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Αριθμητικές Μέθοδοι Πινάκων για τον Υπολογισμό του Μέγιστου Κοινού Διαιρέτη Πολυωνύμων και Εφαρμογές

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Πρόλογος

Θα ήθελα να ευχαριστήσω θερμά την επιβλέπουσσα της διπλωματικής μου κυρία Μαριλένα Μητρούλη για την εμπιστοσύνη , την καθοδήγηση και την συνεχή υποστήριξη της με υλικό για την ολοκλήρωση της παρούσας εργασίας.

Η παρούσα διπλωματική έχει θέμα τη μελέτη αριθμητικών μεθόδων μέσω πινάκων για τον υπολογισμό του Μέγιστου Κοινού Διαιρέτη πολυωνύμων (ΜΚΔ) πολυωνύμων.Έμφαση δίνεται στην QR-Column Pivoting method(QRCP) μέσω της εφαρμογής της στους πίνακες Bézout με χρήση των θεωρημάτων του Barnett για τον (ΜΚΔ) πολυωνύμων.Η παραπάνω μέθοδος μας δίνει μεγάλο πλεονέκτημα καθώς για πίνακες με μεγάλο rank deficiency , συγκριτικά με άλλους μεθόδους όπως η QR-JBJ για Bézout πίνακες που μελετάμε επίσης στην εργασία , ενώ ταυτόχρονα μας μειώνει το χρόνο εκτέλεσης των υπολογισμών δηλαδή την πολυπλοκότητα.

Πιο συγκεκριμμένα , στην πρώτη ενότητα περιγράφουμε τα μαθηματικά εργαλεία όπως οι πίνακες Householder και οι εφαρμογές τους για καλύτερη εμβάνθυση του αναγνώστη στο περιεχόμενο της εργασίας.

Στη δεύτερη ενότητα κάνουμε μια αναλυτική περιγραφή των πινάκων Bézout παρουσιάζοντας τον τρόπο κατασκευής τους , τις ιδιότητες τους καθώς και κάποια αριθμητικά παραδείγματα.

Στην τρίρη ενότητα , παρουσιάζουμε στην αρχή τα θεωρήματα του Barnett για τον (ΜΚΔ) πολυωνύμων,και στη συνέχεια συνδέουμε αυτά τα θεωρήματα με την QRCP method για να δημιουργήσουμε ένα αλγόριθμο υπολογισμού των συντελεστών του (ΜΚΔ) μέσω των πινάκων Bézout όπως τονίσαμε προηγουμένως η μέθοδος αυτή είναι ιδιαίτερα χρήσιμη για πίνακες με μεγάλη rank deficiency . Συνεχίζοντας συγκρίνουμε την QRCP με την μέθοδο QR για Bézout πίνακες. Στο τέλος μέσω αριθμητικών παραδειγμάτων συγκρίνουμε την πολυπλοκότητα των μεθόδων και καταλήγουμε σε συμπεράσματα για τα πλεονέκτηματα και τα μειονέκτηματα των μεθόδων καθώς τη χρήση τους ανάλογα με τα δεδομένα του προβλήματος που αντιμετωπίζουμε.

Στο σημείο αυτό θα ήθελα να ευχαριστήσω θερμά συναργάτες διδακτορικούς φοιτητές (εν εξέλιξη και μη) της κ. Μητρούλη για την πολύτιμη βοήθεια που μου προσφεραν.Αυτοί είναι ονομαστικά και αλφαβητικά κυρία Ρούπα Παρασκευή ,κύριος Τριανταφύλλου Δημήτρης.Ευχαριστώ για την πολύτιμη βοήθεια επίσης των διδακτορικό φοιτητή του κυρίου Βασίλειου Δουγαλή Γρηγόριο Κουνάδη.

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Περίληψη

Ο Μέγιστος Κοινός Διαιρέτης (ΜΚΔ) ενός συνόλου πολυωνύμων έχει αποδειχθεί ότι είναι πολύ σημαντικός για πληθώρα εφαρμογών στα Εφαρμοσμένα Μαθηματικά και την Μηχανική. Αρκετές μέθοδοι έχουν προταθεί για τον υπολογισμό του (ΜΚΔ) πολυωνύμων. Οι περισσότερες από αυτές βασίζονται στον Ευκλείδειο Αλγόριθμο και είναι έτσι σχεδιασμένες έτσι ώστε να επεξεργάζονται 2 πολυώνυμα την φορά και μπορούν να εφαρμοστούν κατά επανάληψη αντί για δύο έχουμε περισσότερα πολυώνυμα. Υπάρχουν πολλές επαρκείς μέθοδοι βασισμένες σε πίνακες οι οποίες μπορούν να υπολογίσουν την τάξη και τους συντελεστές του ΜΚΔ με το να εφαρμόζουν συγκεκριμμένους μετασχηματισμούς σε ένα πίνακα ο οποίος έχει κατασκευαστεί απ' εύθειας από τους συντελεστές των πολυωνύμων που έχουμε.Τα θεωρήματα του Barnett για τον (ΜΚΔ) με χρήση πινάκων Bézout συμπεριλαμβάνει έναν πολύ συμπαγή τρόπο παραμετροποίησης και απεικόνισης του (ΜΚΔ) πολυωνύμων.Η παρούσα εργασία ασχολείται με την εφαρμογή της QR παραγοντοποίησης με οδήγηση κατά στήλες (QRCP) ενός πίνακακαι την επίτευξη σε ένα βαθμό του ΜΚΔ μέσω της τάξης ενός πίνακα ειδικότερα όταν the rank deficiency of the Bézout πίνακα είναι υψηλή.Αρχικά κατασκευάζουμε τον Bézout πίνακα ενός συνόλου πολυωνύμων ,εφαρμόζουμε τα θεωρήματα του Barnett για τον (ΜΚΔ) και στο τέλος εφαρμόζουμε την (QRCP) μέθοδο για να βρούμε τους συντελεστές του (ΜΚΔ).Η μέθοδος αυτή μας δίνει τα μέσα για μια πιο αποτελεσματική εφαρμοφή της κλασσικής QR με λιγότερη πολυπλόκοτητα. Ασχολούμαστε επίσης με τις κλασσικές απεικονίσεις του ΜΚΔ μέσω δομημένων πινάκων όπως η μέθοδος QR Bézout και με την πολυπλοκότητα τους την οποία αναλύουμε θεωρητικά, δίνοντας παραδείγματα.Συγκρίνουμε τις μεθόδους και την πολυπλοκοτητα τους .Τέλος προτείνουμε την χρήση της QR που αποκαλύπτει την τάξη με οδήγηση κατά στήλες για τον υπολογισμό του ΜΚΔ πολυωνύμων.

Abstract

The Greatest Common Divisor (GCD) of a polynomial set is proven to be very important to many applications in applied mathematics and engineering. Several methods have been proposed for the computation of the GCD of sets of polynomials. Most of them are based on the Euclidean algorithm. They are designed to process two polynomials at a time and can be applied iteratively when a set of more than two polynomials is considered . Conversely, there exist efficient matrix-based methods which can compute the degree and the coefficients of the GCD by applying specific transformations to a matrix formed directly from the coefficients of the polynomials of the entire given set. Barnett's theorems about (GCD) through Bezoutians involve Bézout-like matrices and suggest a very compact way of parametrising and representing the GCD of several univariate polynomials. The present work introduces the application of the QR decomposition with column pivoting (QRCP) to a Bézout matrix, achieving the computation of the degree and the coefficients of the GCD through the range of the Bézout matrix ,especially when the rank deficiency of the Bézout matrix is high.In the beginning we construct the Bézoutmatrix of a set of polynomials ,we apply Barnett's theorems and in the end we apply the QRCP method to find the coeffiecients of the GCD. This method provides the means for a more efficient implementation of the classical Bézout-QR method with less computational complexity and without compromising accuracy, and it enriches the existing framework for the computation of the GCD of several polynomials using structured matrices. The classical GCD representations through structured matrices are revisited and their computational complexity is theoretically analyzed and compared. Demonstrative examples explaining the application of each method are given.We compare the methods and their complexity.We propose the use of the *rank revealing QR with column pivoting* for the computation of the GCD of polynomials through Bézout-like matrices which improves the numerical behavior of the existing Bézout-QR algorithms.

Contents

1 Mathematical Tools Introduction to QR Decomposition

1.1 The-QR-Gram-Schmidt (Simple QR)

Let A a matrix $\in \mathbb{C}^{n\times m}(n \geqslant m)$ which its columns $a_1, a_2, ..., a_m$ is linear independent,as follows $rank(A) = m$. Our purpose is to construct an orthonormal system of vectors $q_1, q_2, ...q_m$. such that:

$span\{a_1, a_2, ..., a_m\} = span\{q_1, q_2, ...q_m\}$.

This method is known as Gram-Schmidt orthonormalization,[3] and it is based on:

if $x, y \in \mathbb{C}^n$ two linear independent vectors we symbolize the projection of x in y as $u = \frac{y^*x}{y^*y}y$,then the vector $x - u$ is vertical in y,because :

$$y^*(x-u) = y^*x - \frac{y^*x}{y^*y}y^*y = 0.$$

At the beginning,we normalize the vector a_1 such that:

$$q_1 = \frac{a_1}{||a_1||_2}.$$

Secondly,we construct an orthogonal vector in q_1 $w_2 = a_2 - (q_1^*a_2)q_1$ and we normalize it as $q_2 = \frac{w_2}{||w_2||_2}$. In the third step we construct an orthogonal vector in $span q_1, q_2$ $w_= a_3 - (q_1^*a_3)q_1 - (q_2^*a_3)q_2$ and we normalize as $q_3 = \frac{w_3}{||w_3||_2}$.

We continue the process and we have:

$$q_1 = \frac{a_1}{||a_1||_2}.$$

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}}$$

and

$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}}$$

we continue and we conclude that:

$$q_m = \frac{a_m - \sum_{i=1}^{m-1} r_{im}q_i}{r_{mm}}$$

where

$$r_{ij} = q_i^*a_j(i \lneqq j)$$

and

$$r_{ij} = ||a_m - \sum_{i=1}^{m-1} r_{im} q_i||_2.$$

We define the $n \times m$ matrix

$$Q = [q_1, q_2, ..., q_m]$$

with an orthonormal system of columns and the $m \times m$ upper triangular matrix :

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} \cdots & r_{1,m} \\ 0 & r_{2,2} \cdots & r_{2,m} \\ 0 & \vdots & \vdots \\ 0 & 0 \cdots & r_{m,m} \end{bmatrix}$$

which has elements the cofactors of Gram-Schmidt orthonormalization.

Theorem 1. *The factorization,[3] :*

$$A = QR$$

$$\begin{bmatrix} a_1 & a_2 & ... & a_m \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & ... & q_m \end{bmatrix} \begin{bmatrix} r_{1,1} & r_{1,2} \cdots & r_{1,m} \\ 0 & r_{2,2} \cdots & r_{2,m} \\ 0 & \vdots & \vdots \\ 0 & 0 \cdots & r_{m,m} \end{bmatrix} \quad (1)$$

is called simple QR factorization of A matrix.

Corollary 1. *Every matrix A in*$\mathbb{C}^{n\times m}(n \geqslant m) rank(A) = m$ *has unique QR factorization.*

Proof:

Obviously,the matrix A has QR factorization if and only if Gram-Schmidt orthonormalization is completed successfully.The vector

$$w_j = a_j - \sum_{i=1}^{j-1} r_{ij} q_i$$

is zero. Something like this is not possible because

$$rank(A) = dim[span\{a_1, a_2, ..., a_m\}] = m.$$

```
R(1,1)=||A(:,1)||_2, Q(:,1)=A(:,1)/R(1,1)
for k=2:m
R(1:k-1,k)=Q(1:n,1:k-1)'A(1:n,k)
z=A(1:n,k)-Q(1:n,1:k-1)'A(1:n,k)
R(k,k)=||z||_2
Q(1:n,k)=z/R(k,k),end
```

It is obvious that QR is unique because during the Gram-Schmidt orthonormalization due to the fact that rank(A)=m,vectors q_i and cofactors r_{ij} exist in unique way.

