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Graduate Thesis
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Hardy and Rellich Inequalities on Riemannian Manifolds

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Abstract: The main subject of this graduate thesis is a study of Hardy and Rellich-type inequalities on Riemannian manifolds. The first section serves as a brief introduction to the theory of Riemannian manifolds. The second section offers some background material on Hardy and Rellich inequalities in the much-studied context of Euclidean spaces, aiming to prepare the reader for the third section, which deals with such inequalities in the context of Riemannian manifolds. Finally, Section 4 presents some semi-original results concerning vector fields.

Contents

1 Introduction to Riemannian Geometry

This first section aims to be a brief introduction to the theory of smooth manifolds and Riemannian geometry. The material presented here is standard knowledge covered in many textbooks, so no references will be given throughout. However, it should be noted that most of it has been adapted from [Lee1], [Lee2] and [Burs] (see references).

1.1 Topological Manifolds

The primary objective of this section is to construct generic spaces on which one can extend the concepts and methods of infinitesimal calculus. The cornerstone of calculus is the derivative, and a quick review of its definition reveals that the only thing that seems to be required to define derivatives is that the domain (and co-domain) has a locally linear structure, preferably over some field which is totally ordered (if we hope to have some sense of direction on the space) and complete (so that limits are well-defined). The only such field, up to isomorphism, is of course $\mathbb{R}$, and if we restrict our attention to the finite dimensional case, the choice of linear space is naturally $\mathbb{R}^n$. In addition, before we even think about differentiability, we need to establish a notion of continuity. The weakest structure one can establish on sets that allows us to address questions of continuity and locality is a *topology*.

Definition 1. Let S be a set. A *topology* $\mathcal{O}$ on S is a collection of subsets of S, such that:
(1) $\emptyset, S \in \mathcal{O}$,
(2) if $U, V \in \mathcal{O}$, then $U \cap V \in \mathcal{O}$,
(3) if $U_i \in \mathcal{O},\ i \in I$, then $\bigcup_{i\in I} U_i \in \mathcal{O}$, for any index set I.

The sets that belong to the topology are called *open sets*. A topology is by definition closed under unions and finite intersections. Once we have defined a topology $\mathcal{O}$ on a set S, the pair $(S, \mathcal{O})$ is called a *topological space*. Sometimes we may refer to S as the topological space when no confusion arises or when the topology need not be specified. As already mentioned, this is all that we need in order to define continuity.

Definition 2. Let $(S_1, \mathcal{O}_1)$, $(S_2, \mathcal{O}_2)$ be topological spaces. A map $\phi : S_1 \longrightarrow S_2$ is called *continuous* (with respect to $\mathcal{O}_1$ and $\mathcal{O}_2$), if $\phi^{-1}(\mathcal{O}_2) \subset \mathcal{O}_1$.

This is, in fact, one of several equivalent definitions for continuity. For our purposes, it is the most convenient. By ϕ^{-1} we mean the *preimage* of ϕ and not the inverse (ϕ need not even be invertible). The following is a basic and useful result that concerns the composition of continuous maps.

Theorem 1. If $\phi : S_1 \longrightarrow S_2$ and $\psi : S_2 \longrightarrow S_3$ are continuous maps, so is their composition $\psi \circ \phi : S_1 \longrightarrow S_3$.

Given a topological space $(S, \mathcal{O})$, one can construct a special topology on any subset G of S, inherited directly from $\mathcal{O}$. It is easy to check that $\mathcal{O}_G = \{U \cap G : U \in \mathcal{O}\}$ is indeed a topology, and it is called the *subset topology* inherited from S. It is of special importance because of the following result.

Theorem 2. Let $\phi : S_1 \longrightarrow S_2$ be a continuous map and $G \subset S_1$. If we equip G with the subset topology inherited from S_1, then the *restriction* $\phi|_G : G \longrightarrow S_2$ remains continuous.

The structure-preserving maps of topological spaces are the *homeomorphisms*. They are defined as follows.

Definition 3. Let $(S_1, \mathcal{O}_1)$, $(S_2, \mathcal{O}_2)$ be topological spaces. A map $\phi : S_1 \longrightarrow S_2$ is called a *homeomorphism* if it is bijective and both ϕ and ϕ^{-1} are continuous.

From a more intuitive perspective, a homeomorphism is a map that *continuously distorts* a topological space (no cutting or gluing allowed). They are of central importance in topology, since they preserve *all topological properties*, such as connectedness, compactness, etc. They actually preserve even the topology itself, in the sense that $\phi(\mathcal{O}_1) = \mathcal{O}_2$ and $\phi^{-1}(\mathcal{O}_2) = \mathcal{O}_1$, which is why they are also called *topological isomorphisms*.

We are now fully prepared to start dealing with *topological manifolds*, that is, *locally Euclidean* topological spaces.

Definition 4. A topological space $(M, \mathcal{O})$ is called a *topological manifold* of *dimension* n, if for each point $p \in M$, there is a neighbourhood $U \in \mathcal{O}$ of p and a homeomorphism $x : U \longrightarrow x(U) \subset \mathbb{R}^n$.

In other words, a topological manifold "looks like" Euclidean space around each of its points. Every pair (U, x) as above is called a *chart* and, in particular, the map $x : U \longrightarrow x(U)$ is called the *chart map*. If we break x down to its components $x = (x^1, ..., x^n)$, x^i is called the *i-th coordinate map*. Any collection of charts $\mathcal{A} = \{(U_i, x_i)\}$ such that $\bigcup_i U_i = M$ (i.e the collection covers the manifold) is called an *atlas*.

$\mathbb{R}^n$ has a lot of good topological properties, such as being *Hausdorf* and *second countable*. "Hausdorff" means that each point is considered topologically distinct from the others, in the sense that any two different points have disjoint open neighbourhoods. "Second countable" means that the topology can be produced from countably many sets under unions. One could naively assume that these properties are preserved on topological manifolds through the local homeomorphisms with open subsets of $\mathbb{R}^n$. Unfortunately, the homeomorphisms are only *local*, which means we can only draw weaker conclusions for topological manifolds (they only need to be T_1 and *first countable*). However, all the non-pathological cases that appear in practice are still Hausdorf and second countable, and every manifold from now on will be considered as such. From now on, we will also assume *connectedness*.

The ways that one can can construct charts on topological manifolds is, of course, all but unique. This means some charts may *overlap*. In order to switch between different charts, or, in other words, to *change coordinates*, we use the *chart transition maps*, which we define as follows.

Definition 5. Let M be a topological manifold and (U_1, x_1), (U_2, x_2) be two charts on M with $U_1 \cap U_2 \neq \emptyset$. The maps $x_2 \circ x_1^{-1} : x_1(U_1 \cap U_2) \longrightarrow x_2(U_1 \cap U_2)$ and $x_1 \circ x_2^{-1} : x_2(U_1 \cap U_2) \longrightarrow x_1(U_1 \cap U_2)$ are called the *transition maps* of (U_1, x_1) and (U_2, x_2).

Since the chart maps are by assumption homeomorphisms, the transition maps are always continuous.

1.2 Differentiable Manifolds

Let M be a topological manifold and $\gamma : I \longrightarrow M$ be a path on M (where I is an open interval in $\mathbb{R}$). Assume that in some chart region U_1 with corresponding chart map x_1, the image of the path $x_1 \circ \gamma$ is continuous. If (U_2, x_2) is any other chart that overlaps with the above chart, the image $x_2 \circ \gamma = (x_2 \circ x_1^{-1}) \circ (x_1 \circ \gamma)$ is then *guaranteed* to be continuous in the interval $\gamma^{-1}(U_1 \cap U_2)$, because transition maps are always continuous and the composition of continuous maps is continuous.

However, if $x_1 \circ \gamma$ is *differentiable*, there is absolutely no guarantee that $x_2 \circ \gamma$ in the overlapping region will be, unless of course we require that the transition map be differentiable. This motivates the following definitions.

Definition 6. Two charts (U_1, x_1) and (U_2, x_2) are called C^k*-compatible* if the corresponding transition maps $x_2 \circ x_1^{-1}$ and $x_1 \circ x_2^{-1}$ are C^k-differentiable functions in their respective domains of definition.

Definition 7. An atlas $\mathcal{A}$ is C^k*-compatible* if its charts are pairwise C^k-compatible.

Definition 8. A topological manifold equipped with a C^k-compatible atlas, $(M, \mathcal{O}, \mathcal{A})$, is called a C^k*-manifold*.

Again, when no confusion arises or when the topology and the C^k-compatible atlas need not be specified, we may simply refer to M as the C^k-manifold.

As already mentioned, one can choose charts in many ways, and thus one can obtain many atlases for the same manifold. The same is in general true, if we require that the atlases be C^k-compatible. We can, however, construct C^k-compatible atlases that are *maximal* with respect to inclusion, that is, they contain all the possible charts that they *can* contain, under the restriction that they remain C^k*-compatible*. Such an atlas is called a *maximal* C^k*-compatible atlas*. The following theorem, due to Whitney, will greatly simplify our treatment of differentiable manifolds.

Theorem 3. Let $k \geq 1$ be an integer. Any maximal C^k-compatible atlas of a topological manifold contains a C^∞-compatible subatlas.

This means that, without loss of generality, we can from now on assume that all differentiable manifolds are C^∞-manifolds. Such manifolds are also called *smooth*.

So far we have only defined continuity of maps between manifolds, which only depends on their topology. To define differentiability we need to use all the extra structure we have just introduced, essentially reducing the definition to differentiability of real multivariable functions.

Definition 9. Let $(M_1, \mathcal{O}_1, \mathcal{A}_1)$ and $(M_2, \mathcal{O}_2, \mathcal{A}_2)$ be smooth manifolds. A continuous map $\phi : M_1 \longrightarrow M_2$ is called C^k*-differentiable* if for all $(U_1, x_1) \in \mathcal{A}_1$ and $(U_2, x_2) \in \mathcal{A}_2$, the map $x_2 \circ \phi \circ x_1^{-1} : x_1(U_1 \cap \phi^{-1}(U_2)) \longrightarrow x_2(U_2 \cap \phi(U_1))$ is a C^k-differentiable function.

The smoothness of the transition maps ensures that this is indeed a good definition. Moreover, if someone wishes to *extend* either of the smooth atlases (assuming they are not maximal), they wouldn't ruin the differentiability of the map, because all new charts would be by assumption C^∞-compatible with the existing ones.

Similar to the homeomorphisms, the structure-preserving maps of smooth manifolds are the *diffeomorphisms*.

Definition 10. Let M_1 and M_2 be smooth manifolds. A map $\phi : M_1 \longrightarrow M_2$ that is bijective, smooth and has a smooth inverse is called a *diffeomorphism*.

Intuitively, a diffeomorphism is, again, a continuous distortion, but this time the distortion is required to be smooth as well, in the sense we just defined.

1.3 Multilinear Algebra

So far we have managed to equip manifolds with a differentiable structure, which allows the definition of differentiable maps. The next natural step is to define derivatives of differentiable maps, as well as a notion of a "tangent" vector space at each point of the manifold, as was the case in multi-variable calculus and elementary differential geometry. However, this is not possible unless we first study vector spaces in the more abstract context.

First we will quickly review some basic definitions. A *vector space* over $\mathbb{R}$ (or any other *field*) is a triple $(V, +, \cdot)$, where V is a set, $+ : V \times V \longrightarrow V$ is a binary operation that takes two elements of V to another element of V, and $\cdot : \mathbb{R} \times V \longrightarrow V$ is a binary operation that takes a real number and an element of V to another element of V, such that the usual axioms of *addition* and *scalar multiplication* are satisfied. The elements of a vector space are called *vectors*.

A map $\phi : V_1 \longrightarrow V_2$ between two vector spaces V_1 and V_2 is called *linear* if

$$\phi(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \phi(v_1) + \alpha_2 \phi(v_2)$$

for every $\alpha_1, \alpha_2 \in \mathbb{R}$ and $v_1, v_2 \in V$. Linear maps are extremely important in linear algebra. Moreover, it is easy to show that the composition of two linear maps is again linear. A linear map is also called a *homomorphism*.

Next, we present some more advanced concepts of linear algebra, which are necessary for our treatment of smooth manifolds. We denote $\mathrm{Hom}(V_1, V_2)$ the set of all linear maps from V_1 to V_2. Then we have the following important (yet easy) result.

Theorem 4. Let V_1, V_2 be vector spaces. Then $\text{Hom}(V_1, V_2)$, equipped with the vector operations of V_2, is also a vector space.

A special case of the above is the *dual space* V^* of a vector space V, defined to be the set of all *linear functionals* of V. In other words, $V^* = \text{Hom}(V, \mathbb{R})$. In view of the previous theorem, V^* equipped with the real addition and multiplication is a vector space. If any element $v \in V$ is called a vector, any element $\phi \in V^*$ is called a *covector*.

