THE ELLIPSOIDAL HARMONICS IN SOLVING INVERSE SCATTERING PROBLEMS

ARKOUDIS IOANNIS *

National and Kapodistrian University of Athens
Department of Mathematics

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Abstract

The main subject of this study is the solution of inverse acoustic and electromagnetic scattering problems for ellipsoids using the ellipsoidal harmonics. The scattering problems of time-harmonic acoustic and electromagnetic plane waves by an ellipsoidal scatterer for various boundary conditions imposed on its surface are considered. The study of the ellipsoidal coordinate system, leads to the definition of the ellipsoidal harmonics, which enter in the scattering problems via the low-frequency theory. The methodology which leads to the derivation of low-frequency approximations for ellipsoids is presented. Inverse scattering problems for acoustic and electromagnetic waves for an ellipsoidal scatterer are described. A finite number of measurements of far-field data or near-field data leads to the specification of the size and the orientation of an unknown ellipsoidal scatterer. For the case of penetrable scatterer, physical parameters of its interior are also obtained. Corresponding results for the cases of the sphere and the spheroid are derived, considering them as geometrically degenerate cases of the ellipsoid for appropriate values of its geometrical parameters.

Περίληψη

Η παρούσα διπλωματική εργασία έχει ως κεντρικό θέμα την επίλυση αντστρόφων προβλημάτων σκέδασης ακουστικών και ηλεκτρομαγνητικών κυμάτων για ελλειψοειδή με χρήση των ελλειψοειδών αρμονικών συναρτήσεων. Περιγράφονται τα προβλήματα σκέδασης επίπεδων ακουστικών και ηλεκτρομαγνητικών κυμάτων με αρμονική χρονική εξάρτηση για ελλειψοειδές σκεδαστή με διάφορες συνοριακές συνθήκες. Η μελέτη του ελλειψοειδούς συστήματος συντεταγμένων, οδηγεί στον ορισμό των ελλειψοειδών αρμονικών, οι οποίες υπεισέρχονται στα προβήματα σκέδασης μέσω της θεωρίας χαμηλών συχνοτήτων. Παρουσιάζεται η διαδικασία που ακολουθείται για τον υπολογισμό των προσεγγίσεων χαμηλών συχνοτήτων για ελλειψοειδή. Περιγράφονται τα αντίστροφα προβλήματα σκέδασης για ελλειψοειδείς σκεδαστές. Πεπερασμένος αριθμός μετρήσεων δεδομένων μακρινού πεδίου ή κοντινού πεδίου, οδηγούν στον προσδιορισμό του μεγέθους και του προσανατολισμού ενός αγνώστου ελλειψοειδούς σκεδαστή. Στην περίπτωση διαπερατού σκεδαστή, προσδιορίζονται επιπλέον φυσικές παράμετροι του εσωτερικού του. Αντίστοιχα αποτελέσματα για την περίπτωση της σφαίρας και του σφαιροειδούς υπολογίζονται θεωρώντας τα σχήματα αυτά ως γεωμετρικούς εκφυλισμούς του ελλειψοειδούς για κατάλληλες τιμές των γεωμετρικών παραμέτρων του.

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Advisors

• Evangelia Kotta-Athanasiadou, Assistant Professor (Supervisor),
National and Kapodistrian University of Athens.

• Gerassimos Barbatis, Professor,
National and Kapodistrian University of Athens.

• Nikolaos Bardis, Associate Professor,
Hellenic Military Academy.

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Chapter 0

Introduction

In 1782 Laplace derived the Laplace equation. The solution of this equation led to the theory of the harmonic functions starting with the solutions of Legendre and Laplace in 1785, both of them in spherical geometry which gave rise to the spherical harmonics. When a quantity is independent of orientation and depends only on the distance, it can be described by a sphere which also shows isotropy of the space with respect to this quantity. Nevertheless, when there is dependence on the orientation, then the sphere is replaced by an ellipsoid which shows the anisotropic character of the space. In 1837, in order to study the thermal equilibrium of an ellipsoidal body [25], Lamé developed the theory of the ellipsoidal harmonics by introducing the ellipsoidal coordinate system which was proven really convenient in the study of the ellipsoids. Many years later, Niven studied the Cartesian form of the ellipsoidal harmonics [29] and managed to reduce them to the sphero-conal harmonics as the ellipsoid reduces to a sphere at infinity. A lot of studies about the ellipsoidal harmonics followed after that, since they consist a really useful tool in the study of potential theory problems in the presence of anisotropy. Moreover, the ellipsoid is a good approximation of various shapes such as spheres, spheroids, needles and disks, which makes it appropriate for the mathematical formulation of many physical problems. Thus, the study of ellipsoidal harmonics can be helpful in various areas of science and has a lot of significant applications. For the acoustic wave fields, some basic applications are the sonar and the ultrasound which can be used for the detection of underwater objects and for medical imaging respectively. For the electromagnetic wave fields, since the shape of the brain can be approximated by an ellipsoid, two popular applications are the EEG and EMG in the area of medical imaging and another popular application is the radar for the detection of underground objects. For the case of the elastic waves, there are also a lot of significant applications like structural analysis. Moreover, when the thermal factor enters the problems we have the corresponding applications for the thermoelastic waves. In the present work, we study inverse scattering problems for ellipsoids by using low-frequency theory which allows the introduction of ellipsoidal harmonics into the scattering theory. Specifically, we use the theory of ellipsoidal harmonics in order to calculate the low-frequency approximations which can be used for the specification of geometrical and physical characteristics of an unknown ellipsoidal scatterer.

In chapter 1 we study the ellipsoidal coordinate system and its basic properties. Next, we express the basic differential operators in this system which finally leads to the Laplace equation. The separation of variables leads to the ordinary differential equation known as the Lamé equation with solutions known as Lamé functions. After calculating the polynomials that compose the Lamé functions, we proceed into the definition of the interior and the exterior ellipsoidal harmonics. Next, we study the surface ellipsoidal harmonics and the orthogonality relations that they satisfy. These orthogonality relations are the basic tool for the derivation of the low-frequency approximations for the solutions of the scattering problems for ellipsoidal scatterers. In the last section we derive the general form of Laplace equation in orthogonal curvilinear coordinate systems which allows us to obtain the sphero-conal form of the Laplace equation as well as the spherical form of the Laplace equation. The sphero-conal coordinate system is proved to be important in the study of the ellipsoidal coordinate system, since when the ellipsoid degenerates into a sphere, the ellipsoidal harmonics are reduced into sphero-conal harmonics instead of spherical. This makes the sphero-conal coordinate system to act as an intermediate between the spherical and the ellipsoidal coordinate systems. Finally, we mention the basic properties of the spherical harmonics which help in further understanding some properties of the ellipsoidal harmonics.

In chapter 2 we present the basic acoustic and electromagnetic scattering problems. Next, in order to derive some approximations for the solutions of these problems, we use the low-frequency theory which reduces the scattering problems into a sequence of potential theory problem and allows the usage of the ellipsoidal harmonics for the case of ellipsoidal scatterers. The properties of the ellipsoidal harmonics and mainly the orthogonality of the surface ellipsoidal harmonics are proved to be convenient into the derivation of the approximations for the solutions. Finally, we formulate the scattering problems for the case of the ellipsoidal scatterer and we present the approximations of their solutions that we need in order to solve the corresponding inverse scattering problems.

In chapter 3 we study inverse scattering problems for ellipsoids. Starting with acoustic inverse scattering problems, we present the method that was introduced by Dassios [12]. Next, we proceed to the electromagnetic inverse scattering problems, where we extended Dassios's method for a perfectly conductive ellipsoid [17], to the cases of the lossless dielectric, the lossy dielectric and the impedance ellipsoidal scatterers using far-field data [7]. Finally, we present a method for solving all these inverse electromagnetic scattering problems for ellipsoidal scatterers by using near-field data and in the end we sum up the benefits of this method.

Chapter 1

Ellipsoidal Harmonics

1.1 Ellipsoidal Coordinate System

In almost every coordinate system, a point is specified by the intersection of combination of first and second degree surfaces. For example, in the Cartesian coordinate system it is specified by the intersection of only first degree surfaces which are the three planes and in the spherical coordinate system it is specified by the intersection of a plane, a cone and a sphere. The unique characteristic of the ellipsoidal coordinate system, is that every point is specified by the intersection of 3 non-degenerate second degree surfaces.
A second degree surface is defined by the general quadratic form

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j + \sum_{i=1}^{3} b_ix_i + c = 0 \tag{1.1}$$

with canonical form:

$$\mu_1x_1^2 + \mu_2x_2^2 + \mu_3x_3^2 = 1 \quad . \tag{1.2}$$

Based on the sign of μ_i for $i = 1, 2, 3$ we have the following three cases:

i. For $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$, the canonical form (1.2) defines a triaxial ellipsoid ,

ii. For $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 < 0$, the canonical form (1.2) defines a hyperboloid of one sheet ,

iii. For $\mu_1 > 0$, $\mu_2 < 0$, $\mu_3 < 0$, the canonical form (1.2) defines a hyperboloid of two sheets ,

while every other case leads to a degenerate form. From now on, we will always choose the Cartesian system that reduces the quadratic to its canonical form. The axes and the planes of this system are called principal axes and principal planes respectively.
In opposition to the spherical system which is completely specified by choosing the unit sphere, in the ellipsoidal system the specification of the coordinate system is based on a reference ellipsoid (which establishes the variations in angular dependence).
The reference ellipsoid is given by:

$$\sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2} = 1 \quad , \quad 0 < \alpha_3 < \alpha_2 < \alpha_1 < \infty \tag{1.3}$$

where $\alpha_1, \alpha_2, \alpha_3$ its semi-axes. The squares of the semi-focal distances of the reference ellipsoid are the following:

$$h_1^2 = \alpha_2^2 - \alpha_3^2 \quad , \quad h_2^2 = \alpha_1^2 - \alpha_3^2 \quad , \quad h_3^2 = \alpha_1^2 - \alpha_2^2 \quad . \tag{1.4}$$

It follows that $h_1 < h_2$ and $h_3 < h_2$ but there is no specific relation between h_1 and h_3. The three semi-focal distances satisfy the following relation

$$h_1^2 - h_2^2 + h_3^2 = 0 \quad , \tag{1.5}$$

therefore to characterize the ellipsoidal coordinate system we need two independent semi-focal distances for which we take h_2, h_3. Since these semi-focal distances specify the ellipsoidal system it follows that the foci of the system will be at the fixed points:

$$(\pm h_2, 0, 0) \qquad , \qquad (\pm h_3, 0, 0) \qquad , \qquad (0, \pm h_1, 0) \qquad .$$

Therefore the ellipsoidal system is characterized as con-focal since the foci are located at these six points.
Based on the values of a parameter $\lambda \in \mathbb{R}$ we have the following relation which represents the con-focal family of second degree surfaces:

$$\sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2 - \lambda} = 1 \quad . \tag{1.6}$$

Specifically:

i. For $-\infty < \lambda < \alpha_3^2$, relation (1.6) represents a family of con-focal ellipsoids .

ii. For $\lambda = \alpha_3^2$, it represents the focal ellipse .

iii. For $\alpha_3^2 < \lambda < \alpha_2^2$, it represents a family of con-focal hyperboloids of one sheet (1-hyperboloids).

iv. For $\lambda = \alpha_2^2$ represents the focal hyperbola .

v. For $\alpha_2^2 < \lambda < \alpha_1^2$, it represents a family of con-focal hyperboiloids of two sheets (2-hyperboloids) .

vi. For $\lambda \geqslant \alpha_1^2$, it does not represent a real surface.

Based on [20], we have the following proposition.

Proposition 1.1.1. *For every point (x_1, x_2, x_3), with $x_1x_2x_3 \neq 0$, the cubic polynomial in λ (1.6) has three real roots $\lambda_1, \lambda_2, \lambda_3$, which are ordered as follows:*

$$-\infty < \lambda_3 < \alpha_3^2 < \lambda_2 < \alpha_2^2 < \lambda_1 < \alpha_1^2 \quad .$$

Proof. Consider the polynomial function

$$f(\lambda) = \sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2 - \lambda} = 1 \quad .$$

This function is continuously differentiable and strictly increasing in $D = (-\infty, \alpha_3^2,) \cup (\alpha_3^2, \alpha_2^2) \cup (\alpha_2^2, \alpha_1^2)$. It can be seen by taking $\lim_{\lambda \to \alpha_1^2 \pm} f(\lambda), \lim_{\lambda \to \alpha_2^2 \pm} f(\lambda)$, that the function f has exactly three roots $\lambda_1, \lambda_2, \lambda_3$, in the intervals $(-\infty, \alpha_3^2), (\alpha_3^2, \alpha_2^2)$ and (α_2^2, α_1^2) respectively. □

This one-to-one correspondence between $\mathbb{R}_0 = \{(x_1, x_2, x_3) \,|x_1x_2x_3 \neq 0\}$ and $\mathbb{P} = \left(-\infty, \alpha_3^2\right) \times \left(\alpha_3^2, \alpha_2^2\right) \times \left(\alpha_2^2, \alpha_1^2\right)$ allows the parametrization of $\mathbb{R}_0$ by the vector $(\lambda_1, \lambda_2, \lambda_3)$. In *Geometry and the Imagination* by Hilbert and Cohn-Vossen the ellipsoidal system is described in a geometrical way as the vector $(\lambda_1, \lambda_2, \lambda_3)$ alters. It can easily be concluded that from every point passes exactly one ellipsoid, exactly one hyperboloid of one sheet and exactly one hyperboloid of two sheets as λ varies from $-\infty$ to ∞. These three surfaces have the same foci and constitute the ellipsoidal coordinate system. Note that the three Cartesian planes are singular sets of the ellipsoidal system and this is the reason they are excluded from Proposition 1.1.1.

Proposition 1.1.2. *The confocal ellipsoidal system is orthogonal*

Proof. Consider the arbitrary point $\mathbf{r} = (x_1, x_2, x_3)$ and based on 1.6 let:

$$E(\mathbf{r}) = \sum_{n=1}^{3} \frac{x_n^2}{a_n^2 - \lambda_3} - 1 \quad , \quad \text{for } \lambda_3 \in \left(-\infty, \alpha_3^2\right), \tag{1.7}$$

be the ellipsoid that passes through this point,

$$H_1(\mathbf{r}) = \sum_{n=1}^{3} \frac{x_n^2}{a_n^2 - \lambda_2} - 1 \quad , \quad \text{for } \lambda_2 \in \left(\alpha_3^2, \alpha_2^2\right), \tag{1.8}$$

be the hyperboloid of one sheet that passes through this point and

$$H_2(\mathbf{r}) = \sum_{n=1}^{3} \frac{x_n^2}{a_n^2 - \lambda_1} - 1 \quad , \quad \text{for } \lambda_1 \in \left(\alpha_2^2, \alpha_1^2\right), \tag{1.9}$$

be the hyperboloid of two sheets that passes through this point. The normal vectors to each one of these surfaces are given by their gradients. The corresponding gradients for each one of these surfaces are:

$$\nabla E(\mathbf{r}) = \left(\frac{2x_1}{a_1^2 - \lambda_3}, \frac{2x_2}{a_2^2 - \lambda_3}, \frac{2x_3}{a_3^2 - \lambda_3}\right) \quad , \tag{1.10}$$

$$\nabla H_1(\mathbf{r}) = \left(\frac{2x_1}{a_1^2 - \lambda_2}, \frac{2x_2}{a_2^2 - \lambda_2}, \frac{2x_3}{a_3^2 - \lambda_2}\right) \quad , \tag{1.11}$$

$$\nabla H_2(\mathbf{r}) = \left(\frac{2x_1}{a_1^2 - \lambda_1}, \frac{2x_2}{a_2^2 - \lambda_1}, \frac{2x_3}{a_3^2 - \lambda_1}\right) \quad . \tag{1.12}$$

Therefore since $\mathbf{r}$ belongs on the surfaces E of the ellipsoid and H_1 of the hyperboloid of one sheet:

$$\begin{aligned}
\nabla E(\mathbf{r}) \cdot \nabla H_1(\mathbf{r}) &= \left(\frac{2x_1}{a_1^2 - \lambda_3}, \frac{2x_2}{a_2^2 - \lambda_3}, \frac{2x_3}{a_3^2 - \lambda_3}\right) \cdot \left(\frac{2x_1}{a_1^2 - \lambda_2}, \frac{2x_2}{a_2^2 - \lambda_2}, \frac{2x_3}{a_3^2 - \lambda_2}\right) \\
&= \frac{4x_1^2}{\left(\alpha_1^2 - \lambda_3\right)\left(\alpha_1^2 - \lambda_2\right)} + \frac{4x_2^2}{\left(\alpha_2^2 - \lambda_3\right)\left(\alpha_2^2 - \lambda_2\right)} + \frac{4x_3^2}{\left(\alpha_3^2 - \lambda_3\right)\left(\alpha_3^2 - \lambda_2\right)} \\
= \frac{4}{\lambda_3 - \lambda_2} &\left[\frac{x_1^2(\lambda_3 + \alpha_1^2 - \alpha_1^2 - \lambda_2)}{\left(\alpha_1^2 - \lambda_3\right)\left(\alpha_1^2 - \lambda_2\right)} + \frac{x_2^2(\lambda_3 + \alpha_2^2 - \alpha_2^2 - \lambda_2)}{\left(\alpha_2^2 - \lambda_3\right)\left(\alpha_2^2 - \lambda_2\right)} + \frac{x_3^2(\lambda_3 + \alpha_3^2 - \alpha_3^2 - \lambda_2)}{\left(\alpha_3^2 - \lambda_3\right)\left(\alpha_3^2 - \lambda_2\right)}\right] \\
&= \frac{4}{\lambda_3 - \lambda_2}\left[\sum_{n=1}^{3} \frac{x_n^2}{\left(\alpha_n^2 - \lambda_3\right)} - \sum_{n=1}^{3} \frac{x_n^2}{\left(\alpha_n^2 - \lambda_2\right)}\right] = \frac{4}{\lambda_3 - \lambda_2}(1 - 1) = 0.
\end{aligned}$$

Similarly can be shown that $\nabla E(\mathbf{r}) \cdot \nabla H_2(\mathbf{r}) = \nabla H_1(\mathbf{r}) \cdot \nabla H_2(\mathbf{r}) = 0$. □

Having introduced the possibility of an orthogonal coordinate system based on non-degenerate quadratic surfaces we proceed to define this system and its basic characteristics. Using Lamé's notation [25], let:

$$\rho^2 = \alpha_1^2 - \lambda_3 \quad , \quad \mu^2 = \alpha_2^2 - \lambda_2 \quad , \quad \nu^2 = \alpha_3^2 - \lambda_1 \tag{1.13}$$

be the ellipsoidal coordinates (ρ, μ, ν), which due to the interval of each root λ_n for $n = 1, 2, 3$, they satisfy the following relation:

$$0 \leqslant \nu^2 \leqslant h_3^2 \leqslant \mu^2 \leqslant h_2^2 \leqslant \rho^2 < \infty \tag{1.14}$$

Therefore, from transformation (1.13) and relations (1.6),(1.14), the families of the confocal non-degenerate quadratic surfaces can be written as follows:

i. The family of ellipsoids ((b) in figure (1.1)):

$$\frac{x_1^2}{\rho^2} + \frac{x_2^2}{\rho^2 - h_3^2} + \frac{x_3^2}{\rho^2 - h_2^2} = 1 \quad , \quad \rho^2 \in \left(h_2^2, \infty\right) \quad , \tag{1.15}$$

ii. The family of hyperboloids of one sheet ((c) in figure (1.1)):

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} + \frac{x_3^2}{\mu^2 - h_2^2} = 1 \quad , \quad \mu^2 \in \left(h_3^2, h_2^2\right) \quad , \tag{1.16}$$

iii. The family of hyperboloids of two sheets ((d) in figure (1.1)):

$$\frac{x_1^2}{\nu^2} + \frac{x_2^2}{\nu^2 - h_3^2} + \frac{x_3^2}{\nu^2 - h_2^2} = 1 \quad , \quad \nu^2 \in \left(0, h_3^2\right) \quad , \tag{1.17}$$

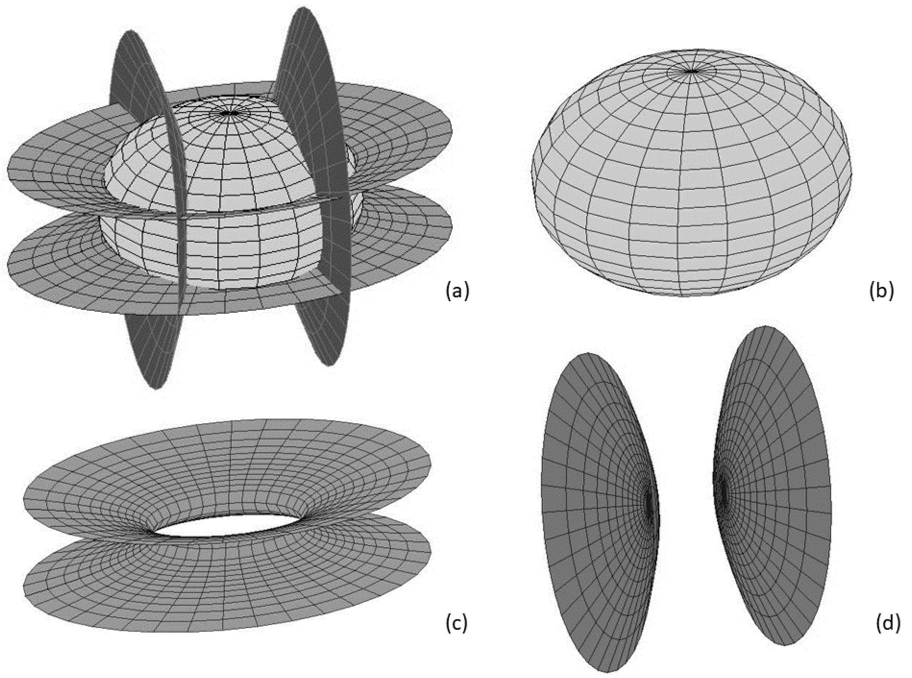


Figure 1.1: (a): the confocal ellipsoidal coordinate system, (b): ellipsoid, (c): hyperboloid of one sheet, (d): hyperboloid of two sheets

From the system of equations (1.15)-(1.17) we obtain the following expressions of the squares of the Cartesian coordinates x_1^2, x_2^2, x_3^2 in terms of the squares of the ellipsoidal coordinates ρ, μ, ν:

$$\begin{aligned} x_1^2 &= \frac{h_1^2}{h_1^2 h_2^2 h_3^2} \rho^2 \mu^2 \nu^2 \quad , \\ x_2^2 &= \frac{h_2^2}{h_1^2 h_2^2 h_3^2} \left(\rho^2 - h_3^2\right)\left(\mu^2 - h_3^2\right)\left(h_3^2 - \nu^2\right) \quad , \\ x_3^2 &= \frac{h_3^2}{h_1^2 h_2^2 h_3^2} \left(\rho^2 - h_2^2\right)\left(h_2^2 - \mu^2\right)\left(h_2^2 - \nu^2\right) \quad , \end{aligned} \tag{1.18}$$

From relations (1.18) and (1.14) we can see the covering of the space from the three non-degenerate quadratic surfaces. For $\nu^2 = 0$ it follows that $x_1 = 0$ and therefore the $x_2 x_3$ plane.

As ν^2 increases, the x_2x_3 plane splits into two planes which bend towards the positive and the negative x_1 axis respectively. These two planes form the two sheets of the family of hyperboloid of two sheets and finally collapse to the exterior of the focal hyperbola at $\nu^2 = h_3^2$. Next, starting at $\mu^2 = h_3^2$, the interior of the focal hyperbola inflates as μ^2 increases forming the family of hyperboloids of one sheet and finally it collapses to the exterior of the focal ellipse $\mu^2 = h_2^2$. Finally, at $\rho^2 = h_2^2$, the interior of the focal ellipsoid inflates as ρ^2 increases forming the family of the ellipsoids which gradually deforms to a sphere at infinity.
The expressions (1.18) show the quadratic symmetry between the Cartesian octants. These symmetries, when expressed in the principal axes x_1, x_2, x_3, are given by the following maps:

$$\begin{aligned}
&(x_1,x_2,x_3) \mapsto (-x_1,x_2,x_3) \quad , \qquad (x_1,x_2,x_3) \mapsto (x_1,-x_2,x_3) \quad , \\
&(x_1,x_2,x_3) \mapsto (x_1,x_2,-x_3) \quad , \qquad (x_1,x_2,x_3) \mapsto (-x_1,-x_2,x_3) \quad , \\
&(x_1,x_2,x_3) \mapsto (-x_1,x_2,-x_3) \quad , \qquad (x_1,x_2,x_3) \mapsto (x_1,-x_2,-x_3) \quad , \\
&(x_1,x_2,x_3) \mapsto (-x_1,-x_2,-x_3) \ ,
\end{aligned} \tag{1.19}$$

Therefore, in order to specify furthermore in which of the eight octants we refer to, we need to adapt some convention rules about the ellipsoidal angular variations. Since ρ is positive these variations depend mainly on the variations of μ, ν as well as the positive and negative branches of $\sqrt{h_3^2 - \nu^2}$, $\sqrt{h_2^2 - \mu^2}$.

Although the expressions of the Cartesian coordinates x_1, x_2, x_3 in terms of the ellipsoidal coordinates ρ, μ, ν are easily deduced, the corresponding expressions of the ellipsoidal coordinates in terms of the Cartesian coordinates are not so easily obtainable. From relations (1.15)-(1.17) we observe that the ellipsoidal coordinates satisfy the same equation:

$$\frac{x_1^2}{\kappa} + \frac{x_2^2}{\kappa - h_3^2} + \frac{x_3^2}{\kappa - h_2^2} = 1 \quad , \quad \kappa \neq 0, h_3^2, h_2^2 \ .$$

This equation can be written as a cubic polynomial which has three distinct roots $\kappa_1, \kappa_2, \kappa_3$, each one of them corresponds to the ellipsoidal coordinates with respect to $\nu^2 \leqslant \mu^2 \leqslant \rho^2$. Therefore, solving this cubic polynomial we can obtain the expressions that give the ellipsoidal coordinates in terms of the Cartesian coordinates.

1.2 Basic Differential Operators in Ellipsoidal Geometry

Based on the previous section and specifically on relation (1.18), the ellipsoidal representation of any point $\mathbf{r} \in \mathbb{R}^3$ is the following:

$$\mathbf{r}(\rho,\mu,\nu) = \left(\frac{\rho\mu\nu}{h_2h_3}, \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1h_3}, \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1h_2} \right) . \tag{1.20}$$

Taking the partial derivatives $\mathbf{r}_\rho, \mathbf{r}_\mu, \mathbf{r}_\nu$ and using the orthogonality relations that they satisfy $\mathbf{r}_\rho \cdot \mathbf{r}_\mu = \mathbf{r}_\rho \cdot \mathbf{r}_\nu = \mathbf{r}_\mu \cdot \mathbf{r}_\nu = 0$, we conclude that the ellipsoidal metric has the following form:

$$(ds)^2 = h_\rho^2 (d\rho)^2 + h_\mu^2 (d\mu)^2 + h_\nu^2 (d\nu)^2 \quad , \tag{1.21}$$

where h_ρ, h_μ, h_ν are the metric coefficients given by:

$$h_\rho = \|\mathbf{r}_\rho\| = \frac{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}}{\sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}} \tag{1.22}$$

$$h_\mu = \|\mathbf{r}_\mu\| = \frac{\sqrt{\rho^2 - \mu^2}\sqrt{\mu^2 - \nu^2}}{\sqrt{h_2^2 - \mu^2}\sqrt{\mu^2 - h_3^2}} \tag{1.23}$$

$$h_\nu = \|\mathbf{r}_\nu\| = \frac{\sqrt{\rho^2 - \nu^2}\sqrt{\mu^2 - \nu^2}}{\sqrt{h_2^2 - \nu^2}\sqrt{h_3^2 - \nu^2}} \tag{1.24}$$

Therefore, the local ellipsoidal coordinate system $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\mu}})$ is defined as follows:

$$\hat{\boldsymbol{\rho}} = \frac{\mathbf{r}_\rho}{\|\mathbf{r}_\rho\|} = \widetilde{T}_\rho \cdot \mathbf{r} = \frac{\rho}{h_\rho} \sum_{n=1}^{3} \frac{x_n}{\rho^2 - \alpha_1^2 + \alpha_n^2} \hat{\mathbf{x}}_{\mathbf{n}}, \tag{1.25}$$

$$\hat{\boldsymbol{\nu}} = \frac{\mathbf{r}_\nu}{\|\mathbf{r}_\nu\|} = \widetilde{T}_\nu \cdot \mathbf{r} = \frac{\nu}{h_\nu} \sum_{n=1}^{3} \frac{x_n}{\nu^2 - \alpha_1^2 + \alpha_n^2} \hat{\mathbf{x}}_{\mathbf{n}}, \tag{1.26}$$

$$\hat{\boldsymbol{\mu}} = \frac{\mathbf{r}_\mu}{\|\mathbf{r}_\mu\|} = \widetilde{T}_\mu \cdot \mathbf{r} = \frac{\mu}{h_\mu} \sum_{n=1}^{3} \frac{x_n}{\mu^2 - \alpha_1^2 + \alpha_n^2} \hat{\mathbf{x}}_{\mathbf{n}}, \tag{1.27}$$

where $\widetilde{T}_\rho, \widetilde{T}_\nu, \widetilde{T}_\mu$ the dyadics that define the Gaussian map at each point.
This local system is dextral with order $\rho \to \nu \to \mu \to \rho$. The identity dyadic in ellipsoidal coordinates has the form:

$$\widetilde{I} = \hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\mu}} \otimes \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\nu}} \otimes \hat{\boldsymbol{\nu}}. \tag{1.28}$$

Taking an arbitrary position vector $\mathbf{r}$ and using relation (1.28) we get:

$$\mathbf{r} \cdot \widetilde{I} = \mathbf{r} \cdot (\hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\mu}} \otimes \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\nu}} \otimes \hat{\boldsymbol{\nu}}). \tag{1.29}$$

Using the basic identity $\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ it follows that:

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\boldsymbol{\rho}})\, \hat{\boldsymbol{\rho}} + (\mathbf{r} \cdot \hat{\boldsymbol{\mu}})\, \hat{\boldsymbol{\mu}} + (\mathbf{r} \cdot \hat{\boldsymbol{\nu}})\, \hat{\boldsymbol{\nu}}. \tag{1.30}$$

Since the arbitrary $\mathbf{r}$ satisfies relation (1.15) for $\rho \in (h_2, \infty)$ we obtain,based on (1.25)-(1.27), the relations:

$$\mathbf{r} \cdot \hat{\boldsymbol{\rho}} = \frac{\rho}{h_\rho} \sum_{n=1}^{3} \frac{x_n^2}{\rho^2 - \alpha_1^2 + \alpha_n^2} = \frac{\rho}{h_\rho}, \tag{1.31}$$

$$\mathbf{r} \cdot \hat{\boldsymbol{\mu}} = \frac{\mu}{h_\mu} \sum_{n=1}^{3} \frac{x_n^2}{\mu^2 - \alpha_1^2 + \alpha_n^2} = \frac{\mu}{h_\mu}, \tag{1.32}$$

$$\mathbf{r} \cdot \hat{\boldsymbol{\nu}} = \frac{\mu}{h_\nu} \sum_{n=1}^{3} \frac{x_n^2}{\nu^2 - \alpha_1^2 + \alpha_n^2} = \frac{\nu}{h_\nu}. \tag{1.33}$$

Therefore, the position vector $\mathbf{r}$ in ellipsoidal coordinates is given by:

$$\mathbf{r} = \frac{\rho}{h_\rho} \hat{\boldsymbol{\rho}} + \frac{\mu}{h_\mu} \hat{\boldsymbol{\mu}} + \frac{\nu}{h_\nu} \hat{\boldsymbol{\nu}} \tag{1.34}$$

The gradient operator in ellipsoidal coordinates has the following expression:

$$\nabla = \frac{\hat{\boldsymbol{\rho}}}{h_\rho} \frac{\partial}{\partial \rho} + \frac{\hat{\boldsymbol{\mu}}}{h_\mu} \frac{\partial}{\partial \mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu} \frac{\partial}{\partial \nu}. \tag{1.35}$$

Let $\mathbf{F}$ be a vector field with ellipsoidal expression:

$$\mathbf{F} = (\mathbf{F} \cdot \hat{\boldsymbol{\rho}})\, \hat{\boldsymbol{\rho}} + (\mathbf{F} \cdot \hat{\boldsymbol{\mu}})\, \hat{\boldsymbol{\mu}} + (\mathbf{F} \cdot \hat{\boldsymbol{\nu}})\, \hat{\boldsymbol{\nu}} = F^\rho \hat{\boldsymbol{\rho}} + F^\mu \hat{\boldsymbol{\mu}} + F^\nu \hat{\boldsymbol{\nu}}, \tag{1.36}$$

then from (1.35) we obtain:

$$\nabla \otimes \mathbf{F} = \frac{\hat{\boldsymbol{\rho}}}{h_\rho} \otimes \frac{\partial \mathbf{F}}{\partial \rho} + \frac{\hat{\boldsymbol{\mu}}}{h_\mu} \otimes \frac{\partial \mathbf{F}}{\partial \mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu} \otimes \frac{\partial \mathbf{F}}{\partial \nu}. \tag{1.37}$$

Since the order of the dextral system is (ρ, ν, μ), the curl of the vector field $\mathbf{F}$ in ellipsoidal coordinates is given by:

$$\begin{aligned} \nabla \times \mathbf{F} = \frac{1}{h_\rho h_\mu h_\nu} \Bigg[& h_\rho \left(\frac{\partial}{\partial \nu} h_\mu F^\mu - \frac{\partial}{\partial \mu} h_\nu F^\nu \right) \hat{\boldsymbol{\rho}} + h_\nu \left(\frac{\partial}{\partial \mu} h_\rho F^\rho - \frac{\partial}{\partial \rho} h_\mu F^\mu \right) \hat{\boldsymbol{\nu}} \\ & + h_\mu \left(\frac{\partial}{\partial \rho} h_\nu F^\nu - \frac{\partial}{\partial \nu} h_\rho F^\rho \right) \hat{\boldsymbol{\mu}} \Bigg] \end{aligned} \tag{1.38}$$

and the divergence in ellipsoidal coordinates is given by:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_\rho h_\mu h_\nu} \left[\frac{\partial}{\partial \rho} \left(h_\mu h_\nu F^\rho \right) + \frac{\partial}{\partial \mu} \left(h_\nu h_\rho F^\mu \right) + \frac{\partial}{\partial \nu} \left(h_\rho h_\mu F^\nu \right) \right]. \tag{1.39}$$

For the Laplacian operator, based on relation $\Delta = \nabla \cdot \nabla$ and the form of the Laplace equation in general curvilinear coordinates, which is proved in the last section of this chapter (1.266), by substituting the metric coefficients (1.22)-(1.24), we obtain:

$$\begin{aligned} \Delta = & \frac{1}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)} \left[\left(\rho^2 - h_3^2\right)\left(\rho^2 - h_2^2\right) \frac{\partial^2}{\partial \rho^2} + \rho \left(2\rho^2 - h_2^2 - h_3^2\right) \frac{\partial}{\partial \rho} \right] \\ & + \frac{1}{(\mu^2 - \rho^2)(\mu^2 - \nu^2)} \left[\left(\mu^2 - h_3^2\right)\left(\mu^2 - h_2^2\right) \frac{\partial^2}{\partial \mu^2} + \mu \left(2\mu^2 - h_2^2 - h_3^2\right) \frac{\partial}{\partial \mu} \right] \\ & + \frac{1}{(\nu^2 - \rho^2)(\nu^2 - \mu^2)} \left[\left(\nu^2 - h_3^2\right)\left(\nu^2 - h_2^2\right) \frac{\partial^2}{\partial \nu^2} + \nu \left(2\nu^2 - h_2^2 - h_3^2\right) \frac{\partial}{\partial \nu} \right] \quad , \end{aligned} \tag{1.40}$$

or equivalently:

$$\begin{aligned} \Delta = & \frac{1}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)} \left(\sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{\partial}{\partial \rho} \right) \left(\sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{\partial}{\partial \rho} \right) \\ & + \frac{1}{(\rho^2 - \mu^2)(\mu^2 - \nu^2)} \left(\sqrt{\mu^2 - h_3^2} \sqrt{h_2^2 - \mu^2} \frac{\partial}{\partial \mu} \right) \left(\sqrt{\mu^2 - h_3^2} \sqrt{h_2^2 - \mu^2} \frac{\partial}{\partial \mu} \right) \\ & + \frac{1}{(\rho^2 - \nu^2)(\mu^2 - \nu^2)} \left(\sqrt{h_3^2 - \nu^2} \sqrt{h_2^2 - \nu^2} \frac{\partial}{\partial \nu} \right) \left(\sqrt{h_3^2 - \nu^2} \sqrt{h_2^2 - \nu^2} \frac{\partial}{\partial \nu} \right) \quad . \end{aligned} \tag{1.41}$$

The study of the Laplacian constitutes the basis of the Ellipsoidal Harmonics since they arise from the solution of the Laplace equation in the ellipsoidal coordinate system. In [25], Lamé introduced the ellipsoidal coordinates in order to study the temperature distribution of an ellipsoid in thermal equilibrium. In order to overcome the difficulties in the solution of the Laplace equation in ellipsoidal coordinates using separation of variables, Lamé introduced the thermometric parameters. Specifically, based on the ellipsoidal form of the Laplace equation (1.41), he introduced the variables (ξ, η, ζ) as follows:

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \sqrt{\rho^2 - h_3^2} \sqrt{\rho^2 - h_2^2} \frac{\partial}{\partial \rho} \quad , \\ \frac{\partial}{\partial \eta} &= \sqrt{\mu^2 - h_3^2} \sqrt{h_2^2 - \mu^2} \frac{\partial}{\partial \mu} \quad , \\ \frac{\partial}{\partial \zeta} &= \sqrt{h_3^2 - \nu^2} \sqrt{h_2^2 - \nu^2} \frac{\partial}{\partial \nu} \end{aligned} \tag{1.42}$$

and used the terminology *Thermometric parameters*. Using the chain rule, these three relations transform into ordinary differential equations with solutions:

$$\begin{aligned} \xi(\rho) &= \int_{h_2}^{\rho} \frac{dt}{\sqrt{t^2 - h_3^2}\sqrt{t^2 - h_2^2}} \quad , \quad h_2 \leq \rho < \infty \quad , \\ \eta(\mu) &= \int_{h_3}^{\mu} \frac{dt}{\sqrt{t^2 - h_3^2}\sqrt{h_2^2 - t^2}} \quad , \quad h_3 \leq \mu \leq h_2 \quad , \\ \zeta(\nu) &= \int_{0}^{\nu} \frac{dt}{\sqrt{h_3^2 - t^2}\sqrt{h_2^2 - t^2}} \quad , \quad 0 \leq \nu \leq h_3 \quad . \end{aligned} \tag{1.43}$$

Therefore, the Laplacian operator has the following expression in terms of (ξ, η, ζ):

$$\Delta = \frac{1}{(\rho^2-\mu^2)(\rho^2-\nu^2)}\frac{\partial^2}{\partial\xi^2} + \frac{1}{(\rho^2-\mu^2)(\mu^2-\nu^2)}\frac{\partial^2}{\partial\eta^2} + \frac{1}{(\rho^2-\nu^2)(\mu^2-\nu^2)}\frac{\partial^2}{\partial\zeta^2} \quad , \tag{1.44}$$

Correspondingly, the Laplace equation in terms of (ξ, η, ζ) has the form:

$$\left(\mu^2-\nu^2\right)\frac{\partial^2 u}{\partial\xi^2} + \left(\rho^2-\nu^2\right)\frac{\partial^2 u}{\partial\eta^2} + \left(\rho^2-\mu^2\right)\frac{\partial^2 u}{\partial\zeta^2} = 0 \quad . \tag{1.45}$$

It is clear that (ξ, η, ζ) are themselves solutions of the Laplace's equation and thus the normal solution $u = (A_1\xi + B_1)(A_2\eta + B_2)(A_3\zeta + B_3)$ satisfies equation (1.45) for appropriate constants A_n, B_n , $n = 1, 2, 3,$. Since (1.45) is inseparable, substituting the normal solution $u(\rho, \mu, \nu) = R(\rho)M(\mu)N(\nu)$ where R depends only on variable ρ, $M(\mu)$ depends only on variable μ and $N(\nu)$ depends only on variable ν, we obtain:

$$\frac{\left(\mu^2-\nu^2\right)}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial\xi^2} + \frac{\left(\rho^2-\nu^2\right)}{M(\mu)}\frac{\partial^2 M(\mu)}{\partial\eta^2} + \frac{\left(\rho^2-\mu^2\right)}{N(\nu)}\frac{\partial^2 N(n)}{\partial\zeta^2} = 0 \quad , \tag{1.46}$$

which can not be broken up by the usual method of separation of variables. If however we use the transformation:

$$\frac{1}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial\xi^2} = \sum a_n\rho^n \quad , \quad \frac{1}{M(\mu)}\frac{\partial^2 M(\mu)}{\partial\eta^2} = \sum b_n\mu^n \quad , \quad \frac{1}{N(\nu)}\frac{\partial^2 N(\nu)}{\partial\zeta^2} = \sum c_n\nu^n \quad , \tag{1.47}$$

and substitute in (1.46) we get:

$$\left(\mu^2-\nu^2\right)\sum a_n\rho^n + \left(\rho^2-\nu^2\right)\sum b_n\mu^n + \left(\rho^2-\mu^2\right)\sum c_n\nu^n = 0 \quad , \tag{1.48}$$

from which we find that the coefficients are zero for all values of n except $n = 0$ and $n = 2$ where $a_0 = -b_0 = c_0$ and $a_2 = -b_2 = c_2$. Thus, the solutions of (1.47) are:

$$\begin{aligned}
\frac{1}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial\xi^2} &= \left(a_0 + a_2\rho^2\right)R(\rho) \ , \\
\frac{1}{M(\mu)}\frac{\partial^2 M(\mu)}{\partial\eta^2} &= -\left(a_0 + a_2\mu^2\right)M(\mu) \ , \\
\frac{1}{N(\nu)}\frac{\partial^2 N(\nu)}{\partial\zeta^2} &= \left(a_0 + a_2\nu^2\right)N(\nu) \ .
\end{aligned}$$

Therefore, equation (1.46) can be broken up into the three equations:

$$\frac{\partial^2 R(\rho)}{\partial\xi^2} + \left(A\rho^2 + B\right)R(\rho) = 0 \quad , \tag{1.49}$$

$$-\frac{\partial^2 M(\mu)}{\partial\eta^2} + \left(A\mu^2 + B\right)M(\mu) = 0 \quad , \tag{1.50}$$

$$\frac{\partial^2 N(\nu)}{\partial\zeta^2} + \left(A\nu^2 + B\right)N(\nu) = 0 \quad , \tag{1.51}$$

Based on relations (1.40),(1.41) and (1.42) the above three equations can be written as:

$$\left(\rho^2-h_3^2\right)\left(\rho^2-h_2^2\right)R''(\rho) + \rho\left(2\rho^2-h_2^2-h_3^2\right)R'(\rho) + \left(A\rho^2+B\right)R(\rho) = 0 \quad , \tag{1.52}$$

$$\left(\mu^2-h_3^2\right)\left(\mu^2-h_2^2\right)M''(\mu) + \mu\left(2\mu^2-h_2^2-h_3^2\right)M'(\mu) + \left(A\mu^2+B\right)M(\mu) = 0 \quad , \tag{1.53}$$

$$\left(\nu^2-h_3^2\right)\left(\nu^2-h_2^2\right)N''(\nu) + \rho\left(2\nu^2-h_2^2-h_3^2\right)N'(\nu) + \left(A\nu^2+B\right)N(\nu) = 0 \quad . \tag{1.54}$$

which are identical with only difference the intervals in which ρ, μ, ν belong to. Therefore the ordinary differential equation that R, M, N satisfy is:

$$\left(x^2 - h_3^2\right)\left(x^2 - h_2^2\right) E''(x) + x\left(2x^2 - h_2^2 - h_3^2\right) E'(x) + \left(Ax^2 + B\right) E(x) = 0 \quad , \tag{1.55}$$

for $x = \rho, \mu, \nu$ in the intervals $(h_2, \infty), (h_3, h_2), (0, h_3)$ respectively.

In order to determine the separation constant A we use the sphero-conal (or conical) system which is studied in the last section. Applying separation of variables for the conical system (the same way we did for the ellipsoidal system), it is observed that the radial part is the same with the radial part of the spherical system while the angular part is the same with the angular part of the ellipsoidal system (which means the part of M and N). Both these parts contain the separation constant A. Using the correspondence of the radial of the conical system to the radial part of spherical system and based on [21], we take $A = -n(n+1)$ which is the separation constant in the radial part of the spherical system. Hence, (1.55) can be written as:

$$\left(x^2 - h_3^2\right)\left(x^2 - h_2^2\right) E''(x) + x\left(2x^2 - h_2^2 - h_3^2\right) E'(x) + \left(B - n(n+1)x^2\right) E(x) = 0 \quad , \tag{1.56}$$

and the separation constant B is taken as $B = (h_2^2 + h_3^2)p$ with p left to be determined in the next section. This equation is called Lamé equations and its solutions are called Lamé functions.

1.3 Lamé Functions

Based on the spherical harmonics (1.330) which are solutions of the Laplace equation, it can be seen that $P_n^m(cos\theta)cosm\phi$ and $P_n^m(cos\theta)sinm\phi$ are rational functions of $\mu\nu$, $\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}$ and $\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}$. This leads, after calculations which can be found in [20], to the separation of Lamé functions into four classes:

- **Class K:** $K = P_n(x)$,
- **Class L:** $L = \sqrt{|x^2 - h_2^3|}P_{n-1}(x)$,
- **Class M:** $M = \sqrt{|x^2 - h_2^2|}P_{n-1}(x)$,
- **Class N:** $N = \sqrt{|x^2 - h_3^2|}\sqrt{|x^2 - h_2^2|}P_{n-2}(x)$,

where $P_n(x) = \alpha_0 x^n + \alpha_1 x^{n-2} + ...$ and $x = \rho, \mu, \nu$.

1.3.1 Lamé Functions of Class K:

Based on the form of Lamé functions of class K (1.3), we denote the polynomial P_n with $K_n(x)$, which has an expansion of the form:

$$K_n(x) = a_0 x^n + a_1 x^{n-2} + a_2 x^{n-4} + ... = \sum_{k=0}^{\infty} a_k x^{n-2k} \quad , \quad a_0 \neq 0 \quad . \tag{1.57}$$

This function must satisfy the Lamé equation (1.55) which equivalently can be written in following form:

$$\left(x^4 - \alpha x^2 + \beta\right) K_n''(x) + \left(2x^3 - \alpha x\right) K_n'(x) + \left[\alpha p - n(n+1)x^2\right] K_n(x) = 0 \; , \tag{1.58}$$

where $\alpha = h_2^2 + h_3^2$, $\beta = h_2^2 h_3^2$ and p the dimensionless parameter via $B = \left(h_2^2 + h_3^2\right)p$. Since:

$$K_n'(x) = \sum_{k=0}^{\infty}(n-2k)a_k x^{n-2k-1} \quad , \quad K_n''(x) = \sum_{k=0}^{\infty}(n-2k)(n-2k-1)a_k x^{n-2k-2} \; , \tag{1.59}$$

by substituting in (1.58), we obtain the following equation:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k)(n-2k-1)a_k x^{n-2k+2} - \alpha\sum_{k=0}^{\infty}(n-2k)(n-2k-1)a_k x^{n-2k} \\
&+\beta\sum_{k=0}^{\infty}(n-2k)(n-2k-1)a_k x^{n-2k-2} + 2\sum_{k=0}^{\infty}(n-2k)a_k x^{n-2k+2} \\
&-\alpha\sum_{k=0}^{\infty}(n-2k)a_k x^{n-2k} + \alpha p\sum_{k=0}^{\infty}a_k x^{n-2k} - n(n+1)\sum_{k=0}^{\infty}a_k x^{n-2k+2} = 0
\end{aligned} \tag{1.60}$$

Turning all the sums into x^{n-2k+2} we obtain:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k)(n-2k-1)a_k x^{n-2k+2} - \alpha\sum_{k=1}^{\infty}(n-2k+2)(n-2k+1)a_{k-1} x^{n-2k+2} \\
&+\beta\sum_{k=2}^{\infty}(n-2k+4)(n-2k+3)a_{k-2} x^{n-2k-2} + 2\sum_{k=0}^{\infty}(n-2k)a_k x^{n-2k+2} \\
&-\alpha\sum_{k=1}^{\infty}(n-2k+2)a_{k-1} x^{n-2k+2} + \alpha p\sum_{k=1}^{\infty}a_{k-1} x^{n-2k+2} - n(n+1)\sum_{k=0}^{\infty}a_k x^{n-2k+2} = 0\ .
\end{aligned} \tag{1.61}$$

From here it is easily concluded that the above equation can be written as:

$$\sum_{k=0}^{\infty} A_k x^{n-2k+2} = 0, \tag{1.62}$$

where

$$A_k = (2k+2)(2n-2k-1)a_{k+1} - \alpha\left[p-(n-2k)^2\right]a_k - \beta(n-2k+2)(n-2k+1)a_{k-1}, \tag{1.63}$$

$$k = 0, 1, 2, \ldots\ .$$

Since the polynomial (1.62) is equivalent to 0, therefore, all the coefficients A_k must be equal to 0.

- For $k = 0$: $[n(n-1)a_0 + 2na_0 - n(n+1)a_0]\,x^{n+2} \Rightarrow 0a_0 = 0$. Therefore, we chose arbitrarily $a_0 = 1$.

- For $k = 1$: $[(n-2)(n-3)a_1 - \alpha n(n-1)a_0 + 2(n-2)a_1 - \alpha n a_0 + \alpha p a_0 - n(n+1)a_1]\,x^n$ $\Rightarrow 2a_1(-2n+1) + \alpha a_0(p-n^2) = 0 \Rightarrow 2(2n-1)a_1 = \alpha\left(p-n^2\right)a_0$.

- For $k = 2$: $4(2n-3)a_2 = \alpha\left[p-(n-2)^2\right]a_1 + \beta n(n-1)a_0$.

$$\vdots$$

- For $k = r$: $2r(2n-2r+1)a_r = \alpha\left[p-(n-2r+2)^2\right]a_{r-1}$ $+\beta(n-2r+4)(n-2r+3)a_{r-2}$.

- For $k = r+1$: $(2r+2)(2n-2r-1)a_{r+1} = \alpha\left[p-(n-2r)^2\right]a_r + \beta(n-2r+2)(n-2r+1)a_{r-1}$.

- For $k = r+2$: $(2r+4)(2n-2r-3)a_{r+2} = \alpha\left[p-(n-2r-2)^2\right]a_{r+1}$ $+\beta(n-2r)(n-2r-1)a_r$.

$$\vdots$$

Therefore the relation that connects the coefficients is the following:

$$2k(2n-2k+1)a_k = \alpha\left[p-(n-2k+2)^2\right]a_{k-1}+\beta(n-2k+4)(n-2k+3)a_{k-2}, \quad k=0,1,2,\ldots \tag{1.64}$$

Choosing:

$$r=\begin{cases}\dfrac{n}{2}\ , \text{ for n even,}\\ \dfrac{n-1}{2}\ , \text{ for n odd,}\end{cases} \tag{1.65}$$

the coefficient of βa_r in the case of $k=r+2$ is vanishing, leaving a relation between a_{r+2} and a_{r+1}. Therefore, choosing the dimensionless parameter p so that $a_{r+1}=0$, it is concluded that $a_{r+2}=a_{r+3}=\ldots=0$ which stops the series of descending in powers of x and leaves us with a polynomial of degree n. Since the dimensionless parameter is determined so $a_{r+1}=a_{r+2}=\ldots=0$, the coefficients $a_0,a_1,\ldots,a_r$ create a $(r+1)\times(r+1)$ linear homogeneous system. In order to find a non trivial solution of the Lamé's equation (1.58) we want to determine the values of the parameter p so that the following system created from the coefficients $a_0,a_1,\ldots,a_r$ will have non trivial solutions:

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 & \ldots & 0 & 0 & 0 \\ & & & \ldots & \ldots & \ldots & & & \\ & & & \ldots & \ldots & \ldots & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & K_{r(r-1)} & K_{rr} & K_{r(r+1)} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & K_{(r+1)r} & K_{(r+1)(r+1)} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_r \end{bmatrix}=0\ , \tag{1.66}$$

where

$$K_{ii}=-\alpha\left[p-(n-2i+2)^2\right]\ ,\quad i=1,2,3,\ldots,r+1, \tag{1.67}$$

$$K_{j(j+1)}=2j(2n-2j+1)\ ,\quad j=1,2,3,\ldots,r, \tag{1.68}$$

$$K_{(j+1)j}=-\beta\,(n-2j+1)\,(n-2j+2)\ ,\quad j=1,2,3,\ldots,r. \tag{1.69}$$

Therefore, the following relation must be satisfied:

$$\begin{vmatrix} K_{11} & K_{12} & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & K_{32} & K_{33} & K_{34} & 0 & \ldots & 0 & 0 & 0 \\ & & & \ldots & \ldots & \ldots & & & \\ & & & \ldots & \ldots & \ldots & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & K_{r(r-1)} & K_{rr} & K_{r(r+1)} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & K_{(r+1)r} & K_{(r+1)(r+1)} \end{vmatrix}=0 \tag{1.70}$$

Based on [20] and some straightforward calculations it is concluded that:

- For $n=0$ we have $r=0$ and based on the relation that connects the coefficients for $k=1$ we determine the parameter p such that $a_1=0$ which leads to $p=0$ and gives the following Lamé function:

$$K_0^1(x)=1\ . \tag{1.71}$$

- For $n=1$ we have $r=0$ and based on the relation that connects the coefficients for $k=1$ we determine the parameter p such that $a_1=0$ which leads to $p=1$ and gives the following Lamé function:

$$K_1^1(x)=x\ . \tag{1.72}$$

- For $n = 2$ we have $r = 1$ and based on the relations that connect the coefficients for $k = 2$ and $k = 1$, in order to have $a_2 = 0$, the paramater takes as values the roots of $\frac{\alpha^2}{6}(p-4)p + 2\beta = 0$:

$$p_1 = 2 + 2\sqrt{1 - 3\frac{\beta}{\alpha^2}} \text{ and } p_2 = 2 - 2\sqrt{1 - 3\frac{\beta}{\alpha^2}} \ , \tag{1.73}$$

which give the following Lamé functions:

$$K_2^1(x) = x^2 + \Lambda - a_1^2 \text{ and } K_2^2(x) = x^2 + \Lambda' - a_1^2 \ , \tag{1.74}$$

where

$$\Lambda = a_1^2 - \frac{h_3^2 + h_2^2}{3} + \frac{\sqrt{h_1^4 + h_3^2 h_2^2}}{3} \ , \tag{1.75}$$

$$\Lambda' = a_1^2 - \frac{h_3^2 + h_2^2}{3} - \frac{\sqrt{h_1^4 + h_3^2 h_2^2}}{3} \ . \tag{1.76}$$

- For $n = 3$, similarly with the previous steps, we have:

$$p_1 = 5 + 2\sqrt{4 - 15\frac{\beta}{\alpha^2}} \text{ and } p_2 = 5 - 2\sqrt{4 - 15\frac{\beta}{\alpha^2}} \ , \tag{1.77}$$

which give the following Lamé functions:

$$K_3^1(x) = x^3 + \left(\Lambda_1 - a_1^2\right) x \text{ and } K_3^2(x) = x^3 + \left(\Lambda_1^{'} - a_1^2\right) x \ , \tag{1.78}$$

where

$$\Lambda_1 = a_1^2 - \frac{2h_3^2 + h_2^2}{5} + \frac{\sqrt{4h_1^4 + h_3^2 h_2^2}}{5} \ , \tag{1.79}$$

$$\Lambda_1^{'} = a_1^2 - \frac{2h_3^2 + 2h_2^2}{5} - \frac{\sqrt{4h_1^4 + h_3^2 h_2^2}}{5} \ . \tag{1.80}$$

1.3.2 Lamé Functions of Class L :

Based on (1.3) the Lamé functions of class L are of the following form:

$$\sqrt{|x^2 - h_3^2|} P_{n-1}(x), \tag{1.81}$$

where in this subsection we denote the polynomial factor P_{n-1} with $L(x)$. This means that we need to study the cases of $x = \rho, x = \mu$ and $x = \nu$ separately because of the square root. Nevertheless, since the polynomial factor of the solution isn't affected by which one of the three cases we study we will assume that $x = \rho$ which leaves us with the form:

$$L_n(x) = \sqrt{x^2 - h_3^2} L(x) \ , \quad n = 0, 1, 2, ... \ , \tag{1.82}$$

where

$$L(x) = a_0 x^{n-1} + a_1 x^{n-3} + a_2 x^{n-5} + ... = \sum_{k=0}^{\infty} a_k x^{n-2k-1} \ . \tag{1.83}$$

$$L_n^{'}(x) = \frac{x}{\sqrt{x^2 - h_3^2}} L(x) + \sqrt{x^2 - h_3^2} L'(x) \ , \tag{1.84}$$

$$L_n^{''}(x) = \left(\frac{-h_3^2}{\sqrt{x^2 - h_3^2}\left(x^2 - h_3^2\right)} \right) L(x) + \left(\frac{x}{\sqrt{x^2 - h_3^2}} \right) L'(x) + \sqrt{x^2 - h_3^2} L''(x). \tag{1.85}$$

Substituting (1.82),(1.84) and (1.85) to the Lamé's equation (1.56) we obtain the following equation:

$$
\begin{aligned}
&\left(x^2-h_3^2\right)\left(x^2-h_2^2\right)\left[\left(\frac{-h_3^2}{\sqrt{x^2-h_3^2}\left(x^2-h_3^2\right)}\right)L(x)+\left(\frac{x}{\sqrt{x^2-h_3^2}}\right)L'(x)+\sqrt{x^2-h_3^2}L''(x)\right]\\
&+x\left(2x^2-h_3^2-h_2^2\right)\left[\frac{x}{\sqrt{x^2-h_3^2}}L(x)+\sqrt{x^2-h_3^2}L'(x)\right]\\
&+\left[\alpha p-n(n+1)x^2\right]\sqrt{x^2-h_3^2}L(x)=0\ ,\quad n=0,1,2,\dots\ .
\end{aligned}
\tag{1.86}
$$

Dividing with $\sqrt{x^2-h_3^2}$ and rearranging the terms the above equation can be written as follows:

$$
\begin{aligned}
&\left(x^2-h^2\right)\left(x^2-h_2^2\right)L''(x)+\left[x\left(\left(x^2-h_2^2\right)+\left(x^2-h_3^2\right)\right)+2x\left(x^2-h_2^2\right)\right]L'(x)\\
&+\left[-\frac{\left(x^2-h_2^2\right)h_3^2}{\left(x^2-h_3^2\right)}+\frac{x^2\left(x^2-h_2^2\right)}{\left(x^2-h_3^2\right)}+x^2+\alpha p-n(n+1)x^2\right]L(x)=0.
\end{aligned}
\tag{1.87}
$$

Equivalently:

$$
\begin{aligned}
&\left(x^4-\alpha x^2+\beta\right)L''(x)+x\left(4x^2-\alpha-2h_2^2\right)L'(x)\\
&+\left[\alpha p-h_2^2-(n-1)(n+2)x^2\right]L(x)=0\quad .
\end{aligned}
\tag{1.88}
$$

The derivatives of the polynomial factor $L(x)$ are the following:

$$
L'(x)=\sum_{k=0}^{\infty}(n-2k-1)a_kx^{n-2k-2}\quad ,\quad L''(x)=\sum_{k=0}^{\infty}(n-2k-1)(n-2k-2)a_kx^{n-2k-3}
\tag{1.89}
$$

Substituting (1.83) and (1.89) into (1.88) we obtain:

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k-1)(n-2k-2)a_kx^{n-2k+1}-\alpha\sum_{k=0}^{\infty}(n-2k-1)(n-2k-2)a_kx^{n-2k-1}+\\
&\beta\sum_{k=0}^{\infty}(n-2k-1)(n-2k-2)a_kx^{n-2k-3}+4\sum_{k=0}^{\infty}(n-2k-1)a_kx^{n-2k+1}\\
&-\alpha\sum_{k=0}^{\infty}(n-2k-1)a_kx^{n-2k-1}-2h_2^2\sum_{k=0}^{\infty}(n-2k-1)a_kx^{n-2k-1}\\
&+\alpha p\sum_{k=0}^{\infty}a_kx^{n-2k-1}-h_2^2\sum_{k=0}^{\infty}a_kx^{n-2k-1}-(n-1)(n+2)\sum_{k=0}^{\infty}a_kx^{n-2k+1}=0
\end{aligned}
\tag{1.90}
$$

Turning them all into series of x^{n-2k+1} the equation takes the following form:

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k-1)(n-2k-2)a_kx^{n-2k+1}-\alpha\sum_{k=1}^{\infty}(n-2k+1)(n-2k)a_{k-1}x^{n-2k+1}\\
&+\beta\sum_{k=2}^{\infty}(n-2k+3)(n-2k+2)a_{k-2}x^{n-2k+1}+4\sum_{k=0}^{\infty}(n-2k-1)a_kx^{n-2k+1}\\
&-\alpha\sum_{k=1}^{\infty}(n-2k+1)a_{k-1}x^{n-2k+1}-2h_2^2\sum_{k=1}^{\infty}(n-2k+1)a_{k-1}x^{n-2k+1}\\
&+\alpha p\sum_{k=1}^{\infty}a_{k-1}x^{n-2k+1}-h_2^2\sum_{k=1}^{\infty}a_{k-1}x^{n-2k+1}-(n-1)(n+2)\sum_{k=0}^{\infty}a_kx^{n-2k+1}=0
\end{aligned}
\quad .
\tag{1.91}
$$

From here it is easily concluded that the above equation can be written in the following form:

$$\sum_{k=0}^{\infty} A_k x^{n-2k+1} = 0 \quad , \tag{1.92}$$

where

$$\begin{aligned} A_k = &-2k(2n-2k+1)a_k + \left[\alpha\left(p-(n-2k+1)^2\right) - h_2^2(2n-4k+3)\right] a_{k-1} \\ &+ \beta(n-2k+3)(n-2k+2)a_{k-2} \ , \quad k = 0,1,2,... \ , \end{aligned} \tag{1.93}$$

Since the series is equal to 0, the coefficients A_k must be equal to 0 for $k = 0, 1, 2, ...$.

- For $k = 0$:
  $[(n-1)(n-2) + 4(n-1) - (n-1)(n+2)]\, a_0 = 0 \Rightarrow 0a_0 = 0$.
  Therefore, we chose arbitrarily $a_0 = 1$.

- For $k = 1$:
  $[(n-3)(n-4) + 4(n-3) - (n-1)(n+2)]\, a_1$
  $- \left[\alpha(n-1)(n-2) + \alpha(n-1) + 2h_2^2(n-1) - \alpha p + h_2^2\right] a_0 = 0$
  $\Rightarrow 2(2n-1)a_1 = \left[\alpha\left(p-(n-1)^2\right) - (2n-1)h_2^2\right] a_0$.

- For $k = 2$:
  $4(2n-3)a_2 = \left[\alpha\left(p-(n-3)^2\right) - (2n-5)h_2^2\right] a_1 + \beta(n-1)(n-2)b_0$.

$$\vdots$$

- For $k = r$:
  $2r(2n-2r+1)a_r = \left[\alpha\left(p-(n-2r+1)^2\right) - (2n-4r+3)h_2^2\right] a_{r-1}$
  $+ \beta(n-2r+3)(n-2r+2)a_{r-2}$.

- For $k = r+1$:
  $2(r+1)(2n-2r-1)a_{r+1} = \left[\alpha\left(p-(n-2r-1)^2\right) - (2n-4r-1)h_2^2\right] a_r$
  $+ \beta(n-2r+1)(n-2r)a_{r-1}$.

$$\vdots$$

Therefore the relation that connects the coefficients is the following:

$$\begin{aligned} 2k(2n-2k+1)a_k = &\left[\alpha\left(p-(n-2k+1)^2\right) - (2n-4k+3)h_2^2\right] a_{k-1} \\ &+ \beta(n-2k+3)(n-2k+2)a_{k-2} \ , \quad k = 0,1,2,... \ . \end{aligned} \tag{1.94}$$

Choosing:

$$r = \begin{cases} \dfrac{n}{2} \ , \ \text{for n even} \\ \dfrac{n+1}{2} \ , \ \text{for n odd} \ , \end{cases} \tag{1.95}$$

the coefficient of βa_{r-1} in the case of $k = r+1$ is vanishing, leaving a relation between the coefficients a_{r+1} and a_r. Therefore, choosing the dimensionless parameter p so that $a_r = 0$, is is concluded that $a_{r+1} = a_{r+2} = ... = 0$ which stops the series from descending in powers of x and leaves us a polynomial of degree $n-1$. Since the dimensionless parameter is determined so $a_{r+1} = a_{r+2} = ... = 0$, the coefficients $a_0, a_1, ..., a_r$ create a $r \times r$ linear homogeneous system. In order to find a non trivial solution of (1.88) we want to determine the values of p so that the following system will have non trivial solutions:

$$\begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 & \dots & 0 & 0 & \\ L_{21} & L_{22} & L_{23} & 0 & 0 & \dots & 0 & 0 & \\ 0 & L_{32} & L_{33} & L_{34} & 0 & \dots & 0 & 0 & \\ & & \dots & \dots & \dots & & & & \\ & & \dots & \dots & \dots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & L_{(r-1)(r-2)} & L_{(r-1)(r-1)} & L_{(r-1)r} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & L_{r(r-1)} & L_{rr} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{r-1} \end{bmatrix} = 0 \;, \tag{1.96}$$

where

$$L_{ii} = -\alpha \left[p - (n-2i+1)^2\right] + (2n-4i+3)h_2^2 \quad i = 1,2,...,r \;, \tag{1.97}$$
$$K_{j(j+1)} = 2j(2n-2j+1) \;, \quad j = 1,2,...,r-1 \;, \tag{1.98}$$
$$L_{(j+1)j} = -\beta(n-2j+1)(n-2j) \;, \quad j = 1,2,...,r-1 \;. \tag{1.99}$$

Therefore, the following relation must be satisfied:

$$\begin{vmatrix} L_{11} & L_{12} & 0 & 0 & 0 & \dots & 0 & 0 & \\ L_{21} & L_{22} & L_{23} & 0 & 0 & \dots & 0 & 0 & \\ 0 & L_{32} & L_{33} & L_{34} & 0 & \dots & 0 & 0 & \\ & & \dots & \dots & \dots & & & & \\ & & \dots & \dots & \dots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & L_{(r-1)(r-2)} & L_{(r-1)(r-1)} & L_{(r-1)r} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & L_{r(r-1)} & L_{rr} \end{vmatrix} = 0 \tag{1.100}$$

Based on [20] and some straightforward calculations it is concluded that:

- For $n = 0$ we observe from (1.83) that there are no Lamé functions of class L and degree 0.

- For $n = 1$ we have $r = 1$ and based on relation that connects the coefficients for the case $k = 1$ we will determine the parameter p so that $a_1 = 0$:

$$2a_1 = \left(\alpha p - h_2^2\right) a_0 \Rightarrow p = \frac{h_2^2}{\alpha} \;, \tag{1.101}$$

  from which it is also concluded that $a_2 = a_3 = ... = 0$.
  Therefore, based on (1.83) the polynomial factor takes the following form:

$$1x^{1-1} = 1 \tag{1.102}$$

  which generates the following Lamé function:

$$L_1^1(x) = \sqrt{x^2 - h_3^2} \tag{1.103}$$

- For $n = 2$ we have $r = 1$ and based on relation that connects the coefficients for $k = 1$ we determine p so that $a_1 = 0$:

$$6a_1 = \left[\alpha(p-1) - 3h_2^2\right] a_0 \Rightarrow p = 3\frac{h_2^2}{a} + 1 \;, \tag{1.104}$$

  from which it is also concluded that $a_2 = a_3 = ... = 0$. Therefore, based on (1.82) and (1.83), we have the following Lamé function:

$$L_2^1(x) = x\sqrt{x^2 - h_3^2} \tag{1.105}$$

- For $n = 3$ we have $r = 2$ and based on relations that connect the coefficients for $k = 1$ and $k = 2$ we determine p so that $a_2 = 0$:

$$12a_2 = (\alpha p - h_2^2)a_1 + 2\beta a_0 \text{ and } 10a_1 = \left[\alpha(p-4) - 5h_2^2\right] a_0 , \tag{1.106}$$

which give:

$$12a_2 = \frac{(ap - h_2^2)(ap - 4a - 5h_2^2)a_0}{10} + 2\beta a_0 . \tag{1.107}$$

Therefore, solving

$$\frac{(ap - h_2^2)(ap - 4a - 5h_2^2)a_0}{10} + 2\beta = 0 , \tag{1.108}$$

for the parameter p we obtain the solutions:

$$p_1 = \frac{1}{\alpha}\left[2\alpha + 3h_2^2 + 2\sqrt{(\alpha + h_2^2)^2 - 5\beta}\right] , \tag{1.109}$$

$$p_2 = \frac{1}{\alpha}\left[2\alpha + 3h_2^2 - 2\sqrt{(\alpha + h_2^2)^2 - 5\beta}\right] . \tag{1.110}$$

These lead to the Lamé functions:

$$L_3^1(x) = \sqrt{x^2 - h_3^2}\left(x^2 + \Lambda_2 - a_1^2\right) , \tag{1.111}$$

$$L_3^2(x) = \sqrt{x^2 - h_3^2}\left(x^2 + \Lambda_2' - a_1^2\right) , \tag{1.112}$$

where

$$\Lambda_2 = a_1^2 - \frac{h_3^2 + 2h_2^2}{5} + \frac{\sqrt{3h_2^4 + h_1^4 + h_3^2h_2^2}}{5} , \tag{1.113}$$

$$\Lambda_2' = a_1^2 - \frac{h_3^2 + 2h_2^2}{5} + \frac{\sqrt{3h_2^4 + h_1^4 + h_3^2h_2^2}}{5} , \tag{1.114}$$

and so on using the relations (1.94) or (1.100).

1.3.3 Lamé Functions of Class M :

Based (1.3) the Lamé functions of class M are of the following form:

$$\sqrt{|x^2 - h_2^2|}P_{n-1}(x) , \tag{1.115}$$

and in this subsection the polynomial factor is denoted with $M(x)$. Similarly with the class L we assume that $x = \rho$ which leaves us with the form:

$$M_n(x) = \sqrt{x^2 - h_2^2}M(x) , \quad n = 0, 1, 2, ... , \tag{1.116}$$

where

$$M(x) = a_0x^{n-1} + a_1x^{n-3} + a_2x^{n-5} + ... = \sum_{k=0}^{\infty} a_kx^{n-2k-1} . \tag{1.117}$$

Substituting (1.116) to the Lamé's equation (1.56) and dividing with $\sqrt{x^2 - h_2^2}$ we obtain the following equation for the polynomial $M(x)$:

$$\begin{aligned}&\left(x^4 - \alpha x^2 + \beta\right) M''(x) + x\left(4x^2 - \alpha - 2h_3^2\right) M'(x)\\ &+ \left[\alpha p - h_3^2 - (n-1)(n+2)x^2\right] M(x) = 0 \quad .\end{aligned} \tag{1.118}$$

Therefore by substituting (1.117) to the equation above we end up with the following relation for the coefficients $a_0, a_1, ...$:

$$\begin{aligned} 2k(2n-2k+1)a_k = &\left[\alpha\left(p-(n-2k+1)^2\right)-(2n-4k+3)h_3^2\right]a_{k-1} \\ &+\beta(n-2k+3)(n-2k+2)a_{k-2} \ , \quad k=0,1,2,... \ . \end{aligned} \tag{1.119}$$

Similarly with the class L, choosing r based on (1.95) and the parameter p so that $a_r = 0$, results to $a_{r+1} = a_{r+2} = ... = 0$. Therefore, a linear homogeneous system similar to (1.96) is obtained for the coefficients $a_0, a_1, ..., a_{r-1}$ and we'll determine the parameter p so that systems determinant is equal to 0 in order for (1.118) to not have trivial solutions. Based on [20] and some straightforward calculations the following Lamé functions are generated:

- For $n = 0$,similarly to the class K, it can be observed that there are no Lamé functions of class M and degree 0.
- For $n = 1$ we have $r = 1$ and from relation (1.119) the parameter p in order for $a_r = 0$ takes the following value:
$$p = \frac{h_3^2}{\alpha} \ , \tag{1.120}$$
and the Lamé function that is generated is:
$$M_1^1(x) = \sqrt{x^2 - h_2^2} \tag{1.121}$$
- For $n = 2$ we have $r = 1$ and from relation (1.119) the parameter p takes the following value:
$$p = 3\frac{h_3^2}{\alpha} + 1 \tag{1.122}$$
and the Lamé function that is generated is:
$$M_2^1(x) = x\sqrt{x^2 - h_2^2} \tag{1.123}$$
- For $n = 3$ we have $r = 2$ and the parameter p is determined by the solution of the following equation:
$$\frac{(\alpha p - h_2^2)(\alpha p - 4\alpha - 5h_2^2)a_0}{10} + 2\beta = 0 \ . \tag{1.124}$$
This gives us the following values for p:
$$p_1 = \frac{1}{\alpha}\left[2\alpha + 3h_3^2 + 2\sqrt{(\alpha + h_3^2)^2 - 5\beta}\right] \ , \tag{1.125}$$
$$p_2 = \frac{1}{\alpha}\left[2\alpha + 3h_3^2 - 2\sqrt{(\alpha + h_3^2)^2 - 5\beta}\right] \ , \tag{1.126}$$
and the Lamé functions that are generated are:
$$M_3^1(x) = \sqrt{x^2 - h_2^2}\left(x^2 + \Lambda_3 - a_1^2\right) \ , \tag{1.127}$$
$$M_3^2(x) = \sqrt{x^2 - h_2^2}\left(x^2 + \Lambda_3' - a_1^2\right) \ , \tag{1.128}$$
where
$$\Lambda_3 = a_1^2 - \frac{2h_3^2 + h_2^2}{5} + \frac{\sqrt{3h_3^4 + h_1^4 + h_3^2 h_2^2}}{5} \ , \tag{1.129}$$
$$\Lambda_3' = a_1^2 - \frac{2h_3^2 + h_2^2}{5} - \frac{\sqrt{3h_3^4 + h_1^4 + h_3^2 h_2^2}}{5} \ . \tag{1.130}$$

and so on using the relation (1.119) or the determinant of the system that they generate.