Now,we present you a pseudocode in Matlab of the QR-Gram-Schmidt:

Example 1.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

k = 2

$$R = \begin{bmatrix} 1.4142 & 0 \\ 0 & 1.0000 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.7071 & 0 \\ 0 & 1.0000 \\ 0.7071 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

k = 2

$$R = \begin{bmatrix} 1.4142 & 1.4142 \\ 0 & 1.7321 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.7071 & 0.5774 \\ 0 & 0.5774 \\ 0.7071 & -0.5774 \end{bmatrix}$$

1.2 The QR-Householder Factorization (Complete QR)

A matrix of nxn dimension which has the following form

$$P = I - \frac{2uu^*}{u^*u}, u \in \mathbb{C}^n$$

it is called Householder matrix or Householder transformation,[4]. The u vector is called Householder vector of P matrix. For every $w \in \mathbb{C}^n$,the vector

$$Pw = w - \frac{2uu^*}{u^*u}w = w - 2\frac{u^*w}{u^*u}u$$

is the reflection of w in the hyperplane of $span\{u^{\perp}\}$. Obviously, Householder matrix P is hermitian and unitary

$$(P^*P = PP^* = I_n).$$

Furthermore, it is a turbulence of I_n matrix

$$rank(\frac{2}{u^*u}uu^*) = rank(uu*) = 1.$$

Let assume that,we want Pw to be a multiple of vector e_1 of standard basis. Then Pw belongs in $span\{e_1\}$ and u belongs in $span\{w, e_1\}$. We write $u = w + ae_1$, $a \in \mathbb{C}$ and we observe that:

$$u^*w = w^*w + \bar{a}e_1^Tw = w^*w + \bar{a}w_1$$

where,

$$w_1$$

is the first element of w which is real number while,

$$u^*u = w^*w + w_1 2Re(a) + |a|^2.$$

Thus,

$$Pw = (I_n - \frac{2}{u^*u}uu^*)w = 1 - 2\frac{w^*w + \bar{a}w_1}{(w*w + 2w_1Re(a) + |a|^2)w - 2a\frac{u^*w}{u^*u}e_1.}$$

The coefficient of w is zero if $a = e^{iarg(w_1)}||w||_2$ and then

$$u = w + e^{iarg(w_1)}||w||_2e_1 \Rightarrow$$

$$Pw = (I_n - 2\frac{uu^*}{u^*u})w = -e^{iarg(w_1)}||w||_2 e_1.$$

In other words, for certain vector w,we construct the vector Householder u and respectively the Householder matrix

$$P = I - \frac{2uu^*}{u^*u}$$

in order Pw vector belongs in

$$span\{e_1\}.$$

For example,
Let

$$w = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

with

$$||w||_2 = 6$$

and

$$w_1 = 3.$$

the householder vector is

$$u = w + e^{i0}||w||_2 e_1 = w + ||w||_2 e_1$$

=

$$\begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$$

and the Householder matrix

$$P = I - \frac{2uu^T}{u^Tu}$$

=

$$\frac{1}{54}$$

$$\begin{bmatrix} -27 & -9 & -45 & -9 \\ -9 & 53 & -5 & -1 \\ -45 & -5 & 29 & -5 \\ -9 & -1 & -5 & 53 \end{bmatrix}$$

and then we have

$$Pw = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

belongs in

$$span\{e_1\}$$

Theorem 2. *[4]*

Let A a matrix $\in \mathbb{C}^{n\times m}$ where $n \geqslant m$ rank(A)=m and as follows columns $a_1, a_2, ..., a_m$ is linear independent. Let P_1 a Householder matrix such that $P_1 a_1$ belongs in span$\{e_1\} \subset \mathbb{C}$. Then,

$$P_1 A = P_1[a_1, a_2, ..., a_m] = [P_1 a_1, P_1 a_2, ..., P_1 a_m]$$

, where

$$P_1 a_1 = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Due to the fact that,the columns of A is linear independent,$P_1 a_2 \notin span\{e_1\}$ because $P_1 a_2$ column has non zero elements under its first element.If $w_2 \in \mathbb{C}^{n-1}$ the non zero vector which comes from the erase of the first element of vector column $P_1 a_2$,then exists $n-1 \times n-1$ Householder matrix P_2 such that $P_2(P_1 a_2)$ is multiple of $e_1 \in \mathbb{C}^{n-1}$.

Thus,we have:

$$\begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$\cdot P_1 a_2$

$=$

$$\begin{bmatrix} * \\ P_2 w_2 \end{bmatrix}$$

=

$$\begin{bmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$P_1 A$ =

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & * \end{bmatrix}$$

It is obvious that

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & P_2 \end{bmatrix}$$

is unitary. Let as assume that $n \geq m$*. We continue the same process and we are able to construct householder matrices* $P_3 \in \mathbb{C}^{(n-2)\times(n-2)}$,$P_4 \in \mathbb{C}^{(n-3)\times(n-3)}$,..., $P_m \in \mathbb{C}^{(n-m+1)\times(n-m+1)}$

such that

$$H_2 = \begin{bmatrix} I_2 & 0 \\ 0 & P_3 \end{bmatrix}$$

and

$$H_3 = \begin{bmatrix} I_3 & 0 \\ 0 & P_4 \end{bmatrix}$$

and

$$H_{m-1} = \begin{bmatrix} I_{m-1} & 0 \\ 0 & P_m \end{bmatrix}$$

$[H_{m-1} ... H_3 H_2 H_1 P_1] A$ =

$$\begin{bmatrix} * & * & * & * & \cdots & * \\ 0 & * & * & * & \cdots & * \\ 0 & 0 & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Now,let

$$\tilde{Q} = [H_{m-1}...H_3H_2H_1P_1]$$

and $\tilde{R}$ the matrix =

$$\begin{bmatrix} * & * & * & * & \cdots & * \\ 0 & * & * & * & \cdots & * \\ 0 & 0 & * & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

which is upper triangular with n-m rows in the last part

we have,

$\tilde{Q}A = \tilde{R}$ *equivalently* $A = \tilde{Q}\tilde{R}$ *where* $\hat{Q} = \tilde{Q}^*$

In other words we have a QR factorization through Householder matrices. $\hat{Q}$ is a unitary $n \times n$ matrix $\tilde{R}$ is $n \times m$ upper triangular matrix.[3] and this factorization is called the Complete QR Factorization. This factorization has major advantages.For example, if $A \in \mathbb{C}^{n \times m}$ $n \geq m$ is order of $r \leq m$ then $\tilde{R}$ has exactly the last n-m rows zero and m-r elements in its major diagonal. In the case of square matrices the simple and and the complete QR are the same methods. Its worth mentioning that Gram-Schmidt orthonormalization can lead to complete QR factorization if we expand the orthonormal system of vectors $q_1, q_2, ..., q_m$ in a orthonormal basis of $\mathbb{C}^m$ and add n-m zero rows in the last part of R matrix. In this case,simple QR is not unique.

Now,we are demonstrating some applications of Householder matrices

Corollary 2. *A Householder matrix* $P = I_n - 2\frac{uu^*}{u^*u}$ *has eigenvalues*

$$||\lambda_i|| = 1$$

or

$$||\lambda_i|| = -1.$$

Proof:

He have: $P = I - 2vv^T$, $||v||_2^2 = 1$. $P = P^T$ because $P^T = (I - 2vv^T)^T = I - 2vv^T = P$

$$PP^T = P^2 = (I - 2uu^T)^2 = I - 4(uu^T)^2 + 4u(u^Tu)u^T = I$$

Now , due to the fact that P matrix is symmetric has real eigenvalues and secondly,because is orthogonal all eigenvalues have

$$||\lambda_i||_2^2 = 1.$$

Thus,

$$||\lambda_i|| = 1$$

or

$$||\lambda_i|| = -1.$$

Because

$$Pu = -u$$

$$\Rightarrow$$

-1 is eigenvalue of P and has eigenvector u which is non-zero and every v vector

$$\in \mathbb{R}^n$$

vertical in u.

$$Pv = (I_n - 2\frac{uu^T}{u^T u})v = v$$

$$\Rightarrow$$

(1,v) same pair.

Corollary 3. *Let $x, y \in \mathbb{C}^n$ non zero vectors.There exists a P Householder matrix such that Px multiple of y.*

Proof:
Let

$$w = \lambda \cdot y$$

with

$$||x|| = ||w||$$

. Let

$$r = w - x$$

and

$$H = \frac{vv^*}{v^*v}$$

. ,where H is the projection matrix.

Let $P = I - 2H$,then we have:

$$Px = x - 2Hx = w - v - 2\frac{vv^*}{v^*v}x = w - x - \frac{vv^*}{v^*v}x - \frac{vv^*}{v^*v}(w - x) = w - \frac{vv^*}{v^*v}(w + x) = w - \frac{(w - x)(w - x^*)}{v^*v}$$

.

Thus, Px multiple of y.

Now,we introduce a pseudocode of Householder transformation and Complete QR Factorization

```
%house1
m=max(abs(x));
      u=x/m;
      suma=0;
      for i=1:n
            suma=suma+u(i)^2;
      end
      i=1;
      while u(i)==0
            i=i+1;
      end
      s=sign(u(i))*sqrt(suma);
      u(1)=u(1)+s;
      s=-m*s;


%QR-house1
smin=min(n(1)-1,n(2));
for k=1:smin
    if sum(abs(A(k:n(1),k)))~=0
      [u(k:n(1)),s]=house1(A(k:n(1),k));
            A(k,k)=s;
      else
            u(k:n(1))=[1;zeros(n(1)-k,1)];
      end

        for i=k+1:n(1)
            A(i,k)=u(i);
        end
     uk(k)=u(k);
      uu=u(k:n(1));
      b=2/(uu*uu');
      for j=k+1:n(2)
            sumi=0;
            for i=k:n(1)
                  sumi=sumi+u(i)*A(i,j);
            end
            s=b*sumi;
            for i=k:n(1)
                 A(i,j)=A(i,j)-s*u(i);
            end
```

```
    end
    for  j=1:n(1)
       sumi=0;
        for  i=k:n(1)
            sumi=sumi+u(i)*Q(i,j);
        end
        s=b*sumi;
        for  i=k:n(1)
            Q(i,j)=Q(i,j)-s*u(i);
     end
end
```

Examples 1.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

k = 1

$$A = \begin{bmatrix} -1.4142 & 0 \\ 0 & 1.0000 \\ 1.0000 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & 1.0000 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$$

k = 2

$$A = \begin{bmatrix} -1.4142 & 0 \\ 0 & -1.0000 \\ 1.0000 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & -1.0000 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$$

Finally,

$$Q = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & -1.0000 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$$

$$R = \begin{bmatrix} -1.4142 & 0 \\ 0 & -1.0000 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

k = 1

$$A = \begin{bmatrix} -1.4142 & -1.4142 \\ 0 & 1.0000 \\ 1.0000 & -1.4142 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ 0 & 1.0000 & 0 \\ -0.7071 & 0 & 0.7071 \end{bmatrix}$$

k = 2

$$A = \begin{bmatrix} -1.4142 & -1.4142 \\ 0 & -1.7321 \\ 1.0000 & -1.0000 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.7071 & 0 & -0.7071 \\ -0.5774 & -0.5774 & 0.5774 \\ -0.4082 & 0.8165 & 0.4082 \end{bmatrix}$$

Finally,

$$Q = \begin{bmatrix} -0.7071 & -0.5774 & -0.4082 \\ 0 & -0.5774 & 0.8165 \\ -0.7071 & 0.5774 & 0.4082 \end{bmatrix}$$

$$R = \begin{bmatrix} -1.4142 & -1.4142 \\ 0 & -1.7321 \\ 0 & 0 \end{bmatrix}$$

1.3 The Real QR Factorization, Examples and Complexity

Since,in the most cases the matrices that we are face are real matrices we are going to present you the Real QR-factorization.

As we present above some useful properties of Householder matrices in $\mathbb{R}$ are:

i) Householder matrix is symmetric.

ii) Householder matrix is orthogonal.

iii) Householder matrix is a reflection matrix.

Reflections are computationally attractive because the can easily constructed and they can be used to introduce zeros in a vector properly.

