On the solvability and the inviscid limit of Cauchy problems for certain classes of the non-linear Schrödinger equation

Doctoral Thesis

Nikolaos Gialelis

Department of Mathematics,
National and Kapodistrian University of Athens,
Athens, Greece

2019

This thesis has been co-financed - via a programme of State Scholarships Foundation (IKY) - by the European Union (European Social Fund - ESF) and Greek national funds through the action entitled “Strengthening Human Resources Research Potential via Doctorate Research” in the framework of the Operational Program “Human Resources Development Program, Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) 2014 - 2020.

Advisory committee

Dimitri J. Frantzeskakis - Department of Physics, National and Kapodistrian University of Athens
Nikolaos I. Karachalios - Department of Mathematics, University of the Aegean
Ioannis G. Stratis (supervisor) - Department of Mathematics, National and Kapodistrian University of Athens

Dissertation Examining Committee

Nicholas D. Alikakos - Department of Mathematics, National and Kapodistrian University of Athens
Gerassimos Barbatis - Department of Mathematics, National and Kapodistrian University of Athens
Panagiotis Gianniotis - Department of Mathematics, National and Kapodistrian University of Athens
Pandelis Dodos - Department of Mathematics, National and Kapodistrian University of Athens
Dimitri J. Frantzeskakis - Department of Physics, National and Kapodistrian University of Athens
Nikolaos I. Karachalios - Department of Mathematics, University of the Aegean
Ioannis G. Stratis - Department of Mathematics, National and Kapodistrian University of Athens

Contents

Abstract

The non-linear (in particular the semi-linear) Schrödinger equation, very often referred to by the acronym NLSE, is a universal model describing the evolution of complex field envelopes in non-linear dispersive media; it appears in a variety of physical contexts, ranging from optics to fluid dynamics and plasma physics, and it has attracted a huge interest from the rigorous mathematical analysis point of view, as well. The importance of the NLSE model is not restrained to the case of conservative systems, but it is also associated to dissipative models. Many of the closely connected to the NLSE pattern formation phenomena, emanate via the genesis of localized structures with finite spatial support, or with sufficiently fast spatial decay, the so-called solitons. Among the various types of waves whose amplitude is modulated, there are two principal kinds of solitons, depending on the category of the non-linearity; in the case of an attractive (or focusing) medium, the non-linearity causes the formation of structures termed "bright solitons", while in the case of a repulsive (or defocusing) medium, the non-linearity generates "dark solitons" (i.e., non-linear solitary waves having the form of localized dips in density, that decay off of a continuous-wave background; if the density of the dip tends to zero, the dark solitons are named "black"; otherwise "grey").

Theoretical physical studies on dark solitons started in 1971, by the work of T. Tsuzuki [45] in the context of Bose-Einstein condensates. Two years later, in [50], V. E. Zakharov and A. B. Shabat demonstrated the complete integrability of the defocusing NLSE utilizing the Inverse Scattering Transform (incidentally, the same authors had shown the integrability of the focusing NLSE in [49]). The progress in the theory after that was very rapid and immense. As for experimental results, the progress was equally impressive: after the "early age" experiments of the 1970s, the "new age", that emerged in the middle of the first decade of the 21st century, is a period of spectacular progress. These led to a vast amount of literature. A detailed presentation of the physical studies (theoretical and experimental) and of the recent progress regarding the defocusing NLSE is contained in [33], that incorporates an extensive bibliography.

Regarding the rigorous mathematical analysis of the NLSE, the books [6], [10], [11], [42], [43] are classical by now. The more recent books [14], [17], [36], also contribute substantially to the field. The reference lists in all these books are representative of the huge interest and amount of research work on the NLSE.

In this doctoral thesis, we are interested in two problems involving the NLSE. The first one is the quest for and the study of a special type of solutions of the defocusing NLSE which do not vanish at the spatial extremity. The second one is the study of the "inviscid limit" of the linearly damped and driven NLSE. Below follows a brief presentation of both the questions that we raise, as well as the conclusions that we reach.

Non-vanishing solutions of the defocusing NLSE. For an interval $J_0 \subseteq \mathbb{R}$ with $0 \in J_0^\circ$, an open $U \subseteq \mathbb{R}^n$, a differentiable $a = (a_{ij})_{i,j=1}^n : U \to \mathbb{C}^{n\times n}$, as well as a twice-differentiable $\zeta : \overline{U} \to \mathbb{C}$ which survives at the boundary or at infinity, we search for a twice-differentiable $u : J_0 \times \overline{U} \to \mathbb{C}$ that solves the n-dimensional initial/"boundary"-value problem for the NLSE with pure power non-linearity

$$\begin{cases} i\frac{\partial u}{\partial t} - \operatorname{div}\left(a^{\mathrm{T}}\nabla(u+\zeta)\right) + \lambda\left(|u+\zeta|^\alpha + r\right)(u+\zeta) = 0, \text{ in } J_0^* \times U \\ u = u_0, \text{ in } \{t=0\}\times\overline{U} \\ u = 0, \text{ in } J_0 \times \partial U, \text{ and } u \xrightarrow{|x|\nearrow\infty} 0, \text{ in } J_0\times\overline{U}, \end{cases} \tag{1}$$

for $\lambda \in \mathbb{R}^*$, $\alpha \in (0,\infty)$ and $r \in \mathbb{R}$. The above problem arises from the search for solutions of the form

$$v(t,x) = e^{-i\lambda r t}\left(u(t,x) + \zeta(x)\right)$$

for the n-dimensional NLSE problem

$$\begin{cases} i\frac{\partial v}{\partial t} - \operatorname{div}\left(a^{\mathrm{T}}\nabla v\right) + \lambda|v|^{\alpha}v = 0, \text{ in } J_0^{*}\times U \\ v = v_0, \text{ in } \{t=0\}\times\overline{U} \\ v \neq 0, \text{ in } J_0\times\partial U, \text{ and } v \nrightarrow 0 \text{ when } |x| \nearrow \infty, \text{ in } J_0\times\overline{U}. \end{cases} \tag{2}$$

If

$$a \equiv \mathrm{id} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3}$$

that is $\operatorname{div}\left(a^{\mathrm{T}}\nabla u\right) \equiv \Delta u$, then the differential equation in (2) is known as "defocusing" (or "self-defocusing") when $\lambda > 0$ and "focusing" (or "self-focusing") when $\lambda < 0$. Here, we extend this definition: if

$$\operatorname{Re}\left(\xi\cdot a\bar{\xi}\right) \geq \theta|\xi|^2, \text{ a.e. in } U, \text{ for every } \xi\in\mathbb{C}^n, \text{ for some } \theta>0 \text{ (uniform ellipticity of } a) \tag{4}$$

and

$$a = \overline{a^{\mathrm{T}}}, \text{ i.e. } a_{ij} = \overline{a_{ji}}, \text{ a.e. in } U \text{ (self-adjointness of } a), \tag{5}$$

then we say that the differential equation in (2) is defocusing if $\lambda>0$ and that it is focusing if $\lambda<0$. It is also direct to check that (5) implies that $\xi\cdot a\bar{\xi} = \overline{\xi\cdot a\bar{\xi}}$ a.e. in U, i.e $\xi\cdot a\bar{\xi}$ is real-valued a.e. in U.

For example, well known solutions of the defocusing problem (2) for $U=\mathbb{R}$, $a\equiv\mathrm{id}$ and $\alpha=2$, are the black solitons

$$v(t,x) = e^{i\lambda t}\tanh\left(\pm\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}x\right).$$

The defocusing problem (1) for $U=\mathbb{R}^n$ with $n=1,2,3$ and

$$\alpha = 2\tau, \text{ for } \begin{cases} \tau\in\mathbb{N}, & \text{if } n=1,2 \\ \tau=1, & \text{if } n=3, \end{cases} \tag{6}$$

was first studied in [19]. Here, we extend the results of the aforementioned paper, not only by weakening the assumptions, but also by considering more general cases of $U\subseteq\mathbb{R}^n$, other than the Euclidean space itself. Moreover, we study the regularity of the solutions.

The above results are included in the papers [24], [25], [23].

The inviscid limit of the linearly damped and driven NLSE. For U, a, λ and α as in (1), as well as for $T\in(0,\infty)$, we approximate a solution v of the n-dimensional initial/"boundary"-value problem for the NLSE with the pure power non-linearity

$$\begin{cases} i\frac{\partial v}{\partial t} - \operatorname{div}\left(a^{\mathrm{T}}\nabla v\right) + \lambda|v|^{\alpha}v = 0, \text{ in } (0,T)\times U \\ v = v_0, \text{ in } \{t=0\}\times\overline{U} \\ v = 0, \text{ in } [0,T)\times\partial U, \end{cases} \tag{7}$$

by a sequence $\{u_m\}_m$ of solutions of the commonly used in applications initial/"boundary" value problems for the linearly damped and driven NLSE

$$\begin{cases} i\frac{\partial u}{\partial t} - \operatorname{div}\left(a^{\mathrm{T}}\nabla u\right) + \lambda|u|^{\alpha}u + i\gamma u = f, \text{ in } (0,T)\times U \\ u = u_0, \text{ in } \{t=0\}\times\overline{U} \\ u = 0, \text{ in } [0,T)\times\partial U, \end{cases} \quad \text{for } \gamma\in(0,\infty), \tag{8}$$

as $\gamma_m \searrow 0$, $f_m \to 0$ and $u_{0_m} \to v_0$ (the convergences will be rigorously interpreted), after we first study the solvability of the above problems. We also estimate the rate of this approximation when $n=1$. In particular, we extract a sufficient relation between the external force f and the constant of damping γ of the form

$$\|f\| = O(\gamma), \text{ as } \gamma \searrow 0,$$

in order to get the aforementioned approximation results.

The above results are included in the paper [22].

Even though the techniques employed here can be used to deal with additional cases of non-linearities, such as the saturated one, we choose to present the results only for the case of pure power non-linearity. This is due to the fact that this kind of non-linearity is the commonest in applications, and also acts as the model case for every other potential non-linearity.

We also emphasize that we present here an alternative technique for the existence results in the weak sense, in both bounded and unbounded sets, that differs from the classical one of "regularized nonlinearities" presented in [10] (see Theorem 3.3.5 therein). As we show here, this technique not only allows us to rigorously derive an estimate for the energy of the solutions, but it also can be applied to derive regularity results for the solutions. We note that the aforementioned energy estimate is formally obtained (this is what is done in a plethora of published papers), by taking the scalar product with $i\frac{\partial u}{\partial t}$. However, for a weak solution, i.e. a solution with values in $H_0^1(U)$, $\frac{\partial u}{\partial t}$ belongs merely to $H^{-1}(U)$, and thus this practice is not justified.

As far as the regularity of the solutions is concerned, we highlight that the applicability of the above technique passes through the exact determination of the dependence of the elliptic regularity estimates from the properties of an appropriate set in which the second-order elliptic problem is considered. That drives us to review the whole regularity theory for a special (yet quite large) class of appropriate sets, which we call sets with uniformly m-Lipschitz boundaries, for $m \in \mathbb{N}$. This review, along with a presentation of useful, already known and new results from the theory of the Sobolev spaces, is included in the introductory first chapter of the present thesis.

Chapter 1

Preliminaries

1.1 Notation

We start with some notation used throughout the thesis:

1. We write C for any positive constant. Such a constant may be explicitly calculated in terms of known quantities and may change from line to line and also within a certain line in a given computation. We also employ the letter $\mathcal{K}$ for any increasing function $\mathcal{K}:[0,\infty]^m \to (0,\infty]$, for some $m\in\mathbb{N}$.

2. When an element appears as subscript in an other element, the first one denotes that the second one depends on it, while its absence designates either independence or "harmless" dependence. We also apply the classic method of writing an element as a function of another one, in order to denote an dependence. The presence of the subscript $\cdot_w$ to a differential operator for "space"-variables indicates that we consider the operator with the weak (i.e. distributional) sense, while the absence indicates differentiations of the ordinary sense.

3. We write U for any non-empty open $\subseteq\mathbb{R}^n$, as well as J for any non-empty open interval. If, in addition, $0\in J^\circ$ we write J_0 for every such interval. We also define $\mathcal{F}(U)$ to be the space of functions with U for their domain.

4. If $u\in\mathcal{F}(U;\mathbb{C})$ and also every derivative, in some sense $\mathcal{S}$, of the kth order ($k\in\mathbb{N}_0$), i.e. every $D^\alpha_{\mathcal{S}}u$, with $\alpha\in\mathbb{N}_0^n$ and $|\alpha|=k$, exists, then $\nabla^k_{\mathcal{S}}u$ stands for the vector of components those derivatives. Moreover, if $u\in\mathcal{F}(U;\mathbb{C}^m)$, for $m\in\mathbb{N}$, and also every $\nabla_{\mathcal{S}}u_j$, for $j=1,\dots,m$, exists, we then write $J_{\mathcal{S}}u$ for the Jacobian matrix, i.e.
$$J_{\mathcal{S}}u \coloneqq \left(\partial^i_{\mathcal{S}}u_j\right)_{j,i=1}^{j=m,i=n} \in \mathcal{F}(U;\mathbb{C}^{m\times n}),$$
as well as $\det(J_{\mathcal{S}}u)$ for its determinant. The Jacobian matrix is essential for the change of variables formula (see, e.g., Theorem 9.52 in [35]), which plays essential role for us here.

5. For every $m\in\mathbb{N}$, $X^m(U)$ stands for the Zhidkov space over U, defined as the Banach space
$$X^m(U)\coloneqq\left\{u\in L^\infty(U)\,\middle|\,\nabla^k_w u\in L^2(U),\ \text{for } k=1,\dots,m\right\},$$
equipped with its natural norm. A typical example is $\tanh\in\bigcap_{m=1}^\infty X^m(\mathbb{R})$. We note that the first version of such spaces over $\mathbb{R}$ is introduced in [51] and a generalization for higher dimensions (along with certain modifications) is done in [52], [18], [20] and [19]. Here, we consider X^m over any U.

6. Following the notation of, e.g., [15] and [44], if $u: J\times U\to\mathbb{C}$, with $u(t,\cdot)\in\mathcal{F}(U)$ for each $t\in J$, then we associate with u the mapping $\mathbf{u}: J\to\mathcal{F}(U)$, defined by $[\mathbf{u}(t)](x)\coloneqq u(t,x)$, for every $x\in U$ and $t\in J$. For the weak derivative (when it exists) of the "time"-variable of a function-space-valued function $\mathbf{u}$, we simply write $\mathbf{u}'$.

1.2 Definitions and basics

Here, we critically review some useful, already known definitions and results, and we provide new ones.

1.2.1 Second-order, symmetric, uniformly elliptic operators

The characterization "uniformly" is used in [15]. Other adverbs also used in the bibliography are, e.g., "strictly" in [26] and "strongly" in [38].

Definition 1.2.1. *For* $a=(a_{ij})_{i,j=1}^n\in L^\infty(U)$ *satisfying (4) and (5), we write*

$$\mathcal{L}_w=\mathcal{L}_w(a,\theta):\left\{u\in L^p(U) \text{ for some } p\in[1,\infty]\,\middle|\,\nabla_w u\in L^2(U)\right\}\to H^{-1}(U)$$

for the linear and bounded operator

$$\langle\mathcal{L}_w u,v\rangle:=\int_U \nabla_w v\cdot a\nabla_w\bar{u}dx=\int_U\sum_{i,j=1}^n a_{ij}\left(\partial_w^i\bar{u}\right)\left(\partial_w^j v\right)dx,$$

$$\text{for every } u\in\left\{u\in L^p(U),\ \text{for some } p\in[1,\infty]\,\middle|\,\nabla_w u\in L^2(U)\right\},\ \text{for every } v\in H_0^1(U).$$

Moreover, we set

$$\mathcal{L}:\left\{u\in L^1_{\text{loc}}(U)\,\middle|\,\nabla_w u\in L^2(U)\right\}^2\to\mathbb{R}$$

for the double-entry form

$$\mathcal{L}[u,v]:=\operatorname{Re}\left(\int_U\nabla_w v\cdot a\nabla_w\bar{u}dx\right)=\operatorname{Re}\left(\int_U\sum_{i,j=1}^n a_{ij}\left(\partial_w^i\bar{u}\right)\left(\partial_w^j v\right)dx\right),$$

$$\text{for every } u,v\in\left\{u\in L^1_{\text{loc}}(U)\,\middle|\,\nabla_w u\in L^2(U)\right\}.$$

Additionally, if $a\in W^{1,\infty}(U)$ *we define*

$$L_w=L_w(a,\theta):\left\{u\in L^1_{\text{loc}}(U)\,\middle|\,\nabla_w^j u\in L^2(U),\ \text{for } j=1,2\right\}\to L^2(U)$$

for the linear operator

$$L_w u:=-\operatorname{div}_w\left(a^{\mathrm{T}}\nabla_w u\right)=\sum_{i,j=1}^n\partial_w^j\left(a_{ji}\left(\partial_w^i u\right)\right),$$

$$\text{for every } u\in\left\{u\in L^1_{\text{loc}}(U)\,\middle|\,\nabla_w^j u\in L^2(U),\ \text{for } j=1,2\right\}.$$

Definition 1.2.2. *For every* $m\in\mathbb{N}_0$ *and* U*, we consider that the space* $H^m(U)\equiv W^{m,2}(U)$ *is equipped with the inner product* $(*,\star)_{H^m(U)}\to\mathbb{C}$ *defined as*

$$(u,v)_{H^m(U)}:=\sum_{0\le|\alpha|\le m}\int_U\left(D_w^\alpha u\right)\left(D_w^\alpha\bar{v}\right)dx,\ \forall\,u,v\in H^m(U).$$

When $m=0$*, we simply write* $(*,\star):=(*,\star)_{H^0(U)}\equiv(*,\star)_{L^2(U)}$.

Remark 1.2.1. *As we will find out below (see, e.g., Lemma 2.3.1), it would be more practical to define the inner product in Definition 1.2.2 as it is done in [38], i.e.*

$$(u,v)_{H^m(U)}:=\sum_{0\le|\alpha|\le m}\int_U\left(D_w^\alpha\bar{u}\right)\left(D_w^\alpha v\right)dx,\ \forall\,u,v\in H^m(U),$$

in order to keep the notation between the real case and the the complex one consistent. However, we avoid doing so, because it is not the commonest practice.

Definition 1.2.3. *We write*

$$\{U_{\mathrm{P}}\}:=\left\{U \text{ satisfies the criterion for the validity of the Poincaré inequality for } H_0^1(U)\right\}.$$

We recall that the Poincaré inequality for the space $H_0^1(U)$ *for some* U *(see, e.g. Theorem 13.19 in [35], or Theorem, Paragraph 6.30 in [1]) implies that there exists* $C=C_U$ *such that*

$$\|u\|_{H^1(U)}\le C\|\nabla_w u\|_{L^2(U)},\ \forall u\in H_0^1(U).$$

Evidently, $C\ge1$*. For every* U_{P}*, we write* $C_{U_{\mathrm{P}}}\ge1$ *for the "smallest" constant of the respective inequality, that is*

$$C_{U_{\mathrm{P}}}:=\inf\left\{C\,\middle|\,\|u\|_{H^1(U)}\le C\|\nabla_w u\|_{L^2(U)},\ \forall u\in H_0^1(U)\right\}\ge1.$$

Proposition 1.2.1. *Let U_{P} be arbitrary. Then, every $\mathcal{L}_w(a,\theta)$ induces an isomorphism from $H_0^1(U)$ onto $H^{-1}(U_{\mathrm{P}})$.*

Proof. Let $\mathcal{L}_w(a,\theta)$ be arbitrary.

Step 1

We write $H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}})$ for the restriction of the vector space $H_0^1(U_{\mathrm{P}})$ ($\equiv H_0^1(U_{\mathrm{P}};\mathbb{C})$) over the field $\mathbb{R}$. We claim that the form $\mathcal{L}[*,\star]$ restricted to $H_0^1(U_{\mathrm{P}})^2$ induces a (real-valued) inner product for $H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}})$. Indeed:

1. $\mathcal{L}[u,v]=\mathcal{L}[v,u]$, for every $u,v\in H_0^1(U_{\mathrm{P}})$: In view of (5), for every such u and v we have

$$\mathcal{L}[u,v] = \mathrm{Re}\left(\int_U \sum_{i,j=1}^n a_{ij}\left(\partial_w^i \bar{u}\right)\left(\partial_w^j v\right)dx\right) =$$

$$= \mathrm{Re}\left(\int_U \sum_{i,j=1}^n \bar{a}_{ij}\left(\partial_w^i u\right)\left(\partial_w^j \bar{v}\right)dx\right) = \mathrm{Re}\left(\int_U \sum_{i,j=1}^n a_{ji}\left(\partial_w^i u\right)\left(\partial_w^j \bar{v}\right)dx\right) =$$

$$= \mathrm{Re}\left(\int_U \sum_{j,i=1}^n a_{ji}\left(\partial_w^j \bar{v}\right)\left(\partial_w^i u\right)dx\right) = \mathcal{L}[v,u].$$

2. The map $\mathcal{L}[\cdot,v] : H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}}) \to \mathbb{R}$ is linear, for every fixed $v\in H_0^1(U_{\mathrm{P}})$: Let such an arbitrary v be fixed. It directly follows that, for every $u_1,u_2\in H_0^1(U_{\mathrm{P}})$ and every $s\in\mathbb{R}$ we have

$$\mathcal{L}[u_1+su_2,v] = \mathcal{L}[u_1,v] + s\mathcal{L}[u_2,v].$$

3. $\mathcal{L}[u,u]>0$, for every $u\in H_0^1(U_{\mathrm{P}})\setminus\{0\}$: In virtue of (4) along with the Poincaré inequality, for every such u we have

$$\mathcal{L}[u,u] \geq \theta\|\nabla_w u\|^2_{L^2(U_{\mathrm{P}})} \geq \mathcal{K}\left(\theta,\frac{1}{C_{U_{\mathrm{P}}}}\right)\|u\|^2_{H^1(U_{\mathrm{P}})} > 0. \tag{1.2.1}$$

We then write

$$\left(H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}}),(\mathcal{L}[\cdot,\cdot])^{\frac{1}{2}}\right),$$

for the respective normed (Banach) space.

Step 2

We fix an arbitrary $f\in H^{-1}(U_{\mathrm{P}})$. Employing a known result concerning the bijective isomerty between the complex dual and the real dual (see, e.g., Theorem 11.22 in [8]), we get that

$$\mathrm{Re}(f)\in\left(H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}})\right)^* \text{ with } \|\mathrm{Re}(f)\|_{\left(H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}})\right)^*} = \|f\|_{H^{-1}(U_{\mathrm{P}})}.$$

Appling the real version of Riesz-Fréchet representation theorem (see, e.g., Proposition 5.5 in [8]) to the linear and bounded functional $\mathrm{Re}(f)$ we get a unique $u\in H_0^1(U_{\mathrm{P}})$, such that

$$\mathrm{Re}(\langle f,v\rangle) = \mathcal{L}[u,v] = \mathrm{Re}(\langle \mathcal{L}_w u,v\rangle),\ \text{for every } v\in H_0^1(U_{\mathrm{P}}) \text{ (in view of (5))} \tag{1.2.2}$$

and also

$$(\mathcal{L}[u,u])^{\frac{1}{2}} = \|\mathrm{Re}(f)\|_{\left(H^1_{0_{\mathbb{R}}}(U_{\mathrm{P}})\right)^*} = \|f\|_{H^{-1}(U_{\mathrm{P}})}. \tag{1.2.3}$$

Setting iv instead of v in (1.2.2), we get

$$\mathrm{Im}(\langle f,v\rangle) = \mathrm{Im}(\langle \mathcal{L}_w u,v\rangle),\ \text{for every } v\in H_0^1(U_{\mathrm{P}}). \tag{1.2.4}$$

Combining (1.2.2) and (1.2.4), we deduce that $f\equiv\mathcal{L}_w u$. Hence, from the arbitrariness of f and the uniqueness of u we deduce that $\mathcal{L}_w : H_0^1(U_{\mathrm{P}}) \to H^{-1}(U_{\mathrm{P}})$ is bijective. Moreover, from (1.2.3) along with (1.2.1), we have, for every $(u,f)\in H_0^1(U_{\mathrm{P}})\times H^{-1}(U_{\mathrm{P}})$ such that $\mathcal{L}_w u = f$, that

$$\|f\|_{H^{-1}(U_{\mathrm{P}})} \leq \mathcal{K}\left(\|a\|_{L^\infty(U_{\mathrm{P}})}\right)\|u\|_{H_0^1(U_{\mathrm{P}})} \text{ and } \|u\|_{H^1(U_{\mathrm{P}})} \leq \mathcal{K}\left(\frac{1}{\theta},C_{U_{\mathrm{P}}}\right)\|f\|_{H^{-1}(U_{\mathrm{P}})}.$$

It follows that both linear operators, $\mathcal{L}_w : H_0^1(U_{\mathrm{P}}) \to H^{-1}(U_{\mathrm{P}})$ and its inverse, are continuous, and the proof is complete.

□

1.2.2 Restriction and extension-by-zero operators

We begin with a definition.

Definition 1.2.4. *For every $U_1 \subseteq U_2$, we write*

$$\mathcal{R}(U_2,U_1) : \mathcal{F}(U_2) \to \mathcal{F}(U_1)$$

for the (linear) restriction-to-U_1 operator, i.e.

$$[(\mathcal{R}(U_2,U_1))\,u]\,(x) \coloneqq u(x),\ \ \forall x \in U_1,\ \ \forall u \in \mathcal{F}(U_2)$$

and also

$$\mathcal{E}_0(U_1,U_2) : \mathcal{F}(U_1) \to \mathcal{F}(U_2)$$

for the (linear) extension-by-zero-to-U_2 operator, i.e.

$$[(\mathcal{E}_0(U_1,U_2))\,u]\,(x) \coloneqq \begin{cases} u(x), & \text{if } x \in U_1 \\ 0, & \text{if } x \in U_2 \setminus U_1, \end{cases} \quad \forall u \in \mathcal{F}(U_1).$$

We further define

$$(\mathcal{R}(U_2,U_1))\,(\mathcal{F}(U_2)) \coloneqq \left\{(\mathcal{R}(U_2,U_1))\,u \,\middle|\, u \in \mathcal{F}(U_2)\right\}$$

and

$$(\mathcal{E}_0(U_1,U_2))\,(\mathcal{F}(U_1)) \coloneqq \left\{(\mathcal{E}_0(U_1,U_2))\,u \,\middle|\, u \in \mathcal{F}(U_1)\right\}.$$

For convenience, in this work we follow the common convention and we use the restriction operators without write them down, for the cases where this practice does not cause any confusion. The following result is basic.

Proposition 1.2.2. *Let $m \in \mathbb{N}_0$, $p \in [1,\infty]$ and $U_1 \subseteq U_2$ be arbitrary. Then $\mathcal{R}(U_2,U_1)$ restricted to $W^{m,p}(U_2)$ maps isometrically into (not onto) $W^{m,p}(U_1)$, with*

$$\begin{gathered}(D_w^\alpha \circ (\mathcal{R}(U_2,U_1)))\,u = ((\mathcal{R}(U_2,U_1)) \circ D_w^\alpha)\,u,\ \ \text{a.e. in } U_1, \\ \text{for every } \alpha \in \mathbb{N}_0^n \text{ with } 0 \le |\alpha| \le m,\end{gathered} \tag{1.2.5}$$

for every $u \in W^{m,p}(U_2)$. Hence, $W^{m,p}(U_2) \hookrightarrow (\mathcal{R}(U_2,U_1))\,(W^{m,p}(U_2))$, if we consider the right-hand space as a normed space equipped with its natural norm.

Proof. Let $u \in W^{m,p}(U_1)$ be arbitrary. Evidently,

$$(((\mathcal{R}(U_2,U_1)) \circ D_w^\alpha)\,u) \in L^p(U_1) \text{ with } \|((\mathcal{R}(U_2,U_1)) \circ D_w^\alpha)\,u\|_{L^p(U_1)} \le \|D_w^\alpha u\|_{L^p(U_2)},$$

for every $\alpha \in \mathbb{N}_0^n$ with $0 \le |\alpha| \le m$. It is only left for us to show (1.2.5) by the definition of the weak derivatives. For every $\psi \in C_c^\infty(U_1)$ and every α as above, we have from

1. the fact that $(D^\alpha \circ (\mathcal{E}_0(U_1,U_2)))\,\psi = ((\mathcal{E}_0(U_1,U_2)) \circ D^\alpha)\,\psi$ everywhere (in U_2), which is direct consequence of the point-wise definition of $\mathcal{E}_0(U_1,U_2)$,

2. the definition of the weak derivatives,

that

$$\begin{aligned}\int_{U_1} ((\mathcal{R}(U_2,U_1))\,u)\,D^\alpha\psi dx &= \int_{U_2} u\,(((\mathcal{E}_0(U_1,U_2)) \circ D^\alpha)\,\psi) dx \overset{1.}{=} \\ \overset{1.}{=} \int_{U_2} u\,((D^\alpha \circ (\mathcal{E}_0(U_1,U_2)))\,\psi) dx &\overset{2.}{=} (-1)^{|\alpha|} \int_{U_2} (D_w^\alpha u)\,((\mathcal{E}_0(U_1,U_2))\,\psi) dx = \\ &= (-1)^{|\alpha|} \int_{U_1} (((\mathcal{R}(U_2,U_1)) \circ D_w^\alpha)\,u)\,\psi dx,\end{aligned}$$

which is the desired result. □

Moreover, in the bibliography the operator $\mathcal{E}_0$ is typically considered for the case $U_2 \equiv \mathbb{R}^n$. Here, we generalize an already known result (see, e.g., Lemma, Paragraph 3.27 in [1]) concerning $\mathcal{E}_0$ restricted to $W_0^{m,p}$-spaces, for every $m \in \mathbb{N}_0$ and every $p \in [1,\infty]$. Apropos the $W_0^{m,p}$-spaces, we first make a note about them, before we state and prove the aforementioned result.

