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Formalizing Constructive Analysis: A Comparison of Minimal Systems and A Study of Uniqueness Principles

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To the memory of my father Syropoulos

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Contents

Preface

In this dissertation we investigate certain aspects of the formalization and axiomatization of parts of constructive mathematics, and specifically constructive analysis. We give here a general description of the mathematics formalized in the systems that we study.

Constructivity in mathematics is understood in many ways. The common feature of all varieties is that, in order to assert that an object (having certain properties) exists, one has to provide a (mental) construction of it. As is observed in [Troelstra1991], "constructivism as a specific viewpoint emerges in the final quarter of the 19th century, and may be regarded as a reaction to the rapidly increasing use of highly abstract concepts and methods in mathematics, a trend exemplified by the works of R. Dedekind and G. Cantor".

Kronecker with his finitistic views, but also the French "empiricists" Baire, Borel, Lebesgue, as well as Poincaré and others, can all be considered as precursors of the first coherent presentation of the constructive standpoint by L. E. J. Brouwer, the founder of intuitionism.

As Brouwer first realized and showed, the demand for constructivity affects the logic inherent in mathematical reasoning. Any reasonable notion of "construction" leads to the rejection of the unrestricted use of the logical law of the excluded middle, as this law allows existential assertions by reductio ad absurdum. In this way, objects can be proved to exist or even defined, without any indication of how to find or construct them.

Constructive mathematics can be considered simply as mathematics with intuitionistic logic, which is just classical logic without the law of the excluded middle. Clearly this standpoint makes constructive mathematics just a portion of classical mathematics, which, however, is not always the case. Constructivity poses serious restrictions to the development of a satisfactory theory for the real numbers and the continuum, as it seems to allow a continuum which is only denumerably infinite. Thus, a constructive theory for the continuum has always been the big challenge for constructivists.

Different answers to this question have been given from the various schools of constructivism. The main three which have made important contributions to constructive analysis, are Brouwer's intuitionism (INT), Markov's Russian Constructive Recursive Mathematics (RUSS) and Bishop's Constructive Mathematics (BISH). We describe very briefly the main tenets of each.

Brouwer's intuitionism rejects the law of the excluded middle for infinite sets, as well as the treating of infinite objects as actual, completed. It proposes a totally new conception of the constructive continuum, by accepting infinitely proceeding sequences (of natural numbers or other already constructed objects) generated by successive arbitrary choices, the *choice sequences*, as legitimate objects. So a theory of the continuum based on the whole classical Baire space, but with intuitionistic logic, is developed. The way that Brouwer found to treat constructively the choice sequences, is captured by the principles of continuity and bar induction. Bar induction is a classically correct principle. But the continuity principle, asserting that every total function on the Baire space is continuous, contradicts classical mathematics. So the resulting theory of the continuum diverges from the classical one.

In Markov's Russian recursive mathematics, the objects of mathematics are algorithms finitely presented. Church's Thesis that every constructive function is recursive is accepted. Markov's principle, a logical principle asserting that if an algorithm cannot fail to converge then it converges, is also assumed.

Bishop's constructive mathematics adopts the standpoint that all mathematical statements should have numerical meaning. This school develops mathematics in a neutral way, compatible with all constructive views, but also with classical mathematics (CLASS).

The relation between these varieties is roughly as follows:

$$\text{BISH} \subseteq \text{INT} \cap \text{RUSS} \cap \text{CLASS}$$

Any two of INT, RUSS, CLASS are incompatible.

All the above varieties agree on the following:

· Natural numbers with mathematical induction are accepted as the undeniable basis for mathematics.
· Integers and rationals are coded by natural numbers.
· The real numbers are represented mostly using Cauchy sequences, almost like in CLASS, with a constructive interpretation of the notion of Cauchy sequence. Other representations are possible as in CLASS.

The investigation in all these branches of constructive mathematics is carried out in a multitude of formal and informal systems and languages, whose relationships remain in many aspects unclear. This problem becomes quite crucial for the development of the relatively new field of constructive reverse mathematics.

In Part 1 we contribute to the endeavor of getting a clearer picture of this area by establishing precise relationships between widely used basic formal systems for constructive analysis. The formal systems that we study here are neutral, they all belong to the common core of all these varieties of constructive analysis and of classical analysis also.

In all branches of constructive analysis, various forms of choice principles, continuity principles and many others are used. Part 2 studies relations between many of them, in their versions having a uniqueness condition, a feature from which interesting properties follow, as well as relations between these principles and non-constructive logical principles, in the spirit of reverse mathematics.

Part 1

CHAPTER 1

Comparison of minimal systems

Introduction

In this chapter we consider particular formal systems of intuitionistic two-sorted arithmetic that are weak, and serve as basis for theories formalizing parts of various branches of constructive analysis. They are all classically correct.

The systems we study have common basic features. They are based on intuitionistic logic and they are formulated in two-sorted first-order languages, with variables for natural numbers and for one-place number-theoretic functions. The primitives of all these formalisms are constants for (different selections of) primitive recursive functions and functionals. They are all theories with equality for natural numbers, which is assumed decidable. Equality for functions is in all of them defined in terms of number equality. λ-abstraction is included in some of them. The function existence principles assumed are all weak, and do not involve real choice.

The differences in the languages, as well as the interplay between the possibilities provided by the languages and the assumed (if any) function existence principles, do not allow, in most cases, to determine directly how these systems relate to each other. Here we present a way to establish a precise relationship between the two most widely used systems among those that we are considering, namely **M** and **EL**, and then apply similar arguments to obtain comparisons in other cases.

1. The formal systems M and EL

1.0.1. Starting with Heyting, intuitionistic logic and arithmetic were formalized as subsystems of the corresponding classical formal theories (see [JRM2009]). On the contrary, Heyting's formalization of Brouwer's set theory (the part of intuitionistic mathematics concerning the continuum and the real numbers) failed to allow comparison with classical mathematics. In order to make such a comparison possible, S. C. Kleene formalized large parts of intuitionistic mathematics, corresponding to mathematical analysis, in the formal system **FIM** ([FIM]), whose language is suitable also for classical analysis. Moreover, **FIM** contains a classically correct subsystem, the *basic system* **B**, which can be extended in two, diverging ways: with the addition of the law of double negation it becomes a system of classical analysis, while with the addition of the continuity principle (which is incompatible with classical mathematics), it becomes a system of intuitionistic analysis.

The *minimal system* **M** is a proper and substantially weaker subsystem of **B**, obtained by omitting bar induction and replacing the countable choice assumed by **B** by a function comprehension principle, which does not involve choice. In the system **M**, S. C. Kleene developed formally, with great detail, the theory of recursive partial functionals. This was a preparatory step towards the formalization of the function realizability interpretation of **FIM** within **B**, by which he obtained a metamathematical consistency proof, relative to the (classically correct) system **B**, of intuitionistic analysis (see [Kleene1969]).

The system of *elementary analysis* **EL** [Troelstra1973] has been developed mostly by A. S. Troelstra, to serve as a formal basis for intuitionistic analysis. It differs from **M** in its arithmetical basis and in the function existence principle it assumes. But like **M**, it represents a classically correct fragment of intuitionistic analysis. It is used in recent work for the formalization of Bishop's constructive analysis, especially in relation to the program of Constructive Reverse Mathematics (see for example [Berger] and [Ishihara]).

1.0.2. **M** and **EL** have many similarities:

· Both are based on two-sorted intuitionistic predicate logic with number and function variables (for one-place number-theoretic functions).

· Both have function and functional constants for (different selections of) only primitive recursive functions and functionals.

· Both have λ-abstraction and function application.

· Both allow definition by primitive recursion (although with different justifications).

Their differences are of two kinds:

· First, they have differences in their languages. **M** has only finitely many function and functional constants, following the paradigm of (the usual presentation of) Peano arithmetic, while **EL** has infinitely many function constants (not including the functional constants of **M**), as it extends the primitive recursive arithmetic **PRA**; **EL** has also a recursor functional (not included in **M**).

· Second, they assume different function existence principles. **M** assumes $\mathrm{AC}_{00}!$ while **EL** assumes QF-AC_{00}, which is a consequence of $\mathrm{AC}_{00}!$.

The two systems were considered more or less equivalent, but their exact relation was unclear. We have found that **EL** is essentially weaker than **M**, and that their difference is captured by the function existence principle $\mathrm{CF_d}$, which is a consequence of $\mathrm{AC_{00}!}$, and asserts that every decidable predicate of natural numbers has a characteristic function. The schema $\mathrm{CF_d}$ makes it possible to avoid explicit decidability hypotheses in the formulation of various principles and theorems. We observed that this schema helps to determine the relation between **M** and **EL**. The argument is as follows. We show first that the system **EL** + $\mathrm{CF_d}$ entails $\mathrm{AC_{00}!}$, and that **EL** does not entail $\mathrm{CF_d}$. These results suggest that **EL** + $\mathrm{CF_d}$ is essentially equivalent to **M**, while **EL** is weaker than **M**. In order to establish these suggested relationships, we have to overcome the differences of the languages. So we extend both systems up to a common language, and we show that the corresponding extensions of **EL** + $\mathrm{CF_d}$ and **M** are conservative and coincide up to trivial notational differences; so we conclude that the suggested relationships hold indeed.

Both systems have λ-abstraction. In [JRMPhD] it is proved that this mechanism can be omitted from **M**, without weakening the system; we show that the same holds for **EL**.

1.1. Language and underlying logic.

1.1.1. The underlying logic of both systems is the two-sorted intuitionistic predicate logic with number and function variables. The languages $\mathcal{L}(\mathbf{M})$ and $\mathcal{L}(\mathbf{EL})$ have a common part consisting of the following:

- logical symbols: $\rightarrow$, $\&$, $\vee, \neg, \forall, \exists$;
- punctuation symbols: commas and parentheses , ( ) ;
- number variables: x, y, z, . . . , intended to range over natural numbers;
- function variables: $\alpha, \beta, \gamma, \ldots$, intended to range over one-place number-theoretic functions (or choice sequences);
- [the set of individual constants, predicate and function and functional symbols (constants) of each language extends in different ways a common part; there are also common as well as different formation rules for the terms (type-0 terms, expressions for natural numbers) and the functors (type-1 terms, expressions for one-place number-theoretic functions (or choice sequences)); these will be included in the description of the non-logical part of the formalisms;]
- the number equality predicate symbol: = .

The quantifiers $\forall$, $\exists$ are used for both sorts.

1.1.2. The logical axioms and rules can be introduced in various ways, for example in a natural deduction or Hilbert-type style. We give here the logical axioms and rules of inference of the formal system of [FIM] (which extends that of [IM]), as we will base our treatment on it.

1a. $A \to (B \to A)$.

1b. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$.

2. From A and $A \to B$ infer B. (modus ponens)

3. $A \to (B \to (A \,\&\, B))$.

4a. $(A \,\&\, B) \to A$.

4b. $(A \,\&\, B) \to B$.

5a. $A \to (A \vee B)$.

5b. $B \to (A \vee B)$.

6. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.

7. $(A \to B) \to ((A \to \neg B) \to \neg A)$.

8^I. $\neg A \to (A \to B)$.

9N. From $C \to A(x)$ infer $C \to \forall x\, A(x)$, x not free in C.

10N. $\forall x\, A(x) \to A(t)$, t a term free for x in A(x).

11N. $A(t) \to \exists x\, A(x)$, t a term free for x in A(x).

12N. From $A(x) \to C$ infer $\exists x\, A(x) \to C$, x not free in C.

9F. From $C \to A(\alpha)$ infer $C \to \forall \alpha\, A(\alpha)$, α not free in C.

10F. $\forall \alpha\, A(\alpha) \to A(u)$, u a functor free for α in $A(\alpha)$.

11F. $A(u) \to \exists \alpha\, A(\alpha)$, u a functor free for α in $A(\alpha)$.

12F. From $A(\alpha) \to C$ infer $\exists \alpha\, A(\alpha) \to C$, α not free in C.

Classical logic is obtained by replacing axiom-schema 8^I by the schema 8° $\neg\neg A \to A$, expressing the double negation law, or, equivalently, by adding the schema $A \vee \neg A$, expressing the law of the excluded middle.

1.1.3. First-order equality in **EL**, and in Heyting arithmetic **HA** (contained in it as we shall see), is treated as part of the logic. As equality axioms are taken (the universal closures of) the axiom

REFL $\mathrm{x} = \mathrm{x}$

and the replacement schema

REPL $\mathrm{A(x)} \;\&\; \mathrm{x} = \mathrm{y} \rightarrow \mathrm{A(y)}$.

In the version of first-order arithmetic of [IM], upon which **M** is based, these are reduced to a finite number of simple axioms, given below, from which REFL and the schema corresponding to REPL are provable.

1.2. Underlying arithmetic.

1.2.1. The systems **M** and **EL** are based on weak systems of two-sorted intuitionistic arithmetic, $\mathbf{IA_1}$ and $\mathbf{HA_1}$, respectively. Both weak theories are based on the two-sorted intuitionistic predicate logic that we described; they are extensions of corresponding versions of first-order intuitionistic arithmetic: $\mathbf{IA_0}$, the version of [IM], by S. C. Kleene, and Heyting arithmetic **HA**, as presented in [TvDI]. The difference between $\mathbf{IA_0}$ and **HA** is that the first one is based on a finite selection of initial functions (has the constants $0, ', +, \cdot$), while the second has infinitely many function symbols, for all the primitive recursive functions; we will consider, more precisely, that it has one function symbol for each primitive recursive description. These two ways of formalizing first-order arithmetic do not differ essentially; it is well-known (see [IM], §74, for a detailed proof) that if $0, ', +, \cdot$ are contained in the formalism, then the addition of function symbols for primitive recursive functions leads to definitional, and so inessential extensions. We note that (any version of) intuitionistic first-order arithmetic is a subtheory of classical first-order arithmetic; the formulation of $\mathbf{IA}_0$ is similar to the usual formulation of classical Peano arithmetic **PA**.

1.2.2. $\mathbf{IA}_0$ is based on first-order intuitionistic predicate logic. Besides the logical symbols, its language $\mathcal{L}(\mathbf{IA_0})$ contains the predicate constant $=$ (equality), the individual constant 0 (zero), the unary function constant $'$ (successor), and the binary function constants $+$ (addition) and $\cdot$ (multiplication).

Terms are defined inductively, as usual:

· the constant 0 and the (number) variables are terms,

· if s, t are terms, then $\mathrm{s}', (\mathrm{s} + \mathrm{t}), (\mathrm{s} \cdot \mathrm{t})$ are terms.

The prime formulas are the equalities of terms: if s, t are terms, then $\mathrm{s} = \mathrm{t}$ is a (prime) formula.

The mathematical axioms of $\mathbf{IA}_0$ are:

13. $\mathrm{A}(0)\ \&\ \forall \mathrm{x}\,(\mathrm{A}(\mathrm{x}) \to \mathrm{A}(\mathrm{x}')) \to \mathrm{A}(\mathrm{x})$. (IND)
14. $\mathrm{x}' = \mathrm{y}' \to \mathrm{x} = \mathrm{y}$.
15. $\neg\, \mathrm{x}' = 0$.
16. $\mathrm{x} = \mathrm{y} \to (\mathrm{x} = \mathrm{z} \to \mathrm{y} = \mathrm{z})$.
17. $\mathrm{x} = \mathrm{y} \to \mathrm{x}' = \mathrm{y}'$.
18. $\mathrm{x} + 0 = \mathrm{x}$.
19. $\mathrm{x} + \mathrm{y}' = (\mathrm{x} + \mathrm{y})'$.
20. $\mathrm{x} \cdot 0 = 0$.
21. $\mathrm{x} \cdot \mathrm{y}' = \mathrm{x} \cdot \mathrm{y} + \mathrm{x}$.

Axiom-schema 13 expresses the principle of mathematical induction; we will refer to it as IND.

Peano arithmetic **PA** is $\mathbf{IA}_0 + \neg\neg\mathrm{A} \to \mathrm{A}$.

1.2.3. Heyting arithmetic **HA** differs from $\mathbf{IA}_0$ in the set of function constants contained in its alphabet. Its language $\mathcal{L}(\mathbf{HA})$ has a constant 0 (zero), a unary function constant S (successor) and one function constant for each primitive recursive description, so it has countably infinitely many function constants $\mathrm{h}_0, \mathrm{h}_1, \mathrm{h}_2, \ldots$.

The term formation rules are adapted accordingly:

- the constant 0 and the (number) variables are terms,
- if $\mathrm{t}_1, \ldots, \mathrm{t}_k$ are terms and h a k-place function constant, then $\mathrm{h}(\mathrm{t}_1, \ldots, \mathrm{t}_k)$ is a term.

The mathematical axioms of **HA** are the axiom-schema of induction IND, the axiom

$$\neg\mathrm{S}(0) = 0,$$

and defining axioms for the function constants, which consist of the equations expressing the corresponding primitive recursive descriptions.

Example. The defining axioms for the binary constant + (addition) are: $\mathrm{x} + 0 = \mathrm{x}$, $\mathrm{x} + \mathrm{S}(\mathrm{y}) = \mathrm{S}(\mathrm{x} + \mathrm{y})$.

1.2.4. $\mathbf{IA}_1$ is described as follows. Its language $\mathcal{L}(\mathbf{IA_1})$[1] has the (finitely many) function and functional constants $\mathrm{f}_0, \ldots \mathrm{f}_p$, where each f_i ($i = 0, \ldots, p$) has k_i number arguments and l_i function arguments. All of them express functions primitive recursive in their arguments (under the intended interpretation). According to the needs of the development of the theory, a different selection of which functions are included in the alphabet may be done, in agreement with the intuitionistic view that no formal system can exhaust the possibilities of

[1] We note that $\mathcal{L}(\mathbf{IA_1})$ is $\mathcal{L}(\mathbf{M})$.

mathematical activity. The particular formal system that we are considering, namely **M**, contains the 27 function(al)s given in a list below; the system of [FIM] contains the first 25 of them, the last two have been added in [Kleene1969]. There are also parentheses serving as constant for function application, and Church's λ for λ-abstraction.

Terms and functors are defined by simultaneous induction as follows:

- the constant 0 and the number variables are terms,
- the function variables and each function constant f_i with $k_i = 1$, $l_i = 0$, are functors,
- if $\mathrm{t}_1, \dots, \mathrm{t}_{k_i}$ are terms and $\mathrm{u}_1, \dots, \mathrm{u}_{l_i}$ functors, then $\mathrm{f}_i(\mathrm{t}_1, \dots, \mathrm{t}_{k_i}, \mathrm{u}_1, \dots, \mathrm{u}_{l_i})$ is a term,
- if u is a functor and t a term, then (u)(t) is a term,
- if x is a number variable and t a term, then λx.t is a functor.

The prime formulas are the equalities of terms: if s, t are terms, then s = t is a (prime) formula.

Equality for functors is introduced by the abbreviation

$$\mathrm{u} = \mathrm{v} \equiv \forall \mathrm{x}\, (\mathrm{u})(\mathrm{x}) = (\mathrm{v})(\mathrm{x}),$$

with x a number variable not free in u or v.

The mathematical axioms of $\mathbf{IA}_1$ are:

- the axiom-schema IND for $\mathcal{L}(\mathbf{IA_1})$;
- the axioms of $\mathbf{IA}_0$ for $=, 0, {}', +, \cdot$;
- the equations expressing the primitive recursive definitions of the additional function(al) constants f_4 - f_{26}; they are of the following forms (corresponding to explicit definition and definition by primitive recursion):
  (a) $\mathrm{f}_i(\mathrm{y}, \mathrm{a}, \alpha) = \mathrm{p}(\mathrm{y}, \mathrm{a}, \alpha)$,
  (b) $\begin{cases} \mathrm{f}_i(0, \mathrm{a}, \alpha) = \mathrm{q}(\mathrm{a}, \alpha) \\ \mathrm{f}_i(\mathrm{y}', \mathrm{a}, \alpha) = \mathrm{r}(\mathrm{y}, \mathrm{f}_i(\mathrm{y}, \mathrm{a}, \alpha), \mathrm{a}, \alpha), \end{cases}$
  where $\mathrm{p}(\mathrm{y}, \mathrm{a}, \alpha)$, $\mathrm{q}(\mathrm{a}, \alpha)$, $\mathrm{r}(\mathrm{y}, \mathrm{z}, \mathrm{a}, \alpha)$ are terms containing only the distinct variables shown and only function constants from $\mathrm{f}_0, \dots, \mathrm{f}_{i-1}$, and $\mathrm{y}, \mathrm{a}, \alpha$ are free for z in $\mathrm{r}(\mathrm{y}, \mathrm{z}, \mathrm{a}, \alpha)$;
- the open equality axiom: $\mathrm{x} = \mathrm{y} \to \alpha(\mathrm{x}) = \alpha(\mathrm{y})$;
- the axiom-schema of λ-conversion: $(\lambda \mathrm{x}.\mathrm{t}(\mathrm{x}))(\mathrm{s}) = \mathrm{t}(\mathrm{s})$, where t(x) is a term and s is free for x in t(x).

We next give the complete list of the function(al) constants of $\mathcal{L}(\mathbf{IA_1})$ with their defining axioms, where a, b are number variables.

- $f_0 \quad \equiv\ 0,$ $\qquad k_0 = 0,\ l_0 = 0,$
- $f_1 \quad \neg\, a' = 0,\ a = b \rightarrow a' = b',\ a' = b' \rightarrow a = b,$ $\qquad k_1 = 1,\ l_1 = 0,$
- $f_2 \quad a + 0 = a,\ a + b' = (a + b)',$ $\qquad k_2 = 2,\ l_2 = 0,$
- $f_3 \quad a \cdot 0 = 0,\ a \cdot b' = a \cdot b + a,$ $\qquad k_3 = 2,\ l_3 = 0,$
- $f_4 \quad a^0 = 1,\ a^{b'} = a^b \cdot a,$ $\qquad k_4 = 2,\ l_4 = 0,$
- $f_5 \quad 0! = 1,\ (a')! = (a!) \cdot a',$ $\qquad k_5 = 1,\ l_5 = 0,$
- $f_6 \quad pd(0) = 0,\ pd(a') = a,$ $\qquad k_6 = 1,\ l_6 = 0,$
- $f_7 \quad a \dot{-} 0 = a,\ a \dot{-} b' = pd(a \dot{-} b),$ $\qquad k_7 = 2,\ l_7 = 0,$
- $f_8 \quad \min(a, b) = b \dot{-} (b \dot{-} a),$ $\qquad k_8 = 2,\ l_8 = 0,$
- $f_9 \quad \max(a, b) = (a \dot{-} b) + b,$ $\qquad k_9 = 2,\ l_9 = 0,$
- $f_{10} \quad \overline{sg}(0) = 1,\ \overline{sg}(a') = 0,$ $\qquad k_{10} = 1,\ l_{10} = 0,$
- $f_{11} \quad sg(0) = 0,\ sg(a') = 1,$ $\qquad k_{11} = 1,\ l_{11} = 0,$
- $f_{12} \quad |a - b| = (a \dot{-} b) + (b \dot{-} a),$ $\qquad k_{12} = 2,\ l_{12} = 0,$
- $f_{13} \quad rm(0, b) = 0,$ $\qquad k_{13} = 2,\ l_{13} = 0,$
  $rm(a', b) = (rm(a, b))' \cdot sg\,|b - (rm(a, b))'|,$
- $f_{14} \quad [\,0/\,b\,] = 0,$ $\qquad k_{14} = 2,\ l_{14} = 0,$
  $[\,a'/\,b\,] = [\,a/\,b\,] + \overline{sg}\,|\,b - (rm(a, b))'\,|,$
- $f_{15} \quad f_{15}(0, \alpha) = 0,\ f_{15}(z', \alpha) = f_{15}(z, \alpha) + \alpha(z),$ $\qquad k_{15} = 1,\ l_{15} = 1,$
  [gives the bounded sum $\Sigma_{y<x}\alpha(y)$]
- $f_{16} \quad f_{16}(0, \alpha) = 1,\ f_{16}(z', \alpha) = f_{16}(z, \alpha) \cdot \alpha(z),$ $\qquad k_{16} = 1,\ l_{16} = 1,$
  [gives the bounded product $\Pi_{y<x}\alpha(y)$]
- $f_{17} \quad f_{17}(0, \alpha) = \alpha(0),\ f_{17}(z', \alpha) = f_8(f_{17}(z, \alpha), \alpha(z')),$ $\qquad k_{17} = 1,\ l_{17} = 1,$
  [gives the bounded minimum $\min_{y\leq x}\alpha(y)$]
- $f_{18} \quad f_{18}(0, \alpha) = \alpha(0),\ f_{18}(z', \alpha) = f_9(f_{18}(z, \alpha), \alpha(z')),$ $\qquad k_{18} = 1,\ l_{18} = 1,$
  [gives the bounded maximum $\max_{y\leq x}\alpha(y)$]
- $f_{19} \quad p_0 = 2,\ p_{i'} = \mu b_{b<p_i!+2}\,[\,p_i < b\ \&\ Pr(b)\,],$ $\qquad k_{19} = 1,\ l_{19} = 0,$
- $f_{20} \quad (a)_i = \mu x_{x<a}\,[\,p_i^x \mid a\ \&\ \neg\, p_i^{x'} \mid a\,],$ $\qquad k_{20} = 2,\ l_{20} = 0,$
- $f_{21} \quad lh(a) = \Sigma_{i<a}\, sg((a)_i),$ $\qquad k_{21} = 1,\ l_{21} = 0,$
- $f_{22} \quad a * b = a \cdot \Pi_{i<lh(b)} p_{lh(a)+i}^{(b)_i},$ $\qquad k_{22} = 2,\ l_{22} = 0,$
- $f_{23} \quad \overline{\alpha}(x) = \Pi_{i<x} p_i^{\alpha(i)+1},$ $\qquad k_{23} = 1,\ l_{23} = 1,$
- $f_{24} \quad \widetilde{\alpha}(x) = \Pi_{i<x} p_i^{\alpha(i)},$ $\qquad k_{24} = 1,\ l_{24} = 1,$
- $f_{25} \quad a \circ b = \Pi_{i<\max(a,b)} p_i^{\max((a)_i, (b)_i)},$ $\qquad k_{25} = 2,\ l_{25} = 0,$
- $f_{26} \quad ccp(0) = 1,\ \ ccp(y') = ccp(y) \cdot p_y^{r(y,\, ccp(y))},$ $\qquad k_{26} = 1,\ l_{26} = 0,$

where the following abbreviations are used:

- $a < b \equiv \exists c(c' + a = b)$;
- $a \leq b \equiv a < b \vee a = b$;
- $a \mid b \equiv \exists c(a \cdot c = b)$;
- $\mu y_{y<z} R(y) \equiv \Sigma_{x<z} \Pi_{y<x'} r(y)$, where r(y) is a term with $\vdash R(y) \leftrightarrow r(y) = 0$, $\vdash r(y) \leq 1$ and x not free in r(y);
- Pr(a) is a prime formula expressing that a is a prime number;
- the term r(y,z) in f_{26} is constructed in [Kleene1969].

NOTATION. We will use the following abbreviation, representing a primitive recursive coding of finite sequences of natural numbers by natural numbers.

For each $k \geq 0$, $\langle x_0, \ldots, x_k \rangle = \mathbf{p_0}^{x_0} \cdot \ldots \cdot \mathbf{p}_k{}^{x_k}$,
where $\mathbf{p}_i$ is the numeral for the i-th prime number p_i. In particular, for the coding of ordered pairs, we have $\langle x, y \rangle = 2^x \cdot 3^y$. We note that the projection functions are provided by f_{20}.

1.2.5. $\mathbf{HA}_1$ is described as follows. Its language $\mathcal{L}(\mathbf{HA_1})$ (note that $\mathcal{L}(\mathbf{HA_1})$ is $\mathcal{L}(\mathbf{EL})$) includes all the function constants of $\mathcal{L}(\mathbf{HA})$. As in $\mathcal{L}(\mathbf{IA_1})$, there are parentheses for function application, and Church's λ for λ-abstraction. There is a functional constant rec, expressing the recursor functional, which corresponds to definition by the schema of primitive recursion.

The terms and functors are defined as in $\mathbf{IA}_1$, with an additional term formation rule for the constant rec: if t, s are terms and u a functor, then rec(t, u, s) is a term.

The mathematical axioms of $\mathbf{HA}_1$ are:

- the axioms of **HA** with
- IND extended for $\mathcal{L}(\mathbf{HA_1})$;
- λ-conversion;
- REC $\begin{cases} \mathrm{rec}(t, u, 0) = t, \\ \mathrm{rec}(t, u, S(s)) = u(\langle \mathrm{rec}(t, u, s), s \rangle), \end{cases}$
  where t, s are terms and u a functor.[2]

1.3. Function existence principles.

1.3.1. The unique existential number quantifier $\exists !y$ is used to express the notion "there exists a unique y such that ..." and it is introduced as an abbreviation ([IM], p. 199):

$$\exists ! y B(y) \equiv \exists y\, [\, B(y)\ \&\ \forall z(B(z) \rightarrow y = z)\,]\,.$$

[2]REC is originally formulated using a pairing function j, which has the additional property of being onto the natural numbers, but this is not an essential feature; for details we refer to [Troelstra1973], 1.3.9, where it is remarked that they "might have used Kleene's $2^x 3^y$".

1.3.2. The *minimal system* **M** is the theory $\mathbf{IA}_1 + \mathrm{AC}_{00}!$, where

$$\mathrm{AC}_{00}! \quad \forall x \exists ! y A(x,y) \to \exists \alpha \forall x A(x, \alpha(x)),$$

where x and α are free for y in A(x, y) and α does not occur free in A(x, y).

The schema $\mathrm{AC}_{00}!$ expresses a countable function comprehension principle. Because of the uniqueness condition in the hypothesis, there is no choice involved. With classical logic, it is equivalent to AC_{00} (like $\mathrm{AC}_{00}!$ without the ! in the hypothesis), expressing the countable numerical choice principle. Constructively, as it is shown in [Weinstein] with a highly non-trivial proof, $\mathrm{AC}_{00}!$ is weaker than AC_{00}.

1.3.3. *Elementary analysis* **EL** is the theory $\mathbf{HA}_1$ + QF-AC_{00}, where

$$\text{QF-AC}_{00} \quad \forall x \exists y A(x,y) \to \exists \alpha \forall x A(x, \alpha(x)),$$

where A(x, y) is a quantifier-free formula, in which x is free for y and α does not occur.