By using a vector space and its dual, one can define more general linear maps, called *tensors*.

Definition 11. Let V be a vector space. An $(r, s) -$ *tensor* T over V is a multilinear map $T : V^{*r} \times V^s \longrightarrow \mathbb{R}$.

Multilinear of course means linear in each of its arguments. So, in other words, a tensor is a multilinear map that takes r covectors and s vectors to a real number.

One of the most useful properties of vector spaces is that their elements can be expanded linearly in terms of a *basis*.

Definition 12. A *(Hamel) basis* of a vector space V is a subset $B \subset V$ such that for each $v \in V$, there exists a unique finite subset $\{v_1, ..., v_n\} \subset B$ and unique real numbers $\alpha_1, ..., \alpha_n \in \mathbb{R}$, such that $v = \alpha_1 v_1 + ... + \alpha_n v_n$.

If we accept the axiom of choice, it follows by Zorn's Lemma, that, in fact, *every* vector space has a basis. If the basis itself has finitely many elements, say n-many, we say that the vector space has *dimension* n (we write this as $\text{dim}V = n$). The real numbers $\alpha_1, ..., \alpha_n \in \mathbb{R}$ are then called the *components* of v with respect to the specific given basis.

Now, it is true that, given a basis for V, one can also construct a special basis for V^*.

Theorem 5. Given a basis $\{e_1, ..., e_n\}$ for a finite-dimensional vector space V, there exists a unique basis $\{\epsilon^1, ..., \epsilon^n\}$ for V^*, such that $\epsilon^i(e_j) = \delta^i_j$.

This is called the *dual basis* of V^* with respect to the basis $\{e_1, ..., e_n\}$ of V. Just as we can define the components of vectors and covectors with respect to a basis and its dual basis, we can also define the components of a tensor.

Definition 13. Let T be an (r, s)-tensor over an n-dimensional vector space V, with a chosen basis $\{e_1, ..., e_n\}$ and dual basis $\{\epsilon^1, ..., \epsilon^n\}$. The *components* of T are the $(r+s)^n$ real numbers $T^{i_1 \dots i_r}_{j_1 \dots j_s} = T(\epsilon_{i_1}, ..., \epsilon_{i_r}, e_{j_1}, ..., e_{j_s})$, where $i_1, ..., i_r, j_1, ..., j_s \in \{1, ..., n\}$.

Because of linearity, we can reconstruct the whole tensor by knowing only the components.

1.4 The Tangent Space

Manifolds in general do not have a vector space structure. However, having some kind of (at least local) vector space structure on the manifold is necessary if we wish to introduce a sense of direction while moving on it, let alone derivatives. In the simple case where the manifold is $\mathbb{R}^n$, the direction of motion along a continuously differentiable *curve* $\gamma : I \longrightarrow \mathbb{R}^n$ at a point $p = \gamma(t)$ is given by the *velocity vector* $\gamma'(t)$. A straightforward attempt to generalise this to arbitrary smooth manifolds fails immediately in the absence of a vector space structure, because the above derivative cannot be defined. One possible solution would be to define the velocity in terms of a chart (U, x) as $(x \circ \gamma)'(t)$, but this is not a good definition since it depends on the chart, and we could have two or more velocity vectors in transition areas. The way to get around this difficulty, on a first glance, may not seem very intuitive, but in fact it is more natural than introducing coordinates.

Definition 14. Let M be a smooth manifold and $\gamma : I \longrightarrow M$ an at least C^1 injective curve. The *velocity* of γ at the point $\gamma(t)$ is the linear functional $\gamma'(t) : C^\infty(M) \longrightarrow \mathbb{R}$, such that $\gamma'(t)[f] = (f \circ \gamma)'(t)$ for each $f \in C^\infty(M)$.

While this definition may seem complicated, it actually matches our perception of velocity better than the one based on coordinates. Before we are even taught about coordinates, we perceive our "velocity", that is, "how fast and in which direction we are moving", as the rate of change of some surrounding parameters along our trajectory. Which specific parameters is of no particular importance, and this is best left unspecified in the definition, and in this one, it actually is. So instead of thinking about velocity as "the rate of change of coordinates along a curve", we simply think of it as "the rate of change along a curve", without specifying a quantity of reference.

With the above in mind, we proceed in defining the *tangent space*.

Definition 15. Let M be a smooth manifold, and let $p \in M$. The *tangent space* of M at p is the set of all velocities $\gamma'(0)$ of any C^∞ curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$, $\epsilon > 0$, such that $\gamma(0) = p$.

Apart from being linear, each velocity vector $\gamma'(0)$ satisfies the *Leibniz rule*

$$\gamma'(0)[fg] = \gamma'(0)[f]g(p) + \gamma'(0)[g]f(p).$$

The opposite is also true: *any* linear functional $\xi_p : C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the Leibniz rule

$$\xi_p[fg] = \xi_p[f]g(p) + \xi_p[g]f(p)$$

can be written as the velocity vector of some curve passing through p at parameter value 0. This allows us to restate the definition of the tangent space from a more algebraic perspective.

Definition 16. Let M be a smooth manifold, and let $p \in M$. The *tangent space* T_pM of M at p is the set of all linear functionals $\xi_p : C^\infty(M) \longrightarrow \mathbb{R}$ satisfying the Leibniz rule $\xi_p[fg] = \xi_p[f]g(p) + \xi_p[g]f(p)$ for every $f, g \in C^\infty(M)$.

The two definitions are, of course, equivalent. The first one is more geometric while the second one is more algebraic. Either of them can be more useful than the other depending on the context in which we use them. In either case, the following is true (though proving it for the algebraic definition is much easier).

Theorem 6. T_pM equipped with pointwise real addition and multiplication is a vector space.

So we have managed to introduce the much wanted vector space structure to each point of the manifold. Unknowingly, we have actually accomplished a lot more. Each vector in this new structure is actually a *directional derivative.* When generalising from $\mathbb{R}^n$ to arbitrary smooth manifolds, *vectors survive as the directional derivatives that they induce.*

This becomes clearer once we represent a tangent vector in terms of a specific chart. Let $\gamma'(0) \in T_pM$ be a tangent vector, and (U, x) be a chart such that $p = \gamma(0) \in U$. Then, for each $f \in C^\infty(M)$,

$$\gamma'(0)[f] = (f \circ \gamma)'(0) = (f \circ x^{-1} \circ x \circ \gamma)'(0) = \frac{\partial (f \circ x^{-1})}{\partial x^i}(x(p)) \cdot (x^i \circ \gamma)'(0).$$

In the above, we have used the *Einstein summation convention*, where repeated up and down indices are summed over all possible values. Now, $(x^i \circ \gamma)'(0)$ is nothing but the derivative of the i-th coordinate map along the curve, which we shall denote $\dot{\gamma}_x^i(0)$. Also, $\frac{\partial(f \circ x^{-1})}{\partial x^i}(x(p))$ is the i-th partial derivative of f with respect to the chart map x, which we shall denote $(\partial f/\partial x^i)_p$. Then we can represent the tangent vector as

$$\gamma'(0) = \dot{\gamma}_x^i(0)\Big(\frac{\partial}{\partial x^i}\Big)_p.$$

Thus, each tangent vector $\xi_p \in T_pM$ can be represented in terms of a local chart (U, x) as

$$\xi_p = \xi_{p,x}^i \Big(\frac{\partial}{\partial x^i}\Big)_p,$$

which looks and behaves exactly like a directional derivative. Moreover, it is clear that $\{(\partial/\partial x^1)_p, ..., (\partial/\partial x^n)_p\}$ constitutes a basis for T_pM, called the *chart induced basis* of T_pM with respect to the chart (U, x). It follows immediately that dim T_pM = dim $M = n$. Each $\xi_{p,x}^i$ is then called the *i-th component* of ξ_p with respect to the chart induced basis. Whenever no confusion arrises or whenever we need not refer to any specific chart, we will use the abbreviation $(\partial_i)_p$ for the i-th basis vector.

Of course, each chart induces a different basis, so in cases we have overlapping charts, we need to know how vector components and bases transform. Let (U, x) and (V, y) be two overlapping charts, $p \in U \cap V$ and $\xi_p \in T_pM$. For the basis vectors, straightforward application of the chain rule yields

$$\Big(\frac{\partial}{\partial x^i}\Big)_p = \Big(\frac{\partial y^j}{\partial x^i}\Big)_p \Big(\frac{\partial}{\partial y^j}\Big)_p.$$

If we choose to represent the vector as $\xi_p = \xi_{p,x}^i \cdot (\partial/\partial x^i)_p = \xi_{p,y}^i \cdot (\partial/\partial y^i)_p$ in the two bases respectively, it follows immediately that the components transform as

$$\xi_{p,x}^i = \Big(\frac{\partial x^i}{\partial y^j}\Big)_p \xi_{p,y}^j.$$

Next, it is useful to consider the dual of the tangent space, which we shall call *cotangent space.*

Definition 17. The vector space $T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R})$ is called the *cotangent space* of M at p. Its elements are then called *cotangent vectors.*

A special case of of cotangent vector is the *derivative* of a real valued function on a smooth manifold.

Definition 18. Let $\phi \in C^\infty(M)$. The *derivative* of ϕ at p is the cotangent vector $(d\phi)_p : T_pM \longrightarrow \mathbb{R}$, such that $(d\phi)_p(\xi_p) = \xi_p[\phi]$ for every tangent vector $\xi_p \in T_pM$.

Since T_p^*M is a vector space, it is natural to equip it with some basis that facilitates calculations in terms of a chart, just as we did for the tangent space. The most useful choice is the dual basis with respect to the chart induced basis of T_pM. It turns out that this basis is given in terms of the derivatives of the coordinate maps.

Theorem 7. Let M be a smooth manifold and (U, x) be a chart. Then $\{(dx^1)_p, ..., (dx^n)_p\}$ is the dual basis of T_p^*M with respect to the chart induced basis $\{(\partial/\partial x^1)_p, ..., (\partial/\partial x^n)_p\}$ of T_pM.

The next step is to find how cotangent vector components transform under change of chart. Again, straightforward calculation yields that if $\omega_p \in T_p^*M$, $(U, x), (V, y)$ are overlapping charts with $p \in U \cap V$ and $\omega_p = \omega_{p,x,i} \cdot (dx^i)_p = \omega_{p,y,i} \cdot (dy^i)_p$, then

$$\omega_{p,x,i} = \left(\frac{\partial y^j}{\partial x^i}\right)_p \omega_{p,y,j},$$

and the basis elements transform accordingly.

1.5 The Tangent Bundle

So now we have defined vectors at each point of a smooth manifold. Next, we could define *vector fields* by just assigning to each point of the manifold a tangent vector. While this isn't wrong, such a definition would lack the ability to define *smoothness* of a vector field, let alone *vector field differentiation*, which is vital for future progress. The only way to get around this is to introduce a *smooth bundle* structure that will smoothly attach tangent spaces to their respective points.

Definition 19. A *smooth bundle* is a triple (N, M, π), where N is a smooth manifold called the *total space*, M is another smooth manifold called the *base space* and $\pi : N \longrightarrow M$ is a smooth surjective map called the *projection map*.

Definition 20. Let (N, M, π) be a smooth bundle. The *fibre* of a point $p \in M$ is the preimage $\pi^{-1}\{p\}$.

Definition 21. Let (N, M, π) be a smooth bundle. A *section* is a map $\sigma : M \longrightarrow N$ such that $\pi \circ \sigma = \mathrm{id}_M$ (in other words, $\sigma(p)$ lies in the fibre of p).

We next consider the disjoint union of all tangent spaces of a smooth n-dimensional manifold $(M, \mathcal{O}, \mathcal{A})$,

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM.$$

We have chosen the disjoint union because we want to keep track of where each tangent vector comes from, for reasons that will become clear in an instant. With this definition, each element of TM is a pair $\xi = (p, \xi_p)$, where $p \in M$ and $\xi_p \in T_pM$.

Defining smooth vector fields requires that we find a way to smoothly project TM onto M. To this end, we define the *natural projection* $\pi : TM \longrightarrow M$, $(p, \xi_p) \longmapsto p$. We equip TM with the topology $\mathcal{O}' = \pi^{-1}(\mathcal{O})$ (which is the smallest topology such that π is continuous), and we introduce the smooth atlas $\mathcal{A}' = \{(TU, \chi_x) : (U, x) \in \mathcal{A}\}$, where $\chi_x : TU \longrightarrow \mathbb{R}^{2n}$ is the chart map such that

$$\chi_x((p, \xi_p)) = (x(p), (dx)_p(\xi_p)),$$

where $(dx)_p(\xi_p) = ((dx^1)_p(\xi_p), ..., (dx^n)_p(\xi_p))$. It then follows that $(TM, \mathcal{O}', \mathcal{A}')$ is a smooth manifold, and one can easily show that in this construction, π is a smooth surjective map. We sum up all the above in the following.