Now,following this short introduction it is time to give the Theorem of The Real-QR factorization,[4].

Theorem 3. *Let $A \in \mathbb{R}^{mxn}$. There is an orthogonal matrix Q_{mxm}, such as*

$$QQ^T = Q^T Q = I$$

and an upper triangular matrix R_{nxn}, in this form

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}$$

such that:

$$A = QR$$

Where Q is the product of

$$Q = H_1 H_2 \cdots H_{n-1}$$

where every H_i is Householder. This factorization of A is called QR factorization.

It is proved that the complexity of QR factorization is

$$O(2mn^2 - \frac{2n^3}{3}).$$

In case of n=m the complexity is:

$$O(2n^3 - \frac{2n^3}{3}) = O(\frac{4n^3}{3})$$

.

Theorem 4. *[3] Let $A \in \mathbb{R}^{mxn}$ with $m \geqslant n$ and $rank(A) = r < n$. Then it is always exists a permutation matrix $P \in \mathbb{R}^{nxn}$ and an orthogonal matrix $Q \in \mathbb{R}^{mxm}$, that is*

$$QQ^T = Q^TQ = I$$

such that:

$$Q^TAP = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \Leftrightarrow AP = QR$$

where $R_{11} \in \mathbb{R}^{rxr}$ an upper triangular matrix with non-zero diagonal elements.

The QR permutation with column pivoting of a matrix A with $rank(A) = r < min\{m, n\}$ has complexity

$$O(2mnr - r^2(m+n) + \frac{2r^3}{3}).$$

In this master thesis we are studying square matrices so in case of m=n we have:

$$O(2rn^2 - (2n)r^2 + \frac{2r^3}{3}).$$

Now,we introduce the pseudocode of the Real QR Factorization with Column Pivoting.

Lets see a simple numerical example:

Example 2. $B = \begin{bmatrix} 0 & +8 & -4 \\ +8 & -4 & 0 \\ -4 & 0 & +1 \end{bmatrix}$.

Then:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0.9759 & 0.2182 \\ -0.8944 & -0.0976 & 0.4364 \\ 0.4472 & -0.1972 & 0.8729 \end{bmatrix}$$

we ascertain that $QQ^T = I_3$ and

$$R = \begin{bmatrix} -8.9443 & 3.5777 & 0.4472 \\ 0 & 8.1976 & -4.0988 \\ 0 & 0 & 0 \end{bmatrix}$$

and finally we ascertain $AP = QR$

1 Construct the P permuted matrix find the column with the max norm from the A matrix,construct the AP_1 matrix,with first column the column we have previoysly mentioned.

2 Construct the Householder matrix H_1 such that $A^{(1)} = H_1AP_1$ with zero elements under the $(1,1)$ element of the matrix. Repeat step one and two for the right low part $(m-1)\times(m-1)$ part of $A^{(1)}$ In matrix $A^{(1)}(2:m,2:n)$ we find the column with max norm we construct the $(n-1)\times(n-1)$ $\hat{P}_2$ matrix and then the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_2 \end{bmatrix}$$

3 After r steps we have zero elements under the diagonal and $A^{(r)} = H_rH_{r-1}...H_2H_1AP_1P_2...P_{r-1}P_r$

and

$$A^{(r)} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$$

1.4 Error Analysis

Remark 1. *In this section,we present you the Roundoff error of QR factorization,[4],[3], that shows the stability of the QR method,[4].*

If $\tilde{R}$ denotes the computed R,then there exists an orthogonal $\tilde{Q}$ such that: $A + E = \tilde{R} \cdot \tilde{Q}$ *The error matrix E satisfies:*

$$\|E\|_F \leq \phi(n)m\|A\|_F$$

where $\|\|_F$ is the Frobenious norm. The $\phi(n)$ is a slowing function of n and m is the machine precision then it can be shown $\phi(n) = 15 - 5n$.The algorithm is stable.

Now,let us see the stability for the 4. We,remind the 4 and the error,[3], as we did above with the Real QR.

Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$, rank$(A = r < n)$.Then always exist one permutation matrix Π of order $n \times n$ and an orthogonal matrix Q of order $m \times m$ such that:

$$Q^T A\Pi = R = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ 0 & \tilde{R}_{22} \end{bmatrix} \begin{matrix} r \\ m-r \end{matrix},$$

$$r \quad n-r$$

If $\tilde{R}_{22}$ is suitably small in norm,then it is reasonable to terminate the reduction and declare A to have rank r. A typical termination criteria might be:

$$\|\tilde{R}_{22}\|_2 \leq \epsilon_1\|A\|_2$$

where

$$\|\|_2$$

is the euclidean norm. for some small machine-dependent parameter $\epsilon 1$.In the view of roundoff properties associated with Householder matrix computation we know that $\tilde{R}$ is the exact R-factor of a matrix A+E,where

$$\|E\|_2 \leq \epsilon_2\|A\|_2$$

$\epsilon_2 = O(u)$.

Corollary 4. *Let A be a matrix in $\mathbb{R}^{m\times n}$ and E be a matrix in $\mathbb{R}^{m\times n}$.Then we have:*

$$\sigma_{max}(A+E) \leqslant \sigma_{max}A + ||E||_2$$

$$\sigma_{max}(A+E) \geqslant \sigma_{max}A - ||E||_2$$

Using the above corollary ,[3], we have

$$\sigma_{k+1}(A+E) = \sigma_{k+1}(\tilde{R}) \leq \|\tilde{R}_{22}\|_2.$$

Since

$$\sigma_{k+1}(A) \leq \sigma_{k+1}(A+E) + \|E\|_2,$$

it follows that

$$\sigma(A)_{k+1} \leq (\epsilon_1 + \epsilon_2)\|A\|_2.$$

In other words, a relative perturbation of $O(\epsilon_1 + \epsilon_2)$ in A yields a rank-r matrix. With this termination criterion,we conclude that QR factorization with column pivoting discovers rank deficiency if $\tilde{R}_{22}$ is small for some $r < n$.However,it does not follow that the matrix $\tilde{R}_{22}$ is small if rank(A)=r.

Remark 2. *$\sigma(A)$ denotes the singular value of a matrix A.*

2 An Introduction to Bézout Matrices

2.1 Definition and Matrix Representation

A Bézout matrix is a special square matrix associated with two polynomials, introduced by Sylvester (1853) and Cayley (1857) and named after Étienne Bézout.

Definition 1. *Let $f(x)$ and $g(x)$ be two polynomials in one variable such that,[5]:*

$$\begin{aligned} f(x) &= \sum_{l=0}^{m} u_l x^l = u_m x^m + u_{m-1} x^{m-1} + ... + u_2 x^2 + u_1 x + u_0 \\ g(x) &= \sum_{l=0}^{n} \upsilon_l x^l = \upsilon_n x^n + \upsilon_{n-1} x^{n-1} + ... + \upsilon_2 x^2 + \upsilon_1 x + \upsilon_0 \end{aligned}$$

with $\deg\{f(x)\} = m$ *and* $\deg\{g(x)\} = n$*, where* $m \geq n$ *and* $(u_m, \upsilon_n) \neq (0, 0)$*. Then, the Bézout matrix associated with the polynomials* $f(x)$ *and* $g(x)$ *and denoted by* $B(f, g)$ *or* $Bez(f(x), g(x)$ *is an* $m \times m$ *symmetric matrix which is constructed from the coefficients of the polynomials as follows:*

$$B = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \\ u_2 & \cdots & u_m & 0 \\ \vdots & & & \vdots \\ u_m & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} \upsilon_0 & \upsilon_1 & \cdots & \upsilon_{m-1} \\ 0 & \upsilon_0 & \cdots & \upsilon_{m-2} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \upsilon_0 \end{bmatrix} -$$

$$- \begin{bmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_m \\ \upsilon_2 & \cdots & \upsilon_m & 0 \\ \vdots & & & \vdots \\ \upsilon_m & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \\ 0 & u_0 & \cdots & u_{m-2} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & u_0 \end{bmatrix}$$

The elements b_{ij} *of Bézout* $B(f, g)$ *matrix are calculated by the following formula:*

$$b_{ij} = \mid u_0 \upsilon_{i+j-1} \mid + \mid u_1 \upsilon_{i+j-2} \mid + ... + \mid u_k \upsilon_{i+j-k-1} \mid$$

$$k = min(i-1, j-1)$$

$u_r = \upsilon_r = 0$ *where* $r > m$ *and*

$$\mid u_r \upsilon_s \mid = u_s \upsilon_r - u_r \upsilon_s$$

Furthermore,there is another equivalent definition of the Bézout matrix,[6]:

$$B(f,g) = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & & \vdots \\ b_{m,1} & \cdots & b_{m,m} \end{bmatrix}$$

and the coefficients are calculated by the equation:

$$\frac{f(x)g(y) - f(y)g(x)}{x-y} = [1, x, x^2, ..., x^{m-1}]B(f,g)[1, y, y^2, ...y^{m-1}] = \sum_{i,j=1}^{m} b_{i,j}x^{i-1}y^{i-1}$$

.

Let J be an antidiagonal matrix such that:

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & 1 & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

and $\tilde{f}(x)$ and $\tilde{g}(x)$ two polynomials such that:

$$\tilde{f}(x) = \sum_{k=0}^{m} u_{m-k}x^k = u_0x^m + u_1x^{m-1} + u_2x^{m-2} + ... + u_{m-2}x^2 + u_{m-1}x + u_m$$

and

$$\tilde{g}(x) = \sum_{k=0}^{n} \upsilon_{n-k}x^k = \upsilon_0x^n + \upsilon_1x^{n-1} + \upsilon_2x^{n-2} + ... + \upsilon_{n-2}x^2 + \upsilon_{n-1}x + \upsilon_n$$

Previously,we have shown that

$$f(x) = \sum_{k=0}^{m} u_kx^k = u_mx^m + u_{m-1}x^{m-1} + u_{m-2}x^{m-2} + ... + u_2x^2 + u_1x + u_0$$

and

$$g(x) = \sum_{k=0}^{n} \upsilon_kx^k = \upsilon_nx^n + \upsilon_{n-1}x^{n-1} + \upsilon_{n-2}x^{n-2} + ... + \upsilon_2x^2 + \upsilon_1x + \upsilon_0$$

.

In other words, the polynomials $\tilde{f}(x)$ and $\tilde{g}(x)$ are the reversed polynomials of f(x) and g(x).

This remark provides us with the following conclusion:

$$B(\tilde{f}(x), \tilde{g}(x)) = J * B(f, g) * J$$

where $B(\tilde{f}(x), \tilde{g}(x))$ is the Bézout matrix of $\tilde{f}(x)$ and $\tilde{g}(x)$.

Considering the case of sets of several polynomials, the following definition of an extended form of the Bézout matrix is given,[5].

Definition 2. *We consider the set of $m + 1$ real univariate polynomials:*

$$\begin{aligned} \mathcal{P}_{m+1,n} &= \Big\{ a(s), b_i(s) \in \mathbb{R}[s],\ i = 1, 2, \ldots, m \text{ with } n = \deg\{a(s)\}, \\ & \quad p = \max_{1 \le i \le m} \big\{ \deg\{b_i(s)\} \big\} \le n \Big\} \end{aligned} \tag{2}$$

Definition 3. *Let $u, v_1, \ldots, v_m$ be $m + 1$ polynomials, with u a polynomial of maximal degree n. Let B_i be the Bézout matrix of polynomials u, v_i, for $i = 1, \ldots, n$. Then the* generalized Bézout *matrix is defined as follows:*

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \in \mathbb{R}^{mn \times n} \tag{3}$$

2.2 Properties of The Bézout Matrix

Some properties of the Bézout matrix are,[**?**],[**?**].

i) $B(f,g) = -B(g,f)$

ii) $B(f,f) = \mathbb{O}$

iii) if $deg\{f(x)\} = deg\{g(x)\} = n$ then the Bézout matrix is non-singular only if and only $f(x)$ $g(x)$ do not have the same roots.

iv)
$$B(af+bw,g) = aB(f,g) + bB(w,g),$$
$$B(f,ag+bs) = aB(f,g) + bB(f,s)$$

$w(x), s(x)$

v) if $deg\{f(x)\} = m$ and $deg\{g(x)\} = n$ then $f(x), g(x) in$ $\mathbb{R}[x]$.

vi) $B(f,g)$ is symmetric for every m, n positive integer or non-negative integer

vii) If $\sum_{k=0}^{m} u_k x^k$ and $\sum_{k=0}^{n} \upsilon_k x^k$ with $deg\{f(x)\} = m$ and $deg\{g(x)\} = n$ then
$$\|B(f,g)\|_2 \leq 2m\|f\|_2\|g\|_2$$

$\|\|_2$ is the euclidean norm.

