Heteroclinic, Homoclinic and Periodic Orbits in Hamiltonian systems for critical values of energy

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To my family

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Abstract: The problem considered is the existence of heteroclinic, homoclinic and periodic orbits of Hamiltonian systems in the classical mechanis case where bounded trajectories $u : \mathbb{R} \to \mathbb{R}^n$ connect points on level sets of the potential function $-W$ and solve the systems : $u_{xx} = W_u(u) - cu_x$ and $u_{xx} = W_u(u)$. These level sets considered have at most a finite number of distinct components.

Περίληψη: Το πρόβλημα που θα μελετήσουμε αφορά την ύπαρξη ετεροκλινών, ομοκλινών και περιοδικών τροχιών Χαμιλτονιανών συστημάτων στην περίπτωση της κλασικής μηχανικής όπου φραγμένες τροχιές $u : \mathbb{R} \to \mathbb{R}^n$ συνδέουν σημεία απο ισοσταθμικά σύνολα της συνάρτησης δυναμικού $-W$ και λύνουν τα συστήματα : $u_{xx} = W_u(u) - cu_x$ καί $u_{xx} = W_u(u)$. Τα εν λόγω ισοσταθμικά σύνολα, έχουν το πολύ πεπερασμένο πλήθος διακριτών συνιστωσών.

Contents

Chapter 1

Introduction

The first part of this dissertation(**Chapter 2**) consists of a presentation of a proof of existence of an heteroclinic orbit that solves the system:

$$u_{xx} = W_u(u) - cu_x$$

with $\Omega = \mathbb{R}$, $W_u = (\frac{\partial W}{\partial u_1}, ..., \frac{\partial W}{\partial u_n})$, $u(x) = (u_1(x), ..., u_n(x))$ and W has a global minimum a^- and a local minimum a^+(and some additional conditions). The proof presented here follows the one in the book: Elliptic Systems of Phase Transition Type(preprint)[6] by Alikakos N.,Fusco G., and Smyrnelis P. The idea that is presented is the constraint minimization, which was first introduced by Alikakos N. and Fusco G. [1] and also applied by Alikakos N. and Katzourakis N. [2]. The last two presented a proof of existence of an heteroclinic connection that was later simplified in [6]. These trajectories are also known as a traveling waves and c denotes their speed.

In **Chapter 3**, the Hamiltonian system:

$$u_{xx} = W_u(u)$$

where the set {W = 0} possesses a finite number of distinct components is considered. The existence of heteroclinic, homoclinic and periodic orbits that connect two distinct components of the set {W = 0} is proven for the vector case as well as the asymptotic convergence of these solutions. The results presented in this chapter derive mostly from the work of Smyrnelis P. and Antonopoulos P. [3] and Fusco G., Gronchi G. and Novaga M. [4]

Finally, in **Chapter 4**, some selected theorems and comments are presented along with some remarks to complete this thesis.

Note that the proofs of some lemmas, propositions or corollaries have been deliberately excluded, partially of wholly. This happens because they are either very technical or engage ideas from previous results. In any case, one can find them in detail in the references mentioned, usually in [2],[3] and [6].

Chapter 2

Heteroclinic connections for double well unbalanced potentials- The travelling wave

Let W be a potential having a global minimum at a^- and a local minimum at a^+ such that: $\mathrm{W}(a^-) < 0 = \mathrm{W}(a^+)$,(see figure 2.1).

The travelling wave problem is:

$$u'' - W_u(u) = -cu', u : \mathbb{R} \to \mathbb{R}^m, c > 0 \tag{2.1}$$

$$\lim_{x \to -\infty} u(x) = a^-, \lim_{x \to +\infty} u(x) = a^+ \tag{2.2}$$

The system has variational structure as the equation is the Euler-Lagrange of the weighted *Action Functional*: $J_c(u) = \int_{\mathbb{R}} (\frac{|\dot{u}|^2}{2} + W(u)) e^{cx} dx$

(cf. Fife and McLeod [5]).

Adopting the following Hypotheses, we will prove that there exists a function $u : \mathbb{R} \to \mathbb{R}^m$ and a unique c > 0 that solve (2.1).

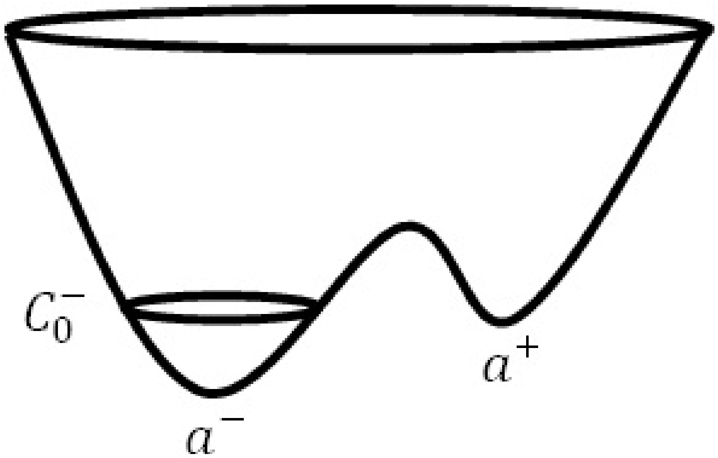

Figure 2.1:

Hypotheses:

H_1 : The potential W : $\mathbb{R}^m \to \mathbb{R}$ is of class C^2, with two minima a^-, a^+ such that: $W(a^-) < 0 = W(a^+)$, W(u) $> W(a^-)$ for every u $\neq a^+$ and $\lim_{|u|\to\infty} W(u) > 0$.

H_2 : {u: W(u) $\leq 0\} = C_0^- \cup \{a^+\}$, dist$(C_0^-, a^+) > 0$ with C_0^- being a strictly convex set with C^2 boundary ∂C_0^-.

H_3 : (i) ∇W·v > 0 on ∂C_0^-, with v being the outward normal on ∂C_0^-.

(ii) The eigenvalues of D^2W $- \epsilon_0 I$ on ∂C_0^- have non-negative real parts.

H_4: (i) There is a $r_0 > 0$ such that the function: $r \to W(a^+ + r\xi)$ is strictly increasing for r $\in (0, r_0]$ and $\xi \in S^{m-1}$

(ii) The function: r $\to W(a^- + r\xi)$ has a strict positive derivative when a^- + rξ $\in C_0^-$. Assuming also that: $B(a^-, r_0) \subset C_0^-$, $\xi \in S^{m-1}$.

Under the Hypotheses H_1 to H_4 we can prove the following theorem:

Theorem 2.1. There exists a function $u : \mathbb{R} \to \mathbb{R}^m$ and a unique $c^* > 0$ that solve the traveling wave problem also satisfying the following statements:

$$J_{c^*}(u) = 0 \tag{2.3}$$

$$c^* = \frac{-W(a^-)}{\int_{\mathbb{R}} |\dot{u}|^2 dx} \tag{2.4}$$

Remark: The speed c^* can also be variationaly characterized as follows:

$$c^* = \sup_{c>0}\{\inf_A J_c(u) < 0\} \tag{2.5}$$

where $A = \{$u $\in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m) :\ni x_u^- < x_u^+$(depending on u) such that: $x \le x_u^- \Rightarrow |u(x) - a^-| \le \frac{r_0}{2}$ and $x \ge x_u^+ \Rightarrow |u(x) - a^+| \le \frac{r_0}{2}\}$.

Before presenting the proof, we will present a replacement lemma that was introduced in [1](cf. section 3.3-Lemma 3.2 and lemma 3.1).
Note that in [1] this lemma gives information about the minimizers of the(unweighted) Action Functional:

$$J_0(u) = \int_{\mathbb{R}}(\tfrac{|\dot{u}|^2}{2} + W(u))dx$$

Which is used in the special case where the potential function W is balanced($W(a^+) = W(a^-) = 0$) and $c = 0$ in order to describe bounded solution of the Euler-Lagrange of $J(u)$. These solution are known as *standing waves*.
However it will also be applied to some parts of this proof as well.

Lemma 2.1: Let α be a point of minimum of W. Let a $<$ b $\in \mathbb{R}$, r $\in \mathbb{R}$ and u $\in W^{1,2}([a,b], \mathbb{R}^m)$ be such that
(i) $0 < \rho_u(a) = \rho_u(b) = r \le \frac{r_0}{2}$
(ii) $r \le \rho_u(x)$ for some x $\in (a, b)$.
Then there exists a function v $\in W^{1,2}([a,b], \mathbb{R}^m)$ such that:

$$\text{v(a)} = \text{u(a)},\ \text{v(b)} = \text{u(b)},$$
$$\rho_v(x) < r \quad \forall\, x \in (a,b),$$
$$\text{J(v)} < \text{J(u) on [a,b]}$$
where $\rho_u(x) = |u(x) - a^-|$ and $\rho_v(x) = |v(x) - a^-|$.

The proof of that lemma derives from a pointwise deformation: $v(x) = a^-\chi_{(-\infty,a]\cup[b,+\infty)} + a_- + rh(\rho(x))\mathbf{n}(x)$ (when $\max_{[a,b]}\rho(x) > r_0$) and when $\max_{[a,b]}\rho(x) \leq r_0$ the proof utilizes another replacement lemma, still introduced in [1], also proved by deforming u pointwise, and states the same results but with the differentiated hypotheses:
(i) $0 < \rho_u(a) = \rho_u(b) = r \leq r_0$
(ii) $r \leq \rho_u(x) \leq r_0$ for every x $\in (a,b)$.

We will prove the existence of a solution by minimizing the *Action Functional* $J_c(u)$ over the class of functions A considering c an arbitrary parameter first. But, minimization cannot be done directly because A is a non-compact space, thus, a minimizing sequence may converge to the trivial minimizers $a^{\pm}$. Also noticing that the equation is translation invariant while the Action Functional is sensitive to translations:

$$J_c(u(x-s)) = e^{cs}J_c(u(x)) \tag{2.6}$$

We understand that if $J_c(u) \neq 0$ the minimizing propery cannot be preserved.
In order to overcome these problems we will minimize the Action over the space:

$$A_L = \{u \in A : x_u^+ \leq L, x_u^- \geq -L, L \geq 1\}$$

considering c a free parameter. This space eliminates the problem of translations.

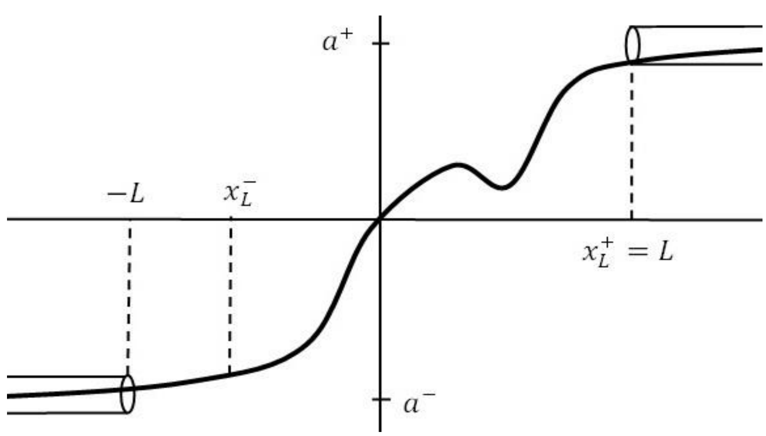

Figure 2.2:

Afterwards, we will prove the existence and uniqueness of a c that satisfy (2.3) and for that c we will show that for L large enough the constraint is not realized. The last one is necessary because it is minimizer to touch the rims of the cylinders above, and then it may not solve the Euler-Lagrange.

