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MASTER THESIS

Trace Theorems for Sobolev spaces

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Abstract

For $\Omega \subset \mathbb{R}^N$ an open set with boundary $\partial\Omega$ satisfying certain smoothness assumptions, we consider the Besov spaces $\mathcal{B}_p^{l-1/p}(\partial\Omega)$ as the trace spaces of the Sobolev spaces $W_p^l(\Omega)$. More specifically, first we consider the traces in $\mathbb{R}^{N-1}$ of functions defined on $\mathbb{R}^N$ and we prove that the trace operator $\mathcal{T} : W_p^1(\mathbb{R}^N) \to \mathcal{L}^p(\mathbb{R}^{N-1})$ satisfies $\mathcal{T}\mathbf{W}_\mathbf{p}^\mathbf{1}(\mathbb{R}_+^\mathbf{N}) = \mathcal{B}_\mathbf{p}^{\mathbf{1-1/p}}(\mathbb{R}^{\mathbf{N-1}})$. Then we prove that $\mathcal{T}\mathbf{W}_\mathbf{p}^\mathbf{l}(\boldsymbol{\Omega}) = \mathcal{B}_\mathbf{p}^{\mathbf{l-1/p}}(\partial\boldsymbol{\Omega})$, where $\Omega \subset \mathbb{R}^N$ an open set with $\mathcal{C}^l$-boundary. For the case $p = 2$, we approach the definition of the trace spaces with two other methods, namely using the Fourier transformation and using the spectral definition given by Auchmuty [3]. Finally, we use the previous results to prove the existence of solution for the Dirichlet problem.

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Contents

Chapter 1

Introduction

In many problems of mathematical physics and the calculus of variations it is not sufficient to deal with classical solutions of differential equations, especially when questions are raised about the regularity of the solutions. In many cases it is necessary to introduce the notion of weak derivatives and work in the so called Sobolev spaces. In 1936–38, S.L.Sobolev introduced spaces of integrable functions having weak derivatives in $\mathcal{L}^p$. These function spaces have turned out to be very useful when studying partial differential equations on smooth and nonsmooth domains and their boundary value problems.

A key component when using Sobolev spaces as a framework to deal with boundary value problems in PDEs is Trace theory. When we work with Sobolev spaces we have a certain knowledge regarding the regularity and smoothness of the solutions. But what about the regularity and smoothness of the functions defining the boundary conditions?
In order to better understand the importance of this question for a domain $\Omega \subset \mathcal{R}^N$ let us consider the following problem:

$$\begin{cases} -\Delta u = \mathcal{F}, & x \in \Omega, \\ u|_{\partial\Omega} = g, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

which is also know as Poisson problem with non-homogeneous Dirichlet boundary conditions and let us suppose that we wish to find a solution $u \in H^s(\Omega)$, where $H^s(\Omega) = W_2^s(\Omega)$.

There are several approaches to the solution of such problem. One approach is to look for a minimizer of the corresponding energy functional. Then the minimizer satisfies the associated Euler-Lagrange equation which coincides with the given problem. For this approach it is essential to know which functions g are admissible. If we think of the trace of a function $u \in H^s(\Omega)$ as an element of $\mathcal{L}^2(\partial\Omega)$ and in a more broad sense as the restriction of u to the boundary of Ω (in fact, it is the restriction if u is continuous) equivalently to problem (1.1) we can write:

$$\begin{cases} -\Delta u = \mathcal{F}, & x \in \Omega, \\ \mathcal{T}u = g, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathcal{T}$ denotes the trace operator.
Thus, it becomes more clear why describing the trace operator's image from a Sobolev space is a fundamental problem. In fact this problem has challenged mathematicians for a long time. It was first solved in the case $p = 2$ via Fourier Transform methods. The solution for $p \neq 2$ and $l = 1$ was given by Gagliardo in the 50's. The general case was sorted out by Besov and Nikolskii. The solution involves the so-called Besov-Nikolskii spaces $\mathcal{B}^l_p(\partial\Omega)$ (in the following chapters we will be referring to them simply as Besov spaces).

A very interesting and recent approach, for the case $p = 2$, involves a spectral representation of the trace spaces and was given by G. Auchmuty in 2006. For this approach, Auchmuty generates an orthonormal basis for $\mathcal{L}^2(\partial\Omega)$ with use of the Steklov eigenfunctions and uses this basis to describe the trace spaces.

In the last chapter, after we present the trace theorems we will go back to the Poisson problem with Dirichlet boundary conditions that motivated us in the first place and we will prove the existence of solutions: first with use of the classical definition of trace spaces (i.e. via Besov spaces) and then with use of the spectral definition as given by Auchmuty.

Lastly, another aspect regarding the importance of describing the trace image of Sobolev spaces lies in the study of Sobolev spaces themselves and not just in the frame of PDEs, since they constitute an individual field of study.

Chapter 2

Preliminaries

We present here a number of results widely used in the following chapters. For further information we refer to [7], [8], [9], [10].

2.1 Some useful results

Lemma 2.1 (Minkowski's inequality). *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Let* $1 \leq p \leq \infty$ *and let* $u, v : X \to [-\infty, \infty]$ *be measurable functions. Then*

$$\|u + v\|_{\mathcal{L}^p} \leq \|u\|_{\mathcal{L}^p} + \|v\|_{\mathcal{L}^p}.$$

Lemma 2.2 (Minkowski's inequality for integrals). *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be two measure spaces. Assume that* μ, ν *are complete and* σ*-finite. Let* $u : X \times Y \to [0, \infty]$ *be a* $(\mathcal{M} \times \mathcal{N})$*-measurable function and let* $1 \leq p \leq \infty$*. Then*

$$\left\| \int_x |u(x, \cdot)| d\mu(x) \right\|_{\mathcal{L}^p(Y, \mathcal{N}, \nu)} \leq \int_x \|u(x, \cdot)\|_{\mathcal{L}^p(Y, \mathcal{N}, \nu)} d\mu(x).$$

Lemma 2.3 (Minkowski's inequality for sums). *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be two measure spaces. Assume that* μ, ν *are complete and* σ*-finite. Let* $u, v : X \times Y \to [0, \infty]$ *be a* $(\mathcal{M} \times \mathcal{N})$*-measurable function and let* $1 \leq p \leq \infty$*. Then*

$$\left(\sum_{x \in X} |u(x) + v(x)|^p \right)^{1/p} \leq \left(\sum_{x \in X} |u(x)|^p \right)^{1/p} + \left(\sum_{x \in X} |v(x)|^p \right)^{1/p}.$$

Theorem 2.1 (Tonelli's Theorem). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and σ-finite and let $u : X \times Y \to [0, \infty]$ be a $(\mathcal{M} \times \mathcal{N})$-measurable function. Then for μ-a.e., $x \in X$ the function $u(x, \cdot)$ is measurable and the function $\int_Y u(\cdot, y) d\nu(y)$ is measurable. Similarly, for ν-a.e. and $y \in Y$ the function $u(\cdot, y)$ is measurable and the function $\int_Y u(x, \cdot) d\mu(y)$ is measurable. Moreover,*

$$\int_X \left(\int_Y u(x,y) d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X u(x,y) d\mu(x) \right) d\nu(y)$$

Theorem 2.2 (Fubini's Theorem). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and let $u : X \times Y \to [-\infty, \infty]$ be $(\mu \times \nu)$-integrable. Then for μ-a.e. $x \in X$ the function $u(x, \cdot)$ is ν-integrable and the function $\int_Y u(\cdot, y) d\nu(y)$ is μ-integrable.*

Theorem 2.3 (Hölder's inequality). *Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$ and and let q be its Hölder conjugate exponent, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. If $u, v : X \to [-\infty, \infty]$ are measurable functions, then*

$$\|uv\|_{\mathcal{L}^1(X)} \leq \|u\|_{\mathcal{L}^p(X)} \|v\|_{\mathcal{L}^q(X)}.$$

Lemma 2.4 (Fatou's lemma). *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $u_n : X \to [0, \infty]$ be a sequence of measurable functions, then*

$$\int_X \liminf_{n \to \infty} u_n d\mu \leq \liminf_{n \to \infty} \int_X u_n d\mu.$$

Theorem 2.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let Y be a metric space, and let $u : X \times Y \to \mathbb{R}$ be a function. Assume that for each fixed $y \in Y$ the function $x \in X \mapsto u(x, y)$ is measurable and that there exists $y_0 \in Y$ such that*

$$\lim_{y \to y_0} u(x, y) = u(x, y_0),$$

for every $x \in X$.
Assume also that there exists an integrable function $g : X \to [0, \infty]$ such that

$$|u(x, y)| \leq g(x),$$

for μ a.e. $x \in X$ and for all $y \in Y$. Then the function $F : Y \to \mathbb{R}$, defined by

$$F(y) = \int_X u(x,y)d\mu(x),\ y \in Y,$$

is well-defined and continuous at y_0.

Theorem 2.5. *Let Y be an interval of $\mathbb{R}$ and assume that for each fixed $x \in X$ the function $y \in Y \mapsto u(x,y)$ is differentiable and that for each fixed $y \in Y$ the functions $x \in X \mapsto u(x,y)$ and $x \in X \mapsto \frac{\partial u}{\partial y}(x,y)$ are measurable. Assume also that for some $y_0 \in Y$ the function $x \in X \mapsto u(x,y_0)$ is integrable and that there exists an integrable function $h : X \to [0,\infty]$ such that*

$$\left|\frac{\partial u}{\partial y}(x,y)\right| \leq h(x)$$

for μ a.e. $x \in X$ and for all $y \in Y$.Then the function $F : Y \to \mathbb{R}$, defined by

$$F(y) = \int_X u(x,y)d\mu(x),\ y \in Y,$$

is well-defined and differentiable, with

$$F'(y) = \int_X \frac{\partial u}{\partial y}(x,y)d\mu(x).$$

Now we need to define the functions called mollifiers.

Definition 2.1. *Let $\phi \in \mathcal{L}^1(\mathbb{R}^N)$ be a non-negative, bounded function with*

$$\mathrm{supp}\phi \subset \overline{\mathrm{B}(0,1)},\quad \int_{\mathrm{\mathbb{R}^N}} \phi(\mathrm{x})\mathrm{dx} = 1. \tag{2.1}$$

For every $\varepsilon > 0$ we define

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^N}\phi\left(\frac{x}{\varepsilon}\right),\ x \in \mathbb{R}^N.$$

The functions ϕ_ε are called mollifiers.
In the case that ϕ is of class $\mathcal{C}_c^\infty$ defined by

$$\phi(x) := \begin{cases} \alpha \exp\left\{\frac{1}{|x|^2-1}\right\}, & \textit{if } |x| < 1 \\ 1, & \textit{if } |x| \geq 1, \end{cases} \tag{2.2}$$

where $\alpha > 0$ is such that (2.1) is satisfied. We call ϕ_ε standard mollifiers.

Remark. *We note that* $\mathrm{supp}\phi_\varepsilon \subset \overline{\mathrm{B}(0,\varepsilon)}$.

Definition 2.2. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $u \in \mathcal{L}^1(\Omega)$. *We consider the open set* $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist(x,}\partial\Omega) > \varepsilon\}$. *For* $x \in \Omega_\varepsilon$ *we define the function* $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$, *which we will call a mollification of u:*

$$u_\varepsilon(x) := (u * \phi_\varepsilon)(x) = \int_\Omega u(x-y)\phi_\varepsilon(y)dy,$$

where ϕ_ε *the standard mollifiers.*

Theorem 2.6. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, let* $\phi \in \mathcal{L}^1(\mathbb{R}^N)$ *be a non-negative bounded function satisfying (2.1), and let* $u \in \mathcal{L}^1(\Omega)$.

1. *If* $u \in \mathcal{C}(\Omega)$, *then* $u_\varepsilon \to u$ *as* $\varepsilon \to 0^+$ *uniformly on compact subsets of* Ω.

2. *For every Lebesque point* $x \in \Omega$ *of* u *we have* $u_\varepsilon(x) \to u(x)$ *as* $\varepsilon \to 0^+$.

3. *If* $1 \leq p \leq \infty$, *then*
$$\|u_\varepsilon\|_{\mathcal{L}^p(\Omega_\varepsilon)} \leq \|u\|_{\mathcal{L}^p(\Omega)},$$
for every $\varepsilon > 0$ *and*
$$\|u_\varepsilon\|_{\mathcal{L}^p(\Omega_\varepsilon)} \to \|u\|_{\mathcal{L}^p(\Omega)} \text{ as } \varepsilon \to 0^+. \tag{2.3}$$

4. *If* $u \in \mathbb{L}^p(\Omega)$, $1 \leq p < \infty$, *then*
$$\lim_{\varepsilon \to 0^+} \left(\int_{\Omega_\varepsilon} |u_\varepsilon - u|^p dx \right)^{1/p} = 0.$$
In particular, for any open set $\Omega' \subset \Omega$ *with* $\mathrm{dist}(\Omega', \partial\Omega) > 0$, $u_\varepsilon \to u$ *in* $\mathcal{L}^p(\Omega')$.

Theorem 2.7. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, let* $\phi \in \mathcal{L}^1(\mathbb{R}^N)$ *be defined as in (2.2) and let* $u \in \mathcal{L}^1(\Omega)$. *Then* $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$ *for all* $0 < \varepsilon < 1$. *Moreover, for every multi-index* α *we have*

$$\frac{\partial^a u_\varepsilon}{\partial x^a}(x) = \left(u * \frac{\partial^a \phi_\varepsilon}{\partial x^a}\right)(x) = \int_{\mathbb{R}^N} \frac{\partial^a \phi_\varepsilon}{\partial x^a}(x-y)u(y)dy,$$

for all $x \in \Omega_\varepsilon$.

Definition 2.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $T : H \to H$ be a symmetric operator, i.e. $\langle Tx, y\rangle = \langle y, Tx\rangle$, where H a Hilbert space. We say that T is non-negative if*

$$\langle Tu, u\rangle \geq 0, \tag{2.4}$$

for all $u \in H$.

Proposition 2.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $T : H \to H$ be a self-adjoint operator. Then T is non-negative if and only if $\sigma(T) \subset [0, \infty]$, where $\sigma(T)$ denotes the spectrum of T.*

Theorem 2.8 (Courant-Rayleigh minmax principle)**.** *Let $\big(H, \langle\cdot,\cdot\rangle\big)$ be a Hilbert space and let $T : H \to H$ be a compact, self-adjoint operator, whose positive eigenvalues are listed in decreasing order $0 \leq \cdots \leq \lambda_n \leq \cdots \leq \lambda_1$. Then*

$$\lambda_n = \min_{\substack{V \subset H \\ \mathrm{dimV=n-1}}} \max_{\substack{x \in V^{\perp} \\ x \neq 0}} \frac{\langle Tx, x\rangle}{\|x\|},$$

where $V \subset H$ is an $(n-1)$-dimensional subspace.

Theorem 2.9 (Riesz's Representation Theorem)**.** *Let M be a bounded linear functional on a Hilbert space H equipped with the inner product $\langle\cdot,\cdot\rangle$. Then there exists some $g \in H$ such that for every $u \in H$*

$$Mu = \langle u, g\rangle.$$

Moreover, $\|M\| = \|g\|$, where $\|\cdot\|$ the inducted norm.