2.3 Complexity of The Bézout Matrix

We have:

$$B_{i,j} = \sum_{k=max(0,i-j)}^{min(i,n-1-j} (u_{i-k}v_{j+1+k} - v_{i-k}u_{j+1+k}).$$

We should calculate $\{B_{i,j}\}$ for every $i \leq j$:

- for $i + j \leq n - 1$ we have:

$$B_{i,j} = \sum_{k=0}^{i} (u_{i-k}v_{j+1+k} - v_{i-k}u_{j+1+k})$$

- for $i + j > n - 1$ we have:

$$A_{i,j} = \sum_{k=0}^{n-1-j} (u_{i-k}v_{j+1+k} - v_{i-k}u_{j+1+k}),$$

Every calculation requires two products and one sum.

Let n odd, the overall number of products in order to calculate $\{B_{i,j}\}$,$i \leq j$ is:

$$2 \cdot \sum_{i=0}^{(n-1)/2} (n - 2i) \cdot (i + 1) + 2 \cdot \sum_{j=(n+1)/2}^{n-1} (n - j) \cdot (2j - n + 1) =$$

$$= \frac{2n^3 + 9n^2 + 10n + 3}{12}$$

and the overall number of sums is:

$$\sum_{i=0}^{(n-1)/2} (n - 2i) \cdot (2i + 1) + \sum_{j=(n+1)/2}^{n-1} (2n - 2j - 1) \cdot (2j - n + 1) =$$

$$= \frac{2n^3 + 3n^2 + 4n + 3}{12}$$

The previous calculation gives the same results in case of an n is even.

Its worth mentioning that,the previous calculation requires

$$O(n^3)$$

flops.We are going to use the advantage that Bézout matrices are symmetric in order to achieve complexity

$$O(n^2)$$

.

Due to the fact that,Bézout matrices are symmetric we need only to find the elements $b_{i,j}$ for $i \leq j$. We use the calculation of 1.1 section and we find the entries for $b_{i,j}$ for $i \leq j$ and we complete for the rest elements($i > j$).

i) In the beginning every element-entry requires 2 products and 1 plus.We have $n^2 + n$ products and $\frac{n^2+n}{2}$ sums.

ii) As we calculating the next elements of matrix,every element above the diagonal when i=j expect from the elements of first row and the elements of last column require 1 sum. The total sums are $\frac{n^2-n}{2}$.

To sum up with, the total number of calculations is: $n^2+n+\frac{n^2+n}{2}+\frac{n^2-n}{2} = \frac{2n^2}{2}$. And finally the total number of flops is $O(n^2)$.

2.4 Numerical Examples

We are presenting some simple examples of how a Bézout matrix is constructed and we introduce you two examples of the Generalized Bézout matrix.

Examples 2. *i)*

$$f(x) = x^2 - 1 = 1x^2 + 0x - 1$$
$$g(x) = x - 1 = 0x^2 + 1x - 1$$

We have:

$$u_2 = 1, u_1 = 0, u_0 = -1$$

and

$$\upsilon_2 = 0, \upsilon_1 = 1, \upsilon_0 = -1$$

We calculate the elements of the matrix:

$$b_{11} = \mid u_0\upsilon_1 \mid = u_1\upsilon_0 - u_0\upsilon_1 = 0 \cdot (-1) - (-1) \cdot 1 = +1$$

$$b_{12} = \mid u_0\upsilon_2 \mid = u_2\upsilon_0 - u_0\upsilon_2 = 1 \cdot (-1) - (-1) \cdot 0 = -1$$
$$b_{21} = \mid u_0\upsilon_2 \mid = u_2\upsilon_0 - u_0\upsilon_2 = 1 \cdot (-1) - (-1) \cdot 0 = -1$$
$$b_{22} = \mid u_0\upsilon_3 \mid + \mid u_1\upsilon_2 \mid = u_2\upsilon_1 - u_1\upsilon_2 = 1 \cdot 1 - 0 \cdot 0 = +1$$

we use $u_3 = \upsilon_3 = 0$
due to the fact that $r = 3 > 2 = m = n$.
The Bézout is:

$$B(f, g) = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

ii)

$$f(x) = x^3 - 8 = 1x^3 + 0x^2 + 0x - 8$$
$$g(x) = x^2 - 4 = 0x^3 + 1x^2 + 0x - 4$$

We have :

$$u_3 = 1, u_2 = 0, u_1 = 0, u_0 = -8$$

and

$$\upsilon_3 = 0, \upsilon_2 = 1, \upsilon_1 = 0, \upsilon_0 = -4$$

We calculate the elements of the matrix:

$$b_{11} =\mid u_0\upsilon_1 \mid= u_1\upsilon_0 - u_0\upsilon_1, (k = 0) = 0 \cdot (-4) - (-8) \cdot 0 = 0$$

$$b_{12} =\mid u_0\upsilon_2 \mid= u_2\upsilon_0 - u_0\upsilon_2, (k = 0) = 0 \cdot (-4) - (-8) \cdot 1 = 8$$

$$b_{13} =\mid u_0\upsilon_3 \mid= u_3\upsilon_0 - u_0\upsilon_3, (k = 0) = 1 \cdot (-4) - (-8) \cdot 0 = -4$$

$$b_{21} =\mid u_0\upsilon_2 \mid= u_2\upsilon_0 - u_0\upsilon_2 (k = 0) = 0 \cdot (-4) - (-8) \cdot 1 = 8$$

$$b_{22} =\mid u_0\upsilon_3 \mid + \mid u_1\upsilon_2 \mid= u_3\upsilon_0 - u_0\upsilon_3 + u_2\upsilon_1 - u_1\upsilon_2, (k = 1) = 1 \cdot (-4) - (-8) \cdot 0 + 0 \cdot 0 - 0 \cdot 1 = -4$$

$$b_{23} =\mid u_0\upsilon_4 \mid + \mid u_1\upsilon_3 \mid= u_3\upsilon_1 - u_1\upsilon_3, (k = 1) = 1 \cdot 0 - 0 \cdot 0 = 0$$

for the b_{23} *we use* $u_4 = \upsilon_4 = 0$
due to the fact that $r = 4 > 3 = m = n$.

$$b_{31} =\mid u_0\upsilon_3 \mid= u_3\upsilon_0 - u_0\upsilon_3, (k = 0) == 1 \cdot (-4) - (-8) \cdot 0 = -4$$

$$b_{32} =\mid u_0\upsilon_4 \mid + \mid u_1\upsilon_3 \mid= u_3\upsilon_1 - u_1\upsilon_3, (k = 1) == 1 \cdot 0 - 0 \cdot 0 = 0$$

$$b_{33} =\mid u_0\upsilon_5 \mid + \mid u_1\upsilon_4 \mid + \mid u_2\upsilon_3 \mid= u_3\upsilon_2 - u_2\upsilon_3, (k = 2) = 1 \cdot 1 - 0 \cdot 0 = 1$$

for b_{33} *we use* $u_4 = \upsilon_4 = 0$ *and* $u_5 = \upsilon_5 = 0$
due to the fact that $r = 4 > 3 = m = n$.

The Bézout matrix is

$$B(f, g) = \begin{bmatrix} 0 & +8 & -4 \\ +8 & -4 & 0 \\ -4 & 0 & +1 \end{bmatrix}$$

iii)

$$f(x) = (x - 1)^3 = 1x^3 - 3x^2 + 3x - 1$$

$$g(x) = (x - 1)(x + 2)(x + 3) = 1x^3 + 4x^2 + 1x - 6$$

We have:

$$u_3 = 1, u_2 = -3, u_1 = 3, u_0 = -1$$

and

$$\upsilon_3 = 1, \upsilon_2 = 4, \upsilon_1 = 1, \upsilon_0 = -6$$

We calculate the elements of the matrix:

$$b_{11} =\mid u_0\upsilon_1 \mid= u_1\upsilon_0 - u_0\upsilon_1, (k = 0) = 3 \cdot (-6) - (-1) \cdot 1 = -17$$

$$b_{12} = \mid u_0 v_2 \mid = u_2 v_0 - u_0 v_2, (k = 0) = -3 \cdot (-6) - (-1) \cdot 4 = 22$$

$$b_{13} = \mid u_0 v_3 \mid = u_3 v_0 - u_0 v_3, (k = 0) = 1 \cdot (-6) - (-1) \cdot 1 = -5$$

$$b_{21} = \mid u_0 v_2 \mid = u_2 v_0 - u_0 v_2, (k = 0) = -3 \cdot (-6) - (-1) \cdot 4 = 22$$

$$b_{22} = \mid u_0 v_3 \mid + \mid u_1 v_2 \mid = u_3 v_0 - u_0 v_3 + u_2 v_1 - u_1 v_2, (k = 1) = -3 \cdot 1 - 4 \cdot 3 + 1 \cdot (-6) - (-1) \cdot 1 = -2$$

$$b_{23} = \mid u_0 v_4 \mid + \mid u_1 v_3 \mid = u_3 v_1 - u_1 v_3, (k = 1) = 1 \cdot 1 - 3 \cdot 1 = -2$$

for the b_{23} *we use that* $u_4 = v_4 = 0$
due to the fact that $r = 4 > 3 = m = n$.

$$b_{31} = \mid u_0 v_3 \mid = u_3 v_0 - u_0 v_3, (k = 0) = 1 \cdot (-6) - (-1) \cdot 1 = -5$$

$$b_{32} = \mid u_0 v_4 \mid + \mid u_1 v_3 \mid = u_3 v_1 - u_1 v_3, (k = 1) = 1 \cdot 1 - 3 \cdot 1 = -2$$

$$b_{33} = \mid u_0 v_5 \mid + \mid u_1 v_4 \mid + \mid u_2 v_3 \mid = u_3 v_2 - u_2 v_3, (k = 2) = 1 \cdot 4 - (-3) \cdot 1 = 7$$

for the b_{33} *we use* $u_4 = v_4 = 0$ *and* $u_5 = v_5 = 0$
due to the fact that $r = 4 > 3 = m = n$.