Proof of Theorem 2.1:

Proposition 2.1: Let L ≥ 1, then the variational problem:

$$\min_{A_L} \int_{\mathbb{R}} (\frac{|\dot{u}|^2}{2} + W(u))e^{cx})dx$$

,

admits a minimizer u_L depending of c > 0.

Proof of proposition 2.1:

Firstly, it can easily be checked that: $J_c(u) > -\infty$, because:

$J_c(u) \geq -\int_{\mathbb{R}} W^-(u)e^{cx}dx \geq -W^-(a^-)\int_{-\infty}^{L} e^{cx}dx = -\frac{W^-(a^-)}{c}e^{cL} \geq -\infty.$

Now, we consider the *affine* function:

$u_{aff}(x) = a^-\chi_{(-\infty,-1]} + \frac{1-x}{2}a^- + \frac{1+x}{2}a^+ + a^+\chi_{[1,+\infty)}$, with $\sigma := J_c(u_{aff}) < +\infty$.

We can consider a minimizing sequence u_n that satisfies $J_c(u_n) \leq \sigma$.

Also, one can check after a few calculation that the following estimate hold:

$$\frac{1}{2}\int_{\mathbb{R}} |\dot{u}|e^{cx}dx \leq J_c(u_{aff}) + \frac{W^-(a^-)}{c}e^{cx} := \sigma'$$

Since $u_n(x) \in A_L$, it is obvious that for $x \in (-\infty, -L] \cup [L, +\infty)$

we have the estimate: $|u_n(x)| \leq \max\{|a^+|, |a^-|\} + \frac{r_0}{2}$.

Considering these, the following useful estimate can also be proven easily:

$|u_n(x)| \leq |u_n(-L)| + \int_{-L}^{x} |\dot{u}|e^{\frac{ct}{2}}e^{\frac{-ct}{2}}dx \leq \max|a^+|, |a^-| + \frac{r_0}{2} + \sqrt{2\sigma'}\int_{-L}^{L} e^{-ct}dt$, and

$|u_n(x) - u_n(y)| = |\int_y^x \dot{u}dx| \leq \int_y^x |\dot{u}|dx \leq \sqrt{\int_y^x |\dot{u}|^2dx}\sqrt{(x-y)} \leq \sqrt{2\sigma'}\sqrt{(x-y)}.$

So, the minimizing sequence is uniformly bounded and equicontinuous on every compact interval. Thus, by the Ascoli-Arzela theorem and a diagonal argument, we can choose a subsequence u_k that converges uniformly on compact intervals to a function $u_L \in C(\mathbb{R}, \mathbb{R}^m)$.
Now, replacing the dx by the absolutely continuous $d\mu(x) = e^{cx}dx$ and taking a subsequence u_{k_j} we have that:

$$u'_{k_j} \rightharpoonup u'_L,$$

By the weak lower semicontinuity it is straightforward that:

$$\liminf \int_{\mathbb{R}} |u'_{k_j}|^2 e^{cx} dx \geq \int_{\mathbb{R}} |u'_L|^2 e^{cx} dx.$$

Finally, applying the fatou lemma to the non-negative measurable sequence:

$W(u_{k_j}) + W^-(a^-)\chi_{(-\infty,L]}$ we obtain that: $J_c(u_L) \leq \liminf J_c(u_{k_j}) = \inf_{A_L} J_c$

this completes the proof of the proposition.

Next, we will prove that inside the cylinders the constraint is not realized.

Proposition 2.2: For the arbitrary c >, the respective minimizer u_L solves the Euler-Lagrange of $J_c(u)$ on every connected component of $\mathbb{R} \setminus${-L,+L} and also satisfy the "boundary" conditions (2.2).

Proof of Proposition 2.2:

Noticing that u_L solves the equation (2.1) in $(-L, +L)$ since it is a minimizer we will prove the following two lemmas to show that (2.1) is also satisfied in $\mathbb{R} \setminus (-L, +L)$.

Lemma 2.2: Let $\rho_L^-(x) := |u_L(x) - \alpha^-|$. Then the equation $\rho_L^-(x) = \frac{r_0}{2}$ has a unique solution $\lambda_L^- \geq -L$. Moreover, the function ρ_L^- is strictly increasing in the interval $(-\infty, \lambda_L^-]$ and $\lim_{x \to -\infty} \rho_L^-(x) = 0$.

Proof:

Lemma 2.1 (cf. [A-F]) is proved by a pointwise deformation of u, therefore it also holds for the functional J_c. Let $\rho_L^-(x) := |u_L(x) - \alpha^-|$. We will prove that $\forall r \in (-\infty, \lambda_L^-)$ the equation $\rho_L^-(x) = r$ has a unique solution.
By lemma 2.1 we have that this equation has at most two solutions.To reach a contradiction, suppose that there exists $x_1 < x_3$ such that $\rho_L^-(x_1) = \rho_L^-(x_3) = r$. Then by lemma 2.1 we have that an x_2 exists such that $\min_{[x_1,x_3]} \rho_L^-(x) = \rho_L(x_2)$.
We also have that $\rho_L^-(x) \geq$ r $\forall$ x $\in (-\infty, x_1]$, since otherwise the equation $\rho_L^-(x) = r - \epsilon$ for $\epsilon > 0$ small enough would have more than two solutions and that would contradict lemma 2.1.
Writing $u_L(x)$ in the polar form: $u_L(x) = \alpha^- + \rho_L^-(x)\mathbf{n}_L^-(x)$, we define the function

$$v(x) := (a^- + \rho_L^-(x_2)\mathbf{n}_L^-(x))\chi_{(-\infty,x_2]} + u_L(x)\chi_{[x_2,+\infty)}$$

It is easy to check that $J_c(v) < J_c(u_L)$, which contradicts the minimality of u_L. The uniqueness of solution has been proved. Moreover, the fact that $\rho_L(x)$ is strictly increasing has been proved as well as that u_L solves the Euler-Lagrange in $(-\infty, \lambda_L^-)$.
Now, we will show that $\lim_{x\to-\infty} \rho_L^-(x) = 0$. Indeed, if $\lim_{x\to-\infty} \rho_L^-(x) > 0$ then by considering the functin $f(x) := (\rho_L^-)^2$, which is bounded by definition, we can see that due to the Hypothesis $\mathbf{H}_4$ and after a few calculations that the bound of f(x) in $(-\infty, \lambda_L^-)$ is violated.

Lemma 2.3: Let $\rho_L^+(x) := |u_L(x) - \alpha^+|$. Then the equation $\rho_L^+(x) = \frac{r_0}{2}$ has a unique solution $\lambda_L^+ \leq L$. Moreover, the function ρ_L^+ is strictly increasing in the interval $[\lambda_L^+, \infty)$ and $\lim_{x\to\infty} \rho_L^+(x) = 0$.

Proof:

Observe that there exists a sequence $x_n \to +\infty$ as $n \to +\infty$ such that $\rho_L^+(x) \to 0$, otherwise $J_c(u_L) = +\infty$. Now we will show that $\forall$ r $\in (0, \frac{r_0}{2}]$ the equation $\rho_L^+ = r$ has a unique solution. Assume there exists $a < b$ such that $\rho_L^+(a) = \rho_L^+(b) = r$. If a $y_2 \in [a, b]$ such that $u_L(y_2) \in \partial C_0^-$, it is obvious that a $y_1 < a$ such that $u_L(y_1)$ $\in \partial C_0^-$. Then, if we define the function:

$$v(x) = P(u_L(x))\chi_{[y_1,y_2]} + u_L(x)\chi_{(-\infty,y_1]\cup[y_2,+\infty)}$$

Where P is the projection onto the (strictly) convex set C_0^-. Projections onto strictly convex sets of Hilbert spaces decrease distances: $|P(a)-P(b)| < |a-b|$, so, $P'(y) < 1$ almost everywhere and thus: $|P'(u_L)u_L'|^2 < |u_L'(x)|^2$ a.e.

The last one gives us: $J_c(v) < J_c(u_L)$. So, the curve cannot intersect C_0^- for x $\in [a,b]$, lemma 2.1 can be applied and since r $\in (0, \frac{r_0}{2}]$ was arbitrary and considering the fact that $\rho_L^+(x) \to 0$ we reach a contradiction. The uniqueness of solutions of the equation $\rho_L^+ = r$ as well as the monotonicity of $\rho_L^+(x)$ are now obvious. The proof of lemma 2.3 is complete.

Now, we introduce the convex set C_a^-.
Consider the set $u : W(u) \leq a$ for $a \in (0, a_0]$, with a_0 small enough. We denote C_a^- the stictly convex set that encloses a^- and being such that $C_a^- \cap B_{(a^-, \frac{r_0}{2})} = \emptyset$. For that sets, we introduce the following results [A-F-S](6)):

Lemma 2.4: For every a $\in (0, a_0)$, there exists a unique $\lambda_L^{a^-} \in (\lambda_L^-, \lambda_L^+)$ such that $u_L(\lambda_L^-) \in \partial C_a^-$ and $u_L(\lambda_L^-) \in C_a^-$ if and only if $x \leq \lambda_L^{a^-}$.

Proof of Lemma 2.4: Suppose, by contradiction that there exists an a $\in (0, a_0)$, and $x_2, x_4 \in \mathbb{R}$ such that $u(x_2)$ and $u(x_4) \in \partial C_a^-$. We have that $u_L(x)$ cannot intersect $\partial B(a^+ . r_0/2)$ in $[x_2, x_4]$ because then, an $x_5 > x_4$ such that $\rho_L^+(x_5) = \rho_L^+(x_4) = r_0/2$ would exists and that would contradict lemma 2.3. Now, let's consider the case when $u_L \notin C_a^-$ for some x $\in [x_2, x_4]$, then by setting

$$v(x) = P(u_L(x))\chi_{[x_2,x_4]} + u_L(x)\chi_{(-\infty,x_2]\cup[x_4,+\infty)}$$

with P being the projection onto the convex set C_a^-, thus, we obtain as before: $J_c(v) < J_c(u_L)$, which is a contradiction to the minimality of u_L. Now, if an x_3 such that $W(x_3) \in C_a^- \setminus \overline{C_0^-}$, then an x_1 such that $W(u_{x_1}) = W(u_{x_3}) < W(u_{x_2})$, and applying the same procedure as before, we reach to a contradiction. Thus, it has to be that $u_L(x) \in C_a^- \; \forall x \in [x_2, x_4]$, but, even in that case, we reach to a contradiction, and that is, because, there exists a $y \in (x_2, x_4)$ such that $u_L'(y) \neq 0$, otherwise $u_L(x)$ would be constant in (x_2, x_4) and that contradicts hypotheses H_3 since $[x_2, x_4] \subset (-L, +L)$ and thus, u_L solve the Euler-Lagrange equation there. For δ small enough we define the function:

$$v_\delta(x) = P_\delta(u_L(x))\chi_{[y,y+\delta]} + u_L(x)\chi_{(-\infty,y]\cup[y+\delta,+\infty)},$$

with P_δ being the projection onto the line passing through $u_L(y)$ and $u_L(y)$, and since the line is also a convex set, then again $J_c(v_\delta) < J_c(u_L)$. The proof of the Lemma is complete.

