_u(x)\xi_u, \xi_u\rangle_u\,dx.$$

Since this holds for every $f \in \mathbf{L}^1(G)$, we conclude that $u(x) = \langle \pi_u(x)\xi_u, \xi_u\rangle$ for locally almost every $x \in G$ and we are done.

□

Corollary 1.2.5. *Let $u \in P(G)$. Then u is locally almost everywhere equal to a continuous function.*

Proof. By 1.2.4, there is a (π, H_π), unitary representation of G and $\xi_u \in H_\pi$ such that $u(x) = \langle \pi(x)\xi_u, \xi_u\rangle$ locally almost everywhere in G and the function $\langle \pi(\cdot)\xi_u.\xi_u\rangle$ is continuous. □

In light of 1.2.5, we will consider u to be continuous for every $u \in P(G)$.

Lemma 1.2.6. *Let $f \in \mathbf{L}^2(G)$, then $f \star \tilde{f} \in P(G)$.*

Proof. Let $x \in G$, then,

$$f \star \tilde{f}(x) = \int f(y)\tilde{f}(y^{-1}x)\,dy = \int f(y)\overline{f(x^{-1}y)}\,dy = \langle f, \lambda(x)f\rangle$$

and so $f \star \tilde{f}$ is of positive type. □

Lemma 1.2.7. *Let $u, v \in P(G)$, then $uv \in P(G)$.*

Proof. By proposition 1.2.4, there exist unitary representations (H_u, π_u), (H_v, π_v) and vectors $\xi_u \in H_u$ and $\xi_v \in H_v$ such that $\langle \pi_u(x)\xi_u, \xi_u\rangle = u(x)$ and $\langle \pi_v(x)\xi_v, \xi_v\rangle = v(x)$ for every $x \in G$. Then,

$$uv(x) = \langle \pi_u(x)\xi_u, \xi_u\rangle\langle \pi_v(x)\xi_v, \xi_v\rangle = \langle (\pi_u \otimes \pi_v)(x)(\xi_u \otimes \xi_v), \xi_u \otimes \xi_v\rangle$$

which is a function of positive type by remark 1.2.3. □

The Fourier-Stieltjes algebra $B(G)$

It is about time that we introduced the main topics of this thesis, the Fourier and the Fourier-Stieltjes algebras of a locally compact group. We will first examine the Fourier-Stieltjes algebra of the group and then define the Fourier algebra as a particular subspace of the former.

Let $\mathbf{\Sigma}(G)$ denote the family of equivalence classes of unitary representations of G up to unitary equivalence and let $\mathcal{S}$ be a subfamily of $\mathbf{\Sigma}(G)$. We will define a Fourier-Stieltjes algebra associated to each such family and denote it by $B_{\mathcal{S}}(G)$. In the case $\mathcal{S} = \mathbf{\Sigma}(G)$, we simply write $B(G)$ and call it the Fourier-Stieltjes algebra of the group.

Let G be a locally compact group and let $\mathcal{S}$ be a subset of $\mathbf{\Sigma}(G)$. Let μ be in $M(G)$ and define

$$\|\mu\|_{\mathcal{S}} = \sup\{\|\pi(\mu)\| : \pi \in \mathcal{S}\}.$$

This is well defined, since π is contractive for every unitary representation of G and so $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|$. Moreover, it is obvious that $\|\cdot\|_{\mathcal{S}}$ is a seminorm on $M(G)$ and by restricting $\|\cdot\|_{\mathcal{S}}$ to $\mathbf{L}^1(G)$ we get a seminorm on $\mathbf{L}^1(G)$. When $\mathcal{S} = \mathbf{\Sigma}(G)$, we write $\|\cdot\|_*$ instead of $\|\cdot\|_{\mathbf{\Sigma}(G)}$.

We will first note some properties of these seminorms.

Proposition 2.0.1. *Let $\mathcal{S}$ be a subset of $\mathbf{\Sigma}(G)$, $\mu, \nu \in M(G)$, $f \in \mathbf{L}^1(G)$ and $x, y \in G$. Then:*

1. $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|$ *and* $\|\mu \star \nu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{S}} \cdot \|\nu\|_{\mathcal{S}}$

2. $\|\mu^*\|_{\mathcal{S}} = \|\mu\|_{\mathcal{S}}$ *and* $\|\mu^* \star \mu\|_{\mathcal{S}} = \|\mu\|_{\mathcal{S}}^2$

3. $\|L_x f\|_{\mathcal{S}} = \|f\|_{\mathcal{S}}$ *and* $\|R_y f\|_{\mathcal{S}} = \Delta(y^{-1})\, \|f\|_{\mathcal{S}}$

Proof. We noted above that $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|$.

We have

$$\|\mu \star \nu\|_{\mathcal{S}} = \sup\{\|\pi(\mu \star \nu)\| : \pi \in \mathcal{S}\} = \sup\{\|\pi(\mu) \cdot \pi(\nu)\| : \pi \in \mathcal{S}\}.$$

Now, for every π, since $\|\pi(\mu) \cdot \pi(\nu)\| \leqslant \|\pi(\mu)\| \cdot \|\pi(\nu)\| \leqslant \|\mu\| \cdot \|\nu\|$ we obtain

$$\|\mu \star \nu\|_{\mathcal{S}} = \sup\{\|\pi(\mu) \cdot \pi(\nu)\| : \pi \in \mathcal{S}\} \leqslant \|\mu\| \cdot \|\nu\|.$$

For every $\pi \in \mathbf{\Sigma}(G)$, we have $\pi(\mu^*) = (\pi(\mu))^*$ and so,

$$\|\mu^*\|_{\mathcal{S}} = \sup\{\|\pi(\mu^*)\| : \pi \in \mathcal{S}\} = \sup\{\|(\pi(\mu))^*\| : \pi \in \mathcal{S}\} = \|\mu\|_{\mathcal{S}},$$

since $\|\pi(\mu)^*\| = \|\pi(\mu)\|$ for every π and μ. Also, $\|\pi(\mu^* \star \mu)\| = \|(\pi(\mu))^* \pi(\mu)\| = \|\pi(\mu)\|^2$ and so $\|\mu^* \star \mu\|_{\mathcal{S}} = \|\mu\|_{\mathcal{S}}^2$.

Now for the last statement, let $\pi \in \mathcal{S}$ and $\xi, \eta \in H_\pi$. Then,

$$\begin{aligned}\langle \pi(L_x f)\xi, \eta\rangle &= \int (L_x f)(y)\langle \pi(y)\xi, \eta\rangle \, dy = \int f(x^{-1}y)\langle \pi(y)\xi, \eta\rangle \, dy \\ &= \int f(y)\langle \pi(xy)\xi, \eta\rangle \, dy = \int f(y)\langle \pi(y)\xi, \pi(x^{-1})\eta\rangle \, dy \\ &= \langle \pi(f)\xi, \pi(x^{-1})\eta\rangle = \langle \pi(x)\pi(f)\xi, \eta\rangle\end{aligned}$$

and this holds for every $\xi, \eta \in H_\pi$, hence $\pi(L_x f) = \pi(x)\pi(f)$. Since $\pi(x)$ is a unitary for every $x \in G$, we have $\|\pi(L_x f)\| = \|\pi(f)\|$ and so $\|L_x f\|_{\mathcal{S}} = \|f\|_{\mathcal{S}}$.

In the same way,

$$\begin{aligned}\langle \pi(R_y f)\xi, \eta\rangle &= \int (R_y f)(x)\langle \pi(x)\xi, \eta\rangle \, dx = \int \Delta(y^{-1}) f(y)\langle \pi(xy^{-1})\xi, \eta\rangle \, dx \\ &= \langle \Delta(y^{-1})\pi(f)\pi(y^{-1})\xi, \eta\rangle\end{aligned}$$

and since this holds for every $\xi, \eta \in H_\pi$, we have $\pi(R_y f) = \Delta(y^{-1})\pi(f)\pi(y^{-1})$ and since $\pi(y^{-1})$ is a unitary, $\|R_y f\|_{\mathcal{S}} = \Delta(y^{-1})\, \|f\|_{\mathcal{S}}$. □

Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$. We define $N_{\mathcal{S}} := \{f \in \mathbf{L}^1(G) : \|f\|_{\mathcal{S}} = 0\}$. Clearly $N_{\mathcal{S}}$ is a closed $*$- ideal in $\mathbf{L}^1(G)$ and so $\mathbf{L}^1(G)/N_{\mathcal{S}}$ is a $*$- algebra and the quotient norm on $\mathbf{L}^1(G)/N_{\mathcal{S}}$ is a $\mathbf{C}^*$- norm, so the completion of $\mathbf{L}^1(G)/N_{\mathcal{S}}$ is a $\mathbf{C}^*$-algebra denoted by $\mathbf{C}^*_{\mathcal{S}}(G)$. If $\mathcal{S} = \mathbf{\Sigma}(G)$, we denote $\mathbf{C}^*_{\mathbf{\Sigma(G)}}(G) = \mathbf{C}^*(G)$ and call it the full group $\mathbf{C}^*$- algebra. If $\mathcal{S} = \{\lambda\}$, where λ is the left regular representation, $\mathbf{C}^*_{\lambda}(G)$ is called the reduced group $\mathbf{C}^*$- algebra and is usually denoted by $\mathbf{C}^*_r(G)$. When $\mathcal{S} = \mathbf{\Sigma}(G)$ we write $\|f\|_*$ instead of $\|f\|_{\mathbf{\Sigma}(G)}$. As we will see, for every $\mathcal{S} \subset \mathbf{\Sigma}(G)$, the space $B_{\mathcal{S}}(G)$ can be naturally identified with the dual space of $\mathbf{C}^*_{\mathcal{S}}(G)$. Note also that for $f \in \mathbf{L}^1(G)$, we have that $\|f + N_{\mathcal{S}}\| = \inf\{\|f + g\|_{\mathcal{S}} : g \in N_{\mathcal{S}}\}$ and for $g \in N_{\mathcal{S}}$ and $\pi \in \mathcal{S}$, we see that $\pi(f + g) = \pi(f)$ and so $\|f + g\|_{\mathcal{S}} = \|f\|_{\mathcal{S}}$ for every $g \in N_{\mathcal{S}}$ and therefore $\|f + N_{\mathcal{S}}\| = \|f\|_{\mathcal{S}}$ for every $f \in \mathbf{L}^1(G)$.

Before defining $B(G)$, we need to note an important property of $\mathbf{C}^*(G)$:

Proposition 2.0.2. *Let $f \in \mathbf{L}^1(G)$ such that $\|f\|_* = 0$. Then $f = 0$. Therefore $\|\cdot\|_*$ is a norm on $\mathbf{L}^1(G)$ and $N_{\mathbf{\Sigma}(G)} = \{0\}$.*

Proof. To see this, consider $f \in \mathbf{L}^1(G)$ with $\|f\|_* = 0$. Then $\pi(f) = 0$ for every unitary representation π of G. In particular, $\lambda(f) = 0$ for the left regular representation of G. Now, there is an approximate unit of $\mathbf{L}^1(G)$, $(g_i)_{i\in I}$, consisting of functions in $C_c(G)$ and then, $g_i \in \mathbf{L}^2(G)$ for every i. By 1.1.10, we know that $\lambda(f)g_i = f \star g_i$ and therefore $f \star g_i = 0$ for every i and $f \star g_i \to f$ in $\mathbf{L}^1(G)$, so $f = 0$ and since $\|\cdot\|_*$ is already a seminorm, it is a norm on $\mathbf{L}^1(G)$. ☐

Remark 2.0.3. *The previous proposition shows that $\mathbf{C}^*(G) = \overline{\mathbf{L}^1(G)}^{\|\cdot\|_*}$ and therefore $\mathbf{L}^1(G)$ is a subset of $\mathbf{C}^*(G)$. Now, by proposition 2.0.1, we know that $\|f\|_* \leqslant \|f\|_1$ for every $f \in \mathbf{L}^1(G)$ and if $i : \mathbf{L}^1(G) \to \mathbf{C}^*(G)$ is the inclusion of $\mathbf{L}^1(G)$ in $\mathbf{C}^*(G)$, then i is continuous and it is a continuous embedding of $\mathbf{L}^1(G)$ in $\mathbf{C}^*(G)$.*

Proposition 2.0.4. *Let π be a unitary representation of G and let H be the corresponding Hilbert space. Then π can be uniquely extended to a non-degenerate $*$- representation of $\mathbf{C}^*(G)$ on H. Moreover, every non-degenerate*

$*$- representation of $\mathbf{C}^*(G)$ determines a unique non-degenerate $*$- representation of $\mathbf{L}^1(G)$ and therefore a unique unitary representation of G, so there is a bijection between the unitary representations of G and the non-degenerate $*$- representations of $\mathbf{C}^*(G)$.

Proof. Let π be a unitary representation of G on H. We denote again by π the corresponding representation of $\mathbf{L}^1(G)$.

Let $a \in \mathbf{C}^*(G)$. We want to define an operator $\pi(a) \in B(H)$.

Let $\xi, \eta \in H$. Since a lies in $\mathbf{C}^*(G)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbf{L}^1(G)$ such that $f_n \to a$ in $\|\cdot\|_*$. We now consider the sequence $(\langle\pi(f_n)\xi, \eta\rangle)_n$ in $\mathbb{C}$. For $m, n \in \mathbb{N}$, $|\langle\pi(f_m)\xi, \eta\rangle - \langle\pi(f_n)\xi, \eta\rangle| = |\langle\pi(f_m - f_n)\xi, \eta\rangle| \leqslant \|\pi(f_m - f_n)\| \cdot \|\xi\| \cdot \|\eta\| \leqslant \|f_m - f_n\|_* \cdot \|\xi\| \cdot \|\eta\|$ and $(f_n)_n$ is Cauchy with respect to $\|\cdot\|_*$ and so $(\langle\pi(f_n)\xi, \eta\rangle)_n$ is a Cauchy sequence in $\mathbb{C}$ and therefore converges in $\mathbb{C}$.

We now define $\psi_a : H \times H \to \mathbb{C}, \quad (\xi, \eta) \mapsto \lim\langle\pi(f_n)\xi, \eta\rangle$. Then, ψ_a is clearly sesquilinear and $|\psi_a(\xi, \eta)| \leqslant M \cdot \|\xi\| \cdot \|\eta\|$, where $M = \max\{\|\pi(f_n)\| : n \in \mathbb{N}\}$, which is finite since $(f_n)_n$ converges with respect to $\|\cdot\|_*$ and so ψ_a is a bounded sesquilinear form on H. Therefore, there exists a unique operator on H, that we denote by $\pi(a)$, such that $\psi_a(\xi, \eta) = \langle\pi(a)\xi, \eta\rangle$, for every $\xi, \eta \in H$ and so $\langle\pi(a)\xi, \eta\rangle = \lim\langle\pi(f_n)\xi, \eta\rangle$ for every $\xi, \eta \in H$. Clearly, if there exists a representation of $\mathbf{C}^*(G)$ extending the representation of $\mathbf{L}^1(G)$ on H it should be the one defined above, so we need to check that it is indeed a non-degenerate $*$- representation of $\mathbf{C}^*(G)$.

Let $a \in \mathbf{C}^*(G)$ and consider $(f_n)_n$ in $\mathbf{L}^1(G)$ such that $f_n \to a$. Since $\mathbf{C}^*(G)$ is a $\mathbf{C}^*$- algebra, $f_n^* \to a^*$ and so for $\xi, \eta \in H$, $\langle\pi(a^*)\xi, \eta\rangle = \lim\langle\pi(f_n^*)\xi, \eta\rangle = \lim\overline{\langle\pi(f_n)\eta, \xi\rangle} = \overline{\langle\pi(a)\eta, \xi\rangle} = \langle(\pi(a))^*\xi, \eta\rangle$ and this holds for every $\xi, \eta \in H$, so $\pi(a^*) = (\pi(a))^*$.

Now, let $a, b \in \mathbf{C}^*(G)$ and $(f_n), (g_n)$ be sequences in $\mathbf{L}^1(G)$ such that $f_n \to a$ and $g_n \to b$. We will first show that $f_n \star g_n \to ab$. To see this, notice that

$$\begin{aligned}
\|ab - f_n \star g_n\|_* &\leqslant \|ab - ag_n\|_* + \|ag_n - f_n \star g_n\|_* \\
&\leqslant \|a\|_* \|b - g_n\|_* + \|g_n\|_* \|a - f_n\|_*
\end{aligned}$$

and since $(\|g_n\|_*)_n$ is bounded, this tends to 0 and $f_n \star g_n \to \omega u$.