1.3.4 Lamé Functions of Class N:

Based on (1.3) the Lamé functions of class N are of the following form:

$$N_n(x) = \sqrt{|x^2 - h_2^2|}\sqrt{|x^2 - h_3^2|}P_{n-2}(x) \ , \tag{1.131}$$

and in this subsection the polynomial factor P_{n-2} is denoted with $N(x)$. For the cases of $x = \rho$, $x = \mu$ and $x = \nu$ we take $\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}$, $\sqrt{h_2^2 - x^2}\sqrt{x^2 - h_3^2}$ and $\sqrt{h_2^2 - x^2}\sqrt{h_3^2 - x^2}$ respectively but since the polynomial factor is unaffected by the choice between the variables ρ, μ and ν, similar with the previous cases, we assume that $x = \rho$ which leaves us with the following form:

$$N_n(x) = \sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}N(x) \ , \tag{1.132}$$

where

$$N(x) = a_0x^{n-2} + a_1x^{n-4} + a_2x^{n-6} + ... = \sum_{k=0}^{\infty} a_kx^{n-2k-2} \ . \tag{1.133}$$

Substituting in (1.56), dividing with $\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}$ and after some straightforward calculations we obtain the following equation:

$$(x^2 - h_2^2)(x^2 - h_3^2)N''(x) + 3x(2x^2 - a)N'(x) + \left[\alpha(p-1) - (n-2)(n+3)x^2\right] N(x) = 0 \ . \tag{1.134}$$

Substituting the derivatives of $N(x)$:

$$N'(x) = \sum_{k=0}^{\infty}(n-2k-2)a_kx^{n-2k-3} \quad \text{and} \quad N''(x) = \sum_{k=0}^{\infty}(n-2k-2)(n-2k-3)a_kx^{n-2k-4} \ , \tag{1.135}$$

into the (1.56) we have:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k-2)(n-2k-3)a_kx^{n-2k} - \alpha\sum_{k=0}^{\infty}(n-2k-2)(n-2k-3)a_kx^{n-2k-2} \\
&+ \beta\sum_{k=0}^{\infty}(n-2k-2)(n-2k-3)a_kx^{n-2k-4} + 6\sum_{k=0}^{\infty}(n-2k-2)a_kx^{n-2k} \\
&- 3\alpha\sum_{k=0}^{\infty}(n-2k-2)a_kx^{n-2k-2} + a(p-1)\sum_{k=0}^{\infty}a_kx^{n-2k-2} - (n-2)(n+3)\sum_{k=0}^{\infty}a_kx^{n-2k} = 0 \ .
\end{aligned} \tag{1.136}$$

Turning all into series of x^{n-2k} we obtain:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(n-2k-2)(n-2k-3)a_kx^{n-2k} - \alpha\sum_{k=1}^{\infty}(n-2k)(n-2k-1)a_{k-1}x^{n-2k} \\
&+ \beta\sum_{k=2}^{\infty}(n-2k+2)(n-2k+1)a_{k-2}x^{n-2k} + 6\sum_{k=0}^{\infty}(n-2k-2)a_kx^{n-2k} \\
&- 3\alpha\sum_{k=1}^{\infty}(n-2k)a_{k-1}x^{n-2k} + \alpha(p-1)\sum_{k=1}^{\infty}a_{k-1}x^{n-2k} - (n-2)(n+3)\sum_{k=0}^{\infty}a_kx^{n-2k} = 0 \ ,
\end{aligned} \tag{1.137}$$

which equivalently can be written as:

$$\sum_{k=0}^{\infty}A_kx^{n-2k} = 0 \ , \tag{1.138}$$

where

$$\begin{aligned} A_k = 2k(2n-2k+1)a_k &= \alpha\left[p-(n-2k+1)^2\right]a_{k-1} \\ &+\beta(n-2k+2)(n-2k+1)a_{k-2}\ , \quad k=,1,2,3\ . \end{aligned} \tag{1.139}$$

Since the polynomial equals to 0, therefore all the coefficients must be equal to 0 which gives us the following relations:

- For $k=0$:
  $0a_0=0$. We chose the non-zero constant a_0 arbitrarily $a_0=1$.

- For $k=1$:
  $2(2n-1)a_1=\alpha\left[p-(n-1)^2\right]a_0$.

- For $k=2$:
  $4(2n-3)a_2=\alpha\left[p-(n-3)^2\right]a_1+\beta(n-2)(n-3)a_0$.

$$\vdots$$

- For $k=r$:
  $$\begin{aligned} 2r(2n-2r+1)a_r &= \alpha\left[p-(n-2r+1)^2\right]a_{r-1} \\ &+\beta(n-2r+2)(n-2r+1)a_{r-2}\ . \end{aligned}$$

- For $k=r+1$:
  $$\begin{aligned} 2(r+1)(2n-2r-1)a_{r+1} &= \alpha\left[p-(n-2r-1)^2\right]a_r \\ &+\beta(n-2r)(n-2r-1)a_{r-1}\ . \end{aligned}$$

$$\vdots$$

Therefore the relation that connects the coefficients is the following:

$$\begin{aligned} 2k(2n-2k+1)a_k &= \alpha\left[p-(n-2k+1)^2\right]a_{k-1} \\ &+\beta(n-2k+2)(n-2k+1)a_{k-2}\ , \quad k=0,1,2,...\ . \end{aligned} \tag{1.140}$$

For

$$r=\begin{cases} \dfrac{n}{2}\ , & \text{for n even} \\ \dfrac{n-1}{2}\ , & \text{for n odd}\ , \end{cases} \tag{1.141}$$

it is easily observed like the previous classes that in the relation that connects the coefficients for the case of $k=r+1$, the coefficient of βa_{r-1} vanishes which leaves us a relation between the coefficients a_{r+1} and a_r. Hence, if we choose the parameter p so that $a_r=0$, it is concluded that $a_{r+1}=a_{r+2}=...=0$ which stops the series from descending in powers of x and leaves us with a polynomial of degree $n-2$. Therefore, the system created by the coefficients a_i for $i=0,1,...,r$ is the following:

$$\begin{bmatrix} N_{11} & N_{12} & 0 & 0 & 0 & \cdots & 0 & 0 & \\ N_{21} & N_{22} & N_{23} & 0 & 0 & \cdots & 0 & 0 & \\ 0 & N_{32} & N_{33} & N_{34} & 0 & \cdots & 0 & 0 & \\ & & \cdots & \cdots & \cdots & & & & \\ & & \cdots & \cdots & \cdots & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & N_{(r-1)(r-2)} & N_{(r-1)(r-1)} & N_{(r-1)r} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & N_{r(r-1)} & N_{rr} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{r-1} \end{bmatrix} = 0\ , \tag{1.142}$$

where

$$N_{ii} = -\alpha\left[p - (n+1-2i)^2\right] , \quad i = 1,2,...,r , \tag{1.143}$$
$$N_{j(j+1)} = 2j(2n+1-2j) , \quad j = 1,2,...,r-1 , \tag{1.144}$$
$$N_{(j+1)j} = -\beta(n-2j)(n-2j-1) , \quad j = 1,2,...,r-1 . \tag{1.145}$$

In order for the system to have no trivial solution and hence the Lamé's equation to have no trivial solution, we need the determinant of the $(r \times r)$ matrix to be equal to 0. From relations (1.140) and (1.142) it is easily concluded via straightforward calculations that:

- For $n = 0 \Rightarrow r = 0$ and we can easily observe that there are no Lamé functions of class N and order 0.
- For $n = 1 \Rightarrow r = 0$ and we can easily observe that there are no Lamé functions of class N and order 1.
- For $n = 2 \Rightarrow r = 1$, from relation (1.140) for the case $k = 1$ we have:

$$6a_1 = \alpha\left[p-1\right]a_0 \Rightarrow p = 1 , \tag{1.146}$$

which based on (1.133) the polynomial factor takes the following form:

$$1x^{2-2} = 1 , \tag{1.147}$$

which generates the Lamé function:

$$N_2^1(x) = \sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2} . \tag{1.148}$$

- For $n = 3 \Rightarrow r = 1$, from relation (1.140) for the case $k = 1$ we have:

$$10a_1 = \alpha(p-4)a_0 \Rightarrow p = 4 , \tag{1.149}$$

which based on (1.133) the polynomial factor takes the following form:

$$1x^{3-2} = x , \tag{1.150}$$

which generates the following Lamé function:

$$N_3^1(x) = x\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2} \tag{1.151}$$

and so on for $n = 0, 1, ...$ using relations (1.140) and the determinant of (1.142). Since we saw how the Lamé functions of the four classes are generated, it is important to mention the following theorem related to the orthogonality that connects them. Next, based on [20], we have the following theorems related to the orthogonality of Lamé functions as well as the number of independent Lamé functions of the same class for which the variable μ is chosen out of convenience.

Lemma 1.3.1. *Let $E_n^i(\mu)$, $E_n^j(\mu)$ be two Lamé functions of degree n that belong to the same class and correspond to two different roots p_i, p_j of the relative polynomial. Then*

$$\int_{h_3}^{h_2} \frac{E_n^i(\mu)E_n^j(\mu)}{\sqrt{\mu^2 - h_3^2}\sqrt{h_2^2 - \mu^2}} d\mu = 0 . \tag{1.152}$$

Proof. For the pairs $\left(p_i,\ E_n^i\right)$ and $\left(p_j,\ E_n^j\right)$ Lamé equation (1.56) after dividing with $\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}$ takes the following two forms:

$$
\begin{aligned}
&-\sqrt{h_2^2-\mu^2}\sqrt{\mu^2-h_3^2}\frac{d^2}{d\mu^2}E_n^i(\mu)+\left[-\frac{\mu\sqrt{h_2^2-\mu^2}}{\sqrt{\mu^2-h_3^2}}+\frac{\mu\sqrt{\mu^2-h_3^2}}{\sqrt{h_2^2-\mu^2}}\right]\frac{d}{d\mu}E_n^i(\mu)\\
&+\frac{\alpha p_i-n(n+1)\mu^2}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}E_n^i(\mu)=0\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sqrt{h_2^2-\mu^2}\sqrt{\mu^2-h_3^2}\frac{d^2}{d\mu^2}E_n^j(\mu)+\left[-\frac{\mu\sqrt{h_2^2-\mu^2}}{\sqrt{\mu^2-h_3^2}}+\frac{\mu\sqrt{\mu^2-h_3^2}}{\sqrt{h_2^2-\mu^2}}\right]\frac{d}{d\mu}E_n^j(\mu)\\
&+\frac{\alpha p_j-n(n+1)\mu^2}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}E_n^j(\mu)=0\ ,
\end{aligned}
\tag{1.153}
$$

which equivalently can be written as:

$$
-\frac{d}{d\mu}\left[\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{d}{d\mu}E_n^i(\mu)\right]+\frac{\alpha p_i-n(n+1)\mu^2}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}E_n^i(\mu)=0\ , \tag{1.154}
$$

and

$$
-\frac{d}{d\mu}\left[\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{d}{d\mu}E_n^j(\mu)\right]+\frac{\alpha p_j-n(n+1)\mu^2}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}E_n^j(\mu)=0\ . \tag{1.155}
$$

Therefore, multiplying (1.154) with E_n^j, (1.155) with E_n^i, subtracting them and then summing and subtracting the term $\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{d}{d\mu}E_n^j(\mu)\frac{d}{d\mu}E_n^i(\mu)$ we obtain the following relation:

$$
\begin{aligned}
&\frac{d}{d\mu}\left[\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\left(E_n^j(\mu)\frac{d}{d\mu}E_n^i(\mu)-E_n^i(\mu)\frac{d}{d\mu}E_n^j(\mu)\right)\right]\\
&=\frac{\alpha(p_i-p_j)E_n^i(\mu)E_n^j(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}\ .
\end{aligned}
\tag{1.156}
$$

Since the choice of convenience was the variable μ, by integrating over its interval which is $[h_3,h_2]$ we have:

$$
\begin{aligned}
&\left[\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\left(E_n^j(\mu)\frac{d}{d\mu}E_n^i(\mu)-E_n^i(\mu)\frac{d}{d\mu}E_n^j(\mu)\right)\right]_{\mu=h_3}^{\mu=h_2}\\
&=\alpha(p_i-p_j)\int_{h_3}^{h_2}\frac{E_n^i(\mu)E_n^j(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}d\mu,
\end{aligned}
\tag{1.157}
$$

from which it can be observed that the left part is equal to 0 and since $p_i\neq p_j$ from the Lemma's hypothesis the above equation leads to relation (1.152) which completes the proof. □

From Lemma 1.3.1 we can immediately see that the separation constant p is real and we can also conclude to the following Lemma

Lemma 1.3.2. *To every separation constant p there corresponds a single independent Lamé function of the first kind.*

Proof. Let E_1 and E_2 two Lamé functions of the same degree and class which correspond to the same separation constant p. Then it can be observed that the right hand of (1.156) vanishes from which it is concluded that:

$$\sqrt{\mu^2 - h_3^2}\sqrt{h_2^2 - \mu^2}\left(E_1(\mu)\frac{d}{d\mu}E_2(\mu) - E_2(\mu)\frac{d}{d\mu}E_1(\mu)\right) = c \ , \tag{1.158}$$

where c is a constant. The expression in the parenthesis is the Wronskian of the two solutions E_1 and E_2 and it will be a polynomial. Therefore, since we have the product of the two square roots which are irrational functions and the Wronskian which is a rational function, it is concluded that $c = 0$. Because of the interval of μ we observe that the Wronskian of E_1 and E_2 must be zero which means that they are linearly dependent. Thus, on a single separation constant p corresponds only one independent Lamé function of the first kind. □

Finally, we have the following lemma related to the independence of Lamé functions.

Lemma 1.3.3. *The set of Lamé functions of the same degree, that belong to the same class, are linearly independent.*

Proof. The proof can be found in ([20], p. 63) □

From Lemmas 1.3.2 and 1.3.3 it is observed that the Lamé functions of the same class are linearly independent. Taking consideration in the difference between the forms of the four classes we have that all Lamé functions are linearly independent. Now, assuming that we pick a degree n which is even. Then, based on (1.65),(1.95) and (1.141) to this degree correspond $\frac{n}{2} + 1$ independent Lamé functions of class K, $\frac{n}{2}$ independent Lamé functions of class L, $\frac{n}{2}$ independent Lamé functions of class M and $\frac{n}{2}$ Lamé functions of class N. Therefore, for a degree n even correspond $2n + 1$ independent Lamé functions distributed in the four classes. Similarly for degree n odd, correspond $\frac{n+1}{2}$ independent functions of class K, $\frac{n+1}{2}$ independent functions of class L, $\frac{n+1}{2}$ independent functions of class M and $\frac{n-1}{2}$ independent functions of class N. Hence for degree n odd also correspond $2n+1$ independent Lamé functions distributed in the four classes. In what follows for Lamé functions we use Lamé's notation E_n^m, where $n = 0, 1, 2, ...$ denotes the degree and $m = 1, 2, ..., 2n + 1$ denotes the order.
The Lamé functions that we studied in this section are called "*interior Lamé functions*" or "*Lamé functions of the first kind*" and are singular at infinity and regular everywhere else. More about the study of Lamé functions can be found in [20] and [21].

1.3.5 Lamé Functions of the Second Kind

In this section we will study the solutions of Lamé equation which are independent of the Lamé functions of the first kind and are regular at infinity. These functions are called "*Lamé functions of the second kind* or "*exterior Lamé functions*". Since equation (1.56) is a second order linear homogeneous O.D.E, if we know a solution of the equation we can find another linearly independent solution as follows:
Let the second solution to be of the following form

$$F_n(x) = \Psi_n(x)E_n(x) \ , \tag{1.159}$$

where $E_n(x)$ a Lamé function of first kind that was studied in the previous section and it is a solution of Lamé equation (1.56). Therefore, substituting the derivatives:

$$F_n^{'}(x) = \Psi_n^{'}(x)E_n(x) + \Psi(x)E_n^{'}(x) \ , \tag{1.160}$$

$$F_n^{''}(x) = \Psi_n^{''}(X)E_n(x) + 2\Psi_n^{'}(x)E_n^{'}(x) + \Psi_n(x)E_n^{''}(x) \ , \tag{1.161}$$

into (1.56), we obtain the following equation:

$$(x^4 - \alpha x^2 + \beta)\Psi_n''(x)E_n(x) + 2(x^4 - \alpha x^2 + \beta)\Psi_n'(x)E_n'(x) + (x^4 - \alpha x^2 + \beta)\Psi_n(x)E_n''(x)$$
$$+ (2x^3 - \alpha x)\Psi_n'(x)E_n(x) + (2x^3 - \alpha x)\Psi_n(x)E_n'(x) + \left[ap - n(n+1)x^2\right]\Psi_n(x)E_n(x) = 0, \tag{1.162}$$

or equivalently:

$$\left\{(x^4 - \alpha x^2 + \beta)E_n''(x) + (2x^3 - \alpha x)E_n'(x) + \left[\alpha p - n(n+1)x^2\right]E_n(x)\right\}\Psi_n(x)$$
$$+ (x^4 - \alpha x^2 + \beta)\Psi_n''(x)E_n(x) + 2(x^4 - \alpha x^2 + \beta)\Psi_n'(x)E_n'(x) + (2x^3 - \alpha x)\Psi_n'(x)E_n(x) = 0, \tag{1.163}$$

and since $E_n(x)$ is a solution of (1.56) we have:

$$(x^4 - \alpha x^2 + \beta)E_n(x)\Psi_n''(x) + \left[2(x^4 - \alpha x^2 + \beta)E_n'(x) + (2x^3 - \alpha x)E_n(x)\right]\Psi_n'(x) = 0\ . \tag{1.164}$$

Multiplying with $\dfrac{E_n(x)}{\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}}$ and substituting $a = h_2^2 + h_3^2$ and $\beta = h_2^2 h_3^2$ the above equation can be written as:

$$\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}E_n^2(x)\Psi_n''(x)$$
$$+ \left[\sqrt{x^2 - h_2^2}\sqrt{x^2 - h_3^2}\left(E_n^2(x)\right)' + \left(\frac{x\sqrt{x^2 - h_2^2}}{\sqrt{x^2 - h_3^2}} + \frac{x\sqrt{x^2 - h_3^2}}{\sqrt{x^2 - h_2^2}}\right)E_n^2(x)\right]\Psi_n'(x) = 0\ , \tag{1.165}$$

or equivalently:

$$\frac{d}{dx}\left[E_n^2(x)\sqrt{x^2 - h_3^2}\sqrt{x^2 - h_2^2}\frac{d}{dx}\Psi_n(x)\right] = 0\ . \tag{1.166}$$

Integrating twice with respect to the variable x, we obtain:

$$\Psi_n(x) = c_1 \int \frac{du}{\left(E_n(u)\right)^2\sqrt{u^2 - h_2^2}\sqrt{u^2 - h_3^2}} + c_2\ . \tag{1.167}$$

Therefore, the second linear independent solution will be:

$$F_n(x) = c_1 E_n(x) \int_{x_0}^{x} \frac{du}{\left(E_n(u)\right)^2\sqrt{u^2 - h_2^2}\sqrt{u^2 - h_3^2}}. \tag{1.168}$$

We note that since $x = \rho, \mu, \nu$ the interval of integration will vary depending on the choice of x with the most interesting case being that of $x = \rho$. Specifically, it is proven that even though the ellipsoid degenerates to a sphere for $\rho \to \infty$, the ellipsoidal harmonics do not reduce to spherical harmonics but into sphero-conal harmonics which keep characteristics of the ellipsoidal system [29]. Nevertheless, the radial part of the sphero-conal coincides with the radial part of the spherical harmonics which due to (1.328) and (1.329) , leads to the following normalization conditions:

$$\lim_{\rho \to \infty} \rho^{-n} E_n(\rho) = 1\ , \tag{1.169}$$

$$\lim_{\rho \to \infty} \rho^{n+1} F_n(\rho) = 1\ , \tag{1.170}$$

from which it is concluded that for very large ρ, we have:

$$F_n(\rho) = c_1 \rho^n \int_{\rho}^{\infty} \frac{du}{u^{2n+2}} = c_1 \rho^n \left[\frac{u^{-2n-1}}{-2n-1}\right]_{\rho}^{\infty} = c_1 \frac{\rho^{-n-1}}{2n+1}\ . \tag{1.171}$$

Therefore, based on (1.170), we have $c_1 = 2n+1$, which leads to the normalization of $F_n(\rho)$ at infinity and the Lamé functions of the second kind are defined as:

$$F_n(\rho) = (2n+1)E_n(\rho)\int_\rho^\infty \frac{du}{(E_n(u))^2\sqrt{u^2-h_2^2}\sqrt{u^2-h_3^2}}\ , \quad \rho > h_2\ .\ n = 0,1,2,...\ , \tag{1.172}$$

In what follows the improper elliptic integral will be denoted as:

$$I_n(\rho) = \int_\rho^\infty \frac{du}{(E_n(u))^2\sqrt{u^2-h_2^2}\sqrt{u^2-h_3^2}}\ , \quad \rho > h_2\ ,\ n = 0,1,2,...\ . \tag{1.173}$$

It is easily concluded based on this section that:

- For $n = 0$:
  $E_0(\rho) = K_0^1(\rho) = 1$ and $F_0(\rho) = \int_\rho^\infty \dfrac{du}{\sqrt{u^2-h_2^2}\sqrt{u^2-h_3^2}}$.

- For $n = 1$:
  $E_1^1(\rho) = \rho$, $E_1^2(\rho)$, $E_1^2(\rho) = \sqrt{\rho^2-h_3^2}$, $E_1^3(\rho) = \sqrt{\rho^2-h_2^2}$,
  and the corresponding Lamé functions of the second kind and degree 1 will be:
  $F_1^1(\rho) = 3\rho\displaystyle\int_\rho^\infty \frac{d\rho}{\rho^2\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}$, $F_1^2(\rho) = 3\sqrt{\rho^2-h_2^2}\displaystyle\int_\rho^\infty \frac{d\rho}{\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}$,
  $F_1^3(\rho) = 3\sqrt{\rho^2-h_3^2}\displaystyle\int_\rho^\infty \frac{d\rho}{\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}$.

  $\vdots$

  and so on for higher degree n.

In the previous section we saw that for every degree n correspond $2n+1$ independent Lamé functions of the first kind. The same goes for the Lamé functions of the second kind since they depend on those of the first kind. Thus, following Lamé's notation we have:

$$F_n^m(\rho) = (2n+1)E_n^m(\rho)\int_\rho^\infty \frac{du}{(E_n^m(u))^2\sqrt{u^2-h_2^2}\sqrt{u^2-h_3^2}}\ , \quad \rho > h_2\ , \tag{1.174}$$

where $n = 0,1,2,...$ denotes the degree, $m = 1,2,...,2n+1$ denotes the order and $I_n^m(\rho)$ denotes the corresponding improper elliptic integral given by:

$$I_n^m(\rho) = \int_\rho^\infty \frac{du}{(E_n^m(u))^2\sqrt{u^2-h_2^2}\sqrt{u^2-h_3^2}}\ , \quad \rho > h_2\ ,\ n = 0,1,2,...\ ,\ m = 1,2,...,2n+1. \tag{1.175}$$

Similarly we can define the cases of $F_n(\mu)$ and $F_n(\nu)$ for their corresponding intervals.

1.3.6 Relations Between Elliptic Integrals

Some basic relations between the elliptic integrals I_1^n and I_0^1 that that will be used in the last chapter are the following:

$$I_1^1(\rho) + I_1^2(\rho) + I_1^3(\rho) = \frac{1}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}\ , \tag{1.176}$$

$$\alpha_1^2 I_1^1(\rho) + \alpha_2^2 I_1^2(\rho) + \alpha_3^2 I_1^3(\rho) = I_0^1(\rho) - \frac{\rho^2-\alpha_1^2}{\rho\sqrt{\rho^2-h_3^2}\sqrt{\rho^2-h_2^2}}\ . \tag{1.177}$$

$$I_1^i(\alpha_1) - I_1^j(\alpha_1) = \frac{\alpha_j^2-\alpha_i^2}{2}\int_0^\infty \frac{dx}{(x+\alpha_i^2)(x+\alpha_j^2)\sqrt{(x+\alpha_1^2)(x+\alpha_2^2)(x+\alpha_3^2)}}\ , \tag{1.178}$$

for $i, j = 1, 2, 3$. Relation (1.176) for $\rho = \alpha_1$ takes the form:

$$\sum_{n=1}^{3} V I_1^n(\alpha_1) = 1 \ . \tag{1.179}$$

Moreover, the incomplete elliptic integral of the first kind is defined as:

$$F(\phi, k) = \int_0^{sin\phi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \ , \tag{1.180}$$

or equivalently

$$F(\phi, \alpha) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - sin^2\alpha sin^2\theta}} \ , \tag{1.181}$$

with $k = sin\alpha$ and the incomplete elliptic integral of the second kind is defined as:

$$E(\phi, k) = \int_0^{sin\phi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt \ , \tag{1.182}$$

or equivalently

$$E(\phi, \alpha) = \int_0^{\phi} \sqrt{1 - sin^2\alpha sin^2\theta} \ d\theta \ , \tag{1.183}$$

where the amplitude ϕ is defined as:

$$\phi = sin^{-1}\frac{h_2}{\rho} \tag{1.184}$$

and the modular angle α is defined as:

$$\alpha = sin^{-1}\frac{h_3}{h_2} \ . \tag{1.185}$$

Then the elliptic integrals I_0^1 and I_1^n can be written in terms of the incomplete elliptic integrals of the first and of the second kind as follows:

$$I_0^1(\rho) = \frac{1}{h_2} F(\phi, \alpha) \ , \tag{1.186}$$

$$I_1^1(\rho) = \frac{1}{h_2 h_3^2} \left(F(\phi, \alpha) - E(\phi, \alpha) \right) \ , \tag{1.187}$$

$$I_1^2(\rho) = \frac{h_2}{h_1^2 h_3^2} E(\phi, \alpha) - \frac{1}{h_2 h_3^2} F(\phi, \alpha) - \frac{1}{h_1^2} \frac{\sqrt{\rho^2 - h_2^2}}{\rho\sqrt{\rho^2 - h_3^2}} \ , \tag{1.188}$$

$$I_1^3(\rho) = -\frac{1}{h_1^2 h_2} E(\phi, \alpha) + \frac{1}{h_1^2} \frac{\sqrt{\rho^2 - h_3^2}}{\rho\sqrt{\rho^2 - h_2^2}} \ . \tag{1.189}$$

1.4 Interior and Exterior Ellipsoidal Harmonics

1.4.1 Interior Ellipsoidal Harmonics

The solution of the Laplace equation in ellipsoidal coordinates will be of the form $R(\rho)M(\mu)N(\nu)$. Hence, based on the separation of variables that led us to Lamé equation (1.55), the "*Interior Ellipsoidal Harmonics*" are defined as follows:

$$\mathbb{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho) E_n^m(\mu) E_n^m(\nu) \ , \tag{1.190}$$

where $n = 0, 1, 2, ...$ denotes the degree and $m = 1, 2, ..., 2n + 1$ denotes the order of the harmonic function. These functions are solutions of the Laplace equation, while $E_n^m(\rho)$, $E_n^m(\mu)$

and $E_n^m(\nu)$ are the Lamé functions that we studied in the previous section which are solutions of the Lamé equation (1.55) in the intervals $[h_2,\infty)$, $[h_3,h_2]$ and $[0,h_3]$ respectively. Based on the calculations of the Lamé functions from the previous section the ellipsoidal harmonics will have the following forms:

- For $n=0$ and $m=1$:

$$\mathbb{E}_0^1(\rho,\mu,\nu)=E_0^1(\rho)E_0^1(\mu)E_0^1(\nu)=K_0^1(\rho)K_0^1(\mu)K_0^1(\nu)=1\ . \tag{1.191}$$

- For $n=1$ and $m=1$:

$$\mathbb{E}_1^1(\rho,\mu,\nu)=E_1^1(\rho)E_1^1(\mu)E_1^1(\nu)=K_1^1(\rho)K_1^1(\mu)K_1^1(\nu)=\rho\mu\nu\ . \tag{1.192}$$

- For $n=1$ and $m=2$:

$$\mathbb{E}_1^2(\rho,\mu,\nu)=E_1^2(\rho)E_1^2(\mu)E_1^2(\nu)=L_1^1(\rho)L_1^1(\mu)L_1^1(\nu)=\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}. \tag{1.193}$$

- For $n=1$ and $m=3$:

$$\mathbb{E}_1^3(\rho,\mu,\nu)=E_1^3(\rho)E_1^3(\mu)E_1^3(\nu)=M_1^1(\rho)M_1^1(\mu)M_1^1(\nu)=\sqrt{\rho^2-h_3^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}. \tag{1.194}$$

- For $n=2$ and $m=1$:

$$\begin{aligned}\mathbb{E}_2^1(\rho,\mu,\nu)&=E_2^1(\rho)E_2^1(\mu)E_2^1(\nu)=K_2^1(\rho)K_2^1(\mu)K_2^1(\nu)\\&=\left(\rho^2+\Lambda-a_1^2\right)\left(\mu^2+\Lambda-a_1^2\right)\left(\nu^2+\Lambda-a_1^2\right)\end{aligned}, \tag{1.195}$$

  where Λ is defined in (1.75).

- For $n=2$ and $m=2$:

$$\begin{aligned}\mathbb{E}_2^2(\rho,\mu,\nu)&=E_2^2(\rho)E_2^2(\mu)E_2^2(\nu)=K_2^2(\rho)K_2^2(\mu)K_2^2(\nu)\\&=\left(\rho^2+\Lambda'-a_1^2\right)\left(\mu^2+\Lambda'-a_1^2\right)\left(\nu^2+\Lambda'-a_1^2\right)\end{aligned}, \tag{1.196}$$

  where Λ' is defined in (1.76).

- For $n=2$ and $m=3$:

$$\begin{aligned}\mathbb{E}_2^3(\rho,\mu,\nu)&=E_2^3(\rho)E_2^3(\mu)E_2^3(\nu)=L_2^1(\rho)L_2^1(\mu)L_2^1(\nu)\\&=\rho\mu\nu\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}\ .\end{aligned} \tag{1.197}$$

- For $n=2$ and $m=4$:

$$\begin{aligned}\mathbb{E}_2^4(\rho,\mu,\nu)&=E_2^4(\rho)E_2^4(\mu)E_2^4(\nu)=M_2^1(\rho)M_2^1(\mu)M_2^1(\nu)\\&=\rho\mu\nu\sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}\ .\end{aligned} \tag{1.198}$$

- For $n=2$ and $m=5$:

$$\begin{aligned}\mathbb{E}_2^5(\rho,\mu,\nu)&=E_2^5(\rho)E_2^5(\mu)E_2^5(\nu)=N_2^1(\rho)N_2^1(\mu)N_2^1(\nu)\\&=\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}\sqrt{h_2^2-\mu^2}\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\nu^2}\sqrt{h_3^2-\nu^2}\ .\end{aligned} \tag{1.199}$$

and so goes on using the Lamé functions of the previous section. Next, the Cartesian forms of some Ellipsoidal Harmonics are presented. These forms will be used in the boundary value problems and applications that are studied in chapters 3 and 4 respectively. In order to calculate these forms we mainly use the relation (1.18) as well as some straightforward calculations in order to obtain the following:

- For $n = 0$ and $m = 1$:
$$\mathbb{E}_0^1(x_1, x_2, x_3) = 1 \ . \tag{1.200}$$

- For $n = 1$ and $m = 1$:
$$\mathbb{E}_1^1(x_1, x_2, x_3) = x_1 h_2 h_3 \ . \tag{1.201}$$

- For $n = 1$ and $m = 2$:
$$\mathbb{E}_1^2(x_1, x_2, x_3) = x_2 h_1 h_3 \ . \tag{1.202}$$

- For $n = 1$ and $m = 3$:
$$\mathbb{E}_1^3(x_1, x_2, x_3) = x_3 h_1 h_2 \ . \tag{1.203}$$

  Hence, (1.201)-(1.203) can be written as:
$$\mathbb{E}_1^m(x_1, x_2, x_3) = \frac{x_m h_1 h_2 h_3}{h_m} \ , \quad n = 1, 2, 3 \ . \tag{1.204}$$

- For $n = 2$ and $m = 1$:
$$\mathbb{E}_2^1(x_1, x_2, x_3) = \left(\Lambda - a_1^2\right)\left(\Lambda - a_2^2\right)\left(\Lambda - a_3^2\right)\left(\sum_{k=1}^{3} \frac{x_k^2}{\Lambda - a_k^2} + 1\right) \ , \tag{1.205}$$

  where Λ has been defined in (1.75).

- For $n = 2$ and $m = 2$:
$$\mathbb{E}_2^2(x_1, x_2, x_3) = \left(\Lambda' - a_1^2\right)\left(\Lambda' - a_2^2\right)\left(\Lambda' - a_3^2\right)\left(\sum_{k=1}^{3} \frac{x_k^2}{\Lambda' - a_k^2} + 1\right) \ , \tag{1.206}$$

  where Λ' has been defined in (1.76).

- For $n = 2$ and $m = 3$:
$$\mathbb{E}_2^3(x_1, x_2, x_3) = x_1 x_2 h_1 h_2 h_3^2 \ . \tag{1.207}$$

- For $n = 2$ and $m = 4$:
$$\mathbb{E}_2^4(x_1, x_2, x_3) = x_1 x_3 h_1 h_2^2 h_3 \ . \tag{1.208}$$

- For $n = 2$ and $m = 5$:
$$\mathbb{E}_2^5(x_1, x_2, x_3) = x_2 x_3 h_1^2 h_2 h_3 \ . \tag{1.209}$$

  Hence, (1.207)-(1.209) can be written as:
$$\mathbb{E}_2^{6-n}(x_1, x_2, x_3) = x_1 x_2 x_3 h_1 h_2 h_3 \frac{h_n}{x_n} \ , \quad n = 1, 2, 3 \ . \tag{1.210}$$

For the Ellipsoidal Harmonics of degree 3 and above in Cartesian and Ellipsoidal coordinates we refer to [20].
Finally, we note that every Ellipsoidal Harmonic is bounded inside any ellipsoid of the corresponding family.

1.4.2 Exterior Ellipsoidal Harmonics

The "*exterior Ellipsoidal Harmonics*" of degree n and order m are defined as:

$$\mathbb{F}_n^m(\rho,\mu,\nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \ , \tag{1.211}$$

or equivalently using relation (1.174), they can be rewritten as:

$$\mathbb{F}_n^m(\rho,\mu,\nu) = (2n+1)I_n^m(\rho)E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) = (2n+1)I_n^m(\rho)\mathbb{E}_n^m(\rho,\mu,\nu) \ , \tag{1.212}$$

for $n = 0, 1, 2, ...$ and $m = 1, 2, ..., 2n+1$ and $I_n^m(\rho)$ the elliptic integral defined in (1.175). As mentioned in the previous section, as we approach infinity the ellipsoidal harmonics degenerate to sphero-conal harmonics (or conical), which means that the ρ part is the same as the radial part of the spherical harmonics. Thus, from relations (1.169) and (1.170) it is concluded that:

$$\mathbb{F}_n^m(\rho,\mu,\nu) = \mathcal{O}\left(\frac{1}{\rho^{n+1}}\right) \ , \quad \rho \to \infty \ . \tag{1.213}$$

Hence, the exterior ellipsoidal harmonics are regular at infinity and are used for the exterior boundary value problems for the Laplace equations.

1.5 Surface Ellipsoidal Harmonics and Orthogonality Relations

1.5.1 Surface Ellipsoidal Harmonics

For the solution of boundary value problems involving ellipsoidal harmonics, it is important to know the orthogonality relations they satisfy as well as the normalization constants for which analytic expression exist only up to degree n=3. Since these orthogonality relations are satisfied over the surface of any confocal ellipsoid, the definition of surface ellipsoidal harmonics is required.

Similarly with the surface spherical harmonics we want to study the radial and angular separation of the Laplacian in the ellipsoidal coordinate system. Therefore, we define the "*surface ellipsoidal harmonics*" as:

$$S_n^m(\mu,\nu) = E_n^m(\mu)E_n^m(\nu) \ , \tag{1.214}$$

for $n = 0, 1, 2, ...$ and $m = 1, 2, ..., 2n+1$. Next, we define the Beltrami operator in the ellipsoidal coordinate system. The Laplace-Beltrami operator in the spherical coordinate system is given by:

$$\mathbb{B} = \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\theta^2} \ . \tag{1.215}$$

It can be proven that it in the Hilbert space $L^2(S^2)$ ($\int_{S^2}|f|^2dS < \infty$) with the inner product:

$$< f, g >= \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\overline{g}(\theta,\phi)sin\theta d\theta d\phi$$

and the norm associated with this inner product:

$$\|f\| = \left(\int_{S^2} |f(\theta,\phi)|^2 \, sin\theta d\theta d\phi\right)^{\frac{1}{2}} \ ,$$

the Beltrami operator is a self-adjoint linear operator with respect to this inner product. It can also be proven that its eigenvectors are the so called spherical harmonics and they form a complete orthonormal basis of this Hilbert space. More about these properties of the spherical harmonics can be found in [21] and they motivate us to define a corresponding Beltrami operator of the ellipsoidal coordinate system and extend these properties in this system. From relation (1.40) by separating the first term which contains $\frac{\partial}{\partial \rho}$ and $\frac{\partial^2}{\partial \rho^2}$ from the second and the third terms which contain the same expression for μ and ν respectively, the Laplacian can be rewritten in the following form:

$$\Delta = \frac{1}{(\rho^2-\mu^2)(\rho^2-\nu^2)}\left[(\rho^2-h_3^2)(\rho^2-h_2^2)\frac{\partial^2}{\partial\rho^2}+\rho(2\rho^2-h_3^2-h_2^2)\frac{\partial}{\partial\rho}+\mathbb{B}_e(\rho)\right] . \tag{1.216}$$

Therefore, the "*ellipsoidal Beltrami operator*" is defined as:

$$\begin{aligned}\mathbb{B}_e(\rho) = &\frac{\rho^2-\nu^2}{(\nu^2-\mu^2)}\left[\left(\mu^2-h_3^2\right)\left(\mu^2-h_2^2\right)\frac{\partial^2}{\partial\mu^2}+\mu\left(2\mu^2-h_2^2-h_3^2\right)\frac{\partial}{\partial\mu}\right]\\ &+\frac{\rho^2-\mu^2}{(\mu^2-\nu^2)}\left[\left(\nu^2-h_3^2\right)\left(\nu^2-h_2^2\right)\frac{\partial^2}{\partial\nu^2}+\nu\left(2\nu^2-h_2^2-h_3^2\right)\frac{\partial}{\partial\nu}\right] ,\end{aligned} \tag{1.217}$$

from which it is easily observed that in contrast to the spherical harmonics where the Beltrami operator is independent of the radial variable r, in the ellipsoidal system the corresponding ellipsoidal Beltrami operator depends on the radial variable ρ. Next, in order to study the eigenspaces of this operator we proceed with the following calculations:

$$\begin{aligned}\mathbb{B}_e(\rho)S_n^m(\mu,\nu) = &\frac{\rho^2-\nu^2}{\nu^2-\mu^2}\left[(\mu^2-h_3^2)(\mu^2-h_2^2)\frac{\partial^2}{\partial\mu^2}+\mu(2\mu^2-h_3^2-h_2^2)\frac{\partial}{\partial\mu}\right]E_n^m(\mu)E_n^m(\nu)\\ &+\frac{\rho^2-\mu^2}{\mu^2-\nu^2}\left[(\nu^2-h_3^2)(\nu^2-h_2^2)\frac{\partial^2}{\partial\nu^2}+\nu(2\nu^2-h_3^2-h_2^2)\frac{\partial}{\partial\nu}\right]E_n^m(\mu)E_n^m(\nu) ,\end{aligned} \tag{1.218}$$

or equivalently:

$$\begin{aligned}\mathbb{B}_e(\rho)S_n^m(\mu,\nu) = &\frac{\rho^2-\nu^2}{\nu^2-\mu^2}E_n^m(\nu)\left[(\mu^2-h_3^2)(\mu^2-h_2^2)\frac{\partial^2}{\partial\mu^2}+\mu(2\mu^2-h_3^2-h_2^2)\frac{\partial}{\partial\mu}\right]E_n^m(\mu)\\ &+\frac{\rho^2-\mu^2}{\mu^2-\nu^2}E_n^m(\mu)\left[(\nu^2-h_3^2)(\nu^2-h_2^2)\frac{\partial^2}{\partial\nu^2}+\nu(2\nu^2-h_3^2-h_2^2)\frac{\partial}{\partial\nu}\right]E_n^m(\nu) .\end{aligned} \tag{1.219}$$

Since $E_n^m(\mu)$ and $E_n^m(\nu)$ are solutions of equation (1.56), they satisfy the relation:

$$(\mu^2-h_3^2)(\mu^2-h_2^2)E_n^{m''}(x)+x(2x^2-h_3^2-h_2^2)E_n^{m''}(x) = \left[n(n+1)x^2-\alpha p_n^m\right]E_n^m(x) . \tag{1.220}$$

Using this relation into (1.219), we obtain the following:

$$\mathbb{B}_e(\rho)S_n^m(\mu,\nu) = E_n^m(\nu)\frac{\rho^2-\nu^2}{\nu^2-\mu^2}\left[n(n+1)\mu^2-\alpha p_n^m\right]E_n^m(\mu)E_n^m(\mu)\frac{\rho^2-\mu^2}{\mu^2-\nu^2}\left[n(n+1)\nu^2-\alpha p_n^m\right]E_n^m(\nu) , \tag{1.221}$$

which equivalently can be written as:

$$\begin{aligned}\mathbb{B}_e(\rho)S_n^m(\mu,\nu) &= E_n^m(\mu)E_n^m(\nu)\left[n(n+1)\left(\frac{\mu^2\rho^2-\mu^2\nu^2}{\nu^2-\mu^2}+\frac{\rho^2\nu^2-\mu^2\nu^2}{\mu^2-\nu^2}\right)-\alpha p_n^m\left(\frac{\rho^2-\nu^2}{\nu^2-\mu^2}+\frac{\rho^2-\mu^2}{\mu^2-\nu^2}\right)\right]\\ &= E_n^m(\mu)E_n^m(\nu)\left[n(n+1)\frac{\rho^2(\mu^2-\nu^2)}{\nu^2-\mu^2}-\alpha p_n^m\frac{\mu^2-\nu^2}{\nu^2-\mu^2}\right] = E_n^m(\mu)E_n^m(\nu)\left[\alpha p_n^m-n(n+1)\rho^2\right] .\end{aligned} \tag{1.222}$$

Therefore, using relation (1.214) and $\alpha = h_2^2+h_3^2$ it is concluded that:

$$\mathbb{B}_e(\rho)S_n^m(\mu,\nu) = \left[(h_2^2+h_3^2)p_n^m-n(n+1)\rho^2\right]S_n^m(\mu,\nu) , \tag{1.223}$$

on any ellipsoidal surface confocal to the reference ellipsoid. This means that for any fixed confocal ellipsoidal surface $\rho = constant$ and any pair (μ, ν) , the ellipsoidal Beltrami operator has eigenvalue the quantity $\left[(h_2^2 + h_3^2)p_n^m - n(n+1)\rho^2\right]$ with corresponding eigenvector the surface ellipsoidal harmonic $S_n^m(\mu, \nu)$. We note that since the parameters p_n^m are different for every pair (μ, ν), it follows that all the eigenvalues of $\mathbb{B}_e(\rho)$ are simple and the corresponding eigenspaces are 1-dimensional. Furthermore, the spectrum of $\mathbb{B}_e(\rho)$ depends on the ellipsoidal surface $\rho = constant$ on which the operator is defined. Next, we proceed with the orthogonality relations of the surface ellipsoidal harmonics (eigenfunctions of $\mathbb{B}_e(\rho)$).

1.5.2 Orthogonality Properties

In order to proceed with the orthogonality properties of the surface ellipsoidal harmonics S_n^m on a surface of an arbitrary ellipsoid $\rho = constant$, first we define the inner product on this surface as:

$$< f, g >= \int_{S_\rho} f(\mu, \nu) g(\mu, \nu) l_\rho(\mu, \nu) d\mu d\nu \ , \tag{1.224}$$

where l_ρ is the weighting function defined as:

$$l_\rho(\mu, \nu) = \frac{1}{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \ . \tag{1.225}$$

Let $\mathbf{r}(\mu, \nu)$ an arbitrary point on an ellipsoidal surface $\rho = constant$. From the definition of the surface element in an orthogonal curvilinear system, the solid angle element is given by:

$$\begin{aligned} l_\rho(\mu,\nu) dS_\rho(\mu,\nu) &= l_\rho(\mu,\nu)\|\mathbf{r}_\mu \times \mathbf{r}_\nu\| d\mu d\nu = l_\rho(\mu,\nu)\sqrt{\|\mathbf{r}_\mu\|^2\|\mathbf{r}_\nu\|^2 - (\mathbf{r}_\mu \cdot \mathbf{r}_\nu)^2} d\mu d\nu \\ &= l_\rho(\mu,\nu)\, \|\mathbf{r}_\mu\|\, \|\mathbf{r}_\nu\|\, d\mu d\nu = l_\rho(\mu,\nu) h_\mu h_\nu d\mu d\nu = \frac{h_\mu h_\nu}{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} d\mu d\nu \\ &= \frac{\mu^2 - \nu^2}{\sqrt{\mu^2 - h_3^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_3^2 - \nu^2}\sqrt{h_2^2 - \nu^2}} d\mu d\nu = d\Omega(\mu,\nu) \ , \end{aligned} \tag{1.226}$$

which is independent of the surface $\rho = constant$, but since it depends on h_2, h_3, it depends on the reference ellipsoid and thats why we denote the complete domain of integration with respect to $d\Omega$ with S_{α_1}. Therefore, we define the ellipsoidal surface element as:

$$dS(\mu, \nu) = \frac{1}{l_\rho(\mu, \nu)} d\Omega(\mu, \nu) = \sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2} d\Omega(\mu, \nu) \ , \tag{1.227}$$

and the ellipsoidal volume element as:

$$dV(\rho, \mu, \nu) = \|\mathbf{r}_\mu \times \mathbf{r}_\nu\|\, \|\mathbf{r}_\rho\|\, d\rho d\mu d\nu = \frac{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}{\sqrt{\rho^2 - h_3^2}\sqrt{\rho^2 - h_2^2}} d\rho d\Omega(\mu, \nu) \ . \tag{1.228}$$

Based on [20] we have the following theorem for the orthogonality of the surface ellipsoidal harmonics

Theorem 1.5.1. *If S_{α_1} denotes the boundary of the reference ellipsoid and S_n^m the surface ellispoidal harmonics defined in (1.214), then*

$$\int_{S_{\alpha_1}} S_n^m(\mu, \nu) S_{n'}^{m'}(\mu, \nu) d\Omega(\mu, \nu) = \gamma_n^m \delta_{nn'} \delta_{mm'} \ , \tag{1.229}$$

for every $n = 0, 1, 2, ...,$ $n' = 0, 1, 2, ...,$ $m = 1, 2, ..., 2n + 1$ and $m' = 1, 2, ..., 2n' + 1$. With γ_n^m we denote the normalization constants.

Proof. Applying the second Green's identity for the interior ellipsoidal harmonics $\mathbb{E}_n^m$ and $\mathbb{E}_{n'}^{m'}$ to the ellipsoidal domain V_ρ which is bounded by the ellipsoidal surface S_ρ for any $\rho = constant$, we obtain:

$$\begin{aligned}&\int_{V_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)\Delta\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)\Delta\mathbb{E}_n^m(\rho,\mu,\nu)\right]dV(\rho,\mu,\nu)\\&=\int_{S_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)\frac{\partial}{\partial n}\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)\frac{\partial}{\partial n}\mathbb{E}_n^m(\rho,\mu,\nu)\right]dS_\rho(\mu,\nu)\ ,\end{aligned}\tag{1.230}$$

where the normal differentiation on the ellipsoidal surface is given by:

$$\begin{aligned}\frac{\partial}{\partial n}&=\hat{\boldsymbol{\rho}}\cdot\nabla=\hat{\boldsymbol{\rho}}\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{h_\rho}\frac{\partial}{\partial\rho},\frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial}{\partial\mu},\frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial}{\partial\nu}\right)=\frac{\|\hat{\boldsymbol{\rho}}\|^2}{h_\rho}\frac{\partial}{\partial\rho}=\frac{1}{h_\rho}\frac{\partial}{\partial\rho}\\&=\frac{\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\frac{\partial}{\partial\rho}=\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}l_\rho(\mu,\nu)\frac{\partial}{\partial\rho}\ .\end{aligned}\tag{1.231}$$

Since $\mathbb{E}_n^m$ and $\mathbb{E}_{n'}^{m'}$ are solutions of the Laplace equation the left hand of (1.230) vanishes leavings us with:

$$\int_{S_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)\frac{\partial}{\partial n}\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)\frac{\partial}{\partial n}\mathbb{E}_n^m(\rho,\mu,\nu)\right]dS_\rho(\mu,\nu)=0\ .\tag{1.232}$$

Using relations (1.231) and (1.226), the equation above can be written as:

$$\begin{aligned}&\int_{S_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)l_\rho(\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)l_\rho(\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_n^m(\rho,\mu,\nu)\right]dS_\rho(\mu,\nu)=0\\&\Rightarrow\int_{S_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_n^m(\rho,\mu,\nu)\right]l_\rho(\mu,\nu)dS_\rho(\mu,\nu)=0\\&\Rightarrow\int_{S_\rho}\left[\mathbb{E}_n^m(\rho,\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)-\mathbb{E}_{n'}^{m'}(\rho,\mu,\nu)\frac{\partial}{\partial\rho}\mathbb{E}_n^m(\rho,\mu,\nu)\right]d\Omega(\mu,\nu)=0\ ,\end{aligned}\tag{1.233}$$

and by using relation (1.190) we obtain:

$$\begin{aligned}&\int_{S_\rho}\left[E_n^m(\rho)\frac{\partial}{\partial\rho}E_{n'}^{m'}(\rho)-E_{n'}^{m'}(\rho)\frac{\partial}{\partial\rho}E_n^m(\rho)\right]E_n^m(\mu)E_n^m(\nu)E_{n'}^{m'}(\mu)E_{n'}^{m'}(\nu)d\Omega(\mu,\nu)=0\\&\Rightarrow\left[E_n^m(\rho)\frac{\partial}{\partial\rho}E_{n'}^{m'}(\rho)-E_{n'}^{m'}(\rho)\frac{\partial}{\partial\rho}E_n^m(\rho)\right]\int_{S_\rho}E_n^m(\mu)E_n^m(\nu)E_{n'}^{m'}(\mu)E_{n'}^{m'}(\nu)d\Omega(\mu,\nu)=0\\&\Rightarrow\left[E_n^m(\rho)\frac{\partial}{\partial\rho}E_{n'}^{m'}(\rho)-E_{n'}^{m'}(\rho)\frac{\partial}{\partial\rho}E_n^m(\rho)\right]\int_{S_\rho}S_n^m(\mu,\nu)S_{n'}^{m'}(\mu,\nu)d\Omega(\mu,\nu)=0\ .\end{aligned}\tag{1.234}$$

We observe that if $E_n^m(\rho)\frac{\partial}{\partial\rho}E_{n'}^{m'}(\rho)-E_{n'}^{m'}(\rho)\frac{\partial}{\partial\rho}E_n^m(\rho)=0$, then E_n^m and $E_{n'}^{m'}$ are linear dependent. Therefore, based on lemmas 1.3.1-1.3.3 this quantity vanishes only for $m=m'$ and $n=n'$. This means that for $n\neq n'$ and $m\neq m'$:

$$\int_{S_{\alpha_1}}S_n^m(\mu,\nu)S_{n'}^{m'}(\mu,\nu)d\Omega(\mu,\nu)=0\ .\tag{1.235}$$

□

The ellipsoidal Beltrami operator is self-adjoint with respect to the inner product (1.224) and his eigenfunctions which are the surface ellipsoidal harmonics (1.214) form a complete orthonormal basis set of the Hilbert space of square-integrable functions, i.e., every smooth function F defined on the surface of the ellipsoid $\rho = \textit{constant}$ can be expressed as linear combination of the surface ellipsoidal harmonics as:

$$F(\mu,\nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} c_n^m S_n^m(\mu,\nu) \ . \tag{1.236}$$

The coefficients c_n^m are given by:

$$c_n^m = \frac{1}{\gamma_n^m} \int_{S_{\alpha_1}} F(\mu,\nu) S_n^m(\mu,\nu) d\Omega(\mu,\nu) \ , \tag{1.237}$$

where γ_n^m the normalization constants given by (1.229):

$$\gamma_n^m = \int_{S_{\alpha_1}} \left[S_n^m(\mu,\nu)\right]^2 d\Omega(\mu,\nu) \ , \tag{1.238}$$

for $n = 0, 1, 2, ...$ and $m = 1, 2, ..., 2n + 1$.

The completeness of the basis and (1.236) is justified via the completeness of the spherical harmonics, since every ellipsoidal harmonic can be written in terms of spherical harmonics and every spherical harmonic can be written in terms of ellipsoidal harmonics. The completeness of the set $\{S_n^m\}$ can also be derived from the corresponding closure relation.

Closure relation

The dirac measure in an orthogonal curvilinear coordinate system which consists of the surfaces $f_2(x_1, x_2, x_3) = q_1$, $f_2(x_1, x_2, x_3) = q_2$ and $f_3(x_1, x_2, x_3) = q_3$, is given by:

$$\delta(\mathbf{r} - \mathbf{r_0}) = \frac{\delta(q_1 - q_{1_0})}{h_{q_1}} \frac{\delta(q_2 - q_{2_0})}{h_{q_2}} \frac{\delta(q_3 - q_{3_0})}{h_{q_3}} \ , \tag{1.239}$$

where d_{q_i} the metric coefficients for $i = 1, 2, 3$. Hence, by (1.22)-(1.24) the dirac measure in the ellipsoidal coordinate system is:

$$\delta(\mathbf{r} - \mathbf{r_0}) = \frac{\delta(\rho - \rho_0)}{h_\rho} \frac{\delta(\mu - \mu_0)}{h_\mu} \frac{\delta(\nu - \nu_0)}{h_\nu} \ , \tag{1.240}$$

where $\mathbf{r} = (\rho, \mu, \nu)$ and $\mathbf{r_0} = (\rho_0, \mu_0, \nu_0)$. On the surface of the ellipsoid $\rho = \rho_0$ the dirac measure becomes:

$$\delta(\mathbf{r} - \mathbf{r_0}) = \frac{\delta(\mu - \mu_0)}{h_\mu} \frac{\delta(\nu - \nu_0)}{h_\nu} \ . \tag{1.241}$$

Thus, by taking a function $F(\mu, \nu)$ smooth enough on the surface of the ellipsoid $\rho = \textit{constant}$ and from the definition of the Dirac delta function, we have the following relation:

$$F(\mu,\nu) = \int_{S_\rho} F(\mu',\nu') \frac{\delta(\mu - \mu')}{h_{\mu'}} \frac{\delta(\nu - \nu')}{h_{\nu'}} d\Omega(\mu',\nu') \ . \tag{1.242}$$

Taking the expansion of F in terms of the basis set $\{S_n^m\}$ based on relations (1.236) and (1.237), we have:

$$F(\mu,\nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} c_n^m S_n^m(\mu,\nu) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \left[\frac{1}{\gamma_n^m} \int_{S_\rho} F(\mu',\nu') S_n^m(\mu',\nu') d\Omega(\mu',\nu') \right] S_n^m(\mu,\nu) \ . \tag{1.243}$$

Interchanging the sum with the integral we obtain:

$$\begin{aligned} F(\mu,\nu) &= \int_{S_\rho} \left[\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} F(\mu',\nu') S_n^m(\mu',\nu') d\Omega(\mu',\nu') \right] S_n^m(\mu,\nu) \\ &= \int_{S_\rho} \left[\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} S_n^m(\mu',\nu') S_n^m(\mu,\nu) \right] F(\mu',\nu') d\Omega(\mu',\nu') \end{aligned} \tag{1.244}$$

From relations (1.242) and (1.244) it can be observed that:

$$\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} S_n^m(\mu',\nu') S_n^m(\mu,\nu) = \frac{\delta(\mu-\mu')}{h_\mu} \frac{\delta(\nu-\nu')}{h_\nu} \ , \tag{1.245}$$

which is the closure relation (or completeness relation) of the basis set of the surface ellipsoidal harmonics $\{S_n^m\}$.

Next, we present the following three orthogonality theorems whose proofs can be found in ([20], p83-86).

Theorem 1.5.2. *For any $n, n' = 0, 1, 2, \ldots$, $m = 1, 2, \ldots, 2n+1$ and $m' = 1, 2, \ldots, 2n'+1$ the following relation holds:*

$$(h_3^2+h_2^2)(p_n^m - p_{n'}^{m'}) \int_{h_3}^{h_2} \frac{E_n^m(\mu) E_{n'}^{m'}(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} d\mu = (n-n')(n+n'+1) \int_{h_3}^{h_2} \int_{h_3}^{h_2} \frac{\mu^2 E_n^m(\mu) E_{n'}^{m'}(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} d\mu. \tag{1.246}$$

Therefore, if $n = n'$, but $p_n^m \neq p_n^{m'}$, then

$$\int_{h_3}^{h_2} \frac{E_n^m(\mu) E_{n'}^{m'}(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} d\mu = 0 \ , \tag{1.247}$$

and if $n \neq n'$, but $p_n^m = p_{n'}^{m'}$, then

$$\int_{h_3}^{h_2} \frac{\mu^2 E_n^m(\mu) E_{n'}^{m'}(\mu)}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} d\mu = 0 \ . \tag{1.248}$$

Theorem 1.5.3. *If the functions E_n^m and $E_{n'}^{m'}$ belong to the same Lamé class and the indices n and n' are either both even or both odd, then*

$$(h_3^2+h_2^2)(p_n^m - p_{n'}^{m'}) \int_0^{h_3} \frac{E_n^m(\nu) E_{n'}^{m'}(\nu)}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} d\nu = (n-n')(n+n'+1) \int_0^{h_3} \frac{\nu^2 E_n^m(\nu) E_{n'}^{m'}(\nu)}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} d\nu \ . \tag{1.249}$$

Therefore, if $n = n'$, but $p_n^m \neq p_n^{m'}$, then

$$\int_0^{h_3} \frac{E_n^m(\nu) E_{n'}^{m'}(\nu)}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} d\nu = 0 \ , \tag{1.250}$$

and if $n = n'$, but $p_n^m \neq p_{n'}^{m'}$, then

$$\int_0^{h_3} \frac{\nu^2 E_n^m(\nu) E_{n'}^{m'}(\nu)}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} d\nu = 0 \ . \tag{1.251}$$

Theorem 1.5.4. *If the functions E_n^m and $E_{n'}^{m'}$ belong to the same Lamé class, the indices n and n' are either both even or both odd, and $n \neq n'$, $p_n^m \neq p_{n'}^{m'}$, then*

$$\int_0^{h_3} \int_{h_3}^{h_2} S_n^m(\mu,\nu) S_{n'}^{m'}(\mu,\nu) d\Omega(\mu,\nu) = 0 \ , \tag{1.252}$$

where S_n^m, $S_{n'}^{m'}$ defined in (1.214) and $d\Omega$ the solid angle element defined in (1.226). When $S_n^m = S_{n'}^{m'}$ we obtain

$$\int_0^{h_3} \int_{h_3}^{h_2} (S_n^m(\mu,\nu))^2 \, d\Omega(\mu,\nu) = \frac{1}{8}\gamma_n^m \ . \tag{1.253}$$

Remarks:

- Theorem 1.5.2 states orthogonality of Lamé fucntions in the interval $[h_3, h_2]$ and theorem 1.5.3 states orthogonality of Lamé functions in the interval $[0, h_3]$
- Theorems 1.5.2 and 1.5.3 are used in order recover orthogonality between surface ellipsoidal harmonics over an octant of an ellipsoidal surface $\rho = constant$ in theorem 1.5.4. This means that because of the symmetry, the integral will have the same value over any other octant and since (1.229) states the values of the normalization constants γ_n^m over the complete ellipsoidal surface it follows that these values are 8 times the value of the integration over one octant.
- The orthogonality relation (1.229) is true for any choice of surface ellipsoidal harmonics but (1.252) is true for specific choices of Lamé functions since the integration is over an octant.

Before we proceed with applications of the ellipsoidal harmonics in scattering theory it is important to study some basic properties of the spherical harmonics since they are frequently mentioned and they consist the basis of the theory of the ellipsoidal harmonics.

1.6 Spherical Harmonics

As the ellipsoidal coordinate system is reduced to the spherical system (for $\mathbf{r} \to \infty$), the ellipsoidal harmonics do not reduce to the spherical harmonics but instead they reduce to the sphero-conal harmonics which preserve the ellipsoidal characteristics [29]. For this reason, in this section we present some basic characteristics of the spherical harmonics as well as the sphero-conal system which help us to further understand the behavior of ellipsoidal coordinate system and its harmonics.

1.6.1 Laplace's Equation in Orthogonal Curvilinear Coordinates

Let

$$f_1(x_1, x_2, x_3) = q_1 \ , \ f_2(x_1, x_2, x_3) = q_2 \ , \ f_3(x_1, x_2, x_3) = q_3 \ , \tag{1.254}$$

three families of surfaces. Since the intersection of two out of these three surfaces is a curve in three-dimensional space which lies on both these surfaces, the two coordinates associated with the surfaces whose the curves lies on will be constant and the third coordinate will vary. This third coordinate will vary continuously along the intersection curve. For example, lets assume that the curve is the intersection of $f_1 = q_1$ and $f_2 = q_2$. This means that q_1 and q_2 are the constant coordinates while q_3 is the variable coordinate. Next, consider a point (q_1, q_2, q_3) on this curve and another point $(q_1, q_2, q_3 + ds_3)$ which is distance ds_3 away from the first one. Then we define the function $h_3(q_1, q_2, q_3)$ as the limit of the ratio $\dfrac{ds_3}{dq_3}$ as dq_3 tends to 0. This relationship in terms of differential can be written as:

$$ds_3 = h_3(q_1, q_2, q_3) dq_3 \ . \tag{1.255}$$

Therefore, with the other two cases we define h_1, h_2, h_3 with the following differential forms:

$$\begin{aligned} ds_1 &= h_1(q_1, q_2, q_3)dq_1 \ , \\ ds_2 &= h_2(q_1, q_2, q_3)dq_2 \ , \\ ds_3 &= h_3(q_1, q_2, q_3)dq_3 \ , \end{aligned} \tag{1.256}$$

At any point these three curves define in general three directions in space. If these three directions are mutually orthogonal at every point then the coordinate system is called orthogonal curvilinear coordinate system. In the present work we study only orthogonal curvilinear coordinate system. Now let $P(q_1, q_2, q_3)$ be a point and Q be any point in a small distance ds from P. Because of the orthogonality it is concluded that the elementary length of the line joining these points is given by:

$$(ds)^2 = (ds_1)^2 + (ds_2)^2 + (ds_3)^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \ . \tag{1.257}$$

For ϕ smooth enough throughout any volume V and Stokes theorem we have:

$$\iiint_V \Delta\phi dxdydz = -\int_S \frac{\partial \phi}{\partial n} dS \ , \tag{1.258}$$

which means that if $\Delta\phi = 0$ throughout the volume V we have:

$$\int_S \frac{\partial \phi}{\partial n} dS = 0 \ . \tag{1.259}$$

Next, we will apply the above theorem to the elementary volume bounded by six surfaces. For these surfaces we take the six surfaces $q_1 = P_1 \pm \frac{1}{2}dq_1$, $q_2 = P_2 \pm \frac{1}{2}dq_2$ and $q_3 = P_3 \pm \frac{1}{2}dq_3$ which bound a small curvilinear parallelepiped about a point with coordinates (P_1, P_2, P_3) and we will calculate the surface integral $\int_S \frac{\partial \phi}{\partial n} dS$ over this closed surface. Starting with the face on which q_1 increases from $P_1 - \frac{1}{2}dq_1$ to $P_1 + \frac{1}{2}dq_1$, q_2 increases from $P_2 - \frac{1}{2}dq_2$ to $P_2 + \frac{1}{2}dq_2$ and q_3 is constant equal to $P_3 - \frac{1}{2}dq_3$ or $P_3 + \frac{1}{2}dq_3$. Therefore, the area of this face is $ds_1 ds_2 = (h_1 dq_1)(h_2 dq_2)$ where h_1 and h_2 take values at the point $(P_1, P_2, P_3 - \frac{1}{2}dq_3)$ or $(P_1, P_2, P_3 + \frac{1}{2}dq_3)$. The normal has the direction which q_3 increases from $P_3 - \frac{1}{2}dq_3$ to $P_3 + \frac{1}{2}dq_3$ and the corresponding length element will be $ds_3 = h_3 dq_3$. Hence, the normal derivative on this face will be $\frac{\partial \phi}{\partial n} = \frac{1}{h_3}\frac{\partial \phi}{\partial q_3}$. Therefore, the contribution of this face to the surface integral of the curvilinear parallelepiped will be:

$$\left(h_1 h_2 \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}_{(P_1, P_2, P_3 + \frac{1}{2}dq_3)} \right) dq_1 dq_2 \ . \tag{1.260}$$

Because of direction of the normal derivative if we choose the point
$(P_1, P_2, P_3 - \frac{1}{2}dq_3)$, the contribution of the corresponding face to the surface integral will be:

$$-\left(h_1 h_2 \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}_{(P_1, P_2, P_3 - \frac{1}{2}dq_3)} \right) dq_1 dq_2 \ . \tag{1.261}$$

Therefore, the two contributions via the total derivative can be summed into:

$$\left[\frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right] dq_1 dq_2 dq_3 \ . \tag{1.262}$$

Similarly from the other two pairs of faces we take obtain:

$$\left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) \right] dq_1 dq_2 dq_3 \ , \tag{1.263}$$

and

$$\left[\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial q_2}\right)\right] dq_1 dq_2 dq_3 \ . \tag{1.264}$$

Hence, the surface integral takes the value:

$$\begin{aligned}\int_S \frac{\partial \phi}{\partial n} dS = &\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q_1}\right)\right] dq_1 dq_2 dq_3 + \left[\frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial q_2}\right)\right] dq_1 dq_2 dq_3 \\ &+ \left[\frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q_3}\right)\right] dq_1 dq_2 dq_3 \ ,\end{aligned} \tag{1.265}$$

thisand surface integral must vanish. Hence, it is concluded that the Laplace equation in curvilinear coordinates is:

$$\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial q_3}\right) = 0 \ . \tag{1.266}$$

Two special cases of orthogonal curvilinear systems are the sphero-conal (or conical) coordinate system and the spherical system.

- **Sphero-conal system:**
  The sphero-conal (or conical) coordinates (r, μ, ν) are given by:

$$x_1 = r\frac{\mu\nu}{h_2 h_3} \ , \quad x_2 = r\frac{\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3} \ , \quad x_3 = r\frac{\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2} \ , \tag{1.267}$$

where $r \in (0, +\infty)$ and $0 < \nu < h_3 < \mu < h_2$. These expressions correspond to the first octant where all the Cartesian coordinates are positive. For the other seven octants the expressions are controlled by the appropriate signs of x_1, x_2, x_3, meaning that we take the appropriate positive or negative branch of the square roots similarly with the ellipsoidal coordinate system.

In the Cartesian coordinate system every point can be represented by the intersection of the three planes. The same goes for every orthogonal curvilinear system, meaning that every point can be represented by the intersection of the three surfaces that form the system.
In order to see the three surfaces which form this orthogonal curvilinear system we take the squares of x_1, x_2, x_3:

$$x_1^2 = r^2\frac{\mu^2\nu^2}{h_2^2 h_3^2} \ , \ x_2^2 = r^2\frac{(\mu^2 - h_3^2)(h_3^2 - \nu^2)}{h_1^2 h_3^2} \ , \ x_3^2 = r^2\frac{(h_2^2 - \mu^2)(h_2^2 - \nu^2)}{h_1^2 h_2^2} \ . \tag{1.268}$$

Summing them and using relation (1.5) we have:

$$x_1^2 + x_2^2 + x_3^2 = r^2 \ . \tag{1.269}$$

which is the surfaces of a spheres of center $(0, 0, 0)$ and radii r. Similarly from (1.268) and (1.5) we have:

$$\frac{x_1^2}{\mu^2} + \frac{x_2^2}{\mu^2 - h_3^2} - \frac{x_3^2}{h_2^2 - \mu^2} = 0 \ , \tag{1.270}$$

which are the elliptic cones with central axes along the x_3 axis and:

$$\frac{x_1^2}{\nu^2} - \frac{x_2^2}{h_3^2 - \nu^2} - \frac{x_3^2}{h_2^2 - \nu^2} = 0 \ , \tag{1.271}$$

which are the also elliptic cones with their central axes along the x_1 axis. Next in order to check the orthogonality of the system we take the partial derivatives:

$$\mathbf{r}_r = \left(\frac{\mu\nu}{h_2 h_3}, \frac{\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}}{h_1 h_3}, \frac{\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1 h_2} \right) , \tag{1.272}$$

$$\mathbf{r}_\mu = \left(\frac{r\nu}{h_2 h_3}, \frac{r\mu\sqrt{h_3^2-\nu^2}}{h_1 h_3\sqrt{\mu^2-h_3^2}}, -\frac{r\mu\sqrt{h_2^2-\nu^2}}{h_1 h_2\sqrt{h_2^2-\mu^2}} \right) , \tag{1.273}$$

$$\mathbf{r}_\nu = \left(\frac{r\mu}{h_2 h_3}, -\frac{r\nu\sqrt{\mu^2-h_3^2}}{h_1 h_3\sqrt{h_3^2-\nu^2}}, -\frac{r\nu\sqrt{h_2^2-\mu^2}}{h_1 h_2\sqrt{h_2^2-\nu^2}} \right) . \tag{1.274}$$

Therefore, taking their inner products and using relation (1.5) we have:

$$\mathbf{r}_r \cdot \mathbf{r}_\mu = \frac{r\mu\nu^2}{h_2^2 h_3^2} + \frac{r\mu(h_3^2-\nu^2)}{h_1^2 h_3^2} - \frac{r\mu(h_2^2-\nu^2)}{h_1^2 h_2^2} = \frac{r\mu\nu^2(h_1^2-h_2^2+h_3^2)+r\mu(h_3^2h_2^2-h_2^2h_3^2)}{h_1^2h_2^2h_3^2} = 0 . \tag{1.275}$$

Similarly can be shown that $\mathbf{r}_r \cdot \mathbf{r}_\nu = 0$. Finally, using (1.5):

$$\mathbf{r}_\mu \cdot \mathbf{r}_\nu = \frac{r^2\mu\nu}{h_2^2h_3^2} - \frac{r^2\mu\nu}{h_1^2h_3^2} + \frac{r^2\mu\nu}{h_1^2h_2^2} = \frac{r^2\mu\nu(h_1^2-h_2^2+h_3^2)}{h_1^2h_2^2h_3^2} = 0 . \tag{1.276}$$

Hence, the system is orthogonal and based on relations (1.22)-(1.24), (1.256) and (1.5) it is concluded that:

$$h_r^2 = 1 ,$$

$$h_\mu^2 = \frac{r^2(\mu^2-\nu^2)}{(\mu^2-h_3^2)(h_2^2-\mu^2)} , \tag{1.277}$$

$$h_\nu^2 = \frac{r^2(\mu^2-\nu^2)}{(h_3^2-\nu^2)(h_2^2-\nu^2)} . \tag{1.278}$$

Substituting the above quantities into (1.261)-(1.263) we obtain:

$$\frac{\partial}{\partial r}\left(\frac{h_\mu h_\nu}{h_r}\frac{\partial}{\partial r}\right) = \frac{\mu^2-\nu^2}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) , \tag{1.279}$$

$$\frac{\partial}{\partial \mu}\left(\frac{h_r h_\nu}{h_\mu}\frac{\partial}{\partial \mu}\right) = \frac{1}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}\frac{\partial}{\partial \mu}\left(\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{\partial}{\partial \mu}\right) , \tag{1.280}$$

$$\frac{\partial}{\partial \nu}\left(\frac{h_r h_\mu}{h_\nu}\frac{\partial}{\partial \nu}\right) = \frac{1}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}\frac{\partial}{\partial \nu}\left(\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}\frac{\partial}{\partial \nu}\right) , \tag{1.281}$$

Therefore the Laplace equation in sphero-conal coordinates takes the following form:

$$\begin{aligned} &\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}{(\mu^2-\nu^2)}\frac{\partial}{\partial \mu}\left(\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{\partial u}{\partial \mu}\right) \\ &+ \frac{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}{(\mu^2-\nu^2)}\frac{\partial}{\partial \nu}\left(\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}\frac{\partial u}{\partial \nu}\right) = 0 , \end{aligned} \tag{1.282}$$

for $u(r,\mu,\nu)$ smooth enough.

A normal solution of (1.282) will be of the form $R(r)M(\mu)N(\nu)$ and by substituting in (1.282) and dividing with it we obtain:

$$\begin{aligned} &\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}}{M(\mu^2-\nu^2)}\frac{\partial}{\partial \mu}\left(\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}\frac{\partial M}{\partial \mu}\right) \\ &+ \frac{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}}{N(\mu^2-\nu^2)}\frac{\partial}{\partial \nu}\left(\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}\frac{\partial N}{\partial \nu}\right) = 0 , \end{aligned} \tag{1.283}$$

from which we can see that the first term is the only term that contains the variable r. Hence in order for the equation to be satisfied the following relation must hold:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = k \ , \tag{1.284}$$

where $k = constant$. Therefore, we have:

$$r^2 R''(r) + 2rR'(r) - 2kR(r) = 0 \ , \tag{1.285}$$

which in order to simplify it's solution similarly to the spherical system we pick based on [21] the second separation constant to be $k = n(n+1)$ for $n = 0, 1, 2, ..$. The other two parts that contain the functions M and N are identical with the corresponding parts of M and N in (1.41) which means that we will end up with the same two equations (1.53) and (1.54) of the ellipsoidal coordinate system. This shows why we chose $A = n(n+1)$ for the case of the ellipsoidal coordinate system.