Theorem 8. Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth manifold and $(TM, \mathcal{O}', \mathcal{A}')$ as above. If $\pi : TM \longrightarrow M$ is the natural projection, then (TM, M, π) is a smooth bundle (called the *tangent bundle*).

Sometimes, even though erroneously, we may refer to the manifold TM itself as the "tangent bundle", assuming all the above structure, when no confusion arises. Likewise, we may refer to the elements of TM as "tangent vectors", and may even let them act on functions, even though they are not functionals, by taking the action of the vector part of the element, of course. These are common abuses of notation that are conventionally used in pretty much every textbook on the subject.

Finally, we consider the *cotangent bundle* of a smooth manifold M,

$$T^*M = \bigsqcup_{p\in M} T_p^*M = \bigcup_{p\in M} \{p\} \times T_p^*M.$$

As with the tangent bundle, we can equip T^*M with a natural projection π on M, a topology and a smooth atlas such that (T^*M, M, π) is again a smooth bundle. The procedure is similar to the one above, so it is omitted.

1.6 Fields

With the tangent bundle structure in mind, we are now ready to define (smooth) vector fields on a smooth manifold.

Definition 22. A *vector field* X on a smooth manifold M is a smooth section $X : M \longrightarrow TM$.

In other words, a vector field is a smooth map that takes a point of the manifold to a tangent element of the tangent space of the *same* point, that is, $X(p) \in \{p\} \times T_pM$.

The set $\Gamma(TM)$ of all vector fields on M is called the *vector field module* of M. Of course, $\Gamma(TM)$ is not called a "module" for nothing. It turns out that $C^\infty(M)$, equipped with point-wise real addition and multiplication, is in fact a *ring*. Then, the following important result justifies the name.

Theorem 9. Let M be a smooth manifold. Then $\Gamma(TM)$ is a $C^\infty(M)$-module.

A module is, roughly speaking, the equivalent of a vector space over a ring (instead of a field), excluding the axioms that don't make sense for rings, of course. It is a weaker structure than a vector space, but they have a lot in common. We have seen that all vector spaces have a basis, given that we accept the axiom of choice. A reasonable question is weather this is true for modules as well. The answer is, unfortunately, no. This means that there are cases where we cannot globally represent a vector field $X \in \Gamma(TM)$ in terms of components $X^1, ..., X^n$ with respect to some basis. To make matters worse, this does not only occur in pathological manifolds that are of no practical interest, but in very simple and fundamental cases as well, such as the spheres $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. This will not stop us, however, from expressing the *restriction* of a vector field X in some chart region in terms of the corresponding chart induced basis, that is,

$$X = X^i \partial_i,$$

where ∂_i is the local vector field such that $\partial_i(p) = (\partial_i)_p$ for each p in the chart region.

In the same way that we defined vector fields we can define *covector fields*, assuming the cotangent bundle structure, of course.

Definition 23. A *covector field* ω on a smooth manifold M is a smooth section $\omega : M \longrightarrow T^*M$.

It can be proved that the set of all covector fields $\Gamma(T^*M)$ is again a $C^\infty(M)$-module, called the *covector field module*.

Finally, we can generalise to *tensor fields*.

Definition 24. Let M be a smooth manifold. An (r, s)*-tensor field* T over M is a $C^\infty(M)$-multilinear map $T : \Gamma(T^*M)^r \times \Gamma(TM)^s \longrightarrow C^\infty(M)$.

1.7 Connections

In the formalism we have developed so far, the vectors are the directional derivatives of scalar functions. Extending this concept of differentiation to vector fields again requires new structure on the manifold, namely a *connection*.

Definition 25. A *connection* ∇ on a smooth manifold M is a map $\nabla : \Gamma(TM)^2 \longrightarrow \Gamma(TM)$, $(X, Y) \longmapsto \nabla_X Y$, such that for every $X_1, X_2, X, Y_1, Y_2, Y \in \Gamma(TM)$ and $f_1, f_2, f \in C^\infty(M)$:
(1) $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$,
(2) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$,
(3) $\nabla_X (fY) = f \cdot \nabla_X Y + X[f] \cdot Y$.

This is a reasonable definition since we would expect from a directional derivative to be linear in the subscript and additive in the argument, and we would also expect it to obey some kind of Leibniz rule in the argument.

The only problem is that such a map is not unique, so one needs to specify it before using it. To this end, it is again useful to introduce some kind of "components" (quote marks used since a connection is not a tensor field), at least locally. To do this, we assume a smooth manifold M with a connection ∇ and vector fields X, Y restricted on some chart region. Then

$$\begin{aligned}\nabla_X Y = \nabla_{X^i \partial_i}(Y^j \partial_j) &= X^i \partial_i Y^j \partial_j + X^i Y^j \nabla_{\partial_i} \partial_j \\ &= X^i \partial_i Y^j \partial_j + X^i Y^j \Gamma^k_{ij} \partial_k,\end{aligned}$$

where $\Gamma^k_{ij} = (\nabla_{\partial_i} \partial_j)^k$ are the *Christoffel symbols* of ∇ with respect to the specific chart. It is clear that specifying the connection is equivalent to specifying the Christoffel symbols for all charts. Direct calculation shows that if $(U, x), (V, y)$ are overlapping charts and the Christoffel symbols ${\Gamma_{(x)}}^k_{ij}$ are known in the first, the Christoffel symbols ${\Gamma_{(y)}}^k_{ij}$ with respect to the new chart must then satisfy the compatibility condition

$${\Gamma_{(y)}}^k_{ij} = \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^j} {\Gamma_{(x)}}^r_{pq} \frac{\partial y^k}{\partial y^r} + \frac{\partial y^k}{\partial y^m} \frac{\partial^2 x^m}{\partial y^i \partial y^j}$$

in the overlap region. Moreover, the above formula clearly shows that the Gammas are not the components of any tensor field, for they would transform differently.

1.8 Parallel Transport

Now that we have a way to differentiate vector fields, it is usefull to introduce the concept of *parallel transport*.

Definition 26. A vector field X on a smooth manifold M with a connection ∇ is said to be *parallely transported* along a smooth curve $\gamma : I \longrightarrow M$ if $\nabla_{\gamma'(t)} X = 0$ for all $t \in I$.

This is actually a very familiar concept. It means that a vector field remains "constant" while moving on some path. The tangent spaces change along the path, of course, so the tangent vectors can't ever be the same, but this is as close as we can get to the original idea. The best example is a *compass*. As long as we don't cross the poles, the direction (and of course, the length) of the needle remains "the same" no matter how we move.

If we use the velocity of the curve itself as the vector field, one can obtain a notion for *uniform straight motion*.

Definition 27. Let M be a smooth manifold with connection ∇. A curve $\gamma : I \longrightarrow M$ is called *auto-parallely transported* if $\nabla_{\gamma'(t)} \gamma'(t) = 0$ for all $t \in I$.

Auto-parallely transported curves are therefore the *straightest possible* curves that we can have on the manifold, with respect to the given connection, at least. One would expect, of course, that we should impose the weaker condition $\nabla_{\gamma'}\gamma' = \mu\gamma'$ to define straightness, but every smooth curve satisfying this condition also satisfies the first under some reparametrisation.

It is useful to have an expression of the auto-parallel equation on local coordinates. Let $\gamma' = \dot{\gamma}^i \partial_i$ be the local component representation of the velocity with respect to some chart. Straightforward substitution yields

$$\dot{\gamma}^i \partial_i \dot{\gamma}^k \partial_k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} \partial_k = 0.$$

Applying the chain rule and separating the components, it follows that

$$\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0.$$

1.9 Torsion and Curvature

Given a connection on a smooth manifold, we can define two important tensor fields, namely *torsion* and *curvature*. First we need the following definition.

Definition 28. Let X, Y be two vector fields on a smooth manifold. The *Lie bracket* of X and Y is the vector field $[X, Y]$ such that for every $\phi \in C^\infty(M)$,

$$[X, Y](\phi) = X(Y(\phi)) - Y(X(\phi)).$$

We first define torsion.

Definition 29. Let M be a smooth manifold with a connection ∇. The *torsion* of M with respect to ∇ is the (1,2)-tensor field $T : T^*M \times (TM)^2 \longrightarrow C^\infty(M)$ such that for every covector field ω and vector fields X, Y,

$$T(\omega, X, Y) = \omega(\nabla_X Y - \nabla_Y X - [X, Y]).$$

A smooth manifold is called *torsion-free* if its torsion is identically zero everywhere. Like every tensor field, we can locally represent it via its components T^k_{ij}.

Next we proceed to curvature.

Definition 30. Let M be a smooth manifold with a connection ∇. The *curvature* of M with respect to ∇ is the (1,3)-tensor field $R : T^*M \times (TM)^3 \longrightarrow C^\infty(M)$ such that for every covector field ω and vector fields X, Y, Z,

$$R(\omega, Z, X, Y) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z).$$

Likewise, we can locally express curvature via its components R^k_{mij}.

1.10 Riemannian Manifolds

So far, none of the structure we have introduced allows us to define length, area, or volume in a manifold. The easiest way to do this is to introduce some kind of inner product on the tangents spaces. This new structure is called a *Riemannian metric*.

Definition 31. A *Riemannian metric* g on a smooth manifold M is a (0,2)-tensor field $g : \Gamma(TM)^2 \longrightarrow C^\infty(M)$ that is symmetric and positive-definite, that is, for every $X, Y \in \Gamma(TM)$,
(1) $g(X,Y) = g(Y,X)$,
(2) $g(X,X) > 0$ iff $X \neq 0$.

Each pair (M,g) is called a *Riemannian manifold*. Sometimes we write $\langle X, Y\rangle$ instead of $g(X,Y)$. It is clear that a metric induces an inner product $g_p : T_pM^2 \longrightarrow \mathbb{R}$ on each tangent space, which we may also denote $\langle\cdot,\cdot\rangle_p$. This way we may also define a *norm* $|\cdot|_p$ on T_pM, such that $|\xi_p|_p = \langle\xi_p,\xi_p\rangle_p^{1/2}$. If $\xi = (p, \xi_p) \in TM$, we will assume $|\xi|$ to mean the norm of the vector part, $|\xi_p|_p$, following our line of conventions.

A very important and useful property of metrics is that they allow us to convert vectors to covectors and vice versa.

Definition 32. Let (M,g) be a Riemannian manifold. The map $\flat : \Gamma(TM) \longrightarrow \Gamma(T^*M)$, $X \longmapsto X^\flat(\cdot) = g(X,\cdot)$ is called the *flat map* of M with respect to g.

It follows that, in coordinates, $X^\flat = g_{ij}X^i dx^j$. This is sometimes called *lowering the indices*, because the components of the new covector are $X_j = (X^\flat)_j = g_{ij}X^i$, so in essence all we have done is use the metric to "lower" the component indices.

From the above and the fact that, by the definition, the matrix of g is invertible, it follows that the flat map is also invertible. We denote its inverse by $\sharp = \flat^{-1} : \Gamma(T^*M) \longrightarrow \Gamma(TM)$, which is called the *sharp map*. Therefore, if ω is a covector field, we obtain a vector field $\omega^\sharp$ with local components $\omega^i = (\omega^\sharp)^i = g^{ij}\omega_j$, where g^{ij} are the components of the inverse of g_{ij}. For the same reason, this is sometimes called *raising the indices*.

We may now proceed with defining length, and more specifically, *length of a curve*.

Definition 33. Let (M, g) be a Riemannian manifold and $\gamma : I \longrightarrow M$ be a smooth curve. The *length* of γ is the number

$$l[\gamma] = \int_I |\gamma'(t)| dt,$$

whenever this integral exists, of course.

We have seen that the straightest possible curves with respect to some connection are the auto-parallely transported curves. Likewise, we may now define the *shortest* (or longest, if they exist) possible curves with respect to a metric.

Definition 34. Let (M, g) be a Riemannian manifold. A curve $\gamma : I \longrightarrow M$ is called a *geodesic* if it is stationary with respect to l, that is, $\delta l[\gamma] = 0$.

A basic knowledge of variational calculus reveals that the local components of a geodesic must satisfy the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^i} - \frac{\partial \mathcal{L}}{\partial \gamma^i} = 0,$$

where $\mathcal{L}(\gamma, \gamma') = |\gamma'| = (g_{ij}\dot{\gamma}^i\dot{\gamma}^j)^{1/2}$. Solving these equations for each component yields the *geodesic equation*

$$\ddot{\gamma}^k + \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})\dot{\gamma}^i\dot{\gamma}^j = 0.$$

This equation has a striking resemblance to the auto-parallel equation in section 1.8. In any "reasonable" geometry, we would expect the notions of straightest and shortest curves to coincide. Given a Riemannian manifold, we can always equip it with a special connection that satisfies the above expectation by setting

$$\Gamma^k_{ij} \equiv \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

for each chart. This is called the *Riemannian* (or *Levi-Civita*) *connection* with respect to the given metric, and it is the standard connection on any Riemannian manifold. Since the Christoffel symbols uniquely determine the connection, the Riemannian connection is unique. Moreover, the following can be shown to be true.