Remark 1.2.2. *We employ the definition*

$$W_0^{0,p}(U) \coloneqq L^p(U) \equiv W^{0,p}(U), \ \forall p \in [1,\infty],$$

which makes sense, since $C_c^\infty(U)$ is dense in $L^p(U)$ with respect to the strong topology, for every $p \in [1,\infty)$, as well as $C_c^\infty(U)$ is dense in $L^p(U)$ with repsect to the weak topology, for every $p \in (1,\infty]$. Of course, the analogous conclusions are true for the $W_0^{m,p}$-spaces (see also Remark 11.15 in [35]).*

Lemma 1.2.1. *Let $U_1 \subset\subset U_2$. Then there exists an open and bounded U such that $U_1 \subset\subset U \subset\subset U_2$ with ∂U being Lipschitz continuous (see, e.g., Definition 9.57 in [35]).*

Proof. We set

$$\delta \coloneqq \frac{\operatorname{dist}(\overline{U_1}, \partial U_2)}{2} > 0$$

and we consider the open cover

$$\{B(x,\delta)\}_{x \in \partial U_1}$$

of ∂U_1. Since ∂U_1 is compact there exists $m \in \mathbb{N}$ and $\{x_j\}_{j=1}^m \subset \partial U_1$ such that

$$\{B(x_j,\delta)\}_{j=1}^m$$

is also an open cover of ∂U_1. Setting

$$U \coloneqq U_1 \cup \bigcup_{j=1}^m B(x_j,\delta),$$

it is direct to check that U has the desired properties. □

Proposition 1.2.3. *Let $m \in \mathbb{N}_0$, $p \in [1,\infty]$ and $U_1 \subseteq U_2$ be arbitrary. Then $\mathcal{E}_0(U_1,U_2)$ restricted to $W_0^{m,p}(U_1)$ maps isometrically into (not onto) $W_0^{m,p}(U_2)$, with*

$$(D_w^\alpha \circ (\mathcal{E}_0(U_1,U_2)))\, u = ((\mathcal{E}_0(U_1,U_2)) \circ D_w^\alpha)\, u, \ \text{a.e. in } U_2, \ \text{for every } \alpha \in \mathbb{N}_0^n \text{ with } 0 \le |\alpha| \le m, \tag{1.2.6}$$

for every $u \in W_0^{m,p}(U_1)$. Hence, $W_0^{m,p}(U_1) \hookrightarrow (\mathcal{E}_0(U_1,U_2))\,(W_0^{m,p}(U_1))$, if we consider the right-hand space as a normed space equipped with its natural norm.

Proof. Let $u \in W_0^{m,p}(U_1)$ be arbitrary and $\{u_k\}_k \subset C_c^\infty(U_1)$ be such that

$$\begin{cases} u_k \to u \text{ in } W^{m,p}(U_1), & \text{if } p \in [1,\infty) \\ u_k \overset{*}{\rightharpoonup} u \text{ in } W^{m,p}(U_1), & \text{if } p = \infty. \end{cases}$$

Evidently,

$$(((\mathcal{E}_0(U_1,U_2)) \circ D_w^\alpha)\, u) \in L^p(U_2) \text{ with } \|((\mathcal{E}_0(U_1,U_2)) \circ D_w^\alpha)\, u\|_{L^p(U_2)} = \|D_w^\alpha u\|_{L^p(U_1)}, \tag{1.2.7}$$

for every $\alpha \in \mathbb{N}_0^n$ with $0 \le |\alpha| \le m$. Moreover, for every α as before, we easily deduce that

$$\int_{U_1} D^\alpha u_k\, v dx \to \int_{U_1} D_w^\alpha u\, v dx, \ \forall v \in L^{\frac{p}{p-1}}(U_1). \tag{1.2.8}$$

Indeed, a direct a way to see this for the case $p \in [1,\infty)$ is by employing

1. the Hölder inequality for $p_1 = p$ and $p_2 = \frac{p}{p-1}$ and
2. the convergence $u_k \to u$ in $W^{m,p}(U_1)$,

in order to get

$$\int_{U_1} (D^\alpha u_k - D_w^\alpha u)\, v dx \overset{1.}{\leq} \|D^\alpha u_k - D_w^\alpha u\|_{L^p(U_1)} \|v\|_{L^{\frac{p}{p-1}}(U_1)} =$$
$$= \|u_k - u\|_{W^{m,p}(U_1)} \|v\|_{L^{\frac{p}{p-1}}(U_1)} \overset{2.}{\to} 0.$$

For the case $p=\infty$, (1.2.8) follows directly from the definition of the weak* convergence. Now, let $\psi \in C_c^\infty(U_2)$ be arbitrary. For every k we fix an open and bounded set $V_k = V_k(\psi, u_k)$ such that[1]

$$(\mathrm{supp}(\psi) \cap \mathrm{supp}(u_k))^\circ = \mathrm{supp}(\psi)^\circ \cap \mathrm{supp}(u_k)^\circ \subset\subset V_k \subset\subset U_1 \cap U_2 = U_1 \subseteq U_2,$$

with ∂V_k being Lipschitz continuous for every k, as Lemma 1.2.1 provides. Hence, from

1. (1.2.8) and
2. the common integration by parts formula (see, e.g., Corollary 9.66 in [35]), applied as many times as needed,

we get, for every α as above, that

$$\int_{U_2} ((\mathcal{E}_0(U_1,U_2))\, u)\ D^\alpha \psi dx = \int_{U_1} u\, D^\alpha \psi dx \overset{1.}{=} \lim_{k \nearrow \infty} \int_{U_1} u_k\ D^\alpha \psi dx =$$
$$= \lim_{k \nearrow \infty} \int_{V_k} u_k\ D^\alpha \psi dx \overset{2.}{=} (-1)^{|\alpha|} \lim_{k \nearrow \infty} \int_{V_k} (D^\alpha u_k)\, \psi dx = (-1)^{|\alpha|} \lim_{k \nearrow \infty} \int_{U_1} (D^\alpha u_k)\, \psi dx \overset{1.}{=}$$
$$\overset{1.}{=} (-1)^{|\alpha|} \int_{U_1} (D_w^\alpha u)\, \psi dx = (-1)^{|\alpha|} \int_{U_2} (((\mathcal{E}_0(U_1,U_2)) \circ D_w^\alpha)\, u)\ \psi dx,$$

thus, we derive the validity of (1.2.6) by the definition of the weak derivatives, since ψ is arbitrary . Therefore, from (1.2.7) we get that

$$(\mathcal{E}_0(U_1,U_2))\, u \in W^{m,p}(U_2) \text{ with } \|(\mathcal{E}_0(U_1,U_2))\, u\|_{W^{m,p}(U_2)} = \|u\|_{W^{m,p}(U_1)}.$$

It is only left to show that $((\mathcal{E}_0(U_1,U_2))\, u) \in W_0^{m,p}(U_2)$. This follows directly from the evident fact that

$$\{(\mathcal{E}_0(U_1,U_2))\, u_k\}_k \subset C_c^\infty(U_2), \text{ and } (\mathcal{E}_0(U_1,U_2))\, u_k \to (\mathcal{E}_0(U_1,U_2))\, u \text{ in } W^{m,p}(U_2).$$

along with the application of the definition of the $W_0^{m,p}$-spaces. □

A direct consequence of Proposition 1.2.3 is the following extension of the definition of the restriction operators to the duals of $W_0^{m,p}$-spaces.

Definition 1.2.5. *For every $m \in \mathbb{N}_0$, $p \in [1,\infty]$ and $U_1 \subseteq U_2$, we define*

$$\mathcal{R}(U_2,U_1) : W^{-m,p}(U_2) \to W^{-m,p}(U_1)$$

by

$$\langle (\mathcal{R}(U_2,U_1))\, f, u \rangle \coloneqq \langle f, (\mathcal{E}_0(U_1,U_2))\, u \rangle\,, \ \ \forall u \in H_0^1(U_2),\ \ \forall f \in W^{-m,p}(U_2).$$

Evidently,

$$\|(\mathcal{R}(U_2,U_1))\, f\|_{W^{-m,p}(U_1)} \leq \|f\|_{W^{-m,p}(U_1)},\ \ \forall f \in W^{-m,p}(U_2),$$

hence, $W^{-m,p}(U_2) \hookrightarrow (\mathcal{R}(U_2,U_1))\,(W^{-m,p}(U_2))$, if we consider the right-hand space as a normed space equipped with its natural norm.

Proposition 1.2.4. *Let $m \in \mathbb{N}_0$, $p \in [1,\infty)$, U and $f_1, f_2 \in W^{-m,p}(U)$. If*

$$(\mathcal{R}(U,V))\, f_1 \equiv (\mathcal{R}(U,V))\, f_2, \text{ for every open } V \subset\subset U \text{ with } \partial V \text{ being Lipschitz continuous,}$$

then $f_1 \equiv f_2$.

[1] We recall that $\mathrm{supp}(u) \coloneqq \overline{\{x \in U \mid u(x) \neq 0\}}$ for every $u \in \mathcal{F}(U)$.

Proof. Let $v\in W_0^{m,p}(U)$ be arbitrary and fix a $\{v_k\}_k\subset C_c^\infty(U)$ such that $v_k\to v$ in $W^{m,p}(U)$. Employing Lemma 1.2.1, for every k we consider open $V_k\subset\subset U$ such that $\operatorname{supp}(v_k)\subset V_k$ and ∂V_k is Lipschitz continuous. Evidently,

$$((\mathcal{R}(U,V_k))\,v_k)\in C_c^\infty(V_k) \text{ and also } ((\mathcal{E}_0\,(V_k,U))\circ(\mathcal{R}(U,V_k)))\,v_k=v_k, \text{ for every } k.$$

Hence, for every k we have

$$\langle(\mathcal{R}(U,V_k))\,f_1,(\mathcal{R}(U,V_k))\,v_k\rangle=\langle(\mathcal{R}(U,V_k))\,f_2,(\mathcal{R}(U,V_k))\,v_k\rangle\Rightarrow$$
$$\Rightarrow\langle f_1,((\mathcal{E}_0\,(V_k,U))\circ(\mathcal{R}(U,V_k)))\,v_k\rangle=\langle f_2,((\mathcal{E}_0\,(V_k,U))\circ(\mathcal{R}(U,V_k)))\,v_k\rangle\Rightarrow\langle f_1,v_k\rangle=\langle f_2,v_k\rangle$$

and the result follows by letting $k\nearrow\infty$, since the convergence in the strong topology implies the convergence in the weak topology, and by the arbitrariness of v. □

1.2.3 Uniformly Lipschitz boundaries

Here, we distinguish certain subsets of the Euclidean space. For the next already known definition (see, e.g., Definition 13.11 in [35]), we recall that

1. $y=\Phi(x)\in\mathbb{R}^n$ are local coordinates (in this case, $x\in\mathbb{R}^n$ are the background coordinates) when Φ is a rigid motion, i.e. an affine transformation of the form $\Phi(x)=x_0+ax$, where $x_0\in\mathbb{R}^n$ and $a\in\mathbb{R}^{n\times n}$ being orthogonal,
2. $f(\bigcup_{i\in\mathcal{I}}U_i)=\bigcup_{i\in\mathcal{I}}f(U_i)$ and $f(\bigcap_{i\in\mathcal{I}}U_i)=\bigcap_{i\in\mathcal{I}}f(U_i)$, for every bijective f,
3. every function $f:\varnothing\to\mathbb{R}$ is just a real constant and
4. x' stands for the $(n-1)$-dimensional vector obtained by removing the n-th component of a given n-dimensional vector x, i.e. $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$.

Definition 1.2.6. *Let $\varepsilon\in(0,\infty]$, $K\in\mathbb{N}$, $L\in[0,\infty)$ and U. We say that ∂U is uniformly Lipschitz of constants ε, K, L and we write $\partial U\in\operatorname{Lip}(\varepsilon,K,L)$ if there exists a locally finite countable open cover $\{U_k\}_k$ of ∂U, such that*

1. *if $x\in\partial U$, then $B(x,\varepsilon)\subseteq U_k$ for some $k\in\mathbb{N}$,*
2. *every collection of $K+1$ of U_k's has empty intersection and*
3. *for every k there exist local coordinates $y_k=\Phi_k(x)$ and a function $\gamma_k:\mathbb{R}^{n-1}\to\mathbb{R}$, such that*
   - *i. γ_k is Lipschitz continuous with $\operatorname{Lip}(\gamma_k)\le L$, uniformly for every k and*
   - *ii. $\Phi_k(U_k\cap U)\,(=\Phi_k(U_k)\cap\Phi_k(U))=\Phi_k(U_k)\cap\{y_k\in\mathbb{R}^n\,|\,y_{n_k}>\gamma_k(y_k{}')\}$.*

Remark 1.2.3. *We recall that, in view of the Rademacher theorem (see, e.g., Theorem 9.14 in [35]), the Lipschitz continuity of every γ_k in the above definition implies that $\nabla\gamma_k$ exists a.e. and, in particular, we can check that $\operatorname{Lip}(\gamma_k)\le L$ implies*[2]

$$\|\nabla\gamma_k\|_{L^\infty(\mathbb{R}^{n-1})}\le L.$$

Indeed, if $y_0\in\mathbb{R}^{n-1}$ is arbitrary, then

$$\lim_{h\searrow0}\frac{\gamma_k(y+hy_0)-\gamma_k(y)-hy_0\cdot\nabla\gamma(y)}{h}=0,\ \textit{for a.e. } y\in\mathbb{R}^{n-1},$$

thus

$$y_0\cdot\nabla\gamma(y)=\lim_{h\searrow0}\frac{\gamma_k(y+hy_0)-\gamma_k(y)}{h}\le L\,|y_0|\,,\ \textit{for a.e. } y\in\mathbb{R}^{n-1}$$

and so, for every y that the above bound holds, we choose $y_0=\nabla\gamma_k(y)$ to get the result.

[2] By the completeness of the Lebesgue measure, we do not mind whether a function is defined in a null set or not, that is why we are allowed to consider that $\nabla\gamma_k\in L^\infty(\mathbb{R}^{n-1})$.

Remark 1.2.4. *For every $U \subseteq \mathbb{R}^n$ such that ∂U is bounded, we have the following equivalence: $\partial U \in \mathrm{Lip}(\varepsilon, K, L)$ if and only if ∂U is Lipschitz continuous (see, e.g., Exercise 13.13 in [35]). We note that we have already used the Lipschitz continuous boundaries in Subsection 1.2.2.*

We note that the uniformly Lipschitz boundaries are also known as "boundaries of minimally smooth domains" (see Section 3.3, Chapter VI in [41]) or "boundaries of domains that satisfy the strong local Lipschitz condition" (see Paragraph 4.9 in [1]). In any case, for those boundaries we have the following well known result (see, e.g., Theorem 13.17 in [35]), concerning the Stein total extension operator (see Paragraph 5.17 in [1] for the definition of such an operator).

Theorem 1.2.1. *Let U with $\partial U \in \mathrm{Lip}(\varepsilon, K, L)$. Then there exists a linear extension operator*

$$\mathcal{E}(U, \mathbb{R}^n) : W^{m,p}(U) \to W^{m,p}(\mathbb{R}^n), \ \forall m \in \mathbb{N}_0, \ \forall p \in [1, \infty],$$

such that, for every $m \in \mathbb{N}_0$, every $p \in [1, \infty]$ and every $u \in W^{m,p}(U)$, we have

$$\|(\mathcal{E}(U, \mathbb{R}^n))\, u\|_{L^p(\mathbb{R}^n)} \leq \mathcal{K}(K) \|u\|_{L^p(U)} \ \text{and}$$
$$\left\|\left(\nabla_w^k \circ (\mathcal{E}(U, \mathbb{R}^n))\right) u\right\|_{L^p(\mathbb{R}^n)} \leq \mathcal{K}(K, L) \sum_{j=0}^{k} \frac{1}{\varepsilon^{k-j}} \left\|\nabla_w^j u\right\|_{L^p(U)}, \ \text{for every } k = 1, \ldots, m, \ \text{if } m \neq 0.$$

Hence, we can write that $W^{m,p}(U) \hookrightarrow (\mathcal{E}(U, \mathbb{R}^n))\,(W^{m,p}(U))$, if we consider a notation similar to Theorem's 1.2.4 and the right-hand space as a normed space equipped with its natural norm.

1.2.4 The continuous Sobolev embeddings

In this subsection, we review the classic Sobolev embeddings. The following result is well known (see, e.g., Corollary 9.13 in [8]).

Theorem 1.2.2. *Let $m \in \mathbb{N}$ and $p \in [1, \infty)$. We have*

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \ \text{for every } q \in \left[p, \frac{np}{n - mp}\right], \ \text{if } n > mp,$$

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \ \text{for every } q \in [p, \infty), \ \text{if } n = mp,$$

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n), \ \text{if } n < mp.$$

In particular, for the case $n < mp$ we have (see, e.g., Paragraph 1.29 in [1] for the definition of the Hölder spaces)

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow C^{\lfloor m - \frac{n}{p} \rfloor, \gamma}\left(\overline{\mathbb{R}^n}\right) \cap C^{\lfloor m - \frac{n}{p} \rfloor - 1, 1}\left(\overline{\mathbb{R}^n}\right), \ \text{for} \ \begin{cases} \gamma = m - \frac{n}{p} - \lfloor m - \frac{n}{p} \rfloor, & \text{if } \left(m - \frac{n}{p}\right) \notin \mathbb{N} \\ \forall \gamma \in (0, 1), & \text{if } \left(m - \frac{n}{p}\right) \in \mathbb{N}, \end{cases}$$

where the above embedding is to be understood modulo the choice of a smooth enough representative.

In view of Proposition 1.2.3, we get a direct consequence of Theorem 1.2.2.

Corollary 1.2.1. *Let $m \in \mathbb{N}$, $p \in [1, \infty)$ and U. We have for every open $V \subseteq U$ that (see Definition 1.2.4)*

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U, V))\,(L^q(U)), \ \text{for every } q \in \left[p, \frac{np}{n - mp}\right], \ \text{if } n > mp,$$

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U, V))\,(L^q(U)), \ \text{for every } q \in [p, \infty), \ \text{if } n = mp,$$

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U, V))\,(L^\infty(U)), \ \text{if } n < mp.$$

In particular, for the case $n<mp$ we have (see, e.g., Paragraph 1.29 in [1] for the definition of the Hölder spaces)

$$W_0^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor,\gamma}(\overline{U})\right)\cap(\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor-1,1}(\overline{U})\right),$$
$$\text{for } \begin{cases}\gamma=m-\frac{n}{p}-\lfloor m-\frac{n}{p}\rfloor, & \text{if } \left(m-\frac{n}{p}\right)\notin\mathbb{N}\\ \forall\gamma\in(0,1), & \text{if } \left(m-\frac{n}{p}\right)\in\mathbb{N},\end{cases}$$

where the right-hand space is considered as a normed space equipped with its natural norm.

All of the above embeddings are scaling invariant, that is the constants of the respective inequalities are uniform, i.e. independent of U. The embeddings are also independent of the choice of V.

Proof. In view of the the evident, scaling invariant embedding $L^p(U) \hookrightarrow (\mathcal{R}(U,V))(L^p(U))$ for every $V \subseteq U$ (Proposition 1.2.2 provides us with a more general and less standard embedding), it suffices to combine Proposition 1.2.3 for $U_1 \equiv U$ and $U_2 \equiv \mathbb{R}^n$ with Theorem 1.2.2. □

Moreover, in view of Theorem 1.2.1, another direct consequence of Theorem 1.2.2 follows.

Corollary 1.2.2. *Let $m\in\mathbb{N}$, $p\in[1,\infty)$ and U with $\partial U\in \mathrm{Lip}(\varepsilon,K,L)$. We have for every open $V\subseteq U$ that*

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q\in\left[p,\frac{np}{n-mp}\right], \text{ if } n>mp,$$

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q\in[p,\infty), \text{ if } n=mp,$$

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))(L^\infty(U)), \text{ if } n<mp.$$

In particular, for the case $n<mp$ we have

$$W^{m,p}(U) \hookrightarrow (\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor,\gamma}(\overline{U})\right)\cap(\mathcal{R}(U,V))\left(C^{\lfloor m-\frac{n}{p}\rfloor-1,1}(\overline{U})\right),$$
$$\text{for } \begin{cases}\gamma=m-\frac{n}{p}-\lfloor m-\frac{n}{p}\rfloor, & \text{if } \left(m-\frac{n}{p}\right)\notin\mathbb{N}\\ \forall\gamma\in(0,1), & \text{if } \left(m-\frac{n}{p}\right)\in\mathbb{N}.\end{cases}$$

All of the above embeddings are scaling dependent, that is the constants of the respective inequalities depend (increasingly) on $\frac{1}{\varepsilon}$, K and L, yet they are independent of the choice of V.

1.2.5 The compact Rellich-Kondrachov embeddings

Here, we provide useful versions of the well known Rellich-Kondrachov compactness theorem. For convenience, we consider only the case $m=1$, since this is the one that we use here.

Proposition 1.2.5. *Let $m\in\mathbb{N}$, $p\in[1,\infty)$ and U. We have for every open $V\subseteq U$ that*

$$W_0^{1,p}(U) \hookrightarrow\hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q\in\left[1,\frac{np}{n-p}\right), \text{ if } n>p \text{ and } |V|<\infty,$$

$$W_0^{1,p}(U) \hookrightarrow\hookrightarrow (\mathcal{R}(U,V))(L^q(U)), \text{ for every } q\in[1,\infty), \text{ if } n=p \text{ and } |V|<\infty,$$

$$W_0^{1,p}(U) \hookrightarrow\hookrightarrow (\mathcal{R}(U,V))\left(C(\overline{U})\right), \text{ if } n<p \text{ and } V \text{ is bounded.}$$

In any case, $W_0^{1,p}(U) \hookrightarrow\hookrightarrow (\mathcal{R}(U,V))(L^p(U))$ for every bounded $V\subseteq U$.

All of the above embeddings are scaling invariant, that is the constants of the respective inequalities are uniform, i.e. independent of U. The embeddings are also independent of the choice of V.

Proof. The case $n>p$ follows directly from Corollary 1.2.1 along with the Ascoli-Arzelá theorem. The case $n=p$ reduces to the case $n>p$, since $|V|<\infty$. As for the case $n>p$, we deal exactly as in the proof of Theorem 12.18, minding to employ Proposition 1.2.3 for the extension to $W^{1,\frac{np}{n-p}}(\mathbb{R}^n)$. □

Employing Corollary 1.2.2 and Theorem 1.2.1 this time, instead of Corollary 1.2.1 and Proposition 1.2.3, respectively, we get the following result.

Proposition 1.2.6. *Let $p\in[1,\infty)$ and U with $\partial U\in \mathrm{Lip}(\varepsilon,K,L)$. We have for every open $V\subseteq U$ that*

$$W^{1,p}(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\,(L^q(U)),\ \textit{for every } q\in\left[1,\frac{np}{n-p}\right),\ \textit{if } n>p \textit{ and } |V|<\infty,$$

$$W^{1,p}(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\,(L^q(U)),\ \textit{for every } q\in[1,\infty),\ \textit{if } n=p \textit{ and } |V|<\infty,$$

$$W^{1,p}(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\left(C\left(\overline{U}\right)\right),\ \textit{if } n<p \textit{ and } V \textit{ is bounded.}$$

In any case, $W^{1,p}(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\,(L^p(U))$ for every bounded $V\subseteq U$.

All of the above embeddings are scaling dependent, that is the constants of the respective inequalities depend (increasingly) on $\frac{1}{\varepsilon}$, K and L, yet they are independent of the choice of V.

1.2.6 Uniformly m-Lipschitz boundaries

In Section 1.2.12, we need to impose a further assumption concerning the regularity of the uniformly Lipschitz boundaries, in order to get the regularity results of the solutions of the second-order elliptic problems.

Definition 1.2.7. *Let $m\in\mathbb{N}$, $\varepsilon\in(0,\infty]$, $K\in\mathbb{N}$, $L\in[0,\infty)$ and U. We say that ∂U is uniformly m-Lipschitz of constants ε, K, L and we write $\partial U\in \mathrm{Lip}^m(\varepsilon,K,L)$ if there exists a locally finite countable open cover $\{U_k\}_k$ of ∂U, such that*

1. *if $x\in\partial U$, then $B(x,\varepsilon)\subseteq U_k$ for some $k\in\mathbb{N}$,*
2. *every collection of $K+1$ of U_k's has empty intersection and*
3. *for every k there exist local coordinates $y_k=\Phi_k(x)$ and a function $\gamma_k:\mathbb{R}^{n-1}\to\mathbb{R}$, such that*
   - *i. $\nabla^{j-1}\gamma_k$ is (globally) Lipschitz continuous, for every $j=1,\ldots,m$ and every k, with*
   $$\max_{j=1,\ldots,m}\left\{\mathrm{Lip}\left(\nabla^{j-1}\gamma_k\right)\right\}\le L,\ \textit{uniformly for every } k,$$
   and
   - *ii. $\Phi_k(U_k\cap U)=\Phi_k(U_k)\cap\left\{y_k\in\mathbb{R}^n\,\middle|\,y_{n_k}>\gamma_k(y_k{}')\right\}$.*

Remark 1.2.5. *We do not assume in Definition 1.2.6 that $\{\gamma_k\}_k$ is a subset of $C^{0,1}\left(\overline{\mathbb{R}^{n-1}}\right)$, nor in Definition 1.2.7 that $\{\gamma_k\}_k$ is a subset of $C^{m-1,1}\left(\overline{\mathbb{R}^{n-1}}\right)$, $m\in\mathbb{N}$. For example, it is obvious that for the simplest (yet non trivial, i.e. $n=1$) case*

$$n=2:\ U=\mathrm{epi}_{\mathbb{S}}(\gamma),\ \textit{i.e. } \partial U\in\mathrm{Lip}(\infty,1,\mathrm{Lip}(\gamma))\ \textit{(evidently, } \{U_k\}_k=\left\{\mathbb{R}^2\right\} \textit{ and } \{\Phi_k\}_k=\{\mathrm{id}\}\textit{)},$$

where

$$\gamma\equiv sin,\ \textit{or } \gamma \textit{ is any real and non trivial polynomial, etc.,}$$

we have that $\gamma\notin C^{0,1}\left(\overline{\mathbb{R}}\right)$ since $\gamma\notin C\left(\overline{\mathbb{R}}\right)$. One could say that we employ the spaces "$C^{0,1}\left(\mathbb{R}^{n-1}\right)$" and "$C^{m-1,1}\left(\mathbb{R}^{n-1}\right)$", respectively, for the aforementioned definitions.

The following trivial result is in fact crucial for Section 2.4.

Proposition 1.2.7. *If U is such that $\partial U\in\mathrm{Lip}^m(\varepsilon,K,L)$, as well as Φ is a rigid motion of the form $\Phi(x):=x_0+\lambda x$, where $x_0\in\mathbb{R}^n$ and $\lambda>1$, then $\partial(\Phi(U))\in\mathrm{Lip}^m(\lambda\varepsilon,K,L)$ also.*

Proof. We set z for the coordinates that the map Φ induces, i.e. $z=\Phi(x)$, for every $x\in\mathbb{R}^n$. It is direct to check that $\{\Phi(U_k)\}_k$ is a locally finite countable open cover of $\partial(\Phi(U))$. In order to obtain the desired result, we argue as follows.