Theorem 2.10. *(Spectral Theorem for compact and self-adjoint operators) Let H be a Hilbert space and $T : H \to H$ a compact and self-adjoint operator on H. Then there exists a finite or infinite sequence of real eigenvalues $\{\lambda_n\}_{n=1}^N$ with $\{\lambda_n\} \neq 0$ and a corresponding orthonormal sequence of eigenfunctions $\{e_n\}_{n=1}^N$ in H such that*

1. *$Te_n = \lambda_n e_n$ for all $1 \leq n \leq N$,*
2. *$\overline{\mathrm{ImT}} = \overline{\mathrm{span}\langle\{\mathrm{e_n}\}_{\mathrm{n=1}}^{\mathrm{N}}\rangle}$,*
3. *if $N = \infty$, that is $\{\lambda_n\}_{n=1}^N$ is infinite, then $\lambda_n \to 0$ as $n \to \infty$.*

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^N$ a bounded set. The space $\mathcal{L}^2(\Omega)$ with the standard norm is a Banach space. Moreover, let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $\mathcal{L}^2(\Omega)$, then given a sequence of numbers $\{c_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} c_n^2 < \infty$ there exists a function $u \in \mathcal{L}^2(\Omega)$ such that*

$$\int_{\Omega} |u(x)|^2 dx = \sum_{n=1}^{\infty} c_n^2, \ c_n = \int_{\Omega} \phi_n(x)u(x)dx.$$

2.2 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set, $s \in \mathbb{N}$, $1 \leq p \leq \infty$. Let $\mathcal{C}_c^{\infty}(\Omega)$ be the space of functions in $\mathcal{C}^{\infty}(\Omega)$ with compact support in Ω, i.e. the space of test functions.

Definition 2.4. *Let $u, v \in \mathcal{L}_{Loc}^1(\Omega)$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ a multi-index. We say that v is the α^{th}-weak partial derivative of u, and we write $D^\alpha u = v$ if*

$$\int_{\Omega} u D^\alpha \phi dx = (-1)^{|\alpha|} \int_{\Omega} v \phi dx,$$

for all $\phi \in \mathcal{C}_c^{\infty}(\Omega)$.

Definition 2.5. *We define the Sobolev Space $W_p^s(\Omega)$ to be the space of all the functions $u \in \mathcal{L}^p(\Omega)$ such that for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq s$, the weak derivative $D^\alpha u$ exists and belongs to $\mathcal{L}^p(\Omega)$. In the special case $p = 2$ we write $H^s(\Omega) = W_2^s(\Omega)$. We also introduce the following norm in $W_p^s(\Omega)$:*

$$\|u\|_{W_p^s(\Omega)} := \left(\sum_{|\alpha| \leq s} \|D^\alpha u\|_{\mathcal{L}^p(\Omega)} \right)^{1/p}, \ p \neq \infty$$

$$\|u\|_{W_\infty^s(\Omega)} := \sum_{|\alpha| \leq s} \|D^\alpha u\|_{\mathcal{L}^\infty(\Omega)}.$$

Definition 2.6. *Let $\{u_m\}_{m=1}^{\infty}$, $\Omega \subset \mathbb{R}^N$ and $u \in W_p^s(\Omega)$. We say that u_m converges to u in $W_p^s(\Omega)$ and write $u_m \to u$ in $W_p^s(\Omega)$ provided that*

$$\lim_{m \to \infty} \|u_m - u\|_{W_p^s(\Omega)} = 0.$$

Definition 2.7. *We denote by $W_0^{s,p}(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W_p^s(\Omega)$ and respectively $H_0^s(\Omega)$ for the case $p=2$.*

Thus we have that $u \in W_0^{s,p}(\Omega)$ for $\Omega \subset \mathbb{R}^N$ if and only if there exist functions $u_m \in \mathcal{C}_c^\infty(\Omega)$ such that $u_m \to u$ in $W_p^s(\Omega)$. We can interpret $W_0^{s,p}(\Omega)$ as comprising all those functions $u \in W_p^s(\Omega)$ such that $D^\alpha u = 0$ on the boundary $\partial\Omega$ for all $|\alpha| \leq s-1$.

Proposition. *For all $s \in \mathbb{N}$, $1 \leq p \leq \infty$, the Sobolev Space $W_p^s(\Omega)$ is a Banach space.*

Proof. Let $\{u_j\}_{j=1}^\infty$ a Cauchy sequence in $W_p^s(\Omega)$. Then for all $|\alpha| \leq s$ $\{D^\alpha u_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{L}^p(\Omega)$ and since $\mathcal{L}^p(\Omega)$ is complete, there exist functions $u, u_a \in \mathcal{L}^p(\Omega)$ such that

$$u_j \to u, \text{as } j \to \infty \text{ in } \mathcal{L}^p(\Omega)$$

$$D^\alpha u_j \to u_a, \text{as } j \to \infty \text{ in } \mathcal{L}^p(\Omega),$$

for all $|\alpha| \leq s$.
Let $\phi \in \mathcal{C}_c^\infty(\Omega)$, then by Hölder's inequality it follows

$$\int_\Omega |(u-u_j)D^\alpha\phi|^p dx \leq \int_\Omega |u-u_j|^p dx \int_\Omega |D^\alpha\phi|^q dx \to 0,$$

where q the conjugate of p.
Hence, $u \in W_p^s(\Omega)$ and $D^\alpha u = u_\alpha$ for all $|\alpha| \leq s$. This means that $D^\alpha u_j \to D^\alpha u$ for all $|\alpha| \leq s$ and as a result we have $u_j \to u$ in $W_p^s(\Omega)$. □

Definition 2.8. *For an arbitrary nonempty set $\Omega \subset \mathbb{R}^N$ we denote by $\mathcal{C}_b(\Omega)$ the Banach space of functions u continuous and bounded on Ω with the norm $\|u\|_{\mathcal{C}(\Omega)} = \sup_{x\in\Omega}|u(x)|$.*

Definition 2.9. *Let $l \in \mathbb{N}$. We denote by $\mathcal{C}_b^l(\Omega)$ the Banach space of functions $u \in \mathcal{C}_b(\Omega)$ such that for all $\alpha \in \mathbb{N}_0^N$ where $|a| = l$ and for all $x \in \Omega$ the derivatives $(D^a u)(x)$ exist and $D^a u \in \mathcal{C}_b(\Omega)$, with the norm*

$$\|u\|_{\mathcal{C}^l(\Omega)} = \|u\|_{\mathcal{C}(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha u\|_{\mathcal{C}(\Omega)}.$$

Lemma 2.5. *Let $1 \leq q < p \leq \infty$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary of class $\mathcal{C}^{0,1}$. Then the embedding*

$$W_p^1(\Omega) \hookrightarrow \mathcal{L}_q(\Omega)$$

is compact.

Lemma 2.6. *Let $l \in \mathbb{N}$, $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary, i.e. of class $\mathcal{C}^{0,1}$. Let also $\phi \in \mathcal{C}_b^l(\Omega)$. Then for all $u \in W_p^l(\Omega)$*

$$\|u\phi\|_{W_p^l(\Omega)} \leq c\|u\|_{W_p^l(\mathrm{supp}\phi\cap\Omega)},$$

where $c > 0$ is independent of u.

Lemma 2.7. *Let $l \in \mathbb{N}$, $1 \leq p \leq \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set having a quasi-resolved boundary[1]. Moreover, let $g = (g_1, \ldots, g_N) : \Omega \to \mathbb{R}^N$, $g_K \in \mathcal{C}^l(\Omega)$, $K = 1, \ldots, N$. We suppose that for all $\alpha \in \mathbb{N}_0^N$, satisfying $1 \leq |\alpha| \leq l$ the derivatives $D^\alpha g_K$ are bounded on Ω and the Jacobian $J_g(x) = \frac{Dg}{Dx}$ is such that $\inf_{x\in\Omega} |\frac{Dg}{Dx}(x)| > 0$. Furthermore, let $g(\Omega)$ be also an open set with a quasi-resolved boundary. Then for all $u \in W_p^l(\Omega)$*

$$c_1\|u\|_{W_p^l(g(\Omega))} \leq \|ug\|_{W_p^l(\Omega)} \leq c_2\|u\|_{W_p^l(g(\Omega))},$$

where $c_1, c_2 > 0$ are independent of u and p.

Next we present two approximation by smooth functions theorems. For the proofs we refer to [5].

Theorem 2.12 (Global approximation by smooth functions)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W_p^l(\Omega)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in \mathcal{C}^\infty(\Omega) \cap W_p^l(\Omega)$ such that*

$$u_m \to u \text{ in } W_p^l(\Omega).$$

Theorem 2.13 (Global approximation by functions smooth up to the boundary)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $\mathcal{C}^1$ boundary. We suppose that $u \in W_p^l(\Omega)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in \mathcal{C}^\infty(\overline{\Omega}) \cap W_p^l(\Omega)$ such that*

$$u_m \to u \text{ in } W_p^l(\Omega).$$

Definition 2.10. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 < p < \infty$. The Sobolev space $\mathcal{L}^{1,p}(\Omega)$ is the space of all functions $u \in \mathcal{L}_{Loc}^1(\Omega)$ whose gradient ∇u (in the sense of distributions) belongs to $\mathcal{L}^p(\Omega)$.*

[1]The precise definition of this notion is given in the next chapter.

2.3 Trace of a function

In this section we present the definition, as well as several results, regarding the Trace operator.
Let $u \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$, $N > 1$. We want to define the trace g of the function u on $\mathbb{R}^M$, where $1 \leq M < N$. In order to do so we will represent each $x \in \mathbb{R}^N$ as $x = (y, z)$,where $y = (x_1, \ldots, x_M) \in \mathbb{R}^M$, $z = (x_{M+1}, \ldots, x_N) \in \mathbb{R}^{N-M}$ and we will consider $\mathbb{R}^M$ as the M-dimensional space of all points (y, z), where $z = (0, \ldots, 0)$ and y runs through any possible values.
If u is continuous we can define the trace to be the restriction of the function on $\mathbb{R}^M$. However, this definition obviously does not make sense for every $u \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$. In order to define the trace g for a function $u \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$ we have to make certain requirements:

1. $g \in \mathcal{L}^1_{Loc}(\mathbb{R}^M)$,
2. if $g \in \mathcal{L}^1_{Loc}(\mathbb{R}^M)$ is a trace of u, then $\psi \in \mathcal{L}^1_{Loc}(\mathbb{R}^M)$ is also a trace of u iff ψ is equivalent to g on $\mathbb{R}^M$,
3. if $g \in \mathcal{L}^1_{Loc}(\mathbb{R}^M)$ is a trace of u and h is equivalent to u on $\mathbb{R}^N$, then g is also a trace of h,
4. if u is continuous then $u(y, 0)$ is a trace of u.

The following definition, as given by Burenkov [1], satisfies all of the requirements above .

Definition 2.11. *Let $u \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$ and $g \in \mathcal{L}^1_{Loc}(\mathbb{R}^M)$. We call the function g a trace of the function u if there exists a function $h \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$ such that h is equivalent to u on $\mathbb{R}^N$ and*

$$h(\cdot, z) \to g(\cdot) \text{ in } \mathcal{L}^1_{Loc}(\mathbb{R}^M) \text{ as } z \to 0.$$

Theorem 2.14. *(Existence of Trace) Let $s, m, N \in \mathbb{N}$, $m < n$ and $1 \leq p \leq \infty$. Then traces on $\mathbb{R}^M$ exist for all $u \in W^s_p(\mathbb{R}^N)$ iff*

$$s > \frac{n-m}{p} \text{ for } 1 < p \leq \infty, \ s \geq N - M \text{ for } p = 1,$$

equivalently, iff

$$W^s_p(\mathbb{R}^{N-M}) \hookrightarrow \mathcal{C}(\mathbb{R}^{N-M}).$$

Proof. We refer to [1]. □

Definition 2.12. *We say that a domain* $\Omega \subset \mathbb{R}^N$ *is a bounded domain with a resolved boundary with the parameters d, D satisfying* $0 < d \leq D < \infty$ *if*

$$\Omega = \{x \in \mathbb{R}^N : \alpha_N < x_N < \phi(\overline{x}), \overline{x} \in A\}, \tag{2.5}$$

where $\mathrm{diam}\Omega \leq \mathrm{D}$, $\overline{x} = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$, $A = \{\overline{x} \in \mathbb{R}^{N-1} : a_i < x_i < b_i, i = 1, \ldots, N-1\}$, $-\infty < a_i < b_i < \infty$ *and*

$$a_N + d \leq \phi(\overline{x}),\ \overline{x} \in A.$$

Let $\Omega \subset \mathbb{R}^N$ be an open set with C^1-boundary. Now we wish to extend Definition 2.11 to the case of $\mathbb{R}^N$, $\mathbb{R}^M$ being replaced with Ω, $\partial\Omega$ respectively.
In order to do so, we will use the same method as in [1]. We start with $\Omega \subset \mathbb{R}^N$ being a bounded domain having the form (2.5), with ϕ of class $\mathcal{C}^1$.
Let $u \in \mathcal{L}^1(\Omega)$. In the spirit of Definition 2.11 we say that the function $g \in \mathcal{L}^1(\Gamma)$, where $\Gamma = \{x \in \mathbb{R}^N : x_N = \phi(\overline{x}), \overline{x} \in A\}$, is a trace of the function u on Γ if there exists a function h equivalent to u on Ω such that

$$h(\cdot + te_N) \to g(\cdot) \text{ in } \mathcal{L}^1(\Gamma) \text{ as } t \to 0^-,$$

where $e_N = (0, \ldots, 0, 1)$.
Since there is no guarantee that the boundary $\partial\Omega$ will be flat near a chosen point $x_0 \in \partial\Omega$, we can use a proper transformation Φ of Ω (with inverse Φ^{-1}), which straightens out $\partial\Omega$ near x_0. Then with use of the transformation Φ of Ω (as chosen in [1]) we have that $g(\Phi^{-1})$ is a trace of $u(\Phi^{-1})$ on $\Phi(\Gamma)$.
Next we suppose that $\Omega \subset \mathbb{R}^N$ is an open set such that for a certain map λ, which is a composition of rotations, reflections and translations, the set $\lambda(\Omega)$ is a bounded domain with $\mathcal{C}^1$-boundary and Γ is such that $\lambda(\Gamma) = \{x \in \mathbb{R}^N : x_N = \phi(\overline{x}), x \in A\}$ (for further details we refer to [1]). In this case we say that g is a trace of u on Γ if $g(\lambda^{-1})$ is a trace of $u(\lambda^{-1})$ on $\lambda(\Gamma)$.
Finally, let $\Omega \subset \mathbb{R}^N$ be an open set with $\mathcal{C}^1$-boundary and let V_j be open parallelepipeds defined as in [1]. Then there exists a partition of unity $\psi_j \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \leq \psi_j \leq 1$, $\mathrm{supp}\psi_j \subset (V_j)_{\frac{d}{2}}$,

$j = 1, \ldots, s,\ \sum_{j=1}^{s} \psi_j = 1$ on Ω and

$$|D^\alpha \psi_j(x)| \le c d^{-|\alpha|},\ x \in \mathbb{R}^N,\ \alpha \in \mathbb{N}_0,\ j = 1, \ldots, s,$$

where $c > 0$ is independent of x, j and d (for the proof we refer to [1]). Having said so, we have the following definition.

Definition 2.13. *Let $\Omega \subset \mathbb{R}^N$ be an open set with $\mathcal{C}^1$-boundary and $u : \Omega \to \mathbb{R}^N$ with $u \in \mathcal{L}^1(B \cap \Omega)$, for each ball $B \subset \mathbb{R}^N$. We suppose that $u = \sum_{j=1}^{s} u_j$, where $\mathrm{supp u_j} \subset \mathrm{V_j}$ and $u_j \in \mathcal{L}^1(V_j \cap \Omega)$. If the functions g_j are traces of the functions u_j on $V_j \cap \partial\Omega$, $j = 1, \ldots, s$, then the function $g = \sum_{j=1}^{s} g_j$ is said to be a trace of the function u on $\partial\Omega$.*

Theorem 2.15 (Trace Theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ bounded with $\mathcal{C}^1$ boundary. Then there exists a bounded linear operator*

$$\mathcal{T} : W_p^1(\Omega) \to \mathcal{L}^p(\partial\Omega)$$

such that

1. $\mathcal{T}u = u|_{\partial\Omega}$
2. $\|\mathcal{T}u\|_{\mathcal{L}^p(\partial\Omega)} \le C\|u\|_{W_p^1(\Omega)}$,

for all $u \in W_p^1(\Omega)$ and $C = C(p, \Omega)$ independent of u.