Theorem 10. Let (M, g) be a Riemannian manifold and ∇ be the Riemannian connection of M with respect to g. Then ∇ is torsion-free and it is *compatible with the metric*, that is, for every $X, Y, Z \in \Gamma(TM)$,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Given a (connected, by assumption) Riemannian manifold (M, g), one can turn it into a metric space by equipping with the *distance function* $\rho : M \times M \longrightarrow [0, \infty)$ such that for each $p, q \in M$,

$$\rho(p, q) = \inf \{l[\gamma] : \gamma \in \mathcal{P}_M[p; q]\},$$

where $\mathcal{P}_M[p; q] = \{\gamma \in C^\infty((0, 1), M) \cap C([0, 1], M) : \gamma(0) = p, \gamma(1) = q\}$. It also turns out that the topology induced by ρ coincides with the initial topology of M.

1.11 Analysis on Riemannian Manifolds

We now have all the ingredients that we need in order to generalise the most familiar objects from multi-variable calculus, such as differential operations and integration. We begin by defining the *differential* of a map.

Definition 35. Let M, N be smooth manifolds and $\phi : M \longrightarrow N$ be a smooth map between them. The *differential* of ϕ at $p \in M$ is the map $(D\phi)_p : T_pM \longrightarrow T_{\phi(p)}N$, $\xi_p \longmapsto (D\phi)_p(\xi_p)$ such that $(D\phi)_p(\xi_p)[f] = \xi_p(f \circ \phi)$ for every $f \in C^\infty(N)$.

The differential is therefore a linear map between vector spaces as was the case in $\mathbb{R}^n$, and it behaves in a similar manner. Since we can identify $T_p\mathbb{R}^k \cong \mathbb{R}^k$, it is easy to check that in the trivial case where $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ the definition essentially coincides with the familiar one.

Next we proceed with the rest of the well-known differential operators. In what follows, let (M, g) be a Riemannian manifold equipped with the Riemannian connection.

Definition 36. The *gradient* of a function $\phi \in C^\infty(M)$ is the vector field

$$\nabla\phi = (d\phi)^\sharp.$$

In components, straightforward calculation reveals that

$$\nabla\phi = g^{ij}(\partial_i\phi)\partial_j.$$

Definition 37. The *divergence* of a vector field $Y \in \Gamma(TM)$ is the function

$$\text{div}Y = \text{trace}(X \longmapsto \nabla_X Y).$$

In components, we have

$$\text{div}Y = (\nabla_{\partial_i} Y)^i = \partial_i Y^i + \Gamma^i_{ij} Y^j$$

and we can easily check that this is well defined, that is, chart-independent.

Definition 38. The *Laplacian* of a function $\phi \in C^\infty(M)$ is the function

$$\Delta\phi = \text{div}\nabla\phi.$$

Again, in components we have

$$\Delta\phi = g^{ij}(\partial_i\partial_j\phi - \Gamma^k_{ij}\partial_k\phi).$$

By locally setting $|g| = \det(g_{ij})$, one can proove the usefull formulas

$$\text{div } Y = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}Y^i), \quad \Delta\phi = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j\phi).$$

Moreover, it is easy to prove the following *product rules*, which we are going to need later:

$$\nabla\phi^a = a\phi^{a-1}\nabla\phi,$$
$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi,$$
$$\text{div}(\phi X) = \langle\nabla\phi, X\rangle + \phi\text{div}X,$$

where $\phi, \psi \in C^\infty(M)$, $X \in \Gamma(TM)$.

Next, we wish to define integration. Unfortunately, this is not possible without imposing some additional requirement on the manifold, namely *orientability.*

Definition 39. Let M be a smooth manifold. M is called an *oriented* manifold if for every pair of overlapping charts $(U, x), (V, y)$, the transition maps both have positive Jacobians, that is, $\det(\partial y/\partial x) > 0$ and $\det(\partial x/\partial y) > 0$.

Now we can define integration on a chart region.

Definition 40. Let (M, g) be an oriented Riemannian manifold. The *integral* of a function $\phi : M \longrightarrow \mathbb{R}$ over a chart region U of a chart (U, x) is the number

$$\int_U \phi \, dv_g = \int_{x(U)} (\phi \circ x^{-1}) \sqrt{|g| \circ x^{-1}} \, d^n x.$$

If change the chart, the integral will not change, so it is well defined. However, orientability is crucial in the above definition. If the manifold was not oriented, changing the chart might change the sign of the integral.

To integrate over the whole manifold, one might think that we should just integrate over all chart regions and then add the integrals. This is wrong, of course, since this does not take into account the fact that charts overlap, so this way we count in some parts of the integral more than once. One way to get around this is via *partitions of unity.*

Definition 41. Let M be a smooth manifold and $\{U_\alpha\}$ be a *locally finite* open cover of M (each point belongs in finitely many sets of the cover), made of chart regions. A family of functions $\{\chi_\alpha : M \longrightarrow [0, 1]\}$ such that $\text{supp}\chi_\alpha \subset U_\alpha$ and $\sum_\alpha \chi_\alpha = 1$ is called a *partition of unity* on M.

Under the topological assumptions we made in section 1.1, a locally finite open cover and a partition of unity always exist.

With this definition in mind, we can now define integration on the whole manifold, since partitions of unity act as *weights* for each chart on any given point, which all add up to 1, meaning we do not overcount or omit anything.

Definition 42. Let (M, g) be an oriented Riemannian manifold and $\{\chi_\alpha\}$ be a partition of unity on M. The *integral* of a function $\phi : M \longrightarrow \mathbb{R}$ over M is the number

$$\int_M \phi \, dv_g = \sum_\alpha \int_{U_\alpha} \chi_\alpha \phi \, dv_g,$$

whenever the RHS is well-defined, of course.

It can be shown that the above definition is indeed a good one, that is, it doesn't depend on the locally finite open cover or the partition of unity.

Of significant importance are the following two results.

Theorem 11. For any compactly supported vector field $X \in \Gamma(TM)$,

$$\int_M \text{div} X dv_g = 0.$$

Theorem 12. For any $\phi, \psi \in C^\infty(M)$ such that at least one is compactly supported,

$$\int_M \phi \Delta \psi dv_g = -\int_M \langle \nabla \phi, \nabla \psi \rangle dv_g,$$

$$\int_M \phi \Delta \psi dv_g = \int_M \psi \Delta \phi dv_g.$$

The first result is a generalisation of the *divergence theorem* and the second is a generalisation of *Green's Identities*.

Since the main subject of this thesis has to do with functional inequalities, it would be extremely useful to have a generalisation of some well known functional inequalities for Riemannian manifolds. Now that we have integration, it is easy to define a *measure*.

Definition 43. Let (M, g) be an oriented Riemannian manifold. The *Hausdoff measure* for $(M, \mathcal{B}(M))$ is the measure $\mathrm{Haus} : \mathcal{B}(M) \longrightarrow [0, \infty]$ such that

$$\mathrm{Haus}(U) = \int_U dv_g$$

for any open subset $U \subset M$.

It is easy to check that the Hausdorff measure is indeed a measure on the Borel sets of the manifold and, therefore, $(M, \mathcal{B}(M)), \mathrm{Haus})$ is a measure space. As such, it inherits all the important results for measurable functions on measure spaces, including the *monotone* and *dominated convergence* theorems, *Fatou's Lemma* and *Hölder's inequality*, which will prove to be irreplaceable tools for our purposes.

2 Hardy and Rellich Inequalities on Euclidean Spaces

The main subject of this thesis is the study of a wide class of integral inequalities that involve a function and its derivatives up to some order, called Hardy and Rellich inequalities. Such inequalities often have the form

$$\int_\Omega X|u|^p \leq c \int_\Omega Y|\nabla^m u|^p$$

where u, X and Y are functions defined on a domain Ω, which for simplicity is assumed to be a domain of some Euclidean space (for the time being), and c is some positive constant. *"Hardy"* applies to the case $m = 1$, *"Rellich"* applies to the case $m = 2$. The other higher-order cases are usually obtained by recursively applying the former two cases, and usually fall under the general term *"higher-order Rellich inequalities"*.

This section aims to offer some background on some of the most important results in this area, and to prepare the reader for the more advanced results of the next section, which deals with such inequalities in the context of Riemannian manifolds.

2.1 Hardy Inequalities

Arguably, the most historic of the Hardy-type Inequalities is the one formulated by Hardy himself in the 1920s, and for this reason we present it first. It was originally stated in an integral form, which may not look the same as the form mentioned above, but it can easily be brought to it. It states the following.

Theorem 13 (Classic Integral Hardy Inequality)**.** Let $f \in L^p(0,\infty)$, $1 < p < \infty$ and $F(x) = \frac{1}{x}\int_0^x f(t)dt$. Then $F \in L^p(0,\infty)$ and

$$\int_0^\infty |F(x)|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p dx.$$

There are several proofs of this result. The one presented below is an adaptation from [BEL].

Proof. Let $p \in (1, \infty)$ and and let $q = \frac{p}{p-1}$ be its conjugate exponent. Also, let $\alpha \in (0, \frac{1}{q})$. Then by Hölder's inequality we have

$$|xF(x)| = \Big| \int_0^x f(t)t^{-\alpha}t^{\alpha}dt \Big| \le \Big[\int_0^x |f(t)t^{\alpha}|^p dt \Big]^{1/p} \Big[\int_0^x t^{-q\alpha}dt \Big]^{1/q}.$$

By evaluating the second integral in the RHS we get

$$|xF(x)| \le \Big[\int_0^x |f(t)|^p t^{p\alpha} dt \Big]^{1/p} \frac{x^{1/q-\alpha}}{(1-q\alpha)^{1/q}},$$

and since $x > 0$,

$$|F(x)| \le (1-q\alpha)^{-1/q} x^{-\alpha-1/p} \Big[\int_0^x |f(t)|^p t^{p\alpha} dt \Big]^{1/p}.$$

Exponentiating both sides by p and integrating over $\mathbb{R}^+$ yields

$$\int_0^\infty |F(x)|^p dx \le (1-q\alpha)^{-p/q} \int_0^\infty \int_0^x |f(t)|^p t^{p\alpha} x^{-p\alpha-1} dt dx.$$

Changing the order of integration in the RHS triangular integral we get

$$\int_0^\infty |F(x)|^p dx \le (1-q\alpha)^{-p/q} \int_0^\infty \int_t^\infty |f(t)|^p t^{p\alpha} x^{-p\alpha-1} dx dt,$$

thus

$$\int_0^\infty |F(x)|^p dx \le (1-q\alpha)^{-p/q} \int_0^\infty |f(t)|^p t^{p\alpha} \int_t^\infty x^{-p\alpha-1} dx dt,$$

and therefore

$$\int_0^\infty |F(x)|^p dx \le (1-q\alpha)^{-p/q} (p\alpha)^{-1} \int_0^\infty |f(t)|^p dt.$$

Since f is L^p integrable, it follows immediately that so is F. Moreover, choosing $\alpha = \frac{1}{pq} \in (0, \frac{1}{q})$, we get $(1-q\alpha)^{-p/q}(p\alpha)^{-1} = q^p$, and consequently

$$\int_0^\infty |F(x)|^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty |f(x)|^p dx.$$

This completes the proof. □

Before moving on, there are two points of interest that should be addressed. First, note that there is nothing special about the choice of the interval $(0, \infty)$. An identical inequality holds if we choose $(-\infty, 0)$ or $\mathbb{R} \setminus \{0\}$ instead. Likewise, we can obtain similar inequalities for any other interval excluding 0. The only thing that might change is the *constant*.

Second, the constant $(\frac{p}{p-1})^p$ that appears in the theorem is *sharp*, that is, it is the *smallest* constant for which the above inequality holds. In other words,

$$\Big(\frac{p}{p-1}\Big)^p = \inf_{f \in L^p(\mathbb{R}^+)} \frac{\int_0^\infty |F(x)|^p dx}{\int_0^\infty |f(x)|^p dx}.$$

Note, also, that the inequality holds as an equality only if $f \equiv 0$. The sharpness of the constant can be proven by choosing a suitable family of test functions (see [BEL]). This thesis, however, is more focused on the analytical methods used to obtain the inequalities themselves, so sharpness will be merely mentioned but not proven.

We now have the following corollary, which brings Hardy's Integral Inequality in a form that is in line with the one mentioned at the beginning of this section.

Corollary 13.1. Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be absolutely continuous with $f' \in L^p(0, \infty)$ for some $p \in (1, \infty)$ and $\lim_{x \to 0^+} f(x) = 0$. Then

$$\int_0^\infty \frac{|f(x)|^p}{|x|^p} dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^\infty |f'(x)|^p dx.$$

Next, we present a generalisation of the above inequality for functions in $\mathbb{R}^n$. The following result is also an adaptation from [BEL].