1. If $z\in\partial(\Phi(U))$, then there exists $x=\Phi^{-1}(z)\in\partial U$, hence $B(x,\varepsilon)\subseteq U_k$ for some k. Therefore, $\Phi(B(x,\varepsilon))\subseteq\Phi(U_k)$, or else $B(z,\lambda\varepsilon)\subseteq\Phi(U_k)$. Indeed, for every $x_1,z_1\in\mathbb{R}^n$ such that $z_1=\Phi(x_1)$, we have
$$\Phi(B(x_1,\varepsilon))=\Phi\big(\big\{x\in\mathbb{R}^n\,\big|\,|x-x_1|<\varepsilon\big\}\big)=\big\{(\Phi(x))\in\mathbb{R}^n\,\big|\,|x-x_1|<\varepsilon\big\}=$$
$$=\big\{z\in\mathbb{R}^n\,\big|\,|\Phi^{-1}(z)-x_1|<\varepsilon\big\}=\left\{z\in\mathbb{R}^n\,\middle|\,\left|-\frac{x_0}{\lambda}+\frac{z}{\lambda}-x_1\right|<\varepsilon\right\}=\big\{z\in\mathbb{R}^n\,\big|\,|z-(x_0+\lambda x_1)|<\lambda\varepsilon\big\}=$$
$$=\big\{z\in\mathbb{R}^n\,\big|\,|z-\Phi(x_1)|<\lambda\varepsilon\big\}=\big\{z\in\mathbb{R}^n\,\big|\,|z-z_1|<\lambda\varepsilon\big\}=B(z_1,\lambda\varepsilon).$$

2. It is direct to check by contradiction that every collection of $K+1$ of $\Phi(U_k)$'s has empty intersection.

3. For every k we consider the local coordinates
$$\widetilde{y}_k=\widetilde{\Phi}_k(z):=\left(\Phi\circ\Phi_k\circ\Phi^{-1}\right)(z),\ \ \forall z\in\mathbb{R}^n$$
(it is straightforward to check that $\widetilde{y}_k$ are indeed local coordinates), as well as $\widetilde{\gamma}_k:\mathbb{R}^{n-1}\to\mathbb{R}$, with
$$\widetilde{\gamma}_k(w):=x_{0_n}+\lambda\gamma_k\left(-\frac{x_0{}'}{\lambda}+\frac{w}{\lambda}\right),\ \ \forall w\in\mathbb{R}^{n-1}.$$
Notice that $\widetilde{y}_k\equiv\Phi(y_k)$ for every k. Now, a direct validation of the definition shows that, for every k, $\widetilde{\gamma}_k$ is Lipschitz continuous with $\mathrm{Lip}(\widetilde{\gamma}_k)\le\mathrm{Lip}(\gamma_k)\le L$. Moreover, if $m\neq1$ we have that
$$\nabla^{j-1}\widetilde{\gamma}_k(w)=\frac{1}{\lambda^{j-2}}\nabla^{j-1}\gamma_k\left(-\frac{x_0{}'}{\lambda}+\frac{w}{\lambda}\right),\ \ \forall w\in\mathbb{R}^n,\ \ \forall j=2,\dots,m,$$
directly from the common Faá di Bruno formula, hence, again by the use of the definition we deduce easily that
$$\max_{j=2,\dots,m}\left\{\nabla^{j-1}\widetilde{\gamma}_k\right\}\le L,\ \ \forall k,$$
since $\lambda>1$. Finally,
$$\Phi_k(U_k\cap U)=\Phi_k(U_k)\cap\{y_{n_k}>\gamma(y_k{}')\}\overset{\Phi}{\Rightarrow}(\Phi\circ\Phi_k)(U_k\cap U)=\Phi(\Phi_k(U_k)\cap\{y_{n_k}>\gamma(y_k{}')\})\Rightarrow$$
$$\Rightarrow\widetilde{\Phi}(\Phi(U_k)\cap\Phi(U))=\widetilde{\Phi}(\Phi(U_k))\cap\{\widetilde{y}_{n_k}>\widetilde{\gamma}_k(\widetilde{y}_k{}')\}.$$
Indeed,
$$\Phi\big(\big\{y_k\in\mathbb{R}^n\,\big|\,y_{n_k}>\gamma_k(y_k{}')\big\}\big)=\big\{\Phi(y_k)\in\mathbb{R}^n\,\big|\,x_{0_n}+\lambda y_{n_k}>x_{0_n}+\lambda\gamma_k(y_k{}')\big\}=$$
$$=\left\{\widetilde{y}_k\in\mathbb{R}^n\,\middle|\,\widetilde{y}_{n_k}>\gamma_k\left(-\frac{x_0{}'}{\lambda}+\frac{\widetilde{y}_k{}'}{\lambda}\right)\right\}=\big\{\widetilde{y}_k\in\mathbb{R}^n\,\big|\,\widetilde{y}_{n_k}>\widetilde{\gamma}_k(\widetilde{y}_k{}')\big\}.$$

□

1.2.7 The Leibniz formula

Here, we slightly generalize a useful, already known result (see, e.g., Theorem 1, Section 5.2 in [15]), concerning the Leibniz rule for a smooth function and a function which belongs to a Sobolev space. Before we state and prove it, we recall that, for every $m\in\mathbb{N}_0$ and every U, $C^m_{\mathrm{B}}(U)$ stands for the Banach space
$$C^m_{\mathrm{B}}(U):=\big\{u\in C^m(U)\,\big|\,D^\alpha u\text{ is bounded everywhere in }U,\text{ for every }0\le|\alpha|\le m\big\},$$
equipped with its natural norm (see, e.g., Paragraph 1.27 in [1]).

Proposition 1.2.8. *Let $m\in\mathbb{N}_0$, $p\in[1,\infty]$ and U. If $\phi\in\bigcap_{m=0}^{\infty}C_{\mathrm{B}}^{m}(U)$ and $u\in W^{m,p}(U)$, then we have that*

1. *$(\phi u)\in W^{m,p}(U)$ also, with*

$$\|\phi u\|_{W^{m,p}(U)}\leq\mathcal{K}\Big(\|\phi\|_{C_{\mathrm{B}}^{m}(U)}\Big)\|u\|_{W^{m,p}(U)} \tag{1.2.9}$$

and

2.

$$D_w^{\alpha}(\phi u)=\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(D^{\beta}\phi\right)\left(D_w^{\alpha-\beta}u\right)\text{ a.e. in } U,\text{ for every }\alpha\in\mathbb{N}_0^n\text{ with }0\leq|\alpha|\leq m. \tag{1.2.10}$$

Proof. Step 1

We easily deduce from

$$\left|\sum_{j=1}^{N}z_j\right|^q\leq C_{N,q}\left(\sum_{j=1}^{N}|z_j|^q\right),\ \forall(z_j)_{j=1}^{N}\subset\mathbb{C}^N,\ \forall N\in\mathbb{N},\ \forall q\in[0,\infty), \tag{1.2.11}$$

that

$$\left(\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(D^{\beta}\phi\right)\left(D_w^{\alpha-\beta}u\right)\right)\in L^p(U),\text{ for every }\alpha\in\mathbb{N}_0^n\text{ with }0\leq|\alpha|\leq m,$$

with

$$\left\|\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(D^{\beta}\phi\right)\left(D_w^{\alpha-\beta}u\right)\right\|_{L^p(U)}\leq\mathcal{K}\Big(\|\phi\|_{C_{\mathrm{B}}^{|\alpha|}(U)}\Big)\|u\|_{W^{|\alpha|,p}(U)},\quad\forall\alpha\in\mathbb{N}_0^n,\ 0\leq|\alpha|\leq m. \tag{1.2.12}$$

We note that inequality (1.2.11) follows directly from applying $N-1$ times the elementary inequality

$$|z_1+z_2|^q\leq C_q\left(|z_1|^q+|z_2|^q\right),\ \forall\,z_1,z_2\in\mathbb{C},\ \forall q\in[0,\infty). \tag{1.2.13}$$

With the previous argument we are done with the case $m=0$. At the next step, we show the result for $m\neq0$ by induction on m, employing of course the estimate in (1.2.12). Before we proceed, we note that

$$(\phi\psi)\in C_c^{\infty}(U),\ \forall\psi\in C_c^{\infty}(U).$$

Step 2α

Let $m=1$. From the estimates (1.2.12) for $m=1$, it suffices to show (1.2.10) for $m=1$. For every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=1$ and every $\psi\in C_c^{\infty}(U)$, we get

$$\int_U\phi u\,D^{\alpha}\psi dx=\int_U u\,\phi D^{\alpha}\psi dx=\int_U u\left(D^{\alpha}(\phi\psi)-\left(D^{\alpha}\phi\right)\psi\right)dx=$$
$$=-\int_U\left(\phi D_w^{\alpha}u+u\left(D^{\alpha}\phi\right)\right)\psi dx,$$

where we employ the definition of the weak derivatives at the last equation. Hence, again from the definition of the weak derivatives, we derive (1.2.10) for $m=1$.

Step 2β

Here follows the induction hypothesis on an arbitrary $m\in\mathbb{N}\smallsetminus\{1\}$: If $\phi\in C_{\mathrm{B}}^{m}(U)$ and $u\in W^{m,p}(U)$, for some $m\in\mathbb{N}_0$, $p\in[1,\infty]$ and U, then $(\phi u)\in W^{m,p}(U)$ also, with

$$\|\phi u\|_{W^{m,p}(U)}\leq\mathcal{K}\Big(\|\phi\|_{C_{\mathrm{B}}^{m}(U)}\Big)\|u\|_{W^{m,p}(U)}$$

and

$$D_w^{\alpha}(\phi u)=\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\left(D^{\beta}\phi\right)\left(D_w^{\alpha-\beta}u\right)\text{ a.e. in } U,\text{ for every }\alpha\in\mathbb{N}_0^n\text{ with }1\leq|\alpha|\leq m.$$

Step 2γ

Now, let $\phi \in C_{\mathrm{B}}^{m+1}(U)$ and $u \in W^{m+1,p}(U)$, for some $m \in \mathbb{N}_0$, $p \in [1,\infty]$ and U. From the estimates (1.2.12) for $m+1$ instead of m, it suffices to show (1.2.10) for $m+1$. For every $\alpha \in \mathbb{N}_0^n$ with $\alpha = \beta + \gamma$ (evidently $|\alpha| = |\beta| + |\gamma|$) where $|\beta| = m$ and $|\gamma| = 1$, as well as every $\psi \in C_c^\infty(U)$, we have from

1. the fact that $(\phi u) \in W^{m,p}(U)$ along with the definition of the weak derivatives,
2. the induction hypothesis,
3. the result for $m=1$ and
4. the fact that operators of the form D_w^ν (for $\nu \in \mathbb{N}_0^n$) commute with each other, that is $D_w^{\nu_1} \circ D_w^{\nu_2} = D_w^{\nu_2} \circ D_w^{\nu_1} = D_w^{\nu_1+\nu_2}$,

that

$$\int_U \phi u\, D^\alpha \psi dx = \int_U \phi u \left(D^{\beta+\gamma}\psi\right) dx \overset{1.}{=} \int_U D_w^\beta(\phi u)\,(D^\gamma \psi) dx \overset{2.}{=}$$

$$\overset{2.}{=} (-1)^{|\beta|} \int_U \sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} (D^\sigma \phi) \left(D_w^{\beta-\sigma} u\right) (D^\gamma \psi) dx \overset{1.}{=}$$

$$\overset{1.}{=} (-1)^{|\beta|+|\gamma|} \int_U D_w^\gamma \left(\sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} (D^\sigma \phi) \left(D_w^{\beta-\sigma} u\right) \right) \psi dx \overset{3.}{=}$$

$$\overset{3.}{=} (-1)^{|\alpha|} \int_U \left(\sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} \left((D^{\sigma+\gamma}\phi) \left(D_w^{\beta-\sigma} u\right) + (D^\sigma \phi) \left(\left(D_w^\gamma \circ D_w^{\beta-\sigma}\right) u \right) \right) \right) \psi dx \overset{4.}{=}$$

$$\overset{4.}{=} (-1)^{|\alpha|} \int_U \left(\sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} \left((D^{\sigma+\gamma}\phi) \left(D_w^{\beta-\sigma} u\right) + (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) \right) \right) \psi dx =$$

$$= (-1)^{|\alpha|} \int_U \left(\sum_{\gamma \le \sigma \le \alpha} \binom{\beta}{\sigma - \gamma} (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) + \sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) \right) \psi dx.$$

For the term inside the parenthesis we have

$$\sum_{\gamma \le \sigma \le \alpha} \binom{\beta}{\sigma - \gamma} (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) + \sum_{0 \le \sigma \le \beta} \binom{\beta}{\sigma} (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) =$$

$$= (D^\alpha \phi)\, u + \phi\, (D_w^\alpha \phi) + \sum_{\gamma \le \sigma \le \beta} \left(\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} \right) (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) =$$

$$= (D^\alpha \phi)\, u + \phi\, (D_w^\alpha \phi) + \sum_{\gamma \le \sigma \le \beta} \binom{\alpha}{\sigma} (D^\sigma \phi) \left(D_w^{\alpha-\sigma} u\right) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} \left(D^\beta \phi\right) \left(D_w^{\alpha-\beta} u\right).$$

Therefore, from the definition of the weak derivatives, we derive (1.2.10) for $m+1$.

□

1.2.8 Change of variables

In Section 1.2.12 we need the following result, concerning both the formula and the bounds of the Sobolev norms under the change of variables. It slightly generalizes an already known one (see, e.g., Theorem 11.57 in [35]), since the new variables do not have to possess a unique, bounded, continuous extension to the closure of their open domain (see also Remark 1.2.5).

Theorem 1.2.3. *Let $m \in \mathbb{N}_0$, $p \in [1,\infty]$, U_1, U_2 and $\Psi : U_2 \to U_1$ be bijective, with $\Phi \coloneqq \Psi^{-1}$. If we assume that there exist $L_1, L_2 \in [0,\infty)$, such that*

i. Φ is Lipschitz continuous with $\mathrm{Lip}(\Phi) \le L_1$ and

ii. if $m \ne 0$, then $\nabla^j \Psi_i$ is Lipschitz continuous for every $j = 0, \ldots, m-1$ and every $i = 1, \ldots, n$, with

$$\max_{\substack{j=0,\ldots,m-1 \\ i=1,\ldots,n}} \left\{ \mathrm{Lip}\left(\nabla^j \Psi_i\right) \right\} \le L_2,$$

then for every $u \in W^{m,p}(U_1)$ *we have that*

1. $u \circ \Psi \in W^{m,p}(U_2)$ *also, with*

$$\begin{aligned} \|u\circ\Psi\|_{L^p(U_2)} &\leq \mathcal{K}(L_1)\|u\|_{L^p(U_1)} \quad \text{and} \\ \left\|\nabla_w^l(u\circ\Psi)\right\|_{L^p(U_2)} &\leq \mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^p(U_1)}, \quad \text{for every } l=1,\ldots,m, \text{ if } m\neq 0 \end{aligned} \tag{1.2.14}$$

and

2. *if* $m \neq 0$, *then*

$$D_w^\alpha(u\circ\Psi) = \sum_{1\leq|\beta|\leq|\alpha|} M_{\alpha,\beta}(\Psi)\left(D_w^\beta u\right)\circ\Psi, \quad \text{a.e. in } U_2, \tag{1.2.15}$$

for every $\alpha \in \mathbb{N}_0^n$ *with* $1 \leq |\alpha| \leq m$, *where*

$$M_{\alpha,\beta}(\Psi) := \alpha! \sum_{s=1}^{|\alpha|} \sum_{p_s(\alpha,\beta)} \prod_{j=1}^{s} \frac{1}{\gamma_j!(\delta_j!)^{|\gamma_j|}} \left(D^{\delta_j}\Psi\right)^{\gamma_j},$$

with $0^0 := 1$, $\gamma_j, \delta_j \in \mathbb{N}_0^n$,

$$\left(D^{\delta_j}\Psi\right)^{\gamma_j} := \prod_{i=1}^{n}\left(D^{\delta_j}\Psi_i\right)^{\gamma_{j_i}},$$

$$p_s(\alpha,\beta) := \left\{(\gamma_1,\ldots,\gamma_s,\delta_1,\ldots,\delta_s) \;\middle|\; |\gamma_j|>0,\ 0\prec\delta_1\prec\ldots\prec\delta_s,\ \sum_{j=1}^{s}\gamma_j=\beta,\ \sum_{j=1}^{s}|\gamma_j|\,\delta_j=\alpha\right\}$$

and $\mu \prec \nu$ *for* $\mu, \nu \in \mathbb{N}_0^n$ *provided one of the following holds:*

(a) $|\mu| < |\nu|$,

(b) $|\mu| = |\nu|$ *and* $\mu_1 < \nu_1$, *or*

(c) $|\mu| = |\nu|$, $\mu_1 = \nu_1, \ldots, \mu_k = \nu_k$ *and* $\mu_{k+1} < \nu_{k+1}$ *for some* $1 \leq k < n$.

Proof. In order to reduce the number of the sub-cases, we only show the results for the case $m \neq 0$, since the concept for the proof of the simpler case $m = 0$ is exactly the same. The only difference is that we use the density of $C_c^\infty(U)$ into $L^p(U)$, for $p \in [1,\infty)$, instead of the Meyers-Serrin theorem in Step 2. Now, the present proof has the following structure: In Step 2 we deal with the case $p \neq \infty$ and in Step 3 with $p = \infty$.

Step 1

The generalized multivariate Faá di Bruno formula (1.2.15) (with the weak derivatives being replaced with the ordinary ones) is already known for every smooth enough functions u and Ψ, and for its proof we refer to [12] (see also [31] for a more compact approach).

Step 2α

If $p \in [1,\infty)$ and $u \in W^{m,p}(U_1)$, from the Meyers-Serrin theorem (see, e.g., Theorem, Paragraph 3.17 in [1], or Theorem 11.24 in [35]) there exists $\{u_k\}_k \subset C^m(U_1) \cap W^{m,p}(U_1)$, such that $u_k \to u$ in $W^{m,p}(U_1)$. Therefore, in view of a well known result (see, e.g., Point (a) of Theorem 4.9. in [8]) along with the classic scheme "consider a subsequence of the subsequence"[3], we deduce that there exists a subsequence $\{u_{k_l}\}_l \subset \{u_k\}_k$, such that

i. $u_{k_l} \to u$ a.e. in U_1 and

ii. $D^\alpha u_{k_l} \to D_w^\alpha u$ a.e. in U_1, for every $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| \leq m$.

Since Φ is Lipschitz continuous, it has the Luzin (N) property (see, e.g., the Exercise 9.54 in [35]), thus

i. $u_{k_l} \circ \Psi \to u \circ \Psi$ a.e. in U_2 and

[3] Formally, this follows by induction on m, but the process is quite trivial and so we omit it.

ii. $M_{\alpha,\beta}(\Psi)\left(D^{\beta}u_{k_l}\right)\circ\Psi \to M_{\alpha,\beta}(\Psi)\left(D^{\beta}_{w}u\right)\circ\Psi$ a.e. in U_2, for every $\alpha,\beta\in\mathbb{N}_0^n$ with $1\le|\alpha|\le m$ and $1\le|\beta|\le|\alpha|$.

Step 2β

Since every $\nabla^j\Psi_i$ ($j=0,\dots,m-1$) is Lipschitz continuous, we have that $u_{k_l}\circ\Psi$ satisfies (1.2.15) a.e. in U_2 if $|\alpha|=m$ and everywhere in U_2 otherwise. Moreover, we have that $u_{k_l}\circ\Psi$ and all its derivatives of order up to $m-1$ are absolutely continuous on all line segments of U_2 that are parallel to the coordinate axes, since the composition of Lipschitz continuous functions is Lipschitz continuous and also the product of bounded and Lipschitz continuous functions is Lipschitz continuous. Thus, it is only left to show that $u_{k_l}\circ\Psi$ and all its derivatives of order up to m belong to $L^p(U_2)$, in order to show that $u_{k_l}\circ\Psi\in W^{m,p}(U_2)$ (see, e.g., Theorem 11.45 (along with Exercise 11.47) in [35], or Theorem 2, Section 1.1.3 in [37]). Indeed, we have that

$$|\det(J\Phi)|\le\mathcal{K}(L_1),\ \text{a.e. in } U_1. \tag{1.2.16}$$

and also that, for every $\alpha,\beta\in\mathbb{N}_0^n$ with $1\le|\alpha|\le m$ and $1\le|\beta|\le|\alpha|$,

$$|M_{\alpha,\beta}(\Psi)|^p\le\mathcal{K}(L_2),\ \text{a.e. in } U_2, \tag{1.2.17}$$

which follows from (1.2.11). Hence, we combine the formula (1.2.15), (1.2.16) and (1.2.17) with the change of variables formula (see, e.g., Theorem 9.52 along with Exercise 9.54 in [35]), to deduce the estimates

$$\|u_{k_l}\circ\Psi\|_{L^p(U_2)}\le\mathcal{K}(L_1)\|u_{k_l}\|_{L^p(U_1)} \text{ and}$$
$$\|D^{\alpha}(u_{k_l}\circ\Psi)\|_{L^p(U_2)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{|\alpha|}\left\|\nabla_w^i u_{k_l}\right\|_{L^p(U_1)},\ \text{for every } \alpha\in\mathbb{N}_0^n \text{ with } 1\le|\alpha|\le m,$$

hence $u_{k_l}\circ\Psi\in W^{m,p}(U_2)$. Additionally, the above estimates also imply (simply by considering the differences in the formula) that

$$\left\|\left(u_{k_{l_1}}\circ\Psi\right)-\left(u_{k_{l_2}}\circ\Psi\right)\right\|_{L^p(U_2)}\le\mathcal{K}(L_1)\left\|u_{k_{l_1}}-u_{k_{l_2}}\right\|_{L^p(U_1)} \text{ and}$$
$$\left\|D^{\alpha}\left(\left(u_{k_{l_1}}\circ\Psi\right)-\left(u_{k_{l_2}}\circ\Psi\right)\right)\right\|_{L^p(U_2)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{|\alpha|}\left\|\nabla_w^i\left(u_{k_{l_1}}-u_{k_{l_2}}\right)\right\|_{L^p(U_1)},$$

for every l_1 and l_2. Since $u_{k_l}\to W^{m,p}(U_1)$, it follows that $\{u_{k_l}\circ\Phi\}_l$ is a Cauchy sequence in $W^{m,p}(U_2)$. In virtue of the completeness of the $W^{m,p}$-spaces, we deal as in Step 2 to obtain a subsequence of $\{u_{k_l}\}_l$, which we still denote as such, and a function $v\in W^{m,p}(U_2)$, such that

i. $u_{k_l}\circ\Psi\to v$ a.e. in U_2 and
ii. $D^{\alpha}(u_{k_l}\circ\Psi)\to D^{\alpha}_w v$ a.e. in U_2, for every $\alpha\in\mathbb{N}_0^n$ with $1\le|\alpha|\le m$.

Hence, $v=u\circ\Psi$ a.e. in U_2, that is $u\circ\Psi\in W^{m,p}(U_2)$. We can also let $l\nearrow\infty$ in the formula (1.2.15) (for u_{k_l} instead of u) to get Point 2..

Step 2γ

As for the estimates (1.2.14), we repeat the first argument of Step 2β to derive

$$\|u\circ\Psi\|_{L^p(U_2)}\le\mathcal{K}(L_1)\|u\|_{L^p(U_1)} \text{ and}$$
$$\|D^{\alpha}_w(u\circ\Psi)\|_{L^p(U_2)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{|\alpha|}\left\|\nabla_w^i u\right\|_{L^p(U_1)},\ \text{for every } \alpha\in\mathbb{N}_0^n \text{ with } 1\le|\alpha|\le m,$$

thus the result follows.

Step 3α

If $u\in W^{m,\infty}(U_1)$, we consider an increasing sequence of bounded subsets $\left\{U_{1_j}\subset\subset U_1\right\}_j$, such that $U_{1_j}\nearrow U_1$. Since Φ is Lipschitz continuous map from U_1 onto U_2, then the metric space $(U_2,|*-\star|)$ preserves all the "metric" properties of $(U_1,|*-\star|)$, hence every $\Phi\left(U_{1_j}\right)$ is open and bounded, as well as $\Phi\big(U_{1_j}\big)\nearrow U_2$. In view of the Step 2 along with the embedding $L^\infty\big(U_{1_j}\big)\hookrightarrow L^p\big(U_{1_j}\big)$ for every $p\in[1,\infty)$ and every j, we deduce that $u\circ\Psi\in W^{m,p}\big(\Phi\big(U_{1_j}\big)\big)$ with

$$\|u\circ\Psi\|_{L^p\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1)\|u\|_{L^p\left(U_{1_j}\right)} \text{ and}$$
$$\left\|\nabla_w^l(u\circ\Psi)\right\|_{L^p\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^p\left(U_{1_j}\right)},\ \forall l=1,\dots,m$$

as well as that the formula (1.2.15) holds for every $\Phi\big(U_{1_j}\big)$ instead of U_2. In view of the latter conclusion along with the fact that $\Phi\big(U_{1_j}\big)\nearrow U_1$, it suffices to show only Point 1., since then the formula (1.2.15) makes sense and its virtue follows easily by contradiction.

Step 3β

Setting[4]

$$u_j:=\Big(\big(\mathcal{E}_0\big(\Phi\big(U_{1_j}\big),U_2\big)\big)\circ\big(\mathcal{R}\big(U_2,\Phi\big(U_{1_j}\big)\big)\big)\Big)\,u\circ\Psi,\ \forall j,$$

as well as

$$u_{\alpha j}:=\Big(\big(\mathcal{E}_0\big(\Phi\big(U_{1_j}\big),U_2\big)\big)\circ\big(\mathcal{R}\big(U_2,\Phi\big(U_{1_j}\big)\big)\big)\Big)\,D_w^\alpha(u\circ\Psi),\ \forall\alpha\in\mathbb{N}_0^n,\ 1\le|\alpha|\le m,\ \forall j$$

and rewriting the above estimates, we have, for j and every $p\in[1,\infty)$, that

$$\|u_j\|_{L^p\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1)\|u\|_{L^p\left(U_{1_j}\right)} \text{ and}$$
$$\|u_{\alpha j}\|_{L^p\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^p\left(U_{1_j}\right)},\ \forall\alpha,\ 1\le|\alpha|\le m.$$

Since the sets appeared in the norms are bounded, we pass to the limit $p\nearrow\infty$ to get

$$\|u_j\|_{L^\infty\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1)\|u\|_{L^\infty(U_1)} \text{ and}$$
$$\|u_{\alpha j}\|_{L^\infty\left(\Phi\left(U_{1_j}\right)\right)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^\infty(U_1)},\ \forall\alpha,\ 1\le|\alpha|\le m,$$

for every j, thus, from the definition of u_j and each $u_{\alpha j}$ we obtain

$$\|u_j\|_{L^\infty(U_2)}\le\mathcal{K}(L_1)\|u\|_{L^\infty(U_1)} \text{ and}$$
$$\|u_{\alpha j}\|_{L^\infty(U_2)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^\infty(U_1)},\ \forall\alpha,\ 1\le|\alpha|\le m,$$

for every j. In virtue of the well known corollary of the Banach-Alaoglu-Bourbaki theorem (see, e.g., Corollary 3.30 in [8]) along with the weak* lower semi-continuity of the L^∞-norm (see, e.g., Point (iii) of Proposition 3.13 in [8]), we deduce that there exist a subsequence of $\{u_j\}_j$, which we still denote as such, and a function $v\in L^\infty(U_2)$ such that

$$u_j\overset{*}{\rightharpoonup}v \text{ in } L^\infty(U_2) \text{ and } \|v\|_{L^\infty(U_2)}\le\mathcal{K}(L_1)\|u\|_{L^\infty(U_1)}. \tag{1.2.18}$$

Dealing again as before, we deduce that, for every α there exist a subsequence of $\{u_{\alpha_j}\}_j$, which we still denote as such, and a function $v_\alpha\in L^\infty(U_2)$ such that

$$u_{\alpha_j}\overset{*}{\rightharpoonup}v_\alpha \text{ in } L^\infty(U_2) \text{ and } \|v_\alpha\|_{L^\infty(U_2)}\le\mathcal{K}(L_1,L_2)\sum_{i=1}^{l}\left\|\nabla_w^i u\right\|_{L^\infty(U_1)}. \tag{1.2.19}$$

Step 3γ

We show that $u\circ\Psi\in L^\infty(U_2)$. Indeed, it suffices to show that $v=u\circ\Psi$ a.e. in U_2. First, we notice that $u\circ\Psi\in L^1_{\text{loc}}(U_2)$, which follows from a direct application of the change of variables formula and the fact that $u\in L^1_{\text{loc}}(U_1)$. Now, let $\psi\in C_c^\infty(U_2)$ be arbitrary and $j_0=j_0(\psi)$ be big enough so that $\text{supp}(\psi)\subset\Phi\big(U_{1_j}\big)$ for every $j\ge j_0$. We then have from

[4] We "cut" at $\partial\Phi\big(U_{1_j}\big)$ and we extend by zero to whole U_2.

1. (1.2.18) and
2. the definition of every u_j along with the fact that $\operatorname{supp}(\psi) \subset \Phi\big(U_{1_j}\big)$ for every $j \geq j_0$,

that

$$\int_{U_2} v\,\psi dx \overset{1.}{=} \lim_{j\nearrow\infty} \int_{U_2} u_j\,\psi dx = \lim_{j\nearrow\infty} \int_{\operatorname{supp}(\psi)} u_j\,\psi dx =$$
$$= \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{\operatorname{supp}(\psi)} u_j\,\psi dx \overset{2.}{=} \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{\operatorname{supp}(\psi)} (u\circ\Psi)\,\psi dx = \int_{U_2} (u\circ\Psi)\,\psi dx$$

and the result follows since ψ is arbitrary (see, e.g., Lemma, Paragraph 3.31 in [1], or Corollary 4.24 in [8]).

Step 3δ

Now, we show that $u\circ\Psi \in W^{m,\infty}(U_2)$ and that the estimates in (1.2.14) hold. Indeed, it suffices to show that $u\circ\Psi$ is m times weakly differentiable in U_2 with $D_w^\alpha(u\circ\Psi) = v_\alpha$ a.e., for every $\alpha\in\mathbb{N}_0^n$ with $1\leq|\alpha|\leq m$. Let α and $\psi\in C_c^\infty(U_2)$ be arbitrary, as well as $j_0 = j_0(\psi)$ be big enough so that $\operatorname{supp}(\psi)\subset\Phi\Big(U_{1_{\alpha_j}}\Big)$ for every $j\geq j_0$. We then have from

1. the fact that $\operatorname{supp}(\psi)\subset\Phi\Big(U_{1_{\alpha_j}}\Big)$ for every $j\geq j_0$,
2. the definition of the weak derivatives,
3. the definition of every u_{α_j} and
4. (1.2.19),

that

$$\int_{U_2} (u\circ\Psi)\,D^\alpha\psi dx = \int_{\operatorname{supp}(\psi)} (u\circ\Psi)\,D^\alpha\psi dx \overset{1.}{=} \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{\Phi\left(U_{1_{\alpha_j}}\right)} (u\circ\Psi)\,D^\alpha\psi dx \overset{2.}{=}$$
$$\overset{2.}{=} (-1)^{|\alpha|} \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{\Phi\left(U_{1_{\alpha_j}}\right)} (D_w^\alpha(u\circ\Psi))\,\psi dx \overset{3.}{=} (-1)^{|\alpha|} \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{\Phi\left(U_{1_{\alpha_j}}\right)} u_{\alpha_j}\,\psi dx \overset{3.}{=}$$
$$\overset{3.}{=} (-1)^{|\alpha|} \lim_{\substack{j\nearrow\infty \\ j\geq j_0}} \int_{U_2} u_{\alpha_j}\,\psi dx \overset{4.}{=} (-1)^{|\alpha|} \int_{U_2} u_\alpha\,\psi dx$$

and the result follows since ψ is arbitrary.

□

1.2.9 Difference quotients

For the regularity results of Section 1.2.12, we employ the classic Nirenberg approach of the difference quotients.