Proof. For the proof we refer to [5]. □

Theorem 2.16. *Let $N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz, let $1 \le p < \infty$, and let $u \in W_p^1(\Omega)$. Then $\mathcal{T}u = 0$ if and only if $u \in W_0^{1,p}(\Omega)$.*

Proof. For the proof we refer to [2]. □

Lastly we present the following Theorem regarding the compactness of the Trace operator. For the proof we refer to [6].

Theorem 2.17. *(Compact Trace Theorem) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $\mathcal{C}^1$ boundary and let $1 \le p < \infty$. Then the trace operator $\mathcal{T} : W_p^1(\Omega) \to \mathcal{L}^p(\partial\Omega)$ is compact.*

2.4 Besov spaces

In order to determine the Trace Spaces of the Sobolev Spaces W_p^s it is necessary to introduce the Besov Spaces $\mathcal{B}_p^s$.

Definition 2.14. *Let* $u : \mathbb{R}^N \to \mathbb{R}$. *We have the following definitions. For all* $h \in \mathbb{R}$, $i = 1, \dots, N$ *and* $x \in \mathbb{R}^N$ *we define*

$$\Delta_i^h u(x) := u(x + he_i) - u(x) = u(x_i', x_i + h) - u(x_i', x_i),$$

where e_i *is the i-th vector of the canonical basis in* $\mathbb{R}$.
For $h \in \mathbb{R}^N$ *and* $\sigma \in \mathbb{N}$ *we define*

$$\Delta_h^\sigma u(x) = \sum_{k=0}^{\sigma} (-1)^{\sigma-k} \binom{\sigma}{k} u(x + kh).$$

Thereinafter, the use of each definition given above will be clear from the context.

Definition 2.15. *Let* $s > 0$, $\sigma \in \mathbb{N}$, $\sigma > s$ *and* $1 \leq p, \theta \leq \infty$. *The function* $u \in \mathcal{L}^1_{Loc}(\Omega)$ *belongs to the Besov space* $\mathcal{B}^s_{p,\theta}(\Omega)$ *if* u *is measurable on* $\mathbb{R}^N$ *and*

$$\|u\|_{\mathcal{B}^s_{p,\theta}(\mathbb{R}^N)} := \|u\|_{\mathcal{L}^p(\mathbb{R}^N)} + |u|_{\mathcal{B}^s_{p,\theta}(\mathbb{R}^N)} < \infty,$$

where

$$|u|_{\mathcal{B}^s_{p,\theta}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \left(\frac{\|\Delta_h^\sigma u\|_{\mathcal{L}^p(\mathbb{R}^N)}}{|h|^l} \right)^\theta \frac{dh}{|h|^N} \right)^{\frac{1}{\theta}},$$

if $1 \leq \theta < \infty$ *and*

$$|u|_{\mathcal{B}^s_\infty(\mathbb{R}^N)} := \sup_{h \in \mathbb{R}^N, h \neq 0} \frac{\|\Delta_h^\sigma u\|_{\mathcal{L}^p(\mathbb{R}^N)}}{|h|^l}.$$

This definition is independent of $\sigma > l$ as the following lemma shows. (For the proofs of the following Lemma and Propositions we refer to [1].)

Lemma 2.8. *Let* $l > 0$, $1 \leq p, \theta \leq \infty$. *Then the norms* $\|\cdot\|_{\mathcal{B}^s_{p,\theta}(\mathbb{R}^N)}$ *corresponding to different* $\sigma \in \mathbb{N}$ *satisfying* $\sigma > l$ *are equivalent.*

Proposition 2.2. *Let* $1 \leq p, \theta \leq \infty$, $s > 0$. *Then the Besov space* $\mathcal{B}^s_{p,\theta}(\Omega)$ *is a Banach space.*

Remark. *For $\theta = s$ we denote $\mathcal{B}^s_{p,p} = \mathcal{B}^s_p$.*

Proposition 2.3. *Let $0 < s < 1$, $1 \le p \le \infty$. For any $u \in \mathcal{L}^1_{Loc}(\mathbb{R}^N)$ let $u_\varepsilon := \phi_\varepsilon * u$, where ϕ_ε is a a standard mollifier. Then*

$$|u_\varepsilon|_{\mathcal{B}^s_p(\mathbb{R}^N)} \le |u|_{\mathcal{B}^s_p(\mathbb{R}^N)},$$

for all $\varepsilon > 0$ and

$$\lim_{\varepsilon\to 0^+} |u_\varepsilon|_{\mathcal{B}^s_p(\mathbb{R}^N)} = |u|_{\mathcal{B}^s_p(\mathbb{R}^N)}.$$

Moreover if $p < \infty$ and $u \in \mathcal{B}^s_p(\mathbb{R}^N)$, then

$$\lim_{\varepsilon\to 0^+} |u_\varepsilon - u|_{\mathcal{B}^s_p(\mathbb{R}^N)} = 0.$$

In particular, if $p < \infty$ then $\mathcal{C}^\infty(\mathbb{R}^N) \cap \mathcal{B}^s_p(\mathbb{R}^N)$ is dense in $\mathcal{B}^s_p(\mathbb{R}^N)$.

Now we need to define the spaces $\mathcal{B}^s_p(\partial\Omega)$, where $s > 0$, $1 \le p \le \infty$. In order to do so, we use similar arguments as in the case of defining the Trace in $\Omega \subset \mathbb{R}^N$ instead of $\mathbb{R}^N$.
If $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\mathcal{C}^1$-boundary and u is defined on Γ, we say that $u \in \mathcal{B}^s_p(\Gamma)$ if $u(\overline{x}, \phi(\overline{x}) \in \mathcal{B}^s_p(A)$ and we set

$$\|u\|_{\mathcal{B}^s_p(\Gamma)} = \|u(\Phi^{-1})\|_{\mathcal{B}^s_p(\Phi(\Gamma))} = \|u(\overline{x}, \phi(\overline{x})\|_{\mathcal{B}^s_p(A)},$$

where Φ, A defined as above.
If $\Omega \subset \mathbb{R}^N$ is such that for a certain map λ, which is a composition of rotations,reflections and translations, the set $\lambda(\Omega)$ is bounded with $\mathcal{C}^1$-boundary, then $u \in \mathcal{B}^s_p(\Gamma)$ if $u(\lambda^{-1}) \in \mathcal{B}^s_p(\lambda(\Gamma))$ and

$$\|u\|_{\mathcal{B}^s_p(\Gamma)} = \|u(\lambda^{-1})\|_{\mathcal{B}^s_p(\lambda(\Gamma))} = \|u(\Lambda)\|_{\mathcal{B}^s_p(\Lambda)},$$

where $\Lambda = \Phi(\lambda)$.
Hence we have the following definition.

Definition 2.16. *Let $s > 0$, $1 \le p \le \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set with with $\mathcal{C}^1$-boundary. We say that $u \in \mathcal{B}^s_p(\partial\Omega)$ if $u\psi_j \in \mathcal{B}^s_p(V_j \cap \partial\Omega)$, $j = 1, \dots, l$ and*

$$\begin{aligned} \|u\|_{\mathcal{B}^s_p(\partial\Omega)} &= \left(\sum_{j=1}^{l} \|u\psi_j\|^p_{\mathcal{B}^s_p(V_j\cap\partial\Omega)}\right)^{1/p} \\ &= \left(\sum_{j=1}^{l} \|u\psi_j(\Lambda^{-1})\|^p_{\mathcal{B}^s_p(\Phi(\Gamma))}\right)^{1/p}, \end{aligned}$$

where $\Lambda_j = \Phi_j(\lambda_j)$ and ψ_j is the partition of unity defined above.

Chapter 3

Trace Spaces of Sobolev Spaces

In this chapter we will define the Trace spaces of the Sobolev spaces with use of the Besov spaces. More specifically we have the following Theorems.

Theorem 3.1. *Let* $1 \leq p \leq \infty$, $l, M, N \in \mathbb{N}$ *with* $M < N$, $l > \frac{N-M}{p}$ *and the trace operator* $\mathcal{T} : W_p^l(\mathbb{R}^N) \to \mathcal{L}^p(\mathbb{R}^M)$*. Then*

$$\mathcal{T}\mathbf{W}_\mathbf{p}^\mathbf{l}(\mathbb{R}^\mathbf{N}) = \mathcal{B}_\mathbf{p}^{\mathbf{l}-\frac{\mathbf{N}-\mathbf{M}}{\mathbf{p}}}(\mathbb{R}^\mathbf{M}).$$

Proof. For the proof we refer to [1]. □

We will prove the theorem for the case $l = 1$ and $M = N - 1$.
For the inclusion $\mathcal{T}W_p^1(\mathbb{R}_+^N) \subset \mathcal{B}_p^{1-1/p}(\mathbb{R}^{N-1})$ it suffices to show that for all $u \in W_p^1(\mathbb{R}_+^N)$ the trace $g = \mathcal{T}u \in \mathcal{B}_p^{1-1/p}(\mathbb{R}^{N-1})$. More specifically we will prove that there exists a constant $C = C(N, p)$ such that

$$\|g\|_{\mathcal{B}_p^{1-1/p}(\mathbb{R}^{N-1})} \leq C\|u\|_{\mathcal{L}^p(\mathbb{R}^N)},$$

for all $u \in \mathcal{L}^1(\mathbb{R}_+^N)$.

Theorem 3.2. *Let* $1 < p < \infty$, $N \geq 2$, $\mathbb{R}_+^N = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0)\}$*. Then,*

$$\mathcal{T}W_p^1(\mathbb{R}_+^N) \subset \mathcal{B}_p^{1-1/p}(\mathbb{R}^{N-1}).$$

Proof. Let $u \in \mathcal{L}^1(\mathbb{R}^N_+) \cap \mathcal{C}^\infty(\mathbb{R}^N)$. For $\Delta_i^h u(x) := u(x+he_i) - u(x) = u(x_i', x_i + h) - u(x_i', x_i)$, we write

$$u(x',0) = u(x',x_N) - (u(x',x_N) - u(x',0)) = u(x',x_N) - \Delta_N^{x_N} u(x',0),$$

for all $x' \in \mathbb{R}^{N-1}$, $x_N > 0$.
By integrating in x_N over the interval $(0,h)$, $h > 0$ we get

$$\int_0^h |u(x',0)| dx_N \leq \int_0^h |u(x',x_N)| dx_N + \int_0^h |\Delta_N^{x_N} u(x',0)| dx_N.$$

Therefore,

$$|u(x',0)| \leq \frac{1}{h} \int_0^h |u(x',x_N)| dx_N + \frac{1}{h} \int_0^h |\Delta_N^{x_N} u(x',0)| dx_N.$$

We replace u with $\Delta_i^h u$

$$\begin{aligned} |\Delta_i^h u(x',0)| &\leq \frac{1}{h} \int_0^h |\Delta_i^h u(x',x_N)| dx_N + \frac{1}{h} \int_0^h |\Delta_N^{x_N} \Delta_i^h u(x',0)| dx_N \\ &\leq \frac{1}{h} \int_0^h |\Delta_i^h u(x',x_N)| dx_N \\ &+ \frac{1}{h} \int_0^h |\Delta_N^{x_N} u(x'+he_i',0)| + |\Delta_N^{x_N} u(x',0)| dx_N, \end{aligned} \tag{3.1}$$

where $e_i = (e_i', 0)$, $i = 1, \ldots, N-1$.
By the Fundamental Theorem of Calculus we have:

- $\Delta_i^h u(x',x_N) = u(x'+he_i', x_N) - u(x',x_N) = \int_0^h \frac{\partial u}{\partial x_i}(x' + \xi e_i', x_N) d\xi$
- $\Delta_N^{x_N} u(x'+e_i',0) = u(x'+he_i', x_N) - u(x'+e_i',0) = \int_0^{x_N} \frac{\partial u}{\partial x_N}(x'+he_i', z) dz$
- $\Delta_N^{x_N} u(x',0) = u(x',x_N) - u(x',0) = \int_0^{x_N} \frac{\partial u}{\partial x_N}(x',z) dz$

Then by 3.1 it follows

$$\begin{aligned} |\Delta_i^h u(x',0)| \leq \frac{1}{h} &\int_0^h \int_0^h \left| \frac{\partial u}{\partial x_i}(x' + \xi e_i', x_N) \right| d\xi dx_N \\ &+ \frac{1}{h} \int_0^h \int_0^{x_N} \left| \frac{\partial u}{\partial x_N}(x' + he_i', z) \right| + \left| \frac{\partial u}{\partial x_N}(x', z) \right| dz dx_N. \end{aligned}$$

For $J = \int_0^h \int_0^{x_N} |\frac{\partial u}{\partial x_N}(x' + he_i', z)| + |\frac{\partial u}{\partial x_N}(x', z)| dz dx_N$ we have

$$J \le \underbrace{\int_0^h \int_0^h \left|\frac{\partial u}{\partial x_N}(x' + he_i', z)\right| + \left|\frac{\partial u}{\partial x_N}(x', z)\right| dz dx_N}_{\mathrm{I}}$$

and the Tonelli-Fubini Theorem yields

$$\begin{aligned} I &= \int_0^h \int_0^h \left|\frac{\partial u}{\partial x_N}(x' + he_i', z)\right| + \left|\frac{\partial u}{\partial x_N}(x', z)\right| dx_N dz \\ &= h \int_0^h \left|\frac{\partial u}{\partial x_N}(x' + he_i', z)\right| + \left|\frac{\partial u}{\partial x_N}(x', z)\right| dz. \end{aligned}$$

Therefore,

$$\begin{aligned} |\Delta_i^h u(x', 0)| \le \frac{1}{h} \int_0^h \int_0^h &\left|\frac{\partial u}{\partial x_i}(x' + \xi e_i', x_N)\right| d\xi dx_N \\ &+ \int_0^h \left|\frac{\partial u}{\partial x_N}(x' + he_i', z)\right| + \left|\frac{\partial u}{\partial x_N}(x', z)\right| dz. \end{aligned}$$

By Minkowski's inequality for integrals we obtain

$$\begin{aligned} \|\Delta_i^h u(x', 0)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} &\le \frac{c}{h} \int_0^h \int_0^h \|\frac{\partial u}{\partial x_i}(x' + \xi e_i', x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} d\xi dx_N \\ &\quad + c \int_0^h \|\frac{\partial u}{\partial x_N}(x' + he_i', z)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} + \|\frac{\partial u}{\partial x_N}(x', z)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dz \\ &= \frac{c}{h} h \int_0^h \|\frac{\partial u}{\partial x_i}(x' + \xi e_i', x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dx_N \\ &\quad + c \int_0^h \|\frac{\partial u}{\partial x_N}(x' + he_i', x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} + \|\frac{\partial u}{\partial x_N}(x', x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dx_N \end{aligned}$$

Hence,

$$\|\Delta_i^h u(\cdot, 0)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} \le c \int_0^h \|\frac{\partial u}{\partial x_i}(\cdot, x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} + \|\frac{\partial u}{\partial x_N}(\cdot, x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dx_N. \tag{3.2}$$

