Annotated Sequent Systems for Linear Temporal Logic

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Abstract

Annotated sequents provide an elegant approach for the design of deductive systems for temporal logics. Their proof theory, however, is notoriously difficult. Until recently it was not even clear how to syntactically show the admissibility of weakening. In this thesis we present a cut-free, finitary sequent system for linear temporal logic, based on annotated sequents. We present proofs for soundness and completeness and also present a purely syntactical proof for the admissibility of weakening in the aforementioned system. Furthermore, we investigate the role of cut in annotated sequent systems.

keywords: annotated sequents, linear temporal logic, weakening, proof-theory, sequent calculus

Περίληψη

Χρησιμοποιώντας ακολουθητές με υποσημειώσεις, μπορούμε να σχεδιάσουμε κατανοητά και απλά συστήματα απαγωγής για χρονικές λογικές. Όμως, οι ακολουθητές με υποσημειώσεις, παρά τη σαφήνεια που προσφέρουν, καθιστούν την απόδειξη συντακτικών ιδιοτήτων πάρα πολύ δύσκολη. Μέχρι πρόσφατα, δεν ήταν καν σαφές πως να αποδείξουμε (με καθαρά συντακτικές μεθόδους) ότι ένα σύστημα ακολουθητών με υποσημειώσεις αποδέχεται την εξασθένηση. Σε αυτή την εργασία παρουσιάζουμε ένα σύστημα ακολουθητών με υποσημειώσεις για τη γραμμική χρονική λογική, το οποίο δε βασίζεται ούτε σε κάποιον κανόνα με άπειρες υποθέσεις ούτε στον κανόνα της τομής. Παρουσιάζουμε αποδείξεις ορθότητας και πληρότητας, καθώς και μία καθαρά συντακτική απόδειξη για την αποδοχή της εξασθένησης στο εν λόγω σύστημα.

λέξεις κλειδιά: ακολουθητές με υποσημειώσεις, γραμμική χρονική λογική, θεωρία αποδείξεων, εξασθένηση, λογισμός ακολουθητών

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Contents

Chapter 1

Introduction

1.1 Overview

The aim of this thesis is to demonstrate how annotated sequent systems can be used to describe linear temporal logic (LTL) and to investigate their proof theory. This thesis consists of 5 Chapters and is mainly based on [2] and [4]. The overview of the chapters is as follows:

Chapter 1 consists of a brief overview of the thesis and some discussion about the proof-theory of annotated sequent systems.

In **Chapter 2** we discuss how a sequent system for LTL can be designed. We define the syntax and semantics of LTL and we see a naive approach for defining a sequent system for LTL. We explain why the naive approach cannot work and finally we present annotations and system LT1.

In **Chapter 3** we prove that system LT1 is complete only if we add a cut rule to it. We also show that in order to define a complete sequent system for LTL, system LT1 needs to be enriched significantly.

In **Chapter 4** we define the notion of histories. We also present system LT2 and give soundness and completeness proofs for it. Finally we prove weakening for LT2 by purely syntactical methods.

In **Chapter 5** we summarize all the results presented in this thesis and also present some open problems.

The chapters can be divided in to two main parts. In the first part (Chapters 2 and 3) we explain the design of the finitary and cut-free sequent system for LTL defined in [2]. In the second part (Chapter 4) we present the system of [2] together with soundness and completeness proofs for it. We also present a purely syntactical proof for the admissibility of weakening in the aforementioned system, which was an open problem from [2].

1.2 On the Proof-Theory of Annotated Sequent Systems

The proof theory of temporal logics, and of modal fixed point logics in general, is notoriously difficult. It is not even clear how to design a finitary deductive system for LTL with nice proof-theoretic properties. In this context, deductive systems featuring infinite long proof branches (together with a global soundness condition) and their cyclic variants have recently obtained much attention, see, for instance, [1, 2, 3, 7, 8, 9].

Brünnler and Lange [2] proposed an elegant formalism with cyclic proofs for LTL using focus games from Lange and Stirling [6] as an inspiration. The main technical feature of their system are annotated sequents, which are employed to derive greatest fixed points. However, some very basic proof-theoretic problems turn out to be surprisingly hard in this setting. For instance, despite the admissibility of several structural rules, including cut, being proved semantically in [2], it remains to prove the same facts by proper proof-theoretic methods. Even the admissibility of weakening, which is quite trivial for most types of sequent calculi, is far from being simple for annotated sequents due to the presence of sequent contexts in the annotations.

As Brünnler and Lange point out [2], the problem with weakening is to be expected since the fact that a certain statement is provable by induction does not imply that a weaker statement is also provable by induction.

In this thesis we present a solution to this open problem and establish the admissibility of weakening by proof-theoretic means.

Chapter 2

Designing a Sequent System for LTL

In this chapter we explore an approach for defining a sound and complete sequent calculus for LTL. Note that we only study the unary fragment of LTL. This is enough to discuss the proof-theoretic problems and also simplifies the presentation.

We start with defining the syntax and semantics for LTL-sequents. Then we recall a naive way of giving an LTL-sequent calculus and explain its shortcomings. We finish this chapter with introducing the sequent calculus LT1, which is based on so-called annotations.

This chapter is taken from [4].

2.1 Syntax and Semantics

We start with a countable set of atomic propositions, which we call Prop. The language of the sequent calculus for LTL, $\mathcal{L}_\mathcal{S}$, is then described by the following grammar:

$$A ::= P \mid \overline{P} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A \mid \bigcirc A$$

where $P \in \mathsf{Prop}$ and $\overline{P}$ denotes the negation of P. In this thesis, we assume right associativity for all binary connectives.

We define the set of sequents, Seq, by:

$$\mathsf{Seq} := \left\{ \Gamma \;\middle|\; \Gamma \text{ is a finite subset of } \mathcal{L}_\mathcal{S} \right\}$$

We will use capital greek letters like $\Gamma,\ \Delta,\ \Sigma,\dots$ for sequents, capital latin letters like $A,\ B,\ C, D,\ \dots$ for $\mathcal{L}_{\mathcal{S}}$-formulas and the letters P, Q for elements of Prop, all of them possibly primed or with subscripts. As usual union is represented by comma, i.e.:

$$\Gamma, \Delta \text{ stands for } \Gamma \cup \Delta$$
$$\Gamma, A \text{ stands for } \Gamma \cup \{A\}$$

We define the negation of an $\mathcal{L}_{\mathcal{S}}$-formula as usual by:

$$\begin{aligned} \overline{\overline{P}} &:= P & \overline{\bigcirc A} &:= \bigcirc \overline{A} \\ \overline{A \vee B} &:= \overline{A} \wedge \overline{B} & \overline{\Box A} &:= \Diamond \overline{A} \\ \overline{A \wedge B} &:= \overline{A} \vee \overline{B} & \overline{\Diamond A} &:= \Box \overline{A} \end{aligned}$$

Let $\Gamma \in \mathsf{Seq}$. We define the following sequents:

$$\bigcirc\Gamma := \{\bigcirc A \mid A \in \Gamma\}$$
$$\Gamma/\bigcirc := \{A \mid \bigcirc A \in \Gamma\}$$

Now we can define the notion of LTL-model and validity.

Definition 1 (LTL-model)**.** An LTL-model or simply a model is a function μ that maps natural numbers to sets of atomic propositions, i.e.:[1]

$$\mu : \mathbb{N} \to \mathcal{P}(\mathsf{Prop})$$

Let μ be a model. Every natural number i represents a point in time and the set $\mu(i)$ represents the facts that hold at the time-point i. The expression $\mu, i \models A$ stands for *model μ satisfies formula A at time-point i.* The relation $\models$ is defined as follows:

Definition 2 (Satisfiability of $\mathcal{L}_{\mathcal{S}}$-formulas)**.** Let μ be a model and let $i \in \mathbb{N}$. We have:

$$\begin{aligned} \mu, i \models P &\iff P \in \mu(i) \\ \mu, i \models \overline{P} &\iff P \notin \mu(i) \\ \mu, i \models A \wedge B &\iff \mu, i \models A \text{ and } \mu, i \models B \\ \mu, i \models A \vee B &\iff \mu, i \models A \text{ or } \mu, i \models B \\ \mu, i \models \Box A &\iff \forall j \geq i(\mu, j \models A) \\ \mu, i \models \Diamond A &\iff \exists j \geq i(\mu, j \models A) \\ \mu, i \models \bigcirc A &\iff \mu, i+1 \models A \end{aligned}$$

[1] $\mathcal{P}$ stands for powerset

A formula A is *valid*, denoted by $\models A$, iff $(\forall\mu)(\forall i)[\mu, i \models A]$, i.e. iff A holds in all models at all time-points. A formula A is satisfiable iff there is a model μ and $i \in \mathbb{N}$ such that $\mu, i \models A$.

Let $\Gamma \in \mathsf{Seq}$. We assume that Γ is semantically equivalent to the disjunction of its elements, i.e. we set

$$\mu, i \models \Gamma \quad \Longleftrightarrow \quad \mu, i \models \bigvee_{A \in \Gamma} A$$

The notions of validity and satisfiability are extended to sequents in the obvious way. If $\Gamma = \emptyset$ we have that Γ is semantically equivalent to the false statement, i.e. the empty sequent is not satisfiable.

2.2 A Naive Approach

We begin with a "naive" sequent calculus for LTL, called $\mathsf{LT}_{\mathsf{naive}}$, which is given in Figure 2.1. This system contains the usual propositional axioms and rules (aid, $\vee$, $\wedge$), the rule for $\bigcirc$ and the rules for unfolding $\Box$ and $\Diamond$.

Let α be a rule of a sequent system. The sequents above α's line will be called the premises of α and the sequent below α's line will be called α's conclusion. We categorize the formulas appearing in α as follows:

- **principal formulas of $\boldsymbol{\alpha}$:** the formulas that are explicitly shown in α's conclusion, i.e. do not belong in Γ. Exception is the $\bigcirc$-rule, where all formulas of $\bigcirc\Gamma$ and only those are principal formulas.

- **side formulas of $\boldsymbol{\alpha}$:** formulas that belong to the sequents Γ, Σ. Exception is the $\bigcirc$-rule where the formulas of Σ and only those are side formulas.

A rule without premises will sometimes be called an axiom. We make no distinction between an axiom's premises and conclusion.

We will use lowercase greek letters like α, β, $\gamma, \ldots$ possibly primed or with subscripts to represent rules. As usual $\mathsf{L} \vdash A$ means that the logic L proves the formula A. When L is clear from the context we may simply write $\vdash A$. By $\mathsf{L} \overset{n}{\vdash} A$ we mean that there is a derivation of A in L with depth at most n.

Soundness of all the rules of system $\mathsf{LT}_{\mathsf{naive}}$ is proved in Theorem 19.

$$\text{aid}\ \frac{}{\Gamma, P, \overline{P}} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\Box\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box A}{\Gamma, \Box A} \qquad \Diamond\ \frac{\Gamma, A, \bigcirc\Diamond A}{\Gamma, \Diamond A} \qquad \bigcirc\ \frac{\Gamma}{\Sigma, \bigcirc\Gamma}$$

Figure 2.1: System $\mathsf{LT}_{\mathsf{naive}}$

But what about completeness? The first observation, see [2], is that system $\mathsf{LT}_{\mathsf{naive}}$ almost works: the only thing that goes wrong is that we cannot derive induction principles as is shown in the following example.

Example 3. The valid sequent

$$\Gamma = \Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \Box P$$

cannot be proved in $\mathsf{LT}_{\mathsf{naive}}$. This sequent is semantically equivalent to the temporal induction axiom

$$P \wedge \Box(P \rightarrow \bigcirc P) \rightarrow \Box P$$

where the connective $\rightarrow$ is interpreted in the standard way.

An attempt to prove Γ in $\mathsf{LT}_{\mathsf{naive}}$ will lead to a derivation like the following:

$$\Box\ \frac{\text{aid}\ \dfrac{}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, P} \qquad \Diamond\ \dfrac{\wedge\ \dfrac{\text{aid}\ \dfrac{}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), P, \overline{P}, \bigcirc\Box P} \qquad \bigcirc\ \dfrac{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \Box P}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), \bigcirc\overline{P}, \overline{P}, \bigcirc\Box P}}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), P \wedge \bigcirc\overline{P}, \overline{P}, \bigcirc\Box P}}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \bigcirc\Box P}}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \Box P}$$

Observe that the endsequent reappears in the top right of the derivation. Hence there is no proof of Γ in $\mathsf{LT}_{\mathsf{naive}}$.

When we try to prove a formula that contains the operator $\Box$, the proof-search will fail like it did in the above example for sequent Γ. However, the obvious idea of just closing a cyclic branch as axiomatic will lead to an unsound system as is illustrated in the next example, see [2].