Definition 1.2.8. *Let U, $i = 1,\ldots,n$ and $\delta>0$. We set $U^{i,\delta} \supsetneq U$ for*

$$U^{i,\delta} \coloneqq \Big\{z\in\mathbb{R}^n \,\Big|\, z = x + he_i,\ \text{for } x\in U \text{ and } h\in(-\delta,\delta)\Big\}.$$

Additionally, we set $U^\delta \supset \overline{U}$ for

$$U^\delta \coloneqq \bigcup_{i=1}^n U^{i,\delta} = \Big\{z\in\mathbb{R}^n \,\Big|\, z = x + y,\ \text{for } x\in U \text{ and } y\in B(0,\delta)\Big\} = U \cup \bigcup_{x\in\partial U} B(x,\delta).$$

Remark 1.2.6. *We recall that $U_\delta \coloneqq \big\{x\in U \,\big|\, \operatorname{dist}(x,\partial U)>\delta\big\}$, for every $\delta>0$. Evidently, $(U_\delta)^\delta \subseteq U$.*

Definition 1.2.9. *Let U, $i = 1,\ldots,n$ and $\delta>0$. For every $\mathbb{R}^n \supseteq A_i \supseteq U^{i,\delta}$ and every $h\in(-\delta,\delta)$ we denote $\cdot^{i,h} : \mathcal{F}(A_i) \to \mathcal{F}(U)$ for*

$$u^{i,h}(x) \coloneqq u(x+he_i),\ \forall x\in U,\ \forall u\in\mathcal{F}(A_i),$$

as well as for every $\mathbb{R}^n \supseteq A_i \supseteq U^{i,\delta}$ and every $h\in(-\delta,\delta)^*$ we write $\partial^{i,h} : \mathcal{F}(A_i) \to \mathcal{F}(U)$ for the ith partial difference h-quotient, i.e.

$$\partial^{i,h} u \coloneqq \frac{u^{i,h} - u}{h},\ \forall u\in\mathcal{F}(A_i).$$

Remark 1.2.7. *For U, i, δ, A_i as in Definition 1.2.9, we can easily derive the formula*

$$\partial^{i,h}(uv) = u^{i,h}\partial^{i,h}v + v\partial^{i,h}u, \ \forall\, u,v \in \mathcal{F}(A_i),\ h \in (-\delta,\delta)^*. \tag{1.2.20}$$

Definition 1.2.10. *Let U and $\delta > 0$. For every $\mathbb{R}^n \supseteq A \supseteq U^\delta$ and every $h \in (-\delta,\delta)^*$ we define $\nabla^h : \mathcal{F}(A) \to \mathcal{F}(U)^n$ by*

$$\nabla^h u \coloneqq \left(\partial^{i,h}u\right)_{i=1}^n, \ \forall u \in \mathcal{F}(A).$$

Remark 1.2.8. *In view of Remark 1.2.6, we can consider (U,U_δ) instead of (A,U) in Definition 1.2.10.*

The following useful result, which generalizes the similar and already known ones (see, e.g., Lemmata 7.23 and 7.24 in [26], or Theorem 3, Section 5.8 in [15], or Lemma 4.13 in [38], or mainly Theorem 11.75 in [35]), is about the properties of the partial difference quotients. Before we proceed, we need a trivial, yet crucial lemma.

Lemma 1.2.2. *Let $p \in [1,\infty]$, U, $i = 1,\dots,n$ and $\delta > 0$. If $u \in L^p\left(U^{i,\delta}\right)$ and $v \in L^{\frac{p}{p-1}}\left(U^{i,\delta}\right)$ such that $\operatorname{supp}(v) \subseteq \overline{U}$, then*

$$\int_U u\left(\partial^{i,h}v\right)dx = -\int_U \left(\partial^{i,-h}u\right)v\,dx, \ \forall h \in (-\delta,\delta)^*. \tag{1.2.21}$$

Proof. First of all, in virtue of the Hölder inequality, (1.2.21) makes sense. Now, for every $h \in (-\delta,\delta)^*$ we have

$$\int_U u\left(\partial^{i,h}v\right)dx = -\int_U \frac{u(x)}{h}v(x)dx + \int_U \frac{u(x)}{h}v(x+he_i)dx.$$

Changing the coordinates x to $x+he_i$, applying the change of variables formula and using the fact that $\operatorname{supp}(v) \subseteq \overline{U}$, we get

$$\int_U \frac{u(x)}{h}v(x+he_i)dx = \int_U \frac{u(x-he_i)}{h}\phi(x)dx$$

and the result follows. □

Proposition 1.2.9. *Let $n \in \mathbb{N} \setminus \{1\}$ and $a \in \mathbb{R}$, as well as $U_1 \subsetneq U_2$ and $\delta > 0$ be such that*[5]

$$U_1 \subsetneq \bigcup_{i=1}^{n-1} U_1^{i,\delta} \subseteq U_2.$$

1. *If $p \in [1,\infty]$ and $u \in \left\{u \in L^1_{\text{loc}}(U_2) \,\middle|\, \nabla_w u \in L^p(U_2)\right\}$, then $\partial^{i,h}u \in L^p(U_1)$, for every $i = 1,\dots,n-1$ and $h \in (-\delta,\delta)^*$, with*

$$\left\|\partial^{i,h}u\right\|_{L^p(U_1)} \le \left\|\partial^i_w u\right\|_{L^p(U_2)}.$$

2. *If $p \in (1,\infty]$, $u \in L^1_{\text{loc}}(U_2)$ and there exists $\delta' \in (0,\delta]$ such that*

$$\left\|\partial^{i,h}u\right\|_{L^p(U_1)} \le C, \ \textit{for every } h \in (-\delta',\delta')^*, \ \textit{for some } i = 1,\dots,n-1,$$

then $\partial^i_w u \in L^p(U_1)$, with

$$\left\|\partial^i_w u\right\|_{L^p(U_1)} \le C, \ \textit{for the same constant as above.}$$

Proof. 1. Let $p \ne \infty$. For every $x \in U_2$ and every $i = 1,\dots,n$, we define

$$u_{x,i} : \left\{t \in \mathbb{R} \,\middle|\, (x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n) \in U_2\right\} \to \mathbb{C}$$

by

$$u_{x,i}(t) \coloneqq u^*(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n),$$

[5] Simple examples of such pairs are two concentric cylinders (=) and two concentric balls (⊂).

where u^* stands for the representative of u that is absolutely continuous on $(n-1)$-a.e. parallel to a coordinate axis line segment of U_2 and whose first-order partial derivatives (in the ordinary sense) belong to $L^p(U_2)$ and agree a.e. with the weak partial derivatives of u (see, e.g., Theorem 11.45 along with Remark 11.46 in [35], or Theorem 1, Section 1.1.3 in [37]). Now, let $i=1,\dots,n-1$ and $h\in(-\delta,\delta)^*$ be arbitrary. From the absolute continuity, we have

$$u_{x,i}(x_i+h)-u_{x,i}(x_i)=\int_{x_i}^{x_i+h} u'_{x,i}(t)dt, \text{ for a.e. } x\in U_1,$$

or else

$$u^*(x+he_i)-u^*(x)=\int_{x_i}^{x_i+h}\left[\partial^i u\right](x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)dt, \text{ for a.e. } x\in U_1.$$

Changing the variable from t to x_i+th we get

$$u^*(x+he_i)-u^*(x)=h\int_0^1\left[\partial^i u\right](x+the_i)dt, \text{ for a.e. } x\in U_1,$$

or else

$$\left[\partial^{i,h}u^*\right](x)=\int_0^1\left[\partial^i_w u\right](x+the_i)dt, \text{ for a.e. } x\in U_1.$$

So, by the Hölder inequality we deduce

$$\left|\left[\partial^{i,h}u^*\right](x)\right|^p\le\int_0^1\left|\left[\partial^i_w u\right](x+the_i)\right|^p dt.$$

Hence by the Tonelli theorem,

$$\left\|\partial^{i,h}u\right\|_{L^p(U_1)}\le\int_{U_1}\int_0^1\left|\left[\partial^i_w u\right](x+the_i)\right|^p dtdx=\int_0^1\int_{U_1}\left|\left[\partial^i_w u\right](x+the_i)\right|^p dxdt.$$

By changing the coordinates x to $x+the_i$ and applying the change of variables formula, we conclude to the desired result.

The case $p=\infty$ follows from the result for $p\neq\infty$, in an analogous manner as in Step 3 of the proof of Theorem 1.2.3 (i.e. considering $\left\{U_{1_j}\subset\subset U_1\right\}_j$ with $U_{1_j}\nearrow U_1$, cutting at ∂U_{1_j} and extending by zero to whole U_1), and so we omit it.

2. Let $p\neq 1$. We consider a sequence $\{h_k\}_k\subset(-\delta',\delta')^*$ with $|h_k|\searrow 0$. Since $\left\|\partial^{i,h_k}u\right\|_{L^p(U_1)}\le C$ for every k, we argue as in Step 3 of the proof of Theorem 1.2.3 in order to find a subsequence $\{h_{k_l}\}_l\subseteq\{h_k\}_k$ and a function $u_i\in L^p(U_1)$, such that

$$\partial^{i,h_{k_l}}u\overset{*}{\rightharpoonup}u_i \text{ in } L^p(U_1) \text{ and } \|u_i\|_{L^p(U_1)}\le C. \tag{1.2.22}$$

Let $\phi\in C_c^\infty(U_1)$ be arbitrary and we consider a subsequence of $\{h_{k_l}\}_l$, which depends on ϕ and we still denote as such, such that

$$|h_{k_l}|<\min\{\delta',\operatorname{dist}(\operatorname{supp}(\phi),\partial U_1)\},\ \forall l.$$

Employing

(a) the dominated convergence theorem,
(b) (1.2.21) and
(c) (1.2.22),

we deduce that

$$\int_{U_1}u\left(\partial^i\phi\right)dx=\int_{\operatorname{supp}(\phi)}u\left(\partial^i\phi\right)dx\overset{(a)}{=}\lim_{l\nearrow\infty}\int_{\operatorname{supp}(\phi)}u\left(\partial^{i,-h_{k_l}}\phi\right)dx\overset{(b)}{=}$$
$$\overset{(b)}{=}-\lim_{l\nearrow\infty}\int_{\operatorname{supp}(\phi)}\left(\partial^{i,h_{k_l}}u\right)\phi dx=-\lim_{l\nearrow\infty}\int_{U_1}\left(\partial^{i,h_{k_l}}u\right)\phi dx\overset{(c)}{=}-\int_{U_1}u_i\,\phi dx$$

and the weak i-partial differentiability follows by the definition, since ϕ is arbitrary.

□

Remark 1.2.9. *For the case $p \neq 1, \infty$ of Point 2. of the above result, we can also employ the reflexivity of L^p-spaces, the well known corollary of the Banach-Alaoglu-Bourbaki theorem (see, e.g., Theorem 3.18 in [8]) and the (sequentially) weak lower semi-continuity of the norm (see, e.g., Point (iii) of Proposition 3.5 in [8]).*

Now, we can easily deduce the following result.

Corollary 1.2.3. *Let U and $\delta > 0$.*

1. *If $p \in [1, \infty]$ and $u \in \{u \in L^1_{\text{loc}}(U^\delta) \mid \nabla_w u \in L^p(U^\delta)\}$, then $\nabla^h u \in L^p(U)$ for every $h \in (-\delta, \delta)^*$, with*

$$\|\nabla^h u\|_{L^p(U)} \leq C \|\nabla_w u\|_{L^p(U^\delta)}.$$

2. *If $p \in (1, \infty]$, $u \in L^p(U^\delta)$ and there exists $\delta' \in (0, \delta]$ such that*

$$\|\nabla^h u\|_{L^p(U)} \leq C, \quad \forall h \in (-\delta', \delta')^*,$$

then, $\nabla_w u \in L^p(U)$, with

$$\|\nabla_w u\|_{L^p(U)} \leq C, \text{ for the same constant as above.}$$

Proof. We only show Point 1. since the other one can be dealt with the use of the same arguments. It also suffices to show the result for $p \neq \infty$. We apply Point 1. of Proposition 1.2.9 for $U_1 = U \times (0, r)$ and $U_2 = U^\delta \times (0, r)$, for some $r > 0$, as well as for the function $v : L^1_{\text{loc}}(U_2)$ with $v(x, x_{n+1}) \coloneqq u(x)$, for every $(x, x_{n+1}) \in U_2$. We note that

$$\begin{cases} \partial^i_w v = \partial^i_w u, & \text{for } i = 1, \ldots, n \\ \partial^{n+1}_w v = 0, \end{cases}$$

hence $v \in W^{1,p}(U_2)$, and so the aforementioned proposition is applicable. Therefore, we deduce that

$$\|\partial^{i,h} v\|_{L^p(U_1)} \leq \|\partial^i_w v\|_{L^p(U_2)}, \quad \forall, i = 1, \ldots, n, \; h \in (-\delta, \delta)^*,$$

thus, from the Tonelli theorem we get the desired result. □

1.2.10 Chain rule

Next result is a slight generalization of a known result (see, e.g., Point (i) of Exercise 11.51 and Exercise 11.52 in [35]). Before we proceed, we refer to Definition 3.70 in [35] for the definition of a purely $\mathcal{H}^1$-unrectifiable set, where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

Proposition 1.2.10. *Let $m \in \mathbb{N}$, $p \in [1, \infty]$, U and $u \in \{u \in L^1_{\text{loc}}(U; \mathbb{R}^m) \mid J_w u \in L^p(U)\}$. If $f : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous with the additional assumption that*

$$\text{the set } \{x \in \mathbb{R}^m \mid f \text{ is not differentiable at } x\} \text{ is purely } \mathcal{H}^1\text{-unrectifiable, when } m \neq 1,$$

then $f \circ u \in \{u \in L^1_{\text{loc}}(U; \mathbb{R}) \mid \nabla_w u \in L^p(U)\}$, with[6]

$$\nabla_w (f \circ u) = ((\nabla f) \circ u) \, J_w u, \text{ a.e. in } U.$$

Proof. First of all, $f \circ u \in L^1_{\text{loc}}(U; \mathbb{R})$, since for every $x, y \in U$ we have

$$(f \circ u)(x) \leq \text{Lip}(f) \, |u(x) - u(y)| + |u(y)| \leq \text{Lip}(f) \, |u(x)| + (\text{Lip}(f) + 1) \, |u(y)|,$$

from the triangle inequality, hence the result follows from the fact that $u \in L^1_{\text{loc}}(U; \mathbb{R}^m)$. It suffices then to show that $(\partial^i_w (f \circ u)) \in L^p(U)$ for every $i = 1, \ldots, n$, with

$$\partial^i_w (f \circ u) = \sum_{j=1}^m \left((\partial^j f) \circ u \right) \left(\partial^i_w u_j \right), \text{ a.e. in } U, \text{ for every } i = 1, \ldots, n. \tag{1.2.23}$$

[6] For every $y \in \mathbb{R}^m$ and every $a \in \mathbb{R}^{m \times n}$, we write $(ya) \in \mathbb{R}^n$ for $ya = a^{\mathrm{T}} y$.

In view of the fact that ∇f is bounded, (1.2.11) and the fact that $\nabla_w u_j \in L^p(U)$ for every $j = 1,\dots,m$, we directly deduce that the right hand of the above formula belongs to $L^p(U)$ also. Therefore, it is only left for us to show (1.2.23) by the definition of the weak derivatives. Indeed, let $\phi \in C_c^\infty(U)$ be arbitrary, and we set

$$\delta = \delta_\phi \coloneqq \frac{\operatorname{dist}(\operatorname{supp}(\phi), U)}{2} > 0,$$

as well as[7]

$$v = v_\phi \in \left\{ u \in L^1_{\mathrm{loc}}\left(\operatorname{supp}(\phi)^{\circ\delta}; \mathbb{R}^m\right) \,\middle|\, J_w u \in L^p\left(\operatorname{supp}(\phi)^{\circ\delta}\right) \right\}$$

to be

$$v = (v_j)_{j=1}^m \coloneqq \left(\left(\mathcal{R}\left(U, \operatorname{supp}(\phi)^{\circ\delta}\right)\right) u_j \right)_{j=1}^m = \left(\mathcal{R}\left(U, \operatorname{supp}(\phi)^{\circ\delta}\right)\right) u.$$

For the case $p = \infty$, we notice that, since $|\operatorname{supp}(\phi)| < \infty$, then $L^\infty\left(\operatorname{supp}(\phi)^{\circ\delta}\right) \hookrightarrow L^q\left(\operatorname{supp}(\phi)^{\circ\delta}\right)$ for every $q \in [1,\infty)$. Hence, employing the notation used for the proof of Proposition 1.2.9, we have, in view of Theorems 3.59 and 3.73 in [35], that

$$\begin{gathered} \partial^i (f \circ v^*) = \sum_{j=1}^m \left((\partial^j f) \circ v^* \right) {v_{j_{x,i}}}', \\ \text{a.e. in } \left\{ t \in \mathbb{R} \,\middle|\, (x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n) \in \operatorname{supp}(\phi)^{\circ\delta} \right\}, \text{ for every } x \in \operatorname{supp}(\phi)^{\circ\delta}, \\ \text{for every } i = 1,\dots,n. \end{gathered} \tag{1.2.24}$$

Now, if $n \neq 1$, for every $x \in \operatorname{supp}(\phi)^\circ$ and every $i = 1,\dots,n$, we set $x_i' \in \mathbb{R}^{n-1}$ for

$$x_i' \coloneqq (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

Moreover, we consider a sequence $\{h_k\}_k \subset (-\delta,\delta)^*$ such that $|h_k| \searrow 0$. From

1. the dominated convergence theorem,
2. Lemma 1.2.2,
3. $v^* = v$ a.e. (in $\operatorname{supp}(\phi)^{\circ\delta}$),

we have, for every $i = 1,\dots,n$, that

$$\begin{gathered} \int_U (f \circ u)\left(\partial^i \phi\right) dx = \int_{\operatorname{supp}(\phi)} (f \circ u)\left(\partial^i \phi\right) dx = \int_{\operatorname{supp}(\phi)} (f \circ v)\left(\partial^i \phi\right) dx \overset{1.}{=} \\ \overset{1.}{=} \lim_{k\nearrow\infty} \int_{\operatorname{supp}(\phi)} (f \circ v)\left(\partial^{i,-h_k}\phi\right) dx \overset{2.}{=} -\lim_{k\nearrow\infty} \int_{\operatorname{supp}(\phi)} \left(\partial^{i,h_k}(f\circ v)\right)\phi dx \overset{3.}{=} \\ \overset{3.}{=} -\lim_{k\nearrow\infty} \int_{\operatorname{supp}(\phi)} \left(\partial^{i,h_k}(f\circ v^*)\right)\phi dx. \end{gathered}$$

If $n = 1$, from

1. the dominated convergence theorem
2. (1.2.24),
3. $v^* = v$ a.e. (in $\operatorname{supp}(\phi)^{\circ\delta}$) with

$$v_j^{*\prime} = \left(\frac{d}{dx}\right)_w v_j, \text{ a.e., for every } j = 1,\dots,m,$$

   and

4. (1.2.5),

[7] That is $\operatorname{supp}(\phi)^{\circ\delta} = (\operatorname{int}(\operatorname{supp}(\phi)))^\delta$ (see Definition 1.2.8).

we get

$$\int_U (f\circ u)\,\phi' dx = -\lim_{k\nearrow\infty}\int_{\mathrm{supp}(\phi)} \frac{f(v^*(x+h_k))-f(v^*(x))}{h_k}\phi(x)dx \overset{1.}{=} -\int_{\mathrm{supp}(\phi)} (f\circ v^*)'\phi dx \overset{2.}{=}$$

$$\overset{2.}{=} -\int_{\mathrm{supp}(\phi)} \left(\sum_{j=1}^m \left((\partial^j f)\circ v^*\right) {v_j^*}'\right)\phi dx \overset{3.}{=} -\int_{\mathrm{supp}(\phi)} \left(\sum_{j=1}^m \left((\partial^j f)\circ v\right)\left(\frac{d}{dx}\right)_w v_j\right)\phi dx \overset{4.}{=}$$

$$\overset{4.}{=} -\int_{\mathrm{supp}(\phi)} \left(\sum_{j=1}^m \left((\partial^j f)\circ u\right)\left(\frac{d}{dx}\right)_w u_j\right)\phi dx = -\int_U \left(\sum_{j=1}^m \left((\partial^j f)\circ u\right)\left(\frac{d}{dx}\right)_w u_j\right)\phi dx$$

and the result follows from the arbitrariness of ϕ. If $n\neq 1$, we set

$$\mathrm{supp}(\phi)^\circ_{x,i} \coloneqq \left\{t\in\mathbb{R}\,\middle|\, (x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\in\mathrm{supp}(\phi)^\circ\right\},\ \forall x\in\mathrm{supp}(\phi)^\circ,\ \forall i=1,\dots,n$$

and from

1. the Fubini theorem,
2. the dominated convergence theorem,
3. (1.2.24) and
4. $v^*=v$ a.e. (in $\mathrm{supp}(\phi)^{\circ^\delta}$) with

   $${v_{j_{x,i}}}'(t) = \left(\frac{d}{dt}\right)_w v_j(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n),$$

   for a.e. $t\in\mathrm{supp}(\phi)^\circ_{x,i}$, for every $i=1,\dots,n$ and every $j=1,\dots,m$, and
5. (1.2.5),

we have, for every $i=1,\dots,n$, that

$$\int_U (f\circ u)\left(\partial^i\phi\right)dx \overset{1.}{=}$$

$$\overset{1.}{=} -\lim_{k\nearrow\infty}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathrm{supp}(\phi)^\circ_{x,i}} \left(\partial^{i,h_k}(f\circ v^*)\right)\phi dt\right)dx_i' \overset{2.\times 2}{=}$$

$$\overset{2.\times 2}{=} -\int_{\mathbb{R}^{n-1}}\left(\int_{\mathrm{supp}(\phi)^\circ_{x,i}} \left(\partial^{i}(f\circ v^*)\right)\phi dt\right)dx_i' \overset{3.}{=}$$

$$\overset{3.}{=} -\int_{\mathbb{R}^{n-1}}\left(\int_{\mathrm{supp}(\phi)^\circ_{x,i}} \left(\sum_{j=1}^m \left(\partial^j f\circ v^*\right){v_{j_{x,i}}}'\right)\phi dt\right)dx_i' \overset{4.}{=}$$

$$\overset{4.}{=} -\int_{\mathbb{R}^{n-1}}\left(\int_{\mathrm{supp}(\phi)^\circ_{x,i}} \left(\sum_{j=1}^m \left(\partial^j f\circ v\right)\left(\frac{d}{dt}\right)_w v_j(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\right)\phi dt\right)dx_i' \overset{5.}{=}$$

$$\overset{5.}{=} -\int_{\mathbb{R}^{n-1}}\left(\int_{\mathrm{supp}(\phi)^\circ_{x,i}} \left(\sum_{j=1}^m \left(\partial^j f\circ u\right)\left(\frac{d}{dt}\right)_w u_j(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\right)\phi dt\right)dx_i' \overset{1.}{=}$$

$$\overset{1.}{=} -\int_{\mathrm{supp}(\phi)}\left(\sum_{j=1}^m \left((\partial^j f)\circ u\right)\left(\partial^i_w u_j\right)\right)\phi dx = -\int_U\left(\sum_{j=1}^m \left((\partial^j f)\circ u\right)\left(\partial^i_w u_j\right)\right)\phi dx.$$

The result then follows since ϕ is arbitrary. □

Corollary 1.2.4. *Let $p\in[1,\infty]$, U and $u\in\left\{u\in L^1_{\mathrm{loc}}(U;\mathbb{C})\,\middle|\,\nabla_w u\in L^p(U)\right\}$. Then*

$$|u|\in\left\{u\in L^1_{\mathrm{loc}}(U;\mathbb{R})\,\middle|\,\nabla_w u\in L^p(U)\right\},\ \textit{with}\ |\nabla_w|u||\le|\nabla_w u|,\ \textit{a.e. in } U. \tag{1.2.25}$$

Proof. Identifying the metric space $(\mathbb{C},|\ast-\star|)$ with the metric space $\left(\mathbb{R}^2,|\ast-\star|\right)$, we may consider $|\cdot|$ as a Lipschitz continuous function from $\mathbb{R}^2$ to $\mathbb{R}$. It is evident (see, e.g., Theorem 3.72 in [35]) that the set

$$\left\{x\in\mathbb{R}^2\,\middle|\, |\cdot| \text{ is not differentiable at } x\right\} = \{0\}$$

is purely $\mathcal{H}^1$-unrectifiable. Hence, $|u|\in\left\{u\in L^1_{\mathrm{loc}}(U;\mathbb{R})\,\middle|\,\nabla_w u\in L^p(U)\right\}$ directly from Proposition 1.2.10. As for the inequality in (1.2.25), directly from

1. (1.2.23) (for $m=2$),
2. the Cauchy-Schwarz inequality and
3. the fact that $\mathrm{Lip}(|\cdot|)=1$ along with Remark 1.2.3,

we get $\left|\partial_w^i|u|\right|\leq\left|\partial^i u\right|$. The desired inequality then follows trivially. □

1.2.11 Cut-off functions

In what follows, we make systematic use of the result below, which concerns cut-off functions.

Proposition 1.2.11. *Let U and $\delta>0$. Then there exists $\phi\in C_c^\infty(\mathbb{R}^n;[0,1])$ such that*

1. $\mathrm{supp}(\phi)\subseteq\overline{U^\delta}$,
2. $\phi\equiv 1$ *in* $\overline{U}$ *and*
3. $\left\|\nabla^k\phi\right\|_{L^\infty(\mathbb{R}^n)}\leq\frac{C_k}{\delta^k}$, *for every* $k\in\mathbb{N}_0$ $(C_0=1)$.

Proof. We consider $\phi=\varphi_\delta*\chi_U$, i.e.

$$\phi(x)=\int_{\mathbb{R}^n}\varphi_\delta(x-y)\chi_U(y)dy=\int_{B(x,\delta)}\varphi_\delta(x-y)\chi_U(y)dy,\ \forall x\in\mathbb{R}^n,$$

where φ_δ stands for the standard mollifier with $\mathrm{supp}(\varphi)\subseteq\overline{B(0,\delta)}$ and also χ_U for the characteristic function of U. It is well known that $\phi\in C^\infty(\mathbb{R}^n)$ with $D^\alpha\phi=D^\alpha\varphi_\delta*\chi_U$, for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\geq 1$. If $x\in\overline{U}$, then $B(x,\delta)\subset U$, thus

$$\phi(x)=\int_{B(x,\delta)}\varphi_\delta(x-y)dy=1,\ \forall x\in\overline{U}.$$

Similarly we can get that $\phi(x)\in[0,1]$ for every $x\in\mathbb{R}^n$, since the same is true for χ_U. If $x\in\overline{U^\delta}^{\mathsf{c}}$, then $B(x,\delta)\cap U=\varnothing$, thus $\phi(x)=0$ for every such x and so $\mathrm{supp}(\phi)\subseteq\overline{U^\delta}$. Lastly, from the Faá di Bruno formula, we have

$$|D^\alpha\phi(x)|\leq\int_{\mathbb{R}^n}|D^\alpha\varphi_\delta(x-y)||\chi_U(y)|dy\leq\left\|\nabla^{|\alpha|}\varphi_\delta\right\|_{L^1(\mathbb{R}^n)}\leq\frac{C_{|\alpha|}}{\delta^{|\alpha|}},\ \forall\alpha\in\mathbb{N}_0^n.$$

□

In view of Proposition 1.2.11, we get a generalization of a well known result that concerns functions of compact support (see, e.g., Lemma 9.5 in [8]), since we drop the assumption of its boundedness. In fact this generalization is not unexpected, since it can shown that the space

$$\left\{u\in W^{m,p}(U)\,\middle|\,\mathrm{supp}(u)\text{ is bounded}\right\}$$

is dense in $W^{m,p}(U)$, for every $m\in\mathbb{N}$ and $p\in[1,\infty)$, by dealing in a similar manner as below, i.e. by considering a sequence of "expanding" cut-off functions.

Proposition 1.2.12. *Let $m\in\mathbb{N}$, $p\in[1,\infty)$, U and $u\in W^{m,p}(U)$. If $\mathrm{dist}(\mathrm{supp}(u),\partial U)>0$, then $u\in W_0^{m,p}(U)$.*

Proof. We define

$$\delta\coloneqq\frac{\mathrm{dist}(\mathrm{supp}(u),\partial U)}{2}>0$$

and we fix a function $\phi\in C_c^\infty(\mathbb{R}^n;[0,1])$, such that

1. $\mathrm{supp}(\phi)\subseteq\overline{\mathrm{supp}(u)^\delta}$,
2. $\phi\equiv 1$ in $\mathrm{supp}(u)$ and
3. $\sum_{j=1}^m\left\|\nabla^j\phi\right\|_{L^\infty(\mathbb{R}^n)}\leq C=C_\delta$.

Moreover, we fix $x_0 \in \text{supp}(u)$ (in fact, we can consider any $x_0 \in \mathbb{R}^n$) and we consider a sequence $\{\phi_k\}_k \subset C_c^\infty(\mathbb{R}^n;[0,1])$, such that

1. $\text{supp}(\phi_k) \subseteq \overline{B(x_0,k+1)}$,
2. $\phi_k \equiv 1$ in $\overline{B(x_0,k)}$ and
3. $\sum_{j=1}^m \left\|\nabla^j \phi_k\right\|_{L^\infty(\mathbb{R}^n)} \le C$, uniformly for every k.