For fixed small $\varepsilon > 0$ we write

$$\int_0^h \|\frac{\partial u}{\partial x_i}(\cdot, x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dx_N = \int_0^h x_N^{\varepsilon} x_N^{-\varepsilon} \|\frac{\partial u}{\partial x_i}(\cdot, x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})} dx_N.$$

With use of Hölder's inequality we get

$$\int_0^h \|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})}dx_N \leq \left(\int_0^h x_N^{-\varepsilon q}dx_N\right)^{1/q}\left(\int_0^h x_N^{\varepsilon p}\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}dx_N\right)^{1/p}$$
$$= ch^{\frac{1}{p}-\varepsilon}\left(\int_0^h x_N^{\varepsilon p}\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}dx_N\right)^{1/p},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Similarly,

$$\int_0^h \|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|_{\mathcal{L}^p(\mathbb{R}^{N-1})}dx_N \leq ch^{1/q-\varepsilon}\left(\int_0^h x_N^{\varepsilon p}\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\right)^{1/p}.$$

Then by (3.2) it follows

$$\|\Delta_i^h u(\cdot,0)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})} \leq ch^{p/q-\varepsilon p}\int_0^h x_N^{\varepsilon p}\left(\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}+\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\right)dx_N.$$

Hence,

$$\int_0^\infty \frac{\|\Delta_i^h u(\cdot,0)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}}{h^p}dh \leq C\int_0^\infty \frac{h^{p/q-\varepsilon p}}{h^p}\int_0^h x_N^{\varepsilon p}\Big(\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}+$$
$$\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\Big)dx_N dh.$$

By Tonelli's Theorem we get

$$\int_0^\infty \frac{\|\Delta_i^h u(\cdot,0)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}}{h^p}dh \leq C\int_0^\infty\int_{x_N}^\infty \frac{h^{p/q-\varepsilon p}}{h^p}x_N^{\varepsilon p}\Big(\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}+$$
$$\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\Big)dh dx_N$$
$$= C\int_0^\infty x_N^{\varepsilon p}\Big(\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}+$$
$$\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\Big)\int_{x_N}^\infty \frac{h^{p/q-\varepsilon p}}{h^p}dh dx_N.$$

Since $\int_{x_N}^\infty \frac{h^{p/q-\varepsilon p}}{h^p}dh = \int_{x_N}^\infty \frac{1}{h^{\varepsilon p+1}}dh = \frac{1}{\varepsilon p}x_N^{-\varepsilon p}$, we have

$$\int_0^\infty \frac{\|\Delta_i^h u(\cdot,0)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}}{h^p}dh \leq \frac{C}{\varepsilon p}\int_0^\infty \left(\|\frac{\partial u}{\partial x_i}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}+\|\frac{\partial u}{\partial x_N}(\cdot,x_N)\|^p_{\mathcal{L}^p(\mathbb{R}^{N-1})}\right)dx_N$$
$$= \frac{C}{\varepsilon p}\left(\|\frac{\partial u}{\partial x_i}\|^p_{\mathcal{L}^p(\mathbb{R}^N_+)}+\|\frac{\partial u}{\partial x_N}\|^p_{\mathcal{L}^p(\mathbb{R}^N_+)}\right).$$

Now we want to remove the assumption that $u \in \mathcal{C}^\infty(\mathbb{R}^N)$.
First we can extend via reflection the function $u \in \mathcal{L}^{1,p}(\mathbb{R}^N_+)$ to the whole $\mathcal{L}^{1,p}(\mathbb{R}^N)$. For $\varepsilon > 0$ we consider the sequence $\{u_\varepsilon\}$ with $u_\varepsilon := u * \phi_\varepsilon$, where ϕ_ε are the standard mollifiers. As $\varepsilon \to 0^+$ we have:

$$u_\varepsilon \to u \text{ in } \mathcal{L}^1_{Loc}(\mathbb{R}^N),\ \nabla u_\varepsilon \to \nabla u \text{ in } \mathcal{L}^p(\mathbb{R}^N) \text{ and}$$
$$u_\varepsilon(\cdot, 0) \to \mathcal{T}u \text{ in } \mathcal{L}^1_{Loc}(\mathbb{R}^{N-1}).$$

We can select a subsequence such that $u_\varepsilon(x', 0) \to \mathcal{T}u(x')$. Then by Fatou's Lemma for

$$I = \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|\mathcal{T}u(x' + he'_i) - \mathcal{T}u(x')|^p}{h^p} dx' dh$$

it follows

$$\begin{aligned} I &\le \liminf_{\varepsilon \to 0+} \int_0^\infty \int_{\mathrm{R}^{\mathrm{N}-1}} \frac{|\mathrm{u}_\varepsilon(\mathrm{x}' + \mathrm{he}'_{\mathrm{i}}, 0) - \mathrm{u}_\varepsilon(\mathrm{x}', 0)|^{\mathrm{p}}}{\mathrm{h}^{\mathrm{p}}} \mathrm{dx}'\mathrm{dh} \\ &\le C(N,p) \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^N_+} |\nabla u_\varepsilon(x)|^p dx = C(N,p) \int_{\mathbb{R}^N_+} |\nabla u(x)|^p dx. \end{aligned}$$

Hence,

$$\mathcal{T}W^1_p(\mathbb{R}^N_+) \subset \mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1}). \tag{3.3}$$

□

For the inverse inclusion $\mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1}) \subset \mathcal{T}W^1_p(\mathbb{R}^N_+)$ we will prove that for all $g \in \mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1})$ there exists a function $u \in W^1_p(\mathbb{R}^N_+)$ such that $\mathcal{T}u = g$ and

$$\|u\|_{W^1_p(\mathbb{R}^N_+)} \le C_2 \|g\|_{\mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1})},$$

where $C_2 = C_2(N,p)$.

Theorem 3.3. *Let* $1 < p < \infty$, $N \ge 2$, $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0)\}$*. Then,*

$$\mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1}) \subset \mathcal{T}W^1_p(\mathbb{R}^N_+).$$

Proof. Let $g \in \mathcal{B}^{1-1/p}_p(\mathbb{R}^{N-1})$. Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^{N-1})$ be such that $\mathrm{supp}\phi \subset \overline{\mathrm{B}_{\mathrm{N}-1}(0,1)}$ and

$$\int_{\mathbb{R}^{N-1}} \phi(x') dx' = 1.$$

For $x_N > 0$ we define

$$\upsilon(x', x_N) := \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \phi\left(\frac{x'-y'}{x_N}\right) g(y')dy',$$

for $x' \in \mathbb{R}^{N-1}$.Then for all $x_N > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} |\upsilon(x', x_N)|^p dx' &\leq \int_{\mathbb{R}^{N-1}} \left|\frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \phi\left(\frac{x'-y'}{x_N}\right) g(y')dy'\right|^p dx' \\ &\leq \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \left|\frac{1}{x_N^{N-1}} \phi\left(\frac{x'-y'}{x_N}\right)\right|^q dy'\right)^{p/q} \int_{\mathbb{R}^{N-1}} |g(y')|^p dy'dx',\end{aligned}$$

where in the last inequality we have used Hölder's inequality.
By Theorem 2.6, where x_N plays the role of ε, we get

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} |\upsilon(x', x_N)|^p dx' &\leq \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} |\phi(x'-y')|^q dy'\right)^{p/q} \int_{\mathbb{R}^{N-1}} |g(y')|^p dy'dx' \\ &\leq \int_{\mathbb{R}^{N-1}} |g(y')|^p dy'.\end{aligned} \tag{3.4}$$

We will use the notation $x' = (x_i'', x_i) \in \mathbb{R}^{N-2} \times \mathbb{R}$, for $i = 1, \ldots, N-1$.
Then by Theorem 2.7 we have

$$\frac{\partial \upsilon}{\partial x_i}(x) = \frac{1}{x_N^N} \int_{\mathbb{R}^{N-1}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) g(y')dy'.$$

Since

$$\int_{\mathbb{R}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) g(y_i'', x_i)dy_i = g(y_i'', x_i) \int_{\mathbb{R}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) dy_i = 0$$

we can write

$$\begin{aligned}\frac{\partial \upsilon}{\partial x_i}(x) &= \frac{1}{x_N^N} \int_{\mathbb{R}^{N-2}} \left(\int_{\mathbb{R}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) g(y')dy_i - \int_{\mathbb{R}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) g(y_i'', x_i)dy_i\right) dy'' \\ &= \frac{1}{x_N^N} \int_{\mathbb{R}^{N-1}} \frac{\partial \phi}{\partial x_i}\left(\frac{x'-y'}{x_N}\right) [g(y') - g(y_i'', x_i)]dy'.\end{aligned}$$

Therefore,

$$\frac{\partial \upsilon}{\partial x_i}(x) = \frac{1}{x_N^N} \int_{\mathbb{R}^{N-1}} \frac{\partial \phi}{\partial x_i}\left(\frac{y'}{x_N}\right) [g(x'-y') - g(x_i'' - y_i'', x_i)]dy'.$$

Since $\mathrm{supp}\phi \subset \overline{\mathrm{B_{N-1}(0,1)}}$ it follows

$$\left|\frac{\partial \upsilon}{\partial x_i}(x)\right| \le \frac{C}{x_N}\int_{B_{N-1}(0,x_N)} |g(x'-y') - g(x_i''-y_i'',x_i)|dy'.$$

We raise both sides to the power p and integrate in x over $\mathbb{R}^N_+$

$$\begin{aligned}\int_{\mathbb{R}^N_+}\left|\frac{\partial \upsilon}{\partial x_i}(x)\right|^p dx &\le C\int_{\mathbb{R}^N_+}\frac{1}{x_N^{Np}}\left(\int_{B_{N-1}(0,x_N)} |g(x'-y') - g(x_i''-y_i'',x_i)|dy'\right)^p dx\\ &\le C\int_{\mathbb{R}^N_+}\frac{1}{x_N^{Np}}|B_{N-1}(0,x_N)|^{p/q}\int_{B_{N-1}(0,x_N)} |g(x'-y') - g(x_i''-y_i'',x_i)|^p dy'dx,\end{aligned}$$

where in the second inequality we have used Hölder's inequality. Then,

$$\int_{\mathbb{R}^N_+}\left|\frac{\partial \upsilon}{\partial x_i}(x)\right|^p dx \le C\int_{\mathbb{R}^N_+}\frac{(x_N^{N-1})^{p-1}}{x_N^{Np}}\int_{B_{N-1}(0,x_N)} |g(x'-y') - g(x_i''-y_i'',x_i)|^p dy'dx,$$

and by Tonelli's theorem we get

$$\int_{\mathbb{R}^N_+}\left|\frac{\partial \upsilon}{\partial x_i}(x)\right|^p dx \le C\underbrace{\int_{\mathbb{R}^N_+}\int_{B_{N-2}(0,x_N)}\int_{-x_N}^{x_N}\frac{|g(x'-y') - g(x_i''-y_i'',x_i)|^p}{x_N^{p+N-1}}dy_i dy_i'' dx}_{\mathrm{I}}. \tag{3.5}$$

For the calculation of I we consider the change of variables $z_i'' = x_i'' - y_i''$ and $z_i = x_i - y_i$. Then,

$$\begin{aligned} I &= \int_{\mathbb{R}^N_+}\int_{B_{N-2}(0,x_N)}\int_{-x_N}^{x_N}\frac{|g(z_i',z_i) - g(z_i'',x_i)|^p}{x_N^{p+N-1}}dz_i dz_i'' dx\\ &= \int_0^\infty\int_{B_{N-2}(0,x_N)}\int_{-x_N}^{x_N}\int_{\mathbb{R}}\int_{\mathbb{R}^{N-2}}\frac{|\Delta_i^{y_i}g(z_I'',x_i-y_i)|^p}{x_N^{p+N-1}}dx_I'' dx_i dz_i dz_i'' dx_N,\end{aligned}$$

where in the second equality we have used Tonelli's theorem. By Hölder's inequality we get

$$\begin{aligned} I &\le C\int_0^\infty \frac{x_N^{N-2}}{x_N^{p+N-1}}\int_{-x_N}^{x_N}\int_{\mathbb{R}^{N-1}}|\Delta_i^{y_i}g(z')|^p dz' dy_i dx_N\\ &= C\underbrace{\int_0^\infty \frac{1}{x_N^{p+1}}\int_{-x_N}^{x_N}\int_{\mathbb{R}^{N-1}}|\Delta_i^{y_i}g(z')|^p dz' dy_i dx_N}_{\mathrm{J}}\end{aligned}$$

and once more with use of Tonelli's theorem we have

$$\begin{aligned} J &= \int_0^\infty \int_{\mathbb{R}^{N-1}} |\Delta_i^{y_i} g(z')|^p \int_{y_i}^\infty \frac{1}{x_N^{p+1}} dz' dy_i dx_N \\ &= C \int_0^\infty \int_{\mathbb{R}^{N-1}} |\Delta_i^{y_i} g(z')|^p \frac{dy_i}{y_i^p} dx_N. \end{aligned}$$

Therefore by 3.5 it follows

$$\int_{\mathbb{R}_+^N} |\frac{\partial v}{\partial x_i}(x)|^p dx \leq C \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|g(x' + he_i') - g(x')|^p}{h^p} dx' dh, \tag{3.6}$$

for all $i = 1, \ldots, N-1$.
In order to estimate $\frac{\partial v}{\partial x_N}$ we write:

$$g(y') - g(x') = \sum_{i=1}^{N-1} \left[g(y_1, \ldots, y_i, x_{i+1}, \ldots, x_{N-1}) - g(y_1, \ldots, y_{i-1}, x_I, \ldots, x_{N-1}) \right].$$

Since $\int_{\mathbb{R}^{N-1}} \phi(x') dx' = 1$ we have

$$\begin{aligned} v(x', x_N) = \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \frac{1}{x_N^{N-1}} \phi\left(\frac{x' - y'}{x_N}\right) &\left[g(y_1, \ldots, y_i, x_{i+1}, \ldots, x_{N-1}) \right. \\ &\left. - g(y_1, \ldots, y_{i-1}, x_I, \ldots, x_{N-1}) \right] dy' + g(x'). \end{aligned}$$

Then by Theorem 2.5 it follows

$$\begin{aligned} \frac{\partial v}{\partial x_N}(x', x_N) = \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial x_N} \left(\frac{1}{x_N^{N-1}} \phi\left(\frac{x' - y'}{x_N}\right) \right) &\left[g(y_1, \ldots, y_i, x_{i+1}, \ldots, x_{N-1}) \right. \\ &\left. - g(y_1, \ldots, y_{i-1}, x_I, \ldots, x_{N-1}) \right] dy'. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right| \leq C \sum_{i=1}^{N-1} \frac{1}{x_N^N} \int_{B_{N-1}(0, x_N)} &\left| g(y_1, \ldots, y_i, x_{i+1}, \ldots, x_{N-1}) \right. \\ &\left. - g(y_1, \ldots, y_{i-1}, x_I, \ldots, x_{N-1}) \right| dy'. \end{aligned}$$

By raising to the power p, integrating over $\mathbb{R}_+^N$ and by following the exact same procedure as for $|\frac{\partial v}{\partial x_i}(x')|$ we obtain:

$$\int_{\mathbb{R}_+^N} \left| \frac{\partial v}{\partial x_N}(x) \right|^p dx \leq C \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|g(z' + he_i') - g(z')|^p}{h^p} dz' dh. \tag{3.7}$$