The Bézout matrix is:

$$B(f, g) = \begin{bmatrix} -17 & 22 & -5 \\ 22 & -20 & -2 \\ -5 & -2 & +7 \end{bmatrix}$$

Now we introduce two examples of generalized Bézout matrix:

iv) Let us consider the next set of three univariate polynomials:

$$\mathcal{P}_{3,3} = \left\{ \begin{array}{lcl} p_1(s) & = & s^3 + 4s^2 + 5s + 2 \\ p_2(s) & = & s^3 - 4s^2 - 3s + 18 \\ p_3(s) & = & s^3 + 12s^2 + 45s + 50 \end{array} \right\} \tag{4}$$

of degree 3.

$$B_1 = Bez\{p_1, p_2\} = \begin{bmatrix} 8 & 8 & 16 \\ 8 & -24 & -6 \\ -16 & 80 & -96 \end{bmatrix} \tag{5}$$

$$B_2 = Bez\{p_1, p_3\} = \begin{bmatrix} -8 & -40 & -48 \\ -40 & -168 & -176 \\ -48 & -176 & -160 \end{bmatrix} \tag{6}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 8 & 8 & -16 \\ 8 & -24 & -80 \\ -16 & 80 & -96 \\ -8 & -40 & -48 \\ -40 & -168 & -176 \\ -48 & -176 & -160 \end{bmatrix} \tag{7}$$

v) Let us consider the next set of three univariate polynomials:

$$\mathcal{P}_{3,3} = \left\{ \begin{array}{rcl} p_1(s) & = & s^3 - 6s^2 + 11s - 6 \\ p_2(s) & = & s^3 - 7s^2 + 14s - 8 \\ p_3(s) & = & s^3 - 8s^2 + 17s - 10 \end{array} \right\} \tag{8}$$

of degree 3.

$$B_1 = Bez\{p_1, p_2\} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -6 \\ 2 & -6 & 4 \end{bmatrix} \tag{9}$$

$$B_2 = Bez\{p_1, p_3\} = \begin{bmatrix} 2 & -6 & 4 \\ -6 & 18 & -12 \\ 4 & -12 & 8 \end{bmatrix} \tag{10}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -6 \\ 2 & -6 & 4 \\ 2 & -6 & 4 \\ -6 & 18 & -12 \\ 4 & -12 & 8 \end{bmatrix} \tag{11}$$

3 Greatest Common Divisor of Polynomials(GCD)through Bézout Matrix

3.1 Barnett's Theorems and Representation of the GCD through Bézout Matrix

The greatest common divisor (GCD) of a polynomial set is proven to be very important to many applications in applied mathematics and engineering. Several methods have been proposed for the computation of the GCD of sets of polynomials. Most of them are based on the Euclidean algorithm. They are designed to process two polynomials at a time and can be applied iteratively when a set of more than two polynomials is considered . Conversely, there exist efficient matrix-based methods which can compute the degree and the coefficients of the GCD by applying specific transformations to a matrix formed directly from the coefficients of the polynomials of the entire given set.The greatest common divisor has a significant role in Control Theory, Network Theory, signal and image processing and in several other areas of mathematics. A number of important invariants for Linear Systems rely on the notion of the greatest common divisor of many polynomials.In some cases,we are not sure if the GCD of a set of polynomials exists or not.In this section we use an algorithm based on the QRCP method and Barnett's Theorems of GCD of polynomials through Bézout matrices in order to find the GCD of a set of real polynomials.
To begin with,we introduce 2 theorems about the greatest common divisor of two polynomials and Bézout matrix and some typical numerical examples.Barnett's theorems provided for the first time an alternative to standard approaches based on the Euclidean algorithm, since the GCD can be found in a single step by solving a system of linear equations.

Theorem 5. *[5] Let $f(s)$ and $g(s)$ two polynomials in one variable as given in Definition 1. The greatest common divisor of the polynomials $f(s)$ and $g(s)$, denoted by $\gcd(f,g)$, is a polynomial with degree $\deg\{\gcd(f,g)\} \leq p$ such that*

$$\dim\{NullSpace\,(B(f,g))\} = \deg\{\gcd(f,g)\} = n - rank(B(f,g)) \qquad (12)$$

Theorem 6. *[5] If $c_1, c_2, ..., c_n$ are the columns of the Bézout matrix $B(f,g)$ with rank $n-k$, then*

i) the last $n-k$ columns, i.e. $c_{k+1}, \ldots, c_n$, are linearly independent, and *ii) every column c_i for $i = 1, 2, \ldots, k$ can be written as a linear combination of $c_{k+1}, \ldots, c_n$:*

$$c_{k-i} = \sum_{j=k+1}^{n} h_{k-i}{}^{(j)} c_j, \quad i = 0, 1, \ldots, k-1 \tag{13}$$

iii) There are $d_1, d_2, \ldots, d_k$ such that $d_j = d_k \cdot h_{k-j+1}^{(k+1)}$ and

$$\begin{bmatrix} d_k \\ d_{k-1} \\ d_{k-2} \\ \vdots \\ d_0 \end{bmatrix} = d_k \begin{bmatrix} 1 \\ h_k^{(k+1)} \\ h_{k-1}^{(k+1)} \\ \vdots \\ h_1^{(k+1)} \end{bmatrix} \tag{14}$$

with d_0 a non-zero real number.

Then, the GCD of the polynomials f and g, denoted by $\gcd(f, g)$, is

$$\gcd(f, g) = d_0 s^k + d_1 s^{k-1} + \ldots + d_{k-1} s + d_k \tag{15}$$

Remark 3. *[5] Let f, g be two polynomials of degree n and p, respectively, and let $k = \max\{n, p\}$. Then $\deg\{\gcd(f, g)\} = k - \operatorname{rank}(B(f, g))$ or equivalently $\operatorname{rank}(B(f, g)) = k - \deg\{\gcd(f, g)\} \leq k$. The equality holds when the polynomials are coprime. Otherwise, $\operatorname{rank}(B(f, g)) < k$, which means that the Bézout matrix is rank deficient.*

Now,we are going to see some examples of how Barnett's theorems apply to Bézout matrices.

Examples 3. *i) We have the following set of polynomials:*

$$f(x) = x^2 - 1$$

$$g(x) = x - 1$$

$$f(x) = x^2 - 1 = (x-1)(x+1) = 1x^2 - 0x - 1$$

,$m = 2$

$$g(x) = x - 1 = 0x^2 + 1x - 1$$

,$n = 1$ their GCD is $x - 1$

$$B(f, g) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Nullspace[B(f,g)] = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$rankB(f,g) = 1$$
$$dim[Nullspace] = 1$$
$$deg(GCD) = 2 - 1 = 1$$
$$deg(GCD) = m - rank(B(f,g)) = dim[NullspaceB(f,g)]$$

.

ii)

$$f(x) = x^3 - 8 = (x-2)(x^2 + 2x + 2) = 1x^3 + 0x^2 + 0x - 8$$
$$g(x) = x^2 - 4 = (x-2)(x+2)$$

$m = 3$, $n = 2$ *their GCD is* $x - 2$

$$B(f,g) = \begin{bmatrix} 0 & 8 & -4 \\ 8 & -4 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$Nullspace[B(f,g)] = \begin{bmatrix} 0.8729 \\ 0.4364 \\ 0.2182 \end{bmatrix}$$

$$rankB(f,g) = 2$$
$$dim[Nullspace] = 1$$
$$deg(GCD) = 3 - 2 = 1$$
$$deg(GCD) = m - rank(B(f,g)) = dim[NullspaceB(f,g)]$$

.

iii)

$$f(x) = (x-1)^3 = 1x^3 - 3x^2 + 3x - 1$$
$$g(x) = (x-1)(x+2)(x+3) = 1x^3 + 4x^2 + 1x - 6$$

$m = 3$, $n = 3$ *their GCD is* $x - 1$

$$B(f,g) = \begin{bmatrix} -17 & 22 & -5 \\ 22 & -20 & -2 \\ -5 & -2 & 7 \end{bmatrix}$$

$$Nullspace[B(f,g)] = \begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}$$

$$rankB(f,g) = 2$$

$$dim[Nullspace] = 1$$

$$deg(GCD) = 3 - 2 = 1$$

$$deg(GCD) = m - rank(B(f,g)) = dim[NullspaceB(f,g)]$$

.

The previous examples demonstrate the application of the first theorem,the following examples are about the second theorem.

i)

$$f(x) = x^2 - 1 = 1x^2 + 0x - 1, m = 2$$

and

$$g(x) = x - 1 = 0x^2 + 1x - 1, n = 1$$

we have:

$$B(f,g) = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

as follows:

$$c_1 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

and

$$c_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

moreover

$$rank(B(f,g)) = 1 = m - k = 2 - 1$$

as follows

$$k = 1$$

- $m - k = 2 - 1 = 1$*column,thus* c_2 *is linearly independent*
- *the* c_1 *column is linear combination of* c_2*, because*

$$c_1 = (-1) \cdot c_2$$

We observe that $c_j = c_2$ *and* ${h_1}^{k+1} = -1$.
because of this:

$$\begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = d_0 \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Thus

$$GCD(f,g) = d_0x - d_0$$

and $d_0 \in \mathbb{R}^*$, *we choose* $d_0 = 1$ *in order to be a monic polynomial and finally we have:*

$$GCD(f,g) = x - 1$$

.

ii)

$$f(x) = x^3 - 8 = (x-2)(x^2+2x+2) = 1x^3 + 0x^2 + 0x - 8, m = 3$$

and

$$g(x) = x^2 - 4 = (x-2)(x+2) = 0x^3 + 1x^2 + 0x - 4, n = 2$$

so:

$$B(f,g) = \begin{bmatrix} 0 & +8 & -4 \\ +8 & -4 & 0 \\ -4 & 0 & +1 \end{bmatrix}$$

as follows:

$$c_1 = \begin{bmatrix} 0 \\ +8 \\ -4 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} +8 \\ -4 \\ 0 \end{bmatrix}$$

and

$$c_3 = \begin{bmatrix} -4 \\ 0 \\ +1 \end{bmatrix}$$

moreover

$$rank(B(f,g)) = 2 = m - k = 3 - 1$$

thus

$$k = 1$$

- *the last* $m-k=3-1=2$ *columns* c_2, c_3 *are linearly independent. Indeed* $\lambda_1,\ \lambda_2$ *:*

$$\lambda_1 \cdot c_2 + \lambda_2 \cdot c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 \cdot \begin{bmatrix} +8 \\ -4 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} -4 \\ 0 \\ +1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4\lambda_1 = 0$$

$$\lambda_2 = 0$$

then

$$\lambda_1 = \lambda_2 = 0$$

- *column* c_1 *is linear combination of* c_2, c_3*, because*

$$c_1 = -2 \cdot c_2 - 4 \cdot c_3$$

We observe that $c_j = c_2, c_{j+1} = c_3$ *and* ${h_1}^{k+1} = -2, {h_2}^{k+1} = -4$*. due to this:*

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = d_0 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}$$

Thus

$$GCD(f,g) = d_0 x - 2d_0$$

because the degree should be $k=1$ *according to the theorem,we choose* $d_0 = 1$ *for the same reason as the previous example and we have:*

$$GCD(f,g) = x - 2$$

.

iii)

$$f(x) = (x-1)^3 = 1x^3 - 3x^2 + 3x - 1, m = 3$$

and

$$g(x) = (x-1)(x+2)(x+3) = 1x^3 + 4x^2 + 1x - 6, n = 3$$

ås follows:

$$B(f,g) = \begin{bmatrix} -17 & 22 & -5 \\ 22 & -20 & -2 \\ -5 & -2 & +7 \end{bmatrix}$$

We have:

$$c_1 = \begin{bmatrix} -17 \\ +22 \\ -5 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} +22 \\ -20 \\ -2 \end{bmatrix}$$

and

$$c_3 = \begin{bmatrix} -5 \\ -2 \\ +7 \end{bmatrix}$$

we find

$$rank(B(f,g)) = 2 = m - k = 3 - 1$$

thus

$$k = 1$$

- *the last* $m - k = 3 - 1 = 2$ *columns, thus* c_2, c_3 *are linearly independent. Indeed* $\lambda_1,\ \lambda_2$ *:*

$$\lambda_1 \cdot c_2 + \lambda_2 \cdot c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 \cdot \begin{bmatrix} +22 \\ -20 \\ -2 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} -5 \\ -2 \\ +7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

  and solving this linear system

$$\lambda_1 = \lambda_2 = 0$$

- *column* c_1 *is linear combination of* c_2, c_3*, because*

$$c_1 = -1 \cdot c_2 - 1 \cdot c_3$$

We have $c_j = c_2, c_{j+1} = c_3$ *and* ${h_1}^{k+1} = -1, {h_2}^{k+1} = -1.$
Thus:

$$\begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = d_0 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

As follows

$$GCD(f,g) = d_0 x - d_0$$

working similarly with the previous examples $d_0 = 1$ *and*

$$GCD(f, g) = x - 1$$

.

An important issue arising from Theorem 6 is the determination of the coefficients of the GCD of the entire set of polynomials.

3.2 The GCD of Polynomials via The QRCP method

In this section, exploiting the rank deficiency property of the Bézout matrix when a non-trivial GCD exists, we propose the application of the rank revealing QR factorization to a Bézout matrix.