The schema QF-AC_{00} expresses a weak principle of countable numerical choice for quantifier-free formulas. This principle does not involve real choice either, since the quantifier-free formulas are decidable, and in this case, existence entails constructively unique existence (of the least such number). For these basic facts we refer to section 2 below.

2. Unique existence and decidability

2.1. The unique existential number quantifier.

2.1.1. Another way to define the unique existential quantifier is by

$$\exists ! y B(y) \equiv \exists y B(y) \mathbin{\&} \forall y \forall z (B(y) \mathbin{\&} B(z) \to y = z).$$

The two definitions are equivalent over intuitionistic predicate logic with equality. In the second definition, unique existence is expressed directly as the conjunction of “exists” and “at most one”, where “at most one” is expressed by

$$\text{(a)} \quad \forall y \forall z (B(y) \mathbin{\&} B(z) \to y = z).$$

Bishop constructivists are using a different (classically equivalent) condition to express “at most one”: they use

$$\text{(b)} \quad \forall y \forall z (y \neq z \to (\neg B(y) \vee \neg B(z))).$$

Although **M** proves (b) → (a), the two interpretations are not constructively equivalent, as we show by the following example.

Example. Consider the formula

$$A(x) \equiv (x = 0 \mathbin{\&} P) \vee (x = 1 \mathbin{\&} \neg P),$$

where P is any formula and x any variable not occurring free in P. We can easily see that for this A(x), condition (a) holds in **M**; however, condition (b) entails $\neg A(0) \vee \neg A(1)$ and hence $\neg P \vee \neg\neg P$, which is unprovable in **M** in general. One easy way to see this, is to consider a continuity principle, say WC! (see [TvDI]), which is consistent with **M** as is well-known. Take as P the formula $\exists x\, \alpha(x) \neq 0$; then using WC! we can prove

$$\neg\forall\alpha\, [\, \neg\exists x\, \alpha(x) \neq 0 \vee \neg\neg\exists x\, \alpha(x) \neq 0\,]\,.$$

Observe however that under either of the assumptions $\exists x A(x)$ or $\forall x(\neg A(x) \vee \neg\neg A(x))$, we have (a) $\leftrightarrow$ (b). Thus either interpretation of "at most one" could be used to unabbreviate the ! in AC_{00}!.

2.2. Uniqueness and decidability.

2.2.1. In intuitionistic arithmetic unique existence (of a natural number satisfying a predicate) and decidability (of natural number predicates) are closely related.[3]

As a consequence, the principles AC_{00}! and QF-AC_{00} are also related in a precise manner, over any reasonable two-sorted intuitionistic arithmetic. The following provide basic facts about the two notions. Using them we determine how AC_{00}! and QF-AC_{00} relate to each other. For most of the proofs we give only some idea of the arguments used, as they are standard, well known facts.

In the following **S** is any of $\mathbf{IA}_0$, **HA**, $\mathbf{IA}_1$ or $\mathbf{HA}_1$, and $\vdash$ denotes provability in **S**.

2.2.2. The decidability of the equality of natural numbers, and hence also of formulas which are quantifier-free or have only bounded number quantifiers, is established by the following lemma.

LEMMA 2.1. $\vdash \forall x\forall y(x = y \vee \neg x = y)$.

PROOF. The decidability of the equality of numbers in intuitionistic arithmetic, unlike in the classical case, needs a non trivial proof using the schema of mathematical induction IND and the axioms of the equality and the successor.

□

The next two lemmas provide a very important property of formulas which are quantifier-free or have only bounded number quantifiers.

[3] In the classical case all these are trivialities, as natural number existence always entails unique existence of a least witness and every formula satisfies the law of excluded middle.

LEMMA 2.2. *For any formula* A *of* **S** *built up from the formulas* $\mathrm{P}_1, \ldots, \mathrm{P}_m$ *by propositional connectives or bounded number quantifiers,*

$$\mathrm{P}_1 \vee \neg \mathrm{P}_1, \ldots, \mathrm{P}_m \vee \neg \mathrm{P}_m \vdash \mathrm{A} \vee \neg \mathrm{A}.$$

PROOF. The proof is by induction on the structure of A. The inductive step for the cases of the bounded existential and universal number quantifiers follows from

$$\forall \mathrm{x}(\mathrm{A}(\mathrm{x}) \vee \neg \mathrm{A}(\mathrm{x})) \vdash \exists \mathrm{y}\,[\,\mathrm{y} < \mathrm{x} \;\&\; \mathrm{A}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{A}(\mathrm{z}))\,] \\ \vee \forall \mathrm{y}\,[\,\mathrm{y} < \mathrm{x} \rightarrow \neg \mathrm{A}(\mathrm{y})\,],$$

which in its turn is proved by formal induction on x.

□

LEMMA 2.3. *For any formula* A *of* **S** *which is quantifier-free or has only bounded number quantifiers (and no function quantifiers),* $\vdash \mathrm{A} \vee \neg \mathrm{A}$.

PROOF. Immediate from the two previous lemmas.

□

2.2.3. The *least (natural) number principle* is expressed in first-order arithmetic by the formula

$$\exists \mathrm{y}\mathrm{B}(\mathrm{y}) \rightarrow \exists \mathrm{y}\,[\,\mathrm{B}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{B}(\mathrm{z}))\,].$$

Unlike in the classical case, it is not provable in intuitionistic arithmetic, as it implies the law of the excluded middle (although its double negation is provable). However, for number predicates that are assumed decidable, the least number principle holds and the least number is unique.

LEMMA 2.4. *In* **S***,*

$$\vdash \forall \mathrm{y}(\mathrm{B}(\mathrm{y}) \vee \neg \mathrm{B}(\mathrm{y})) \rightarrow \;[\,\exists \mathrm{y}\mathrm{B}(\mathrm{y}) \rightarrow \exists ! \mathrm{y}(\mathrm{B}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{B}(\mathrm{z})))\,].$$

PROOF. It follows from:

$$\vdash \forall \mathrm{y}(\mathrm{B}(\mathrm{y}) \vee \neg \mathrm{B}(\mathrm{y})) \rightarrow \;[\,\exists \mathrm{y}\mathrm{B}(\mathrm{y}) \rightarrow \exists \mathrm{y}(\mathrm{B}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{B}(\mathrm{z})))\,]$$

and

$$\vdash \exists \mathrm{y}\,[\,\mathrm{B}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{B}(\mathrm{z}))\,] \\ \rightarrow \exists ! \mathrm{y}\,[\,\mathrm{B}(\mathrm{y}) \;\&\; \forall \mathrm{z}(\mathrm{z} < \mathrm{y} \rightarrow \neg \mathrm{B}(\mathrm{z}))\,].$$

□

The next lemma asserts that, conversely to the previous, uniqueness entails decidability; we have to note that this holds only for numbers, the corresponding property fails for functions or sequences.

LEMMA 2.5. $\exists! y B(y) \vdash B(y) \vee \neg B(y)$.

PROOF. Easy consequence of the decidability of the equality of numbers ([FIM], Lemma 5.6, p. 43).

□

In the next lemma we prove a fact very useful for our purposes.

LEMMA 2.6. $\vdash A \vee \neg A \leftrightarrow \exists! y\, [\, y \leq 1\ \&\ (y = 0 \leftrightarrow A)\,]$.

PROOF. We give the proof in $\mathbf{IA_0}$. We show first the direction $\rightarrow$. CASE A. By A and the logical axiom $A \rightarrow (0 = 0 \rightarrow A)$ (axiom schema 1a) we get $0 = 0 \rightarrow A$; using $\vdash 0 = 0$ we get $A \rightarrow 0 = 0$ in a similar way. Since $\vdash 0 \leq 1$, we get

$$\text{(a)} \quad 0 \leq 1\ \&\ (0 = 0 \leftrightarrow A).$$

Using the assumption A we get

$$\text{(b)} \quad \forall z\, (z \leq 1\ \&\ (z = 0 \leftrightarrow A) \rightarrow 0 = z).$$

From (a) and (b) we get

$$\text{(c)} \quad \exists! y\, [\, y \leq 1\ \&\ (y = 0 \leftrightarrow A)\,].$$

CASE $\neg$A. The argument is similar, using the logical axiom schema 8^I $\neg B \rightarrow (B \rightarrow C)$, the axiom $\neg x' = 0$ and

$$\text{(d)} \quad \vdash z \leq 1 \leftrightarrow (z = 0 \vee z = 1).$$

Direction $\leftarrow$ follows easily from (d).

□

REMARK. In intuitionistic arithmetic (in all versions that we are considering), disjunction can be defined explicitly (see for example [TvDI], p. 127) as

$$A \vee B \equiv \exists y\, [\, (y = 0 \rightarrow A)\ \&\ (y \neq 0 \rightarrow B)\,]\,.$$

The equivalence of Lemma 2.6 corresponds to the following modification, which can also serve for the explicit definition of disjunction (only a slight modification of the related proof in [TvDI], p. 127, is needed):

$$A \vee B \equiv \exists y\, [\, y \leq 1\ \&\ (y = 0 \rightarrow A)\ \&\ (y = 1 \rightarrow B)\,].$$

In the presence in the language of the positivity test function sg defined by the pair of equations $sg(0) = 0$ and $sg(x') = 1$ the two definitions are immediately seen to be provably equivalent over the logical basis without $\vee$. (Only in $\mathbf{IA_0}$ sg is not available, but its addition gives a definitional extension (we will define this later), so we can think as essentially having it in this case too).

According to Lemmas 2.3, 2.4 and 2.5 the unique choice (or non-choice) principle $AC_{00}!$ expresses countable numerical choice for decidable number predicates.

2.2.4. We can now draw a first immediate conclusion about $AC_{00}!$ and QF-AC_{00}.

PROPOSITION 2.7. *Over* $\mathbf{IA_1}$ *(and* $\mathbf{HA_1}$*)* $AC_{00}!$ *entails* QF-AC_{00}.

PROOF. Assume (a) $\forall x \exists y A(x, y)$, where A(x,y) is quantifier-free. Then by Lemma 2.3 we obtain $\vdash \forall y\,[\,A(x, y) \lor \neg A(x, y)\,]$, so by Lemma 2.4 and (a) (after $\forall$x-elimination) we get, after $\forall$x-introduction,

$$\text{(b)}\ \forall x \exists! y (A(x, y)\ \&\ \forall z(z < y \rightarrow \neg A(x, z))).$$

We apply then $AC_{00}!$ to (b); from the resulting formula easily follows (c) $\exists \alpha \forall x A(x, \alpha(x))$, and the proof is completed with $\rightarrow$-introduction discharging (a).

□

3. A characteristic function principle

3.1. The schema CF_d.

3.1.1. Consider the following schema, which asserts that every decidable predicate of natural numbers has a characteristic function and is an immediate consequence of $AC_{00}!$ over $\mathbf{IA}_1$:

$$CF_d \quad \forall x(B(x) \lor \neg B(x)) \rightarrow \exists \beta \forall x\,[\,\beta(x) \leq 1\ \&\ (\beta(x) = 0 \leftrightarrow B(x))\,]\,.$$

Introducing this principle allows to determine the exact relation of $AC_{00}!$ and QF-AC_{00}; and this in its turn will suggest the relation between **M** and **EL**.

PROPOSITION 3.1. *Over* $\mathbf{IA_1}$ *(and* $\mathbf{HA_1}$*),* $AC_{00}!$ *entails* CF_d.

PROOF. Assume (a) $\forall x(B(x) \lor \neg B(x))$. Then by Lemma 2.6 we obtain

$$\text{(b)}\ \forall x \exists! y\,[\,y \leq 1\ \&\ (y = 0 \leftrightarrow B(x))\,].$$

Applying $AC_{00}!$ to (b) gives the conclusion of (the corresponding instance of) CF_d, and the proof is completed with $\rightarrow$-introduction discharging (a).

□

3.1.2. Now we show that the unique choice principle $AC_{00}!$ is equivalent to the conjunction of its two consequences QF-AC_{00} and CF_d over the weak systems of two-sorted arithmetic underlying the two systems **M** and **EL**.

THEOREM 3.2. *Over* $\mathbf{IA_1}$ *(and* $\mathbf{HA_1}$*),* QF-AC_{00}+CF_d *entails* $AC_{00}!$.

PROOF. Assume (a) $\forall x \exists ! y A(x, y)$. By $\forall$-elimination, Lemma 2.5 and $\forall$-introduction (twice) we get

$$\text{(a1)}\ \ \forall x \forall y\, [\, A(x, y) \vee \neg A(x, y)\,]\,;$$

and then by $\forall$-elimination twice (specializing for $(w)_0, (w)_1$) and $\forall$-introduction we have

$$\text{(b)}\ \ \forall w\, [\, A((w)_0, (w)_1) \vee \neg A((w)_0, (w)_1)\,]\,.$$

Applying CF_d to (b) gives

$$\text{(c)}\ \ \exists \beta \forall w\, [\, \beta(w) \leq 1 \ \&\ (\beta(w) = 0 \leftrightarrow A((w)_0, (w)_1))\,]\,.$$

Assume (d) which is (c) without $\exists \beta$, towards $\exists$-elim.; from (a) follows (a1) $\exists y A(x, y)$, so assume, towards $\exists$-elim.,

$$\text{(e)}\ \ A(x, y).$$

After $\forall w$-elim. from (d) (specializing for the pair $\langle x, y \rangle$) with (e) and with $\exists y$-introd., and with $\exists$-elim. disch. (e) and $\forall$-introd. we obtain

$$\text{(f)}\ \ \forall x \exists y \beta(\langle x, y \rangle) = 0.$$

Applying now QF-AC_{00} to (f) gives

$$\text{(g)}\ \ \exists \alpha \forall x \beta(\langle x, \alpha(x) \rangle) = 0,$$

from which finally follows

$$\text{(h)}\ \ \exists \alpha \forall x A(x, \alpha(x))$$

using again (d), and we complete the proof with $\exists$-elim. disch. (d) and $\rightarrow$-introd. disch. (a).

□

COROLLARY 3.3. *Over* $\mathbf{IA_1}$ *(and* $\mathbf{HA_1}$*),* $AC_{00}!$ *is equivalent to* QF-AC_{00} + CF_d.

3.2. Classical models for weak theories of two-sorted arithmetic.

3.2.1. Let $\mathbf{T}$ be the formal theory $\mathbf{IA_1}$+QF-AC_{00}. $\mathbf{T}$ is a classically correct theory, as its axioms, logical and mathematical, are all part of a classical system of analysis (see discussion in [FIM], p. 8). So $\mathbf{T}$ can be extended to a corresponding classical theory $\mathbf{T}^\circ$, by replacing the axiom schema $\neg A \rightarrow (A \rightarrow B)$ by $\neg\neg A \rightarrow A$. We will use $\mathbf{T}^\circ$ to show that $\mathbf{T}$ does not prove CF_d as follows. We will show that the general recursive functions form a classical model of $\mathbf{T}^\circ$, in which CF_d fails. This result is obtained by a classical, non-finitary argument; we note, however, that it is a negative result. For the notion of primitive and general recursive function (or functional, in the presence of function arguments) of number and one-place number-theoretic function arguments that we are using, we note that a function $\phi(x_1, \dots, x_k, \alpha_1, \dots, \alpha_l)$ of k natural numbers and l one-place number-theoretic functions is primitive or general recursive if, as a function of $x_1, \dots, x_k$ it is primitive or general recursive, respectively, uniformly in $\alpha_1, \dots, \alpha_l$; see [FIM], p. 10, and, for the notion of uniformity, see [IM]. For the notion of a structure for the two-sorted first-order language of analysis we refer to S. Simpson's work "Subsystems of Second Order Arithmetic" [Simpson]; here we are using a slight variant of the notion defined there, as in our case, instead of set variables, the variables of the second sort are function variables; we must also have interpretations for the functors formed by λ-abstraction.

THEOREM 3.4. (a) $\mathbf{IA_1} + \mathrm{QF\text{-}AC_{00}}$ *does not prove* $\mathrm{CF_d}$.
(b) **EL** *does not prove* $\mathrm{CF_d}$.

PROOF. (a) Let $\mathbf{T}$ and $\mathbf{T}^\circ$ be as in the discussion preceding the theorem. We consider the structure $\mathcal{GR}$ for the language of $\mathbf{T}^\circ$ which consists of the sets and functions given by (i)-(iii):

(i) The set $\mathbb{N}$ of the natural numbers, that serves as the universe of the first sort, over which the number variables range.

(ii) The subset $\mathcal{GR}$ of the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ consisting of all the general recursive functions from $\mathbb{N}$ to $\mathbb{N}$, which serves as the universe of the second sort, over which the function variables range.

(iii) The function(al)s $f_0, \dots, f_p$ that correspond to the function constants $\mathrm{f}_0, \dots, \mathrm{f}_p$: each f_i, for $i = 0, \dots, p$, is the primitive recursive function(al) obtained by the primitive recursive derivation expressed by the defining axioms of f_i.

The intended model for our language differs from $\mathcal{GR}$ only in having as universe of the second sort the set of all (classical) functions from $\mathbb{N}$ to $\mathbb{N}$. So it is justified to use the same name for the universe of the second sort and the structure itself.

(iv) The interpretation of a term or functor under an assignment into $\mathcal{GR}$ and the notions of satisfaction and truth are as usual. In particular, the interpretation $\mathrm{u}^{\mathcal{GR}}$ in $\mathcal{GR}$ of a functor u of the form λx.t where t is a term, under an assignment v, is given by

$$(\lambda \mathrm{x.t})^{\mathcal{GR}} = \lambda n.\overline{v(\mathrm{x}|n)}(\mathrm{t}),$$

where $\overline{v(\mathrm{x}|n)}$ is the extension to all terms and functors of the assignment which assigns the natural number n to x and agrees with v on all other variables, the λ in the interpretation is the usual (informal) Church's λ, and n ranges over $\mathbb{N}$. Function application (represented by parentheses) is interpreted accordingly. The justification of the fact that all the terms and functors have, under any assignment into $\mathcal{GR}$, interpretations in $\mathcal{GR}$ (of the correct sort) will be given below.

By the fact that the function constants have their intended interpretations the following are obtained:

(v) Each term t with free variables no other than $\mathrm{x}_1,\ldots,\mathrm{x}_k,\alpha_1,\ldots,\alpha_l$, expresses a primitive recursive function of $x_1,\ldots,x_k,\alpha_1,\ldots,\alpha_l$, where $x_1,\ldots,x_k$ are (informal) variables for natural numbers and $\alpha_1,\ldots,\alpha_l$ are (informal) variables for one-place number-theoretic functions ([FIM], Lemma 3.3); under an assignment into $\mathcal{GR}$ the term t is interpreted as usual as a natural number.

(vi) For each functor u with free variables no other than $\mathrm{x}_1,\ldots,\mathrm{x}_k$, $\alpha_1,\ldots,\alpha_l$, the term (u)(x), with x a new number variable, expresses a primitive recursive function of $x,x_1,\ldots,x_k,\alpha_1,\ldots,\alpha_l$ ([FIM], Lemma 3.3); from this it follows that under an assignment into $\mathcal{GR}$, u is interpreted as a function of one number variable which is primitive recursive in the functions assigned to its function variables, and so is interpreted as a general recursive function of one number variable.

From (i)-(vi) it follows that all the terms and functors, under any assignment into $\mathcal{GR}$, have interpretations (of the correct sort) in $\mathcal{GR}$, and that all the logical and mathematical axioms and rules of inference hold in $\mathcal{GR}$. In particular: The axioms defining the function constants all can be seen to hold by an inductive argument, as the function constants are introduced according to the forms given in 1.2.4; in the case of explicit definition the conclusion is immediate, and in the case of definition by primitive recursion the conclusion follows by informal induction on the recursion variable. The axiom schemas for the function quantifiers $\forall\alpha\mathrm{A}(\alpha)\rightarrow\mathrm{A}(\mathrm{u})$ and $\mathrm{A}(\mathrm{u})\rightarrow\exists\alpha\mathrm{A}(\alpha)$ hold, as can be seen using (vi). All other axioms and rules hold as usual. So $\mathcal{GR}$ is a model of $\mathbf{IA_1}$.

We will show now that the principle $\mathrm{QF\text{-}AC_{00}}$ holds in $\mathcal{GR}$. We will use the following fact, shown in [FIM], pp. 27-31:

(vii) For any formula Q which is quantifier-free or has only number quantifiers that are bounded, we can construct a term q, with the same free variables as Q, such that

$$\vdash \mathrm{q}\leq 1 \quad\text{and}\quad \vdash \mathrm{Q}\leftrightarrow \mathrm{q}=0.$$

The construction of q and the proofs are done in $\mathbf{IA_1}$ (which is a subsystem of the formal system of [FIM]), using constants for number-theoretic functions and, for the case of bounded quantifiers, only the functional constants f_{15} and f_{16} representing the finite sum and finite product, respectively.

Let $\mathrm{A(x, y, z, \beta)}$ be a quantifier-free formula with exactly the free variables shown. (In the case of more or fewer free variables the argument is similar). We will show that the corresponding instance of QF-AC_{00}

$$\forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta) \rightarrow \exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}), \mathrm{z}, \beta)$$

is true under any assignment v into $\mathcal{GR}$.

To see this, let $\mathrm{t(x, y, z, \beta)}$ be the term given by (vii), so that we have

$$\vdash \mathrm{A}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta) \leftrightarrow \mathrm{t}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta) = 0;$$

and let $\phi(x, y, z, \beta)$ be the primitive recursive function of x, y, z, β expressed by $\mathrm{t(x, y, z, \beta)}$.

Assume that, under an assignment v into $\mathcal{GR}$, $\forall \mathrm{x} \exists \mathrm{y} \mathrm{A}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \beta)$ is true in $\mathcal{GR}$. Then it is clear that the function ψ which assigns to each natural number x the least (natural number) y such that $\phi(x, y, v(\mathrm{z}), v(\beta)) = 0$ is general recursive and satisfies, for all natural numbers x, $\phi(x, \psi(x), v(\mathrm{z}), v(\beta)) = 0$. From this it follows that $\exists \alpha \forall \mathrm{x} \mathrm{A}(\mathrm{x}, \alpha(\mathrm{x}), \mathrm{z}, \beta)$ is true under v in $\mathcal{GR}$.

We show now that $\mathrm{CF_d}$ does not hold in $\mathcal{GR}$. Let $\mathrm{T(x, y, z)}$ represent the primitive recursive Kleene T-predicate $T(x, y, z) \Leftrightarrow z$ is the code of the computation of the value of the partial recursive function with gödel number x at the argument y. Let $\mathrm{A(x)} \equiv \exists \mathrm{z} \mathrm{T(x, x, z)}$. Then, by the law of the excluded middle, $\forall \mathrm{x}(\mathrm{A(x)} \vee \neg \mathrm{A(x)})$ is true in $\mathcal{GR}$. Applying $\mathrm{CF_d}$ gives under the interpretation the existence in $\mathcal{GR}$ of a function

$$\phi(x) = \begin{cases} 0 & \text{if } \exists z\, T(x, x, z), \\ 1 & \text{if } \neg \exists z\, T(x, x, z), \end{cases}$$

which is not recursive.

(b) The argument is similar to the one for (a). We only have to consider now $\mathcal{GR}$ as a structure with infinitely many functions, corresponding to the function constants for number-theoretic functions of **EL**, and the recursor functional (which, we note, is itself a primitive recursive functional). We also note that property (vii) is obtained for $\mathbf{HA_1}$ too (by use of the recursor in place of f_{15}, f_{16}).

□

Remark. It is well-known that in the presence of $\mathrm{AC}_{00}!$ Church's Thesis contradicts classical logic. The previous theorem makes it clear that this is due to $\mathrm{CF_d}$.

Corollary 3.5. (a) $\mathbf{IA_1} + \text{QF-AC}_{00}$ *is a proper subtheory of* **M**. (b) **EL** *is a proper subtheory of* $\mathbf{EL} + \mathrm{AC}_{00}! = \mathbf{EL} + \mathrm{CF_d}$.

3.2.2. By interpreting the function variables as primitive recursive functions of one number variable we obtain exactly as in the previous theorem a classical model for the weak theory $\mathbf{HA_1}$. In this model the principle QF-AC_{00} does not hold, as it guarantees closure under the notion "general recursive in".

THEOREM 3.6. (a) $\mathbf{HA_1}$ *does not prove* QF-AC_{00}.
(b) $\mathbf{IA_1}$ *does not prove* QF-AC_{00}.

PROOF. (a) $\mathbf{HA_1}$ has a classical model in which the function variables range over the primitive recursive functions (of one number variable). We can see that in this model QF-AC_{00} does not hold, as follows.

Let $A(x, y)$ be a quantifier-free formula. By lemmas 2.3 and 2.4, (vii) in the proof of theorem 3.4 and QF-AC_{00}, we obtain

(a) $\forall x \exists y A(x, y) \to \exists \alpha \forall x (A(x, \alpha(x)) \mathbin{\&} \forall z (z < \alpha(x) \to \neg A(x, z)))$.

Consider the function constant h that represents the characteristic function of the primitive recursive Kleene T-predicate. Let χ be the corresponding informal function and let e be the gödel number of a general recursive, but not primitive recursive, function such as the Ackermann function. Then, if $\mathbf{e}$ is the numeral of e, $\forall x \exists y\, h(\mathbf{e}, x, y) = 0$ is true in the model. By applying QF-AC_{00} we get the corresponding instance of (a), and from this, using the normal form theorem and the primitive recursive extraction function, we obtain finally in the model the existence of a function which is not primitive recursive.

(b) In the place of h of the proof of (a) we use a function variable which we interpret by χ, as h does not belong to $\mathcal{L}(\mathbf{IA_1})$. □

3.2.3. As we already observed, the formal systems **M** and **EL** differ in two ways: they have different sets of function and functional constants and they assume different function existence principles. Introducing the principle CF_d allowed us to clarify the relation between the function existence principles of the two systems, but only if they are considered over the same system of arithmetic.

The equivalence obtained (Corollary 3.3) together with the other similarities of the two systems (among which also is the possibility of definition by primitive recursion, thanks to the presence of the recursor in **EL** and to Lemma 5.3(b) of [FIM] in **M**), suggest that the theories **M** and **EL** + CF_d are essentially equivalent. And also theorem 3.4 suggests that **EL** is essentially weaker than **M**, and their difference is captured by CF_d. It turns out that the relations between these theories are indeed the suggested ones. To justify this, we will give a precise mathematical content to the notion "essentially equivalent" (and "essentially weaker").

4. Introduction of a recursor in M

To show that **M** and **EL** + $\mathrm{CF_d}$ are essentially equivalent we will find a common conservative extension of both. In order to obtain this common extension, we add one by one the missing constants of each system, and show that the corresponding extension is definitional. In this way we reach conservative extensions of the two systems in the same language, which are identical (except for trivial notational differences).

Our treatment is based on [IM], §74, where the (one-sorted) first-order case of definitional extensions is covered, and on [JRMPhD], where the method is applied for a result in the two-sorted case.

The first step in this process is to add a recursor constant to **M**. The notions of conservative extension and of definitional extension will be our main tool. We start by giving their definitions (see also [Troelstra1973]) and then make some essential observations concerning equality and replacement.

4.1. Conservative and definitional extensions.

4.1.1. Definition. Let $\mathbf{S}_1$, $\mathbf{S}_2$ be formal systems based on (many-sorted) intuitionistic logic with equality, and let the language $\mathcal{L}(\mathbf{S}_2)$ of $\mathbf{S}_2$ extend the language $\mathcal{L}(\mathbf{S}_1)$ of $\mathbf{S}_1$, and the theorems of $\mathbf{S}_2$ contain the theorems of $\mathbf{S}_1$. $\mathbf{S}_2$ is a *conservative extension* of $\mathbf{S}_1$ if the theorems of $\mathbf{S}_2$ that are formulas of $\mathbf{S}_1$ are exactly the theorems of $\mathbf{S}_1$.

Definition. Let $\mathbf{S}_1$, $\mathbf{S}_2$ be formal systems with $\mathcal{L}(\mathbf{S}_1)$ contained in $\mathcal{L}(\mathbf{S}_2)$. $\mathbf{S}_2$ is a *definitional extension* of $\mathbf{S}_1$ if there exists an effective mapping (or translation) $'$ which, to each formula E of $\mathbf{S}_2$, assigns a formula E' of $\mathbf{S}_1$ such that:

I. $\mathrm{E}' \equiv \mathrm{E}$, for E a formula of $\mathcal{L}(\mathbf{S}_1)$.
II. $\vdash_{\mathrm{S}_2} \mathrm{E}' \leftrightarrow \mathrm{E}$.
III. If $\Gamma \vdash_{\mathrm{S}_2} \mathrm{E}$ then $\Gamma' \vdash_{\mathrm{S}_1} \mathrm{E}'$.
IV. $'$ commutes with the logical operations of $\mathbf{S}$.

If the addition of a symbol gives a definitional extension, the symbol is called *eliminable (from the extended to the original system)*; conditions I - IV are called *elimination relations*; and we say that the symbol is *added definitionally.*

We observe the following. A definitional extension is a conservative extension. The extended system does not prove more formulas of the original system, and every theorem of the extended system is equivalent (in the extended system) to one of the original. So it is an inessential extension.

This technique is developed in detail in [IM], §74, where in effect **HA** is shown to be an inessential extension of $\mathbf{IA}_0$.

4.2. On equality and replacement.

4.2.1. *Many-sorted intuitionistic predicate logic with equality.* The formal systems that we are studying have been created in order to formalize mathematical theories in which, as is the normal case for mathematical theories, equality is a fundamental extensional relation; so they are based on many-sorted (and specifically two-sorted) intuitionistic predicate logic with equality: a formal system of logic whose alphabet includes a binary symbol = which satisfies, for each sort i (of finitely many sorts $0, \ldots, p$) the following axiom and axiom-schema

$\mathrm{REFL}^i \quad x^i = x^i,$
$\mathrm{REPL}^i \quad x^i = y^i \to (A(x) \to A(y)),$ with x, y free for z in A(z).

For this subject we refer to [Troelstra1973].

4.2.2. *Treatment of equality in the systems under study.* In the varieties of mathematical analysis that we are considering, equality of natural numbers is a primitive concept, intuitively clear and decidable. Equality of number-theoretic functions (and, consequently, of sequences of natural numbers and of reals) is understood extensionally, and is undecidable (its decidability would entail a special case of the law of the excluded middle, the ∀-PEM (known also as WLPO, for "Weak Lesser Principle of Omniscience"), so it is constructively unacceptable.