Now, for $\xi, \eta \in H$,

$$|\langle(\pi(\omega)\pi(u))\xi - \pi(f_n)\pi(g_n)\xi, \eta\rangle| \leqslant$$
$$\|\xi\| \, \|\eta\| \left(\|\pi(\omega)\| \, \|\pi(u - g_n)\| + \|\pi(g_n)\| \, \|\pi(\omega - f_n)\|\right)$$

which tends to 0 and therefore, $\pi(f_n)\pi(g_n) \to \pi(a)\pi(b)$. Now, $\pi(f_n)\pi(g_n) = \pi(f_n \star g_n)$ which tends to $\pi(ab)$ and thus $\pi(ab) = \pi(a)\pi(b)$ for every $a, b \in \mathbf{C}^*(G)$. We have thus shown that π is a $*$- representation of $\mathbf{C}^*(G)$ that extends the representation of $\mathbf{L}^1(G)$ on H and moreover, the extension to $\mathbf{L}^1(G)$ of this representation is non-degenerate and so the representation of $\mathbf{C}^*(G)$ is non-degenerate.

For the converse, every non-degenerate $*$- representation of $\mathbf{C}^*(G)$ defines a unique $*$- representation of $\mathbf{L}^1(G)$ by restriction and notice that since $\mathbf{L}^1(G)$ is dense in $\mathbf{C}^*(G)$ if there were some $\xi \in H$, such that $\pi(\mathbf{L}^1(G))\xi = 0$, then $\pi(a)\xi = 0$ for every $a \in \mathbf{C}^*(G)$ and since π is non-degenerate, we conclude that $\xi = 0$ and therefore π restricted to $\mathbf{L}^1(G)$ is a non-degenerate $*$- representation of $\mathbf{L}^1(G)$. □

Definition 2.0.5. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$, we define $k(\mathcal{S})$ as*

$$k(S) = \{\omega \in \mathbf{C}^*(G) : \pi(\omega) = 0 \quad \forall \pi \in \mathcal{S}\}$$

The next lemma will show that every $\mathbf{C}^*_{\mathcal{S}}(G)$ as defined is in fact a quotient of $\mathbf{C}^*(G)$.

Lemma 2.0.6. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$. We define $\phi : \mathbf{L}^1(G) \to \mathbf{L}^1(G)/N_{\mathcal{S}}, \quad f \mapsto f + N_{\mathcal{S}}$. Then ϕ extends to a surjective $*$- homomorphism from $\mathbf{C}^*(G)$ to $\mathbf{C}^*_{\mathcal{S}}(G)$, with kernel $k(S)$.*

Proof. Let $f \in \mathbf{L}^1(G)$, then $\|f + N_{\mathcal{S}}\| = \|f\|_{\mathcal{S}} \leqslant \|f\|_*$ and therefore it extends to a continuous map from $\mathbf{C}^*(G)$ to $\mathbf{C}^*_{\mathcal{S}}(G)$, which we denote again by ϕ and ϕ is a $*$- homomorphism, since its restriction to $\mathbf{L}^1(G)$ is a $*$- homomorphism and $\mathbf{L}^1(G)$ is a dense *-subalgebra of $\mathbf{C}^*(G)$. Moreover, we have that

$\phi(\mathbf{L}^1(G)) = \mathbf{L}^1(G)/N_{\mathcal{S}}$ which is dense in $\mathbf{C}^*(G)$ and therefore, by 0.1.10, we get that $\phi(\mathbf{C}^*(G)) = \mathbf{C}^*_{\mathcal{S}}(G)$.

Now, let $a \in \mathbf{C}^*(G)$, then $a \in \mathrm{Ker}(\phi)$ if and only if $\phi(f_n) \to 0$ for every sequence $(f_n)_n$ of elements of $\mathbf{L}^1(G)$ converging to a; thus $a \in \mathrm{Ker}(\phi)$ if and only if $\|f_n\|_{\mathcal{S}} \to 0$ for any such $(f_n)_n$ which happens if and only if $\pi(a) = 0$ for every $\pi \in \mathcal{S}$, that is, if and only if $a \in k(\mathcal{S})$. Thus $\mathrm{Ker}(\phi) = k(\mathcal{S})$ and so $\mathbf{C}^*_{\mathcal{S}}$ and $\mathbf{C}^*(G)/k(\mathcal{S})$ are isomorphic as C^*- algebras. From now on, we will use these two interchangeably without further mention. □

We need to define a very important relation between classes of representations of G, that of weak containment.

Definition 2.0.7. *Let $\mathcal{S}, \mathcal{T} \subset \mathbf{\Sigma}(\mathcal{G})$. We say that $\mathcal{S}$ is weakly contained in $\mathcal{T}$ if and only if $k(\mathcal{T}) \subset k(\mathcal{S})$ and we denote it by $\mathcal{S} \prec \mathcal{T}$.*

As the next proposition shows, the notion of weak containment is exactly what we need to describe the dual of $\mathbf{C}^*_{\mathcal{S}}(G)$.

Denote by $P_{\mathcal{S}}(G)$ the set of $u \in P(G)$ such that $\pi_u \prec \mathcal{S}$, where π_u is the representation associated to u that we defined in proposition 1.2.4.

Proposition 2.0.8. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$, then $(\mathbf{C}^*_{\mathcal{S}}(G))^*$ can be identified with $\mathrm{span}(P_{\mathcal{S}}(G))$, via the pairing $\langle f | u \rangle = \int f(x)u(x)\,dx$ for $f \in \mathbf{L}^1(G)$.*

To prove this proposition, we will need a couple of lemmas.

Lemma 2.0.9. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$ and $u \in P(G)$.Then the following are equivalent:*

1. $\pi_u \prec \mathcal{S}$
2. *There exists a positive functional ϕ on $\mathbf{C}^*_{\mathcal{S}}(G)$ such that*

$$\phi(f + k(\mathcal{S})) = \int f(x)u(x)\,dx$$

for $f \in \mathbf{L}^1(G)$, where we identify $\mathbf{C}^*_{\mathcal{S}}(G)$ with $\mathbf{C}^*(G)/k(\mathcal{S})$.

Proof. Let's assume that $u \in P(G)$ and $\pi_u \prec \mathcal{S}$. There is $\xi \in H_u$ such that $u(x) = \langle \pi_u(x)\xi, \xi\rangle$ for every $x \in G$.

We define $\phi : \mathbf{L}^1(G) \to \mathbb{C}, \quad f \mapsto \int f(x)u(x)\,dx$. Then ϕ is well defined since u is bounded and

$$\left|\int f(x)u(x)\,dx\right| = \left|\int f(x)\langle \pi_u(x)\xi, \xi\rangle\,dx\right| = |\langle \pi_u(f)\xi, \xi\rangle| \leqslant \|\xi\|^2 \cdot \|f\|_*$$

and so ϕ extends to a bounded functional on $\mathbf{C}^*(G)$, which we denote again by ϕ.

Let $a \in k(\mathcal{S})$, recall that $\pi_u \prec \mathcal{S}$, so $k(\mathcal{S}) \subset \mathrm{Ker}(\pi_\mathrm{u})$. Now, $a \in k(\mathcal{S})$, so $\pi_u(a) = 0$ and $\phi(a) = \langle \pi_u(a)\xi, \xi\rangle$, so $\phi(a) = 0$. Since ϕ vanishes on $k(\mathcal{S})$, it defines a bounded functional on $\mathbf{C}^*(G)/k(\mathcal{S})$, which we denote by $\tilde{\phi}$, so that

$$\tilde{\phi}(f + k(\mathcal{S})) = \int f(x)u(x)\,dx$$

for $f \in \mathbf{L}^1(G)$. We now need to check that $\tilde{\phi}$ is positive.

At first, consider $a \in \mathbf{C}^*(G)$ and let $(f_n)_n$ be a sequence in $\mathbf{L}^1(G)$ such that $\|f_n - a\|_* \to 0$. Then $f_n + k(\mathcal{S}) \to a + k(\mathcal{S})$ in $\mathbf{C}^*_{\mathcal{S}}(G)$ and thus,

$$\tilde{\phi}(f_n + k(\mathcal{S})) \to \tilde{\phi}(a + k(\mathcal{S}))$$

Then, by definition of $\tilde{\phi}$, this means that $\int f_n(x)u(x)\,dx \to \tilde{\phi}(a + k(\mathcal{S}))$. Now, $u(x) = \langle \pi_u(x)\xi, \xi\rangle$ for every $x \in G$ and so

$$\int f_n(x)u(x)\,dx = \int f_n(x)\langle \pi_u(x)\xi, \xi\rangle\,dx = \langle \pi_u(f_n)\xi, \xi\rangle$$

Now, $f_n \overset{\|\cdot\|_*}{\to} a$, so $\langle \pi_u(f_n)\xi, \xi\rangle \to \langle \pi_u(a)\xi, \xi\rangle$ and therefore

$$\tilde{\phi}(a + k(\mathcal{S})) = \langle \pi_u(a)\xi, \xi\rangle.$$

Now, let $b \in \mathbf{C}^*_{\mathcal{S}}(G)$ be positive, then there exists some $a \in \mathbf{C}^*(G)$ such that $b = a^*a$ and then

$$\tilde{\phi}(b) = \langle \pi_u(b)\xi, \xi\rangle = \langle \pi_u(a^*a)\xi, \xi\rangle = \langle \pi_u(a)\xi, \pi_u(a)\xi\rangle = \|\pi_u(a)\xi\|^2 \geqslant 0.$$

Which proves that $\tilde{\phi}$ is indeed positive.

For the converse, assume that $\phi(f + k(\mathcal{S})) = \int f(x)u(x)\,dx$ determines a well defined bounded positive functional on $\mathbf{C}^*_{\mathcal{S}}(G)$ and define $\tilde{\phi} : \mathbf{C}^*(G) \to \mathbb{C}$ by $\tilde{\phi}(a) = \phi(a + k(\mathcal{S}))$. Then, just as before, $\tilde{\phi}(a) = \langle \pi_u(a)\xi, \xi\rangle$ for every $a \in \mathbf{C}^*(G)$

Now, let $a \in k(\mathcal{S})$. Since $k(\mathcal{S})$ is a two sided closed $*$- ideal in $\mathbf{C}^*(G)$, for every $f \in \mathbf{L}^1(G)$, $f^*a \in k(\mathcal{S})$ and so,

$$0 = \tilde{\phi}(f^*a) = \langle \pi_u(f^*a)\xi, \xi\rangle = \langle \pi_u(a)\xi, \pi_u(f)\xi\rangle$$

and this holds for every $f \in \mathbf{L}^1(G)$.

Recall that ξ is cyclic for π_u, so $\pi_u(a)\xi = 0$ for every $a \in k(\mathcal{S})$. Moreover, for every $f \in \mathbf{L}^1(G)$, we also have that $af \in k(\mathcal{S})$ for every $a \in k(\mathcal{S})$ and so $\pi_u(af)\xi = 0$ and so $\pi_u(a)\pi(f)\xi = 0$ for every $f \in \mathbf{L}^1(G)$ and $\pi_u(a) = 0$ since ξ is cyclic.

We have thus shown that $\pi_u(a) = 0$ for every $a \in k(\mathcal{S})$ and therefore $\pi_u \prec \mathcal{S}$, which is exactly what we wanted.

$\square$

The previous lemma gives a description of a particular class of positive functionals of $\mathbf{C}^*_{\mathcal{S}}(G)$. As we will see, this class turns out to be the whole positive cone of $(\mathbf{C}^*_{\mathcal{S}}(G))^*$.

Lemma 2.0.10. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$ and $\phi \in (\mathbf{C}^*_{\mathcal{S}}(G))^*$ be positive. Then there exists $u \in P_{\mathcal{S}}(G)$ such that $\phi(f + k(\mathcal{S})) = \int f(x)u(x)\,dx$ for every $f \in \mathbf{L}^1(G)$.*

Proof. Let ϕ be a positive functional on $\mathbf{C}^*_{\mathcal{S}}(G)$ and define $\tilde{\phi} : \mathbf{L}^1(G) \to \mathbb{C}$ by $\tilde{\phi}(f) = \phi(f + k(\mathcal{S}))$. Then $\tilde{\phi}$ is linear and

$$|\tilde{\phi}(f)| = |\phi(f + k(\mathcal{S}))| \leqslant \|\phi\| \cdot \|f\|_{\mathcal{S}} \leqslant \|\phi\| \cdot \|f\|_*$$

so $\tilde{\phi} \in (\mathbf{L}^1(G))^*$ and therefore there exists $u \in \mathbf{L}^\infty(G)$ such that $\tilde{\phi}(f) = \int f(x) \cdot u(x)\,dx$ for every $f \in \mathbf{L}^1(G)$.

Let $f \in \mathbf{L}^1(G)$, then $f^* \star f + k(\mathcal{S})$ is a positive element of $\mathbf{C}^*_{\mathcal{S}}(G)$. Moreover, $f^* \star f$ lies in $\mathbf{L}^1(G)$ and $\tilde{\phi}(f^* \star f) = \phi(f^* \star f + k(\mathcal{S})) \geqslant 0$ and therefore

$$\int (f^* \star f)(x) \cdot u(x)\,dx \geqslant 0$$

for every $f \in \mathbf{L}^1(G)$, so $u \in P(G)$. Now it is clear that u defines a positive functional on $\mathbf{C}^*_{\mathcal{S}}(G)$ and so, by lemma 2.0.9, u must lie in $P_{\mathcal{S}}(G)$ and we are done.

$\square$

Now, proposition 2.0.8 follows easily from the previous lemmas: Indeed, it is well known that the dual of a $\mathbf{C}^*$-algebra is spanned by its positive elements and in the case of $\mathbf{C}^*_{\mathcal{S}}(G)$, we just showed that those are exactly the elements of $P_{\mathcal{S}}(G)$, so $(\mathbf{C}^*_{\mathcal{S}}(G))^* = \operatorname{span} P_{\mathcal{S}}(G)$

We will give another description of the dual of $\mathbf{C}^*_{\mathcal{S}}(G)$.

Proposition 2.0.11. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$. Then a function $u : G \to \mathbb{C}$ belongs to $\operatorname{span} P_{\mathcal{S}}(G)$ if and only if there exists a representation $\pi \prec \mathcal{S}$ and $\xi, \eta \in H_\pi$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$.*

Proof. Let $u \in \operatorname{span} P_{\mathcal{S}}(G)$, then $u(x) = \sum_{i=1}^{n} \lambda_i \langle \pi_i(x)\xi_i, \xi_i\rangle$ for every $x \in G$, where $\lambda_i \in \mathbb{C}$, $\pi_i \prec \mathcal{S}$ and $\xi \in H_{\pi_i}$ for every i.

Let $\pi = \bigoplus_{i=1}^n \pi_i$, $\eta = (\lambda_1 \cdot \xi_1, \ldots, \lambda_n \cdot \xi_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. Then $\pi \prec \mathcal{S}$ and $u(x) = \langle \pi(x)\eta, \xi\rangle$ for every $x \in G$.

For the converse, let $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$, where $\pi \prec \mathcal{S}$ and $\xi, \eta \in H_\pi$. Then, by polarization, $u(x) = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \pi(x)(\xi + i^k \cdot \eta), \xi + i^k \cdot \eta\rangle$ and so $u \in \operatorname{span} P_{\mathcal{S}}(G)$. $\square$

The last proposition shows that the dual of $\mathbf{C}^*_{\mathcal{S}}(G)$ can be identified with

$$\{\langle \pi(\cdot)\xi, \eta\rangle : \pi \prec \mathcal{S}, \eta, \xi \in H_\pi\}$$

the set of coefficient functions of G whose associated representation is weakly contained in $\mathcal{S}$. This is exactly how we are going to define $B_{\mathcal{S}}(G)$.

Definition 2.0.12. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$, we define*

$$B_{\mathcal{S}}(G) := \{\langle \pi(\cdot)\xi, \eta\rangle : \pi \prec \mathcal{S}, \eta, \xi \in H_\pi\}$$

and endow it with the norm of the dual of $\mathbf{C}^*_{\mathcal{S}}(G)$. In the case $\mathcal{S} = \mathbf{\Sigma}(G)$ we simply write $B(G)$ and call it the Fourier-Stieltjes algebra of G. We then *have*

$$B(G) = \operatorname{span}\{\langle \pi(\cdot)\xi, \eta\rangle : \pi \in \mathbf{\Sigma}(G), \xi, \eta \in H_\pi\}$$

Remark 2.0.13. *Let $u \in B(G)$. Then $u \in \operatorname{span} P(G)$ and by remark 1.2.2, we see that $\overline{u} \in \operatorname{span} P(G)$ and therefore $\overline{u} \in B(G)$.*

Remark 2.0.14. *By the above definition it is clear that $B_{\mathcal{S}}(G)$ is a subspace of $B(G)$ for any $\mathcal{S} \subset \mathbf{\Sigma}(G)$. As we will soon see, it is actually a closed subspace and the norm we defined on $B_{\mathcal{S}}(G)$ agrees with the one it inherits as a subspace of $B(G)$.*

We will need another lemma.