Example 4. Consider the non-valid sequent $\Delta = \Box P, \bigcirc\Diamond\Box P$. If we could close all the cyclic branches then we would have the following proof for Δ in $\mathsf{LT}_{\mathsf{naive}}$:

$$\dfrac{\bigcirc\,\dfrac{\Diamond\,\dfrac{\Box P, \bigcirc\Diamond\Box P}{\Diamond\Box P}}{P, \bigcirc\Diamond\Box P} \qquad \bigcirc\,\dfrac{\Diamond\,\dfrac{\Box P, \bigcirc\Diamond\Box P}{\Box P, \Diamond\Box P}}{\bigcirc\Box P, \bigcirc\Diamond\Box P}}{\Box P, \bigcirc\Diamond\Box P}\;\Box$$

Hence, a better idea, than simply closing every cyclic branch, is required. Brünnler and Lange's idea ([2]) is to close a cyclic branch if there is a formula such that whenever the $\Box$-rule is applied to it between the two occurrences of the cyclic sequent, the branch is along the right premise. Thus, in Example 3, the rightmost branch of the $\mathsf{LT}_{\mathsf{naive}}$-derivation for Γ would be closed, thus yielding a correct proof. In Example 4, however, the left branch in the derivation for Δ would not be closed and hence this would not be a proof for the non-valid sequent Δ. In order to implement this idea, we have to enrich our syntax with the so-called *annotations*.

2.3 Annotations – System LT1

We define the set of *annotated* formulas, $\mathcal{L}_{\mathsf{ann}}$:

$$\mathcal{L}_{\mathsf{ann}} := \left\{ \Box_\Gamma A, \bigcirc\Box_\Gamma A \;\middle|\; A \in \mathcal{L}_{\mathcal{S}}, \Gamma \in \mathsf{Seq} \right\}$$

In $\Box_\Gamma A$, the sequent Γ is called an annotation. We define the set of annotated sequents:

$$\mathsf{Seq}_{\mathsf{ann}} := \left\{ \Gamma \;\middle|\; \begin{array}{l}\Gamma \text{ is a finite subset of } \mathcal{L}_{\mathcal{S}} \cup \mathcal{L}_{\mathsf{ann}} \text{ that contains} \\ \text{at most one annotated formula}\end{array} \right\}$$

The semantics of $\Box_\Gamma A$ is defined as follows. Let μ be an LTL-model and let $i \in \mathbb{N}$. We have:

$$\mu, i \models \Box_\Gamma A \quad \Longleftrightarrow \quad \forall j \geq i \Big(\big(\forall i \leq k \leq j (\mu, k \models \Gamma) \big) \Longrightarrow \mu, j \models A \Big)$$

System $\mathsf{LT1}$ is given by the axioms and rules in Figure 2.2. For the $\bigcirc$-rule, we assume $\Sigma \in \mathsf{Seq}$, i.e. Σ does not contain annotated formulas. In all the other rules, Γ may contain annotated formulas, when the syntactical restrictions allow it. System $\mathsf{LT1}$ contains all the rules of the system $\mathsf{LT}_{\mathsf{naive}}$ plus the rule

$$\mathsf{aid}\ \frac{}{\Gamma, P, \overline{P}} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\mathsf{rep}\ \frac{}{\Gamma, \Box_\Gamma A} \qquad \mathsf{foc}\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_\Gamma A}{\Gamma, \Box A}$$

$$\Box\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box A}{\Gamma, \Box A} \qquad \Diamond\ \frac{\Gamma, A, \bigcirc\Diamond A}{\Gamma, \Diamond A} \qquad \bigcirc\ \frac{\Gamma}{\Sigma, \bigcirc\Gamma}$$

Figure 2.2: System LT1

foc and the axiom rep. The name of the rule foc stands for focus and implies that we focus on a specific $\Box$-formula, i.e. the formula $\Box A$ in the conclusion of the foc-rule. We focus on this formula by annotating it with its context, i.e. Γ. When the annotated formula appears in exactly the same context (i.e. Γ), then we can close the branch as axiomatic using the axiom rep (the name of this axiom stands for repetition).

The sequent $\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \Box P$ (which could not be proved in system $\mathsf{LT}_{\mathsf{naive}}$) can be derived in LT1 as follows:

$$\mathsf{foc}\ \dfrac{\mathsf{aid}\ \dfrac{}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, P} \qquad \Diamond\ \dfrac{\wedge\ \dfrac{\mathsf{aid}\ \dfrac{}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), P, \overline{P}, \bigcirc\Box_\Gamma P} \qquad \bigcirc\ \dfrac{\mathsf{rep}\ \dfrac{}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \Box_\Gamma P}}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), \bigcirc\overline{P}, \overline{P}, \bigcirc\Box_\Gamma P}}{\bigcirc\Diamond(P \wedge \bigcirc\overline{P}), P \wedge \bigcirc\overline{P}, \overline{P}, \bigcirc\Box_\Gamma P}}{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}, \bigcirc\Box_\Gamma P}}{\underbrace{\Diamond(P \wedge \bigcirc\overline{P}), \overline{P}}_{\Gamma}, \Box P}$$

Soundness for all the rules of system LT1 is proved in Theorem 19. However, as we will see later, system LT1 only is complete if we add a cut rule. We define system $\mathsf{LT1}^{\mathsf{cut}}$ to be LT1 enriched with the rule cut:

$$\mathsf{cut}\ \frac{\Gamma, A \qquad \overline{A}, \Delta}{\Gamma, \Delta}$$

In this rule, A is an $\mathcal{L}_\mathcal{S}$-formula.

Before showing the completeness and incompleteness results we need to show that weakening is syntactically admissible in system $\mathsf{LT1}^{\mathsf{cut}}$.

Definition 5. A derivation satisfies the *next-property* iff any branch from the endsequent to any instance of foc goes through at least one $\bigcirc$-rule.

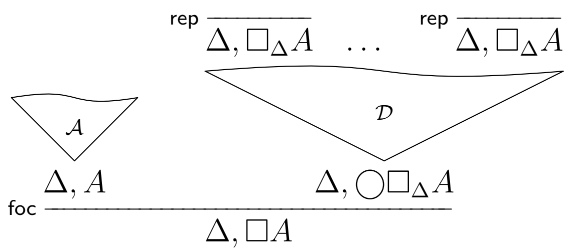
Lemma 6. *Let* $\Gamma \in \mathsf{Seq}$. *If* $\mathsf{LT1}^{\mathsf{cut}} \vdash^{n} \Gamma$, *then there is an* $\mathsf{LT1}^{\mathsf{cut}}$*-proof of* Γ *satisfying the next-property.*

Proof. By induction on n and a case distinction on the last rule.

1. Γ is the conclusion of aid. Then the claim holds trivially.

2. Γ is the conclusion of rep. This case is not possible because of our assumption $\Gamma \in \mathsf{Seq}$.

3. Γ is the conclusion of foc. Then $\Gamma = \Delta, \Box A$ and the given proof $\mathcal{E}$ of Γ has the following form:



First we observe that by the induction hypothesis, there is a proof $\mathcal{A}'$ of Δ, A that satisfies the next-property.

Further we have that in the derivation $\mathcal{D}$

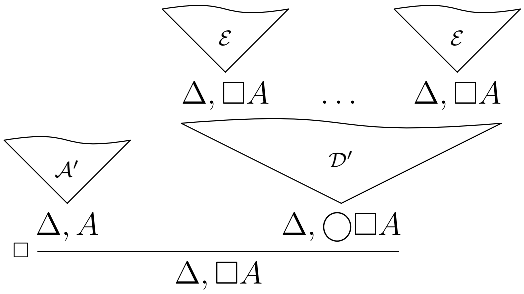
$$\text{any branch from } \Delta, \bigcirc\Box_{\Delta} A \text{ to some } \Delta, \Box_{\Delta} A \text{ goes through a } \bigcirc\text{-rule} \tag{2.1}$$

and

$$\text{there are no instances of foc.} \tag{2.2}$$

Let $\mathcal{D}'$ be the derivation that is obtained from $\mathcal{D}$ by dropping all annotations occurring in threads starting from $\bigcirc\Box_{\Delta} A$, i.e. by replacing sequents of the form $\Sigma, \Box_{\Delta} A$ and $\Sigma, \bigcirc\Box_{\Delta} A$ by $\Sigma, \Box A$ and $\Sigma, \bigcirc\Box A$, rescpectively.

Hence we have the following proof of $\Delta, \Box A$:



This proof satisfies the next-property. Indeed:

(a) The proof $\mathcal{A}'$ satisfies the next-property.

(b) In the derivation $\mathcal{D}'$ every branch from $\Delta, \bigcirc\Box A$ to some $\Delta, \Box A$ goes through at least one $\bigcirc$-rule (because of (2.1)).

(c) The derivation $\mathcal{D}'$ contains no foc-rules (because of (2.2)).

Hence any branch from the conclusion $\Delta, \Box A$ to an instance of foc goes through a $\bigcirc$-rule.

4. In all other cases, the claim follows easily by the induction hypothesis. ⊣

By Lemma 6 we immediately get the following weakening result.

Corollary 7 (Weakening for non-annotated sequents)**.** *For any* $\Gamma, \Delta \in \mathsf{Seq}$ *we have:*

$$\mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma \Longrightarrow \mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma, \Delta$$

Proof. Let $\mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma$. By Lemma 6 we have a proof $\mathcal{D}$ for Γ that satisfies the next-property. We prove the claim by induction on the length of $\mathcal{D}$. Let α be the lowermost rule in $\mathcal{D}$. Since $\mathcal{D}$ satisfies the next-property and $\Gamma \in \mathsf{Seq}$, the rule α can be aid, $\vee, \wedge, \Box, \Diamond$, cut or $\bigcirc$. If α is aid or $\bigcirc$, then claim follows by built-in weaking. Otherwise it follows by the induction hypothesis. ⊣

By a similar proof, we can show invertibiltiy of $\vee$.

Corollary 8 (Invertibility of the $\vee$-rule)**.** *For any* $\Gamma \in \mathsf{Seq}$ *and* $A, B \in \mathcal{L}_{\mathcal{S}}$ *we have:*

$$\mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma, A \vee B \Longrightarrow \mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma, A, B$$

Chapter 3

Completeness and Incompleteness

In this chapter, we show that system $\mathsf{LT1}^{\mathsf{cut}}$ is complete by embedding a complete Hilbert system for LTL in $\mathsf{LT1}^{\mathsf{cut}}$. We also show that the cut-free system $\mathsf{LT1}$ is not complete and that we need more complex annotations to obtain a complete cut-free system for LTL.

This chapter is taken from [4].

3.1 A Hilbert System for LTL

The language $\mathcal{L}_{\mathcal{H}}$ is described by the following grammar:

$$\phi := P \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \bigcirc\phi$$

where $P \in \mathsf{Prop}$. Additionally we will use the following abbreviations:

$$\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi) \qquad \Diamond\phi := \neg\Box\neg\phi$$
$$\phi \to \psi := \neg(\phi \wedge \neg\psi) \qquad \phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$$

We will use the Greek letters $\phi, \psi, \rho, \ldots$ for $\mathcal{L}_{\mathcal{H}}$-formulas possibly primed or with subscripts.

Figure 3.1 shows the Hilbert system Σ_{LTL} for LTL.

Satisfiability of $\mathcal{L}_{\mathcal{H}}$-formulas in an LTL-model is defined as follows:

Axioms:	
(P)	$\vdash \phi$, ϕ is a propositional tautology
(Fun)	$\vdash \bigcirc\neg\phi \leftrightarrow \neg\bigcirc\phi$
($\mathsf{K}_\bigcirc$)	$\vdash \bigcirc(\phi \to \psi) \to (\bigcirc\phi \to \bigcirc\psi)$
(Alw)	$\vdash \Box\phi \to \phi \wedge \bigcirc\Box\phi$
Rules:	
(MP)	$\phi, \phi \to \psi \vdash \psi$
($\mathsf{N}_\bigcirc$)	$\phi \vdash \bigcirc\phi$
(Ind)	$\phi \to \psi, \phi \to \bigcirc\phi \vdash \phi \to \Box\psi$

Figure 3.1: System Σ_{LTL}

Definition 9 (Satisfiability of $\mathcal{L}_\mathcal{H}$-formulas)**.** Let μ be a model and $i \in \mathbb{N}$. We have:

$$\begin{aligned} \mu, i \models P &\iff P \in \mu(i) \\ \mu, i \models \neg\phi &\iff \mu, i \not\models \phi \\ \mu, i \models \phi \wedge \psi &\iff \mu, i \models \phi \text{ and } \mu, i \models \psi \\ \mu, i \models \Box\phi &\iff \forall j \geq i(\mu, j \models \phi) \\ \mu, i \models \bigcirc\phi &\iff \mu, i+1 \models \phi \end{aligned}$$

Validity for $\mathcal{L}_\mathcal{H}$-formulas is defined in the same way as for $\mathcal{L}_\mathcal{S}$-formulas. The following theorem is proved in Chapter 2 of [5].