Theorem 14 (Hardy Inequality in $\mathbb{R}^n$)**.** Let $1 < p < \infty$, $n \in \mathbb{N} \setminus \{p\}$, and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable almost everywhere and $|x|^{n/p-1} f(x) \to 0$ as $|x| \to 0$ if $n < p$ *or* as $|x| \to \infty$ if $n > p$. Then

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} d^n x \leq \Big|\frac{p}{p-n}\Big|^p \int_{\mathbb{R}^n} \Big|\frac{x}{|x|} \cdot \nabla f(x)\Big|^p d^n x.$$

Proof. We switch to polar coordinates by setting $x = r\omega$, where $r = |x|$ and $\omega = x/|x| \in S^{n-1}$. By change of variables the LHS integral becomes

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} d^n x = \int_{S^{n-1}} \int_0^\infty r^{n-p-1} |f(r\omega)|^p dr d\omega.$$

Integrating by parts we get

$$\int_0^\infty r^{n-p-1}|f(r\omega)|^p dr = \Big[\frac{r^{n-p}}{n-p}|f(r\omega)|^p\Big]_{r=0}^\infty - \int_0^\infty \frac{r^{n-p}}{n-p}\frac{\partial}{\partial r}|f(r\omega)|^p dr.$$

Note that, by assumption, the first term of the RHS is non-positive in any case, so we are left with

$$\begin{aligned}\int_0^\infty r^{n-p-1}|f(r\omega)|^p dr &\le -\int_0^\infty \frac{r^{n-p}}{n-p}\frac{\partial}{\partial r}|f(r\omega)|^p dr\\ &\le \frac{1}{|p-n|}\int_0^\infty r^{n-p}\Big|\frac{\partial}{\partial r}|f(r\omega)|^p\Big|dr\\ &\le \Big|\frac{p}{p-n}\Big|\int_0^\infty r^{n-p}|f(r\omega)|^{p-1}\Big|\frac{\partial}{\partial r}f(r\omega)\Big|dr\\ &\le \Big|\frac{p}{p-n}\Big|\Big[\int_0^\infty r^{-p}|f(r\omega)|^p r^{n-1}dr\Big]^{1-1/p}\Big[\int_0^\infty \Big|\frac{\partial}{\partial r}f(r\omega)\Big|^p r^{n-1}dr\Big]^{1/p},\end{aligned}$$

where in the last step we use Hölder's inequality. Now we note that $\partial f/\partial r = x/|x|\cdot\nabla f$, and we integrate both sides over S^{n-1} with respect to ω. Finally, we seperate the RHS integrals applying Hölder's inequality once more, which yields

$$\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x|^p}d^nx \le \Big|\frac{p}{p-n}\Big|\Big[\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x|^p}d^nx\Big]^{1-1/p}\Big[\int_{\mathbb{R}^n}\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^p d^nx\Big]^{1/p},$$

thus

$$\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x|^p}d^nx \le \Big|\frac{p}{p-n}\Big|^p\int_{\mathbb{R}^n}\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^p d^nx.$$

This completes the proof. □

Once again, by a suitable choice of test functions it can be shown that the constant $|\frac{p}{p-n}|^p$ is sharp (see [BEL]).

It is interesting to note that the proof of such inequalities usually relies on other well known functional inequalities. For example, the proofs of the inequalities presented so far both rely heavily on *Hölder's* inequality, with its application being the key step, in fact.

Finally, the previous theorem has the following corollary.

Corollary 14.1. Let $1 < p < \infty$ and $n \in \mathbb{N} \setminus \{p\}$. The inequality

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} d^n x \leq \left|\frac{p}{p-n}\right|^p \int_{\mathbb{R}^n} |\nabla f(x)|^p d^n x$$

holds for all $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ if $n < p$ and for all $f \in C_c^\infty(\mathbb{R}^n)$ if $n > p$.

This last corollary assumes the form in which most of our results will be presented, from now on. By choosing to restrict our attention to compactly supported functions, we essentially eliminate the assumptions about the limit behaviour of the functions at infinity. Although this loses some of the generality of the main result, it is much easier to state and remember.

2.2 Rellich Inequalities

The classic Rellich inequality, formulated by Rellich himself in the 1950s, states that whenever $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and $n \neq 2$,

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^4} d^n x \leq \frac{16}{n^2(n-4)^2} \int_{\mathbb{R}^n} |\Delta u(x)|^2 d^n x.$$

Since Rellich's proof, numerous such inequalities have been proven, including weighted inequalities, improved inequalities with additive terms, as well as more general higher-order L^p Rellich inequalities.

The main objective of this subsection is to prove some particularly powerful results formulated by Davies and Hinz in [DH] that deal with local singularities in Euclidean spaces. In the original paper, which was initially set in a Riemannian manifold context, they are a consequence of several more general auxiliary results that are actually remarkable in their own right. However, for our purposes, it is convenient to sum up the progress of these results in a single lemma, in which the motivation behind each step of the proof becomes more clear.

Lemma 15 (Davies-Hinz)**.** Let $1 < p < \infty$. Also, let $\Omega \subset \mathbb{R}^n$ be open and let $V \in C^2(\Omega)$ be such that $V > 0$, $\Delta V < 0$ and $\Delta V^\delta \leq 0$ for some $\delta > 1$. Then the inequality

$$\int_\Omega |\Delta V||u|^p \leq \frac{p^{2p}}{(1+(p-1)\delta)^p} \int_\Omega \frac{V^p}{|\Delta V|^{p-1}} |\Delta u|^p$$

holds for all $u \in C_c^\infty(\Omega)$.

Proof. Let $u \in C_c^\infty(\Omega)$. Since $\Delta V < 0$,

$$\int_\Omega |\Delta V||u|^p = -\int_\Omega \Delta V |u|^p.$$

By Green's first identity and the fact that $\nabla |u|^p = p|u|^{p-2}u\nabla u$, it follows that

$$\int_\Omega |\Delta V||u|^p = \int_\Omega \nabla V \cdot \nabla |u|^p = p\int_\Omega |u|^{p-2}u\nabla V \cdot \nabla u$$
$$\leq p\int_\Omega |\nabla V||u|^{p-1}|\nabla u|,$$

and Hölder's inequality yields

$$\int_\Omega |\Delta V||u|^p \leq p\Big[\int_\Omega |\Delta V||u|^p\Big]^{1-1/p}\Big[\int_\Omega \frac{|\nabla V|^p}{|\Delta V|^{p-1}}|\nabla u|^p\Big]^{1/p},$$

therefore

$$\int_\Omega |\Delta V||u|^p \leq p^p \int_\Omega \frac{|\nabla V|^p}{|\Delta V|^{p-1}}|\nabla u|^p. \quad (1)$$

By Green's second identity, we have

$$\int_\Omega |\Delta V||u|^p = -\int_\Omega V\Delta |u|^p$$

$$= -\int_\Omega V(p|u|^{p-2}|\nabla u|^2 + p(p-2)|u|^{p-2}|\nabla u|^2 + p|u|^{p-2}u\Delta u)$$
$$= -\int_\Omega V(p(p-1)|u|^{p-2}|\nabla u|^2 + p|u|^{p-2}u\Delta u)$$

thus

$$\int_\Omega |\Delta V||u|^p + p(p-1)\int_\Omega V|u|^{p-2}|\nabla u|^2 \leq p\int_\Omega V|u|^{p-1}|\Delta u|. \quad (2)$$

Since, by assumption, $\Delta V^\delta = \delta(\delta-1)V^{\delta-2}|\nabla V|^2 + \delta V^{\delta-1}\Delta V \leq 0$ and $V > 0$, it follows that $(\delta-1)|\nabla V|^2 \leq V|\Delta V|$. We now apply inequality (1) for $p = 2$ to get

$$(\delta-1)\int_\Omega |\Delta V||u|^2 \leq 4(\delta-1)\int_\Omega \frac{|\nabla V|^2}{|\Delta V|}|\nabla u|^2 \leq 4\int_\Omega V|\nabla u|^2.$$

Replacing u by $|u|^{p/2}$ in the above inequality and taking into account that $|\nabla|u|^{p/2}|^2 = \frac{p^2}{4}|u|^{p-2}|\nabla u|^2$ yields

$$(\delta - 1)\int_\Omega |\Delta V||u|^p \leq p^2 \int_\Omega V|u|^{p-2}|\nabla u|^2. \quad (3)$$

Next we apply (3) and Hölder's inequality on (2) to get

$$\int_\Omega |\Delta V||u|^p + \frac{p-1}{p}(\delta - 1)\int_\Omega |\Delta V||u|^p \leq p\int_\Omega V|u|^{p-1}|\Delta u|$$
$$\leq p\Big[\int_\Omega |\Delta V||u|^p\Big]^{1-1/p}\Big[\int_\Omega \frac{V^p}{|\Delta V|^{p-1}}|\Delta u|^p\Big]^{1/p},$$

therefore

$$\int_\Omega |\Delta V||u|^p \leq \frac{p^{2p}}{(1+(p-1)\delta)^p}\int_\Omega \frac{V^p}{|\Delta V|^{p-1}}|\Delta u|^p.$$

This completes the proof. □

A few comments before we proceed to the main theorems. First, notice that, although not very unintuitive, the proof is significantly more complicated than the proofs of the Hardy-type inequalities that we presented in the previous subsection. When it comes to more modern results, this is the rule rather than the exception.

However, despite the increased complexity, notice that the main tools that were used remain essentially the same: integration by parts, Green's identities and other functional inequalities (particularly Hölder's). Since all these are still valid in a Riemannian manifold context (in fact, the above result was originally set in such a context in [DH]), they will continue to form the backbone of our "toolbox" later when we deal with manifolds.

Now we are ready to state and prove the main results.

Theorem 16 (Davies-Hinz, higher-order L^p Rellich inequality)**.** Let $1 < p < \infty$, $n, m \in \mathbb{N}$ and $\beta > 2$ such that $2+2(m-1)p < \beta < n$. Then the inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^\beta} d^n x \leq c(n,m,p,\beta)\int_{\mathbb{R}^n} \frac{|\Delta^m u(x)|^p}{|x|^{\beta-2mp}} d^n x$$

holds for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, where

$$c(n,m,p,\beta) = \prod_{k=0}^{m-1} \gamma(n, \beta - 2(1+kp), \frac{n-2}{\beta - 2(1+kp)}, p)$$

and

$$\gamma(n,m,p,\beta) = \frac{\beta^2}{m(n-2-m)((\beta-1)p+1)}.$$

Proof. Consider the function $V(x) = |x|^{-\sigma}$ in $\mathbb{R}^n \setminus \{0\}$. Straightforward calculation reveals that

$$\nabla V(x) = -\sigma |x|^{-\sigma-2} x, \quad \Delta V(x) = \sigma(\sigma - (n-2))|x|^{-\sigma-2}.$$

Hence, $\Delta V < 0$ is satisfied for $0 < \sigma < n-2$ and $\Delta V^\delta \leq 0$ is satisfied for $0 \leq \delta \leq (n-2)/\sigma$. In view of the previous lemma, for these values of σ and for $\delta = (n-2)/\sigma$, it follows that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{\sigma+2}} d^n x \leq c^p \int_{\mathbb{R}^n} \frac{|\Delta u(x)|^p}{|x|^{\sigma+2-2p}} d^n x$$

for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, where

$$c = \frac{p^2}{(n-2-\sigma)((p-1)(n-2)+\sigma)}.$$

Setting $\beta = \sigma + 2$ and inductively applying the above inequality completes the proof. □

Theorem 17 (Davies-Hinz, higher-order L^p Hardy-Rellich inequality)**.** Let $1 < p < \infty$, $n, m \in \mathbb{N}$ and $\beta > 2$ such that $2 + 2(m-1)p < \beta < n$. Then the inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^\beta} d^n x \leq \tilde{c}(n,m,p,\beta) \int_{\mathbb{R}^n} \frac{|\nabla \Delta^m u(x)|^p}{|x|^{\beta-(2m+1)p}} d^n x$$

holds for all $u \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, where

$$\tilde{c}(n,m,p,\beta) = \frac{p}{n+2mp-\beta} c(n,m,p,\beta).$$

Proof. Applying inequality (1) from the proof of Lemma 15 for $V(x) = |x|^{-\beta+2+2mp}$ to the previous theorem proves the assertion. □

Note that in proving these results one has to explicitly calculate the constants, essentially. We have omitted the details of these calculations, as they are trivial but rather lengthy to right down. However, the values of the constants that are obtained this way can be shown to be sharp (see [DH]).

2.3 Uncertainty Principles

To close this section, we would like to point out that the broader family of Hardy and Rellich-type inequalities is not restricted to the general form discussed so far. For example, the classic quantum *uncertainty principle* is obtained in a totally analogous manner, as we shall see. Similar inequalities have been formulated in various contexts with numerous applications in physics, statistics, information theory and other fields.