Now, since $p \ne \infty$, in virtue of the Meyers-Serrin theorem we consider $\{u_k\}_k \subset C^m(U) \cap W^{m,p}(U)$ such that $u_k \to u$ in $W^{m,p}(U)$. We set

$$\{v_k\}_k \subset C_c^m(U), \text{ for } v_k \coloneqq ((\mathcal{R}(\mathbb{R}^n,U))\,\phi\phi_k)\,u_k \text{ for every } k,$$

as well as

$$v \coloneqq ((\mathcal{R}(\mathbb{R}^n,U))\,\phi)\,u \equiv u.$$

It then suffices to show that $v_k \to v$ in $W^{m,p}(U)$ (by the definition of $W_0^{m,p}$-spaces). We have that

$$v_k - v = ((\mathcal{R}(\mathbb{R}^n,U))\,\phi\phi_k)\,(u_k - u) + ((\mathcal{R}(\mathbb{R}^n,U))\,((\phi_k - 1)\,\phi))\,u.$$

In virtue of Proposition 1.2.8, both terms of the right side of the above equation belong to $W^{m,p}(U)$, hence

$$\|v_k - v\|_{W^{m,p}(U)} \le \|((\mathcal{R}(\mathbb{R}^n,U))\,\phi\phi_k)\,(u_k-u)\|_{W^{m,p}(U)} + \|((\mathcal{R}(\mathbb{R}^n,U))\,((\phi_k-1)\,\phi))\,u\|_{W^{m,p}(U)}.$$

For the first term, we apply (1.2.9) to obtain

$$\|((\mathcal{R}(\mathbb{R}^n,U))\,\phi\phi_k)\,(u_k-u)\|_{W^{m,p}(U)} \le C\|u_k - u\|_{W^{m,p}(U)} \to 0.$$

As for the second term, we have from

1. $(\mathcal{R}(\mathbb{R}^n, B(x_0,k)))\,(\phi_k - 1) \equiv 0$, for every k,
2. (1.2.9) and
3. $p \ne \infty$ (U may be an unbounded set),

that

$$\|((\mathcal{R}(\mathbb{R}^n,U))\,((\phi_k-1)\,\phi))\,u\|_{W^{m,p}(U)} \overset{1.}{=} \left\|\left(\mathcal{R}\left(\mathbb{R}^n, B(x_0,k)^{\mathsf{c}} \cap U\right)\right)(\phi u)\right\|_{W^{m,p}(B(x_0,k)^{\mathsf{c}}\cap U)} \overset{2.}{\le}$$
$$\overset{2.}{\le} \left\|\left(\mathcal{R}\left(\mathbb{R}^n, B(x_0,k)^{\mathsf{c}} \cap U\right)\right)u\right\|_{W^{m,p}(B(x_0,k)^{\mathsf{c}}\cap U)} \overset{3.}{\to} 0.$$

□

1.2.12 Elliptic regularity theory for m-Lipschitz boundaries

Here, we make a thorough discussion on the elliptic regularity theory, in order to extract certain useful results.

Interior regularity

The crux of this subsection is the application of Corollary 1.2.3 for $p = 2, \infty$.

Theorem 1.2.4. *Let U and $(u,f) \in H^1(U) \times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a \in W^{1,\infty}(U)$ and $f \in L^2(U)$, then*

1. *$u \in H^2(U_\delta)$ for every $\delta > 0$, with*
$$\left\|\nabla_w^2 u\right\|_{L^2(U_\delta)} \le \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{L^2(U)}\right), \ \forall\, 0<\delta'<\delta.$$
2. *$L_w u = f$ a.e. in U_δ, for every $\delta > 0$.*

Proof. Let $0<\delta'<\delta$ be small enough so that $U_\delta \neq \varnothing$, otherwise we have nothing to show.

Step 1

In view of Proposition 1.2.11, we consider a cut-off function $\phi \in C_c^\infty(\mathbb{R}^n;[0,1])$ such that

1. $\mathrm{supp}(\phi)\subseteq \overline{U_{\delta'}}$,
2. $\phi \equiv 1$ in $\overline{U_\delta}$ and
3. $\|\nabla\phi\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{\delta-\delta'}$.

For every $i=1,\dots,n$ and $h\in\left(-\frac{\delta'}{4},\frac{\delta'}{4}\right)^*$, we consider the operators $\partial_1^{i,h}:\mathcal{F}(U)\to\mathcal{F}\left(U_{\frac{\delta'}{4}}\right)$ and $\partial_2^{i,h}:\mathcal{F}\left(U_{\frac{\delta'}{4}}\right)\to\mathcal{F}\left(U_{\frac{\delta'}{2}}\right)$, as well as the operator $\mathcal{V}_{i,h,\phi}:\mathcal{F}(U)\to\mathcal{F}\left(U_{\frac{\delta'}{2}}\right)$ with

$$\mathcal{V}_{i,h,\phi}v \coloneqq \partial_2^{i,-h}\left(\phi^2\left(\partial_1^{i,h}v\right)\right),\ \forall v\in\mathcal{F}(U).$$

In view of (1.2.20) we have

$$\mathcal{V}_{i,h,\phi}v=-\left(\phi^2\right)^{i,-h}\left(\left(\partial_2^{i,-h}\circ\partial_1^{i,h}\right)v\right)-\left(\partial_1^{i,h}u\right)\left(\partial_2^{i,h}\phi^2\right),$$

hence, $\mathcal{V}_{i,h,\phi}:H^1(U)\to H^1\left(U_{\frac{\delta'}{2}}\right)$ with

$$\mathrm{supp}(\mathcal{V}_{i,h,\phi}v)\subseteq\overline{U_{\frac{3\delta'}{4}}},\ \forall v\in H^1(U),$$

since $\mathrm{supp}(\phi)\subseteq\overline{U_{\delta'}}$, thereby, from Proposition 1.2.12, we get that $\mathcal{V}_{i,h,\phi}:H^1(U)\to H_0^1\left(U_{\frac{\delta'}{2}}\right)$. In addition, from Proposition 1.2.3, we conclude that

$$\left(\mathcal{E}_0\left(U_{\frac{\delta'}{2}},U\right)\right)\circ\mathcal{V}_{i,h,\phi}:H^1(U)\to\left\{v\in H_0^1(U)\,\middle|\,\mathrm{supp}(v)\subseteq\overline{U_{\frac{3\delta'}{4}}}\right\},\ \forall\, i,h.$$

Step 2α

For every i we choose $v_i\coloneqq-\left(\left(\mathcal{E}_0\left(U_{\frac{\delta'}{2}},U\right)\right)\circ\mathcal{V}_{i,h,\phi}\right)u$ in the variational equation $\mathcal{L}_w u=f$, thus, in virtue of Lemma 2.3.1, we obtain

$$-\sum_{k,l=1}^n\int_{U_{\frac{3\delta'}{4}}}a_{kl}\left(\partial_w^k\overline{u}\right)\partial_w^l\left(\partial^{i,-h}\left(\phi^2\partial^{i,h}u\right)\right)dx=\int_{U_{\frac{3\delta'}{4}}}\overline{f}v_i dx,$$

and from the change of variables formula we deduce

$$-\sum_{k,l=1}^n\int_{U_{\frac{3\delta'}{4}}}a_{kl}\left(\partial_w^k\overline{u}\right)\partial^{i,-h}\left(\partial_w^l\left(\phi^2\partial^{i,h}u\right)\right)dx=\int_{U_{\frac{3\delta'}{4}}}\overline{f}v_i dx.$$

From (1.2.21), we obtain

$$\sum_{k,l=1}^n\int_{U_{\frac{3\delta'}{4}}}\partial^{i,h}\left(a_{kl}\left(\partial_w^k\overline{u}\right)\right)\left(\partial_w^l\left(\phi^2\partial^{i,h}u\right)\right)dx=\int_{U_{\frac{3\delta'}{4}}}\overline{f}v_i dx,$$

or else

$$\sum_{k,l=1}^n\int_{U_{\delta'}}\partial^{i,h}\left(a_{kl}\left(\partial_w^k\overline{u}\right)\right)\left(\partial_w^l\left(\phi^2\partial^{i,h}u\right)\right)dx=\int_{U_{\frac{3\delta'}{4}}}\overline{f}v_i dx,$$

since $\mathrm{supp}(\phi)\subseteq\overline{U_{\delta'}}$. Considering the real parts in both sides, we get, in virtue of (1.2.20), that

$$\mathrm{Re}(I_1)=\mathrm{Re}\left(\int_{U_{\frac{3\delta'}{4}}}\overline{f}v_i dx\right)=:I_2, \tag{1.2.26}$$

where

$$I_1=I_{11}+I_{12}\coloneqq\sum_{k,l=1}^n\int_{U_{\delta'}}\phi^2 a_{kl}^{i,h}\left(\left(\partial^{i,h}\circ\partial_w^k\right)\overline{u}\right)\left(\left(\partial^{i,h}\circ\partial_w^l\right)u\right)dx+$$

$$+\sum_{k,l=1}^n\int_{U_{\delta'}}\Big[2\phi\left(\partial^l\phi\right)a_{kl}^{i,h}\left(\left(\partial^{i,h}\circ\partial_w^k\right)\overline{u}\right)\left(\partial^{i,h}u\right)+\phi^2\left(\partial^{i,h}a_{kl}\right)\left(\partial_w^k\overline{u}\right)\left(\left(\partial^{i,h}\circ\partial_w^l\right)u\right)+$$

$$+2\phi\left(\partial^l\phi\right)\left(\partial^{i,h}a_{kl}\right)\left(\partial_w^k\overline{u}\right)\left(\partial^{i,h}u\right)\Big]dx.$$

Step 2β

We directly deduce that

$$\mathrm{Re}(I_{11})\geq\theta\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}. \tag{1.2.27}$$

For the second term, we have, from Point 1. of Corollary 1.2.3, that

$$|I_{12}|\leq\mathcal{K}\left(\frac{1}{\delta-\delta'},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\int_{U_{\delta'}}\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\left|\partial^{i,h}u\right|+\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\left|\nabla_w u+\phi\left|\partial^{i,h}u\right|\left|\nabla_w u\right|\right|dx$$

and from the Cauchy inequality with $\frac{\theta}{2}$, we obtain

$$|I_{12}|\leq\frac{\theta}{2}\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}+$$
$$+\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\left(\left\|\partial^{i,h}u\right\|^2_{L^2(U_{\delta'})}+\|\nabla_w u\|^2_{L^2(U_{\delta'})}\right).$$

Hence, from Point 1. of Corollary 1.2.3, we derive

$$|I_{12}|\leq\frac{\theta}{2}\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}+\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\|\nabla_w u\|^2_{L^2(U)}. \tag{1.2.28}$$

Therefore, (1.2.27) and (1.2.28) imply

$$\mathrm{Re}(I_1)\geq\frac{\theta}{2}\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}-\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\|\nabla_w u\|^2_{L^2(U)}. \tag{1.2.29}$$

Step 2γ

From Point 1. of Corollary 1.2.3, we get

$$\|v_i\|_{L^2\left(U_{\frac{3\delta'}{4}}\right)}\leq C\left\|\nabla_w\left(\phi^2\left(\partial^{i,h}u\right)\right)\right\|_{L^2\left(U_{\frac{\delta'}{2}}\right)}=C\left\|\nabla_w\left(\phi^2\left(\partial^{i,h}u\right)\right)\right\|_{L^2(U_{\delta'})}\leq$$
$$\leq\mathcal{K}\left(\frac{1}{\delta-\delta'}\right)\left(\left\|\partial^{i,h}u\right\|_{L^2(U_{\delta'})}+\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|_{L^2(U_{\delta'})}\right)\leq$$
$$\leq\mathcal{K}\left(\frac{1}{\delta-\delta'}\right)\left(\|\nabla_w u\|_{L^2(U)}+\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|_{L^2(U_{\delta'})}\right).$$

Thus, from the Hölder inequality along with the Cauchy inequality with $\frac{\theta}{4}$, we deduce

$$|I_2|\leq\|f\|_{L^2\left(U_{\frac{3\delta'}{4}}\right)}\|v_i\|_{L^2\left(U_{\frac{3\delta'}{4}}\right)}\leq\frac{\theta}{4}\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}+$$
$$+\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta}\right)\left(\|\nabla_w u\|^2_{L^2(U)}+\|f\|^2_{L^2(U)}\right). \tag{1.2.30}$$

Step 2δ

Combing (1.2.26), (1.2.29) and (1.2.30) and the fact that $\phi\equiv1$ in $\overline{U_\delta}$, we obtain

$$\frac{\theta}{4}\left\|\left(\partial^{i,h}\circ\nabla_w\right)u\right\|^2_{L^2(U_\delta)}\leq\frac{\theta}{4}\left\|\phi\left|\left(\partial^{i,h}\circ\nabla_w\right)u\right|\right\|^2_{L^2(U_{\delta'})}\leq$$
$$\leq\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\left(\|\nabla_w u\|^2_{L^2(U)}+\|f\|^2_{L^2(U)}\right),$$

that is

$$\left\|\left(\partial^{i,h}\circ\nabla_w\right)u\right\|^2_{L^2(U_\delta)}\leq\mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\left(\|\nabla_w u\|^2_{L^2(U)}+\|f\|^2_{L^2(U)}\right).$$

From Point 2. of Corollary 1.2.3, we conclude to the desired result.

Step 3

The quantity $L_w u\in L^2(U_\delta)$ is now well defined[8]. From

[8] We can not claim that $L_w u\in L^1_{\mathrm{loc}}(U)$, since the second weak derivatives of u do not exist (as functions) in U.

1. Lemma 2.3.1 and
2. the definition of the weak derivatives,

we deduce, for every $\psi \in C_c^\infty(U_\delta)$, that

$$\int_{U_\delta} \overline{f}\,\psi dx = \int_U \overline{f}\,\left((\mathcal{E}_0(U_\delta,U))\,\psi\right)dx = \overline{(f,(\mathcal{E}_0(U_\delta,U))\,\psi)} \overset{1.}{=} \langle f,(\mathcal{E}_0(U_\delta,U))\,\psi\rangle =$$
$$= \int_U (\nabla_w \circ (\mathcal{E}_0(U_\delta,U)))\,\psi \cdot a\nabla_w \overline{u} dx = \int_{U_\delta} \nabla_w \psi \cdot a\nabla_w \overline{u} dx \overset{2.}{=} -\int_{U_\delta} \operatorname{div}(a\nabla\overline{u})\,\psi dx$$

and thus we get $-\operatorname{div}(a\nabla\overline{u}) = \overline{f}$ a.e. in U_δ, or else $L_w u = f$ a.e. in U_δ, by (5).

□

Remark 1.2.10. *Since every bounded and Lipschitz continuous function on an arbitrary U also belongs to $W^{1,\infty}(U)$, the conclusions of Theorem 1.2.4 is also true for every a_{ij} being Lipschitz continuous instead. We will need this fact later on, so we prove it, along with a bound for the weak derivative: Let $u \in L^\infty(U)$ be Lipschitz with $\operatorname{Lip}(u) \le L$. It suffices to show that $\nabla_w u = \nabla u$, since then we have $\|\nabla_w u\|_{L^\infty(U)} = \|\nabla u\|_{L^\infty(U)} \le L$. We show the claim by the definition of the weak derivatives. Let $\psi \in C_c^\infty(U)$ be arbitrary, set $\delta = \delta_\psi \coloneqq \operatorname{dist}(\operatorname{supp}(\psi),\partial U)$ and consider a sequence $\{h_k\}_k \subset (-\delta,\delta)^*$ such that $|h_k| \searrow 0$. In view of*

1. the dominated convergence theorem and

2. Lemma 1.2.2,

we have, for every $i = 1,\ldots,n$, that

$$\int_U u\left(\partial^i \psi\right)dx = \int_{\operatorname{supp}(\psi)} u\left(\partial^i \psi\right)dx \overset{1.}{=} \lim_{k\nearrow\infty} \int_{\operatorname{supp}(\psi)} u\left(\partial^{i,-h_k}\psi\right)dx \overset{2.}{=}$$
$$\overset{2.}{=} -\lim_{k\nearrow\infty} \int_{\operatorname{supp}(\psi)} \left(\partial^{i,h_k} u\right)\psi dx \overset{1.}{=} -\int_{\operatorname{supp}(\psi)} \left(\partial^i u\right)\psi dx = -\int_U \left(\partial^i u\right)\psi dx.$$

The converse is also true when U is an extension domain for $W^{1,\infty}(U)$, e.g. when $\partial U \in \operatorname{Lip}(\varepsilon,K,L)$. Indeed, in this case it suffices to show the result for whole $\mathbb{R}^n$ only. Let $x,y \in \mathbb{R}^n$ and set $h = (h_1,\ldots,h_n) \coloneqq x-y$. Since $u \in W^{1,\infty}(\mathbb{R}^n)$, then $(\mathcal{R}(\mathbb{R}^n, B(x,|h|))u) \in W^{1,p}(B(x,|h|))$ for every $p \in [1,\infty]$. Hence, we deal as in Proposition 1.2.9 to get

$$u^*(y+h_1e_1)-u^*(y) = h_1 \int_0^1 \left[\partial_w^1 u\right](y+th_1e_1)dt,$$
$$\begin{cases} u^*(y+h_1e_1+h_2e_2)-u^*(y+h_1e_1) = h_2 \int_0^1 \left[\partial_w^2 u\right](y+h_1e_1+th_2e_2)dt \\ \vdots \\ u^*(x)-u^*\left(y+\sum_{i=1}^{n-1} h_ie_i\right) = h_n \int_0^1 \left[\partial_w^n u\right]\left(y+\sum_{i=1}^{n-1} h_ie_i+th_ne_n\right)dt, \end{cases} \quad \text{if } n \neq 1.$$

Summing these equations and applying the Cauchy-Schwarz inequality, we directly deduce the estimate

$$|u^*(x)-u^*(y)| \le \|\nabla_w u\|_{L^\infty(\mathbb{R}^n)} |x-y|,$$

thus $\operatorname{Lip}(u^*) \le \|\nabla_w u\|_{L^\infty(\mathbb{R}^n)}$, from the arbitrariness of x,y.

A direct consequence of the previous theorem follows.

Corollary 1.2.5. *Let $m \in \mathbb{N}\setminus\{1\}$, U and $(u,f) \in H^1(U)\times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, then $u \in H^m(U_\delta)$ for every $\delta > 0$, with*

$$\sum_{j=2}^m \left\|\nabla_w^j u\right\|_{L^2(U_\delta)} \le \mathcal{K}\left(\frac{1}{\delta-\delta'},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)}+\|f\|_{H^{m-2}(U)}\right), \ \forall\, 0<\delta'<\delta.$$

Proof. We show it by induction on m.

Step 1
The case $m = 2$ is nothing but Theorem 1.2.4 itself.

Step 2

Here follows the induction hypothesis on an arbitrary $m \in \mathbb{N} \setminus \{1,2\}$: for every U, $a \in W^{m-1,\infty}(U)$ and $f \in H^{m-2}(U)$, we have that $u \in H^m(U_\delta)$ for every $\delta > 0$, where u solves $\mathcal{L}_w u = f$ (in $H^{-1}(U)$), with

$$\sum_{j=2}^{m} \left\| \nabla_w^j u \right\|_{L^2(U_\delta)} \leq \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U)} \right) \left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)} \right), \ \forall\, 0<\delta'<\delta.$$

Step 3

Now, we assume that $a \in W^{m,\infty}(U)$ and $f \in H^{m-1}(U)$ for some U and also that $u \in H_0^1(U)$ solves $\mathcal{L}_w u = f$. By the induction hypothesis $u \in H^m(U_\delta)$ for every $\delta>0$ with the above estimate. Let $0<\delta''<\delta'<\delta$ be arbitrary and sufficient small (otherwise we have nothing to show). For every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m-1$ and every $\psi \in C_c^\infty(U_{\delta''})$, we set $(-1)^{|\alpha|} \left(\mathcal{E}_0(U_{\delta''}, U) \circ D^\alpha\right) \psi$ in the variational equation $\mathcal{L}_w u = f$. In virtue of the fact that the differential operators of the form D_w^β commute with each other along with the Leibniz rule, we deduce from the induction hypothesis (i.e. $u \in H^m(U_{\delta''})$) and the definition of the weak derivatives that

$$\int_{U_{\delta''}} \nabla \psi \cdot a((\nabla_w \circ D_w^\alpha)\,\overline{u})dx = \int_{U_{\delta''}} \left(D_w^\alpha \overline{f} + I \right) \psi dx,$$

where

$$I = \{\text{sum of terms } C\,(D_w^{\alpha_1} a_{ij})\,(D_w^{\alpha_2} \overline{u})\,, \text{ for } \alpha_1, \alpha_2 \in \mathbb{N}_0^n \text{ such that } 1 \leq |\alpha_1|\,, |\alpha_2| \leq m\},$$

or else

$$(\mathcal{L}_w \circ D_w^\alpha)\, u = D_w^\alpha f + \overline{I}, \text{ in } H^{-1}(U_{\delta''}),$$

since ψ is arbitrary. From the case $m=2$, we derive that $D^\alpha u \in H^2(U_\delta)$ with

$$\left\| \left(\nabla_w^2 \circ D_w^\alpha \right) u \right\|_{L^2(U_\delta)} \leq \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U_{\delta''})} \right) \left(\|\nabla_w u\|_{L^2(U_{\delta''})} + \left\| D_w^\alpha f + \overline{I} \right\|_{L^2(U_{\delta''})} \right),$$

and from the induction hypothesis (i.e. $u \in H^m(U_{\delta''})$ along with the respective estimate) and the evident fact that the norms in subsets are smaller, we get

$$\left\| \left(\nabla_w^2 \circ D_w^\alpha \right) u \right\|_{L^2(U_\delta)} \leq \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{m,\infty}(U)} \right) \left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-1}(U)} \right),$$

or else

$$\left\| \nabla_w^{m+1} u \right\|_{L^2(U_\delta)} \leq \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{m,\infty}(U)} \right) \left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-1}(U)} \right),$$

since α is arbitrary. In virtue of the above estimate for arbitrary δ along with the estimate of the induction hypothesis, we obtain the desired result, that is $u \in H^{m+1}(U_\delta)$ for every $\delta>0$ with

$$\sum_{j=2}^{m+1} \left\| \nabla_w^j u \right\|_{L^2(U_\delta)} \leq \mathcal{K}\left(\frac{1}{\delta-\delta'}, \frac{1}{\theta}, \|a\|_{W^{m,\infty}(U)} \right) \left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-1}(U)} \right), \ \forall 0 < \delta' < \delta.$$

□

Boundary and up-to-boundary regularity

The following results concern the sets with boundaries as in Definition 1.2.7. The crux here is the application of Theorem 1.2.3 for $p=2$, along with Proposition 1.2.9 for $p = 2, \infty$. Before we proceed, we need the following result.

Lemma 1.2.3. *Let U_1, U_2 and $\Phi : U_1 \to U_2$ be bijective and bi-Lipschitz transformation, with $\Psi \coloneqq \Phi^{-1}$ and $\mathrm{Lip}(\Psi) \leq L$, for some $L > 0$. Then, for every second-order uniformly elliptic operator $\mathcal{L}_w(a,\theta) : H^1(U_1) \to H^{-1}(U_1)$ there exists a unique second-order uniformly elliptic operator $\widetilde{\mathcal{L}}_w\left(\widetilde{a}, \widetilde{\theta}\right) : H^1(U_2) \to H^{-1}(U_2)$, with*

$$\widetilde{a}_{kl} = |\det(J\Psi)| \sum_{i,j=1}^{n} (a_{ij} \circ \Psi) \left(\partial^i \Phi_k \circ \Psi \right) \left(\partial^j \Phi_l \circ \Psi \right), \ \textit{a.e. in } U_2, \ \textit{for every } k,l = 1,\ldots,n$$

and

$$\widetilde{\theta}=\frac{\theta|\det(J\Psi)|}{\mathcal{K}(L)},$$

such that

$$\langle \mathcal{L}_w u, v\rangle = \left\langle \widetilde{\mathcal{L}}_w(u\circ\Psi), v\circ\Psi\right\rangle,\ \ \forall\, u\in H^1(U_1)\,, v\in H_0^1(U_1).$$

Proof. First of all, in view of Theorem 1.2.3, $\widetilde{a}\in L^\infty(U_2)$, thus the above statement makes sense. Now, in virtue of Theorem 1.2.3 for $m=1$ along with the change of variables formula, after some trivial calculations we have

$$\int_{U_1}\nabla_w v\cdot a\nabla_w \bar{u}dx=\int_{U_2}\nabla_w(v\circ\Psi)\cdot\widetilde{a}\nabla_w(\bar{u}\circ\Psi)dz,\ \ \forall\, u\in H^1(U_1)\,, v\in H_0^1(U_1).$$

It suffices to show that

$$\mathrm{Re}\big(\xi\cdot\widetilde{a}\bar{\xi}\big)\geq\widetilde{\theta}|\xi|^2,\ \text{ a.e. in } U_2,\ \text{ for every } \xi\in\mathbb{C}^n.$$

Indeed, it is direct to check that

$$\mathrm{Re}\big(\xi\cdot\widetilde{a}\bar{\xi}\big)=|\det(J\Psi)|\mathrm{Re}(\eta\cdot(a\circ\Psi)\,\bar{\eta})\geq\theta|\det(J\Psi)||\eta|^2,\ \text{ for every } \xi\in\mathbb{C}^n \text{ and } \eta=\xi J\Phi.$$

Since $\xi=\eta J\psi$, then $|\xi|\leq\mathcal{K}(L_2)\,|\eta|$, and so

$$\mathrm{Re}\big(\xi\cdot\widetilde{a}\bar{\xi}\big)\geq\widetilde{\theta}|\xi|^2,\ \text{ for every } \xi\in\mathbb{C}^n.$$

□

Now we are ready to show the main result of this subsection, that is the missing part of Theorem 1.2.4.

Theorem 1.2.5. *Let U with $\partial U\in\mathrm{Lip}^2(\varepsilon,K,L)$ and $(u,f)\in H_0^1(U)\times H^{-1}(U)$ be such that $\mathcal{L}_w u=f$. If $a\in W^{1,\infty}(U)$ and $f\in L^2(U)$, then*

1. *$u\in H^2(U\smallsetminus U_\delta)$ for every $\delta\in(0,\varepsilon)$, with*

$$\left\|\nabla_w^2 u\right\|_{L^2(U\smallsetminus U_\delta)}\leq\mathcal{K}\left(\frac{1}{\delta'-\delta},K,L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2(U)}+\|f\|_{L^2(U)}\right),$$

for every $0<\delta<\delta'<\varepsilon$.

2. *$L_w u=f$ a.e. in $U\smallsetminus U_\delta$, for every $\delta\in(0,\varepsilon)$.*

Proof. Steps 1 and 2 are preparatory for Step 3.

Step 1α

We modify the open cover $\{U_k\}_k$ of ∂U in Definition 1.2.7. We set[9]

$$(\partial U)_k:=\left\{x\in\partial U\,\middle|\,B(x,\varepsilon)\subseteq U_k\right\},\ \text{and } U_k\supseteq U_{\delta,k}:=\bigcup_{x\in(\partial U)_k}B(x,\delta),\ \forall\delta\in(0,\varepsilon],\ \forall k.$$

In virtue of condition 1. in the aforementioned definition, $\partial U=\bigcup_k(\partial U)_k$. Hence, for every δ, the collection $\{U_{\delta,k}\}_k$ is also an open cover of ∂U that inherits all of the properties of the definition which characterize the collection $\{U_k\}_k$. Moreover, we can easily deduce that

$$U\smallsetminus U_\delta=U\cap\bigcup_k U_{\delta,k}=\bigcup_k(U_{\delta,k}\cap U),\ \ \forall\delta\in(0,\varepsilon].$$

[9] We naturally consider that $\bigcup_{x\in\varnothing}B_\delta(x)=\varnothing$. However, we can exclude all the vacuous components of the open cover, so that we can assume $(\partial U)_k\neq\varnothing$ for every k.