- **Spherical system:** The spherical coordinates (r, θ, ϕ) are given by:

$$x_1 = rsin\theta cos\phi \ , \tag{1.286}$$
$$x_2 = rsin\theta sin\phi \ , \tag{1.287}$$
$$x_3 = rcos\theta \ , \tag{1.288}$$

where $r \in [0, +\infty)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Similarly to the sphero-conal or conical system by eliminating the two out of the three variables we can obtain the three surfaces and every point can be represented as the intersection of these three surfaces. These surfaces are:
By eliminating the variables θ and ϕ we obtain for $r = constant$ the equation:

$$x_1^2 + x_2^2 + x_3^2 = r^2 \ , \tag{1.289}$$

which is the surface of the sphere of radius r. Eliminating the variables r and ϕ we obtain:

$$z = \frac{1}{tan\phi}\sqrt{x_1^2 + x_2^2} \ , \tag{1.290}$$

which for $\phi = constant$ is the equation of the cone pointing either upward or downward. Eliminating the variables r and θ we obtain:

$$y = xtan\theta \ , \tag{1.291}$$

which for $\theta = constants$ is the half plane since $rsin\phi$ cannot be negative. For the orthogonality we take the partial derivatives:

$$\mathbf{r}_r = (sin\theta cos\phi, sin\theta sin\phi, cos\theta) \ , \tag{1.292}$$
$$\mathbf{r}_\theta = (rcos\theta cos\phi, rcos\theta sin\phi, -rsin\theta) \ , \tag{1.293}$$
$$\mathbf{r}_\phi = (-rsin\theta sin\phi, rsin\theta cos\phi, 0) \ , \tag{1.294}$$

and we calculate:

$$\mathbf{r}_r \cdot \mathbf{r}_\theta = rcos\theta sin\theta cos^2\phi + rcos\theta sin\theta sin^2\phi - rcos\theta sin\theta = 0 \ , \tag{1.295}$$
$$\mathbf{r}_r \cdot \mathbf{r}_\phi = -rsin^2\theta cos\phi sin\phi + rsin^2\theta cos\phi sin\phi = 0 \ , \tag{1.296}$$
$$\mathbf{r}_\theta \cdot \mathbf{r}_\phi = -r^2 cos\theta sin\theta cos\phi sin\phi + r^2 cos\theta sin\theta cos\phi sin\phi = 0 \ . \tag{1.297}$$

Hence, the system is orthogonal and the metric coefficients are:

$$h_r^2 = \|\mathbf{r}_r\|^2 = sin^2\theta cos^2\phi + sin^2\theta sin^2\phi + cos^2\theta = 1 \ , \quad (1.298)$$

$$h_\theta^2 = \|\mathbf{r}_\theta\|^2 = r^2 cos^2\theta(cos^2\phi + sin^2\phi) + r^2 sin^2\theta = r^2 \ , \quad (1.299)$$

$$h_\phi^2 = \|\mathbf{r}_\theta\|^2 = r^2 sin^2\theta(sin^2\phi + cos^2\phi) = r^2 sin^2\theta \ . \quad (1.300)$$

Substituting in (1.266) we obtain the Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \ , \quad (1.301)$$

where $u(r, \theta, \phi)$ is smooth enough.
Equivalently:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\mathbb{B}u = 0 \ , \quad (1.302)$$

where

$$\mathbb{B}u = \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \quad (1.303)$$

the Beltrami operator (or surface Laplacian) on the sphere S^2.

Remark: From these two systems it can be observed that the ellipsoidal coordinate ρ corresponds to the radial part r of the spherical coordinates and the coordinates μ and ν correspond to the angular part θ and ϕ of the spherical coordinates and this correspondence has been frequently used the previous sections i.e. Lamé functions of the second kind and surface ellipsoidal harmonics.

1.6.2 Legendre's Equation

The spherical harmonics arise from the solution of Laplace's equation (1.301) in the spherical coordinate system. Since we want to apply separation of variables we want to find a solution of the form $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ which is called *normal form*. Therefore, by substituting $u = R\Theta\Phi$ in (1.301) and dividing with $R\Theta\Phi$ we obtain:

$$\frac{\partial}{R\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \ , \quad (1.304)$$

where the first term of the equation is the only term that contains the radial variable r. This means that the first term is either equal to a constant or equal to a function of r. If we assume that it is a function of r then in order for equation (1.304) to hold true the other two terms of the equation would have to depend on variable r which cant be true since both terms dont contain this variable. Hence the first term has to be equal to constant in order to satisfy the equation which gives the equation:

$$\frac{\partial}{R\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = k \Leftrightarrow r^2 R'' + 2rR' - kR = 0 \ , \quad (1.305)$$

where $k = constant$. This is a second order Euler differential equation which means that the solution will be of the form r^n. Hence, in order to determine the separation constant k we substitute r^n in (1.305) and we obtain:

$$n(n-1)r^n + 2nr^n - kr^n = 0 \Leftrightarrow r^n\left(n(n-1) + 2n - k\right) = 0 \Leftrightarrow k = n(n+1) \ . \quad (1.306)$$

The same exactly result we have if we choose r^{n-1} meaning we have two independent solutions of (1.305) for the same separation constant $k = n(n+1)$ for $n = 0, 1, 2, ...$ and the general solution of (1.305) will be of the form:

$$R(r) = c_1 r^n + c_2 r^{-n-1} \ . \tag{1.307}$$

The separation constant $k = n(n+1)$ is also used in the radial part of the ellipsoidal coordinate system since the radial part of the conical system which acts as intermediate coincides with the radial part of the spherical system leading us to take the same separation constant.
Replacing $k = n(n+1)$ in (1.305), substituting in (1.304) and multiplying with $sin^2\theta$ we obtain:

$$n(n+1)sin^2\theta + \frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \ . \tag{1.308}$$

It can be observed that similar to the first term of (1.304) the third term contains only the variable ϕ leading us similarly to the radial part to take it equal with a constant m in order for equation to hold true:

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = a \ , \tag{1.309}$$

and because the solutions involving the full range of $\phi \in [0, 2\pi]$ must be periodic we choose the separation variable to be negative , say $a = -m^2$ for m integer which gives us the following equation of $\Phi(\phi)$:

$$\Phi'' + m^2\Phi = 0 \ , \tag{1.310}$$

with solutions of the form

$$\Phi(\phi) = c_1 cosm\phi + c_2 sinm\phi \ , \tag{1.311}$$

or more commonly:

$$\Phi(\phi) = a_m e^{im\phi} \ , \tag{1.312}$$

where c_1,c_2 or a_m arbitrary constants and it can be seen that it is a 2π periodic function.
Finally for $\Theta(\theta)$ by substituting (1.309) into (1.308) we have:

$$\begin{aligned} &n(n+1)sin^2\theta + \frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) - m^2 = 0 \\ &\Leftrightarrow \frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + \left(n(n+1) - \frac{m^2}{sin^2\theta}\right)\Theta = 0 \ . \end{aligned} \tag{1.313}$$

If we write $cos\theta = x$ the above equation becomes:

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right]\Theta = 0 \ . \tag{1.314}$$

This equation is known as associated Legendre's equation and for the specific case of $m = 0$ it becomes:

$$(1-x^2)\Theta'' - 2x\Theta' + n(n+1)\Theta = 0 \ , \tag{1.315}$$

which is known as the classical Legendre's equation. The solutions for $m = 0$ will be functions of $x = cos\theta$ and specifically polynomials known as Legendre's polynomials P_n, while for general m the solutions can be found by differentiating the Legendre polynomials, giving the associated Legendre polynomials P_n^m.

Legendre polynomials:

Equation (1.315) can be rewritten in the following form:

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0 \ , \tag{1.316}$$

where the functions $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ are analytic at x=0 and singular at $x = \pm 1$. For example $\frac{2x}{1-x^2} = 2x\left[1 + x^2 + x^4 + ...\right]$ for $|x| < 1$ and the same goes for $\frac{n(n+1)}{1-x^2}$ which means that the power series solutions of the form:

$$\sum_{k=0}^{\infty} a_k x^k \ , \tag{1.317}$$

will also have radii $R = 1$ and that ± 1 is regular singular point. Substituting this solution in (1.315) and after some straightforward calculation similar to the ones followed in the previous sections for the Lamé functions we obtain an equation of the form:

$$\sum_{k=0}^{\infty} A_k x^k = 0 \ , \tag{1.318}$$

where $A_k = (k+1)(k+2)a_{k+2} + (n-k)(n+k+1)a_k$ for $k = 0, 1, 2, ...$. From this equation it is concluded that $A_k = 0$ which gives the following relation between the constants a_k:

$$a_{k+2} = -\frac{(n-k)(n+k+1)}{(k+1)(k+2)}a_k \ . \tag{1.319}$$

separating the odd term of the sum from the even terms, it is concluded that the resulting solution will be:

$$\Theta(x) = a_0 L_p(x) + a_1 L_q(x) \ , \tag{1.320}$$

where L_p is the even series and L_q is the odd series. For every p non-negative integer, one of the series will terminate while the other remains as an infinite series. Specifically:

- If p is even, then the series L_p terminates, resulting an even polynomial of degree p. The odd series L_q remains as an infinite series which converges for $|x| < 1$ and diverges for $x = \pm 1$ and $x^2 > 1$ and it is a second solution of (1.315).

- If p is odd, then the series L_q terminates, resulting an odd polynomial of degree q. The even series L_p remains as an infinite series which converges for $|x| < 1$, diverges for $x = \pm 1$ and $x^2 > 1$ and it is a second solution of (1.315).

Based on the above the two independent solutions of Legendre's equation (1.315) are [21], [20] :

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k} \ , \quad n = 0, 1, 2, ... \ , \tag{1.321}$$

which are the Legenre's polynomials of the first kind and

$$Q_n(x) = \frac{1}{2}P_n(x) ln\frac{1+x}{1-x} - \sum_{k=0}^{n-1}\frac{1}{k+1}P_k(x)P_{n-k-1}(x) \ , \quad n = 0, 1, 2, ... \ , \tag{1.322}$$

which are the Legendre's polynomials of the second king. The Legendre's polynomials of the first kind consist the finite sum of (1.320) while those of the second kind consist the infinite series of (1.320) which blows up at $x = \pm 1$. The Legendre's polynomials can be defined in various ways. One definition is in terms of Rodrigue's formula:

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n \ , \tag{1.323}$$

and they are normalized so that $P_n(1) = 1$.

Associated Legendre functions

For the general case of $m \neq 0$ it can be observed that by substituting $\Theta = (1-x^2)^{\frac{m}{2}}u(x)$ in (1.315) we have:

$$(1-x^2)u'' - 2(m+1)xu' + (n(n+1) - m(m+1))\, u = 0 \ , \tag{1.324}$$

and by differentiating with respect to x we obtain:

$$(1-x^2)u''' - 2(m+2)xu'' + (n(n+1) - (m+1)(m+2))\, u' = 0 \ . \tag{1.325}$$

Hence, it can easily be observed that if u is a solution of (1.324) for a given n and m, then u' is a solution of the same equation for n and $m+1$. Therefore, if $P_n(x)$ is the solution of (1.324) for $m = 0$, then $P_n^{'}(x)$ is the solution for $m = 1$ and in general $\frac{d^m}{dx^m}P_n(x)$ is the solution for a given n and a general value of m. Hence, the solutions of (1.315) are of the form:

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}\left(P_n(x)\right) \ , \tag{1.326}$$

for $n = 0, 1, 2, ...$ and $m = -n, -n+1, ..., n-1, n]$ and $P_n^m \equiv 0$ for $m > n$. These functions are called *associated Legendre functions of the first kind*. Since Q_n is also solution of (1.315), then:

$$Q_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}\left(Q_n(x)\right) \ , \tag{1.327}$$

are solutions of (1.314) for $n = 0, 1, 2, ..., m = -n, n+1, ..., n-1, n$ and $Q_n^m \equiv 0$ for $m > n$. These functions are the *associated Legendre functions of the second kind*. An associated Legendre function is not polynomial in general and because of the factor $(1-t^2)^{\frac{m}{2}}$, it is only defined on the closed interval $[-1, 1]$. Now combining the solutions r^n and r^{-n-1} of (1.305) , the solution (1.321) of (1.315), and the solution (1.312) of the (1.309) we obtain the independent normal solutions of the Laplace's equation:

$$u_1(r, \theta, \phi) = r^n P_n^m(cos\theta)e^{im\phi} \ , \tag{1.328}$$

$$u_2(r, \theta, \phi) = r^{-n-1} P_n(cos\theta)e^{im\phi} \ . \tag{1.329}$$

Remark: In the present work, x is introduced as the cosine of a real angle and consequently has values in $[-1, 1]$. However, more general cases have been studied in [21].

1.6.3 Spherical Harmonics

The spherical harmonics are the angular part of the solution of Laplace's equation in spherical coordinates $\Theta(\theta)\Phi(\phi)$ and are denoted with $Y_n^m(\theta, \phi)$ where n is the degree and m is the order. Thus, based on (1.312) and (1.326) the spherical harmonics of degree n and order m are of the form:

$$Y_n^m(\theta, \phi) \equiv \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(cos\theta)e^{im\phi} \ , \qquad \text{for } \begin{cases} n = 0, 1, 2, ... \\ m = -n, -n+1, ..., n-1, n \end{cases} . \tag{1.330}$$

These functions are also known as *surface spherical harmonics* but based on (Whittaker and Watson) they are simply called spherical harmonics. If the spherical harmonic of degree n and order m (or surface spherical harmonic) is multiplied with r^n the harmonic homogeneous function $r^n Y_n^m(\theta,\phi)$ which is the normal solution (1.328) of (1.301) is called *solid spherical harmonic of degree n and order m* or simply spherical harmonic of degree n and order m but in order to avoid confusion when we refer to spherical harmonics we mean the surface spherical harmonics (1.330).

The functions Y_n^m have some nice properties but in order to explain them we need to specify the structure of the space $L^2(S^2)$ ($\int_{S^2}|f|^2 dS < \infty$). First we take the inner product on $L^2(S^2)$ given by:

$$< f, g >= \int_{S^2} f\bar{g}dS = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\overline{g(\theta,\phi)}sin\theta d\theta d\phi \ , \tag{1.331}$$

where $f,g \in L^2(S^2)$. With this inner product and the norm associated $\|f\| = \sqrt{< f, f >}$ with this inner product, $L^2(S^2)$ is a Hilbert space. The orthogonality of Y_n^m can be obtained by the orthogonality relations of the associated Legendre functions P_n^m and $e^{im\phi}$. Hence, based on [21] we have:

$$\int_{-1}^{1} P_n^m(x)P_{n'}^m(x)dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'} \ , \tag{1.332}$$

which is an orthogonality relation for the associated Legendre functions of the same order m but different degrees n. For $e^{im\phi}$ we have:

$$\int_0^{2\pi} e^{im\phi}e^{-im\phi}d\phi = 2\pi\delta_{mm'} \ . \tag{1.333}$$

Hence, be defining the *normalized ellipsoidal harmonics* of degree n and order m as:

$$Y_n^m(\theta,\phi) = Y_n^m(\hat{\mathbf{r}}) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^{|m|}(x)e^{im\phi} \ , \tag{1.334}$$

it can easily be concluded that the orthogonality relation of the ellipsoidal harmonics Y_n^m is:

$$\int_{S^2} Y_n^m(\hat{\mathbf{r}})Y_{n'}^{m'*}(\hat{\mathbf{r}})dS(\hat{\mathbf{r}}) = \delta_{nn'}\delta mm', \tag{1.335}$$

where $dS = sin\theta d\theta d\phi$ and $*$ denotes the complex conjugate.
For the completeness of the spherical harmonics Y_n^m we refer to [11] and we have the following completeness relations:
For the associated Legendre functions:

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_n(x)P_n(x') = \frac{2}{2n+1}\delta(x-x') \ . \tag{1.336}$$

For the exponential Φ we have:

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \delta(\phi-\phi') \ . \tag{1.337}$$

For the spherical harmonics the completeness relation is:

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{r}})Y_n^{m*}(\hat{\mathbf{r}}') = \delta(\hat{\mathbf{r}}-\hat{\mathbf{r}}') = \delta(cos\theta - cos\theta')\delta(\phi-\phi') \ . \tag{1.338}$$

In general, the completeness of the spherical harmonics $\{Y_n^m\}$ can be derived straightforward from the completeness of $\{P_n^m(cos\theta)\}$ and $\left\{e^{im\phi}\right\}$ which can be derived from Fourier analysis

and Sturm-Liouville theory respectevily. An alternative way to prove the completeness of the spherical harmonics is via the Peter-Weyl theorem.
Hence, based on the completeness of $Y_n^m(\hat{\mathbf{r}})$ of the $L^2(S^2)$ every square-integrable function defined on S^2 can be expanded as:

$$f(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m Y_n^m(\hat{\mathbf{r}}) \ , \quad \hat{\mathbf{r}} \in S^2 \ , \tag{1.339}$$

where

$$C_n^m = \int_{S^2} f(\hat{\mathbf{r}}) Y_n^{m*}(\hat{\mathbf{r}}) dS(\hat{\mathbf{r}}) \ . \tag{1.340}$$

The spherical harmonics as well as their orthogonality and the completeness, can also be derived from the Laplace-Beltrami operator defined in (1.303). Specifically, by substituting the homogeneous solution (or solid spherical harmonic of degree n and order m) $r^n Y_n^m(\theta, \phi)$ that we found in the previous section in (1.302), we obtain:

$$\begin{aligned} &\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial\left(r^n f\right)}{\partial r}\right] + \frac{1}{r^2}\mathbb{B}(r^n f) = 0 \Leftrightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(n r^{n+1} f\right) + r^{n-2}\mathbb{B}(f) = 0 \\ &\Leftrightarrow r^{n-2} n(n+1) f + r^{n-2}\mathbb{B}(f) = 0 \Leftrightarrow r^{n-2}\left[n(n+1)f + \mathbb{B}(f)\right] = 0 \ , \end{aligned} \tag{1.341}$$

which means that $\Delta(r^n Y_n^m) = 0$ iff $\mathbb{B}(Y_n^m) = -n(n+1)Y_n^m$ which means that the spherical harmonic $Y_n^m(\theta, \phi)$ is eigenfunction of the Beltrami operator for the eigenvalue $-n(n+1)$. The Laplace-Beltrami operator is self-adjoint with respect to the inner product (1.331) which gives the orthogonality of the spherical harmonics. The completeness follows from the fact that the inverse operator of the Laplace-Beltrami is compact. For further study of the spherical harmonics and their properties, we refer to [21] and [11].

Chapter 2

Scattering Theory

The scattering theory studies the interaction of a propagating wave (also called an incident wave) with an obstacle (also called scatterer). The scatterer disturbs the propagating wave and the result of this disturbance is the scattered wave. The scattering problems are separated in the two main categories of direct scattering problems and inverse scattering problems. In direct scattering problems we know the incident wave (or incident field) and physical or geometrical properties of the scatterer and we try to find the scattered wave (or scattered field). In contrast to the direct scattering problems, there are a lot of types of problems under the category of inverse scattering problems. In the present work, we study inverse scattering problems in which we know the incident field and the scattered field and we try to specify the physical and geometrical properties of the scatterer. In this chapter we formulate the direct scattering problems for acoustic and electromagnetic waves and we use the low-frequency theory in order to find approximations for the solutions of these problems. Moreover, the low-frequency theory allows us to introduce the ellipsoidal harmonics into the scattering problems, in order to obtain the corresponding solutions for the case of ellipsoidal scatterers.

A scattering problem can be characterized as a low-frequency problem when $k\alpha \ll 1$, where k the wave number and α the characteristic radius of the scatterer. Lord Rayleigh in 1897 [30] made the assumption that when the wavelength is way bigger that the size of the scatterer, the scattering problem can be dealt as a sequence of perturbations of the corresponding static problem with perturbation parameter the wave number k and when $k = 0$ the scattering problem is reduced to a potential theory problem. This assumption was studied strictly in a series of papers by Stevenson [31] and by Kleinmann [23]. These studies are based on the fact that in the low-frequency area the wave field is an analytic function of the wave number k and therefore can be expressed as a Taylor expansion in a neighborhood of $k = 0$. These series are also known as Rayleigh series. The radius of convergence of these series is called Rayleigh area and inside that area the Rayleigh series converge absolutely and uniformly. The specification of the Rayleigh area has be done for very few scatterers since it is a difficult problem. In the present work we will mainly focus on the low-frequency approximations of the total field and the far-field pattern for each one of the scattering problems.

This theory is applied in the acoustic, electromagnetic , elastic and thermoelastic waves, where for the last two categories of the elastic and thermoelastic waves, the problem becomes more complicated due to the existence of more than one wave numbers. Moreover, the low-frequency theory can also be applied in a chiral enviroment, where the electromagnetic waves are decomposed into right-handed and left-handed wave fields which also propagate with different wave numbers which are not connected linearly. For further study in the application of the low-frequency theory in chiral environment, we refer to [33]. Finally, the application of low-frequency theory into the scattering problems can be extended for the cases of multi-layered ellipsoidal scatterers [3] and [4].

2.1 Acoustic Scattering Theory

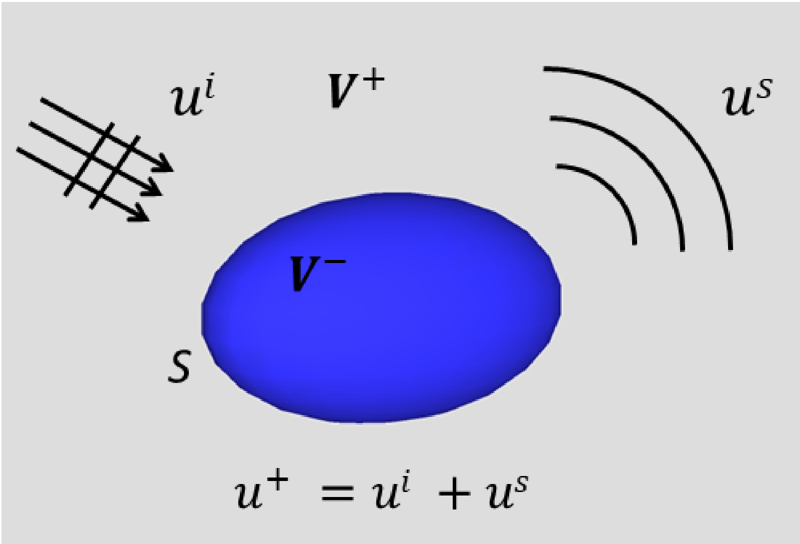


Figure 2.1: Scattering of acoustic waves.

In the present work, it is assumed that an acoustic wave in an irrotational, homogeneous, isotropic and compressible fluid medium is characterized by a scalar function $u(\mathbf{r}, t)$ which is called the *excess acoustic pressure field*. If we denote with $\mathbf{V}(\mathbf{r}, t)$ the velocity of the acoustic wave, then the two basic relations which combine u and $\mathbf{V}$, are the following:

$$\frac{\partial}{\partial t} u(\mathbf{r}, t) = -\frac{1}{\gamma} \nabla \cdot \mathbf{V} \ , \tag{2.1}$$

$$\varrho \frac{\partial}{\partial t} \mathbf{V}(\mathbf{r}, t) = -\nabla u(\mathbf{r}, t) + \delta \nabla \nabla \cdot \mathbf{V}(\mathbf{r}, t) \ , \tag{2.2}$$

where γ denotes the mean compressibility , ϱ denotes the mean mass density and δ denotes the compressional viscocity of the medium which represents losses. Thus, if $\delta = 0$ the medium is considered lossless and if $\delta > 0$, the medium is considered lossy. From relations (2.1) and (2.2), the following equation is derived for u:

$$\frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) = \frac{1}{\gamma \varrho} \Delta u(\mathbf{r}, t) + \frac{\delta}{\varrho} \Delta \left(\frac{\partial}{\partial t} u(\mathbf{r}, t) \right) \ , \tag{2.3}$$

which governs the sound wave propagation and it can easily be observed that for lossless medium of propagation ($\delta = 0$) it is reduced to the classical wave equation:

$$\frac{\partial^2}{\partial t^2} u(\mathbf{r}, t) = c^2 \Delta u(\mathbf{r}, t) \ , \tag{2.4}$$

where $c = \dfrac{1}{\sqrt{\gamma \varrho}}$.

Time Harmonic Waves:
One of the most important case of time varying acoustic waves is the time harmonic (sinusoidal or cosinusoidal) time variation where the excitation of the source varies sinusoidally in time with a single frequency. Specifically, two ways of introducing time harmonic fields are:

- Fourier transform:

$$u(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\mathbf{r}, \omega) e^{-i\omega t d\omega} \ , \tag{2.5}$$

  where $\omega = 2\pi f$ the angular frequency.

- Real-value convention:

$$u(\mathbf{r},t) = Re\left\{u(\mathbf{r})e^{-i\omega t}\right\} \ . \tag{2.6}$$

Hence, by substituting $u(\mathbf{r},t) = u(\mathbf{r})e^{-i\omega t}$ in (2.4) we have:

$$u(\mathbf{r})(-i\omega)^2 e^{-i\omega t} = c^2 \sum_{i=1}^{3} \frac{\partial u(\mathbf{r})}{\partial x_i^2} e^{-i\omega t} \ , \tag{2.7}$$

where $\mathbf{r} = (x_1, x_2, x_3)$. Therefore, by dividing with $e^{-i\omega t}$ we have:

$$-\omega^2 u(\mathbf{r})e^{-i\omega t} = c^2 \sum_{i=1}^{3} \frac{\partial u(\mathbf{r})}{\partial x_i^2} \Leftrightarrow -\omega^2 u(\mathbf{r}) = c^2 \Delta u(\mathbf{r}) \ , \tag{2.8}$$

or equivalently:

$$\Delta u(\mathbf{r}) + k^2 u(\mathbf{r}) = 0 \ , \tag{2.9}$$

with $k = \dfrac{\omega}{c} = \omega\sqrt{\gamma\varrho}$. Equation (2.9) is known as *Helmholtz's equation* (Hermann Ludwig Ferdinand von Helmholtz 1821– 1894). The constant k is the *wave number* and its positive, but in general it can be a complex number with $Imk \geqslant 0$. The constant ϱ is the density defined previously and γ is the compressibility. The fundamental solution of the Helmholtz's equation is:

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{ik\,|\mathbf{r}-\mathbf{r}'|} = \frac{ik}{4\pi}\Phi(\mathbf{r},\mathbf{r}') \ , \tag{2.10}$$

which means that:

$$\left(\Delta_{\mathbf{r}} + k^2\right) G\left(\mathbf{r},\mathbf{r}'\right) = -\frac{4\pi}{ik}\delta\left(\mathbf{r}-\mathbf{r}'\right) \ . \tag{2.11}$$

and it satisfies (2.9) in $\mathbb{R} \setminus \{\mathbf{r}'\}$.

2.1.1 Basic Acoustic Scattering Problems

In order to study the scattering phenomenon it is important to impose some boundary conditions on the surface of the scatterer as well as a radiation condition at infinity. The boundary conditions prescribe the pressure of the acoustic wave on the surface of the scatterer depending on its physical characteristics while the radiation condition specifies the appropriate geometric attenuation of the scattered field and imposes its outgoing character. The radiation condition also provides a necessary condition for the well-posedness of the scattering problem (exterior boundary problem) and specifically the uniqueness of the problem. In what follows we assume that the scatterer is a nonempty bounded open set $D \equiv V^-$, not necessarily simply connected, with boundary $\partial D \equiv S$ sufficiently smooth as to allow the applicability of the Gauss-Green theorems. There are two main categories of scattererers, the penetrable and the impenetrable. When the scatterer is impenetrable, the acoustic field exists only in $\mathbb{R}^3 \setminus \overline{D} \equiv V^+$. When the scatterer is penetrable the incident acoustic wave enters the scatterer which is considered to be another homogeneous and isotropic fluid characterized by different parameters ϱ and γ than those that characterize $\mathbb{R}^3 \setminus \overline{D} \equiv V^+$.

In general, an incident wave or incident field is denoted by u^i and can be either a plane wave or a spherical wave. A plane incident wave is defined as:

$$u^i(\mathbf{r}) = e^{ik\hat{\mathbf{d}}\cdot\mathbf{r}} \ , \tag{2.12}$$

where $\hat{\mathbf{d}}$ the direction of propagation and a spherical incident wave is defined as:

$$u^i_{\mathbf{r_0}} = \frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|} \ , \tag{2.13}$$

where $\mathbf{r_0}$ the source point. In the present work we will mainly focus on the case of plane wave incidence.

The scatterer disturbs the propagation of the incident wave u^i, producing the scattered wave u^s which is always a spherical wave. Hence, the total field u^+ in V^+ is the superposition of the incident wave and the scattered wave:

$$u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \cup S \ . \tag{2.14}$$

The scattered field has to satisfy a radiation condition as $r \to \infty$, where $r = |\mathbf{r}|$, in order to ensure the uniqueness of the solutions of the scattering problems. Due to Sommerfeld (1912), the radiation condition we use is:

$$\lim_{r\to\infty} r \left(\frac{\partial}{\partial r} u^s(\mathbf{r}) - iku^s(\mathbf{r}) \right) = 0 \ , \quad \hat{\mathbf{r}} \in S^2 \ , \tag{2.15}$$

where S^2 the surface of the unit ball in $\mathbb{R}^3$ and the convergence is taken to be uniform over all directions $\hat{\mathbf{r}}$.

For the boundary conditions of the scattering problems, in the present work, we study the following four basic cases:

- *Dirichlet boundary condition:*
  Describes an *acoustically soft scatterer* which offers no resistance to pressure, meaning that yields in such a way as to maintain zero pressure on its boundary. Hence, $u^s = -u^i$ on S which leads to the Dirichlet boundary condition:

$$u^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ . \tag{2.16}$$

- *Neumann boundary condition:*
  Describes an *acoustically hard scatterer* which admits no local displacements and therefore the normal component of the velocity field should vanish. This leads to the Neumann boundary condition:

$$\frac{\partial}{\partial n} u^+(\mathbf{r}) = \hat{\mathbf{n}} \cdot \nabla u^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ . \tag{2.17}$$

- *Robin boundary condition:*
  Describes a *resistive scatterer* which has finite impedance and an intermediate behavior between the acoustically soft and the acoustically hard scatterer. This leads to the Robin boundary condition:

$$\frac{\partial}{\partial n} u^+(\mathbf{r}) + i\frac{\omega \varrho^+}{Z^+} u^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ , \tag{2.18}$$

  where Z^+ is the acoustic impedance measured in units of pressure per unit of velocity. The above three cases are under the category of impenetrable scatterers.

- *Transmission conditions:*
  This case belongs to the category of penetrable scatterers. Hence, the incident wave is transmitted into V^- which results to a total field u^-. The total field u^- satisfies the Helmholtz equation in V^- for the corresponding parameters ϱ^- and γ^-. The two fluids meet at the boundary S which lead to the transmission conditions:

$$\begin{aligned} u^+ &= u^- \ , \qquad \mathbf{r} \in S \ , \\ \frac{\partial u^+}{\partial n} &= \beta \frac{\partial u^-}{\partial n} \ , \quad \mathbf{r} \in S \ , \end{aligned} \tag{2.19}$$

where $\beta = \dfrac{\varrho^+}{\varrho^-}(1 - i\omega\gamma^-\delta^-)$.

Combining all the above we have the following four scattering problems with plane wave incidence of the form (2.12):

1. **Acoustically soft scatterer:**
   Find the solution total field $u^+ \in C^2(V^+) \cap C(V^+ \cup S)$ which satisfies the following boundary value problem:

$$\begin{aligned}
&\Delta u^+(\mathbf{r}) + k^2 u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in V^+ \ , \\
&u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in S \ , \\
&u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}) \ , && \mathbf{r} \in V^+ \cup S \ , \\
&\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}u^s(\mathbf{r}) - iku^s(\mathbf{r})\right) = 0 \ , && \text{uniformly for all directions of } \hat{\mathbf{r}} \in S^2.
\end{aligned} \tag{2.20}$$

2. **Acoustically hard scatterer:**
   Find the solution total field $u^+ \in C^2(V^+) \cap C^1(V^+ \cup S)$ which satisfies the following boundary value problem:

$$\begin{aligned}
&\Delta u^+(\mathbf{r}) + k^2 u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in V^+ \ , \\
&\frac{\partial}{\partial n}u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in S \ , \\
&u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}) \ , && \mathbf{r} \in V^+ \cup S \ , \\
&\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}u^s(\mathbf{r}) - iku^s(\mathbf{r})\right) = 0 \ , && \text{uniformly for all directions of } \hat{\mathbf{r}} \in S^2.
\end{aligned} \tag{2.21}$$

3. **Acoustically resistive scatterer:**
   Find the solution total field $u^+ \in C^2(V^+) \cap C^1(V^+ \cup S)$ which satisfies the following boundary value problem:

$$\begin{aligned}
&\Delta u^+(\mathbf{r}) + k^2 u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in V^+ \ , \\
&\frac{\partial}{\partial n}u^+(\mathbf{r}) + i\frac{\omega\varrho^+}{Z^+}u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in S \ , \\
&u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}) \ , && \mathbf{r} \in V^+ \cup S \ , \\
&\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}u^s(\mathbf{r}) - iku^s(\mathbf{r})\right) = 0 \ , && \text{uniformly for all directions of } \hat{\mathbf{r}} \in S^2.
\end{aligned} \tag{2.22}$$

4. **Penetrable scatterer:**
   Find the total exterior field $u^+ \in C^2(V^+) \cap C^1(V^+ \cup S)$ and the total interior field $u^- \in C^2(V^-) \cap C^1(\overline{V^-})$ which satisfy the following boundary value problem:

$$\begin{aligned}
&\Delta u^\pm(\mathbf{r}) + (k^\pm)^2 u^+(\mathbf{r}) = 0 \ , && \mathbf{r} \in V^\pm \ , \\
&u^+(\mathbf{r}) = u^-(\mathbf{r}) \ , && \mathbf{r} \in S \\
&\frac{\partial}{\partial n}u^+(\mathbf{r}) = \beta\frac{\partial}{\partial n}u^-(\mathbf{r}) \ , && \mathbf{r} \in S \ , \\
&u^+(\mathbf{r}) = u^i(\mathbf{r}) + u^s(\mathbf{r}) \ , && \mathbf{r} \in V^+ \cup S \ , \\
&\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}u^s(\mathbf{r}) - iku^s(\mathbf{r})\right) = 0 \ , && \text{uniformly for all directions of } \hat{\mathbf{r}} \in S^2,
\end{aligned} \tag{2.23}$$

   where k^- is the wave number in the inner region V^-.

Integral Representations

The integral representation of the scattered field u^s is given by:

$$\alpha(\mathbf{r})u^s(\mathbf{r}) = \frac{ik}{4\pi}\int_S \left[u^s(\mathbf{r}')\frac{\partial}{\partial n'}G^+(\mathbf{r},\mathbf{r}') - G^+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}u^s(\mathbf{r}')\right]\, ds(\mathbf{r}') \ , \tag{2.24}$$

where $\partial n' = \partial n(\mathbf{r}')$ and

$$\alpha(\mathbf{r}) = \begin{cases} 0 \ , & \mathbf{r} \in V^- \\ \dfrac{1}{2} \ , & \mathbf{r} \in S \\ 1 \ , & \mathbf{r} \in V^+ \end{cases} \ . \tag{2.25}$$

The proof follows by applying Green theorem for the functions $u^s(\mathbf{r}')$ and $G^+(\mathbf{r},\mathbf{r}')$ in the region $(V^+) \setminus B(\mathbf{r},\varepsilon)$ where $B(\mathbf{r},\varepsilon)$ is a ball with center the point $\mathbf{r}$ and radius ε and by letting $\varepsilon \to 0$ and using the radiation condition.
Also, the integral representation of the incident field is given by:

$$\left(\alpha(\mathbf{r}) - 1\right) u^i(\mathbf{r}) = \frac{ik}{4\pi}\int_S \left[u^i(\mathbf{r}')\frac{\partial}{\partial n'}G^+(\mathbf{r},\mathbf{r}') - G^+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}u^i(\mathbf{r}')\right]\, dS(\mathbf{r}') \ . \tag{2.26}$$

Similarly, the integral representation of the total interior field u^- is given by:

$$\left(\alpha(\mathbf{r}) - 1\right) u^-(\mathbf{r}) = \frac{ik^-}{4\pi}\int_S \left[u^-(\mathbf{r}')\frac{\partial}{\partial n'}G^-(\mathbf{r},\mathbf{r}') - G^-(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}u^-(\mathbf{r}')\right]\, dS(\mathbf{r}') \ . \tag{2.27}$$

Substituting $u^s = u^+ - u^i$ in (2.24) and using (2.26) we obtain:

$$\alpha(\mathbf{r})u^s(\mathbf{r}) = \frac{ik}{4\pi}\int_S \left[u^+(\mathbf{r}')\frac{\partial}{\partial n'}G^+(\mathbf{r},\mathbf{r}') - G^+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}u^+(\mathbf{r}')\right]\, dS(\mathbf{r}') \ . \tag{2.28}$$

due to the fact that a plane incident field is a solution of the Helmholtz equation in $\mathbb{R}^3$. Therefore, the integral representation for the total field is given via (2.28) by:

$$\alpha(\mathbf{r})u^+(\mathbf{r}) = u^i + \frac{ik}{4\pi}\int_S \left[u^+(\mathbf{r}')\frac{\partial}{\partial n'}G^+(\mathbf{r},\mathbf{r}') - G^+(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n'}u^+(\mathbf{r}')\right]\, dS(\mathbf{r}') \ . \tag{2.29}$$

Depending on the conditions on the boundary S the above integral representation can be simplified.

Far-Field

Using the asymptotic relations:

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \mathcal{O}\left(\frac{1}{r}\right) \ , \\ \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} &= \hat{\mathbf{r}} + \mathcal{O}\left(\frac{1}{r}\right) \ , \end{aligned} \tag{2.30}$$

we obtain

$$\begin{aligned} G^+(\mathbf{r},\mathbf{r}') &= e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \\ \nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}') &= -ik\hat{\mathbf{r}}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \\ \nabla_{\mathbf{r}}G^+(\mathbf{r},\mathbf{r}') &= ik\hat{\mathbf{r}}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \end{aligned} \tag{2.31}$$

Substituting the asymptotic forms of G and $\nabla_{\mathbf{r}'}G$ from (2.31) in the integral representation of the scattered field (2.28) we obtain:

$$u^s(\mathbf{r}) = u^\infty(\hat{\mathbf{r}})h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ , \ \hat{\mathbf{r}} \in S^2 \ , \tag{2.32}$$

where the function $u^\infty(\hat{\mathbf{r}})$ is called the *far-field pattern* or *scattering amplitude* of the scattered field and is given by:

$$u^\infty(\hat{\mathbf{r}}) = -\frac{ik}{4\pi}\int_S \left[\frac{\partial}{\partial n'}u^+(\mathbf{r}') + ik\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}'\right)u^+(\mathbf{r}')\right]e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\ dS(\hat{\mathbf{r}'}) \ . \tag{2.33}$$

2.1.2 Low-Frequency Theory in Acoustics

Basic Low-Frequency Theory in Acoustics

Starting with the low-frequency expansions of the acoustic fields, we have:

- *Plane incident wave:*
  The incident field u^i because of its exponential form (2.12) can be written as power series of the wave number k in a neighborhood of $k = 0$:

$$u^i(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(\hat{\mathbf{d}}\cdot\mathbf{r})^n = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^i(\mathbf{r}) \ , \tag{2.34}$$

  where u_n^i are the low-frequency coefficients of the incident field independent of the wave number k. This series converges for all points $\mathbf{r}$.

- *Total exterior field:*
  The total exterior field u^+ in V^+, is an analytic function in a neighborhood of $k = 0$ which was established by Kleinman [23]. Hence, it can be expressed as power series of the wave number k of the form:

$$u^+(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^+(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ , \tag{2.35}$$

  where u_n^+ are the low-frequency coefficients of the exterior total field independent of the wave number k.

- *Total interior field:*
  The total interior field u^- in V^-, similarly to the exterior, is an analytic function of the wave number k in a neighborhood of $k = 0$. Hence, it can be written as the following power series:

$$u^-(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^-(\mathbf{r}) \ , \quad \mathbf{r}\in V^- \ , \tag{2.36}$$

  where u_n^- are the low-frequency coefficients independent of the wave number k.

- *Scattered field:*
  The scattered field u^s, can also be written as power series of the form:

$$u^s(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^s(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ , \tag{2.37}$$

  where u_n^s are the low-frequency coefficients independent of k.

  Remarks:

- We note that with k we denote the wave number at the outer region of the scatterer, meaning that $k^+ \equiv k$.
- For the case of the total interior field the wave number should be k^- but in order to have uniformity for convenience in calculations, the relation $k^- = \eta k$ is used, where $\eta = \sqrt{\dfrac{\gamma^- \varrho^-}{\gamma^+ \varrho^+}}$ the relative index of refraction. The absorption of η from the low-frequency coefficients leads to the expansion in terms of k.

Substituting the low-frequency expansion of the total field (2.35) in the Helmholtz equation (2.9), we have:

$$\Delta \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^+(\mathbf{r}) + k^2 \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^+(\mathbf{r}) - (ik)^2 \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^+(\mathbf{r}) = 0$$
$$\Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^+(\mathbf{r}) - \sum_{n=0}^{\infty} \frac{(ik)^{n+2}}{n!} u_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^+(\mathbf{r}) - \sum_{n=2}^{\infty} \frac{(ik)^n}{(n-2)!} u_{n-2}^+(\mathbf{r}) = 0 \ . \tag{2.38}$$

Therefore by equating to 0 the coefficients of $(ik)^n$, we have:

$$\Delta u_n^+(\mathbf{r}) = n(n-1) u_{n-2}^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \ n = 0, 1, \dots \ . \tag{2.39}$$

For the transmission problems, the low-frequency coefficients of the interior total field u^- will satisfy the Helmholtz equation in the interior V^- for wave number $k^- = \eta k$.
Substituting the low-frequency expansion of the total interior field u^- into the Helmholtz equation gives:

$$\Delta \sum_{n=0}^{\infty} \frac{(ik^-)^n}{n!} u_n^-(\mathbf{r}) + (k^-)^2 \sum_{n=0}^{\infty} \frac{(ik^-)^n}{n!} u_n^-(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^-(\mathbf{r}) - \eta^2 (ik)^2 \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^-(\mathbf{r}) = 0$$
$$\Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^-(\mathbf{r}) - \eta^2 \sum_{n=0}^{\infty} \frac{(ik)^{n+2}}{n!} u_n^-(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \Delta u_n^-(\mathbf{r}) - \eta^2 \sum_{n=2}^{\infty} \frac{(ik)^n}{(n-2)!} u_{n-2}^-(\mathbf{r}) = 0 \ , \tag{2.40}$$

and by equating the coefficients of $(ik)^n$ we obtain:

$$\Delta u_n^-(\mathbf{r}) = n(n-1) \eta^2 u_{n-2}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ , \ n = 0, 1, \dots \ . \tag{2.41}$$

Remarks:

i. Relation (2.39) will also be satisfied by the low-frequency coefficients of the incident field u^i and the scattered field u^s since they are also solutions of the Helmholtz equation.

ii. Thus, the Helmholtz equation is reduced to a sequence of equations for the low-frequency coefficients. Specifically, it can be observed that for $n = 0$ and $n = 1$ the low-frequency coefficients satisfy the Laplace equation and for $n \geqslant 2$ they satisfy a Poisson equation with the non-homogeneous part being known from the previous terms of the sequence of the low-frequency coefficients.

For the boundary conditions we have the following relations:

- *Dirchlet boundary condition:*
  Substituting the low-frequency expansion of the total field in the Dirichlet boundary condition $u^+ = 0$ and equating the coefficients to 0, we obtain:

$$\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^+(\mathbf{r}) = 0 \Leftrightarrow u_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ , \ n = 0, 1, \dots \ . \tag{2.42}$$

- *Neumann boundary condition:*
  Substituting the low-frequency expansion of the total field in the Neumann boundary condition $\frac{\partial u^+}{\partial n} = 0$, we have:

$$\frac{\partial}{\partial n}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\frac{\partial}{\partial n}u_n^+(\mathbf{r}) = 0 \Leftrightarrow \frac{\partial}{\partial n}u_n^+(\mathbf{r}) = 0\ , \quad \mathbf{r}\in S\ , \tag{2.43}$$

  for $n = 0, 1, \ldots$ and $\partial n = \partial n(\mathbf{r})$.

- *Robin boundary condition:*
  Again by substituting the low-frequency expansion of the total field to the Robin boundary condition $\frac{\partial}{\partial n}u^+ + ikRu^+ = 0$ with $R = \frac{1}{Z^+}\sqrt{\frac{\varrho^+}{\gamma^+}}$, we obtain:

$$\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\frac{\partial}{\partial n}u_n^+(\mathbf{r}) + ikR\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\frac{\partial}{\partial n}u_n^+(\mathbf{r}) + R\sum_{n=0}^{\infty}\frac{(ik)^{n+1}}{n!}u_n^+(\mathbf{r}) = 0$$
$$\Leftrightarrow \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\frac{\partial}{\partial n}u_n^+(\mathbf{r}) + R\sum_{n=1}^{\infty}\frac{(ik)^n}{(n-1)!}u_{n-1}^+(\mathbf{r}) = 0\ . \tag{2.44}$$

  Therefore, by equating the coefficients of $(ik)^n$ we obtain the corresponding boundary condition for the low-frequency coefficients:

$$\frac{\partial}{\partial n}u_n^+(\mathbf{r}) + nRu_{n-1}^+(\mathbf{r}) = 0\ , \quad \mathbf{r}\in S\ ,\ n = 1, 2, \ldots\ . \tag{2.45}$$

- *Transmission conditions:*
  Similarly with the previous boundary conditions, we have:

$$\begin{array}{l} u_n^+(\mathbf{r}) = u_n^-(\mathbf{r}) \\ \frac{\partial}{\partial n}u_n^+(\mathbf{r}) = \beta\frac{\partial}{\partial n}u_n^-(\mathbf{r}) \end{array}\ , \quad \mathbf{r}\in S\ . \tag{2.46}$$

Next, for the asymptotic behavior of the low-frequency coefficients the expansion of the fundamental solution is needed. Because of its exponential form, the fundamental solution of the Helmholtz equation in V^+ can be written as:

$$ikG^+(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}|\mathbf{r}-\mathbf{r}'|^n = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}|\mathbf{r}-\mathbf{r}'|^{n-1}\ , \tag{2.47}$$

where the superscript + indicates the wave number $k \equiv k^+$.
For the transmission problems the fundamental solution of the Helmholtz equation in V^- has the expansion:

$$ik^-G^-(\mathbf{r},\mathbf{r}') = \frac{e^{ik^-|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\eta^n|\mathbf{r}-\mathbf{r}'|^n = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\eta^n|\mathbf{r}-\mathbf{r}'|^{n-1}\ , \tag{2.48}$$

where $k^- = \eta k$. Thus, by substituting the low-frequency expansions of the total fields as well as the expansion of the fundamental solution in V^+ (2.47) into the integral representation (2.29) and rearrange the terms via the Cauchy formula:

$$\sum_{n=0}^{\infty}\frac{a_n}{n!}\sum_{n=0}^{\infty}\frac{b_n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}a_k b_{n-m}\ , \tag{2.49}$$

leads to the following form:

$$\alpha(\mathbf{r})\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^+(\mathbf{r})=\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^i(\mathbf{r}) + \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\int_S\left[u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1}-|\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial}{\partial n'}u_{n-m}^+(\mathbf{r}')\right]dS(\mathbf{r}')\ , \tag{2.50}$$

where $\partial n' = \partial n(\mathbf{r}')$. For reasons that will be clear later, by isolating the terms u_n^+ (taking out the case $m = 0$ from the double sum) we have:

$$\alpha(\mathbf{r})\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^+(\mathbf{r})=\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^i(\mathbf{r})+\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\left\{\int_S\left[u_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}-|\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^+(\mathbf{r}')\right]dS(\mathbf{r}') + \sum_{m=1}^{n}\binom{n}{m}\int_S\left[u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1}-|\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial}{\partial n'}u_{n-m}^+(\mathbf{r}')\right]dS(\mathbf{r}')\right\}\ . \tag{2.51}$$

Equating the coefficients of $(ik)^n$ gives:

$$\alpha(\mathbf{r})u_n^+(\mathbf{r}) = f_n^+(\mathbf{r})+\frac{1}{4\pi}\int_S\left[u_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}-|\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^+(\mathbf{r}')\right]dS(\mathbf{r}')\ , \tag{2.52}$$

where

$$f_n^+(\mathbf{r}) = u_n^i(\mathbf{r})+\frac{1}{4\pi}\sum_{m=1}^{n}\binom{n}{m}\int_S\left[u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1}-|\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial}{\partial n'}u_{n-m}^+(\mathbf{r}')\right]dS(\mathbf{r}')\ , \quad n \geqslant 1. \tag{2.53}$$

and $f_0^+ = u_0^i$. It can be observed that f_n^+ is dependent of $u_0^+, u_1^+, ..., u_{n-1}^+$ and independent of u_n^+. Hence, from (2.52) we obtain the decomposition:

$$u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + v_n^+(\mathbf{r})\ , \quad \mathbf{r}\in V^+\ , \tag{2.54}$$

where

$$v_n^+(\mathbf{r}) = \frac{1}{4\pi}\int_S\left[u_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}-|\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^+(\mathbf{r}')\right]dS(\mathbf{r}')\ . \tag{2.55}$$

The function v_n^+ is the combination of a single layer potential and a double layer potential which are solutions of the Laplace equation. Hence, v_n^+ is also a solution of Laplace equation in V^+ with the asymptotic form:

$$v_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right)\ , \quad r\to\infty\ . \tag{2.56}$$

From the asymptotic behavior of v_n^+ it is concluded that the asymptotic form of the low-frequency coefficients of the exterior total field is:

$$u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right)\ , \quad r\to\infty\ , \tag{2.57}$$

and since $f_0^+ = u_0^i$, we have:

$$u_0^+(\mathbf{r}) = 1 + \mathcal{O}\left(\frac{1}{r}\right)\ , \quad r\to\infty\ , \tag{2.58}$$

for plane wave incidence. The non-vanishing part f_n^+ of the asymptotic form of u_n^+, is proven to be a particular solution of (2.39). Specifically, since v_n^+ is regular at infinity it has the integral representation:

$$\alpha(\mathbf{r})v_n^+(\mathbf{r}) = \frac{1}{4\pi}\int_S\left[v_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}-|\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}v_n^+(\mathbf{r}')\right]dS(\mathbf{r}')\ . \tag{2.59}$$

Hence, returning to relation (2.52) and subtracting by parts with (2.59), we have:

$$\alpha(\mathbf{r})f_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + \frac{1}{4\pi}\int_S \left[(u_n^+ - v_n^+)\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1} - |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{n'}(u_n^+ - v_n^+)\right] dS(\mathbf{r}'). \tag{2.60}$$

Therefore, from (2.54), we obtain the integral representation of f_n^+:

$$(\alpha(\mathbf{r})-1)f_n^+(\mathbf{r}) = \frac{1}{4\pi}\int_S \left[f_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1} - |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}f_n^+(\mathbf{r}')\right] dS(\mathbf{r}')\ , \tag{2.61}$$

for $\mathbf{r} \in \mathbb{R}^3$. This leads to the fact that f_n^+ satisfies equation (2.39) as well as the equation:

$$\Delta f_n^+(\mathbf{r}) = 0\ , \quad \mathbf{r} \in V^-\ . \tag{2.62}$$

Based on the above, it can be observed that for every $n = 0, 1, 2, ...$ in the sequence of problems satisfied by the low-frequency coefficients, f_n^+ is dependable on the solutions $u_1^+, ..., u_{n-1}^+$ which means that for every step n it is a known function from the previous steps $0, 1, 2, ..., n-1$. Thus, u_n^+ depends only on the potential function v_n^+.

Finally, for the low-frequency expansion of the scattering amplitude (far-field pattern), by substituting the low-frequency expansion of the exterior total field u^+ and relation:

$$e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(-\hat{\mathbf{r}}\cdot\mathbf{r}')^n\ , \tag{2.63}$$

in the far field pattern (2.33) as well as using (2.49) to rearrange terms, we have [19] :

$$u^\infty(\hat{\mathbf{r}}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^{m+1}\int_S\left[\frac{\partial}{\partial n'}u_{n-m}^+(\mathbf{r}') + ik(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_{n-m}^+(\mathbf{r}')\right](\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\ , \tag{2.64}$$

which can also be written as $u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}})$ to show the dependence on the direction of incidence $\hat{\mathbf{d}}$. Keeping Dassios's notation and separating the even terms (real part) from the odd terms (imaginary part), we have:

$$Re\left\{u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}})\right\} = \sum_{n=1}^{\infty}k^{2n}A_{2n}(\hat{\mathbf{r}}; \hat{\mathbf{d}})\ , \tag{2.65}$$

$$Im\left\{u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}})\right\} = \sum_{n=0}^{\infty}k^{2n+1}A_{2n+1}(\hat{\mathbf{r}}; \hat{\mathbf{d}})\ . \tag{2.66}$$

In what follows we refer to the low-frequency coefficients of the total field and scattered field as "*near-field data*" and to the low-frequency expansion of the scattering amplitude and its coefficients as "*far-field data*". Based on relations (2.39),(2.36), (2.42)-(2.46) and (2.54)-(2.57), to every scattering problem corresponds a sequence of potential theory problems for the low-frequency coefficients.

Acoustically soft scatterer

Based on (2.39) and (2.42) we have the following sequence of exterior boundary value problems for the low-frequency coefficients:

$$\begin{aligned} &\Delta u_n^+(\mathbf{r}) = n(n-1)u_{n-2}^+(\mathbf{r})\ , \quad \mathbf{r} \in V^+\ , \\ &u_n^+(\mathbf{r}) = 0\ , \quad \mathbf{r} \in S\ , \\ &u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right)\ , \quad r \to \infty\ . \end{aligned} \tag{2.67}$$

The specification of the low-frequency coefficients u_n^+, as mentioned in the previous section, can be helped via the decomposition (2.54). Specifically, it is sufficient enough to calculate only v_n^+, where the potential v_n^+ will satisfy the following boundary value problem:

$$\begin{aligned} &\Delta v_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\ &v_n^+(\mathbf{r}) = -f_n^+ \ , \quad \mathbf{r} \in S \ , \\ &v_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.68}$$

For the f_n^+, by replacing the Dirichlet boundary condition into (2.53), we obtain:

$$f_n^+(\mathbf{r}) = u_n^i(\mathbf{r}) - \frac{1}{4\pi}\sum_{m=1}^{n}\binom{n}{m}\int_S |\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial}{\partial n'}u_{n-m}^+(\mathbf{r}')dS(\mathbf{r}') \ , \quad n \geqslant 1 \ . \tag{2.69}$$

Therefore, the first three terms are:

$$f_0^+(\mathbf{r}) = u_0^i \ , \tag{2.70}$$

$$f_1^+(\mathbf{r}) = u_1^i(\mathbf{r}) - \frac{1}{4\pi}\int_S \frac{\partial}{\partial n'}u_0^+(\mathbf{r}')dS(\mathbf{r}') \ , \tag{2.71}$$

$$f_2^+(\mathbf{r}) = u_2^i(\mathbf{r}) - \frac{1}{2\pi}\int_S \frac{\partial}{\partial n'}u_1^+(\mathbf{r}')dS(\mathbf{r}') - \frac{1}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|\frac{\partial}{\partial n'}u_0^+(\mathbf{r}')dS(\mathbf{r}') \ . \tag{2.72}$$

For u_0^i and u_1^i it can be proven via Divergence theorem in V^+ and their asymptotic forms that:

$$\int_S \frac{\partial}{\partial n'}u_0^i(\mathbf{r}')dS(\mathbf{r}') = \int_S \frac{\partial}{\partial n'}u_1^i(\mathbf{r}')dS(\mathbf{r}') = 0 \ . \tag{2.73}$$

Hence, from relations (2.54), (2.73) and $f_0^+ = u_0^i$, we have:

$$\int_S \frac{\partial}{\partial n'}u_0^+(\mathbf{r}')dS(\mathbf{r}') = \int_S \frac{\partial}{\partial n'}f_0^+(\mathbf{r}')dS(\mathbf{r}') + \int_S \frac{\partial}{\partial n'}v_0^+(\mathbf{r}')dS(\mathbf{r}') = \int_S \frac{\partial}{\partial n'}v_0^+(\mathbf{r}')dS(\mathbf{r}') \ . \tag{2.74}$$

Before we proceed, it is necessary to introduce a standard potential function which is known as the *conductor potential* $\phi^c(\mathbf{r}')$ [32]. This function is defined as the solution of the following potential problem:

$$\begin{aligned} &\Delta \phi^c(\mathbf{r}) = 0 \quad \mathbf{r} \in V^+ \ , \\ &\phi^c(\mathbf{r}) = 1 \ , \quad \mathbf{r} \in S \ , \\ &\phi^c(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.75}$$

The capacity C of S in terms of ϕ^c is defined as:

$$C = -\frac{1}{4\pi}\int_S \frac{\partial}{\partial n}\phi^c(\mathbf{r})dS(\mathbf{r}) \ . \tag{2.76}$$

Having introduced the conductor potential ϕ^c, by returning to relation (2.74) and using the boundary condition of (2.75), the following relations are derived which connect u_0^+ to the capacity C:

$$\int_S \frac{\partial}{\partial n'}v_0^+(\mathbf{r}')dS(\mathbf{r}') = \int_S \phi^c(\mathbf{r}')\frac{\partial}{\partial n'}v_0^+(\mathbf{r}')dS(\mathbf{r}') \ , \tag{2.77}$$

which by using Green's second identity, relation (2.73) and the fact that on the boundary S we have $v_0^+ = -f_0^+ = -u_0^i$, it is concluded that:

$$\int_S \phi^c(\mathbf{r}')\frac{\partial}{\partial n'}v_0^+(\mathbf{r}')dS(\mathbf{r}') = \int_S v_0^+(\mathbf{r}')\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') = -\int_S u_0^i(\mathbf{r}')\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') \ . \tag{2.78}$$

For plane wave incidence, since $u_0^i = 1$, from relation (2.75) follows that:

$$\int_S \frac{\partial}{\partial n'} u_0^+(\mathbf{r}')dS(\mathbf{r}') = -\int_S \frac{\partial}{\partial n'} \phi^c(\mathbf{r}')dS(\mathbf{r}') = 4\pi C \ . \tag{2.79}$$

Finally, since ϕ^c is the solution of (2.75), it can be observed that $-\phi^c$ will be a solution of (2.68) for $n = 0$ since $f_0^+ = 1$. Thus, due to uniqueness of the solution, we have:

$$v_0^+(\mathbf{r}) = -\phi^c(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ . \tag{2.80}$$

Based on relation (2.54), the zeroth low-frequency coefficient of the exterior total field is given by:

$$u_0^+(\mathbf{r}) = 1 - \phi^c \ , \tag{2.81}$$

where ϕ^c the conductor potential defined as the solution of (2.75).

For the specification of u_1^+ another basic potential function is needed and specifically a vector valued potential $\mathbf{\Phi}(\mathbf{r}) = (\phi_1, \phi_2, \phi_3)$ known as *polarization potential*, which is the solution of the following potential theory problem:

$$\begin{aligned} &\Delta\mathbf{\Phi}(\mathbf{r}) = \mathbf{0} \ , \quad \mathbf{r} \in V^+ \ , \\ &\mathbf{\Phi} = \mathbf{r} + \mathbf{c} \ , \quad \mathbf{r} \in S \ , \\ &\mathbf{\Phi} = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ , \end{aligned} \tag{2.82}$$

where $\mathbf{c} = (c_1, c_2, c_3)$ is a constant vector chosen so that the following relation is satisfied:

$$\int_S \frac{\partial}{\partial n} \mathbf{\Phi}(\mathbf{r})dS(\mathbf{r}) = \mathbf{0} \ . \tag{2.83}$$

From Gauss Theorem, we have:

$$\begin{aligned} \mathbf{0} &= \int_S \phi^c(\mathbf{r}) \frac{\partial}{\partial n} \mathbf{\Phi}(\mathbf{r})dS(\mathbf{r}) = \int_S \mathbf{\Phi}(\mathbf{r}) \frac{\partial}{\partial n} \phi^c(\mathbf{r})dS(\mathbf{r}) \\ &= \int_S (\mathbf{r} + \mathbf{c}) \frac{\partial}{\partial n} \phi^c(\mathbf{r})dS(\mathbf{r}) = \int_S \mathbf{r} \frac{\partial}{\partial n} \phi^c(\mathbf{r})dS(\mathbf{r}) - 4\pi C\mathbf{c} \ , \end{aligned} \tag{2.84}$$

which leads to the constant vector $\mathbf{c}$ in terms of the conductor potential ϕ^c:

$$\mathbf{c} = \frac{1}{4\pi C} \int_S \mathbf{r} \frac{\partial}{\partial n} \phi^c(\mathbf{r})dS(\mathbf{r}) \ . \tag{2.85}$$

For the first coefficient, from relations (2.71) and (2.79), we have:

$$f_1^+(\mathbf{r}) = u_1^i(\mathbf{r}) - \frac{1}{4\pi} \int_S \frac{\partial}{\partial n'} u_0^+(\mathbf{r}')dS(\mathbf{r}') = (\hat{\mathbf{d}} \cdot \mathbf{r}) - C \ . \tag{2.86}$$

This means that the potential v_1^+ takes the value $C - (\mathbf{r} \cdot \hat{\mathbf{d}})$ on the boundary (since $v_1^+ = -f_1^+$ on S). Moreover, since $\phi^c = 1$ and $\mathbf{\Phi} = \mathbf{r} + \mathbf{c}$ on S, we have the following relation:

$$\phi^c(\mathbf{r}) \left(C + \hat{\mathbf{d}} \cdot \mathbf{c}\right) - \hat{\mathbf{d}} \cdot \mathbf{\Phi}(\mathbf{r}) = C - \hat{\mathbf{d}} \cdot \mathbf{r} \ , \tag{2.87}$$

Hence, since ϕ^c is the solution of (2.75) and $\mathbf{\Phi}$ is the solution of (2.82), then based on the above relation we have that $\phi^c(\mathbf{r}) \left(C + \hat{\mathbf{d}} \cdot \mathbf{c}\right) - \hat{\mathbf{d}} \cdot \mathbf{\Phi}(\mathbf{r})$ is the solution of (2.68) for $n = 1$, which due to uniqueness, gives:

$$v_1^+(\mathbf{r}) = \phi^c(\mathbf{r}) \left(C + \hat{\mathbf{d}} \cdot \mathbf{c}\right) - \hat{\mathbf{d}} \cdot \mathbf{\Phi}(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ . \tag{2.88}$$

Therefore, based on the above and the decomposition (2.54), the first order low-frequency approximation of the exterior total field is given by:

$$u_1^+(\mathbf{r}) = C(\phi^c(\mathbf{r}) - 1) + \hat{\mathbf{d}} \cdot (\mathbf{r} + \mathbf{c}\phi^c(\mathbf{r}) - \mathbf{\Phi}(\mathbf{r})) \tag{2.89}$$

Far-field data:
For the scattering amplitude by substituting the Dirichlet boundary condition in (2.64) we obtain:

$$u^\infty(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(ik)^{n+1}}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+1} \int_S (\hat{\mathbf{r}} \cdot \mathbf{r}')^m \frac{\partial}{\partial n'} u_{n-m}^+(\mathbf{r}') dS(\mathbf{r}') \ . \tag{2.90}$$

It can be easily observed that the scattering amplitude is of the form:

$$u^\infty(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = ikA_1(\hat{\mathbf{r}}, \hat{\mathbf{d}}) + k^2 A_2(\hat{\mathbf{r}}, \hat{\mathbf{d}}) + ik^3 A_3(\hat{\mathbf{r}}, \hat{\mathbf{d}}) + \mathcal{O}(k^4) \ . \tag{2.91}$$

Similarly to the near-field data, the main idea is to express $A_1,\ A_2,\ A_3$ in terms of $\phi^c,\ \mathbf{\Phi}$ and C.

Thus, for $n = m = 0$ in (2.90), the coefficient of ik is:

$$A_1(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = -\frac{1}{4\pi} \int_S \frac{\partial}{\partial n'} u_0^+(\mathbf{r}') dS(\mathbf{r}') = -C \ , \tag{2.92}$$

where C is the capacity defined in (2.76).
For $n = 1$ in (2.90), the coefficient of k^2 is:

$$A_2(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = \frac{1}{4\pi} \int_S \frac{\partial}{\partial n'} u_1^+(\mathbf{r}') dS(\mathbf{r}') - \frac{1}{4\pi} \int_S (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} u_0^+(\mathbf{r}') dS(\mathbf{r}') \ , \tag{2.93}$$

where using the calculated u_0^+ and u_1^+ from (2.81) and (2.89) as well as the definitions of the capacity (2.76) and of the constant vector $\mathbf{c}$ (2.85), leads to the form:

$$A_2(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = -C^2 - C\hat{\mathbf{d}} \cdot \mathbf{c} + \frac{1}{4\pi} \int_S (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') = -C^2 + C(\hat{\mathbf{r}} - \hat{\mathbf{d}}) \cdot \mathbf{c} \ . \tag{2.94}$$

For $n = 2$ in (2.90), the coefficient of ik^3 is:

$$A_3(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = \frac{1}{8\pi} \int_S \left[\frac{\partial}{\partial n'} u_2^+(\mathbf{r}') - 2(\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} u_1^+(\mathbf{r}') + (\hat{\mathbf{r}} \cdot \mathbf{r}')^2 \frac{\partial}{\partial n'} u_0^+(\mathbf{r}') \right] dS(\mathbf{r}') \ . \tag{2.95}$$

For the expression of A_3 in terms of the capacity C, the constant vector $\mathbf{c}$ and a potential function, it is necessary to introduce two basic quantities. Specifically, the following integral in terms of the potential $\mathbf{\Phi}$ defined in (2.82) (Schiffer and Szegö 1949):

$$\widetilde{\mathbf{Q}} = -\int_S \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Phi}(\mathbf{r}) dS(\mathbf{r}) \ , \tag{2.96}$$

is known as the *polarization tensor* and the tensor given by:

$$\widetilde{\mathbf{P}} = \widetilde{\mathbf{Q}} + \left|V^-\right| \widetilde{\mathbf{I}} \ , \tag{2.97}$$

where $|V^-|$ is the volume of domain V^-, is known as the *electric polarizability tensor* (Keller 1972). With the two necessary quantities $\widetilde{\mathbf{Q}}, \widetilde{\mathbf{P}}$ defined, by splitting (2.95) into three integrals I_1, I_2, I_3 the calculations can be more convenient. Starting with the first integral I_1, we have:

$$I_1 = \frac{1}{8\pi} \int_S \frac{\partial}{\partial n'} u_2^+(\mathbf{r}) dS(\mathbf{r}') = \frac{1}{8\pi} \int_S \frac{\partial}{\partial n'} f_2^+(\mathbf{r}) dS(\mathbf{r}') + \frac{1}{8\pi} \int_S \frac{\partial}{\partial n'} v_2^+(\mathbf{r}) dS(\mathbf{r}') \ . \tag{2.98}$$

For the f_2^+, assuming plane wave incidence and based on (2.72) as well as the capacity definition (2.76) and the definition of $\mathbf{c}$ (2.85), we get:

$$f_2^+(\mathbf{r}) = (\hat{\mathbf{d}} \cdot \mathbf{r})^2 + \frac{1}{4\pi} \int_S |\mathbf{r} - \mathbf{r}'| \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') + 2C^2 + 2C\hat{\mathbf{d}} \cdot \mathbf{c} \ . \tag{2.99}$$

From Gauss theorem on the integral of f_2^+, we obtain:

$$\begin{aligned} &\int_S \frac{\partial}{\partial n'} \left[(\hat{\mathbf{d}} \cdot \mathbf{r}')^2 + \frac{1}{4\pi} \int_S |\mathbf{r} - \mathbf{r}'| \frac{\partial}{\partial n''} \phi^c(\mathbf{r}'') \right] dS(\mathbf{r}') = \int_D \Delta_{\mathbf{r}'} \left[(\hat{\mathbf{d}} \cdot \mathbf{r}')^2 + \frac{1}{4\pi} \int_S |\mathbf{r} - \mathbf{r}'| \frac{\partial}{\partial n''} \phi^c(\mathbf{r}'') \right] dS(\mathbf{r}'') \\ &= \int_D \left[2 + \frac{1}{4\pi} \int_S \frac{2}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial n''} \phi^c(\mathbf{r}'') dS(\mathbf{r}'') \right] dV(\mathbf{r}') \end{aligned} \tag{2.100}$$

From Green's identity and the fact that ϕ^c and $|\mathbf{r} - \mathbf{r}'|^{-1}$ are solutions of Laplace equation as well as the boundary condition of $\phi^c = 1$ on S, we get:

$$\begin{aligned} \int_S \frac{\partial}{\partial n'} f_2^+(\mathbf{r}') dS(\mathbf{r}') &= 2 \int_D \left[1 + \frac{1}{4\pi} \int_S \phi^c(\mathbf{r}'') \frac{\partial}{\partial n''} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}'') \right] dV(\mathbf{r}') \\ &= 2 \int_D \left[1 + \frac{1}{4\pi} \int_S \frac{\partial}{\partial n''} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}'') \right] dV(\mathbf{r}') \ , \end{aligned} \tag{2.101}$$

which from Gauss theorem vanishes, leaving:

$$I_1 = \frac{1}{8\pi} \int_S \frac{\partial}{\partial n'} v_2^+(\mathbf{r}') dS(\mathbf{r}') \ . \tag{2.102}$$

Again using the boundary conditions $\phi^c = 1, v_2^+ = -f_2^+$ on S and Green's theorem in V^+ we have:

$$\begin{aligned} I_1 &= \frac{1}{8\pi} \int_S \phi^c \frac{\partial}{\partial n'} v_2^+(\mathbf{r}') dS(\mathbf{r}') = \frac{1}{8\pi} \int_S v_2^+(\mathbf{r}') \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') \\ &= -\frac{1}{8\pi} \int_S f_2^+(\mathbf{r}') \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') \ . \end{aligned} \tag{2.103}$$

Substituting the form of f_2^+ in terms of conductor potential and capacity (2.99) and relation (2.79), we have:

$$\begin{aligned} I_1 &= -\frac{1}{8\pi} \int_S \left[(\hat{\mathbf{d}} \cdot \hat{\mathbf{r}}')^2 + \frac{1}{4\pi} \int_S |\mathbf{r}' - \mathbf{r}''| \frac{\partial}{\partial n''} \phi^c(\mathbf{r}'') dS(\mathbf{r}'') + 2C^2 + 2C\hat{\mathbf{d}} \cdot \mathbf{c} \right] \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') \\ &= -\frac{1}{8\pi} \int_S (\hat{\mathbf{d}} \cdot \hat{\mathbf{r}}')^2 \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') + \frac{1}{32\pi^2} \int_S \int_S |\mathbf{r}' - \mathbf{r}''| \frac{\partial}{\partial n''} \phi^c(\mathbf{r}'') dS(\mathbf{r}'') \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') dS(\mathbf{r}') \\ &\quad + C^3 + C^2 \hat{\mathbf{d}} \cdot \mathbf{c} \ , \end{aligned} \tag{2.104}$$

which is an expression only in terms of the conductor potential ϕ^c, the capacity C and the constant vector $\mathbf{c}$ defined in (2.85).

For the second integral I_2 by replacing the low-frequency coefficient u_1^+ with its formula (2.89) which is in terms of the capacity C and the potential functions $\phi^c, \mathbf{\Phi}$, leads to:

$$\begin{aligned} I_2 &= -\frac{1}{4\pi} \int_S (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} u_1^+(\mathbf{r}') dS(\mathbf{r}') = -\frac{1}{4\pi} \int_S (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} \left[(C + \hat{\mathbf{d}} \cdot \mathbf{c}) \phi^c(\mathbf{r}') + \hat{\mathbf{d}} \cdot \mathbf{r}' - \hat{\mathbf{d}} \cdot \mathbf{\Phi}(\mathbf{r}') \right] dS(\mathbf{r}') \\ &= -\frac{1}{4\pi} \int_S \left[\left(C + \hat{\mathbf{d}} \cdot \mathbf{c} \right) (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} \phi^c(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \frac{\partial}{\partial n'} (\hat{\mathbf{d}} \cdot \mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{r}') \hat{\mathbf{d}} \cdot \frac{\partial}{\partial n'} \mathbf{\Phi}(\mathbf{r}') \right] dS(\mathbf{r}') \end{aligned} \tag{2.105}$$

In order to proceed the following relations are needed:

$$\frac{\partial}{\partial n'}(\hat{\mathbf{d}}\cdot\mathbf{r}') = \hat{\mathbf{n}}'\cdot\hat{\mathbf{d}} , \quad \mathbf{r}'\in S \tag{2.106}$$

$$\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial}{\partial n'}(\hat{\mathbf{d}}\cdot\mathbf{r}')dS(\mathbf{r}') = |V^-|(\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}) = |V^-|\widetilde{\mathbf{I}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} , \tag{2.107}$$

$$\frac{1}{4\pi}\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') = \frac{1}{4\pi}4\pi C\mathbf{c}\cdot\hat{\mathbf{r}} = C(\mathbf{c}\cdot\hat{\mathbf{r}}) , \tag{2.108}$$

$$\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')\left(\hat{\mathbf{d}}\cdot\frac{\partial}{\partial n'}\boldsymbol{\Phi}(\mathbf{r}')\right)dS(\mathbf{r}') = -\widetilde{\mathbf{Q}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} . \tag{2.109}$$

where the two last relations are derived directly from the definitions of $\mathbf{c}$ (2.85) and electric polarization tensor $\widetilde{\mathbf{Q}}$ respectively as well as the double dot product. Substituting these relations in (2.105) leads to:

$$I_2 = -\left(C + \hat{\mathbf{d}}\cdot\mathbf{c}\right)C(\mathbf{c}\cdot\hat{\mathbf{r}}) - \widetilde{\mathbf{Q}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} - |V^-|\widetilde{\mathbf{I}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} , \tag{2.110}$$

which based on the definition of polarizability tensor can be rewritten as:

$$I_2 = -\left(C + \hat{\mathbf{d}}\cdot\mathbf{c}\right)C(\mathbf{c}\cdot\hat{\mathbf{r}}) - \frac{1}{4\pi}\widetilde{\mathbf{P}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} , \tag{2.111}$$

where $\widetilde{\mathbf{P}} : \hat{\mathbf{d}}\otimes\mathbf{r} = \hat{\mathbf{d}}\cdot\widetilde{\mathbf{P}}\cdot\hat{\mathbf{r}}$.

For the third integral I_3 we have:

$$\begin{aligned} I_3 = \frac{1}{8\pi}\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')^2\frac{\partial}{\partial n'}u_0^+(\mathbf{r}')dS(\mathbf{r}') &= \frac{1}{8\pi}\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')^2\frac{\partial}{\partial n'}(1-\phi^c(\mathbf{r}'))dS(\mathbf{r}') \\ &= -\frac{1}{8\pi}\int_S (\hat{\mathbf{r}}\cdot\mathbf{r}')^2\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') . \end{aligned} \tag{2.112}$$

Having the expressions of the three integrals in terms of the capacity C, the vector constant $\mathbf{c}$, the conductor potential ϕ^c and the polarizability tensor $\widetilde{\mathbf{P}}$, by substituting into (2.95), we obtain:

$$\begin{aligned} A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = &-\frac{1}{8\pi}\int_S \left[(\hat{\mathbf{d}}\cdot\hat{\mathbf{r}}')^2 + \left(\hat{\mathbf{r}}\cdot\mathbf{r}'\right)^2\right]\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') \\ &- \frac{1}{32\pi^2}\int_S\int_S |\mathbf{r}'-\mathbf{r}''|\frac{\partial}{\partial n''}\phi^c(\mathbf{r}'')dS(\mathbf{r}'')\frac{\partial}{\partial n'}\phi^c(\mathbf{r}')dS(\mathbf{r}') \\ &+ C^3 + C^2(\hat{\mathbf{d}}-\hat{\mathbf{r}})\cdot\mathbf{c} - C(\hat{\mathbf{r}}\cdot\mathbf{c})(\hat{\mathbf{d}}\cdot\mathbf{c}) - \frac{1}{4\pi}\widetilde{\mathbf{P}} : \hat{\mathbf{d}}\otimes\hat{\mathbf{r}} . \end{aligned} \tag{2.113}$$

Thus, we have the first three coefficients of the scattering amplitude $u^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}})$ in terms of $C, \mathbf{c}, \phi^c, \widetilde{\mathbf{P}}$.