Lemma 2.0.15. *Let X be a Banach space and Y a closed subspace of X. Then $(X/Y)^*$ is isometrically isomorphic to the subspace of X^* consisting of the functionals of X vanishing on Y.*

Proof. Let $\phi \in (X/Y)^*$. We define $\tilde{\phi} : X \to \mathbb{C}, \quad \phi(x+Y)$. Then $\tilde{\phi}$ is linear and $|\tilde{\phi}(x)| = |\phi(x+Y)| \leqslant \|\phi\| \cdot \|x+Y\| \leqslant \|\phi\| \cdot \|x\|$. So $\tilde{\phi}$ is bounded with $\left\|\tilde{\phi}\right\| \leqslant \|\phi\|$ and so we can define $T : (X/Y)^* \to X^*, \quad \phi \mapsto \tilde{\phi}$. Then T is clearly linear with $\|T\| \leqslant 1$ and $\tilde{\phi}$ vanishes on Y.

Now consider $\psi \in X^*$ vanishing on Y and define $\phi : X/Y \to \mathbb{C}$ by $\phi(x+Y) = \psi(x)$. Then ϕ is well defined since ψ vanishes on Y and it is clearly linear. Let $x+Y \in X/Y$, then $|\phi(x+Y)| = |\psi(x)| \leqslant \|\psi\| \cdot \|x\|$. Let $z \in X$ be such that $z+Y = x+Y$ and therefore $z-x \in Y$, then $\phi(x+Y) = \psi(z) = \psi(x-(x-z))$ and so $|\phi(x+Y)| \leqslant \|\psi\| \cdot \|x-y\|$ for every $y \in Y$, so $|\phi(x+Y)| \leqslant \|\psi\| \cdot \|x+Y\|$ and $\phi \in (X/Y)^*$ with $\|\phi\| \leqslant \|\psi\|$. Clearly $\tilde{\phi} = \psi$ and so for every $\phi \in (X/Y)^*$ we have $\left\|\tilde{\phi}\right\| \leqslant \|\phi\| \leqslant \left\|\tilde{\phi}\right\|$ and so T is an isometry. ◻

We are going to use this lemma to make the following remark, that we outlined earlier:

Remark 2.0.16. *Let $\mathcal{S} \subset \mathbf{\Sigma}(G)$, then the norm we previously defined on $B_{\mathcal{S}}(G)$ is the same as the one $B_{\mathcal{S}}(G)$ inherits as a subspace of $B(G)$.*

Proof. Let $u \in B_{\mathcal{S}}(G)$, then u defines a functional on $\mathbf{C}^*_{\mathcal{S}}(G)$ that we also denote by u and $\|u\|$ is by definition the norm of the functional as an element of $(\mathbf{C}^*_{\mathcal{S}}(G))^* = (\mathbf{C}^*(G)/k(\mathcal{S}))^*$, so by the previous lemma, u defines a unique functional on $\mathbf{C}^*(G)$, call it $\tilde{u}$, of the same norm as u and vanishing on $k(\mathcal{S})$. By identifying u with $\tilde{u}$, we can regard $B_{\mathcal{S}}(G)$ as a closed subspace of $B(G)$ and then, for $u \in B_{\mathcal{S}}(G)$,

$$\begin{aligned}\|u\|_{B(G)} = &\sup\{|\langle a + k(S)\big|u\rangle| : \|a + k(S)\| \leqslant 1\} = \\ &\sup\{|\langle u\big|f + k(S)\rangle| : \|f + k(S)\| \leqslant 1, f \in \mathbf{L}^1(G) = \\ &\sup\{\left|\int f(x)\cdot u(x)\,dx\right| : f \in \mathbf{L}^1(G), \|f\|_{\mathcal{S}} \leqslant 1\}.\end{aligned}$$

In particular, if $\mathcal{S} = \{\lambda\}$ and $u \in B_\lambda(G)$, then

$$\|u\| = \sup\{\left|\int f(x)\cdot u(x)\,dx\right| : f \in \mathbf{L}^1(G),\ \|\lambda(f)\| \leqslant 1\}.$$

□

Lemma 2.0.17. *Let $\mathcal{S}, \mathcal{T} \subset \mathbf{\Sigma}(G)$. Then the following are equivalent:*

1. *$\mathcal{S} \prec \mathcal{T}$*
2. *For every $f \in \mathbf{L}^1(G)$, $\|f\|_{\mathcal{S}} \leqslant \|f\|_{\mathcal{T}}$*
3. *For every $\mu \in M(G)$, $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$*

Proof. We first assume that $\mathcal{S} \prec \mathcal{T}$. This means that $k(\mathcal{T}) \subset k(\mathcal{S})$. Now notice that $\|f\|_{\mathcal{S}} = \inf\{\|f + a\|_* : a \in k(\mathcal{S})\} \leqslant \inf\{\|f + a\|_* : a \in k(\mathcal{T})\} = \|f\|_{\mathcal{T}}$ and therefore 1 implies 2.

For the opposite direction, assume that $\|f\|_{\mathcal{S}} \leqslant \|f\|_{\mathcal{T}}$ for every $f \in \mathbf{L}^1(G)$ and let $a \in k(\mathcal{T})$. We need to show that $a \in k(\mathcal{S})$. Let $(f_n)_n$ be a sequence in $\mathbf{L}^1(G)$ converging to a with respect to $\|\cdot\|_*$. Then,

$$f_n + k(\mathcal{T}) \to a + k(\mathcal{T}) = 0 + k(\mathcal{T})$$

and so

$$\|f_n + k(\mathcal{T})\| = \|f_n\|_{\mathcal{T}} \to 0$$

At the same time, $f_n + k(\mathcal{S}) \to a + k(\mathcal{S})$ and $\|f_n + k(\mathcal{S})\| = \|f_n\|_{\mathcal{S}} \leqslant \|f_n\|_{\mathcal{T}}$ which converges to 0, so $f_n + k(\mathcal{T}) \to 0$ and so $a \in k(\mathcal{T})$ and 1 implies 2.

$\mathbf{L}^1(G)$ is a subspace of $M(G)$ and so clearly 3 implies 2.

Now suppose that 2 holds and let $u \in M(G)$. We need to show that $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$

Let $\mathcal{V}$ be a neighborhood basis of the identity and for each $V \in \mathcal{V}$ let g_V be a non-negative continuous function on G that is supported on V and such that $\|g_V\|_1 = 1$. Let $f_V = \mu \star g_V$. Then f_V lies in $\mathbf{L}^1(G)$ and we will show that for every bounded and continuous function h, we have

$$\int f_V(x)h(x)\,dx \to \int h(x)\,d\mu(x).$$

To see this, let h be a continuous bounded function on G. Then,

$$\begin{aligned}
&\left|\int g_V(x)h(x)\,dx - \int h(x)\,d\mu(x)\right| \\
&= \left|\iint h(x)f_V(y^{-1}x)\,dxd\mu(y) - \iint g_V(x)h(y)\,dxd\mu(y)\right|
\end{aligned}$$

since $\int g_V(x)\,dx = 1$ and then

$$\begin{aligned}
\left|\int g_V(x)h(x)\,dx - \int h(x)\,d\mu(x)\right| &= \left|\iint (h(yx)g_V(x) - g_V(x)h(y))\,dxd\mu(y)\right| \\
&= \left|\iint g_V(x)(h(yx) - h(y))\,dxd\mu(y)\right| \\
&+ \left|\iint_V g_V(x)(h(yx) - h(y))\,dxd\mu(y)\right| \\
&\leqslant \int \sup_{x\in V}\{|h(yx) - h(y)|\}\,d|\mu|(y)\,.
\end{aligned}$$

Now, let

$$\phi_V : G \to \mathbb{C}, \quad y \mapsto \sup_{x\in V}\{|h(yx) - h(y)|\}.$$

Then, for every $y \in G$, we have that $\phi_V(y) \to 0$, as V tends to $\{0\}$, since h is continuous at 0 and $|\phi_V(y)| \leqslant 2\|h\|_\infty$ and the constant function $2\|h\|_\infty$ is integrable with respect to $|\mu|$, since $\mu \in M(G)$ and therefore, from the dominated convergence theorem, we get

$$\int \phi_V(y)\, d|\mu|(y) \to 0$$

and therefore,

$$\int f_V(x)h(x)\, dx \to \int h(x)\, d\mu(x)$$

as V tends to $\{e\}$.

Now, let $\pi \in \mathbf{\Sigma}(G)$ and $\xi, \eta \in H_\pi$. Then,

$$\langle \pi(f_V)\xi, \eta\rangle = \int f_V(x)\langle \pi(x)\xi, \eta\rangle\, dx$$

and $\langle \pi(\cdot)\xi, \eta\rangle$ is bounded and continuous, so

$$\langle \pi(f_V)\xi, \eta\rangle \to \int \langle \pi(x)\xi, \eta\rangle\, d\mu(x) = \langle \pi(\mu)\xi, \eta\rangle$$

and therefore $\pi(f_V) \to \pi(\mu)$ in the weak operator topology.

Moreover, for every $\pi \in \mathbf{\Sigma}(G)$, we have

$$\|\pi(\mu)\| = \sup\{|\langle \pi(\mu)\xi, \eta\rangle| : \|\xi\|, \|\eta\| \leqslant 1\}$$

and so $\|\pi(\mu)\| = \lim \|\pi(f_V)\|$ and $\|\mu\|_{\mathcal{S}} = \sup_{\pi \in \mathcal{S}} \lim \|\pi(f_V)\|$.

Moreover,

$$\|\pi(f_V)\| = \|\pi(\mu \star g_V)\| \leqslant \|\pi(\mu)\| \cdot \|\pi(g_V)\| \leqslant \|\pi(\mu)\| \cdot \|g_V\|_1 = \|\pi(\mu)\|$$

and therefore, $\|\pi(\mu)\| = \sup_{V \in \mathcal{V}} \|\pi(f_V)\|$.

Now, for every $\pi \in \mathcal{S}$,

$$\begin{aligned} \|\pi(\mu)\| = \sup_{V \in \mathcal{V}} \|\pi(f_V)\| \leqslant \sup_{V \in \mathcal{V}} \|f_V\|_{\mathcal{S}} \leqslant \\ \sup_{V \in \mathcal{V}} \|f_V\|_{\mathcal{T}} = \sup_{V \in \mathcal{V}} \sup_{\pi \in \mathcal{T}} \|\pi(f_V)\| = \|\mu\|_{\mathcal{S}} \end{aligned}$$

and since this holds for every $\pi \in \mathcal{S}$, we conclude that $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$ for every $\mu \in M(G)$ and we are done.

$\square$

Remark 2.0.18. *Let $u \in B_{\mathcal{S}}(G)$, then u defines a bounded linear functional on $\mathbf{C}^*_{\mathcal{S}}(G)$ and therefore, we can consider the functional $u^* \in (\mathbf{C}^*_{\mathcal{S}}(G))^*$, defined in 0.1.24, so $u^* \in B_{\mathcal{S}}(G)$. For every $f \in \mathbf{L}^1(G)$, we have*

$$\int f(x) \cdot u^*(x)\, dx = \overline{\int f^*(x) \cdot u(x)\, dx} = \overline{\int \Delta(x^{-1}) \cdot \overline{f(x^{-1})} \cdot u(x)\, dx} = \int f(x) \cdot \overline{u(x^{-1})}\, dx.$$

This holds for every $f \in \mathbf{L}^1(G)$ and clearly if we define $\tilde{u}(x) = \overline{u(x^{-1})}$, $x \in G$, then $\tilde{u} \in \mathbf{L}^\infty(G)$ and so $u^* = \tilde{u}$ almost everywhere. They are both continuous, so $u^* = \tilde{u}$ everywhere on G and so $\tilde{u} \in B_{\mathcal{S}}(G)$.

Now, let $u \in B_{\mathcal{S}}(G)$ such that $u = u^* = \tilde{u}$, then, using the Jordan decomposition of a functional on a $\mathbf{C}^*$-algebra (proposition 0.1.25), there exist unique $u_1, u_2 \in (\mathbf{C}^*_{\mathcal{S}}(G))^*_+$ such that $u = u_1 - u_2$ and $\|u\| = \|u_1\| + \|u_2\|$ and so $u_1, u_2 \in B_{\mathcal{S}}(G)$. In the same way, we can find $u'_1, u'_2 \in B(G)$ positive, such that $u = u'_1 + u'_2$ and $\|u\| = \|u'_1\| + \|u'_2\|$. But $u_1, u_2 \in B(G)$ and the Jordan decomposition is unique, so $u_1 = u'_1$ and $u_2 = u'_2$ and therefore, the components of the Jordan decomposition of $u \in B_{\mathcal{S}}(G)$, when considered in $B(G)$, belong again to $B_{\mathcal{S}}(G)$. Moreover, if $u = u^* \in B_{\mathcal{S}}$ and $u = u_+ - u_-$, where $u_+, u_- \in P_{\mathcal{S}}(G)$ such that $\|u\| = \|u_+\| + \|u_-\|$, then $|u| = u_+ + u_-$ lies in$B_{\mathcal{S}}(G)$.

Having defined the universal enveloping von Neumann algebra of a $\mathbf{C}^*$-algebra in 0.3.2, we give an analogous definition for locally compact groups.

Definition 2.0.19. *Let G be a locally compact group. We define the universal enveloping von Neumann algebra of G to be the universal enveloping von Neumann algebra of the full group $\mathbf{C}^*$-algebra $\mathbf{C}^*(G)$.*

We have already seen that $B(G)$ can be naturally identified with $(\mathbf{C}^*(G))^*$ and the universal enveloping von Neumann algebra of G can be identified with $(\mathbf{C}^*(G))^{**}$, so there is a natural identification between $(B(G))^*$ and the universal enveloping von Neumann algebra of the group.

We will let $\mathcal{M}(\omega)$ be the universal enveloping von Neumann algebra of G and denote the corresponding representation by $\omega : \mathbf{C}^*(G) \to \mathcal{M}(\omega)$. Using this representation of $\mathbf{C}^*(G)$ we will obtain a very useful insight on the norm of $B(G)$.

Before that, let us make this identification a little more specific.

Remark 2.0.20. *By theorem 0.3.2, the map $\tilde{\pi} : (\mathbf{C}^*(G))^* \to \mathcal{M}_\omega$ is an isometric isomorphism, where $\omega : \mathbf{C}^*(G) \to B(H_\omega)$ denotes the universal representation of $\mathbf{C}^*(G)$ and $\mathcal{M}_\omega = (\omega(\mathbf{C}^*(G)))''$. Moreover, if $i : \mathbf{C}^*(G) \to (\mathbf{C}^*(G))^{**}$ is the natural inclusion map, we know that $\tilde{\pi} \circ i = \omega$(0.3.1). Now, let $a \in \mathbf{C}^*(G)$ and $u \in B(G) = (\mathbf{C}^*(G))^*$. Then,*

$$\langle \omega(a) | u \rangle = \langle u | a \rangle$$

and in particular, for $f \in \mathbf{L}^1(G) \subset \mathbf{C}^*(G)$, this means that

$$\langle \omega(f) | u \rangle = \int f(x) u(x)\, dx$$

since this is how we have defined the duality between $\mathbf{C}^*(G)$ and $B(G)$ (2.0.8).