Reflecting this use of equality, in the formalisms under study the constant = represents type-0 (number) equality. Type-1 (function) equality, represented by the same symbol, is defined in terms of type-0 equality, and is introduced by the abbreviation

$$u = v \equiv \forall x\, u(x) = v(x),$$

where u, v are functors and x is not free in u or v.

The axioms of equality of both types are REFL^i, REPL^i, $i = 0, 1$. By EQ we denote all these axioms.

It is possible (and very useful for our investigation) to reduce the equality axioms to simpler (and in some cases only finitely many) axioms, as follows (we refer to systems with only function constants as non-logical symbols; the case of predicate constants is treated similarly):

(A) By equality axioms for $=$ are meant the axioms

$$\begin{aligned} &x = x,\\ &x = y \to (x = z \to y = z). \end{aligned}$$

(B) By equality axioms for a function symbol f of k number and l function arguments are meant the $k + l$ formulas

$$\begin{aligned} x = y \to\ & f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k, \alpha_1, \ldots, \alpha_l) =\\ & f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k, \alpha_1, \ldots, \alpha_l), \quad i = 1, \ldots, k,\\ \alpha = \beta \to\ & f(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \ldots, \alpha_l) =\\ & f(x_1, \ldots, x_k, \alpha_1, \ldots, \alpha_{i-1}, \beta, \alpha_{i+1}, \ldots, \alpha_l), \quad i = 1, \ldots, l. \end{aligned}$$

(C) The axioms EQ of a two-sorted formal system with type-0 equality as a primitive and type-1 equality defined as above, and with only function constants in its alphabet, are provable from the following instances or consequences of them:
1. The equality axioms for $=$.
2. The equality axioms for the function constants of its alphabet.
3. The open equality axiom $x = y \to \alpha(x) = \alpha(y)$.
For this subject, we refer to [IM].

Thanks to the fact that the function constants of **M** are introduced successively via the primitive recursive description of the corresponding functions, the equality axioms for these function constants are provable in **M**; the proofs are by use of IND (see [FIM], p. 20).

In the case of **HA** and **EL**, the axioms by EQ are all introduced from the beginning, but it is easy to see that it suffices to include (C) 1, 3, and the equality axiom for the successor: $x = y \to S(x) = S(y)$, and then prove successively, in parallel with the introduction of each function constant to the formalism, the corresponding equality axioms.

About the constant rec of **EL**: the equality axioms are provable by the method of Lemma 5.1, [FIM], p. 20, using IND.

About the constant rec that will be introduced in **M**: we will see that the equality axioms for rec are also provable in the extended system.

4.2.3. *The replacement theorem.* Lemma 4.2, p. 16 of [FIM], gives the replacement theorem for **M**. Since the proviso of the lemma is satisfied in the case of **EL** as a consequence of the preceding paragraph, Lemma 4.2 of [FIM] provides the replacement theorem for **EL**. The same holds for the system that we will obtain by adding a recursor to **M**.

We note that the replacement theorem requires the equality axioms only for the function symbols that have the specified occurrence to be replaced within their scope, in the formula in which the replacement takes place. We refer also to [IM], §73(D) for this argument.

So, in order to prove the instance of replacement

$$\forall\overrightarrow{\alpha}\,\forall\overrightarrow{x}\;r = s \;\vdash\; E_r \leftrightarrow E_s,$$

where E_s is the result of replacing in E_r a specified occurrence of r by s, and where $\overrightarrow{\alpha}$, $\overrightarrow{x}$ are all the free variables of r or s which belong to a λ-prefix or quantifier having the specified occurrence of r within its scope, it suffices to have the equality axioms for the function symbols occurring in E_r and having the specified occurrence of r within their scope.

4.3. Introducing a recursor in M.

4.3.1. We will add now to **M** a recursor functional, and we will prove that the resulting extension is definitional, and so the system is not essentially strengthened. Let $\mathbf{S}_1$ be the *minimal system* of analysis **M**, and $\mathbf{S}_2$ be the system **M** + Rec, obtained by adding to **M** the functional constant rec in its alphabet, together with the corresponding term formation rule "if t, s are terms and u a functor, then rec(t, u, s) is a term" and the following axiom, defining it:

$$\text{Rec} \quad A(x, \alpha, y, \text{rec}(x, \alpha, y)),$$

where $A(x, \alpha, y, w)$ is the formula

$$\exists\beta\,[\,\beta(0) = x \;\&\; \forall z\,\beta(z+1) = \alpha(\langle\beta(z), z\rangle) \;\&\; \beta(y) = w\,]\,.$$

The new constant rec represents then the recursor functional, which corresponds to definition by the schema of primitive recursion.

Remark. We could have introduced the new functional constant rec in **M** by the pair of equations REC, that define it in **EL**, and consider it as the f_{27}, extending the list of constants of **M**. Observe that the equations REC have one of the forms of the definitions of the constants f_i. We rejected this choice, because the presence of rec would make redundant all the constants of the list except the first four. Observe also that **EL** could avoid the infinite list of constants thanks to rec.

Lemma 5.3(b) of [FIM] provides definition by primitive recursion in **M**, so we based our definition directly on it. We followed [IM], §74, in introducing a new function symbol by a formula for which the formalism proves that it has a functional character. We note also that some of the formal systems that we will consider (**BIM**, **WKV**, **H**) have definition by primitive recursion as an axiom, in forms very similar to Lemma 5.3(b) of [FIM].

As we will see the two ways of introducing the new constant are equivalent (see also [IM], p. 416).

4.3.2. *Interderivability of* Rec *and* REC. The constant rec could have been defined by the following pair of equations, as is done in the case of **EL**:

$$\text{REC} \quad \begin{cases} \mathrm{rec}(\mathrm{t}, \mathrm{u}, 0) = \mathrm{t}, \\ \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{S}(\mathrm{s})) = \mathrm{u}(\langle \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s}), \mathrm{s} \rangle), \end{cases}$$

where t, s are terms, S the successor function, u a functor. This is justified by the fact that Rec and REC are interderivable over **M** with rec added in its symbolism, as we show now.

Instead of considering the axioms REC as they are formulated in **EL**, we consider the following equivalent (by logic) formulation:

$$\text{REC} \quad \begin{cases} \mathrm{rec}(\mathrm{x}, \alpha, 0) = \mathrm{x}, \\ \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}') = \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y} \rangle), \end{cases}$$

(where the symbol $'$ of **M** is used in place of S for the successor).

We show that $\mathbf{S}_2 = \mathbf{M} + \text{Rec}$ is equivalent with $\mathbf{S}'_2 = \mathbf{M} + \text{REC}$, in the sense that every instance of REC is provable in $\mathbf{S}_2$ and vice versa. In fact, we show that REC and Rec are interderivable in the above sense over $\mathbf{IA_1}$ (and $\mathbf{HA_1}$). So, in the case of **EL** and other systems, it is also immaterial which of the two "definitions" of the constant rec is considered.

Remark. From the definition of rec by REC, the equality axioms for rec become provable by the method of [FIM], Lemma 5.1 (see 4.2.2, on the treatment of equality).

Lemma A. *Every instance of* REC *is provable in* $\mathbf{S} = \mathbf{IA_1} + \text{Rec}$ *(and in* $\mathbf{HA_1} + \text{Rec}$*).*

Proof. (i) We show (a) $\vdash_{\mathrm{s}} \mathrm{rec}(\mathrm{x}, \alpha, 0) = \mathrm{x}$.
By Rec,

$$\text{(b)} \;\; \mathrm{A}(\mathrm{x}, \alpha, 0, \mathrm{rec}(\mathrm{x}, \alpha, 0))$$

is an axiom of **S**. Assume

$$\text{(c)} \;\; \beta(0) = \mathrm{x} \;\&\; \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z} \rangle) \;\&\; \beta(0) = \mathrm{rec}(\mathrm{x}, \alpha, 0).$$

From (c), by Ax. 16 (transitivity of $=$) we get

$$\text{(d)} \;\; \mathrm{rec}(\mathrm{x}, \alpha, 0) = \mathrm{x}.$$

After $\exists\beta$-elimination discharging (c), we get (a).

(ii). We show (a) $\vdash_{\mathrm{s}} \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}') = \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y} \rangle)$.
By Rec, we obtain the following:

$$\text{(b)} \;\; \exists \beta \, [\, \beta(0) = \mathrm{x} \;\&\; \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z} \rangle) \;\&\; \beta(\mathrm{y}) = \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}) \,] \,.$$

(c) $\exists\gamma\,[\,\gamma(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\,\gamma(\mathrm{z}') = \alpha(\langle\gamma(\mathrm{z}), \mathrm{z}\rangle)\ \&\ \gamma(\mathrm{y}') = \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}')\,]$.

Assume (b1) which is (b) without $\exists\beta$ and (c1) which is (c) without $\exists\gamma$. Then, using IND and the replacement property for $=$ we get

$$\text{(d)}\ \ \beta = \gamma.$$

By (b1) and replacement, we get

$$\text{(e)}\ \ \beta(\mathrm{y}') = \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y}\rangle),$$

and from (e), (d) and (c1) (the last conjunct), we get

$$\text{(f)}\ \ \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}') = \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y}\rangle).$$

After $\exists\beta$, $\exists\gamma$-eliminations discharging (b1), (c1), we get (a).

□

LEMMA B. *The axiom* Rec *is provable in* $\mathbf{S}' = \mathbf{IA_1} + \mathrm{REC}$ *(and in* $\mathbf{HA_1} + \mathrm{REC}$*).*

PROOF. Using the axiom-schema of λ-reduction for the functor $\lambda \mathrm{z}.\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{z})$ we have:

(a) $(\lambda \mathrm{z}.\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{z}))(0)$	$= \mathrm{rec}(\mathrm{x}, \alpha, 0)$	by λ-red.
	$= \mathrm{x}$	by REC
(b) $(\lambda \mathrm{z}.\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{z}))(\mathrm{y}')$	$= \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}')$	by λ-red.
	$= \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y}\rangle)$	by REC
	$= \alpha(\langle(\lambda \mathrm{z}.\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{z}))(\mathrm{y}), \mathrm{y}\rangle)$	by λ-red. and repl. for $\langle\ ,\ \rangle, \alpha$
(c) $(\lambda \mathrm{z}.\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{z}))(\mathrm{y})$	$= \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y})$	by λ-red.

After $\forall$-introduction to (b), with (a) and (c), and then with $\exists\beta$-introduction, we get

$$\exists\beta\,[\,\beta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\beta(\mathrm{z}') = \alpha(\langle\beta(\mathrm{z}), \mathrm{z}\rangle)\ \&\ \beta(\mathrm{y}) = \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y})\,],$$

which is $\mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}))$.

□

4.3.3. We will show that $\mathbf{S}_2$ is a definitional extension of $\mathbf{S}_1$.

NOTATION. (a) In the following, by $\vdash_1$ and $\vdash_2$ we denote provability in $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively.

(b) The unique existential function quantifier is introduced as an abbreviation by

$$\exists!\beta \mathrm{C}(\beta) \equiv \exists\beta\,[\,\mathrm{C}(\beta)\ \&\ \forall\gamma\,(\mathrm{C}(\gamma) \to \beta = \gamma)\,].$$

Remark. With the help of the unique existential function quantifier, we can formulate compactly the following version of $\mathrm{AC}_{00}!$:

$$\forall x \exists ! y A(x, y) \rightarrow \exists ! \alpha \forall x A(x, \alpha(x)).$$

Although this schema is apparently stronger than $\mathrm{AC}_{00}!$, it is easily shown that it is a consequence of it, hence equivalent (over two-sorted intuitionistic logic with equality). We will use this version in some proofs.

In the following we will use the Lemma 5.3(b) of [FIM], which is proved in **M** and justifies "definition by primitive recursion" in this formal theory.

Lemma 5.3(b) ([FIM]). Let y, z be distinct number variables, and α a function variable. Let q, $r(y, z)$ be terms not containing α free, with α and y free for z in $r(y, z)$. Then

$$\vdash \exists \alpha\, [\, \alpha(0) = q \;\&\; \forall y\, \alpha(y') = r(y, \alpha(y))\,]\,.$$

Lemma 4.1. $\vdash_1 \exists ! \beta\, [\, \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle)\,]\,.$

Proof. From [FIM], Lemma 5.3(b), we have

$$\text{(a)} \quad \vdash_1 \exists \beta\, [\, \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle)\,].$$

Assume

$$\text{(b)} \quad \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle).$$

Assume

$$\text{(c)} \quad \gamma(0) = x \;\&\; \forall z\, \gamma(z') = \alpha(\langle \gamma(z), z\rangle).$$

We show $\beta = \gamma$ (which abbreviates $\forall z\, (\beta(z) = \gamma(z))$ using IND:
By (b), (c), we get

$$\text{(d)} \quad \beta(0) = \gamma(0).$$

Assume

$$\text{(e)} \quad \beta(z) = \gamma(z).$$

Then by (b), (c) (the second conjunct of each, after $\forall$-eliminations), we get using the replacement property of equality,

$$\text{(f)} \quad \beta(z') = \alpha(\langle \beta(z), z\rangle) = \alpha(\langle \gamma(z), z\rangle) = \gamma(z').$$

So, after $\rightarrow$-introduction discharging (e) and $\forall z$-introduction, we get

$$\text{(g)} \quad \forall z\, (\beta(z) = \gamma(z) \rightarrow \beta(z') = \gamma(z')),$$

so by (d), (g) with IND, we get $\beta = \gamma$. From this, with $\rightarrow$-introduction discharging (c) and then $\forall\gamma$-introduction, &-introduction (with (b)) and $\exists\beta$-introduction, after $\exists\beta$-elimination discharging (b), we get the lemma.

□

LEMMA 4.2. $\vdash_1 \forall y \exists ! m\, A(x, \alpha, y, m)$.

PROOF. From [FIM], Lemma 5.3(b) again, we have

$$\text{(a1)} \quad \vdash_1 \exists \beta\, [\, \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle)\,]\,.$$

Assume

$$\text{(a2)} \quad \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle).$$

Then by the reflexivity of equality we get

$$\text{(b)} \quad \beta(0) = x \;\&\; \forall z\, \beta(z') = \alpha(\langle \beta(z), z\rangle) \;\&\; \beta(y) = \beta(y),$$

so with $\exists$m- and then $\exists\gamma$-introduction, we get

$$\text{(c)} \quad \exists \gamma \exists m\, [\, \gamma(0) = x \;\&\; \forall z\, \gamma(z') = \alpha(\langle \gamma(z), z\rangle) \;\&\; \gamma(y) = m\,]\,.$$

Assume

$$\text{(d1)} \quad \exists m\, [\, \gamma(0) = x \;\&\; \forall z\, \gamma(z') = \alpha(\langle \gamma(z), z\rangle) \;\&\; \gamma(y) = m\,]\,.$$

Assume

$$\text{(d2)} \quad \gamma(0) = x \;\&\; \forall z\, \gamma(z') = \alpha(\langle \gamma(z), z\rangle) \;\&\; \gamma(y) = m.$$

By $\exists\gamma$-introduction, from (d2) we get

$$\text{(e)} \quad A(x, \alpha, y, m)$$

Assume now

$$\text{(f)} \quad A(x, \alpha, y, n),$$

so we get

$$\text{(f1)} \quad \exists \delta\, [\, \delta(0) = x \;\&\; \forall z\, \delta(z') = \alpha(\langle \delta(z), z\rangle) \;\&\; \delta(y) = n\,]\,.$$

Assume also

$$\text{(g)} \quad \delta(0) = x \;\&\; \forall z\, \delta(z') = \alpha(\langle \delta(z), z\rangle) \;\&\; \delta(y) = n.$$

By Lemma 4.1, (d2) and (g) we get $m = n$, so after $\exists\delta$-elimination discharging (g), and then $\rightarrow$-introduction discharging (f) and $\forall$n-introduction we get

$$\text{(h)} \quad \forall n (A(x, \alpha, y, n) \rightarrow m = n).$$

By (e) and (h) with &- and $\exists$m-introductions, after $\exists$m, $\exists\gamma$, $\exists\beta$-eliminations discharging (d2), (d1), (a2) and $\forall$y-introduction, we get the lemma.

□

LEMMA 4.3. $\vdash_2 \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}) = \mathrm{z} \leftrightarrow \mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{z})$.

PROOF. The formula (a) $\mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}))$ is an axiom of $\mathbf{S}_2$.

(i) Assume $\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}) = \mathrm{z}$. From this, (a) and the replacement property of equality (which as is observed in (D) §73 of [IM] requires only the predicate calculus with the equality axioms for $=$ and the function and predicate symbols of $\mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{z})$ - we refer to §73 of [IM] for the treatment of equality) we get $\mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{z})$.

(ii) Assume $\mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{z})$. From this and (a), applying lemma 4.2, we obtain $\mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}) = \mathrm{z}$.

□

NOTATION. Let t be a term. Let all the free number variables of t be among $\mathrm{x}_0, \ldots, \mathrm{x}_k$, and let w be a number variable not occurring in t. For the rest of this section t^{w} will be the result of replacing in t, for each $i = 0, \ldots, k$, each free occurrence of x_i by an occurrence of the term $(\mathrm{w})_i$. The same notation is used for functors too. Since the exponential will not appear in the proofs, there is no chance of confusion by the use of this notation.

LEMMA 4.4. *Let* t, s *be terms and* u *a functor of* $\mathbf{S_1}$. *Let* $\mathrm{x}_0, \ldots, \mathrm{x}_k$ *include all the number variables occurring free in* t, u *or* s, *let* w *be a number variable not occurring in* t, u, s *and* γ *a function variable free for* v *in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$, *not occurring free in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$. *Then*

$$\vdash_1 \exists! \gamma \forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

PROOF. Let z be a number variable different from $\mathrm{w}, \mathrm{x}_0, \ldots, \mathrm{x}_k$. By Lemma 4.2 we get

$$\vdash_1 \forall \mathrm{x} \forall \alpha \forall \mathrm{y} \exists! \mathrm{z}\, \mathrm{A}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{z}),$$

so, by specializing for t, u, s with the corresponding $\forall$-eliminations, we get

$$\vdash_1 \exists! \mathrm{z}\, \mathrm{A}(\mathrm{t}, \mathrm{u}, \mathrm{s}, \mathrm{z}),$$

and then

$$\vdash_1 \forall \mathrm{x}_0 \ldots \forall \mathrm{x}_\mathrm{k} \exists! \mathrm{z}\, \mathrm{A}(\mathrm{t}, \mathrm{u}, \mathrm{s}, \mathrm{z}).$$

From this, after specializing for each $i = 0, \ldots, k$ for $(\mathrm{w})_i$ with $\forall \mathrm{x}_i$-elimination, we get

$$\vdash_1 \exists! \mathrm{z}\, \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{z}).$$

From this we get

$$\vdash_1 \forall \mathrm{w} \exists! \mathrm{z}\, \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{z}),$$

and by $\mathrm{AC}_{00}!$ with the Remark in the beginning of 4.3.3 we get

$$\vdash_1 \exists! \gamma \forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

□

NOTATION. (i) The notation $\mathrm{E}[\,\mathrm{a}\,]$ is used to indicate some specified occurrences of a term or functor a in an expression E. We will also use similarly $\mathrm{E}[\,\mathrm{a}_1, \dots, \mathrm{a}_k\,]$ for $k > 0$ to indicate some specified occurrences of k terms or functors. This notation may leave some ambiguity regarding the indicated occurrences, but in each case we will explain its use.

(ii) We use $\alpha(\mathrm{x}_0, \dots, \mathrm{x}_k)$ as an abbreviation for $\alpha(\langle \mathrm{x}_0, \dots, \mathrm{x}_k \rangle)$.

LEMMA 4.5. *Let* t, s *be terms and* u *a functor of* $\mathbf{S_2}$*, let* $\mathrm{x}_0, \dots, \mathrm{x}_k$ *be all the number variables occurring free in* t, u *or* s*, and let* w *be a number variable not occurring in* $\mathrm{A}(\mathrm{t}, \mathrm{u}, \mathrm{s}, \mathrm{v})$*. Let* $\mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]$ *be a formula of* $\mathbf{S_2}$ *in which* $\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)$ *is not within the scope of some function quantifier* $\forall\alpha$ *or* $\exists\alpha$*, where* α *is a function variable occurring free in* t, u *or* s *or* α *is* γ*, and* γ *is new for* $\mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$*, is free for* v *in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$*, and does not occur free in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$*, and where* $\mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$ *is obtained by replacing in* $\mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]$ *each of the (specified) occurrences of* $\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)$ *by an occurrence of* $\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})$*.*

Then

$$\vdash_2 \mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,] \leftrightarrow \exists\gamma[\,\forall\mathrm{w}\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \;\&\; \mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]\,].$$

PROOF. (i) Assume (a) $\mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$. By Lemma 4.4 we have

$$\text{(b)}\quad \vdash_2 \exists\gamma\forall\mathrm{w}\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

Assume

$$\text{(c)}\quad \forall\mathrm{w}\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

Then, by Lemma 4.3,

$$\forall\mathrm{w}\,\mathrm{rec}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}) = \gamma(\mathrm{w}),$$

so by specializing for $\mathrm{w} = \langle \mathrm{x}_0, \dots, \mathrm{x}_k \rangle$ we get

$$\text{(d)}\quad \forall\mathrm{x}_0 \dots \forall\mathrm{x}_\mathrm{k}\ \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s}) = \gamma(\mathrm{x}_0, \dots, \mathrm{x}_k).$$

Since the occurrences of $\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)$ are not within the scope of some function quantifier $\forall\alpha$ or $\exists\alpha$ where α occurs free in s, t, or u or α is γ, by the replacement theorem, from (a) we get

$$\text{(e)}\ \mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,],$$

so with (c) and &-introduction and then $\exists\gamma$-introduction and $\exists\gamma$-elim. discharging (c), after $\to$-introduction we get the "$\to$" case from (a).

(ii) Assume

$$\text{(a)}\ \forall\mathrm{w}\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w}))$$

and

$$\text{(b)}\ \mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,].$$

By (a), specializing for $\mathrm{w} = \langle \mathrm{x}_0, \dots, \mathrm{x}_k \rangle$, we get $\mathrm{A}(\mathrm{t}, \mathrm{u}, \mathrm{s}, \gamma(\mathrm{x}_0, \dots, \mathrm{x}_k))$, and by Lemma 4.3, with $\forall$-introductions we get

$$\text{(c)}\ \ \forall \mathrm{x}_0 \dots \forall \mathrm{x}_\mathrm{k}\, \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s}) = \gamma(\mathrm{x}_0, \dots, \mathrm{x}_k).$$

By the conditions on the bindings due to function quantifiers, the replacement theorem applies, so from (b), (c) we get $\mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$, and after $\exists\gamma$-eliminations discharging (a) and (b), and with $\rightarrow$-introduction, we get the "$\leftarrow$" case.

□

TERMINOLOGY. A term of the form rec(t, u, s) is called a *rec-term.* A term in which the constant rec does not occur is called a *rec-less term.* A term rec(t, u, s) where rec does not occur in t, u, s is called a *rec-plain term.* An occurrence of the constant rec in a formal expression is called a *rec-occurrence.*

LEMMA 4.6. *To each formula* E *of* $\mathbf{S_2}$ *there can be correlated a formula* E′ *of* $\mathbf{S_1}$, *called the principal rec-less transform of* E, *in such a way that the elimination relations* I, II *hold, no free variables are introduced or removed, and the logical operators of the two-sorted predicate calculus are preserved (elimination relation* IV*).*

PROOF. The definition of E′ is done by induction on the number g of occurrences of the logical operators in E. The basis of the induction consists in giving the definition for E prime; this is done by induction on the number q of occurrences of rec-terms in E.

CASE E is rec-less: Then E′ shall be E.

CASE E has $q > 0$ rec-occurrences: Let rec(t, u, s) be the first (the leftmost) occurrence of a rec-plain term, so that $\mathrm{E} \equiv \mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$, and let $\mathrm{x}_0, \dots, \mathrm{x}_k$ be all the free number variables of this occurrence, w a number variable and γ a function variable, such that they are new for both $\mathrm{E} \equiv \mathrm{E}[\,\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})\,]$ and $\mathrm{A}(\mathrm{t}, \mathrm{u}, \mathrm{s}, \mathrm{v})$. Then we define

$$\mathrm{E}' \equiv \exists\gamma[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^\mathrm{w}, \mathrm{u}^\mathrm{w}, \mathrm{s}^\mathrm{w}, \gamma(\mathrm{w}))\ \&\ [\,\mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]\,]'\,],$$

where $\mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]$ is the result of replacing in E the specified (first) occurrence of rec(t, u, s) by an occurrence of $\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)$. We observe that $\mathrm{E}[\,\gamma(\mathrm{x}_0, \dots, \mathrm{x}_k)\,]$ is prime and contains $q - 1$ occurrences of rec-terms. About the choice of the bound variables γ and w, as well as the possibly necessary changes of the bound variables of A(x, α, y, w) to make the substitutions of t^w, u^w, s^w free, we note that all permissible choices lead to congruent formulas.

The condition that the logical operators are preserved determines in a unique way the definition of ′ for all formulas of $\mathbf{S_2}$; the cases are the following:

- $(\neg A)' \equiv \neg (A')$,
- $(A \circ B)' \equiv A' \circ B'$, for $\circ \equiv \to, \&, \vee$,
- $(QxA(x))' \equiv Qx(A(x))'$, for $Q \equiv \forall, \exists$,
- $(Q\alpha A(\alpha))' \equiv Q\alpha(A(\alpha))'$, for $Q \equiv \forall, \exists$.

We immediately see that elimination relations I and IV hold. Elimination relation II is now proved easily by induction on the number of logical operators in E. The basis of the induction is the case of E prime, and is proved (easily) by induction on the number of rec-occurrences in E, using Lemma 4.5 and replacement.

□

We still need to prove elimination relation III, so we have to show: If $\Gamma \vdash_{S_2} E$ then $\Gamma' \vdash_{S_1} E'$. The proof depends on a sequence of lemmas, as follows. We will use the version with function variables of Lemma 25 [IM], p. 408. This lemma provides useful consequences of unique existence for number variables in intuitionistic predicate logic with equality. Corresponding consequences have been obtained by S. C. Kleene for function variables, in the two-sorted case (in a manuscript mentioned in [JRMPhD]); we can use this version thanks to Lemma 4.4. The versions of [IM] *181 - *190 with a function variable instead of v are mentioned as $^*181^F$ - $^*188^F$, $^*189^F_n$, $^*190^F_n$, and there are also cases $^*189^F_f$, $^*190^F_f$, $^*189^N_f$, $^*190^N_f$, with variables whose sorts will be obvious from the notation. We will state the needed cases in the place we use them.

Lemma 4.7. *Let* $\mathrm{rec}(t, u, s)$ *be any specified occurrence of a rec-plain term in a prime formula* E *of* $\mathbf{S_2}$*, so that we have* $E \equiv E[\,\mathrm{rec}(t, u, s)\,]$*. Then*

$$\text{(a)} \quad \vdash_1 E' \leftrightarrow \exists\gamma[\forall w A(t^w, u^w, s^w, \gamma(w)) \;\&\; [\,E[\,\gamma(x_0, \ldots, x_k)\,]\,]'\,],$$

where the conditions on the variables are as in the definition of $'$*.*

Proof. The proof is by induction on the number q of occurrences of rec-terms in the prime formula E.

We will use the following case of the functional version of Lemma 25 of [IM], p.408, and the functional version of *78, [IM], p. 162:

$$^*190^F_f \quad \vdash \exists\beta\,[\,F(\beta) \;\&\; \exists\alpha\, D(\alpha, \beta)\,] \leftrightarrow \exists\alpha\exists\beta\,[\,F(\beta) \;\&\; D(\alpha, \beta)\,],$$

where α does not occur free in $F(\beta)$.

$^*78^F_f \quad \vdash \exists\alpha\exists\beta\, D(\alpha, \beta) \leftrightarrow \exists\beta\exists\alpha\, D(\alpha, \beta)$.

Case $q = 0$ or $q = 1$: is trivial.

Case $q > 1$: Assume that Lemma 4.7 holds for prime formulas having $q-1$ rec-occurrences, and let E be a prime formula with q rec-occurrences.

Let $\mathrm{rec}(t,u,s)$ be a specified occurrence of a rec-plain term in E, so $E \equiv E[\,\mathrm{rec}(t,u,s)\,]$. If $\mathrm{rec}(t,u,s)$ is the first occurrence of a rec-plain term in E, then (a) holds by the definition of $'$. If $\mathrm{rec}(t,u,s)$ is not the first occurrence of a rec-plain term, then let $\mathrm{rec}(t_1,u_1,s_1)$ be the first such, so that $E \equiv E[\,\mathrm{rec}(t_1,u_1,s_1),\mathrm{rec}(t,u,s)\,]$.

Then, by the definition of $'$, we have

$$\text{(b)}\quad E' \equiv \exists\gamma[\forall w A(t_1^w,u_1^w,s_1^w,\gamma(w))\ \&\ [\,E[\,\gamma(x_0,\dots,x_k),\mathrm{rec}(t,u,s)\,]\,]'\,],$$

where $x_0,\dots,x_k$ are all the free number variables of t_1,u_1,s_1. Since $E[\,\gamma(x_0,\dots,x_k),\mathrm{rec}(t,u,s)\,]$ has $q-1$ rec-occurrences, by the inductive hypothesis we have

$$\text{(c)}\quad \vdash_1 [\,E[\,\gamma(x_0,\dots,x_k),\mathrm{rec}(t,u,s)\,]\,]' \leftrightarrow \exists\delta[\,\forall w A(t^w,u^w,s^w,\delta(w)) \\ \&\ [\,E[\,\gamma(x_0,\dots,x_k),\delta(y_0,\dots,y_l)\,]\,]'\,],$$

where $y_0,\dots,y_l$ are the free number variables of t, u, s. From (b), (c), by the replacement theorem (for equivalence) we get

$$\text{(d)}\quad \vdash_1 E' \leftrightarrow \exists\gamma[\forall w A(t_1^w,u_1^w,s_1^w,\gamma(w))\ \&\ \exists\delta[\,\forall w A(t^w,u^w,s^w,\delta(w)) \\ \&\ [\,E[\,\gamma(x_0,\dots,x_k),\delta(y_0,\dots,y_l)\,]\,]'\,].$$

By (d), $^*190_f^F$, $^*78_f^F$, we get

$$\text{(e)}\quad \vdash_1 E' \leftrightarrow \exists\delta[\,\forall w A(t^w,u^w,s^w,\delta(w))\ \&\ \exists\gamma[\,\forall w A(t_1^w,u_1^w,s_1^w,\gamma(w)) \\ \&\ [\,E[\,\gamma(x_0,\dots,x_k),\delta(y_0,\dots,y_l)\,]\,]'\,].$$

By the inductive hypothesis again, we have

$$\text{(f)}\quad \vdash_1 [\,E[\,\mathrm{rec}(t_1,u_1,s_1),\delta(y_0,\dots,y_l)\,]\,]' \leftrightarrow \\ \exists\gamma[\,\forall w A(t_1^w,u_1^w,s_1^w,\gamma(w))\ \&\ [\,E[\,\gamma(x_0,\dots,x_k),\delta(y_0,\dots,y_l\,]\,]'\,].$$

The left part of the $\leftrightarrow$ in (f) is just the result of replacing the specified occurrence $\mathrm{rec}(t,u,s)$ by an occurrence of $\delta(y_0,\dots,y_l)$ in E. So, by (e), (f) and the replacement theorem we get (a congruent of) (a).