Acoustically hard scatterer

Based on (2.39) and (2.43) we have the following sequence of exterior boundary value problems for the low-frequency coefficients:

$$\begin{aligned} &\Delta u_n^+(\mathbf{r}) = n(n-1)u_{n-2}^+(\mathbf{r}) , && \mathbf{r}\in V^+ , \\ &\frac{\partial}{\partial n}u_n^+(\mathbf{r}) = 0 , && \mathbf{r}\in S , \\ &u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) , && r\to\infty . \end{aligned} \tag{2.114}$$

Working similarly to the case of the acoustically soft scatterer, by using the decomposition (2.54), the low-frequency coefficients of the exterior total field u_n^+ are specified for every n from the terms v_n^+ of the decomposition. By replacing the Neumann boundary condition into (2.55) and (2.59) we have:

$$f_n^+(\mathbf{r}) = u_n^i(\mathbf{r}) + \frac{1}{4\pi}\sum_{m=2}^{n}\binom{n}{m}\int_S u_{n-m}^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1}dS(\mathbf{r}') \ , \quad n \geqslant 2 \ , \ \mathbf{r} \in \mathbb{R}^3 \ , \tag{2.115}$$

$$v_n^+(\mathbf{r}) = \frac{1}{4\pi}\int_S v_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}dS(\mathbf{r}') \ , \quad \mathbf{r} \in V^+ \ , \tag{2.116}$$

where $f_0^+ = u_0^i$ and $f_1^+ = u_1^i$.

The function v_n^+ is a double layer potential and solution of Laplace equation which decays as $\frac{1}{r^2}$ as $r \to \infty$. Thus, the functions v_n^+ for $n = 0, 1, ...$ are specified as the unique solutions of the following sequence of potential theory problems:

$$\begin{aligned} &\Delta v_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\ &\frac{\partial}{\partial n} v_n^+(\mathbf{r}) = -\frac{\partial}{\partial n} f_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \\ &v_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.117}$$

By Green's theorem in V^+ we obtain:

$$v_n^+(\mathbf{r}) = \frac{1}{4\pi}\int_S v_n^+(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1}dS(\mathbf{r}') = \frac{1}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}v_n^+(\mathbf{r}')dS(\mathbf{r}') \ . \tag{2.118}$$

Thus, for $n = 0$ and plane wave incidence we have $\frac{\partial}{\partial n}v_0^+(\mathbf{r}) = -\frac{\partial}{\partial n}u_0^i(\mathbf{r}) = 0$ on S. This means that we have an exterior Neumann boundary valued problem which is satisfied from any constant function. From the asymptotic forms it is derived $v_0^+ = 0$. Hence, the zeroth low-frequency coefficient of the exterior total field is:

$$u_0^+(\mathbf{r}) = 1 \ , \quad \mathbf{r} \in V^+ \cup S \ . \tag{2.119}$$

For the first low-frequency coefficient u_1^+, a vector valued potential is needed like $\mathbf{\Phi}$ but for the Neumann boundary condition. Hence, this vector valued potential known as *virtual mass potential*, is defined as the unique solution of the potential theory problem:

$$\begin{aligned} &\Delta \mathbf{\Psi}(\mathbf{r}) = \mathbf{0} \ , \quad \mathbf{r} \in V^+ \ , \\ &\frac{\partial}{\partial n}\mathbf{\Psi}(\mathbf{r}) = \hat{\mathbf{n}} \ , \quad \mathbf{r} \in S \ , \\ &\mathbf{\Psi}(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ , \end{aligned} \tag{2.120}$$

and is used to define the quantity:

$$\widetilde{\mathbf{W}} = -\int_S \hat{\mathbf{n}}\mathbf{\Psi}(\mathbf{r})dS(\mathbf{r}) \ , \tag{2.121}$$

which is known as the *virtual mass tensor* (Taylor 1928, Schiffer and Szegö 1949). From this tensor another quantity is defined from the formula:

$$\widetilde{\mathbf{M}} = \widetilde{\mathbf{W}} + |V-|\widetilde{\mathbf{I}} \ , \tag{2.122}$$

which is known as the *magnetic polarizability tensor* (Keller 1972).

Having defined the necessary quantities, by using the boundary condition of (2.117), we have that:

$$\frac{\partial}{\partial n}v_1^+(\mathbf{r}) = -\frac{\partial}{\partial n}f_1^+(\mathbf{r}) - \frac{\partial}{\partial n}(\hat{\mathbf{d}}\cdot\mathbf{r}) = -\hat{\mathbf{n}}\cdot\hat{\mathbf{d}}\ , \quad \in S\ . \tag{2.123}$$

It can be seen that since $\mathbf{\Psi}$ is the solution of (2.120), hence $-\hat{\mathbf{d}}\cdot\mathbf{\Psi}$ is a solution of (2.117) for $n=1$, which due to uniqueness leads to the fact that:

$$v_1^+(\mathbf{r}) = -\hat{\mathbf{d}}\cdot\mathbf{\Psi}(\mathbf{r})\ , \quad \mathbf{r}\in V^+\ , \tag{2.124}$$

Substituting v_1^+ and f_1^+ in (2.54), gives the first coefficient of the low-frequency expansion of the total field in terms of the vector valued potential $\mathbf{\Psi}$:

$$u_1^+(\mathbf{r}) = f_1^+(\mathbf{r}) + v_1^+(\mathbf{r}) = \hat{\mathbf{d}}\cdot\mathbf{r} - \hat{\mathbf{d}}\cdot\mathbf{\Psi}(\mathbf{r})\ , \quad \mathbf{r}\in V^+\ . \tag{2.125}$$

Far-field data:
For the acoustically hard scatterer, by substituting into (2.64) the Neumann boundary relation, the scattering amplitude becomes:

$$u^\infty(\hat{\mathbf{r}}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+2}}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^{m+1}\int_S(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{r}}\cdot\mathbf{r}')^m u_{n-m}^+(\mathbf{r}')dS(\mathbf{r}')\ . \tag{2.126}$$

Since for $n=0$ we start with the coefficient of k^2 which is the A_2, we assume that the coefficient of ik is $A_1=0$.

For $n=0=m$ in (2.126) and the decomposition (2.54), we have:

$$\begin{aligned} A_2(\hat{\mathbf{r}};\hat{\mathbf{d}}) &= -\frac{1}{4\pi}(ik)^2\int_S(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_0^+(\mathbf{r}')dS(\mathbf{r}') \\ &= \frac{1}{4\pi}k^2\int_S(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_0^i(\mathbf{r}')dS(\mathbf{r}') + \frac{1}{4\pi}k^2\int_S(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')v_0^+(\mathbf{r}')dS(\mathbf{r}')\ . \end{aligned} \tag{2.127}$$

For plane wave incidence it is clear from divergence theorem that the first integral vanishes since $u_0^i=1$. For the second integral it was shown above that $v_0^+=0$. Thus, both integrals vanish in (2.127), which leads to $A_2=0$. Therefore, since $A_1=A_2=0$, the scattering amplitude for the acoustically hard scatterer has the following form:

$$u^\infty(\hat{\mathbf{r}}) = ik^3A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) + \mathcal{O}(k^4)\ . \tag{2.128}$$

What's left is to calculate the leading order coefficient A_3 in terms of the quantities $\mathbf{\Psi}$, $\widetilde{\mathbf{W}}$, $\widetilde{\mathbf{M}}$. For $n=1$ in (2.126), we have:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}})u_1^+(\mathbf{r}')dS(\mathbf{r}') - \frac{1}{4\pi}\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\mathbf{r}')u_0^+(\mathbf{r}')dS(\mathbf{r}')\ . \tag{2.129}$$

Substituting (2.119) and (2.125) into (2.129) leads to:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}})\left[(\hat{\mathbf{d}}\cdot\mathbf{r}') - \hat{\mathbf{d}}\cdot\mathbf{\Psi}(\mathbf{r}')\right]dS(\mathbf{r}') - \frac{1}{4\pi}\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\mathbf{r}')dS(\mathbf{r}')\ . \tag{2.130}$$

In order to simplify this formula, the following relations are needed:

$$\int_S(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}})(\hat{\mathbf{d}}\cdot\mathbf{r}')dS(\mathbf{r}') = \int_S\frac{\partial}{\partial n'}(\mathbf{r}'\cdot\hat{\mathbf{r}})(\hat{\mathbf{d}}\cdot\mathbf{r}')dS(\mathbf{r}') = \int_S(\mathbf{r}'\cdot\hat{\mathbf{r}})\frac{\partial}{\partial n'}(\hat{\mathbf{d}}\cdot\mathbf{r}')dS(\mathbf{r}') = (\hat{\mathbf{r}}\cdot\hat{\mathbf{d}})|V^-|\ , \tag{2.131}$$

$$\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{r}})(\hat{\mathbf{d}}\cdot\mathbf{\Psi}(\mathbf{r}'))dS(\mathbf{r}') = \widetilde{W}:\hat{\mathbf{d}}\otimes\hat{\mathbf{r}}\ , \tag{2.132}$$

where for the second relation the double dot product has been used. From these relations and the definition of virtual mass $\widetilde{\mathbf{M}}$ (2.122), we have:

$$A_3(\hat{\mathbf{r}},\hat{\mathbf{d}}) = \frac{1}{4\pi}\left[\widetilde{\mathbf{M}}:\hat{\mathbf{d}}\otimes\hat{\mathbf{r}} - |V^-|\right]\ , \tag{2.133}$$

which is an expression in terms of the magnetic polarizability tensor.

The Acoustic Impedance

In the present work we won't study the general solution of the acoustic impedance since the corresponding inverse scattering problem for the ellipsoids is not been studied. Nevertheless, the steps followed are similar to the previous cases and the results obtained are ([19], p. 90-93):

$$u_0^+(\mathbf{r}) = f_0^+(\mathbf{r}) + v_0^+(\mathbf{r}) = 1 \ , \quad \mathbf{r} \in V^+ \ , \tag{2.134}$$

$$u_1^+(\mathbf{r}) = \hat{\mathbf{d}} \cdot \mathbf{r} - R\psi^c(\mathbf{r}) - \hat{\mathbf{d}} \cdot \mathbf{\Psi}(\mathbf{r}) \ , \tag{2.135}$$

where $\mathbf{\Psi}$ the virtual mass potential defined in (2.120) and ψ^c is a basic potential function ψ^c defined as the solution of the following boundary value problem:

$$\begin{aligned} &\Delta\psi^c(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\ &\frac{\partial}{\partial n}\psi^c(\mathbf{r}) = 1 \ , \quad \mathbf{r} \in S \ , \\ &\psi^c(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.136}$$

Far-field data:

$$A_1(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = 0 \ , \tag{2.137}$$

$$A_2(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = -\frac{R|S|}{4\pi} \ , \tag{2.138}$$

where $|S|$ is the surface area of the boundary S.

Transmission problem

Based on (2.39),(2.41) and (2.46) we have the following sequence of transmission problems for the low-frequency coefficients:

$$\begin{aligned} &\Delta u_n^+(\mathbf{r}) = n(n-1)u_{n-2}^+(\mathbf{r}) \ , && \mathbf{r} \in V^+ \ , \\ &\Delta u_n^-(\mathbf{r}) = n(n-1)\eta^2 u_{n-2}^-(\mathbf{r}) \ , && \mathbf{r} \in V^- \ , \\ &u_n^+(\mathbf{r}) = u_n^-(\mathbf{r}) \ , && \mathbf{r} \in S \ , \\ &\frac{\partial}{\partial n}u_n^+(\mathbf{r}) = \beta\frac{\partial}{\partial n}u_n^-(\mathbf{r}) \ , && \mathbf{r} \in S \ , \\ &u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \ , && r \to \infty \ . \end{aligned} \tag{2.139}$$

In order to study the transmission problem, a decomposition for the low-frequency coefficients of the interior total field u_n^- is needed, similarly to (2.54). Specifically, by substituting the low-frequency expansion of u^- and the expansion of the fundamental solution of Helmholtz operator in V^- (2.48) into (2.27) and following the same process with the exterior total field (2.47)-(2.54), which is to rearrange the terms of the sums using the Cauchy formula (2.49) and equate the coefficients of $(ik)^n$, leads to the following relation for the low-frequency coefficients u_n^- [19].

$$(\alpha(\mathbf{r}) - 1)u_n^-(\mathbf{r}) = f_n^-(\mathbf{r}) + \frac{1}{4\pi}\int_S \left[u_n^-(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{-1} - |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^-(\mathbf{r}')\right] dS(\mathbf{r}') \ , \tag{2.140}$$

where

$$f_n^-(\mathbf{r}) = \frac{1}{4\pi}\sum_{m=1}^{n}\binom{n}{m}\eta^m \int_S \left[u_{n-m}^-(\mathbf{r}')\frac{\partial}{\partial n'}|\mathbf{r}-\mathbf{r}'|^{m-1} - |\mathbf{r}-\mathbf{r}'|^{m-1}\frac{\partial}{\partial n'}u_{n-m}^-(\mathbf{r}')\right] dS(\mathbf{r}') \ , \tag{2.141}$$

for $n \geqslant 1$ and $f_0^- = 0$. Thus , since for $\mathbf{r} \in V^-$ we have $\alpha(\mathbf{r}) = 0$, it is concluded that that relation (2.140) can be rewritten in the following form:

$$u_n^-(\mathbf{r}) = -f_n^-(\mathbf{r}) + v_n^-(\mathbf{r}) \ , \tag{2.142}$$

where

$$v_n^- = \int_S \left[|\mathbf{r} - \mathbf{r}'|^{-1} \frac{\partial}{\partial n'} u_n^-(\mathbf{r}') - u_n^-(\mathbf{r}') \frac{\partial}{\partial n'} |\mathbf{r} - \mathbf{r}'|^{-1} \right] dS(\mathbf{r}') \ . \tag{2.143}$$

It is clear that v_n^- is a combination of single and double layer potentials. Therefore, it is a solution of Laplace equation in V^- which means that it satisfies an interior boundary problem for the Laplace equation and does not have an asymptotic form since its regular only in V^-. It can be observed that for every step n the term f_n^- of (2.142) is independent of u_n^- which means that the specification of u_n^- depends only on the specification of v_n^-.

For $k \to 0$ the following identity is obtained [19]:

$$(\alpha(\mathbf{r}) - 1)\, v_n^-(\mathbf{r}) = \frac{1}{4\pi} \int_S \left[v_n^-(\mathbf{r}') \frac{\partial}{\partial n'} |\mathbf{r} - \mathbf{r}'|^{-1} - |\mathbf{r} - \mathbf{r}'|^{-1} \frac{\partial}{\partial n'} v_n^-(\mathbf{r}') \right] dS(\mathbf{r}') \ , \tag{2.144}$$

for $\mathbf{r} \in \mathbb{R}^3$. Subtracting this formula from (2.140) and using the decomposition (2.142), leads to the integral representation of f_n^- as follows:

$$\alpha(\mathbf{r}) f_n^-(\mathbf{r}) = \frac{1}{4\pi} \int_S \left[f_n^-(\mathbf{r}') \frac{\partial}{\partial n'} |\mathbf{r} - \mathbf{r}'|^{-1} - |\mathbf{r} - \mathbf{r}'|^{-1} \frac{\partial}{\partial n'} f_n^-(\mathbf{r}') \right] dS(\mathbf{r}') \ . \tag{2.145}$$

Therefore, based on decomposition (2.142) and the integral representation (2.145), f_n^- satisfies the following equations:

$$\Delta f_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \tag{2.146}$$

$$\Delta f_n^-(\mathbf{r}) = -n(n-1)\eta^2 f_{n-2}^- \ , \quad \mathbf{r} \in V^- \ . \tag{2.147}$$

For the lossless transmission problem where the compressional viscocity $\delta^- = 0$ in V^-, we have the following sequence of boundary conditions for the low-frequency coefficients:

$$u_n^+(\mathbf{r}) = u_n^-(\mathbf{r}) \ , \qquad \frac{\partial}{\partial n} u_n^+(\mathbf{r}) = \beta \frac{\partial}{\partial n} u_n^-(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \tag{2.148}$$

where $\beta = \frac{\varrho^+}{\varrho^-}$ and $\beta\eta^2 = \frac{\gamma^-}{\gamma^+}$. Based on the decomposition (2.142) and relations (2.141) and (2.143) as well as the corresponding relations of the exterior field (2.53),(2.59) and (2.54), the specification of $u_n^\pm$ can be obtained by solving the following transmission problem:

$$\begin{aligned}
&\Delta v_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\
&\Delta v_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^- \ , \\
&v_n^+(\mathbf{r}) = v_n^-(\mathbf{r}) - (f_n^-(\mathbf{r}) + f_n^+(\mathbf{r})) \ , \quad \mathbf{r} \in S \ , \\
&\frac{\partial}{\partial n} v_n^+(\mathbf{r}) = \beta \frac{\partial}{\partial n} v_n^-(\mathbf{r}) - \left(\beta \frac{\partial}{\partial n} f_n^-(\mathbf{r}) + \frac{\partial}{\partial n} f_n^+(\mathbf{r}) \right) \ , \quad \mathbf{r} \in S \ , \\
&v_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ .
\end{aligned} \tag{2.149}$$

Hence, for every step n, since $f_n^\pm$ are independent of $u_n^\pm$, the specification of $u_n^\pm$ is reduced to the specification of $v_n^\pm$ which are solutions of (2.149). Nevertheless, for the transmission problems there is an alternate way of calculating $u_n^\pm$. Specifically, for the potential v_n^+, by substituting the transmission conditions (2.148) into the integral representation of v_n^+ (2.55), we get:

$$v_n^+(\mathbf{r}) = -f_n^-(\mathbf{r}) + \frac{1-\beta}{4\pi} \int_S |\mathbf{r} - \mathbf{r}'|^{-1} \frac{\partial}{\partial n'} u_n^-(\mathbf{r}') dS(\mathbf{r}') \ , \quad \mathbf{r} \in V^+ \ . \tag{2.150}$$

Similarly, for $v_n^-(\mathbf{r})$, we have:

$$v_n^-(\mathbf{r}) = f_n^+(\mathbf{r}) + \frac{1-\beta}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^-(\mathbf{r}')dS(\mathbf{r}') \ , \quad \mathbf{r}\in V^- \ . \tag{2.151}$$

Thus, the decompositions (2.54) (2.142) take the following forms:

$$u_n^+(\mathbf{r}) = f_n^+(\mathbf{r}) - f_n^-(\mathbf{r}) + (1-\beta)w_n^+(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ , \tag{2.152}$$

$$u_n^-(\mathbf{r}) = f_n^+(\mathbf{r}) - f_n^-(\mathbf{r}) + (1-\beta)w_n^-(\mathbf{r}) \ , \quad \mathbf{r}\in V^- \ , \tag{2.153}$$

with

$$w_n^\pm(\mathbf{r}) = \frac{1}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|^{-1}\frac{\partial}{\partial n'}u_n^-(\mathbf{r}')dS(\mathbf{r}') \ , \quad \mathbf{r}\in V^\pm \ . \tag{2.154}$$

It can be observed that for $\beta = 1$ the terms with $w_n^\pm$ vanish. This leaves the determination of $u_n^\pm$ to the determination of $f_n^\pm$ which does not require the solutions of potential theory problems.

For $\beta \neq 1$, the specification of the coefficients $u_n^\pm$ depends on the specification of $w_n^\pm$ which is a solution of Laplace equation. Specifically, based on (2.152) and (2.153), $w_n^\pm$ is the solution of the following potential theory problem:

$$\begin{aligned}
&\Delta w_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^+ \ , \\
&\Delta w_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^- \ , \\
&w_n^+(\mathbf{r}) = w_n^-(\mathbf{r}) \ , \quad \mathbf{r}\in S \ , \\
&\frac{\partial}{\partial n}w_n^+(\mathbf{r}) + \frac{\partial}{\partial n}\left(f_n^+(\mathbf{r}) - f_n^-(\mathbf{r})\right) = \beta\frac{\partial}{\partial n}w_n^-(\mathbf{r}) \ , \quad \mathbf{r}\in S \ , \\
&w_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r\to\infty \ .
\end{aligned} \tag{2.155}$$

Thus, the calculation of $u_n^\pm$ is reduced either in the calculation of $v_n^\pm$ via the sequence of potential theory problems (2.149), either in the calculation of $w_n^\pm$ via the sequence of potential theory problems (2.155).

Again the main idea is the expressions of the low-frequency coefficients in terms of basic potential functions. In what follows, it is assumed that we have plane wave incidence.

For $n = 0$, from (2.53) and (2.141), we have:

$$f_0^+(\mathbf{r}) = u_0^i(\mathbf{r}) = 1 \ , \quad \mathbf{r}\in V^+ \ , \tag{2.156}$$

$$f_0^-(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^- \ . \tag{2.157}$$

Therefore, it is concluded that:

$$u_0^+(\mathbf{r}) = 1 \ , \quad \mathbf{r}\in V^+ \ , \tag{2.158}$$

$$u_0^-(\mathbf{r}) = 1 \ , \quad \mathbf{r}\in V^- \ . \tag{2.159}$$

For $n = 1$ in (2.53), we have:

$$f_1^+(\mathbf{r}) = \hat{\mathbf{d}}\cdot\mathbf{r} - \frac{1}{4\pi}\int_S \frac{\partial}{\partial n'}u_0^+(\mathbf{r}')dS(\mathbf{r}') = \hat{\mathbf{d}}\cdot\mathbf{r} - \frac{1}{4\pi}\int_{V^+}\Delta u_0^+(\mathbf{r}')dS(\mathbf{r}') = \hat{\mathbf{d}}\cdot\mathbf{r} = u_1^i(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ . \tag{2.160}$$

For $n = 1$ in (2.141), we obtain:

$$f_1^-(\mathbf{r}) = -\frac{\eta}{4\pi}\int_S \frac{\partial}{\partial n'}u_0^-(\mathbf{r}')dS(\mathbf{r}') = -\frac{\eta}{4\pi}\int_{V^-}\Delta u_0^-(\mathbf{r}')dV(\mathbf{r}') = 0 \ , \quad \mathbf{r}\in V^- \ . \tag{2.161}$$

The above relations lead to the following forms:

$$u_1^+(\mathbf{r}) = \hat{\mathbf{d}}\cdot\mathbf{r} + v_1^+(\mathbf{r}) = \hat{\mathbf{d}}\cdot\mathbf{r} + (1-\beta)w_1^+(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ , \tag{2.162}$$

$$u_1^-(\mathbf{r}) = v_1^-(\mathbf{r}) = \hat{\mathbf{d}}\cdot\mathbf{r} + (1-\beta)w_1^-(\mathbf{r}) \ , \quad \mathbf{r}\in V^- \ , \tag{2.163}$$

where $v_1^\pm$ and $w_1^\pm$ are the solutions of the problems (2.149) and (2.155) for $n = 1$. What's left is to express the coefficients $u_1^\pm$ in terms of basic potential functions. For this reason, it is necessary to introduce the generalized polarization potentials. Specifically, the vector fields $\mathbf{v}^\pm$ associated with the general polarizability tensor, are the solutions of the following transmission problem:

$$\begin{aligned}
&\Delta \mathbf{v}^+(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^+ \ , \\
&\Delta \mathbf{v}^-(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^- \ , \\
&\mathbf{v}^+(\mathbf{r};\beta) = \mathbf{v}^-(\mathbf{r};\beta) + \mathbf{r} \ , \quad \mathbf{r} \in S \ , \\
&\frac{\partial}{\partial n}\mathbf{v}^-(\mathbf{r};\beta) = \beta\frac{\partial}{\partial n}\mathbf{v}^-(\mathbf{r};\beta) + \hat{\mathbf{n}} \ , \quad \mathbf{r} \in S \ , \\
&\mathbf{v}^+(\mathbf{r};\beta) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ .
\end{aligned} \tag{2.164}$$

Based on this vector field, the *general polarizability tensor* is defined as:

$$\widetilde{\mathbf{X}}(\beta) = (1-\beta)\int_S \hat{\mathbf{n}} \otimes \mathbf{v}^-(\mathbf{r};\beta) dS(\mathbf{r}) \ . \tag{2.165}$$

Based on [19] the vector fields $\mathbf{v}^+$ and $\mathbf{v}^-$ are decomposed into the following forms in V^+ and V^- respectively:

$$\mathbf{v}^+(\mathbf{r};\beta) = \mathbf{\Phi}^+(\mathbf{r};\beta) + \mathbf{\Psi}^+(\mathbf{r};\beta) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.166}$$
$$\mathbf{v}^-(\mathbf{r};\beta) = \mathbf{\Phi}^-(\mathbf{r};\beta) + \mathbf{\Psi}^-(\mathbf{r};\beta) \ , \quad \mathbf{r} \in V^- \ , \tag{2.167}$$

where $\mathbf{\Phi}^\pm$ satisfy the following problem:

$$\begin{aligned}
&\Delta \mathbf{\Phi}^+(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^+ \ , \\
&\Delta \mathbf{\Phi}^-(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^- \ , \\
&\mathbf{\Phi}^+(\mathbf{r};\beta) = \mathbf{\Phi}^-(\mathbf{r};\beta) + \mathbf{r} \ , \mathbf{r} \in S \ , \\
&\frac{\partial}{\partial n}\mathbf{\Phi}^+(\mathbf{r};\beta) = \beta\frac{\partial}{\partial n}\mathbf{\Phi}^-(\mathbf{r};\beta) \ , \quad \mathbf{r} \in S \ , \\
&\mathbf{\Phi}^+(\mathbf{r};\beta) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ ,
\end{aligned} \tag{2.168}$$

and $\mathbf{\Psi}^\pm$ satisfy the problem:

$$\begin{aligned}
&\Delta \mathbf{\Psi}^+(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^+ \ , \\
&\Delta \mathbf{\Psi}^-(\mathbf{r};\beta) = \mathbf{0} \ , \quad \mathbf{r} \in V^- \ , \\
&\mathbf{\Psi}^+(\mathbf{r};\beta) = \mathbf{\Psi}^-(\mathbf{r};\beta) \ , \quad \mathbf{r} \in S \ , \\
&\frac{\partial}{\partial n}\mathbf{\Psi}^+(\mathbf{r};\beta) = \beta\frac{\partial}{\partial n}\mathbf{\Psi}^-(\mathbf{r};\beta) + \hat{\mathbf{n}} \ , \quad \mathbf{r} \in S \ , \\
&\mathbf{\Psi}^+(\mathbf{r};\beta) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ .
\end{aligned} \tag{2.169}$$

As it can be observed, the reason for this decomposition is to separate the two non-homogeneous transmission conditions of (2.164) into two different problems of one non-homogeneous transmission condition (2.168) and (2.120).

In order to express the general polarizability tensor $\widetilde{\mathbf{X}}$ in terms of $\mathbf{\Phi}$ and $\mathbf{\Psi}$, the following relations are needed:

$$\int_S \hat{\mathbf{n}} dS(\mathbf{r}) = |V^-|\widetilde{\mathbf{I}} \ , \tag{2.170}$$

$$\int_s \hat{\mathbf{n}} \otimes \mathbf{\Phi}^+(\mathbf{r};\beta) dS(\mathbf{r}) = -\beta \int_S \hat{\mathbf{n}} \otimes \mathbf{\Psi}^+(\mathbf{r};\beta) dS(\mathbf{r}) \ , \tag{2.171}$$

$$\int_s \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Psi}^+(\mathbf{r};\beta) dS(\mathbf{r}) = -\frac{1}{\beta} \int_S \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Phi}^+(\mathbf{r};\beta) dS(\mathbf{r}) \ , \tag{2.172}$$

which lead to the following expression:

$$\widetilde{\mathbf{X}}(\beta) = (1-\beta^2) \int_S \hat{\mathbf{n}} \otimes \mathbf{\Psi}^+(\mathbf{r};\beta) dS(\mathbf{r}) - (1-\beta)|V^-|\widetilde{\mathbf{I}} \ , \tag{2.173}$$

$$\widetilde{\mathbf{X}}(\beta) = \left(\frac{1-\beta}{\beta}\right)^2 \int_S \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Phi}^+(\mathbf{r};\beta) dS(\mathbf{r}) - \frac{1-\beta}{\beta}|V^-|\widetilde{\mathbf{I}} \ . \tag{2.174}$$

Having defined the necessary potentials, the next step is to express $u_1^\pm$ in terms of these potentials. Starting with $v_1^\pm$, it was shown that $f_1^+ = \hat{\mathbf{d}} \cdot \mathbf{r}$ and $f_1^- = 0$ on S, which leads to the transmission conditions of (2.149) to take the form:

$$v_1^+(\mathbf{r}) = v_1^-(\mathbf{r}) - \hat{\mathbf{d}} \cdot \mathbf{r} \ , \qquad \frac{\partial}{\partial n} v_1^+(\mathbf{r}) = \beta \frac{\partial}{\partial n} v_1^-(\mathbf{r}) - \hat{\mathbf{d}} \cdot \hat{\mathbf{n}} \ , \quad \mathbf{r} \in S. \tag{2.175}$$

Hence, since $\mathbf{v}^\pm$ are the solutions of (2.164), this means that $-\hat{\mathbf{d}} \cdot \mathbf{v}^\pm$ are solutions of (2.149) for $n = 1$, which due to uniqueness leads to the conclusion that:

$$v_1^\pm(\mathbf{r}) = -\hat{\mathbf{d}} \cdot \mathbf{v}^\pm(\mathbf{r};\beta) \ , \quad \mathbf{r} \in V^\pm \ . \tag{2.176}$$

Similarly for the $w_1^\pm$, for $n = 1$ in (2.155), we have the following transmission conditions:

$$w_1^+(\mathbf{r}) = w_1^-(\mathbf{r}) \ , \qquad \frac{\partial}{\partial n} w_1^+(\mathbf{r}) = \beta \frac{\partial}{\partial n} w_1^-(\mathbf{r}) - \hat{\mathbf{n}} \cdot \hat{\mathbf{d}} \ , \quad \mathbf{r} \in S \ . \tag{2.177}$$

Thus, since $\mathbf{\Psi}^\pm$ is the solution of (2.169), then $-\hat{\mathbf{d}} \cdot \mathbf{\Psi}^\pm(\mathbf{r};\beta)$ is the solution of (2.155) for $n = 1$, which due to uniqueness of the solution, leads to:

$$w_1^\pm(\mathbf{r}) = -\hat{\mathbf{d}} \cdot \mathbf{\Psi}^\pm(\mathbf{r};\beta) \ , \quad \mathbf{r} \in V^\pm \ . \tag{2.178}$$

Therefore, by substituting $v_1^\pm$ or $w_1^\pm$ into (2.162) and (2.163) respectively, we obtain the first low-frequency coefficients of the exterior and the interior total fields in terms of the generalized potentials:

$$u_1^\pm(\mathbf{r}) = \hat{\mathbf{d}} \cdot \mathbf{r} - (1-\beta)\hat{\mathbf{d}} \cdot \mathbf{\Psi}^\pm(\mathbf{r};\beta) \ , \quad \mathbf{r} \in V^\pm \ , \tag{2.179}$$

or

$$u_1^+(\mathbf{r}) = \hat{\mathbf{d}} \cdot \mathbf{r} - \hat{\mathbf{d}} \cdot \mathbf{v}^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.180}$$

$$u_1^-(\mathbf{r}) = -\hat{\mathbf{d}} \cdot \mathbf{v}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ . \tag{2.181}$$

Far-field data:
The scattering amplitude (2.64), by substituting the transmission conditions (2.46), takes the following form [19]:

$$u^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \sum_{n=0}^{\infty} \left[k^{2n+2} A_{2n+2}(\hat{\mathbf{r}};\hat{\mathbf{d}}) + i k^{2n+3} A_{2n+3}(\hat{\mathbf{r}};\hat{\mathbf{d}}) \right] \ , \tag{2.182}$$

where

$$A_{2n+2}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(-1)^n}{4\pi(2n)!}\left\{2n\beta\eta^2\int_{V^-} u^-_{2n-1}(\mathbf{r}')dV(\mathbf{r}')\right.$$
$$\left.+\sum_{m=0}^{2n}\binom{2n}{m}(-1)^{m+1}\int_S\left[\frac{(\hat{\mathbf{r}}\cdot\mathbf{r}')^{m+1}}{m+1}\frac{\partial}{\partial n'}u^+_{2n-m}(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{r}}\cdot\mathbf{r}')^m u^+_{2n-m}(\mathbf{r}')\right]dS(\mathbf{r}')\right\}\ , \tag{2.183}$$

and

$$A_{2n+3}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(-1)^n}{4\pi(2n+1)!}\left\{(2n+1)\beta\eta^2\int_{V^-} u^-_{2n}(\mathbf{r}')dV(\mathbf{r}')\right.$$
$$\left.+\sum_{m=0}^{2n+1}\binom{2n+1}{m}(-1)^{m+1}\int_S\left[\frac{(\hat{\mathbf{r}}\cdot\mathbf{r}')^{m+1}}{m+1}\frac{\partial}{\partial n'}u^+_{2n+1-m}(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{r}}\cdot\mathbf{r}')^m u^+_{2n+1-m}(\mathbf{r}')\right]dS(\mathbf{r}')\right\}\ , \tag{2.184}$$

Therefore, for $n = 0$ in (2.183), we have:

$$A_2(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\int_S\left[(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_0^+(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial}{\partial n'}u_0^+(\mathbf{r}')\right]dS(\mathbf{r}')\ . \tag{2.185}$$

From the transmission conditions and relation $\frac{\partial}{\partial n'}(\hat{\mathbf{r}}\cdot\mathbf{r}') = \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}'$, we get:

$$A_2(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\int_S\left[u_0^-(\mathbf{r}')\frac{\partial}{\partial n'}(\hat{\mathbf{r}}\cdot\mathbf{r}') - (\hat{\mathbf{r}}\cdot\mathbf{r}')\beta\frac{\partial}{\partial n'}u_0^-(\mathbf{r}')\right]dS(\mathbf{r}')\ , \tag{2.186}$$

From Green's theorem in V^- we have:

$$A_2(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1-\beta}{4\pi}\int_S(\hat{\mathbf{r}}\cdot\mathbf{r}')\beta\frac{\partial}{\partial n'}u_0^-(\mathbf{r}')dS(\mathbf{r}') = 0\ , \tag{2.187}$$

where the vanishing to 0 is derived from the value of $u_0^- = 1$ in $\overline{V}^-$. Therefore, the scattering amplitude is of the form:

$$u^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}}) = ik^3A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) + \mathcal{O}(k^4)\ . \tag{2.188}$$

For the coefficient A_3, by substituting $n = 0$ into (2.184) we obtain:

$$\begin{aligned}A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = &\frac{\beta\eta^2}{4\pi}\int_{V^-}u_0^-(\mathbf{r}')dV(\mathbf{r}') + \frac{1}{4\pi}\int_S\left[(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')u_1^+(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\mathbf{r}')\frac{\partial}{\partial n'}u_1^+(\mathbf{r}')\right]dS(\mathbf{r}')\\ &+\frac{1}{4\pi}\int_S\left[\frac{(\hat{\mathbf{r}}\cdot\mathbf{r}')^{\mathbf{2}}}{2}\frac{\partial}{\partial n'}u_0^+(\mathbf{r}') - (\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{r}}\cdot\mathbf{r}')u_0^+(\mathbf{r}')\right]dS(\mathbf{r}')\ ,\end{aligned} \tag{2.189}$$

which can be simplified even further using the following relations:

$$\int_{V^-}dV(\mathbf{r}') = |V^-|\ , \tag{2.190}$$

$$\int_S(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{d}}\cdot\mathbf{r}')dS(\mathbf{r}')dS(\mathbf{r}') = |V^-|\widetilde{\mathbf{I}} : (\hat{\mathbf{r}}\otimes\hat{\mathbf{d}})\ , \tag{2.191}$$

$$\int_S(\mathbf{r}\cdot\hat{\mathbf{n}}')(\hat{\mathbf{d}}\cdot\mathbf{v}^+(\mathbf{r}'))dS(\mathbf{r}') = \hat{\mathbf{r}}\cdot\int_S(\hat{\mathbf{n}}'\mathbf{v}^-)dS(\mathbf{r}')\cdot\hat{\mathbf{d}} = \widetilde{\mathbf{X}} : \hat{\mathbf{r}}\otimes\hat{\mathbf{d}}\ , \tag{2.192}$$

where for the last relation, the double dot product for dyadics has been used, as well as the definition of the general polarizability tensor $\widetilde{\mathbf{X}}$ (2.165). Thus, the coefficient A_3 can be rewritten as:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = -\frac{1}{4\pi}\left[(1-\beta\eta^2)|V^-| + \hat{\mathbf{r}}\cdot\widetilde{\mathbf{X}}(\beta)\cdot\hat{\mathbf{d}}\right]\ . \tag{2.193}$$

Remark: The transmission problems are separated into lossless transmission problems (compressional viscocity $\delta^- = 0$) and lossy transmission problem (compressional viscocity $\delta^- > 0$. For the lossless transmission case the constants β and η are given by:

$$\beta = \frac{\varrho^+}{\varrho^-} \ , \quad \beta\eta^2 = \frac{\gamma^-}{\gamma^+} \ , \tag{2.194}$$

where $\gamma^{\pm}$ the compressibility in $V^{\pm}$.

For the lossy transmission case the constants are given by:

$$\beta = \frac{\varrho^+}{\varrho^-}\left(1 - ik\frac{\delta^-\gamma^-}{\sqrt{\varrho^+\gamma^+}}\right) \ , \quad \eta^2 = \frac{\gamma^-\varrho^-}{\gamma^+\varrho^+}\left(1 - ik\frac{\delta^-\gamma^-}{\sqrt{\varrho^+\gamma^+}}\right)^{-1} \ . \tag{2.195}$$

Thus, there are no significant differences in the low-frequency coefficients calculated above, except for the values of β and η.

2.2 Electromagnetic Scattering Theory

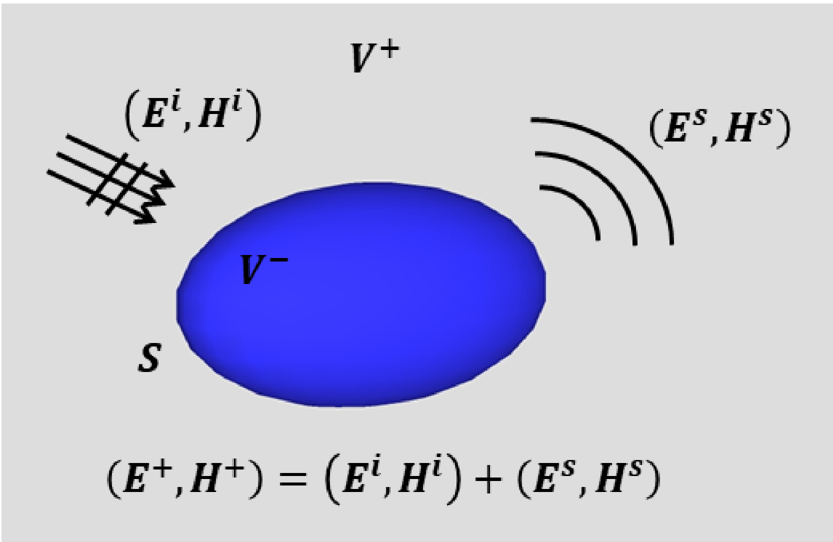


Figure 2.2: Scattering of electromagnetic waves.

The basic fields in electromagnetic theory are the electric field $\boldsymbol{\mathcal{E}}$ measured in units of force per unit charge, the electric displacement $\boldsymbol{\mathcal{D}}$ measured in units of charge per unit area, the magnetic field $\boldsymbol{\mathcal{H}}$ measured in units of charge per unit length per unit time and the magnetic induction $\boldsymbol{\mathcal{B}}$ measured in units of mass per unit charge per unit time. These fields are connected via the Maxwell equations:

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{E}}(\mathbf{r}, t) &= -\frac{\partial \boldsymbol{\mathcal{B}}(\mathbf{r}, t)}{\partial t} \ , \\ \nabla \times \boldsymbol{\mathcal{H}}(\mathbf{r}, t) &= \frac{\partial \boldsymbol{\mathcal{D}}(\mathbf{r}, t)}{\partial t} + \boldsymbol{\mathcal{J}}(\mathbf{r}, t) \ , \end{aligned} \tag{2.196}$$

where $\boldsymbol{\mathcal{J}}$ is the conduction current density measured in units of charge per unit area per unit time. Equations (2.196) are the differential forms of the Faraday's law and the Maxwell-Amber law respectively. Assuming time-harmonic dependence of the form:

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \ , \ \boldsymbol{\mathcal{H}}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{-i\omega t} \\ \boldsymbol{\mathcal{B}}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r})e^{-i\omega t} \ , \ \boldsymbol{\mathcal{D}}(\mathbf{r}, t) = \mathbf{D}(\mathbf{r})e^{-i\omega t} \ , \ \boldsymbol{\mathcal{J}}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t} \ , \end{aligned} \tag{2.197}$$

with ω the angular frequency and substituting (2.197) in (2.196) we obtain the corresponding spatial form of Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mathbf{B}(\mathbf{r}) \ , \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \ . \end{aligned} \tag{2.198}$$

For a linear, homogeneous and isotropic medium, such as the media in our work, the following constitutive relations are valid:

$$\mathbf{D}(\mathbf{r}) = \varepsilon\mathbf{E}(\mathbf{r}) \ , \ \mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r}) \ , \ \mathbf{J}(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r}) \ , \tag{2.199}$$

where ε is the electric permittivity measured in capacity per unit length, μ is the magnetic permeability measured in inductance per unit length and σ is the conductivity measured in capacity per unit length per unit time. A medium with $\sigma \in (0,\infty)$ is called dielectric, a medium with $\sigma = 0$ is called non-conductive or lossless and a mediun with $\sigma \to \infty$ is called a perfect conductor. Substituting (2.199) in (2.198) we obtain the equations:

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mu\mathbf{H}(\mathbf{r}) \ , \\ \nabla \times \mathbf{H}(\mathbf{r}) &= \left(-i\omega\varepsilon + \sigma\right)\mathbf{E}(\mathbf{r}) \ , \end{aligned} \tag{2.200}$$

from which we obtain that both the electric and magnetic fields are solenoidal:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0 \ . \tag{2.201}$$

since the divergence of the curl of any vector field is always zero.
Taking the curl operator on each one of the equations (2.200) and using the other one as well, we obtain:

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) &= \left(\varepsilon\mu\omega^2 + i\mu\sigma\omega\right)\mathbf{E}(\mathbf{r}) \ , \\ \nabla \times \nabla \times \mathbf{H}(\mathbf{r}) &= \left(\varepsilon\mu\omega^2 + i\mu\sigma\omega\right)\mathbf{H}(\mathbf{r}) \ . \end{aligned} \tag{2.202}$$

A plane incident electromagnetic wave $(\mathbf{E}^i, \ \mathbf{H}^i)$ is of the form:

$$\mathbf{E^i}(\mathbf{r}) = \hat{\mathbf{p}}e^{ik\hat{\mathbf{d}}\cdot\mathbf{r}} \ , \tag{2.203}$$

$$\mathbf{H^i}(\mathbf{r}) = Y^{+}\hat{\mathbf{q}}e^{ik\hat{\mathbf{d}}\cdot\mathbf{r}} \ , \tag{2.204}$$

where k is the wave number, $\hat{\mathbf{d}}$ is the direction of propagation, $\hat{\mathbf{p}}$ is the electric polarization and $\hat{\mathbf{q}}$ is the magnetic polarization (with $\hat{\mathbf{d}} \cdot \hat{\mathbf{p}} = \hat{\mathbf{d}} \cdot \hat{\mathbf{q}} = \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} = 0$).
The constant $Y^{\pm}$ is the characteristic admittance of $V^{\pm}$ and is given by:

$$Y^{\pm} = \frac{1}{Z^{\pm}} = \frac{\sqrt{\varepsilon^{\pm}\mu^{\pm}\omega^2 + i\mu^{\pm}\sigma^{\pm}\omega}}{\mu^{\pm}\omega} \ , \tag{2.205}$$

where $Z^{\pm}$ is the characteristic impedance of $V^{\pm}$, measured in inductance per unit length and is given by:

$$Z^{\pm} = \frac{\mu\omega}{\sqrt{\varepsilon^{\pm}\mu^{\pm}\omega^2 + i\mu^{\pm}\sigma^{\pm}\omega}} \ . \tag{2.206}$$

The square root chosen, implies that $Im(Z^{\pm}) \leq 0$ and $Im(Y^{\pm}) \geq 0$.
Substitution of (2.203) in (2.202) leads to:

$$k^2 = \varepsilon^{+}\mu^{+}\omega^2 + i\mu^{+}\sigma^{+}\omega \ . \tag{2.207}$$

Therefore, we define the wave number k as:

$$k = \sqrt{\varepsilon^{+}\mu^{+}\omega^2 + i\mu^{+}\sigma^{+}\omega} \ , \tag{2.208}$$

where the branch of the square root with $Im(k) \geq 0$ has been chosen.
Substituting (2.208) in (2.202) and using the identity $\nabla \times \nabla \times = \nabla(\nabla \cdot) - \triangle$ as well as (2.201) we obtain:

$$\begin{aligned} \Delta \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) &= \mathbf{0} \ , \\ \Delta \mathbf{H}(\mathbf{r}) + k^2 \mathbf{H}(\mathbf{r}) &= \mathbf{0} \ . \end{aligned} \tag{2.209}$$

Therefore, the fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the vector Helmholtz equation with wave number k given by (2.208), which means that all Cartesian components of $\mathbf{E}$ and $\mathbf{H}$ satisfy the scalar Helmholtz equation with wave number k.
In the present work, it assumed that the medium of propagation is lossless and thus $\sigma^+ = 0$. Using (2.208), (2.206) and (2.205) the Maxwell equations (2.200) can be written as:

$$\nabla \times \mathbf{E}(\mathbf{r}) = ikZ^+ \mathbf{H}(\mathbf{r}) \ , \ \nabla \times \mathbf{H}(\mathbf{r}) = -ikY^+ \mathbf{E}(\mathbf{r}) \ . \tag{2.210}$$

The fundamental dyadic solution of the vector Helmholtz equation, is given by:

$$\widetilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2} \left(\nabla_{\mathbf{r}} \otimes \nabla_{\mathbf{r}} + k^2 \widetilde{\mathbf{I}} \right) G(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2} \left(\nabla_{\mathbf{r}} \otimes \nabla_{\mathbf{r}} + k^2 \widetilde{\mathbf{I}} \right) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{ik|\mathbf{r}-\mathbf{r}'|} \ , \tag{2.211}$$

where $\widetilde{\mathbf{I}}$ is the identity dyadic in $\mathbb{R}^3$ and G is given by (2.10). This means that:

$$\nabla_{\mathbf{r}} \times \left(\nabla_{\mathbf{r}} \times \widetilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right) - k^2 \widetilde{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \frac{4\pi}{ik} \delta(\mathbf{r} - \mathbf{r}') \widetilde{\mathbf{I}} \, . \tag{2.212}$$

In eletromagnetic scattering, the scatterer disturbs the propagation of the incident electromagnetic wave $\left(\mathbf{E}^{\mathbf{i}}, \mathbf{H}^{\mathbf{i}}\right)$ producing the scattered field $(\mathbf{E}^{\mathrm{s}}, \mathbf{H}^{\mathrm{s}})$. The total field $(\mathbf{E}^+, \mathbf{H}^+)$ is the superposition of the incident and the scattered field:

$$\mathbf{E}^+(\mathbf{r}) = \mathbf{E}^{\mathbf{i}}(\mathbf{r}) + \mathbf{E}^{\mathrm{s}}(\mathbf{r}) \quad , \qquad \mathbf{H}^+(\mathbf{r}) = \mathbf{H}^{\mathbf{i}}(\mathbf{r}) + \mathbf{H}^{\mathrm{s}}(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \cup S \ . \tag{2.213}$$

Similarly to the acoustic scattering, the scatterers are separated into penetrable scatterers and impenetrable scatterers.
For the case of an impenetrable scatterer the following boundary conditions can be imposed on the surface of the scatterer:

- *Perfectly conductive surface:*
  Describes a scatterer where the normal component of the magnetic field and the tangential component of the electric field vanish on S:

$$\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) = \mathbf{0} \ , \ \hat{\mathbf{n}} \cdot \mathbf{H}^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ . \tag{2.214}$$

- *Impedance surface:*

$$\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) \right) = -Z_s Z^+ \left(\hat{\mathbf{n}} \times \mathbf{H}^+(\mathbf{r}) \right) \ , \quad \mathbf{r} \in S \tag{2.215}$$

  or equivalently

$$\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) = Z_s Z^+ \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^+(\mathbf{r}) \right) \ , \quad \mathbf{r} \in S \ , \tag{2.216}$$

  where Z_s denotes the surface impedance relative to the characteristic impedance Z^+ of the medium.

  For the case of a penetrable scatterer the following transmission conditions that secure the continuity of the tangential component of the electric field and the continuity of the normal component of the displacement field can be imposed on the surface of the scatterer:

- *Transmission conditions:*

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) &= \hat{\mathbf{n}} \times \mathbf{E}^-(\mathbf{r}) \ , \qquad \mathbf{r} \in S \ , \\ Y^+ \hat{\mathbf{n}} \cdot \mathbf{E}^+(\mathbf{r}) &= \eta Y^- \hat{\mathbf{n}} \cdot \mathbf{E}^-(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \end{aligned} \tag{2.217}$$

where η is the relative index of refraction given by $\eta = k^+/k^-$. Equivalently, the above conditions of continuity can be replaced by the following conditions:

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{H}^+(\mathbf{r}) &= \hat{\mathbf{n}} \times \mathbf{H}^-(\mathbf{r}) \ , \qquad \mathbf{r} \in S \ , \\ Z^+ \hat{\mathbf{n}} \cdot \mathbf{H}^+(\mathbf{r}) &= \eta Z^- \hat{\mathbf{n}} \cdot \mathbf{H}^-(\mathbf{r}) \ , \quad \mathbf{r} \in S \ . \end{aligned} \tag{2.218}$$

Also, the scattered field satisfies the Silver-Müller radiation conditions at infinity (Müller 1948, Silver 1949):

$$\begin{aligned} &\lim_{r \to \infty} \left[\mathbf{r} \times \left(\nabla \times \mathbf{E^s} + ikr\mathbf{E^s} \right) \right] = \mathbf{0} \ , \\ &\lim_{r \to \infty} \left[\mathbf{r} \times \left(\nabla \times \mathbf{H^s} + ikr\mathbf{H^s} \right) \right] = \mathbf{0} \ , \end{aligned} \tag{2.219}$$

uniformly for all directions $\hat{\mathbf{r}} \in S^2$. The radiation conditions (2.219) can also be written as:

$$\begin{aligned} Z^+ \hat{\mathbf{r}} \times \mathbf{H^s} + \mathbf{E^s} &= o\left(\frac{1}{r}\right) \ , \\ Y^+ \hat{\mathbf{r}} \times \mathbf{E^s} - \mathbf{H^s} &= o\left(\frac{1}{r}\right) \ . \end{aligned} \tag{2.220}$$

2.2.1 Basic Electromagnetic Scattering Problems

Combining the field equations, the conditions on the boundary and the radiation conditions we are now in position to formulate the basic three electromegnetic scattering problems for plane wave incidence:

1. **Perfect Conductor :**
For given $\mathbf{E^i}, \mathbf{H^i}$ and ω, ε^+, μ^+ (hence Z^+, Y^+ and k) find the total fields $\mathbf{E}^+$, $\mathbf{H}^+ \in C^1\left(V^+\right) \cap C\left(V^+ \cup S\right)$ that solve the following boundary value problem:

$$\begin{aligned} &\nabla \times \mathbf{E}^+(\mathbf{r}) = i\omega\mu^+ \mathbf{H}^+(\mathbf{r}) \quad , \quad \nabla \times \mathbf{H}^+(\mathbf{r}) = -i\omega\varepsilon^+ \mathbf{E}^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \\ &\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) = \mathbf{0} \quad , \qquad \hat{\mathbf{n}} \cdot \mathbf{H}^+(\mathbf{r}) = 0 \ , \qquad \mathbf{r} \in S \ , \\ &\mathbf{E}^+(\mathbf{r}) = \mathbf{E^i}(\mathbf{r}) + \mathbf{E^s}(\mathbf{r}) \quad , \quad \mathbf{H}^+(\mathbf{r}) = \mathbf{H^i}(\mathbf{r}) + \mathbf{H^s}(\mathbf{r}) \ , \qquad \mathbf{r} \in V^+ \cup S \ , \\ &Z^+ \hat{\mathbf{r}} \times \mathbf{H^s} + \mathbf{E^s} = o\left(\frac{1}{r}\right) \quad , \quad Y^+ \hat{\mathbf{r}} \times \mathbf{E^s} - \mathbf{H^s} = o\left(\frac{1}{r}\right) , \end{aligned} \tag{2.221}$$

uniformly for all directions $\hat{\mathbf{r}} \in S^2$.

2. **Impedance scatterer :**
For given $\mathbf{E^i}, \mathbf{H^i}$ and ω, ε^+, μ^+ (hence Z^+, Y^+ and k) find the total fields $\mathbf{E}^+$, $\mathbf{H}^+ \in C^1\left(V^+\right) \cap C\left(V^+ \cup S\right)$ that solve the following boundary value problem:

$$\begin{aligned} &\nabla \times \mathbf{E}^+(\mathbf{r}) = i\omega\mu^+ \mathbf{H}^+(\mathbf{r}) \quad , \quad \nabla \times \mathbf{H}^+(\mathbf{r}) = -i\omega\varepsilon^+ \mathbf{E}^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \\ &\quad \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r})\right) = -Z_s Z \left(\hat{\mathbf{n}} \times \mathbf{H}^+(\mathbf{r})\right) , \qquad \mathbf{r} \in S \ , \\ &\mathbf{E}^+(\mathbf{r}) = \mathbf{E^i}(\mathbf{r}) + \mathbf{E^s}(\mathbf{r}) \quad , \quad \mathbf{H}^+(\mathbf{r}) = \mathbf{H^i}(\mathbf{r}) + \mathbf{H^s}(\mathbf{r}) \ , \qquad \mathbf{r} \in V^+ \cup S \ , \\ &Z^+ \hat{\mathbf{r}} \times \mathbf{H^s} + \mathbf{E^s} = o\left(\frac{1}{r}\right) \quad , \quad Y^+ \hat{\mathbf{r}} \times \mathbf{E^s} - \mathbf{H^s} = o\left(\frac{1}{r}\right) , \end{aligned} \tag{2.222}$$

uniformly for all directions $\hat{\mathbf{r}} \in S^2$.

3. **Penetrable Scatterer :**
For given $\mathbf{E^i}, \mathbf{H^i}$, ω, ε^+, μ^+, ε^-, μ^-, σ^- (hence Z^+, Y^+, k, Z^-, Y^-, k^-, η) find the total exterior fields $\mathbf{E}^+$, $\mathbf{H}^+ \in C^1(V^+) \cap C(V^+ \cup S)$ and the total interior fields $\mathbf{E}^-$, $\mathbf{H}^- \in C^1(V^-) \cap C\left(\overline{V^-}\right)$ that solve the following boundary value problem:

$$\begin{aligned}
&\nabla \times \mathbf{E}^+(\mathbf{r}) = i\omega\mu^+\mathbf{H}^+(\mathbf{r}) \quad , \quad \nabla \times \mathbf{H}^+(\mathbf{r}) = -i\omega\varepsilon^+\mathbf{E}^+(\mathbf{r}) \ , && \mathbf{r} \in V^+ \ ,\\
&\nabla \times \mathbf{E}^-(\mathbf{r}) = i\omega\mu^-\mathbf{H}^-(\mathbf{r}) \ , \quad \nabla \times \mathbf{H}^-(\mathbf{r}) = \left(-i\omega\varepsilon^- + \sigma^-\right)\mathbf{E}^-(\mathbf{r}) \ , && \mathbf{r} \in V^- \ ,\\
&\hat{\mathbf{n}} \times \mathbf{E}^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{E}^-(\mathbf{r}) \quad , \quad Y^+\hat{\mathbf{n}} \cdot \mathbf{E}^+(\mathbf{r}) = \eta Y^-\hat{\mathbf{n}} \cdot \mathbf{E}^-(\mathbf{r}) \ , && \mathbf{r} \in S \ ,\\
&\hat{\mathbf{n}} \times \mathbf{H}^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}^-(\mathbf{r}) \quad , \quad Z^+\hat{\mathbf{n}} \cdot \mathbf{H}^+(\mathbf{r}) = \eta Z^-\hat{\mathbf{n}} \cdot \mathbf{H}^-(\mathbf{r}) \ , && \mathbf{r} \in S \ ,\\
&\mathbf{E}^+(\mathbf{r}) = \mathbf{E^i}(\mathbf{r}) + \mathbf{E^s}(\mathbf{r}) \quad , \quad \mathbf{H}^+(\mathbf{r}) = \mathbf{H^i}(\mathbf{r}) + \mathbf{H^s}(\mathbf{r}) \ , && \mathbf{r} \in V^+ \cup S \ ,\\
&Z^+\hat{\mathbf{r}} \times \mathbf{H^s} + \mathbf{E^s} = o\left(\frac{1}{r}\right) \quad , \quad Y^+\hat{\mathbf{r}} \times \mathbf{E^s} - \mathbf{H^s} = o\left(\frac{1}{r}\right) ,
\end{aligned} \tag{2.223}$$

uniformly for all directions $\hat{\mathbf{r}} \in S^2$.

2.2.2 Integral Representations

The Stratton-Chu integral representation of the scattered electromagnetic field $(\mathbf{E}^s, \mathbf{H}^s)$ is given by [19]:

$$\begin{aligned}
\alpha(\mathbf{r})\mathbf{E^s}(\mathbf{r}) = \frac{ik}{4\pi}\int_S &\{ikZ^+G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H^s}(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\mathbf{n}' \cdot \mathbf{E^s}(\mathbf{r}')\right)\\
&- \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{E^s}(\mathbf{r}')\right)\}\, dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.224}$$

$$\begin{aligned}
\alpha(\mathbf{r})\mathbf{H^s}(\mathbf{r}) = \frac{ik}{4\pi}\int_S &\{-ikY^+G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E^s}(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\mathbf{n}' \cdot \mathbf{H^s}(\mathbf{r}')\right)\\
&- \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{H^s}(\mathbf{r}')\right)\}\, dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.225}$$

for $\mathbf{r} \in \mathbb{R}^3$, with $\alpha(\mathbf{r})$ given by (2.25). Also, the integral representation of the interior electromagnetic field $(\mathbf{E}^-, \mathbf{H}^-)$ is given by:

$$\begin{aligned}
(\alpha(\mathbf{r}) - 1)\,\mathbf{E}^-(\mathbf{r}) = \frac{ik^-}{4\pi}\int_S &\{ik^-Z^-G^-(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H}^-(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'}G^-(\mathbf{r},\mathbf{r}')\right)\left(\mathbf{n}' \cdot \mathbf{E}^-(\mathbf{r}')\right)\\
&- \left(\nabla_{\mathbf{r}'}G^-(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{E}^-(\mathbf{r}')\right)\}\, dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.226}$$

$$\begin{aligned}
(\alpha(\mathbf{r}) - 1)\,\mathbf{H}^-(\mathbf{r}) = \frac{ik^-}{4\pi}\int_S &\{-ik^-Y^-G^-(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E}^-(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'}G^-(\mathbf{r},\mathbf{r}')\right)\left(\mathbf{n}' \cdot \mathbf{H}^-(\mathbf{r}')\right)\\
&- \left(\nabla_{\mathbf{r}'}G^-(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{H}^-(\mathbf{r}')\right)\}\, dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.227}$$

for $\mathbf{r} \in \mathbb{R}^3$. Similarly, the integral representation of the incident field $(\mathbf{E^i}, \mathbf{H^i})$ is given by:

$$\begin{aligned}
(\alpha(\mathbf{r}) - 1)\,\mathbf{E^i}(\mathbf{r}) = \frac{ik^-}{4\pi}\int_S &\{ikZ^+G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H^i}(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\mathbf{n}' \cdot \mathbf{E^i}(\mathbf{r}')\right)\\
&- \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{E^i}(\mathbf{r}')\right)\}\, dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.228}$$

$$(\alpha(\mathbf{r}) - 1)\, \mathbf{H}^{\mathbf{i}}(\mathbf{r}) = \frac{ik^-}{4\pi} \int_S \{-ikY^+ G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E}^{\mathbf{i}}(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \left(\mathbf{n}' \cdot \mathbf{H}^-(\mathbf{r}')\right) - \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{H}^{\mathbf{i}}(\mathbf{r}')\right)\} \, dS(\mathbf{r}') \,, \tag{2.229}$$

for $\mathbf{r} \in \mathbb{R}^3$. Therefore, from (2.224)-(2.228) and (2.225)-(2.229) the integral representation of the total exterior electromagnetic field $(\mathbf{E}^+, \mathbf{H}^+)$ is given by:

$$\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r}) = \mathbf{E}^{\mathbf{i}}(\mathbf{r}) + \frac{ik}{4\pi} \int_S \{ikZ^+ G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H}^+(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \left(\mathbf{n}' \cdot \mathbf{E}^+(\mathbf{r}')\right) - \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{E}^+(\mathbf{r}')\right)\} \, dS(\mathbf{r}') \,, \tag{2.230}$$

$$\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r}) = \mathbf{H}^{\mathbf{i}}(\mathbf{r}) + \frac{ik}{4\pi} \int_S \{-ikY^+ G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E}^+(\mathbf{r}')) + \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \left(\mathbf{n}' \cdot \mathbf{H}^+(\mathbf{r}')\right) - \left(\nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\right) \times \left(\mathbf{n}' \times \mathbf{H}^+(\mathbf{r}')\right)\} \, ds(\mathbf{r}') \,, \tag{2.231}$$

for $\mathbf{r} \in \mathbb{R}^3$. Taking the curl operator in (2.226)-(2.227) and (2.230)-(2.231) and using the Maxwell equations we obtain:

$$\begin{aligned}(\alpha(\mathbf{r}) - 1)\, \mathbf{E}^-(\mathbf{r}) &= \frac{ik^-}{4\pi} \nabla \times \int_S G^-(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E}^-(\mathbf{r}'))\, ds(\mathbf{r}') \\ &- \frac{Z^-}{4\pi} \nabla \times \left[\nabla \times \int_S G^-(\mathbf{r},\mathbf{r}') \left(\mathbf{n}' \times \mathbf{H}^-(\mathbf{r}')\right)\, dS(\mathbf{r}')\right] \,,\end{aligned} \tag{2.232}$$

$$\begin{aligned}(\alpha(\mathbf{r}) - 1)\, \mathbf{H}^-(\mathbf{r}) &= \frac{ik^-}{4\pi} \nabla \times \int_S G^-(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H}^-(\mathbf{r}'))\, ds(\mathbf{r}') \\ &- \frac{Y^-}{4\pi} \nabla \times \left[\nabla \times \int_S G^-(\mathbf{r},\mathbf{r}') \left(\mathbf{n}' \times \mathbf{E}^-(\mathbf{r}')\right)\, dS(\mathbf{r}')\right] \,,\end{aligned} \tag{2.233}$$

as well as

$$\begin{aligned}\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r}) &= \mathbf{E}^{\mathbf{i}}(\mathbf{r}) + \frac{ik}{4\pi} \nabla \times \int_S G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{E}^+(\mathbf{r}'))\, ds(\mathbf{r}') \\ &- \frac{Z^+}{4\pi} \nabla \times \left[\nabla \times \int_S G^+(\mathbf{r},\mathbf{r}') \left(\mathbf{n}' \times \mathbf{H}^+(\mathbf{r}')\right)\, dS(\mathbf{r}')\right] \,,\end{aligned} \tag{2.234}$$

$$\begin{aligned}\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r}) &= \mathbf{H}^{\mathbf{i}}(\mathbf{r}) + \frac{ik}{4\pi} \nabla \times \int_S G^+(\mathbf{r},\mathbf{r}')(\mathbf{n}' \times \mathbf{H}^+(\mathbf{r}'))\, ds(\mathbf{r}') \\ &- \frac{Y^+}{4\pi} \nabla \times \left[\nabla \times \int_S G^+(\mathbf{r},\mathbf{r}') \left(\mathbf{n}' \times \mathbf{E}^+(\mathbf{r}')\right)\, dS(\mathbf{r}')\right] \,,\end{aligned} \tag{2.235}$$

for $\mathbf{r} \in \mathbb{R}^3 \setminus S$.

2.2.3 Far-Field

Substituting the asymptotic forms of G and $\nabla_{\mathbf{r}'} G$ in the integral representations of the scattered field (2.234) and (2.235) we obtain:

$$\mathbf{E}^{\mathbf{s}}(\mathbf{r}) = \mathbf{E}^{\infty}(\hat{\mathbf{r}}) h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \,,\ r \to \infty \,,\ \hat{\mathbf{r}} \in S^2 \,, \tag{2.236}$$

$$\mathbf{H}^{\mathrm{s}}(\mathbf{r}) = \mathbf{H}^{\infty}(\hat{\mathbf{r}})h(kr) + \mathcal{O}\left(\frac{1}{r^2}\right) \ , \ r \to \infty \ , \ \hat{\mathbf{r}} \in S^2 \ , \tag{2.237}$$

where the functions $\mathbf{E}^{\infty}(\hat{\mathbf{r}})$ and $\mathbf{H}^{\infty}(\hat{\mathbf{r}})$ are the electric far-field pattern or electric scattering amplitude and the magnetic far-field pattern or magnetic scattering amplitude respectively, given by:

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}}) = \frac{k^2}{4\pi}\int_S \left[-Z^+\left(\hat{\mathbf{n}}' \times \mathbf{H}^+(\mathbf{r}')\right) + \left(\hat{\mathbf{n}} \cdot \mathbf{E}^+(\mathbf{r}')\right)\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \left(\hat{\mathbf{n}}' \times \mathbf{E}^+(\mathbf{r}')\right)\right] e^{ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\hat{\mathbf{r}}) \ , \tag{2.238}$$

$$\mathbf{H}^{\infty}(\hat{\mathbf{r}}) = \frac{k^2}{4\pi}\int_S \left[Y^+\left(\hat{\mathbf{n}}' \times \mathbf{E}^+(\mathbf{r}')\right) + \left(\hat{\mathbf{n}} \cdot \mathbf{H}^+(\mathbf{r}')\right)\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \left(\hat{\mathbf{n}}' \times \mathbf{H}^+(\mathbf{r}')\right)\right] e^{ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\hat{\mathbf{r}}) \ . \tag{2.239}$$

For the low-frequency study, the following forms of the scattering amplitudes are proved to be more convinient [19]:

$$\begin{aligned}\mathbf{E}^{\infty}(\hat{\mathbf{r}}) = &-\frac{ik^3}{4\pi}\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_S \left[-Z^+\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \mathbf{H}^+(\mathbf{r}')) + \hat{\mathbf{n}}' \cdot \mathbf{E}^+(\mathbf{r}')\right] \mathbf{r}' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\mathbf{r}')\right] \\ &-\frac{ik^3}{4\pi}\hat{\mathbf{r}} \times \int_S \left[\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \mathbf{E}^+(\mathbf{r}')) + Z^+\hat{\mathbf{n}}' \cdot \mathbf{H}^+(\mathbf{r}')\right] \mathbf{r}' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\mathbf{r}')\end{aligned} \tag{2.240}$$

$$\begin{aligned}\mathbf{H}^{\infty}(\hat{\mathbf{r}}) = &-\frac{ik^3}{4\pi}\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \int_S \left[Y^+\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \mathbf{E}^+(\mathbf{r}')) + \hat{\mathbf{n}}' \cdot \mathbf{H}^+(\mathbf{r}')\right] \mathbf{r}' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\mathbf{r}')\right] \\ &-\frac{ik^3}{4\pi}\hat{\mathbf{r}} \times \int_S \left[\hat{\mathbf{r}} \cdot (\hat{\mathbf{n}}' \times \mathbf{H}^+(\mathbf{r}')) - Y^+\hat{\mathbf{n}}' \cdot \mathbf{E}^+(\mathbf{r}')\right] \mathbf{r}' e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \ dS(\mathbf{r}')\end{aligned} \tag{2.241}$$

2.2.4 Low-Frequency Theory in Electromagnetics

The low-frequency expansions for the electromagnetic waves are:

- *Plane incident wave:*
  Since in the present work we mainly study the case of plane wave incidence, the incident fields (2.203) and (2.204) because of their exponential forms can be written as power series of the wave number k as follows:

$$\mathbf{E}^i(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} E_n^i(\mathbf{r}) \ , \quad \mathbf{r} \in \mathbb{R}^3 \ , \tag{2.242}$$

$$\mathbf{H}^i(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^i(\mathbf{r}) \ , \quad \mathbf{r} \in \mathbb{R}^3 \ , \tag{2.243}$$

  where $\mathbf{E}_n^i$ and $\mathbf{H}_n^i$ the low-frequency coefficients independent of the wave number k.

- *Total exterior electromagnetic field:*
  The total exterior electromagnetic field it is an analytic function in a neighborhood of $k = 0$ which was established by Kleinmann [23]. Therefore, it can be written as power series of the wave number k in a neighborhood of $k = 0$:

$$\mathbf{E}^+(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.244}$$

$$\mathbf{H}^+(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.245}$$

  where $\mathbf{E}_n^+$ and $\mathbf{H}_n^+$ the low-frequency coefficients independent of the wave number k.

- *Total interior electromagnetic field:*
  For the transmission problems, the total interior electromagnetic field, similarly to the total exterior electromagnetic field, has the following low-frequency expansion:

$$\mathbf{E}^-(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^-(\mathbf{r}) , \quad \mathbf{r} \in V^- , \tag{2.246}$$

$$\mathbf{H}^-(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^-(\mathbf{r}) , \quad \mathbf{r} \in V^- , \tag{2.247}$$

  where $\mathbf{E}_n^-$ and $\mathbf{H}_n^-$ the low-frequency coefficients independent of the wave number k.

- *Electromagnetic scattered field:*
  Similarly to the total field and the incident field, the electromagnetic scattered field can also be written as power series of the wave number k in a neighborhood of $k = 0$:

$$\mathbf{E}^s(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^s(\mathbf{r}) , \quad \mathbf{r} \in V^+ , \tag{2.248}$$

$$\mathbf{H}^s(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^s(\mathbf{r}) , \quad \mathbf{r} \in V^+ , \tag{2.249}$$

  where $\mathbf{E}_n^-$ and $\mathbf{H}_n^-$ the low-frequency coefficients independent of k.

Substituting the low-frequency expansion of the exterior total electromagnetic field into the Maxwell equations (2.210), we obtain:

$$\begin{aligned} \nabla \times \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) &= Z^+ ik \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) = Z^+ \sum_{n=0}^{\infty} \frac{(ik)^{n+1}}{n!} \mathbf{H}_n^+(\mathbf{r}) \\ &= Z^+ \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \mathbf{H}_{n-1}^+(\mathbf{r}) = nZ^+ \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_{n-1}^+(\mathbf{r}) , \quad \mathbf{r} \in V^+ , \end{aligned} \tag{2.250}$$

and

$$\begin{aligned} \nabla \times \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) &= -Y^+ ik \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) = -Y^+ \sum_{n=0}^{\infty} \frac{(ik)^{n+1}}{n!} \mathbf{E}_n^+(\mathbf{r}) \\ &= -Y^+ \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \mathbf{E}_{n-1}^+(\mathbf{r}) = -nY^+ \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_{n-1}^+(\mathbf{r}) , \quad \mathbf{r} \in V^+ . \end{aligned} \tag{2.251}$$

Also

$$\begin{aligned} \nabla \cdot \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) = 0 , \quad \mathbf{r} \in V^+ , \\ \nabla \cdot \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) = 0 , \quad \mathbf{r} \in V^+ . \end{aligned} \tag{2.252}$$

Therefore, equating the coefficients of $(ik)^n$ leads to the following sequence of equations:

$$\begin{aligned} &\nabla \times \mathbf{E}_n^+(\mathbf{r}) = nZ^+ \mathbf{H}_{n-1}^+(\mathbf{r}) , \quad \mathbf{r} \in V^+ , \\ &\nabla \times \mathbf{H}_n^+(\mathbf{r}) = -nY^+ \mathbf{E}_{n-1}^+(\mathbf{r}) , \quad \mathbf{r} \in V^+ , \\ &\nabla \cdot \mathbf{E}_n^+(\mathbf{r}) = \nabla \cdot \mathbf{H}_n^+(\mathbf{r}) = 0 , \quad \mathbf{r} \in V^+ , \end{aligned} \tag{2.253}$$

for $n = 0, 1, 2, ...$
Remark: The scattered electromagnetic field $\mathbf{E}^+$, $\mathbf{H}^+$ satisfy the Silver-Müller radiation condition (2.219) as $r \to \infty$ uniformly for all directions of $\hat{\mathbf{r}}$ which ensures the well-posedness of the problem.