Theorem 10 (Soundness and Completeness)**.** *The system Σ_{LTL} is sound and complete with respect to* LTL*-models, i.e. we have for all $\mathcal{L}_\mathcal{H}$-formulas ϕ:*

$$\models \phi \iff \Sigma_{\mathsf{LTL}} \vdash \phi$$

3.2 System LT1 + cut is Complete

In this section we will show that the system $\mathsf{LT1}^{\mathsf{cut}}$ is complete by embedding system Σ_{LTL} to system $\mathsf{LT1}^{\mathsf{cut}}$. Before proving the embedding we need some auxiliary definitions and lemmata.

As a first step we extend axiom aid to all $\mathcal{L}_\mathcal{S}$-formulas.

Lemma 11. *[Extension of axiom* aid *to $\mathcal{L}_\mathcal{S}$-formulas] Let $\Gamma \in$* Seq *and let $A \in \mathcal{L}_\mathcal{S}$. Then we have the following in system $\mathsf{LT1}^{\mathsf{cut}}$:*

$$\vdash \Gamma, A, \overline{A}$$

Proof. The proof is by induction on the structure of A. We distinguish the following cases:

1. $A = P \in \mathsf{Prop}$. Then we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$\dfrac{}{\Gamma, P, \overline{P}}\ \mathsf{aid}$$

2. $A = \overline{P}$ for $P \in \mathsf{Prop}$. It holds that $\overline{A} = P$. Then we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$\dfrac{}{\Gamma, \overline{P}, P}\ \mathsf{aid}$$

3. $A = B \vee C$. It holds that $\overline{A} = \overline{B} \wedge \overline{C}$. Then we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, B, C, \overline{B}}\ \text{i.h.} \quad \dfrac{}{\Gamma, B, C, \overline{C}}\ \text{i.h.}}{\Gamma, B, C, \overline{B} \wedge \overline{C}}\ \wedge}{\Gamma, B \vee C, \overline{B} \wedge \overline{C}}\ \vee$$

4. $A = B \wedge C$. Similar to case 3.

5. $A = \bigcirc B$. It holds that $\overline{A} = \bigcirc \overline{B}$. Then we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$\dfrac{\dfrac{}{\Gamma/\bigcirc, B, \overline{B}}\ \text{i.h.}}{\Gamma, \bigcirc B, \bigcirc \overline{B}}\ \bigcirc$$

6. $A = \Box B$. It holds $\overline{A} = \Diamond \overline{B}$. Then we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B, \bigcirc\Diamond\overline{B}, \overline{B}}\ \text{i.h.}}{B, \Diamond\overline{B}}\ \Diamond \quad \dfrac{\dfrac{\dfrac{}{\Box_{\Diamond\overline{B}} B, \Diamond\overline{B}}\ \mathsf{rep}}{\bigcirc\Box_{\Diamond\overline{B}} B, \bigcirc\Diamond\overline{B}, \overline{B}}\ \bigcirc}{\bigcirc\Box_{\Diamond\overline{B}} B, \Diamond\overline{B}}\ \Diamond}{\Box B, \Diamond\overline{B}}\ \mathsf{foc}}{\Gamma, \Box B, \Diamond\overline{B}}\ \text{Corollary 7}$$

7. The case $A = \Diamond B$ simply is dual to the case $A = \Box B$.

$\dashv$

An easy induction on the structure of the formula A also yields the following lemma.

Lemma 12. *Let $A \in \mathcal{L}_{\mathcal{S}}$. It holds that:*

$$\overline{\overline{A}} = A$$

Proof. By induction on A. We distinguish the following cases:

1. $A \equiv P \in \mathsf{Prop}$. Then the claim holds by definition.
2. $A \equiv \overline{P} \in \mathsf{Prop}$. We have:
$$\overline{\overline{A}} = \overline{\overline{\overline{P}}} = \overline{P} = A$$
3. $A \equiv B \wedge C$. We have:
$$\overline{\overline{A}} = \overline{\overline{B \wedge C}} = \overline{\overline{B} \vee \overline{C}} = \overline{\overline{B}} \wedge \overline{\overline{C}} \stackrel{\text{i.h.}}{=} B \wedge C = A$$
4. $A \equiv B \vee C$. We have:
$$\overline{\overline{A}} = \overline{\overline{B \vee C}} = \overline{\overline{B} \wedge \overline{C}} = \overline{\overline{B}} \vee \overline{\overline{C}} \stackrel{\text{i.h.}}{=} B \vee C = A$$
5. $A \equiv \Box B$. We have:
$$\overline{\overline{A}} = \overline{\overline{\Box B}} = \overline{\Diamond \overline{B}} = \Box \overline{\overline{B}} \stackrel{\text{i.h.}}{=} \Box B = A$$
6. $A \equiv \Diamond B$. We have:
$$\overline{\overline{A}} = \overline{\overline{\Diamond B}} = \overline{\Box \overline{B}} = \Diamond \overline{\overline{B}} \stackrel{\text{i.h.}}{=} \Diamond B = A$$
7. $A \equiv \bigcirc B$. We have:
$$\overline{\overline{A}} = \overline{\overline{\bigcirc B}} = \overline{\bigcirc \overline{B}} = \bigcirc \overline{\overline{B}} \stackrel{\text{i.h.}}{=} \bigcirc B = A \qquad \dashv$$

Now we define two translation functions between the languages $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{S}}$. We define the function $\sigma : \mathcal{L}_{\mathcal{S}} \to \mathcal{L}_{\mathcal{H}}$ inductively:

$$\begin{aligned} \sigma(P) &= P & \sigma(\overline{P}) &= \neg P \\ \sigma(A \wedge B) &= \sigma(A) \wedge \sigma(B) & \sigma(A \vee B) &= \sigma(A) \vee \sigma(B) \\ \sigma(\Box A) &= \Box \sigma(A) & \sigma(\Diamond A) &= \Diamond \sigma(A) \\ \sigma(\bigcirc A) &= \bigcirc \sigma(A) \end{aligned}$$

We define the function $\tau : \mathcal{L}_{\mathcal{H}} \to \mathcal{L}_{\mathcal{S}}$ inductively:

$$\tau(P) = P \qquad \tau(\Box\phi) = \Box\tau(\phi)$$
$$\tau(\neg\phi) = \overline{\tau(\phi)} \qquad \tau(\bigcirc\phi) = \bigcirc\tau(\phi)$$
$$\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$$

Some simple calculations show that the function τ behaves as expected with respect to the propositional connectives.

Lemma 13. *Let $\phi, \psi \in \mathcal{L}_{\mathcal{H}}$. It holds:*

(1) $\tau(\phi \to \psi) = \overline{\tau(\phi)} \vee \tau(\psi)$

(2) $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$

(3) $\tau(\phi \leftrightarrow \psi) = \left(\overline{\tau(\phi)} \vee \tau(\psi)\right) \wedge \left(\overline{\tau(\psi)} \vee \tau(\phi)\right)$

(4) $\tau(\Diamond\phi) = \Diamond\tau(\phi)$

Proof. 1. We have:

$$\begin{aligned}\tau(\phi \to \psi) &= \tau(\neg(\phi \wedge \neg\psi)) = \overline{\tau(\phi \wedge \neg\psi)} = \overline{\tau(\phi) \wedge \tau(\neg\psi)} \\ &= \overline{\tau(\phi)} \vee \overline{\tau(\neg\psi)} = \overline{\tau(\phi)} \vee \overline{\overline{\tau(\psi)}} \overset{\text{Lemma 12}}{=} \overline{\tau(\phi)} \vee \tau(\psi)\end{aligned}$$

2. We have:

$$\begin{aligned}\tau(\phi \vee \psi) &= \tau(\neg(\neg\phi \wedge \neg\psi)) = \overline{\tau(\neg\phi \wedge \neg\psi)} = \overline{\tau(\neg\phi) \wedge \tau(\neg\psi)} \\ &= \overline{\tau(\neg\phi)} \vee \overline{\tau(\neg\psi)} = \overline{\overline{\tau(\phi)}} \vee \overline{\overline{\tau(\psi)}} \overset{\text{Lemma 12}}{=} \tau(\phi) \vee \tau(\psi)\end{aligned}$$

3. We have:

$$\begin{aligned}\tau(\phi \leftrightarrow \psi) &= \tau\big((\phi \to \psi) \wedge (\psi \to \phi)\big) = \tau(\phi \to \psi) \wedge \tau(\psi \to \phi) \overset{(1)}{=} \\ &\left(\overline{\tau(\phi)} \vee \tau(\psi)\right) \wedge \left(\tau(\phi) \vee \overline{\tau(\psi)}\right)\end{aligned}$$

4. We have:

$$\begin{aligned}\tau(\Diamond\phi) &= \tau(\neg\Box\neg\phi) = \overline{\tau(\Box\neg\phi)} = \overline{\Box\tau(\neg\phi)} = \Diamond\overline{\tau(\neg\phi)} \\ &= \Diamond\overline{\overline{\tau(\phi)}} \overset{\text{Lemma 12}}{=} \Diamond\tau(\phi)\end{aligned}$$

⊣

It is straightforward to show that τ is the inverse of σ.

Lemma 14. *Let $A \in \mathcal{L}_{\mathcal{S}}$. It holds:*

$$\tau(\sigma(A)) = A$$

Proof. By induction on A. We distinguish the following cases:

1. $A = P$. Then we have:

$$\tau(\sigma(A)) = \tau(\sigma(P)) = \tau(P) = P = A$$

2. $A = \overline{P}$. Then we have:

$$\tau(\sigma(A)) = \tau(\sigma(\overline{P})) = \tau(\neg P) = \overline{\tau(P)} = \overline{P} = A$$

3. $A = B \wedge C$. Then we have:

$$\begin{aligned}\tau(\sigma(A)) =& \tau(\sigma(B \wedge C)) = \tau(\sigma(B) \wedge \sigma(C)) \\ =& \tau(\sigma(B)) \wedge \tau(\sigma(C)) \overset{\text{i.h.}}{=} B \wedge C = A\end{aligned}$$

4. $A = B \vee C$. Then we have:

$$\begin{aligned}\tau(\sigma(A)) &= \tau(\sigma(B \vee C)) = \tau(\sigma(B) \vee \sigma(C)) \\ &\overset{\text{Lemma 13(2)}}{=} \tau(\sigma(B)) \vee \tau(\sigma(C)) \overset{\text{i.h.}}{=} B \vee C = A\end{aligned}$$

5. $A = \Box B$. Then we have:

$$\tau(\sigma(A)) = \tau(\sigma(\Box B)) = \tau(\Box\sigma(B)) = \Box\tau(\sigma(B)) \overset{\text{i.h.}}{=} \Box B = A$$

6. $A = \Diamond B$. Then we have:

$$\tau(\sigma(A)) = \tau(\sigma(\Diamond B)) = \tau(\Diamond\sigma(B)) \overset{\text{Lemma 13(4)}}{=} \Diamond\tau(\sigma(B)) \overset{\text{i.h.}}{=} \Diamond B$$

7. $A = \bigcirc B$. Then we have:

$$\tau(\sigma(A)) = \tau(\sigma(\bigcirc B)) = \tau(\bigcirc\sigma(B)) = \bigcirc\tau(\sigma(B)) \overset{\text{i.h.}}{=} \bigcirc B = A \qquad \dashv$$

Now we can prove the embedding Lemma:

Lemma 15 (Embedding of Σ_{LTL} into $\mathsf{LT1}^{\mathsf{cut}}$)**.**

$$\Big(\forall \phi \in \mathcal{L}_{\mathcal{H}}\Big)\Big[\Sigma_{\mathsf{LTL}} \vdash \phi \Longrightarrow \mathsf{LT1}^{\mathsf{cut}} \vdash \tau(\phi)\Big]$$

Proof. By induction on the length of the derivation $\Sigma_{\mathsf{LTL}} \vdash \phi$. We distinguish the following cases:

1. ϕ is a propositional tautology. Then $\tau(\phi)$ is also a propositional tautology. Hence, clearly $\mathsf{LT1}^{\mathsf{cut}} \vdash \tau(\phi)$.