□

Lemma 4.8. *Let* $E \equiv E[\,\mathrm{rec}(t,u,s)\,]$ *be any formula of* $\mathbf{S_2}$ *such that* $\mathrm{rec}(t,u,s)$ *is a specified occurrence of a rec-plain term in* E*, which is not in the scope of any quantifier binding a free variable of* $\mathrm{rec}(t,u,s)$*. Then, if* $x_0,\dots,x_k$ *are all the free number variables of* t,u,s *and* γ *a function variable not occurring in* E*, then*

$$\vdash_1 E' \leftrightarrow \exists\gamma[\,\forall w A(t^w,u^w,s^w,\gamma(w))\ \&\ [\,E[\,\gamma(x_0,\dots,x_k)\,]\,]'\,].$$

PROOF. The proof is by induction on the complexity of the formula E.

CASE E prime. Apply Lemma 4.7.

CASE E composite. Let r be the term $\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)$. Then $\mathrm{E}[\mathrm{r}]$ will be of one of the forms $\neg\mathrm{B}[\mathrm{r}]$, $\mathrm{B}[\mathrm{r}] \,\&\, \mathrm{C}$, $\mathrm{B} \,\&\, \mathrm{C}[\mathrm{r}]$, $\mathrm{B}[\mathrm{r}] \vee \mathrm{C}$, $\mathrm{B} \vee \mathrm{C}[\mathrm{r}]$, $\mathrm{B}[\mathrm{r}] \rightarrow \mathrm{C}$, $\mathrm{B} \rightarrow \mathrm{C}[\mathrm{r}]$, $\forall \mathrm{x}\, \mathrm{B}(\mathrm{x})[\mathrm{r}]$, $\exists \mathrm{x}\, \mathrm{B}(\mathrm{x})[\mathrm{r}]$, $\forall \alpha\, \mathrm{B}(\alpha)[\mathrm{r}]$, $\exists \alpha\, \mathrm{B}(\alpha)[\mathrm{r}]$. By Lemma 4.4 we have

$$\vdash_1 \exists! \gamma \forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

So for each form we can apply the corresponding cases of the functional version of Lemma 25 of [IM], p. 408, that we mention before Lemma 4.7. We note that the conditions on the variables in the statement of the present lemma and in the definition of the translation $'$ allow applying each corresponding case.

We treat here the case $\forall \alpha\, \mathrm{B}(\alpha)[\mathrm{r}]$. We will use the following case of the functional version of Lemma 25 of [IM].

$^*189^{\mathrm{F}}_{\mathrm{f}}$ $\exists! \beta \mathrm{F}(\beta) \vdash \exists \beta\, [\mathrm{F}(\beta) \,\&\, \forall \alpha\, \mathrm{D}(\alpha, \beta)] \leftrightarrow \forall \alpha \exists \beta\, [\mathrm{F}(\beta) \,\&\, \mathrm{D}(\alpha, \beta)]$,

where α does not occur free in $\mathrm{F}(\beta)$.

CASE $\mathrm{E} \equiv \forall \alpha\, \mathrm{B}(\alpha) \equiv \forall \alpha\, \mathrm{B}(\alpha)\, [\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]$,
where $\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})$ is the occurrence to be eliminated, so α does not occur free in t, u or s. We have

$$\mathrm{E}' \equiv \forall \alpha\, [\mathrm{B}(\alpha)\, [\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]]'.$$

By the induction hypothesis,

$$\vdash_1 [\mathrm{B}(\alpha)\, [\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]]' \leftrightarrow \exists \gamma\, [\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \\ \&\ [\mathrm{B}(\alpha)\, [\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'].$$

By this we get

$$\vdash_1 \mathrm{E}' \leftrightarrow \forall \alpha \exists \gamma\, [\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \,\&\, [\mathrm{B}(\alpha)\, [\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'].$$

Since α does not occur free in $\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w}))$, by $^*189^{\mathrm{F}}_{\mathrm{f}}$ with Lemma 4.4 we get

$$\vdash_1 \mathrm{E}' \leftrightarrow \exists \gamma\, [\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \,\&\, \forall \alpha [\mathrm{B}(\alpha)\, [\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'],$$

from which we get the lemma for this case.

□

LEMMA 4.9. *Let* E *be a formula of* $\mathbf{S_2}$ *in which the rec-plain term* $\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})$ *has some occurrences such that no free (number or function) variable becomes bound in* E *by a universal or existential quantifier. Let* $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]$ *be the result of replacing in* E *one or more specified occurrences of* $\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})$ *by* $\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)$*, where* $\mathrm{x}_0, \ldots, \mathrm{x}_k$ *are all the* *free number variables of* t, u, s *and* γ *is a function variable not occurring in* E. *Then*

$$\vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \ \& \ [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'].$$

PROOF. The proof is by induction on the number of occurrences of rec(t, u, s) that are replaced (by $\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)$). If one occurrence is replaced, then the lemma follows by Lemma 4.8.

Assume the lemma holds if $q-1$ occurrences are replaced in some formula. Let $\mathrm{E} \equiv \mathrm{E}[\mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]$ indicate q specified occurrences of rec(t, u, s) in E, and let $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]$ be the result of replacing these q specified occurrences by $\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)$. By Lemma 4.8 we have

$$\vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \ \& \ [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]]'],$$

where $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]$ is the result of replacing in E the first of the q specified occurrences of rec(t, u, s) by $\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)$. Then, by the inductive hypothesis,

$$\vdash_1 [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]]' \leftrightarrow \\ \exists\delta[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \delta(\mathrm{w})) \ \& \ [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \delta(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'],$$

where $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \delta(\mathrm{x}_0, \ldots, \mathrm{x}_k)]$ is the result of replacing in $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \mathrm{rec}(\mathrm{t}, \mathrm{u}, \mathrm{s})]$ the other $q-1$ specified occurrences of rec(t, u, s) by $\delta(\mathrm{x}_0, \ldots, \mathrm{x}_k)$.

By the replacement theorem we get then

$$\vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \ \& \ \exists\delta[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \delta(\mathrm{w})) \\ \& \ [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \delta(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'].$$

We will use the following case of the functional version of Lemma 25 of [IM], p.408:

$$^{*}183^{\mathrm{F}} \quad \exists!\beta\, \mathrm{F}(\beta) \vdash \exists\alpha\, [\mathrm{F}(\alpha) \ \& \ \mathrm{C}(\alpha, \alpha)] \leftrightarrow \\ \exists\alpha\, [\mathrm{F}(\alpha) \ \& \ \exists\beta\, [\mathrm{F}(\beta) \ \& \ \mathrm{C}(\alpha, \beta)]],$$

where α does not occur free in $\mathrm{F}(\beta)$ and is free for β in $\mathrm{F}(\beta)$ and in $\mathrm{C}(\alpha, \beta)$.

Then by $^{*}183^{\mathrm{F}}$ and Lemma 4.4

$$\vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma[\forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})) \ \& \ [\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]]'],$$

and, since $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k), \gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]$ is just $\mathrm{E}[\gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k)]$, we get the result.

□

In the next lemma we will use the following case of the functional version of Lemma 25 of [IM], p. 408:

$$^{*}182^{\mathrm{F}} \quad \exists!\beta \mathrm{F}(\beta), \forall\beta\, \mathrm{C}(\beta) \vdash \exists\beta\, [\, \mathrm{F}(\beta)\ \&\ \mathrm{C}(\beta)\,]\,.$$

LEMMA 4.10. *If* E *is any axiom of* $\mathbf{S_2}$*, then* $\vdash_1$ E$'$.

PROOF. (i) If E is an axiom of $\mathbf{S_2}$ by an axiom-schema of the propositional logic, then E$'$ is equivalent in $\mathbf{S_1}$ to an axiom of $\mathbf{S_1}$ by the same axiom-schema. This follows by the fact that, by its definition, the translation $'$ preserves the logical operators, together with the fact that the translations of different instances of the same formula may differ only in their bound variables, so they are congruent, hence equivalent. Using the replacement theorem for equivalence, it follows that the translation of any propositional axiom is equivalent to an axiom by the same schema, so we get $\vdash_1$ E$'$.

(ii) The logical axioms for the quantifiers need a different treatment. We give the proof for the case 10N, and the other cases follow by the same method.

CASE Axiom-schema 10N: $\mathrm{E} \equiv \forall \mathrm{x}\, \mathrm{B}(\mathrm{x}) \to \mathrm{B}(\mathrm{r})$, where r is a term of $\mathbf{S_2}$ free for x in B(x). Then $\mathrm{E}' \equiv \forall \mathrm{x}\, [\, \mathrm{B}(\mathrm{x})\,]' \to [\, \mathrm{B}(\mathrm{r})\,]'$.

Ia. If x has no free occurrences in B(x), or r is rec-less and x does not occur free in any rec-occurrence, we can easily see that, by simply choosing the same bound variables at corresponding steps in the elimination processes in B(x) and B(r), we get a formula $\forall \mathrm{x}\, \mathrm{C}(\mathrm{x}) \to \mathrm{C}(\mathrm{r})$ which is congruent to E$'$ and is an axiom of $\mathbf{S_1}$ by 10N, so $\vdash_1$ E$'$.

Ib. In the case that x may occur free in some rec-occurrences of B(x) and r is rec-less, we show first

$$\text{(A)} \quad \vdash_1 [\, \mathrm{B}(\mathrm{x})\,]'(\mathrm{x}/\mathrm{r}) \leftrightarrow [\, \mathrm{B}(\mathrm{r})\,]'.$$

This is proved by induction on the number g of the logical operators in B(x). The case of B(x) prime ($g = 0$) gives the basis, and is obtained by induction on the number q of the rec-occurrences in B(x) as follows.

BASIS for $q = 0$: Trivial.

INDUCTIVE STEP: Let rec(t(x), u(x), s(x)) be the first rec-occurrence of a rec-plain term in the formula B(x), where x occurs free in t(x), u(x) or s(x) (if there is no such rec-occurrence, the result follows by the argument of Ia). Let $\mathrm{x}_0, \ldots, \mathrm{x}_k, \mathrm{x}$ be all the free number variables of t(x), u(x), s(x). Then, by eliminating this rec-occurrence, by Lemma 4.7 we get

$$\text{(a)} \quad \vdash_1 \ [\, \mathrm{B}(\mathrm{x})\,]' \leftrightarrow \exists\gamma\, [\, \forall \mathrm{w}(\mathrm{A}(\mathrm{t}(\mathrm{x})^{\mathrm{w}}, \mathrm{u}(\mathrm{x})^{\mathrm{w}}, \mathrm{s}(\mathrm{x})^{\mathrm{w}}, \gamma(\mathrm{w})) \\ \&\ [\, \mathrm{B}(\mathrm{x})\, [\, \gamma(\mathrm{x}_0, \ldots, \mathrm{x}_k, \mathrm{x})]\,]'\,],$$

where all the bound variables in the displayed formula are chosen so that r is free for x.

For simplicity, let r have only one number variable not included in $x_0, \ldots, x_k, x$, say z, so that $r = r(x, z)$ (in case that x does not occur free in r, the proof is by a similar argument). By eliminating the corresponding rec-occurrence from B(r) we get

$$\text{(b)} \quad \vdash_1 \ [\,B(r)\,]' \leftrightarrow \exists\delta\,[\,\forall w(A(t(r)^w, u(r)^w, s(r)^w, \delta(w)) \\ \& \ [\,B(r)\,[\,\delta(x_0, \ldots, x_k, x, z)\,]\,]'].$$

Consider now

$$\text{(a1)} \quad \vdash_1 \ [\,B(x)\,]'(x/r) \leftrightarrow \exists\gamma\,[\,\forall w(A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \\ \& \ [\,B(x)\,[\,\gamma(x_0, \ldots, x_k, x)]\,]'(x/r)\,],$$

which is obtained from (a) by the substitution (x/r), using the axiom-schema 10N of $\mathbf{S_1}$.

Assume

$$\text{(a2)} \quad [\,B(x)\,]'(x/r).$$

We will obtain $[\,B(r)\,]'$ as follows.

From (a1) and (a2) we get the conclusion of (a1), so we can assume

$$\text{(a3)} \quad \forall w(A(t(x)^w, u(x)^w, s(x)^w, \gamma(w)) \ \& \ [\,B(x)\,[\,\gamma(x_0, \ldots, x_k, x)\,]\,]'(x/r)\,.$$

From the inductive hypothesis, we have

$$\text{(a4)} \quad \vdash_1 [\,B(x)\,[\,\gamma(x_0, \ldots, x_k, x)]\,]'(x/r) \leftrightarrow [\,B(r)\,[\,\gamma(x_0, \ldots, x_k, r(x, z))\,]\,]'.$$

From Lemma 4.4 we have

$$\vdash_1 \exists!\delta\forall w\, A(t(r)^w, u(r)^w, s(r)^w, \delta(w)),$$

so we can assume

$$\text{(a5)} \quad \forall w\, A(t(r)^w, u(r)^w, s(r)^w, \delta(w)).$$

From the first conjunct of (a3) and (a5) we can obtain using Lemma 4.2

$$\text{(c)} \quad \forall x_0 \ldots \forall x_k \forall x \forall z\, \gamma(x_0, \ldots, x_k, r(x, z)) = \delta(x_0, \ldots, x_k, x, z),$$

by corresponding $\forall$w-eliminations and then $\forall$-introductions.

We will use now the following

FACT. For any formula $D[\,x\,]$ of $\mathbf{S}_2$ we can show that if s, t are rec-less terms, $\overrightarrow{x}$ contains all the free number variables of s and t, and no free function variable of s or t becomes bound in $D[\,x\,]$, then

$$\forall\overrightarrow{x}\, s = t \vdash_1 [\,D[\,s\,]\,]' \leftrightarrow [\,D[\,t\,]\,]'.$$

The proof is by induction on the number of logical operators in D[x]. The basis (case of prime formulas) is obtained by induction on the number q of rec-occurrences in the formula, by use of the replacement theorem for $\mathbf{S_1}$.

By the above fact and (c), we get now

(d) $[\,\mathrm{B(r)}\,[\,\delta(\mathrm{x}_0,\dots,\mathrm{x}_k,\mathrm{x},\mathrm{z})]\,]' \leftrightarrow [\,\mathrm{B(r)}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k,\mathrm{r}(\mathrm{x},\mathrm{z}))]\,]'$.

From the second conjunct of (a3) with (a4) and with (d) we get

$$\text{(e)}\quad [\,\mathrm{B(r)}\,[\,\delta(\mathrm{x}_0,\dots,\mathrm{x}_k,\mathrm{x},\mathrm{z})]\,]'.$$

Now from (a5) and (e) with $\exists\delta$-introduction we get the right part of (b) and finally $[\,\mathrm{B(r)}\,]'$.
The other direction of the equivalence of (A) is obtained similarly.
The case of composite formulas (inductive step for $g > 0$) follows easily.
From (A) follows immediately that $\mathrm{E}' \equiv \forall\mathrm{x}\,[\,\mathrm{B(x)}\,]' \to [\,\mathrm{B(r)}\,]'$ is a congruent of an axiom of $\mathbf{S_1}$ by the same axiom-schema, so $\vdash_1 \mathrm{E}'$.

II. If r has some rec-occurrences, the result is obtained by an induction on the number q of these occurrences.

Basis for $q = 0$ is given by the second case of Ia and Ib.

Inductive Step: If r contains q ($q > 0$) rec-occurrences, we consider the first rec-plain occurrence in r, say rec(t, u, s), so

$$\text{(a)}\quad \mathrm{E} \equiv \mathrm{E}(\mathrm{r}\,[\,\mathrm{rec}(\mathrm{t},\mathrm{u},\mathrm{s})\,]\,) \equiv \forall\mathrm{x}\,\mathrm{B(x)} \to \mathrm{B}(\mathrm{r}\,[\,\mathrm{rec}(\mathrm{t},\mathrm{u},\mathrm{s})\,]).$$

By Lemma 4.9,

$$\text{(b)}\quad \vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma\,[\,\forall\mathrm{w}\mathrm{A}(\mathrm{t}^{\mathrm{w}},\mathrm{u}^{\mathrm{w}},\mathrm{s}^{\mathrm{w}},\gamma(\mathrm{w}))\ \&\ [\,\mathrm{E}(\mathrm{r}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,]\,)\,]'\,],$$

with γ new for E and $\mathrm{x}_0,\dots,\mathrm{x}_k$ as usual. By the inductive hypothesis,

$$\text{(c)}\quad \vdash_1 [\,\forall\mathrm{x}\,\mathrm{B(x)} \to \mathrm{B}(\mathrm{r}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])\,]'.$$

But $\mathrm{E}(\mathrm{r}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,]\,)$ is just $\forall\mathrm{x}\,\mathrm{B(x)} \to \mathrm{B}(\mathrm{r}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])$, so by Lemma 4.4 with $^*182^{\mathrm{F}}$, from (b), (c) we get $\vdash_1 \mathrm{E}'$.

(iii) The axiom-schema of induction (Ax. 13, IND). Consider

$$\mathrm{E} \equiv \mathrm{A}(0)\ \&\ \forall\mathrm{x}(\mathrm{A(x)} \to \mathrm{A}(\mathrm{x}')) \to \mathrm{A(x)}.$$

Then

$$\mathrm{E}' \equiv [\,\mathrm{A}(0)\,]'\ \&\ \forall\mathrm{x}([\,\mathrm{A(x)}\,]' \to [\,\mathrm{A}(\mathrm{x}')\,]') \to [\,\mathrm{A(x)}\,]',$$

and the result follows by the arguments of (ii), Ia and Ib.

(iv) The axiom-schema of λ-reduction.

$$\mathrm{E} \equiv (\lambda\mathrm{x}.\mathrm{r(x)})(\mathrm{p}) = \mathrm{r(p)},$$

where r(x), p are terms of $\mathbf{S_2}$ and p is free for x in r(x).

If r(x), p are rec-less, then E′ is E and is an axiom of $\mathbf{S_1}$ by the same schema.

If there are q ($q > 0$) rec-occurrences in total in r(x) and p, we consider the first occurrence of a rec-plain term in r(x) (in case that r(x) is rec-less, then consider the first such in p), say rec(t, u, s), with free number variables the $\mathrm{x}_0, \dots, \mathrm{x}_k$, and γ new. Then

$$\mathrm{E} \equiv \mathrm{E}[\,\mathrm{rec(t,u,s)}\,] \equiv (\lambda\mathrm{x.r(x)}[\,\mathrm{rec(t,u,s)}\,])(\mathrm{p}) = (\mathrm{r(x)}[\,\mathrm{rec(t,u,s)}\,])(\mathrm{x/p}).$$

By Lemma 4.9,

$$\text{(a)}\quad \vdash_1 \mathrm{E}' \leftrightarrow \exists\gamma[\forall\mathrm{w}\mathrm{A}(\mathrm{t^w},\mathrm{u^w},\mathrm{s^w},\gamma(\mathrm{w}))\ \&\ [\,(\lambda\mathrm{x.r(x)}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])(\mathrm{p}) = (\mathrm{r(x)}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])(\mathrm{x/p})\,]'\,].$$

But the instance of λ-reduction shown in (a)[4] has $q-1$ rec-occurrences in the terms involved, so the induction hypothesis applies and we obtain

$$\text{(b)}\quad \vdash_1 [\,(\lambda\mathrm{x.r(x)}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])(\mathrm{p}) = (\mathrm{r(x)}\,[\,\gamma(\mathrm{x}_0,\dots,\mathrm{x}_k)\,])(\mathrm{x/p})\,]'.$$

By Lemma 4.4 with $^*182^{\mathrm{F}}$, we get now from (a) and (b) that $\vdash_1 \mathrm{E}'$.

In case that the rec-occurrence to be eliminated is in the term p, the argument is similar.

(v) The axiom-schema $\mathrm{AC}_{00}!$.

$$\mathrm{E} \equiv \forall\mathrm{x}\exists\mathrm{y}\,[\,\mathrm{A(x,y)}\ \&\ \forall\mathrm{z}(\mathrm{A(x,z)} \rightarrow \mathrm{y}=\mathrm{z})\,] \rightarrow \exists\alpha\forall\mathrm{x}\mathrm{A}(\mathrm{x},\alpha(\mathrm{x})).$$

Then

$$\mathrm{E}' \equiv \forall\mathrm{x}\exists\mathrm{y}\,[\,[\,\mathrm{A(x,y)}\,]'\ \&\ \forall\mathrm{z}([\,\mathrm{A(x,z)}\,]' \rightarrow \mathrm{y}=\mathrm{z})\,] \rightarrow \exists\alpha\forall\mathrm{x}\,[\,\mathrm{A}(\mathrm{x},\alpha(\mathrm{x}))\,]',$$

which by the arguments of (ii), Ia and Ib is (congruent to) an axiom of $\mathbf{S_1}$ by the same axiom-schema, so $\vdash_1 \mathrm{E}'$.

(vi) The (open) axiom Rec. Let E be

$$\exists\beta\,[\,\beta(0)=\mathrm{x}\ \&\ \forall\mathrm{z}\,\beta(\mathrm{z}')=\alpha(\langle\beta(\mathrm{z}),\mathrm{z}\rangle)\ \&\ \beta(\mathrm{y})=\mathrm{rec}(\mathrm{x},\alpha,\mathrm{y})\,].$$

Then E′ shall be

$$\text{(a)}\quad \exists\beta\,[\,\beta(0)=\mathrm{x}\ \&\ \forall\mathrm{z}\,\beta(\mathrm{z}')=\alpha(\langle\beta(\mathrm{z}),\mathrm{z}\rangle)\ \&\ [\,\beta(\mathrm{y})=\mathrm{rec}(\mathrm{x},\alpha,\mathrm{y})\,]'\,].$$

We have

$$\text{(b)}\quad \vdash_1 [\,\beta(\mathrm{y})=\mathrm{rec}(\mathrm{x},\alpha,\mathrm{y})\,]' \leftrightarrow \exists\gamma\,[\,\forall\mathrm{w}\mathrm{A}(\mathrm{x^w},\alpha^{\mathrm{w}},\mathrm{y^w},\gamma(\mathrm{w}))\ \&\ \beta(\mathrm{y})=\gamma(\mathrm{x},\mathrm{y})\,].$$

By Lemma 4.1,

$$\text{(c)}\quad \vdash_1 \exists!\beta\,[\,\beta(0)=\mathrm{x}\ \&\ \forall\mathrm{z}\,\beta(\mathrm{z}')=\alpha(\langle\beta(\mathrm{z}),\mathrm{z}\rangle)\,].$$

[4] We observe that p is free for x in r(x), as, if x occurs free in r(x), then x is one of the variables $\mathrm{x}_0, \dots, \mathrm{x}_k$.

By Lemma 4.4,

$$\text{(d)}\quad \vdash_1 \exists!\gamma\forall \mathrm{w}\mathrm{A}(\mathrm{x}^{\mathrm{w}}, \alpha^{\mathrm{w}}, \mathrm{y}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

Assume

$$\text{(c1)}\quad \beta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)$$
$$\&\ \forall\delta(\delta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \delta(\mathrm{z}') = \alpha(\langle \delta(\mathrm{z}), \mathrm{z}\rangle) \to \beta = \delta).$$

Assume

$$\text{(d1)}\quad \forall \mathrm{w}\mathrm{A}(\mathrm{x}^{\mathrm{w}}, \alpha^{\mathrm{w}}, \mathrm{y}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

Then

$$\exists\delta\,[\,\delta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \delta(\mathrm{z}') = \alpha(\langle \delta(\mathrm{z}), \mathrm{z}\rangle)\ \&\ \delta(\mathrm{y}) = \gamma(\mathrm{x}, \mathrm{y})\,].$$

Assume

$$\text{(d2)}\quad \delta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \delta(\mathrm{z}') = \alpha(\langle \delta(\mathrm{z}), \mathrm{z}\rangle)\ \&\ \delta(\mathrm{y}) = \gamma(\mathrm{x}, \mathrm{y}).$$

By (c1) and (d2) we get $\beta = \delta$, so

$$\text{(e)}\quad \beta(\mathrm{y}) = \gamma(\mathrm{x}, \mathrm{y}).$$

After $\exists\delta$-elimination discharging (d2), we get from (d1), (e),

$$\text{(f)}\quad \forall \mathrm{w}\mathrm{A}(\mathrm{x}^{\mathrm{w}}, \alpha^{\mathrm{w}}, \mathrm{y}^{\mathrm{w}}, \gamma(\mathrm{w}))\ \&\ \beta(\mathrm{y}) = \gamma(\mathrm{x}, \mathrm{y}).$$

And after $\exists\gamma$-introduction, $\exists\gamma$-elimination discharging (d1) we get from (f) with (c1)

$$\text{(g)}\quad \beta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)$$
$$\&\ \exists\gamma\,[\,\forall \mathrm{w}\mathrm{A}(\mathrm{x}^{\mathrm{w}}, \alpha^{\mathrm{w}}, \mathrm{y}^{\mathrm{w}}, \gamma(\mathrm{w}))\ \&\ \beta(\mathrm{y}) = \gamma(\mathrm{x}, \mathrm{y})\,].$$

After $\exists\beta$-introduction, $\exists\beta$-elimination discharging (c1), we get (a) in $\mathbf{S_1}$ (using (b)).

(vii) The remaining axioms are finitely many axioms not containing the constant rec, and they are also axioms of $\mathbf{S_1}$.

□

LEMMA 4.11. *If* E *is an immediate consequence of* F *(* F *and* G*) in* $\mathbf{S_2}$*, then* E′ *is an immediate consequence of* F′ *(* F′ *and* G′*) in* $\mathbf{S_1}$*.*

PROOF. Since by the definition of the translation ′ the logical operators are preserved and since, by Lemma 4.6, no free variables are introduced or removed, and also since congruent formulas are equivalent, it follows that to each instance of a rule of $\mathbf{S_2}$ corresponds an instance of the same rule in $\mathbf{S_1}$.

□

We conclude that if $\Gamma \vdash_2 \mathrm{E}$, where Γ is a list of formulas and E is a formula of $\mathbf{S_2}$, then $\Gamma' \vdash_1 \mathrm{E}'$. So elimination relation III is satisfied.

5. Comparison of M and EL

5.1. Introduction of the other function(al) constants.

5.1.1. Having the recursor constant rec in the formalism, it is immediate that any constant for a function, in whose primitive recursive derivation are used only functions with names already in the symbolism, can be added definitionally. The needed translation is trivial, it amounts just to the replacement of each occurrence of the new constant by the corresponding (longer) term provided by the formalism.

More concretely, constants for all the primitive recursive functions can be added definitionally, successively according to their primitive recursive descriptions, as follows:

· For the initial functions and for functions defined by composition from functions for which we already have constants, it is very easy to find terms expressing them.

· For the case of definition by primitive recursion we use the constant rec: for example if $f(x, 0) = g(x)$ and $f(x, y + 1) = h(f(y), y, x)$, we introduce a new constant f_j by $\mathrm{f}_j(\mathrm{x}, \mathrm{y}) = \mathrm{rec}(\mathrm{g}(\mathrm{x}), \lambda \mathrm{z}.\mathrm{h}((\mathrm{z})_0, (\mathrm{z})_1, \mathrm{x}), \mathrm{y})$, if we have already in our symbolism g and h for g and h, respectively.

We note that in this way, not only functions, but also functionals can be added in a formalism having a recursor, like **EL** or **M** + Rec. We also note that the equality axioms for the new constants become provable.

5.2. Comparison of M and EL.

5.2.1. Let $\mathbf{M}^+$ be obtained by adding to **M** + Rec all the (infinitely many) function constants of **HA**, with their defining axioms, extending also all axiom-schemata to the new language.

Let $\mathbf{EL}^+$ be obtained by adding to **EL** all the (finitely many) functional constants of **M**, with their defining axioms, extending also all axiom-schemata to the new language.

We see that the languages of the extended systems $\mathbf{M}^+$ and $\mathbf{EL}^+$ coincide (with trivial differences). Using the relations between the function existence principles that we have obtained as well as the equivalence of the different definitions of the recursor constant, we arrive at the following:

$$\begin{aligned}\mathbf{EL}^+ + \mathrm{CF_d} &= \mathbf{HA}_1 + \text{fin.list}(\mathbf{M}) + \text{QF-AC}_{00} + \mathrm{CF_d} = \\ &\mathbf{IA}_1 + \mathrm{Rec} + \text{inf.list}(\mathbf{HA}) + \mathrm{AC}_{00}! = \mathbf{M}^+.\end{aligned}$$

THEOREM 5.1. $\mathbf{EL}^+ + \mathrm{CF_d}$ *is a conservative (in fact definitional) extension of* **M**.

PROOF. It suffices to observe that every proof in $\mathbf{EL}^+ + \mathrm{CF_d}$ is done in a finite subsystem of it, so in a definitional extension of **M**. □

THEOREM 5.2. $\mathbf{M}^+$ *is a conservative (in fact definitional) extension of* $\mathbf{EL} + \mathrm{CF_d}$.

COROLLARY 5.3. *The systems* $\mathbf{M}^+$ *and* $\mathbf{EL}^+ + \mathrm{CF_d}$ *essentially coincide, so* $\mathbf{M}$ *and* $\mathbf{EL} + \mathrm{CF_d}$ *are essentially equivalent, in the sense that they have a common conservative extension obtained by definitional extensions.*

TERMINOLOGY. We say that $\mathbf{M}$ and $\mathbf{EL} + \mathrm{CF_d}$ are *definitionally equivalent.*

In [Troelstra1974] p. 585, a result of N. Goodman is mentioned and used, stating that $\mathbf{EL_1}$ is conservative over $\mathbf{HA}$, where $\mathbf{EL_1}$ is $\mathbf{EL}$ + $\mathrm{AC_{01}}$, where $\mathrm{AC_{01}}$ (which entails $\mathrm{AC_{00}}!$) is the countable choice assumed in **FIM**. It follows that $\mathbf{EL}$ + $\mathrm{CF_d}$ is conservative over $\mathbf{HA}$. With our previous results, we obtain the following.