For the transmission problems, the total interior electromagnetic field $\mathbf{E}^-$, $\mathbf{H}^-$ in V^-, satisfies the Maxwell equations:

$$\begin{aligned} &\nabla \times \mathbf{E}^-(\mathbf{r}) = ik^- Z^- \mathbf{H}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \\ &\nabla \times \mathbf{H}^-(\mathbf{r}) = -ik^- Y^- \mathbf{E}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \\ &\nabla \cdot \mathbf{E}^-(\mathbf{r}) = \nabla \cdot \mathbf{E}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ , \end{aligned} \tag{2.254}$$

where

$$Z^- = \sqrt{\frac{\mu^-}{\varepsilon^-}} \frac{1}{\sqrt{1 + i\frac{\sigma^-}{\varepsilon^-\omega^-}}} \ , \quad Y^- = \frac{\varepsilon^- \sqrt{1 + i\frac{\sigma^-}{\varepsilon^-\omega^-}}}{\sqrt{\mu^-}} \ , \quad k^- = \eta k = k\sqrt{\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}}\sqrt{1 + i\frac{\sigma^-}{\varepsilon^-\omega^-}} \tag{2.255}$$

After some straightforward calculations, the Maxwell equations in terms of k, Y^+ and Z^+ become:

$$\begin{aligned} &\nabla \times \mathbf{E}^-(\mathbf{r}) = ik\frac{\mu^-}{\mu^+} Z^+ \mathbf{H}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ , \\ &\nabla \times \mathbf{H}^-(\mathbf{r}) = -ik\frac{\varepsilon^-}{\varepsilon^+} Y^+ \left(1 + \frac{i\sigma^-\varepsilon^+}{k\varepsilon^-} Z^+\right) \mathbf{E}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ , \\ &\nabla \cdot \mathbf{E}^-(\mathbf{r}) = \nabla \cdot \mathbf{E}^-(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^- \ , \end{aligned} \tag{2.256}$$

which by substituting the low-frequency expansions and equating the coefficients of $(ik)^n$, similarly with the Maxwell equations in the exterior, we obtain the following sequence of equations for the low-frequency coefficients:

$$\begin{aligned} &\nabla \times \mathbf{E}_n^-(\mathbf{r}) = n\frac{\mu^-}{\mu^+} Z^+ \mathbf{H}_{n-1}^-(\mathbf{r}) \ , \quad \mathbf{r} \in V^- \ , \\ &\nabla \times \mathbf{H}_n^-(\mathbf{r}) = -n\frac{\varepsilon^-}{\varepsilon^+} Y^+ \mathbf{E}_{n-1}^-(\mathbf{r}) + \sigma^- \mathbf{E}_n^- \ , \quad \mathbf{r} \in V^- \ , \\ &\nabla \cdot \mathbf{E}_n^-(\mathbf{r}) = \nabla \cdot \mathbf{H}_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^- \ , \end{aligned} \tag{2.257}$$

From Gauss theorem we also have:

$$\int_S \hat{\mathbf{n}} \cdot \mathbf{E}_n^\pm = \int_S \hat{\mathbf{n}} \cdot \mathbf{H}_n^\pm = 0 \ , \tag{2.258}$$

which will be used frequently in what follows. For the boundary conditions we have the following relations:

- *Perfect conductor:*
  Substituting the low-frequency expansion of the total electromagnetic field into the boundary conditions $\hat{\mathbf{n}} \times \mathbf{E}^+ = 0$, $\hat{\mathbf{n}} \cdot \mathbf{H}^+ = 0$, we have:

$$\begin{aligned} &\hat{\mathbf{n}} \times \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ , \\ &\hat{\mathbf{n}} \cdot \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) = 0 \Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \hat{\mathbf{n}} \cdot \mathbf{H}_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ . \end{aligned} \tag{2.259}$$

Equating the coefficients of $(ik)^n$ to 0, leads to the following boundary conditions for the low-frequency coefficients:

$$\hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) = 0 \quad , \quad \hat{\mathbf{n}} \cdot \mathbf{H}_n^+(\mathbf{r}) = 0 \ , \qquad \mathbf{r} \in S \ . \tag{2.260}$$

for $n = 0, 1, \dots$.

- *Impedance boundary conditions:*
  Substituting the low-frequency expansion of the total field in the impedance boundary condition $\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}^+) = -Z_s Z^+ \hat{\mathbf{n}} \times \mathbf{H}^+$, we obtain:

$$\begin{aligned} &\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{E}_n^+(\mathbf{r}) \right) = -Z_s Z^+ \hat{\mathbf{n}} \times \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathbf{H}_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \\ &\Leftrightarrow \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) \right) = -\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} Z_s Z^+ \hat{\mathbf{n}} \times \mathbf{H}_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ . \end{aligned} \tag{2.261}$$

  Equating the coefficients of $(ik)^n$ leads to the following boundary conditions for the low-frequency coefficients:

$$\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) \right) = -Z_s Z^+ \hat{\mathbf{n}} \times \mathbf{H}_n^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ . \tag{2.262}$$

- *Transmission conditions:*
  Substituting the low-frequency expansions of the total electromagnetic field into the transmission conditions and equating the coefficients of ik, we have similarly with the previous boundary conditions:

$$\begin{aligned} &\hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{E}_n^-(\mathbf{r}) \quad , \qquad \hat{\mathbf{n}} \times \mathbf{H}_n^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}_n^-(\mathbf{r}) \ , \\ &Y^+ \hat{\mathbf{n}} \cdot \mathbf{E}_n^+(\mathbf{r}) = \eta Y^- \hat{\mathbf{n}} \cdot \mathbf{E}_n^-(\mathbf{r}) \quad , \quad Z^+ \hat{\mathbf{n}} \cdot \mathbf{H}_n^+(\mathbf{r}) = \eta Z^- \hat{\mathbf{n}} \cdot \mathbf{H}_n^-(\mathbf{r}) \ , \end{aligned} \quad \mathbf{r} \in S \ , \tag{2.263}$$

  which equivalently after some calculations can be rewritten as:

$$\begin{aligned} &\hat{\mathbf{n}} \times \mathbf{E}_n^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{E}_n^-(\mathbf{r}) \ , \\ &\hat{\mathbf{n}} \times \mathbf{H}_n^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{H}_n^-(\mathbf{r}) \ , \\ &\sigma^- \hat{\mathbf{n}} \cdot \mathbf{E}_n^+(\mathbf{r}) = n Y^+ \left(\frac{\varepsilon^-}{\varepsilon^+} \hat{\mathbf{n}} \cdot \mathbf{E}_{n-1}^-(\mathbf{r}) - \hat{\mathbf{n}} \cdot \mathbf{E}_{n-1}^+(\mathbf{r}) \right) \ , \\ &\mu^+ \hat{\mathbf{n}} \cdot \mathbf{H}_n^+(\mathbf{r}) = \mu^- \hat{\mathbf{n}} \cdot \mathbf{H}_n^-(\mathbf{r}) \ , \end{aligned} \quad \mathbf{r} \in S \ , \tag{2.264}$$

The main idea, similarly to the acoustic problems, is to find a decomposition for $\mathbf{E}^+$, $\mathbf{H}^+$ similar to (2.54) with one part depending only on the previous steps of the sequence of the problems for the low-frequency coefficients and the other part to be a solution of a potential theory problem. This reduces the sequence of problems for the low-frequency coefficients into corresponding potential theory problems.

Substituting the low-frequency expansion of the total electromagnetic field and the expansion of the fundamental solution (2.47) into the Stratton-Chu integral representations (2.230), (2.231) and using the Cauchy formula (2.49) to rearrange terms as well as relation $\Delta_{\mathbf{r}} |\mathbf{r} - \mathbf{r}'| = 2|\mathbf{r} - \mathbf{r}'|^{-1}$, leads to the following decompositions for the low-frequency coefficients $\mathbf{E}_n^+$ and $\mathbf{H}_n^+$:

$$\alpha(\mathbf{r}) \mathbf{E}_n^+(\mathbf{r}) = \mathbf{F}_{en}^+(\mathbf{r}) + \mathbf{U}_{en}^+(\mathbf{r}) \ , \tag{2.265}$$

$$\alpha(\mathbf{r}) \mathbf{H}_n^+(\mathbf{r}) = \mathbf{F}_{mn}^+(\mathbf{r}) + \mathbf{U}_{mn}^+(\mathbf{r}) \ , \tag{2.266}$$

where

$$\begin{aligned}\mathbf{F}^+_{en}(\mathbf{r}) = E^i_n(\mathbf{r}) + \frac{Z^+}{4\pi}\sum_{m=1}^{n}\binom{n}{m} m \int_S |\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}' \times \mathbf{H}^+_{n-m}(\mathbf{r}')\right) dS(\mathbf{r}') \\ + \sum_{m=2}^{n}\binom{n}{m}\frac{(m-1)}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|^{m-3}\left[(\mathbf{r}-\mathbf{r}') \times \left(\hat{\mathbf{n}}' \times \mathbf{E}^+_{n-m}(\mathbf{r}')\right)\right. \\ \left. -(\mathbf{r}-\mathbf{r}')\left(\hat{\mathbf{n}}' \cdot \mathbf{E}^+_{n-m}(\mathbf{r}')\right)\right] dS(\mathbf{r}') \ ,\end{aligned} \tag{2.267}$$

$$\begin{aligned}\mathbf{F}^+_{mn}(\mathbf{r}) = \mathbf{H}^+_n(\mathbf{r}) - \frac{Y^+}{4\pi}\sum_{m=1}^{n}\binom{n}{m} m \int_S |\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}' \times \mathbf{E}^+_{n-m}(\mathbf{r}')\right) dS(\mathbf{r}') \\ + \sum_{m=2}^{n}\binom{n}{m}\frac{(m-1)}{4\pi}\int_S |\mathbf{r}-\mathbf{r}'|^{m-3}\left[(\mathbf{r}-\mathbf{r}') \times \left(\hat{\mathbf{n}}' \times \mathbf{H}^+_{n-m}(\mathbf{r}')\right) - \right. \\ \left. (\mathbf{r}-\mathbf{r}')\left(\hat{\mathbf{n}}' \times \mathbf{H}^+_{n-m}(\mathbf{r}')\right)\right] dS(\mathbf{r}') \ ,\end{aligned} \tag{2.268}$$

$$\mathbf{U}^+_{en}(\mathbf{r}) = \frac{1}{4\pi}\nabla \times \int_S \frac{\hat{\mathbf{n}}' \times \mathbf{E}^+_n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dS(\mathbf{r}') - \frac{1}{4\pi}\nabla \int_S \frac{\hat{\mathbf{n}}' \cdot \mathbf{E}^+_n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dS(\mathbf{r}') \ , \tag{2.269}$$

and

$$\mathbf{U}^+_{mn}(\mathbf{r}) = \frac{1}{4\pi}\nabla \times \int_S \frac{\hat{\mathbf{n}}' \times \mathbf{H}^+_n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dS(\mathbf{r}') - \frac{1}{4\pi}\nabla \int_S \frac{\hat{\mathbf{n}}' \cdot \mathbf{H}^+_n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dS(\mathbf{r}') \ , \tag{2.270}$$

with the convention:

$$\sum_{m=p}^{q} (\ ) = 0 \ , \quad \text{if } q < p \ , \tag{2.271}$$

which leads to:

$$\mathbf{F}^+_{e0}(\mathbf{r}) = \mathbf{E}^i_0(\mathbf{r}) \ , \tag{2.272}$$

$$\mathbf{F}^+_{m0}(\mathbf{r}) = \mathbf{H}^i_0(\mathbf{r}) \ . \tag{2.273}$$

It can be seen that $\mathbf{U}^+_{en}$, $\mathbf{U}^+_{mn}$ are vector valued functions associated with single layer potentials, which means that the determination of these functions leads to potential theory problems: From all the above, it is concluded that the asymptotic forms for the low-frequency coefficients $\mathbf{E}^+_n$ and $\mathbf{H}^+_n$ based on (2.265) and (2.266), are of the form:

$$\mathbf{E}^+_n(\mathbf{r}) = \mathbf{F}^+_{en}(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ , \tag{2.274}$$

$$\mathbf{H}^+_n(\mathbf{r}) = \mathbf{F}^+_{mn}(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \tag{2.275}$$

This decomposition, similarly to the decomposition (2.54) of acoustics, is useful due to the fact that on every step n the vector fields $\mathbf{F}^+_{en}$ and $\mathbf{F}^+_{mn}$ depend only on the low-frequency coefficients $\mathbf{E}^+_0, ..., \mathbf{E}^+_{n-1}$ and $\mathbf{H}^+_0, ..., \mathbf{H}^+_{n-1}$ respectively and they are independent of $\mathbf{E}^+_n$ and $\mathbf{H}^+_n$ similarly to f^+_n in the case of acoustics. Therefore the specification of $\mathbf{E}^+_n$, $\mathbf{H}^+_n$ for every step $n = 0, 1, 2, ...$ depends only on the specification of the vector valued functions $\mathbf{U}^+_{en}$ and $\mathbf{U}^+_{mn}$. For the transmission problems where the calculation of the low-frequency coefficients of the total interior electromagnetic field is needed, working with the Straton-Chu integral representation of the interior electromagnetic field (2.226)-(2.227), the same way to the total exterior electromagnetic field, leads to a corresponding decomposition of $\mathbf{E}^-$ and $\mathbf{H}^-$:

$$(\alpha(\mathbf{r})-1)\,\mathbf{E}_n^-(\mathbf{r}) = \mathbf{F}_{en}^-(\mathbf{r}) + U_{en}^-(\mathbf{r}) \ , \tag{2.276}$$

$$(\alpha(\mathbf{r})-1)\,\mathbf{H}_n^-(\mathbf{r}) = \mathbf{F}_{mn}^-(\mathbf{r}) + U_{mn}^-(\mathbf{r}) \ , \tag{2.277}$$

where

$$\begin{aligned}\mathbf{F}_{en}^-(\mathbf{r}) &= \frac{Z^-}{4\pi}\sum_{m=1}^{n}\binom{n}{m} m\eta^m \int_S |\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^-(\mathbf{r}')\right) dS(\mathbf{r}') \\ &+ \frac{1}{4\pi}\sum_{m=2}^{n}\binom{n}{m}(m-1)\eta^m \int_S |\mathbf{r}-\mathbf{r}'|^{m-3}\left[(\mathbf{r}-\mathbf{r}')\times\left(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^-(\mathbf{r}')\right)\right. \\ &\qquad \left. -(\mathbf{r}-\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^-(\mathbf{r}')\right)\right] dS(\mathbf{r}') \ ,\end{aligned} \tag{2.278}$$

$$\begin{aligned}\mathbf{F}_{mn}^-(\mathbf{r}) &= -\frac{Y^-}{4\pi}\sum_{m=1}^{n}\binom{n}{m} m\eta^m \int_S |\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^-(\mathbf{r}')\right) dS(\mathbf{r}') \\ &+ \frac{1}{4\pi}\sum_{m=2}^{n}\binom{n}{m}(m-1)\eta^m \int_S \left[\mathbf{r}-\mathbf{r}'|^{m-3}(\mathbf{r}-\mathbf{r}')\times\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^-(\mathbf{r}')\right)\right. \\ &\qquad \left. -\ (\mathbf{r}-\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^-(\mathbf{r}')\right)\right] dS(\mathbf{r}') \ ,\end{aligned} \tag{2.279}$$

$$\mathbf{U}_{en}^-(\mathbf{r}) = \frac{1}{4\pi}\nabla\times\int_S \frac{\hat{\mathbf{n}}'\times\mathbf{E}_n^-(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') - \frac{1}{4\pi}\nabla\int_S \frac{\hat{\mathbf{n}}'\cdot\mathbf{E}_n^-(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') \ , \tag{2.280}$$

and

$$\mathbf{U}_{mn}^-(\mathbf{r}) = \frac{1}{4\pi}\nabla\times\int_S \frac{\hat{\mathbf{n}}'\times\mathbf{H}_n^-(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') - \frac{1}{4\pi}\nabla\int_S \frac{\hat{\mathbf{n}}'\cdot\mathbf{H}_n^-(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') \ , \tag{2.281}$$

and based on the convention (2.271), we have:

$$\mathbf{F}_{e0}^- = \mathbf{F}_{m0}^- = 0 \ . \tag{2.282}$$

Finally, based on [19] , for the calculation of low-frequency expansion of the scattering amplitude, the most convenient forms are (2.240) and (2.241), since with these forms it becomes clear that the lowest terms are of order k^3. Therefore, substituting the low-frequency expansions of $\mathbf{E}^+$ and $\mathbf{H}^+$ into (2.240) and (2.241) respectively, we obtain:

$$\begin{aligned}\mathbf{E}^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}}) &= -\frac{ik^3}{4\pi}\hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[-Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}H_n^+(\mathbf{r}')\right)\right.\right. \\ &\qquad \left.\left. +\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{E}_n^+(\mathbf{r}')\right]\mathbf{r}'\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}(\hat{\mathbf{r}}\cdot\mathbf{r}')^n dS(\mathbf{r}')\right\} \\ &- \frac{ik^3}{4\pi}\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{E}_n^+(\mathbf{r}')\right) + Z^+\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{H}_n^+(\mathbf{r}')\right]\mathbf{r}'\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}(\hat{\mathbf{r}}\cdot\mathbf{r}')^n dS(\mathbf{r}') \ ,\end{aligned} \tag{2.283}$$

or equivalently:

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[-Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}H_n^+(\mathbf{r}')\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(-1)^n(\hat{\mathbf{r}}\cdot\mathbf{r}')^n\right)\right.\right.$$
$$\left.\left.+\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{E}_n^+(\mathbf{r}')\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(-1)^n(\hat{\mathbf{r}}\cdot\mathbf{r}')^n\right]\mathbf{r}'dS(\mathbf{r}')\right\}$$
$$+\frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{E}_n^+(\mathbf{r}')\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(-1)^n(\hat{\mathbf{r}}\cdot\mathbf{r}')^n\right)\right.$$
$$\left.+Z^+\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathbf{H}_n^+(\mathbf{r}')\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}(-1)^n(\hat{\mathbf{r}}\cdot\mathbf{r}')^n\right]\mathbf{r}'dS(\mathbf{r}') \tag{2.284}$$

Applying Cauchy formula (2.49) it becomes:

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[-Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}H_{m-n}^+(\mathbf{r}')(-1)^m(\hat{\mathbf{r}}\cdot\mathbf{r}')^m\right)\right.\right.$$
$$\left.\left.+\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{\infty}\binom{n}{m}\mathbf{E}_{m-n}^+(\mathbf{r}')(-1)^m(\hat{\mathbf{r}}\cdot\mathbf{r}')^m\right]\mathbf{r}'dS(\mathbf{r}')\right\}$$
$$+\frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{\infty}\binom{n}{m}\mathbf{E}_{m-n}^+(\mathbf{r}')(-1)^m(\hat{\mathbf{r}}\cdot\mathbf{r}')^m\right)\right.$$
$$\left.+Z^+\hat{\mathbf{n}}'\cdot\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{\infty}\binom{n}{m}\mathbf{H}_{m-n}^+(\mathbf{r}')(-1)^m(\hat{\mathbf{r}}\cdot\mathbf{r}')^m\right]\mathbf{r}'dS(\mathbf{r}')\ , \tag{2.285}$$

or equivalently:

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+3}}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^+(\mathbf{r}')\right.\right.\right.$$
$$\left.\left.-Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^+(\mathbf{r}')\right)\right]\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\right] \tag{2.286}$$
$$\left.+\hat{\mathbf{r}}\times\int_S\left[Z^+(\hat{\mathbf{n}}'\cdot\mathbf{H}_{n-m}^+(\mathbf{r}'))+\hat{\mathbf{r}}\cdot(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^+(\mathbf{r}'))\right]\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\right\}\ .$$

Similarly we work for $\mathbf{H}^{\infty}$ and obtain:

$$\mathbf{H}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+3}}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{n}}'\cdot\mathbf{H}_{n-m}^+(\mathbf{r}')\right.\right.\right.$$
$$\left.\left.+Y^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^+(\mathbf{r}')\right)\right]\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\right] \tag{2.287}$$
$$\left.+\hat{\mathbf{r}}\times\int_S\left[-Y^+(\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^+(\mathbf{r}'))+\hat{\mathbf{r}}\cdot(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^+(\mathbf{r}'))\right]\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\right\}\ ,$$

and they satisfy the following relations ([19], p. 58):

$$\hat{\mathbf{r}}\cdot\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \hat{\mathbf{r}}\cdot\mathbf{H}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = 0\ , \tag{2.288}$$
$$\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = -Z^+\hat{\mathbf{r}}\times\mathbf{H}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}})\ , \tag{2.289}$$
$$\mathbf{H}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) = Y^+\hat{\mathbf{r}}\times\mathbf{E}^{\infty}(\hat{\mathbf{r}};\hat{\mathbf{d}}) \tag{2.290}$$

Remark: Even though we used the integral representations (2.240) and (2.241) out of convenience, it can be proven [19] that all the integral representations are equivalent.

Similarly to the acoustic problems, the main idea is to express the low-frequency coefficients of the total exterior and interior electromagnetic field (near-field data) and the coefficients of the low-frequency expansion of the scattering amplitude (far-field data) in terms of the basic potential functions that we defined during the study of acoustic problems.

Perfect Conductor

Based on (2.253) and (2.260) we have the following sequence of exterior boundary value problems for the low-frequency coefficients:

$$\begin{aligned} \nabla\times\mathbf{E}_n^+(\mathbf{r}) &= nZ^+\mathbf{H}_{n-1}^+(\mathbf{r}) \quad , \quad \nabla\times\mathbf{H}_n^+(\mathbf{r}) = -nY^+\mathbf{E}_{n-1}^+(\mathbf{r}) \ , \quad \mathbf{r}\in V^+ \ , \\ \nabla\cdot\mathbf{E}_n^+(\mathbf{r}) &= 0 \quad , \quad \nabla\cdot\mathbf{H}_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^+ \ , \\ \hat{\mathbf{n}}\times\mathbf{E}_n^+(\mathbf{r}) &= 0 \quad , \quad \hat{\mathbf{n}}\cdot\mathbf{H}_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in S \ , \\ \mathbf{E}_n^+(\mathbf{r}) &= \mathbf{F}_{en}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \quad , \quad \mathbf{H}_n^+(\mathbf{r}) = \mathbf{F}_{mn}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r\to\infty \ . \end{aligned} \tag{2.291}$$

Substituting the boundary conditions into (2.267) and (2.268) leads to the forms:

$$\begin{aligned} \mathbf{F}_{en}^+(\mathbf{r}) = \mathbf{E}_n^i(\mathbf{r}) &+ \frac{Z^+}{4\pi}\sum_{m=1}^{n}\binom{n}{n} m\int_S |\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^+(\mathbf{r}')\right)dS(\mathbf{r}') \\ &- \frac{1}{4\pi}\sum_{m=2}^{n}\binom{n}{m}(m-1)|\mathbf{r}-\mathbf{r}'|^{m-3}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^+(\mathbf{r}')\right)(\mathbf{r}-\mathbf{r}')dS(\mathbf{r}') \ , \end{aligned} \tag{2.292}$$

and

$$\begin{aligned} \mathbf{F}_{mn}^+(\mathbf{r}) &= \mathbf{H}_n^i(\mathbf{r}) \\ &+ \frac{1}{4\pi}\sum_{m=2}^{n}\binom{n}{m}(m-1)\int_S |\mathbf{r}-\mathbf{r}'|^{m-3}(\mathbf{r}-\mathbf{r}')\times\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^+(\mathbf{r}')\right)dS(\mathbf{r}') \ . \end{aligned} \tag{2.293}$$

The potentials $\mathbf{U}_{en}^+$ and $\mathbf{U}_{mn}^+$, via the boundary conditions of the perfect conductor (2.260), take the following forms:

$$\mathbf{U}_{en}^+(\mathbf{r}) = -\frac{1}{4\pi}\nabla\int_S \frac{\hat{\mathbf{n}}'\cdot\mathbf{E}_n^+(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') \ , \tag{2.294}$$

$$\mathbf{U}_{mn}^+(\mathbf{r}) = \frac{1}{4\pi}\nabla\times\int_S \frac{\hat{\mathbf{n}}'\times\mathbf{H}_n^+(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') \ . \tag{2.295}$$

As mentioned previously, the determination of the coefficients $\mathbf{E}_n^+$ and $\mathbf{H}_n^+$ for every step $n = 0, 1, 2, ...$ depends completely on the determination of functions $\mathbf{U}_{en}^+$, $\mathbf{U}_{mn}^+$. Thus, in order to determine these functions, we define the scalar functions ϕ_{en}^+ and ϕ_{mn}^-. Specifically:

• *Scalar function* ϕ_{en}^+ *and coefficients* $\mathbf{U}_{en}^+$:
From the definitions of (2.269) it can be easily observed that it can be written as the gradient of a scalar function which will be shown later that it is a potential function. Thus, the scalar function ϕ_{en}^+ is defined as:

$$\phi_{en}^+(\mathbf{r}) = \frac{1}{4\pi}\int_S \frac{\hat{\mathbf{n}}'\cdot\mathbf{E}_n^+(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dS(\mathbf{r}') \ , \quad \mathbf{r}\in\mathbb{R}^3 \ . \tag{2.296}$$

Due to of the asymptotic form of $|\mathbf{r}-\mathbf{r}'|$:

$$|\mathbf{r}-\mathbf{r}'| = r - \hat{\mathbf{r}}\cdot\mathbf{r}' + \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r\to\infty \ , \tag{2.297}$$

the asymptotic form of $\mathbf{E}_n^+$ (2.274) and relation (2.258), it can be concluded that the asymptotic form of ϕ_{en}^+ is:

$$\phi_{en}^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ . \tag{2.298}$$

It can be seen that ϕ_{en}^+ is a single layer potential of density $\hat{\mathbf{n}} \cdot \mathbf{E}_n^+$ which is a solution of Laplace equation in V^+, and from Divergence theorem we have:

$$\int_S \frac{\partial}{\partial n'} \phi_{en}^+(\mathbf{r}')dS(\mathbf{r}') = 0 \ . \tag{2.299}$$

From the jump relation of single layer potentials we also have:

$$\left[\frac{\partial}{\partial n'}\phi_{e(n-1)}^+(\mathbf{r})\right]^+ - \left[\frac{\partial}{\partial n'}\phi_{e(n-1)}^+(\mathbf{r})\right]^- = -\hat{\mathbf{n}} \cdot \mathbf{E}_{n-1}^+(\mathbf{r}) \ , \tag{2.300}$$

where $[\ \ ]^\pm$ denotes the limits from V^+ and V^- respectively.

Hence, from the definition (2.269) and (2.296), the decomposition of $\mathbf{E}_n^+$ in V^+ takes the following form:

$$\mathbf{E}_n^+(\mathbf{r}) = \mathbf{F}_{en}^+(\mathbf{r}) - \nabla\phi_{en}^+(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ . \tag{2.301}$$

Moreover, from (2.258) and (2.299) we obtain:

$$\int_S \hat{\mathbf{n}}' \cdot \mathbf{F}_{en}^+(\mathbf{r}')dS(\mathbf{r}') = 0 \ . \tag{2.302}$$

Based on the above properties of ϕ_{en}^+ and the decomposition (2.301), for every step n the determination of $\mathbf{E}_n^+$ is reduced to the determination of the unique solution of the potential theory problem:

$$\begin{aligned}
&\Delta\phi_{en}^+(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\
&\hat{\mathbf{n}} \times \nabla\phi_{en}^+(\mathbf{r}) = \hat{\mathbf{n}} \times \mathbf{F}_{en}^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \\
&\phi_{en}^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ .
\end{aligned} \tag{2.303}$$

• *Scalar functionw ϕ_{mn}^- and coefficients $\mathbf{U}_{mn}^+$:*
In contrast to $\mathbf{U}_{en}^+$, the coefficients $\mathbf{U}_{mn}^+$ can not be expressed as the gradient of a scalar potential function. This obstacle can be overcome [19], [31] using the idea of Stevenson of adding and subtracting a known auxiliary function $\mathbf{h}_n^+$, with the difference $\mathbf{U}_{mn}^+ - \mathbf{h}_n^+$ being expressible as the gradient of a scalar potential. Based on this idea, the scalar potential function ϕ_{mn}^- is defined as the solution of the following boundary value problem for the Laplace equation:

$$\begin{aligned}
&\Delta\phi_{mn}^-(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^- \ , \\
&\frac{\partial}{\partial n}\phi_{mn}^-(\mathbf{r}) = -nY^+\hat{\mathbf{n}} \cdot \mathbf{E}_{n-1}^+(\mathbf{r}) \ , \quad \mathbf{r} \in S \ .
\end{aligned} \tag{2.304}$$

From the jump relation (2.300) the boundary condition can be rewritten as:

$$\frac{\partial}{\partial n}\phi_{mn}^-(\mathbf{r}) = nY^+\left(\left[\frac{\partial}{\partial n}\phi_{e(n-1)}^+(\mathbf{r})\right]^+ - \left[\frac{\partial}{\partial n}\phi_{e(n-1)}^+(\mathbf{r})\right]^-\right) \ , \quad \mathbf{r} \in S \ , \tag{2.305}$$

and also based on relation (2.258) we have:

$$\int_S \frac{\partial}{\partial n}\phi_{mn}^-(\mathbf{r})dS(\mathbf{r}) = -nY^+\int_S \hat{\mathbf{n}} \cdot \mathbf{E}_{n-1}^+(\mathbf{r})dS(\mathbf{r}) = 0 \ , \tag{2.306}$$

which leads to the fact that the interior Neumann boundary problem for the Laplace equation (2.304) is satisfied for every constant function.

With the scalar function ϕ^+_{mn} defined, what is left based on Stevenson's idea, is the definition of the auxiliary function $\mathbf{h}^+_n$. Based on [19] , this auxiliary function can be defined as:

$$\mathbf{h}^+_n(\mathbf{r}) = \frac{1}{4\pi}\nabla \times \int_S \frac{\phi^-_{mn}(\mathbf{r}')\hat{\mathbf{n}}'}{|\mathbf{r}-\mathbf{r}'|} dS(\mathbf{r}') \ , \quad \mathbf{r} \in V^+ \ , \tag{2.307}$$

and we have the following Lemma [19] :

Lemma 2.2.1. *There exists a potential function $\phi^+_{mn}(\mathbf{r})$ in V^+ which decays as $\frac{1}{r}$ as $r \to \infty$, such that*

$$\mathbf{U}^+_{mn}(\mathbf{r}) - \mathbf{h}^+_n(\mathbf{r}) = -\nabla\phi^+_{mn}(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ . \tag{2.308}$$

Proof. The proof can be found in ([19] p. 109) □

Thus, the decomposition after the introduction of the auxiliary function $\mathbf{h}^+_n$ and the Lemma 2.2.1 takes the following forms:

$$\mathbf{H}^+_n(\mathbf{r}) = \mathbf{F}^+_{mn}(\mathbf{r}) + \mathbf{h}^+_n(\mathbf{r}) + \mathbf{U}^+_{mn}(\mathbf{r}) - \mathbf{h}^+_n(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.309}$$

or

$$\mathbf{H}^+_n(\mathbf{r}) = \mathbf{F}^+_{mn}(\mathbf{r}) + \mathbf{h}^+_n(\mathbf{r}) - \nabla\phi^+_{mn}(\mathbf{r}) \ , \quad \mathbf{r} \in V^+ \ . \tag{2.310}$$

Taking the inner product $\hat{\mathbf{n}}\cdot$ on both parts of (2.310), integrating over the surface S and using relation (2.258), leads to the following boundary condition for ϕ^+_{mn}:

$$\frac{\partial}{\partial n}\phi^+_{mn}(\mathbf{r}) = \hat{\mathbf{n}} \cdot \mathbf{F}^+_{mn}(\mathbf{r}) + \hat{\mathbf{n}} \cdot \mathbf{h}^+_n(\mathbf{r}) \ , \quad \mathbf{r} \in S \ . \tag{2.311}$$

Thus, the scalar function ϕ^+_{mn} is the solution of the exterior boundary value problem for Laplace equation:

$$\begin{aligned} &\Delta\phi^+_{mn}(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\ &\frac{\partial}{\partial n}\phi^+_{mn}(\mathbf{r}) = \hat{\mathbf{n}} \cdot \mathbf{F}^+_{mn}(\mathbf{r}) + \hat{\mathbf{n}} \cdot \mathbf{h}^+_n(\mathbf{r}) \ , \quad \mathbf{r} \in S \ , \\ &\phi^+_{mn}(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.312}$$

Based on all the above, from the decomposition (2.265), the determination of $\mathbf{E}^+_n$ depends on the solution of the scalar problem (2.303) and based on the decomposition (2.310) and the definition of auxiliary function (2.307), the determination of $\mathbf{H}^+_n$ depends on the solutions of the two scalar problems (2.304) and (2.312).

For $n = 0$ and relations (2.272), (2.273) we have:

$$\mathbf{F}^+_{e0}(\mathbf{r}) = \mathbf{E}^i_0(\mathbf{r}) = \hat{\mathbf{p}} = \nabla(\hat{\mathbf{p}} \cdot \mathbf{r}) = \nabla\left[\hat{\mathbf{p}} \cdot (\mathbf{r} + \mathbf{c})\right] \ , \tag{2.313}$$

$$\mathbf{F}^+_{m0}(\mathbf{r}) = \mathbf{H}^i_0(\mathbf{r}) = Y^+\hat{\mathbf{q}} \ , \tag{2.314}$$

where $\mathbf{c}$ a constant vector. The potential function ϕ^+_{e0} is determined as the unique solution of the following exterior boundary value problem:

$$\begin{aligned} &\Delta\phi^+_{e0}(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in V^+ \ , \\ &\phi^+_{e0}(\mathbf{r}) = \hat{\mathbf{p}} \cdot (\mathbf{r} + \mathbf{c}) \ , \quad \mathbf{r} \in S \ , \\ &\int_S \frac{\partial}{\partial n}\phi^+_{e0}(\mathbf{r}) dS(\mathbf{r}) = \int_S \hat{\mathbf{n}} \cdot \mathbf{F}^+_{e0}(\mathbf{r}) dS(\mathbf{r}) = \int_S \hat{\mathbf{n}} \cdot \hat{\mathbf{p}} dS(\mathbf{r}) = 0 \ , \quad \mathbf{r} \in S \ , \\ &\phi^+_{e0}(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ , \end{aligned} \tag{2.315}$$

where the boundary conditions are derived from relations (2.258), (2.296), (2.301) and (2.303). What's left is to determine ϕ_{e0}^{+} in terms of basic potential functions we defined in the previous section of acoustic problems. Thus, returning to the vector valued potential $\mathbf{\Phi}$ defined in the previous section as the unique solution of the potential theory problem (2.82), it is concluded that $\hat{\mathbf{p}}\cdot\mathbf{\Phi}$ will be a solution of (2.315). Therefore, because of the uniqueness of the solution for (2.315) we have:

$$\phi_{e0}^{+}(\mathbf{r}) = \hat{\mathbf{p}}\cdot\mathbf{\Phi}(\mathbf{r}) \ , \quad \mathbf{r}\in V^{+} \ . \tag{2.316}$$

Thus, the zeroth low-frequency coefficient of the exterior total electric field $\mathbf{E}_0^{+}$, based on relations (2.301), (2.272) and (2.316), is given by:

$$\mathbf{E}_0^{+}(\mathbf{r}) = \hat{\mathbf{p}} - \nabla\left(\hat{\mathbf{p}}\cdot\mathbf{\Phi}\right) \ , \quad \mathbf{r}\in V^{+} \ , \tag{2.317}$$

where $\mathbf{\Phi}$ the vector valued potential defined as the solution of (2.82). For the determination of $\mathbf{H}_0^{+}$, based on the decomposition (2.266) or (2.310) and the relation (2.273), we have that $\mathbf{F}_{m0}^{+} = \mathbf{H}_0^{i} = Y^{+}\hat{\mathbf{q}}$, leaving only the determination of $\mathbf{h}_0^{+}$ and the determination of the scalar function ϕ_{mn}^{+}.

The auxiliary function $\mathbf{h}_0^{+}$, based on its definition (2.307), is depending on the specification of the potential function ϕ_{m0}^{-}. This potential function is the solution of the interior Neumann boundary value problem for the Laplace equation (2.312) for $n=0$. This means that ϕ_{m0}^{-} equals a constant and since $\mathbf{h}_n^{+}$ is not uniquely defined, the constant can be chosen to be equal to 0 which leads to $h_0^{+}=0$. Hence the decomposition (2.310) becomes:

$$\mathbf{H}_0^{+}(\mathbf{r}) = Y^{+}\hat{\mathbf{q}} - \nabla\phi_{m0}^{+}(\mathbf{r}) \ , \quad \mathbf{r}\in V^{+} \ . \tag{2.318}$$

Finally, the scalar potential ϕ_{m0}^{+}, for $n=0$ in (2.312), is defined as the solution of the potential theory problem:

$$\begin{aligned} &\Delta\phi_{m0}^{+}(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^{+} \ , \\ &\frac{\partial}{\partial n}\phi_{m0}^{+}(\mathbf{r}) = \hat{\mathbf{n}}\cdot\mathbf{H}_0^{i}(\mathbf{r}) = Y^{+}\hat{\mathbf{n}}\cdot\hat{\mathbf{q}} \ , \quad \mathbf{r}\in S \ , \\ &\phi_{m0}^{+}(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r\to\infty \ . \end{aligned} \tag{2.319}$$

From Divergence theorem on V^{+}, we have:

$$\int_S \frac{\partial}{\partial n'}\phi_{m0}^{+}(\mathbf{r}')dS(\mathbf{r}') = Y^{+}\int_S \hat{\mathbf{n}}'\cdot\hat{\mathbf{q}}dS(\mathbf{r}') = 0 \ , \tag{2.320}$$

which shows rapid decay of ϕ_{m0}^{+} and leads to the asymptotic form:

$$\phi_{m0}^{+}(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r\to\infty \ . \tag{2.321}$$

Hence, returning to the vector valued potential $\mathbf{\Psi}$ defined in the previous section as the solution of the potential theory problem (2.120), it is concluded that $Y^{+}\hat{\mathbf{q}}\cdot\mathbf{\Psi}$ is the solution of (2.319) with the asymptotic form of (2.321). Due to the asymptotic forms, it is concluded that the scalar potential ϕ_{m0}^{+} is given by:

$$\phi_{m0}^{+}(\mathbf{r}) = Y^{+}\hat{\mathbf{q}}\cdot\mathbf{\Psi}(\mathbf{r}) \ , \quad \mathbf{r}\in V^{+} \ . \tag{2.322}$$

Based on relations (2.318) and (2.322), the zeroth low-frequency coefficient of the exterior magnetic total field $\mathbf{H}_0^{+}$ is of the following form:

$$\mathbf{H}_0^{+}(\mathbf{r}) = Y^{+}\hat{\mathbf{q}} - Y^{+}\nabla\left(\hat{\mathbf{q}}\cdot\mathbf{\Psi}(\mathbf{r})\right) \ , \quad \mathbf{r}\in V^{+} \ . \tag{2.323}$$

With this relation both $\mathbf{E}_0^+$ and $\mathbf{H}_0^+$ are expressed in terms of basic potential functions associated with the electric polarizability tensor $\widetilde{\mathbf{P}}$ and the magnetic polarizability tensor $\widetilde{\mathbf{M}}$. *Far-field data*:
For the scattering amplitude, the idea is again to express both $\mathbf{E}^\infty$ and $\mathbf{H}^\infty$ in terms of basic potential functions defined in the previous section of acoustic problems. Substituting the boundary conditions (2.260) into (2.286) and (2.287), leads to the following forms:

$$\begin{aligned}\mathbf{E}^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(ik)^{n+3}}{n!}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^+(\mathbf{r}')\right.\right.\\ \left.\left.-Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^+(\mathbf{r}')\right)\right]\mathbf{r}'(\hat{\mathbf{r}}\cdot\mathbf{r}')^m dS(\mathbf{r}')\right\}\ ,\end{aligned} \tag{2.324}$$

and from relation (2.290) $\mathbf{H}^\infty = Y^+\hat{\mathbf{r}}\times\mathbf{E}^\infty$. It can be observed that the leading order coefficient of $\mathbf{E}^\infty$ is the A_3. Therefore, for $n = 0$ in (2.324), we have:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{n}}'\cdot\mathbf{E}_0^+(\mathbf{r}') - Z^+\hat{\mathbf{r}}\cdot\left(\hat{\mathbf{n}}'\times\mathbf{H}_0^+(\mathbf{r}')\right)\right]\mathbf{r}'dS(\mathbf{r}')\right\}\ . \tag{2.325}$$

Thus, by substituting the coefficients $\mathbf{E}_0^+$ and $\mathbf{H}_0^+$ from relations (2.317) and (2.323), we obtain:

$$\begin{aligned}A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \hat{\mathbf{r}}\times\left\{\hat{\mathbf{r}}\times\int_S\left[\hat{\mathbf{n}}'\cdot\hat{\mathbf{p}} - \hat{\mathbf{n}}'\cdot\nabla\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Phi}(\mathbf{r}')\right)\right.\right.\\ \left.\left.-Z^+\hat{\mathbf{r}}\cdot\left(Y^+\hat{\mathbf{n}}'\times\hat{\mathbf{q}} - Y^+\hat{\mathbf{n}}'\times\nabla\left(\hat{\mathbf{q}}\cdot\boldsymbol{\Psi}(\mathbf{r}')\right)\right)\right]\mathbf{r}'dS(\mathbf{r}')\right\}\ .\end{aligned} \tag{2.326}$$

which can be simplified even further using the relations:

$$\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{p}})\mathbf{r}'dS(\mathbf{r}') = \hat{\mathbf{p}}|V^-|\ , \tag{2.327}$$

$$\int_S\hat{\mathbf{n}}'\cdot\nabla\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Phi}(\mathbf{r}')\right)\mathbf{r}'dS(\mathbf{r}') = \int_S\frac{\partial}{\partial n'}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Phi}(\mathbf{r}')\right)\mathbf{r}'dS(\mathbf{r}') = -\hat{\mathbf{p}}\cdot\widetilde{\mathbf{Q}}\ . \tag{2.328}$$

(2.329)

Thus, from the definitions of electric polarizability tensor (2.97) and of magnetic polarizability tensor (2.122) , A_3 takes the following form:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{P}}\cdot\hat{\mathbf{p}} - \widetilde{\mathbf{M}}\cdot\hat{\mathbf{q}}\right)\ , \tag{2.330}$$

and together with (2.290), we have the leading order terms of $\mathbf{E}^\infty$ and $\mathbf{H}^\infty$ expressed in terms of the electric and the magnetic polarizabilty tensors.

Impedance Problem

Based on (2.253) and (2.262) we have the following sequence of exterior boundary value problems for the low-frequency coefficients:

$$\begin{aligned}\nabla\times\mathbf{E}_n^+(\mathbf{r}) = nZ^+\mathbf{H}_{n-1}^+(\mathbf{r})\quad &,\quad \nabla\times\mathbf{H}_n^+(\mathbf{r}) = -nY^+\mathbf{E}_{n-1}^+(\mathbf{r})\ ,\quad \mathbf{r}\in V^+\ ,\\ \nabla\cdot\mathbf{E}_n^+(\mathbf{r}) = 0\quad &,\quad \nabla\cdot\mathbf{H}_n^+(\mathbf{r}) = 0\ ,\quad \mathbf{r}\in V^+\ ,\\ \hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\mathbf{E}_n^+(\mathbf{r})\right) &= -Z_sZ^+\hat{\mathbf{n}}\times\mathbf{H}_n^+(\mathbf{r})\ ,\quad \mathbf{r}\in S\ ,\\ \mathbf{E}_n^+(\mathbf{r}) = \mathbf{F}_{en}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right)\quad &,\quad \mathbf{H}_n^+(\mathbf{r}) = \mathbf{F}_{mn}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right)\ ,\quad r\to\infty\ .\end{aligned} \tag{2.331}$$

In the case of the impedance problem the functions $\mathbf{U}_{en}^+$ defined in (2.269) and $\mathbf{U}_{mn}^+$ defined in (2.270) can not be reduced to the solution of scalar potentials. Nevertheless, the zeroth low-frequency coefficients $\mathbf{E}_0^+(\mathbf{r})$ and $\mathbf{H}_0^+(\mathbf{r})$ can be obtained [19], [20]

Transmission Problem

Based on (2.253),(2.257) and (2.263) we have the following sequence of exterior boundary value problems for the low-frequency coefficients:

$$\begin{aligned}
\nabla\times\mathbf{E}_n^+(\mathbf{r}) &= nZ^+\mathbf{H}_{n-1}^+(\mathbf{r}) \quad , \quad \nabla\times\mathbf{H}_n^+(\mathbf{r}) = -nY^+\mathbf{E}_{n-1}^+(\mathbf{r}) \ , \qquad \mathbf{r}\in V^+ \ , \\
\nabla\times\mathbf{E}_n^-(\mathbf{r}) &= n\frac{\mu^-}{\mu^+}Z^+\mathbf{H}_{n-1}^-(\mathbf{r}) \ , \ \nabla\times\mathbf{H}_n^-(\mathbf{r}) = -n\frac{\varepsilon^-}{\varepsilon^+}Y^+\mathbf{E}_{n-1}^-(\mathbf{r}) + \sigma^-\mathbf{E}_n^-(\mathbf{r}) \ , \ \mathbf{r}\in V^- \ , \\
\nabla\cdot\mathbf{E}_n^+(\mathbf{r}) &= 0 \quad , \quad \nabla\cdot\mathbf{H}_n^+(\mathbf{r}) = 0 \ , \qquad \mathbf{r}\in V^+ \ , \\
\nabla\cdot\mathbf{E}_n^-(\mathbf{r}) &= 0 \quad , \quad \nabla\cdot\mathbf{H}_n^-(\mathbf{r}) = 0 \ , \qquad \mathbf{r}\in V^- \ , \\
\hat{\mathbf{n}}\times\mathbf{E}_n^+(\mathbf{r}) &= \hat{\mathbf{n}}\times\mathbf{E}_n^-(\mathbf{r}) \quad , \quad \hat{\mathbf{n}}\times\mathbf{H}_n^+(\mathbf{r}) = \hat{\mathbf{n}}\times\mathbf{H}_n^-(\mathbf{r}) \ , \qquad \mathbf{r}\in S \ , \\
Y^+\hat{\mathbf{n}}\cdot\mathbf{E}_n^+(\mathbf{r}) &= \eta Y^-\hat{\mathbf{n}}\cdot\mathbf{E}_n^-(\mathbf{r}) \quad , \quad Z^+\hat{\mathbf{n}}\cdot\mathbf{H}_n^+(\mathbf{r}) = \eta Z^-\hat{\mathbf{n}}\cdot\mathbf{H}_n^-(\mathbf{r}) \ , \quad \mathbf{r}\in S \ , \\
\mathbf{E}_n^+(\mathbf{r}) &= \mathbf{F}_{en}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \quad , \quad \mathbf{H}_n^+(\mathbf{r}) = \mathbf{F}_{mn}^+(\mathbf{r}) + \mathcal{O}\left(\frac{1}{r}\right) \ , \qquad r\to\infty \ .
\end{aligned} \tag{2.332}$$

The transmission problems for electromagnetic waves, depending on the values of conductivity σ^-, are separated in the two categories of lossless and lossy transmission problems. For $\sigma^- = 0$ we have a lossless transmission problem, while for $\sigma^- > 0$ we have a lossy transmission problem which as $\sigma^- \to \infty$ the problem coincides with the case of the perfect conductor. Moreover, it will be shown later in this subsection, that the zeroth low-frequency coefficient $\mathbf{E}_0^+$ for $\sigma^- \neq 0$, is independent of the conductivity σ^- which makes the coefficient to coincide with that of the perfect conductor. More details about the conductivity and how it affects the low-frequency coefficients of electromagnetic field, can be found in [19] . For the transmission problems, the Stratton-Chu integral representations (2.230)-(2.231) need some rearrangements in order to take a more convenient form before substituting the low-frequency expansions in order to derive the relations between the low-frequency coefficients. Thus, by substituting the transmission conditions (2.263) into the Straton-Chu integral representations of $\mathbf{E}^+$ and $\mathbf{H}^+$ (2.230),(2.231) and adding and subtracting terms ([19], p. 146) the integral representations take the following forms:

$$\begin{aligned}
\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r}) &= \mathbf{E}^i(\mathbf{r}) + \frac{ik}{4\pi}\int_S \left[ik^-Z^-G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right) + \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)\right. \\
&\left. - \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\times\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right)\right] dS(\mathbf{r}') + \frac{ik}{4\pi}\left(ikZ^+ - ik^-Z^-\right)\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right) dS(\mathbf{r}') \\
&+ \frac{ik}{4\pi}\left(\eta Y^-Z^+ - 1\right)\int_S \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right) dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.333}$$

and

$$\begin{aligned}
\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r}) &= \mathbf{H}^i(\mathbf{r}) + \frac{ik}{4\pi}\int_S \left[ik^-Y^-G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right) + \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\hat{\mathbf{n}}'\cdot\mathbf{H}^-(\mathbf{r}')\right)\right. \\
&\left. - \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\times\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right)\right] dS(\mathbf{r}') + \frac{ik}{4\pi}\left(ik^-Y^- - ikY^+\right)\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right) dS(\mathbf{r}') \\
&+ \frac{ik}{4\pi}\left(\eta Z^-Y^+ - 1\right)\int_S \left(\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\right)\left(\hat{\mathbf{n}}'\cdot\mathbf{H}^-(\mathbf{r}')\right) dS(\mathbf{r}') \ ,
\end{aligned} \tag{2.334}$$

Using the vector Helmholtz equations satisfied by $\mathbf{E}^-$ and $\mathbf{H}^-$ (2.209) together with $\nabla\cdot\mathbf{E}^+=0$ and $\nabla\cdot\mathbf{H}^+=0$ leads to the following relation ([19] p. 146):

$$\begin{aligned}&\frac{ik}{4\pi}\int_S\big[G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times(\nabla_{\mathbf{r}'}\times\mathbf{u}(\mathbf{r}'))\right)+\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{u}(\mathbf{r}')\right)\\&-\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\times\left(\hat{\mathbf{n}}'\times\mathbf{u}(\mathbf{r}')\right)\big]\,dS(\mathbf{r}')=(\alpha(\mathbf{r})-1)\mathbf{u}(\mathbf{r})+\frac{(ik)^3}{4\pi}(1-\eta^2)\int_{V^-}\mathbf{u}(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\ ,\end{aligned}\tag{2.335}$$

where $\mathbf{u}(\mathbf{r})=\mathbf{E}^+,\ \mathbf{H}^+$. Thus, by subtracting (2.335) to (2.333) and (2.334) leads to the forms:

$$\begin{aligned}&\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r})+(1-\alpha(\mathbf{r}))\mathbf{E}^-(\mathbf{r})=\mathbf{E}^i(\mathbf{r})+\frac{(ik)^3}{4\pi}(1-\eta^2)\int_{V^-}\mathbf{E}^-(\mathbf{r})G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\\&+\frac{ik}{4\pi}\left(ikZ^+-ik^-Z^-\right)\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\&+\frac{ik}{4\pi}\left(\eta Y^-Z^+-1\right)\int_S\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned}\tag{2.336}$$

and

$$\begin{aligned}&\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r})+(1-\alpha(\mathbf{r}))\mathbf{H}^-(\mathbf{r})=\mathbf{H}^i(\mathbf{r})+\frac{(ik)^3}{4\pi}(1-\eta^2)\int_{V^-}\mathbf{H}^-(\mathbf{r})G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\\&+\frac{ik}{4\pi}\left(ik^-Y^--ikY^+\right)\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\&+\frac{ik}{4\pi}\left(\eta Z^-Y^+-1\right)\int_S\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ .\end{aligned}\tag{2.337}$$

These forms are proved more convenient for the further substitution of the low-frequency expansions and the derivation of the relations for the low-frequency coefficients $\mathbf{E}_n^\pm$ and $\mathbf{H}_n^\pm$.

• **Lossless transmission problem:**
For the lossless case, we have $\sigma^-=0$ which leads to:

$$\eta^2=\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\ ,\quad \eta Y^-=\frac{\varepsilon^-}{\varepsilon^+}Y^+\ ,\quad \eta Z^-=\frac{\mu^-}{\mu^+}Z^+\ .\tag{2.338}$$

Hence, the transmission conditions for the low-frequency coefficients (2.263) become:

$$\begin{aligned}\hat{\mathbf{n}}\times\mathbf{E}_n^+(\mathbf{r})=\hat{\mathbf{n}}\times\mathbf{E}_n^-(\mathbf{r})\quad&,\quad\hat{\mathbf{n}}\times\mathbf{H}_n^+(\mathbf{r})=\hat{\mathbf{n}}\times\mathbf{H}_n^-(\mathbf{r})\ ,\\ \varepsilon^+\hat{\mathbf{n}}\cdot\mathbf{E}_n^+(\mathbf{r})=\varepsilon^-\hat{\mathbf{n}}\cdot\mathbf{E}_n^-(\mathbf{r})\quad&,\quad\mu^+\hat{\mathbf{n}}\cdot\mathbf{H}_n^+(\mathbf{r})=\mu^-\hat{\mathbf{n}}\cdot\mathbf{H}_n^-(\mathbf{r})\ ,\end{aligned}\qquad \mathbf{r}\in S\ .\tag{2.339}$$

Thus by substituting these quantities to (2.336) and (2.337) as well as (2.338) and $k^-=\eta k$, we obtain:

$$\begin{aligned}&\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r})+(1-\alpha(\mathbf{r}))\mathbf{E}^-(\mathbf{r})=\mathbf{E}^i(\mathbf{r})+\frac{(ik)^3}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\int_{V^-}\mathbf{E}^-(\mathbf{r})G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')\\&+\frac{(ik)^2}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)Z^+\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\&-\frac{ik}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S\nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned}\tag{2.340}$$

and

$$\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{H}^-(\mathbf{r}) = \mathbf{H}^i(\mathbf{r}) + \frac{(ik)^3}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\int_{V^-}\mathbf{H}^-(\mathbf{r})G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')$$
$$-\frac{(ik)^2}{4\pi}Y^+\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')$$
$$-\frac{ik}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\int_S \nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ . \tag{2.341}$$

Substituting the low-frequency expansions (2.246), (2.247) and the expansion of $G^+(\mathbf{r},\mathbf{r}')$ (2.47), using the Cauchy formula (2.49) to rearrange terms and then equating the coefficients of $(ik)^n$, gives the following relations for the low-frequency coefficients:

$$\begin{aligned}&\alpha(\mathbf{r})\mathbf{E}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{E}_n^-(\mathbf{r}) = \mathbf{E}_n^i(\mathbf{r})\\
&+\frac{1}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\sum_{m=2}^{n}\binom{n}{m}m(m-1)\int_{V^-}|\mathbf{r}-\mathbf{r}'|^{m-3}\mathbf{E}_{n-m}^-(\mathbf{r}')dV(\mathbf{r}')\\
&+\frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)Z^+\sum_{m=1}^{n}\binom{n}{m}m\int_S|\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\
&-\frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\sum_{m=0}^{n}\binom{n}{m}\int_S\nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.342}$$

and

$$\begin{aligned}&\alpha(\mathbf{r})\mathbf{H}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{H}_n^-(\mathbf{r}) = \mathbf{H}_n^i(\mathbf{r})\\
&+\frac{1}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\sum_{m=2}^{n}\binom{n}{m}m(m-1)\int_{V^-}|\mathbf{r}-\mathbf{r}'|^{m-3}\mathbf{H}_{n-m}^-(\mathbf{r}')dV(\mathbf{r}')\\
&-\frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)Y^+\sum_{m=1}^{n}\binom{n}{m}m\int_S|\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\
&-\frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\sum_{m=0}^{n}\binom{n}{m}\int_S\nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{H}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.343}$$

From these two forms the following decompositions are derived:

$$\alpha(\mathbf{r})\mathbf{E}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{E}_n^-(\mathbf{r}) = \mathbf{F}_{en}(\mathbf{r}) + \nabla\phi_n(\mathbf{r})\ , \tag{2.344}$$
$$\alpha(\mathbf{r})\mathbf{H}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{H}_n^-(\mathbf{r}) = \mathbf{F}_{mn}(\mathbf{r}) + \nabla\psi_n(\mathbf{r})\ , \tag{2.345}$$

where

$$\begin{aligned}\mathbf{F}_{en} &= \mathbf{E}_n^i(\mathbf{r}) + \frac{1}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\sum_{m=2}^{n}\binom{n}{m}m(m-1)\int_{V^-}|\mathbf{r}-\mathbf{r}'|^{m-3}\mathbf{E}_{n-m}^-(\mathbf{r}')dV(\mathbf{r}')\\
&+\frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)Z^+\sum_{m=1}^{n}\binom{n}{m}m\int_S|\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{H}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\
&-\frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\sum_{m=1}^{n}\binom{n}{m}\int_S\nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.346}$$

$$\begin{aligned}\mathbf{F}_{mn} = \mathbf{H}_n^i(\mathbf{r}) + \frac{1}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\sum_{m=2}^{n}\binom{n}{m}m(m-1)\int_{V^-}|\mathbf{r}-\mathbf{r}'|^{m-3}\mathbf{H}_{n-m}^-(\mathbf{r}')dV(\mathbf{r}')\\ -\frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)Y^+\sum_{m=1}^{n}\binom{n}{m}m\int_S|\mathbf{r}-\mathbf{r}'|^{m-2}\left(\hat{\mathbf{n}}'\times\mathbf{E}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\ -\frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\sum_{m=1}^{n}\binom{n}{m}\int_S\nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{H}_{n-m}^-(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.347}$$

$$\phi_n(\mathbf{r}) = \frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S|\mathbf{r}-\mathbf{r}'|^{-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}_n^-(\mathbf{r}')\right)dS(\mathbf{r}')\ , \tag{2.348}$$

and

$$\psi_n(\mathbf{r}) = \frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\int_S|\mathbf{r}-\mathbf{r}'|^{-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{H}_n^-(\mathbf{r}')\right)dS(\mathbf{r}')\ . \tag{2.349}$$

The functions $\mathbf{F}_{en}$, $\mathbf{F}_{mn}$ are continuous in $\mathbb{R}^3$ but not continuous differentiable on S and based on the convention (2.271) we have:

$$\mathbf{F}_{e0}(\mathbf{r}) = \mathbf{E}_0^i(\mathbf{r}) = \hat{\mathbf{p}}\ , \tag{2.350}$$

$$\mathbf{F}_{m0}(\mathbf{r}) = \mathbf{H}_0^i(\mathbf{r}) = Y^+\hat{\mathbf{q}}\ . \tag{2.351}$$

The main idea behind this decomposition is to separate (2.342) and (2.343) into a part which on every step $n = 0, 1, ...$ depends on the low-frequency coefficients of the previous steps (meaning the terms $\mathbf{F}_{en}$ and $\mathbf{F}_{mn}$) and one term depending on the low-frequency coefficients of the step n (meaning the terms ϕ_n and ψ_n). It can also be observed that ϕ_n and ψ_n are single layer potentials, meaning that they satisfy the Laplace equation and with relation (2.258), they have the following asymptotic forms:

$$\phi_n(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad r\to\infty\ , \tag{2.352}$$

$$\psi_n(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad r\to\infty\ . \tag{2.353}$$

Based on the values of $\alpha(\mathbf{r})$ in $V^\pm$ and S, corresponding decompositions for $\mathbf{E}_n^\pm$ and $\mathbf{H}_n^\pm$ can be derived from (2.342) and (2.343):

$$\mathbf{E}_n^\pm(\mathbf{r}) = \mathbf{F}_{en}^\pm(\mathbf{r}) + \nabla\phi_n^\pm(\mathbf{r})\ , \quad \mathbf{r}\in V^\pm\ , \tag{2.354}$$

$$\mathbf{H}_n^\pm(\mathbf{r}) = \mathbf{F}_{mn}^\pm(\mathbf{r}) + \nabla\psi_n^\pm(\mathbf{r})\ , \quad \mathbf{r}\in V^\pm\ . \tag{2.355}$$

Thus, the determination of the low-frequency coefficients of total electromagnetic field is depending on the determination of the scalar potentials $\phi_n^\pm$ and $\psi_n^\pm$ which based on the above, are the solutions of the following transmission problems:

$$\begin{aligned}&\Delta\phi_n^+(\mathbf{r}) = 0\ , \quad \mathbf{r}\in V^+\ ,\\ &\Delta\phi_n^-(\mathbf{r}) = 0\ , \quad \mathbf{r}\in V^-\ ,\\ &\hat{\mathbf{n}}\times\nabla\left(\phi_n^+(\mathbf{r})-\phi_n^-(\mathbf{r})\right) = \mathbf{0}\ , \quad \mathbf{r}\in S\ ,\\ &\frac{\partial}{\partial n}\phi_n^+(\mathbf{r}) - \frac{\varepsilon^-}{\varepsilon^+}\frac{\partial}{\partial n}\phi_n^-(\mathbf{r}) = -\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{n}}\cdot\mathbf{F}_{en}(\mathbf{r})\ , \quad \mathbf{r}\in S\ ,\\ &\phi_n(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad r\to\infty\ ,\end{aligned} \tag{2.356}$$

and

$$\begin{aligned}
&\Delta\psi_n^+(\mathbf{r}) = 0\ , \quad \mathbf{r} \in V^+\ , \\
&\Delta\psi_n^-(\mathbf{r}) = 0\ , \quad \mathbf{r} \in V^-\ , \\
&\hat{\mathbf{n}} \times \nabla\left(\psi_n^+(\mathbf{r}) - \psi_n^-(\mathbf{r})\right) = \mathbf{0}\ , \quad \mathbf{r} \in S\ , \\
&\frac{\partial}{\partial n}\psi_n^+(\mathbf{r}) - \frac{\mu^-}{\mu^+}\frac{\partial}{\partial n}\psi_n^-(\mathbf{r}) = -\left(1 - \frac{\mu^-}{\mu^+}\right)\hat{\mathbf{n}} \cdot \mathbf{F}_{mn}(\mathbf{r})\ , \quad \mathbf{r} \in S\ , \\
&\psi_n(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad r \to \infty\ ,
\end{aligned} \tag{2.357}$$

where $\mathbf{F}_{en} \equiv \mathbf{F}_{en}^\pm$ and $\mathbf{F}_{mn} \equiv \mathbf{F}_{mn}^\pm$ since $\mathbf{F}_{mn}$, $\mathbf{F}_{en}$ are continuous in $\mathbb{R}^3$. The conditions on the boundary S are derived from the decompositions (2.354)-(2.355) and the transmission conditions (2.339).

For $n = 0$ the potential theory problems (2.356) and (2.357), based on (2.272) and (2.273), have the following boundary conditions:

$$\begin{aligned}
&\hat{\mathbf{n}} \times \nabla\left(\phi_0^+(\mathbf{r}) - \phi_0^-(\mathbf{r})\right) = \mathbf{0}\ , \quad \mathbf{r} \in S\ , \\
&\frac{\partial}{\partial n}\phi_0^+(\mathbf{r}) - \frac{\varepsilon^-}{\varepsilon^+}\frac{\partial}{\partial n}\phi_0^-(\mathbf{r}) = -\left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{p}} \cdot \hat{\mathbf{n}}\ , \quad \mathbf{r} \in S\ ,
\end{aligned} \tag{2.358}$$

and

$$\begin{aligned}
&\hat{\mathbf{n}} \times \nabla\left(\psi_0^+(\mathbf{r}) - \psi_0^-(\mathbf{r})\right) = \mathbf{0}\ , \quad \mathbf{r} \in S\ , \\
&\frac{\partial}{\partial n}\psi_0^+(\mathbf{r}) - \frac{\mu^-}{\mu^+}\frac{\partial}{\partial n}\psi_0^-(\mathbf{r}) = -\left(1 - \frac{\mu^-}{\mu^+}\right)Y^+\left(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}\right)\ , \quad \mathbf{r} \in S\ ,
\end{aligned} \tag{2.359}$$

Thus, returning to the general polarizability potential $\mathbf{\Psi}^\pm$ defined in the previous section as the solution of (2.169), it is straightforward that $-\left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{p}} \cdot \mathbf{\Psi}^\pm$ and $-Y^+\left(1 - \frac{\mu^-}{\mu^+}\right)\hat{\mathbf{q}} \cdot \mathbf{\Psi}^\pm$ for $\beta_\varepsilon = \frac{\varepsilon^-}{\varepsilon^+}$ and $\beta_\mu = \frac{\mu^-}{\mu^+}$, are solutions of (2.356) and (2.357) respectively for $n = 0$ (it can be observed from (2.358) and (2.359)). Hence, we have:

$$\phi_0^\pm(\mathbf{r}) = -\left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{p}} \cdot \mathbf{\Psi}^\pm\left(\mathbf{r}; \frac{\varepsilon^-}{\varepsilon^+}\right)\ , \tag{2.360}$$

$$\psi_0^\pm(\mathbf{r}) = -Y^+\left(1 - \frac{\mu^-}{\mu^+}\right)\hat{\mathbf{q}} \cdot \mathbf{\Psi}^\pm\left(\mathbf{r}; \frac{\mu^-}{\mu^+}\right)\ . \tag{2.361}$$

Therefore, based on relations (2.272), (2.273) and the decompositions (2.354)-(2.355) we have:

$$\mathbf{E}_0^\pm(\mathbf{r}) = \hat{\mathbf{p}} - \left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right)\nabla\left[\hat{\mathbf{p}} \cdot \mathbf{\Psi}^\pm\left(\mathbf{r}; \frac{\varepsilon^-}{\varepsilon^+}\right)\right]\ , \tag{2.362}$$

$$\mathbf{H}_0^\pm(\mathbf{r}) = Y^+\left\{\hat{\mathbf{q}} - \left(1 - \frac{\mu^-}{\mu^+}\right)\nabla\left[\hat{\mathbf{q}} \cdot \mathbf{\Psi}^\pm\left(\mathbf{r}; \frac{\mu^-}{\mu^+}\right)\right]\right\}\ . \tag{2.363}$$

Far-field data:
Substituting the transmission conditions (2.339) into the low-frequency expansion of the scattering amplitudes $\mathbf{E}^\infty$ (2.286) and $\mathbf{H}^\infty$ (2.287), we have:

$$\begin{aligned}
\mathbf{E}^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi}\hat{\mathbf{r}} \times \Bigg[&\frac{\varepsilon^-}{\varepsilon^+}\hat{\mathbf{r}} \times \int_S \left(\hat{\mathbf{n}}' \cdot \mathbf{E}_0^-(\mathbf{r}')\right)\mathbf{r}' dS(\mathbf{r}') - Z^+\hat{\mathbf{r}} \times \int_S \hat{\mathbf{r}} \cdot \left(\hat{\mathbf{n}}' \times \mathbf{H}_0^-(\mathbf{r}')\right)\mathbf{r}' dS(\mathbf{r}') \\
&+ Z^+\frac{\mu^-}{\mu^+}\int_S \left(\hat{\mathbf{n}}' \times \mathbf{H}_0^-(\mathbf{r}')\right)\mathbf{r}' dS(\mathbf{r}') + \int_S \hat{\mathbf{r}} \cdot \left(\hat{\mathbf{n}}' \times \mathbf{E}_0^-(\mathbf{r}')\right)\mathbf{r}' dS(\mathbf{r}')\Bigg] + \mathcal{O}(k^4)
\end{aligned} \tag{2.364}$$

Thus, by substituting the low-frequency coefficients $\mathbf{E}_0^-$ from (2.362) and $\mathbf{H}_0^-$ from (2.363), the leading order coefficient A_3 becomes:

$$\begin{aligned} A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\Bigg\{ & \hat{\mathbf{r}}\times\int_S\left[\frac{\varepsilon^-}{\varepsilon^+}\hat{\mathbf{n}}'\cdot\hat{\mathbf{p}}-\hat{\mathbf{r}}\cdot(\hat{\mathbf{n}}'\times\hat{\mathbf{q}})\right]\mathbf{r}'dS(\mathbf{r}')+\int_S\left[\frac{\mu^-}{\mu^+}\hat{\mathbf{n}}'\cdot\hat{\mathbf{q}}+\hat{\mathbf{r}}\cdot(\hat{\mathbf{n}}'\times\hat{\mathbf{p}})\right]\mathbf{r}'dS(\mathbf{r}') \\ & -\frac{\varepsilon^-}{\varepsilon^+}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{r}}\times\int_S\hat{\mathbf{n}}'\cdot\nabla_{\mathbf{r}'}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\mathbf{r}'dS(\mathbf{r}') \\ & +\left(1-\frac{\mu^-}{\mu^+}\right)\hat{\mathbf{r}}\times\int_S\hat{\mathbf{r}}\cdot\left[\hat{\mathbf{n}}'\times\nabla_{\mathbf{r}'}\left(\hat{\mathbf{q}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\mu^-}{\mu^+}\right)\right)\right]\mathbf{r}'dS(\mathbf{r}') \\ & -\frac{\mu^-}{\mu^+}\left(1-\frac{\mu^-}{\mu^+}\right)\int_S\hat{\mathbf{n}}'\cdot\nabla_{\mathbf{r}'}\left(\hat{\mathbf{q}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\mu^-}{\mu^+}\right)\right)\mathbf{r}'dS(\mathbf{r}') \\ & -\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S\hat{\mathbf{r}}\cdot\left[\hat{\mathbf{n}}'\times\nabla_{\mathbf{r}'}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\right]\mathbf{r}'dS(\mathbf{r}')\Bigg\}\ , \end{aligned} \tag{2.365}$$

or equivalently:

which can be simplified even further. Starting with the terms that contain $\hat{\mathbf{p}}$, we have the following relations:

$$\int_S(\hat{\mathbf{n}}'\cdot\hat{\mathbf{p}})\,\mathbf{r}'dS(\mathbf{r}')=\hat{\mathbf{p}}|V^-|\ , \tag{2.366}$$

$$\int_S\mathbf{r}'\cdot(\hat{\mathbf{n}}'\times\hat{\mathbf{p}})\,\mathbf{r}'dS(\mathbf{r}')=-\hat{\mathbf{r}}\times\hat{\mathbf{p}}|V^-|\ , \tag{2.367}$$

$$\begin{aligned} &\int_S\hat{\mathbf{n}}'\cdot\nabla_{\mathbf{r}'}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\mathbf{r}'dS(\mathbf{r}')=\int_S\left(\hat{\mathbf{p}}\cdot\frac{\partial}{\partial n'}\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\mathbf{r}'dS(\mathbf{r}') \\ &=\hat{\mathbf{p}}\cdot\int_S\frac{\partial}{\partial n'}\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\mathbf{r}'dS(\mathbf{r}')=\hat{\mathbf{p}}\cdot\int_S\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\frac{\partial}{\partial n'}\mathbf{r}'dS(\mathbf{r}') \\ &=\hat{\mathbf{p}}\cdot\int_S\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{n}}'dS(\mathbf{r}')=\int_S\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{n}}'\right)dS(\mathbf{r}')\ , \end{aligned} \tag{2.368}$$

and from identity ([19], p. 190) we have also the following relation:

$$\int_S\hat{\mathbf{r}}\cdot\left[\hat{\mathbf{n}}'\times\nabla_{\mathbf{r}'}\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\right]\mathbf{r}'dS(\mathbf{r}')=\int_S\left(\hat{\mathbf{p}}\cdot\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)\hat{\mathbf{n}}'\right)dS(\mathbf{r}')\ . \tag{2.369}$$

Using the above relations the same way for the terms with $\hat{\mathbf{q}}$, the coefficient A_3 can be rewritten as:

$$\begin{aligned} A_3(\hat{\mathbf{r}};\hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi}\hat{\mathbf{r}}\times\Bigg\{ & \hat{\mathbf{r}}\times\left[-\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)|V^-|\widetilde{\mathbf{I}}+\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)^2\int_S\hat{\mathbf{n}}'\otimes\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\varepsilon^-}{\varepsilon^+}\right)dS(\mathbf{r}')\right]\cdot\hat{\mathbf{p}} \\ & +\left[-\left(1-\frac{\mu^-}{\mu^+}\right)|V^-|\widetilde{\mathbf{I}}+\left(1-\frac{\mu^-}{\mu^+}\right)^2\int_S\hat{\mathbf{n}}'\otimes\boldsymbol{\Psi}^-\left(\mathbf{r}';\frac{\mu^-}{\mu^+}\right)dS(\mathbf{r}')\right]\cdot\hat{\mathbf{q}}\Bigg\} \end{aligned} \tag{2.370}$$

Finally, from the following identity for the general polarizability tensor:

$$\widetilde{\mathbf{X}}(\beta)=(1-\beta^2)\int_S\hat{\mathbf{n}}\boldsymbol{\Psi}^+(\mathbf{r};\beta)dS(\mathbf{r})-(1-\beta)|V^-|\widetilde{\mathbf{I}}\ , \tag{2.371}$$

the leading order coefficient A_3 becomes:

$$A_3(\hat{\mathbf{r}};\hat{\mathbf{d}})=\frac{(ik)^3}{4\pi}\left[\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{X}}\left(\frac{\varepsilon^-}{\varepsilon^+}\right)\right)\cdot\hat{\mathbf{p}}+\left(\hat{\mathbf{r}}\times\widetilde{\mathbf{X}}\left(\frac{\mu^-}{\mu^+}\right)\right)\cdot\hat{\mathbf{q}}\right]\ , \tag{2.372}$$

The leading order coefficient of $\mathbf{H}^{\infty}$ can be obtained in terms of the general polarizability tensor $\widetilde{\mathbf{X}}$ from relation (2.290).

• **Lossy transmission problem:**
For the lossy transmission case the conductivity σ^- is positive and as it tends to infinity, the problem coincides to the case of the perfect conductor in which we have infinite conductivity. In contrast to the acoustic case, where the cases of lossless and lossy transmission problems were not studied separately, for electromagnetic waves a different approach is needed for the lossy case than what was followed for the lossless case.