For $x = (x', x_N) \in \mathbb{R}^N_+$ we define $u(x) := e^{-\frac{x_N}{p}} \upsilon(x)$.
Then by (3.4) and Tonelli's theorem we get

$$\begin{aligned}\int_{\mathbb{R}^N_+} |u(x)|^p dx &= \int_0^\infty e^{-x_N} \int_{\mathbb{R}^{N-1}} |\upsilon(x', x_N)|^p dx' dx_N \\ &\leq \int_0^\infty e^{-x_N} dx_N \int_{\mathbb{R}^{N-1}} |g(x')|^p dx' \\ &= \int_{\mathbb{R}^{N-1}} |g(x')|^p dx'.\end{aligned}$$

since for $i = 1, \ldots, N-1$ we have

$$\left|\frac{\partial u}{\partial x_i}(x)\right| = \left|e^{-\frac{x_N}{p}} \frac{\partial \upsilon}{\partial x_i}(x)\right| \leq \left|\frac{\partial \upsilon}{\partial x_i}(x)\right|,$$

by (3.6) it follows

$$\int_{\mathbb{R}^N_+} \left|\frac{\partial u}{\partial x_i}(x)\right|^p dx \leq C \int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|g(x' + he_i') - g(x')|^p}{h^p} dx' dh.$$

Moreover,

$$\frac{\partial u}{\partial x_N}(x) = e^{-\frac{x_N}{p}} \frac{\partial \upsilon}{\partial x_N}(x) - \frac{1}{p} e^{-\frac{x_N}{p}} \upsilon(x),$$

so by (3.4) and (3.7) we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N_+} \left|\frac{\partial u}{\partial x_N}(x)\right|^p dx\right)^{1/p} &\leq \left(\int_{\mathbb{R}^N_+} \left|\frac{\partial \upsilon}{\partial x_N}(x)\right|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^N_+} \left|\frac{1}{p} e^{-\frac{x_N}{p}} \upsilon(x)\right|^p dx\right)^{1/p} \\ &\leq C \Bigg\{ \left(\int_0^\infty \int_{\mathbb{R}^{N-1}} \frac{|g(z' + he_i') - g(z')|^p}{h^p} dz' dh\right)^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^{N-1}} |g(x')|^p dx'\right)^{1/p} \Bigg\} < \infty.\end{aligned}$$

In other words $u \in W^1_p(\mathbb{R}^N_+)$ and

$$\|u\|_{W^1_p(\mathbb{R}^N_+)} \leq C \|g\|_{B^{1-1/p}_p(\mathbb{R}^{N-1})}. \tag{3.8}$$

It remains to show that $\mathcal{T} u = g$.
By Proposition (2.3) for $p \neq \infty$ we know that there exists a sequence

$\{g_n\}$ in $\mathcal{C}^\infty(\mathbb{R}^{N-1}) \cap B_p^{1-1/p}(\mathbb{R}^{N-1})$ such that $\|g - g_n\|_{B_p^{1-1/p}(\mathbb{R}^{N-1})} \to 0$.
Let $u_n := e^{\frac{-x_N}{p}} \upsilon_n(x)$ be the corresponding sequence. We have that $\upsilon_n \in \mathcal{C}^0(\overline{\mathbb{R}^N_+})$ with $\upsilon_n(x', 0) = g_n(x')$, which follows from the fact that $\upsilon_n = \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \phi\big(\frac{x'-y'}{x_N}\big) g_n(y') dy'$ is defined via convolusion with parameter $\varepsilon = \frac{1}{x_N}$ (as $\varepsilon \to 0$).
By (3.8) we have that $u_n \to u$ in $W_p^1(\mathbb{R}^N_+)$ and since $\mathcal{T} : W_p^1(\mathbb{R}^N_+) \to \mathcal{L}^p(\mathbb{R}^{N-1})$ is a continuous operator we have that $\mathcal{T}u = g$.
Therefore,

$$B_p^{1-1/p}(\mathbb{R}^{N-1}) \subset \mathcal{T} W_p^1(\mathbb{R}^N_+). \tag{3.9}$$

□

Therefore by (3.3), (3.9) it follows

$$\mathcal{T}\mathbf{W}^{\mathbf{1}}_{\mathbf{p}}(\mathbb{R}^{\mathbf{N}}_+) = \mathbf{B}^{\mathbf{1-1/p}}_{\mathbf{p}}(\mathbb{R}^{\mathbf{N-1}}).$$

Now we wish to define the trace spaces for the case in which $\mathbb{R}^N$ is replaced with $\Omega \subset \mathbb{R}^N$ an open set. Once more we will divide the proof in two parts. For the inclusion $\mathcal{T}W_p^l(\Omega) \subset \mathcal{B}_p^{l-1/p}(\partial\Omega)$ we will prove that for all $u \in W_p^l(\Omega)$ there exists a trace $g \in \mathcal{B}_p^{l-1/p}(\partial\Omega,)$, $l > 1/p$.

Theorem 3.4. *Let $l \in \mathbb{N}$, $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ an open set with $\mathcal{C}^l-$boundary. Then, for $l > 1/p$ we have*

$$\mathcal{T}W_p^l(\Omega) \subset \mathcal{B}_p^{l-1/p}(\partial\Omega). \tag{3.10}$$

Proof. Let V_j be parallelepipeds defined as by Burenkov [1] (pg.149) and partition of unity $\psi_j \in \mathcal{C}_0^\infty(V_j)$, $j = 1, \ldots, s$.
First we will prove that the trace g_j of $u\psi_j$ exists on $V_j \cap \partial\Omega$ and $g_j \in \mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)$, $j = 1, \ldots, s$.
By Lemma 2.6 we get:

$$\|u\psi_j\|_{W_p^l(\Omega)} \le c_1 \|u\|_{W_p^l(\mathrm{supp}\psi_j \cap \Omega)} \le c_1 \|u\|_{W_p^l(V_j \cap \Omega)}, \tag{3.11}$$

where $c_1 > 0$ independent of u. In other words we have that $u\psi_j \in W_p^l(V_j \cap \Omega)$, for all $j = 1, \ldots, s$.
By Lemma 2.7 it follows

$$\begin{aligned} \|(u\psi_j)(\Lambda_j^{-1})\|_{W_p^l(\Lambda_j(V_j\cap\Omega))} &\le c_2 \|(u\psi_j)(\Lambda_j^{-1})\Lambda_j\|_{W_p^l(V_j\cap\Omega)} \\ &= c_2 \|u\psi_j\|_{W_p^l(V_j\cap\Omega)}, \end{aligned} \tag{3.12}$$

where $\Lambda_j = \Phi_j(\lambda_j)$ (see Definition 2.16). Therefore we have that $(u\psi_j)(\Lambda_j^{-1}) \in W_p^l(\Lambda_j(V_j \cap \Omega))$. We extend by 0 to $\mathbb{R}^N$ the function $(u\psi_j)(\Lambda_j^{-1})$ and for this extension we have that $(u\psi_j)(\Lambda_j^{-1}) \in W_p^l(\mathbb{R}^N)$, since $\mathrm{supp}(\psi_j)(\Lambda_j^{-1}) \subset \Lambda_j(V_j \cap \Omega)$.
Then by Theorem 2.14 we know that there exists a trace h_j (of the extended function) on $\mathbb{R}^{N-1}$ and therefore on $\Lambda_j(V_j \cap \partial\Omega)$. That means that $g_j := h_j(\Lambda_j)$ is a trace of $u\psi_j$ on $V_j \cap \partial\Omega$ and as a result by Definition 2.13 it follows that $g := \sum_{j=1}^{s} g_j$ is a trace of u on $\partial\Omega$ (since $\sum_{j=1}^{s} u\psi_j = u \sum_{j=1}^{s} \psi_j = u$).
For $l > 1/p$ we have that

$$\|g_j\|_{\mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)} = \|h_j\|_{\mathcal{B}_p^{l-1/p}(\Lambda_j(V_j \cap \partial\Omega))} \leq \|h_j\|_{\mathcal{B}_p^{l-1/p}(\mathbb{R}^{N-1})} \leq c_3 \|u\psi_j\|_{\mathcal{B}_p^{l-1/p}(V_j \cap \Omega)},$$

where in the last inequality we have used (3.12).
Then by Definition 2.16 it follows

$$\begin{aligned} \|g\|_{\mathcal{B}_p^{l-1/p}(\partial\Omega)} &= \left(\sum_{j=1}^{s} \|g\psi_j\|^p_{\mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)} \right)^{\frac{1}{p}} \\ &= \left(\sum_{j=1}^{s} \|g_j\|^p_{\mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)} \right)^{\frac{1}{p}} \\ &\leq c_3 \left(\sum_{j=1}^{s} \|u\psi_j\|^p_{W_p^l(V_j \cap \Omega)} \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore, Lemma 2.6 gives us

$$\|g\|_{\mathcal{B}_p^{l-1/p}(\partial\Omega)} \leq c_4 \left(\sum_{j=1}^{s} \|u\|^p_{W_p^l(V_j \cap \Omega)} \right)^{\frac{1}{p}} \leq c_5 \left(\sum_{j=1}^{s} \{ \|u\|^p_{\mathcal{L}^p(V_j \cap \Omega)} + \sum_{|a|=l} \|\partial^a u\|^p_{\mathcal{L}^p(V_j \cap \Omega)} \} \right)^{\frac{1}{p}},$$

where in the last inequality we have used an equivalent norm to the standard one in $W_p^l(V_j \cap \Omega)$.

With use of Minkowski's inequality for sums we get

$$\begin{aligned}
\|g\|_{\mathcal{B}_p^{l-1/p}(\partial\Omega)} &\leq c_5 \left\{ \left(\sum_{j=1}^{s} \|u\|^p_{\mathcal{L}^p(V_j\cap\Omega)} \right)^{\frac{1}{p}} + \left(\sum_{|a|=l} \sum_{j=1}^{s} \|\partial^a u\|^p_{\mathcal{L}^p(V_j\cap\Omega)} \right)^{\frac{1}{p}} \right\} \\
&\leq c_5 \left(s^{\frac{1}{p}} \|u\|^p_{\mathcal{L}^p(\Omega)} + \sum_{|a|=l} s^{\frac{1}{p}} \|\partial^a u\|^p_{\mathcal{L}^p(\Omega)} \right) \\
&\leq c_6 s^{\frac{1}{p}} \|u\|_{W_p^l(\Omega)}.
\end{aligned}$$

Hence, it follows that $g \in \mathcal{B}_p^{l-1/p}(\partial\Omega)$.

$\square$

For the inverse inclusion it suffices to show that for all $g \in \mathcal{B}_p^{l-1/p}(\partial\Omega)$ there exists a function $u \in W_p^l(\Omega)$ such that g is a trace of u on $\partial\Omega$. Hence we have the following theorem.

Theorem 3.5. *Let $l \in \mathbb{N}$, $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ an open set with $\mathcal{C}^l-$boundary. Then, for $l > 1/p$ we have*

$$\mathcal{B}_p^{l-1/p}(\partial\Omega) \subset \mathcal{T}W_p^l(\Omega). \tag{3.13}$$

Proof. Let $g \in \mathcal{B}_p^{l-1/p}(\partial\Omega)$. We consider the functions $g_j = (g\psi_j)(\Lambda_j^{-1})$ on $\Lambda_j(V_j \cap \partial\Omega)$ and we extend them by 0 to $\mathbb{R}^{N-1}$. We define $E : \mathcal{B}_p^{l-1/p}(\partial\Omega) \to W_p^l(\Omega)$ with

$$Eg = \sum_{j=1}^{s} \Big(E_0\big((g\psi_j)(\Lambda_j^{-1})\big)\Big)(\Lambda_j),$$

where E_0 is a modification of the extension operator such that

$$\mathrm{supp(E_0g_j(\Lambda_j)) \subset V_j \cap \Omega} \text{ and } \mathrm{supp(E_0g_j) \subset \Lambda_j(V_j \cap \Omega)}.$$

Then by Lemma 2.7 it follows

$$\begin{aligned}
\|E_0 g_j(\Lambda)\|_{W_p^l(\mathbb{R}^N)} &= \|(E_0 g_j)(\Lambda_j)\|_{W_p^l(V_j\cap\Omega)} \\
&\leq M_1 \|E_0 g_j\|_{W_p^l(\Lambda_j(V_j\cap\Omega))} \\
&= M_1 \|E_0 g_j\|_{W_p^l(\mathbb{R}^N)},
\end{aligned}$$

where M_3 is independent of g and j.
We have that g_j is a trace of $E_0 g_j$ on $\mathbb{R}^{N-1}$ (see details in [1]), therefore $g_j(\Lambda_j^{-1}) = (g\psi_j)(\Lambda_j^{-1})$ on $\Lambda_j(V_j \cap \partial\Omega)$. Then $\big((g\psi_j)(\Lambda_j^{-1})\big)(\Lambda_j) = g\psi_j$ is a trace of $\big(E_0(g\psi_j)(\Lambda_j^{-1})\big)(\Lambda_j)$ on $V_j \cap \partial\Omega$ and by definition 3 $g = \sum\limits_{j=1}^{s} g\psi_j$ is a trace of $\sum\limits_{j=1}^{s} \big(E_0(g\psi_j)(\Lambda^{-1})\big)(\Lambda_j) = Eg$ on $\sum\limits_{j=1}^{s} V_j \cap \partial\Omega = \partial\Omega$.
We have

$$\begin{aligned}
\|E_0 g_j\|_{W_p^l(\mathbb{R}^N)} &\le M_2 \|g_j\|_{\mathcal{B}_p^{l-1/p}(\mathbb{R}^{N-1})} \\
&\le M_3 \|g_j\|_{\mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)} \\
&\le M_4 \|g\psi_j(\Lambda_j^{-1})\|_{\mathcal{B}_p^{l-1/p}(\Lambda_j(V_j \cap \partial\Omega))},
\end{aligned}$$

where the last two inequalities follow from Lemma 2.5 and 2.7 respectively and M_2, M_3, M_4 are constants independent of g, j.
Then

$$\begin{aligned}
\|Eg\|_{W_p^l(\mathbb{R}^N)} &= \Big\| \sum_{j=1}^{s} (E_0 g_j(\Lambda_j)) \Big\|_{W_p^l(\mathbb{R}^N)} \\
&\le M_3 \sum_{j=1}^{s} \|E_0 g_j\|_{W_p^l(\mathbb{R}^N)} \\
&\le M_3 M_4 \sum_{j=1}^{s} \|g\psi_j\|_{\mathcal{B}_p^{l-1/p}(V_j \cap \partial\Omega)}
\end{aligned}$$

Hence,

$$\|Eg\|_{W_p^l(\mathbb{R}^N)} \le M_5 \|g\|_{\mathcal{B}_p^{l-1/p}(\partial\Omega)}.$$

This means that the extension operator E is bounded (and linear) and since $\mathcal{T}Eg = g$ we have that $\mathcal{B}_p^{l-1/p}(\partial\Omega) \subset \mathcal{T}W_p^l(\Omega)$. □

Therefore it follows

$$\mathcal{T}\mathbf{W}_\mathbf{p}^\mathbf{l}(\boldsymbol{\Omega}) = \mathcal{B}_\mathbf{p}^{\mathbf{l-1/p}}(\partial\boldsymbol{\Omega}).$$

Remark. *For $l = 1$ and $p \neq \infty$ we have that $\mathcal{T}W_1^1(\Omega) = \mathcal{L}^1(\partial\Omega)$ (for the proof we refer to [1]).*
For the case $p = \infty$ let us consider for simplicity that $l = 1$, i.e. the space $W_\infty^1(\Omega)$. Then since Ω is open, bounded and with $\partial\Omega$ of class $\mathcal{C}^1$ it follows that $W_\infty^1(\Omega)$ is the space of all Lipschitz functions (this follows from the characterization of $W_\infty^1(\Omega)$ in [5]).
Now let $u : \Omega \to \mathbb{R}$ be such that $u \in W_\infty^1(\Omega)$. Since u is bounded and Lipschitz, it admits a unique Lipschitz extension to $\overline{\Omega}$. Hence, the trace of u can be defined as the restriction of the extension of u to $\partial\Omega$. Thus, the trace of u is a Lipschitz function on $\partial\Omega$.
Therefore, it follows that $\mathcal{T}W_\infty^1(\Omega) \subset Lip(\partial\Omega)$, where $Lip(\partial\Omega)$ the space of Lipschitz continuous functions defined on $\partial\Omega$. As a matter of fact $\mathcal{T}W_\infty^1(\Omega) = Lip(\partial\Omega)$.
Indeed let $u \in Lip(\partial\Omega)$, then as proven in [11], u can be extended to a Lipschitz function on the whole $\mathbb{R}^N$ therefore on $\overline{\Omega}$ and we have that $W_\infty^1(\Omega)$ is exactly the space of all Lipschitz functions defined on $\overline{\Omega}$.