PROPOSITION 5.4. $\mathbf{M}^+$ *is a conservative extension of* $\mathbf{HA}$.

PROPOSITION 5.5. $\mathbf{M}$ *is a conservative extension of* $\mathbf{IA_0}$.

6. Elimination of the symbol λ from EL

6.1. Description of the problem.

6.1.1. From [JRMPhD] it is known that λ can be eliminated from the formal systems that S. C. Kleene set up to formalize parts of intuitionistic analysis, including $\mathbf{M}$. In this proof, the principle $\mathrm{AC_{00}}!$ is used in a substantial way, so this proof is not valid in the case of $\mathbf{EL}$. Here we obtain the corresponding result for $\mathbf{EL}$ by modifying part of this proof.

Let $\mathbf{EL} - \lambda$ be the formal system obtained from $\mathbf{EL}$ by omitting the symbol λ, the corresponding functor formation rule and the axiom-schema of λ-reduction.

Let $\mathbf{S_1}$ be $\mathbf{EL} - \lambda$ and $\mathbf{S_2}$ be $\mathbf{EL}$.

By $\vdash_1$, $\vdash_2$ we denote provability in $\mathbf{S_1}$, $\mathbf{S_2}$, respectively.

As axioms for the recursor constant rec we include in both systems the following:

$$\mathrm{REC}\quad \begin{cases} \mathrm{rec}(\mathrm{x}, \alpha, 0) = \mathrm{x}, \\ \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{S}(\mathrm{y})) = \alpha(\langle \mathrm{rec}(\mathrm{x}, \alpha, \mathrm{y}), \mathrm{y}\rangle). \end{cases}$$

Over both systems, this version of the axioms REC is equivalent by logic to the one used in the definition of $\mathbf{EL}$, with terms and a functor in the places of the number variables and the function variable, respectively.

Both systems $\mathbf{S_1}$ and $\mathbf{S_2}$ include the quantifier-free axiom-schema of numerical choice. We choose to include in both systems a term-version of this principle which is a special case of QF-AC_{00}, but is, as we shall prove, over both systems, equivalent with it:

$$QF_t\text{-}AC_{00} \quad \forall x \exists y\, t(\langle x, y\rangle) = 0 \to \exists \alpha \forall x\, t(\langle x, \alpha(x)\rangle) = 0,$$

where x is free for y in $t(\langle x, y\rangle)$ and α does not occur in $t(\langle x, y\rangle)$.

PROPOSITION. *Over* $\mathbf{HA_1}$ *(and* $\mathbf{IA_1}$*),* QF-AC_{00} *is interderivable with* QF_t-AC_{00}.

PROOF. We first observe that QF_t-AC_{00} is a special case of QF-AC_{00}. The converse is obtained as follows:

Using Fact (vii) of the proof of theorem 3.4, which as we noted there holds also in **EL**, for any quantifier-free formula $A(x, y)$, we can find a term $t(x, y)$ with the same free variables as $A(x, y)$, such that

$$\vdash t(x, y) \leq 1 \ \text{ and } \ \vdash A(x, y) \leftrightarrow t(x, y) = 0.$$

We consider then the term $s(w) \equiv t((w)_0, (w)_1)$, for which we obtain

$$\vdash A(x, y) \leftrightarrow s(\langle x, y\rangle) = 0.$$

Replacing A(x,y) in the hypothesis of QF-AC_{00} by $s(\langle x, y\rangle) = 0$ allows to apply QF_t-AC_{00} and get easily the result.

□

6.2. Eliminating λ from EL.

6.2.1. To obtain λ-eliminability in [JRMPhD] a translation is defined and then it is shown that the elimination relations are satisfied. The translation is the following (the terminology used corresponds to the one we gave for the case of the eliminability of rec but of course now E' is defined differently) :

If P is any prime formula of $\mathbf{S_2}$, then:

· If P has no λ's, then P' is P.

· Otherwise, if $\lambda x.s(x)$ is the first (free) λ-occurrence in P, in which case we use the notation $P[\lambda x.s(x)]$, then

$$P' \equiv \exists \alpha [\forall x [s(x) = \alpha(x)]' \,\&\, [P[\alpha]]'],$$

where $P[\alpha]$ is obtained from P by replacing the occurrence $\lambda x.s(x)$ by α.

The only point of the proof in [JRMPhD] where $AC_{00}!$ is used is in order to obtain

$$\vdash_1 \exists! \alpha \forall x [t(x) = \alpha(x)]'.$$

Here we present some lemmas by which we obtain this result without using $AC_{00}!$, so, after treating the axioms of **EL** not included in the systems of [JRMPhD], we obtain λ-eliminability for **EL**.

REMARK 1. Consider $\mathrm{E} \equiv \forall \mathrm{x} \mathrm{A}(\mathrm{x}) \rightarrow \mathrm{A}(\mathrm{t})$, where $\mathrm{A}(\mathrm{x})$ is any formula of $\mathbf{S_2}$ (with t free for x in $\mathrm{A}(\mathrm{x})$), an axiom of $\mathbf{S_2}$ by the axiom-schema 10N. In the case that t has no λ's, we obtain that $\vdash_1 \mathrm{E}'$, where $\mathrm{E}' \equiv \forall \mathrm{x} [\mathrm{A}(\mathrm{x})]' \rightarrow [\mathrm{A}(\mathrm{t})]'$. We can see this as follows: By choosing the same bound variables for corresponding steps in the elimination process in $\mathrm{A}(\mathrm{x})$ and $\mathrm{A}(\mathrm{t})$, we obtain a formula $\forall \mathrm{x}\, \mathrm{B}(\mathrm{x}) \rightarrow \mathrm{B}(\mathrm{t})$ for the same t and with t free for x in $\mathrm{B}(\mathrm{x})$, but this is an axiom of $\mathbf{S_1}$ which is a congruent of E'.

The general case can be shown only after more results have been obtained. But the present case is now available with the above justification. We will make use of it in the following proofs.

The same argument applies for the $\forall\alpha$-elimination schema, the axiom-schema 10F.

REMARK 2. We use the following notation: If u is a specified occurrence of a functor u in a formula E (or term t), then we write $\mathrm{E}[\mathrm{u}]$ $(\mathrm{t}[\mathrm{u}])$ to indicate it, so $\mathrm{E} \equiv \mathrm{E}[\mathrm{u}]$. When this occurrence is replaced by a function variable, say α, we write $\mathrm{E}[\alpha]$ to denote the result of replacing the occurrence u in E by an occurrence of α; the same for terms.

LEMMA 6.1. *Let* $\mathrm{r}(\mathrm{x})$ *be any term of* $\mathbf{S_2}$ *with* α *not free in it. Then*

$$\vdash_1 \exists\alpha \forall \mathrm{x} [\mathrm{r}(\mathrm{x}) = \alpha(\mathrm{x})]'.$$

PROOF. The proof is by induction on the number q of λ's in $\mathrm{r}(\mathrm{x})$.

CASE $q = 0$. $\mathrm{r}(\mathrm{x})$ has no λ's. Then we have

$$\text{(a)} \quad \vdash_1 \exists! \mathrm{w}\, \mathrm{r}(\mathrm{x}) = \mathrm{w},$$

so

$$\text{(b)} \quad \vdash_1 \forall \mathrm{x} \exists \mathrm{w}\, \mathrm{r}(\mathrm{x}) = \mathrm{w},$$

Applying QF-AC_{00} to (b) we get

$$\text{(c)} \quad \vdash_1 \exists\alpha\forall \mathrm{x}\, \mathrm{r}(\mathrm{x}) = \alpha(\mathrm{x}).$$

The lemma follows from (c) and the definition of the translation $'$.

CASE $q > 0$. Let $\mathrm{r}(\mathrm{x})$ have $q > 0$ λ's, and let $\lambda \mathrm{z}.\mathrm{s}(\mathrm{z}, \mathrm{x})$ be the first λ-occurrence in $\mathrm{r}(\mathrm{x})$, so that

$$\mathrm{r}(\mathrm{x}) \equiv \mathrm{r}(\mathrm{x})[\lambda \mathrm{z}.\mathrm{s}(\mathrm{z}, \mathrm{x})].$$

Then, if

$$\mathrm{E} \equiv \exists\alpha\forall \mathrm{x}\, \mathrm{r}(\mathrm{x}) = \alpha(\mathrm{x}),$$

we have

$$\text{(a)} \quad \mathrm{E}' \equiv \exists\alpha\forall \mathrm{x}\exists\beta [\forall \mathrm{z} [\mathrm{s}(\mathrm{z}, \mathrm{x}) = \beta(\mathrm{z})]' \;\&\; [\mathrm{r}(\mathrm{x})[\beta] = \alpha(\mathrm{x})]'].$$

Assume

$$\text{(b)}\ \ \forall z\,[\,s(z,x) = \beta(z)\,]'$$

and

$$\text{(c)}\ \ \forall x\,[\,r(x)[\,\beta\,] = \alpha(x)\,]'.$$

Then, after ∀x-elimination, we get

$$\text{(d)}\ \ \forall z\,[\,s(z,x) = \beta(z)\,]'\ \&\ [\,r(x)[\,\beta\,] = \alpha(x)\,]'.$$

By the induction hypothesis, we have

$$\text{(e)}\ \ \vdash_1 \exists\beta\forall z\,[\,s(z,x) = \beta(z)\,]'$$

and

$$\text{(f)}\ \ \vdash_1 \exists\alpha\forall x\,[\,r(x)[\,\beta\,] = \alpha(x)\,]'.$$

So with $\exists\beta$-introduction to (d) we get

$$\text{(g)}\ \ \exists\beta\,[\forall z\,[\,s(z,x) = \beta(z)\,]'\ \&\ [\,r(x)[\,\beta\,] = \alpha(x)\,]'\,],$$

from (e) with $\exists\beta$-elimination discharging (b), and with ∀x-introduction (since x is not free in (c)) and then $\exists\alpha$-introduction, with $\exists\alpha$-elimination from (f) discharging (c), we get the right part of the $\leftrightarrow$ in (a), so $\vdash_1 E'$. □

LEMMA 6.2. *Let* t *be any term of* $\mathbf{S_2}$ *with* z *not free in it. Then*

$$\vdash_1 \exists! z\,[\,t = z\,]'.$$

PROOF. The proof is by induction on the number q of λ's in t.

CASE $q = 0$. t has no λ's. Then

$$\vdash_1 \exists! z\ t = z,$$

and the lemma follows by the definition of the translation ′.

CASE $q > 0$. Let t have $q > 0$ λ's, and let $\lambda x.s(x)$ be the first λ-occurrence in t, so that

$$t \equiv t\,[\,\lambda x.s(x)\,]\,.$$

Then,

$$[\,t = z\,]' \equiv \exists\alpha\,[\forall x\,[\,s(x) = \alpha(x)\,]'\ \&\ \ [\,t[\,\alpha\,] = z\,]'\,]\,.$$

Assume

$$\text{(a)}\ \ \forall x\,[\,s(x) = \alpha(x)\,]'$$

and

$$\text{(b)}\ \ [\,t[\,\alpha\,] = z\,]'.$$

By Lemma 6.1,

$$\text{(c)}\ \ \vdash_1 \exists\alpha\forall x\,[\,s(x) = \alpha(x)\,]'$$

and by the inductive hypothesis

$$\text{(d)}\ \ \vdash_1 \exists! z\,[\,t[\,\alpha\,] = z\,]'.$$

By (a), (b), with $\exists\alpha$-introduction we get

$$\text{(e)}\quad [\,\mathrm{t}=\mathrm{z}\,]'.$$

Assume

$$\text{(f)}\quad [\,\mathrm{t}=\mathrm{y}\,]' \equiv \exists\beta\,[\forall \mathrm{x}\,[\,\mathrm{s}(\mathrm{x})=\beta(\mathrm{x})\,]' \;\&\; [\,\mathrm{t}[\,\beta\,]=\mathrm{y}\,]'\,]\,.$$

Assume

$$\text{(g1)}\quad \forall \mathrm{x}\,[\,\mathrm{s}(\mathrm{x})=\beta(\mathrm{x})\,]'$$

and

$$\text{(g2)}\quad [\,\mathrm{t}[\,\beta\,]=\mathrm{y}\,]'.$$

From (a) and (g1) we get $\alpha=\beta$ as follows: From (a) and (g1), by $\forall$-elimination, we get

$$\text{(h1)}\quad [\,\mathrm{s}(\mathrm{x})=\alpha(\mathrm{x})\,]'$$

and

$$\text{(h2)}\quad [\,\mathrm{s}(\mathrm{x})=\beta(\mathrm{x})\,]',$$

respectively. By the inductive hypothesis we have

$$\vdash_1 \exists!\mathrm{z}\,[\,\mathrm{s}(\mathrm{x})=\mathrm{z}\,]',$$

so by (h1), (h2) we get $\alpha(\mathrm{x})=\beta(\mathrm{x})$, so $\forall \mathrm{x}\,\alpha(\mathrm{x})=\beta(\mathrm{x})$ (since x is not free in (a), (g1)), so

$$\text{(i)}\quad \alpha=\beta.$$

By the replacement theorem, from (g2), (i), we get

$$\text{(j)}\quad [\,\mathrm{t}[\,\alpha\,]=\mathrm{y}\,]'.$$

From (b), (j), (d), we get y = z, so after $\exists\beta$-elimination from (f), we get from (e), (f)

$$[\,\mathrm{t}=\mathrm{z}\,]' \;\&\; \forall \mathrm{y}([\,\mathrm{t}=\mathrm{y}\,]' \rightarrow \mathrm{y}=\mathrm{z}),$$

so we get the lemma, after completing the $\exists\alpha$, $\exists$z-eliminations from (c), (d).

□

LEMMA 6.3. *Let* t(x) *be any term of* $\mathbf{S_2}$ *with* α *not free in it. Then*

$$\vdash_1 \exists!\alpha\,\forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x})=\alpha(\mathrm{x})\,]'.$$

PROOF. By Lemma 6.1, we have $\vdash_1 \exists\alpha\,[\forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x})=\alpha(\mathrm{x})\,]'$.
Assume

$$\text{(a)}\quad \forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x})=\alpha(\mathrm{x})\,]'$$

and

$$\text{(b)}\quad \forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x})=\beta(\mathrm{x})\,]'.$$

Using Remark 1, we get

$$\text{(c)}\quad [\,\mathrm{t}(\mathrm{x})=\alpha(\mathrm{x})\,]'$$

and

$$\text{(d)} \quad [\,\mathrm{t}(\mathrm{x}) = \beta(\mathrm{x})\,]'.$$

By Lemma 6.2, (c), (d), we get $\alpha(\mathrm{x}) = \beta(\mathrm{x})$, so $\forall \mathrm{x}\, \alpha(\mathrm{x}) = \beta(\mathrm{x})$ (from (a), (b)), so

$$\text{(e)} \quad \alpha = \beta.$$

So from (a), (b), (e), we get

$$\text{(a)} \quad \forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x}) = \alpha(\mathrm{x})\,]' \;\&\; \forall \beta\,[\forall \mathrm{x}\,[\,\mathrm{t}(\mathrm{x}) = \beta(\mathrm{x})\,]' \to \alpha = \beta\,]\,.$$

So after $\exists\alpha$-introduction, we get (with $\exists\alpha$-elimination) the lemma.

□

We have also to show that the axioms and axiom-schemas of **EL** that are not included in the systems of [JRMPhD] become under the translation theorems of $\mathbf{EL} - \lambda$.

· The case of the axioms REC is trivial, since there are no λ-occurrences in them.

· The case of the axiom-schema $\mathrm{QF_t}$-$\mathrm{AC_{00}}$ is treated as follows.

LEMMA 6.4. *Let*

$$\mathrm{E} \equiv \forall \mathrm{x} \exists \mathrm{y}\, \mathrm{t}(\langle \mathrm{x}, \mathrm{y} \rangle) = 0 \to \exists \alpha \forall \mathrm{x}\, \mathrm{t}(\langle \mathrm{x}, \alpha(\mathrm{x}) \rangle) = 0$$

be an instance of $\mathrm{QF_t}$-$\mathrm{AC_{00}}$ *in* $\mathbf{S_2}$. *Then* $\vdash_1 \mathrm{E}'$.

PROOF. We have that

$$\mathrm{E}' \equiv \forall \mathrm{x} \exists \mathrm{y}\; [\,\mathrm{t}\,(\langle \mathrm{x}, \mathrm{y} \rangle) = 0\,]' \;\to \exists \alpha \forall \mathrm{x}\; [\,\mathrm{t}(\langle \mathrm{x}, \alpha(\mathrm{x}) \rangle) = 0\,]'.$$

By Lemma 6.1,

$$(a) \quad \vdash_1 \exists \beta\, \forall \mathrm{w}\,[\,\mathrm{t}(\mathrm{w}) = \beta(\mathrm{w})\,]'.$$

Assume

$$\text{(a1)} \quad \forall \mathrm{w}\,[\,\mathrm{t}(\mathrm{w}) = \beta(\mathrm{w})\,]'.$$

Assume

$$\text{(b)} \quad \forall \mathrm{x} \exists \mathrm{y} [\,\mathrm{t}(\langle \mathrm{x}, \mathrm{y} \rangle) = 0\,]'\,,$$

so we get

$$\text{(c)} \quad \exists \mathrm{y}\,[\,\mathrm{t}(\langle \mathrm{x}, \mathrm{y} \rangle) = 0\,]'\,.$$

Assume

$$\text{(c1)} \quad [\,\mathrm{t}(\langle \mathrm{x}, \mathrm{y} \rangle) = 0\,]'\,.$$

By (a1) with $\forall$-elimination,

$$\text{(a2)} \quad [\,\mathrm{t}(\langle \mathrm{x}, \mathrm{y} \rangle) = \beta(\langle \mathrm{x}, \mathrm{y} \rangle)\,]'\,.$$

By (c1), (a2), Lemma 6.2, $\beta(\langle \mathrm{x}, \mathrm{y} \rangle) = 0$, so

$$\text{(d)} \quad \exists \mathrm{y}\, \beta(\langle \mathrm{x}, \mathrm{y} \rangle) = 0$$

from (c) with ∃-elimination. So

$$\text{(e)}\ \ \forall x \exists y\, \beta(\langle x, y\rangle) = 0$$

(x not free in (b)). Now by QF_t-AC_{00},

$$\text{(f)}\ \ \exists \alpha \forall x\, \beta(\langle x, \alpha(x)\rangle) = 0.$$

Assume

$$\text{(g)}\ \ \forall x\, \beta(\langle x, \alpha(x)\rangle) = 0.$$

By (a1), ∀-elimination, and with an induction on the number of λ's in t using Lemma 6.3, we get

$$\text{(h)}\ \ \forall x\,[\, t(\langle x, \alpha(x)\rangle) = 0\,]',$$

so

$$\text{(i)}\ \ \exists \alpha \forall x\,[\, t(\langle x, \alpha(x)\rangle) = 0\,]',$$

from (f) with $\exists\alpha$-elimination. So from (b) we get E′. After completing the $\exists\beta$-elimination from (a) we get $\vdash_1$ E′.

□

Before stating the concluding result, we make an observation motivated by the question of Iris Loeb whether this method of elimination would have worked for **M**: all arguments in this section apply equally well for the system $\mathbf{IA}_1$ + QF-AC_{00} in the place of **EL**. As suggested by J. Rand Moschovakis, we include the corresponding elimination result in the final theorem of the section.

THEOREM 6.5. *(a) The system* **EL** *is a definitional extension of the system* $\mathbf{EL} - \lambda$.

(b) The system $\mathbf{IA_1}$ + QF-AC_{00} *is a definitional extension of the system* $\mathbf{IA_1}$ + QF-$AC_{00} - \lambda$.

7. The formal systems BIM, H and WKV

7.1. The formal system BIM. The formal system **BIM** of Basic Intuitionistic Mathematics has been introduced by W. Veldman to serve as a basis for the development of intuitionistic mathematics, and especially intuitionistic reverse mathematics. We give its description, as presented in the paper "Brouwer's Approximate Fixed-Point Theorem is Equivalent to Brouwer's Fan Theorem" [Veldman].

BIM is based on two-sorted intuitionistic predicate logic, with number and function variables, as are all the systems in the present study. It has a numerical constant 0, unary function constants $\underline{0}$ for the function with constant value 0, S for the successor and K and L for the projection functions, and a binary function constant J for the pairing function. Terms and function terms (correspond to the functors with the terminology we use) are as usual. There are equality symbols $=_0$ for numerical terms and $=_1$ for function terms. The function equality is in fact defined in terms of number equality, as the following axiom is assumed:

Axiom of Extensionality : $\forall\alpha\forall\beta\,[\,\alpha =_1 \beta \leftrightarrow \forall \mathrm{n}\,[\,\alpha(\mathrm{n}) =_0 \beta(\mathrm{n})\,]\,]$.

Assumed are the *Axioms on the function constants*:

$\forall \mathrm{n}\,[\neg(\mathrm{S}(\mathrm{n}) = 0)\,]$, $\forall \mathrm{m}\forall \mathrm{n}[\,\mathrm{S}(\mathrm{m}) = \mathrm{S}(\mathrm{n}) \to \mathrm{m} = \mathrm{n}\,], \forall \mathrm{n}[\,\underline{0}(\mathrm{n}) = 0\,]$,

$\forall \mathrm{m}\forall \mathrm{n}\,[\,\mathrm{K}(\mathrm{J}(\mathrm{m},\mathrm{n})) = \mathrm{m}\ \&\ \mathrm{L}(\mathrm{J}(\mathrm{m},\mathrm{n})) = \mathrm{n}\,]$.

Also the following function existence principles are assumed, all of them expressed by single axioms:

Composition :
$\forall\alpha\forall\beta\exists\gamma\forall \mathrm{n}\,[\,\gamma(\mathrm{n}) = \alpha(\beta(\mathrm{n}))\,]$
Primitive Recursion :
$\forall\alpha\forall\beta\exists\gamma\forall \mathrm{m}\forall \mathrm{n}\,[\,\gamma(\mathrm{m},0) = \alpha(\mathrm{m})\ \&\ \gamma(\mathrm{m},\mathrm{S}(\mathrm{n})) = \beta(\mathrm{m},\mathrm{n},\gamma(\mathrm{m},\mathrm{n}))\,]$
Unbounded Search :
$\forall\alpha\,[\,\forall \mathrm{m}\exists \mathrm{n}\,\alpha(\mathrm{m},\mathrm{n}) = 0 \to \exists\gamma\forall \mathrm{m}\,\alpha(\mathrm{m},\gamma(\mathrm{m})) = 0\,]$

The *Axiom-schema of Induction* is also assumed:

$$[\mathrm{A}(0)\ \&\ \forall \mathrm{n}\,(\mathrm{A}(\mathrm{n}) \to \mathrm{A}(\mathrm{S}(\mathrm{n})))] \to \forall \mathrm{n}\mathrm{A}(\mathrm{n}).$$

It is also assumed that constants for primitive recursive functions and their defining equations are added to the system. These additions are definitional; the method exposed in [IM] §74, applies directly in this case.

In order to examine the relation of this system with the others that we have considered, we first observe the following:

PROPOSITION 7.1. *Over* **BIM**, *the schema* $\mathrm{CF_d}$ *entails* $\mathrm{AC_{00}}!$.

PROOF. The proof is similar to that of Theorem 3.2 since we have the decidability of number equality (proved by the axiom-schema of Induction), but using the constants J, K, L instead of the pairing and projection functions of $\mathbf{IA_1}$, and the axiom of Unbounded Search instead of QF-$\mathrm{AC_{00}}$.

□

In fact, over **BIM** without the axiom of Unbounded Search, the conjunction of the axiom of Unbounded Search with the schema $\mathrm{CF_d}$ is equivalent to $\mathrm{AC_{00}}!$. The one direction is given by the preceding proposition, and for the converse, $\mathrm{AC_{00}}!$ entails $\mathrm{CF_d}$ like in Proposition 3.1, and it is easy to see that $\mathrm{AC_{00}}!$ entails the axiom of Unbounded

Search if we add (conservatively) addition $+$ to **BIM** so that $<$ can be expressed efficiently and adapt the proof of Proposition 2.7.

The result of [JRMPhD] on λ-eliminability applies to any formal system of the sort that we are studying which has function constants for $0, ', +, \cdot$ and assumes $AC_{00}!$. The system **BIM** can be extended definitionally to a system **BIM′**, so that it includes all these constants. Consequently, the λ symbol with the related formation rule and axiom-schema can be added definitionally to the system **BIM′** + CF_d. Consider now the system obtained by the addition of λ, **BIM′** + $CF_d + \lambda$. In this system, thanks to the presence of λ, we can easily obtain Lemma 5.3(b) of [FIM] from the Axiom of Primitive Recursion. By the same method that the recursor constant rec is added to **M**, all the additional function and functional constants of **M** can be added definitionally, successively according to their primitive recursive descriptions, to this extended system. Let $\mathbf{BIM}^+$ be the result of adding to **BIM** λ and all the additional constants of **M**, with their formation rules and axioms. Let $\mathbf{M}^j$ be the trivial extension of **M** obtained by adding the constants J, K, L (these additions are also definitional). Then $\mathbf{BIM}^+$ + CF_d and $\mathbf{M}^j$ coincide up to trivial differences, and we have then the following:

Theorem 7.2. $\mathbf{M}^\mathbf{j}$ *is a definitional extension of* **BIM** + CF_d.

We conclude that **BIM** + CF_d and **M** are essentially equivalent, and also essentially equivalent with **EL** + CF_d. It is remarkable that, in developing the theory within **BIM**, Veldman defined a set of natural numbers to be decidable if it has a characteristic function, as this means that CF_d is implicitly assumed.

Theorem 7.3. **BIM** *does not prove* CF_d.

Proof. This is obtained as in the case of **EL**, Theorem 3.4, interpreting the function variables as varying over the general recursive functions.

□

7.2. The formal system H. In the paper "Transfinite Induction and Bar Induction of Types Zero and One and the Role of Continuity in Intuitionistic Analysis" of Howard and Kreisel ([Howard-Kreisel]), the formal system of *elementary intuitionistic analysis* **H** is used. **H** differs from **BIM** only in that it does not assume the axiom of unbounded search (it has also another difference, which is inessential: it does not use only one-place number-theoretic functions like **BIM**).

Like $\mathbf{IA}_1$ and $\mathbf{HA}_1$, **H** has a classical model consisting of the primitive recursive functions. Essentially **H** is a proper subtheory of **BIM**.

7.3. The formal system WKV. In the paper "Equivalents of the (Weak) Fan Theorem" [Loeb], Iris Loeb presents and uses the formal system **WKV** (for "Weak Kleene-Vesley"). This is again based on two-sorted intuitionistic predicate logic with number and function variables, and constants 0, S, +, ·, =, j (pairing) and j_1, j_2 (projections). The system has also the symbol λ and parentheses for function application, and corresponding formation rules for terms and functors. The axioms are the usual for the function constants and number equality, the λ-conversion axiom-schema, the axiom-schema of mathematical induction, the axiom-schema of primitive recursion in the version

$$\exists\beta\,[\,\beta(0) = \mathrm{t}\ \&\ \forall \mathrm{y}\,\beta(\mathrm{S}(\mathrm{y})) = \mathrm{r}(\mathrm{j}(\mathrm{y}, \beta(\mathrm{y})))\,]\,,$$

where t is a term and r a functor. The axiom-schema $AC_{00}!$ is also assumed. We observe that the assumed version of the axiom-schema of primitive recursion is very similar to Lemma 5.3(b) of [FIM]; in fact, as it is easily seen, these schemas are equivalent (modulo the pairing) over both **WKV** and **M**. Using the method of the addition of the recursor constant rec to **M**, all constants of **M** can, successively according to their primitive recursive descriptions, be added definitionally to **WKV**. So, if we consider the trivial extension $\mathbf{M}^j$ of **M** by the pairing and projections of **WKV** with their axioms, we conclude that the two systems are essentially equivalent.

THEOREM 7.4. $\mathbf{M}^{\mathrm{j}}$ *is a definitional extension of* **WKV**.

8. Concluding observations

We close Part 1 by collecting the main results following from the arguments we presented.

I. In relation to the small classical models that the weak constructive systems we studied admit:

(a) The systems **H**, $\mathbf{IA}_1$, $\mathbf{HA}_1$, $\mathbf{IA}_1$ + Rec have the primitive recursive functions as a classical model.

(b) The systems resulting from adding QF-AC_{00} to the systems of (a) as well as **BIM**, have the general recursive functions as a classical model.

(c) Adding CF_d to the systems of (b) gives stronger systems, which do not have small classical models consisting of recursive functions.

II. The systems of each group are essentially equivalent between them:

(a) $\mathbf{IA}_1$ + Rec, $\mathbf{HA}_1$.
(b) $\mathbf{IA}_1$ + QF-AC_{00}, **EL**.

(c) **M**, **WKV**, **EL** + $\mathrm{CF_d}$, **BIM** + $\mathrm{CF_d}$, **H** + $\mathrm{AC_{00}}$!.
(d) **BIM**, **H** + *Unbounded Search*.

III. We observe also the following.

(a) Adding QF-$\mathrm{AC_{00}}$ to any of **H**, $\mathbf{IA}_1$, $\mathbf{HA}_1$ gives stronger systems.

(b) Adding QF-$\mathrm{AC_{00}}$ to any of **H** + $\mathrm{CF_d}$, $\mathbf{IA}_1$ + $\mathrm{CF_d}$, $\mathbf{HA}_1$ + $\mathrm{CF_d}$ gives stronger systems (J. Rand Moschovakis, [JRM-GV2012]).

(c) Adding $\mathrm{CF_d}$ to any of $\mathbf{HA}_1$, **EL**, **BIM** gives stronger systems.

Almost all results follow easily from the arguments presented so far. The only point that needs some more justification is the relation in II(b). We give it next.