2. ϕ is an instance of the axiom (Fun). That means there is a $\psi \in \mathcal{L}_{\mathcal{H}}$ such that $\phi = \neg\bigcirc\psi \leftrightarrow \bigcirc\neg\psi$. From Lemma 12 and Lemma 13 we get:

   $$\tau(\phi) = \left(\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}\right) \wedge \left(\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}\right)$$

   So, we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

   $$\dfrac{\dfrac{\dfrac{}{\bigcirc\tau(\psi), \bigcirc\overline{\tau(\psi)}}\,\text{Lemma 11}}{\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}}\,\vee \qquad \dfrac{\dfrac{}{\bigcirc\tau(\psi), \bigcirc\overline{\tau(\psi)}}\,\text{Lemma 11}}{\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}}\,\vee}{\left(\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}\right) \wedge \left(\bigcirc\tau(\psi) \vee \bigcirc\overline{\tau(\psi)}\right)}\,\wedge$$

3. ϕ is an instance of axiom $(\mathsf{K}_{\bigcirc})$. That means there are $\psi, \rho \in \mathcal{L}_{\mathcal{H}}$ such that $\phi = \bigcirc(\psi \to \rho) \to \bigcirc\psi \to \bigcirc\rho$. By Lemma 12 and Lemma 13 we get:

   $$\tau(\phi) = \bigcirc\left(\tau(\psi) \wedge \overline{\tau(\rho)}\right) \vee \bigcirc\overline{\tau(\psi)} \vee \bigcirc\tau(\rho)$$

   So, we have the following derivation:

   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\tau(\psi), \overline{\tau(\psi)}, \tau(\rho)}\,\text{Lemma 11} \qquad \dfrac{}{\overline{\tau(\rho)}, \overline{\tau(\psi)}, \tau(\rho)}\,\text{Lemma 11}}{\tau(\psi) \wedge \overline{\tau(\rho)}, \overline{\tau(\psi)}, \tau(\rho)}\,\wedge}{\bigcirc\left(\tau(\psi) \wedge \overline{\tau(\rho)}\right), \bigcirc\overline{\tau(\psi)}, \bigcirc\tau(\rho)}\,\bigcirc}{\bigcirc\left(\tau(\psi) \wedge \overline{\tau(\rho)}\right) \vee \bigcirc\overline{\tau(\psi)}, \bigcirc\tau(\rho)}\,\vee}{\bigcirc\left(\tau(\psi) \wedge \overline{\tau(\rho)}\right) \vee \bigcirc\overline{\tau(\psi)} \vee \bigcirc\tau(\rho)}\,\vee$$

4. ϕ is an instance of the axiom (Alw). That means there is a $\psi \in \mathcal{L}_{\mathcal{H}}$ such that $\phi = \Box\psi \to \psi \wedge \bigcirc\Box\psi$. From Lemma 12 and Lemma 13 we get:

   $$\tau(\phi) = \Diamond\overline{\tau(\psi)} \vee \left(\tau(\psi) \wedge \bigcirc\Box\tau(\psi)\right)$$

   So, we have the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$
\vee\;\dfrac{\Diamond\;\dfrac{\wedge\;\dfrac{\text{Lemma 11}\;\dfrac{}{\overline{\tau(\psi)}, \bigcirc\Diamond\overline{\tau(\psi)}, \tau(\psi)} \qquad \text{Lemma 11}\;\dfrac{}{\overline{\tau(\psi)}, \bigcirc\Diamond\overline{\tau(\psi)}, \bigcirc\Box\tau(\psi)}}{\overline{\tau(\psi)}, \bigcirc\Diamond\overline{\tau(\psi)}, \tau(\psi) \wedge \bigcirc\Box\tau(\psi)}}{\Diamond\overline{\tau(\psi)}, \tau(\psi) \wedge \bigcirc\Box\tau(\psi)}}{\Diamond\overline{\tau(\psi)} \vee \big(\tau(\psi) \wedge \bigcirc\Box\tau(\psi)\big)}
$$

5. ϕ is the conclusion of an application of (MP). That means there is a $\psi \in \mathcal{L}_{\mathcal{H}}$ such that $\Sigma_{\mathsf{LTL}} \vdash \psi$ and $\Sigma_{\mathsf{LTL}} \vdash \psi \to \phi$. Then we find the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$
\mathsf{cut}\;\dfrac{\text{i.h.}\;\dfrac{}{\tau(\psi)} \qquad \text{Corollary 8}\;\dfrac{\text{Lemma 13 (1)}\;\dfrac{\text{i.h.}\;\dfrac{}{\tau(\psi \to \phi)}}{\overline{\tau(\psi)} \vee \tau(\phi)}}{\overline{\tau(\psi)}, \tau(\phi)}}{\tau(\phi)}
$$

6. ϕ is the conclusion of an application of $(\mathsf{N}_{\bigcirc})$. That means that there is a $\psi \in \mathcal{L}_{\mathcal{H}}$ such that $\phi = \bigcirc\psi$ and that $\Sigma_{\mathsf{LTL}} \vdash \psi$. In $\mathsf{LT1}^{\mathsf{cut}}$ we have the following derivation:

$$
\bigcirc\;\dfrac{\text{i.h.}\;\dfrac{}{\tau(\psi)}}{\bigcirc\tau(\psi)}
$$

And since $\bigcirc\tau(\psi) = \tau(\bigcirc\psi) = \tau(\phi)$ we have that $\mathsf{LT1}^{\mathsf{cut}} \vdash \tau(\phi)$.

7. ϕ is the conclusion of an application of (Ind). That means there are $\psi, \rho \in \mathcal{L}_{\mathcal{H}}$ such that $\phi = \psi \to \Box\rho$, and that $\Sigma_{\mathsf{LTL}} \vdash \psi \to \rho$ and $\Sigma_{\mathsf{LTL}} \vdash \psi \to \bigcirc\psi$. By Lemma 13 (1) we have:

$$
\begin{aligned}
\tau(\phi) = \tau(\psi \to \Box\rho) &= \overline{\tau(\psi)} \vee \Box\tau(\rho) \\
\tau(\psi \to \rho) &= \overline{\tau(\psi)} \vee \tau(\rho) \\
\tau(\psi \to \bigcirc\psi) &= \overline{\tau(\psi)} \vee \bigcirc\tau(\psi)
\end{aligned}
$$

We find the following derivation in $\mathsf{LT1}^{\mathsf{cut}}$:

$$
\dfrac{\dfrac{\text{Corollary 8}\;\dfrac{\text{i.h.}\;\dfrac{}{\overline{\tau(\psi)} \vee \tau(\rho)}}{\overline{\tau(\psi)}, \tau(\rho)} \qquad \mathsf{cut}\;\dfrac{\text{Corollary 8}\;\dfrac{\text{i.h.}\;\dfrac{}{\overline{\tau(\psi)} \vee \bigcirc\tau(\psi)}}{\overline{\tau(\psi)}, \bigcirc\tau(\psi)} \qquad \dfrac{\mathsf{rep}\;\dfrac{}{\overline{\tau(\psi)}, \Box_{\overline{\tau(\psi)}}\tau(\rho)}}{\bigcirc\overline{\tau(\psi)}, \bigcirc\Box_{\overline{\tau(\psi)}}\tau(\rho)}}{\overline{\tau(\psi)}, \bigcirc\Box_{\overline{\tau(\psi)}}\tau(\rho)}}{\overline{\tau(\psi)}, \Box\tau(\rho)}\;\mathsf{foc}}{\overline{\tau(\psi)} \vee \Box\tau(\rho)}\;\vee
$$

$\dashv$

To establish completeness of $\mathsf{LT1}^{\mathsf{cut}}$, we need the following lemma.

Lemma 16. *Let $A \in \mathcal{L}_{\mathcal{S}}$, let μ be a model and let $i \in \mathbb{N}$. It holds:*

$$\mu, i \models A \quad \Longrightarrow \quad \mu, i \models \sigma(A)$$

Proof. By induction on the structure of A. We distinguish the following cases:

1. $A = P \in \mathsf{Prop}$. It holds that $\sigma(A) = \sigma(P) = P = A$, hence the claim holds trivially.

2. $A = \overline{P}$ for $P \in \mathsf{Prop}$. It holds that $\sigma(A) = \sigma(\overline{P}) = \neg\sigma(P) = \neg P$. So, we have:

$$\mu, i \models A \Longrightarrow \mu, i \models \overline{P} \Longrightarrow P \notin \mu(i) \Longrightarrow \mu, i \models \neg P \Longrightarrow \mu, i \models \sigma(A)$$

3. $A = B \wedge C$. It holds that $\sigma(A) = \sigma(B) \wedge \sigma(C)$. Then, we have:

$$\mu, i \models A \Longrightarrow \mu, i \models B \wedge C \Longrightarrow \big(\mu, i \models B \text{ and } \mu, i \models C\big) \overset{\text{i.h.}}{\Longrightarrow} \big(\mu, i \models \sigma(B) \text{ and } \mu, i \models \sigma(C)\big) \Longrightarrow \big(\mu, i \models \sigma(B) \wedge \sigma(C)\big) \Longrightarrow \mu, i \models \sigma(B \wedge C) \Longrightarrow \mu, i \models \sigma(A)$$

4. $A = B \vee C$. It holds that $\sigma(A) = \sigma(B) \vee \sigma(C)$. Then, we have:

$$\mu, i \models A \Longrightarrow \mu, i \models B \vee C \Longrightarrow \big(\mu, i \models B \text{ or } \mu, i \models C\big) \overset{\text{i.h.}}{\Longrightarrow} \big(\mu, i \models \sigma(B) \text{ or } \mu, i \models \sigma(C)\big) \Longrightarrow \mu, i \models \sigma(B) \vee \sigma(C) \Longrightarrow \mu, i \models \sigma(B \vee C) \Longrightarrow \mu, i \models \sigma(A)$$

5. $A = \Box B$. It holds that $\sigma(A) = \Box\sigma(B)$. Then, we have:

$$\mu, i \models A \Longrightarrow \mu, i \models \Box B \Longrightarrow (\forall j \geq i)\big[\mu, j \models B\big] \overset{\text{i.h.}}{=} (\forall j \geq i)\big[\mu, j \models \sigma(B)\big] \Longrightarrow \mu, i \models \Box\sigma(B) \Longrightarrow \mu, i \models \sigma(A)$$

6. $A = \Diamond B$. It holds that $\sigma(A) = \Diamond\sigma(B)$. Then, we have:

$$\mu, i \models A \Longrightarrow \mu, i \models \Diamond B \Longrightarrow (\exists j \geq i)\big[\mu, j \models B\big] \overset{\text{i.h.}}{=} (\exists j \geq i)\big[\mu, j \models \sigma(B)\big] \Longrightarrow \mu, i \models \Diamond\sigma(B) \Longrightarrow \mu, i \models \sigma(A)$$

$\dashv$

Finally we can prove completeness of system $\mathsf{LT1}^{\mathsf{cut}}$.

Theorem 17. *System $\mathsf{LT1}^{\mathsf{cut}}$ is complete for $\mathcal{L}_{\mathcal{S}}$-formulas, i.e. every $A \in \mathcal{L}_{\mathcal{S}}$ we have:*

$$\models A \quad \Longrightarrow \quad \mathsf{LT1}^{\mathsf{cut}} \vdash A$$

Proof. Let $A \in \mathcal{L}_{\mathcal{S}}$. We have:

$$\models A \Longrightarrow (\forall \mu)(\forall i \in \mathbb{N})\Big[\mu, i \models A\Big] \overset{\text{Lemma 16}}{\Longrightarrow}$$
$$(\forall \mu)(\forall i \in \mathbb{N})\Big[\mu, i \models \sigma(A)\Big] \Longrightarrow \models \sigma(A) \overset{\text{Theorem 10}}{\Longrightarrow}$$
$$\Sigma_{\mathsf{LTL}} \vdash \sigma(A) \overset{\text{Lemma 15}}{\Longrightarrow} \mathsf{LT1}^{\mathsf{cut}} \vdash \tau(\sigma(A)) \overset{\text{Lemma 14}}{\Longrightarrow} \mathsf{LT1}^{\mathsf{cut}} \vdash A$$

⊣

And as a corollary we have that system $\mathsf{LT1}^{\mathsf{cut}}$ is complete for (not annotated) sequents.

Corollary 18. *System $\mathsf{LT1}^{\mathsf{cut}}$ is complete for sequents, i.e.:*

$$\Big(\forall \Gamma \in \mathsf{Seq}\Big)\Big[\models \Gamma \Longrightarrow \mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma \Big]$$

Proof. Let $\Gamma \in \mathsf{Seq}$. It holds:

$$\models \Gamma \Longrightarrow \models \bigvee_{A \in \Gamma} A \overset{\text{Theorem 17}}{\Longrightarrow} \mathsf{LT1}^{\mathsf{cut}} \vdash \bigvee_{A \in \Gamma} A \overset{\text{Corollary } 8^*}{\Longrightarrow} \mathsf{LT1}^{\mathsf{cut}} \vdash \Gamma$$

Where Corollary 8^* means that we apply Corollary 8 several times. ⊣

3.3 System LT1 is not Complete

Now we show that if we remove the rule cut from system $\mathsf{LT1}^{\mathsf{cut}}$, the resulting system (i.e. LT1) is not complete.

Let Γ be the following valid sequent:

$$\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box(P \vee \bigcirc P)$$

Γ is semantically equivalent to the following $\mathcal{L}_{\mathcal{H}}$-formula:

$$P \wedge \Box(P \to \bigcirc\bigcirc P) \to \Box(P \vee \bigcirc P)$$

which expresses a valid induction statement in LTL.