Chapter 4

The case $\mathrm{p} = 2$: $\mathrm{H^s}$ spaces and their traces

As mentioned above, when $p = 2$ we denote $H^s = W_2^s$. We will see in the following sections that for this case we can skip the classical definition of the Trace Spaces, that is with use of Besov spaces, and use alternative ways to define the trace spaces.

4.1 Characterization via Fourier transform

A way to characterize the trace spaces is with use of Fourier transform. In order to do so we will need several useful definitions and results which are presented bellow.

Definition 4.1. *We call Schwartz class and write $S(\mathbb{R}^N)$ the class of functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that for any multi-index α and any $k \in \mathbb{N}$*

$$\sup_{x\in\mathbb{R}^N} (1 + |x|)^k |\partial^\alpha \phi(x)| < \infty.$$

Definition 4.2. *(Fourier Transform) Let $u \in S(\mathbb{R}^N)$. We define the Fourier transformation $\mathscr{F} : u \mapsto \widehat{u}$ by*

$$\mathscr{F}\{u(x)\} = \widehat{u}(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} dx.$$

It is known that if $u \in S(\mathbb{R}^N)$ then $\widehat{u} \in S(\mathbb{R}^N)$. Therefore $\mathscr{F} : S(\mathbb{R}^N) \to S(\mathbb{R}^N)$ is a linear operator. For the inverse transformation $\mathscr{F}^{-1} : S(\mathbb{R}^N) \to S(\mathbb{R}^N)$ we have the following formula:

$$u(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \widehat{u}(\xi) e^{ix\cdot\xi} d\xi.$$

The Fourier transform can be extended by continuity to $\mathcal{L}^2(\mathbb{R}^N)$ and in fact $\mathscr{F} : \mathcal{L}^2(\mathbb{R}^N) \to \mathcal{L}^2(\mathbb{R}^N)$ is a unitary operator :

$$\int_{\mathbb{R}^N} |u(x)|^2 dx = \int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2 d\xi,$$

that is

$$\|u\|_{\mathcal{L}^2(\mathbb{R}^N)} = \|\mathscr{F} u\|_{\mathcal{L}^2(\mathbb{R}^N)},$$

which is also known as Parseval's equality.
Now we will use the Fourier transform in order to characterize the Sobolev spaces $H^s(\Omega)$.
For the derivatives $\partial^\alpha u(x)$ we have

$$\mathscr{F}\{\partial^\alpha u(x)\} = \widehat{\partial^\alpha u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{u}(\xi).$$

Hence,

$$\|u\|^2_{H^l(\mathbb{R}^N)} = \sum_{|\alpha|\leq l} \int_{\mathbb{R}^N} |\partial^\alpha u|^2 dx = \int_{\mathbb{R}^N} \Big(\sum_{|\alpha|\leq l} |\xi^\alpha|^2 \Big) |\widehat{u}(\xi)|^2 d\xi. \tag{4.1}$$

Here we notice that if we expand $A(\xi) = (1 + |\xi|^2)^l$ by the binomial expansion, that is $A(\xi) = (1 + |\xi|^2)^l = \sum_{|k|\leq l} \binom{l}{k} \big(|\xi|^2\big)^k$, we can choose appropriate constants c_1, c_2 such that

$$c_1(1 + |\xi|^2)^l \leq \sum_{|\alpha|\leq l} |\xi^\alpha|^2 \leq c_2(1 + |\xi|^2)^l,$$

which when applied in (4.1) gives us

$$c_1 \int_{\mathbb{R}^N} (1 + |\xi|^2)^l |\widehat{u}(\xi)|^2 d\xi \leq \|u\|^2_{H^l(\mathbb{R}^N)} \leq c_2 \int_{\mathbb{R}^N} (1 + |\xi|^2)^l |\widehat{u}(\xi)|^2 d\xi.$$

Thus, the norm $\big(\int_{\mathbb{R}^N} (1+|\xi|^2)^l |\widehat{u}(\xi)|^2 d\xi \big)^{\frac{1}{2}}$ is equivalent to the standard one in $W^l_p(\mathbb{R}^N)$. With that being said we have the following definition of $H^s(\mathbb{R}^N)$ spaces.

Definition 4.3. *Let $u \in \mathcal{L}^2(\mathbb{R}^N)$ and $s \in \mathbb{R}$. We say that $u \in H^s(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N}(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi < \infty$.*

As we have proven in the previous chapter, Besov spaces allow us to define the traces of Sobolev spaces and naturally this also applies in the case of $H^s = W_2^s$ spaces. However, the previous representation of H^s spaces via Fourier transform allows us to have an additional approach on how to define the trace spaces. More specifically, we have the following theorems.

Theorem 4.1. *Let $u \in H^s(\mathbb{R}^N)$ and $s > 1/2$. Then the trace operator $\mathcal{T} : \mathcal{C}_0^\infty(\mathbb{R}^N) \to \mathcal{C}_0^\infty(\mathbb{R}^{N-1})$ can be extended uniquely by continuity to a linear continuous operator $\mathcal{T} : H^s(\mathbb{R}^N) \to H^{s-1/2}(\mathbb{R}^{N-1})$. In particular,*

$$\|\mathcal{T}u\|_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq C\|u\|_{H^s(\mathbb{R}^N)}.$$

Proof. Let $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Let $x' = (x_1, x_2, ..., x_{N-1}) \in \mathbb{R}^{N-1}$ then we have the inverse Fourier transform formula:

$$u(x) = u(x', x_N) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} \widehat{u}(\xi', \xi_N)e^{ix_N\xi_N}e^{ix'\cdot\xi'}d\xi' d\xi_N.$$

By definition of the Trace operator we get

$$\mathcal{T}u(x') = u(x', 0) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} \widehat{u}(\xi', \xi_N)e^{ix'\cdot\xi'}d\xi' d\xi_N.$$

That is

$$\begin{aligned} u(x', 0) &= (2\pi)^{-N/2}\int_{\mathbb{R}^N} \widehat{u}(\xi', \xi_N)e^{ix'\cdot\xi'}d\xi' d\xi_N \\ &= (2\pi)^{-N/2}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^{N-1}} \widehat{u}(\xi', \xi_N)e^{ix'\cdot\xi'}d\xi' d\xi_N \end{aligned}$$

and by Fubini-Tonelli Theorem we get

$$u(x', 0) = (2\pi)^{-N/2}\int_{\mathbb{R}^{N-1}}\int_{-\infty}^{+\infty} \widehat{u}(\xi', \xi_N)d\xi_N e^{ix'\cdot\xi'}d\xi'.$$

We write

$$\mathcal{T}u(x') = (2\pi)^{-(N-1)/2}\int_{\mathbb{R}^{N-1}}\Big(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\widehat{u}(\xi', \xi_N)d\xi_N\Big)e^{ix'\cdot\xi_i'}d\xi,$$

which by the inverse Fourier formula gives us

$$\widehat{\mathcal{T}u}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{u}(\xi', \xi_N) d\xi_N.$$

We have

$$\begin{aligned} |\widehat{\mathcal{T}u}(\xi')|^2 &= \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}\widehat{u}(\xi', \xi_N) d\xi_N \right|^2 \\ &\leq \underbrace{\int_{-\infty}^{+\infty} |\widehat{u}(\xi)|^2(1+|\xi|^2)^s d\xi_N}_{I_1} \underbrace{\int_{-\infty}^{+\infty} (1+|\xi'|^2+\xi_N^2)^{-s} d\xi_N}_{I_2}, \end{aligned} \tag{4.2}$$

where in the last inequality we have used Hölder's inequality.
For the calculation of I_2 we set $1+|\xi'|^2 = \alpha^2$ and get

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} \frac{1}{(\alpha^2+\xi_N^2)^s} d\xi_N = \int_{-\infty}^{+\infty} \frac{1}{\alpha^{2s}(1+(\frac{\xi_N}{\alpha})^2)^s} d\xi_N \\ &\overset{y=\frac{\xi_N}{\alpha}}{=} \frac{\alpha}{\alpha^{2s}} \int_{-\infty}^{+\infty} \frac{1}{(1+y^2)^s} dy = c_s \alpha^{1-2s}, \end{aligned}$$

where $c_s = \int_{-\infty}^{+\infty} \frac{1}{(1+y^2)^s} < \infty$ since $s > \frac{1}{2}$.
Therefore,

$$I_2 = c_s(1+|\xi'|^2)^{1-2s}.$$

Then by (4.2) we have

$$(1+|\xi'|^2)^{s-1/2}|\widehat{\mathcal{T}u}(\xi')|^2 \leq c_s \int_{-\infty}^{+\infty} |\widehat{u}(\xi)|^2(1+|\xi|^2)^s d\xi_N.$$

We integrate over $\mathbb{R}^{N-1}$

$$\int_{\mathbb{R}^{N-1}} (1+|\xi'|^2)^{s-1/2}|\widehat{\mathcal{T}u}(\xi')|^2 d\xi' \leq c_s \int_{\mathbb{R}^N} |\widehat{u}(\xi)|^2(1+|\xi|^2)^s d\xi.$$

Thus for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$

$$\|\mathcal{T}u\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \|u\|^2_{H^s(\mathbb{R}^N)}. \tag{4.3}$$

We note that the norm $\|u(\xi)\| = (\int_{\mathbb{R}^N}(1+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi)^{1/2}$ is equivalent to the standard norm in $H^s(\mathbb{R}^N)$.

Now we will extend the previous result for functions u in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ to functions u in $H^s(\mathbb{R}^N)$.
In order to do so, we will use the fact that $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, as follows.
Let $u \in H^s(\mathbb{R}^N)$. Then there exists $\{u_j\}_{j=1}^\infty$ in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$\|u_j - u\|_{H^s(\mathbb{R}^N)} \to 0 \ , \ j \to \infty \ .$$

Then (4.3) gives us that

$$\|\mathcal{T}u_m - \mathcal{T}u_l\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \|u_m - u_l\|^2_{H^s(\mathbb{R}^N)} \to 0 \ ,$$

as $m, l \to \infty$.

In other words $\{\mathcal{T}u_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^{s-1/2}(\mathbb{R}^{N-1})$ and since $H^{s-1/2}(\mathbb{R}^{N-1})$ is complete in $\mathbb{R}^{N-1}$ there exists a limit $v \in H^{s-1/2}(\mathbb{R}^{N-1})$, $\mathcal{T}u_j \to v$, $j \to \infty$ in $H^{s-1/2}(\mathbb{R}^{N-1})$ and by definition we set $v = \mathcal{T}u$.
We note that v does not depend on the choice of the sequence $\{u_j\}$.
Indeed, let $\{w_j\}_{j=1}^\infty$ in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that $\|w_j - u\|_{H^s(\mathbb{R}^N)} \to 0$ as $j \to \infty$. Then, as above, $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence and for $w \in H^{s-1/2}(\mathbb{R}^{N-1})$ such that $\mathcal{T}w_j \to w$, $j \to \infty$, we have:

$$\|w_j - u_j\|_{H^s(\mathbb{R}^N)} \leq \|w_j - u\|_{H^s(\mathbb{R}^N)} + \|u - u_j\|_{H^s(\mathbb{R}^N)} \to 0,$$

as $j \to \infty$. Hence,

$$\|\mathcal{T}w_j - \mathcal{T}u_j\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} = \|\mathcal{T}(w_j - u_j)\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \|w_j - u_j\|_{H^s(\mathbb{R}^N)} \to 0.$$

Since $u_j \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ for all $j \in \mathbb{N}$, by 4.3 it follows

$$\|\mathcal{T}u_j\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \|u_j\|^2_{H^s(\mathbb{R}^N)}.$$

We take the limit

$$\lim_{j\to\infty} \|\mathcal{T}u_j\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \lim_{j\to\infty}, \|u_j\|^2_{H^s(\mathbb{R}^N)}$$

which gives us

$$\|\mathcal{T}u\|^2_{H^{s-1/2}(\mathbb{R}^{N-1})} \leq c_s \|u\|^2_{H^s(\mathbb{R}^N)},$$

with $u \in H^s(\mathbb{R}^N)$.

$\square$

As a result of the previous Theorem we have

$$\mathbf{H^{s-1/2}}(\mathbb{R}^{\mathbf{N-1}}) \subset \mathcal{T}\mathbf{H^s}(\mathbb{R}^{\mathbf{N}}).$$

For the inverse inclusion we have the following Theorem.

Theorem 4.2 (Extension Theorem). *For $k \in \mathbb{Z}_+$, $s > k + 1/2$ we set $H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1}) = H^{s-1/2}(\mathbb{R}^{N-1}) \times H^{s-3/2}(\mathbb{R}^{N-1}) \times \cdots \times H^{s-k-1/2}(\mathbb{R}^{N-1})$. Then there exists a linear continuous operator $E : H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1}) \longrightarrow H^s(\mathbb{R}^N)$ such that if $u = Eg$ with $g = (g_0, g_1, \cdots, g_k) \in H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1})$ then $g_j = \mathcal{T}_j u$, for all $j = 0, 1, \cdots, k$, where $\mathcal{T}_j := \mathcal{T} \circ \frac{\partial^j}{\partial x_N^j} : H^s(\mathbb{R}^N) \to H^{s-j-1/2}(\mathbb{R}^{N-1})$.*
Moreover,

$$\|Eg\|^2_{H^s(\mathbb{R}^N)} \leq c\|g\|^2_{H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1})} := c\sum_{j=0}^{k} \|g_j\|^2_{H^{\langle s-j-1/2\rangle}(\mathbb{R}^{N-1})}$$

Proof. Let $h \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leq h(t) \leq 1$ and $h(t) = 1$ for $|t| \leq 1$. For $\xi' \in \mathbb{R}^{N-1}$, $x_N \in \mathbb{R}$ we consider:

$$V(\xi', x_N) = \sum_{j=0}^{k} \frac{1}{j!} x_N^j \widehat{g}_j(\xi') h\left(x_N\sqrt{1 + |\xi'|^2}\right),$$

where $\widehat{g}_j(\xi')$ is the Fourier transform of $g_j(x')$ over $g = (g_0, \ldots, g_k) \in H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1})$.
We will prove that $V(\xi', x_N)$ is the Fourier transform of a function $u(x', x_N) \in H^s(\mathbb{R}^N)$ with respect to x'. For $x_N = 0$ we have

$$V(\xi', 0) = \widehat{g}_0(\xi') \quad \text{and} \quad \tfrac{\partial^j}{\partial x_N} V(\xi', 0) = \widehat{g}_j(\xi').$$

We use the following identities regarding the Fourier transform:

- $\mathscr{F}\{x_N^j g(x_N)\} = i^j \widehat{g}^{(j)}(\xi_N) = i^j \frac{\partial^j}{\partial \xi_N^j} \widehat{g}(\xi_N)$
- $\mathscr{F}\{g(\alpha x_N)\} = \frac{1}{\alpha}\widehat{g}(\frac{\xi_N}{\alpha})$, $\alpha \in \mathbb{R}$
- $\mathscr{F}\{x_N^j g(\alpha x_N)\} = i^j \frac{1}{\alpha^{j+1}} \widehat{g}^{(j)}(\frac{\xi_N}{\alpha})$, $\alpha \in \mathbb{R}$.