Summarizing the previous lemmas we get that:

(i) For every a positive and small u_L exits C_a^- precisely once at $x = \lambda_L^{a-}$, $u_L^{-1}(C_a^-) = (-\infty, \lambda_L^{a-}]$,
(ii) $u_L^{-1}(B(a^+, r_0/2)) = (\lambda_L^+, +\infty)$,
(iii) $x \in [\lambda_L^{a-}, \lambda_L^+] \implies W(u_L(x)) \geq \alpha$
(iv) $\lambda_L^{\pm}$ are well defined as the unique values of x that u_L crosses the spheres $\partial B(a^{\pm}, r_0/2)$.

By (iii), we can, intuitively say that if $[\lambda_L^{a-}, \lambda_L^+]$ is large enough, then there would be a problem with the fact that $J_c(u) \leq 0$. Therefore, it is reasonable to expect that an upper bound on $|\lambda_L^+ - \lambda_L^{a^-}|$ holds. This upper bound should also depend on c, since if c is very large, then the minimizer would have to enter $B(a^+, r_0/2)$ quickly, so that $J_c(u_L)$ remains non-positive(for the c that this is possible) and if the opposite happens, then we can allow the minimizer to enter $B(a^+, r_0/2)$ not that fast.

To write these down in a formalistic way, we will prove the following two lemmas, that also provide us an L-independent bound on $|\lambda_L^+ - \lambda_L^{a^-}|$ and on $|\lambda_L^{a^-} - \lambda_L^-|$.

Lemma 2.5: For all a $\in (0, a_0], L \geq 1$ and $c > 0$ such that $J_c(u_L) \leq 0$ the following estimate holds:

$$|\lambda_L^+ - \lambda_L^{a^-}| \leq \frac{1}{c} ln(1 + \frac{W^-(a^-)}{a}) := \Lambda_a^+ \tag{2.7}$$

Proof of Lemma 2.5:

We have the identity: $J_c(u_L) = -\int_{-\infty}^{\lambda_L^{0-}} W^-(u_L)e^{cx}dx + \int_{\lambda_L^{0-}}^{+\infty} W^+(u_L)e^{cx}dx + \frac{1}{2}\int_{\mathbb{R}} |u_L'|^2 e^{cx}dx$.

We have that $W(u_L) \geq \alpha$ on $[\lambda_L^{a-}, \lambda_L^{a-}]$, $W^-(u_L) \leq W^-(a^-)$, also, the following estimates hold:

$$\int_{-\infty}^{\lambda_L^{0-}} W^-(u_L)e^{cx}dx \leq \frac{W^-(a^-)}{c} e^{c\lambda_L^{a-}}$$

.

$$\int_{\lambda_L^{0-}}^{+\infty} W^+(u_L)e^{cx}dx \geq \int_{\lambda_L^{a-}}^{\lambda_L^+} W^+(u_L)e^{cx}dx \geq \frac{\alpha}{c}[e^{c\lambda_L^+} - e^{c\lambda_L^{a-}}],$$

$$d_a \leq |u_L(\lambda_L^{a-}) - u_L(\lambda_L^+)| \leq \int_{\lambda_L^{a-}}^{\lambda_L^+} |\dot{u}|dx \leq \sqrt{(\int_{\lambda_L^{a-}}^{\lambda_L^+} e^{-cx}dx)(\int_{\lambda_L^{a-}}^{\lambda_L^+} |\dot{u}|^2 dx)}$$

Where d_a denotes the distance between C_a^- and $B(a^+, r_0/2)$.

Therefore:

$$0 \geq J_c(u_L) \geq -\frac{W^-(a^-)}{c}e^{c\lambda_L^{a-}} + \frac{\alpha}{c}[e^{c\lambda_L^+} - e^{c\lambda_L^{a-}}] + \frac{cd_a^2}{2(e^{-c\lambda_L^{a-}} - e^{-c\lambda_L^+})} \geq e^{c\lambda_L^{a-}}\Big(- \frac{W^-(a^-)}{c} + \frac{\alpha}{c}(e^{c(\lambda_L^+ - \lambda_L^{a-}}) + \frac{cd_a^2}{2(1 - e^{-c(\lambda_L^+ - \lambda L^{a-})})}\Big) \geq \frac{e^{c\lambda_L^{a-}}}{c}\Big(- \Big(\frac{W^-(a^-)}{\alpha} + 1\Big) + e^{c(\lambda_L^+ - \lambda_L^{a^-})}\Big).$$

Now, the desired inequality is straightforward.

Lemma 2.6: For all a $\in (0, a_0], L \geq 1$ and $c > 0$ such tha $J_c(u_L) \leq 0$ the following estimate holds:

$$|\lambda_L^{a^-} - \lambda_L^-| \leq \frac{1}{w^*}\{cR_{\max}^a + \sqrt{(cR_{\max}^a)^2 + 2w^*[(R_{\max}^a - \frac{r_0}{2})]}\} := \Lambda_a^- \tag{2.8}$$

where $R_{\max}^a := \max_{u \in \partial C_a^-} |u - a^-|$

and $w^* := \min\{\frac{d}{dt}\|_{t=r} W(a^- + t\xi) : \frac{r_0}{2} \leq r \leq R_{\max}^a, |\xi| = 1, a^- + r\xi \in C_a^-\}$

Proof of Lemma 2.6:

Considering the polar form: $u_L = a^- + \rho_L^-(x)\mathbf{n}(x)$ of u. By Hypotheses H_4 and since u_L satisfies the Euler-Lagrange equation (2.1) on $(\lambda_L^-, \lambda_L^{a-}] \subseteq (-L, L]$ we have the following inequality(with $\rho(x)$ declaring $\rho_L^-(x)$):

$$\rho'' + c\rho' = \rho|(n_L^-)'|^2 + \nabla W(a^- + \rho n_L^-) \geq w^* > 0$$

If we integrate the above inequality from λ_L^- to x such that: $\lambda_L^- < x < \lambda_L^{a-}$ and take into account that $\rho'(x) \geq 0$ for x a little bit greater than λ_L^-, then we obtain:

$$\rho'(x) + c\rho(x) \geq w^*(x - \lambda_L^-)$$

and integrating once more from λ_L^- to an x greater than this and lesser than λ_L^{a-} and taking into account that $0 \leq \rho(x) \leq R_{\max}^a$ we obtain:

$$(R_{\max}^a - r_0/2) + (\lambda_L^{a-} - \lambda_L^-)cR_{\max}^a \geq \frac{w^*}{2}(\lambda_L^{a-} - \lambda_L^-)^2$$

From which the desired inequality follows.
Moreover, since $\lim_{x \to -\infty} \rho'(x) \geq 0$ integrating the first inequality from -∞ to an $x < \lambda_L^{a-}$ we get that $\rho'(x) > 0$ in $(-\infty, \lambda_L]$,i.e. ρ is strictly increasing there.

Combining the above, which all bounds are L-independent, the following corollary can be established:

Colollary 2.1: For all a $\in (0, a_0], L \geq 1$ and $c > 0$ such that $J_c(u_L) \leq 0$ the following estimate holds:

$$|\lambda_L^+ - \lambda_L^-| \leq \Lambda = \textit{an L-independent bound.}$$

So far, we have proved, for every c > 0 and L > 1 a minimizer(depending on c and L) exists, and for c such that $J_c(u_L) \leq 0$ then the time that u_L crosses the spheres $B_{(a^-, r_0/2)}$ and $B_{(a^+, r_0/2)}$ is bounded by a bound that only depends on c.

From now on, the (depended on c)minimizer u_L will be denoted as: $u_{L,c}$.

Now, we proceed to determine the speed c.

To do this, we will, among others, prove that $\forall \in [\frac{c_0}{2}, 2c_0]$ uniform bounds for the minizizers $u_{L,c}$, afterwards for a fixed c_0 we will prove that the function: $c \to J_c(u_{L,c_0})$ is continuous.

Proposition 2.3: Let $L > 1$ and $c_0 >$, both fixed. Then, there exists a constant $k > 0$ such that for every $c \in [\frac{c_0}{2}, 2c_0]$ the following results are true:

(i) $|u_{L,c}(x)| \leq k$, for every $x \in \mathbb{R}$,
(ii) The minimizers $u_{L,c}$ are equicontinuous on bounded intervals for c $\in [\frac{c_0}{2}, 2c_0]$,
(iii) $|u'_{L,c}(x)| \leq k$, for every $x \in \mathbb{R} \setminus \{-L, +L\}$,
(iv) $|u_{L,c}(x) - a^+| \leq ke^{-cx}, |W(u_{L,c}(x))| \leq ke^{-2cx}$ and $|u'_{L,c}(x)| \leq ke^{-cx}$ $\forall x > L$.

Proof:

The first two statements((i)-(ii)) can easily be proved by **proposition 2.1**
To prove the third one(iii), we set: $v_{L,c}(x) := e^{cx/2}u_{L,c}(x)$. After a few calculations, we have that $\forall x \in \mathbb{R} \setminus [-L - \epsilon, L + \epsilon]$ (for $\epsilon > 0$): $v''_{L,c}(x) = e^{cx/2}W_u(u_{L,c}(x)) + (c^2/4)u_{L,c}(x)$ and utilizing the first two statements we deduce that: $|v'_{L,c}| \leq Me^{cx/2}$ so $|u'_{L,c}| \leq M$, for a positive constant M.

Now, we note that: $\frac{d}{dx}(W(u_{L,c}(x)) - \frac{1}{2}|u'_{L,c}(x)|^2) = c|u'_{L,c}|^2 \geq 0$, Thus, the function $W(u_{L,c}(x)) - \frac{1}{2}|u'_{L,c}(x)|^2)$ is non-decreasing, so the one sided limits at -L, +L exist. Integrating $\frac{d}{dx}(W(u_{L,c}(x)) - \frac{1}{2}|u'_{L,c}(x)|^2) = c|u'_{L,c}|^2 \geq 0$ we obtain a uniform bound of $|u'_{L,c}|$ and again, utilizing the monotonicity we get the uniform bound of $u'_{L,c}$ in $\mathbb{R} \setminus \{-L, +L\}$.

To prove the last statement(iv), we need to notice Hypotheses H_4 implies $\rho'' + c\rho' \geq 0$ for x > , where $\rho = \rho_{L,c}(x) := |u_{L,c}(x) - a^+|$ and utilizing (iii) we obtain $\rho'_{L,c}(x) \leq Me^{-cx}$, for a positive constant M.
Integrating the last equation, we obtain $|u_{L,c}(x) - a^+| \leq ke^{-cx/2}$.

Now integrating equation 2.1 over the interval $[x, +\infty)$, for arbitrary x $> L$ we get the second inequality.
As for the third inequality, we introduce the function: $s_{L,c}(x) := e^{cx/2}(u_{L,c} - a^+)$, and deduce that $s'_{L,c} \leq Me^{-cx/2}$ $\forall$ x > L. Now, after a few calculations we can easily prove: $|u'_{L,c}| \leq ke^{-cx}$.

Corollary 2.2: For every L > 1 and $c_0 > 1$ both fixed. The function: $c \rightarrow .$ $J_c(u_{L,c_0})$ is continuous in the interval $(c_0/2, 2c_0)$.