The following derivation is an attempt to prove Γ in LT1.

$$
\text{foc}\;\dfrac{\text{∨}\;\dfrac{\text{aid}\;\dfrac{}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P, \bigcirc P}}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P \vee \bigcirc P} \qquad \Diamond\;\dfrac{\wedge\;\dfrac{\text{aid}\;\dfrac{}{\overline{P}, P, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta P} \qquad \mathcal{D}}{\overline{P}, P \wedge \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)}}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)}}{\underbrace{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P})}_{\Delta}, \Box(P \vee \bigcirc P)}
$$

$$
\mathcal{D}\left\{ \bigcirc\;\dfrac{\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)}{\overline{P}, \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \right.
$$

The reason we cannot prove sequent Γ in system LT1 is that in system LT1 it is impossible to "get rid of an annotation". So, in the above proof-attempt for Γ there is no way we can drop Δ from sequent $\bigcirc\overline{P}, \Diamond(P\wedge\bigcirc\bigcirc\overline{P}), \Box_\Delta(P\vee\bigcirc P)$, which is the sequent on the top of the derivation $\mathcal{D}$. So, the only way to prove an annotated sequent is to reach either axiom rep or axiom aid from it. In our case it is impossible to reach rep from $\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)$, since it would require the application of a $\bigcirc$-rule, which is not possible (the conclusion of a $\bigcirc$-rule cannot contain a formula of the form $\Box_\Delta A$). It is also impossible to reach aid from $\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)$ since the sequent $\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P})$ is not valid and our system is sound. Thus, proof-search for Γ in LT1 fails and, therefore, system LT1 is not complete.

We can tackle the problem of "being unable to get rid of annotations" by introducing a new rule. For any $\Delta \in \mathsf{Seq}$ we define the following rule:

$$
\Box_\Delta\;\dfrac{\Gamma, A \qquad \Gamma, \bigcirc\Box_\Delta A}{\Gamma, \Box_\Delta A}
$$

Rule $\Box_\Delta$ is sound with respect to LTL-models for any Δ. Indeed, let μ be a model and let $i \in \mathbb{N}$. It holds:

$$
\begin{gathered}
\mu, i \models \Gamma, \bigcirc\Box_\Delta A \text{ and } \mu, i \models \Gamma, A \Longrightarrow \\
(\mu, i \models \Gamma \text{ or } \mu, i \models \bigcirc\Box_\Delta A) \text{ and } (\mu, i \models \Gamma \text{ or } \mu, i \models A) \Longrightarrow \\
\mu, i \models \Gamma \text{ or } (\mu, i \models A \text{ and } \mu, i \models \bigcirc\Box_\Delta A) \Longrightarrow \\
\mu, i \models \Gamma \text{ or } (\mu, i \models A \text{ and } \mu, i+1 \models \Box_\Delta A) \Longrightarrow \\
\mu, i \models \Gamma \text{ or } \Big(\mu, i \models A \text{ and} \\
(\forall j \geq i+1)\Big[(\forall i+1 \leq k \leq j)[\mu, k \models \Delta] \Longrightarrow \mu, j \models A\Big]\Big) \Longrightarrow \\
\mu, i \models \Gamma \text{ or } (\forall j \geq i)\Big[(\forall i \leq k \leq j)[\mu, k \models \Delta] \Longrightarrow \mu, j \models A\Big]\Big) \Longrightarrow
\end{gathered}
$$

$$\begin{aligned}&\mu, i \models \Gamma \text{ or } \mu, i \models \Box_\Delta A \Longrightarrow\\ &\qquad \mu, i \models \Gamma, \Box_\Delta A\end{aligned}$$

Hence we have that rule $\Box_\Delta$ preserves validity.

As we can see, rule $\Box_\Delta$ allows us to drop the annotation from an annotated sequent (the left premise of the rule $\Box_\Delta$ is an unannotated sequent). We define $\mathsf{LT1}^+$ to be system $\mathsf{LT1}$ plus the rule $\Box_\Delta$ for any $\Delta \in \mathsf{Seq}$.

In system $\mathsf{LT1}^+$ we can prove the sequent $\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box(P \vee \bigcirc P)$ as follows:

$$\mathsf{foc}\,\dfrac{\vee\,\dfrac{\mathsf{aid}\,\dfrac{}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P, \bigcirc P}}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P \vee \bigcirc P} \qquad \Diamond\,\dfrac{\wedge\,\dfrac{\mathsf{aid}\,\dfrac{}{\overline{P}, P, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \qquad \mathcal{D}_1}{\overline{P}, P \wedge \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)}}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)}}{\underbrace{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P})}_{\Delta}, \Box(P \vee \bigcirc P)}$$

$$\mathcal{D}_1 \left\{ \bigcirc\,\dfrac{\Box_\Delta\,\dfrac{\vee\,\dfrac{\bigcirc\,\dfrac{\mathsf{aid}\,\dfrac{}{\overline{P}, P}}{\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P, \bigcirc P}}{\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), P \vee \bigcirc P} \qquad \Diamond\,\dfrac{\mathcal{D}_2}{\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)}}{\bigcirc\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)}}{\overline{P}, \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \right.$$

$$\mathcal{D}_2 \left\{ \wedge\,\dfrac{\bigcirc\,\dfrac{\mathsf{rep}\,\dfrac{}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)}}{\bigcirc\overline{P}, P, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \qquad \mathcal{D}_3}{\bigcirc\overline{P}, P \wedge \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \right.$$

$$\mathcal{D}_3 \left\{ \bigcirc\,\dfrac{\mathsf{rep}\,\dfrac{}{\overline{P}, \Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \Box_\Delta(P \vee \bigcirc P)}}{\bigcirc\overline{P}, \bigcirc\bigcirc\overline{P}, \bigcirc\Diamond(P \wedge \bigcirc\bigcirc\overline{P}), \bigcirc\Box_\Delta(P \vee \bigcirc P)} \right.$$

However, $\mathsf{LT1}^+$ is still too simple: it fails to prove all valid sequents. Take for example the following valid sequent, which we call Σ:

$$\bigcirc\Box(\overline{P} \vee \overline{Q}), \Diamond(\bigcirc P \wedge \bigcirc Q)$$

where P, Q are different elements of Prop. We show that Σ cannot be derived in $\mathsf{LT1}^+$.

We set $C = \overline{P} \vee \overline{Q}$ and $D = \bigcirc P \wedge \bigcirc Q$. A proof-attempt for Σ in $\mathsf{LT1}^+$ is as follows:

$$
\wedge\,\dfrac{\bigcirc\,\dfrac{\mathcal{D}_1(P,Q)}{\bigcirc\Box C,\bigcirc\Diamond D,\bigcirc P}\qquad \bigcirc\,\dfrac{\mathcal{D}_1(Q,P)}{\bigcirc\Box C,\bigcirc\Diamond D,\bigcirc Q}}{\Diamond\,\dfrac{\bigcirc\Box C,\bigcirc\Diamond D, D}{\bigcirc\Box C,\Diamond D}}
$$

where $\mathcal{D}_1(P,Q)$ is

$$
\mathsf{foc}\,\dfrac{\vee\,\dfrac{\mathsf{aid}\,\dfrac{}{\overline{P},\overline{Q},\Diamond D, P}}{C,\Diamond D, P}\qquad \Diamond\,\dfrac{\wedge\,\dfrac{\bigcirc\,\dfrac{\mathsf{rep}\,\dfrac{}{\Diamond D, P, \Box_{\Diamond D,P} C}}{\bigcirc\Diamond D,\bigcirc P, P,\bigcirc\Box_{\Diamond D,P} C}\qquad \bigcirc\,\dfrac{\mathcal{D}_2(P,Q)}{\bigcirc\Diamond D,\bigcirc Q, P,\bigcirc\Box_{\Diamond D,P} C}}{\bigcirc\Diamond D, D, P,\bigcirc\Box_{\Diamond D,P} C}}{\Diamond D, P,\bigcirc\Box_{\Diamond D,P} C}}{\Box C,\Diamond D, P}
$$

and $\mathcal{D}_2(P,Q)$ is

$$
\Box_{\Diamond D,P}\,\dfrac{\vee\,\dfrac{\mathsf{aid}\,\dfrac{}{\Diamond D, Q,\overline{P},\overline{Q}}}{\Diamond D, Q, C}\qquad \Diamond\,\dfrac{\wedge\,\dfrac{\bigcirc\,\dfrac{\mathsf{rep}\,\dfrac{}{\Diamond D, P, \Box_{\Diamond D,P} C}}{\bigcirc\Diamond D,\bigcirc P, Q,\bigcirc\Box_{\Diamond D,P} C}\qquad \bigcirc\,\dfrac{\Diamond D, Q, \Box_{\Diamond D,P} C}{\bigcirc\Diamond D,\bigcirc Q, Q,\bigcirc\Box_{\Diamond D,P} C}}{\bigcirc\Diamond D, D, Q,\bigcirc\Box_{\Diamond D,P} C}}{\Diamond D, Q,\bigcirc\Box_{\Diamond D,P} C}}{\Diamond D, Q,\Box_{\Diamond D,P} C}
$$

The reason we cannot prove Σ in $\mathsf{LT1}^+$ is that it is impossible to prove the sequent $\Diamond D, Q, \Box_{\Diamond D,P} C$ in $\mathsf{LT1}^+$. Since $\Diamond D, Q, \Box_{\Diamond D,P} C$ is no instance of axiom rep, it is natural to try to prove it by applying the $\Box_{\Diamond,P}$ rule first. However, as we can see in derivation $\mathcal{D}_2(P,Q)$ this leads again to sequent $\Diamond D, Q, \Box_{\Diamond D,P} C$. Hence proof-search for Σ fails, i.e. $\mathsf{LT1}^+$ cannot be complete. However, the fact that an annotated sequent, i.e. $\Diamond D, Q, \Box_{\Diamond D,P} C$, is cyclic in $\mathcal{D}_2(P,Q)$ gives us a hint for how we should improve the principle for closing cyclic branches. What if we keep a set of sequents rather than a single sequent in the annotation? This is the approach of Brünnler and Lange [2], which we study in the next chapter.

Chapter 4

Histories

So far, all our systems could only store one sequent in the annotation. In the previous chapter we have seen that this not enough to define a complete cut-free sequent system for LTL. Brünnler and Lange [2] present a system that is very similar to LT1. The only difference is that their annotations contain sets of sequents rather than single sequents. Based on their approach, we present a system LT2 together with a soundness and completeness proof. We also present a purely syntactical proof for weakening. This is a solution to an open problem of Brünnler and Lange.

This chapter is based on [2] and [4].

4.1 System LT2

We start by defining the set of histories (finite sets of sequents), which we call His:

$$\mathsf{His} = \left\{ H \;\middle|\; H \text{ is a finite subset of } \mathsf{Seq} \right\}$$

We will use the letters H and G for histories, possibly primed or with subscripts. We also define the set $\mathcal{L}_{\mathsf{his}}$:

$$\mathcal{L}_{\mathsf{his}} := \left\{ \Box_H A, \bigcirc\Box_H A \;\middle|\; A \in \mathcal{L}_{\mathcal{S}}, H \in \mathsf{His} \right\}$$

We will refer to $\mathcal{L}_{\mathsf{his}}$-formulas as formulas with histories, or when there is no danger of confusion as annotated formulas. We also define the set of sequents with histories:

$$\mathsf{Seq}_{\mathsf{his}} := \Big\{ \Gamma \;\Big|\; \Gamma \text{ is a finite subset of } \mathcal{L}_{\mathcal{S}} \cup \mathcal{L}_{\mathsf{his}} \text{ that contains at most one } \mathcal{L}_{\mathsf{his}}\text{-formula} \Big\}$$

We assume that a history is semantically equivalent to the conjunction of its elements, i.e. for a model μ and $i \in \mathbb{N}$:

$$\mu, i \models H \iff \mu, i \models \bigwedge_{\Gamma \in H} \Gamma \iff \mu, i \models \bigwedge_{\Gamma \in H} \bigvee_{A \in \Gamma} A$$

If $H = \emptyset$ we have that H is semantically equivalent to true, i.e. the empty history is valid.