As a representative of this important class of inequalities, we present a variant of the classic Heisenberg-Pauli-Weyl (HPW) uncertainty principle in Euclidean spaces of arbitrary dimension.

Theorem 18 (HPW uncertainty principle)**.** Let $u \in C_c^\infty(\mathbb{R}^n)$. Then

$$\frac{n^2}{4}\Big[\int_{\mathbb{R}^n} |u(x)|^2 d^n x\Big]^2 \leq \int_{\mathbb{R}^n} |x|^2 |u(x)|^2 d^n x \int_{\mathbb{R}^n} |\nabla u(x)|^2 d^n x.$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^n)$. By the product rule $(\nabla \cdot x)u^2 = \nabla \cdot (u^2 x) - x \cdot \nabla u^2$ and the fact that $\nabla \cdot x = n$, it follows that

$$n \int_{\mathbb{R}^n} |u(x)|^2 d^n x = \int_{\mathbb{R}^n} (\nabla \cdot x) u^2(x) d^n x = -\int_{\mathbb{R}^n} x \cdot \nabla u^2(x) d^n x,$$

where in the last step we used the divergence theorem and the fact that u vanishes outside a sufficiently large bounded set. Thus

$$n \int_{\mathbb{R}^n} |u(x)|^2 d^n x = -2 \int_{\mathbb{R}^n} u(x) x \cdot \nabla u(x) d^n x \leq 2 \int_{\mathbb{R}^n} |x||u(x)||\nabla u(x)| d^n x.$$

Now we apply the Cauchy-Schwarz inequality to the RHS integral to get

$$\frac{n}{2} \int_{\mathbb{R}^n} |u(x)|^2 d^n x \leq \Big[\int_{\mathbb{R}^n} |x|^2 |u(x)|^2 d^n x\Big]^{1/2} \Big[\int_{\mathbb{R}^n} |\nabla u(x)|^2 d^n x\Big]^{1/2},$$

therefore

$$\frac{n^2}{4}\Big[\int_{\mathbb{R}^n} |u(x)|^2 d^n x\Big]^2 \leq \int_{\mathbb{R}^n} |x|^2 |u(x)|^2 d^n x \int_{\mathbb{R}^n} |\nabla u(x)|^2 d^n x.$$

This completes the proof. □

3 Hardy and Rellich Inequalities on Riemannian Manifolds

We are ready to start dealing with Hardy and Rellich inequalities in the context of Riemannian manifolds. But before we begin, let us make a few general remarks.

Since the inequalities of our interest are *integral* inequalities, we specialise the term "Riemannian manifold" to be such that integrals as presented in 1.11 are well defined. For the rest of this section, when we say, for example, that M is a Riemannian manifold, we assume that M is a connected, Hausdorff, second countable and orientable Riemannian manifold.

As already mentioned, we will continue to rely heavily on the divergence theorem and Green's identities, as well as on product rules of the usual differential operators. However, it is important to remember that these results still hold because we implicitly assume a Riemannian manifold to be equipped with the standard Riemannian connection. Otherwise, the picture could be very different.

Another thing that requires attention is that this subject lies in the intersection of Analysis and Differential Geometry. There are some minor "communication problems" between these two fields that need to be dealt with before we proceed any further. In particular, while matters of regularity are of significant (and sometimes central) importance for analysts, most differential geometers would rather consider everything to be C^∞-differentiable and turn their attention to other matters, as is evident in most Differential Geometry textbooks. The entire first section of the present is, in fact, written in such a fashion, although the author has tried not to toss regularity completely. However, a large part of the theory of smooth manifolds is, in a more or less obvious way, still valid for not-so-smooth functions and vector fields, and for the rest of this thesis, their assumed regularity will be explicitly stated.

Additionally, we may define differential operators in the *weak sense* as follows: we say that a function u on M has weak gradient ∇u if

$$\int_M \langle H, \nabla u \rangle dv_g = -\int_M u \mathrm{div} H dv_g$$

for all vector fields $H \in C_c^1(M)$, and that a vector field X has weak divergence div$X$ if

$$\int_M \varphi \text{divX} dv_g = -\int_M \langle \nabla\varphi, X\rangle dv_g$$

for all $\varphi \in C_c^1(M)$.

Last but not least, we define the *Sobolev space* $W^{1,p}(M)$ to be the space of all functions $u \in L^p(M)$ such that the weak gradient ∇u exists and also belongs to $L^p(M)$, equipped with the norm

$$\|u\|_{W^{1,p}(M)} = \Big(\int_M |u|^p dv_g + \int_M |\nabla u|^p dv_g\Big)^{1/p}.$$

3.1 Hardy Inequalities

In [DD], D'Ambrosio and Dipierro prove a series of interesting results concerning Hardy inequalities on Riemannian manifolds. These results are quite remarkable due to their generality and the fact that they offer a criterion that characterises the manifolds themselves.

In what follows, (M, g) is a Riemannian manifold and Ω is an open subset of M. Let us begin with the following definition.

Definition 44. Let $p > 1$. The *p-Laplacian* of a function $u \in W^{1,p}_{loc}(\Omega)$ is the function

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$$

defined in weak sense, that is, the equality

$$\int_\Omega \varphi \Delta_p u dv_g = -\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle dv_g$$

holds for all $\varphi \in C_c^1(\Omega)$. The function u is called *p-subharmonic* on Ω if $\Delta_p u \geq 0$, and *p-superharmonic* if $\Delta_p u \leq 0$.

Notice that the classic Laplacian is the case $p = 2$. To prove the main results, we need the following lemma, which is actually a Hardy-type inequality by itself.

Lemma 19 (D'Ambrosio-Dipierro))**.** Let $h \in L^1_{loc}(\Omega)$ be a vector field and let $A \in L^1_{loc}(\Omega)$ be such that $0 \leq A \leq \text{div} h$ in weak sense and $|h|^p/A^{p-1} \in L^1_{loc}(\Omega)$ for some $p \geq 1$. Then the inequality

$$\int_\Omega |u|^p A dv_g \leq p^p \int_\Omega \frac{|h|^p}{A^{p-1}} |\nabla u|^p dv_g$$

holds for all $u \in C_c^1(\Omega)$.

Proof. By assumption, we have that

$$\int_\Omega |u|^p A dv_g \leq \int_\Omega |u|^p \mathrm{div} h dv_g.$$

From the product rule of the divergence and the divergence theorem, it follows that

$$\int_\Omega |u|^p A dv_g \leq - \int_\Omega \langle \nabla |u|^p, h \rangle dv_g = -p \int_\Omega |u|^{p-2} u \langle \nabla u, h \rangle dv_g$$

$$\leq p \int_\Omega |u|^{p-1} |\nabla u| |h| dv_g.$$

Applying Hölder's inequality on the RHS yields

$$\int_\Omega |u|^p A dv_g \leq p \Big[\int_\Omega |u|^p A dv_g \Big]^{1-1/p} \Big[\int_\Omega \frac{|h|^p}{A^{p-1}} |\nabla u|^p dv_g \Big]^{1/p},$$

and since both sides are non-negative it follows that

$$\int_\Omega |u|^p A dv_g \leq p^p \int_\Omega \frac{|h|^p}{A^{p-1}} |\nabla u|^p dv_g.$$

This completes the proof. □

By using the above result alone one can derive other similar results of general nature. For example, taking the limit case $A = \mathrm{div} h$ proves the following.

Corollary 19.1. Let $h \in L^1_{loc}(\Omega)$ be a vector field such that $\mathrm{div} h \geq 0$ in weak sense and $|h|^p / |\mathrm{div} h|^{p-1} \in L^1_{loc}(\Omega)$ for some $p \geq 1$. Then the inequality

$$\int_\Omega |u|^p \mathrm{div} h dv_g \leq p^p \int_\Omega \frac{|h|^p}{|\mathrm{div} h|^{p-1}} |\nabla u|^p dv_g$$

holds for all $u \in C^1_c(\Omega)$.

Another interesting case is obtained if we take $h = \nabla V$ and $A = \mathrm{div} h = \Delta V$, where V is such that the assumptions of the lemma are met. The outcome is strikingly similar to the lemma of Davies and Hinz in 2.2.

Corollary 19.2. Let $V \in C^2(\Omega)$ be such that $\Delta V \geq 0$ and $|\nabla V|^p/|\Delta V|^{p-1} \in L^1_{loc}(\Omega)$ for some $p \geq 1$. Then the inequality

$$\int_\Omega |u|^p \Delta V dv_g \leq p^p \int_\Omega \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p dv_g$$

holds for all $u \in C^1_c(\Omega)$.

We are now ready to state and prove the main result of this subsection.

Theorem 20 (D'Ambrosio-Dipierro, L^p Hardy Inequality)**.** Let $\rho \in W^{1,p}_{loc}(\Omega)$ be such that $\rho \geq 0$ and $\Delta_p \rho \leq 0$ in weak sense for some $p > 1$. Then $|\nabla \rho|^p/\rho^p \in L^1_{loc}(\Omega)$ and the inequality

$$\Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|\nabla \rho|^p}{\rho^p} |u|^p dv_g \leq \int_\Omega |\nabla u|^p dv_g$$

holds for all $u \in C^1_c(\Omega)$.

Proof. Let $0 < \delta < 1$ and set $\rho_\delta = \rho + \delta$. Consider, respectively, the vector field and the scalar function

$$h = -\frac{|\nabla \rho_\delta|^{p-2} \nabla \rho_\delta}{\rho_\delta^{p-1}} \quad , \quad A = (p-1) \frac{|\nabla \rho_\delta|^p}{\rho_\delta^p}.$$

Since $1/\rho_\delta \leq 1/\delta$ and $\nabla \rho \in L^p_{loc}(\Omega)$, it follows that $h \in L^1_{loc}(\Omega)$ and $A \in L^1_{loc}(\Omega)$. Moreover, notice that

$$\frac{|h|^p}{A^{p-1}} = \frac{1}{(p-1)^{p-1}} \in L^1_{loc}(\Omega)$$

for all $p > 1$.

In order to apply Lemma 19 for h and A, we still need to show that $A \leq \mathrm{div} h$ in weak sense on Ω. To this end, it suffices to show that

$$(p-1) \int_\Omega \frac{|\nabla \rho_\delta|^p}{\rho_\delta^p} \varphi dv_g \leq \int_\Omega \frac{|\nabla \rho_\delta|^{p-2} \langle \nabla \rho_\delta, \nabla \varphi \rangle}{\rho_\delta^{p-1}} dv_g \quad (1)$$

for all non-negative $\varphi \in C^1_c(\Omega)$.

As the reader might already suspect, the proof of (1) will be based on the fact that ρ (and consequently, ρ_δ) is p-superharmonic. In weak language, this means that

$$\int_\Omega |\nabla \rho_\delta|^{p-2} \langle \nabla \rho_\delta, \nabla \varphi \rangle dv_g \geq 0 \quad (2)$$

for all $\varphi \in C_c^1(\Omega)$. If ρ was of class C^1, putting $\varphi/\rho_\delta^{p-1}$ in the place of φ in (2) would do the trick. Unfortunately, ρ is only assumed to be a Sobolev function, and this simple argument doesn't work. To work our way around this difficulty, we exploit the fact that any non-negative Sobolev function defined on a relatively compact set can be approximated by a sequence of smooth functions.

Let us fix a non-negative $\varphi \in C_c^1(\Omega)$, and put $K = \text{supp}\varphi$. Since Ω is open and $K \subset \Omega$ is compact, there exists an open and relatively compact set U such that $K \subset U \subset \Omega$. Since

$$\ln \rho_\delta \in L^p_{loc}(\Omega) \text{ and } |\nabla \ln \rho_\delta| = \frac{|\nabla \rho_\delta|}{\rho_\delta} \leq \frac{|\nabla \rho|}{\delta} \in L^p_{loc}(\Omega),$$

it follows that $\ln \rho_\delta \in W^{1,p}_{loc}(\Omega) \subset W^{1,p}(U)$. Hence there exists a sequence of functions $\phi_n \in C^\infty(U), n \in \mathbb{N}$ such that $\phi_n \to \ln \rho_\delta$ a.e., $\|\phi_n - \ln \rho_\delta\|_{W^{1,p}(U)} \to 0$ and $\phi_n \geq \ln \delta$. Setting $\psi_n = e^{\phi_n}$, we have that $\psi_n \in C^\infty(U)$, $\psi_n \geq \delta$, $\psi_n \to \rho_\delta$ a.e and $\| \ln \psi_n - \ln \rho_\delta\|_{W^{1,p}(U)} \to 0$, that is

$$\int_U | \ln \psi_n - \ln \rho_\delta|^p dv_g \to 0 \text{ and } \int_U \Big| \frac{\nabla \psi_n}{\psi_n} - \frac{\nabla \rho_\delta}{\rho_\delta}\Big|^p dv_g \to 0$$

as $n \to \infty$.