Remark 2.0.21. *Let $x \in G$ and consider δ_x, the Dirac measure at x. Let $(g_V)_{V \in \mathcal{V}}$ be an approximate identity of $\mathbf{L}^1(G)$ as the one in the proof of lemma 2.0.17 and let $u \in B(G)$. Then, since u is continuous and bounded, we have proved in the course of the proof of lemma 2.0.17, that*

$$\int (\mu \star g_V)(y) u(y)\, dy \to \int u(y)\, d\mu(y)$$

with respect to V. Now, for $\mu = \delta_x$, if we set $f_V = \delta_x \star g_V$ for every $V \in \mathcal{V}$, we get

$$\int f_V(y) u(y)\, dy \to \int u(y) d\delta_x(y) = u(x).$$

Now, in remark 2.0.20, we saw that for every $f \in \mathbf{L}^1(G)$ and $u \in B(G)$, we have

$$\langle \omega(f) | u \rangle = \int f(y) u(y)\, dy$$

and therefore, for every $V \in \mathcal{V}$, we have

$$\langle \omega(f_V) | u \rangle = \int f_V(y) u(y)\, dy \to \int u(y)\, d\delta_x(y) = u(x). \qquad (*)$$

Now, if $\xi, \eta \in H_\omega$, then the function $\langle \omega(\cdot)\xi, \eta \rangle$ is continuous and bounded and therefore,

$$\int f_V(y)\langle \omega(y)\xi, \eta \rangle \, dy = \langle \omega(f_V)\xi, \eta \rangle \to \int \langle \omega(y)\xi, \eta \rangle \, d\delta_x(y) = \langle \omega(x)\xi, \eta \rangle$$

with respect to V. This means that $\omega(f_V) \to \omega(x)$ in the weak operator operator topology. Moreover, for every $V \in \mathcal{V}$, we have that $\|\omega(f_V)\| \leqslant \|f_V\|_1 = 1$, by 1.1.7 and since the weak operator topology and ultraweak topology coincide on bounded subsets of $B(H)$ ([12] proposition 2.7.19.), we get that $\omega(f_V) \to \omega(x)$ ultraweakly and hence,

$$\langle \omega(f_V) \big| u \rangle \to \langle \omega(x) \big| u \rangle$$

for every $u \in B(G)$. Now, as we saw, $\langle \omega(f_V) \big| u \rangle = \int f_V(y)u(y) \, dy \to u(x)$ and therefore, we get that

$$\langle \omega(x) \big| u \rangle = u(x).$$

For what follows, we will need a few definitions.

Definition 2.0.22. *Let A be a* **C***- *algebra, $f \in A^*$ and $x \in A$. Then we define $x \cdot f \in A^*$ by $(x \cdot f)(a) := f(x \cdot a)$ for every $a \in A$.*

Clearly $x \cdot f$ is linear and if $a \in A$, $|(x \cdot f)(a)| = |f(x \cdot a)| \leqslant \|f\| \cdot \|x \cdot a\| \leqslant \|f\| \cdot \|x\| \cdot \|a\|$ and so $x \cdot f$ is bounded with $\|x \cdot f\| \leqslant \|f\| \cdot \|x\|$, so the above definition makes sense.

Definition 2.0.23. *Let A be* **C***- *algebra and $a \in A$. Then a is called a partial isometry if a^*a and aa^* are both projections. The projection a^*a is called the source projection and aa^* is called the range projection.*

In general, **C***- algebras do not need to contain non-trivial projections (projections different than the 0 and the identity) and so they need not contain non trivial isometries. The situation is rather different for von Neumann algebras. Von Neumann algebras are in fact equal to the norm closure of the span of their projections ([12] proposition 2.8.12.), so they contain many projections and many partial isometries. We will need the following lemma, related to partial isometries in a von Neumann algebra, whose proof we omit and can be found in [2] (Theorem 12.2.4).

Lemma 2.0.24. *Let $\mathcal{M}$ be a von Neumann algebra and $\phi \in \mathcal{M}_*$ an element of its predual. Then there exists a unique positive functional $|\phi| \in \mathcal{M}_*$ and $v \in \mathcal{M}$ partial isometry, such that $\phi = v \cdot |\phi|$.*

Proposition 2.0.25. *Let $u \in B(G)$ and suppose that $u(x) = \langle \pi(x)\xi, \eta\rangle, \quad x \in G$ for some $\pi \in \mathbf{\Sigma}(G)$ and $\xi, \eta \in H_\pi$. Then $\|u\| \leqslant \|\xi\| \cdot \|\eta\|$. Moreover, if $u \in B_{\mathcal{S}}(G)$ for some $\mathcal{S} \subset \mathbf{\Sigma}(G)$, then there exist $\pi \prec \mathcal{S}$ and $\xi, \eta \in H_\pi$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$.*

Proof. Let $u \in B(G)$ and suppose that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$. Then, since $B(G)$ is identified with $(\mathbf{C}^*(G))^*$ and $\mathbf{L}^1(G)$ is dense in $\mathbf{C}^*(G)$, $\|u\| = \sup\{|\langle u|f\rangle| : f \in \mathbf{L}^1(G), \|f\|_* \leqslant 1\} =$
$\sup\{|\langle \pi(f)\xi, \eta\rangle| : f \in \mathbf{L}^1(G), \|f\|_* \leqslant 1\}$.

Now, $|\langle \pi(f)\xi, \eta\rangle| \leqslant \|\pi(f)\| \cdot \|\xi\| \cdot \|\eta\| \leqslant \|f\|_* \cdot \|\xi\| \cdot \|\eta\| \leqslant \|\xi\| \cdot \|\eta\|$ and this holds for every $f \in \mathbf{L}^1(G)$ with $\|f\|_* \leqslant 1$ and therefore we conclude that $\|u\| \leqslant \|\xi\| \cdot \|\eta\|$.

Now, let $\omega : \mathbf{C}^*(G) \to \mathcal{M}_\omega$ be the representation of $\mathbf{C}^*(G)$ on its universal enveloping von Neumann algebra and by our usual abuse of notation, we denote by ω the corresponding representation of G as well.

Since $u \in B(G)$, we have that $u \in (\mathbf{C}^*(G))^*$. Then, as we saw in theorem 0.3.2, there exist $x, y \in H_\omega$ such that $u = \omega_{x,y}$ and in particular, $u \in \mathcal{M}_*$. Now, by lemma 2.0.24, there are $|u| \in (\mathcal{M}_*)_+$ and $V \in \mathcal{M}_\omega$ partial isometry, such that $u = V \cdot |u|$ and $\|u\| = \||u|\|$. Then, $|u| \in (\mathcal{M}^*)_+$, so $|u| \in B(G) = (\mathbf{C}^*)^*$ and by theorem 0.3.2 there exists $\eta \in H_\omega$ such that $|u|(a) = \langle \omega(a)\eta, \eta\rangle$ for every $a \in \mathbf{C}^*(G)$.

Now, $|u|$ is in $B(G)$ and it is a positive functional, so $|u| \in P(G)$ and so there exist $\pi \in \mathbf{\Sigma}(G)$ and $\eta \in H_\pi$ cyclic for π such that $|u|(x) = \langle \pi(x)\eta, \eta\rangle$ for every $x \in G$. Let $x \in G$, then $u(x) = \langle u|\omega(x)\rangle = \langle V \cdot |u||\omega(x)\rangle = \langle |u||V \cdot \omega(x)\rangle$.

Now, for every $a \in \mathbf{C}^*(G), \langle |u| | a\rangle = \langle \pi(a)\eta, \eta\rangle = \langle \tilde{\pi} \circ \omega(a)\eta, \eta\rangle$ and since $\pi(\mathbf{C}^*(G)$ is ultraweakly dense in $\mathcal{M}_\omega$ by Kaplansky's density theorem, we conclude that $\langle |u| | T\rangle = \langle \tilde{\pi}(T)\eta, \eta\rangle$ for every $T \in \mathcal{M}_\omega$.Therefore, $\langle |u| | V \cdot \omega(x)\rangle = \langle \tilde{\pi}(V \cdot \omega(x))\eta, \eta\rangle = \langle \pi(x) \cdot \tilde{\pi}(V)\eta, \eta\rangle$. Set $\xi = \tilde{\pi}(V)\eta$, then $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$.

Then, we already showed that $\|u\| \leqslant \|\xi\| \cdot \|\eta\|$. Moreover, $\|u\| = \||u|\| = \|\eta\|^2$ and $\xi = V\eta$ and V is a partial isometry, so V is contractive and $\|\xi\| \leqslant \|\eta\|$ and so $\|u\| \leqslant \|\xi\| \cdot \|\eta\| \leqslant \|\eta\|^2 = \|u\|$ and so $\|u\| = \|\xi\| \cdot \|\eta\|$.

We just need to check that $\pi \prec \mathcal{S}$. Let $a \in k(\mathcal{S})$, we need to show that $\pi(a) = 0$. $|u| \in B_{\mathcal{S}}(G)$, so $\langle |u| | a\rangle = 0$ and so $\langle \pi(a)\eta, \eta\rangle = 0$ for every $a \in k(\mathcal{S})$. Let $b \in \mathbf{C}^*(G)$, then $b^* \cdot a \in k(\mathcal{S})$ and $\langle \pi(a)\eta, \pi(b)\eta\rangle = 0$ for every $b \in \mathbf{C}^*(G)$ and η is cyclic, so $\pi(a)\eta = 0$ for every $a \in k(\mathcal{S})$. Now, for every $b \in \mathbf{C}^*(G), a \cdot b \in k(\mathcal{S})$ and $\pi(a \cdot b)\eta = 0$ and so $\pi(a) \cdot \pi(b)\eta = 0$ for every $b \in \mathbf{C}^*(G)$ and η cyclic, so $\pi(a) = 0$ for every $a \in k(\mathcal{S})$ and we are done. □

This property will prove to be very useful for the remainder of this thesis. For the time being, we give two immediate corollaries.

Corollary 2.0.26. *For a function $u \in B(G)$, proposition 2.0.25 shows that*

$$\|u\| = \min\{\|\xi\| \, \|\eta\| : u(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle, (\pi, H_\pi) \in \mathbf{\Sigma}(G) and \xi, \eta \in H_\pi\}.$$

Corollary 2.0.27. *Let $u \in B(G)$, then $\|u\|_\infty \leqslant \|u\|_{B(G)}$*

Proof. At first, notice that since $u \in B(G)$ there exists $\pi \in \mathbf{\Sigma}(G)$ and $\xi, \eta \in H_\pi$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$ and $|u(x)| \leqslant \|\pi(x)\| \cdot \|\xi\| \cdot \|\eta\| = \|\xi\| \cdot \|\eta\|$ since $\pi(x)$ is a unitary and so $u \in \mathbf{L}^\infty(G)$. Now, $\mathbf{L}^\infty(G)$ can be identified with the dual of $\mathbf{L}^1(G)$ with the pairing $\langle \phi | f\rangle = \int f(x) \cdot \phi(x)\, dx$ for $\phi \in \mathbf{L}^\infty(G)$ and $f \in \mathbf{L}^1(G)$ and so $\|u\|_\infty = \sup\{\left|\int f(x) \cdot u(x)\, dx\right| : \|f\|_1 \leqslant 1\}$.

On the other hand, $\|u\|_{B(G)} = \sup\{\left|\int f(x) \cdot u(x)\, dx\right| : f \in \mathbf{L}^1(G), \|f\|_* \leqslant 1\}$ and $\|f\|_* \leqslant \|f\|_1$, so for any $f \in \mathbf{L}^1(G)$ with $\|f\|_1 \leqslant 1$, we have that $\|f\|_* \leqslant 1$ and therefore $\|u\|_\infty \leqslant \|u\|_{B(G)}$.

□

Remark 2.0.28. *Let $u \in B(G)$. By remark 2.0.13, we know that $\overline{u} \in B(G)$. By proposition 2.0.25, there is (π, H) unitary representation of G and $\xi, \eta \in H$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$ and $\|u\| = \|\xi\|\,\|\eta\|$. Then, $\overline{u}(x) = \langle \eta, \pi(x)\xi\rangle$ for every $x \in G$ and again by proposition 2.0.25, we have $\|\overline{u}\| \leqslant \|\xi\|\,\|\eta\| = \|u\|$. Since $\overline{\overline{u}} = u$, we conclude that $\|\overline{u}\| = \|u\|$ for every $u \in B(G)$.*

So far we have been calling $B(G)$ the Fourier-Stieltjes algebra of G, but we have not yet justified the term "algebra". As we will soon see, $B(G)$ is in fact a Banach algebra and contains $B_\lambda(G)$ as a closed ideal.

To prove this, we will need a result of independent interest, known as Fell's absorption principle.

Theorem 2.0.29. *(Fell's absorption principle) Let G be a locally compact group, λ its left regular representation and (π, H) a unitary representation of G. Then $\pi \otimes \lambda$ is unitarily equivalent to $\bigoplus_{i\in I} \lambda_i$, where I is an index set for a basis of H and $\lambda_i = \lambda$ for every i. In particular, $\pi \otimes \lambda \prec \lambda$.*

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis of H, we claim that there exists a unitary $W : \mathbf{L}^2(G) \otimes H \to \bigoplus_{i\in I} H_i$, where $H_i = H$ for every i, such that $W(\pi \otimes \lambda)(x) = (\oplus_{i\in I}\lambda)(x)W$ for every $x \in G$. To see this, consider the space of strongly measurable functions $f : G \to H$, such that $\int \|f(x)\|^2 \, dx < \infty$, equipped with the norm $\|\cdot\|$, where $\|f\| = \int \|f(x)\|^2 \, dx$, which we denote by $\mathbf{L}^2(G, H)$[1]. We will first show that $\mathbf{L}^2(G) \otimes H$ is isometrically isomorphic with $\mathbf{L}^2(G, H)$. Let $f \in \mathbf{L}^2(G)$ and $\xi \in H$ and define $T(f \otimes \xi) \in \mathbf{L}^2(G, H)$ by $T(f \otimes \xi)(x) = f(x)\xi$ for every $x \in G$. Then

$$\|T(f \otimes \xi)\|^2 = \int \|f(x)\xi\|^2 \, dx = \int |f(x)|^2 \cdot \|\xi\|^2 \, dx =$$
$$\|\xi\|^2 \int |f(x)|^2 \, dx = \|\xi\|^2 \cdot \|f\|^2 = \|f \otimes \xi\|^2$$

and extend T linearly to the span of simple tensors. Recall that $(e_i)_{i\in I}$ is an orthonormal basis for H and consider $i, j \in I$ and $f, g \in \mathbf{L}^2(G)$, then

[1]For a definition of strong measurability and an exposition of Bochner integral, we refer to [8](chapter III)

$\langle f \otimes e_i, g \otimes e_j \rangle = \langle f, g \rangle \cdot \langle e_i, e_j \rangle = \langle f, g \rangle \cdot \delta_{ij}$ and so, if $i \neq j$,

$$\|f \otimes e_i + g \otimes e_j\|^2 = \|f \otimes e_i\|^2 + \|g \otimes e_2\|^2 .$$

Set $u = f \otimes e_1 + g \otimes e_2$, then

$$\begin{aligned} \|Tu\|^2 &= \int \|f(x)e_i + g(x)e_j\|^2 \, dx = \int \left(|f(x)|^2 \|e_i\|^2 + |g(x)|^2 \|e_j\|^2 \right) dx \\ &= \|f \otimes e_1\|^2 + \|g \otimes e_2\|^2 = \|u\|^2 \end{aligned}$$

In the same way, we can see that T is isometric on $A = \operatorname{span}\{f \otimes e_i : i \in I\}$ and A is dense on $\mathbf{L}^2(G) \otimes H$, so we can extend T to an isometric linear map from $\mathbf{L}^2(G) \otimes H$ to $\mathbf{L}^2(G, H)$, which we denote again by T. Now notice that simple integrable functions lie in the image of T and those are dense in $\mathbf{L}^2(G, H)$, so T is onto and therefore it is an isometric isomorphism.

We now define $U : \mathbf{L}^2(G, H) \to \mathbf{L}^2(G, H)$ by $(Uf)(x) = \pi(x)f(x)$. Then, $\|(Uf)(x)\|^2 = \int \|\pi(x)f(x)\|^2 \, dx = \int \|f(x)\|^2 \, dx = \|f\|^2$, since $\pi(x)$ is a unitary for every $x \in G$ and so U is an isometry. Moreover, if $f \in \mathbf{L}^2(G)$, consider $g : G \to H$ with $g(x) = \pi(x^{-1})f(x)$. Clearly $g \in \mathbf{L}^2(G, H)$ and $Ug = f$, so U is onto and therefore it is an isometric isomorphism.

Now, let $x \in G$ and consider $T\lambda(x)T^{-1} : \mathbf{L}^2(G, H) \to \mathbf{L}^2(G, H)$. First, notice that since $C_c(G)$ is dense in $\mathbf{L}^2(G)$ and so, if

$$B = \operatorname{span}\{f \otimes \xi : f \in C_c(G),\, \xi \in H\}$$

then B is dense in $\mathbf{L}^2(G) \otimes H$ and since T is an isomorphism, $T(B)$ is dense in $\mathbf{L}^2(G, H)$.