The semantics of $\Box_H A$ is defined like that of $\Box_\Gamma A$. We have:

$$\mu, i \models \Box_H A \iff \forall j \geq i\Big(\big(\forall i \leq k \leq j(\mu, k \models H)\big) \Longrightarrow \mu, j \models A\Big)$$

In Figure 4.1 we present system $\mathsf{LT2}$. Again we assume $\Sigma \in \mathsf{Seq}$ in the $\bigcirc$-rule. In all the other rules we have $\Gamma \in \mathsf{Seq}_{\mathsf{his}}$, whenever the syntactical restrictions allow it.

$$\mathsf{aid}\ \frac{}{\Gamma, P, \overline{P}} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\mathsf{rep}\ \frac{}{\Gamma, \Box_{H,\Gamma} A} \qquad \mathsf{foc}\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{\{\Gamma\}} A}{\Gamma, \Box A} \qquad \Box_H\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{H,\Gamma} A}{\Gamma, \Box_H A}$$

$$\Box\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box A}{\Gamma, \Box A} \qquad \Diamond\ \frac{\Gamma, A, \bigcirc\Diamond A}{\Gamma, \Diamond A} \qquad \bigcirc\ \frac{\Gamma}{\Sigma, \bigcirc\Gamma}$$

Figure 4.1: System $\mathsf{LT2}$

Whenever rule $\Box_\Delta$ is applied in an $\mathsf{LT1}^+$-proof, the sequent in the annotation does not change. In system $\mathsf{LT2}$, an application of rule $\Box_H$ leads to a new sequent being added to the annotation (history). Thus, rule $\Box_H$ is a generalization of rule $\Box_\Delta$, which implies that system $\mathsf{LT1}^+$ is a subsystem of $\mathsf{LT2}$.

System $\mathsf{LT2}$ proves the sequent:

$$\Sigma = \bigcirc\Box(\overline{P} \vee \overline{Q}), \Diamond(\bigcirc P \wedge \bigcirc Q)$$

which as we saw in chapter 3 cannot be proved in $\mathsf{LT1}^+$. Hence, system $\mathsf{LT1}^+$ is proper subsystem of system $\mathsf{LT2}$. We now present the proof of Σ in $\mathsf{LT2}$. As before we set $D = \bigcirc P \wedge \bigcirc Q$ and $C = \overline{P} \vee \overline{Q}$. Moreover, we sometimes write, e.g., $\Box_{\Delta,\Gamma}$ for $\Box_{\{\Delta,\Gamma\}}$.

$$\Diamond\,\cfrac{\wedge\,\cfrac{\bigcirc\,\cfrac{\mathcal{D}_1(P,Q)}{\bigcirc\Box C, \bigcirc\Diamond D, \bigcirc P} \qquad \bigcirc\,\cfrac{\mathcal{D}_1(Q,P)}{\bigcirc\Box C, \bigcirc\Diamond D, \bigcirc Q}}{\bigcirc\Box C, \bigcirc\Diamond D, D}}{\bigcirc\Box C, \Diamond D}$$

where $\mathcal{D}_1(P,Q)$ is

$$\mathsf{foc}\,\cfrac{\vee\,\cfrac{\mathsf{aid}\,\cfrac{}{\overline{P}, \overline{Q}, \Diamond D, P}}{C, \Diamond D, P} \qquad \Diamond\,\cfrac{\wedge\,\cfrac{\bigcirc\,\cfrac{\mathsf{rep}\,\cfrac{}{\Diamond D, P, \Box_{\{\Diamond D,P\}}C}}{\bigcirc\Diamond D, \bigcirc P, P, \bigcirc\Box_{\{\Diamond D,P\}}C} \qquad \bigcirc\,\cfrac{\mathcal{D}_2(P,Q)}{\bigcirc\Diamond D, \bigcirc Q, P, \bigcirc\Box_{\{\Diamond D,P\}}C}}{\bigcirc\Diamond D, D, P, \bigcirc\Box_{\{\Diamond D,P\}}C}}{\Diamond D, P, \bigcirc\Box_{\{\Diamond D,P\}}C}}{\Box C, \Diamond D, P}$$

$\mathcal{D}_2(P,Q)$ is

$$\Box_{\{\Diamond D,P\}}\,\cfrac{\vee\,\cfrac{\mathsf{aid}\,\cfrac{}{\Diamond D, Q, \overline{P}, \overline{Q}}}{\Diamond D, Q, C} \qquad \Diamond\,\cfrac{\mathcal{D}_3(P,Q)}{\Diamond D, Q, \bigcirc\Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C}}{\Diamond D, Q, \Box_{\{\Diamond D,P\}}C}$$

and $\mathcal{D}_3(P,Q)$ is

$$\wedge\,\cfrac{\bigcirc\,\cfrac{\mathsf{rep}\,\cfrac{}{\Diamond D, P, \Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C}}{\bigcirc\Diamond D, \bigcirc P, Q, \bigcirc\Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C} \qquad \bigcirc\,\cfrac{\mathsf{rep}\,\cfrac{}{\Diamond D, Q, \Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C}}{\bigcirc\Diamond D, \bigcirc Q, Q, \bigcirc\Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C}}{\bigcirc\Diamond D, D, Q, \bigcirc\Box_{\{\Diamond D,P\},\{\Diamond D,Q\}}C}$$

4.2 Soundness and Completeness of System LT2

Brünnler and Lange [2] defined a system for linear temporal logic using annotated sequents. System $\mathsf{LT2}$ is defined using the ideas from Brünnler and

Lange but has some small differences in comparison to the original system from [2]. Due to these differences it is easier to prove weakening for system $\mathsf{LT2}$, but it is no longer possible to apply Brünnler and Lange's idea for proving completeness. For this reason we define another system, that strictly follows the lines of Brünnler and Lange [2]. This system is $\mathsf{LT2}'$ and is presented in Figure 4.2. For $P \in \mathsf{Prop}$ we define $\pm P$ to be either P or $\neg P$. As we will see later the completeness of $\mathsf{LT2}$ follows from that of $\mathsf{LT2}$'.

The essential difference between systems $\mathsf{LT2}$ and $\mathsf{LT2}'$ is the definition of the $\bigcirc$-rule (compare figures 4.1 and 4.2). The foc rule is also defined differently between systems $\mathsf{LT2}$ and $\mathsf{LT2}'$, but as we will see later this is not an important difference.

$$\mathsf{aid}\ \frac{}{\Gamma, P, \overline{P}} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \wedge\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$\mathsf{rep}\ \frac{}{\Gamma, \Box_{H,\Gamma} A} \qquad \mathsf{foc}\ \frac{\Gamma, \Box_{\emptyset} A}{\Gamma, \Box A} \qquad \Box_H\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{H,\Gamma} A}{\Gamma, \Box_H A}$$

$$\Box\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box A}{\Gamma, \Box A} \qquad \Diamond\ \frac{\Gamma, A, \bigcirc\Diamond A}{\Gamma, \Diamond A} \qquad \bigcirc\ \frac{\Gamma}{\pm P_1, \pm P_2 \ldots \pm P_n, \bigcirc\Gamma}$$

Condition:
The result of the $\bigcirc$-rule cannot be an instance of aid

Figure 4.2: System $\mathsf{LT2}'$

Theorem 19 (Soundness)**.** *System $\mathsf{LT2}'$ is sound with respect to LTL-models, i.e. for all $\Gamma \in \mathsf{Seq}$ we have:*

$$\mathsf{LT2}' \vdash \Gamma \Longrightarrow \models \Gamma$$

Proof. By induction on the length of the derivation $\mathsf{LT2}' \vdash \Gamma$. We distinguish the following cases:

1. Γ is an instance of aid. That means there exists an atomic proposition P and a sequent Γ' such that $\Gamma = \Gamma', P, \overline{P}$. Let μ be a model and let

$i \in \omega$. The following statements are equivalent:

$$\mu, i \models \Gamma$$
$$\mu, i \models \Gamma', P, \overline{P}$$
$$\mu, i \models \Gamma' \text{ or } \mu, i \models P \text{ or } \mu, i \models \overline{P}$$
$$\mu, i \models \Gamma' \text{ or } \ P \in \mu(i) \text{ or } P \notin \mu(i)$$

The last statement is obviously true, so Γ is valid.

2. Γ is the conclusion of an application of the rule $\vee$. That means there exist $A, B \in \mathcal{L}_{\mathcal{S}}$ and a sequent Γ' such that $\mathsf{LT2}' \vdash \Gamma', A, B$ and $\Gamma = \Gamma', A \vee B$. Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\mu, i \models \Gamma', A \vee B$$
$$\mu, i \models \Gamma' \text{ or } \mu, i \models A \vee B$$
$$\mu, i \models \Gamma' \text{ or } \mu,i \models A \text{ or } \mu, i \models B$$

The last statement is true due to i.h., so Γ is valid.

3. Γ is the conclusion of an application of the rule $\wedge$. That means there exist $A, B \in \mathcal{L}_{\mathcal{S}}$ and a sequent Γ' such that $\mathsf{LT2}' \vdash \Gamma', A$ and $\mathsf{LT2}' \vdash \Gamma', B$ and $\Gamma = \Gamma', A \wedge B$. Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\mu, i \models \Gamma', A \wedge B$$
$$\mu, i \models \Gamma' \text{ or } \mu, i \models A \wedge B$$
$$\mu, i \models \Gamma' \text{ or } (\mu,i \models A \text{ and } \mu, i \models B)$$
$$(\mu, i \models \Gamma' \text{ or } \mu, i \models A) \text{ and } (\mu, i \models \Gamma' \text{ or } \mu, i \models B)$$
$$\mu, i \models \Gamma', A \text{ and } \mu, i \models \Gamma', B$$

The last statement is true due to i.h., so Γ is valid.

4. Γ is an instance of axiom rep. Then there exists $H \in \mathsf{His}$, $\Gamma' \in \mathsf{Seq}$ and a formula $B \in \mathcal{L}_{\mathcal{S}}$ such that $\Gamma = \Gamma', \Box_{H,\Gamma'} B$. Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\mu, i \models \Gamma', \Box_{H,\Gamma'} B$$
$$\mu, i \models \Gamma' \text{ or } \Box_{H,\Gamma'} B$$

But if $\mu, i \not\models \Gamma'$ we have that $(\forall j \geq i)(\exists i \leq k \leq j)[\mu, k \not\models H \wedge \Gamma']$. So, $\mu, i \models \Box_{H,\Gamma'} B$. Thus Γ is valid.

5. Γ is the conclusion of an application of the rule foc. That means there exists a sequent Δ and a formula $B \in \mathcal{L}_S$ such that $\Gamma = \Delta, \Box B$. Let μ be a model and $i \in \mathbb{N}$. By i.h. we have:

$$\begin{aligned}
&\mu, i \models \Delta, \Box_\emptyset B && \iff \\
&\mu, i \models \Delta \text{ or } \mu, i \models \Box_\emptyset B && \iff \\
&\mu, i \models \Delta \text{ or } (\forall j \geq i)\Big[\forall i \leq k \leq j\big(\mu, k \models \emptyset\big) \implies \mu, j \models B\big)\Big] && \overset{\emptyset \text{ is valid as a history}}{\iff} \\
&\mu, i \models \Delta \text{ or } \forall j \geq i\Big(\mu, j \models B\Big) && \iff \\
&\mu, i \models \Delta \text{ or } \mu, i \models \Box B
\end{aligned}$$

which implies that Γ is valid.

6. Γ is the conclusion of an application of the rule $\Box_H$ for some $H \in \mathsf{His}$. That means there exists $\Delta \in \mathsf{Seq}$, and $B \in \mathcal{L}_S$ such that $\Gamma = \Delta, \Box_H B$, $\mathsf{LT2}' \vdash \Delta, B$ and $\mathsf{LT2}' \vdash \Delta, \bigcirc\Box_{H,\Delta} B$. By i.h. we get:

$$\models \Delta, B \tag{4.1}$$
$$\models \Delta, \bigcirc\Box_{H,\Delta} B \tag{4.2}$$

Suppose that Γ is not valid. That means there exists a model μ such that $\mu, 0 \not\models \Gamma$, i.e. $\mu, 0 \not\models \Delta \wedge \Box_H B$. So, there exists an $i \in \mathbb{N}$ such that:

$$\mu, i \not\models B \tag{4.3}$$
$$\forall 0 \leq j \leq i\big(\mu, j \models H\big) \tag{4.4}$$

Since $\mu, 0 \not\models \Delta$, by 4.1 we get that $\mu, 0 \models B$. Therefore it must be $i \geq 1$. By 4.2 we get that $\mu, 0 \models \bigcirc\Box_{H,\Delta} B$, i.e. $\mu, 1 \models \Box_{H,\Delta} B$. If we have $\mu, 1 \not\models \Delta$ we get by 4.1 that $\mu, 1 \models B$ and by 4.2 that $\mu, 2 \models \Box_{H,\Delta} B$. On the other hand, if we have $\mu, 1 \models \Delta$ then, since by 4.4 we have $\mu, 1 \models H$, by $\mu, 1 \models \Box_{H,\Delta} B$ we get $\mu, 1 \models B$ and $\mu, 2 \models \Box_{H,\Delta} B$. Therefore in all cases we get that $\mu, 1 \models B$ and that $\mu, 2 \models \Box_{H,\Delta} B$. So $i \geq 2$. Repeating the same argument ad infinitum we can prove that i is greater than any finite natural number, i.e. there is no natural number i such that $\mu, i \not\models B$, which contradicts 4.3. Thus Γ is valid.