In proving that the recursor constant rec can be added definitionally to **M**, the axiom-schema $\mathrm{AC_{00}}$! has been used in two cases (except from the point where it is shown that the translation of $\mathrm{AC_{00}}$! itself is a theorem of **M**): in the proof of Lemma 5.3(b) of [FIM], and in the proof of Lemma 4.4. We will prove these two lemmas by using only QF-$\mathrm{AC_{00}}$. In this way, together with the observation that, thanks to the presence of λ, QF-$\mathrm{AC_{00}}$ is equivalent over $\mathbf{IA}_1$ with the single axiom

$$\forall\alpha\,[\,\forall \mathrm{x}\exists \mathrm{y}\,\alpha(\langle \mathrm{x},\mathrm{y}\rangle)=0\rightarrow\exists\gamma\forall \mathrm{x}\,\alpha(\langle \mathrm{x},\gamma(\mathrm{x})\rangle)=0\,],$$

we obtain the fact that rec can be added definitionally to the system $\mathbf{IA}_1$ + QF-$\mathrm{AC_{00}}$.

LEMMA 8.1. *In* $\mathbf{IA_1}$ + QF-$\mathrm{AC_{00}}$,

$$\vdash\exists\beta\,[\,\beta(0)=\mathrm{x}\ \&\ \forall \mathrm{z}\,\beta(\mathrm{z}')=\alpha(\langle\beta(\mathrm{z}),\mathrm{z}\rangle)\,].$$

PROOF. We slightly modify the proof of Lemma 5.3(b) of [FIM] as follows.
Let

$$\mathrm{P}(\mathrm{x},\alpha,\mathrm{y},\mathrm{v})\equiv(\mathrm{v})_0=\mathrm{x}\ \&\ \forall \mathrm{i}<\mathrm{y}\ (\mathrm{v})_{\mathrm{i}'}=\alpha(\langle(\mathrm{v})_\mathrm{i},\mathrm{i}\rangle).$$

By formal induction (IND) on y we show first

$$\text{(a)}\quad\vdash\forall \mathrm{y}\exists \mathrm{v}\,\mathrm{P}(\mathrm{x},\alpha,\mathrm{y},\mathrm{v})$$

as follows:

BASIS. We get $\exists \mathrm{v}\,\mathrm{P}(\mathrm{x},\alpha,0,\mathrm{v})$ by "setting" $\mathrm{v}=\mathrm{p}_0^{\mathrm{x}}$.

INDUCTIVE STEP. Assuming $\mathrm{P}(\mathrm{x},\alpha,\mathrm{y},\mathrm{w})$ and "setting"

$$\mathrm{v}=\Pi_{\mathrm{i}\le \mathrm{y}}\mathrm{p}_\mathrm{i}^{(\mathrm{w})_\mathrm{i}}*\mathrm{p}_{\mathrm{y}'}^{\alpha(\langle(\mathrm{w})_\mathrm{y},\mathrm{y}\rangle)},$$

since then

$$\forall \mathrm{i}\le \mathrm{y}\ (\mathrm{v})_\mathrm{i}=(\mathrm{w})_\mathrm{i}\ \&\ (\mathrm{v})_{\mathrm{y}'}=\alpha(\langle(\mathrm{w})_\mathrm{y},\mathrm{y}\rangle),$$

we get

$$\exists \mathrm{v}\, \mathrm{P}(\mathrm{x}, \alpha, \mathrm{y}', \mathrm{v})$$

from the inductive hypothesis.

Using f_{15}, $\mathrm{P}(\mathrm{x}, \alpha, \mathrm{y}, \mathrm{v})$ is equivalent over $\mathbf{IA}_1$ to a quantifier-free formula (with the same free variables). So we can apply QF-AC_{00} to (a) and get

$$\text{(b)}\quad \exists \gamma \forall \mathrm{y}\, \mathrm{P}(\mathrm{x}, \alpha, \mathrm{y}, \gamma(\mathrm{y})).$$

Assume now

$$\text{(c)}\quad \forall \mathrm{y}\, \mathrm{P}(\mathrm{x}, \alpha, \mathrm{y}, \gamma(\mathrm{y})).$$

Define $\beta = \lambda \mathrm{y}.\gamma(\mathrm{y})_{\mathrm{y}}$ (justified by Lemma 5.3(a) of [FIM]). Then we can show

$$\text{(d)}\quad \exists \beta\, [\, \beta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)\,]$$

(for the second conjunct in (d), specialize from (c) for z' and $\mathrm{i} = \mathrm{z}$ ($\mathrm{z} < \mathrm{z}'$), and get $\gamma(\mathrm{z}')_{\mathrm{z}'} = \alpha(\langle \gamma(\mathrm{z}')_{\mathrm{z}}, \mathrm{z}\rangle)$. But $\gamma(\mathrm{z}')_{\mathrm{z}} = \gamma(\mathrm{z})_{\mathrm{z}}$ (by induction on z, using (c)), so $\beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)$).

□

Lemma 8.2. *In* $\mathbf{IA_1}$ + QF-AC_{00},

$$\vdash \exists ! \beta\, [\, \beta(0) = \mathrm{x}\ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = \alpha(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)\,]\,.$$

Proof. Same as the proof of Lemma 4.1.

□

In the next lemma the notation and abbreviations are as in Lemma 4.4.

Lemma 8.3. *Let* t, s *be terms and* u *a functor of* $\mathbf{IA_1}$. *Let* $\mathrm{x}_0, \ldots, \mathrm{x}_k$ *include all the number variables occurring free in* t, u *or* s, *let* w *be a number variable not occurring in* t, u, s, *distinct from* $\mathrm{x}_0, \ldots, \mathrm{x}_k$, *and* γ *a function variable free for* v *in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$, *not occurring free in* $\mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v})$. *Then in* $\mathbf{IA_1}$ + QF-AC_{00}

$$\vdash_1 \exists ! \gamma \forall \mathrm{w} \mathrm{A}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \gamma(\mathrm{w})).$$

Proof. We want to show

$$\vdash \exists ! \gamma \forall \mathrm{w} \exists \beta\, [\, \beta(0) = \mathrm{t}^{\mathrm{w}}\ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = (\mathrm{u}^{\mathrm{w}})(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)\ \&\ \beta(\mathrm{s}^{\mathrm{w}}) = \gamma(\mathrm{w}))\,]\,.$$

From (a) in the proof of Lemma 1.1 we have

$$\vdash \forall \mathrm{w} \exists \mathrm{v}\, \mathrm{P}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \mathrm{v}).$$

Applying QF-AC_{00} we get

$$\text{(a)}\quad \vdash \exists \delta \forall \mathrm{w}\, \mathrm{P}(\mathrm{t}^{\mathrm{w}}, \mathrm{u}^{\mathrm{w}}, \mathrm{s}^{\mathrm{w}}, \delta(\mathrm{w})).$$

Assume

$$\text{(a1)}\quad \forall \mathrm{w}\, [\, (\delta(\mathrm{w}))_0 = \mathrm{t}^{\mathrm{w}}\ \&\ \forall \mathrm{i} < \mathrm{s}^{\mathrm{w}} (\delta(\mathrm{w}))_{\mathrm{i}'} = (\mathrm{u}^{\mathrm{w}})(\langle (\delta(\mathrm{w}))_{\mathrm{i}}, \mathrm{i}\rangle)\,]\,,$$

and let

$$(*)\quad \gamma = \lambda \mathrm{w}.(\delta(\mathrm{w}))_{\mathrm{s}^{\mathrm{w}}}.$$

By (a1), we get

$$\text{(a2)}\quad (\delta(\mathrm{w}))_0 = \mathrm{t}^{\mathrm{w}} \ \&\ \forall \mathrm{i} < \mathrm{s}^{\mathrm{w}} (\delta(\mathrm{w}))_{\mathrm{i}'} = (\mathrm{u}^{\mathrm{w}})(\langle(\delta(\mathrm{w}))_{\mathrm{i}}, \mathrm{i}\rangle).$$

From Lemma 1.2,

$$\text{(b)}\quad \vdash \forall \mathrm{w} \exists! \beta\, [\, \beta(0) = \mathrm{t}^{\mathrm{w}} \ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = (\mathrm{u}^{\mathrm{w}})(\langle \beta(\mathrm{z}), \mathrm{z}\rangle)\,]\,.$$

Assume

$$\text{(b1)}\quad \beta(0) = \mathrm{t}^{\mathrm{w}} \ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = (\mathrm{u}^{\mathrm{w}})(\langle \beta(\mathrm{z}), \mathrm{z}\rangle).$$

Then

$$\text{(c)}\quad \beta(\mathrm{s}^{\mathrm{w}}) = (\delta(\mathrm{w}))_{\mathrm{s}^{\mathrm{w}}}$$

(to get (c) we prove $\forall \mathrm{i} \leq \mathrm{s}^{\mathrm{w}}\ \beta(\mathrm{i}) = (\delta(\mathrm{w}))_{\mathrm{i}}$ by formal induction (IND) on i), so

$$\text{(d)}\quad \gamma(\mathrm{w}) = \beta(\mathrm{s}^{\mathrm{w}})\,.$$

By $\rightarrow$-elimination discharging (b1) with $\forall\beta$- and $\forall$w-introductions we get

$$\text{(e)}\quad \forall \mathrm{w} \forall \beta\, [\, \text{(b1)} \rightarrow \gamma(\mathrm{w}) = \beta(\mathrm{s}^{\mathrm{w}})\,].$$

From (b) and (e) we get

$$\text{(f)}\quad \forall \mathrm{w} \exists \beta\, [\, \beta(0) = \mathrm{t}^{\mathrm{w}} \ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = (\mathrm{u}^{\mathrm{w}})(\langle \beta(\mathrm{z}), \mathrm{z}\rangle) \ \&\ \beta(\mathrm{s}^{\mathrm{w}}) = \gamma(\mathrm{w})\,]\,.$$

Assume

$$\text{(g)}\quad \forall \mathrm{w} \exists \beta\, [\, \beta(0) = \mathrm{t}^{\mathrm{w}} \ \&\ \forall \mathrm{z}\, \beta(\mathrm{z}') = (\mathrm{u}^{\mathrm{w}})(\langle \beta(\mathrm{z}), \mathrm{z}\rangle) \ \&\ \beta(\mathrm{s}^{\mathrm{w}}) = \varepsilon(\mathrm{w})\,]\,.$$

Then from (e) and (f) we get easily

$$\text{(h)}\quad \forall \mathrm{w}\, \varepsilon(\mathrm{w}) = \gamma(\mathrm{w}).$$

So from (g) we get (i) $\varepsilon = \gamma$, and then $\forall\varepsilon$ ((g) $\rightarrow$ (i)). So with (f) and $\exists\gamma$-introduction we get the lemma, after completing the $\exists\gamma$, $\exists\delta$-eliminations discharging $(*)$ and (a1), respectively.

□

Part 2

CHAPTER 2

A study of uniqueness principles

Introduction

Intuitionistic logic does not determine in a unique way the interpretation of quantifiers (and combinations of them). Consequently, several principles are introduced, reflecting the particular ways of reasoning accepted by the various tendencies of constructivism. Among them are versions of countable choice, continuity principles, versions of Church's Thesis and many other.

Several of these principles have a hypothesis of the form $\forall\exists$, and are usually formulated in two versions, one for numbers, with the existential quantifier being a number quantifier, and the corresponding one for functions. In general, the versions for numbers are weaker than the ones for functions. In some cases, the versions for functions are considered according to some constructivists' views as unnecessarily strong, or even as problematic as some compatibility questions arise. One such example is the strong continuous choice principle CC_1 assumed by **FIM**, and other formal systems for analysis. S. C. Kleene proposed this schema to formulate Brouwer's principle of continuous choice. While the number version CC_0 of this principle is compatible with Kripke's Schema, a principle reflecting Brouwer's "creative subject" arguments, the functional version CC_1 contradicts it.

But there is a remarkable phenomenon that we meet: When uniqueness versions are considered, namely when in the hypotheses the existential quantifier is replaced by the corresponding unique existential quantifier $\exists!$, the function and number versions become equivalent (over a weak theory like **M**). We note that under the constructive interpretation, the combination $\forall x\exists y$ implicitly asserts that, for each x, a particular y can be specified, such that a certain property of the pair (x,y) holds.

For an example, consider the principle of countable choice

$$\mathrm{AC}_{01}\quad \forall \mathrm{x}\exists\alpha \mathrm{A}(\mathrm{x},\alpha) \rightarrow \exists\alpha\forall \mathrm{x}\mathrm{A}(\mathrm{x},\lambda \mathrm{y}.\alpha(\langle \mathrm{x},\mathrm{y}\rangle)),$$

where x, y are distinct and x is free for α in $\mathrm{A}(\mathrm{x},\alpha)$. This is the countable choice assumed by **FIM**. AC_{00} is the corresponding number version and it is weaker than AC_{01} (for this result we refer to the related observation in [Howard-Kreisel], p. 347). Consider the uniqueness versions $AC_{01}!$ and $AC_{00}!$. In [JRMPhD], J. R. Moschovakis proved that they are equivalent over **M**. The situation is similar for CC_1 and the weaker CC_0. Their uniqueness versions are equivalent over **M** ([FIM], p. 89, and [JRMPhD]).

In this chapter we show similar results for the uniqueness versions of Troelstra's Generalized Continuity principle, of the principle of Weak Continuity and of the principle of Strong Extensionality. We consider also some non-constructive principles like Markov's Principle, and special cases of the principle of the excluded middle, we observe that they are equivalent to versions with some uniqueness condition and we obtain various derivability and underivability results. In the same spirit we consider a uniqueness version of Vesley's schema and finally we combine some of our results with the principle of Independence of Premise.

1. Generalized Continuity

The strong continuous choice principle CC_1 ([FIM] p. 73) and its number version CC_0 assert that every total function from Baire space (with the topology of the initial segments) to Baire space or the natural numbers is continuous and guarantee the existence of a neighborhood function, a function which "recognizes" when a finite initial segment of a choice sequence suffices to determine the value of the function at this choice sequence, and then provides the value. In [FIM] p. 74, it is shown that strong continuity applies as well to functions whose domain is any spread (a characteristically intuitionistic notion of set that corresponds to closed subsets of Baire space). But it cannot be extended to functions whose domain is an arbitrary species (the intuitionistic analog of a set defined by a property) of choice sequences. A. S. Troelstra ([Troelstra1973], 3.3.9) formulated the Generalized Continuity principle, by extending strong continuity to functions whose domain is expressed by formulas of a certain syntactic form, and used it to characterize Kleene's function realizability.

To state the principle of Generalized Continuity we need the notion of *almost negative formula*:

A formula is called *almost negative* if it contains no $\vee$ and no $\exists$, except in parts of the form $\exists \mathrm{x}\, \mathrm{P}(\mathrm{x})$ with $\mathrm{P}(\mathrm{x})$ prime or of the form $\exists \alpha\, \mathrm{P}(\alpha)$ with $\mathrm{P}(\alpha)$ prime.

We will use also the following definitions and abbreviations related to the neighborhood functions. S. C. Kleene defined the following partial recursive functions of σ and α in order to code continuous functionals from sequences to numbers and from sequences to sequences, respectively:

$$\{\sigma\}(\alpha) \simeq \sigma(\overline{\alpha}(\mu y\, \sigma(\overline{\alpha}(y)) > 0)) \dot{-} 1,$$

and

$$\{\sigma\}[\,\alpha\,] \simeq \lambda t.\sigma(2^{t+1} * \overline{\alpha}(y_t)) \dot{-} 1,$$

where $y_t \simeq \mu y\, \sigma(2^{t+1} * \overline{\alpha}(y)) > 0$.

Abbreviations: we follow the formulations of [FIM] and we add only the condition $\sigma(1) = 0$ where 1 is the code number of the empty sequence to the second abbreviation below, so that we agree essentially with Troelstra's versions.

$\{\sigma\}(\alpha)\downarrow$ & $A(\alpha, \{\sigma\}(\alpha))$ is an abbreviation for

$$\exists y\, [\, \sigma(\overline{\alpha}(y)) > 0 \;\&\; \forall x(\sigma(\overline{\alpha}(x)) > 0 \rightarrow y = x) \;\&\; A(\alpha, \sigma(\overline{\alpha}(y)) \dot{-} 1)\,],$$

and $\{\sigma\}[\,\alpha\,]\downarrow$ & $A(\alpha, \{\sigma\}[\,\alpha\,])$ is an abbreviation for

$$\forall t \exists! y \sigma(2^{t+1} * \overline{\alpha}(y)) > 0 \;\&\; \sigma(1) = 0 \;\&$$
$$\forall \beta\, [\, \forall t \exists y \sigma(2^{t+1} * \overline{\alpha}(y)) = \beta(t) + 1 \rightarrow A(\alpha, \beta)\,].$$

The principle of Generalized Continuity is expressed by the following schema:

$$GC_1 \quad \forall \alpha\, [\, A(\alpha) \rightarrow \exists \beta\, B(\alpha, \beta)\,]$$
$$\rightarrow \exists \sigma \forall \alpha\, [\, A(\alpha) \rightarrow \{\sigma\}\,[\,\alpha\,]\downarrow \;\&\; B(\alpha, \{\sigma\}\,[\,\alpha\,])\,],$$

where $A(\alpha)$ is almost negative and β does not occur free in $A(\alpha)$, and σ does not occur free in $A(\alpha) \rightarrow \exists \beta\, B(\alpha, \beta)$ and σ, α are free for β in $B(\alpha, \beta)$.

We will show that the uniqueness version $GC_1!$ which is like GC_1 with $\exists! \beta$ instead of $\exists \beta$, and the number version

$$GC_0! \quad \forall \alpha\, [\, A(\alpha) \rightarrow \exists! x\, B(\alpha, x)\,]$$
$$\rightarrow \exists \sigma \forall \alpha\, [\, A(\alpha) \rightarrow \{\sigma\}(\alpha)\downarrow \;\&\; B(\alpha, \{\sigma\}(\alpha))\,],$$

where $A(\alpha)$ is almost negative and x does not occur free in $A(\alpha)$, and and σ does not occur free in $A(\alpha) \rightarrow \exists! x\, B(\alpha, x)$ and σ, α are free for x in $B(\alpha, x)$, are equivalent over **M**.

In [FIM] p. 89, S. C. Kleene remarks that the uniqueness version of the strong principle of continuous choice for functions $CC_1!$ (his $^*27.1'$) is derivable from the strong continuity principle for numbers CC_0 (his $^*27.2$), and gives a hint for the proof. In fact, using the same hint,

$CC_1!$ can be derived in **M** from the weaker principle for numbers with uniqueness $CC_0!$. This proof can be found in [JRMPhD].

We use the same method to get the corresponding result for the case of generalized continuity.

PROPOSITION 1.1. $GC_1!$ *is derivable in* **M** *from* $GC_0!$.

PROOF. We give here a description of a formal proof in **M** of the above fact. First from

$$\text{(a)}\ \ \forall\alpha\,[\,A(\alpha)\to\exists!\beta\,B(\alpha,\beta)\,]$$

we prove

$$\text{(b)}\ \ \forall\alpha\,[\,A(\alpha)\to\forall x\exists!y\exists\beta\,[\,B(\alpha,\beta)\ \&\ \beta(x)=y\ \&\ \forall\gamma(B(\alpha,\gamma)\to\beta=\gamma)]]$$

as follows.

Assume $A(\alpha)$. Then from (a), specializing for α and with modus ponens, we can assume for $\exists$-elimination

$$\text{(a1)}\ \ B(\alpha,\beta)\ \&\ \forall\gamma(B(\alpha,\gamma)\to\beta=\gamma).$$

So taking the conjunction with $\beta(x)=\beta(x)$, we have

$$\text{(a2)}\ \ B(\alpha,\beta)\ \&\ \beta(x)=\beta(x)\ \&\ \forall\gamma(B(\alpha,\gamma)\to\beta=\gamma),$$

so, by $\exists$-introduction, we have

$$\text{(a3)}\ \ \exists\delta\,[\,B(\alpha,\delta)\ \&\ \delta(x)=\beta(x)\ \&\ \forall\gamma(B(\alpha,\gamma)\to\delta=\gamma)\,].$$

Assume now

$$\text{(a4)}\ \ \exists\delta\,[\,B(\alpha,\delta)\ \&\ \delta(x)=z\ \&\ \forall\gamma(B(\alpha,\gamma)\to\delta=\gamma)\,].$$

Assume for $\exists$-elimination

$$\text{(a5)}\ \ B(\alpha,\delta)\ \&\ \delta(x)=z\ \&\ \forall\gamma(B(\alpha,\gamma)\to\delta=\gamma).$$

By (a2), we have $\forall\gamma(B(\alpha,\gamma)\to\beta=\gamma)$, so by $B(\alpha,\delta)$ (from (a5)) we get $\beta=\delta$, so $\beta(x)=\delta(x)$, but by (a5) $\delta(x)=z$, so

$$\text{(a6)}\ \ \beta(x)=z.$$

So, completing the $\exists$-elimination in (a5) and with $\forall$-introduction we get

$$\text{(a7)}\ \ \forall z(\exists\delta\,[\,B(\alpha,\delta)\ \&\ \delta(x)=z\ \&\ \forall\gamma(B(\alpha,\gamma)\to\delta=\gamma)\,]\to\beta(x)=z).$$

So by (a3), (a7) and $\exists$-introduction we get

$$\text{(a8)}\ \ \exists!y\exists\delta\,[\,B(\alpha,\delta)\ \&\ \delta(x)=y\ \&\ \forall\gamma(B(\alpha,\gamma)\to\delta=\gamma)].$$

By $\forall$-introduction and changing the bound variable δ to β we get the conclusion of the implication of (b), and with $\to$ and $\forall$-introduction we get (b).

By (b), following Kleene's hint in FIM, we specialize for $\lambda t.\alpha(t+1)$ and $\alpha(0)$ and then with $\forall$-introduction we get

$$\text{(c)}\ \ \forall\alpha\,[\,A'(\alpha) \to \exists! y B'(\alpha, y)\,],$$

where

$$A'(\alpha) \equiv A(\lambda t.\alpha(t+1))$$

and

$$B'(\alpha, y) \equiv \exists\beta\,[\,B(\lambda t.\alpha(t+1), \beta)\ \&\ \beta(\alpha(0)) = y\ \&\ \forall\gamma(B(\lambda t.\alpha(t+1), \gamma) \to \beta = \gamma)\,]\,.$$

Since $A'(\alpha)$ is still almost negative (like $A(\alpha)$), we can apply $GC_0!$ and obtain

$$\text{(d)}\ \ \exists\sigma\forall\alpha\,[\,A'(\alpha) \to \{\sigma\}(\alpha)\!\downarrow\ \&\ B'(\alpha, \{\sigma\}(\alpha))\,]\,.$$

Using this σ we will establish the conclusion of $GC_1!$. So we assume for $\exists$-elimination

$$\text{(e)}\ \ \forall\alpha\,[\,A'(\alpha) \to \{\sigma\}(\alpha)\!\downarrow\ \&\ B'(\alpha, \{\sigma\}(\alpha))\,]\,.$$

We define now σ' by cases, which is justified by FIM, Lemma 5.5:

$$\sigma'(s) = \begin{cases} 0, & \text{if } lh(s) = 0, \\ \sigma(\Pi_{i < lh(s) \dot{-} 1} p_i^{(s)_i}), & \text{otherwise.} \end{cases}$$

We then have, for any y,

$$\sigma'(\overline{\alpha}(y+1)) = \sigma(\overline{\alpha}(y)).$$

We will show that this σ' is the one needed (as the σ) for the conclusion of $GC_1!$. By (e), we have

$$\text{(f)}\ \ \forall\alpha\,[\,A'(\alpha) \to \exists y\,[\,\sigma'(\overline{\alpha}(y+1)) > 0\ \&\ \forall x(\sigma'(\overline{\alpha}(x+1)) > 0 \to y = x)\ \&\ B'(\alpha, \sigma'(\overline{\alpha}(y+1)) \dot{-} 1)\,]\,].$$

Assume now $A(\alpha)$. Put, for $\exists$-elimination, in other words define, again as justified by FIM, Lemma 5.5,

$$\delta(y) = \begin{cases} n, & \text{if } y = 0, \\ \alpha(y \dot{-} 1), & \text{if } y > 0, \end{cases}$$

where n is a number variable. We have then $\alpha = \lambda t.\delta(t+1)$, and we get $A'(\delta)$. Now we first specialize for this δ from (f), and then we can prove, completing finally the $\exists$-elimination for δ and with $\forall$-introduction for n,

$$\text{(g)}\ \ \forall n \exists y[\sigma'(2^{n+1} * \overline{\alpha}(y)) > 0\ \&\ \forall x(\sigma'(2^{n+1} * \overline{\alpha}(x)) > 0 \to y = x)\ \&\ \exists\beta[B(\alpha, \beta)\ \&\ \beta(n) = \sigma'(2^{n+1} * \overline{\alpha}(y)) \dot{-} 1\ \&\ \forall\gamma(B(\alpha, \gamma) \to \beta = \gamma)]].$$

Now from (g) we have

$$\text{(h)}\ \ \forall n\exists! y\sigma'(2^{n+1} * \overline{\alpha}(y)) > 0.$$

From (g) also, we can prove that if β_1 is the β that exists for any n and β_0 the one for n=0, then $\beta_1 = \beta_0$. And from this fact and assuming

$$\text{(i)}\ \ \forall t\exists y\sigma'(2^{t+1} * \overline{\alpha}(y)) = \beta(t) + 1$$

we get $B(\alpha, \beta)$. So we get

$$\text{(j)}\ \ \forall\beta\,[\,\forall t\exists y\sigma'(2^{t+1} * \overline{\alpha}(y)) = \beta(t) + 1 \to B(\alpha, \beta)\,]\,.$$

By (h), (j) and the definition of σ', with $\forall$ and $\to$-introductions and completing the $\exists$- eliminations we establish finally the conclusion of $GC_1!$.

□

PROPOSITION 1.2. $GC_0!$ *is derivable in* **M** *from* $GC_1!$.

PROOF. The proof is similar to the one by which CC_0 (*27.2) is obtained from CC_1 (*27.1) in [FIM], p. 73.

□

2. Weak Continuity

Weak continuity, which follows from strong continuity, asserts only the continuity of functions on Baire space, without providing a neighborhood function. Weak continuity (for numbers) with uniqueness, can be expressed by

$$WC_0!\quad \forall\alpha\exists! x A(\alpha, x) \to \forall\alpha\exists x\exists y\forall\beta\,[\,\overline{\alpha}(x) = \overline{\beta}(x) \to A(\beta, y)\,]\,.$$

The functional version of the above principle is:

$$WC_1!\quad \forall\alpha\exists!\beta A(\alpha, \beta) \to \forall\alpha\forall x\exists y\exists z\forall\gamma\,[\,\overline{\alpha}(y) = \overline{\gamma}(y) \to \exists\beta(A(\gamma, \beta)\ \&\ \beta(x) = z)\,]\,.$$

In fact, these two versions are equivalent over the minimal system **M**, as the following proposition shows.

PROPOSITION 2.1. *The schemata* $WC_0!$ *and* $WC_1!$ *are equivalent over* **M**.

PROOF. (i) We show first that $WC_0!$ entails $WC_1!$ over **M**. Assume $\forall\alpha\exists!\beta A(\alpha, \beta)$. From this we get easily

$$\forall x\forall\alpha\exists! z\exists\beta\,[\,A(\alpha, \beta)\ \&\ \beta(x) = z\,]\,.$$

Specializing for x and applying then $WC_0!$ to the formula

$$B(\alpha, x, z) \equiv \exists\beta\,[\,A(\alpha, \beta)\ \&\ \beta(x) = z\,]$$

we get

$$\forall\alpha\exists y\exists z\forall\gamma\,[\,\overline{\alpha}(y) = \overline{\gamma}(y) \rightarrow \exists\beta\,[\,A(\gamma,\beta)\ \&\ \beta(x) = z\,]\,],$$

and then the conclusion of $WC_1!$.

(ii) We show next that $WC_1!$ entails $WC_0!$.
Assume $\forall\alpha\exists!xA(\alpha, x)$, which abbreviates

$$\forall\alpha\exists x\,[\,A(\alpha, x)\ \&\ \forall y(A(\alpha, y) \rightarrow x = y)\,].$$

Assume, towards $\exists$-elimination,

$$A(\alpha, x)\ \&\ \forall y(A(\alpha, y) \rightarrow x = y).$$

From this, considering x as the value at 0 of the function $\lambda t.x$, we have

$$\text{(a)}\ \ A(\alpha, (\lambda t.x)(0)\,)\ \&\ \forall z\,(\lambda t.x)(z) = (\lambda t.x)(0)$$

and also

$$\text{(b)}\ \ \forall y(A(\alpha, y) \rightarrow (\lambda t.x)(0) = y).$$

Assume

$$\text{(c)}\ \ A(\alpha, \gamma(0)\,)\ \&\ \forall z\,\gamma(z) = \gamma(0).$$

From (b) and (c) we obtain

$$\text{(d)}\ \ (\lambda t.x)(0) = \gamma(0)\ \&\ \forall z\,\gamma(z) = (\lambda t.x)(0),$$

so (discharging (c) and using $\forall\gamma$-introduction)

$$\text{(e)}\ \ \forall\gamma\,[\,A(\alpha, \gamma(0)\,)\ \&\ \forall z\,\gamma(z) = \gamma(0) \rightarrow \gamma = \lambda t.x\,].$$

By (a) and (e) with $\exists\beta$-introduction and then completing the $\exists$-elimination and with $\forall\alpha$-introduction we get finally

$$\text{(f)}\ \ \forall\alpha\exists!\beta\,[\,A(\alpha, \beta(0))\ \&\ \forall z\beta(z) = \beta(0)\,].$$

Applying $WC_1!$ to (f), we get

$$\text{(g)}\ \ \forall\alpha\forall x\exists y\exists w\forall\beta\,[\,\overline{\alpha}(y) = \overline{\beta}(y) \rightarrow \exists\gamma\,[\,A(\beta, \gamma(0))\ \&\ \forall z\,\gamma(z) = \gamma(0)\ \&\ \gamma(x) = w\,]\,].$$

We consider now α and $x = 0$ and, towards $\exists$-eliminations, assume

$$\text{(h)}\ \ \forall\beta\,[\,\overline{\alpha}(y) = \overline{\beta}(y) \rightarrow \exists\gamma\,[\,A(\beta, \gamma(0))\ \&\ \forall z\,\gamma(z) = \gamma(0)\ \&\ \gamma(0) = w\,]\,].$$

From (h), for any β, assuming $\overline{\alpha}(y) = \overline{\beta}(y)$, we obtain
$\exists\gamma\,[\,A(\beta, \gamma(0))\ \&\ \gamma(0) = w\,]$ and from this $A(\beta, w)$. So we get

$$\text{(i)}\ \ \forall\beta\,[\,\overline{\alpha}(y) = \overline{\beta}(y) \rightarrow A(\beta, w)\,].$$

By completing the $\exists$-eliminations and with $\forall\alpha$-introduction, we get the conclusion of $WC_0!$.