The proof of this result is conducted by utilizing the previous estimates and applying the dominated convergence theorem (cf. Alikakos and Katzourakis [2]).

Corollary 2.3: Consider a fixed L>1, and a sequence c_n that converges to c^* and $\forall n$ $c_n \in (c_0/2, 2c_0)$. If $J_{c_n}(u_{L,c_n}) \leq 0$ $\forall n$, then $J_{c^*}(u_{L,c^*}) \leq 0$.

Proof: According to the previous proposition, the sequences u_{L,c_n} and $u'_{L,c}e^{c_n x/2}$ are uniformly bounded in $\mathbb{R}^m$ and $L^2(\mathbb{R},\mathbb{R}^m)$ respectively, with the first one being also equicontinuous on compact intervals.

By the Ascoli-Arzela theorem and a diagonal argument, we can take a subse quence of u_{L,c_n} that converges to $u^*_{L,c}$ on compact intervals. There is also a subsequence of $u'_{L,c_n}e^{c_n x/2}$ that $\rightharpoonup v^*$ weakly in L^2, for a $v^* \in L^2_{loc}(\mathbb{R}, \mathbb{R}^m)$. By the lower semicontinuity of the L^2 norm we have that:

$$\int_{\mathbb{R}} |v^*(x)|^2 e^{c^*} x dx \leq \liminf \int_{\mathbb{R}} |u'_n(x)|^2 e^{c_n x} dx$$

It can also be proved that $(u^*)' = v^*$.

Finally, thanks to the second inequality of (iv) of the previous proposition, we can use the dominated convergence theorem and conclude that:

$$\int_{\mathbb{R}} W(u^*(x))e^{c^*x}dx = \lim_{n'} \int_{\mathbb{R}} W(u_{n'}(x))e^{c^*x}dx$$

Where n' denotes the subsequence. It is clear now that, $J_{c*}(u_{L,c^*}) \leq 0$.

The proof of Corollary 2.3 is complete.

Now we will prove the existence of a unique c>0 such that: $J_c(u_{L,c}) = \inf_{A_L} J_{c^*} = 0, \forall L > \Lambda$.

Firstly, we will use a definition for the set of speeds established by Steffen Heinze, who introduced this on his Ph.D thesis in Heidelberg in 1989:

$$C := \{c > 0 : \exists L \geq 1 : J_c(u_{L,c}) < 0\}.$$

The following results have been proved in [2]

Lemma 2.7: The set C' is non-empty, open has
a supremum such that:

$$\sup C \leq \sqrt{2W^-(a^-)}(d_0^-)^{-1},$$

where $d_0^- = dist(C_0^-, r_0/2)$.

Proof of Lemma 2.7:

We will prove the lemma using the following set of "speeds":

$$C' = \{c : \exists \text{ v} \in A_L \text{: } J_c(v) < 0\}$$

Firstly, we prove that C' is open, then we prove that is upper bounded, and finally we will prove the desired estimate.

Let c_0 *in* C, then $J_{c_0}(u_{L,c_0}) < 0$ and by corollary 2.2 we have that for c close to c_0: $J_c(u_{L,c_0}) < 0$, so C is open. Now, for the affine function introduced in proposition 2.1 we have that: $u_{aff} \in \cap_{L\geq 1} A_L$. We introduce now the C^1 function:

$$f : (0, +\infty) \to \mathbb{R},\ f(c) := e^{-c}\Big(- \tfrac{W^-(a^-)}{c} + e^{2c}J_0^+(u_{aff})\Big),$$

where:

$$J_0^+(u_{aff}) := \int_{\mathbb{R}}(\frac{|\dot{u}_{aff}|^2}{2} + W^+(u_{aff}))dx.$$

We also have that: $\lim_{c\to 0} f(c) = -\infty$, f'(c) > 0, $\lim_{c\to\infty} f(c) = +\infty$ and after a few calculations, one can check that:

$$f(c) \geq J_c(u_{aff})$$

therefore, a unique c_0 such that $f(c_0) = 0$. So, $(0, c_0) \subseteq C$ (and $C \neq \emptyset$).

Since u_L is a minimizer then $\forall$ v $\in$ C and by lemma 2.5 the following inequalities hold:

$$0 > J_c(v) \geq J_c(u_L) \geq \Big(-\frac{W^-(a^-)}{c} + \frac{cd_a^2}{2(1 - e^{-c(\lambda_L^+ - \lambda L^{a_-})})}\Big)$$

which implies that: $0 \geq c^2 d_a^2 - 2W^-(a^-)$.

Finally, by passing to the limit $a \to 0$ we conclude the desired bound.

Lemma 2.8: If $c^* = \sup C$, then $J_{c^*}(u_{L,c^*}) = 0 \ \forall L \geq \Lambda$.

Proof of Lemma 2.8:

First,we get a sequence $c_m \subset C'$ such that $c_m \to c^*$ as $m \to +\infty$. Now, by the definition of C we have that there exists a sequence L_m such that:

$$J_{c_m}(u_{L_m,c_m}) < 0$$

By the L-independent bound proved in corollary 2.1 we have:

$$\lambda_{L_m}^+ - \lambda_{L_m}^- \leq \Lambda$$

Notice that these happen for every m $\in \mathbb{N}$. Now, observe that $\lambda_{L_m}^+ \leq L_m$ by definition. If, $\lambda_{L_m}^+ < L_m$ then by translating u_L to the right, $u_{L_m,c_m}(x) \to u_{L_m,c_m}(x - \epsilon)$ (for ϵ positive), we reach to a contradiction since $u_{L,c}$ is the minimizers and:

$$J_{c_m}(u_{L_m,c_m(x)}) = e^{c_m \epsilon} J_c(u_{L_m,c_m}(x - \epsilon)) < J_{c_m}(u_{L_m,c_m}(x - \epsilon)) < 0$$

So, we must have $\lambda^+_{L_m} = L_m$. Now, since $u_{L_m} \in A_L \ \forall L \geq L_m$, we can choose L_m such that $L_m \geq \Lambda$. Translating u_{L_m,c_m} by $+L_m$ $(u_{L_m,c_m}(x) \to u_{L_m,c_m}(x+L_m))$ we get that the last one is in A_L and since u_{L,c_m} is a minimizer in A_Λ we have the inequality:

$$J_{c_m}(u_{\Lambda,c_m}) \leq J_{c_m}(u_{L_m,c_m}(:+L_m)) = e^{-c_m\Lambda_m} J_{c^*}(u_{L_m,c_m}).$$

.

Finally, taking the limit $m \to +\infty$ we obtain the inequality:

$$J_{c_m}(u_{\Lambda,c_m}) \leq 0$$

and since C' is an open and bounded subset of $\mathbb{R}$ we have that $c := \sup C' \notin C'$, thus :

$$J_{c_m}(u_{L,c_m}) = 0, \, \forall L \geq \Lambda.$$

Proof of existence of solution:

Let an L $> \Lambda$. We have that $J_{c*}(u_{L,c^*}) = 0$, u_{L,c^*} cannot touch both because: $|\lambda^+_L - \lambda^-_L| < \Lambda$. If it touches one cylinder, we can translate the solution by δ small enough in order not to avoid the rim. Therefore u_{L,c^*} is still a minimizer, and thus it satisfies the Euler-Lagrange equation $u'' - W_u(u) = -c^*u'$ along with the asymptotic limits(proposition 2.2). The existence is proved.

Proposition 2.4:(Uniqueness of the speed)

There exists precisely one speed c^* such that (u_{L,c^*}, c^*) solves (1).

Proof:

Let (u,c) be a solution to the equation: $u'' - W_u(u) = -cu'$, after a few calculations we obtain:

$$\frac{|u'|^2}{2} + W(u) = e^{-cx}\left(\frac{e^{cx}}{c}\left(W(u) - \frac{|\dot{u}|^2}{2}\right)\right)'$$

Now, let (u_1, c_1^*),(u_2, c_2^*) solutions to (2.1), with u_1, u_2 minimizing and $0 \leq c_1 < c_2$ without loss of generality. In the identity above if we replace c with c_2, u with u_2 and multiply by $e^{c_1 x}$ we have after a few calculations th identity:

$$c_1^* J_{c_1^*}(u_2, (-t,t)) = (c_1^* - c_2^*) \int_{-t}^{t} |\dot{u}|^2 e^{c_1^* x} dx + e^{-c_1^* x}\left[\frac{e^{c_1^* x}}{c}\left(W(u_2) - \frac{|\dot{u_2}|^2}{2}\right)\right]_{-t}^{t}$$

Letting $t \to +\infty$ and taking into account the estimates in (iii) and (iv) of proposition 2.3, the fact that: $J_{c_2^*}(u_2) = 0$ and $c_1^* < c_2^*$ we finally obtain:

$$c_1^* J_{c_1^*}(u_2) = (c_1^* - c_2^*) \int_{\mathbb{R}} |u_2'|^2 e^{c_1^* x} dx < 0$$

which is a contradiction due to the non-negativity of $c_1^* J_{c_1^*}(u_2)$.

The uniqueness of c is now proved.

The variational characterization:

Note that: $C'' = \{c : \exists \text{ v} \in A\text{: } J_c(v) < 0\} = C = \{c > 0 : \exists L \geq 1 : J_c(u_{L,c}) < 0\}$. So $c^* = \sup C = \sup C'$ and thus, $c^* = \sup_{c>0}\{\inf_A J_c(v) < 0\}$. We have that: $0 = J_{c^*}(u) = \min_{A_L^*} J_{c^*} \geq \inf_A J_{c^*}(v) \geq 0$, for $L^* \geq \Lambda + \delta$ (for $\delta > 0$).

Lastly, we multiply the equation (1) and integrate all over $\mathbb{R}$ and obtain

$$c^* = \frac{-W(a^-)}{\int_{\mathbb{R}} |\dot{u}|^2 dx}.$$

The theorem is now proved.

Remark: In this thesis, we noticed that Hypotheses $H_4(ii)$ can be deduced by the strict convexity assumption mentioned in hypotheses H_2. Indeed, for a fixed $\xi \in S^{m-1}$ considering the twice continuously differentiable $f(r) = W(a^- + r\xi)$ on $[0, r_\xi)$, where: $r_\xi = sup\{r > 0 : a^- + r\xi \in C_0^-\}$ we notice that:

$$f'(r) = f'(r) - f'(0) = \int_0^r \xi^T D^2 W(a^-) \xi dx > 0$$

since $f'(0) = 0$ and $D^2W(a^-)$ is positive definite.

Chapter 3

Heteroclinic, Homoclinic and Periodic orbits as critical points of the Action Functional

In this section, we consider the existence of bounded non trivial minimizers of W. After presenting some requirements that need to be satisfied and proving them, we show the existence of a heteroclinic, homoclinic and periodic orbits considering respective hypothesis on W and also prove asymptotic convergence of these non-periodic orbits.

This part of this thesis follows [3].

The system we are interested to is: $u''(x) = \nabla W(u(x))$ (1)

First we introduce respective results happening in the scalar O.D.E. case and also exist in literature:

Let W be the potential C^2 function, such that W > 0 in (a^-, a^+) and W $= 0$ on a^- and a^+. Then the following statements are true.