Step 1β

We write

$$x \underset{\Psi_{1_k}}{\overset{\Phi_{1_k}}{\rightleftarrows}} y_k \underset{\Psi_{2_k}}{\overset{\Phi_{2_k}}{\rightleftarrows}} z_k\,,$$

where x stands for the background coordinates, y_k is as in Definition 1.2.7 and z_k stands for the coordinates that "straighten out the boundary", i.e.

$$z_k' = y_k' \text{ and } z_{n_k} = y_{n_k} - \gamma_k(y_k')\,.$$

We notice that since every Φ_{1_k} and Ψ_{1_k} are rigid motions, then they are also isometries, as well as $|\det(J\Phi_{1_k})| = |\det(J\Psi_{1_k})| = 1$, for every k. Moreover, $\det(J\Phi_{2_k}) = \det(J\Psi_{2_k}) = 1$ for every k, and also, in virtue of condition 3. in the aforementioned definition, it is direct to check that every Φ_{2_k} and Ψ_{2_k} are Lipschitz continuous with $\mathrm{Lip}(\Phi_{2_k}) \le \mathcal{K}(L)$ and $\mathrm{Lip}(\Psi_{2_k}) \le \mathcal{K}(L)$. Therefore, setting $\Phi_k := \Phi_{2_k} \circ \Phi_{1_k}$ and $\Psi_k := \Psi_{1_k} \circ \Psi_{2_k}$ for every k, we deduce that $|\det(J\Phi_k)| = |\det(J\Psi_k)| = 1$ for every k (a direct application of the chain rule), as well as that every Φ_k is a bi-$\mathcal{K}(L)$-Lipschitz transformation with $\Psi_k = \Phi_k^{-1}$, that is

$$\frac{|x_1 - x_2|}{\mathcal{K}(L)} \le |\Phi_k(x_1) - \Phi_k(x_2)| \le \mathcal{K}(L)\,|x_1 - x_2|\,,\ \ \forall\, x_1, x_2 \in \mathbb{R}^n,\ \ \forall k.$$

Step 2α

Let $0 < \delta < \delta' < \varepsilon$ and $d := \frac{\varepsilon - \delta'}{4}$. We consider the sets

$$U_{\delta,k} \subsetneq U_{\delta',k} \subsetneq (U_{\delta',k})^d = (U_{\varepsilon,k})_{3d} \subsetneq (U_{\delta',k})^{2d} = (U_{\varepsilon,k})_{2d} \subsetneq (U_{\delta',k})^{3d} = (U_{\varepsilon,k})_d \subsetneq U_{\varepsilon,k},\ \ \forall k,$$

hence

$$\Phi_k(U_{\delta,k}) \subsetneq \Phi_k(U_{\delta',k}) \subsetneq \Phi_k\big((U_{\varepsilon,k})_{3d}\big) \subsetneq \Phi_k\big((U_{\varepsilon,k})_{2d}\big) \subsetneq \Phi_k\big((U_{\varepsilon,k})_d\big) \subsetneq \Phi_k(U_{\varepsilon,k}),\ \ \forall k.$$

In view of Step 1β, we deduce that

$$\frac{\delta' - \delta}{\mathcal{K}(L)} = \frac{\mathrm{dist}(U_{\delta,k}, \partial U_{\delta',k})}{\mathcal{K}(L)} \le \mathrm{dist}(\Phi_k(U_{\delta,k}), \partial\Phi_k(U_{\delta',k})),\ \text{uniformly for every } k,$$

as well as

$$\begin{cases} \mathrm{dist}\big(\Phi_k(U_{\delta',k}), \partial\Phi_k\big((U_{\varepsilon,k})_{3d}\big)\big) \le \mathcal{K}(1+L)d =: d' \\ \mathrm{dist}\big(\Phi_k\big((U_{\varepsilon,k})_{3d}, \partial\Phi_k\big((U_{\varepsilon,k})_{2d}\big)\big)\big) \le d' \\ \mathrm{dist}\big(\Phi_k\big((U_{\varepsilon,k})_{2d}, \partial\Phi_k\big((U_{\varepsilon,k})_{d}\big)\big)\big) \le d' \\ \mathrm{dist}\big(\Phi_k\big((U_{\varepsilon,k})_{d}, \partial\Phi_k((U_{\varepsilon,k}))\big)\big) \le\ d', \end{cases} \qquad \text{uniformly for every } k.$$

For every k we choose a cut-off function $\phi_k \in C_c^\infty(\mathbb{R}^n; [0,1])$ such that

1. $\mathrm{supp}(\phi_k) \subseteq \overline{\Phi_k(U_{\delta',k})}$,
2. $\phi_k \equiv 1$ in $\overline{\Phi_k(U_{\delta,k})}$ and
3. $\|\nabla\phi_k\|_{L^\infty(\mathbb{R}^n)} \le \frac{\mathcal{K}(L)}{\delta' - \delta}$, uniformly for every k.

Now, for every k we consider the sets

$$\Phi_k(U_{\delta,k} \cap U) \subsetneq \Phi_k(U_{\delta',k} \cap U) \subsetneq \Phi_k\big((U_{\varepsilon,k})_{3d} \cap U\big) \subsetneq$$
$$\subsetneq \Phi_k\big((U_{\varepsilon,k})_{2d} \cap U\big) \subsetneq \Phi_k\big((U_{\varepsilon,k})_{d} \cap U\big) \subsetneq \Phi_k(U_{\varepsilon,k} \cap U)$$

and for convenience we denote the above group of sets as

$$D_{1_k} \subsetneq D_{2_k} \subsetneq D_{3_k} \subsetneq D_{4_k} \subsetneq D_{5_k} \subsetneq D_{6_k},\ \ \forall k.$$

Step 2β

If $n\neq 1$, for every $i=1,\dots,n-1$ and $h\in(-d',d')^*$, we consider the operators

$$\partial_{1_k}^{i,h}:\mathcal{F}(D_{6_k})\to\mathcal{F}(D_{5_k}) \text{ and } \partial_{2_k}^{i,h}:\mathcal{F}(D_{5_k})\to\mathcal{F}(D_{4_k}), \text{ for every } k,$$

as well as the operators

$$\mathcal{V}_{i,h,\phi_k}:\mathcal{F}(D_{6_k})\to\mathcal{F}(D_{4_k}),\ \forall k.$$

with

$$\mathcal{V}_{i,h,\phi_k}v:=\partial_{2_k}^{i,-h}\Big(\phi_k^2\left(\partial_{1_k}^{i,h}v\right)\Big),\ \forall v\in\mathcal{F}(D_{6_k}),\ \forall k.$$

We can now deal as in Step 1 of the proof of Theorem 1.2.4, to show that

$$(\mathcal{E}_0(D_{4_k},D_{6_k}))\circ\mathcal{V}_{i,h,\phi_k}:\Big\{v\in H^1(D_{6_k})\Big|\,[Tv]\,(D_{6_k}\cap\{z_n=0\})=\{0\}\Big\}\to$$
$$\to\left\{v\in H_0^1(D_{6_k})\,\middle|\,\mathrm{supp}(v)\subseteq\overline{D_{3_k}}\right\},\ \forall\, i,h,k.$$

Step 2γ

Since $\mathcal{L}_w u=f$, then $((\mathcal{R}(U,U_{\varepsilon,k}\cap U))\circ\mathcal{L}_w)\,u=(\mathcal{R}\,(U,U_{\varepsilon,k}\cap U))\,f$ for every k also (see Definition 1.2.5). Therefore, in virtue of Lemma 1.2.3, we deduce that

$$\left\langle\widetilde{\mathcal{L}}_{k_w}(u\circ\Psi_k),v\right\rangle=\langle f\circ\Psi_k,v\rangle\,,\ \forall v\in H_0^1(D_{6_k}),\ \forall k,$$

where

$$\widetilde{a}_{k_{rs}}=\sum_{i,j=1}^{n}(a_{ij}\circ\Psi_k)\left(\partial^i\Phi_{k_r}\circ\Psi_k\right)\left(\partial^j\Phi_{k_s}\circ\Psi_k\right),\ \text{a.e. in } D_{6_k},\ \text{for every } r,s=1,\dots,n$$

and

$$\widetilde{\theta}_k=\frac{\theta}{\mathcal{K}(L)},$$

for every k. In view of Theorem 1.2.3, the second part of Remark 1.2.10 and Proposition 1.2.8, we directly obtain that $\widetilde{a}_k\in W^{1,\infty}(D_{6_k})$ for every k, with

$$\|\widetilde{a}_k\|_{W^{1,\infty}(D_{6_k})}\leq\mathcal{K}(L)\|a\|_{W^{1,\infty}(U_{\varepsilon,k}\cap U)}\leq\mathcal{K}(L)\|a\|_{W^{1,\infty}(U)},\ \forall k.$$

Additionally[10], $u\circ\Psi_k\in H^1(D_{5_k})$ and $f_k\circ\Psi\in L^2(D_{5_k})$ for every k, with

$$\|\nabla_w(u\circ\Psi_k)\|_{L^2(D_{5_k})}\leq\mathcal{K}(L)\|\nabla_w u\|_{L^2((U_{\varepsilon,k})_d\cap U)}$$

and

$$\|f\circ\Psi_k\|_{L^2(D_{5_k})}\leq\mathcal{K}(L)\|f\|_{L^2((U_{\varepsilon,k})_d\cap U)}.$$

Step 3α

In $n\neq 1$, for every k and $i=1,\dots,n-1$, we choose $v_i:=-((\mathcal{E}_0(D_{4_k},D_{6_k}))\circ\mathcal{V}_{i,h,\phi_k})\,u$ into the variational equation $\widetilde{\mathcal{L}}_{k_w}(u\circ\Psi)=f\circ\Psi$ in $H^{-1}(D_{6_k})$, and then we deal exactly as in the proof of Theorem 1.2.4, minding to apply Proposition 1.2.9 instead of Corollary 1.2.3, to get that $D_w^\alpha(u\circ\Psi_k)\in L^2(D_{1_k})$ for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=2$ and $\alpha\neq(0,\dots,0,2)$, and every k, with

$$\|D_w^\alpha(u\circ\Psi_k)\|_{L^2(D_{1_k})}\leq K\left(\frac{1}{\delta'-\delta},\frac{1}{\theta},\|\widetilde{a}_k\|_{W^{1,\infty}(D_{6_k})}\right)\times$$
$$\times\Big(\|\nabla_w(u\circ\Psi_k)\|_{L^2(D_{5_k})}+\|f\circ\Psi_k\|_{L^2(D_{5_k})}\Big),$$

or else

$$\|D_w^\alpha(u\circ\Psi_k)\|_{L^2(D_{1_k})}\leq\mathcal{K}\left(\frac{1}{\delta'-\delta},L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\Big(\|\nabla_w u\|_{L^2((U_{\varepsilon,k})_d\cap U)}+\|f\|_{L^2((U_{\varepsilon,k})_d\cap U)}\Big),\ \forall k.$$

10 See Step 3δ below for the reason why we estimate these terms in D_{5_k} and not in whole D_{6_k}.

Step 3β

For $\alpha=(0,\dots,0,2)$, we first notice that the uniform ellipticity condition implies that

$$|\widetilde{a}_{nn}|\ge \mathrm{Re}(\widetilde{a}_{nn})\ge\widetilde{\theta}>0, \text{ a.e. in } D_{6_k},$$

and so $\widetilde{a}_{nn}\neq 0$, a.e. in D_{6_k}. Moreover, since $\partial U\in \mathrm{Lip}(\varepsilon,K,L)$, then α_{nn} is (bounded and) Lipschitz continuous (from the second part of Remark 1.2.10). Thereby, we return to each $\widetilde{\mathcal{L}}_{k_w}(u\circ\Psi_k)=f\circ\Psi_k$, where we put

$$\frac{\mathcal{E}_0(D_{1_k},D_{6_k})\psi_k}{\overline{\widetilde{a}_{nn}}}, \text{ for arbitrary } \psi_k\in C_c^\infty(D_{1_k}), \text{ for every } k.$$

Therefore, by the definition of the weak derivatives, we have that

$$\int_{D_{1_k}}\widetilde{a}_{nn}\left(\partial_w^n(\overline{u}\circ\Psi_k)\right)\left(\partial^n\left(\frac{\psi}{\overline{\widetilde{a}_{nn}}}\right)\right)dz=\int_{D_{1_k}}\left(\sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^n\partial_w^j\left(\widetilde{a}_{ij}\left(\partial_w^i(\overline{u}\circ\Psi_k)\right)\right)+\overline{f}\right)\frac{\psi}{\overline{\widetilde{a}_{nn}}}dz,$$

or else

$$\int_{D_{1_k}}\left(\partial_w^n(\overline{u}\circ\Psi_k)\right)\left(\partial^n\psi\right)dz=$$

$$=\int_{D_{1_k}}\left(\sum_{\substack{i,j=1\\(i,j)\neq(n,n)}}^n\partial_w^j\left(\widetilde{a}_{ij}\left(\partial_w^i(\overline{u}\circ\Psi_k)\right)\right)+\overline{f}+\left(\partial^n\widetilde{a}_{nn}\right)\left(\partial_w^n(\overline{u}\circ\Psi_k)\right)\right)\frac{\psi}{\widetilde{a}_{nn}}dz,$$

for every $\psi_k\in C_c^\infty(D_{1_k})$ and every k. Therefore, again from the definition of the weak derivative, we deduce that $\partial_w^n(u\circ\Psi_k)$ is weakly n-partially differentiable and $D_w^\alpha(u\circ\Psi_k)\in L^2(D_{1_k})$ for every k, with

$$\|D_w^\alpha(u\circ\Psi_k)\|_{L^2(D_{1_k})}\le\mathcal{K}\left(\frac{1}{\delta'-\delta},L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\left(\|\nabla_w u\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}+\|f\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}\right),\ \forall k.$$

We note that, in the particular case $n=1$, the above estimate has the form

$$\|D_w^\alpha(u\circ\Psi_k)\|_{L^2(D_{1_k})}\le\mathcal{K}\left(\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\left(\|\nabla_w u\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}+\|f\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}\right),\ \forall k,$$

but that does not make any difference for us.

Step 3γ

Combining the estimates of Step 3α and 3β, we obtain

$$\left\|\nabla_w^2(u\circ\Psi_k)\right\|_{L^2(D_{1_k})}\le\mathcal{K}\left(\frac{1}{\delta'-\delta},L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\left(\|\nabla_w u\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}+\|f\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}\right),\ \forall k.$$

We also add the missing terms to both sides of the above estimate and we multiply by $\mathcal{K}(L)$ to derive

$$\mathcal{K}(L)\sum_{j=1}^2\left\|\nabla_w^j(u\circ\Psi_k)\right\|_{L^2(D_{1_k})}\le\mathcal{K}\left(\frac{1}{\delta'-\delta},L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\left(\|\nabla_w u\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}+\|f\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}\right),\ \forall k.$$

Now, from the fact that every Φ_{1_k} is rigid motions, combined with condition 3. in Definition 1.2.7 that implies $\nabla\Phi_{2_{k_i}}$ is Lipschitz continuous with $\mathrm{Lip}\left(\nabla\Phi_{2_{k_i}}\right)\le\mathcal{K}(L)$, for every $i=1,\dots,n$ and k, we deduce easily by the chain rule that every $\nabla\Phi_{k_i}$ is Lipschitz continuous with $\mathrm{Lip}(\nabla\Phi_{k_i})\le\mathcal{K}(L)$. Thus, in view of Theorem 1.2.3, we obtain

$$\left\|\nabla_w^2 u\right\|_{L^2(U_{\delta,k}\cap U)}\le\mathcal{K}\left(\frac{1}{\delta-\delta'},L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\right)\times$$
$$\times\left(\|\nabla_w u\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}+\|f\|_{L^2\left((U_{\varepsilon,k})_d\cap U\right)}\right),\ \forall k.$$

Step 3δ

It is only left to use the above bound to estimate $\|\nabla_w^2 u\|_{L^2(U\smallsetminus U_\delta)}$. Employing Proposition 1.2.11, for every k we consider a cut-off function $\phi_k \in C_c^\infty(\mathbb{R}^n;[0,1])$, such that

1. $\operatorname{supp}(\phi_k)\subseteq \overline{U_{\varepsilon,k}}$ and
2. $\phi_k \equiv 1$ in $\overline{(U_{\varepsilon,k})_d}$.

From condition 2. in Definition 1.2.7 (see also Step 1α), we have that every collection of $K+1$ of $\operatorname{supp}(\phi_k)\cap U$'s has empty intersection, that is, for every $x\in U\smallsetminus U_\varepsilon$ at most K of $\phi_k(x)$ are non-zero. From

1. the fact the indefinite integral of a non-negative measurable function (with respect to a measure) is a measure itself (which is a direct consequence of the Beppo Levi theorem),
2. the latter estimate,
3. the Beppo Levi theorem, or alternatively the Tonelli theorem for the counting measure in a subset of $\mathbb{Z}$ (where every k exists) and the Lebesgue measure in $U\smallsetminus U_\varepsilon$, and
4. the fact that $\sum_k \phi_k \le K$ in $U\smallsetminus U_\varepsilon$,

in this order, we then get

$$\begin{aligned}
\int_{U\smallsetminus U_\delta} |\nabla_w^2 u|^2 dx &= \int_{\bigcup_k (U_{\delta,k}\cap U)} |\nabla_w^2 u|^2 dx \overset{1.}{\le} \sum_k \int_{U_{\delta,k}\cap U} |\nabla_w^2 u|^2 dx \overset{2.}{\le} \\
&\overset{2.}{\le} \mathcal{K}\Big(\frac{1}{\delta'-\delta}, L, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\Big) \sum_k \int_{(U_{\varepsilon,k})_d\cap U} |\nabla_w u|^2 + |f|^2 dx \le \\
&\le \mathcal{K}\Big(\frac{1}{\delta'-\delta}, L, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\Big) \sum_k \int_{U\smallsetminus U_\varepsilon} \phi_k \left(|\nabla_w u|^2 + |f|^2\right) dx \overset{3.}{=} \\
&\overset{3.}{=} \mathcal{K}\Big(\frac{1}{\delta'-\delta}, L, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\Big) \int_{U\smallsetminus U_\varepsilon} \left(|\nabla_w u|^2 + |f|^2\right) \sum_k \phi_k dx \overset{4.}{\le} \\
&\overset{4.}{\le} \mathcal{K}\Big(\frac{1}{\delta'-\delta}, K, L, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\Big) \int_{U\smallsetminus U_\varepsilon} |\nabla_w u|^2 + |f|^2 dx,
\end{aligned}$$

thereby the desired estimate follows. As far as point 2. is concerned, it follows exactly as in Step 3 of Theorem 1.2.4.

□

Remark 1.2.11. *U being an extension set is essential for the proof of Theorem 1.2.5.*

Now, we combine Theorem 1.2.4 and Theorem 1.2.5 for the regularity up to boundary.

Proposition 1.2.13. *Let U with $\partial U \in \operatorname{Lip}^2(\varepsilon,K,L)$ and $(u,f)\in H_0^1(U)\times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a\in W^{1,\infty}(U)$ and $f\in L^2(U)$, then*

1. *$u\in H^2(U)\cap H_0^1(U)$ with*

$$\|\nabla_w^2 u\|_{L^2(U)} \le \mathcal{K}\Big(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{1,\infty}(U)}\Big)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{L^2(U)}\right).$$

2. *$L_w u = f$ a.e. in U.*

Proof. It suffices to notice that, by fixing $0<\delta_1<\delta<\delta_2<\varepsilon$ we have $\delta-\delta_1 = C\varepsilon$ and $\delta_2-\delta = C\varepsilon$. □

The higher regularity result then follows.

Corollary 1.2.6. *Let $m\in\mathbb{N}\smallsetminus\{1\}$, U with $\partial U\in\operatorname{Lip}^m(\varepsilon,K,L)$ and $(u,f)\in H_0^1(U)\times H^{-1}(U)$ be such that $\mathcal{L}_w u = f$. If $a\in W^{m-1,\infty}(U)$ and $f\in H^{m-2}(U)$, then $u\in H^m(U)\cap H_0^1(U)$, with*

$$\sum_{j=2}^m \|\nabla_w^j u\|_{L^2(U)} \le \mathcal{K}\Big(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U)}\Big)\left(\|\nabla_w u\|_{L^2(U)} + \|f\|_{H^{m-2}(U)}\right).$$

Proof. We show it by induction on m.

Step 1

The case $m=2$ is nothing but Proposition 1.2.13 itself.

Step 2

Here follows the induction hypothesis on an arbitrary $m\in\mathbb{N}\setminus\{1,2\}$: for every U with $\partial U\in \mathrm{Lip}^m(\varepsilon,K,L)$, $a\in W^{m-1,\infty}(U)$ and $f\in H^{m-2}(U)$, we have that $u\in H^m(U)\cap H_0^1(U)$, where u solves $\mathcal{L}_w u=f$, with

$$\sum_{j=2}^{m}\left\|\nabla_w^j u\right\|_{L^2(U)}\le\mathcal{K}\Big(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)}+\|f\|_{H^{m-2}(U)}\Big).$$

Step 3

Now, we assume that $a\in W^{m,\infty}(U)$ and $f\in H^{m-1}(U)$ for some U with $\partial U\in\mathrm{Lip}^{m+1}(\varepsilon,K,L)$, and also that $u\in H_0^1(U)$ solves $\mathcal{L}_w u=f$. By the induction hypothesis $u\in H^m(U)$ with the above estimate. For every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=m-1$ and every $\psi\in C_c^\infty(U)$, we set $(-1)^{|\alpha|}D^\alpha\psi$ in the variational equation $\mathcal{L}_w u=f$. In virtue of the fact that the differential operators of the form D_w^β commute with each other along with the Leibniz rule, we deduce from the induction hypothesis (i.e. $u\in H^m(U)$) and the definition of the weak derivatives that

$$\int_U\nabla\psi\cdot a((\nabla_w\circ D_w^\alpha)\,\bar{u})dx=\int_U\left(D_w^\alpha\bar{f}+I\right)\psi dx,$$

where

$$I=\{\text{sum of terms } C\,(D_w^{\alpha_1}a_{ij})\,(D_w^{\alpha_2}\bar{u})\,,\ \text{for } \alpha_1,\alpha_2\in\mathbb{N}_0^n \text{ such that } 1\le|\alpha_1|\,,|\alpha_2|\le m\},$$

or else

$$(\mathcal{L}_w\circ D_w^\alpha)\,u=D_w^\alpha f+\bar{I},\ \text{ in } H^{-1}(U),$$

since ψ is arbitrary. From the case $m=2$, we derive that $D^\alpha u\in H^2(U)$ with

$$\left\|\left(\nabla_w^2\circ D_w^\alpha\right)u\right\|_{L^2(U)}\le\mathcal{K}\Big(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{1,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)}+\left\|D_w^\alpha f+\bar{I}\right\|_{L^2(U)}\Big),$$

and from the induction hypothesis (i.e. $u\in H^m(U)$ along with the respective estimate), we get

$$\left\|\left(\nabla_w^2\circ D_w^\alpha\right)u\right\|_{L^2(U)}\le\mathcal{K}\Big(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{m,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)}+\|f\|_{H^{m-1}(U)}\Big),$$

or else

$$\left\|\nabla_w^{m+1}u\right\|_{L^2(U)}\le\mathcal{K}\Big(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{m,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)}+\|f\|_{H^{m-1}(U)}\Big),$$

since α is arbitrary. In virtue of the above estimate along with the estimate of the induction hypothesis, we obtain the desired result, that is $u\in H^{m+1}(U)$ with

$$\sum_{j=2}^{m+1}\left\|\nabla_w^j u\right\|_{L^2(U_\delta)}\le\mathcal{K}\Big(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{m,\infty}(U)}\Big)\Big(\|\nabla_w u\|_{L^2(U)}+\|f\|_{H^{m-1}(U)}\Big).$$

□

Remark 1.2.12. *The estimate in Corollary 1.2.6 can be generalized in order for us to consider functions with non-zero trace, i.e.* $Tu=g\not\equiv 0$. *Indeed, it suffices to generalize Theorem 3.37 in [38] for the extension-in-U operator* $\widetilde{\cdot}$ *of the trace operator* T, *where* $\partial U\in\mathrm{Lip}^m(\varepsilon,K,L)$, *and then to set the difference* $u-\widetilde{T}g$ *into the estimate. In fact we can generalize the result for the space* $\{u\in L^1_{\mathrm{loc}}(U)\,|\,\nabla_w^j u\in L^2(U),\ \text{for } j=1,\dots,m,\ \text{and } [Tu]\,(\partial U)=g\}$, *e.g. for* $X_0^m(U)$, *by defining the trace operator only locally. However, such approaches exceed the scope of the present study.*

Remark 1.2.13. *In the classic reference books (see, e.g, [26], or [15], or [38]), the second-order uniformly elliptic operator in* $H^1(U)$ *has the general form* $\mathcal{L}_w^{\mathrm{g}}=\mathcal{L}_w+b\cdot\nabla_w u+cu$, *for* $b=(b_i)_{i=1}^n\in L^\infty(U)$ *and* $c\in L^\infty(U)$, *which in general does not induces a symmetric bilinear form nor an isomorphism from* $H_0^1(U_{\mathrm{P}})$ *onto* $H^{-1}(U_{\mathrm{P}})$ *(see Definition 1.2.3). However, we note that the elliptic regularity results of a solution* $u\in H^1(U)$ *of* $\mathcal{L}_w u=f$ *in* $H^{-1}(U)$ *appeared in Subsection 1.2.12 and Subsection 1.2.12 are trivially true also for* $\mathcal{L}_w^{\mathrm{g}}$, *since all we have to do is to consider* $f^{\mathrm{g}}=f-\bar{b}\cdot\nabla_w\bar{u}-\bar{c}\bar{u}$ *instead of* f *in the variational equation.*

A priori estimates

In this section, we are interested in the sets U_{P} (see Definition 1.2.3) with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^m(\varepsilon, K, L)$.

Theorem 1.2.6. *Let $m \in \mathbb{N}$, U_{P} with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^m(\varepsilon, K, L)$ and $\mathcal{L}_w(a,\theta)$ with $a \in W^{m-1,\infty}(U_{\mathrm{P}})$. Then,*

1. *$\mathcal{L}_w$ induces an isomorphism from $H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$ onto $H^{m-2}(U_{\mathrm{P}})$ and*
2. *for $m \neq 1$ and every $u \in H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$ we have*

$$\sum_{j=2}^{m} \left\| \nabla_w^j u \right\|_{L^2(U_{\mathrm{P}})} \leq \mathcal{K}\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U_{\mathrm{P}})} \right) \left(\|\nabla_w u\|_{L^2(U_{\mathrm{P}})} + \|L_w u\|_{H^{m-2}(U_{\mathrm{P}})} \right).$$

Proof. Point 1. follows easily from the combination of Proposition 1.2.1 with Corollary 1.2.6, and Point 2. follows from Point 1. along with the estimate in Corollary 1.2.6. □

Proposition 1.2.14. *Let $m \in \mathbb{N} \setminus \{1\}$, U_{P} with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^m(\varepsilon, K, L)$, $\mathcal{L}_w(a,\theta)$ with $a \in W^{m-1,\infty}(U_{\mathrm{P}})$ and $u \in H^m(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$. If*

$$\left(L_w^j u \right) \in H_0^1(U_{\mathrm{P}}), \ \forall j = 0, \ldots, \left\lfloor \frac{m}{2} \right\rfloor - 1 \ \textit{(compatibility conditions)},$$

then we have

$$\sum_{j=2}^{m} \left\| \nabla_w^j u \right\|_{L^2(U_{\mathrm{P}})} \leq \mathcal{K}\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U_{\mathrm{P}})} \right) \times$$
$$\times \left(\sum_{\substack{j \in \mathbb{N}_0, \\ 2j+1 \leq m}} \left\| \left(\nabla_w \circ L_w^j \right) u \right\|_{L^2(U_{\mathrm{P}})} + \sum_{\substack{j \in \mathbb{N}, \\ 2j \leq m}} \left\| L_w^j u \right\|_{L^2(U_{\mathrm{P}})} \right).$$

Proof. We show the desired estimate by induction on m, only for even m, i.e. $m = 2k$ for some $k \in \mathbb{N}$, since the other case follows analogously. Therefore, we want to show that

$$\sum_{j=2}^{2k} \left\| \nabla_w^j u \right\|_{L^2(U_{\mathrm{P}})} \leq \mathcal{K}\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{m-1,\infty}(U_{\mathrm{P}})} \right) \left(\sum_{j=0}^{k-1} \left\| \left(\nabla_w \circ L_w^j \right) u \right\|_{L^2(U_{\mathrm{P}})} + \sum_{j=1}^{k} \left\| L_w^j u \right\|_{L^2(U_{\mathrm{P}})} \right).$$

Step 1

The case $k = 1$ follows directly from the estimate in Theorem 1.2.6.

Step 2

Here follows the induction hypothesis on an arbitrary $k \in \mathbb{N} \setminus \{1\}$: for every U_{P} with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^{2k}(\varepsilon, K, L)$, every $L_w(a,\theta)$ with $a \in W^{2k-1,\infty}(U_{\mathrm{P}})$ and every $u \in H^{2k}(U) \cap H_0^1(U)$, if

$$\left(L_w^j u \right) \in H_0^1(U_{\mathrm{P}}), \ \forall j = 0, \ldots, k-1,$$

then we have

$$\sum_{j=2}^{2k} \left\| \nabla_w^j u \right\|_{L^2(U_{\mathrm{P}})} \leq \mathcal{K}\left(\frac{1}{\varepsilon}, K, L, \frac{1}{\theta}, \|a\|_{W^{2k-1,\infty}(U_{\mathrm{P}})} \right) \times$$
$$\times \left(\sum_{j=0}^{k-1} \left\| \left(\nabla_w \circ L_w^j \right) u \right\|_{L^2(U_{\mathrm{P}})} + \sum_{j=1}^{k} \left\| L_w^j u \right\|_{L^2(U_{\mathrm{P}})} \right).$$

Step 3

Now, let $L_w(a,\theta)$ with $a \in W^{2k+1,\infty}(U_{\mathrm{P}})$ and $u \in H^{2k+2}(U) \cap H_0^1(U)$ be arbitrary, for some arbitrary U_{P} with $\partial U_{\mathrm{P}} \in \mathrm{Lip}^{2k+2}(\varepsilon, K, L)$, as well as

$$\left(L_w^j u \right) \in H_0^1(U_{\mathrm{P}}), \ \forall j = 0, \ldots, k.$$

Then $(L_w u) \in H^{2k}(U_{\mathrm{P}}) \cap H_0^1(U_{\mathrm{P}})$ and

$$\left(L_w^j (L_w u) \right) \in H_0^1(U_{\mathrm{P}}), \ \forall j = 0, \ldots, k-1,$$

hence by the induction hypothesis, we obtain

$$\sum_{j=2}^{2k}\left\|\left(\nabla_w^j\circ L_w\right)u\right\|_{L^2(U_{\mathrm{P}})}\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k-1,\infty}(U_{\mathrm{P}})}\right)\times$$
$$\times\left(\sum_{j=0}^{k-1}\left\|\left(\nabla_w\circ L_w^{j+1}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=1}^{k}\left\|L_w^{j+1}u\right\|_{L^2(U_{\mathrm{P}})}\right),$$

or else

$$\sum_{j=2}^{2k}\left\|\left(\nabla_w^j\circ L_w\right)u\right\|_{L^2(U_{\mathrm{P}})}\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\times$$
$$\times\left(\sum_{j=1}^{k}\left\|\left(\nabla_w\circ L_w^{j}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=2}^{k+1}\left\|L_w^{j}u\right\|_{L^2(U_{\mathrm{P}})}\right).$$

We add the missing terms, i.e. $\|\nabla_w u\|_{L^2(U_{\mathrm{P}})}+\|L_w u\|_{L^2(U_{\mathrm{P}})}+\|\left(\nabla_w\circ L_w\right)u\|_{L^2(U_{\mathrm{P}})}$, to both sides, to get

$$\|\nabla_w u\|_{L^2(U_{\mathrm{P}})}+\|L_w u\|_{H^{2k}(U_{\mathrm{P}})}\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\times$$
$$\times\left(\sum_{j=1}^{k}\left\|\left(\nabla_w\circ L_w^{j}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=2}^{k+1}\left\|L_w^{j}u\right\|_{L^2(U_{\mathrm{P}})}\right)+\|\nabla_w u\|_{L^2(U_{\mathrm{P}})}+$$
$$+\|L_w u\|_{L^2(U_{\mathrm{P}})}+\|\left(\nabla_w\circ L_w\right)u\|_{L^2(U_{\mathrm{P}})},$$

that is

$$\|\nabla_w u\|_{L^2(U_{\mathrm{P}})}+\|L_w u\|_{H^{2k}(U_{\mathrm{P}})}\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\times$$
$$\times\left(\sum_{j=0}^{k}\left\|\left(\nabla_w\circ L_w^{j}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=1}^{k+1}\left\|L_w^{j}u\right\|_{L^2(U_{\mathrm{P}})}\right).$$

We then write

$$\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\left(\|\nabla_w u\|_{L^2(U_{\mathrm{P}})}+\|L_w u\|_{H^{2k}(U_{\mathrm{P}})}\right)\le$$
$$\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\left(\sum_{j=0}^{k}\left\|\left(\nabla_w\circ L_w^{j}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=1}^{k+1}\left\|L_w^{j}u\right\|_{L^2(U_{\mathrm{P}})}\right)$$

and we employ the estimate in Theorem 1.2.6 for $m=2k+2$ to obtain the desired result, that is

$$\sum_{j=2}^{2k+2}\left\|\nabla_w^j u\right\|_{L^2(U_{\mathrm{P}})}\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\frac{1}{\theta},\|a\|_{W^{2k+1,\infty}(U_{\mathrm{P}})}\right)\times$$
$$\times\left(\sum_{j=0}^{k}\left\|\left(\nabla_w\circ L_w^{j}\right)u\right\|_{L^2(U_{\mathrm{P}})}+\sum_{j=1}^{k+1}\left\|L_w^{j}u\right\|_{L^2(U_{\mathrm{P}})}\right).$$

□

Remark 1.2.14. *A plethora of sets U that appear in the applications do not belong in the class of the results of this subsection, that is*

$$U\notin\{U_{\mathrm{P}}\ \textit{with}\ \partial U_{\mathrm{P}}\in\mathrm{Lip}^m(\varepsilon,K,L)\},$$

e.g.