□

Remark. Given a function defined on Baire space, together with a (not necessarily optimal) neighborhood function provided by strong continuity, we eventually know a particular (finite) initial segment of each choice sequence which suffices to determine the value of the function for that argument. If the function is only weakly continuous, then the property which guarantees that an initial segment is sufficient, namely the condition

$$\forall\beta\,[\,\overline{\alpha}(x) = \overline{\beta}(x) \to A(\beta, y)\,]\,,$$

is just monotone in x. We give an example, showing that monotone existence does not entail decidability over **M** and more concretely, that **M** does not prove the schema $\exists y\, B(y) \to \forall y(B(y) \vee \neg B(y))$ for $B(y)$ monotone.

Example. Consider the formula

$$A(x) \equiv (x = 0 \;\&\; P) \vee (x > 0),$$

where P is any formula not containing x free. Clearly we have $\exists x A(x)$ and $\forall x \forall y (A(x) \;\&\; x \leq y \to A(y))$. Assuming that **M** entails the above schema, we get $\forall x(A(x) \vee \neg A(x))$ and specializing for x=0 we have $A(0) \vee \neg A(0)$, from which follows $P \vee \neg P$; but then **M** would prove $P \vee \neg P$ for any P, which is impossible.

3. Strong Extensionality

The need for the notion of strong extensionality is due to the fact that, constructively, denying the equality of two (infinite) sequences of natural numbers or of two (one-place) number-theoretic functions is weaker than providing a witness guaranteeing their inequality.

The condition that the total functionals from Baire space to the natural numbers or to the Baire space are strongly extensional is expressed by the two following schemas, respectively:

$$\mathrm{SE}_0! \quad \forall\alpha\exists! x A(\alpha, x) \to \forall\alpha\forall\beta\forall x\forall y\,[\,A(\alpha, x) \;\&\; A(\beta, y) \;\&\; x \neq y \to \exists z\; \alpha(z) \neq \beta(z)\,]\,.$$

$$\mathrm{SE}_1! \quad \forall\alpha\exists!\beta A(\alpha, \beta) \to \forall\alpha\forall\beta\forall\gamma\forall\delta\forall x\,[\,A(\alpha, \gamma) \;\&\; A(\beta, \delta) \;\&\; \gamma(x) \neq \delta(x) \to \exists y\; \alpha(y) \neq \beta(y)\,]\,.$$

The apparently stronger principle for functions $\mathrm{SE}_1!$ is in fact equivalent to the principle for numbers $\mathrm{SE}_0!$, as the following proof shows.

PROPOSITION 3.1. *The schemata* $\mathrm{SE}_0!$ *and* $\mathrm{SE}_1!$ *are equivalent over* **M**.

PROOF. 1. We show that $\mathrm{SE}_0!$ entails $\mathrm{SE}_1!$ over **M**.
Assume $\forall\alpha\exists!\beta \mathrm{A}(\alpha,\beta)$, which is an abbreviation for

$$\text{(a)}\ \ \forall\alpha\exists\beta\,[\,\mathrm{A}(\alpha,\beta)\ \&\ \forall\gamma(\mathrm{A}(\alpha,\gamma)\to\beta=\gamma)\,]\,.$$

We observe that from (a) we can prove (see below)

$$\text{(b)}\ \ \forall \mathrm{x}\forall\alpha\exists!\mathrm{y}\exists\beta\,[\,\mathrm{A}(\alpha,\beta)\ \&\ \beta(\mathrm{x})=\mathrm{y}\,]\,.$$

Let $\mathrm{B}(\alpha,\mathrm{x},\mathrm{y})\equiv\exists\varepsilon\,[\,\mathrm{A}(\alpha,\varepsilon)\ \&\ \varepsilon(\mathrm{x})=\mathrm{y}\,]$ (we use ε instead of β, so that we need to introduce fewer new variables in the following steps). From (b), specializing for x and changing β to ε, we get

$$\text{(c)}\ \ \forall\alpha\exists!\mathrm{y}\mathrm{B}(\alpha,\mathrm{x},\mathrm{y}).$$

So we can apply $\mathrm{SE}_0!$ and we get

$$\text{(d)}\ \ \forall\alpha\forall\beta\forall\mathrm{y}\forall\mathrm{z}\,[\,\mathrm{B}(\alpha,\mathrm{x},\mathrm{y})\ \&\ \mathrm{B}(\beta,\mathrm{x},\mathrm{z})\ \&\ \mathrm{y}\neq\mathrm{z}\to\exists\mathrm{w}\ \alpha(\mathrm{w})\neq\beta(\mathrm{w})\,]\,.$$

Now, in order to obtain (the considered instance of) $\mathrm{SE}_1!$, it suffices to prove $\exists\mathrm{y}\,\alpha(\mathrm{y})\neq\beta(\mathrm{y})$, assuming

$$\text{(e)}\ \ \mathrm{A}(\alpha,\gamma)\ \&\ \mathrm{A}(\beta,\delta)\ \&\ \gamma(\mathrm{x})\neq\delta(\mathrm{x}).$$

So, assume (e). From $\mathrm{A}(\alpha,\gamma)$ we have $\mathrm{A}(\alpha,\gamma)\ \&\ \gamma(\mathrm{x})=\gamma(\mathrm{x})$ and with $\exists$-introduction,

$$\text{(e1)}\ \ \exists\varepsilon\,[\,\mathrm{A}(\alpha,\varepsilon)\ \&\ \varepsilon(\mathrm{x})=\gamma(\mathrm{x})\,]\,.$$

Similarly, from $\mathrm{A}(\beta,\delta)$ we have

$$\text{(e2)}\ \ \exists\varepsilon\,[\,\mathrm{A}(\beta,\varepsilon)\ \&\ \varepsilon(\mathrm{x})=\delta(\mathrm{x})\,]\,.$$

So by (e), (e1), (e2) we have

$$\text{(f)}\ \ \mathrm{B}(\alpha,\mathrm{x},\gamma(\mathrm{x}))\ \&\ \mathrm{B}(\beta,\mathrm{x},\delta(\mathrm{x}))\ \&\ \gamma(\mathrm{x})\neq\delta(\mathrm{x}).$$

Now, from (d), specializing for $\alpha,\beta,\gamma(\mathrm{x}),\delta(\mathrm{x})$ and (f), we obtain $\exists\mathrm{w}\,\alpha(\mathrm{w})\neq\beta(\mathrm{w})$, and by changing w to y, finally $\exists\mathrm{y}\,\alpha(\mathrm{y})\neq\beta(\mathrm{y})$.

[*Proof* of (b) from (a). Assume (a), and after $\forall\alpha$-elimination assume towards $\exists$-elimination

$$\text{(i)}\ \ \mathrm{A}(\alpha,\beta)\ \&\ \forall\gamma\,(\mathrm{A}(\alpha,\gamma)\to\beta=\gamma).$$

So, we have $\mathrm{A}(\alpha,\beta)$; from this we have $\mathrm{A}(\alpha,\beta)\ \&\ \beta(\mathrm{x})=\beta(\mathrm{x})$, and with $\exists$-introduction, finally

$$\text{(ii)}\ \ \exists\delta\,[\,\mathrm{A}(\alpha,\delta)\ \&\ \delta(\mathrm{x})=\beta(\mathrm{x})\,]\,.$$

Now assume

$$\text{(iii)}\ \ \exists\delta\,[\,\mathrm{A}(\alpha,\delta)\ \&\ \delta(\mathrm{x})=\mathrm{z}\,]\,.$$

Towards ∃-elimination, assume

$$\text{(iv)}\ \ \mathrm{A}(\alpha,\delta)\ \&\ \delta(\mathrm{x})=\mathrm{z}.$$

From (i) we have $\forall\gamma\,(\mathrm{A}(\alpha,\gamma)\to\beta=\gamma)$ and from (iv) $\mathrm{A}(\alpha,\delta)$, so we get $\beta=\delta$, and so $\beta(\mathrm{x})=\delta(\mathrm{x})$. But from (iv) $\delta(\mathrm{x})=\mathrm{z}$, so $\beta(\mathrm{x})=\mathrm{z}$. So by →-introduction (discharging iii)) and completing the ∃-elimination of (iv) and with → and ∀-introductions, we get

$$\text{(v)}\ \ \forall\mathrm{z}\,(\exists\delta\,[\,\mathrm{A}(\alpha,\delta)\ \&\ \delta(\mathrm{x})=\mathrm{z}\,]\to\beta(\mathrm{x})=\mathrm{z}).$$

Now from (ii) and (v) with ∃-introduction, we have

$$\text{(vi)}\ \ \exists!\mathrm{y}\exists\delta\,[\,\mathrm{A}(\alpha,\delta)\ \&\ \delta(\mathrm{x})=\mathrm{y}\,]\,.$$

By changing δ to β and completing the ∃-elimination of (i), and with ∀-introductions we get (b).]

2. We show now that $\mathrm{SE}_1!$ entails $\mathrm{SE}_0!$ over **M**.
Assume $\forall\alpha\exists!\mathrm{x}\mathrm{A}(\alpha,\mathrm{x})$, which is an abbreviation for

$$\text{(a)}\ \ \forall\alpha\exists\mathrm{x}\,[\,\mathrm{A}(\alpha,\mathrm{x})\ \&\ \forall\mathrm{y}(\mathrm{A}(\alpha,\mathrm{y})\to\mathrm{x}=\mathrm{y})\,]\,.$$

We observe that from (a) we can prove (see below)

$$\text{(b)}\ \ \forall\alpha\exists!\beta\,[\,\mathrm{A}(\alpha,\beta(0))\ \&\ \forall\mathrm{z}\beta(\mathrm{z})=\beta(0)\,]\,.$$

Let

$$\mathrm{B}(\alpha,\beta)\equiv\mathrm{A}(\alpha,\beta(0))\ \&\ \forall\mathrm{z}\,\beta(\mathrm{z})=\beta(0).$$

Then by (b) we have $\forall\alpha\exists!\beta\,\mathrm{B}(\alpha,\beta)$; so applying $\mathrm{SE}_1!$ we have

$$\text{(c)}\ \forall\alpha\forall\beta\forall\gamma\forall\delta\forall\mathrm{x}\,[\,\mathrm{B}(\alpha,\gamma)\ \&\ \mathrm{B}(\beta,\delta)\ \&\ \gamma(\mathrm{x})\neq\delta(\mathrm{x})\to\exists\mathrm{y}\ \alpha(\mathrm{y})\neq\beta(\mathrm{y})\,]\,.$$

Now, in order to obtain the instance of $\mathrm{SE}_0!$ that we are considering, it suffices to prove $\exists\mathrm{z}\,\alpha(\mathrm{z})\neq\beta(\mathrm{z})$, assuming

$$\text{(d)}\ \ \mathrm{A}(\alpha,\mathrm{x})\ \&\ \mathrm{A}(\beta,\mathrm{y})\ \&\ \mathrm{x}\neq\mathrm{y}.$$

So, assume (d). From $\mathrm{A}(\alpha,\mathrm{x})$ we get

$$\text{(d1)}\ \ \mathrm{A}(\alpha,(\lambda\mathrm{t.x})(0))\ \&\ \forall\mathrm{z}\,(\lambda\mathrm{t.x})(\mathrm{z})=(\lambda\mathrm{t.x})(0).$$

From $\mathrm{A}(\beta,\mathrm{y})$ we get

$$\text{(d2)}\ \ \mathrm{A}(\beta,(\lambda\mathrm{t.y})(0))\ \&\ \forall\mathrm{z}\,(\lambda\mathrm{t.y})(\mathrm{z})=(\lambda\mathrm{t.y})(0).$$

So by (d), (d1), (d2) we have

$$\text{(e)}\ \ \mathrm{B}(\alpha,\lambda\mathrm{t.x})\ \&\ \mathrm{B}(\beta,\lambda\mathrm{t.y})\ \&\ (\lambda\mathrm{t.x})(\mathrm{z})\neq(\lambda\mathrm{t.y})(\mathrm{z}).$$

From (c), specializing for $\alpha,\beta,\lambda\mathrm{t.x},\lambda\mathrm{t.y}$ and z and with (e), we get $\exists\mathrm{y}\,\alpha(\mathrm{y})\neq\beta(\mathrm{y})$, and by changing y to z we get $\exists\mathrm{z}\,\alpha(\mathrm{z})\neq\beta(\mathrm{z})$.

[*Proof* of (b) from (a). Assume (a), and after $\forall\alpha$-elimination assume towards $\exists$-elimination

$$\text{(i)}\ \ \mathrm{A}(\alpha, \mathrm{x})\ \&\ \forall \mathrm{y}\,(\mathrm{A}(\alpha, \mathrm{y}) \to \mathrm{x} = \mathrm{y}),$$

from which we have

$$\text{(ii)}\ \ \mathrm{A}(\alpha, (\lambda \mathrm{t.x})(0))\ \&\ \forall \mathrm{y}\,(\mathrm{A}(\alpha, \mathrm{y}) \to \mathrm{x} = \mathrm{y}).$$

So we have $\mathrm{A}(\alpha, (\lambda \mathrm{t.x})(0))$ and from this

$$\text{(iii)}\ \ \mathrm{A}(\alpha, (\lambda \mathrm{t.x})(0))\ \&\ \forall \mathrm{z}\,(\lambda \mathrm{t.x})(\mathrm{z}) = (\lambda \mathrm{t.x})(0).$$

Assume

$$\text{(iv)}\ \ \mathrm{A}(\alpha, \gamma(0))\ \&\ \forall \mathrm{z}\,\gamma(\mathrm{z}) = \gamma(0).$$

So we have $\mathrm{A}(\alpha, \gamma(0))$. But by (i), $\forall \mathrm{y}\,(\mathrm{A}(\alpha, \mathrm{y}) \to \mathrm{x} = \mathrm{y})$, so $\mathrm{x} = \gamma(0)$. By (iv) $\forall \mathrm{z}\,\gamma(\mathrm{z}) = \gamma(0)$, so $\forall \mathrm{z}\,\gamma(\mathrm{z}) = \mathrm{x} = (\lambda \mathrm{t.x})(\mathrm{z})$, so $\gamma = \lambda \mathrm{t.x}$. So

$$\text{(v)}\ \ \forall \gamma\,[\,(\mathrm{A}(\alpha, \gamma(0))\ \&\ \forall \mathrm{z}\,\gamma(\mathrm{z}) = \gamma(0)) \to \lambda \mathrm{t.x} = \gamma\,].$$

So by (iii), (v) and $\exists$-introduction we get

$$\text{(vi)}\ \ \exists!\beta\,[\,\mathrm{A}(\alpha, \beta(0))\ \&\ \forall \mathrm{z}\,\beta(\mathrm{z}) = \beta(0)\,].$$

Completing the $\exists$-elimination of (i) and with $\forall$-introduction we obtain (b).]

□

4. Markov's Principle

Markov's principle is a necessary assumption for the constructive recursive mathematics of Markov's school. It asserts roughly that, for decidable properties of numbers, indirect proofs of existence are acceptable. Over intuitionistic analysis **FIM**, Markov's principle is neither provable nor refutable, as Kleene's function realizability validates it, while typed realizabilities such as Kleene's special realizability validate its negation.

In the context of **M**, where every decidable predicate has a characteristic function, Markov's Principle is expressed by

$$\mathrm{MP}_1\ \ \forall \alpha\,[\,\neg\neg\exists \mathrm{x}\,\alpha(\mathrm{x}) = 0 \to \exists \mathrm{x}\,\alpha(\mathrm{x}) = 0\,].$$

We consider also the version with uniqueness:

$$\mathrm{MP}_1!\ \ \forall \alpha\,[\,\neg\neg\exists! \mathrm{x}\,\alpha(\mathrm{x}) = 0 \to \exists \mathrm{x}\,\alpha(\mathrm{x}) = 0\,].$$

Proposition 4.1. MP_1 *and* $MP_1!$ *are equivalent over* **M**.

Proof. 1. MP_1 entails $MP_1!$ by logic, as follows. From $\exists! x\, \alpha(x) = 0$ we get (easily) $\exists x\, \alpha(x) = 0$, and so, assuming (a) $\neg\neg\exists! x\, \alpha(x) = 0$ we get (b) $\neg\neg\exists x\, \alpha(x) = 0$. Now from MP_1, specializing for α and with (b) and modus ponens, we get $\exists x\, \alpha(x) = 0$. So with $\rightarrow$-introduction discharging (a) and with $\forall\alpha$-introduction we get $MP_1!$.

2. $MP_1!$ entails MP_1 as follows. In intuitionistic arithmetic we cannot prove the least number principle (which says that if there exists a natural number n with the property $P(n)$ then there exists a least such natural number) but we can prove its double negation (see IM, p.190). So in particular we have

$$\text{(a)}\ \ \neg\neg\,[\,\exists x\, \alpha(x) = 0 \rightarrow \exists x\, (\alpha(x) = 0\ \&\ \forall y < x\, \alpha(y) \neq 0)\,]\,.$$

From (a) and the schema $\neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$ which is provable in intuitionistic propositional logic, we get

$$\text{(b)}\ \ \neg\neg\exists x\, \alpha(x) = 0 \rightarrow \neg\neg\exists x(\alpha(x) = 0\ \&\ \forall y < x\, \alpha(y) \neq 0).$$

We define now $\gamma(x) = 0$ if $\alpha(x) = 0\ \&\ \forall y < x\, \alpha(y) \neq 0$ and 1 otherwise, so from $\exists x\, \alpha(x) = 0$ we get $\exists! x\, \gamma(x) = 0$ (by Lemma 2.4, Ch. 1). So we get

$$\text{(c)}\ \ \neg\neg\exists x\, \alpha(x) = 0 \rightarrow \neg\neg\exists! x\, \gamma(x) = 0.$$

Now we assume (d) $\neg\neg\exists x\, \alpha(x) = 0$. From (c), (d) and modus ponens and then $MP_1!$ (after specializing for γ) and modus ponens we get

$$\text{(e)}\ \ \exists x\, (\alpha(x) = 0\ \&\ \forall y < x\, \alpha(y) \neq 0)$$

and from this $\exists x\, \alpha(x) = 0$. So, with $\rightarrow$-introduction discharging (d) and with $\forall\alpha$-introduction we get MP_1.

□

5. Other non constructive principles

Bishop called the principle, valid in classical mathematics, according to which *either all elements of a certain set A have property P or there exists an element of A with property (not P)*, the *principle of omniscience*. The simplest form of this principle, the *limited principle of omniscience*, and the *weak limited principle of omniscience*, are respectively:

$$\text{LPO}\ \ \forall\alpha\,[\,\exists x\, \alpha(x) = 0 \vee \forall x\, \alpha(x) \neq 0\,]$$

and

$$\text{WLPO}\ \ \forall\alpha\,[\,\neg\exists x\, \alpha(x) \neq 0 \vee \neg\neg\exists x\, \alpha(x) \neq 0\,]\,.$$

Both are special cases of the constructively (and intuitionistically) invalid principle of excluded middle

$$\text{PEM} \quad \mathrm{A} \vee \neg \mathrm{A}.$$

Since LPO is equivalent to $\forall\alpha\,[\,\exists x\,\alpha(x) = 0 \vee \neg\exists x\,\alpha(x) = 0\,]$ and WLPO is equivalent to $\forall\alpha\,[\,\forall x\,\alpha(x) = 0 \vee \neg\forall x\,\alpha(x) = 0\,]$, they are also called $\exists$-PEM and $\forall$-PEM, respectively.

In intuitionistic arithmetic **HA** and in the minimal system **M**, disjunction is explicitly definable by

$$\mathrm{A} \vee \mathrm{B} \equiv \exists y\,[\,(y = 0 \to \mathrm{A})\ \&\ (y \neq 0 \to \mathrm{B})\,]\,.$$

From this we get[1] that $\mathrm{A} \vee \neg\mathrm{A} \leftrightarrow \exists!y\,[\,y \leq 1\ \&\ (y = 0 \leftrightarrow \mathrm{A})\,]$ and so we have equivalent uniqueness expressions PEM!, LPO!, WLPO! for the above principles:

$$\begin{array}{ll} \text{PEM!} & \exists!y\,[\,y \leq 1\ \&\ (y = 0 \leftrightarrow \mathrm{A})\,]\,, \\ \text{LPO!} & \forall\alpha\exists!y\,[\,y \leq 1\ \&\ (y = 0 \leftrightarrow \exists x\,\alpha(x) = 0)\,]\,, \\ \text{WLPO!} & \forall\alpha\exists!y\,[\,y \leq 1\ \&\ (y = 0 \leftrightarrow \forall x\,\alpha(x) = 0)\,]\,. \end{array}$$

It is immediate that LPO implies WLPO and MP_1, and that MP_1 with WLPO implies LPO. We can see that some of the converses of the above implications cannot be proved in **M**, as in the presence of more powerful principles we have the following results.

Proposition 5.1.
(a) $\mathbf{M} + \mathrm{WC}_0! \vdash \neg\mathrm{WLPO}$.
(b) $\mathbf{M} + \mathrm{WC}_0! \vdash \neg\mathrm{LPO}$.

Proof. (a) This result is obtained in [FIM] (p.84) using a disjunctive form of CC_0 and in [TvDI] (p.209) using a disjunctive form of WC_0 in the place of $\mathrm{WC}_0!$. Here we argue as follows. Assume WLPO!. For $\alpha \equiv \lambda t.0$, we get from $\mathrm{WC}_0!$ an m and (exactly one) y such that $(*)\ \forall\beta[\overline{\lambda t.0}(m) = \overline{\beta}(m) \to (y = 0 \leftrightarrow \forall x\beta(x) = 0)]$. But then y=0 and for $\beta \equiv \lambda t.\overline{sg}(m \dot{-} t)$ $(*)$ does not hold.

(b) Immediate from (a) and the fact that LPO implies WLPO.

□

Neither MP_1 nor its negation is provable in $\mathbf{M} + \mathrm{GC}_1$.

Proposition 5.2.
(a) $\mathbf{M} + \mathrm{GC}_1 \nvdash \mathrm{MP}_1$.
(b) $\mathbf{M} + \mathrm{GC}_1 \nvdash \neg\mathrm{MP}_1$.

[1]This is Lemma 2.6 of Chapter 1.

PROOF. (a) By Troelstra's characterization of Kleene's function realizability ([Troelstra1973], pp. 209-211), we have that if $\mathbf{M}^{+}$+GC_1 proves MP_1 then $\mathbf{M}^{+}$ proves a formula expressing that MP_1 is realizable. Observe that MP_1 is almost negative. By Lemma 3.3.8 of [Troelstra1973], p. 209, each almost negative formula is equivalent (in **EL** and so in $\mathbf{M}^{+}$) to a formula expressing that it is realizable. But then $\mathbf{M}^{+}$ would prove MP_1, which is not the case, as its negation is (classically) special-realizable ([FIM], p. 131).

(b) GC_1 and MP_1 are (classically) realizable, for Kleene's function realizability.

□

PROPOSITION 5.3.
(a) $\mathbf{M} + \mathrm{GC}_1 \nvdash \mathrm{MP}_1 \rightarrow \mathrm{WLPO}$.
(b) $\mathbf{M} + \mathrm{GC}_1 \nvdash \mathrm{MP}_1 \rightarrow \mathrm{LPO}$.

PROOF. (a) By Proposition 5.1(a) and the fact that GC_1 entails $\mathrm{WC}_0!$ over $\mathbf{M}$, if $\mathbf{M} + \mathrm{GC}_1 \vdash \mathrm{MP}_1 \rightarrow \mathrm{WLPO}$, we get by logic that $\mathbf{M} + \mathrm{GC}_1 \vdash \neg\mathrm{MP}_1$. It is known that GC_1 is realizable (for the functional realizability of Kleene, see [FIM]), so from the soundness theorem for this notion of realizability we would then have that $\neg\mathrm{MP}_1$ is also realizable, which is impossible since MP_1 is (classically) realizable. So (a) holds.

(b) Immediate from (a).

□

REMARKS. (a) With an argument using the ^{G}realizability of J. R. Moschovakis ([JRM1971]]), under which MP_1 is not realizable, we can also obtain that $\mathbf{M} + \neg\mathrm{MP}_1 \nvdash \mathrm{GC}_1$.

(b) All the above underivability results hold if we take $\mathrm{GC}_0!$ in the place of GC_1, by the fact that GC_1 entails $\mathrm{GC}_0!$ over $\mathbf{M}$.

6. Further results

Let $\mathrm{MP}_1(\alpha)$ be the formula $\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \rightarrow \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$, and let $\mathrm{LPO}!(\alpha)$ be LPO! with the $\forall\alpha$ deleted, and $\mathrm{WLPO}!(\alpha)$ similarly. As a corollary to Proposition 5.3 of the previous section we get that

PROPOSITION 6.1.
(a) $\mathbf{M} + \mathrm{GC}_1 \nvdash \forall\alpha\, [\, \mathrm{MP}_1(\alpha) \rightarrow \mathrm{WLPO}!(\alpha)\,]$ *and*
(b) $\mathbf{M} + \mathrm{GC}_1 \nvdash \forall\alpha\, [\, \mathrm{MP}_1(\alpha) \rightarrow \mathrm{LPO}!(\alpha)\,]$.

We will give now an argument of J. R. Moschovakis which, using a strengthening of the above proposition together with ^{G}realizability ([JRM1971]), shows that $\mathrm{CC}_0!$ does not entail $\mathrm{GC}_0!$ over **M**.

PROPOSITION 6.2.
(a) $\mathbf{M} + \mathrm{GC}_0! \vdash \neg\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{WLPO!}(\alpha)\,]$.
(b) $\mathbf{M} + \mathrm{GC}_0! \vdash \neg\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{LPO!}(\alpha)\,]$.

PROOF. We give the proof of (b). MP_1 is an almost negative formula, so

$$\mathbf{M} + \mathrm{GC}_0! \vdash \forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{LPO!}(\alpha)\,] \to \exists\sigma\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to (\{\sigma\}(\alpha){\downarrow}\ \&\ (\{\sigma\}(\alpha) = 0 \leftrightarrow \exists \mathrm{x}\,\alpha(\mathrm{x}) = 0))\,].$$

Assume $\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{LPO!}(\alpha)\,]$. Consider a σ that satisfies the conclusion. Since $\mathrm{MP}_1(\lambda\mathrm{t}.1)$ holds, $\{\sigma\}(\lambda\mathrm{t}.1)$ is defined, and since $\neg\exists\mathrm{x}(\lambda\mathrm{t}.1)(\mathrm{x}) = 0$, we have that $\{\sigma\}(\lambda\mathrm{t}.1) \neq 0$. Also we have that for some m $\{\sigma\}(\lambda\mathrm{t}.1) = \sigma(\overline{\lambda\mathrm{t}.1}(\mathrm{m}))\dot{-}1$; so if $\alpha = \lambda\mathrm{t}.\mathrm{sg}(\mathrm{m}\dot{-}\mathrm{t})$ then we have $\{\sigma\}(\alpha) \neq 0$ and $\alpha(\mathrm{m}) = 0$, which implies that $\{\sigma\}(\alpha) = 0$. So we get a contradiction, so $\neg\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{LPO!}(\alpha)\,]$.

The proof of (a) is similar; take $\lambda\mathrm{t}.0$ instead of $\lambda\mathrm{t}.1$ and take $\alpha = \lambda\mathrm{t}.\overline{\mathrm{sg}}(\mathrm{m}\dot{-}\mathrm{t})$ for the counterexample. (Also (b) follows by logic from (a)).

□

PROPOSITION 6.3. $\forall\alpha\,[\,\mathrm{MP}_1(\alpha) \to \mathrm{LPO!}(\alpha)\,]$ *is* ^{G}*realizable.*

PROOF. Classically, the formula

$$\forall\alpha\,[\,(\neg\neg\exists\mathrm{x}\,\alpha(\mathrm{x}) = 0 \to \exists\mathrm{x}\,\alpha(\mathrm{x}) = 0) \to \exists\mathrm{y}\,(\mathrm{y} = 0 \to \exists\mathrm{x}\,\alpha(\mathrm{x}) = 0)\ \&\ (\mathrm{y} \neq 0 \to \neg\exists\mathrm{x}\,\alpha(\mathrm{x}) = 0)\,]$$

is ^{G}realized by

$$\Lambda\alpha\Lambda\rho\langle\lambda s\,\alpha(x_0), \langle\Lambda\tau\{\rho\}[\,\Lambda\pi\lambda t\,0\,], \Lambda\tau\Lambda\pi\lambda t\,0\rangle\rangle,$$

where $x_0 = (\{\rho\}[\,\Lambda\pi\lambda t\,0\,](0))_0$.

□

COROLLARY 6.4. $\mathrm{GC}_0!$ *is not* ^{G}*realizable.*

PROOF. By Propositions 6.2 and 6.3.

□

COROLLARY 6.5. $\mathbf{M} + \mathrm{CC}_0! \nvdash \mathrm{GC}_0!$.

PROOF. By Corollary 6.4 together with the fact that $\mathrm{CC}_0!$ is ^{G}realizable by [JRM1971].

□

We obtain now some more results about the relations among some of the principles with uniqueness under discussion.

PROPOSITION 6.6. *Over the minimal system* $\mathbf{M}$, $\mathrm{WC}_0!$ *entails* $\mathrm{SE}_0!$.