It follows that the Fourier transform of $V(\xi', x_N)$ with respect to x_N is

$$\begin{aligned}\mathscr{F}\{V(\xi', x_N)\} &= \sum_{j=0}^{k} \frac{1}{j!}\widehat{g}_j(\xi')\mathscr{F}\left\{x_N^j h\left(x_N\sqrt{1+|\xi'|^2}\right)\right\} \\ &= \sum_{j=0}^{k} \frac{i^j}{j!}\widehat{g}_j(\xi')(1+|\xi'|^2)^{-\frac{j+1}{2}}\widehat{h}^{(j)}\Big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\Big)\end{aligned}$$

For $\widehat{u}(\xi) = V(\xi', \xi_N)(= \mathscr{F}\{V(\xi', x_N)\})$ we have that $u \in H^s(\mathbb{R}^N)$. Indeed,

$$\begin{aligned}\|u\|^2_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\widehat{u(\xi)}|^2(1+|\xi|^2)^s d\xi \\ &\leq c\sum_{j=0}^{k} \underbrace{\int_{\mathbb{R}^N} |\widehat{g}_j(\xi')|^2(1+|\xi'|^2)^{j-1}\Big|\widehat{h}^{(j)}\Big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\Big)\Big|^2(1+|\xi|^2)^s d\xi}_{\mathrm{I}}\,.\end{aligned}$$

We write

$$\begin{aligned}I &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} |\widehat{g}_j(\xi')|^2(1+|\xi'|^2)^{j-1}\Big|\widehat{h}^{(j)}\Big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\Big)\Big|^2(1+|\xi|^2)^s d\xi_n d\xi' \\ &= \int_{\mathbb{R}^{N-1}} |\widehat{g}_j(\xi')|^2(1+|\xi'|^2)^{j-1}\underbrace{\int_{\mathbb{R}}\Big|\widehat{h}^{(j)}\Big(\frac{\xi_N}{\sqrt{1+|\xi'|^2}}\Big)\Big|^2(1+|\xi|^2)^s d\xi_N}_{I_1}\, d\xi'.\end{aligned}$$

For the calculation of I_1 we set $\tau = \frac{\xi_N}{\sqrt{1+|\xi'|^2}}$. We have $d\xi_N = \sqrt{1+|\xi'|^2}d\tau$ and $1+|\xi|^2 = 1+|\xi'|^2+\xi_N^2 = 1+|\xi'|^2+\tau^2(1+|\xi'|^2) = (1+|\xi'|^2)(1+\tau^2)$, then

$$\begin{aligned}I_1 &= \int_{\mathbb{R}} |\widehat{h}^{(j)}(\tau)|^2(1+|\xi'|^2)^s(1+\tau^2)^s(1+|\xi'|^2)^{1/2}d\tau \\ &= (1+|\xi'|^2)^{s+1/2}\underbrace{\int_{\mathbb{R}} |\widehat{h}^{(j)}(\tau)|^2(1+\tau^2)^s d\tau}_{C(j,s)},\end{aligned}$$

where $C(j,s)<\infty$ since $\widehat{h}\in\mathcal{S}(\mathbb{R})$ for $h\in\mathcal{C}_0^\infty(\mathbb{R})$. Hence,

$$\begin{aligned}\|u\|^2_{H^s(\mathbb{R}^N)} &\le C\sum_{j=0}^k\int_{\mathbb{R}^{N-1}}|\widehat{g_j}(\xi')|^2(1+|\xi'|^2)^{s-j-1/2}d\xi' \\ &= C\sum_{j=0}^k\|g_j\|^2_{H^{s-j-1/2}(\mathbb{R}^{N-1})}.\end{aligned}$$

Therefore the operator $E:H^{\langle s-1/2\rangle}(\mathbb{R}^{N-1})\longrightarrow H^s(\mathbb{R}^N)$ with $Eg=u$ is linear and continuous and we now prove that $\mathcal{T}u_j=g_j$, for all $j=0,1,\dots,k$. This is equivalent to proving that the Fourier Transform of $\frac{\partial^j u}{\partial x_N^j}$ with respect to x' is $\widehat{g_j}(\xi')$. Denote by $\mathscr{F}'$ the Fourier Transform on the variable x'. We have $\mathscr{F}'(\frac{\partial^j u}{\partial x_N^j})=\frac{\partial^j\mathscr{F}'u}{\partial x_N^j}=\frac{\partial^j V(\xi',x_N)}{\partial x_N^j}=\widehat{g_j}(\xi')$, as resumed. $\square$

As a result we have $\mathcal{T}\mathbf{H^s}(\mathbb{R}^\mathbf{N})\subset\mathbf{H^{s-1/2}}(\mathbb{R}^\mathbf{N-1})$.
Therefore,

$$\mathcal{T}\mathbf{H^s}(\mathbb{R}^\mathbf{N})=\mathbf{H^{s-1/2}}(\mathbb{R}^\mathbf{N-1}).$$

4.2 Auchmuty's method

Let Ω be a bounded region in $\mathbb{R}^N$ and let its boundary $\partial\Omega$ be a finite union of disjoint closed Lipschitz surfaces, each surface having finite surface area and unique outward normal $\nu(\cdot)$ defined almost everywhere.
We consider the inner product:

$$\langle u,v\rangle_\theta=\int_\Omega\nabla u\nabla v dx+\int_{\partial\Omega}uv d\sigma$$

and the corresponding norm $\|\cdot\|_\theta$ which is equivalent to the standard one in $H^1(\Omega)$.
A function $u\in H^1(\Omega)$ is called harmonic in Ω provided that it is a solution of Laplace's equation in the weak sense. Namely:

$$\int_\Omega\nabla u\nabla\phi dx=0, \tag{4.4}$$

for all $\phi \in \mathcal{C}_c^1(\Omega)$, or equivalently for all $\phi \in H_0^1(\Omega)$. We define $\mathcal{H}(\Omega)$ to be the space of all harmonic functions u in Ω.
When Ω is defined as above we have by definition $\overline{\mathcal{C}_c^1}(\Omega) = H_0^1(\Omega)$ and by (1) we have $(\mathcal{C}_c^1(\Omega))^\perp = \mathcal{H}(\Omega)$.
Then by the Projection Theorem on the Hilbert space $H^1(\Omega)$ endowed with $\langle \cdot, \cdot \rangle_\partial$ it follows

$$H^1(\Omega) = \overline{\mathcal{C}_c^1(\Omega)} \oplus_\theta (\mathcal{C}_c^1(\Omega))^\perp$$

Therefore,

$$H^1(\Omega) = H_0^1(\Omega) \oplus_\theta \mathcal{H}(\Omega), \tag{4.5}$$

where $\oplus_\theta$ is a θ-orthogonal direct sum.
Auchmuty's method gives us a spectral definition of the trace spaces $H_A^s(\partial\Omega)$, $s \in \mathbb{R}$, where the subscript A is here used to emphasize the fact here the definition is given for this method. The idea is based on the fact that the harmonic Steklov eigenfunctions provide an orthogonal basis in $\mathcal{H}(\Omega)$ as well as an orthogonal basis in $\mathcal{L}^2(\partial\Omega, d\sigma)$. More specifically we can define an orthonormal basis in $\mathcal{L}^2(\partial\Omega, d\sigma)$ by means of the Steklov eigenfunctions and then use this basis to represent the trace spaces.
In the end the definition of $H_A^s(\partial\Omega)$ spaces is reduced to certain summability conditions regarding the harmonic Steklov coefficients which, as described above, help us represent the functions $g \in H_A^s(\partial\Omega)$.

4.2.1 The harmonic Steklov eigenproblem

Let Ω be a region in $\mathbb{R}^N$ defined as above. We consider the boundary value problem:

$$\begin{cases} \Delta s = 0, & x \in \Omega \\ \mathcal{D}_\nu s = \delta s, & x \in \partial\Omega \end{cases} \tag{4.6}$$

for $\delta \in \mathbb{R}$. Recall that $D_\nu s$ denotes the normal derivative of s on $\partial\Omega$. In order to find the weak formulation of (4.6) we argue as follows. We assume that $s \in H^1(\Omega)$ is a classical solution to (4.6). Then by integrating over Ω we get

$$\int_\Omega \Delta s v dx = 0,$$

for all $\upsilon \in H^1(\Omega)$. By Green's Formula it follows

$$\int_\Omega \nabla s \nabla \upsilon dx = \int_{\partial\Omega} \upsilon \mathcal{D}_\nu s d\sigma,$$

Finally, by using the boundary condition we get the weak form of (4.6):

$$\int_\Omega \nabla s \nabla \upsilon dx = \delta \int_{\partial\Omega} \upsilon s d\sigma, \tag{4.7}$$

for all $\upsilon \in H^1(\Omega)$. For $\upsilon = s$ we get that $\delta \geq 0$.
We rewrite (4) in the following form

$$\int_\Omega \nabla s \nabla \upsilon dx + \int_{\partial\Omega} \upsilon s d\sigma = (\delta + 1) \int_{\partial\Omega} \upsilon s d\sigma. \tag{4.8}$$

We consider the Laplace operator Δ as an operator from $H^1(\Omega)$ to its dual $H^1(\Omega)'$ defined by $\Delta[s][\upsilon] = -\int_\Omega \nabla s \nabla \upsilon dx$, for all $\upsilon \in H^1(\Omega)$ and the operator $I : H^1(\Omega) \to H^1(\Omega)'$ defined ny $I[s][\upsilon] = \int_{\partial\Omega} s\upsilon d\sigma$, for all $\upsilon \in H^1(\Omega)$.
The operator $I - \Delta : H^1(\Omega) \to H^1(\Omega)'$ with $(I - \Delta)[s][\upsilon] = \int_\Omega \nabla s \nabla \upsilon dx + \int_{\partial\Omega} s\upsilon d\sigma = \langle s, \upsilon \rangle_\theta$ is an isometry and there exists $(I - \Delta)^{-1}$ (by Riesz's Representation Theorem).
We also define the operator $J : \mathcal{L}^2(\partial\Omega) \to H^1(\Omega)'$ with $J[s][\upsilon] = \int_{\partial\Omega} s\upsilon d\sigma$, for all $\upsilon \in \mathcal{L}^2(\partial\Omega)$, and lastly we consider the trace operator $\mathcal{T} : H^1(\Omega) \to \mathcal{L}^2(\partial\Omega)$.
Then (4.8) can be written as

$$(I - \Delta)s = (\delta + 1)(J \circ \mathcal{T})s.$$

Therefore,

$$(I - \Delta)^{-1} \circ J \circ \mathcal{T} s = \frac{1}{\delta + 1} s.$$

Note that

$$H^1(\Omega) \xrightarrow{\mathcal{T}} \mathcal{L}^2(\partial\Omega) \xrightarrow{J} H^1(\Omega)' \xrightarrow{(I-\Delta)^{-1}} H^1(\Omega).$$

Hence we can define the operator $M : H^1(\Omega) \to H^1(\Omega)$ with $M = (I - \Delta)^{-1} \circ J \circ \mathcal{T}$.
Then the weak form (4.7) of the Steklov eigenproblem is equivalent to

$$Ms = \mu s, \tag{4.9}$$

where $\mu = \frac{1}{\delta+1} > 0$.
We notice that M is compact by definition, since the trace operator $\mathcal{T} : H^1(\Omega) \to \mathcal{L}^2(\partial\Omega)$ is compact, and also M is non-negative.
Let $s \in \mathrm{KerM}$, then

$$(I - \Delta)^{-1} \circ J \circ \mathcal{T} s = 0,$$

which, since $(I - \Delta)$ is an isometry, is equivalent to

$$J \circ \mathcal{T} s = 0$$

and by definition of J

$$\mathcal{T} s = 0.$$

By Theorem (2.16) we have that $\mathcal{T} s = 0$ iff $s \in H^1_0(\Omega)$. Therefore, $\mathrm{KerM} = \mathrm{H}^1_0(\Omega)$.
Hence, by (4.5) we obtain

$$H^1(\Omega) = \mathrm{KerM} \oplus_\theta \mathcal{H}(\Omega). \tag{4.10}$$

Moreover, we have that the codimension of KerM, $\mathrm{codimKerM} = \dim(\mathrm{H}^1(\Omega)/\mathcal{H}(\Omega))$, is infinite and since M is a non-negative, compact, self-adjoint operator in a Hilbert space we have that the spectrum $\sigma(M)$ is discrete and $\sigma(M) \setminus \{0\}$ consists of eigenvalues $\{\mu_j\}$ of finite multiplicity with $\mu_j \to 0$, as $j \to \infty$.
More specifically by the Courant-Raylegh minmax principle we have

$$\begin{aligned}
\mu_j &= \max_{\substack{V \subset \mathcal{H}(\Omega) \\ \mathrm{dimV=j}}} \min_{s \in V \setminus \{0\}} \frac{\langle Ms, s\rangle_\theta}{\|s\|_\theta} \\
&= \max_{\substack{V \subset \mathcal{H}(\Omega) \\ \mathrm{dimV=j}}} \min_{s \in V \setminus \{0\}} \frac{\langle (I - \Delta^{-1}) \circ J \circ \mathcal{T} s, s\rangle_\theta}{\|s\|_\theta} \\
&= \max_{\substack{V \subset \mathcal{H}(\Omega) \\ V \cap H^1_0(\Omega) = \{0\} \\ \mathrm{dimV=j}}} \min_{s \neq 0} \frac{(I - \Delta) \circ (I - \Delta)^{-1} \circ J[\mathcal{T} s][s]}{\|s\|_\theta}
\end{aligned}$$

Therefore for the Steklov eigenvalues $\delta_j = \frac{1}{\mu_j} - 1$ we have

$$\delta_j = \min_{\substack{V \subset \mathcal{H}(\Omega) \\ V \cap H^1_0(\Omega) = \{0\} \\ \mathrm{dimV=j}}} \max_{s \neq 0} \frac{\int_\Omega |\nabla s|^2 dx}{\int_{\partial\Omega} s^2 d\sigma}.$$

By the Spectral Theorem for compact and self-adjoint operators there exists an orthonormal set of eigenfunctions $\{s_j\}_{j=0}^{\infty}$ corresponding to the eigenvalues $\{\mu_j\}_{j=0}^{\infty}$ such that

$$(\mathrm{KerM})^{\perp} = \overline{\mathrm{span}\langle \mathrm{s_j} : \mathrm{j} \geq 0\rangle}.$$

Then by (4.10) it follows that $S = \{s_j : j \geq 0\}$ is an orthonormal basis for $\mathcal{H}(\Omega)$.

Definition 4.4. *We call a Steklov expansion an expression of the form* $u(x) = \sum_{j=0}^{\infty} c_j s_j$ *, where* $c_j := \langle u, s_j\rangle_\theta$ *, for* $u \in H^1(\Omega)$*.*

Since $S = \{s_j : j \geq 0\}$ is an orthonormal basis in $\mathcal{H}(\Omega)$ by Theorem (2.11) (which implies the existence of limiting function iff the coefficients are square summable) we have that a Steklov expression represents a H^1-harmonic function on Ω iff $\sum_{j=0}^{\infty} |c_j|^2 < \infty$.