Near-field data:
Substituting in the integral representations (2.336) and (2.337) the relations that connect Z^+ and Z^- as well as η^2 and Y^- (2.206)-(2.205) in order to show the dependence on the conductivity σ^- (which shows the clearly the effect of the losses) and the wave number $k \equiv k^+$, the integral representations can be rewritten as:

$$\alpha(\mathbf{r})\mathbf{E}^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{E}^-(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + \frac{(ik)^3}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\int_{V^-}\mathbf{E}^-(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')$$
$$+\frac{(ik)^2}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)Z^+\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')$$
$$-\frac{ik}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S \nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')$$
$$+\sigma^- Z^+\left[\frac{(ik)^2}{4\pi}\frac{\mu^-}{\mu^+}\int_{V^-}\mathbf{E}^-(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}') - \frac{1}{4\pi}\int_S \nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\right], \tag{2.373}$$

and

$$\alpha(\mathbf{r})\mathbf{H}^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{H}^-(\mathbf{r}) = \mathbf{H}^i(\mathbf{r}) + \frac{(ik)^3}{4\pi}\left(1-\frac{\mu^-\varepsilon^-}{\mu^+\varepsilon^+}\right)\int_{V^-}\mathbf{H}^-(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}')$$
$$-\frac{(ik)^2}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)Y^+\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')$$
$$-\frac{ik}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\int_S \nabla_{\mathbf{r}'}G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{H}^-(\mathbf{r}')\right)dS(\mathbf{r}')$$
$$+\sigma^-\left[\frac{(ik)^2}{4\pi}\frac{\mu^-}{\mu^+}Z^+\int_{V^-}\mathbf{H}^-(\mathbf{r}')G^+(\mathbf{r},\mathbf{r}')dV(\mathbf{r}') - \frac{ik}{4\pi}\int_S G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\times\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\right]. \tag{2.374}$$

Based on the Maxwell equations satisfied by the low-frequency coefficients (2.256), we have the following relations for $n=0$:

$$\nabla\times\mathbf{E}_0^-(\mathbf{r}) = \mathbf{0}\,, \quad \mathbf{r}\in V^-\,, \tag{2.375}$$

which is derived from the convention (2.271) and shows that $\mathbf{E}_0^-$ is irrotational and therefore there is a scalar functions f such that $\mathbf{E}_0^- = \nabla f$. From the transmission conditions for the low-frequency coefficients (2.264) , it is derived that:

$$\hat{\mathbf{n}}\cdot\mathbf{E}_0^-(\mathbf{r}) = \hat{\mathbf{n}}\cdot\nabla f(\mathbf{r}) = 0\,, \quad \mathbf{r}\in S\,. \tag{2.376}$$

Applying the divergence theorem in V^-, shows that f is a potential function in V^-. Thus, having an interior Neumann boundary value problem for the Laplace equation satisfied by f, leads to the conclusion that f equals to a constant in V^-, Therefore, the coefficient $\mathbf{E}_0^-$ vanishes in V^- as the gradient of a constant function. From the transmission conditions (2.264) we have:

$$\hat{\mathbf{n}}\times\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{n}}\times\mathbf{E}_0^-(\mathbf{r}) = \mathbf{0}\,, \quad \mathbf{r}\in S\,, \tag{2.377}$$

which leads to the fact that $\mathbf{E}_0^+$ in V^+ for the lossy case ($\sigma^- > 0$) coincides with that of the perfect conductor. The vanishing of $\mathbf{E}_0^-$ can also be used to derive a convenient decomposition similar to the lossless case. Specifically, we have:

$$\begin{aligned}&\int_S \nabla_{\mathbf{r}'} G^+(\mathbf{r},\mathbf{r}')\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-(\mathbf{r}')\right)dS(\mathbf{r}')\\ &=\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{n+1-m}\int_S \nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-_{n+1-m}(\mathbf{r}')\right)dS(\mathbf{r}')\ .\end{aligned} \tag{2.378}$$

Substituting the above into the integral representations (2.373) and (2.374) leads to the following decompositions:

$$\alpha(\mathbf{r})\mathbf{E}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{E}_n^-(\mathbf{r}) = \mathbf{F}_{en}(\mathbf{r}) + \sigma^-\mathbf{G}_{en}(\mathbf{r}) + \nabla\phi_n(\mathbf{r})\ , \tag{2.379}$$
$$\alpha(\mathbf{r})\mathbf{H}_n^+(\mathbf{r}) + (1-\alpha(\mathbf{r}))\mathbf{H}_n^-(\mathbf{r}) = \mathbf{F}_{mn}(\mathbf{r}) + \sigma^-\mathbf{G}_{mn}(\mathbf{r}) + \nabla\psi_n(\mathbf{r})\ , \tag{2.380}$$

where $\mathbf{F}_{en}$ and $\mathbf{F}_{mn}$ are defined the same as (2.346) and (2.347) respectively and

$$\begin{aligned}\mathbf{G}_{en}(\mathbf{r}) &= \frac{1}{4\pi}\frac{\mu^-}{\mu^+}Z^+\sum_{m=1}^{n-1}\binom{n}{m}m\int_{V^-}|\mathbf{r}-\mathbf{r}'|^{m-2}\mathbf{E}^-_{n-m}(\mathbf{r}')dV(\mathbf{r}')\\ &-\frac{1}{4\pi}Z^+\sum_{m=1}^{n}\binom{n}{m}\frac{1}{n+1-m}\int_S \nabla_{\mathbf{r}'}|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\cdot\mathbf{E}^-_{n+1-m}(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.381}$$

$$\begin{aligned}\mathbf{G}_{mn}(\mathbf{r}) &= \frac{1}{4\pi}\frac{\mu^-}{\mu^+}Z^+\sum_{m=1}^{n}\binom{n}{m}m\int_V^-|\mathbf{r}-\mathbf{r}'|^{m-2}\mathbf{H}_{n-m}(\mathbf{r}')dV(\mathbf{r}')\\ &-\frac{1}{4\pi}\sum_{m=0}^{n-1}\binom{n}{m}\int_S|\mathbf{r}-\mathbf{r}'|^{m-1}\left(\hat{\mathbf{n}}'\times\mathbf{E}^-_{n-m}(\mathbf{r}')\right)dS(\mathbf{r}')\ ,\end{aligned} \tag{2.382}$$

$$\phi_n(\mathbf{r}) = \frac{1}{4\pi}\left(1-\frac{\varepsilon^-}{\varepsilon^+}\right)\int_S|\mathbf{r}-\mathbf{r}'|^{-1}\hat{\mathbf{n}}'\cdot\mathbf{E}_n^-(\mathbf{r}')dS(\mathbf{r}') - \frac{\sigma^-Z^+}{4\pi(n+1)}\int_S|\mathbf{r}-\mathbf{r}'|^{-1}\hat{\mathbf{n}}'\cdot\mathbf{E}^-_{n+1}(\mathbf{r}')dS(\mathbf{r}') \tag{2.383}$$

and

$$\psi_n(\mathbf{r}) = \frac{1}{4\pi}\left(1-\frac{\mu^-}{\mu^+}\right)\int_S|\mathbf{r}-\mathbf{r}'|^{-1}\hat{\mathbf{n}}'\cdot\mathbf{H}_n^-(\mathbf{r}')dS(\mathbf{r}')\ . \tag{2.384}$$

It is noted that $\mathbf{F}_{en}$ is continuous in $\mathbb{R}^3$ and $\mathbf{G}_{en}$ is continuous differentiable in $\mathbb{R}^3$ which together with the values of $\alpha(\mathbf{r})$ in $V^\pm$, leads to the following forms of the decompositions (2.379) and (2.380):

$$\mathbf{E}_n^\pm(\mathbf{r}) = \mathbf{F}_{en}^\pm(\mathbf{r}) + \sigma^-\mathbf{G}_{en}^\pm + \nabla\phi_n^\pm\ , \quad \mathbf{r}\in V^\pm \tag{2.385}$$
$$\mathbf{H}_n^\pm(\mathbf{r}) = \mathbf{F}_{mn}^\pm(\mathbf{r}) + \sigma^-\mathbf{G}_{mn}^\pm + \nabla\psi_n^\pm \quad \mathbf{r}\in V^\pm. \tag{2.386}$$

For every step $n = 0, 1, 2, \dots$ in the sequence of transmission problems for the low-frequency coefficients, the functions $\mathbf{F}_{en}^\pm$ and $\mathbf{F}_{mn}^\pm$ are dependent on terms of the previous steps up to $n-1$. Moreover, it will be shown that $\mathbf{G}_{en}^\pm$ and $\mathbf{G}_{mn}^\pm$ are functions also considered known on every step n. Thus, the determination of the low-frequency coefficients of the total exterior and total interior electromagnetic field $\mathbf{E}_n^\pm$, $\mathbf{H}_n^\pm$ is reduced to the determination of the scalar potential functions $\phi_n^\pm$ and $\psi_n^\pm$. Starting with the electric field, based on the convention (2.271), we have that $\mathbf{G}_{e0} = 0$ and for $n = 1$ it can be observed that the second term of $\mathbf{G}_{e1}$ vanishes which means that on every step n it depends only on the low-frequency coefficients $\mathbf{E}_0^-, \dots, \mathbf{E}_{n-1}^-$

and not on $\mathbf{E}_n^-$. Therefore, what's left for the determination of $\mathbf{E}_n^\pm$, is the determination of $\phi_n^\pm$. The scalar function ϕ_n^+ (ϕ_n in V^+) is a single layer potential which means that it satisfies the Laplace equation with asymptotic form:

$$\phi_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{2.387}$$

Based on the continuity of $\mathbf{F}_{en}$ and $\mathbf{G}_{en}^\pm$ on the boundary S as well as the transmission conditions (2.263), we have the following boundary condition from (2.385) and (2.258):

$$\int_S \left[\hat{\mathbf{n}}\cdot\mathbf{E}_n^+(\mathbf{r}) - \hat{\mathbf{n}}\cdot\mathbf{E}_n^-(\mathbf{r})\right] dS(\mathbf{r}) = \int_S \hat{\mathbf{n}}\cdot\nabla\left(\phi_n^+(\mathbf{r}) - \phi_n^-(\mathbf{r})\right) dS(\mathbf{r}') = 0 \ , \quad \mathbf{r}\in S \ , \tag{2.388}$$

$$\int_S \left[\hat{\mathbf{n}}\times\mathbf{E}_n^+(\mathbf{r}) - \hat{\mathbf{n}}\times\mathbf{E}_n^-(\mathbf{r})\right] dS(\mathbf{r}) = \int_S \hat{\mathbf{n}}\times\nabla\left(\phi_n^+(\mathbf{r}) - \phi_n^-(\mathbf{r})\right) dS(\mathbf{r}') = \mathbf{0} \ , \quad \mathbf{r}\in S \ , \tag{2.389}$$

which leads to the fact that $\phi_n^+ = \phi_n^- + c$ where c is a constant. From the transmission conditions (2.264) and relation (2.385) it is concluded that:

$$\hat{\mathbf{n}}\cdot\nabla\phi_n^-(\mathbf{r}) = -\hat{\mathbf{n}}\cdot\mathbf{F}_{en}^-(\mathbf{r}) - \sigma^-\hat{\mathbf{n}}\cdot\mathbf{G}_{en}^-(\mathbf{r}) + \frac{n}{\sigma^-}Y^+\left(\frac{\varepsilon^-}{\varepsilon^+}\hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^-(\mathbf{r}) - \hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^+(\mathbf{r})\right) \ . \tag{2.390}$$

From divergence theorem we have that $\int_S \phi_n^- dS = \int_S \Delta\phi_n^- dV = 0$ which together with (2.258) leads to relations:

$$\int_S \left[\hat{\mathbf{n}}\cdot\mathbf{F}_{en}^-(\mathbf{r}) + \sigma^-\hat{\mathbf{n}}\cdot\mathbf{G}_{en}^-(\mathbf{r}) - \frac{n}{\sigma^-}Y^+\left(\frac{\varepsilon^-}{\varepsilon^+}\hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^-(\mathbf{r}) - \hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^+(\mathbf{r})\right)\right] dS(\mathbf{r}) = 0 \ . \tag{2.391}$$

Thus, we have an interior Neumann boundary value problem for ϕ_n^- which has a solution in V^-. This together with the relation (2.388) and the divergence theorem leads to:

$$\int_S \frac{\partial}{\partial n}\phi_n^+(\mathbf{r}) = \int_S \frac{\partial}{\partial}\phi_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in S \ , \tag{2.392}$$

which ensures the asymptotic form (2.387) of ϕ_n^+. From the above relations it is concluded that the potentials $\phi_n^\pm$ satisfy the following problems:

$$\begin{aligned} &\Delta\phi_n^-(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^- \ , \\ &\frac{\partial}{\partial n}\phi_n^-(\mathbf{r}) = -\hat{\mathbf{n}}\cdot\mathbf{F}_{en}^-(\mathbf{r}) - \sigma^-\hat{\mathbf{n}}\cdot\mathbf{G}_{en}^-(\mathbf{r}) + \frac{n}{\sigma^-}Y^+\left(\frac{\varepsilon^-}{\varepsilon^+}\hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^-(\mathbf{r}) - \hat{\mathbf{n}}\cdot\mathbf{E}_{n-1}^+(\mathbf{r})\right) \ , \quad \mathbf{r}\in S \ , \end{aligned} \tag{2.393}$$

and

$$\begin{aligned} &\Delta\phi_n^+(\mathbf{r}) = 0 \ , \quad \mathbf{r}\in V^+ \ , \\ &\phi_n^+(\mathbf{r}) = \phi_n^-(\mathbf{r}) + c \ , \quad \mathbf{r}\in S \ , \\ &\phi_n^+(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r^2}\right) \ . \end{aligned} \tag{2.394}$$

Hence, the determination of $\mathbf{E}_0^\pm$ depends on the solution of (2.393) and (2.394) for $n = 0$. The boundary conditions of these two problems for $n = 0$ are:

$$\frac{\partial}{\partial n}\phi_0^-(\mathbf{r}) = -\hat{\mathbf{n}}\cdot\hat{\mathbf{p}} \ , \quad \mathbf{r}\in S \tag{2.395}$$

and

$$\phi_0^+(\mathbf{r}) = -\hat{\mathbf{p}}\cdot\mathbf{r} + c \quad \mathbf{r}\in S \ . \tag{2.396}$$

It is straightforward that the solution of the interior Neumann problem in V^- (2.393) for $n = 0$, is $\phi_0^- = -\hat{\mathbf{p}} \cdot \mathbf{r}$ since $\dfrac{\partial}{\partial n}(\hat{\mathbf{p}} \cdot \mathbf{r}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{p}}$.

For ϕ_0^+, returning to the basic vector valued potential function $\mathbf{\Phi}$ defined in the previous section as the solution of (2.82), it can be observed that $-\hat{\mathbf{p}} \cdot \mathbf{\Phi}$ is the solution of (2.394) for $c = -\hat{\mathbf{p}} \cdot \mathbf{c}$ and $n = 0$, where $\mathbf{c}$ is defined in (2.85). This leads to the fact that $\phi_0^+ = -\hat{\mathbf{p}} \cdot \mathbf{\Phi}$. Thus, based on (2.272), (2.385) and the fact that $\mathbf{G}_{e0}^{\pm} = 0$ from convention (2.271), the low-frequency coefficients $\mathbf{E}_0^{\pm}$ in terms of the vector valued potential $\mathbf{\Phi}$ are given by:

$$\mathbf{E}_0^-(\mathbf{r}) = \hat{\mathbf{p}} - \nabla(\hat{\mathbf{p}} \cdot \mathbf{r}) = \mathbf{0} \ , \quad \mathbf{r} \in V^- \ , \tag{2.397}$$

$$\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{p}} - \nabla(\hat{\mathbf{p}} \cdot \mathbf{\Phi}(\mathbf{r})) \ , \quad \mathbf{r} \in V^+ \ , \tag{2.398}$$

where $\mathbf{E}_0^+$ is the same with the case of the perfect conductor and confirms that the zeroth coefficient of the electric total field is independent of the conductivity σ^-. For the low-frequency coefficients $\mathbf{H}_0^{\pm}$, it is clear from (2.271) that $\mathbf{G}_{mn}^{\pm} = 0$. Moreover, since $\mathbf{F}_{m0}^{\pm}$ are defined the same as the lossless case. Thus, from relation (2.273), we get from (2.386):

$$\mathbf{H}_0^{\pm}(\mathbf{r}) = Y^+ \hat{\mathbf{q}} + \nabla \psi_n^{\pm}(\mathbf{r}) \ , \quad \mathbf{r} \in V^{\pm} \ , \tag{2.399}$$

which is exactly the same with the lossless case since $\psi_n^{\pm}$ and $\mathbf{F}_{mn}$ are the same by their definition. Thus, the low-frequency coefficients $\mathbf{H}_0^{\pm}$ are given by (2.363):

$$\mathbf{H}_0^{\pm}(\mathbf{r}) = Y^+ \left\{ \hat{\mathbf{q}} - \left(1 - \frac{\mu^-}{\mu^+}\right) \nabla \left[\hat{\mathbf{q}} \cdot \mathbf{\Psi}^{\pm} \left(\mathbf{r}; \frac{\mu^-}{\mu^+} \right) \right] \right\} \ , \quad \mathbf{r} \in V^{\pm} \ , \tag{2.400}$$

where $\mathbf{\Psi}^{\pm}$ is the general polarizability potential defined as the solution of (2.169).

Far-field data:

For the leading order term A_3 of the scattering amplitude $\mathbf{E}^{\infty}$ given in (2.286) for $n = 0$, by substituting (2.397) and working similarly with the lossy case, we obtain:

$$\begin{aligned} A_3(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi} \hat{\mathbf{r}} \times \Bigg\{ \hat{\mathbf{r}} \times \left[-\left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right) |V^-| \widetilde{\mathbf{I}} + \left(1 - \frac{\varepsilon^-}{\varepsilon^+}\right)^2 \int_S \hat{\mathbf{n}}' \otimes \mathbf{\Phi}^- \left(\mathbf{r}'; \frac{\varepsilon^-}{\varepsilon^+} \right) dS(\mathbf{r}') \right] \cdot \hat{\mathbf{p}} \\ + \left[-\left(1 - \frac{\mu^-}{\mu^+}\right) |V^-| \widetilde{\mathbf{I}} + \left(1 - \frac{\mu^-}{\mu^+}\right)^2 \int_S \hat{\mathbf{n}}' \otimes \mathbf{\Psi}^- \left(\mathbf{r}'; \frac{\mu^-}{\mu^+} \right) dS(\mathbf{r}') \right] \cdot \hat{\mathbf{q}} \Bigg\} \ , \end{aligned} \tag{2.401}$$

which due to the definition of (2.96), (2.97) and (2.173) and the identities:

$$\int_S \hat{\mathbf{n}} \otimes \mathbf{\Phi}^+(\mathbf{r}; \beta) dS(\mathbf{r}) = -\beta \int_S \hat{\mathbf{n}} \otimes \mathbf{\Psi}^+(\mathbf{r}; \beta) dS(\mathbf{r}) \ , \tag{2.402}$$

$$\int_S \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Psi}^+(\mathbf{r}; \beta) dS(\mathbf{r}) = -\frac{1}{\beta} \int_S \mathbf{r} \otimes \frac{\partial}{\partial n} \mathbf{\Phi}^+(\mathbf{r}; \beta) dS(\mathbf{r}) \ , \tag{2.403}$$

A_3 can be rewritten as:

$$A_3(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = \frac{(ik)^3}{4\pi} \left[\mathbf{r} \times \left(\mathbf{r} \times \widetilde{\mathbf{P}} \right) \cdot \hat{\mathbf{p}} + \hat{\mathbf{r}} \times \widetilde{\mathbf{X}} \left(\frac{\mu^-}{\mu^+} \right) \cdot \hat{\mathbf{q}} \right] \ , \tag{2.404}$$

which is an expression in terms of the general polarizability tensor $\widetilde{\mathbf{X}}$ and the electric polarizability tensor $\widetilde{\mathbf{P}}$.

The leading order term of $\mathbf{H}^{\infty}$ in terms of the polarizibility tensors can be obtain easily from relation (2.290).

2.3 The Ellipsoidal Harmonics in the Low-frequency Problems

Based on the previous sections of this chapter, it is concluded that via low-frequency theory, the scattering problems can be reduced to potential theory problems which leads to the introduction of harmonic functions. Moreover, for ellipsoidal scatterers, it is convenient to introduce the ellipsoidal harmonic functions into our problems. In this section, the main purpose is to derive the low-frequency coefficients of the total field and the scattering amplitude in terms of the ellipsoidal harmonics. This may seem complicated, but due to the fact that all the low-frequency coefficients were expressed in terms of basic potential functions and the quantities related to them, it suffices to express these potentials and the corresponding quantities in terms of the ellipsoidal harmonics.

2.3.1 Conductor Potential

The conductor potential ϕ^c defined in (2.75), satisfies the following boundary valued problem in terms of the ellipsoidal coordinates:

$$\begin{aligned} &\Delta\phi^c(\rho,\mu,\nu) = 0 \ , \quad \rho > \alpha_1 \ , \\ &\phi^c(\rho,\mu,\nu) = 1 \ , \quad \rho = \alpha_1 \ , \\ &\phi(\mathbf{r}) = \mathcal{O}\left(\frac{1}{r}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.405}$$

Since it is an exterior boundary valued problem for the Laplce, the solution can be expressed in terms of the exterior ellipsoidal harmonics as follows:

$$\phi^c(\rho,\mu,\nu) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^m \mathbb{F}_n^m(\rho,\mu,\nu) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^m (2n+1) I_n^m(\rho) E_n^m(\rho) E_n^m(\mu) E_n^m(\nu) \ , \tag{2.406}$$

which on the boundary satisfies the relation:

$$1 = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^m (2n+1) I_n^m(\alpha_1) E_n^m(\alpha_1) E_n^m(\mu) E_n^m(\nu) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^m (2n+1) I_n^m(\alpha_1) E_n^m(\alpha_1) S_n^m(\mu,\nu) \ . \tag{2.407}$$

From relation (to anaptygma twn epifaneiakwn pou dinei ton syntelesth), we have:

$$A_n^m = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1) E_n^m(\alpha_1)} \int_{S_{\alpha_1}} S_n^m(\mu,\nu) d\Omega(\mu,\nu) \ , \tag{2.408}$$

where

$$\gamma_n^m = \int_{S_{\alpha_1}} \left[S_n^m(\mu,\nu)\right]^2 d\Omega(\mu,\nu) \ , \quad n = 0,1,2,\dots \ , \ m = 1,2,\dots,2n+1. \tag{2.409}$$

From relation (2.408), since $S_0^1 = 1$, we have that:

$$A_n^m = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1) E_n^m(\alpha_1)} \int_{S_{\alpha_1}} S_0^1(\mu,\nu) S_n^m(\mu,\nu) d\Omega(\mu,\nu) \ , \tag{2.410}$$

which means that due to the orthogonality of the surface ellipsoidal harmonics the only non vanishing coefficient A_n^m is for $n = 0$ and $m = 1$. which from relation (1.237), takes the following value:

$$A_0^1 = \frac{1}{I_0^1(\alpha_1)} \ . \tag{2.411}$$

Returning to the expansion (2.406), for $n = 0$ and $m = 1$ we have:

$$\phi^c(\rho,\mu,\nu) = A_0^1 I_0^1(\rho) = \frac{I_0^1(\rho)}{I_0^1(\alpha_1)} \tag{2.412}$$

The capacity C of the ellipsoidal conductor, due to relation (2.76), is given by:

$$C = -\frac{1}{4\pi}\int_{S_{\alpha_1}} \frac{\partial}{\partial n}\phi^c(\alpha_1,\mu,\nu)dS(\mu,\nu) \ , \tag{2.413}$$

where (1.231) and $E_0^1(\rho) = 1$:

$$\begin{aligned}
\frac{\partial}{\partial n}\phi^c(\alpha_1,\mu,\nu) &= \frac{\partial}{\partial n}\phi^c(\rho,\mu,\nu)\Big|_{\rho=\alpha_1} = \frac{\partial}{\partial n}\left(\frac{I_0^1(\rho)}{I_0^1(\alpha_1)}\right)\Bigg|_{\rho=\alpha_1} \\
&= \frac{1}{I_0^1(\alpha_1)}\left(\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}\,l_\rho(\mu,\nu)\frac{\partial}{\partial\rho}I_0^1(\rho)\right)\Bigg|_{\rho=\alpha_1} \\
&= \frac{-1}{I_0^1(\alpha_1)}\left(\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}\,l_\rho(\mu,\nu)\frac{1}{\left(E_0^1(\rho)\right)^2\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}\right)\Bigg|_{\rho=\alpha_1} \\
&= -\frac{l_{\alpha_1}(\mu,\nu)}{I_0^1(\alpha_1)} \ .
\end{aligned} \tag{2.414}$$

Thus, the capacity C takes the following form:

$$C = \frac{1}{4\pi I_0^1(\alpha_1)}\int_{S_{\alpha_1}} l_{\alpha_1}(\mu,\nu)dS(\mu,\nu) = \frac{1}{I_0^1(\alpha_1)} \ . \tag{2.415}$$

2.3.2 Polarization Potential

The vector valued potential defined in (2.82), satisfies the following boundary value problem:

$$\begin{aligned}
&\Delta\mathbf{\Phi}(\rho,\mu,\nu) = 0 \ , \quad \rho > \alpha_1 \ , \\
&\mathbf{\Phi}(\rho,\mu,\nu) = \mathbf{r} + \mathbf{c} \ , \quad \rho = \alpha_1 \ , \\
&\mathbf{\Phi}(\rho,\mu,\nu) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ ,
\end{aligned} \tag{2.416}$$

where the constant $\mathbf{c}$ is chosen such that:

$$\int_{S_{\alpha_1}} \frac{\partial}{\partial n}\mathbf{\Phi}(\rho,\mu,\nu)dS(\mu,\nu) = \mathbf{0} \ , \tag{2.417}$$

which ensures the well-posedness of the problem. Moreover, based on relation (2.85), the constant vector $\mathbf{c}$ in terms of the conductor potential ϕ^c is given by the following relation:

$$\mathbf{c} = \frac{1}{4\pi C}\int_{S_{\alpha_1}} \mathbf{r}\frac{\partial}{\partial n}\phi^c(\rho,\mu,\nu)dS(\mu,\nu) \ . \tag{2.418}$$

Hence, based on (2.414) and (2.415), we have:

$$\mathbf{c} = \frac{1}{4\pi}\int_{S_{\alpha_1}} \mathbf{r}\,l_{\alpha_1}(\mu,\nu)dS(\mu,\nu) = \frac{1}{4\pi}\int_{S_{\alpha_1}} \mathbf{r}\,d\Omega(\mu,\nu) \ . \tag{2.419}$$

Using the following relation:

$$\mathbf{r}|_{\rho=\alpha_1} = \sum_{m=1}^{3}\frac{\alpha_m h_m}{h_1h_2h_3}S_1^m(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}} \ , \tag{2.420}$$

we have:

$$\mathbf{c}\cdot\hat{\mathbf{x}}_{\mathbf{m}} = \frac{\alpha_m h_m}{4\pi h_1 h_2 h_3}\int_{S_{\alpha_1}} S_1^m(\mu,\nu)S_0^m(\mu,\nu)d\Omega(\mu,\nu) = 0\ . \tag{2.421}$$

Returning to the exterior potential theory problem (2.416), we take the solution as an expansion of exterior ellipsoidal harmonics $\mathbb{F}_n^m$:

$$\mathbf{\Phi}(\rho,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} A_n^m \mathbb{F}_n^m(\rho,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} A_n^m (2n+1) I_n^m(\rho)\mathbb{E}_n^m(\rho,\mu,\nu), \tag{2.422}$$

where the sum of n starts from $n = 1$ instead of $n = 0$ because of the asymptotic form of $\mathbf{\Phi}$. Since the vector coefficient $\mathbf{c}$ vanishes for the ellipsoidal scatterer, the boundary condition becomes $\mathbf{\Phi} = \mathbf{r}$. Thus on the surface $\rho = \alpha_1$, we have:

$$\mathbf{r} = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \mathbf{A}_n^m (2n+1) I_n^m(\alpha_1)E_n^m(\alpha_1)S_n^m(\mu,\nu)\ , \tag{2.423}$$

from which it can be observed that the properties of the vector are transfered into the coefficients $\mathbf{A}_n^m$.

Due to orthogonality, we have:

$$\mathbf{A}_n^m = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1)E_n^m(\alpha_1)}\int_{S_{\alpha_1}} \mathbf{r} S_n^m(\mu,\nu)d\Omega(\mu,\nu) \tag{2.424}$$

Hence, by taking the inner product $\cdot\hat{\mathbf{x}}_{\mathbf{m}}$ (the projection over the axis x_m) for $m = 1, 2, 3$, we have:

$$\mathbf{A}_n^m\cdot\hat{\mathbf{x}}_{\mathbf{1}} = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1)E_n^m(\alpha_1)}\int_{S_{\alpha_1}} \frac{\alpha_1\mu\nu}{h_2h_3} S_n^m(\mu,\nu)d\Omega(\mu,\nu)\ , \tag{2.425}$$

$$\mathbf{A}_n^m\cdot\hat{\mathbf{x}}_{\mathbf{2}} = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1)E_n^m(\alpha_1)}\int_{S_{\alpha_1}} \frac{\alpha_2\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}}{h_1h_3} S_n^m(\mu,\nu)d\Omega(\mu,\nu)\ , \tag{2.426}$$

$$\mathbf{A}_n^m\cdot\hat{\mathbf{x}}_{\mathbf{3}} = \frac{1}{\gamma_n^m (2n+1) I_n^m(\alpha_1)E_n^m(\alpha_1)}\int_{S_{\alpha_1}} \frac{\alpha_3\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1h_2} S_n^m(\mu,\nu)d\Omega(\mu,\nu)\ . \tag{2.427}$$

Since $S_1^1 = \mu\nu$, $S_1^2 = \sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}$ and $S_1^3 = \sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}$ (1.214), due to orthogonality of surface ellipsoidal harmonics $S_n^m(\mu,\nu)$, we have that the above projections do not vanish only for $n = 1$ and $m = 1, 2, 3$ in (2.425)-(2.427) respectively.

Thus, we have:

$$\mathbf{A}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m}} = \frac{\alpha_m h_m}{h_1h_2h_3 3 I_1^m(\alpha_1)E_1^m(\alpha_1)}\ , \quad m = 1, 2, 3\ , \tag{2.428}$$

and it can easily observed that due to orthogonality of S_n^m, we have $\mathbf{A}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m'}} = 0$, for $m \neq m'$. Returning to (2.416), for $n = 1$, we have:

$$\begin{aligned}\mathbf{\Phi}(\rho,\mu,\nu) &= \mathbf{A}_1^m 3 I_1^m(\rho)\mathbb{E}_1^m(\rho,\mu,\nu) = \mathbf{A}_1^1 3 I_1^1(\rho)\mathbb{E}_1^1(\rho,\mu,\nu) + \mathbf{A}_1^2 3 I_1^2(\rho)\mathbb{E}_1^2(\rho,\mu,\nu) + \mathbf{A}_1^3 3 I_1^3(\rho)\mathbb{E}_1^3(\rho,\mu,\nu)\\ &= \mathbf{A}_1^1 3 I_1^1(\rho)\rho\mu\nu + \mathbf{A}_1^2 3 I_1^2(\rho)\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2} + \mathbf{A}_1^3 I_1^3(\rho)\sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}\\ &= \left(\frac{I_1^1(\rho)}{I_1^1(\alpha_1)}\frac{\rho\mu\nu\alpha_1}{h_2h_3}, \frac{I_1^2(\rho)}{I_1^2(\alpha_1)}\frac{\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}}{h_1h_3}, \frac{I_1^3(\rho)}{I_1^3(\alpha_1)}\frac{\sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1h_2}\right)\ ,\end{aligned} \tag{2.429}$$

which can be rewritten as:

$$\mathbf{\Phi}(\rho,\mu,\nu)=\mathbf{r}\cdot\sum_{m=1}^{3}\frac{I_1^m(\rho)}{I_1^m(\alpha_1)}\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}=\sum_{m=1}^{3}\frac{I_1^m(\rho)}{I_1^m(\alpha_1)}x_m\otimes\hat{\mathbf{x}}_{\mathbf{m}}\ , \tag{2.430}$$

where $\mathbf{r}=(x_1,x_2,x_3)$. From this relation and $\left.\frac{\partial}{\partial n}\right|_{\rho=\alpha_1}=\alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)\frac{\partial}{\partial\rho}$, we obtain:

$$\left.\frac{\partial}{\partial n}\mathbf{\Phi}(\rho,\mu,\nu)\right|_{\rho=\alpha_1}=\alpha_3\alpha_2 l_{\alpha_1}(\mu,\nu)\sum_{m=1}^{3}\frac{1}{I_1^m(\alpha_1)}\left(\frac{\partial}{\partial\rho}(I_1^m(\rho)x_m)\right)\hat{\mathbf{x}}_{\mathbf{m}}\Bigg|_{\rho=\alpha_1} \tag{2.431}$$

Hence, for $m=1$, the first component becomes:

$$\begin{aligned}&\frac{\alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)}{I_1^1(\alpha_1)}\left(x_1\frac{\partial}{\partial\rho}I_1^1(\rho)+I_1^1(\rho)\frac{\partial}{\partial\rho}x_1\right)\Bigg|_{\rho=\alpha_1}\\&=\frac{\alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)}{I_1^1(\alpha_1)}\left(-\frac{\mu\nu}{h_2h_3\rho\sqrt{\rho^2-h_2^2}\sqrt{\rho^2-h_3^2}}+I_1^1(\rho)\frac{\mu\nu}{h_2h_3}\right)\Bigg|_{\rho=\alpha_1}\\&=\frac{\alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)}{I_1^1(\alpha_1)}\left(I_1^1(\alpha_1)-\frac{1}{\alpha_1\alpha_2\alpha_3}\right)\frac{S_1^1(\mu,\nu)}{h_2h_3}\ ,\end{aligned} \tag{2.432}$$

and working similarly for $m=2$ and $m=3$ we can obtain the other two components. Moreover, the vector $\mathbf{r}=(x_1,x_2,x_3)$ on the surface of the ellipsoid S_{α_1}, takes the following form:

$$\begin{aligned}\mathbf{r}|_{\rho=\alpha_1}&=\left(\frac{\alpha_1\mu\nu}{h_2h_3},\frac{\alpha_2\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2}}{h_1h_3},\frac{\alpha_3\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2}}{h_1h_2}\right)\\&=\sum_{m=1}^{3}\frac{\alpha_m h_m}{h_1h_2h_3}S_1^m(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}}=\sum_{m=1}^{3}\frac{h_mE_1^m(\alpha_1)}{h_1h_2h_3}S_1^m(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}}\ .\end{aligned} \tag{2.433}$$

Thus, based on the above, the polarization tensor $\widetilde{\mathbf{Q}}$ for ellipsoidal scatterers of surface S_{α_1} can be derived by substituting (2.430) into (2.96) and using the above relations to obtain:

$$\begin{aligned}\widetilde{\mathbf{Q}}&=-\int_{S_{\alpha_1}}\mathbf{r}\otimes\frac{\partial}{\partial n}\mathbf{\Phi}(\rho,\mu,\nu)dS(\mu,\nu)\\&=\frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m}{\alpha_m}\left(\frac{1}{\alpha_1\alpha_2\alpha_3I_1^m(\alpha_1)}-1\right)\int_{S_{\alpha_1}}S_1^m(\mu,\nu)\mathbf{r}\otimes\hat{\mathbf{x}}_{\mathbf{m}}d\Omega(\mu,\nu)\\&=\frac{\alpha_1\alpha_2\alpha_3}{h_1^2h_2^2h_3^2}\sum_{m=1}^{3}h_m^2\left(\frac{1}{\alpha_1\alpha_2\alpha_3I_1^m(\alpha_1)}-1\right)\int_{S_{\alpha_1}}\left(S_1^m(\mu,\nu)\right)^2\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}d\Omega(\mu,\nu)\\&=\frac{\alpha_1\alpha_2\alpha_3}{h_1^2h_2^2h_3^2}\sum_{m=1}^{3}h_m^2\left(\frac{1}{\alpha_1\alpha_2\alpha_3I_1^m(\alpha_1)}-1\right)\gamma_1^m\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}\\&=\frac{4\pi}{3}\alpha_1\alpha_2\alpha_3\sum_{m=1}^{3}\left(\frac{1}{\alpha_1\alpha_2\alpha_3I_1^m(\alpha_1)}-1\right)\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}\ ,\end{aligned} \tag{2.434}$$

where γ_1^m the normalization constant defined in (1.253),(1.229) which is proven to be [20] $\gamma_1^m=\dfrac{4\pi h_1^2h_2^2h_3^2}{3h_m^2}$ for $m=1,2,3$.

Finally, the electric polarizability tensor defined in (2.97), for ellipsoidal scatterers, takes the following form based on (2.434):

$$\widetilde{\mathbf{P}}=\frac{4\pi}{3}\sum_{m=1}^{3}\frac{1}{I_1^m(\alpha_1)}\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}\ . \tag{2.435}$$

2.3.3 Virtual Mass Potential

The virtual mass potential defined in (2.121), satisfies the following boundary value problem in terms of ellipsoidal coordinates:

$$\begin{aligned} &\Delta \mathbf{\Psi}(\rho,\mu,\nu) = 0 \ , \quad \rho > \alpha_1 \ , \\ &\frac{\partial}{\partial n}\mathbf{\Psi}(\rho,\mu,\nu) = \hat{\boldsymbol{\rho}} \ , \quad \rho = \alpha_1 \ , \\ &\mathbf{\Psi}(\rho,\mu,\nu) = \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad r \to \infty \ . \end{aligned} \tag{2.436}$$

Similarly to the previous cases, the potential for this exterior problem can be expressed in terms of the exterior ellipsoidal harmonics:

$$\mathbf{\Psi}(\rho,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \mathbf{A}_n^m \mathbb{F}_n^m(\rho,\mu,\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \mathbf{A}_n^m (2n+1) I_n^m(\rho)\mathbb{E}(\rho,\mu,\nu) \ , \tag{2.437}$$

Based on relation $\frac{\partial}{\partial n} = \alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)\frac{\partial}{\partial \rho}$, on the surface $\rho = \alpha_1$, we have:

$$\hat{\boldsymbol{\rho}} = \frac{\partial}{\partial n}\mathbf{\Psi}(\rho,\mu,\nu)\bigg|_{\rho=\alpha_1} = \alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{A}_n^m S_n^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}I_n^m(\rho)E_n^m(\rho)\right)\bigg|_{\rho=\alpha_1} \ , \tag{2.438}$$

where $l_{\alpha_1}(\mu,\nu) = \frac{1}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}$ the weighting function defined in (1.225).
Based on relation (1.25) and (2.433), we also have:

$$\hat{\boldsymbol{\rho}}|_{\rho=\alpha_1} = \frac{\alpha_1\alpha_2\alpha_3}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}\sum_{m=1}^{3}\frac{x_m}{\alpha_m^2}\hat{\mathbf{x}}_{\mathbf{m}} = \frac{\alpha_1\alpha_2\alpha_3}{\sqrt{\alpha_1^2-\mu^2}\sqrt{\alpha_1^2-\nu^2}}\sum_{m=1}^{3}\frac{h_m S_1^m}{h_1h_2h_3\alpha_m}\hat{\mathbf{x}}_{\mathbf{m}} \ , \tag{2.439}$$

Based on the above and the relation $\frac{\partial}{\partial n} = \alpha_2\alpha_3 l_{\alpha_1}(\mu,\nu)\frac{\partial}{\partial \rho}$, the boundary relation (2.438) can be rewritten as:

$$\frac{\alpha_1\alpha_2\alpha_3}{h_1h_2h_3}\sum_{m=1}^{3}\frac{h_m S_1^m}{\alpha_m}\hat{\mathbf{x}}_{\mathbf{m}} = 3\alpha_2\alpha_3\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{A}_n^m S_n^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}I_n^m(\rho)E_n^m(\rho)\right)\bigg|_{\rho=\alpha_1} \ , \tag{2.440}$$

where

$$A_n^m = \frac{\alpha_1\alpha_2\alpha_3}{3\gamma_n^m\alpha_2\alpha_3\left(\frac{\partial}{\partial\rho}I_n^m(\rho)E_n^m(\rho)\right)\bigg|_{\rho=\alpha_1}}\sum_{m=1}^{3}\int_{S_{\alpha_1}}\frac{h_m S_n^m}{h_1h_2h_3\alpha_m}S_n^m(\mu,\nu)d\Omega(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}} \tag{2.441}$$

Due to orthogonality of the surface ellipsoidal harmonics, the only non vanishing coefficients are $\mathbf{A}_1^m$ for $m = 1,2,3$ which are given by:

$$\mathbf{A}_1^m = \frac{\alpha_1\alpha_2\alpha_3 h_m}{3\gamma_1^m\alpha_2\alpha_3 h_1h_2h_3\alpha_m\left(\frac{\partial}{\partial\rho}I_1^m(\rho)E_1^m(\rho)\right)\bigg|_{\rho=\alpha_1}}\int_{S_{\alpha_1}}\left(S_1^m(\mu,\nu)\right)^2 d\Omega(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}} \ , \tag{2.442}$$

which after the calculation of the partial derivative and since $\gamma_1^m = \int_{S_{\alpha_1}}(S_1^m)^2 d\Omega$, can be rewritten as:

$$\mathbf{A}_1^m = \frac{\alpha_1\alpha_2\alpha_3 h_m}{3h_1h_2h_3\left(\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1) - 1\right)}\hat{\mathbf{x}}_{\mathbf{m}} \ , \quad m = 1,2,3 \ . \tag{2.443}$$

Hence, the virtual mass potential for the ellipsoid, based on (1.201)-(1.203), is given by:

$$\mathbf{\Psi}(\rho,\mu,\nu)=\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 h_m 3I_1^m(\rho)\mathbb{E}_1^m(\rho,\mu,\nu)}{3h_1h_2h_3\left(\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1\right)}\hat{\mathbf{x}}_{\mathbf{m}}=\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\rho)x_m}{\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1}\hat{\mathbf{x}}_{\mathbf{m}}\ . \tag{2.444}$$

Finally, the virtual mass tensor $\widetilde{\mathbf{W}}$ defined in (2.121), for the case of the ellipsoid, is given by:

$$\begin{aligned}\widetilde{\mathbf{W}}&=-\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 h_m 3I_1^m(\rho)\mathbb{E}_1^m(\rho,\mu,\nu)}{3h_1h_2h_3\left(\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1\right)}\hat{\mathbf{x}}_{\mathbf{m}}=\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\rho)x_m}{\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1}\int_{S_{\alpha_1}}x_m\hat{\boldsymbol{\rho}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}dS(\mu,\nu)\\&=\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\rho)}{\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1}\int_{h_2\leqslant\rho\leqslant\alpha_1}\nabla\otimes x_m\hat{\mathbf{x}}_{\mathbf{m}}dV(\rho,\mu,\nu)=\frac{4\pi\alpha_1^2\alpha_2^2\alpha_3^2}{3}\sum_{m=1}^{3}\frac{I_1^m(\alpha_1)}{1-\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}\ ,\end{aligned} \tag{2.445}$$

for $m=1,2,3$ and the magnetic polarizability tensor $\widetilde{\mathbf{M}}$ defined in (2.122), is given by:

$$\widetilde{\mathbf{M}}=\frac{4\pi}{3}\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3}{1-\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_{\mathbf{m}}\otimes\hat{\mathbf{x}}_{\mathbf{m}}\ . \tag{2.446}$$

2.3.4 Generalized polarization potentials

The vector fields $\mathbf{v}^{\pm}$ associated with the general polarizability tensor, for the case of the ellipsoidal scatterer, the problem (2.164) is written as:

$$\begin{aligned}&\Delta\mathbf{v}^+(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho>\alpha_1\ ,\\&\Delta\mathbf{v}^-(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho<\alpha_1\ ,\\&\mathbf{v}^+(\rho,\mu,\nu;\beta)=\mathbf{v}^-(\rho,\mu,\nu;\beta)+\mathbf{r}\ ,\quad \rho=\alpha_1\ ,\\&\frac{\partial}{\partial n}\mathbf{v}^+(\rho,\mu,\nu;\beta)=\beta\frac{\partial}{\partial n}\mathbf{v}^-(\rho,\mu,\nu;\beta)+\hat{\boldsymbol{\rho}}\ ,\rho=\alpha_1\ ,\\&\mathbf{v}^+(\rho,\mu,\nu;\beta)=\mathcal{O}\left(\frac{1}{r^2}\right)\ ,\quad r\to\infty\ .\end{aligned} \tag{2.447}$$

The vector fields $\mathbf{\Phi}^{\pm}$, $\mathbf{\Psi}^{\pm}$ introduced from the decompositions (2.166) and (2.167) of $\mathbf{v}^{\pm}$, for case of the ellipsoid, based on (2.168) and (2.169), they are solutions of the following transmission problems:

$$\begin{aligned}&\Delta\mathbf{\Phi}^+(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho>\alpha_1\ ,\\&\Delta\mathbf{\Phi}^-(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho<\alpha_1\ ,\\&\mathbf{\Phi}^+(\rho,\mu,\nu;\beta)=\mathbf{\Phi}^-(\rho,\mu,\nu;\beta)+\mathbf{r}\ ,\quad \rho=\alpha_1\ ,\\&\frac{\partial}{\partial n}\mathbf{\Phi}^+(\rho,\mu,\nu;\beta)=\beta\frac{\partial}{\partial n}\mathbf{\Phi}^-(\rho,\mu,\nu;\beta)\ ,\rho=\alpha_1\ ,\\&\mathbf{\Phi}^+(\rho,\mu,\nu;\beta)=\mathcal{O}\left(\frac{1}{r^2}\right)\ ,\quad r\to\infty\ ,\end{aligned} \tag{2.448}$$

and

$$\begin{aligned}&\Delta\mathbf{\Psi}^+(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho>\alpha_1\ ,\\&\Delta\mathbf{\Psi}^-(\rho,\mu,\nu;\beta)=\mathbf{0}\ ,\quad \rho<\alpha_1\ ,\\&\mathbf{\Psi}^+(\rho,\mu,\nu;\beta)=\mathbf{\Phi}^-(\rho,\mu,\nu;\beta)\ ,\quad \rho=\alpha_1\ ,\\&\frac{\partial}{\partial n}\mathbf{\Phi}^+(\rho,\mu,\nu;\beta)=\beta\frac{\partial}{\partial n}\mathbf{\Phi}^-(\rho,\mu,\nu;\beta)+\hat{\boldsymbol{\rho}}\ ,\rho=\alpha_1\ ,\\&\mathbf{\Psi}^+(\rho,\mu,\nu;\beta)=\mathcal{O}\left(\frac{1}{r^2}\right)\ ,\quad r\to\infty\ .\end{aligned} \tag{2.449}$$

Starting with the vector potentials $\mathbf{\Phi}^{\pm}$, the potential $\mathbf{\Phi}^{-}$ can be expanded in terms of the interior ellipsoidal harmonics and the potential $\mathbf{\Phi}^{+}$ in terms of the exterior ellipsoidal harmonics as follows:

$$\mathbf{\Phi}^{+}(\rho,\mu,\nu;\beta)=\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{A}_n^m\mathbb{F}_n^m(\rho,\mu,\nu)\ ,\quad \rho>\alpha_1\ , \tag{2.450}$$

$$\mathbf{\Phi}^{-}(\rho,\mu,\nu;\beta)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\mathbf{B}_n^m\mathbb{E}_n^m(\rho,\mu,\nu)\ ,\quad \rho<\alpha_1\ , \tag{2.451}$$

where the exterior potential starts from $n=1$ because of its asymptotic form.

Hence, based on relation $\frac{\partial}{\partial n}=\alpha_2\alpha_3 l)S_{\alpha_1}(\mu,\nu)\frac{\partial}{\partial\rho}$ and (2.433), the transmission conditions take the following forms:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{A}_n^m F_n^m(\alpha_1)S_n^m(\mu,\nu)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\mathbf{B}_n^m E_n^m(\alpha_1)S_n^m(\mu,\nu)+\sum_{m=1}^{3}\frac{h_m E_1^m(\alpha_1)}{h_1h_2h_3}S_1^m(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{m}}\ , \tag{2.452}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\mathbf{A}_n^m S_n^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}F_n^m(\rho)\right)\Bigg|_{\rho=\alpha_1}=\beta\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\mathbf{B}_n^m S_n^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}E_n^m(\rho)\right)\Bigg|_{\rho=\alpha_1}\ . \tag{2.453}$$

From the second transmission condition, it can easily be observed that the sum on the right hand of the equation start from $n=0$, while the sum on the left hand starts from $n=1$, which due to the orthogonality of the surface ellipsoidal harmonics, leads to the fact that $\mathbf{B}_0^1=\mathbf{0}$.

For $n=1$ the system of equations (2.452) and (2.453) becomes:

$$\mathbf{A}_1^m F_1^m(\alpha_1)S_1^m(\mu,\nu)=\mathbf{B}_1^m E_1^m(\alpha_1)S_1^m(\mu,\nu)+\frac{1}{h_2h_3}E_1^m(\alpha_1)S_1^m(\mu,\nu)\hat{\mathbf{x}}_{\mathbf{1}}\ , \tag{2.454}$$

$$\mathbf{A}_1^m S_1^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}F_1^m(\rho)\right)\Bigg|_{\rho=\alpha_1}=\mathbf{B}_1^m S_1^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}E_1^m(\rho)\right)\Bigg|_{\rho=\alpha_1}\ , \tag{2.455}$$

for $m=1,2,3$. From the orthogonality of the surface ellipsoidal harmonics, it is concluded that $\mathbf{A}_1^m$ and $\mathbf{B}_1^m$ are given from the following system:

$$\mathbf{A}_1^m F_1^m(\alpha_1)=\mathbf{B}_1^m E_1^m(\alpha_1)+\frac{h_m}{h_1h_2h_3}E_1^m(\alpha_1)\hat{\mathbf{x}}_{\mathbf{m}}\ , \tag{2.456}$$

$$\mathbf{A}_1^m\left(\frac{\partial}{\partial\rho}F_1^m(\rho)\right)\Bigg|_{\rho=\alpha_1}=\beta\mathbf{B}_1^m\left(\frac{\partial}{\partial\rho}E_1^m(\rho)\right)\Bigg|_{\rho=\alpha_1}\ , \tag{2.457}$$

for $m=1,2,3$. This system of the components can be written in matrix form as follows:

$$\begin{bmatrix} F_1^m(\alpha_1) & -E_1^m(\alpha_1) \\ \left(\frac{\partial}{\partial\rho}F_1^m(\rho)\right)\Big|_{\rho=\alpha_1} & -\beta\left(\frac{\partial}{\partial\rho}E_1^m(\rho)\right)\Big|_{\rho=\alpha_1} \end{bmatrix}\begin{bmatrix}\mathbf{A}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m}} \\ \mathbf{B}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m}}\end{bmatrix}=\begin{bmatrix}\frac{h_m}{h_1h_2h_3}E_1^m(\alpha_1) \\ 0\end{bmatrix}\ , \tag{2.458}$$

or equivalently:

$$\begin{bmatrix} 3\alpha_m I_1^m(\alpha_1) & -\alpha_m \\ 3\left(\frac{\alpha_1}{\alpha_m}I_1^m(\alpha_1)-\frac{1}{\alpha_m\alpha_2\alpha_3}\right) & -\beta\frac{\alpha_1}{\alpha_m} \end{bmatrix}\begin{bmatrix}\mathbf{A}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m}} \\ \mathbf{B}_1^m\cdot\hat{\mathbf{x}}_{\mathbf{m}}\end{bmatrix}=\begin{bmatrix}\frac{\alpha_m h_m}{h_1h_2h_3} \\ 0\end{bmatrix}\ , \tag{2.459}$$

with solution given by:

$$\begin{bmatrix} \mathbf{A}_1^m \cdot \hat{\mathbf{x}}_\mathbf{m} \\ \mathbf{B}_1^m \cdot \hat{\mathbf{x}}_\mathbf{m} \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha_1\alpha_2\alpha_3\beta h_m}{3h_1h_2h_3\left[1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)\right]} \\ \\ \dfrac{h_m\left[\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1\right]}{h_1h_2h_3\left[1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)\right]} \end{bmatrix}, \tag{2.460}$$

and it can be observed that $\mathbf{A}_1^m \cdot \hat{\mathbf{x}}_{\mathbf{m}'} = \mathbf{B}_1^m \cdot \hat{\mathbf{x}}_{\mathbf{m}'} = \mathbf{0}$ for $m \neq m'$.

For $n \geqslant 2$, the last term of the first transmission condition ($E_1^m S_1^m$) vanishes due to orthogonality, leaving us with:

$$\mathbf{A}_n^m F_n^m(\alpha_1) S_n^m(\mu,\nu) = \mathbf{B}_n^m E_n^m(\alpha_1) S_n^m(\mu,\nu)\ , \tag{2.461}$$

$$\mathbf{A}_1^m S_1^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}F_n^m(\rho)\right)\bigg|_{\rho=\alpha_1} = \mathbf{B}_n^m S_n^m(\mu,\nu)\left(\frac{\partial}{\partial\rho}E_n^m(\rho)\right)\bigg|_{\rho=\alpha_1}, \tag{2.462}$$

for $m = 1, 2, ..., 2n+1$. This system is a homogeneous system with non-vanishing determinant which leads to $\mathbf{A}_n^m = \mathbf{B}_n^m = \mathbf{0}$ for $n = 2, 3, ...$ and $m = 1, 2, ..., 2n+1$. Thus , the only non-vanishing coefficients are for $n = 1$ and $m = 1, 2, 3$. Substituting the coefficients (2.460) into (2.450) and (2.451) correspondingly, leads to:

$$\mathbf{\Phi}^+(\rho,\mu,\nu;\beta) = \mathbf{r}\cdot\sum_{m=1}^{3}\frac{\beta\alpha_1\alpha_2\alpha_3 I_1^m(\rho)}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho > \alpha_1\ , \tag{2.463}$$

$$\mathbf{\Phi}^-(\rho,\mu,\nu;\beta) = \mathbf{r}\cdot\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)-1}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho < \alpha_1\ . \tag{2.464}$$

Working exactly the same way for $\mathbf{\Psi}^\pm$ we obtain the following solutions [20] :

$$\mathbf{\Psi}^+(\rho,\mu,\nu;\beta) = -\mathbf{r}\cdot\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\rho)}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho > \alpha_1\ , \tag{2.465}$$

$$\mathbf{\Psi}^+(\rho,\mu,\nu;\beta) = -\mathbf{r}\cdot\sum_{m=1}^{3}\frac{\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho < \alpha_1\ . \tag{2.466}$$

Substituting (2.463)-(2.466) into (2.166) and (2.167) correspondingly, we have:

$$\mathbf{v}^+(\rho,\mu,\nu;\beta) = \mathbf{r}\cdot\sum_{m=1}^{3}\frac{(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\rho)}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho > \alpha_1\ , \tag{2.467}$$

$$\mathbf{v}^-(\rho,\mu,\nu;\beta) = -\mathbf{r}\cdot\sum_{m=1}^{3}\frac{1}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ , \quad \rho < \alpha_1\ . \tag{2.468}$$

Thus, based on relation (2.165), the general polarizability tensor $\widetilde{\mathbf{X}}$ for the ellipsoid is given by:

$$\begin{aligned} \widetilde{\mathbf{X}} &= (\beta-1)\int_{S_{\alpha_1}}\hat{\boldsymbol{\rho}}\otimes\mathbf{r}\,dS(\mu,\nu)\cdot\sum_{m=1}^{3}\frac{1}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m} \\ &= (\beta-1)\int_{\rho\leqslant\alpha_1}\nabla\otimes\mathbf{r}\,dV(\rho,\mu,\nu)\cdot\sum_{m=1}^{3}\frac{1}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m} \\ &= \frac{4\pi}{3}\sum_{m=1}^{3}\frac{(\beta-1)\alpha_1\alpha_2\alpha_3}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^m(\alpha_1)}\hat{\mathbf{x}}_\mathbf{m}\otimes\hat{\mathbf{x}}_\mathbf{m}\ . \end{aligned} \tag{2.469}$$

With all the basic potentials calculated for the case of the ellipsoid of surface $\rho = \alpha_1$, it is now possible to calculate the low-frequency coefficients of the total fields and the scattering amplitudes for each scattering problems we studied in this chapter, but for the case of ellipsoidal scatterer, since all the low-frequency coefficients have been expressed in terms of these basic potential functions.

2.4 Low-Frequency Formulas for Ellipsoids

In this section, we calculate the basic potential functions for the ellipsoidal coordinate system and from them we derive the low-frequency coefficients for the ellipsoidal scatterers.

Acoustically soft ellipsoid

The zeroth-low frequency coefficient of the total field u_0^+ for the case of the ellipsoid, based on (2.81) and (2.412):

$$u_0^+(\mathbf{r}) = 1 - \phi^c = 1 - \frac{I_0^1(\rho)}{I_0^1(\alpha_1)} , \quad \rho > \alpha_1 . \tag{2.470}$$

The first low-frequency coefficient of the total field u_1^+, based on(2.89), (2.415) and (2.430) is given by:

$$u_1^+(\mathbf{r}) = \frac{1}{I_0^1(\alpha_1)} \left(\frac{I_0^1(\rho)}{I_0^1(\alpha_1)} - 1 \right) + \sum_{n=1}^{3} \left(1 - \frac{I_1^n(\rho)}{I_1^n(\alpha_1)} \right) \hat{\mathbf{d}} \cdot \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \mathbf{r} , \quad \rho > \alpha_1 . \tag{2.471}$$

For the scattering amplitude, based on (2.92),(2.94), (2.415) and the fact that $\mathbf{c}$ defined in (2.85) vanishes for the ellipsoid, we have:

$$u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = -ik \frac{1}{I_0^1(\alpha_1)} - k^2 \left(\frac{1}{I_0^1(\alpha_1)} \right)^2 + \mathcal{O}(k^3) , \quad k \to 0 . \tag{2.472}$$

Acoustically hard ellipsoid:

Based on (2.119), (2.125), (2.444) and (2.133) , we have:

$$u_0^+(\mathbf{r}) = 1 , \quad \rho > \alpha_1 , \tag{2.473}$$

$$u_1^+(\mathbf{r}) = \hat{\mathbf{d}} \cdot [\mathbf{r} - \mathbf{V}(\mathbf{r})] = \mathbf{r} \cdot \sum_{n=1}^{3} \left(1 + \frac{\alpha_1\alpha_2\alpha_3 I_1^n(\rho)}{1 - \alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \right) \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{d}} , \quad \rho > \alpha_1 , \tag{2.474}$$

$$u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = ik^3 A_3(\hat{\mathbf{r}}; \hat{\mathbf{d}}) + \mathcal{O}(k^4) = ik^3 \frac{\alpha_1\alpha_2\alpha_3}{3} \left[\sum_{n=1}^{3} \frac{\hat{\mathbf{d}} \cdot \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{r}}}{1 - \alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} - 1 \right] + \mathcal{O}(k^4) , \quad k \to 0 . \tag{2.475}$$

Ellipsoidal acoustic impedance:

Based on (2.134) and (2.135), we obtain:

$$u_0^+(\mathbf{r}) = 1 , \quad \rho > \alpha_1 , \tag{2.476}$$

$$\begin{aligned} u_1^+(\mathbf{r}) &= \hat{\mathbf{d}} \cdot [\mathbf{r} - \boldsymbol{\Psi}(\mathbf{r})] - R\psi^c(\mathbf{r}) \\ &= \mathbf{r} \cdot \sum_{n=1}^{3} \left(1 + \frac{\alpha_1\alpha_2\alpha_3 I_1^n(\rho)}{1 - \alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \right) \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{d}} - R\psi^c , \quad \rho > \alpha_1 , \end{aligned} \tag{2.477}$$

where ψ^c defined as the solution of (2.136) which is not trivial. The scattering amplitude, based on (2.138), is given by:

$$u^\infty(\hat{\mathbf{r}}; \hat{\mathbf{d}}) = k^2 A_2(\hat{\mathbf{r}}; \hat{\mathbf{d}}) + \mathcal{O}(k^3) = -k^2 \frac{R|S|}{4\pi} , \quad k \to 0 . \tag{2.478}$$

Acoustic transmission problem for the ellipsoid

Based on relations (2.158), (2.159), (2.179), (2.465) and (2.466), we obtain:

$$u_0^+(\mathbf{r}) = 1 \,, \quad \rho > \alpha_1 \,, \tag{2.479}$$

$$u_0^-(\mathbf{r}) = 1 \,, \quad h_2 \leqslant \rho < \alpha_1 \,, \tag{2.480}$$

$$u_1^+(\mathbf{r}) = \mathbf{r} \cdot \sum_{n=1}^{3} \left[1 - \frac{(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^n(\rho)}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \right] \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{d}} \,, \quad \rho > \alpha_1 \,, \tag{2.481}$$

$$u_1^-(\mathbf{r}) = \sum_{n=1}^{3} \frac{\hat{\mathbf{d}} \cdot \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \mathbf{r}}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \,, \quad h_2 \leqslant \rho < \alpha_1 \,, \tag{2.482}$$

The scattering amplitude, based on (2.193) and (2.469), we get:

$$\begin{aligned} u^\infty(\hat{\mathbf{r}};\hat{\mathbf{d}}) &= -ik^3 \frac{1}{4\pi} \left[(1-\beta\eta^2)|V^-| + \hat{\mathbf{r}} \cdot \widetilde{\mathbf{X}}(\beta) \cdot \hat{\mathbf{d}} \right] \\ &= ik^3 \frac{\alpha_1\alpha_2\alpha_3}{3} \left[(\beta\eta^2-1) - (\beta-1) \sum_{n=1}^{3} \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{d}}}{1+(\beta-1)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \right] + \mathcal{O}(k^4) \,, \quad k \to 0 \,. \end{aligned} \tag{2.483}$$

The only changes to the low-frequency coefficients given above, between the lossless and the lossy transmission problems, are the values of β and η based on (2.194) and (2.195).

Ellipsoidal perfect conductor:

Based on relations (2.317), (2.323), (2.416), (2.444):

$$\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{p}} - \hat{\mathbf{p}} \cdot \sum_{n=1}^{3} \frac{1}{I_1^n(\alpha_1)} \left[I_1^n(\rho) \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2-\alpha_1^2+\alpha_n^2)\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] \,, \tag{2.484}$$

$$\mathbf{H}_0^+(\mathbf{r}) = Y^+\hat{\mathbf{q}} + Y^+\hat{\mathbf{q}} \cdot \sum_{n=1}^{3} \frac{\alpha_1\alpha_2\alpha_3}{1-\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \left[I_1^n(\rho) \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} \right. \tag{2.485}$$

$$\left. - \frac{x_n}{(\rho^2-\alpha_1^2+\alpha_n^2)\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] \,, \tag{2.486}$$

for $\rho > \alpha_1$. The scattering amplitude based on (2.330), becomes:

$$\begin{aligned} \mathbf{E}^\infty(\hat{\mathbf{r}}) = -ik^3 \frac{\alpha_1\alpha_2\alpha_3}{3} \sum_{n=1}^{3} & \left[\frac{1}{\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) \, (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) \right. \\ & \left. - \frac{1}{1-\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) \, (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) \right] + \mathcal{O}(k^5) \,, \quad k \to 0 \,. \end{aligned} \tag{2.487}$$

$$\begin{aligned} \mathbf{H}^\infty(\hat{\mathbf{r}}) = -ik^3 \frac{\alpha_1\alpha_2\alpha_3}{3} Y^+ \sum_{n=1}^{3} & \left[\frac{1}{1-\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) \, (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) \right. \\ & \left. + \frac{1}{\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) \, (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) \right] + \mathcal{O}(k^5) \,, \quad k \to 0 \,. \end{aligned} \tag{2.488}$$

The impedance problem for the ellipsoid:

The impendance problem as mentioned in the theory does not have a general solution. Nevertheless the zeroth low-frequency coefficient can can be found [19]:

$$\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{p}} - \hat{\mathbf{p}} \cdot \sum_{n=1}^{3} \frac{1}{I_1^n(\alpha_1)} \left[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2 - \alpha_1^2 + \alpha_n^2)\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] , \tag{2.489}$$

$$\mathbf{H}_0^+(\mathbf{r}) = Y^+\hat{\mathbf{q}} + Y^+\hat{\mathbf{q}} \cdot \sum_{n=1}^{3} \frac{1}{I_1^n(\alpha_1)} \left[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2 - \alpha_1^2 + \alpha_n^2)\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] , \tag{2.490}$$

for $\rho > \alpha_1$. The scattering amplitude is given by [19] :

$$\mathbf{E}^\infty(\hat{\mathbf{r}}) = -\frac{ik^3}{3} \sum_{n=1}^{3} \frac{1}{I_1^n(\alpha_1)} \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) + (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) \right] + \mathcal{O}(k^4) , \quad k \to 0 , \tag{2.491}$$

$$\mathbf{H}^\infty(\hat{\mathbf{r}}) = \frac{ik^3}{3} Y^+ \sum_{n=1}^{3} \frac{1}{I_1^n(\alpha_1)} \left[(\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) - \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) \right] + \mathcal{O}(k^4) , \quad k \to 0 \ . \ . \tag{2.492}$$

Lossless transmission problem for the ellipsoid:

Based on relations (2.362), (2.363), (2.465) and (2.466), we have:

$$\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{p}} - \hat{\mathbf{p}} \cdot \sum_{n=1}^{3} \frac{(\mu^+\eta^2 - \mu^-)\alpha_1\alpha_2\alpha_3}{(\mu^+\eta^2 - \mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1) + \mu^-} \left[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2 - \alpha_1^2 + \alpha_n^2)\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] , \quad \rho > \alpha_1 \tag{2.493}$$

$$\mathbf{E}_0^-(\mathbf{r}) = \hat{\mathbf{p}} \cdot \sum_{n=1}^{3} \frac{\mu^-}{(\mu^+\eta^2 - \mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1) + \mu^-} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} , \quad h_2 \leqslant \rho < \alpha_1 , \tag{2.494}$$

$$\mathbf{H}_0^+(\mathbf{r}) = Y^+\hat{\mathbf{q}} + Y^+\hat{\mathbf{q}} \cdot \sum_{n=1}^{3} \frac{(\mu^+ - \mu^-)\alpha_1\alpha_2\alpha_3}{\mu^+ - (\mu^+ - \mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \left[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2 - \alpha_1^2 + \alpha_n^2)\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\boldsymbol{\rho}} \right] , \quad \rho > \alpha_1 \tag{2.495}$$

$$\mathbf{H}_0^-(\mathbf{r}) = Y^+\hat{\mathbf{q}} \cdot \sum_{n=1}^{3} \frac{\mu^+}{\mu^+ - (\mu^+ - \mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)} \hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n} , \quad h_2 \leqslant \rho < \alpha_1 . \tag{2.496}$$

The scattering amplitude based on (2.372), for the case of the ellipsoid can be rewritten as:

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}}) = -ik^3\frac{\alpha_1\alpha_2\alpha_3}{3}\sum_{n=1}^{3}\Bigg[\frac{(\mu^+\eta^2-\mu^-)}{(\mu^+\eta^2-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)+\mu^-}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{p}})$$
$$+\frac{\mu^+-\mu^-}{(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)-\mu^+}(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{q}})\Bigg]+\mathcal{O}(k^4)\ ,\quad k\to 0\ . \tag{2.497}$$

$$\mathbf{H}^{\infty}(\hat{\mathbf{r}}) = ik^3\frac{\alpha_1\alpha_2\alpha_3}{3}Y^+\sum_{n=1}^{3}\Bigg[\frac{(\mu^+\eta^2-\mu^-)}{(\mu^+\eta^2-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)+\mu^-}(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{p}})$$
$$-\frac{\mu^+-\mu^-}{(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)-\mu^+}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{q}})\Bigg]+\mathcal{O}(k^4)\ ,\quad k\to 0\ . \tag{2.498}$$

(2.499)

Lossy transmission problem for the ellipsoid:

Based on (2.398),(2.397), (2.400), (2.404), (2.416), (2.465) and (2.466), we get:

$$\mathbf{E}_0^+(\mathbf{r}) = \hat{\mathbf{p}} - \hat{\mathbf{p}}\cdot\sum_{n=1}^{3}\frac{1}{I_1^n(\alpha_1)}\left[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n}\otimes\hat{\mathbf{x}}_\mathbf{n} - \frac{x_n}{(\rho^2-\alpha_1^2+\alpha_n^2)\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\hat{\mathbf{x}}_\mathbf{n}\otimes\hat{\boldsymbol{\rho}}\right]\ ,\quad \rho>\alpha_1\ , \tag{2.500}$$

$$\mathbf{E}_0^-(\mathbf{r}) = \mathbf{0}\ ,\quad h_2\leqslant\rho<\alpha_1\ , \tag{2.501}$$

$$\mathbf{H}_0^+(\mathbf{r}) = Y^+\hat{\mathbf{q}} + Y^+\hat{\mathbf{q}}\cdot\sum_{n=1}^{3}\frac{(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3}{\mu^+-(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)}\Bigg[I_1^n(\rho)\hat{\mathbf{x}}_\mathbf{n}\otimes\hat{\mathbf{x}}_\mathbf{n}-$$
$$-\frac{x_n}{(\rho^2-\alpha_1^2+\alpha_n^2)\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}}\hat{\mathbf{x}}_\mathbf{n}\otimes\hat{\boldsymbol{\rho}}\Bigg]\ ,\quad\rho>\alpha_1\ , \tag{2.502}$$

$$\mathbf{H}_0^-(\mathbf{r}) = Y^+\hat{\mathbf{q}}\cdot\sum_{n=1}^{3}\frac{\mu^+}{\mu^+-(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)}\hat{\mathbf{x}}_\mathbf{n}\otimes\hat{\mathbf{x}}_\mathbf{n}\ ,\quad h_2\leqslant\rho<\alpha_1\ . \tag{2.503}$$

The scattering amplitude is given by (2.404) and (2.469):

$$\mathbf{E}^{\infty}(\hat{\mathbf{r}}) = -ik^3\frac{\alpha_1\alpha_2\alpha_3}{3}\sum_{n=1}^{3}\Bigg[\frac{1}{\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{p}})$$
$$+\frac{\mu^+-\mu^-}{(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)-\mu^+}(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{q}})\Bigg]+\mathcal{O}(k^4)\ ,\quad k\to 0\ . \tag{2.504}$$

$$\mathbf{H}^{\infty}(\hat{\mathbf{r}}) = ik^3\frac{\alpha_1\alpha_2\alpha_3}{3}Y^+\sum_{n=1}^{3}\Bigg[\frac{1}{\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)}(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{p}})$$
$$-\frac{\mu^+-\mu^-}{(\mu^+-\mu^-)\alpha_1\alpha_2\alpha_3 I_1^n(\alpha_1)-\mu^+}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n}\cdot\hat{\mathbf{q}})\Bigg]+\mathcal{O}(k^4)\ ,\quad k\to 0\ . \tag{2.505}$$

Remark: The sphere, the spheroid, the needle and the disk can be considered as geometrically degenerate forms of the ellipsoid for appropriate values of $\alpha_1, \alpha_2, \alpha_3$. Specifically, for the case of the sphere, the corresponding formulas can easily be obtained from the above formulas by substituting the corresponding forms of the elliptic integrals (1.175) that are given as follows:

Sphere

$$\alpha_1=\alpha_2=\alpha_3\quad ,\quad h_2=h_3=0\ ,\quad I_0^1(\rho)=\frac{1}{\rho}\ ,\quad I_1^n(\rho)=\frac{1}{3\rho^3}\ ,\quad n=1,2,3\ . \tag{2.506}$$

Chapter 3

Inverse Scattering Problems for Ellipsoids

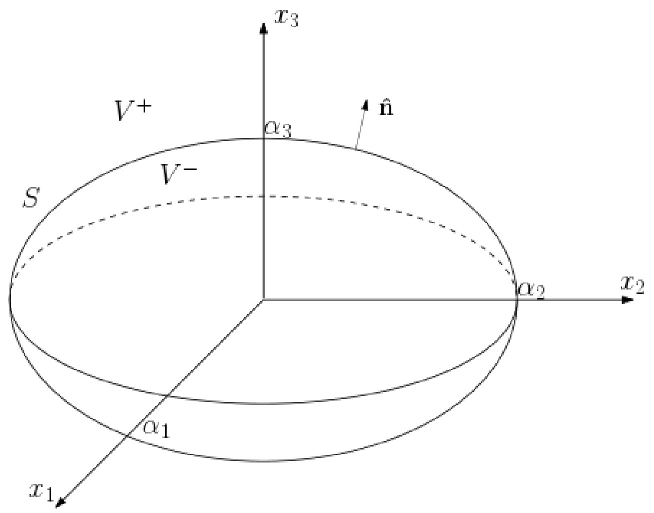
In this chapter we will study a type of inverse scattering problems in which the incident field (plane wave incidence assumed) and the scattered field are known and we try to determine the size and the orientation of the ellipsoidal scatterer as well as physical parameters of the interior of the ellipsoidal scatterer for the case of the transmission scattering problems. In order to solve these problems we use the low-frequency approximations calculated in the previous sections which are separated in near-field data (exterior total field) and the far-field data (scattering amplitude).

In a series of papers starting with [12], Dassios and his colleagues studied solved these inverse problems using far-field data. Specifically, for the acoustic waves, they used the low-frequency expansions of the scattering amplitude in order to find the size and the orientation of the acoustically soft [12], the acoustically hard [24] and the penetrable [16] ellipsoidal scatterers. Later, in collaboration with R. Lucas they solved the corresponding electromagnetic inverse scattering problem for the perfectly conductive ellipsoidal scatterer [17] using far-field data. In 2017, C. E. Athanasiadis, E. S. Athanasiadou, S. Zoi and I. Arkoudis extended this method for the lossless and the lossy dielectric ellipsoids [7]. Moreover, we suggested a method for solving this type of inverse scattering problems using near-field data. A simplification of this method was applied the same year in [8] for the case of an acoustic two-layered ellipsoid.

The inverse scattering problems, in general, are nonlinear and not well posed due to the fact that small pertrubations can lead to big errors in the results. The nonlinearity enters in our problems via the Euler angles as well as the elliptic integrals which connect the exterior with the interior ellipsoidal harmonics. Nevertheless, there is some kind of uniqueness, in the sense that if the scattering amplitudes of two scatterers coincide for all directions and polarizations of incidence, then the two scatterers are identical. Moreover, the a-priori knowledge of the ellipsoidal shape of the scatterer in our problems can secure the well posedness of our problems [16], since we can use the ellipsoidal formulas that are derived in the last section of chapter 2. By taking measurements of far-field data or near-field data and by constructing a measurement matrix via them, we can use its eigenvectors to specify the orientation of the ellipsoid and its eigenvalues to obtain the semi-axes of the scatterer and theferore its size. Specifically, since the measurement matrix is real and symmetric, its eigenvalues depend continuously on the elements of the matrix, which means that small perturbations on the elements (measured data) result small perturbations on the eigenvalues (Wilkinson). Then by the implicit function theorem we obtain that the perturbations on the semi-axes will be small as well. The perturbations on the eigenvectors of the measurement matrix depend on the perturbation on the elements as well as to the spacing of the eigenvalues. Since we have already seen that the perturbations on the eigenvalues are small, we obtain that the perturbation on the eigenvectors will be small too which means that the perturbations on the Euler angles and therefore on the orientation of the ellipsoidal scatterer will stay small [17]. For further study of the well-posedness of inverse scattering problems, we refer to [10], [11], [22] and [27].

3.1 Inverse Scattering Problems for Acoustic Waves

In this section we present the method followed by Dassios and Lucas for the solution of the inverse problems for the acoustically soft, acoustically hard and acoustically penetrable ellipsoids, using far-field data.



Let V^+ be the exterior and V^- be the interior region of an ellipsoidal scatterer of surface S with equation:

$$\sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2} = 1 \ , \tag{3.1}$$

where $\alpha_1 > \alpha_2 > \alpha_3 > 0$ are the semi-axes of the ellipsoidal scatterer and $\hat{\mathbf{n}}$ the outward unit normal vector on S. The superscripts $+, -$ will denote parameters or functions in the regions V^+, V^- respectively. In the ellipsoidal coordinate system the surface S is defined by $\rho = \alpha_1$, the exterior by $\rho > \alpha_1$ and the interior by $\sqrt{\alpha_1^2 - \alpha_3^2} \leq \rho < \alpha_1$. The purpose of this section is to determine the size and the orientation of this ellipsoid using far-field data or near-field data, assuming that we know the center of the scatterer.

In what follows, with the superscripts $(D), (N), (T)$ we will denote the acoustically soft (2.67) , acoustically hard (2.114) and acoustically penetrable (2.139) cases respectively.