7. Γ is the conclusion of an application of the rule $\Diamond$. That means there is $A \in \mathcal{L}_S$ and sequent Γ' such that $\Gamma = \Gamma', \Diamond A$ and $\mathsf{LT2}' \vdash \Gamma', A, \bigcirc\Diamond A$.

Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\begin{gathered}\mu, i \models \Gamma', \Diamond A \\ \mu, i \models \Gamma' \text{ or } (\exists j \geq i)[\mu, j \models A] \\ \mu, i \models \Gamma' \text{ or } \mu, i \models A \text{ or } (\exists j \geq i+1)[\mu, j \models A] \\ \mu, i \models \Gamma' \text{ or } \mu, i \models A \text{ or } \mu, i+1 \models \Diamond A \\ \mu, i \models \Gamma' \text{ or } \mu, i \models A \text{ or } \mu, i \models \bigcirc\Diamond A\end{gathered}$$

The last statement is true due to i.h., so Γ is valid.

8. Γ is the conclusion of an application of the rule $\bigcirc$. That means that

$$\Gamma = \pm P_1, \ldots, \pm P_n, \bigcirc\Gamma'$$

and that $\mathsf{LT2}' \vdash \Gamma'$. Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\begin{gathered}\mu, i \models \pm P_1, \ldots, \pm P_n, \bigcirc\Gamma' \\ \mu, i \models \pm P_1, \ldots, \pm P_n \text{ or } \mu, i \models \bigcirc\Gamma' \\ \mu, i \models \pm P_1, \ldots, \pm P_n \text{ or } \mu, i+1 \models \Gamma'\end{gathered}$$

The last statement is true due to i.h., so Γ is valid.

9. Γ is the conclusion of an application of the rule $\Box$. That means that $\Gamma = \Delta, \Box A$ and that $\mathsf{LT2}' \vdash \Delta, A$ and $\mathsf{LT2}' \vdash \Delta, \bigcirc\Box A$. Let μ be a model and $i \in \mathbb{N}$. The following statements are equivalent:

$$\begin{gathered}\mu, i \models \Delta, \Box A \\ \mu, i \models \Delta \text{ or } \mu, i \models \Box A \\ \mu, i \models \Delta \text{ or } (\forall j \geq i)[\mu, j \models \Box A] \\ \mu, i \models \Delta \text{ or } \big(\mu, i \models A \text{ and } (\forall j \geq i+1)[\mu, j \models A]\big) \\ \mu, i \models \Delta \text{ or } \big(\mu, i \models A \text{ and } \mu, i \models \bigcirc\Box A\big)\end{gathered}$$

The last statement is true due to i.h., therefore Γ is valid. $\dashv$

For the completeness proof we will need the following definitions.

Definition 20 (Subformulas of a Formula)**.** The set sf is defined recursively for all $\mathcal{L}_\mathcal{S}$-formulas as follows:

$$\begin{aligned}
\mathsf{sf}(P) &:= \{P, \overline{P}\}\\
\mathsf{sf}(\overline{P}) &:= \{P, \overline{P}\}\\
\mathsf{sf}(A \wedge B) &:= \{A \wedge B, \overline{A} \vee \overline{B}\} \cup \mathsf{sf}(A) \cup \mathsf{sf}(B)\\
\mathsf{sf}(A \vee B) &:= \{A \vee B, \overline{A} \wedge \overline{B}\} \cup \mathsf{sf}(A) \cup \mathsf{sf}(B)\\
\mathsf{sf}(\Box A) &:= \{\Box A, \Diamond \overline{A}\} \cup \mathsf{sf}(A)\\
\mathsf{sf}(\Diamond A) &:= \{\Diamond A, \Box \overline{A}\} \cup \mathsf{sf}(A)\\
\mathsf{sf}(\bigcirc A) &:= \{\bigcirc A, \bigcirc \overline{A}\} \cup \mathsf{sf}(A)
\end{aligned}$$

Definition 21 (Closure of a Formula)**.** Let $A \in \mathcal{L}_\mathcal{S}$. We define the following set:

$$\begin{aligned}
\mathsf{cl}(A) :=& \mathsf{sf}(A) \cup \{\Box B, \bigcirc\Box B \mid \Box B \in \mathsf{sf}(A)\}\\
&\cup \{\Diamond B, \bigcirc\Diamond B \mid \Diamond B \in \mathsf{sf}(A)\}
\end{aligned}$$

Let $\Gamma \in \mathsf{Seq}$. We set:

$$\mathsf{cl}(\Gamma) := \bigcup_{A \in \Gamma} \mathsf{cl}(A)$$

We can easily prove that for all $\Gamma \in \mathsf{Seq}$, $\mathsf{cl}(\Gamma)$ is finite.

We will prove completeness for $\mathsf{LT2}'$ via a more restricted system $\mathsf{LT2}''$, which we define now. It proves statements of the form $\Gamma : l$, where $\Gamma \in \mathsf{Seq}$ and l is a finite list which contains all $\Box$-formulas that occur in Γ. The rules of $\mathsf{LT2}''$ are just like those of $\mathsf{LT2}$ and they simply pass on the list from the conclusion to all premises. The only exception is the foc-rule. We want to focus on the $\Box$-formula that occurs earliest in the list, so the foc-rule for system $\mathsf{LT2}''$ is defined as follows:

$$\mathsf{foc}\ \frac{\Gamma, \Box_\emptyset A : l_1, l_2, \Box A}{\Gamma, \Box A : l_1, \Box A, l_2}\ \text{no } \Box\text{-formula in } \Gamma \text{ ocurrs in } l_1$$

This ensures that each $\Box$-formula which keeps occuring in a branch will be annotated eventually.

Theorem 22 (Completeness)**.** *System* $\mathsf{LT2}'$ *is complete with respect to* LTL*-models, i.e. for all* $\Gamma \in \mathsf{Seq}$ *we have:*

$$\models \Gamma \Longrightarrow \mathsf{LT2}' \vdash \Gamma$$

Proof. It suffices to prove completeness for system $\mathsf{LT2}''$. Then we can get completeness for $\mathsf{LT2}'$ by simply dropping all the lists. So we will show that:

$$\models \Gamma \Longrightarrow \mathsf{LT2}'' \vdash \Gamma : l(\Gamma)$$

where $l(\Gamma)$ is a list containing all the $\Box$-formulas in $\mathsf{cl}(\Gamma)$. We assume that the formulas in $l(\Gamma)$ are listed according to some fixed enumeration of $\mathcal{L}_{\mathcal{S}}$-formulas. We will show the contrapositive of the above sentence, i.e.:

$$\mathsf{LT2}'' \nvdash \Gamma : l(\Gamma) \Longrightarrow \not\models \Gamma$$

Assume that $\mathsf{LT2}'' \nvdash \Gamma : l(\Gamma)$ we will find a countermodel for sequent Γ, i.e. we will prove that $\not\models \Gamma$. We build a possibly infinite derivation with the conclusion $\Gamma : l(\Gamma)$ applying the rules of $\mathsf{LT2}''$, repeating the following 3 steps ad infinitum:

1. apply rules $\mathsf{aid}, \wedge, \vee, \Diamond, \bigcirc$ as long as possible,
2. apply the rule foc if possible,
3. apply rules $\Box, \Box_H, \mathsf{rep}$ as long as possible.

Notice that with this strategy only subsets of the closure of the endsequent will enter the histories. By assumption the above algorithm will not yield a proof, and thus there will be either

1) a finite branch ending in a leaf to which no rule applies

or

2) an infinite branch.

In both cases define a sequence π of sequents with length $|\pi| \leq \omega$ such that $\pi(i)$ contains exactly those formulas and annotated formulas which occur in some sequent along this branch between the i-th and (i + 1)-th application of the $\bigcirc$-rule. In the second case this sequence is infinite and we define the model μ as $\mu(i) = \{P|\overline{P} \in \pi(i)\}$. In the first case this sequence is finite, with $\pi(n)$ its last element, and we define the model μ as before but with $\mu(i) = \{P|\overline{P} \in \pi(n)\}$ for $i > n$. We will prove the following claim:

Claim: For all $i < |\pi|$ and all $A \in \mathcal{L}_{\mathcal{S}}$ we have:

$$A \in \pi(i) \Longrightarrow \mu, i \not\models A$$

We prove the claim by induction on the structure of A. We distinguish the following cases:

(i) A is an atomic proposition. Then the claim is true since by assumption no element of π can be axiomatic.

(ii) A is the negation of an atomic proposition. Then the claim is true by definition.

(iii) $A = B \vee C$. Assume that $A \in \pi(i)$. Then by step 1 of our algorithm and by definition of the $\vee$-rule it must be that $B \in \pi(i)$ and $C \in \pi(i)$. By i.h. we have that $\mu, i \not\models B$ and that $\mu, i \not\models C$, i.e. $\mu, i \not\models B \vee C$.

(iv) $A = B \wedge C$. Assume that $A \in \pi(i)$. Then by step 1 of our algorithm and by definition of the $\wedge$-rule it must be that $B \in \pi(i)$ or $C \in \pi(i)$. By i.h. we have that $\mu, i \not\models B$ or that $\mu, i \not\models C$, i.e. $\mu, i \not\models B \wedge C$.

(v) $A = \bigcirc B$. Assume that $A \in \pi(i)$. If π is finite and $\pi(n)$ is its last element, then no $\bigcirc$-formula can occur in $\pi(n)$ because of our algorithm, and thus $i < n$. But then $B \in \pi(i+1)$ follows by the $\bigcirc$-rule. By i.h. we have that $\mu, i+1 \not\models B$, i.e. $\mu, i \not\models A$. The same holds for the case where π is infinite.

(vi) $A = \Diamond B$. Assume that $A \in \pi(i)$. By our strategy and the definition of the $\bigcirc$- and the $\Diamond$-rule we have that for all $j \geq i$, $B \in \pi(j)$. Hence, by i.h. we get that for all $j \geq i$, $\mu, j \not\models B$, i.e. $\mu, i \not\models \Diamond B$.

(vii) $A = \Box B$. Assume that $A \in \pi(i)$. We will show that there exists some $j \geq i$ such that $B \in \pi(j)$. Assume otherwise that for all $j \geq i$ we have $B \notin \pi(j)$. This means that every time a $\Box$-rule is applied to formula $\Box B$ or a $\Box_H$-rule is applied to a formula $\Box_H B$ the branch which our π is following should contain the right premise of the $\Box/\Box_H$-rule. Hence, we have that for all $j \geq i$ either $\Box B \in \pi(j)$ or, for some $H \in \mathsf{His}$, $\Box_H B \in \pi(j)$. Thus, by our strategy this branch cannot end in an irreducible leaf and hence has to be infinite. Notice that a formula of the form $\Box C$ cannot be annotated for ever. Since the $\Box_H$ rule always adds context to the history and there are only finitely many subsets of the closure of the endsequent, some branch would eventually be an instance of the rep axiom, i.e. at some point we will have:

$$\mathsf{rep}\ \frac{}{\Delta, \Box_{H,\Delta} C}$$

for some Δ and H, which cannot occur in our infinite branch. Thus, by our strategy and the foc-rule, every $\Box$-formula which occurs in every $\pi(j)$ for $j \geq i$ is eventually annotated and later unannotated. Thus, we in particular have $\Box_H B \in \pi(j)$ for some annotation H and some $j \geq i$. The only way to drop the annotation is by taking the branch along the left premise of the $\Box_H$-rule. Thus there is a $j \geq i$ such that $B \in \pi(j)$. By i.h. we have that $\mu, j \not\models B$, which implies that $\mu, i \not\models \Box B$, i.e. $\mu, i \not\models A$.

By the claim we have for all $A \in \pi(0)$ $\mu, 0 \not\models A$. Since $\Gamma \subseteq \pi(0)$ we have that for all $A \in \Gamma$, $\mu, 0 \not\models A$. Hence, $\not\models \Gamma$ and the proof of the theorem is completed. ⊣

By theorems 19 and 22 we get the following corollary.

Corollary 23. *System* LT2 *is sound and complete with respect to* LTL*-models.*

Proof. Since, all the rules of LT2′ are very similar to those of LT2, soundness follows from Theorem 19.

For completeness it suffices to embed LT2′ to LT2, since by Theorem 22, LT2′ is complete. We will do this by transforming each LT2′-proof to an LT2-proof.