1. *certain unbounded sets such as exterior domains with uniformly regular boundaries, hypo/epigraphs of "$C^{m,1}\left(\mathbb{R}^{n-1}\right)$"-functions and the whole Euclidean space, belong to the class*

$$\{U\ \textit{with}\ \partial U\in\mathrm{Lip}^m(\varepsilon,K,L)\}\setminus\{U_{\mathrm{P}}\},$$

2. *sets with corners that satisfy the criterion for the Poincaré inequality, such as a bounded triangle, belong to*

$$\{U_{\mathrm{P}}\}\smallsetminus\big\{U \text{ with } \partial U\in\mathrm{Lip}^2(\varepsilon,K,L)\big\},$$

3. *any combination of the previous examples.*

We can generalize the a priori estimates of the previous results for U as above (for the case of U being as in the second example, the whole Theorem 1.2.6 can be generalized), by considering a sequence

$$\{U_k\}_k\subset\{U_{\mathrm{P}} \text{ with } \partial U_{\mathrm{P}}\in\mathrm{Lip}^m(\varepsilon,K,L)\},$$

such that $U_k\nearrow U$ and

$$\sup_k\left\{\max\left\{\frac{1}{\varepsilon_k},K_k,L_k\right\}\right\}<\infty.$$

This approach, which is similar to already known ones concerning certain cases of sets (see, e.g., Section 3 in [30] for the case where U is bounded and convex), lies beyond the scope of this work.

Chapter 2

Non-vanishing solutions of the defocusing NLSE

2.1 Introduction

The problem (1) for $U=\mathbb{R}^n$ with $n=1,2,3$ and α being as in (6), along with more general cases of non-linearity, has been studied in [19]. There, it is shown that if $r=-\rho^\tau$ with $\rho>0$, as well as $\zeta\in C_{\mathrm{B}}^{k+1}\big(\overline{\mathbb{R}^n}\big)$, $\nabla_w\zeta\in H^{k+1}(\mathbb{R}^n)$, with $k=1$ if $n=1$ and $k=2$ if $n=2,3$, and additionally $\Big(|\zeta|^2-\rho\Big)\in L^2(\mathbb{R}^n)$, then (1) is globally and uniquely solvable in $H^1(\mathbb{R}^n)$.

Here, we extend the results of the aforementioned paper, not only by weakening the assumptions, but also by considering more general cases of $U\subseteq\mathbb{R}^n$, other than the Euclidean space itself. Moreover, we study the regularity of the solutions.

This chapter is organized as follows: In Section 2.2, we rigorously formulate the problem and provide properties of the operators and the quantities that appear. Local existence, uniqueness, globality and conservation of energy of the solutions is considered in Section 2.3. In particular, for the case of bounded[1] U, we first show (see Theorem 2.3.1) local existence for every

$$\alpha\in\begin{cases}(0,\infty), & \text{if } n=1,2\\ \left(0,\frac{4}{n-2}\right), & \text{otherwise,}\end{cases} \tag{2.1.1}$$

every $r\in\mathbb{R}$ and every U, if $\zeta\in H^1(U)\cap L^{\alpha+2}(U)$ (notice that $H^1(U)\cap L^{\alpha+2}(U)\equiv H^1(U)$ if $\partial U\in \mathrm{Lip}(\varepsilon,K,L)$). We also show (see Theorem 2.3.2) local existence for every α as in (6), every $r=-\rho^{2\tau}$ with $\rho>0$ and every U, if $\zeta\in X^1(U)$. Local existence in unbounded sets is studied next (see Theorem 2.3.3). For this purpose, we employ a technique which is based on the extension-by-zero of certain approximations of solutions, each one of which is considered in a bigger bounded open set than the domain of the previous one, in order to extend Theorem 2.3.2 for any unbounded U, if $\zeta\in X^1(U)$ and $(|\zeta|-\rho)\in L^2(U)$. Uniqueness and globality is also provided for certain cases of α and U (see Proposition 2.3.1), as well as the conservation of energy of the solutions (see Proposition 2.3.2) and a consequence of this, concerning the well posedness of the problem (see Corollary 2.3.1). Moreover, the regularity of the solutions of Theorem 2.3.2 and Theorem 2.3.3 for the cases

$$\begin{cases}\tau\in\mathbb{N}, & \text{if } n=1\\ \tau=1, & \text{if } n=2,\end{cases} \tag{2.1.2}$$

is studied in Section 2.4, where the high-order regularity in bounded sets (see Corollary 2.4.4) and in whole Euclidean space (see Theorem 2.4.2) is shown. The crux for obtaining the latter result is the application of the estimate in Proposition 1.2.14.

2.2 Formulation of the problem

In this section, we state the problem (1) rigorously.

[1] We note that all of the results concerning the case of bounded U, can also be applied to $H^1_{\mathrm{per}}(\mathbb{R}^n)$.

2.2.1 A general non-linear operator

We assume that α is as in (2.1.1), $\zeta \in L^2(U) \cap L^{\alpha+2}(U)$ for an arbitrary U and $r \in \mathbb{R}$. In virtue of the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$ (see Corollary 1.2.1), it is direct to derive that

$$\left(|u+\zeta|^{\alpha}(u+\zeta)\right) \in L^{\frac{\alpha+2}{\alpha+1}}(U), \ \forall u \in H_0^1(U).$$

Moreover, from

1. the Hölder inequality for $p_1 = \frac{\alpha+2}{\alpha+1}$ and $p_2 = \alpha+2$ and also $p_1, p_2 = 2$, as well as
2. the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$,

we get, for every $u, v \in H_0^1(U)$, that

$$\left|\int_U \left(|u+\zeta|^{\alpha}+r\right)(u+\zeta)\bar{v}dx\right| \leq \mathcal{K}\left(\|u\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}, \|\zeta\|_{L^2(U)}\right)\|v\|_{H^1(U)}. \tag{2.2.1}$$

Hence, for every $\lambda \in \mathbb{R}^*$ we define $g_\lambda : H_0^1(U) \to Y_\alpha(U) \coloneqq L^{\frac{\alpha+2}{\alpha+1}}(U) + L^2(U) \hookrightarrow H^{-1}(U)$ to be the non-linear and bounded (if we see it as an operator that maps to $H^{-1}(U)$) operator such that

$$g_\lambda(u; \alpha, \zeta, r) \coloneqq \lambda\left(|u+\zeta|^{\alpha}+r\right)\left(\bar{u}+\bar{\zeta}\right), \ \forall u \in H_0^1(U),$$

or else

$$\langle g_\lambda(u), v\rangle = \lambda \int_U \left(|u+\zeta|^{\alpha}+r\right)\left(\bar{u}+\bar{\zeta}\right) v dx, \ \forall u, v \in H_0^1(U).$$

For the above operator we have the following estimate, for the proof of which we need a lemma.

Lemma 2.2.1. *If* $\alpha \in [0, \infty)$*, then*

$$\left\||u|^{\alpha}u-|v|^{\alpha}v\right\|_{L^{\frac{\alpha+2}{\alpha+1}}(U)} \leq C\left(\|u\|^{\alpha}_{L^{\alpha+2}(U)}+\|v\|^{\alpha}_{L^{\alpha+2}(U)}\right)\|u-v\|_{L^{\alpha+2}(U)}, \ \forall u, v \in L^{\alpha+2}(U). \tag{2.2.2}$$

Proof. It is direct application of the elementary inequality

$$\left||z_1|^q z_1-|z_2|^q z_2\right| \leq C_q |z_1-z_2|\left(|z_1|^q+|z_2|^q\right), \ \forall z_1, z_2 \in \mathbb{C}, \ \forall q \in [0, \infty), \tag{2.2.3}$$

the Hölder inequality for $p_1 = \alpha+1$ and $p_2 = \frac{\alpha+1}{\alpha}$ and (1.2.13). □

We also need the following Gagliardo-Nirenberg interpolation inequality (see, e.g., Theorem 1.3.7 in [10]): for α being as in (2.1.1) we have

$$\|u\|_{L^{\alpha+2}(\mathbb{R}^n)} \leq C\|\nabla_w u\|^{\frac{n\alpha}{2(\alpha+2)}}_{L^2(\mathbb{R}^n)}\|u\|^{1-\frac{n\alpha}{2(\alpha+2)}}_{L^2(\mathbb{R}^n)}, \ \forall u \in C_c^{\infty}(\mathbb{R}^n),$$

or else

$$\|u\|_{L^{\alpha+2}(U)} \leq C\|\nabla_w u\|^{\frac{n\alpha}{2(\alpha+2)}}_{L^2(U)}\|u\|^{1-\frac{n\alpha}{2(\alpha+2)}}_{L^2(U)}, \ \forall u \in H_0^1(U). \tag{2.2.4}$$

Proposition 2.2.1. *For every* $u, v \in H_0^1(U)$ *we have*

$$\|g_\lambda(u)-g_\lambda(v)\|_{Y_\alpha(U)} \leq \mathcal{K}\left(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}\right)\left(\|u-v\|^{1-\frac{n\alpha}{2(\alpha+2)}}_{L^2(U)}+\|u-v\|_{L^2(U)}\right). \tag{2.2.5}$$

Proof. From (2.2.2) and the scaling invariant embedding $H_0^1(U) \hookrightarrow L^{\alpha+2}(U)$ we get

$$\|g_\lambda(u)-g_\lambda(v)\|_{Y_\alpha(U)} \leq C\left(\|u\|^{\alpha}_{H^1(U)}+\|v\|^{\alpha}_{H^1(U)}+\|\zeta\|^{\alpha}_{L^{\alpha+2}(U)}\right)\|u-v\|_{L^{\alpha+2}(U)}+|r|\,\|u-v\|_{L^2(U)},$$

or else

$$\|g_\lambda(u)-g_\lambda(v)\|_{Y_\alpha(U)} \leq \mathcal{K}\left(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}\right)\left(\|u-v\|_{L^{\alpha+2}(U)}+\|u-v\|_{L^2(U)}\right),$$

hence, employing (2.2.4) we get the desired estimate. □

Now, we further assume that $\zeta \in H^1(U)$ and we consider an arbitrary $\mathcal{L}_w(a,\theta)$ as in Definition 1.2.1. We define $\mathcal{N}_\lambda[\ast,\star] : H_0^1(U)^2 \to \mathbb{C}$ to be the form which is associated with the operator $\mathcal{L}_w(\cdot+\zeta)+g_\lambda$, such that

$$\mathcal{N}_\lambda[u,v] := \langle \mathcal{L}_w(u+\zeta), v\rangle + \langle g_\lambda(u), v\rangle, \ \forall\, u,v \in H_0^1(U).$$

We then restate the problem (1): for every $u_0 \in H_0^1(U)$, we seek a solution

$$\mathbf{u} = \mathbf{u}_{J_0} \in L^\infty\big(J_0; H_0^1(U)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U)\big)$$

of

$$\begin{cases} \langle i\mathbf{u}', v\rangle + \mathcal{N}_\lambda[\mathbf{u}, v] = 0, \text{ for every } v \in H_0^1(U), \text{ a.e. in } J_0 \\ \mathbf{u}(0) = u_0. \end{cases} \tag{2.2.6}$$

We note that we obtain the following estimate for the form $\mathcal{N}_\lambda$,

$$|\mathcal{N}_\lambda[u,v]| \le \mathcal{K}\Big(\|a\|_{L^\infty(U)}, \|u\|_{H^1(U)}, \|\zeta\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}\Big)\|v\|_{H^1(U)}, \ \forall\, u,v \in H_0^1(U), \tag{2.2.7}$$

directly from the Hölder inequality ($p_1 = p_2 = 2$) and (2.2.1).

We further define the energy functional $E_\lambda : H_0^1(U) \to [-\infty,\infty]$ by

$$E_\lambda(\cdot;\alpha,\zeta,r) := \frac{1}{2}\mathcal{L}[\cdot+\zeta,\cdot+\zeta] + G_\lambda(\cdot;\alpha,\zeta,r),$$

where $G_\lambda : H_0^1(U) \to [-\infty,\infty]$, with

$$G_\lambda(\cdot;\alpha,\zeta,r) := \lambda \int_U V(|\cdot+\zeta|;\alpha,r)dx,$$

where $V : [0,\infty) \to [0,\infty)$ is defined as

$$V(x;\alpha,r) := \frac{1}{\alpha+2}x^{\alpha+2} + \frac{1}{2}rx^2 + \frac{\alpha}{2(\alpha+2)}|r|^{\frac{\alpha+2}{\alpha}}. \tag{2.2.8}$$

It is direct to check that for every $C_\alpha > \alpha+2$ we have

$$x^{\alpha+2} \le C_\alpha V(x), \ \forall x \ge \left(\frac{C_\alpha |r|}{C_\alpha - (\alpha+2)}\right)^{\frac{1}{\alpha}} > |r|^{\frac{1}{\alpha}}. \tag{2.2.9}$$

For the functional G_λ we have the following estimates.

Proposition 2.2.2. *For every $u,v \in H_0^1(U)$ we have*

$$\begin{aligned} |G_\lambda(u) - G_\lambda(v)| \le \mathcal{K}\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}, \|\zeta\|_{L^2(U)}\Big) \times \\ \times \left(\|u-v\|_{L^2(U)}^{1-\frac{n\alpha}{2(\alpha+2)}} + \|u-v\|_{L^2(U)}\right) \end{aligned} \tag{2.2.10}$$

and

$$G(u) \le \mathcal{K}\Big(\|u\|_{H^1(U)}, \|\zeta\|_{L^{\alpha+2}(U)}, \|\zeta\|_{L^2(U)}, |U|\Big). \tag{2.2.11}$$

Proof. We notice that, for every fixed $x \in U$ we have

$$\begin{aligned} V(|u(x)+\zeta(x)|) - V(|v(x)+\zeta(x)|) = \int_0^1 \frac{d}{ds} V(|su(x)+(1-s)\,v(x)+\zeta(x)|)ds = \\ = \int_0^1 |su(x)+(1-s)\,v(x)+\zeta(x)|^\alpha \mathrm{Re}[(su(x)+(1-s)\,v(x)+\zeta(x))\,(\bar{u}(x)-\bar{v}(x))]ds + \\ + \int_0^1 r\mathrm{Re}[(su(x)+(1-s)\,v(x)+\zeta(x))\,(\bar{u}(x)-\bar{v}(x))]ds = \\ = \mathrm{Re}\Big(\int_0^1 \big(|su(x)+(1-s)\,v(x)+\zeta(x)|^\alpha + r\big)\,(su(x)+(1-s)\,v(x)+\zeta(x))\,(\bar{u}(x)-\bar{v}(x))ds\Big), \end{aligned}$$

that is

$$V(|u+\zeta|)-V(|v+\zeta|)=\mathrm{Re}\Big(\int_0^1 \left(|su+(1-s)\,v+\zeta|^{\alpha}+r\right)(su+(1-s)\,v+\zeta)\,(\bar{u}-\bar{v})ds\Big),\ \text{in } U. \quad (2.2.12)$$

Hence

$$\begin{aligned}|V(|u+\zeta|)-V(|v+\zeta|)| &\leq \int_0^1 \Big(|su+(1-s)\,v+\zeta|^{\alpha+1}+|r|\,|su+(1-s)\,v+\zeta|\Big)|u-v|ds\leq\\ &\leq C\int_0^1 \Big(s^{\alpha+1}|u|^{\alpha+1}+(1-s)^{\alpha+1}|v|^{\alpha+1}+|\zeta|^{\alpha+1}+|u|+|v|+|\zeta|\Big)|u-v|ds\leq\\ &\leq C\Big(|u|^{\alpha+1}+|v|^{\alpha+1}+|\zeta|^{\alpha+1}+|u|+|v|+|\zeta|\Big)|u-v|,\end{aligned}$$

and so

$$\begin{aligned}|G_\lambda(u)-G_\lambda(v)| &\leq |\lambda|\int_U |V(|u+\zeta|)-V(|v+\zeta|)|dx\leq\\ &\leq C\int_U \Big(|u|^{\alpha+1}+|v|^{\alpha+1}+|\zeta|^{\alpha+1}+|u|+|v|+|\zeta|\Big)|u-v|dx,\end{aligned}$$

From the Hölder inequality for $p_1=\frac{\alpha+2}{\alpha+1}$ and $p_2=\alpha+2$ and also for $p_1,p_2=2$, as well as the scaling invariant embedding $H_0^1(U)\hookrightarrow L^{\alpha+2}(U)$, we get

$$\begin{aligned}|G_\lambda(u)-G_\lambda(v)| &\leq \mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^{\alpha+2}(U)},\|\zeta\|_{L^2(U)}\Big)\times\\ &\times\Big(\|u-v\|_{L^{\alpha+2}(U)}+\|u-v\|_{L^2(U)}\Big),\end{aligned}$$

thus (2.2.10) follows from (2.2.4).

As for (2.2.11), we have that

$$|G_\lambda(0)|=|\lambda|\int_U V(|\zeta|)dx\leq\mathcal{K}\Big(\|\zeta\|_{L^{\alpha+2}(U)},\|\zeta\|_{L^2(U)},|U|\Big).$$

Then the result follows from (2.2.10) and the triangle inequality. □

Remark 2.2.1. *A similar proof for Proposition 2.2.2 would be via the validation that g_λ is the Gateaux derivative of G_λ.*

2.2.2 A special case of the non-linear operator

The operator g_λ is not useful for the study of the problem in unbounded domains. Here we see how we can overcome this drawback, by considering different assumptions.

First, we assume that $\zeta\in L^2(U)\cap L^{\alpha+2}(U)\cap L^\infty(U)$ and we extract some properties concerning the aforementioned operator.

Proposition 2.2.3. *Let $u,v\in H_0^1(U)$.*

1. *If $n=1$, then*

$$\|g_\lambda(u)-g_\lambda(v)\|_{L^2(U)}\leq\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\|u-v\|_{L^2(U)}. \quad (2.2.13)$$

2. *If $n=2$, then*

$$\begin{aligned}\|g_\lambda(u)-g_\lambda(v)\|_{L^2(U)} &\leq\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ &\times\Big(\|u-v\|_{L^4(U)}+\|u-v\|_{L^2(U)}\Big)\end{aligned} \quad (2.2.14)$$

and

$$\begin{aligned}\|g_\lambda(u)-g_\lambda(v)\|_{L^2(U)} &\leq\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ &\times\Big(\|u-v\|^{\frac12}_{L^2(U)}+\|u-v\|_{L^2(U)}\Big).\end{aligned} \quad (2.2.15)$$

3. If $n=3$ and $\alpha=2$, then

$$\|g_\lambda(u)-g_\lambda(v)\|_{Y_2(U)}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times \times\Big(\|u-v\|_{L^4(U)}+\|u-v\|_{L^2(U)}\Big) \tag{2.2.16}$$

and

$$\|g_\lambda(u)-g_\lambda(v)\|_{Y_2(U)}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times \times\Big(\|u-v\|_{L^2(U)}^{\frac14}+\|u-v\|_{L^2(U)}\Big). \tag{2.2.17}$$

Proof. Let $n=1,2$. By simple application of (2.2.3), we get

$$\int_U|g_\lambda(u)-g_\lambda(v)|^2dx\le C\int_U\left(|u|^{2\alpha}+|v|^{2\alpha}\right)|u-v|^2dx+C\left(\|\zeta\|_{L^\infty(U)}^{2\alpha}+1\right)\|u-v\|_{L^2(U)}^2.$$

For $n=1$, we employ the scaling invariant embedding $H_0^1(U)\hookrightarrow L^\infty(U)$ to derive (2.2.13). For $n=2$, we get (2.2.14) from the Hölder inequality ($p_1=p_2=2$) and the scaling invariant $H_0^1(U)\hookrightarrow L^\vartheta(U)$, for $\vartheta\in[2,\infty)$. (2.2.15) follows from (2.2.14) and (2.2.4). As for $n=3$ and $\alpha=2$, we have

$$g_\lambda(u)-g_\lambda(v)\coloneqq\lambda\left(I_1+I_2\right),$$

where

$$I_1=|u|^2\bar{u}-|v|^2\bar{v}\text{ and }I_2=2\bar{\zeta}\left(|u|^2-|v|^2\right)+\zeta\left(\bar{u}^2-\bar{v}^2\right)+\left(2|\zeta|^2+r\right)(u-v)+\bar{\zeta}^2\,(u-v).$$

Directly from (2.2.2) we get that $I_1\in L^{\frac{\alpha+2}{\alpha+1}}(U)$ with

$$\|I_1\|_{L^{\frac{\alpha+2}{\alpha+1}}(U)}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)}\Big)\|u-v\|_{L^4(U)}.$$

Additionally, from the Hölder inequality ($p_1=p_2=2$) and the scaling invariant embedding $H_0^1(U)\hookrightarrow L^\vartheta(U)$, for $\vartheta\in[2,6]$ we have

$$\int_U|I_2|^2dx\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\int_U|u-v|^2\left(|u|^2+|v|^2\right)+|u-v|^2dx\le \le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\left(\|u-v\|_{L^4(U)}^2+\|u-v\|_{L^2(U)}^2\right),$$

hence (2.2.16) follows. We obtain (2.2.17) from (2.2.16) along with (2.2.4). □

We further notice that, by dealing as for (2.2.16), we also can have that

$$\|g_\lambda(u)-g_\lambda(v)\|_{L^{p_1}\left(J;L^{\frac43}(U)\right)+L^{p_2}(J;L^2(U))}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times \times\Big(\|u-v\|_{L^{p_1}(J;L^4(U))}+\|u-v\|_{L^{p_2}(J;L^2(U))}\Big), \tag{2.2.18}$$

for every $u,v\in H_0^1(U)$ and $p_1,p_2\in[1,\infty]$, if $n=3$ and $\alpha=2$.

Proposition 2.2.4. *Let $u,v\in H_0^1(U)$,*

$$\alpha=2\tau\ \textit{for}\ \begin{cases}\tau\in\mathbb{N}, & \textit{if } n=1,2\\ \tau=1, & \textit{if } n=3\end{cases}\ \textit{and}\ r=-\rho^{2\tau}\ \textit{for}\ \rho\in(0,\infty), \tag{2.2.19}$$

as well as $(|\zeta|-\rho)\in L^2(U)$.

1. If $n=1,2$, then

$$\|g_\lambda(u)\|_{L^2(U)}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big). \tag{2.2.20}$$

2. If $n=3$, then

$$\|g_\lambda(u)\|_{Y_2(U)}\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big). \tag{2.2.21}$$

Proof. We notice that $g_\lambda(0)=\lambda\Big(|\zeta|^{2\tau}-\rho^{2\tau}\Big)\bar{\zeta}$, which belongs to $L^2(U)$. Indeed, by expanding via

$$a^n-b^n=(a-b)\big(a^{n-1}+a^{n-2}b+\ldots+ab^{n-2}+b^{n-1}\big), \tag{2.2.22}$$

we get

$$\|g_\lambda(0)\|_{L^2(U)}\leq\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big).$$

The results then follow from Proposition 2.2.3 and the triangle inequality. □

Let us now notice that ζ being in $L^2(U)\cap L^{\alpha+2}(U)$ plays no essential role at any of the above results. Hence, for

$$\alpha,\ r \text{ as in (2.2.19) and } \zeta\in L^\infty(U) \text{ with } (|\zeta|-\rho)\in L^2(U),$$

and for every $\lambda\in\mathbb{R}^*$, we define

$$\widetilde{g}_\lambda : H_0^1(U)\to\begin{cases}L^2(U), & \text{if } n=1,2\\ Y_2(U), & \text{if } n=3\end{cases}$$

by

$$\widetilde{g}_\lambda(u;\tau,\zeta,\rho)\coloneqq\lambda\left(|u+\zeta|^{2\tau}-\rho^{2\tau}\right)\left(\bar{u}+\bar{\zeta}\right),\ \ \forall u\in H_0^1(U),$$

or else

$$\langle\widetilde{g}_\lambda(u),v\rangle\coloneqq\lambda\int_U\left(|u+\zeta|^{2\tau}-\rho^{2\tau}\right)\left(\bar{u}+\bar{\zeta}\right)v\,dx,\ \ \forall u,v\in H_0^1(U),$$

which satisfies the estimates from (2.2.13) to (2.2.18), as well as (2.2.20) and (2.2.21).

Now, we further assume that $\zeta\in X^1(U)$.