(1) If a^- and a^+ are critical points of W, then there exists a C^2 function: $u : \mathbb{R} \to (a^-, a^+)$ that solves $u'' = W'(u)$ and satisfies: $\lim_{x\to\pm\infty} u(x) = a^\pm$. This is the case of the heteroclinic connection.

(2) If $W'(a^-) = 0$ and $Wa^+ \neq 0$ the there exists a unique even solution $u : \mathbb{R} \to (a^-, a^+]$ of $u'' = W'(u)$ such that $\lim_{x\to\pm\infty} u(x) = a^-$ and $u(0) = a^+$. This is the homoclinic connection case.

(3) If $W'(a^\pm) \neq 0$ then there exists a periodic solution $u : \mathbb{R} \to [a^-, a^+]$ such that $u(0) = a^-, u(T/2) = a^+$ and for every x $\in \mathbb{R}$ we have: $u(x+T) = u(x)$, $u(x + T/2) = u(-x + T/2)$, for some T > 0.

We will prove the above results in the vector case by proving an abstract theorem assuming, that $\theta\Omega$ is partitioned into two disjoint compact sets $A^\pm$, where Ω is a connected component of the set: $\{W > 0\}$ and then particularizing it in the cases when: $\nabla W(A^\pm) = 0, \nabla W(A^-) = 0$ and $\nabla W(A^+) \neq 0$ and $\nabla W(A^\pm) \neq 0$ respectively. After that, we will give some additional hypotheses on W that are sufficient to give us asymptotic convergence.

3.1 Conditions for existence of bounded local minimizers and preliminaries

Assume $W \in C^2(R, R^m)$ a potential not necessarily non-negative. Let $\Omega \subsetneq \mathbb{R}^m$ a conenected component of the set $\{u \in \mathbb{R}^m : W(u) > 0\}$. We can see that $W = 0$ on $\partial\Omega$ and consider the sets:

$$\begin{aligned} \partial\Omega_0 &:= \{u \in \partial\Omega : \nabla W(u) = 0\} \\ \partial\Omega_{\neq} &:= \{u \in \partial\Omega : \nabla W(u) \neq 0\} \\ Z &:= \{u \in \mathbb{R}^m : W(u) = 0\} \end{aligned}$$

Heteroclinic orbit: Assume $A^{\pm}$ are two closed and disjoint subsets of $\partial\Omega_0$, we say that a bounded solution $u \in C^2(\mathbb{R}, \Omega)$to the system (1) such that $d(u(x), A^{\pm}) \to 0$ when $x \to \pm\infty$ is an heteroclinic orbit connecting $A^{\pm}$.

Homoclinic orbit: Assume $A^{\pm}$ are two closed and disjoint subsets of $\partial\Omega$ such that $A^{-} \subseteq \partial\Omega_0$ and $A^{+} \subseteq \partial\Omega_{\neq}$, we say that a bounded solution $u \in C^2(\mathbb{R}, \overline{\Omega})$to the system (1) such that:
(1) $d(u(x), A^{\pm}) \to 0$ when $x \to \pm\infty$
(2) $u(0) \in \partial\Omega_{\neq}$
(3) $d(u(x), A^{-}) \to 0$ as x $\to \pm\infty$
is an homoclinic orbit.

Periodic Orbit Let $a^{\pm}$ points in $\partial\Omega_{\neq}$, a solution $u \in C^2(\mathbb{R}, \overline{\Omega})$ of the system (1) such that:
(1) $u(0) = a^{-}, u(T/2) = a^{+}$
(2) $\forall x \in \mathbb{R} : u(x + T) = u(x), u(x + T/2) = u(-x + T/2)$, for some T > 0.
(3) $u(x) \in \Omega \iff x \neq k\frac{T}{2}, k \in \mathbb{Z}$

By theory of O.D.E. we know that the Hamiltonian of the system $u''(x) - W_u(u(x))$ $H = \frac{1}{2}|u'(x)|^2 - W(u(x))$ is constant along solutions. For homoclinic and periodic orbits I can easily be proved that: $H = 0$ since $u'(0) = 0$, $W(u(0)) = 0$ As for heteroclinic orbits, the equipartition relation ($H = 0$) is proved as follows:

Equipartition of energy: Let $u \in C^2(\mathbb{R}, \overline{\Omega})$ be a bounded solution to the system (1) such that: $d(u(x), \partial\Omega_0) \to 0$ as $x \to -\infty$. Then $H = 0$.

Proof of equipartition: We immediately have that $H \geq 0$ since $W(u(x)) \to 0$ as $x \to -\infty$. Since, $W(u(x)) \to 0$ as $x \to -\infty$, we deduce that $H \geq 0$.

Now, if $H > 0$, we can see that for the function $f(x) := |u(x)|^2$ the following statement is true:

$$f''(x) = 2|u'(x)|^2 + 2u(x)\nabla W(u(x)) = 4H + 4W(u(x)) + 2u(x)\nabla W(u(x))$$

We have by assumption $\nabla W(u(x)) \to 0$ when $x \to -\infty$ we deduce that $f''(x) \geq 2H$ for $x < a$, with a negative and $|a| \gg 1$. What we get now is that $f'(x) \to -\infty$ and $f(x) \to +\infty$ as $x \to -\infty$ which is a contradiction.

The equipartition of energy is proved.
Note that $u(x) \in \Omega \cup \partial\Omega_{\neq}$ if the solution is not constant, since that if there was an x_0 such that $W(u(x_0)) = 0$ and $\nabla W(u(x_0)) = 0$ then $u'(x_0) = 0$ and by the uniqueness of O.D.E. follows that $u(x) = u(x_0)$.

Note also, that the previous result does not remain true if we assume only that:

$$\lim_{x \to -\infty} d(u(x), \partial\Omega) \to 0$$

cf. remark 3 from paragraph 2 in [3] .

It is also easy to see that of $W \geq 0$, then every solution u to the system (1) such that $J_{\mathbb{R}}(u) < \infty$ satisfies $H = 0$:

$$J_{\mathbb{R}}(u) = \int_{\mathbb{R}}(2W(u(x)) + H)dx = \int_{\mathbb{R}}(|u'(x)|^2 - H)dx < \infty \iff H = 0$$

We now establish some propositions as sufficient conditions for the existence of bounded local minimizers.

Proposition 3.1: If there exists a local minimizer $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ for system (1), then the potential W has a global minimum which is supposed to be 0 without loss of generality. In addition, $J_{\mathbb{R}}(u) < \infty$, $\lim_{|x|\to\infty} W(u(x)) = 0$ and $\lim_{|x|\to\infty} d(u(x), Z) = 0$, with Z being the set of zeros of W.

Proof of proposition 3.1: First, we notice that there is a sequence x_n such that $x_n \to +\infty$ as $n \to +\infty$ such that $u(x_n) \to b \in \mathbb{R}^m$, otherwise, u would be unbounded. Suppose by contradiction that W does not have a global minimum, then there exists an $a \in R$ and an $\epsilon > 0$ such that $W(a) + \epsilon \leq \min\{W(v) : |v| \leq ||u||_{L^\infty(\mathbb{R},\mathbb{R}^m)}\}$. Now for n large enough we define the sequence of functions:

$$\upsilon_n(x) = ((1 + x_0 - x)u(x_0) + (x - x_0))a\chi_{[x_0,x_0+1]} + a\chi_{[x_0+1,x_n-1]} + ((x_n - x)a$$
$$+(x - x_n + 1)u(x_n))\chi_{[x_n-1,x_n]}$$

Since u is bounded by the hypotheses, and the intervals when u in engaged in the integral of action have finite measure, we deduce that there is an $M > 0$ independent of n such that:

$$J_{[x_0,x_n]}(\upsilon_n) \leq W(a)(x_n - x_0 - 2) + M$$

also:

$$J_{[x_0,x_n](u)} \geq (W(a) + \epsilon)(x_n - x_0)$$

Now, since $u(x)$ and $\upsilon_n(x)$ coincide at x_0 and x_n we can extend υ_n so that $\upsilon_n = u(x)$ $\forall x \in \mathbb{R} \setminus ((-\infty, x_0] \cup [x_n, +\infty))$ and by the minimality of u we can deduce the following inequality for the integral of Action in $[x_0, x_n]$:

$$(W(a) + \epsilon)(x_n - x_0) \leq W(a)(x_n - x_0 - 2) + M$$

and a straightforward result is that: $\epsilon(x_n - x_0) \leq -2W(a) + M$ and we have reached a contradiction because this is impossible for n large enough.
So, the W has indeed a global minimum. We assume, without loss of generality that $\min_{\mathbb{R}^m} W(x) = 0$

To prove the remaining statements of the proposition, we consider the sequence of functions defined above with $a \in \mathbb{R}$ such that $W(a) = \min_{\mathbb{R}^m} W(x) = 0$
The sequence $J_{[x_0,x_n]}(u(x))$ is uniformly bounded because $\forall n \in \mathbb{N}$ is bounded by $J_{[x_0,x_n]}(\upsilon_n) < M$, so $J_{\mathbb{R}}(u(x)) < \infty$. By similar arguments we deduce the same result in $(-\infty, x_0]$. It is straightforward now that $J_{\mathbb{R}}(u(x)) < \infty$. Moreover, $u(x)$ is $\frac{1}{2} - Holder$ continuous,(and so uniformly continuous):

$$\forall y \geq x \text{ we have: } |u(x) - u(y)| = \int_x^y |\dot{u}| dx \leq \sqrt{\int_x^y |\dot{u}|^2 dx}\sqrt{\int_x^y dx} \leq$$
$$\sqrt{\int_x^y (|\dot{u}|^2 + 2W(u(x)))dx}\sqrt{|y - x|} \leq \sqrt{2J_{\mathbb{R}}(u)}\sqrt{|y - x|}$$

Lastly, if $W(u(x))$ or $d(u(x), Z)$ does not converge to 0 as $|x| \to \infty$ then a sequence $x_n \to +\infty$ or $-\infty$ and a $b \in \mathbb{R}$ such that $u(x) \to b$ and $W(b) > 0$. Due to the uniform continuity of u we have that there exists an $N > 0$ and a $\delta > 0$ such that

$$\forall n > N \text{ and } \forall x \in [x_n - \delta, x_{n+\delta}]\text{: } W(u(x)) \geq W(b)/2$$

.

$$\text{Therefore: } \forall n > N \ J_{[x_n-\delta, x_n+\delta]}(u) \geq \delta W(u(x)).$$

and now, for n large enough, and passing to a subsequence if necessary we can see that the intervals $[x_n - \delta, x_n + \delta]$ are disjoint, and that contradicts $J_{\mathbb{R}}(u) < \infty$ and the proof is complete.

Now, assuming that 0 is the global minimum of W. We define the equivalence relation $u \sim \upsilon$ on $Z = u \in \mathbb{R}^m : W(u) = 0$, if and only if there exists a curve $\gamma \in W^{1,2}([a,b]), \mathbb{R}^m)$ such that $\gamma([a,b]) \in Z$, and $\gamma(a) = u$ and $\gamma(b) = \upsilon$.

Proposition 3.2: Let W be a potential such that $\min_{\mathbb{R}^m} W = 0$. If $u \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ is a local minimizer for system (1) and if there exists two sequences $x_n \to -\infty$ and $y_n \to +\infty$ such that $u(x_n) \to a^-$ and $u(y_n) \to a^+$, with $a^\pm \in Z$ and being connected by a path that belongs to Z, then u is constant.