3.1.1 Acoustically Soft Ellipsoid

The scattering amplitude for the case of the acoustically soft ellipsoid is given by (2.472). Letting $\hat{\mathbf{r}} = \hat{\mathbf{d}}$ and taking the real part of the low frequency approximation of the far-field pattern for the acoustically soft ellipsoid we obtain [12] :

$$-\mathrm{Re}\left\{u^{\infty,(D)}(\hat{\mathbf{d}},\hat{\mathbf{d}})\right\} = k^2 \frac{1}{(I_0^1)^2} + k^4 \left[T - \frac{1}{3(I_0^1)^2}\left(i_1^2\alpha_1^2 + i_2^2\alpha_2^2 + i_3^2\alpha_3^2\right)\right] + \mathcal{O}(k^6) \ , \ k \to 0 \ , \tag{3.2}$$

where

$$T = -\frac{1}{\left(I_0^1\right)^4} + \frac{5}{9\left(I_0^1\right)^2}\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right) - \frac{2}{3\left(I_0^1\right)^3}\left(\alpha_1^4 I_1^1 + \alpha_2^4 I_1^2 + \alpha_3^4 I_1^3\right) \tag{3.3}$$

and $\hat{\mathbf{d}} = (i_1, i_2, i_3)$. We will use the $A_2^{(D)}$ and $A_4^{(D)}$ leading two coefficients of (3.2) that can be written as:

$$A_2^{(D)}(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = -\frac{1}{\left(I_0^1\right)^2} \ , \tag{3.4}$$

$$A_4^{(D)}(\hat{\mathbf{d}}, \hat{\mathbf{d}}) = -\left[T - \frac{1}{3\left(I_0^1\right)^2}\left(\hat{\mathbf{d}}^\top A \hat{\mathbf{d}}\right)\right] \ , \tag{3.5}$$

with

$$A = diag\,(A_n) \quad , \quad A_n = \alpha_n^2 \quad , \tag{3.6}$$

for $n = 1, 2, 3$. We consider two Cartesian systems with the same origin and their corresponding orthonormal bases $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ and $\{\hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3\}$. The system $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ coincides with the principal directions of the unknown ellipsoid while the system $\{\hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3\}$ is a known reference system. In order to specify the orientation and the size of the ellipsoid we will transform the $\mathbf{x}'_\mathbf{i}$ system to the $\mathbf{x}_\mathbf{i}$. To achieve this transformation we will use an orthogonal rotation matrix P whose elements are given in terms of the Euler angles (α, β, γ) as follows [28] :

$$P = \begin{bmatrix} cos\alpha\ cos\gamma - cos\beta\ sin\alpha\ sin\gamma & sin\alpha\ cos\gamma + cos\beta\ cos\alpha\ sin\gamma & sin\beta\ sin\gamma \\ -cos\alpha\ sin\gamma - cos\beta\ sin\alpha\ cos\gamma & -sin\alpha\ sin\gamma + cos\beta\ cos\alpha\ cos\gamma & sin\beta\ cos\gamma \\ sin\beta\ sin\alpha & -sin\beta\ cos\alpha & cos\beta \end{bmatrix} \tag{3.7}$$

Therefore, the vector $\hat{\mathbf{d}}$ satisfies the rotation relation:

$$\hat{\mathbf{d}} = P\hat{\mathbf{d}}' \ . \tag{3.8}$$

Applying rotation (3.8) in (3.5) we obtain:

$$A_4^{(D)}(\hat{\mathbf{d}}, \hat{\mathbf{d}}) = -\left[T - \frac{1}{3\left(I_0^1\right)^2}\left(\hat{\mathbf{d}'}^\top P^\top A P \hat{\mathbf{d}'}\right)\right] \ , \tag{3.9}$$

We take one measurement of $-A_2$ for any direction $\hat{\mathbf{d}}$ and six measurements of $-A_4$ for six directions as follows:

$$\begin{aligned} &\hat{\mathbf{d}}'_\mathbf{1} = \hat{\mathbf{x}}'_\mathbf{1} \quad , \quad \hat{\mathbf{d}}'_\mathbf{2} = \hat{\mathbf{x}}'_\mathbf{2} \quad , \quad \hat{\mathbf{d}}'_\mathbf{3} = \hat{\mathbf{x}}'_\mathbf{3} \\ &\hat{\mathbf{d}}'_\mathbf{4} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_\mathbf{1} + \hat{\mathbf{x}}'_\mathbf{2}\right) \quad , \quad \hat{\mathbf{d}}'_\mathbf{5} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_\mathbf{1} + \hat{\mathbf{x}}'_\mathbf{3}\right) \quad , \quad \hat{\mathbf{d}}'_\mathbf{6} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_\mathbf{2} + \hat{\mathbf{x}}'_\mathbf{3}\right) \quad . \end{aligned} \tag{3.10}$$

that can also be written as:

$$\begin{cases} \hat{\mathbf{d}}'_\mathbf{i} = \hat{\mathbf{x}}'_\mathbf{i} \ , \ \text{for} \ \ i = 1, 2, 3 \ , \\ \hat{\mathbf{d}}'_\mathbf{i+j+1} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_\mathbf{i} + \hat{\mathbf{x}}'_\mathbf{j}\right) \ , \ \text{for} \ \ i, j = 1, 2, 3 \ , \ \text{with} \ \ i \neq j \ . \end{cases} \tag{3.11}$$

Therefore, the measurements are the following:

$$\begin{aligned}
m_0 &= -A_2^{(D)}(\hat{\mathbf{r}}', \hat{\mathbf{d}}') = \frac{1}{\left(I_0^1\right)^2} \;, \\
m_1 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{1}}, \hat{\mathbf{d}}'_{\mathbf{1}}) = T - \frac{m_0}{3}\mathbf{P}_{\mathbf{1}}^\top A \mathbf{P}_{\mathbf{1}} \;, \\
m_2 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{2}}, \hat{\mathbf{d}}'_{\mathbf{2}}) = T - \frac{m_0}{3}\mathbf{P}_{\mathbf{2}}^\top A \mathbf{P}_{\mathbf{2}} \;, \\
m_3 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{3}}, \hat{\mathbf{d}}'_{\mathbf{3}}) = T - \frac{m_0}{3}\mathbf{P}_{\mathbf{3}}^\top A \mathbf{P}_{\mathbf{3}} \;, \\
m_4 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{4}}, \hat{\mathbf{d}}'_{\mathbf{4}}) = T - \frac{m_0}{3}\frac{1}{2}\left(\mathbf{P}_{\mathbf{1}}^\top + \mathbf{P}_{\mathbf{2}}^\top\right) A \left(\mathbf{P}_{\mathbf{1}} + \mathbf{P}_{\mathbf{2}}\right) \\
&\qquad\qquad\qquad\quad = T - \frac{m_0}{6}\left(\mathbf{P}_{\mathbf{1}}^\top A \mathbf{P}_{\mathbf{1}} + 2\mathbf{P}_{\mathbf{1}}^\top A \mathbf{P}_{\mathbf{2}} + \mathbf{P}_{\mathbf{2}}^\top A \mathbf{P}_{\mathbf{2}}\right) \;, \\
m_5 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{5}}, \hat{\mathbf{d}}'_{\mathbf{5}}) = T - \frac{m_0}{6}\left(\mathbf{P}_{\mathbf{1}}^\top A \mathbf{P}_{\mathbf{1}} + 2\mathbf{P}_{\mathbf{1}}^\top A \mathbf{P}_{\mathbf{3}} + \mathbf{P}_{\mathbf{3}}^\top A \mathbf{P}_{\mathbf{3}}\right) \;, \\
m_6 &= -A_4^{(D)}(\hat{\mathbf{d}}'_{\mathbf{5}}, \hat{\mathbf{d}}'_{\mathbf{5}}) = T - \frac{m_0}{6}\left(\mathbf{P}_{\mathbf{2}}^\top A \mathbf{P}_{\mathbf{2}} + 2\mathbf{P}_{\mathbf{2}}^\top A \mathbf{P}_{\mathbf{3}} + \mathbf{P}_{\mathbf{3}}^\top A \mathbf{P}_{\mathbf{3}}\right) \;,
\end{aligned} \tag{3.12}$$

where $\mathbf{P_i} = P\hat{\mathbf{x}}'_{\mathbf{i}}$ is the i-th column of matrix P and $\mathbf{P}_{\mathbf{i}}^\top = \hat{\mathbf{x}}'^\top_{\mathbf{i}} P^\top$ is the i-th row of matrix $P^\top$ and where the relation

$$\mathbf{P}_{\mathbf{i}}^\top A \mathbf{P_j} = \mathbf{P}_{\mathbf{j}}^\top A \mathbf{P_i} \;, \tag{3.13}$$

for $i, j = 1, 2, 3$, due to the fact that A is a diagonal matrix, has been used.
We construct a measurement matrix $M^{(D)} = (M_{ij}^{(D)})$ for $i, j = 1, 2, 3$, with elements of the form:

$$M_{ij}^{(D)} = \mathbf{P}_{\mathbf{i}}^\top \left(A - \frac{3T}{m_0} I\right) \mathbf{P_j} \;. \tag{3.14}$$

Relation (3.14) due to (3.13) can be written in matrix form as follows:

$$M^{(D)} = P^\top \left(A - \frac{3T}{m_0} I\right) P \quad \Leftrightarrow PM^{(D)}P^\top = A - \frac{3T}{m_0} I \quad . \tag{3.15}$$

Therefore, the measurement matrix $M^{(D)}$ can be given in terms of $m_1, ..., m_6$ as follows:

$$M^{(D)} = \frac{3}{2m_0}\begin{bmatrix} -2m_1 & (m_1 + m_2 - 2m_4) & (m_1 + m_3 - 2m_5) \\ (m_1 + m_2 - 2m_4) & -2m_2 & (m_2 + m_3 - 2m_6) \\ (m_1 + m_3 - 2m_5) & (m_2 + m_3 - 2m_6) & -2m_3 \end{bmatrix} \tag{3.16}$$

where the orthogonality of P has been used (i.e. $\mathbf{P}_{\mathbf{i}}^\top \mathbf{P_i} = 1$ and $\mathbf{P}_{\mathbf{i}}^\top \mathbf{P_j} = 0$ for $i \neq j$).
Since $M^{(D)}$ is real and symmetric it has three real eigenvalues $\lambda_1^{(D)}, \lambda_2^{(D)}, \lambda_3^{(D)}$ and three orthonormal eigenvectors $\mathbf{v_1}^{(D)}, \mathbf{v_2}^{(D)}, \mathbf{v_3}^{(D)}$. Therefore, due to the orthogonal similarity relation (3.15) we obtain:

$$\lambda_n^{(D)} = A_n - \frac{3T}{m_0} \;, \tag{3.17}$$

$$\mathbf{v_n}^{(D)} = (P_{n1}, P_{n2}, P_{n3})^{(D)} \;, \tag{3.18}$$

for $n = 1, 2, 3$, where P_{n1}, P_{n2}, P_{n3} are the elements of the n^{th} row of the rotation matrix P and the superscript (D) denotes the acoustically soft case that we study in this subsection. From the system of equations (3.18) that connect the eigenevectors of matrix M with the elements of matrix P, we obtain the elements of matrix P and therefore we can specify the Euler angles by using the following relations [28], [20], [28] :

$$\alpha = sin^{-1}\left(\frac{P_{31}}{\sqrt{1 - P_{33}^2}}\right) \;,\; \beta = sin^{-1}\left(\sqrt{1 - P_{33}^2}\right) \;,\; \gamma = sin^{-1}\left(\frac{P_{13}}{\sqrt{1 - P_{33}^2}}\right) \quad . \tag{3.19}$$

These angles show the orientation of the ellipsoid.
From the system of equations (3.17) that connect the eigenvalues of matrix M with the diagonal elements of matrix $A - \frac{3T}{m_0} I$ we obtain:

$$\lambda_n^{(D)} = \alpha_n^2 - \frac{3T}{m_0} \quad \Leftrightarrow \quad a_n^2 = \lambda_n^{(D)} + \frac{3T}{m_0} \tag{3.20}$$

for $n = 1, 2, 3$. Therefore, in order to specify the semi-axes α_n of the ellipsoid we need to calculate T. Substituting in the definition of I_0^1 (1.175), the semi-axes α_n from (3.20) and using the m_0 measurement of (3.12) we obtain:

$$\int_0^\infty \frac{dx}{\sqrt{x + \lambda_1^{(D)} + \frac{3T}{m_0}}\sqrt{x + \lambda_2^{(D)} + \frac{3T}{m_0}}\sqrt{x + \lambda_3^{(D)} + \frac{3T}{m_0}}} = \frac{2}{\sqrt{m_0}} \ . \tag{3.21}$$

Using the transformation

$$x = \frac{(\lambda_1^{(D)} - \lambda_3^{(D)})}{t^2} - (\lambda_1^{(D)} + \frac{3T}{m_0}) \ , \tag{3.22}$$

relation (3.21) can be written in its canonical form:

$$F(\phi_0, \alpha_0) = \int_0^{sin\phi_0} \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2 sin^2\alpha_0}} = \sqrt{\frac{\lambda_1^{(D)} - \lambda_3^{(D)}}{m_0}} \ , \tag{3.23}$$

where $F(\phi_0, \alpha_0)$ is the imcomplete elliptic integral of the first kind (1.180),(1.181), with

$$\phi_0 = sin^{-1}\sqrt{\frac{\lambda_1^{(D)} - \lambda_3^{(D)}}{\lambda_1^{(D)} + \frac{3T}{m_0}}} \ , \tag{3.24}$$

$$\alpha_0 = sin^{-1}\sqrt{\frac{\lambda_1^{(D)} - \lambda_2^{(D)}}{\lambda_1^{(D)} - \lambda_3^{(D)}}} \ . \tag{3.25}$$

From (3.24) we obtain T as follows:

$$T = \frac{1}{3} m_0 \left[(\lambda_1^{(D)} - \lambda_3^{(D)}) cot^2\phi_0 - \lambda_3^{(D)} \right] \ . \tag{3.26}$$

Therefore, having found T we obtain the semi-axes α_n for $n = 1, 2, 3$ from (3.20) as follows:

$$\alpha_1^2 = (\lambda_1^{(D)} - \lambda_3^{(D)}) cot^2\phi_0 + \left(\lambda_1^{(D)} - \lambda_3^{(D)}\right) \ , \tag{3.27}$$

$$\alpha_2^2 = (\lambda_1^{(D)} - \lambda_3^{(D)}) cot^2\phi_0 + \left(\lambda_2^{(D)} - \lambda_3^{(D)}\right) \ , \tag{3.28}$$

$$\alpha_1^2 = (\lambda_1^{(D)} - \lambda_3^{(D)}) cot^2\phi_0 \tag{3.29}$$

and therefore we obtain the size of the ellipsoid .

For the geometrically degenerate case of the spheroid, assuming $\alpha_2 = \alpha_3$, we only need to determine the two semi-axes α_1, α_2 as well as the two Euler angles α, β. Letting $\gamma = 0$ and denoting the corresponding rotation matrix with $P_{(0)}$ we obtain from the measurements m_1, m_2, m_3, the following relations from the elements $M_{ii} = \mathbf{P_{i}}_{(0)}^{\top} A \mathbf{P_{i}}_{(0)}$, for $i = 1, 2, 3$, by replacing P with $P_{(0)}$:

$$\alpha_1^2 cos^2\alpha + \alpha_2^2 sin^2\alpha = \frac{3}{m_0}(T - m_1) \ , \tag{3.30}$$

$$\alpha_1^2 sin^2\alpha + \alpha_2^2 cos^2\alpha = \frac{3}{m_0}(T - m_2) \ , \tag{3.31}$$

$$\alpha_2^2 = \frac{3}{m_0}(T - m_3) \ . \tag{3.32}$$

From this system we obtain the semi-axes α_1, α_2 in terms of T as well as the Euler angle α:

$$\alpha_1^2 = \frac{3}{m_0}(T - m_1 - m_2 + m_3) \ , \tag{3.33}$$

$$\alpha_2^2 = \frac{3}{m_0}(T - m_3) \ , \tag{3.34}$$

$$sin^2\alpha = \frac{(m_2 - m_3)}{(m_1 + m_2 - 2m_3)} \ . \tag{3.35}$$

In order to obtain the second Euler angle β we proceed as follows. Letting $\gamma = \frac{\pi}{2}$ and denoting the corresponding rotation matrix with $P_{(\pi/2)}$, since the measurement m_1 is independent of this rotation, we obtain from $\mathbf{P_1}_{(0)}^\top A\mathbf{P_1}_{(0)} = \mathbf{P_1}_{(\pi/2)}^\top A\mathbf{P_1}_{(\pi/2)}$ and (3.30):

$$\alpha_1^2 cos^2\beta sin^2\alpha + \alpha_2^2(cos^2\alpha sin^2\beta sin^2\alpha) = \frac{3}{m_0}(T - m_1) \tag{3.36}$$

and therefore using (3.33)-(3.35), we obtain the Euler angle β:

$$cos^2\beta = \frac{m_1 - m_3}{m_2 - m_3} \ . \tag{3.37}$$

In order to specify T, from $m_0 = 1/(I_0^1)^2$ we obtain:

$$I_0^1 = \begin{cases} \dfrac{1}{\sqrt{\alpha_1^2 - \alpha_2^2}} cosh^{-1}\left(\dfrac{\alpha_1}{\alpha_2}\right) & , \quad \alpha_1 > \alpha_2 \\ \dfrac{1}{\sqrt{\alpha_2^2 - \alpha_1^2}} cos^{-1}\left(\dfrac{\alpha_1}{\alpha_2}\right) & , \quad \alpha_1 < \alpha_2 \end{cases} \tag{3.38}$$

and

$$T = \frac{[(\delta + 1)m_3 - m_1 - m2]}{(\delta - 1)} \ , \tag{3.39}$$

with

$$\delta = \begin{cases} cosh^2\left(\dfrac{\sqrt{3(2m_3 - m_1 - m_2)}}{m_0}\right) & , \quad \alpha_1 > \alpha_2 \\ cos^2\left(\dfrac{\sqrt{3(m_1 + m_2 - 2m_3)}}{m_0}\right) & , \quad \alpha_1 < \alpha_2 \end{cases} \ . \tag{3.40}$$

Having found T we obtain the semi-axes from (3.33)-(3.34).
For the geometrically degenerate case of the sphere $\alpha_1 = \alpha_2 = \alpha_3$ there is no need to specify the orientation due to the symmetry of the sphere and we only need to determine the radius of the sphere α_1. From the measurement $m_0 = 1/(I_0^1)^2$ we can directly specify the radius α_1 since $I_0^1 = \frac{1}{\alpha_1}$ for the case of the sphere.
Therefore, for the case of the spheroid only four out of the seven measurements are needed for the specification of its size and orientation and for the case of the sphere only one measurement is needed for the specification of its size. Nevertheless, if there is no a-priori knowledge of the actual shape of the scatterer we need to take all seven measurements, construct the measurement matrix $M^{(D)}$ and find its eigenvalues in order to see whether there are two or three equal eigenvalues that will lead to the conclusion that the scatterer is a spheroid or a sphere respectively.

3.1.2 Acoustically Hard Ellipsoid

The low-frequency expansion of the far-field pattern has the form (2.475). For the acoustically hard (rigid) ellipsoid, the coefficients $A_1^{(N)}$ and $A_2^{(N)}$ are equal to zero. We will use the leading order coefficient of the imaginary part of the low-frequency expansion which is the coefficient $A_3^{(N)}$. Letting $\hat{\mathbf{r}} = \hat{\mathbf{d}}$ in (2.475), the leading order coefficient becomes:

$$A_3^{(N)}(\hat{\mathbf{d}}, \hat{\mathbf{d}}) = \frac{V}{3}\left[-1 + \hat{\mathbf{d}}^\top H \hat{\mathbf{d}}\,\right] , \tag{3.41}$$

where $V = \alpha_1\alpha_2\alpha_3$ and $H = diag(H_n)$ for $n = 1, 2, 3$ with

$$H_n = \frac{1}{1 - V I_1^n} , \tag{3.42}$$

with $I_1^n \equiv I_1^n(\alpha_1)$. Similarly to the acoustically soft case we consider two Cartesian systems and use the orthogonal matrix P given by (3.7) to transform the known system $\mathbf{x'_i}$ to the unknown system $\mathbf{x_i}$. Therefore, using the rotation relation (3.8) in (3.41) we obtain:

$$A_3^{(N)}(\hat{\mathbf{d}}', \hat{\mathbf{d}}') = \frac{V}{3}\left[-1 + \hat{\mathbf{d}}'^\top P^\top H P \hat{\mathbf{d}}'\,\right] . \tag{3.43}$$

We take six measurements of $A_3^{(N)}(\hat{\mathbf{d}}', \hat{\mathbf{d}}')$ for six directions of propagation $\hat{\mathbf{d}}'$ as follows:

$$\begin{cases} \hat{\mathbf{d}}'_{\mathbf{i}} = \hat{\mathbf{x}}'_{\mathbf{i}} \ , \ \text{ for } \ i = 1, 2, 3 \ , \\ \hat{\mathbf{d}}'_{\mathbf{i+j+1}} = \dfrac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_{\mathbf{i}} + \hat{\mathbf{x}}'_{\mathbf{j}}\right) \ , \ \text{ for } \ i, j = 1, 2, 3 \ , \ \text{ with } \ i \neq j \ . \end{cases} \tag{3.44}$$

Therefore, the measurements are the following:

$$\begin{aligned} m_1 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{1}}, \hat{\mathbf{d}}'_{\mathbf{1}}) = \frac{V}{3}\left[-1 + \mathbf{P_1^\top} H \mathbf{P_1}\right] , \\ m_2 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{2}}, \hat{\mathbf{d}}'_{\mathbf{2}}) = \frac{V}{3}\left[-1 + \mathbf{P_2^\top} H \mathbf{P_2}\right] , \\ m_3 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{3}}, \hat{\mathbf{d}}'_{\mathbf{3}}) = \frac{V}{3}\left[-1 + \mathbf{P_3^\top} H \mathbf{P_3}\right] , \\ m_4 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{4}}, \hat{\mathbf{d}}'_{\mathbf{4}}) = \frac{V}{3}\left[-1 + \frac{1}{2}\left(\mathbf{P_1^\top} H \mathbf{P_1} + 2\mathbf{P_1^\top} H \mathbf{P_2} + \mathbf{P_2^\top} H \mathbf{P_2}\right)\right] , \\ m_5 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{5}}, \hat{\mathbf{d}}'_{\mathbf{5}}) = \frac{V}{3}\left[-1 + \frac{1}{2}\left(\mathbf{P_1^\top} H \mathbf{P_1} + 2\mathbf{P_1^\top} H \mathbf{P_3} + \mathbf{P_3^\top} H \mathbf{P_3}\right)\right] , \\ m_6 &= A_3^{(N)}(\hat{\mathbf{d}}'_{\mathbf{6}}, \hat{\mathbf{d}}'_{\mathbf{6}}) = \frac{V}{3}\left[-1 + \frac{1}{2}\left(\mathbf{P_2^\top} H \mathbf{P_2} + 2\mathbf{P_2^\top} H \mathbf{P_3} + \mathbf{P_3^\top} H \mathbf{P_3}\right)\right] , \end{aligned} \tag{3.45}$$

where the relation

$$\mathbf{P_i^\top} H \mathbf{P_j} = \mathbf{P_j^\top} H \mathbf{P_i} , \tag{3.46}$$

due to H being a diagonal matrix has been used. We construct a measurement matrix $M^{(N)} = (M_{ij}^{(N)})$ for $i, j = 1, 2, 3$, with elements of the form:

$$M_{ij}^{(N)} = \frac{V}{3}\mathbf{P_i^\top}\left(H - I\right)\mathbf{P_j} . \tag{3.47}$$

Relation (3.47) due to (3.46) can be written in matrix form as follows:

$$M^{(N)} = \frac{V}{3}P^\top\left(H - I\right)P \quad \Leftrightarrow PM^{(N)}P^\top = \frac{V}{3}(H - I) \quad . \tag{3.48}$$

Therefore, the measurement matrix M can be given in terms of $m_1, ..., m_6$ as follows:

$$M^{(N)} = \frac{1}{2}\begin{bmatrix} 2m_1 & (2m_4 - m_1 - m_2) & (2m_5 - m_1 - m_3) \\ (2m_4 - m_1 - m_2) & 2m_2 & (2m_6 - m_2 - m_3) \\ (2m_5 - m_1 - m_3) & (2m_6 - m_2 - m_3) & 2m_3 \end{bmatrix} \quad (3.49)$$

where the orthogonality of P has been used (i.e. $\mathbf{P_i}^\top \mathbf{P_i} = 1$ and $\mathbf{P_i}^\top \mathbf{P_j} = 0$ for $i \neq j$).
Since M is real and symmetric it has three real eigenvalues $\lambda_1^{(N)}, \lambda_2^{(N)}, \lambda_3^{(N)}$ and three orthonormal eigenvectors $\mathbf{v_1}^{(N)}, \mathbf{v_2}^{(N)}, \mathbf{v_3}^{(N)}$. Therefore, due to the orthogonal similarity relation (3.48) we obtain:

$$\lambda_n^{(N)} = \frac{V}{3}(H_n - 1) \; , \quad (3.50)$$

$$\mathbf{v_n}^{(N)} = (P_{n1}, P_{n2}, P_{n3})^{(N)} \; , \quad (3.51)$$

for $n = 1, 2, 3$, where P_{n1}, P_{n2}, P_{n3} are the elements of the n^{th} row of the rotation matrix P and the superscript (N) denotes the acoustically hard case that we study in this subsection. From the system of equations (3.51) that connect the eigenevectors of matrix $M^{(N)}$ with the elements of matrix P, we obtain the elements of matrix P and therefore we can specify the Euler angles by using the relations (3.19) These angles show the orientation of the ellipsoid.
From the system of equations (3.50) that connect the eigenvalues of matrix $M^{(N)}$ with the diagonal elements of matrix $H - I$ we obtain:

$$\lambda_n^{(N)} = \frac{V}{3}\left(\frac{1}{1 - VI_1^n} - 1\right) \quad (3.52)$$

$$(3.53)$$

for $n = 1, 2, 3$, which after simple calculations can also be written as:

$$VI_1^n = \frac{3\lambda_n^{(N)}}{3\lambda_n^{(N)} + V} \; . \quad (3.54)$$

for $n = 1, 2, 3$. At this point both V and I_1^n are unknown. Using the identity:

$$\sum_{n=1}^{3} VI_1^n = 1 \; , \quad (3.55)$$

we obtain via (3.54):

$$\frac{3\lambda_1^{(N)}}{3\lambda_1^{(N)} + V} + \frac{3\lambda_2^{(N)}}{3\lambda_2^{(N)} + V} + \frac{3\lambda_3^{(N)}}{3\lambda_3^{(N)} + V} = 1 \quad (3.56)$$

The left hand side of (3.56) is a function of V which will be denoted with $f(V)$. The function $f(V)$ from the value $f(0) = 3$ decreases monotonically to zero, therefore it has one root that can be found numerically or by solving a cubic equation of V in terms of known formulas. Therefore, after finding V, we obtain the elliptic integrals I_1^n, for $n = 1, 2, 3$, from (3.54). Next, we denote the right hand side of (3.54) as M_n for $n = 1, 2, 3$ respectively (which are known quantities from this point forward). In the case that the eigenvalues of the measurement matrix are distinct $\lambda_1 \neq \lambda_2 \neq \lambda_3$ we obtain via (1.178) that $\alpha_1 \neq \alpha_2 \neq \alpha_3$. We write the elliptic integrals I_1^n in terms of the canonical elliptic integrals (1.175), (1.186)-(1.189) and obtain from (3.54) the following relations [24] :

$$\begin{aligned} &\frac{1}{lm}\left[F(\theta, m) - E(\theta, m)\right] = M_1 \; , \\ &\frac{1}{lm(1-m)}\left[E(\theta, m) - (1-m)F(\theta, m) - \frac{msin\theta cos\theta}{\sqrt{1 - msin^2\theta}}\right] = M_2 \; , \\ &\frac{1}{l(1-m)}\left[tan\theta\sqrt{1 - msin^2\theta} - E(\theta, m)\right] = M_3 \; , \end{aligned} \quad (3.57)$$

where

$$sin^2\theta = \frac{a_1^2 - a_3^2}{a_1^2} = \left(\frac{h_2}{\alpha_1}\right)^2 , \quad m = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 - \alpha_3^2} = \left(\frac{h_3}{h_2}\right)^2 , \quad l = \left[\left(\frac{a_1}{V^{1/3}}\right)^2 - \left(\frac{a_3}{V^{1/3}}\right)^2\right]^{3/2} . \tag{3.58}$$

The system (3.57) can be simplified by using $M_1 + M_2 + M_3 = 1$ which is valid due to (3.54). Therefore, adding equations (3.57) and solving for l we obtain:

$$l = \frac{1}{1-m}\left(tan\theta\sqrt{1 - msin^2\theta} - \frac{sin\theta cos\theta}{\sqrt{1 - msin^2\theta}}\right) \equiv \frac{\Delta}{1-m} . \tag{3.59}$$

Then eliminating l from (3.57) via (3.59) we obtain:

$$\begin{aligned} \frac{1-m}{M_1\Delta m}[F(\theta,m) - E(\theta,m)] &= \frac{1}{M_2\Delta m}\left(E(\theta,m) - (1-m)F(\theta,m) - \frac{msin\theta cos\theta}{\sqrt{1 - msin^2\theta}}\right) , \\ \frac{1-m}{M_1\Delta m}[F(\theta,m) - E(\theta,m)] &= \frac{1}{M_3\Delta}\left(tan\theta\sqrt{1 - msin^2\theta} - E(\theta,m)\right) . \end{aligned} \tag{3.60}$$

which can be solved numerically (Matlab) for θ, m. Then l is obtained from (3.59). Denoting by θ_0, m_0, l_0 the solution of system (3.57) (or respectively of (3.60) and (3.59)) the semi-axes are given by:

$$\alpha_1 = (l_0)^{1/3}|sin\theta_0| , \quad \alpha_2 = \alpha_1\sqrt{1 - m_0 sin^2\theta_0} , \quad \alpha_3 = \alpha_1|cos\theta_0| . \tag{3.61}$$

We note that in the case that $\lambda_1^{(N)} = \lambda_2^{(N)} = \lambda_3^{(N)}$ we obtain $\alpha_1 = \alpha_2 = \alpha_3$ due to relation (1.178) and therefore the ellipsoid is a sphere of radius $\alpha_1 = V^{1/3}$. In the case that two out of the three eigenvalues $\lambda_n^{(N)}$ are equal then we can proceed as follows: Assume without loss of generality that $\lambda_1^{(N)} \neq \lambda_2^{(N)} = \lambda_3^{(N)}$ which means that $\alpha_1 \neq \alpha_2 = \alpha_3$ due to (1.178). Set $q = \alpha_1/\alpha_2$. Then $q > 1$ or $q < 1$ for $M_1 > M_2$ or $M_1 < M_2$ respectively. From (3.54) we obtain:

$$\frac{1}{1-q^2}\left(1 - \frac{qcos^{-1}q}{\sqrt{1-q^2}}\right) = M_1 , \quad \text{for } \frac{1}{3} < M_1 < 1, \text{ if } \frac{M_1}{M_2} > 1 , \tag{3.62}$$

$$\frac{1}{q^2-1}\left(\frac{qcosh^{-1}q}{\sqrt{q^2-1}} - 1\right) = M_1 , \quad \text{for } 0 < M_1 < \frac{1}{3}, \text{ if } \frac{M_1}{M_2} < 1 , \tag{3.63}$$

where equation (3.62) or (3.63) can be solved numerically for q. If q_0 is the solution then the semi-axes are given by:

$$\alpha_1 = \left(q_0^2 V\right)^{1/3} , \ \alpha_2 = \alpha_3 = \left(\frac{V}{q_0}\right)^{1/3} . \tag{3.64}$$

3.1.3 Acoustically Penetrable Ellipsoid

In the penetrable case it is possible to specify physical parameters of the interior of the ellipsoid additionally to the specification of the orientation and the size of the ellipsoid.
We have $B = \mathcal{B} - 1$, $C = \mathcal{C} - 1$, with $\mathcal{B} = \frac{\rho^+}{\rho^-}$, $\mathcal{C} = \mathcal{B}\eta^2 = \frac{\gamma^+}{\gamma^-}$ and $\eta = \frac{k^+}{k^-}$ where $\rho^\pm$, $\gamma^\pm$, $k^\pm$ the mass densities, the compressibilities and the wave numbers in $V^\pm$ respectively. The low-frequency expansion of the far-field pattern for the penetrable ellipsoid, for either the lossless or the lossy transmission problem, is given by formula (2.483). In this case as mentioned in the previous chapter, the coefficients $A_1^{(T)}$ and $A_2^{(T)}$ are equal to zero. We will use the leading order coefficient of the imaginary part of the low-frequency expansion which is the coefficient $A_3^{(T)}$ that for the acoustically penetrable ellipsoid has the form:

$$A_3^{(T)}(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = \frac{V}{3}\left[C - B\sum_{n=1}^{3}\frac{i_n o_n}{1 + BVI_1^n}\right], \tag{3.65}$$

where $\hat{\mathbf{r}} = (o_1, o_2, o_3)$, $\hat{\mathbf{d}} = (i_1, i_2, i_3)$ and $V = \alpha_1\alpha_2\alpha_3$. Relation (3.65) can be written in matrix form as:

$$A_3^{(T)}(\hat{\mathbf{r}}, \hat{\mathbf{d}}) = \frac{V}{3}\left[C - B\hat{\mathbf{d}}^\top \mathcal{A}\hat{\mathbf{r}}\right] , \tag{3.66}$$

with $\mathcal{A} = diag(\mathcal{A}_n)$ for $n = 1, 2, 3$ with

$$\mathcal{A}_n = \frac{1}{1 + BVI_1^n} . \tag{3.67}$$

Similarly to the previous cases of the soft and the hard ellipsoids, we consider two Cartesian systems and use the orthogonal matrix P given by (3.7) to transform the known system $\mathbf{x}'_\mathbf{i}$ to the unknown system $\mathbf{x_i}$. Therefore, using the rotation relation (3.8) for $\hat{\mathbf{d}}$ and the corresponding rotation relation for $\hat{\mathbf{r}}$ in (3.66) we obtain:

$$A_3^{(T)}(\hat{\mathbf{r}}', \hat{\mathbf{d}}') = \frac{V}{3}\left[C - B\hat{\mathbf{d}}'^\top P^\top \mathcal{A}P\hat{\mathbf{r}}'\right] . \tag{3.68}$$

Letting $\hat{\mathbf{r}}' = -\hat{\mathbf{d}}'$, we will take six measurements of $A_3^{(T)}(-\hat{\mathbf{d}}', \hat{\mathbf{d}}')$ for six directions as follows:

$$\begin{cases} \hat{\mathbf{d}}'_\mathbf{i} = \hat{\mathbf{x}}'_\mathbf{i} \ , \ \text{for} \ \ i = 1,2,3 \ , \\ \hat{\mathbf{d}}'_\mathbf{i+j+1} = \dfrac{1}{\sqrt{2}}\left(\hat{\mathbf{x}}'_\mathbf{i} + \hat{\mathbf{x}}'_\mathbf{j}\right) \ , \ \text{for} \ \ i,j = 1,2,3 \ , \ \text{with} \ \ i \neq j \ . \end{cases} \tag{3.69}$$

as well as one additional measurement for an opposite direction. Therefore, the measurements are the following:

$$\begin{aligned}
m_1 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{1}, \hat{\mathbf{d}}'_\mathbf{1}) = \frac{V}{3}\left[C + B\mathbf{P_1}^\top \mathcal{A}\mathbf{P_1}\right] , \\
m_2 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{2}, \hat{\mathbf{d}}'_\mathbf{2}) = \frac{V}{3}\left[C + B\mathbf{P_2}^\top \mathcal{A}\mathbf{P_2}\right] , \\
m_3 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{3}, \hat{\mathbf{d}}'_\mathbf{3}) = \frac{V}{3}\left[C + B\mathbf{P_3}^\top \mathcal{A}\mathbf{P_3}\right] , \\
m_4 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{4}, \hat{\mathbf{d}}'_\mathbf{4}) = \frac{V}{3}\left[C + \frac{1}{2}B\left(\mathbf{P_1}^\top \mathcal{A}\mathbf{P_1} + 2\mathbf{P_1}^\top \mathcal{A}\mathbf{P_2} + \mathbf{P_2}^\top \mathcal{A}\mathbf{P_2}\right)\right] , \\
m_5 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{5}, \hat{\mathbf{d}}'_\mathbf{5}) = \frac{V}{3}\left[C + \frac{1}{2}B\left(\mathbf{P_1}^\top \mathcal{A}\mathbf{P_1} + 2\mathbf{P_1}^\top \mathcal{A}\mathbf{P_3} + \mathbf{P_3}^\top \mathcal{A}\mathbf{P_3}\right)\right] , \\
m_6 &= A_3^{(T)}(-\hat{\mathbf{d}}'_\mathbf{6}, \hat{\mathbf{d}}'_\mathbf{6}) = \frac{V}{3}\left[C + \frac{1}{2}B\left(\mathbf{P_2}^\top \mathcal{A}\mathbf{P_2} + 2\mathbf{P_2}^\top \mathcal{A}\mathbf{P_3} + \mathbf{P_3}^\top \mathcal{A}\mathbf{P_3}\right)\right] , \\
m_7 &= A_3^{(T)}(\hat{\mathbf{d}}'_\mathbf{1}, \hat{\mathbf{d}}'_\mathbf{1}) \ \ = \frac{V}{3}\left[C - B\mathbf{P_1}^\top \mathcal{A}\mathbf{P_1}\right] ,
\end{aligned} \tag{3.70}$$

where the relation

$$\mathbf{P_i}^\top \mathcal{A}\mathbf{P_j} = \mathbf{P_j}^\top \mathcal{A}\mathbf{P_i} , \tag{3.71}$$

due to $\mathcal{A}$ being a diagonal matrix has been used. From the measurements m_1 and m_7 we obtain:

$$V = \frac{3}{2C}(m_1 + m_7) . \tag{3.72}$$

We construct a measurement matrix $M^{(T)} = (M_{ij}^{(T)})$ for $i, j = 1, 2, 3$, with elements of the form:

$$M_{ij}^{(T)} = V\mathbf{P_i}^\top \left(B\mathcal{A} + CI\right)\mathbf{P_j} . \tag{3.73}$$

Relation (3.73) due to (3.46) can be written in matrix form as follows:

$$M^{(T)} = VP^\top \left(B\mathcal{A} + CI\right)P \quad \Leftrightarrow PM^{(T)}P^\top = V(B\mathcal{A} + CI) \quad . \tag{3.74}$$

Therefore, the measurement matrix M can be given in terms of $m_1, ..., m_6$ as follows:

$$M^{(T)} = \frac{3}{2}\begin{bmatrix} 2m_1 & (2m_4 - m_1 - m_2) & (2m_5 - m_1 - m_3) \\ (2m_4 - m_1 - m_2) & 2m_2 & (2m_6 - m_2 - m_3) \\ (2m_5 - m_1 - m_3) & (2m_6 - m_2 - m_3) & 2m_3 \end{bmatrix} \tag{3.75}$$

where the orthogonality of P has been used (i.e. $\mathbf{P_i}^\top \mathbf{P_i} = 1$ and $\mathbf{P_i}^\top \mathbf{P_j} = 0$ for $i \neq j$).
Since M is real and symmetric it has three real eigenvalues $\lambda_1^{(T)}, \lambda_2^{(T)}, \lambda_3^{(T)}$ and three orthonormal eigenvectors $\mathbf{v_1}^{(T)}, \mathbf{v_2}^{(T)}, \mathbf{v_3}^{(T)}$. Therefore, due to the orthogonal similarity relation (3.48) we obtain:

$$\lambda_n^{(T)} = V(BA_n + C) \ , \tag{3.76}$$

$$\mathbf{v_n}^{(T)} = (P_{n1}, P_{n2}, P_{n3})^{(T)} \ , \tag{3.77}$$

for $n = 1, 2, 3$, where P_{n1}, P_{n2}, P_{n3} are the elements of the n^{th} row of the rotation matrix P and the superscript (T) denotes the acoustically penetrable case that we study in this subsection. From the system of equations (3.77) that connect the eigenevectors of matrix $M^{(T)}$ with the elements of matrix P, we obtain the elements of matrix P and therefore we can specify the Euler angles by using the relations (3.19) These angles show the orientation of the ellipsoid.
From the system of equations (3.76) that connect the eigenvalues of matrix $M^{(T)}$ with the diagonal elements of matrix $BA + CI$ we obtain:

$$\lambda_n^{(T)} = V\left(B\frac{1}{1 + BVI_1^n} + C\right) \tag{3.78}$$

for $n = 1, 2, 3$, which after simple calculations can also be written as:

$$VI_1^n = \frac{V}{\lambda_n^{(T)} - VC} - \frac{1}{B} \ , \tag{3.79}$$

for $n = 1, 2, 3$. Using the identity:

$$\sum_{n=1}^{3} VI_1^n = 1 \ , \tag{3.80}$$

we obtain via (3.79):

$$\sum_{n=1}^{3} \frac{1}{\lambda_n^{(T)} - VC} = \frac{B+3}{VB} \ . \tag{3.81}$$

At this point we have equations (3.72), (3.79), (3.81) to specify the semi-axes $\alpha_1, \alpha_2, \alpha_3$ as well as one of B or C. We assume that one of the parameters B or C is known and proceed as follows: If C is known, then from (3.72) we obtain V and then via (3.81) we obtain B. If B is known, then (since VC is known from (3.79)) we obtain V from (3.81) and then from (3.79) we obtain C.
Therefore, since now B, C, V and I_1^n for $n = 1, 2, 3$ are known we can find the semi-axes α_n (for $n = 1, 2, 3$) and therefore the size of the ellipsoid by following the same process as in the previous section for the acoustically hard ellipsoid where in this case the known quantities M_n (for $n = 1, 2, 3$) are given by the right hand side of (3.79) for $n = 1, 2, 3$ respectively.

3.2 Electromagnetic Waves

Let V^+ be the exterior and V^- be the interior region of an ellipsoidal scatterer of surface S with equation:

$$\sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2} = 1 \ , \tag{3.82}$$

where $\alpha_1 > \alpha_2 > \alpha_3 > 0$ are the semi-axes of the ellipsoidal scatterer and $\hat{\mathbf{n}}$ the outward unit normal vector on S. The superscripts $+, -$ will denote parameters or functions in the regions V^+, V^- respectively. In the ellipsoidal coordinate system the surface S is defined by $\rho = \alpha_1$, the exterior by $\rho > \alpha_1$ and the interior by $\sqrt{\alpha_1^2 - \alpha_3^2} \leq \rho < \alpha_1$. (figure 3.1).

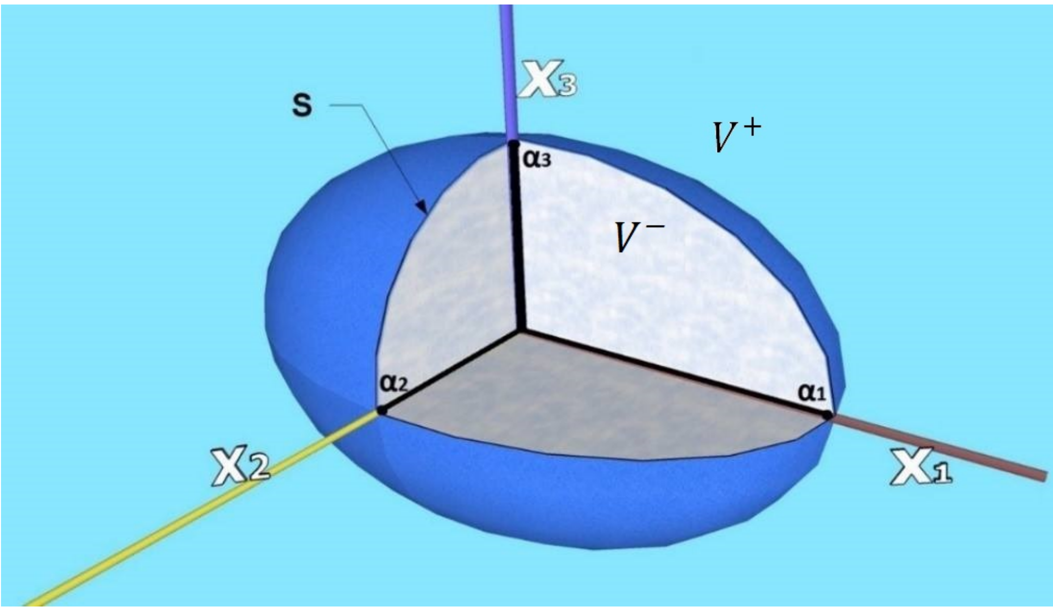


Figure 3.1: Ellipsoid of surface S.

The purpose of this section is to determine the size and the orientation of this ellipsoid using far-field data or near-field data, assuming that we know the center of the scatterer.

3.2.1 Far-Field Method

We will study the inverse scattering problems of electromagnetic waves from an impedance, a lossless dielectric and a lossy dielectric ellipsoid. The case of a perfectly conductive ellipsoid has been studied in [17]. In what follows the superscript $(I), (D), (L)$ will refer to the impedance (2.331), the lossless dielectric ($\sigma^- = 0$) and the lossy ($\sigma^- > 0$) dielectric ellipsoid respectively (2.332). Moreover, the superscript (B) will denote all the above mentioned cases, therefore $B = I, D, L$. The low-frequency expansion of the electric far-field patterns for the impedance, the lossless dielectric and the lossy dielectric ellipsoids respectively for $k \to 0$ are given by (2.491),(2.497) and (2.504):

$$\mathbf{E}^{\infty(B)}(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) = -ik^3 \frac{V}{3} \sum_{n=1}^{3} \left[W_n^{(B)} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) + T_n^{(B)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n}) (\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) \right] + \mathcal{O}(k^4) \ , \tag{3.83}$$

for $B = I, D, L$ with

$$W_n^{(D)} = \frac{\mu^+\eta^2 - \mu^-}{(\mu^+\eta^2 - \mu^-)\, V I_1^n + \mu^-}\ ,\ W_n^{(L)} = W_n^{(I)} = \frac{1}{V I_1^n}\ , \tag{3.84}$$

$$T_n^{(D)} = T_n^{(L)} = \frac{(\mu^+ - \mu^-)}{(\mu^+ - \mu^-)\, V I_1^n - \mu^+}\ ,\ T_n^{(I)} = \frac{1}{V}\ , \tag{3.85}$$

where $V = \alpha_1\alpha_2\alpha_3$, η the relative index of refraction and I_1^n the elliptic integral of degree 1 and order $n = 1, 2, 3$, given in (1.175):

$$I_1^n = \frac{1}{2}\int_0^\infty \frac{du}{(u+\alpha_n^2)\sqrt{u+\alpha_1^2}\sqrt{u+\alpha_2^2}\sqrt{u+\alpha_3^2}}\ . \tag{3.86}$$

with

$$I_1^1 + I_1^2 + I_1^3 = \frac{1}{V}\ . \tag{3.87}$$

The far-field patterns for $k \to 0$ can be written as:

$$\mathbf{E}^{\infty,(B)}(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) = ik^3 \mathbf{f}^{(B)}\left(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}\right) + \mathcal{O}\left(k^4\right)\quad , \tag{3.88}$$

for $B = I, D, L$, with the leading-order coefficients given by:

$$\mathbf{f}^{(B)}\left(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}\right) = \frac{V}{3}\hat{\mathbf{r}} \times \left[\left(W^{(B)}\hat{\mathbf{p}}\right) \times \hat{\mathbf{r}} - T^{(B)}\left(\hat{\mathbf{d}} \times \hat{\mathbf{p}}\right)\right]\quad , \tag{3.89}$$

where

$$W^{(B)} = diag\left(W_n^{(B)}\right)\ ,\ T^{(B)} = diag\left(T_n^{(B)}\right) \tag{3.90}$$

for $B = I, D, L$.

Next, we consider two cartesian systems with the same origin and their corresponding orthonormal bases $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ and $\{\hat{\mathbf{x}}_1', \hat{\mathbf{x}}_2', \hat{\mathbf{x}}_3'\}$. The system $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ coincides with the principal directions of the unknown ellipsoid while the system $\{\hat{\mathbf{x}}_1', \hat{\mathbf{x}}_2', \hat{\mathbf{x}}_3'\}$ is a known reference system. In order to specify the orientation and the size of the ellipsoid we transform the $\mathbf{x_i'}$ system to the $\mathbf{x_i}$ with the use of the orthogonal rotation matrix P whose elements are given in terms of the Euler angles (α, β, γ) as follows [28], [20]:

$$P = \begin{bmatrix} cos\alpha\ cos\gamma - cos\beta\ sin\alpha\ sin\gamma & sin\alpha\ cos\gamma + cos\beta\ cos\alpha\ sin\gamma & sin\beta\ sin\gamma \\ -cos\alpha\ sin\gamma - cos\beta\ sin\alpha\ cos\gamma & -sin\alpha\ sin\gamma + cos\beta\ cos\alpha\ cos\gamma & sin\beta\ cos\gamma \\ sin\beta\ sin\alpha & -sin\beta\ cos\alpha & cos\beta \end{bmatrix} \tag{3.91}$$

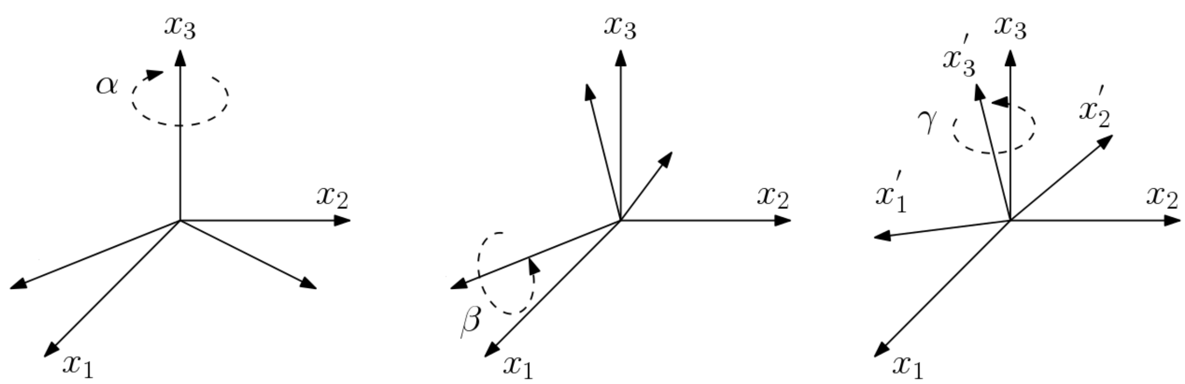


Figure 3.2: Euler angles.

Therefore, vectors $\mathbf{r}, \hat{\mathbf{r}}, \hat{\mathbf{d}}, \hat{\mathbf{p}}$ satisfy the following rotation relations:

$$\mathbf{r} = P\mathbf{r}' \quad , \quad \hat{\mathbf{r}} = P\hat{\mathbf{r}}' \quad , \quad \hat{\mathbf{d}} = P\hat{\mathbf{d}}' \quad , \quad \hat{\mathbf{p}} = P\hat{\mathbf{p}}' \quad . \tag{3.92}$$

Inserting the above relations for the directions in the coefficient $\mathbf{f}^{(B)}$ we transform them from the $\mathbf{x_i'}$ system to the $\mathbf{x_i}$. Then by multiplying $\mathbf{f}^{(B)}$ with $P^\top$, we go back to the reference system $\mathbf{x_i'}$ where we will take the measurements. The superscript $\top$ denotes transposition and $P^\top = P^{-1}$ since P is orthogonal. Therefore, let $\mathbf{g}^{(B)}$ for $(B) = (I), (D), (L)$ given by:

$$\mathbf{g}^{(B)}\left(\hat{\mathbf{r}}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}'\right) := P^\top \mathbf{f}^{(B)}\left(P\hat{\mathbf{r}}'; P\hat{\mathbf{d}}', P\hat{\mathbf{p}}'\right) = P^\top \mathbf{f}^{(B)}\left(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}\right). \tag{3.93}$$

Then

$$\mathbf{g}^{(B)}\left(\hat{\mathbf{r}}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}'\right) = \frac{V}{3} P^\top \left\{ P\hat{\mathbf{r}}' \times \left[W^{(B)} P\hat{\mathbf{p}}' \times P\hat{\mathbf{r}}' - T^{(B)}\left(P\hat{\mathbf{d}}' \times P\hat{\mathbf{p}}'\right)\right]\right\} \quad . \tag{3.94}$$

Letting the directions

$$\hat{\mathbf{r}}' = \varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}' \quad , \quad \hat{\mathbf{d}}' = \hat{\mathbf{x}}_\mathbf{i}' \quad , \quad \hat{\mathbf{p}}' = \hat{\mathbf{x}}_\mathbf{j}' \quad , \tag{3.95}$$

where ε_{ijk} the permutation symbol with $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = 1$ and $\varepsilon_{ikj} = \varepsilon_{jik} = \varepsilon_{kji} = -1$, for $1 \le i, j, k \le 3$ with i, j, k distinct, we get:

$$\mathbf{g}^{(B)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{i}', \hat{\mathbf{x}}_\mathbf{j}'\right) = \frac{V}{3}\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}' \times \left[P^\top W^{(B)} P\hat{\mathbf{x}}_\mathbf{j}' \times \varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}' - P^\top T^{(B)} P\left(\hat{\mathbf{x}}_\mathbf{i}' \times \hat{\mathbf{x}}_\mathbf{j}'\right)\right] \quad . \tag{3.96}$$

After calculations we obtain:

$$\begin{aligned} \mathbf{g}^{(B)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{i}', \hat{\mathbf{x}}_\mathbf{j}'\right) = \frac{V}{3}\Big[&\left(\mathbf{P_i}^\top W^{(B)}\mathbf{P_j} + \varepsilon_{ijk}\mathbf{P_j}^\top T^{(B)}\mathbf{P_k}\right)\hat{\mathbf{x}}_\mathbf{i}' \\ &+ \left(\mathbf{P_j}^\top W^{(B)}\mathbf{P_j} - \varepsilon_{ijk}\mathbf{P_i}^\top T^{(B)}\mathbf{P_k}\right)\hat{\mathbf{x}}_\mathbf{j}'\Big], \end{aligned} \tag{3.97}$$

where $\mathbf{P_j} = P\hat{\mathbf{x}}_\mathbf{j}'$ the j-th column of matrix P and $\mathbf{P_j}^\top = \hat{\mathbf{x}}_\mathbf{j}'^\top P^\top$ the j-th row of matrix $P^\top$. Since $W^{(B)}$ are diagonal we obtain the following relations for $1 \le i, j \le 3$:

$$\mathbf{P_i}^\top W^{(B)}\mathbf{P_j} = \mathbf{P_j}^\top W^{(B)}\mathbf{P_i} \quad . \tag{3.98}$$

We take measurements of the vectors $\mathbf{g}^{(B)}$ for direction combinations along the axes $\mathbf{x_i}'$. Specifically we use 6 direction combinations to solve for $\mathbf{P_i}^\top W^{(B)}\mathbf{P_j}$, for $B = I, D, L$ as follows:

$$\frac{2V}{3}\mathbf{P_j}^\top W^{(B)}\mathbf{P_j} = \left[\mathbf{g}^{(B)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{i}', \hat{\mathbf{x}}_\mathbf{j}'\right) + \mathbf{g}^{(B)}\left(\varepsilon_{kji}\hat{\mathbf{x}}_\mathbf{i}'; \hat{\mathbf{x}}_\mathbf{k}', \hat{\mathbf{x}}_\mathbf{j}'\right)\right] \cdot \hat{\mathbf{x}}_\mathbf{j}' \tag{3.99}$$

$$\begin{aligned} \frac{2V}{3}\mathbf{P_i}^\top W^{(B)}\mathbf{P_j} &= \left[2\mathbf{g}^{(B)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{i}', \hat{\mathbf{x}}_\mathbf{j}'\right) - \mathbf{g}^{(B)}\left(\varepsilon_{jik}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{j}', \hat{\mathbf{x}}_\mathbf{i}'\right) + \mathbf{g}^{(B)}\left(\varepsilon_{kij}\hat{\mathbf{x}}_\mathbf{j}; \hat{\mathbf{x}}_\mathbf{k}', \hat{\mathbf{x}}_\mathbf{i}'\right)\right] \cdot \hat{\mathbf{x}}_\mathbf{i}' \\ &= \left[2\mathbf{g}^{(B)}\left(\varepsilon_{jik}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{j}', \hat{\mathbf{x}}_\mathbf{i}'\right) - \mathbf{g}^{(B)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}_\mathbf{k}'; \hat{\mathbf{x}}_\mathbf{i}', \hat{\mathbf{x}}_\mathbf{j}'\right) + \mathbf{g}^{(B)}\left(\varepsilon_{kji}\hat{\mathbf{x}}_\mathbf{i}'; \hat{\mathbf{x}}_\mathbf{k}', \hat{\mathbf{x}}_\mathbf{j}'\right)\right] \cdot \hat{\mathbf{x}}_\mathbf{j}'. \end{aligned} \tag{3.100}$$

We construct the measurement matrices $M^{(B)}$ for the impedance, the lossless dielectric and the lossy dielectric cases respectively as follows:

$$M^{(B)} = (M_{ij}^{(B)}) \quad , \quad M_{ij}^{(B)} = \frac{2V}{3}\mathbf{P_i}^\top W^{(B)}\mathbf{P_j} \tag{3.101}$$

for $1 \le i, j \le 3$. The above set of equations can be written in matrix form as:

$$M^{(B)} = \frac{2V}{3} P^\top W^{(B)} P \quad , \tag{3.102}$$

or since the rotation matrix P is orthogonal:

$$PM^{(B)}P^{\top} = \frac{2V}{3}W^{(B)} \quad , \tag{3.103}$$

which is an orthogonal similarity relation between the measurement matrices $M^{(B)}$ and the diagonal matrices $W^{(B)}$ for $B = I, D, L$.
The measurement matrices $M^{(B)}$ are real and symmetric and each has three real eigenvalues $\lambda_1^{(B)}, \lambda_2^{(B)}, \lambda_3^{(B)}$ and three corresponding orthonormal eigenvectors $\mathbf{v}_1^{(B)}, \mathbf{v}_2^{(B)}, \mathbf{v}_3^{(B)}$ respectively. Therefore based on the orthogonal similarity relation, we conclude that for $n = 1, 2, 3$:

$$\lambda_n^{(B)} = \frac{2V}{3}W_n^{(B)} \tag{3.104}$$

$$\mathbf{v_n}^{(B)} = (P_{n1}, P_{n2}, P_{n3})^{(B)} \ , \tag{3.105}$$

for $n = 1, 2, 3$ and $B = I, D, L$, where P_{n1}, P_{n2}, P_{n3} are the elements of the n^{th} row of the rotation matrix P and the superscript (B) on the right hand side of (3.105) denotes the elements of the rotation matrix P corresponding to the cases $B = I, D, L$ respectively. From the elements of the three orthogonal eigenvectors $\mathbf{v}_1^{(B)}, \mathbf{v}_2^{(B)}, \mathbf{v}_3^{(B)}$ we obtain the elements of matrix P for $B = I, D, L$ respectively and therefore we can specify the Euler angles by using the following relations [20], [28]:

$$\alpha = sin^{-1}\left(\frac{P_{31}}{\sqrt{1-P_{33}^2}}\right) \ , \ \beta = sin^{-1}\left(\sqrt{1-P_{33}^2}\right) \ , \ \gamma = sin^{-1}\left(\frac{P_{13}}{\sqrt{1-P_{33}^2}}\right) \ . \tag{3.106}$$

These angles show the orientation of the ellipsoid in the impedance, the lossless and the lossy cases respectively.
From the system of equations (3.104) that connect the eigenvalues $\lambda_n^{(B)}$ with the elements $W_n^{(B)}$ for $n = 1, 2, 3$ we obtain via (3.84):

$$\lambda_n^{(I)} = \frac{2}{3I_1^n} \ , \tag{3.107}$$

$$\lambda_n^{(D)} = \frac{2V}{3}\left[\frac{\mu^+\eta^2 - \mu^-}{(\mu^+\eta^2 - \mu^-)\,VI_1^n + \mu^-}\right] \ , \tag{3.108}$$

$$\lambda_n^{(L)} = \frac{2}{3I_1^n} \ . \tag{3.109}$$

From these equations we can specify the semi-axes $\alpha_1, \alpha_2, \alpha_3$ and therefore we can specify the size of the ellipsoid for the impedance, the lossless dielectric and the lossy dielectric cases respectively. Specifically, for the case of the lossless dielectric ellipsoid, using relation (3.87) and solving relation (3.108) for $V^{(D)}$ we obtain:

$$V^{(D)} = \frac{3(\mu^+\eta^2 + 2\mu^-)\lambda_1^{(D)}\lambda_2^{(D)}\lambda_3^{(D)}}{2(\mu^+\eta^2 - \mu^-)(\lambda_1^{(D)}\lambda_2^{(D)} + \lambda_2^{(D)}\lambda_3^{(D)} + \lambda_1^{(D)}\lambda_3^{(D)})} \quad , \tag{3.110}$$

which gives the relation:

$$I_1^n = \frac{2}{3\lambda_n^{(D)}} - \frac{2\mu^-(\lambda_1^{(D)}\lambda_2^{(D)} + \lambda_2^{(D)}\lambda_3^{(D)} + \lambda_1^{(D)}\lambda_3)}{3(\mu^+\eta^2 + 2\mu^-)\lambda_1^{(D)}\lambda_2^{(D)}\lambda_3^{(D)}} \quad . \tag{3.111}$$

For the cases of the impedance and the lossy dielectric ellipsoid, we obtain I_1^n from (3.107) and (3.109) respectively as follows:

$$I_1^n = \frac{2}{3\lambda_n^{(I)}} \ , \tag{3.112}$$

$$I_1^n = \frac{2}{3\lambda_n^{(L)}} \;, \tag{3.113}$$

for $n = 1, 2, 3$, and therefore using relation (3.87) we obtain:

$$V^{(I)} = \frac{1}{\left(\sum_{n=1}^{3} I_1^n\right)} = \frac{3}{2}\left(\frac{1}{\lambda_1^{(I)}} + \frac{1}{\lambda_2^{(I)}} + \frac{1}{\lambda_3^{(I)}}\right)^{-1} , \tag{3.114}$$

$$V^{(D)} = \frac{1}{\left(\sum_{n=1}^{3} I_1^n\right)} = \frac{3}{2}\left(\frac{1}{\lambda_1^{(D)}} + \frac{1}{\lambda_2^{(D)}} + \frac{1}{\lambda_3^{(D)}}\right)^{-1} , \tag{3.115}$$

Having found $V^{(B)}$ and I_1^n for the impedance, the lossless and the lossy dielectric cases respectively, we can specify the semi-axes α_n, for $n = 1, 2, 3$, as follows:
We denote with $M_n^{(B)}$ the quantities :

$$V^{(B)} I_1^n =: M_n^{(B)} \;, \tag{3.116}$$

for $n = 1, 2, 3$, which are now known.
If the eigenvalues $\lambda_n^{(B)}$ are distinct then we obtain $\alpha_1^{(B)} \neq \alpha_2^{(B)} \neq \alpha_3^{(B)}$ from (1.178). We write the elliptic integrals I_1^n in terms of the canonical elliptic integrals (1.175), (1.186)-(1.189) and obtain:

$$\begin{aligned} &\frac{1}{lm}\left[F(\theta,m) - E(\theta,m)\right] = M_1^{(B)} \;, \\ &\frac{1}{lm(1-m)}\left[E(\theta,m) - (1-m)F(\theta,m) - \frac{msin\theta cos\theta}{\sqrt{1-msin^2\theta}}\right] = M_2^{(B)} \;, \\ &\frac{1}{l(1-m)}\left[tan\theta\sqrt{1-msin^2\theta} - E(\theta,m)\right] = M_3^{(B)} \;, \end{aligned} \tag{3.117}$$

where

$$sin^2\theta = \frac{a_1^2 - a_3^2}{a_1^2} = \left(\frac{h_2}{\alpha_1}\right)^2, \quad m = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 - \alpha_3^2} = \left(\frac{h_3}{h_2}\right)^2, \quad l = \left[\left(\frac{a_1}{V^{(B)1/3}}\right)^2 - \left(\frac{a_3}{V^{(B)1/3}}\right)^2\right]^{3/2} , \tag{3.118}$$

for $B = I, D, L$. In order to solve the system (3.117) numerically, we make use of the relation $M_1 + M_2 + M_3 = 1$ which is valid from (3.87) and (3.116). Specifically, by adding the three equations and solving for l we obtain:

$$l = \frac{1}{1-m}\left(tan\theta\sqrt{1-msin^2\theta} - \frac{sin\theta cos\theta}{\sqrt{1-msin^2\theta}}\right) \equiv \frac{\Delta}{1-m} \;. \tag{3.119}$$

Then eliminating l from (3.117) via (3.119), dividing each of the three equations with its corresponding right hand and then equating the resulted first with the resulted second equation and the resulted first with the resulted third equation, we obtain the system of equations:

$$\begin{aligned} &\frac{1-m}{M_1\Delta m}\left[F(\theta,m) - E(\theta,m)\right] = \frac{1}{M_2\Delta m}\left(E(\theta,m) - (1-m)F(\theta,m) - \frac{msin\theta cos\theta}{\sqrt{1-msin^2\theta}}\right) , \\ &\frac{1-m}{M_1\Delta m}\left[F(\theta,m) - E(\theta,m)\right] = \frac{1}{M_3\Delta}\left(tan\theta\sqrt{1-msin^2\theta} - E(\theta,m)\right) \;. \end{aligned} \tag{3.120}$$

which can be solved numerically (Matlab) for θ, m. Then l is obtained from (3.119). Denoting by θ_0, m_0, l_0 the solution of system (3.117) (or respectively of (3.120) and (3.119) the semi-axes are given by:

$$\alpha_1 = (l_0)^{1/3}|sin\theta_0| \;, \quad \alpha_2 = \alpha_1\sqrt{1-m_0 sin^2\theta_0} \;, \quad \alpha_3 = \alpha_1|cos\theta_0| \;. \tag{3.121}$$

If $\lambda_1^{(B)} = \lambda_2^{(B)} = \lambda_3^{(B)}$ then $\alpha_1 = \alpha_2 = \alpha_3$ due to (1.178) and therefore the ellipsoid is a sphere of radius $\alpha_1 = \left(V^{(B)}\right)^{1/3}$ for the impedance, the lossless dielectric and the lossy dielectric cases respectively.
If two out of the three eigenvalues $\lambda_n^{(B)}$ are equal then we can proceed as follows: We assume without loss of generality that $\lambda_1^{(B)} \neq \lambda_2^{(B)} = \lambda_3^{(B)}$ which means that $\alpha_1 \neq \alpha_2 = \alpha_3$ due to (1.178) and we set $q = \alpha_1/\alpha_2$. Then $q > 1$ or $q < 1$ for $\lambda_1^{(B)} > \lambda_2^{(B)}$ or $\lambda_1^{(B)} < \lambda_2^{(B)}$ respectively. From (3.116):

$$\frac{1}{1-q^2}\left(1 - \frac{qcos^{-1}q}{\sqrt{1-q^2}}\right) = M_1^{(B)} \ , \quad \text{for } \frac{1}{3} < M_1^{(B)} < 1, \text{ if } \frac{M_1^{(B)}}{M_2^{(B)}} > 1 \tag{3.122}$$

$$\frac{1}{q^2-1}\left(\frac{qcosh^{-1}q}{\sqrt{q^2-1}} - 1\right) = M_1^{(B)} \ , \quad \text{for } 0 < M_1^{(B)} < \frac{1}{3}, \text{ if } \frac{M_1^{(B)}}{M_2^{(B)}} < 1 \ , \tag{3.123}$$

where equation (3.122) or (3.123) can be solved numerically for q. If q_0 is the solution then the semi-axes are given by:

$$\alpha_1 = \left(q_0^2 V^{(B)}\right)^{1/3} \quad , \quad \alpha_2 = \alpha_3 = \left(\frac{V^{(B)}}{q_0}\right)^{1/3} . \tag{3.124}$$

for $B = I, D, L$ for the impedance, the lossless dielectric and the lossy dielectric cases respectively.

We note here that the inverse problem can also be solved using directions of observation in the forward and backward directions of propagation instead of the directions of magnetic polarization as in the method described above. In this case, letting

$$\hat{\mathbf{r}}' = c\hat{\mathbf{x}}'_{\mathbf{i}} \ \text{(with } c = \pm 1) \quad , \quad \hat{\mathbf{d}}' = \hat{\mathbf{x}}'_{\mathbf{i}} \quad , \quad \hat{\mathbf{p}}' = \hat{\mathbf{x}}'_{\mathbf{j}} \quad , \tag{3.125}$$

for $1 \leq i, j \leq 3$, we take:

$$\mathbf{g}^{(B)}\left(c\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{j}}\right) = \frac{V}{3}\left[\left(\mathbf{P}_{\mathbf{j}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}} + c\mathbf{P}_{\mathbf{k}}^{\top} T^{(B)} \mathbf{P}_{\mathbf{k}}\right)\hat{\mathbf{x}}'_{\mathbf{j}} + \left(\mathbf{P}_{\mathbf{k}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}} - c\mathbf{P}_{\mathbf{j}}^{\top} T^{(B)} \mathbf{P}_{\mathbf{k}}\right)\hat{\mathbf{x}}'_{\mathbf{k}}\right] . \tag{3.126}$$

We take measurements for direction combinations similarly to the previous method and solve for $\mathbf{P}_{\mathbf{k}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}}$ (for $B = I, D, L$ respectively) as follows:

$$\frac{2V}{3}\mathbf{P}_{\mathbf{j}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}} = \left[\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}_{\mathbf{j}}\right) + \mathbf{g}^{(B)}\left(-\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{j}}\right)\right] \cdot \hat{\mathbf{x}}'_{\mathbf{j}} \tag{3.127}$$

$$\begin{aligned}\frac{2V}{3}\mathbf{P}_{\mathbf{k}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}} &= \left[\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{j}}\right) + \mathbf{g}^{(B)}\left(-\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{j}}\right)\right] \cdot \hat{\mathbf{x}}'_{\mathbf{k}} \\ &= \left[\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{k}}\right) + \mathbf{g}^{(B)}\left(-\hat{\mathbf{x}}'_{\mathbf{i}}; \hat{\mathbf{x}}'_{\mathbf{i}}, \hat{\mathbf{x}}'_{\mathbf{k}}\right)\right] \cdot \hat{\mathbf{x}}'_{\mathbf{j}}\end{aligned} \tag{3.128}$$

We construct the measurement matrices:

$$M^{(B)} = (M_{kj}^{(B)}) \quad M_{kj}^{(B)} = \mathbf{P}_{\mathbf{k}}^{\top} W^{(B)} \mathbf{P}_{\mathbf{j}} \tag{3.129}$$

for $B = I, D, L$ and use their eigenvalues and eigenvectors to specify the orientation and the size of the impedance, the lossless and the lossy dielectric ellipsoid respectively as previously.
Furthermore, the inverse problem can also be solved using directions of observation in the directions of electric polarization. In this case, letting

$$\hat{\mathbf{r}}' = \hat{\mathbf{x}}'_{\mathbf{j}} \quad , \quad \hat{\mathbf{d}}' = \hat{\mathbf{x}}'_{\mathbf{i}} \quad , \quad \hat{\mathbf{p}}' = \hat{\mathbf{x}}'_{\mathbf{j}} \quad , \tag{3.130}$$

we take:

$$\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{i}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)=\frac{V}{3}\left[\left(\mathbf{P}_{\mathbf{i}}^{\top}W^{(B)}\mathbf{P}_{\mathbf{j}}+c\mathbf{P}_{\mathbf{k}}^{\top}T^{(B)}\mathbf{P}_{\mathbf{k}}\right)\hat{\mathbf{x}}'_{\mathbf{i}}+\left(\mathbf{P}_{\mathbf{k}}^{\top}W^{(B)}\mathbf{P}_{\mathbf{j}}-c\mathbf{P}_{\mathbf{i}}^{\top}T\mathbf{P}_{\mathbf{k}}\right)\hat{\mathbf{x}}'_{\mathbf{k}}\right], \tag{3.131}$$

We take measurements for direction combinations similarly to the previous method and solve for $\mathbf{P_i}^{\top}T^{(B)}\mathbf{P_k}$ as follows:

$$\begin{aligned}\frac{2V}{3}\mathbf{P}_{\mathbf{k}}^{\top}T^{(B)}\mathbf{P}_{\mathbf{k}}&=\left[\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{i}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)-\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{k}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)\right]\cdot\hat{\mathbf{x}}'_{\mathbf{i}}\\&=\left[\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{i}};\hat{\mathbf{x}}'_{\mathbf{j}},\hat{\mathbf{x}}'_{\mathbf{i}}\right)-\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{i}};\hat{\mathbf{x}}'_{\mathbf{k}},\hat{\mathbf{x}}'_{\mathbf{i}}\right)\right]\cdot\hat{\mathbf{x}}'_{\mathbf{j}}\end{aligned} \tag{3.132}$$

$$\begin{aligned}\frac{2V}{3}\mathbf{P}_{\mathbf{i}}^{\top}T^{(B)}\mathbf{P}_{\mathbf{k}}&=\left[2\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{k}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)-\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{i}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)\right]\cdot\hat{\mathbf{x}}'_{\mathbf{k}}\\&=\left[2\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{i}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)-\mathbf{g}^{(B)}\left(\hat{\mathbf{x}}'_{\mathbf{j}};\hat{\mathbf{x}}'_{\mathbf{k}},\hat{\mathbf{x}}'_{\mathbf{j}}\right)\right]\cdot\hat{\mathbf{x}}'_{\mathbf{i}}\end{aligned} \tag{3.133}$$

We construct the measurement matrices:

$$M^{(B)}=(M_{ik}^{(B)})\quad M_{ik}^{(B)}=\mathbf{P}_{\mathbf{i}}^{\top}T^{(B)}\mathbf{P}_{\mathbf{k}} \tag{3.134}$$

and use their eigenvalues and eigenvectors to specify the orientation and the size of the lossless and the lossy dielectric ellipsoid respectively as previously.

3.2.2 Near-Field Method

In this subsection we present a method for solving the inverse electromagnetic scattering problems for the cases of the perfectly conductive, the impedance, the lossless dielectric and the lossy dielectric ellipsoids, using near-filed data. Next, we suggest a method to determine the electric permitivity ε^- and the magnetic permeability μ^- of V^- for the cases of lossless and lossy dielectric ellispoids. Moreover, we study the cases of the sphere and spheroid as geometrically degenerate cases of the ellipsoid and we present a way to predict when the scatterer belongs to one of these geometrically degenerate cases from the first three measurements. This can help us reduce the number of needed measurements and can prevent unneeded calculations. Finally, we present an alternative option for the set of measured near-field data and we comment on the advantages and the disadvantages of this option. Finally, we summarize the near-field method presented in this section in an algorithm and we make some remarks.

Notation

In what follows, the superscript $(C),(I),(D),(L)$ will refer to the perfectly conductive, the impedance, the lossless dielectric ($\sigma^-=0$) and the lossy ($\sigma^->0$) dielectric ellipsoid respectively. Moreover, the superscript (K) will denote all cases except of the lossless dielectric one and the superscript (A) will denote all four cases. Therefore $K=C,I,L$ and $A=K,D$.