Let $f \in T(B)$. Notice that f is continuous with compact support. For $x \in G$, consider $T(\lambda(x) \otimes Id_H)T^{-1} : \mathbf{L}^2(G, H) \to \mathbf{L}^2(G, H)$ and let $y \in G$. Now, $f \in T(A)$ and so there exist $f_1, \ldots, f_n \in C_c(G)$ and $\xi_1, \ldots, \xi_n \in H$ such that $f = T(\sum_{i=1}^{n} f_i \otimes \xi)$ Then,

$$\begin{aligned}
(T(\lambda(x) \otimes Id_H)T^{-1}f)(y) &= (T(\lambda(x) \otimes Id_H)(\sum_{i=1}^{n} f_i \otimes \xi_i))(y) \\
&= (T(\sum_{i=1}^{n}(L_x f_i) \otimes \xi_i))(y) \\
&= \sum_{i=1}^{n}(L_x f_i)(y)\xi_i = \sum_{i=1}^{n} f_i(x^{-1}y)\xi_i \\
&= f(x^{-1}y)
\end{aligned}$$

By a slight abuse of notation, we identify $\mathbf{L}^2(G, H)$ with $\mathbf{L}^2(G)\otimes H$. Then, if we consider U acting on $\mathbf{L}^2(G) \otimes H$ this time, we get that

$$U(\lambda \otimes 1_H)(x) = (\lambda \otimes \pi)(x)U$$

where 1_H stands for the trivial representation of G on H and U is an isometry onto and thus a unitary, so $\lambda \otimes \pi$ is unitarily equivalent to $\lambda \otimes 1_H$ and this is unitarily equivalent to $\oplus_{i\in I}\lambda$, which is exactly what we wanted.

Now, notice that

$$\oplus_{i\in I}\lambda \prec \lambda.$$

Indeed, for $a \in \mathbf{C}^*(G)$, it is evident that $a \in k(\lambda)$ if and only if $a \in k\left(\oplus_{i\in I}\lambda\right)$ and therefore,

$$\lambda \otimes \pi \prec \oplus_{i\in I}\lambda \prec \lambda.$$

□

Proposition 2.0.30. *Let G be a locally compact group. Then $B(G)$ equipped with pointwise multiplication and the norm we defined earlier is a commutative unital Banach algebra, containing $B_\lambda(G)$ as a closed ideal.*

Proof. Let $u, v \in B(G)$, then, by proposition 2.0.25, we can find $\pi_1, \pi_2 \in \mathbf{\Sigma}(G)$ and $\xi_i, \eta_i \in H_{\pi_i}$ for $i = 1, 2$ such that $u(x) = \langle \pi_1(x)\xi_1, \eta_1\rangle$ and $v(x) = \langle \pi_2(x)\xi_2, \eta_2\rangle$ for every $x \in G$ and $\|u\| = \|\xi_1\| \cdot \|\eta_1\|$, $\|v\| = \|\xi_2\| \cdot \|\eta_2\|$. Then,

$$(u + v)(x) = \langle \pi_1(x)\xi_1, \eta_1\rangle + \langle \pi_2(x), \eta_2\rangle = \langle (\pi_1 \oplus \pi_2)(x)(\xi_1, \eta_1), (\xi_2, \eta_2)\rangle$$

and therefore $u + v \in B(G)$. For the product, we have that

$$(u \cdot v)(x) = \langle \pi_1(x)\xi_1, \eta_1 \rangle \cdot \langle \pi_2(x)\xi_2, \eta_2 \rangle = \langle (\pi_1 \otimes \pi_2)(x)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle$$

and so $u \cdot v \in B(G)$ and $B(G)$ is indeed an algebra. Now, by 2.0.25, we have

$$\|u \cdot v\| \leqslant \|\xi_1 \otimes \xi_2\| \cdot \|\eta_1 \otimes \eta_2\| = \|\xi_1\| \cdot \|\xi_2\| \cdot \|\eta_1\| \cdot \|\eta_2\| = \|u\| \cdot \|v\|$$

and so the norm on $B(G)$ is submultiplicative. Moreover, $B(G)$ is isometrically isomorphic to the dual of $\mathbf{C}^*(G)$ and therefore it is complete, so $B(G)$ is a Banach algebra.

Also, notice that if H is a Hilbert space, $\xi \in H$ a unit vector and 1_G the trivial representation of G on H, then $\langle 1_G(x)\xi, \xi \rangle = 1$ for every $x \in G$ and so if $u(x) = 1$ for every $x \in G$, then u lies in $B(G)$ and it is clearly a unit for $B(G)$. Moreover, multiplication is clearly commutative and therefore $B(G)$ is a commutative unital Banach algebra.

To see that $B_\lambda(G)$ is a closed ideal in $B(G)$, notice that $B_\lambda(G)$ is identified with the dual of the reduced group $\mathbf{C}^*$-algebra and so it is complete and therefore closed in $B(G)$. It it is clearly a vector subspace of $B(G)$, so it is a Banach subspace of $B(G)$. All that is left is to show that $B_\lambda(G)$ is an ideal.

Let $u \in B_\lambda(G)$ and $v \in B(G)$. Then we can find $\pi \prec \lambda$ and $\sigma \in \mathbf{\Sigma}(G)$ such that $u(x) = \langle \pi(x)\xi, \eta \rangle$ and $v(x) = \langle \sigma(x)k, \rho \rangle$ for some $\xi, \eta \in H_\pi$ and $k, \rho \in H_\sigma$. Then, $(u \cdot v)(x) = \langle (\pi \otimes \sigma)(x)(\xi \otimes k, \eta \otimes \rho \rangle$ for every $x \in G$ and so it suffices to show that $\pi \otimes \sigma \prec \lambda$, which is clear from theorem 2.0.29 and we are done.

□

Lemma 2.0.31. *Let $\mu \in M(G)$ and $\mathcal{S}$ be a class of representations of G. Then,*

$$\|\mu\|_{\mathcal{S}} = \sup \left\{ \left| \int u(x)\, d\mu(x) \right| : u \in B_{\mathcal{S}}(G), \|u\| \leqslant 1 \right\}$$

Proof. By the definition of $\|\cdot\|_{\mathcal{S}}$, we have

$$\|\mu\|_{\mathcal{S}} = \sup \{ \|\pi(\mu)\| : \pi \in \mathcal{S} \}$$

Then, by lemma 2.0.17, we know that $\|\pi(\mu)\| \leqslant \|\mu\|_{\mathcal{S}}$ for every $\pi \prec \mathcal{S}$ and therefore,

$$\begin{aligned}\|\mu\|_{\mathcal{S}} &= \sup\{\|\pi(\mu)\| : \pi \prec \mathcal{S}\} \\ &= \sup_{\pi\prec\mathcal{S}} \sup\{|\langle \pi(\mu)\xi, \eta\rangle| : \xi, \eta \in H_\pi \text{with } \|\xi\|, \|\eta\| = 1\}\end{aligned}$$

Now, by proposition 2.0.25, we know that the functions $u \in B_{\mathcal{S}}(G)$ with $\|u\| \leqslant 1$ are exactly the functions of the form $u(x) = \langle \pi(x)\xi, \eta\rangle$ for $\pi \prec \mathcal{S}$ and $\xi, \eta \in H_\pi$ with $\|\xi\| \cdot \|\eta\| \leqslant 1$ and notice that if $\|\xi\| \cdot \|\eta\| = 1$, we can assume that $\|\xi\| = \|\eta\| = 1$, since $u(x) = \frac{u(x)}{\|\xi\|\cdot\|\eta\|} = \langle \pi(x)\frac{\xi}{\|\xi\|}, \frac{\eta}{\|\eta\|}\rangle$ and then

$$\begin{aligned}&\sup_{\pi\prec\mathcal{S}} \sup\{|\langle \pi(\mu)\xi, \eta\rangle| : \xi, \eta \in H_\pi \text{with } \|\xi\|, \|\eta\| \leqslant 1\} \\ &= \sup_{\pi\prec\mathcal{S}} \sup\{|\langle \pi(\mu)\xi, \eta\rangle| : \xi, \eta \in H_\pi \text{with } \|\xi\|, \|\eta\| = 1\} \\ &= \sup\left\{\left|\int u(x)\, d\mu(x)\right| : u \in B_{\mathcal{S}}(G), \|u\| = 1\right\} \\ &= \sup\left\{\left|\int u(x)\, d\mu(x)\right| : u \in B_{\mathcal{S}}(G), \|u\| \leqslant 1\right\}\end{aligned}$$

and therefore,

$$\|\mu\|_{\mathcal{S}} = \sup\left\{\left|\int u(x)\, d\mu(x)\right| : u \in B_{\mathcal{S}}(G), \|u\| \leqslant 1\right\}$$

as we wanted.

□

Proposition 2.0.32. *Let $\mathcal{S}$ be a class of unitary representations of G and let $u \in B_{\mathcal{S}}(G)$. Then,*

$$\|u\| = \sup\left\{\left|\sum_{j=1}^{n} u(x_j)\right| : c_j \in \mathbb{C}, x_j \in G, \left\|\sum_{j=1}^{n} c_j\delta_{x_j}\right\|_{\mathcal{S}} \leqslant 1\right\}$$

Proof. Let ω denote the universal representation of $\mathbf{C}^*(G)$ and let $\mathcal{M}(\omega)$ be the universal enveloping von Neumann algebra of $\mathbf{C}^*(G)$. Then, by theorem 0.3.2, we know that $\mathcal{M}(\omega)$ can be identified with the double dual of $\mathbf{C}^*(G)$ and therefore with the dual of $B(G)$, so

$$\|u\| = \sup\{|\langle a|u\rangle| : a \in \mathcal{M}(\omega), \|a\| \leqslant 1\}.$$

Now, let $\mu \in M(G)$ and define

$$\phi_\mu : B(G) \to \mathbb{C}, \quad u \mapsto \int u(x)\, d\mu(x).$$

Then ϕ_μ is clearly linear. Let $u \in B(G)$. Then, by proposition 2.0.25, there is some $\pi \in \mathbf{\Sigma}(G)$ and $\xi, \eta \in H_\pi$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for every $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$.

Then,

$$\phi_\mu(u) = \int u(x)\, d\mu(x) = \int \langle \pi(x)\xi, \eta\rangle\, d\mu(x) = \langle \pi(\mu)\xi, \eta\rangle$$

and therefore,

$$|\phi_\mu(u)| \leqslant \|\pi(\mu)\| \cdot \|\xi\| \cdot \|\eta\| \leqslant \|\mu\| \cdot \|u\|$$

so ϕ_μ is bounded with $\|\phi_\mu\| \leqslant \|\mu\|$.

Let $\mathcal{V}$ be a neighborhood basis of the identity and for each $V \in \mathcal{V}$, let g_V be non-negative continuous function that is supported in V and such that $\|g_V\|_1 = 1$. Let $f_V = \mu \star g_V$. Then, we have already seen in the proof of lemma 2.0.17 that for every $h \in C(G)$ that is bounded, we have

$$\int h(x) f_V(x)\, dx \to \int h(x)\, d\mu(x)$$

and since u is bounded and continuous, it is evident that,

$$\int u(x) f_V(x)\, dx \to \int u(x)\, d\mu(x) = \phi_\mu(u)\,.$$

Now, $f_V \in \mathbf{L}^1(G)$ and therefore,

$$\langle \omega(f_V) | u\rangle = \int f_V(x) u(x)\, dx,$$

so

$$\langle \omega(f_V) | u\rangle \to \phi_\mu(u). \tag{*}$$

Notice that ϕ_μ lies in the dual of $B(G)$ and by identifying $(B(G))^*$ with $\mathcal{M}(\omega)$, we can consider ϕ_μ to be an element of $\mathcal{M}(\omega)$. After making this identification, relation (*) shows that $\omega(f_V) \xrightarrow{w^*} \phi_\mu$.

At the same time, if $\xi', \eta' \in H_\omega$ and $V \in \mathcal{V}$, then

$$\langle \omega(f_V)\xi', \eta' \rangle = \int f_V(x)\langle \omega(x)\xi', \eta' \rangle \, dx$$

and this converges to $\int \langle \omega(x)\xi', \eta' \rangle \, d\mu(x) = \langle \omega(\mu)\xi', \eta' \rangle$ and therefore

$$\omega(f_V) \overset{\text{WOT}}{\to} \omega(\mu)$$

and in particular, $\omega(\mu) \in \mathcal{M}(\omega)$.

Now, $\omega(f_V) \to \omega(\mu)$ in the weak operator topology and in fact in the ultraweak topology, since $\|f_V\| \leqslant \|\omega(\mu)\|$ for every V and the ultraweak and the weak operator topology agree on the ball of radius $\|\omega(\mu)\|$ (0.2.11). Therefore, $\omega(f_V) \overset{w*}{\to} \omega(\mu)$, but we have already shown that $\omega(f_V) \overset{w*}{\to} \phi_\mu$ and therefore,

$$\omega(\mu) = \phi_\mu.$$

In particular,

$$\langle \omega(\delta_x) \big| u \rangle = u(x)$$

for every $x \in G$.

Now, by 1.1.9, we see that

$$(\operatorname{span} \omega(G))'' = (\omega(\mathbf{L}^1(G)))''$$

and therefore, by Kaplansky's density theorem, if A is the linear span of $\omega(G)$, we have

$$\begin{aligned} \|u\| = &\sup \left\{ |\langle a \big| u \rangle| : a \in A, \|a\| \leqslant 1 \right\} = \\ &\sup \left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| : x_j \in G, \left\| \omega(\sum_{j=1}^n c_j \delta_{x_j}) \right\| \leqslant 1, \, n \in \mathbb{N} \right\} \end{aligned}$$

Now, notice that if we set $\mu = \sum_{j=1}^n c_j \delta_{x_j}$, then

$$\begin{aligned} \|\omega(\mu)\| = &\sup\{ |\langle \omega(\mu) \big| u \rangle| : u \in B(G), \|u\| \leqslant 1 \} = \\ &\sup \left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| : u \in B(G), \|u\| \leqslant 1 \right\} = \\ &\sup \left\{ |\langle \mu \big| u \rangle| : u \in B(G), \|u\| \leqslant 1 \right\} = \\ &\|\mu\|_{\boldsymbol{\Sigma}(\mathcal{G})} \end{aligned}$$

by lemma 2.0.31 and so we get

$$\|u\| = \sup\left\{\left|\sum_{j=1}^{n} c_j u(x_j)\right| : x_j \in G, \left\|\sum_{j=1}^{n} c_j \delta_{x_j}\right\|_{\mathbf{\Sigma}(G)} \leqslant 1\right\}$$

Now, let G_d be the same group G, but this time endowed with the discrete topology. Then G_d is a locally compact group and every unitary representation of G is again a unitary representation of G_d and therefore, $B(G) \subset B(G_d)$ and $B_{\mathcal{S}}(G) \subset B_{\mathcal{S}}(G_d)$.

Then, notice that

$$\|u\|_{B(G_d)} = \sup\{|\langle u|f\rangle| : f \in C_c(G_d), \|f\|_{\mathbf{C}^*(G_d)} \leqslant 1\}$$

and since $u \in B_{\mathcal{S}}(G_d)$, we get

$$\|u\|_{B(G_d)} = \sup\{|\langle u|f\rangle| : f \in C_c(G_d), \|f\|_{\mathcal{S}} \leqslant 1\}$$

Now, G_d is discrete, so $f \in C_c(G_d)$ if and only if $f = \sum_{j=1}^{n} c_j \delta_{x_j}$, where $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $x_j \in G$ and then,

$$\|u\|_{B(G_d)} = \sup\left\{\left|\sum_{j=1}^{n} c_j u(x_j)\right| : n \in \mathbb{N}, c_j \in \mathbb{C} \text{ s.t. } \left\|\sum_{j=1}^{n} c_j \delta_{x_j}\right\|_{\mathcal{S}} \leqslant 1\right\}$$

Recall that $\mathcal{S}$ is contained in $\mathbf{\Sigma}(G)$, so $\|u\|_{B_{\mathcal{S}}(G_d)} = \|u\|_{B_{\mathbf{\Sigma}(G)}(G_d)}$ and therefore,

$$\sup\left\{\left|\sum_{j=1}^{n} c_j u(x_j)\right| : n \in \mathbb{N}, c_j \in \mathbb{C} \text{ s.t. } \left\|\sum_{j=1}^{n} c_j \delta_{x_j}\right\|_{\mathcal{S}} \leqslant 1\right\} =$$

$$\sup\left\{\left|\sum_{j=1}^{n} c_j u(x_j)\right| : x_j \in G, \left\|\sum_{j=1}^{n} c_j \delta_{x_j}\right\|_{\mathbf{\Sigma}(G)} \leqslant 1\right\}$$

and so,

$$\|u\| = \sup\left\{\left|\sum_{j=1}^{n} c_j u(x_j)\right| : n \in \mathbb{N}, c_j \in \mathbb{C} \text{ s.t. } \left\|\sum_{j=1}^{n} c_j \delta_{x_j}\right\|_{\mathcal{S}} \leqslant 1\right\}$$

as we wanted.