Proof:

Assume that $\gamma \in W^{1,2}([0,l], \mathbb{R}^m)$ be a path connecting $a^- = \gamma(0)$ and $a^+ = \gamma(l)$ in Z.
For n large enough we define the sequence of functions:

$$\upsilon_n(x) = ((1 + x_n - x)u(x_n) + (x - x_n)a^-)\chi_{[x_n, x_n+1]} + \gamma\big(\tfrac{l(x-x_n-1)}{y_n-x_n-2}\big)\chi_{[x_n+1, y_n-1]} + \\ ((y_n - x)a^+ + (x - y_n + 1)u(y_n))\chi_{[y_n-1, x_n]}$$

After a few computations we get that:

$$J_{[x_n, y_n]}(\upsilon_n) = o(1) + \tfrac{l^2}{2(y_n-x_n-2)^2} \int_{x_n+1}^{y_n-1} |\gamma'(\tfrac{l(x-x_n-1)}{y_n-x_n-2})|^2 dx = \\ o(1) + \tfrac{l}{2(y_n-x_n-2)} \int_{x_n+1}^{y_n-1} |\gamma'(y)|^2 dy = o(1)$$

By constraction of the sequences we have: $u(x_n) = \upsilon_n(x_n)$ and $u(y_n) = \upsilon_n(y_n)$. As in proposition 3.1 we deduce by the minimality of u that $J_{[x_n, y_n]}(u) = o(1)$. Letting n go to infity we deduce that $J_{\mathbb{R}}(u) = 0$, thus, u is constant.

A notable case, is the Ginzburg-Landau potential $W(u) = \frac{1}{4}(|u|^2 - 1)^2$, where for $m > 2$ the zero level set is the unit sphere which is path-connected. Thus, by the previous proposition, the only case where we have a non-trivial local minimizer is the case where $n = 1$, otherwise, the minimizers are constant. To be more precise, the solution $u : \mathbb{R} \to \mathbb{R}^m$, $u(x) = (tanh(x/\sqrt{2}, 0, ..., 0)$ connects $(\pm 1, 0, ..., 0)$ and is minimal only when $m = 1$.

Corollary 3.1: If W is a potential such that $\min_{R^m} W = 0$ then every nontrivial, bounded, local minimizer is an heteroclinic connection.

Proof:

Let u be a nontrivial, bounded, local minimizer. By the equipartition relation, we know that u takes its values in a connected component Ω of the set $\{W > 0\}$. Let $A^{\pm}$ be the set of limit points of u at $\pm\infty$, these are compact because u is bounded. By proposition 3.1 we have: $A^{\pm} \subseteq \partial\Omega$ and because u is non-constant we have that $A^{\pm}$ are disjoint. Lastly, $d(u(x), A^{\pm}) \to 0$ at $\pm\infty$, therefore, u is an heteroclinic connection.

Note that **the converse is not necessarily true**. We will constract now a non-negative potential $H \in C^{\infty}(\mathbb{R}^2, \mathbb{R})$ which vanishes only at the points $a^{\pm}$, and being such that the matrix $D^2H(a^{\pm})$ is positive definite.
Consider the Ginzburg-Landau potential $W(u) = \frac{1}{4}(|u|^2-1)^2$ with $u \in \mathbb{R}^2 \sim \mathbb{C}$. After a few calculation we conclude that the Action of the solution $u(x) = (tanh(x/\sqrt{2}), 0)$ in the interval $[-R, R]$ is:

$$J_{[-R,R]}(u) = \sqrt{2}\Big[|u(R)| - \tfrac{|(u(R))|^3}{3}\Big] \to \tfrac{2\sqrt{2}}{3} \text{ as } R \to +\infty$$

Now, defining the map: $\upsilon(x) = -u(R)e^{i\frac{\pi(x+R)}{2R}}$ for $x \in [-R, R]$, we get that: $\upsilon(\pm R) = u(\pm R)$ by construction and

$$J_{[-R,R]} = 2RW(W(u(R))) + |u(R)|^2\tfrac{\pi^2}{4R} \to 0 \text{ as } R \to +\infty$$

Thus, for R large enough we have that $J_{[-R,R]}(u) > J_{[-R,R]}(\upsilon)$. Now, we have to modify the potential W in order to make the zero level set not connected, since otherwise the result would be trivial by the previous proposition.
For a fixed R large enough and an $\epsilon > 0$ small enough we take a function $\phi(t) \in C^k(\mathbb{R}, [0, +\infty))$ where $k \in \mathbb{N}$ and $k \geq 2$ that satisfies:

$$\phi(t) = 0 \text{ if } t \leq |u(R)|^2 + \epsilon \text{ and } \phi(t) = 1 \text{ if } 1 - \epsilon \leq t.$$

with $\epsilon < \frac{1-|u(R)|^2}{2}$ and positive.

Now, we set the potential function: $H(u_1, u_2) = W(u) + u_2^2\phi(|u|^2)$ that satisfies $H(u_1, 0) = W(u_1, 0)$ and $\frac{\partial H}{\partial u_2}(u_1, 0) = 0$.
Finally, after a few calculations we conclude that:

$$u'' = \nabla H(u)$$

The proof is complete.

The Main Theorem:

We will present now a central theorem, results about Heteroclinic, Homoclinic and Periodic Orbits are obtained, depending on what hypothesis hold on $A^{\pm}$.

Firstly, assume that:

H_1 : The potential $W \in C^2(R^m, R)$ is such that $\partial\Omega$ is partitioned into two disjoint compact sets $A^{\pm}$. Additionally $R^m \setminus \Omega$ is partitioned into two disjoint closed sets $F^{\pm}$ with $\partial F^{\pm} = A^{\pm}$.

H_2 : $\liminf_{u\in\Omega} W(u) > 0$, if Ω is not bounded.

*The C^2 smoothness of W is only needed in the proof of Homoclinic and Heteroclinic Orbits connecting portions of $A^{\pm}$ asymptotically, while in the proof of this theorem we only need W to be continuous and in some cases of other results we need it to be C^1.

Theorem 3.1: Assume $W : \mathbb{R}^m \to \mathbb{R}$ satisfies H_1 and H_2. Then $J_{\mathbb{R}}(\overline{u})$ admits a minimizers $\overline{u} \in A$ such that:

$$J_{\mathbb{R}}(u) = \min_{u\in A} J_{\mathbb{R}}(u) < \infty$$

$$\lim_{x\to\pm\infty} d(\overline{u}(x), A^{\pm}) = 0$$

where **d** denotes the Euclidean distance and:

$A = \{u \in W^{1,2}_{loc}(\mathbb{R}, \overline{\Omega}) : d(u(x), A^{-}) \leq \overline{q}\ \forall x \leq x^{-}_{u} \text{ and } d(u(x), A^{+}) \leq \overline{u}\ \forall x \geq x^{+}_{u}\}$.

Proof: We first notice that the affine function:

$$u_0(x) = a^{-}\chi_{(\infty,0]} + (a^{-} + x(a^{+} - a^{-}))\chi_{[0,1]} + a^{+}\chi_{[1,+\infty)}$$

has finite Action: $J_{\mathbb{R}}(u_0) < +\infty$, thus $J_{\mathbb{R}}(\overline{u}(x)) \leq J_{\mathbb{R}}(u_0) < \infty$ since $u_0(x)$ is a minimizer and:

$$\inf_{A_b} J_{\mathbb{R}}(u) = \inf_{A} J_{\mathbb{R}}(u) < +\infty$$

where:

$$A_b = A \cap \{J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u)\}$$

For $A^* = A^+ or A^-$ and q, q' such that: $0 < q' < q/2 < q < \overline{q}$, we define $U_q^{q'}$ the set of maps $\in W^{1,2}$ that satisfy:

$$d(u(a), A^*) \geq q \text{ and } d(u(b), A^*) \leq q'.$$

We also define: $\upsilon_u : [\beta - 1, \beta] \to \overline{\Omega}$ setting: $\upsilon_u(x) = a^* + (x - \beta + 1)(u(\beta) - a^*)$

with $a^* \in A^*$ such that $d(u(\beta), A^*) = d(u(\beta), a^*)$

Lemma 3.1: For each $q \in (0, \overline{u}]$ there exists $q' \in (0, q/2)$ such that:

$$J_{[a,b]}(u) \geq J_{[\beta-1,\beta]}(\upsilon_u)$$

for $u \in U_q^{q'}$ for $A^* = A^\pm$.

Proof: We define:

$$f(q) = \min\{W(u) : u \in \Omega, q \leq d(u, A^*) \geq \overline{q}\}, q \in (0, \overline{q}]$$

and

$$F(q) = \max\{W(u) : u \in \Omega, d(u, A^*) \geq \overline{q}\}, q \in (0, \overline{q}]$$

and obtain:

$$J_{[a,b]}(u) \geq \textstyle\int_a^\beta \sqrt{2W(u)}|\dot{u}|dx \geq \sqrt{2\phi(q/2)}(q/2)$$

$$J_{\beta-1,\beta}(\upsilon_u) \leq F(q') + \tfrac{1}{2}|\upsilon(\beta) - a^*|^2 \leq F(q') + \tfrac{1}{2}(q')^2$$

Thus, for $q \in (0, \overline{q}]$ and $q' \in (0, q/2)$ small enough we obtain the inequality:

$$F(q') + \tfrac{1}{2}(q')^2 < \sqrt{2\phi(q/2)}(q/2)$$

We will utilize a minimizing sequence on the functional of Action and this lemma, together with some other results that will be proved straightaway, give us information about how the sequence will be chosen. This will give us the information we need to prove the existence of a “suitable” minimizer.

These can be understood by noticing:

1) For $(x_1, x_2) \subset (\infty, x_u^-)$ and $x_2 - x_1 \geq 2\frac{J_{\mathbb{R}}(u_0)}{f(q')}$ the function $d(u(x), A^-)$ cannot be greater or equal than q' for all $x \in (x_1, x_2)$, because if it was we would reach a contradiction:

$$\phi(q')(x_2 - x_1) \leq J_{(x_1,x_2)}(u) \leq J_{\mathbb{R}}(u_0)$$

A similar result holds also for A^+ in $(x_1, x_2) \subset (x_u^+, +\infty)$ with $x_2 - x_1 \geq 2\frac{J_{\mathbb{R}}(u_0)}{f(q')}$

2) For every $u \in A_b$ there exists a function $\bar{u}$ and an $M > 0$ such that

$$||\bar{u}|| \leq M$$
$$J_{\mathbb{R}}(\bar{u}) \leq J_{\mathbb{R}}(u)$$

This is proved by noticing by the previous result that there exists an $x_0 \in (-\infty, x_u^-)$ such that $d(u(x_0), A^-) < q'$. Moreover, due to the finite Action of u we know that for every $\epsilon > 0$ there exist an x_ϵ such that $d(u(x_\epsilon), A^-) < \epsilon$. Let $\overline{q}'$ be the number given by lemma 3.1.(we can, indeed choose it as small as we want or decrease it). Thus, there exists an:

$$\overline{x}'_u = \max\{x : d(u(x_0), A^-) \leq \overline{q}'\}$$

Assume that there exists an $a \in (-\infty, \overline{x}'_u)$ such that:

$$d(u(a), A^-) = \overline{q}$$

and defining the function:

$$\bar{u}(x) = a^- \chi_{(-\infty, \overline{x}_u^- - 1]} + \upsilon_u(x)\chi_{[\overline{x}_u^- - 1, \overline{x}_u^-]} + u(x)\chi_{[\overline{x}_u^-, +\infty)}$$

where $\upsilon_u : [\overline{x}_u^- - 1, \overline{x}_u^-] \to \overline{\Omega}$ is the function defined just before Lemma 3.1.