PROOF. Assume the hypothesis of an instance of $\mathrm{SE}_0!$

$$(a)\ \ \forall\alpha\exists!\mathrm{x}\,\mathrm{A}(\alpha,\mathrm{x}),$$

and also assume

$$(\mathrm{b})\ \ \mathrm{A}(\alpha,\mathrm{y})\ \&\ \mathrm{A}(\beta,\mathrm{z})\ \&\ \mathrm{y}\neq\mathrm{z}.$$

By (a) we obtain the conclusion of $\mathrm{WC}_0!$, and by specializing for α, we obtain

$$(\mathrm{c})\ \ \exists\mathrm{x}\exists\mathrm{y}\forall\beta\,[\,\overline{\alpha}(\mathrm{x})=\overline{\beta}(\mathrm{x})\rightarrow\mathrm{A}(\beta,\mathrm{y})\,]\,.$$

Towards $\exists$-eliminations, assume

$$(\mathrm{d})\ \ \forall\beta\,[\,\overline{\alpha}(\mathrm{u})=\overline{\beta}(\mathrm{u})\rightarrow\mathrm{A}(\beta,\mathrm{v})\,]\,.$$

Then, for $\beta\equiv\alpha$, we have $\mathrm{A}(\alpha,\mathrm{v})$, so by (a) and $\mathrm{A}(\alpha,\mathrm{y})$ (from (b)), we have (e) $\mathrm{y}=\mathrm{v}$. By (e) and $\mathrm{y}\neq\mathrm{z}$ (from (b)), we have (f) $\mathrm{z}\neq\mathrm{v}$. Now, assuming $\mathrm{A}(\beta,\mathrm{v})$, by (a) and $\mathrm{A}(\beta,\mathrm{z})$ (from (b)), we have $\mathrm{v}=\mathrm{z}$, which contradicts (f), so we have $\neg\mathrm{A}(\beta,\mathrm{v})$ and so, by (d), we have $\overline{\alpha}(\mathrm{u})\neq\overline{\beta}(\mathrm{u})$. So we obtain $\exists\mathrm{w}<\mathrm{u}\ \alpha(\mathrm{w})\neq\beta(\mathrm{w})$, so, finally we get

$$(\mathrm{g})\ \ \exists\mathrm{w}\,\alpha(\mathrm{w})\neq\beta(\mathrm{w}).$$

Now, with $\exists$-eliminations discharging (d), and an $\rightarrow$-introduction discharging (b), we have that (b) $\rightarrow$ (g), and then with $\forall$-introductions in (b) $\rightarrow$ (g) and an $\rightarrow$-introduction discharging (a), we obtain the considered instance of $\mathrm{SE}_0!$.

□

It is well-known that MP_1 entails $\mathrm{SE}_0!$. But from the above theorem, the converse does not hold in $\mathbf{M}$.

COROLLARY 6.7. $\mathbf{M}+\mathrm{SE}_0!\nvdash\mathrm{MP}_1$.

PROOF. $\mathbf{M}+\mathrm{WC}_0!\nvdash\mathrm{MP}_1$, because $\mathrm{WC}_0!$ is ^{G}realizable, although MP_1 is not.

□

The following schema expresses the strong extensionality of partial functionals on almost negative subsets of their domains:

$$\mathrm{PSE}_0!\ \ \forall\alpha\,[\,\mathrm{A}(\alpha)\rightarrow\exists!\mathrm{y}\mathrm{B}(\alpha,\mathrm{y})\,]$$
$$\rightarrow\forall\alpha\forall\beta\forall\mathrm{y}\forall\mathrm{z}\,[\,\mathrm{A}(\alpha)\ \&\ \mathrm{B}(\alpha,\mathrm{y})\ \&\ \mathrm{B}(\beta,\mathrm{z})\ \&\ \mathrm{y}\neq\mathrm{z}\rightarrow\exists\mathrm{x}\ \alpha(\mathrm{x})\neq\beta(\mathrm{x})\,]$$

where $\mathrm{A}(\alpha)$ is almost negative.

PROPOSITION 6.8. *Over* **M**, $\mathrm{PSE_0!}$ *is equivalent to* $\mathrm{MP_1}$,

PROOF. We show first that $\mathrm{MP_1}$ entails $\mathrm{PSE_0!}$.
Assume

$$\text{(a)}\quad A(\alpha) \rightarrow \exists!yB(\alpha, y)$$

and

$$\text{(b)}\quad A(\alpha)\ \&\ B(\alpha, y)\ \&\ B(\beta, z)\ \&\ y \neq z.$$

From these we get (c) $\exists!yB(\alpha, y)$. Assume (d) $\neg\exists x\, \alpha(x) \neq \beta(x)$. We get then $\forall x\, \alpha(x) = \beta(x)$ so $\alpha = \beta$, and so by replacement from $B(\alpha, y)$ (which follows from (b) by &-elimination) we have (e) $B(\beta, y)$, which with (c) gives (f) $z = y$, contradicting (b). So with $\rightarrow$-introduction discharging (d) we get (g) $\neg\neg\exists x\alpha(x) \neq \beta(x)$, whence $\exists x\, \alpha(x) \neq \beta(x)$ by $\mathrm{MP_1}$.

We show now that $\mathrm{PSE_0!}$ entails $\mathrm{MP_1}$. Consider the formulas

$$A(\alpha) \equiv \neg\forall x\, \alpha(x) = 0$$

and

$$B(\alpha, y) \equiv (\neg\forall x\, \alpha(x) = 0\ \&\ y = 0) \vee (\forall x\, \alpha(x) = 0\ \&\ y = 1).$$

We have then (a) $A(\alpha) \rightarrow \exists!y\, B(\alpha, y)$. Assume $A(\alpha)$; then we have $B(\alpha, 0)$. Also we have $B(\lambda t.0, 1)$. So we have

$$\text{(b)}\quad A(\alpha)\ \&\ B(\alpha, 0)\ \&\ B(\lambda t.0, 1)\ \&\ 0 \neq 1,$$

and by (a) with $\mathrm{PSE_0!}$ we get $\exists x\, \alpha(x) \neq (\lambda t.0)(x)$, so $\exists x\, \alpha(x) \neq 0$. So we have

$$\text{(c)}\quad \forall\alpha\, [\neg\forall x\, \alpha(x) = 0 \rightarrow \exists x\, \alpha(x) \neq 0\,].$$

From (c), starting with specializing for $\lambda t.\overline{sg}(\alpha(t))$ we can easily get $\mathrm{MP_1}$.

□

7. Vesley's Schema

7.1. In his paper "A palatable substitute for Kripke's Schema" ([Vesley]), R. E. Vesley showed that it is possible to avoid Kripke's Schema[2] in the proofs of the results for which Brouwer was using the "creative subject" arguments; he employed instead the following axiom schema:

$$\text{VS}\quad \forall w\, [\,\mathrm{Seq}(w) \rightarrow \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w\ \&\ \neg A(\alpha))\,] \rightarrow [\,\forall\alpha(\neg A(\alpha) \rightarrow \exists\beta B(\alpha, \beta)) \rightarrow \forall\alpha\exists\beta(\neg A(\alpha) \rightarrow B(\alpha, \beta))\,],$$

[2]Kripke's Schema is introduced in order to formulate Brouwer's "creative subject" arguments, see [TvDI], pp. 234-241.

where[3] β is not free in A, and where the hypothesis of the main implication expresses that $\neg A(\alpha)$ defines a dense subset of the Baire space. He proved that the system **FIM**+VS extends consistently **FIM** and, among other things, that $\neg MP_1$ is a theorem of this system.

VS expresses a classically correct principle. Vesley omitted any continuity condition. Here we are considering the following uniqueness version of VS, which, unlike VS, is not classically correct:

$$\text{VS!}\quad \forall w\,[\,\mathrm{Seq}(w) \to \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w \;\&\; \neg A(\alpha))\,] \to [\,\forall\alpha(\neg A(\alpha) \to \exists!\beta B(\alpha,\beta)) \to \exists\tau\forall\alpha\,[\,\{\tau\}[\,\alpha\,]\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha,\{\tau\}[\,\alpha\,]))\,]\,],$$

where β is not free in A, which also guarantees the continuity of the function whose existence is asserted by the principle.

VS! can be consistently added to **M**, as it is immediately derivable from VS and CC_1. The resulting system **M**+VS! is an extension of **M**+CC_1! (CC_1! is obtained from VS! if we let $A(\alpha)$ be $\neg\forall x\,\alpha(x) = \alpha(x)$ for example) in which $\neg MP_1$ is derivable, as is shown next.

In fact we will need only the version of VS! for number-valued functions:

$$\text{VS}_0!\quad \forall w\,[\,\mathrm{Seq}(w) \to \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w \;\&\; \neg A(\alpha))\,] \to [\,\forall\alpha(\neg A(\alpha) \to \exists! b B(\alpha,b)) \to \exists\tau\forall\alpha\,[\,\{\tau\}(\alpha)\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha,\{\tau\}(\alpha)))\,]\,],$$

where b does not occur free in A. But, like in the cases of CC_1! and GC_1!, the uniqueness condition ensures the equivalence of the two versions, for number-valued and for function-valued functions. To see this, we first observe that if $\neg A(\alpha)$ is dense, then also $\neg A(\lambda t.\alpha(t+1))$ is: take any sequence number $w = \langle w_0{+}1, \ldots, w_m{+}1\rangle$. Then for the sequence number $w^* = \langle w_1{+}1, \ldots, w_m{+}1\rangle$, from the density of $\neg A(\alpha)$ there is a sequence say α^* for which $\overline{\alpha^*}(\mathrm{lh}(w^*)) = w^*$ and $\neg A(\alpha^*)$. So for $\alpha = \langle w_0{+}1\rangle * \alpha^*$ we have $\overline{\alpha}(\mathrm{lh}(w)) = w$ and $\neg A(\lambda t.\alpha(t+1))$. Once we have this, we can show the desired equivalence by the same method as for the corresponding equivalence in the cases of CC_1! and GC_1! (see previous paragraphs).

We show now how to derive $\neg MP_1$ in **M**+VS!. Consider the following version of Markov's Principle:

$$MP_1!!\quad \forall\alpha\,[\,\neg\neg\exists! x\,\alpha(x) = 0 \to \exists! x\,\alpha(x) = 0\,].$$

For this, we have the following result.

[3]Seq(w) abbreviates $w > 0 \;\&\; \forall i_{i<\mathrm{lh}(a)}(a)_i > 0$ and expresses that w is the code number of a finite sequence.

LEMMA 7.1. $\mathrm{MP}_1!!$ *and* MP_1 *are equivalent over* $\mathbf{M}$.

PROOF. We have shown previously that MP_1 is equivalent to $\mathrm{MP}_1!$. Now we show that $\mathrm{MP}_1!!$ is equivalent to $\mathrm{MP}_1!$. The implication $\mathrm{MP}_1!! \to \mathrm{MP}_1!$ is trivial. We give a sketch of proof for the converse implication. Assume (a) $\neg\neg\exists!\mathrm{x}\alpha(\mathrm{x}) = 0$. By $\mathrm{MP}_1!$ we get $\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$. To show $\exists!\mathrm{x}\, \alpha(\mathrm{x}) = 0$ we assume (b) $\alpha(\mathrm{x}) = 0$ and we will show (c) $\forall \mathrm{y}(\alpha(\mathrm{y}) = 0 \to \mathrm{y} = \mathrm{x})$. We assume $\alpha(\mathrm{y}) = 0$ and $\mathrm{x} \neq \mathrm{y}$. From this and (b) we can prove $\neg\exists!\mathrm{x}\, \alpha(\mathrm{x}) = 0$, so from (a) we get $\mathrm{x} = \mathrm{y}$ and finally (c).

□

LEMMA 7.2. *Over* $\mathbf{M}$,

$$\mathrm{MP}_1!! \to \forall\alpha\,[\,\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \exists!\mathrm{x}(\alpha(\mathrm{x}) = 0\ \&\ \forall \mathrm{y}(\mathrm{y} < \mathrm{x} \to \alpha(\mathrm{y}) \neq 0))\,]\,.$$

PROOF. Assume $\neg\neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0$. From this follows similarly to previous cases

$$\neg\neg\exists!\mathrm{x}\,[\,\alpha(\mathrm{x}) = 0\ \&\ \forall \mathrm{y}(\mathrm{y} < \mathrm{x} \to \alpha(\mathrm{y}) \neq 0)\,]\,.$$

Let $\gamma(\mathrm{x}) = 0$ if $\alpha(\mathrm{x}) = 0\ \&\ \forall \mathrm{y}(\mathrm{y} < \mathrm{x} \to \alpha(\mathrm{y}) \neq 0)$ and 1 otherwise. Then we have $\neg\neg\exists!\mathrm{x}\, \gamma(\mathrm{x}) = 0$ and, by $\mathrm{MP}_1!!$, $\exists!\mathrm{x}\, \gamma(\mathrm{x}) = 0$.

□

THEOREM 7.3. $\mathbf{M} + \mathrm{VS}! \vdash \neg\mathrm{MP}_1$.

PROOF. It easy to see that $\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$ is dense. So, assuming $\mathrm{MP}_1!!$, by Lemma 7.2 and $\mathrm{VS}_0!$ we get

$$\text{(a)}\quad \exists\tau\forall\alpha\,[\,\{\tau\}(\alpha)\!\downarrow\ \&\ (\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \gamma(\{\tau\}(\alpha)) = 0)\,]$$

for the γ of Lemma 7.2. Since $\forall \mathrm{x}(\gamma(\mathrm{x}) = 0 \to \alpha(\mathrm{x}) = 0)$, from (a) we have

$$\text{(b)}\quad \exists\tau\forall\alpha\,[\,\{\tau\}(\alpha)\!\downarrow\ \&\ (\neg\neg\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \alpha(\{\tau\}(\alpha)) = 0)\,]\,.$$

We repeat now Vesley's argument and we get $\neg\mathrm{MP}_1!!$ or, by Lemma 7.1, $\neg\mathrm{MP}_1$.

□

We will see now that the continuous total extension of a partial function with a negative dense domain asserted by VS! is unique over $\mathbf{M}$ (in the sense that any modulus of continuity functional which extends the given partial function leads to the same extension).

THEOREM 7.4. *In* $\mathbf{M} + \mathrm{VS}_0!$

$$\vdash \forall w\,[\,\mathrm{Seq}(w) \to \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w \;\&\; \neg A(\alpha))\,] \to$$
$$[\,\forall\alpha(\neg A(\alpha) \to \exists! b B(\alpha, b)) \to \exists\tau\forall\alpha\,[\,\{\tau\}(\alpha)\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha, \{\tau\}(\alpha)))$$
$$\&\; \forall\sigma[\forall\alpha[\,\{\sigma\}(\alpha)\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha, \{\sigma\}(\alpha)))] \to \forall\alpha\{\sigma\}(\alpha) = \{\tau\}(\alpha)\,]\,]\,],$$

where b *does not occur free in* A.

PROOF. We assume the two hypotheses of the formal theorem that we want to prove. Then, by $\mathrm{VS}_0!$ we can assume for $\exists\tau$-elimination,

$$\text{(a)}\;\; \forall\alpha\,[\,\{\tau\}(\alpha)\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha, \{\tau\}(\alpha)))\,]\,.$$

Assume also

$$\text{(b)}\;\; \forall\alpha\,[\,\{\sigma\}(\alpha)\!\downarrow \;\&\; (\neg A(\alpha) \to B(\alpha, \{\sigma\}(\alpha)))\,]\,.$$

Consider now any α; we will show (*c*) $\{\tau\}(\alpha) = \{\sigma\}(\alpha)$. Since $\{\tau\}(\alpha)\!\downarrow$ and $\{\sigma\}(\alpha)\!\downarrow$, we get (unique) y_1, y_2 with $\tau(\overline{\alpha}(y_1)) > 0$ and $\sigma(\overline{\alpha}(y_2)) > 0$.

Since $\mathrm{Seq}(\overline{\alpha}(y_1))$ holds, from the density of $\neg A(\alpha)$ we obtain an α_1 with $\neg A(\alpha_1)$ and $\overline{\alpha}_1(y_1) = \overline{\alpha}(y_1)$; so (d) $\tau(\overline{\alpha}_1(y_1)) > 0$. And similarly an α_2 with $\neg A(\alpha_2)$ and $\overline{\alpha}_2(y_2) = \overline{\alpha}(y_2)$, so that (e) $\sigma(\overline{\alpha}_2(y_2)) > 0$.

Now we argue by cases. If $y_1 \le y_2$, then we have

$$\text{(f)}\;\; \overline{\alpha}_1(y_1) = \overline{\alpha}(y_1) \sqsubseteq \overline{\alpha}(y_2) = \overline{\alpha}_2(y_2).$$

Now by (d), (f) and the fact that $\{\tau\}(\alpha_2)\!\downarrow$, we get (*g*) $\{\tau\}(\alpha_1) = \{\tau\}(\alpha_2)$. By $\neg A(\alpha_2)$ and the $\exists!$b of the second hypothesis we get (*h*) $\{\tau\}(\alpha_2) = \{\sigma\}(\alpha_2)$. By (g), (h) and the fact that $\{\tau\}(\alpha) = \{\tau\}(\alpha_1)$ and $\{\sigma\}(\alpha) = \{\sigma\}(\alpha_2)$, we get finally $\{\tau\}(\alpha) = \{\sigma\}(\alpha)$.

The case $y_2 < y_1$ is treated similarly, so we have (c) and then the theorem follows by logic.

□

7.2. A Weak Continuity Alternative of VS!. We consider the following version of Vesley's Schema with uniqueness, which ensures the existence of a weakly continuous total extension of the given partial function:

$$\text{WVS!}\;\; \forall w\,[\,\mathrm{Seq}(w) \to \exists\alpha(\overline{\alpha}(\mathrm{lh}(w)) = w \;\&\; \neg A(\alpha))\,] \to [\,\forall\alpha(\neg A(\alpha)$$
$$\to \exists! b\, B(\alpha, b)) \to \forall\alpha\exists m\exists b\forall\beta\,[\,\overline{\beta}(m) = \overline{\alpha}(m) \to (\neg A(\beta) \to B(\beta, b))\,]\,],$$

where b does not occur free in A.

THEOREM 7.5. $\mathbf{M} + \mathrm{WVS!} \vdash \neg\mathrm{MP}_1$.

PROOF. Following the proof of Theorem 7.3 but considering WVS! instead of VS!, we assume $\mathrm{MP}_1!!$ and we get

$$\text{(a)}\ \forall\alpha\exists\mathrm{m}\exists\mathrm{x}\forall\beta\,[\,\overline{\beta}(\mathrm{m}) = \overline{\alpha}(\mathrm{m}) \to (\neg\neg\exists!\mathrm{x}\beta(\mathrm{x}) = 0 \to \beta(\mathrm{x}) = 0)\,]\,.$$

Letting $\alpha = \lambda\mathrm{t}.1$ we get from (a) $\mathrm{m}_1, \mathrm{x}_1$ with

$$\text{(b)}\ \forall\beta\,[\,\overline{\beta}(\mathrm{m}_1) = \overline{\alpha}(\mathrm{m}_1) \to (\neg\neg\exists!\mathrm{x}\,\beta(\mathrm{x}) = 0 \to \beta(\mathrm{x}_1) = 0)\,]\,.$$

We define now α_1 by *(c)* $\alpha_1(\mathrm{x}) = 0$ if $\mathrm{x} = \max(\mathrm{m}_1, \mathrm{x}_1) + 1$ and 1 otherwise. So we have $\overline{\alpha}_1(\mathrm{m}_1) = \overline{\alpha}(\mathrm{m}_1)$ and $\neg\neg\exists!\mathrm{x}\,\alpha_1(\mathrm{x}) = 0$ so $\alpha_1(\mathrm{x}_1) = 0$ by (b), but $\alpha_1(\mathrm{x}_1) = 1$ by (c). So we have a contradiction which gives us finally $\neg\mathrm{MP}_1$.

□

REMARKS. (a) We include the above proof because the same argument can show that in Vesley's original proof of the negation of Markov's Principle from Vesley's Schema it suffices to use the weak continuity principle instead of the strong one of **FIM**.

(b) The weak continuity condition of WVS! suffices also for ensuring the uniqueness of the weakly continuous total extension of the given partial function. We can see this with an argument very similar to the one of Theorem 7.4:

Assume that we have two functions that coincide on the negative dense domain described in WVS!. Consider an argument α. The weak continuity condition gives moduli of continuity $\mathrm{m}_1, \mathrm{m}_2$ and values $\mathrm{b}_1, \mathrm{b}_2$ respectively for the two functions at α. The density condition gives then α_1, α_2 with $\overline{\alpha}_1(\mathrm{m}_1) = \overline{\alpha}(\mathrm{m}_1)$, $\overline{\alpha}_2(\mathrm{m}_2) = \overline{\alpha}(\mathrm{m}_2)$ and $\neg\mathrm{A}(\alpha_1)$, $\neg\mathrm{A}(\alpha_2)$. Arguing now by cases for $\mathrm{m}_1, \mathrm{m}_2$ and with the basic remark that $\overline{\alpha}_1(\mathrm{m}_1)$ and $\overline{\alpha}_2(\mathrm{m}_2)$ are initial segments of the same sequence α and so comparable, we can get that $\mathrm{b}_1 = \mathrm{b}_2$.

7.3. Consequences of VS! on some non constructive local principles. In a previous section we have seen that using continuity principles we can get underivability results concerning non constructive principles, and even formal negations of global and local non constructive principles. From VS! (and just by the same proofs from WVS!) we obtain the following related results.

PROPOSITION 7.6.
(a) $\mathbf{M} + \mathrm{VS!} \vdash \neg\forall\alpha\,[\,\mathrm{WLPO}(\alpha) \to \mathrm{MP}_1(\alpha)\,]\,.$
(b) $\mathbf{M} + \mathrm{VS!} \vdash \neg\forall\alpha\,[\,\mathrm{WLPO}(\alpha) \to \mathrm{LPO}(\alpha)\,]\,.$

PROOF. (a) In **M** we can very easily see that

$$\vdash \forall\alpha[(\neg\neg\exists x\,\alpha(x)=0 \vee \neg\exists x\,\alpha(x)=0) \rightarrow (\neg\neg\exists x\alpha(x)=0 \rightarrow \exists x\,\alpha(x)=0)] \\ \rightarrow \forall\alpha\,[\,\neg\neg\exists x\,\alpha(x)=0 \rightarrow \exists x\,\alpha(x)=0\,],$$

which is just $\forall\alpha\,[\,\mathrm{WLPO}(\alpha) \rightarrow \mathrm{MP}_1(\alpha)\,] \rightarrow \mathrm{MP}_1$. But from Theorem 7.3 we have $\mathbf{M}+\mathrm{VS}! \vdash \neg\mathrm{MP}_1$, so we get (a).

(b) In **M** we can also very easily see that

$$\vdash \forall\alpha\,[\,(\neg\neg\exists x\,\alpha(x)=0 \vee \neg\exists x\,\alpha(x)=0) \rightarrow (\neg\exists x\,\alpha(x)=0 \vee \exists x\,\alpha(x)=0)\,] \\ \rightarrow \forall\alpha\,[\,\neg\neg\exists\,\alpha(x)=0 \rightarrow \exists x\,\alpha(x)=0\,],$$

which is $\forall\alpha\,[\,\mathrm{WLPO}(\alpha) \rightarrow \mathrm{LPO}(\alpha)\,] \rightarrow \mathrm{MP}_1$, so again from the fact that $\mathbf{M}+\mathrm{VS}! \vdash \neg\mathrm{MP}_1$ we get (b).

□

Since $\mathbf{M}+\mathrm{GC}_1! \nvdash \mathrm{MP}_1$, for the system $\mathbf{M}+\mathrm{GC}_1!$ we have as a corollary of the above proofs the following weaker results.

PROPOSITION 7.7.
(a) $\mathbf{M}+\mathrm{GC}_1! \nvdash \forall\alpha\,[\,\mathrm{WLPO}(\alpha) \rightarrow \mathrm{MP}_1(\alpha)\,]$.
(b) $\mathbf{M}+\mathrm{GC}_1! \nvdash \forall\alpha\,[\,\mathrm{WLPO}(\alpha) \rightarrow \mathrm{LPO}(\alpha)\,]$.

8. On some extensions of intuitionistic analysis

Intuitionistic analysis is compatible with principles and sentences which are not theorems of it. One fact asserting this is that the various realizability notions which serve as interpretations of formal systems for analysis, validate also different sentences, unprovable in the corresponding systems: Kleene's functional realizability validates Markov's Principle (with classical reasoning) and Troelstra's Generalized Continuity Principle, while versions of (implicitly or explicitly) typed realizability validate the negation of Markov's Principle, Vesley's Schema, a weak form of Church's Thesis, and the Independence of Premise Principle (stated below) for example.

This situation suggests the possibility of diverging extensions of intuitionistic analysis, from which different pictures of the intuitionistic continuum emerge. And so the problem of the comparison of such extensions arises. In her recent paper "Unavoidable sequences in constructive analysis" [JRM2010], J. R. Moschovakis formulates this question, and proposes comparing in detail the theories $\mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1 + \mathrm{GC}_1$ and $\mathbf{FIM} + \mathrm{VS}$ from a reverse mathematics perspective, observing that MP_1 and VS are both classically correct principles. The following could be considered as a first step towards this direction, as well as towards charting the area of the constructive theories for the continuum.

8.1. Independence of premise. In our following arguments we will make use of the principle of Independence of Premise, so we state it and show some consequences of it. The Independence of Premise Principle that we are considering here is the following schema:

$$\mathrm{IP} \qquad (\neg \mathrm{A} \to \exists \beta \mathrm{B}(\beta)) \to \exists \beta (\neg \mathrm{A} \to \mathrm{B}(\beta)),$$

where β does not occur free in A.

IP is consistent with **FIM**, since it is special-realizable and also ^{G}realizable, but **FIM** does not entail this principle, as S. C. Kleene has shown[4], but also as it follows from the next propositions. Vesley's Schema is derivable in **M** from IP, but Troelstra's Generalized Continuity is inconsistent with IP. This is already known, but we can see it also in the following simple way.

Proposition 8.1. $\mathbf{M} + \mathrm{IP} \vdash \forall \alpha \, [\, \mathrm{MP}_1(\alpha) \to \mathrm{LPO}(\alpha) \,]$.

Proof. Assume

$$\neg\neg \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0.$$

Then using IP, we obtain

$$\exists \mathrm{x}(\neg\neg \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \alpha(\mathrm{x}) = 0).$$

Now assume

$$(*) \quad \neg\neg \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \to \alpha(\mathrm{x}) = 0.$$

Since we have $\alpha(\mathrm{x}) = 0 \vee \alpha(\mathrm{x}) \neq 0$, we argue by cases and get

$$\exists \mathrm{x}\, \alpha(\mathrm{x}) = 0 \vee \neg \exists \mathrm{x}\, \alpha(\mathrm{x}) = 0$$

(in the first case the conclusion is immediate, in the second obtained by $(*)$).

□

Corollary 8.2. $\mathbf{M} + \mathrm{GC}_1! + \mathrm{IP} \vdash 0 = 1$.

Proof. We have already seen that

$$\mathbf{M} + \mathrm{GC}_1! \vdash \neg \forall \alpha \, [\, \mathrm{MP}_1(\alpha) \to \mathrm{LPO}(\alpha) \,].$$

From this and the previous proposition we get the corollary.

□

In the next two corollaries we see how to get $\neg \mathrm{MP}_1$ from IP and weak continuity almost immediately.

[4]S. C. Kleene in his paper "Logical Calculus and Realizability" gives a classical proof that a special disjunctive instance of IP is not realizable in the sense of **FIM**.

COROLLARY 8.3. $\mathbf{M} + \mathrm{IP} \vdash \mathrm{MP}_1 \to \mathrm{LPO}$.

PROOF. By Proposition 8.1, using

$$\forall\alpha(\mathrm{A}(\alpha) \to \mathrm{B}(\alpha)) \to (\forall\alpha\mathrm{A}(\alpha) \to \forall\alpha\mathrm{B}(\alpha)).$$

□

COROLLARY 8.4. $\mathbf{M} + \mathrm{IP} + \mathrm{WC}_0! \vdash \neg\mathrm{MP}_1$.

PROOF. By Corollary 8.3 and the fact that $\mathbf{M} + \mathrm{WC}_0! \vdash \neg\mathrm{LPO}$. □

REMARK. From the above corollary we can see immediately that **FIM** does not entail IP, otherwise **FIM** would prove $\neg\mathrm{MP}_1$.

In the previous chapter we derived in $\mathbf{M}$ the negation of Markov's Principle from Vesley's Schema and weak continuity. We give here another very short proof of the same fact, using arguments very similar to the ones of this paragraph.

PROPOSITION 8.5. $\mathbf{M} + \mathrm{VS} \vdash \mathrm{MP}_1 \to \mathrm{LPO}$.

PROOF. Assume

$$\forall\alpha\,[\,\neg\neg\exists \mathrm{x}\,\alpha(\mathrm{x}) = 0 \to \exists \mathrm{x}\,\alpha(\mathrm{x}) = 0\,].$$

Then by VS, we obtain

$$\forall\alpha\,\exists \mathrm{x}\,[\,\neg\neg\exists \mathrm{x}\,\alpha(\mathrm{x}) = 0 \to \alpha(\mathrm{x}) = 0\,].$$

Arguing for each α by cases as in Proposition 8.1, we get

$$\forall\alpha\,[\,\exists \mathrm{x}\alpha(\mathrm{x}) = 0 \vee \neg\exists \mathrm{x}\alpha(\mathrm{x}) = 0\,].$$

□

COROLLARY 8.6. $\mathbf{M} + \mathrm{WC}_0! + \mathrm{VS} \vdash \neg\mathrm{MP}_1$.

PROOF. By Proposition 8.5 and the fact that $\mathbf{M} + \mathrm{WC}_0! \vdash \neg\mathrm{LPO}$. □

8.2. A theory between FIM and each of $\mathbf{FIM} + \mathrm{VS}$ and $\mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1 + \mathrm{GC}_1$. Although IP is inconsistent with GC_1 as we have seen, it is yet an open question if VS is consistent or not with GC_1 (over $\mathbf{M}$). But because of the presence of MP_1, it is clear that the two theories $\mathbf{M} + \mathrm{BI}_1 + \mathrm{MP}_1 + \mathrm{GC}_1$ and $\mathbf{FIM} + \mathrm{VS}$ are divergent. Their common part is not just **FIM**, but a larger theory, as the following show.

We consider the following version of generalized continuity:

$$\mathrm{GC}^- \quad \forall\alpha\,[\,\neg\mathrm{A}(\alpha) \to \exists\beta\mathrm{B}(\alpha,\beta)\,] \to \exists\sigma\forall\alpha\,[\,\neg\mathrm{A}(\alpha) \to \{\sigma\}[\,\alpha\,]\!\downarrow\ \&\ \mathrm{B}(\alpha,\{\sigma\}[\,\alpha\,])\,],$$

where the domain of continuity is a negation instead of almost negative as in GC_1.

GC^- can be consistently added to **FIM** + VS, as it is immediately derivable from IP and CC_1. The interesting feature of this principle is that it allows us to exploit two facts: (1) every negative formula[5] is equivalent over **M** to a negation and (2) every negative formula is almost negative. If we call GC^n the common special case of GC^- and of GC_1 where the domain of continuity is defined by a negative formula, we obtain the theory **FIM** + GC^n, which lies between **FIM** and its two divergent extensions.

[5]A formula is negative if it does not contain $\vee$ or $\exists$.

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