For a far more comprehensive exposition of $B(G)$, the interested reader is referred to [6].

$\square$

The Fourier algebra $A(G)$

3.1 Definition of $A(G)$

We are going to introduce the second topic of this thesis, the Fourier algebra $A(G)$ of a locally compact group G. As we will see, the Fourier algebra is a particular subspace of $B(G)$, in fact an ideal of $B(G)$ and it is often considered to be an analogue of the Fourier transform of $\mathbf{L}^1(G)$, when the group G is not abelian. Before defining $A(G)$, we will need some additional groundwork.

Lemma 3.1.1. *Let $f, g \in L^2(G)$, then $f * \tilde{g} \in B_\lambda(G)$ and $\|f * \tilde{g}\| \leqslant \|f\|_2 \|g\|_2$*

Proof. Let $x \in G$, then

$$(f * \tilde{g})(x) = \int f(y)\tilde{g}(y^{-1}x)\,dy = \int f(y)\bar{g}(x^{-1}y)\,dy = \langle f, \lambda(x)g\rangle$$

and if $v(x) = \langle \lambda(x)g, f\rangle$, then $v \in B_\lambda(G)$ and $f * \tilde{g} = \bar{v}$, so $f * \tilde{g} \in B_\lambda(G)$. $\square$

Lemma 3.1.2. *Let $C \subset G$ be compact and $U \subset G$ be open, such that $C \subset U$. Then there exists a $u \in B(G) \cap C_c(G)$ with* $\mathrm{supp(u)} \subset U$ *such that $0 \leqslant u \leqslant 1$ and $u|_C = 1$.*

Proof. We begin with the following observation: Let W be a neighborhood of the identity, then there exists a symmetric relatively compact neighborhood of the identity V such that $V \cdot V \subset W$.

To see this, notice that the multiplication map $m : G \times G \to G$, $(x, y) \mapsto xy$ is continuous at (e, e) and since $e \in W$ and W is open, the set $m^{-1}(W)$ is an open subset of $G \times G$ and $(e, e) \in m^{-1}(W)$, so there exists an open relatively compact symmetric neighborhood V of e such that $V \times V \subset m^{-1}(W)$, so that $VV \subset W$.

Now, for every $g \in C$ the set $g^{-1}U$ is an open neighborhood of the identity, so there exists an open relatively compact symmetric neighborhood V_g of the identity such that $V_g V_g \subset g^{-1}U$. In exactly the same way, we can find a symmetric relatively compact neighborhood of the identity U_g such that $U_g U_g \subset V_g$. Now $C \subset \bigcup_{g \in C} gU_g$ and C is compact, so there exist $g_1, \ldots g_n \in C$ such that $C \subset \bigcup_{i=1}^{n} g_i U_{g_i}$. Letting $V = \bigcap_{i=1}^{n} U_{g_i}$, we have a relatively compact symmetric neighborhood of the identity. Now, let $g \in C$. Then, there exists $i \in \{1, \ldots, n\}$ such that $g \in g_i U_{g_i}$, so $g = g_i x$ for some $x \in U_{g_i}$.

Thus $gVV = g_i xVV \subset g_i V_{g_i} V_{g_i} \subset U$, so that $CVV \subset U$.

Now let V be as above and define $u : G \to \mathbf{C}$ by

$$u = \frac{1}{\lambda(V)} \mathbf{1}_{CV} * \widetilde{\mathbf{1}_V}.$$

From the previous lemma, $u \in B(G)$ and u is continuous, since $\mathbf{1}_{CV} \in L^1(G)$ and $\widetilde{\mathbf{1}_\mathbf{V}} \in L^\infty(G)$. Now, let $x \in G$, then,

$$\begin{aligned}
u(x) &= \frac{1}{\lambda(V)} (\mathbf{1}_{CV} \star \widetilde{\mathbf{1}_V})(x) = \frac{1}{\lambda(V)} \int \mathbf{1}_{\mathbf{CV}}(y) \widetilde{\mathbf{1}_\mathbf{V}}(y^{-1}x)\, dy \\
&= \frac{1}{\lambda(V)} \int \mathbf{1}_{CV}(y) \mathbf{1}_V(x^{-1}y)\, dy = \frac{1}{\lambda(V)} \int (\mathbf{1}_{CV} \mathbf{1}_{xV})(y)\, dy \\
&= \frac{\lambda(CV \cap xV)}{\lambda(V)}.
\end{aligned}$$

From this expression, it is evident that $0 \leqslant u(x) \leqslant 1\, \forall x \in G$. Moreover, lemma 3.1.1 shows that $(\mathbf{1}_{CV} \star \widetilde{\mathbf{1}_V})(x) = \langle \mathbf{1}_{CV}, \lambda(x) \mathbf{1}_V \rangle$ for every $x \in G$ and therefore $\mathbf{1}_{CV} \star \widetilde{\mathbf{1}V} \in B(G)$ and so u lies in $B(G)$. Furthermore, let $x \notin CV^2$ and assume that there is a $g \in G$ such that $g \in CV \cap xV$. Then, there are $c \in C$ and $v, w \in V$ such that $g = cv$ and $g = xw$, hence $x = cvw^{-1} \in CV^2$, which contradicts our assumption on x. So $CV \cap xV = \varnothing$ and thus $u(x) = 0$

for every $x \notin CV^2$. This means that

$$G \backslash CV^2 \subset \{x \in G : u(x) = 0\}$$

and therefore,

$$\{x \in G : u(x) \neq 0\} \subset CV^2 \subset C\overline{V^2}.$$

Now,

$$V^2 \subset U_{g_1}$$

and $\overline{U_{g_1}}$ is compact, so $\overline{V^2}$ is compact, as a closed subset of the compact $\overline{U_{g_1}}$. Therefore,

$$\operatorname{supp} u = \overline{\{x \in G : u(x) \neq 0\}} \subset C\overline{V^2}$$

and so, $\operatorname{supp} u$ is compact and $u \in P(G) \cap C_c(G)$.

□

Proposition 3.1.3. *Let G be a locally compact group. We define the following subsets of $B(G)$.*

$$\begin{aligned}
M_1 &= \{f \star \tilde{g} : f, g \in C_c(G)\} \\
M_2 &= \{f \star \tilde{f} : f \in C_c(G)\} \\
M_3 &= \{f \star \tilde{g} : f, g \in \mathbf{L}^\infty(G) \textit{ with compact support}\} \\
M_4 &= \{h \star \tilde{h} : h \in \mathbf{L}^\infty(G) \textit{ with compact support}\} \\
M_5 &= B(G) \cap C_c(G) \\
M_6 &= P(G) \cap C_c(G) \\
M_7 &= P(G) \cap \mathbf{L}^2(G) \\
M_8 &= \{f \star \tilde{g} : f, g \in \mathbf{L}^2(G)\} \\
M_9 &= \{f \star \tilde{f} : f \in \mathbf{L}^2(G)\}
\end{aligned}$$

Let E_j denote the linear span of M_j for every j. Then all E_j have the same closure in the norm of $B(G)$.

Proof. By polarization, it is clear that $E_1 = E_2$, that $E_3 = E_4$ and that $E_8 = E_9$. Moreover, clearly $M_2 \subset M_4$ and therefore $E_1 = E_2 \subset E_3 = E_4$.

To see that $M_4 \subset M_5$, notice that if $h \in \mathbf{L}^\infty(G)$ has compact support, then $h \in \mathbf{L}^1(G)$ and $\tilde{h} \in \mathbf{L}^\infty(G)$ and therefore, by [4] (proposition 2.39), we get that $h \star \tilde{h} \in C(G)$ and $\operatorname{supp}(h \star \tilde{h}) \subset \operatorname{supp} h \cdot \operatorname{supp} \tilde{h}$ which is compact, since both h and $\tilde{h}$ have compact supports. Moreover, by lemma 3.1.1, since $h \in \mathbf{L}^2(G)$, we have that

$$h \star \tilde{h} \in B_\lambda(G) \subset B(G)$$

and therefore,

$$h \star \tilde{h} \in B(G) \cap C_c(G)$$

and $M_4 \subset M_5$.

To see that $E_5 = E_6$, first notice that $M_6 \subset M_5$ and therefore $E_6 \subset E_5$.

For the converse, let $u \in M_5 = B(G) \cap C_c(G)$. By the definition of $B(G)$, we know that there exist $u_1, u_2, u_3, u_4 \in P(G)$ such that $u = u_1 - u_2 + i(u_3 - u_4)$. Now, $u \in C_c(G)$ and therefore, by lemma 3.1.2 there is a $v \in P(G) \cap C_c(G)$ such that $v|_{\operatorname{supp} u} = 1$. Then,

$$u = uv = vu_1 - vu_2 + iv(u_3 - u_4)$$

and $vu_i \in P(G) \cap C_c(G)$ for every i, by lemma 1.2.7 and so $u \in E_6$ and $E_5 = E_6$.

The fact that $E_7 \subset E_8$ follows from [2](theorem 13.8.6).

To finish the proof, we will show that E_1 is dense in E_9. Once we prove that, we will know that $\overline{E_1} \subset \cdots \subset \overline{E_9} = \overline{E_1}$ and we will be done.

To see this, let $f, g \in \mathbf{L}^2(G)$ and $\epsilon > 0$. Then, since $C_c(G)$ is dense in $\mathbf{L}^2(G)$, we can find $f_1, g_1 \in C_c(G)$ such that $\|f - f_1\|_2 < 2$ and $\|g - g_1\|_2 < \epsilon$.

Then, we have that

$$\|f \star \tilde{g} - f_1 \star \tilde{g}_1\|_{B(G)} \leqslant \|f \star \tilde{g} - f_1 \star \tilde{g}\|_{B(G)} + \|f_1 \star \tilde{g} - f_1 \star \tilde{g}_1\|_{B(G)} = \\ \|(f - f_1) \star \tilde{g}\|_{B(G)} + \left\| f_1 \star \widetilde{(g - g_1)} \right\|_{B(G)}$$

and therefore, by lemma 3.1.1, we see that

$$\|f \star \tilde{g} - f_1 \star \tilde{g_1}\|_{B(G)} \leqslant \|f - f_1\|_2 \|g\|_2 + \|f_1\|_2 \|g - g_1\|_2 \leqslant \\ \epsilon \|g\|_2 + (\|f\|_2 + \epsilon)\epsilon$$

which tends to 0 as ϵ tends to 0 and therefore, E_1 is dense in E_9 and we are done.

□

Definition 3.1.4. *We define the closure of any of the E_i's in the norm of $B(G)$ to be the Fourier algebra of G and we denote it by $A(G)$.*

We just called $A(G)$ an algebra and the next proposition justifies the use of this word.

Proposition 3.1.5. *Let G be a locally compact group. Then $A(G)$ is a closed ideal in $B(G)$. In particular, $A(G)$ is a closed subalgebra of $B(G)$.*

Proof. Let $u \in B(G) \cap C_c(G)$ and $v \in B(G)$. Then, $vu \in B(G)$, since $B(G)$ is an algebra and $uv \in C_c(G)$, since both u, v are continuous and u has compact support, so $vu \in B(G) \cap C_c(G)$. Moreover, $B(G) \cap C_c(G)$ is clearly a linear subspace of $B(G)$ and therefore it is an ideal in $B(G)$.

Now, let $u \in A(G)$. Then, by proposition 3.1.3, there is a $(u_n)_{n\in\mathbb{N}}$ in $B(G) \cap C_c(G)$ such that $u_n \to u$ in the norm of $B(G)$. Let $v \in B(G)$. Then, $vu_n \to vu$ and $vu_n \in B(G) \cap C_c(G)$ for every $n \in \mathbb{N}$, so $vu \in \overline{B(G) \cap C_c(G)} = A(G)$ (Proposition 3.1.3) and therefore $A(G)$ is an ideal in $B(G)$ and by definition it is closed. In particular, $A(G)$ is a closed subalgebra of $B(G)$. □

Proposition 3.1.6. *Let G be a locally compact group. Then $A(G) \subset C_0(G)$ and $A(G)$ is in fact uniformly dense in $C_0(G)$.*

Proof. Let $u \in A(G)$, then, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $B(G) \cap C_c(G)$ such that $u_n \to u$ in the norm of $B(G)$. Then, by corollary 2.0.27, we know that $u_n \to u$ with respect to $\|\cdot\|_\infty$ and $u \in \overline{C_c(G)}^{\|\cdot\|_\infty} = C_0(G)$, so

$$A(G) \subset C_0(G).$$

To see that $A(G)$ is uniformly dense in $C_0(G)$ notice first that for every $u \in A(G)$, we have that $\overline{u} \in A(G)$. Indeed, let $(u_n)_n$ be a sequence in $B(G) \cap C_c(G)$ such that $u_n \to u$. Then, $\overline{v_n} \in B(G) \cap C_c(G)$ for every $n \in \mathbb{N}$. and by remark 2.0.28, we see that $\|\overline{v} - \overline{u_n}\| = \|v - u_n\|$ which tends to 0 as n tends to infinity, so $\overline{u_n} \to \overline{v}$ and therefore $\overline{v} \in A(G)$. Since $A(G)$ is closed under conjugation, we see that $A(G)$ is a $*$-subalgebra of $C_0(G)$. Moreover, by lemma 3.1.2, it is clear that $A(G)$ separates the points of G and that for every $x \in G$, there is $f \in A(G)$ such that $f(x) \neq 0$ and therefore, by the Stone-Weierstrass theorem, $A(G)$ is $\|\cdot\|_\infty$-dense in the C^*-algebra $C_0(G)$. $\square$

We showed in proposition 3.1.3 that $A(G) = \overline{\text{span}\{f \star \tilde{g} : f, g \in \mathbf{L}^2(G)\}}^{\|\cdot\|_{B(G)}}$. In fact, it turns out that there is no need for taking linear span or closure in the norm of $B(G)$ and we actually have the following:

Theorem 3.1.7 (Characterization of $A(G)$)**.** *Let G be a locally compact group, then*

$$A(G) = \{f \star \tilde{g} : f, g \in \mathbf{L}^2(G)\}.$$

This theorem was proved by Eymard in [3] and the proof relied heavily on the theory of locally compact groups. A different approach is to use von Neumann algebra theory and show that vN(G) is in standard form on $\mathbf{L}^2(G)$, as it is done in [11].

3.2 The spectrum of $A(G)$

In this section, we are going to identify the spectrum of $A(G)$. As we will see, the spectrum is homeomorphic to the group G and therefore, "$A(G)$ remembers the group G."

Theorem 3.2.1. *Let G be a locally compact group G and $A(G)$ its Fourier algebra. Then, $\sigma(A(G))$ is homeomorphic to G, where $\sigma(A(G))$ denotes the spectrum of $A(G)$.*

The homeomorphism we will use is the map $T : G \to \sigma(A(G))$, $x \mapsto \phi_x$, where $\phi_x(u) = u(x)$ for all $u \in A(G)$. We will need a couple of lemmas.

Lemma 3.2.2. *Let $f \in A(G)$ and $x \in G$ such that $f(x) = 0$. Then, for every $\epsilon > 0$ there exists an open neighborhood $V \subset G$ of x and $g \in A(G) \cap C_c(G)$ such that $g|_V = 0$ and $\|f - g\|_{A(G)} < \epsilon$.*

Before proving this lemma, we will need some additional terminology.

Definition 3.2.3. *Let $\phi \in (A(G))^*$. We define the support supp(ϕ) of ϕ, to be the subset of G such that*

$$G \backslash \operatorname{supp} \phi = \bigcup \{V \subset G \text{ open: } u \in A(G) \cap C_c(G), \operatorname{supp} u \subset V \Rightarrow \phi(u) = 0\}.$$

For a function $u : G \to \mathbb{C}$, we define $Z(u) = \{x \in G : u(x) = 0\}$.

Definition 3.2.4. *Let E be a closed subset of G.*
(i) We call E a set of synthesis *if for every $\tau \in (A(G))^*$ and $u \in A(G)$, the relation* $\operatorname{supp} \tau \subset E \subset Z(u)$ *implies $\tau(u) = 0$.*

(ii) We call E a set of local synthesis *if for every $\tau \in (A(G))^*$ and $u \in A(G)$ with compact support, the relation* $\operatorname{supp} \tau \subset E \subset Z(u)$ *implies $\tau(u) = 0$.*

This definition may seem out of the blue at this point, but the next proposition will show that the notion of local synthesis is exactly what we need for lemma 3.2.2.