It follows that:

$$|\bar{u}(x) - a^-| \leq \overline{q}' < \overline{q} \text{ for every } x \in (-\infty, \overline{x}_u^-]$$

$$J_{(\infty, \overline{x}_u^-)}(\bar{u}) = J_{(\overline{x}_u^- - 1, \overline{x}_u^-)}(\upsilon_u) \leq J_{[a, \overline{x}_u^-]}(u) \leq J_{(-\infty, \overline{x}_u^-]}(u)$$

Similarly, we can prove that there exists an $\overline{x}_u^+$ such that:

$$\overline{x}_u^+ = \min\{x : d(u(x), A^+) \leq q'\}.$$

By this way, one can also conclude that $\overline{u}$ can be constructed such that:

$$d(\overline{u}(x), A^+) \leq \overline{q} \text{ for every } x \in [\overline{x}, +\infty)$$

Now, from $w = \min\{W(u) : u \in \Omega, d(u, \partial\Omega) \geq \overline{q}'\}$ and by the definition of $\overline{x}_u^{\pm}$:

$$w(\overline{x}_u^+ - \overline{x}_u^-) \leq J_{(\overline{x}_u^-, \overline{x}_u^+)}(u) \leq J_{\mathbb{R}}(u_0)$$

and now for $x \in [\overline{x}_u^-, \overline{x}_u^+]$ we obtain:

$$|\bar{u}(x) - \bar{u}(\overline{x}_u^-)| = |u(x) - u(\overline{x}_u^-)| = (\sqrt{x - \overline{x}_u^-}) \int_{\overline{x}_u^-}^{x} |\dot{u}|^2 dx \leq \sqrt{(\tfrac{2}{w})} J_{\mathbb{R}}(u_0) < +\infty$$

and the boundness of $\overline{u}_(x)$ has been proved.

Note that, if $d(u(x), A^-) < \overline{q}\ \forall x \in (-\infty, \overline{x}_-)$ then we can get analogous results, for example a uniform bound of $(x_u^+ - x_u^-)$ with $w = \min\{W(u) : u \in \Omega, d(u, \partial\Omega) \geq \overline{q}\}$. Then, for $\overline{u} = u$ the rest of the results are proven the same way and the respective inequality is trivial.

Now, for each $u \in A_b$ we have:

$$\lim_{x \to \pm\infty} d(u(x), A^{\pm}) = 0$$

This can easily be proved by noticing that if there was a sequence $x_n \to +\infty$ and a $q_0 \in (0, \overline{q})$ such that $d(u(x_n), A^+) \geq q_0$ for every n. Now, u is uniformly continuous since it belongs to S_b, so, there exists a $\delta > 0$ such that:

$$d(u(x), A^+) \geq \tfrac{q_0}{2}$$

for every $x \in (x_n - \delta, x_n + \delta)$ for every $n > 0$ large enough with $\delta > 0$ independent of n.
Therefore, we obtain:

$$J_{(x_n - \delta, x_n + \delta)}(u) \geq 2\delta f(\tfrac{q_0}{2})$$

for every n large enough. Considering a subsequence of necessary we can assume that the intervals $(x_n - \delta, x_{n+\delta})\ \forall n\ in \mathbb{N}$ are disjoint are contradict and that contradicts $J_{\mathbb{R}}(u) < +\infty$.

Now, we note that:

$$|u(x_2) - u(x_1)| = (\sqrt{x_2 - x_1}) \int_{x_1}^{x_2} |\dot{u}|^2 dx \leq \sqrt{(\tfrac{2}{w})} J_{\mathbb{R}}(u_0)$$

holds for every $u \in A_b$, so A_b is an equicontinuous set of functions.
Thus, if $u_n \subset A_b$ is a minimizing sequence then we can also assume that $u_k \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$ is uniformly bounded, i.e. equibounded.

By the translation invariance of $J_{\mathbb{R}}(u)$ we can assume that:

$$\overline{x}^-_{u_n} = 0, \, \overline{x}^+_{u_n} \leq \frac{J_{\mathbb{R}}(u_0)}{w} \text{ for every } n \in \mathbb{N}$$

Considering a subsequence if necessary, and still call it u_n, we have that:

1) $u_n \rightarrowtail \overline{u}$ uniformly on compact intervals, with $\overline{u}$ being continuous. This follows from the embedding:

i)

$$||u||_{L^2([-K,K],\mathbb{R}^m)} \leq \sqrt{2K}||u||_{C([-K,K])}$$

in $L^2_{loc}(\mathbb{R}, \mathbb{R}^m)$ for every $k \in \mathbb{N}$, this follows from the Ascoli-Arzela theorem and a diagonal argument on it.

ii) $\dot{u_k}$ converges weakly in $L^2(\mathbb{R}, \mathbb{R}^m)$ to some $\upsilon \in L^2(\mathbb{R}, \mathbb{R}^m)$, since the uniform bound:

$$\int_{\mathbb{R}} |\dot{u_k}|^2 dx \leq 2J_{\mathbb{R}}(u_0).$$

By the fact that the derivative operator is weakly closed, we deduce that $\dot{\overline{u}} = \upsilon$, thus, $\overline{u} \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m)$.

By the lower semicontinuity of L^2 norm we get:

$$\liminf \int_{\mathbb{R}} |\dot{u_k}|^2 dx \geq \int_{\mathbb{R}} |\dot{u}|^2 dx$$

also, u_k converges pointwise to $\overline{u}$, so we can apply the Fatou's lemma to the sequence of non-negative functions:

$$\liminf \int_{\mathbb{R}} W(u_k) dx \geq \int_{\mathbb{R}} W(\overline{u}) dx$$

Finally, by adding these inequalities we obtain:

$$J_{\mathbb{R}}(\overline{u}) \leq \liminf \int_{\mathbb{R}} \frac{|\dot{u_k}|}{2}^2 dx + \liminf \int_{\mathbb{R}} W(u_k) dx \leq \lim J_{\mathbb{R}}(u_k) = \inf_{A_b} J_{\mathbb{R}}(u) \leq J_{\mathbb{R}}(u_0)$$

Therefore, u is a minimizer belonging to A_b that satisfies the "boundary" conditions.

Heteroclinic, Homoclinic and Periodic orbits

Now, taking into account the previous theorem, we will consider some additional conditions on $\partial\Omega$, to find out some cases in which heteroclinic, homoclinic and periodic orbits exist.

First, we present a basic proposition that states(for W continuously differentiable) the Euler-Lagrange equation is satisfied for an interval where $W(u(x)) > 0$

Proposition 3.3. There exists L^- and L^+ with: $-\infty < L^- < 0 < L^+ < +\infty$ such that: $(L^-, L^+) = x \in \mathbb{R} : \overline{u}(x) \in \Omega$ and if L^-(respectively L^+) $\in \mathbb{R}$ then $\forall\, x \leq L^-$ we have $u(x) = u(L^-) \in A^-$, with a respective result for $x \geq L^+ \in A^+$. Moreover, on (L^-, L^+) the minimizer $\overline{u}$(from the previous theorem) satisties the Euler-Lagrange equation:

$$u''(x) = \nabla W(u(x))$$

and the equipartition relation:

$$\frac{|u'|^2}{2} = W(u(x)).$$

By the translation invariance of $J(u(x))$, we can assume without loss of generality that $\overline{u}(0) \in \Omega$. Also, we define:

$$L^- = \inf\{x < 0 : \overline{u}((x, 0]) \subset \Omega\}$$

$$L^+ = \sup\{x > 0 : \overline{u}([0, x)) \subset \Omega\}$$

Taking into account proposition 3.2, since u is a minimizer we deduce that $\overline{u}$ is constant on the intervals $(-\infty, L^-], [L^+, +\infty)$, so it solves the Euler-Lagrange equation on them, and by its minimality we have that $\overline{u}$ solves the Euler-Lagrange also in (L^-, L^+).
As for the Hamiltonian:

$$H := \tfrac{1}{2}|\overline{u}'|^2 - W(\overline{u}(x))$$

we have that if $L^+ = +\infty$ *and/or* $L^- = -\infty$ then $H = 0$. This can be seen by noticing:

$$+\infty > J_{(L^-, L^+)}(\overline{u}) = \int_{L^-}^{L^+} (\tfrac{1}{2}|\overline{u}'|^2 + W(\overline{u}(x)))dx = \int_{L^-}^{L^+} (H + 2W(\overline{u}(x)))dx,$$

if $(L^+ - L^-) = +\infty$, the boundness of $J(\overline{u})$ above holds if $H = 0$.

Now, for $L^+ < +\infty$ and $L^- > -\infty$ we define the functions:

$$\upsilon_k = \overline{u}(x)_{\chi(-\infty,0]} + \overline{u}(x/k)_{\chi[0,+\infty)}$$

with $k > 1$. We calculate:

$$J_{(0,+\infty)}(\upsilon) = \int_0^{kL^+}(\frac{1}{2k^2}|\dot{\overline{u}}(x/k)|^2 + W(\overline{u}(x/k)))dx = \int_0^{L^+}(\frac{1}{2k}|\dot{\overline{u}}(t)|^2 + kW(\overline{u}(t)))dt = \int_0^{L^+}(\frac{H}{k} + (\frac{k^2+1}{2})W(\overline{u}(t)))dt = \frac{HL^+}{k} + \int_0^{L^+}(\frac{k^2+1}{2})W(\overline{u}(t)))dt.$$

We note that since $\overline{u}$ is a local minimizer and coincides with υ for $x \leq 0$ the folllowing inequality holds:

$$J_{(0,+\infty)}(\overline{u}) \leq J_{(0,+\infty)}(\upsilon) \Rightarrow HL^+ \leq (k-1)\int_0^{L^+} W(\overline{u}(x))dx$$

If we let $k \to 1$ we get that $H \leq 0$.

We also see, that $\lim_{x\to L^\pm} W(\overline{u}(x)) = 0$ implies $H \geq 0$. Therefore, $H = 0$.

From now on, the proofs of most Theorems, corollaries and propositions will not be explicitly written but descriptive. However, all of them can be found in [3](A-S).

Now, we assume that the one of the following conditions holds for $A^* = A^+$ or A^-:

H_e**:** $\nabla W(u) = 0$, $\forall u \in A^*$
H_o**:** $\nabla W(u) \neq 0$, $\forall u \in A^*$

Depending on which one of these conditions holds on A^- or A^+ we will distinguish the cases where an heteroclinic, homoclinic or periodic orbit exists.

Theorem 3.2(Heteroclinic orbit): If H_e holds on both A^+, A^-, and W is C^2 smooth, then the minimizer $\overline{u}$ constructed in Theorem 3.1 is an heteroclinic connection that solves the Euler-Lagrange on $\mathbb{R}$.
Moreover, $L^\pm = \pm\infty$ and if $W \geq 0$, then $\overline{u}$ is also a local minimizer.