Theorem 7. *[2] Let $B \in \mathbb{R}^{n\times n}$ and $\operatorname{rank}(B) = r < n$, where B is a Bézout matrix as defined in (1). Then, there always exist a permutation matrix Π of order n and a $n \times n$ orthogonal matrix Q*

$$Q^T B\Pi = R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \qquad (16)$$

$$r \quad n-r$$

where R_{11} is an $r\times r$ upper triangular matrix with non-zero diagonal elements. Furthermore, if $B\Pi = [\widehat{b}_{c_1}, \widehat{b}_{c_2}, \ldots, \widehat{b}_{c_n}]$ and $Q = [q_1, \ldots, q_n]$ presented in column form, then

$$\widehat{b}_{c_k} = \sum_{i=1}^{\min\{r,k\}} r_{ik} q_i \in \operatorname{span}\{q_1, \ldots, q_r\}, \quad k = 1, 2, \ldots, n \qquad (17)$$

which implies that $\operatorname{range}(B) = \operatorname{span}\{q_1, \ldots, q_r\}$.

Remark 4. *[2]*

Considering the values of r_{ik} in (17) as the values of $h_{k-1}^{(j)}$ in (13), we can directly obtain the coefficients d_i of the $\gcd(f, g)$ *through the Bézout-QRCP method. (This is fully demonstrated in Example 1). The application of the QRCP method to Bézout matrices simultaneously reveals the rank and an orthogonal base for the range of the Bézout matrix. Thus, by following Theorem 6 the coefficients of the GCD can easily be determined in a more efficient way.*

Remark 5. *Theorems 5, 6, and 7 also hold for the generalized Bézout matrix.*

Now,we are demonstrating a pseudo code of GCD computation through QRCP method,[2]

Input: Real polynomials u(x) and v(x) of degrees n and m with $n \geq m$ tolerance ϵ.
Output:An GCD for u(x) and v(x).

1 Form the vectors u and v with the coefficients of the u(x) and v(x) respectively.

2 Compute B=Bez(u,v).

3 Perform QR decomposition with column pivoting in B ,BP=QR,where P is a permuted matrix.

4 Apply 5 and 6,find the rank of BP and the degree of the GCD of the polynomials u(x) and v(x).

5 Use 7 and solve an upper triangular linear system.

6 Compute the coefficients of GCD of the of the polynomials u(x) and v(x) from the solution of the previous step.

- *Bézout-Computation of the GCD through QR factorization with column pivoting -QRCP factorization,[2]*

  Since the $mn \times n$ Bézout B matrix is always rank deficient when a non-trivial GCD exists, it is more efficient to extract the coefficients h_i appeared in (13) using Remark 4, which indicates that the coefficients of the GCD of the polynomials can be derived from the QRCP factorization of the Bézout matrix. The complexity of the QRCP factorization is
$$O\left(2mn^2r - r^2(mn+n) + \frac{2r^3}{3}\right) \tag{18}$$
  flops , where r is the rank of B, which is less than the flops required by the classical QR factorization. The appropriate correspondence of the columns of the original and the permuted matrix, which reveal the GCD coefficients (Remark 4), is symbolically implemented. In the case where the rank deficiency of B is high, the Bézout-QRCP method becomes more efficient. When $m \simeq n$ the required flops are about $O\left(2n^3r - n^2r^2\right)$

3.3 Demonstrative Examples

The following examples demonstrates the steps of the current Bézout-QRCP method for computing the GCD of set of many polynomials,[2].

Examples 4. *i) We consider the pair of real univariate polynomials of degree 5:*

$$\mathcal{P}_{2,5} = \left\{ \begin{array}{lcl} p_1(s) & = & s^5 - 24s^4 + 208s^3 - 786s^2 + 1231s - 630 \\ p_2(s) & = & s^5 - 23s^4 + 195s^3 - 745s^2 + 1244s - 672 \end{array} \right\} \quad (19)$$

The exact GCD is $s^2 - 8s + 7$. The Bézout matrix of the given polynomials in the set $\mathcal{P}_{2,5}$ is

$$\begin{aligned} B = Bez\{p_1, p_2\} &= \begin{bmatrix} -1 & 13 & -41 & -13 & 42 \\ 13 & -145 & 185 & 1585 & -1638 \\ -41 & 185 & 3275 & -20345 & 16926 \\ -13 & 1585 & -20345 & 77615 & -58842 \\ 42 & -1638 & 16926 & -58842 & 43512 \end{bmatrix} \quad (20) \\ &= \begin{bmatrix} b_{c_1} & b_{c_2} & b_{c_3} & b_{c_4} & b_{c_5} \end{bmatrix} \end{aligned}$$

where b_{c_i}, $i = 1, 2, \ldots, 5$ are the columns of the initial Bézout matrix $B \in \mathbb{R}^{5\times 5}$.

The following factorization is achieved by applying the QR factorization with column pivoting (QRCP) to B, such that

$$B\,\Pi = Q\,R \quad (21)$$

where

$$\begin{aligned} Q &= \begin{bmatrix} -0.0001306 & 0.017252 & 0.12062 & 0.52198 & -0.84421 \\ 0.015928 & -0.23579 & -0.85628 & -0.32276 & -0.32674 \\ -0.20444 & 0.83472 & 0.029962 & -0.44344 & -0.25281 \\ 0.77995 & -0.13851 & 0.31873 & -0.46068 & -0.24225 \\ -0.5913 & -0.47767 & 0.38697 & -0.46314 & -0.24074 \end{bmatrix} \quad (22) \\ &= \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix} \end{aligned}$$

$$R = \begin{bmatrix} 99513 & -26543 & 2164.6 & -75109 & -26.384 \\ 0 & -2577.6 & 751.71 & 1881.4 & -55.567 \\ 0 & 0 & 2.6078 & -2.2362 & -0.37162 \\ 0 & 0 & 0 & 7.2816\cdot 10^{-12} & 2.0961\cdot 10^{-14} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (23)$$

and

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{24}$$

After applying the QRCP factorization, the permuted Bézout matrix $B_{perm} = B \cdot \Pi$ *is*

$$B_{perm} = \begin{bmatrix} -13 & -41 & 13 & 42 & -1 \\ 1585 & 185 & -145 & -1638 & 13 \\ -20345 & 3275 & 185 & 16926 & -41 \\ 77615 & -20345 & 1585 & -58842 & -13 \\ -58842 & 16926 & -1638 & 43512 & 42 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} \widehat{b}_{c_1} & \widehat{b}_{c_2} & \widehat{b}_{c_3} & \widehat{b}_{c_4} & \widehat{b}_{c_5} \end{bmatrix} = \begin{bmatrix} b_{c_4} & b_{c_3} & b_{c_2} & b_{c_5} & b_{c_1} \end{bmatrix}$$

The lowest right 2×2 *part of* R *is considered to be zero and, thus, QRCP indicates that* $r = \text{rank}(B) = 3$ *and* $\deg\{\gcd(\mathcal{P}_{2,5})\} = 5 - 3 = 2$.

From Theorem 6 we know that the last 3 columns of the initial Bézout matrix B *in (20), i.e.* b_{c_3}, b_{c_4}, *and* b_{c_5}, *are linear independent. Therefore, the first two columns of* B, b_{c_1} *and* b_{c_2}, *can be written as a linear combination of* b_{c_3}, b_{c_4} *and* b_{c_5}. *Thus, from (13) in Theorem 6 we have:*

$$b_{c_2} = h_2^{(3)} b_{c_3} + h_2^{(4)} b_{c_4} + h_2^{(5)} b_{c_5} \tag{26}$$

$$b_{c_1} = h_1^{(3)} b_{c_3} + h_1^{(4)} b_{c_4} + h_1^{(5)} b_{c_5} \tag{27}$$

Let $d_0 s^2 + d_1 s + d_2$ *be the GCD of the polynomials. The coefficients* $h_2^{(3)}$ *and* $h_1^{(3)}$ *give the coefficients* d_1 *and* d_0, *respectively, and the constant term* d_2 *is 1.*

Using QRCP, the coefficients $h_2^{(3)}$ *and* $h_1^{(3)}$ *of the GCD are derived from the correspondence of the columns of* B *and* B_{perm}. *According to Theorem 7, the columns* q_1, q_2 *and* q_3 *of* Q *generate the range of* B_{perm}. *From (17) we have:*

$$\begin{aligned} \widehat{b}_{c_1} &= b_{c_4} = R_{11}\, q_1 \\ \widehat{b}_{c_2} &= b_{c_3} = R_{12}\, q_1 + R_{22}\, q_2 \\ \widehat{b}_{c_3} &= c_{c_2} = R_{13}\, q_1 + R_{23}\, q_2 + R_{33}\, q_3 \\ \widehat{b}_{c_4} &= b_{c_5} = R_{14}\, q_1 + R_{24}\, q_2 + R_{34}\, q_3 \\ \widehat{b}_{c_5} &= b_{c_1} = R_{15}\, q_1 + R_{25}\, q_2 + R_{35}\, q_3 \end{aligned} \tag{28}$$

Since the columns b_{c_2} *and* b_{c_1} *of the initial Bézout matrix* B *correspond to* $\widehat{b}_{c_3}$ *and* $\widehat{b}_{c_5}$ *of the permuted Bézout matrix* B_{perm}*, respectively, it is necessary to express the columns* $\widehat{b}_{c_3}$ *and* $\widehat{b}_{c_5}$ *as linear combinations of the columns* $\widehat{b}_{c_1}$*,* $\widehat{b}_{c_2}$ *and* $\widehat{b}_{c_4}$*. Since each column* q_i*,* $i = 1, 2, 3$ *is given by an analytic formula as the solution of the lower triangular system, formed from the first, the second, and the fourth equation of (28), we symbolically substitute in the third and the fifth equation of (28) and we obtain:*

$$\widehat{b}_{c_3} = R_{13}\, q_1 + R_{23}\, q_2 + R_{33}\, q_3 \quad (29)$$

$$\widehat{b}_{c_5} = R_{15}\, q_1 + R_{25}\, q_2 + R_{35}\, q_3 \quad (30)$$

Therefore, we conclude that

$$\widehat{b}_{c_3} = -1.14282712402397\,\widehat{b}_{c_2} - 1.16326188998929\,\widehat{b}_{c_1} - 1.16617476075485\,\widehat{b}_{c_4}$$
$$\widehat{b}_{c_5} = 0.142855765690511\,\widehat{b}_{c_2} + 0.163268401615028\,\widehat{b}_{c_1} + 0.166183704498703\,\widehat{b}_{c_4}$$

and from the correspondence of the columns of B *and* B_{perm} *we have:*

$$b_{c_2} = \widehat{b}_{c_3} = -1.14282712402397\, b_{c_3} - 1.16326188998929\, b_{c_4} - 1.16617476075485\, b_{c_5}$$
$$b_{c_1} = \widehat{b}_{c_5} = 0.142855765690511\, b_{c_3} + 0.163268401615028\, b_{c_4} + 0.166183704498703\, b_{c_5}$$

Thus,

$$h_2^{(3)} = -1.14282712402397 \quad \textit{and} \quad h_1^{(3)} = 0.142855765690511$$

and we obtain the quadratic polynomial:

$$0.142855765690511\, s^2 - 1.14282712402397\, s + 1$$

If we convert it to a monic polynomial, dividing by 0.142855765690511*, we finally compute the GCD of the polynomials in* $\mathcal{P}_{2,5}$*. That is*

$$\gcd(\mathcal{P}_{2,5}) = 1.0\, s^2 - 7.999866988216918\, s + 7.000067481815496 \quad (31)$$

ii) We consider the pair of real univariate polynomials of degree 5:

$$\mathcal{P} = \left\{ \begin{array}{lcl} p_1(s) & = & s^5 + s^4 - 37s^3 + 16s^2 + 97s - 10 \\ p_2(s) & = & s^5 - 13s^4 + 53s^3 - 72s^2 + 45s - 50 \end{array} \right\}$$

The GCD is $s^2 - 7s + 10$*. The Bézout matrix of the given polynomials in the set* $\mathcal{P}$ *is:*

$$B = Bez\{p_1, p_2\} = \begin{bmatrix} 14 & -90 & 88 & 52 & 40 \\ -90 & 516 & -84 & -1266 & 180 \\ 88 & -84 & -3082 & 6986 & -2380 \\ 52 & -1266 & 6986 & -10084 & 1520 \\ 40 & 180 & -2380 & 1520 & 4400 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ *are the columns of the initial Bézout matrix* B.