Proposition 3.2.5. *Let E be a closed subset of G. Then E is a set of synthesis if and only if for every $f \in A(G)$ such that $f|_E = 0$, f can be approximated in the norm of $A(G)$ by functions in $A(G) \cap C_c(G)$ vanishing in a neighbourhood of E.*

Proof. Let's first assume that E is a set of synthesis and let $f \in A(G)$ be such that $f|E = 0$.

Also, let

$$J(E) = \overline{\operatorname{span}}\{g \in A(G) \cap C_c(G) : g \text{ vanishes in a neighborhood of } E\}.$$

Consider a functional $\tau \in (A(G))^*$ that vanishes on $J(E)$. Then, by the definition of $\operatorname{supp} \tau$, it is evident that $\operatorname{supp} \tau \subset E$ and since E is a set of synthesis we have $\tau(f) = 0$ and therefore $f \in J(E)$.

For the opposite direction, assume that every $f \in A(G)$ vanishing on E lies in $J(E)$ and let $\tau \in (A(G))^*$ and $f \in A(G)$ be such that $\operatorname{supp}\tau \subset E \subset Z(f)$.

Then, there is a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \in A(G) \cap C_c(G)$ for every $n \in \mathbb{N}$ such that g_n vanishes on a neighborhood of E and g_n converges to f, so $\tau(g_n) \to \tau(f)$. In this case, each g_n vanishes on a neighborhood of E and so, g_n vanishes on a neighborhood of $\operatorname{supp}\tau$ and since g_n has compact support, $\tau(g_n) = 0$ by the definition of the support of τ and this holds for every $n \in \mathbb{N}$, so $\tau(f) = 0$ and thus E is a set of synthesis. □

This proposition shows that what we need to prove for Lemma 3.2.2 is exactly the fact that $\{x\}$ is a set of synthesis for every $x \in G$.

There is a characterisation of sets of local synthesis, completely analogous to the one of Proposition 3.2.5.

Proposition 3.2.6. *Let E be a closed subset of G. Then E is a set of local synthesis if and only if every $f \in A(G) \cap C_c(G)$ that vanishes in E can be approximated by functions in $A(G) \cap C_c(G)$ that vanish in a neighborhood of E.*

Proof. The proof is essentially the same as the one of proposition 3.2.5 and so we omit it. □

Definition 3.2.7. *Let $\tau \in (A(G))^*$ and $f \in A(G)$. We define $f \cdot \tau \in (A(G))^*$ by $f \cdot \tau(u) = \tau(f \cdot u)$ for every $u \in A(G)$.*

For every $u \in A(G)$ we have $\|\tau(f \cdot u)\| \leqslant \|\tau\| \cdot \|f \cdot u\| \leqslant \|\tau\| \cdot \|f\| \cdot \|u\|$ and so $f \cdot \tau$ is indeed bounded with $\|f \cdot u\| \leqslant \|\tau\| \cdot \|f\|$.

Lemma 3.2.8. *Let $E \subset G$ be a set of local synthesis and $\tau \in (A(G))^*$ with $\operatorname{supp}\tau \subset E$ and let $f \in A(G)$ with $f|_E = 1$. Then $f \cdot \tau = \tau$.*

Proof. Since $A(G) \cap C_c(G)$ is dense in $A(G)$, we need to show that for every $u \in A(G) \cap C_c(G)$, we have $f \cdot \tau(u) = \tau(u)$, or equivalently,$\tau(u(1 - f)) = 0$.

Notice that $(1 - f)u \in A(G) \cap C_c(G)$ and $(1 - f)u$ vanishes on E, so

$$\tau((1 - f)u) = 0$$

since E is a set of local synthesis. □

The proof of lemma 3.2.2 will be completed with the following proposition:

Proposition 3.2.9. *Let E be a compact subset of G. Then E is a set of local synthesis if and only if it is a set of synthesis.*

Proof. Every set of synthesis is a set of local synthesis, so we need to prove that for a compact set, local synthesis implies synthesis. In that direction, consider $\tau \in (A(G))^*$ and $u \in A(G)$ be such that $\operatorname{supp} \tau \subset E \subset Z(u)$. We need to prove that $\tau(u) = 0$.

Since E is compact, by lemma 3.1.2 there is $f \in A(G) \cap C_c(G)$ such that $f|_E = 1$.

Now, consider $f \cdot u$. Then, $f \cdot u \in A(G)$ and has compact support. Moreover, $fu|_E = 0$ and since E is a local synthesis set, $\tau(fu) = 0$ and so $(f \cdot \tau)(u) = 0$.

Then, by lemma 3.2.8, we know that $f \cdot \tau = \tau$ and so $\tau(u) = 0$ as we wanted.

□

To prove lemma 3.2.2, since $\{x\}$ is compact for every $x \in G$, we need to prove that $\{x\}$ is a set of local synthesis.

Proof. Consider $f \in A(G) \cap C_c(G)$ vanishing on x. We will show that f can be approximated by functions in $A(G) \cap C_c(G)$ vanishing on a neighborhood of $\{x\}$.

Let $\epsilon > 0$, we want to find some $h \in A(G) \cap C_c(G)$ such that h vanishes on a neighborhood of x and $\|f - h\| < \epsilon$. Let $W = \{y \in G : ||f - R_y f|| < \epsilon\}$. Then W is open, since the map $G \to A(G)$, $x \mapsto R_x f$ is continuous and clearly $e \in W$. Now, since G is locally compact, we can find a relatively compact open neighborhood of e, that we call U. Let $W' = W \cap U$. Then W' is open, contains the identity, it is relatively compact and $\|f - R_y f\| < \epsilon$ for every $y \in W'$.

Now, let V be an open neighborhood of the identity such that $|f(xy)| < \epsilon$ for all $y \in V$ and such that $xV \subset W'$ (we can find such a neighborhood, since f is continuous at x and $f(x) = 0$). Since $V \subset \overline{V} \subset \overline{W'}$ and $\overline{W'}$ is compact,

$0 < \lambda(V) < \infty$ and since the Haar measure is inner regular on open sets, we can find $K \subset V$ compact, such that

$$\lambda(K) \geqslant \lambda(V) - \epsilon\lambda(V). \tag{*}$$

Now, we define $u = \frac{1}{\lambda(K)}\mathbf{1}_K$, $g = \mathbf{1}_{xV}f$ and $h = (f-g) * \tilde{u}$. Since $\tilde{u} \in \mathbf{L}^1(G)$ and $f-g \in \mathbf{L}^\infty(G)$, the function $h = (f-g)*\tilde{u}$ is continuous (see [4]proposition 2.39.) and also, $f - g = \mathbf{1}_{G\setminus xV}f \in \mathbf{L}^2(G)$, since $f \in C_c(G)$ and u is also in $\mathbf{L}^2(G)$, since it is in $\mathbf{L}^\infty(G)$ and has compact support, so, from proposition 3.1.3 we get that $h \in A(G)$. As one can imagine, h will be the function we are searching for. Expanding the definition of h we get

$$h(z) = ((f-g) * \tilde{u})(z) = \frac{1}{\lambda(K)} \int \mathbf{1}_{G\setminus xV}(y) f(y) \mathbf{1}_K(z^{-1}y)\, dy.$$

Now, let $z \in G$ such that $h(z) \neq 0$. Then, there must exist $y \in G$ such that $y \in \operatorname{supp} f$ and $z^{-1}y \in K$. Therefore, $z^{-1} \in Ky^{-1} \subset K \cdot (\operatorname{supp} f)^{-1}$. So, we have shown that $\{z^{-1} : h(z) \neq 0\} \subset K \cdot (\operatorname{supp} f)^{-1}$ and thus,

$$\{z \in G : h(z) \neq 0\} \subset (\operatorname{supp} f) \cdot K^{-1}$$

and $(\operatorname{supp} f) \cdot K^{-1}$ is compact and therefore closed, so

$$\operatorname{supp} h = \overline{\{z \in G : h(z) \neq 0\}} \subset (\operatorname{supp} f) \cdot K^{-1}.$$

Therefore, $\operatorname{supp} h$ is compact and $h \in C_c(G)$.

Now, let $z \in G$ such that $zK \subset xV$. Then, $z^{-1}y \in K$ implies that $y \in zK \subset xV$ and therefore, whenever $\mathbf{1}_K(z^{-1}y) \neq 0$, we have that $\mathbf{1}_{G/xV}(y)$ vanishes and therefore the integral defining h is zero and $h(x) = 0$.

Clearly, since $K \subset V$, we have that $xK \subset xV$. Recall that the multiplication map $m : G \times G \to G$ is continuous and therefore $m^{-1}(xV)$ is an open subset of $G \times G$ containing $\{x\} \times K$. Therefore, for each $k \in K$ we can find U_k, W_k open neighborhoods of k and x respectively, such that $W_k \times U_k \subset m^{-1}(xV)$.

Now, K is compact and therefore we can find $k_1, \cdots, k_n \in K$ such that $K \subset \bigcup_{i=1}^n U_{k_i}$. Set $W_0 = \bigcap_{i=1}^n W_{k_i}$. Then, W_0 is an open neighborhood of x

and for every $k \in K$, it is evident that $W_0 \times \{k\} \subset m^{-1}(xV)$ and therefore $W_0K \subset xV$ and $h(z) = 0$ for every $z \in W_0$, so h vanishes in a neighborhood of x.

To finish the proof, we have to estimate $\|f - h\|$.

$\|f - h\| = \|f - (f - g) * \tilde{u}\| = \|f - f * \tilde{u} + g * \tilde{u}\| \leqslant \|f - f * \tilde{u}\| + \|g * \tilde{u}\|$.

Now, by lemma 3.1.1, we have $\|g * \tilde{u}\| \leqslant \|g\|_2 \|u\|_2$ and

$$\|g\|_2^2 = \int |\mathbf{1}_{xV} f(y)|^2 \, dy = \int_{xV} |f(y)|^2 \, dy < \epsilon^2 \lambda(xV) = \epsilon^2 \lambda(V),$$

since $|f(y) < \epsilon$ for $y \in xV$ by the choice of V and thus,

$$\|g\|_2 \leqslant \epsilon(\lambda(V))^{\frac{1}{2}}.$$

Now,

$$\|u\|_2^2 = \int (\frac{1}{\lambda(K)})^2 \mathbf{1}_K(y)^2 \, dy \leqslant \frac{1}{\lambda(K)} \leqslant \frac{1}{\lambda(V)} \frac{1}{1 - \epsilon}$$

from (*) and therefore,

$$\|u\|_2 \leqslant \sqrt{\frac{1}{\lambda(V)} \frac{1}{1 - \epsilon}}$$

so

$$\|u\|_2 \, \|g\|_2 \leqslant \epsilon \sqrt{\frac{1}{1 - \epsilon}}.$$

Now, let $g \in \mathbf{L}^1(G)$ such that $\|g\|_* \leqslant 1$. We want to estimate

$$\left| \int g(x)(f(x) - f * \tilde{u}(x)) \, dx \right|.$$

Then,

$$\begin{aligned}
&\left|\int g(x)(f(x) - f * \tilde{u}(x))\,dx\right| = \\
&\left|\int g(x)f(x)\,dx - \int g(x)\int f(y)\tilde{u}(y^{-1}x)\,dydx\right| = \\
&\left|\int g(x)f(x) - \int\int g(x)f(y)\frac{1}{\lambda(K)}\mathbf{1}_K(x^{-1}y)\,dydx\right| = \\
&\left|\int\int_K \frac{1}{\lambda(K)}g(x)f(x)\,dydx - \frac{1}{\lambda(K)}\int\int_K g(x)f(xy)\,dydx\right| = \\
&\left|\frac{1}{\lambda(K)}\int\int_K (g(x)f(x) - g(x)f(xy))\,dydx\right| \leqslant \\
&\frac{1}{\lambda(K)}\int\int |g(x)(f(x) - R_yf(x))\mathbf{1}_K(y)\,|\,dxdy
\end{aligned}$$

Now, for $y \in K$, we have that $\|f - R_yf\| \leqslant \epsilon$ and so,

$$\left|\int g(x)(f(x) - R_yf(x))\,dx\right| \leqslant \|g\|_*\|f - R_yf\| \leqslant \epsilon$$

and so

$$\|f - f * \tilde{u}\| \leqslant \int_K \frac{\epsilon}{\lambda(K)}\,dx = \epsilon.$$

Now, putting everything together, we have that

$$\|f - h\| \leqslant \epsilon + \epsilon\sqrt{\frac{1}{1-\epsilon}}$$

which tends to 0 as ϵ tends to 0 and we are done.

□

Lemma 3.2.10. *Let X be a Banach space and $\phi, \psi \in X^*$ such that* $\mathrm{Ker}(\phi) \subset \mathrm{Ker}(\psi)$*. Then $\psi = \lambda\phi$ for some $\lambda \in \mathbb{C}$*

Proof. If $\psi = 0$ then this clearly holds for $\lambda = 0$.

Now assume that $\psi \neq 0$. Then $\phi \neq 0$ and there exist unique $x_0, y_0 \in X$ with $\phi(x_0) = \psi(y_0) = 1$ and such that $X = \mathrm{Ker}(\phi) \oplus \mathrm{span}\{x_0\} = \mathrm{Ker}(\psi) \oplus \mathrm{span}\{y_0\}$

Let $x \in X$, then there exist unique $x_1, y_1 \in X$ and $\lambda, \mu \in \mathbb{C}$ such that $x = x_1 + \lambda \cdot x_0 = y_1 + \mu \cdot y_0$.

Now,

$$\phi(x) = \phi(x_1) + \lambda\phi(x_0) = \lambda$$

and

$$\psi(x) = \psi(y_1) + \mu\psi(y_0) = \mu$$

Then,

$$\psi(x) = \psi(x_1) + \lambda\psi(x_0) = \lambda\psi(x_0) = \phi(x)\psi(x_0)$$

and this holds for any $x \in X$, so we are done. $\square$

We are now ready to prove theorem 3.2.1.

Proof. Let $x \in G$ and let $\phi_x : A(G) \to \mathbf{C}$, $u \mapsto u(x)$. Then, $\phi_x : A(G) \to \mathbf{C}$ is obviously an algebra homomorphism and lemma 3.1.2 for the compact $\{x\}$ implies that there is $u \in A(G)$ such that $u(x) = 1$. In particular, $u(x) \neq 0$, so $\phi_x \in \sigma(A(G))$. So the map $T : G \to \sigma(A(G)), \quad x \mapsto \phi_x$ is well defined and obviously injective.

Let $(x_i)_{i\in I}$ be a net in G converging to $x \in G$. Then, if $u \in A(G)$ we have

$$T(x_i)(u) = \phi_{x_i}(u) = u(x_i) \to u(x) = \phi_x(u) = T(x)(u),$$

since u is continuous. Thus, $T(x_i) \overset{w*}{\to} T(x)$, so T is continuous with respect to the topology of $\sigma(A(G))$.

Now, let's assume that T is not onto, so there exists $\phi \in \sigma(A(G))$ such that $\phi \neq \phi_x \, \forall x \in G$ and so, for each $x \in G$, we can find a $f_x \in A(G)$ such that $\phi(f_x) \neq \phi_x(f_x)$.

We will show that we can choose f_x such that $\phi_x(f_x) = 0$ and $\phi(f_x) = 1$ for every $x \in A(G)$. Notice that if there is $u \in A(G)$ such that $\phi_x(u) = 0$ and $\phi(u) \neq 0$, then if we set $f_x = \frac{u}{\phi(u)}$, we have $\phi_x(f_x) = 0$ and $\phi(f_x) = 1$, so we need to show that there is $u \in A(G)$ such that $\phi_x(u) = 0$ and $\phi(u) \neq 0$.

Assume that this is not possible, so there is $x \in G$ such that for every $u \in A(G)$ with $\phi_x(u) = 0$ we have $\phi(u) = 0$ and so $\mathrm{Ker}(\phi_{\mathrm{x}}) \subset \mathrm{Ker}(\phi)$.

Then, by lemma 3.2.10, there is some $\lambda \in \mathbb{C}$ such that $\phi = \lambda\phi_x$.

Now, let $u \in A(G)$, then,

$$\phi(u^2) = (\phi(u))^2 = (\lambda\phi_x(u))^2 = \lambda^2(\phi_x(u))^2$$

and also,

$$\phi(u^2) = \lambda\phi_x(u^2) = \lambda(\phi_x(u))^2$$

so

$$\lambda(\phi_x(u))^2 = \lambda^2(\phi_x(u))^2$$

and this holds for every $u \in A(G)$, so we get $\lambda = \lambda^2$ and therefore λ can be either 0 or 1. If $\lambda = 0$, then $\phi = 0$ which is impossible, since $\phi \in \sigma(A(G))$, so $\lambda = 1$ and therefore

$$\phi = \phi_x .$$

But we have assumed that $\phi \neq \phi_x$ and we have reached a contradiction. Therefore, for every $x \in G$ there exists a function $f_x \in A(G)$ such that $\phi_x(f_x) = 0$ and $\phi(f_x) = 1$.