Proof:

The fact that $L^\pm = \pm\infty$ is proved by noting that since the equipartition of energy holds, the O.D.E.: $u''(x) = \nabla W(u(x))$ equipped with the facts: $u'(L^+) = 0$, $\nabla W(\overline{u}(x)) = 0$ has a solution $u(x) = u(L^+)$ which is unique(the same holds if we replace L^+ with L^-).

Now, suppose that $W \geq 0$, and let $z \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m)$ a map such that: $z(x) = \overline{u}(x)$ $\forall x \in \mathbb{R} \setminus [a, b]$ for some $a < b$. If $z(x) \in \Omega$ $\forall x \in \mathbb{R}$ the we immediately have by the definition of minimizer that $J_{\mathbb{R}}(\overline{u}(x)) = J_{\mathbb{R}}(z(x))$. In any other case, suppose that there exists an $s \in \mathbb{R}$ such that $z(s) \notin \overline{\Omega}$. Assume without loss of generality that $z(s) \in F^-$ and set: $s^- = \max\{s \in \mathbb{R} : z(s) \in F^-\}$. Now, if a $t > s^-$ such that $z(t) \in F^+$ exists, we set: $s^+ = \min\{t > s^- : z(t) \in F^+\}$, and since $s^- < s^+$, $z(s^\pm) \in \partial F^\pm$, $z([s^-, s^+]) = \Omega$ we immediately have that: $J_{\mathbb{R}}(\overline{u}) \leq J_{[s^-,s^+]}(z) \leq J_{\mathbb{R}}(z)$, and the last inequality remains true even when $s^- = -\infty$ and/or $s^+ = +\infty$. Finally, since $z \in A$ was arbitrary, we get that $\overline{u}$ is a local minimizer.

The proof of the Theorem is complete.

Note that, for $Z = \{a_1, ..., a_k\}$, with k being a positive integer greater than 1, if we set $A^- = a_i$ for any $i \in \{1, ...k\}$, and $A^+ = Z \setminus \{a_i\}$, then from the previous theorem we can easily deduce that there exists an $a_j \in Z \setminus \{a_i\}$ and a solution to the Euler-Lagrange that connects these two.

Theorem 3.3(Homoclinic Orbit) If H_e and H_o hold on A^- and A^+ respectively, then there exists an even function υ such that: υ is a homoclinic to A^- connection solving the Euler-Lagrange on $\mathbb{R}$ with $\upsilon(x) \in \Omega$ $\forall$ $x \neq 0$ and $\upsilon(x) \in A^+$ iff $x = 0$.

Moreover, it minimizes the Action functional in the class:

$A_{H_o} = \{u \in W^{1,2}_{loc}(\mathbb{R}, \overline{\Omega}) : d(u(x), A^-) \leq \overline{q}$ for $|x| \gg 1$, and $u(0) \in A^-\}$

The proof of Theorem 3.3 states that if $L^+ = +\infty$, then since the equipartition relation holds, for $x > 0$ large enough we would have that there exists an $\epsilon > 0$ such that:

$$\tfrac{d^2 W(\overline{u})}{dx^2}(x) = |\nabla W(\overline{u})|^2 + D^2 W(\overline{u}(x))(\overline{u}', \overline{u}') \geq \epsilon > 0$$

.

Therefore, $W(\overline{u}(x)) \to +\infty$ as $x \to +\infty$ which is a contradiction, so L^+ must be finite. Now, defining the function:

$$\upsilon(x) = \overline{u}(x + L^+)\chi_{(-\infty,0]} + \overline{u}(-x + L^+)\chi_{[0,+\infty)}$$

It is easy to see that $\upsilon \in A_{H_o}$.

The derivative of υ exists and vanishes due to the equipartition relation and by symmetry we have that υ twice continuously differentiable and satisfies the Euler-Lagrange on all $\mathbb{R}$, minimizes the Action in the class A_{H_o} and also: $J_{\mathbb{R}}(\upsilon) = 2J_{\mathbb{R}}(\upsilon)$.

Theorem 3.4(Periodic Orbit): If H_o holds on $A^{\pm}$, then there exists an even T-periodic function υ that solves the Euler-Lagrange on $\mathbb{R}$ with:
$\upsilon(x+T/2) = \upsilon(-x+T/2), \upsilon(0) \in A^-, \upsilon(T/2) \in A^+$ and $\upsilon(x) \in \Omega$ iff $x \notin \frac{T}{2}\mathbb{Z}$

and can be variationally characterized as follows:

$$J_{[0,T/2]}(\upsilon) = \min\{J_{[0,l]}(u) : u \in W^{1,2}([0,l],\Omega), u(0) \in A^-, u(l) \in A^+, l > 0\}$$

Proof description: As in Theorem 3.3, it is first proven that: $|L^{\pm}| < +\infty$
Now, for $T := 2(T^+ - T^-)$ we define the function:

$$\upsilon(x) = \overline{u}(x + L^-)\chi_{[0,T/2]} + \overline{u}(-x + 2L^+ - L^-)\chi_{[T/2,T]}$$

where $\overline{u}$ is the minimizer given in the main Theorem. We extend it periodically and it is easy to check that $\overline{u}$ has the stated properties, as well as that is twice continuously differentiable, satisfies the Euler-Lagrange and last but not least, it can be variationally characterized as declared.

Finally, it has been proved in [3] (A-S) that if A^- is a twice continuously differentiable compact orientable surface with a unit normal vector $\mathbf{n}$ and if W satisfies H_e and $\partial^2 W(u)/\partial n^2$ on A^- then the connection approaches A^- **exponentially and the limit exists**. An observation that is utilized to prove these, is that if A^- has positive diameter and $u(x)$ approaches A^- like a spiral then the curve would have infinite length, thus $|u'(x)|$ will not be integrable. Similarly for A^+.

Chapter 4

Epilogue: Remarks and complementary material

Solutions to the problem: $u'' = \nabla W - cu'$, $u(\pm) = a^{\pm}$ with: $c > 0$, $W(a^+) = 0 > W(a^-)$ are known as *travelling waves*, while solutions to the respective problem with $c = 0 = W(a^+) = W(a^-)$ are known as *standing waves*. This is because they can be viewed as special solution of the form: $U(z - ct) = u(z, t)$ to the diffusion system with gradient structure:

$$u_t = u_{zz} - \nabla W(u),\ u = u(z, t) : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^n$$

By equation (2.4) we can see that there is a linear dependence between c and $W(a^-)$.

The existence of standing waves have been proven in various ways. A proof via a contrained minimization, as in the traveling wave presented in this thesis, is constructed by Alikakos N. and Fusco G. in [1](A-F) as we said in chapter 2. Also, there exists a very similar proof that states the existence of such solutions when the potential has exactly two global minima, this proof is presented in [6](A-F-S).

A second proof that is presented here, is a proof invented by Antonopoulos P. and Smyrnelis P. in [3] (A-S), which is simpler but also more general than the one in [1] or in [6], since it not only proves the existence of such solutions where the zero set of the potential has several disjoint components, but also when it is connected, and not just single points(when the components of the zero set are writen with capital letters we mean they are sets - and with small letters we mean they are single points).

Another proof of existence of standing waves that doesn't utilize the idea of the constrained minimization can be found in [6](A-F-S) and in [4](F-G-N). The proof in in [6], is actually a special case of a more general result presented in [4], and at the same time, it is more general than the one in [3] for the case when the zero set of W is finite. The one in [4] states that there exists an heteroclinic connection between two disjoint connected components of the zero set of W, where the critical points are allowed to be included in these components and the last of them can have positive diameter. Both of these proofs discuss, as in [3] the existence of a connection were W has several global minima along with the possibility that the solution u connects these minima at finite time. Moreover they allow W to be just continuous and decay to 0 at infinity provided that there exists a non-negative function $\gamma(s)$ and an $r_0 > 0$ such that:

$$\sqrt{W(u)} \geq \gamma(|u|) \text{ for } |u| \geq r_0 \text{ and } \int_{r_0}^{+\infty} \gamma(s)ds = +\infty$$

The fact that this condition is good enough to provide us the results we want is justified by utilizing the Jacobi functional:

$$A_{[a,b]}(u(x)) = \int_a^b \sqrt{2W(u(x))}|u'(x)|dx$$

We can easily check that for integrals with infinite length, if the Jacobi functional is finite on them and the previous hypotheses on W holds, then u is bounded.

Note that, in the case when the number of components of the zero set of W is greater than two, then a connection between any of these two components does not exist in general. For example, consider the case when $W(u) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and has just three global minima $a_1 < a_2 < a_3$, then it is easy to prove that an heteroclinic connection between a_1 and a_3 does not exist since it would certainly have to pass through a_2 which contradicts the uniqueness of O.D.E.

For the case when the zero set of W is a finite set, $Z = \{u \in \mathbb{R}^m : W = 0\} = \{a_1, ..., a_k\}$, a sufficient condition for the existence of an orbit that connects a_i and a_j. for $i \neq j \in \{1, 2, ...k\}$ is:

$$\sigma_{ij} < \sigma_{ih} + \sigma_{hj}, \forall a_h \in Z \setminus \{a_i, a_j\}, \text{ where } \sigma_{ij} = \inf_{u \in A_{ij}} J(u)$$

where $J(u)$ is the Action fuctional and

$$A_{ij} = \{u \in W^{1,2}_{loc}((l_u^-, l_u^+), \mathbb{R}^m) : -\infty \leq l_u^- < l_u^+ \leq +\infty, \lim_{x \to l_u^-} u(x) = a_i, \\ \lim_{x \to l_u^+} u(x) = a_j\}.$$

This condition and its proof can be found in [6](A-F-S).
As far as I know the question if this condition is also necessary is an open problem yet.

The proof of existence of a traveling wave, utilizes some lemmas from the proof of existence of a standing wave, introduced in [1] as well as [2], therefore these two proofs have some thing in common. However, they have some major differences.

In the standing wave case, the proof is much simpler, since there it is straightforward that the solution u will exit one cylinder and enter the second one in finite time, while in the traveling wave case this is more complicated because c has to be chosen in a way such that the total energy, i.e. Action functional, is equal to zero. Otherwise we are not sure that we can construct a uniform bound.

Another thing is, that due to the fact that $W(a^-) < 0$ if the hypotheses on a^- was not $H_4(ii)$ and just $H_4(i)$ as it is stated on both on the standing wave, then it is not guaranteed that the minimizer would converge to a^- as $x \to -\infty$ if $B(a^-, r_0) \subset C_0^-$, thus, we need hypotheses $H_4(ii)$.

Also, in the traveling wave case, the solution has to pass through some points of the zero set of W ($W = 0$), so, to avoid the possibility that the minimizer converges to an equilibrium solution, we assume Hypothesis H_3 in order for these points not to be critical points (in this case, however, equation 2.3 would be satified). By the convexity hypothesis in H_3, we can make manipulations and overcome these difficulties.

Finally, note that since the wave profiles $u(x)$ in the balanced case of W are not unique in general, since if W is symmetric to the line that connects the two minima, then at least two solutions of the standing wave problem exist, each one of them lies completely in one of the respective half-planes. Therefore, we do not expect a unique solution for the traveling wave problem.

$\sim$ The End $\sim$

References

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