Now set $\varphi_n = \varphi/\psi_n^{p-1}$ for all $n \in \mathbb{N}$, which is a sequence of non-negative C^1 test functions on U with $\text{supp}\varphi_n = K$. These functions can be smoothly extended on Ω without changing the support, taking $\varphi_n|_{\Omega\setminus U} \equiv 0$. Using φ_n as a test function on (2), we get

$$0 \leq \int_\Omega |\nabla \rho|^{p-2} \langle \nabla \rho, \nabla \varphi_n \rangle dv_g = \int_\Omega |\nabla \rho|^{p-2} \langle \nabla \rho, \nabla \Big(\frac{\varphi}{\psi_n^{p-1}}\Big) \rangle dv_g.$$

Since

$$\nabla \Big(\frac{\varphi}{\psi_n^{p-1}}\Big) = \frac{\nabla \varphi}{\psi_n^{p-1}} - (p-1)\frac{\nabla \psi_n}{\psi_n^p}\varphi,$$

it follows that

$$(p-1)\int_\Omega \frac{|\nabla \rho|^{p-2}\langle \nabla \rho, \nabla \psi_n\rangle}{\psi_n^p}\varphi dv_g \leq \int_\Omega \frac{|\nabla \rho|^{p-2}\langle \nabla \rho, \nabla \varphi\rangle}{\psi_n^{p-1}} dv_g. \quad (3)$$

For the RHS, we have

$$\Big|\frac{|\nabla \rho|^{p-2}\langle \nabla \rho, \nabla \varphi\rangle}{\psi_n^{p-1}}\Big| \leq \frac{|\nabla \rho|^{p-1}}{\delta^{p-1}} |\nabla \varphi| \in L^1(\Omega),$$

thus by dominated convergence

$$\int_\Omega \frac{|\nabla\rho|^{p-2}\langle\nabla\rho,\nabla\varphi\rangle}{\psi_n^{p-1}}dv_g \to \int_\Omega \frac{|\nabla\rho|^{p-2}\langle\nabla\rho,\nabla\varphi\rangle}{\rho_\delta^{p-1}}dv_g \text{ as } n\to\infty.$$

For the LHS, we have that

$$\int_\Omega \frac{|\nabla\rho|^{p-2}\langle\nabla\rho,\nabla\psi_n\rangle}{\psi_n^p}\varphi dv_g = \int_\Omega \langle \frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1}}, \frac{\nabla\psi_n}{\psi_n}\rangle \varphi dv_g.$$

Since

$$\frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1}} \to \frac{|\nabla\rho|^{p-2}\nabla\rho}{\rho_\delta^{p-1}} \text{ a.e.}$$

and since

$$\Big|\frac{|\nabla\rho|^{p-2}\nabla\rho}{\psi_n^{p-1}}\Big| \le \frac{|\nabla\rho|^{p-1}}{\delta^{p-1}} \in L^{\frac{p}{p-1}}(\Omega)\subset L^1(\Omega)$$

and

$$\int_U \Big|\frac{\nabla\psi_n}{\psi_n} - \frac{\nabla\rho_\delta}{\rho_\delta}\Big|^p dv_g \to 0,$$

it follows by dominated convergence that

$$\int_\Omega \frac{|\nabla\rho|^{p-2}\langle\nabla\rho,\nabla\psi_n\rangle}{\psi_n^p}\varphi dv_g \to \int_\Omega \frac{|\nabla\rho|^p}{\rho_\delta^p}\varphi dv_g \text{ as } n\to\infty,$$

so taking the limit in (3) yields (1).

Applying Lemma 19 for A and h yields

$$\Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|\nabla\rho|^p}{\rho_\delta^p}|u|^p dv_g \le \int_\Omega |\nabla u|^p dv_g,$$

and Fatou's lemma for $\delta\to 0$ completes the proof. □

The constant is sharp if and only if some additional assumptions are satisfied (see [DD]).

The above result has the following generalisation.

Theorem 21 (D'Ambrosio-Dipierro, Weighted L^p Hardy Inequality)**.** Let $\rho\in W^{1,p}_{loc}(\Omega)$ be such that $\rho\ge 0$ and $(p-1-\alpha)\Delta_p\rho\le 0$ in weak sense on Ω for some $p>1$ and some $\alpha\in\mathbb{R}$, and $|\nabla\rho|^p/\rho^{p-\alpha}, \rho^\alpha \in L^1_{loc}(\Omega)$. Then the inequality

$$\Big|\frac{p-1-\alpha}{p}\Big|^p \int_\Omega \frac{|\nabla\rho|^p}{\rho^{p-\alpha}}|u|^p dv_g \le \int_\Omega \rho^\alpha |\nabla u|^p dv_g$$

holds for all $u\in C^1_c(\Omega)$.

Proof. Similar to the proof of the previous theorem, apply Lemma 19 for

$$h = -(p-1-\alpha)\frac{|\nabla\rho_\delta|^{p-2}\nabla\rho_\delta}{\rho_\delta^{p-1-\alpha}} \quad , \quad A = (p-1-\alpha)^2\frac{|\nabla\rho_\delta|^p}{\rho_\delta^{p-\alpha}},$$

and take the cases $\alpha < p-1$ and $\alpha > p-1$. In either case, dominated convergence and Fatou's lemma complete the proof. □

It is worth considering the special case of the above where $\alpha = p+q$, for some $q > -1$. What we obtain is a *Caccioppoli*-type inequality for p-subharmonic functions.

Corollary 21.1 (D'Ambrosio-Dipierro, L^p Caccioppoli-type Inequality)**.** Let $\rho \in W^{1,p}_{loc}(\Omega)$ be such that $\rho \geq 0$ and $\Delta_p\rho \geq 0$ in weak sense on Ω for some $p > 1$ and $|\nabla\rho|^p\rho^q, \rho^{p+q} \in L^1_{loc}(\Omega)$ for some $q > -1$. Then the inequality

$$\left(\frac{q+1}{p}\right)^p \int_\Omega |\nabla\rho|^p\rho^q|u|^p dv_g \leq \int_\Omega \rho^{q+p}|\nabla u|^p dv_g$$

holds for all $u \in C^1_c(\Omega)$.

The Hardy inequality is particularly useful since it allows us to obtain a characterisation for manifolds. We will show that a manifold is *p-hyperbolic* if and only if a particular type of inequality holds for it.

Definition 45. A Riemannian manifold M is said to be *p-hyperbolic* if there exists a positive and symmetric Green function $G_x(\cdot)$ for the p-Laplacian for some $x \in M$ (that is, $-\Delta_p G_x = \delta_x$). If this is not the case, it is called *p-parabolic.*

The special case $M = \mathbb{R}^n$ is a good example of a p-hyperbolic manifold. For $p < n$, the fundamental solution with pole at $\xi \in \mathbb{R}^n$ is given by

$$G_\xi(x) = c(p,n)|x-\xi|^{\frac{p-n}{p-1}}, \; x \in \mathbb{R}^n \setminus \{\xi\},$$

which is both positive and symmetric (see, for example [FP]). Hence $\mathbb{R}^n$ is p-hyperbolic for all $p < n$.

Before we obtain the criterion of the characterisation, we need the following (for the proof, see [T], also mentioned in [DD]).

Lemma 22. Let $p > 1$. A Riemannian manifold (M, g) is p-parabolic if and only if there exists a sequence of functions $u_k \in C^\infty(M)$ such that $0 \leq u_k \leq 1$, $u_k \to 1$ uniformly on every compact subset of M and $\int_M |\nabla u_k|^p dv_g \to 0$.

Now we are ready to state and prove the result.

Theorem 23. A Riemannian manifold (M, g) is p-hyperbolic if and only if there exists a non-negative non-trivial function $f \in L^1_{loc}(M \setminus \{x\})$ for some $x \in M$ such that the inequality

$$\int_M f|u|^p dv_g \leq \int_M |\nabla u|^p dv_g$$

holds for all $u \in C_c^\infty(M \setminus \{x\})$.

Proof. First, assume that M is p-hyperbolic, and let G_x be the non-negative Green function such that $-\Delta_p G_x = \delta_x$. We know that $G_x \in W^{1,p}_{loc}(M)$ and that $\Delta_p G_x = 0$ on $M \setminus \{x\}$. By Theorem 20, it follows that $|\nabla G_x|^p / G_x^p \in L^1_{loc}(M \setminus \{x\})$, and that the inequality holds for

$$f = \Big(\frac{p-1}{p}\Big)^p \frac{|\nabla G_x|^p}{G_x^p}.$$

Conversely, assume that M is p-parabolic and the inequality holds at the same time. Then there exists a sequence of functions $u_k \in C^\infty(M)$ such that $0 \leq u_k \leq 1$, $u_k \to 1$ uniformly on every compact subset of M and $\int_M |\nabla u_k|^p dv_g \to 0$. It follows by Fatou's Lemma that

$$0 \leq \int_K f dv_g \leq \liminf_{k\to\infty} \int_K f|u_k|^p dv_g \leq \liminf_{k\to\infty} \int_K |\nabla u_k|^p dv_g = 0$$

for every compact subset of $K \subset M \setminus \{x\}$, thus $f = 0$ almost everywhere, a contradiction. $\square$

It is worth noting that the above also holds for functions $u \in C_c^\infty(M)$ and non-negative non-trivial $f \in L^1_{loc}(M)$, provided that $p < \dim M$. This extension is based on the fact that, in this case,

$$\overline{C_c^\infty(M)}^{W^{1,p}} = \overline{C_c^\infty(M \setminus \{x\})}^{W^{1,p}}.$$

This can be shown by choosing a suitable sequence of $C_c^\infty(M \setminus \{x\})$ functions that approximates any given $C_c^\infty(M) \cap W^{1,p}(M)$ function in the $W^{1,p}$ sense.

Though not as numerous as in the Euclidean case, there exist quite a few similar results written by other authors (see [KO], [X] and [C], to name a few). The results of [DD] where mostly chosen because they, unlike most others, where set in the more general L^p setting, and also because of they offer a criterion for the characterisation of p-hyperbolic manifolds.

3.2 Rellich Inequalities

Next, we move on to Rellich inequalities. Such results are quite rare for Riemannian manifolds, and their proofs are rather complicated. Our main objective in this subsection will be to prove an L^2 Rellich inequality by Kombe and Ozaydin from [KO].

Before we prove the result we are going to need the following Hardy-type inequality from [C]. Let (M, g) be a non-compact, complete Riemannian manifold.

Lemma 24 (Carron, L^2 Hardy Inequality)**.** Let $\alpha, C \in \mathbb{R}$ be such that $C + \alpha - 1 > 0$ and $\rho \in C^2(M)$ be such that $|\nabla\rho| = 1$ and $\Delta\rho \geq C/\rho$. Then the inequality

$$\Big(\frac{C+\alpha-1}{2}\Big)^2 \int_M \rho^{\alpha-2}|u|^2 dv_g \leq \int_M \rho^\alpha |\nabla u|^2 dv_g$$

holds for all $u \in C_c^\infty(M \setminus \rho^{-1}\{0\})$.

The proof is similar to the other Hardy-type inequalities we have seen so far and will be omitted.

We proceed to the main result.

Theorem 25 (Kombe-Ozaydin, L^2 Rellich Inequality)**.** Assume that $\dim M \geq 2$ and let $\rho \in C^2(M)$ be such that $|\nabla\rho| = 1$ and $\Delta\rho \geq C/\rho$ for some $C > 0$. Then the inequality

$$\frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16} \int_M \rho^{\alpha-4}|u|^2 dv_g \leq \int_M \rho^\alpha |\Delta u|^2 dv_g$$

holds for all $u \in C_c^\infty(M \setminus \rho^{-1}\{0\})$, $\alpha < 2$ and $\alpha > 3 - C$.