Now, $\phi_x(f_x) = 0$ and this means that $f_x(x) = 0$ for every $x \in G$, so, by lemma 3.2.2, there exists a sequence $(g_n^x)_{n\in\mathbf{N}}$ in $(C_c(G) \cap A(G))$ and for every n there exists an open neighborhood of x, which we denote by V_n^x, such that $g_n^x|_{V_n^x} = 0$.

Now, $|\phi(f_x) - \phi(g_n^x)| \leqslant \|\phi\| \, \|f_x - g_n^x\|$ and so, for large enough n, we have that $\phi(g_n^x) \neq 0$ and $g_n^x|_{V_n^x} = 0$. Thus, for each $x \in G$, we can find $g_x \in C_c(G) \cap A(G)$ such that $\phi(g_x) \neq 0$ and g_x vanishes in a neighborhood V_x of x. Let $f_0 \in C_c(G) \cap A(G)$ be such that $\phi(f_0) = 1$ and let $K = \operatorname{supp} f_0$. Then, K is compact and $K \subset \bigcup_{x\in K} V_x$ so there exist $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Let $f = f_0 \prod_{i=1}^n f_{x_i}$. Then, $f \in A(G)$ and $\phi(f) = \phi(f_0) \prod_{i=1}^n \phi(f_{x_i}) \neq 0$.

Now, let $x \in G$. If $x \in K$, then there exists $i \in \{1, \ldots, n\}$ such that $x \in V_{x_i}$ and thus $f_{x_i}(x) = 0$ and so $f(x) = 0$. On the other hand, if $x \notin K$, then $f_0(x) = 0$ and so $f(x) = 0$. In either case, $f(x) = 0$ and so $f = 0$, but $\phi(f) \neq 0$, which is a contradiction, since ϕ is linear. Thus $\phi = \phi_x$ for some $x \in G$ and hence T is onto.

To finish the proof, consider a net $(\phi_{x_i})_{i\in I}$ in $\sigma(A(G))$ and $x \in G$ such that $\phi_{x_i} \to \phi_x$ and let's assume that $x_i \nrightarrow x$. In this case, we can find an open neighborhood V of x and a subnet $(x_j)_{j\in J}$ of $(x_i)_{i\in I}$, such that $x_j \notin V$ for every $j \in J$. Now, from lemma 3.1.2, we can find $u \in A(G)$ such that

$u(x) = 1$ and $u(y) = 0$ for every $y \in G \backslash V$ and so, $u(x_j) = 0$ for every $j \in J$ and $u(x) = 1$, but $\phi_{x_j} \to \phi_x$ and so $u(x_j) \to u(x)$, which is absurd, so $x_i \to x$ and thus T is a homeomorphism. $\square$

3.3 The dual of $A(G)$

Having described the spectrum of A(G), we are now going to describe its dual. As the next theorem shows, its dual is isometrically isomorphic to another operator algebra associated to G, the so called group von Neumann algebra, vN(G).

Of course we will first define vN(G).

Definition 3.3.1 (Group von Neumann algebra)**.** *Let G be a locally compact group and let λ denote the left regular representation of G. Then the group von Neumann algebra is defined to be the second commutant of $\lambda(\mathbf{L}^1(G)) \subset B(\mathbf{L}^2(G))$ and we denote it by* vN(G).

Definition 3.3.2. *Let $u \in B(G)$. We define a function $\check{u} : G \to \mathbb{C}$ by $\check{u}(x) = u(x^{-1})$ for every $x \in G$.*

Proposition 3.3.3. *Let $u \in B(G)$. Then $\check{u} \in B(G)$ and the map $B(G) \to B(G), \quad u \mapsto \check{u}$ is a linear isometry.*

Proof. Let $u \in B(G)$. Then, by 2.0.25, there is (π, H_π) unitary representation of G and $\xi, \eta \in H_\pi$ such that $u(x) = \langle \pi(x)\xi, \eta \rangle$ for every $x \in G$ and such that $\|u\| = \|\xi\| \, \|\eta\|$. Then, we have that

$$\check{u}(x) = u(x^{-1}) = \langle \pi(x^{-1})\xi, \eta \rangle = \overline{\langle \pi(x)\eta, \xi \rangle},$$

for every $x \in G$. Now, by 2.0.12, the function $\langle \pi(\cdot)\eta, \xi \rangle$ clearly lies in $B(G)$ and by 2.0.13, we get that $\overline{\langle \pi(\cdot)\eta, \xi \rangle} \in B(G)$ and therefore $\check{u} \in B(G)$.

Moreover, by 2.0.25, we get that $\|\check{u}\| \leqslant \|\xi\| \, \|\eta\| = \|u\|$. It is clear that $\check{\check{u}} = u$ and therefore $\|\check{u}\| = \|u\|$ for every $u \in B(G)$. The map that sends u to $\check{u}$ is clearly linear, so it is a linear isometry on $B(G)$. $\square$

Proposition 3.3.4. *Let $u \in A(G)$, then $\check{u} \in A(G)$.*

Proof. Let $u \in B(G) \cap C_c(G)$. Then $\check{u}$ clearly lies in $B(G) \cap C_c(G)$. Now, consider $u \in A(G)$. Then, by 3.1.3, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $B(G) \cap C_c(G)$ such that u_n converges to u in the norm of $B(G)$. Now, by 3.3.3, we get that $\check{u}_n \to \check{u}$ in the norm of $B(G)$ and $\check{u}_n \in A(G)$ for every $n \in \mathbb{N}$, so $\check{u} \in A(G)$ for every $u \in A(G)$.

□

Theorem 3.3.5. *Let G be a locally compact group.*
We define $T : (A(G))^* \to \mathrm{vN(G)}$ by $\phi \mapsto T_\phi$, where

$$\langle T_\phi(f), g\rangle = \phi(\overset{\frown}{f \star \tilde{g}}) = \phi(\overline{g} \star \check{f})$$

for every $f, g \in \mathbf{L}^2(G)$. Then, T is well defined and it is an isometric isomorphism between $(A(G))^*$ and $\mathrm{vN(G)}$ and has the following properties:

1. *If $u = \sum_{j=1}^{\infty} g_j \star \check{f}_j$, where $f_j, g_j \in \mathbf{L}^2(G)$ are such that $\sum_{j=1}^{\infty} \|f_j\|_2 \|g_j\|_2 < \infty$, then $\phi(u) = \sum_{j=1}^{\infty} u(g_j * \check{f}_j) = \sum_{j=1}^{\infty}\langle T_\phi f_j, \overline{g_j}\rangle_2$*
2. *If $\mu \in M(G)$ and $\phi_\mu \in (A(G))^*$ is such that $\phi_\mu(u) = \int u(x)\, d\mu(x)$ for every $u \in A(G)$, then $T_{\phi_\mu} = \lambda_G(\mu)$.*
3. *T is a homeomorphism for the w^* topology on $(A(G))^*$ and the ultra-weak topology on $\mathrm{vN(G)}$.*

Proof. Let $\phi \in (A(G))^*$. We define $\psi : \mathbf{L}^2(G) \times \mathbf{L}^2(G) \to \mathbf{C}$ $(f, g) \mapsto \phi(\overset{\frown}{f * \tilde{g}})$. Then, ψ is obviously sesquilinear and $|\psi(f, g)| \leqslant \|\phi\| \|f \star \tilde{g}\| \leqslant \|\phi\| \|f\|_2 \|g\|_2$ and so ψ is a bounded sesquilinear form on $\mathbf{L}^2(G)$, with $\|\psi\| \leqslant \|\phi\|$ and so there exists a unique $T_\phi \in B(\mathbf{L}^2(G))$ such that $\psi(f, g) = \langle T_\phi f, g\rangle$ and thus $\langle T_\phi f, g\rangle = \phi(\overset{\frown}{f * \tilde{g}})$ and $\|T_\phi\| \leqslant \|\phi\|$. We want to show that $T_\phi \in \mathrm{vN(G)} = (\lambda(\mathbf{L}^1(G))''$, where λ is the left regular representation of G, so we need to show that T_ϕ commutes with the commutant of the left regular representation. Now, it is proved in ([11] proposition VII.3.1) that $(\lambda(\mathbf{L}^1(G))' = \rho(\mathbf{L}^1(G))$, where ρ is the right regular representation of G, so we need to show that T_ϕ commutes with the operators $\rho(h)$ for $h \in \mathbf{L}^1(G)$. To see this, let $h \in \mathbf{L}^1(G)$ and $f, g \in C_c(G)$. We will show that $\langle T_\phi(\rho(h)f), g\rangle = \langle \rho(h)T_\phi f, g\rangle$.

Indeed, we have

$$\begin{aligned}\langle T_\phi(\rho(h)f), g\rangle = \phi(\overline{g} \star \widecheck{(f \star h)}) = \phi(\overline{g} \star \check{h} \star \check{f}) = \phi(\overline{g \star \tilde{h}} \star \check{f}) = \\ \langle T_\phi f, g \star \tilde{h}\rangle = \int (T_\phi f)(x)\overline{(g \star \tilde{h})}(x)\, dx = \\ \int (T_\phi f)(x) \int \overline{g(y)\tilde{h}(y^{-1}x)}\, dydx = \\ \iint (T_\phi f)(x)\overline{g}(y)h(x^{-1}y)\, dydx = \int \overline{g}(y)((T_\phi f) \star h)(y)\, dy = \\ \langle (T_\phi f) \star h, g\rangle = \langle \rho(h)(T_\phi f), g\rangle\end{aligned}$$

and since this holds for every $f, g \in C_c(G)$, we conclude that $\rho(h)T_\phi = T_\phi\rho(h)$ for every $h \in \mathbf{L}^1(G)$ and we are done.

Thus, the map $T : (A(G))^* \to \mathrm{vN(G)}, \quad \phi \mapsto T_\phi$, is well defined, $\|T_\phi\| \leqslant \|\phi\|$ and T is obviously linear.

Now, let $u = \sum_{j=1}^\infty g_j \star \check{f}_j$, where $\sum_{j=1}^\infty \|f_j\|_2 \|g_j\|_2 < \infty$. First of all, for every $j \in \mathbf{N}$, we have that $g_j \star \check{f}_j = g_j \star \tilde{\overline{f_j}}$ and $g_j, \overline{f_j} \in \mathbf{L}^2(G)$, so, by proposition 3.1.3, we see that $g_j \star \check{f}_j \in A(G)$. Now, $\sum_{j=1}^\infty \|g_j \star \check{f}_j\| \leqslant \sum_{j=1}^\infty \|f_j\|_2 \|g_j\|_2 < \infty$ and since $A(G)$ is complete, u is in $A(G)$ and so $\phi(u)$ is well defined. Now, let $S_n = \sum_{j=1}^n g_j \star \check{f}_j$, for every $n \in \mathbf{N}$. Then $S_n \in A(G)$ and $S_n \to u$ and ϕ is continuous, so $\phi(S_n) \to \phi(u)$. Now,

$$\phi(S_n) = \sum_{j=1}^n \phi(g_j \star \check{f}_j) = \sum_{j=1}^n \phi(\widecheck{f_j \star \tilde{\overline{g_j}}}) = \sum_{j=1}^n \langle T_\phi f_j, \overline{g_j}\rangle.$$

So, $\sum_{j=1}^n \langle T_\phi f_j, \overline{g_j}\rangle \to \phi(u)$ and so $\phi(u) = \sum_{j=1}^\infty \langle T_\phi f_j, \overline{g_j}\rangle$, which proves 1.

Now, for 2, consider $\mu \in M(G)$. Then,

$$\langle T_{\phi_\mu} f, g\rangle = \phi_\mu(\widecheck{f \star \tilde{g}}) = \int \widecheck{(f \star \tilde{g})}(x)\, d\mu(x)$$

and thus,

$$\begin{aligned}\langle T_{\phi_\mu} f, g\rangle = \iint f(y)\tilde{g}(y^{-1}x^{-1})\,dyd\mu(x) = \\ \iint f(y)\overline{g}(xy)\,dyd\mu(x) = \iint f(x^{-1}y)\overline{g}(y)\,dyd\mu(x) = \\ \int\langle\lambda(x)f, g\rangle\,d\mu(x) = \langle\lambda_G(\mu)f, g\rangle\end{aligned}$$

and this holds for every $f, g \in \mathbf{L}^2(G)$, so $\lambda_G(\mu) = T_{\phi_\mu}$, which proves 2.

Next, we want to prove that T is onto. Let us start with an operator $A \in \mathrm{vN(G)}$. We are going to construct a functional $\phi_A \in (A(G))^*$ such that $T_{\phi_A} = A$. Let us first consider $f, g \in C_c(G)$. Then, of course, $f \star \tilde{g} \in A(G)$ and since $A \in \mathrm{vN(G)}$, we know that A commutes with the right regular representation and so, $A(f \star \tilde{g}) = A(\rho(\tilde{g})f) = \rho(\tilde{g})Af = (Af) \star \tilde{g}$. Now, $(Af) \star \tilde{g}$ is continuous, so it makes sense to consider $(Af \star \tilde{g})(e)$ and so we define $\phi_A : E_1 \to \mathbf{C}$, by $u \mapsto A(\check{u})(e)$. Notice that

$$(Af \star \tilde{g})(e) = \int Af(y)\tilde{g}(y^{-1})\,dy = \int Af(y)\overline{g}(y)\,dy = \langle Af, g\rangle.$$

Consider $u = \sum_{j=1}^n f_j \star \tilde{g}_j \in E_1$. Since $A \in \mathrm{vN(G)}$, by Kaplansky's density theorem (0.2.12), there exists a net $(\lambda(h_i))_{i\in I}$ with $h_i \in C_c(G)$ for every i and $\|\lambda(h_i)\| \leqslant \|A\|\,\forall i \in I$ such that $\lambda(h_i) \overset{SOT}{\to} A$ and so, for every $j \in \{1, \dots, n\}$, we have that $\langle\lambda(h_i)f_j, g_j\rangle \to \langle Af_j, g_j\rangle$ and so,

$$\sum_{j=1}^n\langle\lambda(h_i)f_j, g_j\rangle \to \sum_{j=1}^n\langle Af_j, g_j\rangle = \sum_{j=1}^n \phi_A(\widecheck{f_j \star \tilde{g}_j}) = \phi_A(\check{u}).$$

Now, from 2, we have that $\langle\lambda(h_i)f_j, g_j\rangle = \phi_{h_i}(\widecheck{f_j \star \tilde{g}_j})$ (recall that by the discussion right after the definition, 0.4.20, we can consider $\mathbf{L}^1(G)$ as a subspace of $M(G)$). So, $\sum_{j=1}^n\langle\lambda(h_i)f_j, g_j\rangle = \phi_{h_i}(\check{u})$ and then, $|\phi_{h_i}(\check{u})| \leqslant \|h_i\|\|\check{u}\| \leqslant \|A\|\|u\|$ and so, $|\phi_A(\check{u})| \leqslant \|A\|\|\check{u}\|$ and since $\check{E}_1 = E_1$, we conclude that ϕ_A is bounded on E_1 and thus it has a unique extension to $A(G)$, again denoted by ϕ_A, that satisfies $\|\phi_A\| \leqslant \|A\|$.

Now that we have defined a way to associate to each $A \in \mathrm{vN(G)}$ a functional in $(A(G))^*$, we must check that this association is in fact inverse to our previous map. With that goal in mind, consider $A \in \mathrm{vN(G)}$ and $f, g \in C_c(G)$. Then, by definition, $\langle T_{\phi_A} f, g\rangle = \phi_A(\widetilde{f \star \tilde{g}}) = \langle Af, g\rangle$ and since this holds for any $f, g \in C_c(G)$, we obtain $T_{\phi_A} = A$ as claimed.

Let $A \in \mathrm{vN(G)}$, then, $\|\phi_A\| \leqslant \|A\|$ and $\|A\| = \|T_{\phi_A}\| \leqslant \|\phi_A\| \leqslant \|A\|$, so, we see that, $\|\phi_A\| = \|A\|$ for every $A \in \mathrm{vN(G)}$ and so, $T : (A(G))^* \to \mathrm{vN(G)}, \quad \phi \mapsto T_\phi$ is an isometry onto and therefore the dual space of $A(G)$ is isometrically isomorphic to vN(G).

Moreover, by 0.2.17, we see that $A(G)$ is isometrically isomorphic to the predual of vN(G) and therefore T is a homeomorphism for the w^* and the ultraweak topologies, which proves 3.

$\square$

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