The following factorization is achieved by applying the QR factorization with column pivoting to B*:*

$$B\,\Pi = Q\,R$$

where

$$Q = \begin{bmatrix} -0.00013064 & 0.017252 & 0.12062 & 0.52198 & -0.84421 \\ 0.015928 & -0.23579 & -0.85628 & -0.32276 & -0.32674 \\ -0.20444 & 0.83472 & 0.029962 & -0.44344 & -0.25281 \\ 0.77995 & -0.13851 & 0.31873 & -0.46068 & -0.24225 \\ -0.5913 & -0.47767 & 0.38697 & -0.46314 & -0.24074 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}$$

$$R = \begin{bmatrix} -1.9426 & 0.2052 & 0.7684 & -0.0949 & -0.0021 \\ 0 & 0.4812 & -0.1723 & 0.0229 & 0.0015 \\ 0 & 0 & -0.1402 & 0.0981 & -0.0140 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \Pi = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The permuted Bézout matrix $B_{perm} = B \cdot \Pi$ *after the QRCP factorization is*

$$B_{per} = \begin{bmatrix} 52 & 40 & 13 & -90 & 14 \\ -1266 & 180 & -145 & 516 & -90 \\ 6986 & -2380 & 185 & -84 & 88 \\ -10084 & 1520 & 1585 & -1266 & 52 \\ 1520 & 4400 & -1638 & 180 & 40 \end{bmatrix} = \begin{bmatrix} a_{c_4} & a_{c_5} & a_{c_3} & a_{c_2} & a_{c_1} \end{bmatrix}$$

Now the first Barnett's theorem indicates that $rank(B) = 3$ *and* $degree(GCD) = 5 - 3 = 2$.

From Theorem 6 we get that the last 3 columns of the initial Bézout matrix B, $c_3,\ c_4,\ c_5$, *are linear independent and thus the other two columns of* B, $c_1,\ c_2$, *can be written as a linear combination of* $c_3,\ c_4,\ c_5$, *as indicates formula 13 of Theorem 6:*

$$c_2 = h_2^{(3)} c_3 + h_2^{(4)} c_4 + h_2^{(5)} c_5$$

$$c_1 = h_1^{(3)} c_3 + h_1^{(4)} c_4 + h_1^{(5)} c_5$$

The coefficients $h_2^{(3)}$ *and* $h_1^{(3)}$ *give the coefficients of* x *and* x^2 *respectively, and the constant term of the GCD of the polynomials is 1 (formuls 13).*

The coefficients $h_2^{(3)}$ *and* $h_1^{(3)}$ *of the GCD will be derived from the correspondence of the columns of* B *and* B_{perm}. *Theorem 7 denotes that the columns* $q_1,\ q_2,\ q_3$ *of* Q *generate the range of* B. *Thus,* a_{c_2} *and* a_{c_1} *are written as linear combination of* $a_{c_3},\ a_{c_4},\ a_{c_5}$. *From equation 17 of Theorem 7 it holds that*

$$a_{c_1} = R_{11}q_1$$

$$a_{c_2} = R_{12}q_1 + R_{22}q_2$$

$$a_{c_3} = R_{13}q_1 + R_{23}q_2 + R_{33}q_3$$

Solving symbolically the previous under triangular system with respect to $q_1,\ q_2,\ q_3$ *and substituting to*

$$a_{c_4} = R_{14}q_1 + R_{24}q_2 + R_{34}q_3$$

and

$$a_{c_5} = R_{15}q_1 + R_{25}q_2 + R_{35}q_3$$

we conclude that

$$a_{c_2} = -0.697a_{c_3} - 0.8537a_{c_4} - 2.504a_{c_5}$$

and

$$a_{c_1} = 0.098a_{c_3} - 0.35099a_{c_4} + 0.0384a_{c_5}$$

From the correspondence of the columns of the initial and permuted matrices we get that

$$d_k h_2^{(3)} = -0.697$$

and

$$d_k h_1^{(3)} = 0.098.$$

Thus $GCD = 0.098x^2 - 0.697x + 1$ *and making the GCD monic polynomial by dividing with* 0.100 *we finally compute the greatest common divisor of the polynomials:*

$$GCD = 1.000x^2 - 7.1122x + 10.204$$

Thus

$$GCD = x^2 - 7x + 10$$

.

iii) Let us consider the next set of three univariate polynomials:

$$\mathcal{P}_{3,3} = \left\{ \begin{array}{lcl} p_1(s) & = & s^3 - 6s^2 + 11s - 6 \\ p_2(s) & = & s^3 - 7s^2 + 14s - 8 \\ p_3(s) & = & s^3 - 8s^2 + 17s - 10 \end{array} \right\} \tag{32}$$

of degree 3*. Their exact GCD is* $s^2 - 3s + 2$.

The generalized Bézout matrix of the given polynomials in the set $\mathcal{P}_{3,3}$ *is*

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -6 \\ 2 & -6 & 4 \\ 2 & -6 & 4 \\ -6 & 18 & -12 \\ 4 & -12 & 8 \end{bmatrix} = \begin{bmatrix} b_{c_1} & b_{c_2} & b_{c_3} \end{bmatrix} \tag{33}$$

where

$$B_1 = Bez\{p_1, p_2\} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -6 \\ 2 & -6 & 4 \end{bmatrix} \tag{34}$$

and

$$B_2 = Bez\{p_1, p_3\} = \begin{bmatrix} 2 & -6 & 4 \\ -6 & 18 & -12 \\ 4 & -12 & 8 \end{bmatrix} \tag{35}$$

and $c_1,\ c_2,\ c_3$ *are the columns of* B.

We apply the QRCP factorization to B*, such that*

$$B\,\Pi = Q\,R$$

where

$$\begin{array}{rcl} Q & = & \begin{bmatrix} -0.1195 & -0.9008 & 0.0782 & 0.1362 & 0.0739 & 0.3796 \\ 0.3586 & 0.1955 & -0.1646 & 0.2821 & -0.5820 & 0.6228 \\ -0.2390 & -0.0528 & -0.9655 & -0.0125 & 0.0871 & -0.0140 \\ -0.2390 & 0.1333 & 0.0583 & 0.9320 & 0.1821 & -0.1407 \\ 0.7171 & 0.0188 & -0.1214 & 0.0791 & 0.6675 & 0.1371 \\ -0.4781 & 0.3597 & 0.1285 & -0.1636 & 0.4118 & 0.6552 \end{bmatrix} \\ & = & \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix} \end{array} \tag{36}$$

$$R = \begin{bmatrix} 25.0998 & -16.7332 & -8.3666 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \textit{ and } \Pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{37}$$

where q_1, q_2, q_3, q_4, *and* q_5 *are the columns of* Q. *The lowest right* 5×2 *part of* R *is zero and thus, QRCP indicates that* $r = \text{rank}(B) = 1$. *The degree of the GCD is* $\deg\{\gcd(\mathcal{P}_{3,3})\} = 3 - r = 2$.

Theorem 6 denotes that the last column b_3 *of the initial Bézout matrix* B *in (33) is linear independent and the other columns* b_1 *and* b_2 *are multiples of* b_3. *Working similarly with Example 1 we conclude that:*

$$\gcd(\mathcal{P}_{3,3}) = s^2 - 3.0000s + 2.0000 \tag{38}$$

iv) *Let us consider the next set of three univariate polynomials:*

$$\mathcal{P}_{3,3} = \left\{ \begin{array}{rcl} p_1(s) & = & s^3 + 4s^2 + 5s + 2 \\ p_2(s) & = & s^3 - 4s^2 - 3s + 18 \\ p_3(s) & = & s^3 + 12s^2 + 45s + 50 \end{array} \right\} \tag{39}$$

of degree 3. *Their exact GCD is* $s + 2$.

The generalized Bézout matrix of the given polynomials in the set $\mathcal{P}_{3,3}$ *is*

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 8 & 8 & -16 \\ 8 & -24 & -80 \\ -16 & 80 & -96 \\ -8 & -40 & -48 \\ -40 & -168 & -176 \\ -48 & -176 & -160 \end{bmatrix} = \begin{bmatrix} b_{c_1} & b_{c_2} & b_{c_3} \end{bmatrix} \tag{40}$$

where

$$B_1 = Bez\{p_1, p_2\} = \begin{bmatrix} 8 & 8 & 16 \\ 8 & -24 & -6 \\ -16 & 80 & -96 \end{bmatrix} \tag{41}$$

and

$$B_2 = Bez\{p_1, p_3\} = \begin{bmatrix} -8 & -40 & -48 \\ -40 & -168 & -176 \\ -48 & -176 & -160 \end{bmatrix} \qquad (42)$$

and $c_1,\ c_2,\ c_3$ *are the columns of* B.

We apply the QRCP factorization to B, such that

$$B\,\Pi = Q\,R$$

where

$$Q = \begin{bmatrix} -5.8521e-002 & -1.1420e-001 & 3.5384e-001 & -2.0436e-001 & -6.9655e-001 \\ -2.9260e-001 & -1.9632e-001 & 8.0675e-001 & -6.1589e-002 & 1.0594e-001 \\ -3.5112e-001 & -8.7254e-001 & -3.3968e-001 & -1.3878e-017 & -2.7756e-016 \\ -1.7556e-001 & 3.2079e-002 & 9.9074e-002 & 9.6373e-001 & -1.2922e-001 \\ -6.4373e-001 & 2.4252e-001 & 4.2460e-002 & -1.2354e-001 & 4.9631e-001 \\ -5.8521e-001 & 3.5672e-001 & -3.1138e-001 & -1.0200e-001 & -4.9049e-001 \end{bmatrix}$$
$$= \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}$$

$$R = \begin{bmatrix} 273.408 & 252.8089 & 58.0524 \\ 0 & -62.6711 & -31.3366 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \Pi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (44)$$

where $q_1,\ q_2,\ q_3,\ q_4,$ *and* q_5 *and* q_6 *are the columns of* Q. *The lowest right* 5×2 *part of* R *is zero and thus, QRCP indicates that* $r = \text{rank}(B) = 2$. *The degree of the GCD is* $\deg\{\gcd(\mathcal{P}_{3,3})\} = 3 - 2 = 1$.

Theorem 6 denotes that the first column b_1 *of the initial Bézout matrix* B *is linear independent and the other columns* b_2 *and* b_3 *are multiples of* b_1. *Working similarly with Example 1 we conclude that:*

$$\gcd(\mathcal{P}_{3,3}) = 1.000s + 2.000 \qquad (45)$$

Remark 6. *From,Numerical Linear Algebra,[1], we know that the following system has this solution:* $\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{m2} \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & u_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Now,we have this system of equations

$$x_1 = \frac{b_1}{u_{11}}$$

.

we continue and we conclude that:

$$x_i = \frac{b_i - \sum_{j=i+1}^{m} u_{ij} r_j}{u_{ii}}$$

$$i = 2, 3, ...$$

As,we notice in 28 the solution of this system reminds us the backward substitution of a linear triangular system as we know from the Numerical Linear Algebra.We conclude that for the case of $n = 1, 2, 3$ *we use the same equations in order to solve the system immediately through symbolical packages.However,we are studying if we can expand the system of solutions for* $n \geq 4$.

3.4 Comparison of Methods

An alternative way of specifying the GCD of the polynomials is through the following theorem,[7]

Theorem 8. *Let B be the generalized Bézout matrix of $m+1$ polynomials. If $J\,B\,J = Q\,R$ is the QR factorization of $J\,B\,J$, where J a permutation matrix with ones in its anti-diagonal and zeros elsewhere, then the last non-zero row of R gives the coefficients of the GCD of the polynomials.*

Now,we are demonstrating a pseudo code of GCD computation through JBJ method

---

Input: Real polynomials u(x) and v(x) of degrees n and m with $n \geq m$ tolerance ϵ.
Output:An GCD for u(x) and v(x).

1 Form the vectors u and v with the coefficients of the u(x) and v(x) respectively.

2 Compute B=Bez(u,v).

3 Compute the JBJ matrix where J is a permuted matrix.

4 Compute the QR decomposition of JBJ :QR=JBJ.

5 Compute the coefficients of GCD of the of the polynomials u(x) and v(x) from the last non-vanishing column of R.

The following examples demonstrates the steps of the current Bézout-JBJ method for computing the GCD of set of many polynomials.