4.2.2 Spectral representation of the trace and the extension operator

The Steklov eigenfunctions s_j (as described above) have $\mathcal{L}^2$ traces on the boundary $\partial\Omega$ whenever Ω is defined as in the beggining of this section.
We now define

$$\hat{s_j}(x) := \sqrt{1+\delta_j}\mathcal{T} s_j(x), \text{ for } x \in \partial\Omega \text{ and } j \geq 0.$$

Then, as proven in [4], $\hat{S} = \{\hat{s_j} : j \geq 0\}$ is an orthonormal basis in $\mathcal{L}^2(\partial\Omega, d\sigma)$.
Since $\mathcal{T}$ is continuous for all $u \in H^1(\Omega)$ we have

$$\mathcal{T}u = \mathcal{T}\sum_{j=0}^{\infty}\langle u, s_j\rangle_\theta s_j = \sum_{j=0}^{\infty}\langle u, s_j\rangle_\theta \mathcal{T} s_j,$$

that is

$$\mathcal{T}u = \sum_{j=0}^{\infty}(1+\delta_j)^{-1/2}\langle u, s_j\rangle_\theta \hat{s_j},$$

for all $u \in H^1(\Omega)$.
For all $f, g \in \mathcal{L}^2(\partial\Omega, d\sigma)$ we consider the inner product:

$$\langle g, f \rangle_{\partial\Omega} = \int_{\partial\Omega} g f d\sigma,$$

We suppose now that $g = \mathcal{T}u$ for some $u \in H^1(\Omega)$. Then $g \in \mathcal{L}^2(\partial\Omega, d\sigma)$ and we have $g = \sum_{j=0}^{\infty} g_j \hat{s_j}(x)$, where $g_j = \langle g, \hat{s_j} \rangle_{\partial\Omega}$.
By the Riesz-Fischer Theorem it follows that $g \in \mathcal{L}^2(\partial\Omega, d\sigma)$ iff we have $\sum_{j=0}^{\infty} |g_j|^2 < \infty$.
We now define the extension operator $E : A \to \mathcal{H}(\Omega)$ with

$$Eg = \sum_{j=0}^{\infty} (1 + \delta_j)^{1/2} g_j s_j,$$

where $A \subset \mathcal{L}^2(\partial\Omega, d\sigma)$ the subspace of all the functions $u \in \mathcal{L}^2(\partial\Omega, d\sigma)$ such that $\sum_{j=0}^{\infty} (1 + \delta_j)|g_j|^2 < \infty$.

4.2.3 The $H_A^s(\partial\Omega)$ spaces

As described at the beginning of this section, in Auchmuty [3] the $H_A^s(\partial\Omega)$ spaces are defined as the subspaces of $\mathcal{L}^2(\partial\Omega, d\sigma)$ of functions whose Steklov harmonic coefficients satisfy certain summability conditions. More specifically, we have the following definition.

Definition 4.5. *For $s \geq 0$ we define $H_A^s(\partial\Omega)$ as the subspace of all functions $g \in \mathcal{L}^2(\partial\Omega, d\sigma)$ with Steklov expansion satisfying $\sum_{j=0}^{\infty} (1 + \delta_j)^{2s} |g_j|^2 < \infty$.*

We also define the s-inner product and s-norm on $H^s(\partial\Omega)$:

$$\langle g, f \rangle_{s,\partial\Omega} := \sum_{j=0}^{\infty} (1 + \delta_j)^{2s} g_j f_j$$
$$\|g\|_{s,\partial\Omega}^2 := \sum_{j=0}^{\infty} (1 + \delta_j)^{2s} g_j^2$$

Proposition. *For $s = \frac{1}{2}$ the space $H_A^{1/2}(\partial\Omega)$ coincides with the space $H^{1/2}(\partial\Omega)$, that is*

$$H_A^{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega).$$

Proof. We have proven that $\mathcal{T}H^1(\Omega) = H^{1/2}(\partial\Omega)$ (result of Theorem 3.4 and Theorem 3.5). Therefore for the inclusion $H_A^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ it suffices to show that for all $g \in H_A^{1/2}(\partial\Omega)$ there exists $u \in H^1(\Omega)$ such that $\mathcal{T}u = g$.
For the inverse inclusion $H^{1/2}(\partial\Omega) \subset H_A^{1/2}(\partial\Omega)$ we will prove that $\mathcal{T}u \in H_A^{1/2}(\partial\Omega)$ for all $u \in H^1(\Omega)$, which implies that $H^{1/2}(\partial\Omega) = \mathcal{T}H^1(\Omega) \subset H_A^{1/2}(\partial\Omega)$.
Let $g \in H_A^{1/2}(\partial\Omega)$, then we have that $g = \sum_{j=0}^{\infty} g_j \hat{s}_j$. We consider the extension operator E and we define $u = Eg = \sum_{j=o}^{\infty}(1+\delta_j)^{1/2} g_j s_j$. Then $u \in \mathcal{H}(\Omega)$ since $\sum_{j=0}^{\infty}(1+\delta_j)|g_j|^2 < \infty$, by definition of $H_A^{1/2}(\partial\Omega)$.
Since the trace operator $\mathcal{T} : H^1(\Omega) \to \mathcal{L}^2(\partial\Omega)$ is continuous we have

$$\begin{aligned}\mathcal{T}u = \mathcal{T}\sum_{j=o}^{\infty}(1+\delta_j)^{1/2} g_j s_j &= \sum_{j=o}^{\infty}(1+\delta_j)^{1/2} g_j \mathcal{T}s_j \\ &= \sum_{j=o}^{\infty}(1+\delta_j)^{1/2} g_j (1+\delta_j)^{-1/2}\hat{s}_j = \sum_{j=o}^{\infty} g_j \hat{s}_j.\end{aligned}$$

Hence,

$$H_A^{1/2}(\partial\Omega) \subset H^{1/2}(\partial\Omega). \tag{4.11}$$

Let $u \in H^1(\Omega)$, then $u = \sum_{j=0}^{\infty}\langle u, s_j\rangle_\theta s_j$ and

$$\mathcal{T}u = \mathcal{T}\sum_{j=0}^{\infty}\langle u, s_j\rangle_\theta s_j = \sum_{j=0}^{\infty}\langle u, s_j\rangle_\theta \mathcal{T}s_j.$$

We write

$$\mathcal{T}u = \sum_{j=0}^{\infty}\frac{\langle u, s_j\rangle_\theta}{(1+\delta_j)^{1/2}}(1+\delta_j)^{1/2}\mathcal{T}s_j = \sum_{j=0}^{\infty}\frac{\langle u, s_j\rangle_\theta}{(1+\delta_j)^{1/2}}\hat{s}_j.$$

According to the definition of $H_A^{1/2}(\partial\Omega)$ in order to prove that $\mathcal{T}u \in H_A^{1/2}(\partial\Omega)$ we have to check the summability condition for the Steklov coefficient:

$$\sum_{j=o}^{\infty}(1+\delta_j)^{2\frac{1}{2}}\left|\frac{\langle u, s_j\rangle_\theta}{(1+\delta_j)^{1/2}}\right|^2 = \sum_{j=o}^{\infty}\langle u, s_j\rangle_\theta^2 < \infty,$$

since $u \in H^1(\Omega)$. Hence,

$$H^{1/2}(\partial\Omega) \subset H_A^{1/2}(\partial\Omega). \tag{4.12}$$

By (4.11),(4.12) we obtain

$$H_A^{1/2}(\partial\Omega) = H^{1/2}(\partial\Omega).$$

□

Remark. *We note that for the definition of the spaces $H_A^s(\partial\Omega)$ the boundary $\partial\Omega$ is required to be minimally smooth for the Steklov eigenfunctions-eigenvalues analysis to hold. Contrary, for the classical definition of the trace spaces $H^s(\partial\Omega)$ via Besov spaces or Fourier analysis (for the case p=2) it is required $\partial\Omega$ to be $\mathcal{C}^l$, $l \in \mathbb{N}$, with $l > s$.*

Chapter 5

Existence of solution for the Poisson Problem with Dirichlet boundary conditions

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $\mathcal{F} \in \mathcal{L}^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$. We consider the following Problem:

$$\begin{cases} -\Delta u = \mathcal{F}, & x \in \Omega, \\ \mathcal{T}u = g, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

which is known as the Poisson Problem with Dirichlet boundary conditions and will be understood in the weak sense as follows.
We say that $u \in H^1(\Omega)$ is a weak solution to problem (5.1) iff

$$\int_\Omega \nabla u \overline{\nabla \phi} dx = \int_\Omega \mathcal{F} \overline{\phi} dx, \tag{5.2}$$

for all $\phi \in H_0^1(\Omega)$, and $\mathcal{T}u = g$ in $\partial\Omega$.

Let $\Phi \in H^1(\Omega)$ be a real-valued function such that $\mathcal{T}\Phi = g$ and let $u \in H^1(\Omega)$ be a weak solution to problem (5.1), as described above. Then for $v(x) := u(x) - \Phi(x)$ it follows that $\mathcal{T}v = 0$ and

$$\int_\Omega \nabla v \overline{\nabla \phi} dx = \int_\Omega f \overline{\phi} dx, \tag{5.3}$$

for all $\phi \in H_0^1(\Omega)$ and $f = \mathcal{F} + \Delta\Phi$, where $\Delta\Phi$ is an element of $H^{-1}(\Omega)$ defined by $\langle \Delta\Phi, \phi \rangle = -\int_\Omega \nabla\Phi \nabla\phi dx$, for all $\phi \in H_0^1(\Omega)$.
We can interpret the fact that $\mathcal{T}v = 0$ in the sense that $v \in H_0^1(\Omega)$.

Definition 5.1. *Let* $\Omega \subset \mathbb{R}^N$ *as above. We consider the Poisson problem with homogeneous Dirichlet boundary conditions :*

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \mathcal{T} u = 0, & x \in \partial\Omega, \end{cases} \tag{5.4}$$

with $f \in H^{-1}(\Omega)$*. A function* $v \in H_0^1(\Omega)$ *is called a weak solution to the problem (5.4) iff it satisfies (5.3) for all* $\phi \in H_0^1(\Omega)$*.*

5.1 Existence via the classical definition of H^s spaces

Theorem 5.1. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary. Then for all* $f \in H^{-1}(\Omega)$ *there exists a unique weak solution* $v \in H_0^1(\Omega)$ *to problem* (5.4)*.*

Proof. The form $\langle v, \phi \rangle := \int_\Omega \nabla v \overline{\nabla \phi} dx$ defines an inner product in $H_0^1(\Omega)$ for all $v, \phi \in H_0^1(\Omega)$ and the corresponding norm is equivalent to the standard norm $\| \cdot \|_{\mathcal{H}^1(\Omega)}$.
Indeed since Ω is bounded Friedrichs inequality holds:

$$\int_\Omega |v|^2 dx \leq C_\Omega \int_\Omega |\nabla v|^2 dx$$

which implies that

$$\int_\Omega |v|^2 + |\nabla v|^2 dx \leq (C_\Omega + 1) \int_\Omega |\nabla v|^2 dx \ ,$$

for all $v \in H_0^1(\Omega)$.
Hence,

$$\|v\|_{\mathcal{H}^1(\Omega)} \leq C_\Omega \|v\|_{\langle,\rangle}.$$

By Riesz's Representation Theorem there exists a unique function $v \in \mathcal{H}_0^1(\Omega)$ such that

$$f(\phi) = \langle v, \phi \rangle,$$

for all $\phi \in \mathcal{H}_0^1(\Omega)$ and

$$\|f\| = \|v\|,$$

which means that $v \in H_0^1(\Omega)$ is a weak solution to (5.4).
Moreover the solution is unique. Indeed, let $v_1, v_2 \in H_0^1(\Omega)$ both be weak solutions to problem (5.4), then

$$\int_\Omega \nabla(v_1 - v_2)\overline{\nabla\phi}dx = 0,$$

for all $\phi \in H_0^1(\Omega)$.
Therefore,

$$v_1 - v_2 = c,$$

for some $c \in \mathbb{R}$.
Since the trace operator $\mathcal{T}$ is linear we have that

$$\mathcal{T}(v_1 - v_2) = \mathcal{T}(v_1) - \mathcal{T}(v_2) = 0.$$

Hence,

$$v_1 - v_2 = 0.$$

□

We now return to the Poisson problem with non-homogeneous Dirichlet boundary conditions.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\mathcal{C}^l$ boundary. Let $\mathcal{F} \in \mathcal{L}^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exists unique weak solution $u \in H^1(\Omega)$ to the problem (5.1).*

Proof. By Theorem 3.5 for $g \in H^{1/2}(\partial\Omega)$ there exists an extension $\Phi \in H^1(\Omega)$ with $\Phi = \mathcal{E}_\Omega g$ such that $\mathcal{T}\Phi = g$.
We consider $f = \mathcal{F} + \Delta\Phi \in H^{-1}(\Omega)$, with $\Delta\Phi$ defined as above. By Theorem 5.1 there exists a unique solution $v \in H_0^1(\Omega)$ to Problem (5.4).
Then the function $u := v + \Phi \in H^1(\Omega)$ is such that

$$\mathcal{T}u = \mathcal{T}(v + \Phi) = \mathcal{T}v + \mathcal{T}\Phi = g$$

and

$$\int_\Omega \nabla u\overline{\nabla\phi}dx = \int_\Omega \nabla v\overline{\nabla\phi}dx + \int_\Omega \nabla\Phi\overline{\nabla\phi}dx,$$

for all $\phi \in H_0^1(\Omega)$.
Then by (5.3) we obtain

$$\int_\Omega \nabla u\overline{\nabla\phi}dx = \int_\Omega f\overline{\nabla\phi}dx + \int_\Omega \nabla\Phi\overline{\nabla\phi}dx,$$

for all $\phi \in H_0^1(\Omega)$.
Since $f = \mathcal{F} + \Delta\Phi$ it follows that

$$\int_\Omega \nabla u \overline{\nabla \phi} dx = \int_\Omega \mathcal{F}\overline{\phi} dx,$$

for all $\phi \in H_0^1(\Omega)$. By Definition 1 we have that u is a solution to the problem (5.1).
We note that the solution is unique. This follows by the uniqueness of the problem (5.4) : if we consider u_1, u_2 both to be solutions for the problem (5.1), then for $w := u_1 - u_2$ from (1) we have

$$\int_\Omega \nabla w \overline{\nabla \phi} dx = 0,$$

for all $\phi \in H_0^1(\Omega)$. Finally, since $\mathcal{T}w = \mathcal{T}(u_1 - u_2) = \mathcal{T}u_1 - \mathcal{T}u_2 = 0$ we have $w = u_1 - u_2 = 0$. □

5.2 With use of Auchmuty's Definition

Another way to prove that the problem (5.1) has a unique solution is by using the spectral definition of $\mathcal{H}_A^{1/2}(\partial\Omega) = \mathcal{H}^{1/2}(\partial\Omega)$ given by Auchmuty [3].
Since $\mathcal{T}u = g$ we know that $g \in \mathcal{H}^{1/2}(\partial\Omega)$. Then by Auchmuty's definition we have

$$g = \sum_{j=1}^{\infty} g_j \hat{s_j}, \text{ where } g_j = \langle g_j, \hat{s_j} \rangle_{\partial\Omega} \text{ and } \sum_{j=1}^{\infty} (1 + \delta_j |g_j|^2) < \infty.$$

Then the the extension u of g to Ω is such that $\mathcal{T}u = g$ and

$$u = \mathcal{E}g = \sum_{j=1}^{\infty} (1 + \delta_j)^{1/2} g_j s_j \in \mathcal{H}(\Omega) \subset \mathcal{H}^1(\Omega).$$

Moreover, it is the unique weak solution to the Dirichlet problem, i.e. to the Poisson problem with homogeneous dirichlet boundary conditions.

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