Proof. First, notice that

$$\Delta\rho^{\alpha-2} = \operatorname{div}(\nabla\rho^{\alpha-2}) = \operatorname{div}((\alpha-2)\rho^{\alpha-3}\nabla\rho)$$

$$= (\alpha-2)(\alpha-3)\rho^{\alpha-4}|\nabla\rho|^2 + (\alpha-2)\rho^{\alpha-3}\Delta\rho.$$

Since $|\nabla\rho| = 1$ and $\Delta\rho \geq C/\rho$, for $\alpha < 2$ it follows that

$$\Delta\rho^{\alpha-2} \leq (\alpha-2)(\alpha-3+C)\rho^{\alpha-4}.$$

Multiplying both sides by u^2 and integrating over M yields

$$\int_M u^2 \Delta \rho^{\alpha-2} dv_g \leq (\alpha-2)(\alpha-3+C) \int_M \rho^{\alpha-4} u^2 dv_g,$$

while by Green's second identity, we have that

$$\int_M u^2 \Delta \rho^{\alpha-2} dv_g = \int_M \rho^{\alpha-2} \Delta u^2 dv_g = \int_M \rho^{\alpha-2}(2|\nabla u|^2 + 2u\Delta u) dv_g.$$

Combining the last two relations yields

$$2\int_M \rho^{\alpha-2}|\nabla u|^2 dv_g - (\alpha-2)(\alpha-3+C)\int_M \rho^{\alpha-4}u^2 dv_g$$

$$\leq -2\int_M u\Delta u \rho^{\alpha-2}.$$

Since by Carron's Hardy inequality we know that

$$\Big(\frac{C+\alpha-1}{2}\Big)^2 \int_M \rho^{\alpha-2}|u|^2 dv_g \leq \int_M \rho^{\alpha}|\nabla u|^2 dv_g,$$

it follows that

$$\frac{(C+\alpha-3)(C-\alpha+1)}{4}\int_M \rho^{\alpha-4}|u|^2 dv_g \leq -\int_M u\Delta u \rho^{\alpha-2} dv_g$$

$$\leq \int_M |u||\Delta u|\rho^{\alpha/2-2}\rho^{\alpha/2} dv_g.$$

Finally, we apply the Cauchy-Schwarz inequality to the RHS to get

$$\frac{(C+\alpha-3)(C-\alpha+1)}{4}\int_M \rho^{\alpha-4}|u|^2 dv_g$$

$$\leq \Big[\int_M \rho^{\alpha-4}|u|^2 dv_g\Big]^{1/2}\Big[\int_M \rho^{\alpha}|\Delta u|^2\Big]^{1/2} dv_g.$$

For $\alpha > 3 - C$ both sides are non-negative, and taking the square of both sides gives

$$\frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16}\int_M \rho^{\alpha-4}|u|^2 dv_g \leq \int_M \rho^{\alpha}|\Delta u|^2 dv_g.$$

This completes the proof. □

As for higher-order Rellich inequalities and L^p Rellich inequalities, they are significantly harder to obtain and even more rare to find. In [B], Barbatis has obtained, under a simple geometric assumption, an improved L^p higher-order inequality of the form

$$\int_\Omega \frac{|\Delta^{m/2}u|^p}{\rho^\gamma} dv_g \geq A(m,\gamma) \int_\Omega \frac{|u|^p}{\rho^{\gamma+mp}} dv_g + B(m,\gamma) \int_\Omega V_i |u|^p dv_g,$$

where $\rho(\cdot) = \mathrm{dist}(\cdot, K)$, $u \in C_c^\infty(\Omega \setminus K)$, $m, \gamma \in \mathbb{N}$ and V_i involves some suitably iterated logarithmic functions. The geometric assumption, in particular, is that the distance function should be such that

$$\rho\Delta\rho \geq k - 1$$

in $\Omega \setminus K$. This assumption is, for example, satisfied in Euclidean spaces and Cartan-Hadamard manifolds, that is, simply connected geodesically complete non-compact manifolds with non-positive sectional curvature.

The sectional curvature is obtained from the Riemannian curvature as follows: if X and Y are two linearly independent vector fields, the sectional curvature is the function

$$S(X,Y) = \frac{R_{kmij}X^kY^mX^iY^j}{X^aX_aY^bY_b - X^cY_c} = \frac{\tilde{R}(X,Y,X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}.$$

3.3 Uncertainty Principles

In [K], Kristaly proved a generalisation of the classic HPW uncertainty principle for Riemannian manifolds, and more specifically that

$$\frac{n^2}{4}\Big[\int_M |u|^2 dv_g\Big]^2 \leq \int_M \rho_p^2 |u|^2 dv_g \int_M |\nabla u|^2 dv_g$$

for all $u \in C^\infty(M)$ and $p \in M$, where $\rho_p(\cdot) = \mathrm{dist}(\cdot, p)$ and (M, g) has dimension n and satisfies some additional assumptions.

To be more precise, he proved that if M has non-positive sectional curvature, the inequality holds and the constant is sharp. Moreover, he proved that if M has non-negative sectional curvature, the inequality holds if and only if M is isometric to some Euclidean space. We will not prove these results.

Instead, we are going to prove two different results. The first one is a straightforward consequence of the Caccioppoli-type inequality of section

3.1. Let (M, g) be a Riemannian manifold and $\Omega \subset M$ be open.

Corollary 21.1.1. (Uncertainty Principle) Let $p, q > 0$ be such that $1/p + 1/q = 1$, and let $\rho \in C^2(\Omega)$ be such that $\rho \geq 0$, $\Delta_p \rho \geq 0$. Then the inequality

$$\frac{1}{2}\int_\Omega |\nabla \rho||u|^2 dv_g \leq \Big[\int_\Omega \rho^q |u|^q dv_g\Big]^{1/q}\Big[\int_\Omega |\nabla u|^p\Big]^{1/p}$$

holds for all $u \in C_c^\infty(\Omega)$.

Proof. By Hölder's inequality we have

$$\Big[\int_\Omega \rho^q |u|^q dv_g\Big]^{1/q}\Big[\int_\Omega |\nabla u|^p\Big]^{1/p} \geq \int_\Omega \rho|u||\nabla u| dv_g = \frac{1}{2}\int_\Omega \rho|\nabla u^2| dv_g$$

and by the Caccioppoli inequality 21.1 for $p = 1$ and $q = 0$ it follows that

$$\int_\Omega \rho|\nabla u^2| dv_g \geq \int_\Omega |\nabla \rho||u|^2 dv_g.$$

This completes the proof. □

The other result is an L^p generalisation of the classic HPW uncertainty principle for manifolds.

Theorem 26 (L^p Uncertainty Principle)**.** Let X be a C^1 vector field on Ω such that $\mathrm{div} X \geq 0$, and let $p > 1$. Then the inequality

$$\int_\Omega \mathrm{div} X |u|^p dv_g \leq p\Big[\int_\Omega |X|^{\frac{p}{p-1}}|u|^p dv_g\Big]^{\frac{p-1}{p}}\Big[\int_\Omega |\nabla u|^p dv_g\Big]^{1/p}$$

holds for all $u \in C_c^1(\Omega)$.

Proof. As with the proof of the classic HPW inequality, we use the product rule for the divergence and the divergence theorem to obtain

$$\begin{aligned}\int_\Omega \mathrm{div} X |u|^p dv_g &= -\int_\Omega \langle X, \nabla |u|^p\rangle dv_g = -p\int_\Omega |u|^{p-1}\langle X, \nabla |u|\rangle dv_g \\ &\leq p\int_\Omega |X||u|^{p-1}|\nabla u| dv_g.\end{aligned}$$

Applying Hölder's inequality on the RHS yields

$$\int_\Omega \mathrm{div} X |u|^p dv_g \leq p\Big[\int_\Omega |X|^{\frac{p}{p-1}}|u|^p dv_g\Big]^{\frac{p-1}{p}}\Big[\int_\Omega |\nabla u|^p dv_g\Big]^{1/p}.$$

This completes the proof. □

Note that the classic HPW uncertainty principle is the very special case where $\Omega = \mathbb{R}^n$, $X(x) = x$ and $p = 2$.

For similar inequalities, the reader may also see [KO] and [X].

4 Inequalities for Vector Fields

In this section we present some results that offer estimates for vector fields instead of scalar functions. These may be considered to be modifications of inequalities that we have already encountered, so they cannot be considered entirely original, but the fact that they deal with vector fields is in itself a novel aspect.

4.1 Preliminaries

Let M be a Riemannian manifold. From the definition of the gradient in 1.11, we know that $\nabla f = (df)^\sharp$. We are going to apply this to the specific case where $f = \langle X, Y\rangle$ for two vector fields X and Y.

Assuming the Riemannian connection, we have that

$$d\langle X, Y\rangle(\xi) = \xi\langle X, Y\rangle = \langle \nabla_\xi X, Y\rangle + \langle X, \nabla_\xi Y\rangle.$$

If we define the *Jacobian* of X to be the (0,2)-tensor field ∇X such that, for all vector fields ξ, η, $\nabla X(\xi, \eta) = \langle \nabla_\xi X, \eta\rangle$, it follows that

$$d\langle X, Y\rangle(\cdot) = \nabla X(\cdot, Y) + \nabla Y(\cdot, X),$$

and therefore we obtain the product rule

$$\begin{aligned}\nabla\langle X, Y\rangle &= \nabla X(\cdot, Y)^\sharp + \nabla Y(\cdot, X)^\sharp \\ &= \nabla X \cdot Y + \nabla Y \cdot X,\end{aligned}$$

where we use the notation $\nabla X \cdot Y = \nabla X(\cdot, Y)^\sharp$.

$\nabla X \cdot Y$ is of course linear in Y, and therefore bounded at each point of the manifold (since the tangent space is finite-dimensional), with norm

$$\|\nabla X\|(p) = \sup_{\xi_p \in T_pM\setminus\{0\}} \frac{|\nabla X \cdot \xi_p|}{|\xi_p|}$$

for each point $p \in M$.

Combining this with the product rule, we obtain the estimate

$$|\nabla\langle X, Y\rangle| \le \|\nabla X\||Y| + \|\nabla Y\||X|.$$

An important application of this is the special case $X = Y$, from which we obtain $|\nabla|X|| \le \|\nabla X\|$.

4.2 A Hardy Inequality

Following a process similar to [DD], we obtain an L^p Hardy-type inequality for vector fields. The only thing that changes is that the estimate now is given in terms of the *Jacobian* of the vector field. This is totally analogous to the scalar case where the estimate is given with respect to the gradient.

Theorem 27 (L^p Hardy Inequality for Vector Fields)**.** Let Y be a C^1 vector field on Ω be such that $\text{div}Y \geq 0$ and $|Y|^p/|\text{div}Y|^{p-1} \in L^1_{loc}(\Omega)$ for some $p \geq 1$. Then the inequality

$$\int_\Omega \text{div}Y|X|^p dv_g \leq p^p \int_\Omega \frac{|Y|^p}{|\text{div}Y|^{p-1}} \|\nabla X\|^p dv_g$$

holds for all compactly supported C^1 vector fields.

Proof. From the product rule of the divergence and the divergence theorem, it follows that

$$\int_\Omega \text{div}Y|X|^p dv_g = -\int_\Omega \langle \nabla |X|^p, Y\rangle dv_g = -p\int_\Omega |X|^{p-1}\langle \nabla |X|, h\rangle dv_g$$
$$\leq p\int_\Omega |Y||X|^{p-1}\|\nabla X\| dv_g.$$

Applying Hölder's inequality on the RHS yields

$$\int_\Omega \text{div}Y|X|^p dv_g \leq p\Big[\int_\Omega \text{div}Y|X|^p dv_g\Big]^{1-1/p}\Big[\int_\Omega \frac{|Y|^p}{|\text{div}Y|^{p-1}}\|\nabla X\|^p dv_g\Big]^{1/p},$$

and since both sides are non-negative it follows that

$$\int_\Omega \text{div}Y|X|^p dv_g \leq p^p \int_\Omega \frac{|Y|^p}{|\text{div}Y|^{p-1}} \|\nabla X\|^p dv_g.$$

This completes the proof. □

Since $|X|$ is C^1 almost everywhere, this may actually be considered an application of Lemma 19.

4.3 An Uncertainty Principle

In a similar fashion we may obtain an analogue of the generalised HPW uncertainty principle for vector fields.

Theorem 28 (L^p Uncertainty Principle for Vector Fields)**.** Let Y be a C^1 vector field on Ω such that $\text{div}Y \geq 0$, and let $p > 1$. Then the inequality

$$\int_\Omega \text{div}Y|X|^p dv_g \leq p\Big[\int_\Omega |Y|^{\frac{p}{p-1}}|X|^p dv_g\Big]^{\frac{p-1}{p}}\Big[\int_\Omega \|\nabla X\|^p dv_g\Big]^{1/p}$$

holds for all compactly supported C^1 vector fields.

Proof. We have

$$\int_\Omega \text{div}Y|X|^p dv_g = -\int_\Omega \langle Y, \nabla|X|^p\rangle dv_g = -p\int_\Omega |X|^{p-1}\langle Y, \nabla|X|\rangle dv_g$$

$$\leq p\int_\Omega |Y||X|^{p-1}\|\nabla X\| dv_g.$$

Applying Hölder's inequality on the RHS yields

$$\int_\Omega \text{div}Y|X|^p dv_g \leq p\Big[\int_\Omega |Y|^{\frac{p}{p-1}}|X|^p dv_g\Big]^{\frac{p-1}{p}}\Big[\int_\Omega \|\nabla X\|^p dv_g\Big]^{1/p}.$$

This completes the proof. □

As a final remark, note that in general we can do the same for any Hardy or Sobolev-type inequality, converting it to a vector field version of its initial scalar form.

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