Examples 5. *i)*

$$f(x) = (x+1)^5 = 1x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1, m = 5$$

and

$$g(x) = (x-1)(x+1) = 0x^5 + 0x^4 + 0x^3 + 1x^2 + 0x - 1, n = 2$$

we have:

$$Bezout = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 5 & -1 & -5 \\ 1 & 5 & 9 & -5 & -10 \\ 0 & -1 & -5 & -15 & -11 \\ -1 & -5 & -10 & -11 & -5 \end{bmatrix}$$

$$J_5 B J_5 = \begin{bmatrix} -5 & 11 & -10 & 5 & -1 \\ 11 & -15 & 5 & -1 & 0 \\ -10 & 5 & 9 & -5 & 1 \\ 5 & -1 & -5 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.5703 & -0.4441 & -0.4159 & -0.3233 & 0.4472 \\ 0.7777 & -0.0328 & -0.2993 & -0.3233 & 0.4472 \\ -0.2592 & 0.7994 & 0.1770 & -0.2498 & 0.4472 \\ 0.0518 & -0.3964 & 0.7989 & 0.0441 & 0.4472 \\ 0 & 0.0739 & -0.2607 & 0.8524 & 0.4472 \end{bmatrix}$$

$$R = \begin{bmatrix} -19.2873 & 6.9994 & -2.2813 & 0.311114.2581 & \\ 0 & 13.5280 & -6.5813 & 1.2435 & -8.1903 \\ 0 & 0 & -1.8660 & 0.5928 & 1.2732 \\ 0 & 0 & 0 & 0.0735 & -0.0735 \\ 0 & 0 & 0 & 0 & -0.0000 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and now an alternative way of the GCD computation of 4 .

ii) *We consider the pair of real univariate polynomials of degree 5:*

$$\mathcal{P} = \left\{ \begin{array}{lcl} p_1(s) & = & s^5 + s^4 - 37s^3 + 16s^2 + 97s - 10 \\ p_2(s) & = & s^5 - 13s^4 + 53s^3 - 72s^2 + 45s - 50 \end{array} \right\}$$

The GCD is $s^2 - 7s + 10$. *The Bézout matrix of the given polynomials in the set* $\mathcal{P}$ *is:*

$$B = Bez\{p_1, p_2\} = \begin{bmatrix} 14 & -90 & 88 & 52 & 40 \\ -90 & 516 & -84 & -1266 & 180 \\ 88 & -84 & -3082 & 6986 & -2380 \\ 52 & -1266 & 6986 & -10084 & 1520 \\ 40 & 180 & -2380 & 1520 & 4400 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ *are the columns of the initial Bézout matrix* B.

Now the J permuted matrix is:

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.8411 & 0.4915 & -0.1609 & -0.0320 & 0.1554 \\ -0.2905 & -0.7583 & -0.4981 & -0.0565 & 0.2987 \\ 0.4549 & 0.4172 & -0.5699 & -0.0759 & 0.5371 \\ -0.0344 & -0.0961 & 0.6258 & 0.0336 & 0.7725 \\ -0.0076 & 0.0078 & -0.0981 & 0.9944 & 0.0369 \end{bmatrix}$$

$$R = \begin{bmatrix} -1.9426 & 0.2052 & 0.7684 & -0.0949 & -0.0021 \\ 0 & 0.4812 & -0.1723 & 0.0229 & 0.0015 \\ 0 & 0 & -0.1402 & 0.0981 & -0.0140 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the permuted matrix:

$$B_{per} = \begin{bmatrix} 4400 & 1520 & -2380 & 180 & 40 \\ 1520 & -10084 & 6986 & -1266 & 52 \\ -2380 & 6986 & -3082 & -84 & 88 \\ 180 & -1266 & -84 & 516 & -90 \\ 40 & 52 & 88 & -90 & 14 \end{bmatrix}$$

Now if we divide the last non vanishing row with -0.1402 we have that the gcd is

$$GCD = s^2 - 7s + 10$$

.

Remark 7. *The complexity of the previous factorization, [2],for a $mn \times n$ Bézout matrix is $O\big(2n^2(mn - \frac{n}{3})\big)$ flops and, if $m \simeq n$, the required flops are about $O(2n^4)$.*

Remark 8. *Remarks upon the computational complexity of the methods.[2]*

The Bézout QRCP exploits the rank deficiency $n-r$ of the matrices, which is equal to the degree of the GCD of the polynomials. Thus, the higher the GCD degree is (i.e higher rank deficiency of the Bézout matrix) the more efficient the method becomes. If the rank of the Bézout matrix r is significantly *less than the maximum degree n of the polynomials, then the complexity of the Bézout QRCP method is one order less comparing to the complexity of the classical Bézout QR. However,as we notice in the table above in section 3.4 if the rank deficiency is not high we can choose either the QRCP or the QR-JBJ,it depends on the problem that we are challenging.*

Example 3. *In the table below we summarize the results obtained regarding the numerical relative error for the computed GCD of the polynomial sets in Example 4*

Table 1: Numerical relative error for the GCD of Example 4 ii).

Algorithm	Tolerance	Rel. Error
Bézout-QRCP	$10^{-10} - 10^{-16}$	$O(10^{-13})$
Bézout-QR	$10^{-10} - 10^{-16}$	$O(10^{-12})$

The tolerance indicates the different levels of precision (numerical accuracy) where a number is considered to be zero. For the particular sets of polynomials a tolerance between 10^{-10} and 10^{-16} was selected .

3.5 Conclusions

We proposed the application of the QR factorization with column pivoting to a Bézout matrix in order to compute the coefficients of the GCD of sets of several polynomials in a more efficient way.We also presented an overview of the most frequently applied structured matrix-based representations:i) the Bézout QR. As the number of the polynomials in the set decreases the Bézout QRCP becomes more efficient. The Bézout QRCP exploits the rank deficiency $n-r$ of the matrices, which is equal to the degree of the GCD of the polynomials. Thus, the higher the GCD degree is (i.e higher rank deficiency of the Bézout matrix) the more efficient the method becomes. If the rank of the Bézout matrix r is significantly less than the maximum degree n of the polynomials, then the complexity of the Bézout QRCP method is one order less comparing to the complexity of the classical Bézout QR. However,as we notice in the table above in section 3.4 if the rank deficiency is not high we can choose either the QRCP or the QR-JBJ,it depends on the problem that we are challenging. The study of the approximate GCD case is also a topic of great interest. A thorough comparison among the existing methods and possible extension of the QRCP method to the approximate case is under consideration. Furthermore, a proper framework for the algebraic and geometric properties of the GCD of sets of many polynomials in a multidimensional space is currently under study in order to define and evaluate exact or approximate multivariate GCDs given by the QRCP method. This is a challenging problem for further research, because several real-time applications, such as image and signal processing, rely on GCD methods where multivariate polynomials (especially in two variables) are used.

Appendix

In this section,we present you the Matlab Codes that we use

```
function [Q,R]=qr_gs(A)

 [n,m]=size(A);
 R(1,1)=norm(A(:,1));
 Q(:,1)=A(:,1)/R(1,1);
 for k=2:m
    R(1:k-1,k)=Q(1:n,1:k-1)'*A(1:n,k);
    z=A(1:n,k)-Q(1:n,1:k-1)*R(1:k-1,k);
    R(k,k)=norm(z);
    Q(1:n,k)=z/R(k,k);
 end
 end

function [Q,R]=qrfq(A)
% function qrfq calls the function house1

 n=size(A);
 Q=eye(n(1),n(1));
 smin=min(n(1)-1,n(2));
 for k=1:smin
     if sum(abs(A(k:n(1),k)))~=0
       [u(k:n(1)),s]=house1(A(k:n(1),k));
            A(k,k)=s;
        else
            u(k:n(1))=[1;zeros(n(1)-k,1)];
        end

         for i=k+1:n(1)
            A(i,k)=u(i);
         end
      uk(k)=u(k);
        uu=u(k:n(1));
        b=2/(uu*uu');
        for j=k+1:n(2)
            sumi=0;
            for i=k:n(1)
                sumi=sumi+u(i)*A(i,j);
            end
```

```
            s=b*sumi;
            for i=k:n(1)
                A(i,j)=A(i,j)-s*u(i);
            end
        end

        for j=1:n(1)
           sumi=0;
            for i=k:n(1)
                sumi=sumi+u(i)*Q(i,j);
            end
            s=b*sumi;
            for i=k:n(1)
                Q(i,j)=Q(i,j)-s*u(i);
         end
    end

  end

  for j=1:n(2)
    for i=j+1:n(1)
         A(i,j)=0;
     end
  end
  Q=Q';
  s=min(n(1),n(2));
  R=zeros(size(A));
  R(1:s,1:s)=A(1:s,1:s);


function [u,s]=house1(x)
    n=length(x);
    m=max(abs(x));
    if m~=0
        u=x/m;
        suma=0;
        for i=1:n
            suma=suma+u(i)^2;
        end
        i=1;
        while u(i)==0
```

```
            i=i+1;
        end
        s=sign(u(i))*sqrt(suma);
        u(1)=u(1)+s;
        s=-m*s;
    else
       u=zeros(n,1);
        s=0;
    end

    R = A;
    Q = eye(m);

% I am going to use a permutation matrix.
    P = eye(n);

% Compute the norms.
    for i = 1 : n
      colnorm(i) = R(:,i)'*R(:,i)
    end

%Swapping procedure
    for i=1:n

    %Find max col norm
       maxcolnorm = colnorm(i); perms = i;
       for  j = i + 1 : n
           if (colnorm(j) $>$ maxcolnorm)
               perms = j;
               maxcolnorm = colnorm(j);
           end
       end

    %Break
       if ( colnorm(perms) == 0 )
           break;
       end

    %Swap P
      temp = P(:, i);
       P(:, i) = P (:, perms)
```

```
        P(:, perms) = temp

    %Swap R
        temp = R(:, i);
        R(:, i) = R (:, perms)
        R(:, perms) = temp

    %Swap colnorm
        colnorm = colnorm*P

    % Get the Householder vector from get_house.
        v = gethouse(R(:,i),i,m)

    % Apply the transformation to R from the left.
        R = R-v*(v'*R)

    % And also apply it to Q from the right.
        Q = Q - (Q*v)*v'

    %Norm downdate
        if i~=n
            colnorm(i+1:n) = colnorm(i+1:n) - $R(i, i+1 :
                n)^2$
        end

    end

% Get the Householder vector from get_house.
    v = gethouse(R(:,n),n,m)

% Apply the transformation to R from the left.
    R = R- v   *(v' * R)

% And also apply it to Q from the right.
    Q = Q - (Q * v)* v'
    R = R*P'; % put the columns back to its original
        order!

function [v] = gethouse(x,i,j)

% Initialization.
```

```
    n = length(x);
    v = zeros(n,1);

% Copy that part of x to be worked on to the
    corresponding positions in v.
    v(i:j) = x(i:j);

% Compute the proper Householder vector.
    v(i) = v(i) - norm(x(i:j));

% Normalize the result so that H = I - v*v'. Includes
    an error check for
% the trivial reflection.

    if ((v'*v) $>$ 0)
       $v = v*sqrt(2/(v'*v))$;
    end



    function B =bezoutmatrix(u,v)
    n=length(u)-1;
    m=length(v)-1;
    if m<n
    v=[zeros(1,n-m) v];
    end
    if m>n
    temp=u;
    u=v;
    v=temp;
    n=length(u)-1;
    m=length(v)-1;
    v=[zeros(1,n-m) v];
    end
     B=zeros(n);
     for i=1:n
     for j=1:n
     mij=min([i,n+1-j]);
    for k=1:mij
    B(i,j)=B(i,j)+u(j+1+k-1)*v(i+1-k) -u(i+1-k)*v(j+1+k
       -1);
```

```
end
end
  end
```

Also,there are two commands in Matlab : [Q,R]=qr(A), [Q,R,P]=qr(A) for the QR permutation of a A matrix without and with column pivoting respectively.

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