Based on relations $\mathbf{E}_n^s=\mathbf{E}_n^+-\mathbf{E}_n^i$ and $\mathbf{H}_n^s=\mathbf{H}_n^+-\mathbf{H}_n^i$ as well as relations (2.484), (2.489), (2.493) and (2.500), the zeroth low-frequency approximation of the electric scattered field is given by:

$$\mathbf{E}_{\mathbf{0}}^{\mathbf{s}(K)}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})=-\hat{\mathbf{p}}\cdot\sum_{n=1}^{3}\frac{1}{I_1^n}\left[I_1^n(\rho)\hat{\mathbf{x}}_{\mathbf{n}}\otimes\hat{\mathbf{x}}_{\mathbf{n}}-\frac{x_n}{J_n(\rho)}\hat{\mathbf{x}}_{\mathbf{n}}\otimes\hat{\boldsymbol{\rho}}\right]\quad. \tag{3.135}$$

The zeroth low-frequency approximation for the lossless dielectric ellipsoid is given by:

$$\mathbf{E}_{\mathbf{0}}^{\mathbf{s}(D)}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})=-\hat{\mathbf{p}}\cdot\sum_{n=1}^{3}\frac{(\mu^+\eta^2-\mu^-)V}{(\mu^+\eta^2-\mu^-)VI_1^n+\mu^-}\left[I_1^n(\rho)\hat{\mathbf{x}}_{\mathbf{n}}\otimes\hat{\mathbf{x}}_{\mathbf{n}}-\frac{x_n}{J_n(\rho)}\hat{\mathbf{x}}_{\mathbf{n}}\otimes\hat{\boldsymbol{\rho}}\right], \tag{3.136}$$

where $\otimes$ denotes dyadic product, $\mathbf{r} = (x_1, x_2, x_3)$ is the observation vector and $\hat{\boldsymbol{\rho}}$ is the outward unit normal vector which plays the role of radial vector in ellipsoidal coordinates and is given by:

$$\hat{\boldsymbol{\rho}} = \frac{\rho}{h_\rho} \sum_{n=1}^{3} \frac{\hat{\mathbf{x}}_\mathbf{n} \otimes \hat{\mathbf{x}}_\mathbf{n}}{\left(\rho^2 - \alpha_1^2 + \alpha_n^2\right)} \cdot \mathbf{r} \quad , \tag{3.137}$$

with

$$h_\rho = \|\mathbf{r}_\rho\| = \frac{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}}{\sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}} \tag{3.138}$$

be the ellipsoidal metric coefficient and

$$J_n(\rho) = \left(\rho^2 - \alpha_1^2 + \alpha_n^2\right) \sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2} \quad . \tag{3.139}$$

Also, I_1^n is the elliptic integral of degree 1 and order $n = 1, 2, 3$, where:

$$I_1^n(\rho) = \frac{1}{2} \int_{\rho^2 - \alpha_1^2}^{\infty} \frac{du}{\left(u + \alpha_n^2\right) \sqrt{u + \alpha_1^2}\sqrt{u + \alpha_2^2}\sqrt{u + \alpha_3^2}} \quad , \tag{3.140}$$

with

$$I_1^1(\rho) + I_1^2(\rho) + I_1^3(\rho) = \frac{1}{\rho\sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}} \quad , \tag{3.141}$$

and $I_1^n = I_1^n(\alpha_1)$ with

$$I_1^1 + I_1^2 + I_1^3 = \frac{1}{V} \quad , \tag{3.142}$$

where $V = \alpha_1\alpha_2\alpha_3$.

Taking into account that $k^+ = \omega\sqrt{\mu^+\varepsilon^+}$ and $k^- = \omega\sqrt{\mu^-\varepsilon^-}\sqrt{1 + i\frac{\sigma^-}{\varepsilon^-\omega}}$ with k^- be the wave number in V^-, the relative index of refraction η is given by $\eta = \frac{k^-}{k^+} = \frac{\sqrt{\mu^-\varepsilon^-}}{\sqrt{\mu^+\varepsilon^+}}\sqrt{1 + i\frac{\sigma^-}{\varepsilon^-\omega}}$.

Inverse Problem

We study the inverse problems of electromagnetic scattering from a perfectly conducting, an impedance, a lossless dielectric and a lossy dielectric ellipsoid by using near-field data. We suggest a method in order to determine the semi-axes as well as the orientation of the ellipsoid in each case. For this method, we use eight measurements of the zeroth low-frequency approximation of the electric scattered field. We consider two cartesian systems with the same origin and their corresponding orthonormal bases $\{\hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{2}, \hat{\mathbf{x}}_\mathbf{3}\}$ and $\{\hat{\mathbf{x}}'_\mathbf{1}, \hat{\mathbf{x}}'_\mathbf{2}, \hat{\mathbf{x}}'_\mathbf{3}\}$. The system $\{\hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{2}, \hat{\mathbf{x}}_\mathbf{3}\}$ coincides with the principal directions of the unknown ellipsoid while the system $\{\hat{\mathbf{x}}'_\mathbf{1}, \hat{\mathbf{x}}'_\mathbf{2}, \hat{\mathbf{x}}'_\mathbf{3}\}$ is a known reference system. In order to specify the orientation and the size of the ellipsoid we transform the $\mathbf{x}'_\mathbf{i}$ system to the $\mathbf{x}_\mathbf{i}$ with the use of an orthogonal 3×3 rotation matrix P whose elements are given in terms of the Euler angles α, β, γ as follows [20], [28]

$$P = \begin{bmatrix} cos\alpha\ cos\gamma - cos\beta\ sin\alpha\ sin\gamma & sin\alpha\ cos\gamma + cos\beta\ cos\alpha\ sin\gamma & sin\beta\ sin\gamma \\ -cos\alpha\ sin\gamma - cos\beta\ sin\alpha\ cos\gamma & -sin\alpha\ sin\gamma + cos\beta\ cos\alpha\ cos\gamma & sin\beta\ cos\gamma \\ sin\beta\ sin\alpha & -sin\beta\ cos\alpha & cos\beta \end{bmatrix} \tag{3.143}$$

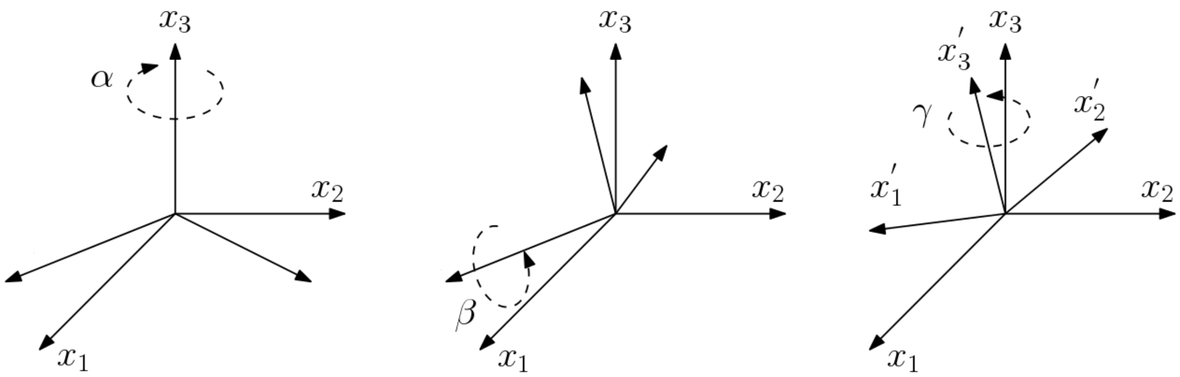


Figure 3.3: Euler angles.

Therefore, the vectors $\mathbf{r}, \hat{\mathbf{d}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}, \hat{\boldsymbol{\rho}}$ satisfy the rotation relations:

$$\mathbf{r} = P\mathbf{r}' \quad , \quad \hat{\mathbf{d}} = P\hat{\mathbf{d}}' \quad , \quad \hat{\mathbf{p}} = P\hat{\mathbf{p}}' \quad , \quad \hat{\mathbf{q}} = P\hat{\mathbf{q}}' \quad , \quad \hat{\boldsymbol{\rho}} = P\hat{\boldsymbol{\rho}}' \quad . \tag{3.144}$$

The zeroth low-frequency approximation of the electric scattered field can be written from (3.135) and (3.136) equivalently as a 3×1 column vector in the form:

$$\mathbf{E}_{\mathbf{0}}^{\mathbf{s}\,(A)}(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) = W^{(A)}\hat{\mathbf{p}} + \left(\mathbf{r}^{\top} Z^{(A)} \hat{\mathbf{p}}\right) \hat{\boldsymbol{\rho}} \quad , \tag{3.145}$$

where $W^{(A)} = diag(W_n^{(A)})$ and $Z^{(A)} = diag(Z_n^{(A)})$ for $n = 1, 2, 3$ and $A = K, D$, are diagonal 3×3 matrices with diagonal elements given by:

$$W_n^{(K)} = -\frac{1}{I_1^n} I_1^n(\rho) \quad , \quad W_n^{(D)} = -\frac{(\mu^+\eta^2 - \mu^-)V}{(\mu^+\eta^2 - \mu^-)V I_1^n + \mu^-} I_1^n(\rho) \quad , \tag{3.146}$$

$$Z_n^{(K)} = \frac{1}{I_1^n J_n(\rho)} \quad , \quad Z_n^{(D)} = \frac{(\mu^+\eta^2 - \mu^-)V}{\left[(\mu^+\eta^2 - \mu^-)V I_1^n + \mu^-\right] J_n(\rho)} \quad , \tag{3.147}$$

and the superscript $\top$ denotes transposition. Using the rotation relations (3.144) for $\mathbf{r}, \hat{\mathbf{d}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\rho}}$ we transform them from the $\mathbf{x_i'}$ system to the $\mathbf{x_i}$. Inserting them in $\mathbf{E}_{\mathbf{0}}^{\mathbf{s}\,(A)}$ and then multiplying the result with $P^{\top}$, we go back to the reference system $\mathbf{x_i'}$ where we will take the measurements. Let

$$\mathbf{S}^{(A)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') := P^{\top}\mathbf{E}_{\mathbf{0}}^{\mathbf{s}\,(A)}(P\mathbf{r}'; P\hat{\mathbf{d}}', P\hat{\mathbf{p}}') = P^{\top}\mathbf{E}_{\mathbf{0}}^{\mathbf{s}\,(A)}(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}}) \quad . \tag{3.148}$$

Then

$$\mathbf{S}^{(A)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = P^{\top} W^{(A)} P\hat{\mathbf{p}}' + \left(\mathbf{r}'^{\top} P^{\top} Z^{(A)} P\hat{\mathbf{p}}'\right) \hat{\boldsymbol{\rho}}' \quad . \tag{3.149}$$

where the relation $P^{\top} = P^{-1}$ due to P being an orthogonal matrix has been used.

Our method starts by taking a point $\mathbf{r_1'} = (x_1', x_2', x_3')$ with ellipsoidal coordinates (ρ_0, μ_0, ν_0), with $\alpha_1 < \rho_0 < \infty$ and $x_n' \neq 0,\ n = 1, 2, 3$. Also, we consider the points:

$$\mathbf{r_2'} = (-x_1', x_2', x_3') \quad , \quad \mathbf{r_3'} = (x_1', -x_2', x_3') \quad , \quad \mathbf{r_4'} = (x_1', x_2', -x_3') \quad , \tag{3.150}$$

which belong on the surface of the ellipsoid ($\rho = \rho_0$):

$$\frac{{x_1'}^2}{\rho_0^2} + \frac{{x_2'}^2}{\rho_0^2 - h_3^2} + \frac{{x_3'}^2}{\rho_0^2 - h_2^2} = 1 \quad , \tag{3.151}$$

since they are symmetric to $\mathbf{r_1'}$ over the principal planes of the reference system. We take three measurements of the zeroth low-frequency approximation of the electric scattered field in the reference system at $\mathbf{r_1'}$ for electric polarization $\mathbf{p}'$ along the axes $\hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{3}'$ respectively, three measurements at $\mathbf{r_3'}$ for electric polarization $\hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{3}'$ respectively and two measurements at $\mathbf{r_4'}$ for electric polarization $\hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{3}'$ respectively. Note that the direction of propagation does not appear in the formulas (3.145) and (3.149) and therefore for any electric polarization along an axis in each measurement there are two possible directions of propagation along the axes such that $\hat{\mathbf{d}}' \cdot \hat{\mathbf{p}}' = 0$. Let the measured electric near-field data be taken as follows:

$$\begin{array}{lll} \mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}') \quad , & \mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{2}') \quad , & \mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{3}') \quad , \\ \mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}') \quad , & \mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{2}') \quad , & \mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{3}') \quad , \\ \mathbf{S}^{(A)}(\mathbf{r_4'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}') \quad , & \mathbf{S}^{(A)}(\mathbf{r_4'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{3}') \quad . & \end{array} \tag{3.152}$$

Taking the inner product of $\mathbf{S}^{(A)}$ with the axis $\hat{\mathbf{x}}_\mathbf{i}'$ we obtain from (3.150) and (3.137) the following scalar quantity:

$$\hat{\mathbf{x}}_\mathbf{i}' \cdot \mathbf{S}^{(A)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = \mathbf{P}_\mathbf{i}^\top W^{(A)} P \hat{\mathbf{p}}' + \frac{\rho x_i'}{h_\rho(\rho^2 - \alpha_1^2 + \alpha_i^2)} (\mathbf{r}'^\top P^\top Z^{(A)} P \hat{\mathbf{p}}') \quad , \tag{3.153}$$

for $i = 1, 2, 3$, with $\mathbf{P}_\mathbf{i}^\top = (P\hat{\mathbf{x}}_\mathbf{i}')^\top$ be the i^{th} row of $P^\top$ and x_i' be the i^{th} coordinate of the observation vector $\mathbf{r}'$. From (3.150) and (3.153), after some calculations, we obtain:

$$\hat{\mathbf{x}}_\mathbf{i}' \cdot \Big(\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') - \mathbf{S}^{(A)}(\mathbf{r_{k+1}'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \Big) = \frac{2\rho_0 x_i' x_k'}{h_{\rho_0}({\rho_0}^2 - {\alpha_1}^2 + {\alpha_i}^2)} \mathbf{P}_\mathbf{k}^\top Z^{(A)} P \hat{\mathbf{p}}' \quad , \tag{3.154}$$

for $i, k = 1, 2, 3$ with $i \neq k$.
From (3.154) and three measurements taken for the same polarization, one at $\mathbf{r_1'}$, one at $\mathbf{r_3'}$ and one at $\mathbf{r_4'}$, we can calculate $\dfrac{h_2}{\rho_0}$, $\dfrac{h_3}{\rho_0}$ and $\dfrac{h_3}{h_2}$ as follows :

$$\left(\frac{h_2}{\rho_0} \right)^2 = 1 - \frac{\hat{\mathbf{x}}_\mathbf{1}' \cdot \Big(\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') - \mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \Big) x_3'}{\hat{\mathbf{x}}_\mathbf{3}' \cdot \Big(\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') - \mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \Big) x_1'} \quad , \tag{3.155}$$

$$\left(\frac{h_3}{\rho_0} \right)^2 = 1 - \frac{\hat{\mathbf{x}}_\mathbf{1}' \cdot \Big(\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') - \mathbf{S}^{(A)}(\mathbf{r_4'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \Big) x_2'}{\hat{\mathbf{x}}_\mathbf{2}' \cdot \Big(\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') - \mathbf{S}^{(A)}(\mathbf{r_4'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \Big) x_1'} \quad , \tag{3.156}$$

From (3.155)-(3.156) we obtain the quantity $\left(\dfrac{h_3}{h_2} \right)^2$.
Therefore, from the near-field data (3.152) we can calculate the quantity h_2/ρ_0 using the two measurements $\mathbf{S}^{(A)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}')$ and $\mathbf{S}^{(A)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}')$ and then calculate the quantities h_3/ρ_0 and h_3/h_2 using additionally the measurement $\mathbf{S}^{(A)}(\mathbf{r_4'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{1}')$. Note that the three measurements in (3.152) with electric polarization $\hat{\mathbf{x}}_\mathbf{3}'$ taken from the same three points would also work. We denote:

$$y := \frac{h_2}{\rho_0} \quad , \quad \phi := \frac{h_3}{\rho_0} \quad , \quad \kappa := \frac{h_3}{h_2} \quad . \tag{3.157}$$

The quantities y, ϕ and κ can directly categorize the ellipsoid to its geometrically degenerate forms of the sphere, the prolate spheroid and the oblate spheroid. More details about this are presented in section 5. Using relation (3.157) in (3.151), we solve for ρ_0 and obtain:

$$\rho_0 = \sqrt{{x_1'}^2 + \frac{{x_2'}^2}{1 - \phi^2} + \frac{{x_3'}^2}{1 - y^2}} \quad . \tag{3.158}$$

Therefore, $h_2 = y\rho_0$ and $h_3 = \phi\rho_0$ are now also known.
Next, in order to determine the Euler angles and the semi-axes of the ellipsoid we take the inner product of $\mathbf{S}^{(A)}$ with the observation vector $\mathbf{r'}$ and obtain the following scalar quantity:

$$\mathbf{r'} \cdot \mathbf{S}^{(A)}(\mathbf{r'}; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = \mathbf{r'}^{\top} P^{\top} \left(W^{(A)} + \frac{\rho}{h_\rho} Z^{(A)} \right) P \hat{\mathbf{p}}' \quad , \tag{3.159}$$

where the relation $\mathbf{r'} \cdot \hat{\boldsymbol{\rho}}' = \dfrac{\rho}{h_\rho}$ has been used (1.25) .
Therefore, from (3.159) and the measured near-field data (3.152) we obtain:

$$\begin{aligned}
m_1^{(A)} &:= \mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) = (+x'_1 \mathbf{P_1}^{\top} + x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_1}, \\
m_2^{(A)} &:= \mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2) = (+x'_1 \mathbf{P_1}^{\top} + x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_2}, \\
m_3^{(A)} &:= \mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3) = (+x'_1 \mathbf{P_1}^{\top} + x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_3}, \\
m_4^{(A)} &:= \mathbf{r'_3} \cdot \mathbf{S}^{(A)}(\mathbf{r'_3}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) = (+x'_1 \mathbf{P_1}^{\top} - x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_1}, \\
m_5^{(A)} &:= \mathbf{r'_3} \cdot \mathbf{S}^{(A)}(\mathbf{r'_3}; \hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2) = (+x'_1 \mathbf{P_1}^{\top} - x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_2}, \\
m_6^{(A)} &:= \mathbf{r'_3} \cdot \mathbf{S}^{(A)}(\mathbf{r'_3}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3) = (+x'_1 \mathbf{P_1}^{\top} - x'_2 \mathbf{P_2}^{\top} + x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_3}, \\
m_7^{(A)} &:= \mathbf{r'_4} \cdot \mathbf{S}^{(A)}(\mathbf{r'_4}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) = (+x'_1 \mathbf{P_1}^{\top} + x'_2 \mathbf{P_2}^{\top} - x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_1}, \\
m_8^{(A)} &:= \mathbf{r'_4} \cdot \mathbf{S}^{(A)}(\mathbf{r'_4}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3) = (+x'_1 \mathbf{P_1}^{\top} + x'_2 \mathbf{P_2}^{\top} - x'_3 \mathbf{P_3}^{\top}) \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_3},
\end{aligned} \tag{3.160}$$

with $\mathbf{P_i} = P\hat{\mathbf{x}}'_{\mathbf{i}}$ be the i^{th} column of matrix P. Now, we construct the 3×3 measurement matrix $M^{(A)} = (M_{ij}^{(A)})$ for $i, j = 1, 2, 3$, with elements as follows:

$$M_{ij}^{(A)} = \mathbf{P_i}^{\top} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_j} \quad . \tag{3.161}$$

Since $W^{(A)} + \dfrac{\rho_0}{h_{\rho_0}} Z^{(A)}$ is a diagonal matrix we obtain:

$$\mathbf{P_i}^{\top} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_j} = \mathbf{P_j}^{\top} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P_i} \quad , \tag{3.162}$$

for $i, j = 1, 2, 3$. From (3.150) and (3.161)-(3.162) the elements $M_{ij}^{(A)}$ of the measurement matrix $M^{(A)}$ can be given by:

$$\begin{aligned}
M_{ij}^{(A)} = M_{ji}^{(A)} &= \left(\mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_{\mathbf{i}}) - \mathbf{r'_{j+1}} \cdot \mathbf{S}^{(A)}(\mathbf{r'_{j+1}}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_{\mathbf{i}}) \right) / \left(2x'_j \right) && (3.163) \\
&= \left(\mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_{\mathbf{j}}) - \mathbf{r'_{i+1}} \cdot \mathbf{S}^{(A)}(\mathbf{r'_{i+1}}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_{\mathbf{j}}) \right) / \left(2x'_i \right) && (3.164)
\end{aligned}$$

for $i, j = 1, 2, 3$, with $\hat{\mathbf{d}}'$ such that $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_{\mathbf{i}} = 0$ in (3.163) and $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_{\mathbf{j}} = 0$ in (3.164). These elements can also be obtained by other combinations of measurements due to the relation $\mathbf{r'_1} - (\mathbf{r'_2} + \mathbf{r'_3} + \mathbf{r'_4}) = \mathbf{0}$.

Specifically, due to the measured near-field data (3.152) and therefore the quantities (38), the measurement matrix $M^{(A)}$ can be given in terms of $m_1^{(A)}, ..., m_8^{(A)}$ by:

$$M^{(A)} = \begin{bmatrix} \frac{m_4^{(A)} + m_7^{(A)}}{2x_1'} & \frac{m_1^{(A)} - m_4^{(A)}}{2x_2'} & \frac{m_1^{(A)} - m_7^{(A)}}{2x_3'} \\ \frac{m_1^{(A)} - m_4^{(A)}}{2x_2'} & \frac{m_2^{(A)} - m_5^{(A)}}{2x_2'} & \frac{m_3^{(A)} - m_6^{(A)}}{2x_2'} \\ \frac{m_1^{(A)} - m_7^{(A)}}{2x_3'} & \frac{m_3^{(A)} - m_6^{(A)}}{2x_2'} & \frac{m_3^{(A)} - m_8^{(A)}}{2x_3'} \end{bmatrix} \quad . \tag{3.165}$$

The set of equations (3.161) can be written in matrix form as:

$$M^{(A)} = P^\top \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) P \quad , \tag{3.166}$$

or since the rotation matrix P is orthogonal:

$$P M^{(A)} P^\top = W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \quad , \tag{3.167}$$

which is an orthogonal similarity relation between the measurement matrix $M^{(A)}$ and the diagonal matrix $W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)}$. Since $M^{(A)}$ is real and symmetric it has three real eigenvalues $\lambda_1^{(A)}, \lambda_2^{(A)}, \lambda_3^{(A)}$ and three corresponding orthonormal eigenvectors $\mathbf{v_1}^{(A)}, \mathbf{v_2}^{(A)}, \mathbf{v_3}^{(A)}$ respectively. Therefore, based on the orthogonal similarity relation, we conclude that :

$$\lambda_n^{(A)} = W_n^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_n^{(A)} \quad , \tag{3.168}$$

$$\mathbf{v_n}^{(A)} = (P_{n1}, P_{n2}, P_{n3})^{(A)} \quad , \tag{3.169}$$

for $n = 1, 2, 3$ and $A = K, D$, where P_{n1}, P_{n2}, P_{n3} are the elements of the n^{th} row of the rotation matrix P and the superscript (A) on the right hand side of (3.169) denotes the elements of the rotation matrix P corresponding to the cases $A = K, D$. By the system of equations (3.169), from the elements of the three orthogonal eigenvectors we obtain the elements of matrix P and therefore we can specify the Euler angles by using the following relations [20], [28]

$$\alpha = sin^{-1} \left(\frac{P_{31}}{\sqrt{1 - P_{33}^2}} \right) \ , \ \beta = sin^{-1} \left(\sqrt{1 - P_{33}^2} \right) \ , \ \gamma = sin^{-1} \left(\frac{P_{13}}{\sqrt{1 - P_{33}^2}} \right) \quad . \tag{3.170}$$

Therefore, for $P_{13}^{(A)}$, $P_{31}^{(A)}$ and $P_{33}^{(A)}$ from (3.169) we obtain the Euler angles $\alpha^{(A)}$, $\beta^{(A)}$ and $\gamma^{(A)}$ from (3.170) for each case. These angles show the orientation of the ellipsoid.
From the system of equations (3.168) that connect the eigenvalues $\lambda_n^{(A)}$ with the diagonal elements of the diagonal matrix $W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)}$, we obtain from (3.146)-(3.147) for the cases of the perfectly conductive, the impedance and the lossy dielectric ellipsoid:

$$\lambda_n^{(K)} = \frac{1}{I_1^n} \left(\frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right) \quad , \tag{3.171}$$

for $n = 1, 2, 3$ and similarly for the lossless dielectric ellipsoid:

$$\lambda_n^{(D)} = \frac{(\mu^+ \eta^2 - \mu^-) V}{(\mu^+ \eta^2 - \mu^-) V I_1^n + \mu^-} \left(\frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right) \quad , \tag{3.172}$$

for $n = 1, 2, 3$. Solving each equation of system (3.171) or (3.172) for the corresponding I_1^n, summing all three of them, multiplying both sides with V and using relation (3.142), we get the following for the cases of the perfectly conductive, the impedance and the lossy dielectric ellipsoid :

$$V^{(K)} = \frac{1}{L^{(K)}} \tag{3.173}$$

and similarly for the case of the lossless dielectric ellipsoid:

$$V^{(D)} = \frac{1 + \dfrac{3\mu^-}{\mu^+\eta^2 - \mu^-}}{L^{(D)}} \quad , \tag{3.174}$$

with

$$L^{(A)} := \sum_{n=1}^{3} \frac{1}{\lambda_n^{(A)}} \left(\frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right) \quad , \tag{3.175}$$

be the index of electric measured data, for $A = K, D$. In order to calculate the right hand side of (3.173) or (3.174), we calculate $\dfrac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0)$ making use of the relations:

$$x_1'^2 h_2^2 h_3^2 = \rho^2 \mu^2 \nu^2 \quad , \tag{3.176}$$

$$\sum_{n=1}^{3} h_n^2 x_n'^2 + 2x_1'^2 h_3^2 + h_2^2 h_3^2 = \rho^2\mu^2 + \mu^2\nu^2 + \nu^2\rho^2 \quad , \tag{3.177}$$

as well as the relations that connect I_1^n with the standard elliptic integrals of the first (E) and the second kind (F) ([20] p.381) and obtain the following:

$$\frac{\rho_0}{h_{\rho_0} J_1(\rho_0)} - I_1^1(\rho_0) = \frac{\rho_0 \sqrt{\rho_0^2 - h_2^2}\sqrt{\rho_0^2 - h_3^2}}{\rho_0^2 \left(\rho_0^4 - (\sum_{n=1}^3 h_n^2 x_n'^2 + 2x_1'^2 h_3^2 + h_2^2 h_3^2) + 2\dfrac{x_1'^2 h_2^2 h_3^2}{\rho_0^2} \right)} - \frac{F(y,\kappa) - E(y,\kappa)}{h_2 h_3^2} \quad , \tag{3.178}$$

$$\begin{aligned}\frac{\rho_0}{h_{\rho_0} J_2(\rho_0)} - I_1^2(\rho_0) &= \frac{\rho_0 \sqrt{\rho_0^2 - h_2^2}\sqrt{\rho_0^2 - h_3^2}}{(\rho_0^2 - h_3^2) \left(\rho_0^4 - (\sum_{n=1}^3 h_n^2 x_n'^2 + 2x_1'^2 h_3^2 + h_2^2 h_3^2) + 2\dfrac{x_1'^2 h_2^2 h_3^2}{\rho_0^2} \right)} \\ &- \frac{h_2 E(y,\kappa)}{(h_2^2 - h_3^2) h_3^2} + \frac{F(y,\kappa)}{h_2 h_3^2} + \frac{\sqrt{\rho_0^2 - h_2^2}}{(h_2^2 - h_3^2)\rho_0\sqrt{\rho_0^2 - h_3^2}} \quad ,\end{aligned} \tag{3.179}$$

$$\begin{aligned}\frac{\rho_0}{h_{\rho_0} J_3(\rho_0)} - I_1^3(\rho_0) &= \frac{\rho_0 \sqrt{\rho_0^2 - h_2^2}\sqrt{\rho_0^2 - h_3^2}}{(\rho_0^2 - h_2^2) \left(\rho_0^4 - (\sum_{n=1}^3 h_n^2 x_n'^2 + 2x_1'^2 h_3^2 + h_2^2 h_3^2) + 2\dfrac{x_1'^2 h_2^2 h_3^2}{\rho_0^2} \right)} \\ &+ \frac{E(y,\kappa)}{(h_2^2 - h_3^2) h_2} - \frac{\sqrt{\rho_0^2 - h_3^2}}{(h_2^2 - h_3^2)\rho_0\sqrt{\rho_0^2 - h_2^2}} \quad ,\end{aligned} \tag{3.180}$$

where x_n', ρ_0, y, κ, h_2, h_3 are known. Therefore, we can calculate V from relation (3.173) or (3.174) for each case respectively. Finally, writing V as follows:

$$V = \alpha_1\alpha_2\alpha_3 = \alpha_1\sqrt{\alpha_1^2 - h_2^2}\sqrt{\alpha_1^2 - h_3^2} \quad , \tag{3.181}$$

or equivalently:

$$\alpha_1^6 - (h_2^2 + h_3^2)\alpha_1^4 + (h_2^2 h_3^2)\alpha_1^2 - V^2 = 0 \quad , \tag{3.182}$$

we can solve for α_1 and obtain the size of the ellipsoid for each case respectively since $\alpha_2 = \sqrt{\alpha_1^2 - h_3^2}$ and $\alpha_3 = \sqrt{\alpha_1^2 - h_2^2}$.

We note that this method can be similarly applied using the zeroth coefficient of the low frequency expansion of the magnetic scattered field instead of the electric scattered field.

Physical Parameters

Previously, the orientation as well as the size of the ellipsoid were determined. For the case of the lossless dielectric ellipsoid based on relation (3.174), in order to determine V and therefore α_1 we need to know the physical parameters μ^+, μ^- and η. In this subsubsection we additionally use the leading order term of the low-frequency expansion of the magnetic scattered field in order to additionally determine physical parameters of the interior of the ellipsoid. Relation (3.181) from (3.174) can be written as:

$$\alpha_1 \sqrt{\alpha_1^2 - h_2^2}\sqrt{\alpha_1^2 - h_3^2} = \frac{1 + \dfrac{3\mu^-}{\mu^+\eta^2 - \mu^-}}{L^{(D)}} . \tag{3.183}$$

Solving this polynomial equation, we obtain α_1 according to the physical parameters μ^+, μ^- and η. The zeroth low-frequency approximation of the magnetic scattered field, based on $\mathbf{H}_n^s = \mathbf{H}_n^+ - \mathbf{H}_n^i$ as well as the relations (2.485),(2.490),(2.495) and (2.502), can be written as:

$$\mathbf{H_0^s}^{(D)}(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{q}}) = Q^{(D)}\hat{\mathbf{q}} + (\mathbf{r}^\top B^{(D)}\hat{\mathbf{q}})\hat{\boldsymbol{\rho}} \quad , \tag{3.184}$$

where $Q^{(D)} = diag(Q_n^{(D)})$ and $B^{(D)} = diag(B_n^{(D)})$ for $n = 1, 2, 3$ are diagonal 3×3 matrices with diagonal elements given by:

$$Q_n^{(D)} = Y^+ \frac{(\mu^+ - \mu^-)V}{\mu^+ - (\mu^+ - \mu^-)VI_1^n} I_1^n(\rho) \quad , \tag{3.185}$$

$$B_n^{(D)} = -Y^+ \frac{(\mu^+ - \mu^-)V}{[\mu^+ - (\mu^+ - \mu^-)VI_1^n]\, J_n(\rho)} \quad . \tag{3.186}$$

Using the rotation relations (3.144) for the vectors $\mathbf{r}, \hat{\mathbf{d}}, \hat{\mathbf{p}}, \hat{\boldsymbol{\rho}}$ we transform them from the $\mathbf{x_i'}$ system to the $\mathbf{x_i}$. Inserting them in $\mathbf{H_0^s}^{(D)}$ and then by multiplying the result with $P^\top$ we go back to the reference system $\mathbf{x_i'}$ where we will take the measured near-field data. Let

$$\mathbf{N}^{(D)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{q}}') := P^\top \mathbf{H_0^s}^{(D)}(P\mathbf{r}'; P\hat{\mathbf{d}}', P\hat{\mathbf{q}}') = P^\top \mathbf{H_0^s}^{(D)}(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{q}}) \quad . \tag{3.187}$$

Then

$$\mathbf{N}^{(D)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{q}}') = P^\top Q^{(D)} P\hat{\mathbf{q}}' + \left(\mathbf{r}'^\top P^\top B^{(D)} P\hat{\mathbf{q}}'\right)\hat{\boldsymbol{\rho}}' \quad . \tag{3.188}$$

Taking measurements at the same points of observation as we did for the electric scattered field in the previous section, we construct a measurement matrix using the magnetic near-field data as follows:

$$\begin{aligned}
\mathsf{m}_1^{(D)} &:= \mathbf{r_1'} \cdot \mathbf{N}^{(D)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{1}') \quad , & \mathsf{m}_2^{(D)} &:= \mathbf{r_1'} \cdot \mathbf{N}^{(D)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{2}') \quad , \\
\mathsf{m}_3^{(D)} &:= \mathbf{r_1'} \cdot \mathbf{N}^{(D)}(\mathbf{r_1'}; \hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{3}') \quad , & \mathsf{m}_4^{(D)} &:= \mathbf{r_3'} \cdot \mathbf{N}^{(D)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{1}') \quad , \\
\mathsf{m}_5^{(D)} &:= \mathbf{r_3'} \cdot \mathbf{N}^{(D)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{3}', \hat{\mathbf{x}}_\mathbf{2}') \quad , & \mathsf{m}_6^{(D)} &:= \mathbf{r_3'} \cdot \mathbf{N}^{(D)}(\mathbf{r_3'}; \hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{3}') \quad , \\
\mathsf{m}_7^{(D)} &:= \mathbf{r_4'} \cdot \mathbf{N}^{(D)}(\mathbf{r_4'}; \hat{\mathbf{x}}_\mathbf{2}', \hat{\mathbf{x}}_\mathbf{1}') \quad , & \mathsf{m}_8^{(D)} &:= \mathbf{r_4'} \cdot \mathbf{N}^{(D)}(\mathbf{r_4'}; \hat{\mathbf{x}}_\mathbf{1}', \hat{\mathbf{x}}_\mathbf{3}') \quad .
\end{aligned} \tag{3.189}$$

The 3×3 measurement matrix $\mathcal{M}^{(D)} = \left(\mathcal{M}_{ij}^{(D)}\right)$, for $i, j = 1, 2, 3$ with elements:

$$\mathcal{M}_{ij}^{(D)} = \mathbf{P_i}^\top \left(Q^{(D)} + \frac{\rho_0}{h_{\rho_0}} B^{(D)} \right) \mathbf{P_j} \quad , \tag{3.190}$$

is given in terms of $\mathsf{m}_1^{(D)}, ..., \mathsf{m}_8^{(D)}$ from (3.189)-(3.190), exactly like the measurement matrix $M^{(D)}$ was given in terms of $m_1^{(D)}, ..., m_8^{(D)}$ in (3.165). The set of equations (3.190) can be written in matrix form as:

$$\mathcal{M}^{(D)} = P^\top \left(Q^{(D)} + \frac{\rho_0}{h_{\rho_0}} B^{(D)} \right) P \tag{3.191}$$

and therefore we have an orthogonal similarity relation between the matrices $\mathcal{M}^{(D)}$ and $Q^{(D)} + \frac{\rho_0}{h_{\rho_0}} B^{(D)}$. Let $\kappa_n^{(D)}$ for $n = 1, 2, 3$, be the eigenvalues of the measurement matrix $\mathcal{M}^{(D)}$. Then similarly to the previous section the system of the eigenvalues for the case of the lossless dielectric ellipsoid is given by:

$$\kappa_n^{(D)} = \frac{Y^+(\mu^+ - \mu^-)V}{\mu^+ - (\mu^+ - \mu^-)V I_1^n} \left(I_1^n(\rho_0) - \frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} \right) \quad . \tag{3.192}$$

From the systems of equations (3.172) and (3.192), using relation (**??**), we can obtain two relations, one by suppressing I_1^n and one by suppressing $\frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0)$, as follows:

$$1 = \sum_{n=1}^{3} \frac{\left(\mu^-(\mu^+ - \mu^-)Y^+\lambda_n^{(D)} + \mu^+(\mu^+\eta^2 - \mu^-)\kappa_n^{(D)}\right)}{(\mu^+\eta^2 - \mu^-)(\mu^+ - \mu^-)\left(\kappa_n^{(D)} - Y^+\lambda_n^{(D)}\right)} \quad , \tag{3.193}$$

$$\frac{L^{(D)}}{Y^+ \mathsf{L}^{(D)}} = \frac{(\mu^+\eta^2 + 2\mu^-)(\mu^- - \mu^+)}{(\mu^+\eta^2 - \mu^-)(2\mu^+ + \mu^-)} \quad , \tag{3.194}$$

with

$$\mathsf{L}^{(D)} := \sum_{n=1}^{3} \frac{1}{\kappa_n^{(D)}} \left(\frac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right) \tag{3.195}$$

be the index of magnetic measured data. The system of equations (3.193)-(3.194) can be solved for μ^- and η using matlab. By finding these two physical parameters, we specify α_1 from (3.181) and therefore the size of the ellipsoid in the lossless dielectric case.

We note here that for the case of the lossy dielectric ellipsoid, the zeroth order coefficient $\mathbf{E_0^+}$ is independent of σ^- and is identical to that of the perfect conductor. Therefore, we can specify V from relation (3.173) as well as the size of the ellipsoid without the need of knowing the physical parameter μ^- that appears in the zeroth approximation of the magnetic scattered field. Nevertheless, we can apply the method described in this section to specify μ^- for this case similarly and obtain:

$$\mu^- = \mu^+ \left(1 - \sum_{n=1}^{3} \frac{\kappa_n^{(L)}}{\kappa_n^{(L)} - Y^+\lambda_n^{(L)}} \right) \quad , \tag{3.196}$$

with $\kappa_n^{(L)}$ be the eigenvalues of the corresponding $\mathcal{M}^{(L)}$ measurement matrix whose formula is given by (3.191) with the superscript (D) be replaced by (L) and where the matrices $Q^{(L)}$ and $B^{(L)}$ are identical to the ones of the lossless dielectric case (2.495),(2.502).

Geometrically Degenerate Forms

In this subsubsection we will study the cases of spheroids and spheres as geometrically degenerate forms of the ellipsoid. Specifically, for a sphere we have the following relations:

$$\alpha_1 = \alpha_2 = \alpha_3 \quad , \quad h_2 = h_3 = \mu = \nu = 0 \quad , \quad I_1^n(\rho) = \frac{1}{3\rho^3} \ , \ n = 1, 2, 3. \tag{3.197}$$

For a spheroid we have the following relations:

$$\text{prolate spheroid:} \quad \alpha_1 > \alpha_2 = \alpha_3 \quad , \quad h_2 = h_3 \quad , \tag{3.198}$$

$$\text{oblate spheroid:} \quad \alpha_1 = \alpha_2 > \alpha_3 \quad , \quad h_3 = 0 \quad . \tag{3.199}$$

Based on relations (3.155)-(3.157) we can categorize the unknown ellipsoid to one of these forms using three measurements. Specifically, start by taking two measurements at $\mathbf{r_1'}, \mathbf{r_3'}$ with the same polarization and calculate $y = h_2/\rho_0$ from relation (3.155). For $y = 0$, then $h_2 = 0$ and therefore from relation (1.15) it is concluded that $h_3 = 0$ and the ellipsoid deforms into a sphere. If $y \neq 0$ then continue by taking additionally one measurement at $\mathbf{r_4'}$ with the same polarization as the previous two measurements and calculate $\phi = h_3/\rho_0$ from relation (3.156). For $\phi = 0$ then $h_3 = 0$ and therefore the ellipsoid deforms into an oblate spheroid. If $\phi \neq 0$ then continue by finding $\kappa = h_3/h_2$. For $\kappa = 1$ the ellipsoid deforms into a prolate spheroid. Note here that since near-field data are used, it is assumed that $\rho_0 < \infty$. For $\rho \to \infty$ it is known that $r := |\mathbf{r}| \approx \rho$.

For the case of the sphere we only need to determine the size and specifically the radius α_1. There is no need to determine the orientation because of the symmetry of the sphere. Also, for the sphere it is known that $\rho = r$ as well as $\hat{\boldsymbol{\rho}} = \hat{\mathbf{r}}$. Therefore, the zeroth low-frequency approximation of the electric scattered field can be written from (3.145)-(3.147) and (3.197) as follows:

$$\mathbf{E_0^s}^{(A)}(\mathbf{r}, \hat{\mathbf{d}}, \hat{\mathbf{p}}) = W^{(A)}\hat{\mathbf{p}} + \left(\mathbf{r}^\top Z^{(A)}\hat{\mathbf{p}}\right)\hat{\mathbf{r}} \quad , \tag{3.200}$$

with

$$W^{(K)} = -\frac{\alpha_1^3}{r^3} I \quad , \quad W^{(D)} = -\frac{\alpha_1^3}{r^3}\frac{(\mu^+\eta^2 - \mu^-)}{(\mu^+\eta^2 + 2\mu^-)} I \quad , \tag{3.201}$$

$$Z^{(K)} = \frac{3\alpha_1^3}{r^4} I \quad , \quad Z^{(D)} = \frac{3\alpha_1^3}{r^4}\frac{(\mu^+\eta^2 - \mu^-)}{(\mu^+\eta^2 + 2\mu^-)} I \quad , \tag{3.202}$$

be diagonal 3×3 matrices and $I = diag(1)$ be the 3×3 identity matrix. Using the rotation relations (**??**) for the vectors $\mathbf{r}, \hat{\mathbf{r}}, \hat{\mathbf{d}}, \hat{\mathbf{p}}$ we transform them from the $\mathbf{x_i'}$ system to the $\mathbf{x_i}$. Inserting them into $\mathbf{E_0^s}^{(A)}$ and then multiplying the result with $P^\top$ we go back to the reference system. From relations (3.148) and (3.200)-(3.202) we obtain:

$$\mathbf{S}^{(A)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = W^{(A)}\hat{\mathbf{p}}' + \left(\mathbf{r}'^\top Z^{(A)}\hat{\mathbf{p}}'\right)\hat{\mathbf{r}}' \quad , \tag{3.203}$$

for $A = K, D$. Therefore, from (3.203) it is seen that $\mathbf{S}^{(A)}$ is independent of the rotation matrix P and there is no need to specify the orientation in the case of the sphere. Next, taking the inner product of $\mathbf{S}^{(A)}$ with $\hat{\mathbf{x}}_\mathbf{i}'$ we obtain for the perfectly conductive, the impedance and the lossy dielectric sphere:

$$\hat{\mathbf{x}}_\mathbf{i}' \cdot \mathbf{S}^{(K)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = \frac{\alpha_1^3}{r^3}\left(\frac{3}{r^2}(\mathbf{r}' \cdot \hat{\mathbf{p}}')x_i' - p_i'\right) \tag{3.204}$$

and similarly for the lossless dielectric sphere:

$$\hat{\mathbf{x}}_\mathbf{i}' \cdot \mathbf{S}^{(D)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') = \frac{\alpha_1^3}{r^3}\frac{(\mu^+\eta^2 - \mu^-)}{(\mu^+\eta^2 + 2\mu^-)}\left(\frac{3}{r^2}(\mathbf{r}' \cdot \hat{\mathbf{p}}')x_i' - p_i'\right) \quad , \tag{3.205}$$

for $i = 1, 2, 3$, with p'_i be the i^{th} coordinate of the electric polarization $\hat{\mathbf{p}}'$. Therefore, the radius α_1 can be specified from one measurement as follows. From (3.204) the radius for the perfectly conductive, the impedance and the lossy dielectic sphere is given by:

$$\alpha_1^{(K)} = \left[\frac{r^3 \left(\hat{\mathbf{x}}'_{\mathbf{i}} \cdot \mathbf{S}^{(K)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \right)}{\left(\frac{3}{r^2} (\mathbf{r}' \cdot \hat{\mathbf{p}}') x'_i - p'_i \right)} \right]^{1/3} \tag{3.206}$$

and similarly from (3.205) the radius for the lossless dielectric sphere is given by:

$$\alpha_1^{(D)} = \left[\frac{r^3 \left(\hat{\mathbf{x}}'_{\mathbf{i}} \cdot \mathbf{S}^{(D)}(\mathbf{r}'; \hat{\mathbf{d}}', \hat{\mathbf{p}}') \right)}{\frac{(\mu^+ \eta^2 - \mu^-)}{(\mu^+ \eta^2 + 2\mu^-)} \left[\left(\frac{3}{r^2} (\mathbf{r}' \cdot \hat{\mathbf{p}}') x'_i - p'_i \right) \right]} \right]^{1/3}, \tag{3.207}$$

where $r := |\mathbf{r}| = |\mathbf{r}'| = \sum_{n=1}^{3} x'^2_n$ from rotation relation (3.144) since P is orthogonal. Consequently, for the case of the sphere the measurements are reduced to two. We start by taking the two measurements included in (3.155) in order to calculate h_2/ρ_0 and after categorizing the ellipsoid into a sphere, we use one of these two measurements again in order to specify the radius α_1 from relation (3.206) or (3.207). We note that the method using all eight measurements of (3.152) described previously can also be applied for the case of the sphere. Specifically, for the eigenvalues of the measurement matrix $M^{(A)}$ it is valid that $\lambda_1^{(A)} = \lambda_2^{(A)} = \lambda_3^{(A)}$ from (3.168) and (3.201)-(3.202) as well as $\alpha_1 = V^{1/3}$ from (3.181) and (3.197).

For the case of the spheroid we need to specify two of the semi-axes in order to determine the size. Also, due to its rotational symmetry we need to specify two of the Euler angles in order to determine the orientation. Specifically, for the case of the prolate spheroid we need to specify the two semi-axes α_1, α_2 and the two Euler angles α, β. The axis of the spheroid that is fixed by the angles α, β is independent to any rotation by the Euler angle γ. Letting $\gamma = 0$ in (3.143), the orthogonal rotation matrix takes the form:

$$P_{(0)} = \begin{bmatrix} cos\alpha & sin\alpha & 0 \\ -cos\beta\ sin\alpha & cos\beta\ cos\alpha & sin\beta \\ sin\beta\ sin\alpha & -sin\beta\ cos\alpha & cos\beta \end{bmatrix} \tag{3.208}$$

Therefore, taking additionally to the three measurements that were used for the categorization, the three measurements that are used for $m_2^{(A)}$, $m_3^{(A)}$ and $m_5^{(A)}$ in (3.160), we obtain from the elements $\mathbf{P}_{\mathbf{i}}^{\top} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P}_{\mathbf{i}}$ for $i = 1, 2, 3$, by replacing P with $P_{(0)}$, the following system:

$$\frac{m_4^{(A)} + m_7^{(A)}}{2x'_1} = \left(W_1^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_1^{(A)} \right) cos^2\alpha + \left(W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} \right) sin^2\alpha, \tag{3.209}$$

$$\frac{m_2^{(A)} - m_5^{(A)}}{2x'_2} = \left(W_1^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_1^{(A)} \right) sin^2\alpha + \left(W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} \right) cos^2\alpha, \tag{3.210}$$

$$\frac{m_3^{(A)}}{x'_3} = W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)}, \tag{3.211}$$

where the relation $W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} = W_3^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_3^{(A)}$ due to (3.198) has been used. Solving the system (3.209)-(3.211) we obtain the quantities $W_n^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_n^{(A)}$, $n = 1, 2, 3$, as well as the

Euler angle α:

$$W_1^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_1^{(A)} = \frac{m_4^{(A)} + m_7^{(A)}}{2x_1'} + \frac{m_2^{(A)} - m_5^{(A)}}{2x_2'} - \frac{m_3^{(A)}}{x_3'} \quad , \tag{3.212}$$

$$W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} = W_3^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_3^{(A)} = \frac{m_3^{(A)}}{x_3'} \quad , \tag{3.213}$$

$$sin^2\alpha^{(A)} = \frac{1}{2}\left(\frac{\dfrac{m_2^{(A)} - m_5^{(A)}}{x_2'} - \dfrac{2m_3^{(A)}}{x_3'}}{\dfrac{m_4^{(A)} + m_7^{(A)}}{2x_1'} + \dfrac{m_2^{(A)} - m_5^{(A)}}{2x_2'} - \dfrac{2m_3^{(A)}}{x_3'}} \right) , \tag{3.214}$$

where the superscript (A) on the left hand side of (3.214) denotes the Euler angle corresponding to the cases $A = K, D$.
Next, letting $\gamma = \pi/2$ the rotation matrix P takes the form:

$$P_{(\frac{\pi}{2})} = \begin{bmatrix} -cos\beta\ sin\alpha & cos\beta\ cos\alpha & sin\beta \\ -cos\alpha & -sin\alpha & 0 \\ sin\beta\ sin\alpha & -sin\beta\ cos\alpha & cos\beta \end{bmatrix} \quad . \tag{3.215}$$

Since the values of $m_4^{(A)}$, $m_7^{(A)}$ are still the same, we obtain the following relation:

$$\mathbf{P}^{\top}_{\mathbf{1}\,(\frac{\pi}{2})} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P}_{\mathbf{1}(\frac{\pi}{2})} = \mathbf{P}^{\top}_{\mathbf{1}\,(0)} \left(W^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z^{(A)} \right) \mathbf{P}_{\mathbf{1}(0)}$$

or equivalently

$$\begin{aligned} &\left(W_1^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_1^{(A)} \right) cos^2\beta\ sin^2\alpha + \left(W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} \right) (cos^2\alpha + sin^2\beta\ sin^2\alpha) \\ &= \left(W_1^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_1^{(A)} \right) cos^2\alpha + \left(W_2^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_2^{(A)} \right) sin^2\alpha. \end{aligned} \tag{3.216}$$

Solving for β we obtain:

$$cos^2\beta^{(A)} = \frac{\dfrac{m_4^{(A)} + m_7^{(A)}}{x_1'} - \dfrac{2m_3^{(A)}}{x_3'}}{\dfrac{m_2^{(A)} - m_5^{(A)}}{x_2'} - \dfrac{2m_3^{(A)}}{x_3'}} \quad , \tag{3.217}$$

where relation (3.214) has been used and where the superscript (A) on the left hand side of (3.217) denotes the Euler angle corresponding to the cases $A = K, D$. Therefore, from (3.214) and (3.217) the orientation of the prolate spheroid is specified. Next, using relation (3.146) or (3.147) for the known due to (3.212)-(3.213) quantities $C_n^{(A)} := W_n^{(A)} + \frac{\rho_0}{h_{\rho_0}} Z_n^{(A)}$, solving each equation for the corresponding I_1^n, summing all three of them, multiplying both sides with V and using relation (**??**), we obtain for the perfectly conductive, the impedance and the lossy dielectric prolate spheroid:

$$V^{(K)} = \frac{1}{\sum_{n=1}^{3} \dfrac{1}{C_n^{(K)}} \left(\dfrac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right)} \tag{3.218}$$

and similarly for the case of the lossless dielectric prolate spheroid:

$$V^{(D)} = \frac{1 + \dfrac{3\mu^-}{\mu^+\eta^2 - \mu^-}}{\sum_{n=1}^{3} \dfrac{1}{C_n^{(D)}} \left(\dfrac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0) \right)} \quad , \tag{3.219}$$

where $\dfrac{1}{C_2^{(A)}} \left(\dfrac{\rho_0}{h_{\rho_0} J_2(\rho_0)} - I_1^2(\rho_0) \right) = \dfrac{1}{C_3^{(A)}} \left(\dfrac{\rho_0}{h_{\rho_0} J_3(\rho_0)} - I_1^3(\rho_0) \right)$, for $A = K, D$, due to relation (3.198). In order to calculate the right hand side of (3.218) or (3.219), we calculate $\dfrac{\rho_0}{h_{\rho_0} J_n(\rho_0)} - I_1^n(\rho_0)$ making use of (3.176)-(3.177) and the formulas of the $I_1^n(\rho)$ for the prolate spheroid and obtain:

$$\frac{\rho_0}{h_{\rho_0} J_1(\rho_0)} - I_1^1(\rho_0) = \frac{\rho_0^2 - h_2^2}{\rho_0 \left(\rho_0^4 - h_2^2 (2{x'_1}^2 + {x'_2}^2 + {x'_3}^2) - h_2^4 + 2\dfrac{{x'_1}^2 h_2^4}{\rho_0^2} \right)} - \frac{1}{2h_2^3} \ln\left(\frac{\rho_0 + h_2}{\rho_0 - h_2} \right) + \frac{1}{h_2^2 \rho_0} \quad , \tag{3.220}$$

$$\frac{\rho_0}{h_{\rho_0} J_2(\rho_0)} - I_1^2(\rho_0) = \frac{\rho_0}{\rho_0^4 - h_2^2 (2{x'_1}^2 + {x'_2}^2 + {x'_3}^2) - h_2^4 + 2\dfrac{{x'_1}^2 h_2^4}{\rho_0^2}} + \frac{1}{4h_2^2} \ln\left(\frac{\rho_0 + h_2}{\rho_0 - h_2} \right) - \frac{\rho_0}{2h_2^2(\rho_0^2 - h_2^2)} \quad , \tag{3.221}$$

where x'_n, ρ_0 and h_2 are known. Therefore, we can calculate V from (3.218) or (3.219) for each case respectively. Finally, writing V as in (3.181) and using (3.198) we obtain:

$$\alpha_1^6 - 2h_2^2 \alpha_1^4 + h_2^4 \alpha_1^2 - V^2 = 0 \quad , \tag{3.222}$$

which can be solved for α_1 and therefore we can specify the size of the prolate spheroid for each case respectively since $\alpha_2 = \alpha_3 = \sqrt{\alpha_1^2 - h_2^2}$.
Consequently, for the case of the prolate spheroid the measurements are reduced to six. We start by taking the three measurements included in (3.155)-(3.156) in order to calculate h_2/ρ_0, h_3/ρ_0 and h_3/h_2 and after categorizing the ellipsoid into a prolate spheroid we take additionally the three measurements used for $m_2^{(A)}$, $m_3^{(A)}$, $m_5^{(A)}$ in (3.160) in order to specify its size and orientation from (3.212)-(3.214) and (3.217)-(3.222). We note that the method using all eight measurements of (3.152) described previously can also be applied for the case of the prolate spheroid. Specifically, for the eigenvalues of the measurement matrix $M^{(A)}$ it is valid that $\lambda_2^{(A)} = \lambda_3^{(A)}$ from (3.168) and (3.213).

Similarly we work for the case of the oblate spheroid, where after calculating h_2 and V we solve for α_1 the equation:

$$\alpha_1^6 - h_2^2 \alpha_1^4 - V^2 = 0 \quad . \tag{3.223}$$

Note that if the scatterer is a priori known to be a sphere then one measurement suffice to specify its radius α_1 from (3.206) or (3.207). Also, if the scatterer is a priori known to be a spheroid then the measurements are reduced to five since the measurement $\mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1)$ used for the categorization in (3.155)-(3.157) is not being used for the specification of its size and orientation.

Measurement Reduction

In this subsubsection we suggest an alternative option for the set of the measured near-field data needed in order to specify the orientation and the size of the ellipsoid. This option has the advantage of reducing the number of needed measurements from eight to six but has the disadvantage of increasing the number of points from which we take the measurements from three to six as well as the disadvantage that the additional three points used are dependent on the first three measurements and are found after calculating the quantities ρ_0, h_2, h_3.
Similarly, we start by taking a point $\mathbf{r'_1} = (x'_1, x'_2, x'_3)$ with ellipsoidal coordinates (ρ_0, μ_0, ν_0), with $\alpha_1 < \rho_0 < \infty$ and $x'_n \neq 0,\ n = 1, 2, 3$. Also, we consider the points $\mathbf{r'_3}, \mathbf{r'_4}$ given in (3.150). From these points we take the following three measurements:

$$\mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) \quad , \quad \mathbf{S}^{(A)}(\mathbf{r'_3}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) \quad , \quad \mathbf{S}^{(A)}(\mathbf{r'_4}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) \tag{3.224}$$

and use them to to calculate the quantities y, ϕ, κ from relations (3.155)-(3.157) as well as the quantities ρ_0, h_2, h_3 via (3.157)-(3.158). Next, we consider the points:

$$\mathbf{y'_1} = (\rho_0, 0, 0) \quad , \quad \mathbf{y'_2} = (0, \sqrt{\rho_0^2 - h_3^2}, 0) \quad , \quad \mathbf{y'_3} = (0, 0, \sqrt{\rho_0^2 - h_2^2}) \quad , \tag{3.225}$$

which can be written equivalently as $\mathbf{y'_i} = \sqrt{\rho_0^2 - \alpha_1^2 + \alpha_i^2}\, \hat{\mathbf{x}}'_\mathbf{i}$ for $i = 1, 2, 3$ respectively and which belong on the surface S_0 of the ellipsoid ($\rho = \rho_0$) given by (3.151). From the three new points (3.225) we take the following three measurements:

$$\mathbf{S}^{(A)}(\mathbf{y'_1}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3)\ ,\ \mathbf{S}^{(A)}(\mathbf{y'_2}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1)\ ,\ \mathbf{S}^{(A)}(\mathbf{y'_3}; \hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2)\ . \tag{3.226}$$

From (3.153) and the measured near-field data (3.226) we obtain the quantities:

$$\begin{aligned}
\tilde{m}_1^{(A)} &:= \hat{\mathbf{x}}'_2 \cdot \mathbf{S}^{(A)}(\mathbf{y'_1}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3) = \mathbf{P_2}^\top W^{(A)} \mathbf{P_3} \quad , \\
\tilde{m}_2^{(A)} &:= \hat{\mathbf{x}}'_3 \cdot \mathbf{S}^{(A)}(\mathbf{y'_1}; \hat{\mathbf{x}}'_2, \hat{\mathbf{x}}'_3) = \mathbf{P_3}^\top W^{(A)} \mathbf{P_3} \quad , \\
\tilde{m}_3^{(A)} &:= \hat{\mathbf{x}}'_1 \cdot \mathbf{S}^{(A)}(\mathbf{y'_2}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) = \mathbf{P_1}^\top W^{(A)} \mathbf{P_1} \quad , \\
\tilde{m}_4^{(A)} &:= \hat{\mathbf{x}}'_3 \cdot \mathbf{S}^{(A)}(\mathbf{y'_2}; \hat{\mathbf{x}}'_3, \hat{\mathbf{x}}'_1) = \mathbf{P_3}^\top W^{(A)} \mathbf{P_1} \quad , \\
\tilde{m}_5^{(A)} &:= \hat{\mathbf{x}}'_1 \cdot \mathbf{S}^{(A)}(\mathbf{y'_3}; \hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2) = \mathbf{P_1}^\top W^{(A)} \mathbf{P_2} \quad , \\
\tilde{m}_6^{(A)} &:= \hat{\mathbf{x}}'_2 \cdot \mathbf{S}^{(A)}(\mathbf{y'_3}; \hat{\mathbf{x}}'_1, \hat{\mathbf{x}}'_2) = \mathbf{P_2}^\top W^{(A)} \mathbf{P_2} \quad .
\end{aligned} \tag{3.227}$$

Therefore, from the three measurements (3.226) we can construct the 3×3 real and symmetric measurement matrix $\tilde{M}^{(A)} = P^\top W^{(A)} P$ with elements $\tilde{M}_{ij}^{(A)}$ given by:

$$\tilde{M}_{ij}^{(A)} = \mathbf{P_i}^\top W^{(A)} \mathbf{P_j} \quad , \tag{3.228}$$

in terms of $\tilde{m}_1^{(A)}, ..., \tilde{m}_6^{(A)}$ as follows:

$$\tilde{M}^{(A)} = \begin{bmatrix} \tilde{m}_3^{(A)} & \tilde{m}_5^{(A)} & \tilde{m}_4^{(A)} \\ \tilde{m}_5^{(A)} & \tilde{m}_6^{(A)} & \tilde{m}_1^{(A)} \\ \tilde{m}_4^{(A)} & \tilde{m}_1^{(A)} & \tilde{m}_2^{(A)} \end{bmatrix} \quad .$$

Next, we find the eigenvalues and the corresponding eigenvectors of the measurement matrix $\tilde{M}^{(A)}$ and the process from this point forward continues following the exact same steps described in the Inverse Problem subsubsection that we presented the near-field method, in order to specify the orientation and the size of the ellipsoid.

Specifically, from the eigenvalues $\tilde{\lambda}_n^{(A)}$ of the measurement matrix $\tilde{M}^{(A)}$ we obtain for the perfectly conductive, the impedance and the lossy dielectric ellipsoid:

$$V^{(K)} = -\frac{1}{\sum_{n=1}^{3} \frac{1}{\tilde{\lambda}_n^{(K)}} I_1^n(\rho_0)} \quad ,$$

and similarly for the lossless dielectric ellipsoid:

$$V^{(D)} = \frac{\dfrac{1+3\mu^-}{\mu^- - \mu^+\eta^2}}{\sum_{n=1}^{3} \frac{1}{\tilde{\lambda}_n^{(D)}} I_1^n(\rho_0)}$$

and therefore after calculating V in each case, we can specify the size of the corresponding ellipsoid by solving equation (3.182).

Conclusion

In this subsubsection we make some remarks referring to the variety of possible combinations for the needed near-field data. Moreover, we summarize the near-field method presented in this section in an algorithm and we give the corresponding flow chart.

We note that the calculation of the quantities $y = h_2/\rho_0$, $\phi = h_3/\rho_0$ and $\kappa = h_3/h_2$, that are used for the calculation of the quantities ρ_0, h_2, h_3 as well as for the categorization of the ellipsoidal scattered to a sphere, a prolate spheroid or an oblate spheroid, can be made using a variety of possible combinations of electric near-field data. This can be seen from relations (3.155)-(3.156), where three measurements taken for the same electric polarization, one at $\mathbf{r'_1}$, one at $\mathbf{r'_3}$ and one at $\mathbf{r'_4}$, suffice. Moreover, if we consider the points:

$$\mathbf{r'_5} = (x'_1, -x'_2, -x'_3) \;\; , \;\; \mathbf{r'_6} = (-x'_1, x'_2, -x'_3) \;\; , \;\; \mathbf{r'_7} = (-x'_1, -x'_2, x'_3) \;\; , \tag{3.229}$$

which belong on the surface of the ellipsoid $\rho = \rho_0$ since they are symmetric to $\mathbf{r'_1}$ over the principal axes of the reference system, we note that the measurement taken from $\mathbf{r'_3}$ could be replaced by a measurement taken from $\mathbf{r'_6}$ and the measurement taken from $\mathbf{r'_4}$ could be replaced by a measurement taken from $\mathbf{r'_7}$, providing that the electric polarization is the same for all the three used measurements. Also, we note that these quantities could also be calculated using magnetic near-field data in a similar way.
Moreover, the measurement matrix $M^{(A)}$ with elements $M_{ij}^{(A)}$ given by (3.161) whose eigenvalues and eigenvectors are used for the specification of the orientation and the size of the ellipsoid, can be constructed using a variety of possible combinations of electric near-field data. Some of these possible combinations are given in (3.163)-(3.164) as well as in the following relations:

$$M_{ij}^{(A)} = M_{ji}^{(A)} = \Big(\mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{i}) + \mathbf{r'_{j+4}} \cdot \mathbf{S}^{(A)}(\mathbf{r'_{j+4}}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{i})\Big) \,/\, \big(2x'_j\big) \tag{3.230}$$

$$= \Big(\mathbf{r'_1} \cdot \mathbf{S}^{(A)}(\mathbf{r'_1}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{j}) + \mathbf{r'_{i+4}} \cdot \mathbf{S}^{(A)}(\mathbf{r'_{i+4}}; \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{j})\Big) \,/\, \big(2x'_i\big) \quad , \tag{3.231}$$

for $i, j = 1, 2, 3$, with $\hat{\mathbf{d}}'$ such that $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_\mathbf{i} = 0$ in (3.230) and $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_\mathbf{j} = 0$ in (3.231). These elements can also be obtained using other combinations of measurements due to the relation $\mathbf{r'_1} - (\mathbf{r'_2} + \mathbf{r'_3} + \mathbf{r'_4}) = \mathbf{r'_1} + (\mathbf{r'_5} + \mathbf{r'_6} + \mathbf{r'_7}) = \mathbf{0}$.
Furthermore, after calculating the quantities ρ_0, h_2, h_3, the measurement matrix $\tilde{M}$ with elements $\tilde{M}_{ij}^{(A)}$ given by (3.228) can be obtained from a variety of possible combinations of electric near-field data as follows:

$$\tilde{M}_{ij}^{(A)} = \tilde{M}_{ji}^{(A)} = \hat{\mathbf{x}}'_\mathbf{j} \cdot \mathbf{S}^{(A)}(\mathbf{y'_k}, \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{i}) \tag{3.232}$$

$$= \hat{\mathbf{x}}'_\mathbf{i} \cdot \mathbf{S}^{(A)}(\mathbf{y'_k}, \hat{\mathbf{d}}', \hat{\mathbf{x}}'_\mathbf{j}) \quad , \tag{3.233}$$

for $i, j, k = 1, 2, 3$ with $k \neq j$ and $\hat{\mathbf{d}}'$ such that $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_{\mathbf{i}} = 0$ in (3.232) and with $k \neq i$ and $\hat{\mathbf{d}}'$ such that $\hat{\mathbf{d}}' \cdot \hat{\mathbf{x}}'_{\mathbf{j}} = 0$ in (3.233).

Next, we present an algorithm that summarizes the method described in this section.
Step 1: Choose a point $\mathbf{r'_1} = (x'_1, x'_2, x'_3)$ with ellipsoidal coordinates (ρ_0, μ_0, ν_0), with $\alpha_1 < \rho_0 < \infty$ and $x'_n \neq 0,\ n = 1, 2, 3$. Based on $\mathbf{r'_1}$ consider the points $\mathbf{r'_2}$, $\mathbf{r'_3}$, $\mathbf{r'_4}$ given by (3.150) or the points $\mathbf{r'_5}$, $\mathbf{r'_6}$, $\mathbf{r'_7}$ given by (3.229). Continue to Step 2.
Step 2: Take two measurements, one at $\mathbf{r'_1}$ and one at $\mathbf{r'_3}$ (or $\mathbf{r'_6}$), for the same electric polarization and use them to calculate the quantity $y = h_2/\rho_0$ from (3.155). If $y = 0$ then categorize the scatterer into a sphere and go directly to Step 4a. If $y \neq 0$, take additionally one measurement at $\mathbf{r'_4}$ (or $\mathbf{r'_7}$) for the same electric polarization as the previous two measurements and calculate the quantities $\phi = h_3/\rho_0$ and $\kappa = h_3/h_2$ from (3.156)-(3.157). If $\phi = 0$ then categorize the scatterer into an oblate spheroid. If $\phi \neq 0$ and $\kappa = 1$ then categorize the scatterer into a prolate spheroid. If the scatterer is categorized into a spheroid continue to Step 3 and then continue to Step 4b or to Step 4c. If the scatterer is an ellipsoid continue to Step 3 and then continue to Step 4d or to Step 4e.
Step 3: Using the known from the previous step quantities y, ϕ, κ, calculate ρ_0 from (3.158). Then calculate the semi-interfocal distances h_2 and h_3 from (3.157).
Step 4a: Use one of the first two measurements taken in Step 2, to calculate the sphere's radius from relation (3.206) or (3.207).
Step 4b: Take three more measurements from an appropriate combination of the points $\mathbf{r'_1}, ..., \mathbf{r'_7}$ and electric polarizations along the axes and use them along with the three measurements taken in Step 2, to specify the semi-axe α_1 and therefore the size of the spheroid as well as its orientation from (3.212)-(3.214) and (3.217)-(3.223).
Step 4c: Consider the points $\mathbf{y'_1}, \mathbf{y'_2}, \mathbf{y'_3}$ given by (3.225). Take three measurements from an appropriate combination of these points and electric polarizations along the axes and use them to specify the semi-axe α_1 and therefore the size of the spheroid as well as its orientation.
Step 4d: Take five more measurements from an appropriate combination of the points $\mathbf{r'_1}, ..., \mathbf{r'_7}$ and electric polarizations along the axes and use them along with the three measurements taken in Step 2 in order to construct the measurement matrix $M^{(A)}$ with elements given by (3.161). Continue to Step 5.
Step 4e: Consider the points $\mathbf{y'_1}, \mathbf{y'_2}, \mathbf{y'_3}$ given by (3.225). Take three measurements from an appropriate combination of these points and electric polarizations along the axes and use them in order to construct the measurement matrix $\tilde{M}^{(A)}$ with elements given by (3.228). Continue to Step 5.
Step 5: Find the eigenvalues and the corresponding orthonormal eigenvectors of the measurement matrix $M^{(A)}$ due to Step 4d (or of the measurement matrix $\tilde{M}^{(A)}$ due to Step 4e). Continue to Step 6.
Step 6: From the eigenvectors and the relations (3.169) and (3.170) find the Euler angles and therefore the orientation of the ellipsoid. Continue to Step 7.
Step 7: From the eigenvalues and the known (due to Step 2 and Step 3) quantities y, κ, ρ_0, h_2, h_3, calculate V. Next, Specify the semi-axe α_1 from relation (3.182) and therefore specify the size of the ellipsoid.
The above algorithm is depicted in a flow chart in the last page.

In conclusion, the near-field method presented in this section gives us the opportunity to use measurements taken from convenient points of observation and for convenient directions of propagation and polarization along the axes. Moreover, it allows us to categorize the ellipsoid to its geometrically degenerate forms of the sphere, the prolate spheroid and the oblate spheroid using two measurements for the case of the sphere or three measurements for the case of the spheroid. The possibility of categorizing the ellipsoidal scatterer to one of these geometrically degenerate forms gives us the opportunity to reduce the amount of measurements needed to find the size and the orientation. Furthermore, this method allows us to specify the orientation and the size of the lossy dielectric ellipsoid using electric near-field data without the need of knowing the physical parameters of the ellipsoid. Finally, it gives us the opportunity to specify additionally physical parameters of the ellipsoid using both the electric and the magnetic near-field data in the lossless and the lossy dielectric cases.

Bibliography

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover, (1964).

[2] T. S. Angell and R. E. Kleinman, Polarizability Tensors in Low-Frequency Inverse Scattering, Radio Sci. 22, 1120 (1987).

[3] C. Athanasiadis, K. Kiriaki, Low-Frequency Scattering by an Ellipsoidal Dielectric with a Confocal Ellipsoidal Perfect Conductor Core, Mathematica Balcanica , 3, pp .370-389, (1989).

[4] C. Athanasiadis, The Multi-Layered Ellipsoid with a Soft Core in the Presence of a Low-Frequency Acoustic Wave, Q. J. Mech. Appl. Math., vol. 47, pp. 441-159, (1994).

[5] C. Athanasiadis, P. Martin, I. G. Stratis, On Spherical-Wave Scattering by a Spherical Scatterer and Related Near-Field Inverse Problems. IMA J. Appl. Math., vol. 66 , pp.539-549, (2001).

[6] C. Athanasiadis, Wave Propagation Elements and Applications. (2015) (HOU , Patras, Greece).

[7] C. E. Athanasiadis, E. Athanasiadou, S. Zoi and I. Arkoudis, An Inverse Scattering Problem for the Dielectric Ellipsoid, International Conference of Numerical Analysis and Applied Mathematics, AIP Conference Proceedings 1978, 470055 (2018); doi: 10.1063/1.5044125, (2018)

[8] C. E. Athanasiadis, E. Athanasiadou and I. Arkoudis, Detecting a Layered Ellipsoid Solving a Near-Field Inverse Acoustic Scattering Problem, Journal of Applied Mathematics and Bioinformatics, vol.8, no.2, 2018, 29-46, ISSN: 1792-6602 (print), 1792-6939 (online) Scienpress Ltd, (2018).

[9] T. Apostolopoulos, K. Kiriaki and D. Polyzos: The Inverse Scattering Problem for a Rigid Ellipsoid in Linear Elasticity, Inverse Problems, vol. 6 , p.1, (1990).

[10] F. Cakoni and D. Colton, Qualitative Methods in Inverse Scattering Theory, Springer, Series on Interaction of Mechanics and Mathematics, (2006).

[11] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Second Edition, ed. Springer, Berlin Heidelberg, (1998).

[12] G. Dassios, The Inverse Scattering Problem for the Soft Ellipsoid, Journal of Mathematical Physics, 28, pp. 2858-2862, (1987).

[13] G. Dassios, K. Kiriaki, Size, Orientation, and Thickness Identification of an Ellipsoidal Shell. workshop on inverse problems and imaging, Glasgow 1989, Proceedings, pp. 38-48, Glasgow, (1991).

[14] G. Dassios and A. Charalambopoulos , Inverse Scattering via Low-frequency Moments, Journal of Mathematical Physics, 32: 4206–4216, (1992).

[15] G. Dassios and G. Kamvyssas, Point Source Excitation in Direct and Inverse Scattering: The Soft and the Hard Sphere, IMA J. Appl. Math. 55:67–84, (1995).

[16] G. Dassios and R. Lucas, Inverse Scattering for the Penetrable Ellipsoid and Ellipsoidal boss, Journal of the Acoustical Society of America, vol. 99, pp. 1877-1882, (1996).

[17] G. Dassios and R. Lucas, Electromagnetic Imaging of Ellipsoids and Ellipsoidal Bosses, Quarterly Journal of Mechanics and Applied Mathematics, vol. 51, pp. 413-426, (1998).

[18] G. Dassios, Scattering of Acoustic waves by a Coated Pressure-Release Ellipsoid. Journal of the Acoustical Society of America, 70, pp. 176-185 , (1998).

[19] G. Dassios and R. Kleinman, Low Frequency Scattering, Oxford University Press, Oxford, (2000).

[20] G. Dassios, Ellipsoidal Harmonics: Theory and Applications (Cambridge: Cambridge University Press), (2012).

[21] E.W. Hobson. The Theory of Spherical and Ellipsoidal Harmonics. Cambridge: Cambridge University Press, (1931).

[22] A. Kirsch and N. Grinberg, The Factorization Method For Inverse Problems, Oxford University Press (2008).

[23] R. E. Kleinman, The Rayleigh region. Proceedings of the IEEE, 53:848–856, (1965).

[24] R. J. Lucas, An Inverse Problem in Low-Frequency Scattering by a Rigid Ellipsoid, J. Acoust. Soc. Am., vol. 95, pp. 2330-2333 (1994).

[25] M. G. Lamé, Sur les Surfaces Isothermes dans les Corps Solides Homogènes en équilibre de Température. Journal de Mathématiques Pures et Appliquées, 2:147–183, (1837).

[26] W. D. MacMillan, The Theory of the Potential. New York: Dover, (1958).

[27] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, (2003).

[28] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Volume I. New York: McGraw-Hill, (1953).

[29] W. D. Niven. On Ellipsoidal Harmonics. Philosophical Transactions of the Royal Society of London A., 182:231–278, (1891).

[30] Lord Rayleigh, On the Incidence of Aerial and Electric Waves upon Small Obstacles in the Form of Ellipsoids or Elliptic Cylinders and on the Passage of Electric Waves through a Circular Aperture in a Conducting Screen. Philosophical Magazine, 44:28–52, (1897).

[31] A. F. Stevenson, Solution of Electromagnetic Scattering Problems as Power Series in the Ratio (Dimension of Scatterer)/Wavelength, Journal of Applied Physics, 24:1134–1142, (1953).

[32] J. A. Stratton, Electromagnetic Theory. New York: McGraw-Hill, (1941)

[33] K. Skourogiannis, Phd thesis, NKUA, (2014).

[34] I. Todhunter, An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions. London: MacMillan, (1875).