Let $\mathcal{D}$ be an LT2′-proof. Let α be an instance of an LT2′-rule in $\mathcal{D}$. If $\alpha \neq$ foc then α is an instance of an LT2-rule too (observe that the LT2′-$\bigcirc$ rule is a special case of the LT2-$\bigcirc$ rule). Assume that α is an instance of foc, as below:

$$\text{foc}\ \frac{\Gamma, \Box_\emptyset A}{\Gamma, \Box A}$$

First we replace every occurrence of $\Box_\emptyset A$ with $\Box A$ in all branches starting from α and then we remove rule α from $\mathcal{D}$ (by compressing the conclusion and the premise). Assume that $\Box_\emptyset A$ was used in the conclusion of a $\Box_H$-rule in $\mathcal{D}$, as follows:

$$\Box_\emptyset\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{\{\Gamma\}} A}{\Gamma, \Box_\emptyset A}$$

Then we can transform the above rule to an LT2-foc rule very easily:

$$\text{foc}\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{\{\Gamma\}} A}{\Gamma, \Box A}$$

All the other rules are fine after our transformations. We repeat the same procedure for all the $\mathsf{LT2}'$-foc occurrences in $\mathcal{D}$. In this way we can transform $\mathcal{D}$ to an $\mathsf{LT2}$-proof, so the embedding from $\mathsf{LT2}'$ to $\mathsf{LT2}$ is completed. ⊣

Remark 24. It is easy to see that no one of the sequent systems presented in this thesis is complete for annotated sequents. Take for example the valid sequent $\Box_{\{\{P\}\}}P$, where $\{\{P\}\}$ is a history that contains only the sequent $\{P\}$, i.e. a sequent with one element. Proof search in $\mathsf{LT2}'$ fails as we can see in the following proof-attempt:

$$\dfrac{\dfrac{}{\Box_{\{\{P\}\}}P, P}\,\mathsf{rep} \qquad \dfrac{\Box_{\{\{P\}\}}P}{\bigcirc\Box_{\{\{P\}\}}P}\,\bigcirc}{\Box_{\{\{P\}\}}P}\,\Box_{\{\{P\}\}}$$

Since $\mathsf{LT2}'$ is stronger than all the other systems we presented it is not difficult to prove that also $\mathsf{LT1}$, $\mathsf{LT1}^+$ and $\mathsf{LT2}$ are not complete for annotated sequents. Brünnler and Lange in [2] present another variation of system $\mathsf{LT2}'$ that is complete for annotated (and not annotated) sequents.

4.3 Weakening for LT2

We would like to show weakening for not annotated sequents in $\mathsf{LT2}$, i.e. we for Γ and $\Sigma \in \mathsf{Seq}$ we would like to have a syntactical proof for the following implication:

$$\mathsf{LT2} \vdash \Gamma \Longrightarrow \mathsf{LT2} \vdash \Gamma, \Sigma$$

In sequent systems the above implication is typically shown by induction on the depth of the proof-tree. This can be done provided that weakening permutes over every rule of the system.

As it is pointed out in [2] weakening does not permute over the foc-rule. Permuting weakening over the foc-rule would require a rule that is not even sound:

$$\dfrac{\dfrac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{\{\Gamma\}}A}{\Gamma, \Box A}\,\mathsf{foc}}{\Gamma, \Box A, \Sigma}\,\text{weakening}$$

$$\downarrow$$

$$\dfrac{\text{weakening}\ \dfrac{\Gamma, A}{\Gamma, A, \Sigma} \qquad \dfrac{\Gamma, \bigcirc\Box_{\{\Gamma\}} A}{\Gamma, \bigcirc\Box_{\{\Gamma,\Sigma\}} A, \Sigma}\ \text{not sound!}}{\Gamma, \Box A, \Sigma}\ \mathsf{foc}$$

As Brünnler and Lange [2] point out the problem is to be expected. The foc-rule incorporates an induction principle and the fact that a certain statement is provable by induction does not imply that a weaker statement is also provable by induction. Here we present a solution to this problem for LT2.

We use the same approach as for LT1. That is we show weakening as a corollary of the next-property. However, the presence of histories in LT2 requires some care. We will need the following lemma.

Lemma 25. *Let H and $G \in$ His. It holds:*

1. *If* LT2 $\vdash^n \Gamma, \Box_H A$*, then* LT2 $\vdash^n \Gamma, \Box_{H,G} A$*.*
2. *If* LT2 $\vdash^n \Gamma, \bigcirc\Box_H A$*, then* LT2 $\vdash^n \Gamma, \bigcirc\Box_{H,G} A$*.*

Proof. We prove 1 and 2 simultaneously by induction on n.

1. Let α be the last rule applied to obtain $\Gamma, \Box_H A$. We distinguish cases depending on α.

- α cannot be foc.
- If $\Box_H A$ is the principal formula of α then α can be either rep or $\Box_{H,G}$. If α is rep then the claim is obvious. If α is $\Box_H$ then the claim follows by i.h. 2.
- If $\Box_H A$ is a side formula of α then the claim follows by i.h. 1.

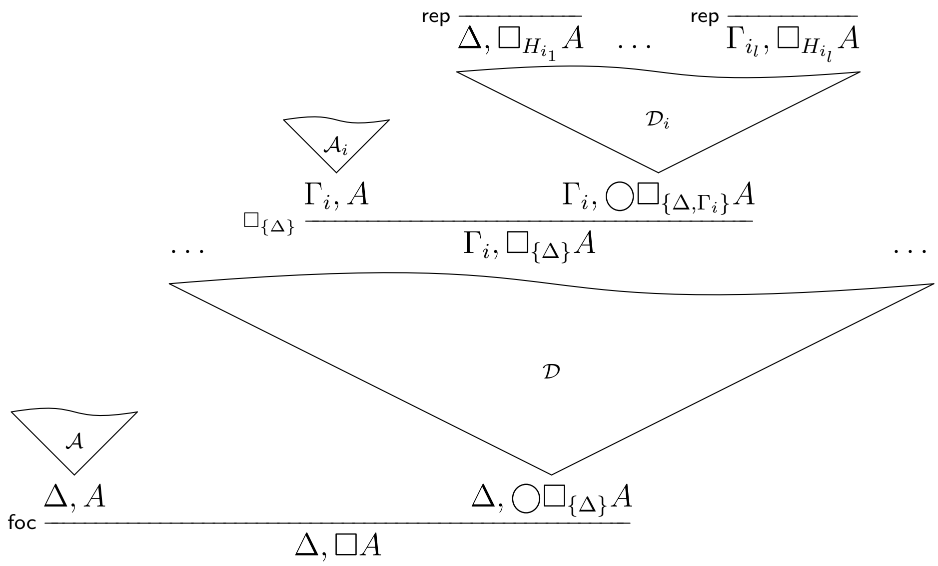
2. Let α be the last rule applied to obtain $\Gamma, \bigcirc\Box_H A$. We distinguish cases depending on α.

- α cannot be foc.
- If $\bigcirc\Box_H A$ is the principal formula of α then α can be only rule $\bigcirc$. Then the claim follows by i.h. 1.
- If $\bigcirc\Box_H A$ is a side formula of α then the claim follows by i.h. 2. ⊣

The analogue of Lemma 6 for system LT2 is the following lemma:

Lemma 26. *Let* $\Gamma \in \mathsf{Seq}$. *If* $\mathsf{LT2} \overset{n}{\vdash} \Gamma$, *then there is an* $\mathsf{LT2}$*-proof of* Γ *satisfying the next-property.*

Proof. Again the proof is by induction on n and a case distinction on the last rule. We only show the case for foc. Then $\Gamma = \Delta, \Box A$ and the given proof of Γ has the following form:



We have that in the derivation $\mathcal{D}$:

$$\begin{array}{l}\text{any branch from } \Delta, \bigcirc\Box_{\{\Delta\}} A \text{ to some } \Gamma_i, \Box_{\{\Delta\}} A \\ \text{goes through a } \bigcirc\text{-rule}\end{array} \tag{4.5}$$

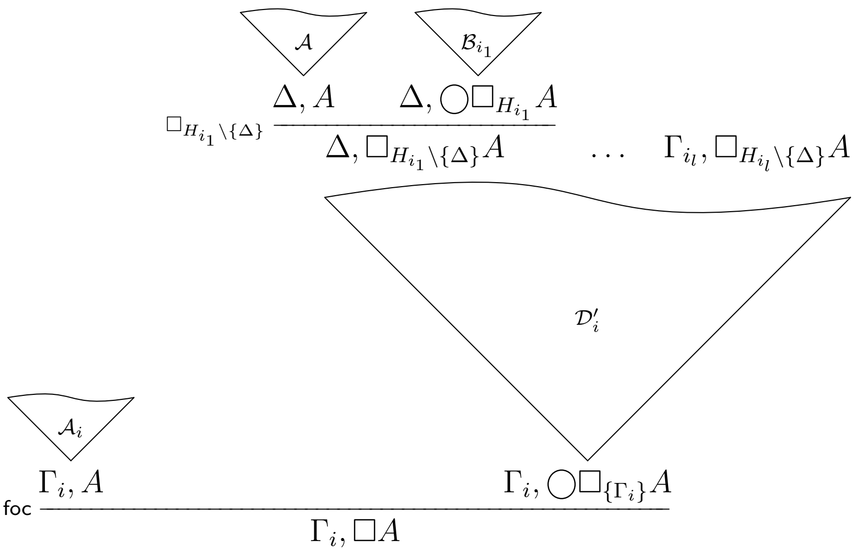
and

$$\text{there are no instances of } \mathsf{foc} \tag{4.6}$$

Furthermore, we observe that if $\Delta \in H_{i_k}$ (that is when $\Delta, \Box_{H_{i_k}} A$ is an instance of rep), then from $\mathsf{LT2} \vdash \Delta, \bigcirc\Box_{\{\Delta\}} A$ and Lemma 25 we get a proof $\mathcal{B}_{i_k}$ for $\Delta, \bigcirc\Box_{H_{i_k}} A$.

We let $\mathcal{D}'_i$ be the the derivation that results from $\mathcal{D}_i$ by deleting Δ from all histories occurring in threads starting from $\bigcirc\Box_{\{\Delta,\Gamma_i\}} A$.

Hence we obtain the following proofs of $\Gamma_i, \Box A$, which we denote by $\mathcal{C}_i$.



Now we proceed as follows:

1. We apply the induction hypothesis to $\mathcal{A}$, which yields a proof $\mathcal{A}'$ of Δ, A that satisfies the next-property.
2. We let $\mathcal{D}'$ be the derivation that results from $\mathcal{D}$ by dropping the annotation Δ in the threads starting from $\bigcirc\Box_{\{\Delta\}} A$.

We find that in the derivation $\mathcal{D}'$,

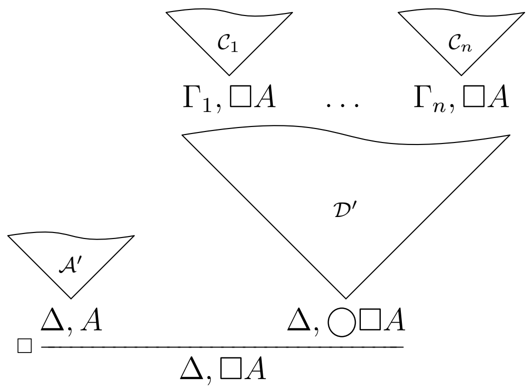
any branch from $\Delta, \bigcirc\Box A$ to some $\Gamma_i, \Box A$ goes through a $\bigcirc$-rule

because of (4.5) and

there are no instances of foc

because of (4.6).

Finally we obtain the following proof of $\Delta, \Box A$.



This proof satisfies the next-property. Indeed, we have:

1. the proof $\mathcal{A}'$ satisfies the next-property;
2. any branch from $\Delta, \bigcirc\Box A$ to some $\Gamma_i, \Box A$ goes through a $\bigcirc$-rule;
3. the derivation $\mathcal{D}'$ does not contain instances of foc.

Hence any branch from the conclusion $\Delta, \Box A$ to an instance of foc goes through a $\bigcirc$-rule. $\dashv$

We get weakening for $\mathsf{LT2}$ as a corollary of Lemma 26. The proof is the same as for Corollary 7.

Corollary 27 (Weakening for non-annotated sequents)**.** *For any* $\Gamma, \Delta \in \mathsf{Seq}$ *we have:*

$$\mathsf{LT2} \vdash \Gamma \quad \Longrightarrow \quad \mathsf{LT2} \vdash \Gamma, \Delta$$

Chapter 5

Summary-Open Problems

In this thesis we explored the use of annotated sequent systems for linear temporal logic LTL. The first finitary cut-free annotated sequent system for LTL was introduced by Brünnler and Lange in [2]. In this thesis we explored the design of this system and proved, by presenting a series of examples, that the system of [2] is as simple as a cut-free sequent system for LTL can be. We also showed that if we use the cut rule, then the system of Brünnler and Lange can be made much simpler. This provides a very nice and instructive example on the role of cut in proofs of induction statements. We also presented the soundness and completeness proof for the system of [2]. As we mentioned before, the proof-theory of systems like the one of Brünnler and Lange is notoriously difficult. In this thesis we presented a purely syntactical proof for the admissibility of weakening for an annotated sequent system for LTL that has all the nice properties (finitary, cut-free) of Brünnler and Lange's system. Unfortunately our idea for proving the admissibility of weakening cannot be applied in proving cut elimination, which is an important proof-theoretical problem for LTL and fixed-point logics in general that still remains open.

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