Remark 2.2.2. *Since $\zeta\in X^1(U)$ and $(|\zeta|-\rho)\in L^2(U)$, then, in virtue of Corollary 1.2.4, we have that $(|\zeta|-\rho)\in H^1(U)$.*

Considering also an arbitrary $\mathcal{L}_w(a,\theta)$ as in Definition 1.2.1, we define $\widetilde{\mathcal{N}}_\lambda[*,\star]:H_0^1(U)^2\to\mathbb{C}$ to be the form which is associated with the operator $\mathcal{L}_w(\cdot+\zeta)+\widetilde{g}_\lambda$, such that

$$\widetilde{\mathcal{N}}_\lambda[u,v]\coloneqq\langle\mathcal{L}_w(u+\zeta),v\rangle+\langle\widetilde{g}_\lambda(u),v\rangle,\ \ \forall\, u,v\in H_0^1(U).$$

Now, the problem (1) becomes: for every $u_0\in H_0^1(U)$, we seek for a solution

$$\mathbf{u}=\mathbf{u}_{J_0}\in L^\infty\big(J_0;H_0^1(U)\big)\cap W^{1,\infty}\big(J_0;H^{-1}(U)\big)$$

of

$$\begin{cases}\langle i\mathbf{u}',v\rangle+\widetilde{\mathcal{N}}_\lambda[\mathbf{u},v]=0, \text{ for every } v\in H_0^1(U), \text{ a.e. in } J_0\\ \mathbf{u}(0)=u_0.\end{cases} \tag{2.2.23}$$

Directly from (2.2.20) and (2.2.21) and the Hölder inequality, we derive the following estimate

$$\Big|\widetilde{\mathcal{N}}_\lambda[u,v]\Big|\leq\mathcal{K}\Big(\|a\|_{L^\infty(U)},\|u\|_{H^1(U)},\|\zeta\|_{X^1(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\|v\|_{H^1(U)},\ \forall\, u,v\in H_0^1(U). \tag{2.2.24}$$

We also define the respective energy functional $\widetilde{E}_\lambda:H_0^1(U)\to[-\infty,\infty]$ by

$$\widetilde{E}_\lambda(\cdot;\tau,\zeta,\rho)\coloneqq\frac{1}{2}\mathcal{L}[\cdot+\zeta,\cdot+\zeta]+\widetilde{G}_\lambda(\cdot;\tau,\zeta,\rho),$$

where $\widetilde{G}_\lambda:H_0^1(U)\to[-\infty,\infty]$, with

$$\widetilde{G}_\lambda(\cdot;\tau,\zeta,\rho)\coloneqq\lambda\int_U V(|\cdot+\zeta|;\tau,\rho)dx.$$

Proposition 2.2.5. *For every $u,v\in H_0^1(U)$ we have*

$$\left|\widetilde{G}_\lambda(u)-\widetilde{G}_\lambda(v)\right|\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times \\ \times\|u-v\|_{L^2(U)},\ \text{if } n=1,2 \tag{2.2.25}$$

and

$$\left|\widetilde{G}_\lambda(u)-\widetilde{G}_\lambda(v)\right|\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times \\ \times\Big(\|u-v\|_{L^4(U)}+\|u-v\|_{L^2(U)}\Big),\ \text{if } n=3, \tag{2.2.26}$$

or else

$$\left|\widetilde{G}_\lambda(u)-\widetilde{G}_\lambda(v)\right|\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times \\ \times\Big(\|u-v\|_{L^2(U)}^{\frac14}+\|u-v\|_{L^2(U)}\Big),\ \text{if } n=3, \tag{2.2.27}$$

as well as

$$\left|\widetilde{G}_\lambda(u)\right|\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big). \tag{2.2.28}$$

Proof. The equation (2.2.12) now becomes

$$V(|u+\zeta|)-V(|v+\zeta|)=\mathrm{Re}\bigg(\int_0^1\Big(|su+(1-s)\,v+\zeta|^{2\tau}-\rho^{2\tau}\Big)\,(su+(1-s)\,v+\zeta)\,(\bar{u}-\bar{v})ds\bigg),\ \text{in } U,$$

and expanding via (2.2.22) we deduce

$$V(|u+\zeta|)-V(|v+\zeta|)=\mathrm{Re}\bigg(\int_0^1\Big(|su+(1-s)\,v+\zeta|^2-\rho^2\Big)\,(su+(1-s)\,v+\zeta)\,(\bar{u}-\bar{v})\times \\ \times\Big(|su+(1-s)\,v+\zeta|^{2(\tau-1)}+\rho^2|su+(1-s)\,v+\zeta|^{2(\tau-2)}+\cdots+ \\ +\rho^{2(\tau-2)}|su+(1-s)\,v+\zeta|^2+\rho^{2(\tau-1)}\Big)ds\bigg).$$

Setting $w:=su+(1-s)\,v$ and further expanding the term $|w+\zeta|^2$, we easily derive that

$$\left|\Big(|w+\zeta|^2-\rho^2\Big)\,(w+\zeta)\right|=\left|\Big(|w|^2+2\mathrm{Re}(\bar{\zeta}w)+|\zeta|^2-\rho^2\Big)\,(w+\zeta)\right|\le \\ \le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|w|^2+|w|+||\zeta|-\rho|\Big)\,(|w|+|\zeta|)\le \\ \le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|w|^3+|w|^2+|w|+||\zeta|-\rho|\Big)\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|w|^3+||\zeta|-\rho|\Big),$$

as well as

$$\left||w+\zeta|^{2(\tau-1)}+\rho^2|w+\zeta|^{2(\tau-2)}+\ldots+\rho^{2(\tau-2)}|w+\zeta|^2+\rho^{2(\tau-1)}\right|\le C\left||w+\zeta|^{2(\tau-1)}+1\right|\le \\ \le C\left||w|^{2(\tau-1)}+|\zeta|^{2(\tau-1)}+1\right|\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|w|^{2(\tau-1)}+1\Big),$$

hence

$$\left|\Big(|w+\zeta|^2-\rho^2\Big)\,(w+\zeta)\right|\left||w+\zeta|^{2(\tau-1)}+\rho^2|w+\zeta|^{2(\tau-2)}+\ldots+\rho^{2(\tau-2)}|w+\zeta|^2+\rho^{2(\tau-1)}\right|\le \\ \le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|w|^{2\tau+1}+||\zeta|-\rho|\Big)\le \\ \le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(s^{2\tau+1}|u|^{2\tau+1}+(1-s)^{2\tau+1}|v|^{2\tau+1}+||\zeta|-\rho|\Big),$$

and so

$$|V(|u+\zeta|)-V(|v+\zeta|)|\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\Big(|u|^{2\tau+1}+|v|^{2\tau+1}+||\zeta|-\rho|\Big)\,|u-v|,$$

or else

$$\left|\widetilde{G}_\lambda(u)-\widetilde{G}_\lambda(v)\right|\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\int_U\Big(|u|^{2\tau+1}+|v|^{2\tau+1}+||\zeta|-\rho|\Big)\,|u-v|dx.$$

For $n=1,2$ we employ the Hölder inequality for $p_1=p_2=2$ and the scaling invariant embedding $H_0^1(U)\hookrightarrow L^{4\tau+2}(U)$ to get (2.2.25), while, for $n=3$ and $\tau=1$, we get (2.2.26) from the Hölder inequality for $p_1=\frac{4}{3}$ and $p_2=4$ and also for $p_1=p_2=2$, as well as from the scaling invariant embedding $H_0^1(U)\hookrightarrow L^4(U)$. (2.2.27) follows from (2.2.26) along with (2.2.4).

As far as (2.2.28) is concerned, it suffices to show that

$$\left|\widetilde{G}_\lambda(0)\right|\le\mathcal{K}\Big(\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big).$$

Indeed,

$$\left|\widetilde{G}_\lambda(0)\right|\le|\lambda|\int_U V(|\zeta|;\tau,\rho)dx=C\int_U\frac{1}{2(\tau+1)}|\zeta|^{2(\tau+1)}-\frac{1}{2}\rho^{2\tau}|\zeta|^2+\frac{\tau}{2(\tau+1)}\rho^{2(\tau+1)}dx,$$

thus we employ the identity

$$a^{n+1}-a(n+1)b^n+nb^{n+1}=(a-b)^2\left(a^{n-1}+2a^{n-2}b+\ldots+(n-1)ab^{n-2}+nb^{n-1}\right)$$

to obtain the desired estimate. □

It then follows that $\widetilde{E}_\lambda,\widetilde{G}_\lambda:H_0^1(U)\to(-\infty,\infty)$.

2.2.3 The non-linear operator on restrictions

Here, we make a note concerning the ability to define the operators and the functionals of the previous subsections on restrictions of functions.

The scaling invariant Sobolev embeddings are the crux for the definition of the operators g_λ and $\widetilde{g}_\lambda$, as well as the functionals E_λ, G_λ and $\widetilde{E}_\lambda$, $\widetilde{G}_\lambda$ on $H_0^1(U)$, for every U. Hence, in virtue of Corollary 1.2.1, by defining these operators and functionals for every $u\in H_0^1(U)$ for some arbitrary U, we can also consider them defined for every $((\mathcal{R}(U,V))u)\in H^1(V)$ for every open $V\subseteq U$ (note that we have $((\mathcal{R}(U,V))u)\in H^1(V)$ for every open $V\subseteq U$ from Proposition 1.2.2). This means that we do not need to impose any regularity on ∂V in order to consider the scaling dependent Sobolev embeddings of Corollary 1.2.2. However, the above conclusion is evidently not true for the bounds that are obtained with the use of (2.2.4), for which we need to employ the extension operators of Theorem 1.2.1.

For example, in order to get (2.2.1) for every $u,v\in H_0^1(U)$ for arbitrary U, we use the scaling invariant embedding $H_0^1(U)\hookrightarrow L^{\alpha+2}(U)$. Let $V\subseteq U$ be open. Instead of trying to apply an embedding of the form $H^1(V)\hookrightarrow L^{\alpha+2}(V)$ in order to get (2.2.1) for every $((\mathcal{R}(U,V))u)\in H^1(V)$ and every $v\in H_0^1(V)$, we simply notice the trivial fact that

$$\|((\mathcal{R}(U,V))u)\|_{L^{\alpha+2}(V)}\le\|u\|_{L^{\alpha+2}(U)}\le C\|u\|_{H^1(U)},\ \forall V\subseteq U,\ \forall u\in H_0^1(U).$$

Thus, dealing exactly as above (and focusing only on the results of Subsection 2.2.2 which we use later), we get, for every U and every $u,v\in H_0^1(U)$, that

$$\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)):H_0^1(U)\to\begin{cases}L^2(V), & \text{if } n=1,2\\ Y_2(V), & \text{if } n=3,\end{cases}\ \text{for every open } V\subseteq U \tag{2.2.29}$$

and

$$\widetilde{E}_\lambda\circ(\mathcal{R}(U,V)),\widetilde{G}_\lambda\circ(\mathcal{R}(U,V)):H_0^1(U)\to(-\infty,\infty),\ \text{for every open } V\subseteq U \tag{2.2.30}$$

are well defined, with

$$\begin{aligned}&\|(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)-(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)(v)\|_{L^2(V)}\le\\&\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\&\times\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)},\ \text{if } n=1,\end{aligned} \tag{2.2.31}$$

$$\begin{aligned}&\|(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)-(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)(v)\|_{L^2(V)}\le\\&\le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\&\times\Big(\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^4(V)}+\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)}\Big),\ \text{if } n=2,\end{aligned} \tag{2.2.32}$$

$$\begin{aligned}\|(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)-(\widetilde{g}_\lambda\circ(\mathcal{R}(U,V)))(u)(v)\|_{Y_2(V)}\le\\ \le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)}\Big)\times\\ \times\Big(\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^4(V)}+\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)}\Big),\text{ if } n=3,\end{aligned}\tag{2.2.33}$$

as well as

$$\begin{aligned}\Big|\big(\widetilde{G}_\lambda\circ(\mathcal{R}(U,V))\big)(u)-\big(\widetilde{G}_\lambda\circ(\mathcal{R}(U,V))\big)(v)\Big|\le\\ \le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times\\ \times\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(V)},\text{ if } n=1,2\end{aligned}\tag{2.2.34}$$

and

$$\begin{aligned}\Big|\big(\widetilde{G}_\lambda\circ(\mathcal{R}(U,V))\big)(u)-\big(\widetilde{G}_\lambda\circ(\mathcal{R}(U,V))\big)(v)\Big|\le\\ \le\mathcal{K}\Big(\|u\|_{H^1(U)},\|v\|_{H^1(U)},\|\zeta\|_{L^\infty(U)},\||\zeta|-\rho\|_{L^2(U)}\Big)\times\\ \times\Big(\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^4(U)}+\|(\mathcal{R}(U,V))u-(\mathcal{R}(U,V))v\|_{L^2(U)}\Big),\text{ if } n=3,\end{aligned}\tag{2.2.35}$$

for every open $V\subseteq U$.

2.3 Weak solutions

In this section we study the weak solvability of (2.2.6) and (2.2.23) and the properties of the weak solutions, for the defocusing case $\lambda>0$. Note that we then have

$$E_\lambda,G_\lambda:H_0^1(U)\to[0,\infty]\text{ and }\widetilde{E}_\lambda,\widetilde{G}_\lambda:H_0^1(U)\to[0,\infty).\tag{2.3.1}$$

2.3.1 Local existence results

The following local existence results are somewhat "strong" ones, in the sense that the interval of existence of a solution does not depend on the initial datum. Before we proceed, we state and prove some preliminary lemmata.

Lemma 2.3.1. *For every $f\in H^{-1}(U)$ there exists $\{f_j\}_{j=0}^n\subset L^2(U)$ such that*

$$\langle f,v\rangle=\int_U v\overline{f_0}+\sum_{j=1}^n\left(\partial^j v\right)\overline{f_j}dx,\ \forall v\in H_0^1(U)$$

and, in particular, we have

$$(v,f)=\langle f,v\rangle,\ \forall v\in H_0^1(U),\ \forall f\in L^2(U).$$

Proof. The first result follows from a direct application the complex version of Riesz-Fréchet representation theorem (see, e.g., Proposition 11.27 in [8]). The second is a direct consequence of the first one. ☐

Lemma 2.3.2. *Let J be bounded, $\mathcal{X}_1$ be a Banach space and $\mathcal{X}_2$ be a Banach space with the Radon-Nikodym property with respect to the Lebesgue measure in $(J,\mathscr{B}(J))$.*

1. *Let $\{\mathbf{u}_k\}_k\subset L^\infty(J;\mathcal{X}_1)$ and $\mathbf{u}:J\to\mathcal{X}_1$ with $\mathbf{u}_k(t)\rightharpoonup\mathbf{u}(t)$ in $\mathcal{X}_1$, for a.e. $t\in J$. If $\|\mathbf{u}_k\|_{L^\infty(J;\mathcal{X}_1)}\le C$ uniformly for every k, then $\mathbf{u}\in L^\infty(J;\mathcal{X}_1)$ with $\|\mathbf{u}\|_{L^\infty(J;\mathcal{X}_1)}\le C$, where C is the same in both inequalities.*

2. *Let $\{\mathbf{u}_k\}_k\cup\{\mathbf{u}\}\subset L^\infty(J;\mathcal{X}_2^*)$ with $\mathbf{u}_k\overset{*}{\rightharpoonup}\mathbf{u}$ in $L^\infty(J;\mathcal{X}_2^*)$[2]. If $\|\mathbf{u}_k\|_{L^\infty(J;\mathcal{X}_2^*)}\le C$ uniformly for every k, then $\|\mathbf{u}\|_{L^\infty(J;\mathcal{X}_2^*)}\le C$, where C is the same in both inequalities.*

[2] That is, $\mathbf{u}_k\overset{*}{\rightharpoonup}\mathbf{u}$ in $\sigma\big(L^\infty(J;\mathcal{X}_2^*),L^1(J;\mathcal{X}_2)\big)$. Note that $L^\infty(J;\mathcal{X}_2^*)\cong\big(L^1(J;\mathcal{X}_2)\big)^*$ (see, e.g., Theorem 1, Section 1, Chapter IV in [13]).

Proof. For Point 1., we derive that $\|\mathbf{u}(t)\|_{\mathcal{X}_1} \leq C$, for a.e. $t \in J$, from the (sequentially) weak lower semi-continuity of the norm, hence the result follows. As for Point 2., let $v \in \mathcal{X}_2$ be such that $\|v\|_{\mathcal{X}_2} \leq 1$ and set $\mathbf{v} : J \to \mathcal{X}_2$ the constant function with $\mathbf{v}(t) \coloneqq v$, for all $t \in J$. We have

$$\int_s^{s+h} \langle \mathbf{u}_k, \mathbf{v} \rangle \, dt \leq Ch, \text{ for a.e. } s \in J^\circ \text{ and every sufficiently small } h > 0$$

Considering the limit $\mathbf{u}_k \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(J; \mathcal{X}_2^*)$, dividing both parts by h and then letting $h \searrow 0$, we get, from the Lebesgue differentiation theorem, that $\langle \mathbf{u}(s), v \rangle \leq C$, for a.e. $s \in J^\circ$. Since v arbitrary, the proof is complete. □

Lemma 2.3.3. *Let $z_0 \in \mathbb{C}^n$ and $z : J_0 \to \mathbb{C}^n$ be the unique, maximal solution of the initial-value problem*

$$\begin{cases} z'(t) = iF(z(t)), & \forall t \in J_0^* \\ z(0) = z_0, \end{cases}$$

for an appropriate function F (e.g. locally Lipschitz). If $z_0 \in \mathbb{R}^n$ and $\overline{F}(z) = F(\bar{z})$, then J_0 is symmetric around 0 and also $z(t) = \bar{z}(-t)$, for all $t \in J_0$.

Proof. We define $-J_0 \coloneqq \{ t \in \mathbb{R} \mid -t \in J_0 \}$ and also $y : -J_0 \to \mathbb{C}^n$ with $y(t) \coloneqq \bar{z}(-t)$, for all $t \in -J_0$. Since $z_0 \in \mathbb{R}^n$ and $\overline{F}(z) = F(\bar{z})$, we can easily see that y solves the above problem (in $-J_0$). Hence $-J_0 \subseteq J_0$, since z is the maximal solution. Therefore, J_0 is symmetric around 0. We can now define the function $x : J_0 \to \mathbb{C}^n$ as $x(t) \coloneqq \bar{z}(-t)$, for all $t \in J_0$ and we deduce that x also solves the problem (in J_0). Hence, $\bar{z}(-t) = x(t) = z(t)$, for all $t \in J_0$, since z is unique. □

Lemma 2.3.4. *Let $m \in \mathbb{N}$, $p \in [1, \infty]$, U_1, U_2, $\phi \in C_c^\infty(U_1)$ and $u \in W_0^{m,p}(U_2)$. If we set*

$$\varphi \coloneqq (\mathcal{R}(U_1, U_1 \cap U_2))\,\phi \text{ and } v \coloneqq (\mathcal{R}(U_2, U_1 \cap U_2))\,u,$$

then

$$(\varphi v) \in W_0^{m,p}(U_1 \cap U_2), \text{ with } \|\varphi v\|_{W^{m,p}(U_1 \cap U_2)} \leq \mathcal{K}\Big(\|\phi\|_{C_{\mathrm{B}}^m(U_1)}\Big) \|u\|_{W^{m,p}(U_2)}.$$

Proof. We assume that $U_1 \cap U_2 \neq \varnothing$, otherwise we have nothing to show (see also Point 3. before Definition 1.2.6). In view of Proposition 1.2.8, we derive that

$$(\varphi v) \in W^{m,p}(U_1 \cap U_2), \text{ with } \|\varphi v\|_{W^{m,p}(U_1 \cap U_2)} \leq \mathcal{K}\Big(\|\varphi\|_{C_{\mathrm{B}}^m(U_1 \cap U_2)}\Big) \|v\|_{W^{m,p}(U_1 \cap U_2)},$$

hence

$$\|\varphi v\|_{W^{m,p}(U_1 \cap U_2)} \leq \mathcal{K}\Big(\|\phi\|_{C_{\mathrm{B}}^m(U_1)}\Big) \|u\|_{W^{m,p}(U_2)}.$$

Now, we consider $\{u_k\}_k \subset C_c^\infty(U_2)$, such that $u_k \to u$ in $W^{m,p}(U_2)$ and in analogous manner we set

$$v_k \coloneqq (\mathcal{R}(U_2, U_1 \cap U_2))\, u_k, \ \forall k.$$

Evidently,

$$(\varphi v_k) \in C^m(U_1 \cap U_2).$$

In particular, we can deduce, by employing trivial arguments, that

$$(\varphi v_k) \in C_c^m(U_1 \cap U_2).$$

Applying (1.2.9), we derive that

$$\begin{aligned} \|\varphi v_k - \varphi v\|_{W^{m,p}(U_1 \cap U_2)} = \|\varphi\,(v_k - v)\|_{W^{m,p}(U_1 \cap U_2)} \leq C \|v_k - v\|_{W^{m,p}(U_1 \cap U_2)} \leq \\ \leq C \|u_k - u\|_{W^{m,p}(U_2)} \to 0 \end{aligned}$$

and the desired result follows from the definition of $W_0^{m,p}$-spaces. □

Lemma 2.3.5. *Let U be arbitrary. For every $a=(a)_{i,j=1}^{n}\in L^{\infty}(U)$ satisfying (4) and (5), an equivalent norm for the normed space $\left(H^1_{\mathbb{R}}(U),\|\cdot\|_{H^1(U)}\right)$ is*

$$\left(\|\cdot\|^2_{L^2(U)}+\mathcal{L}[\cdot,\cdot]\right)^{\frac{1}{2}},$$

where $H^1_{\mathbb{R}}(U)$ is defined in an analogous manner as $H^1_{0_{\mathbb{R}}}(U)$ in the proof of Proposition 1.2.1.

Proof. We argue as in the proof of Proposition 1.2.1 to validate that

$$\mathrm{Re}((*,\star))+\mathcal{L}[*,\star]$$

is indeed an inner product for $H^1_{\mathbb{R}}(U)$. Hence,

$$\left(\|\cdot\|^2_{L^2(U)}+\mathcal{L}[\cdot,\cdot]\right)^{\frac{1}{2}}$$

is a norm for the aforementioned space. The equivalence follows from the fact that

$$\theta\|\nabla_w u\|^2_{L^2(U)}\le\mathcal{L}[u,u]\le\|a\|_{L^{\infty}(U)}\|\nabla_w u\|^2_{L^2(U)},\ \ \forall u\in H^1(U).$$

□

Theorem 2.3.1. *If U is bounded and $u_0\in H^1_0(U)$, then for every bounded J_0 there exists a solution of (2.2.6), such that*

$$\|\mathbf{u}\|_{L^{\infty}(J_0;H^1(U))}+\|\mathbf{u}'\|_{L^{\infty}(J_0;H^{-1}(U))}\le\mathcal{K}\Big(\|u_0\|_{H^1(U)},\|\zeta\|_{H^1(U)},\|\zeta\|_{L^{\alpha+2}(U)},\ ,\frac{1}{\theta},\|a\|_{L^{\infty}(U)},|U|\Big) \tag{2.3.2}$$

and also

$$E_{\lambda}(\mathbf{u})\le E_{\lambda}(u_0)\ \text{ everywhere in } \overline{J_0}. \tag{2.3.3}$$

Moreover, if u_0 and ζ are real-valued, then the above solution satisfies $\mathbf{u}(t)=\overline{\mathbf{u}}(-t)$, for every $t\in\overline{J_0}$ with $|t|\le\mathrm{dist}\,(0,\partial J_0)$.

Proof. Here, we use the notation

$$\widetilde{\mathcal{K}}:=\mathcal{K}\Big(\|u_0\|_{H^1(U)},\|\zeta\|_{H^1(U)},\|\zeta\|_{L^{\alpha+2}(U)},\frac{1}{\theta},\|a\|_{L^{\infty}(U)},|U|\Big).$$

Now, based on

1. the fact that $U\in\{U_{\mathrm{P}}\}$ (see Definition 1.2.3) and also $H^1_0(U;\mathbb{R})\hookrightarrow\hookrightarrow L^2(U;\mathbb{R})$ (see Proposition 1.2.6),
2. the Fredholm theory and
3. the fact that the field $\mathbb{C}$ can be regarded as a vector space over the field $\mathbb{R}$,

we deduce that the complete set of eigenfunctions for the operator $\mathcal{L}_w$ restricted to $H^1_0(U;\mathbb{R})$, is an orthogonal basis of both $H^1_0(U;\mathbb{C})$ and $L^2(U;\mathbb{C})$. Let $\{w_k\}_{k=1}^{\infty}\subset H^1_0(U;\mathbb{R})$ be the aforementioned basis, appropriately normalized so that $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(U;\mathbb{C})$. We then employ the standard Faedo-Galerkin method.

Step 1α

For every $m\in\mathbb{N}$ we define $d_m\in C^{\infty}(J_{0_m};\mathbb{C}^m)$, with $d_m(t):=\left(d^k_m(t)\right)_{k=1}^{m}$, to be the unique maximal solution of the initial-value problem

$$\begin{cases} {d_m}'(t)=F_m(d_m(t)),\ \ \forall t\in J^*_{0_m}\\ d_m(0)=((w_k,u_0))_{k=1}^{m}\ \ (=(\langle u_0,w_k\rangle)_{k=1}^{m},\ \text{in view of Lemma 2.3.1}),\end{cases}$$

where $F_m \in C^\infty\big(\mathbb{R}^{2m};\mathbb{C}^m\big)$ with

$$F_m^k(z) \coloneqq i\mathcal{N}_\lambda[\sum_{l=1}^m z_l w_l, w_k], \text{ for every } z \coloneqq (z_l)_{l=1}^m \in \mathbb{C}^m, \text{ for every } k=1,\dots,m.$$

We note that the smoothness of F_m follows from directly applying (N times, for arbitrary $N\in\mathbb{N}$) the common Leibniz integral rule. Now, we define $\mathbf{u}_m \in C^\infty\big(J_{0_m}; H_0^1(U;\mathbb{C})\big)$, with

$$\mathbf{u}_m(t) \coloneqq \sum_{k=1}^m \overline{d_m^k}(t) w_k.$$

In view of Lemma 2.3.1, it is direct to verify that

$$\langle i\mathbf{u}_m{}', w_k\rangle + \mathcal{N}_\lambda[\mathbf{u}_m, w_k] = 0 \text{ everywhere in } J_{0_m}, \text{ for every } k=1,\dots,m. \tag{2.3.4}$$

Step 1β

By its definition and the Bessel-Parseval identity, we have

$$\mathbf{u}_m(0) \to u_0 \text{ in } L^2(U) \text{ and } \|\mathbf{u}_m(0)\|_{L^2(U)} \le \|u_0\|_{L^2(U)}. \tag{2.3.5}$$

Furthermore, we can argue as in Step 3. of the proof of Theorem 2, Section 6.5 in [15] to deduce

$$\mathbf{u}_m(0) \to u_0 \text{ in } \Big(H^1_{0_\mathbb{R}}(U), (\mathcal{L}[\cdot,\cdot])^{\frac{1}{2}}\Big) \text{ and } (\mathcal{L}[\mathbf{u}_m(0),\mathbf{u}_m(0)])^{\frac{1}{2}} \le (\mathcal{L}[u_0,u_0])^{\frac{1}{2}}. \tag{2.3.6}$$

For the definition of the normed space $\Big(H^1_{0_\mathbb{R}}(U), (\mathcal{L}[\cdot,\cdot])^{\frac{1}{2}}\Big)$ see the proof of Proposition 1.2.1. Now, these facts has two immediate consequences: First, the bound in (2.3.6) implies

$$\theta^{\frac{1}{2}}\|\nabla_w(\mathbf{u}_m(0))\|_{L^2(U)} \le (\mathcal{L}[\mathbf{u}_m(0),\mathbf{u}_m(0)])^{\frac{1}{2}} \le (\mathcal{L}[u_0,u_0])^{\frac{1}{2}} \le \|a\|^{\frac{1}{2}}_{L^\infty(U)}\|\nabla_w u_0\|_{L^2(U)},$$

that is

$$\|\nabla_w(\mathbf{u}_m(0))\|_{L^2(U)} \le \mathcal{K}\Big(\frac{1}{\theta}, \|a\|_{L^\infty(U)}\Big)\|\nabla_w u_0\|_{L^2(U)}.$$

This bound and the respective one in (2.3.5) give us

$$\|\mathbf{u}_m(0)\|_{H^1(U)} \le \mathcal{K}\Big(\frac{1}{\theta}, \|a\|_{L^\infty(U)}\Big)\|u_0\|_{H^1(U)},$$

hence, in view of (2.2.11), we derive

$$E_\lambda(\mathbf{u}_m(0)) \le \widetilde{\mathcal{K}}. \tag{2.3.7}$$

Second, the convergence in (2.3.6) states that

$$\mathcal{L}[\mathbf{u}_m(0)-u_0, \mathbf{u}_m(0)-u_0] \to 0,$$

hence, (4) implies that

$$\nabla_w(\mathbf{u}_m(0)) \to \nabla_w u_0 \text{ in } L^2(U), \text{ and so } \mathbf{u}_m(0) \to u_0 \text{ in } H_0^1(U). \tag{2.3.8}$$

We now expand

$$\mathcal{L}[\mathbf{u}_m(0)+\zeta, \mathbf{u}_m(0)+\zeta] = \mathcal{L}[\mathbf{u}_m(0),\mathbf{u}_m(0)] + 2\mathcal{L}[\mathbf{u}_m(0),\zeta] + \mathcal{L}[\zeta,\zeta].$$

The first term converges to $\mathcal{L}[u_0,u_0]$ directly from the convergence in (2.3.6), while the second term goes to $2\mathcal{L}[u_0,\zeta]$ from the Hölder inequality ($p_1=p_2=2$) along with the convergence in (2.3.8). Hence

$$\mathcal{L}[\mathbf{u}_m(0)+\zeta, \mathbf{u}_m(0)+\zeta] \to \mathcal{L}[u_0+\zeta, u_0+\zeta].$$

Moreover, from (2.2.10) and the convergence in (2.3.5), we get

$$G_\lambda(\mathbf{u}_m(0)) \to G_\lambda(u_0).$$

Combining the last two convergences, we conclude to

$$E_\lambda(\mathbf{u}_m(0)) \to E_\lambda(u_0). \tag{2.3.9}$$

Step 2α

We multiply the variational equation in (2.3.4) by ${d_m^k}'(t)$, sum for $k=1,\dots,m$, and consider the real parts of both sides, to obtain

$$\frac{d}{dt}E_\lambda(\mathbf{u}_m)=0,\ \text{that is } E_\lambda(\mathbf{u}_m)=E_\lambda(\mathbf{u}_m(0))\ \text{everywhere in } J_{0_m}, \tag{2.3.10}$$

hence, from (2.3.7) we get

$$E_\lambda(\mathbf{u}_m)\le\widetilde{K}\ \text{everywhere in } J_{0_m},\ \text{uniformly for every } m\in\mathbb{N},$$

and so, from (2.3.1) we deduce

$$\mathcal{L}[\mathbf{u}_m+\zeta,\mathbf{u}_m+\zeta]\le\widetilde{K},$$

which implies

$$\|\nabla_w\mathbf{u}_m\|_{L^2(U)}\le\widetilde{K}.$$

Thus $J_{0_m}\equiv\mathbb{R}$ for every m, as well as

$$\|\mathbf{u}_m\|_{H^1(U)}\le\widetilde{K}\ \text{everywhere in } \mathbb{R},\ \text{uniformly for every } m\in\mathbb{N},$$

since for the Poincaré constant C_U we have $C_U=\mathcal{K}(|U|)$, therefore

$$\|\mathbf{u}_m\|_{L^\infty(\mathbb{R};H^1(U))}\le\widetilde{\mathcal{K}},\ \ \forall m\in\mathbb{N}. \tag{2.3.11}$$

Step 2β

We fix an arbitrary $v\in H_0^1(U)$ with $\|v\|_{H^1(U)}\le1$ and write $v=\mathcal{P}v\oplus(\mathcal{I}-\mathcal{P})v$, where $\mathcal{P}$ is the projection in $\operatorname{span}\{w_k\}_{k=1}^m$. Since $\mathbf{u}_m'\in\operatorname{span}\{w_k\}_{k=1}^m$ and $\mathcal{N}_\lambda[h,g]$ is linear for g, from the variational equation in (2.3.4) we get that

$$\langle i{\mathbf{u}_m}',v\rangle=-\mathcal{N}_\lambda[\mathbf{u}_m,\mathcal{P}v].$$

Applying (2.2.7) and (2.3.11) we derive

$$|\langle i{\mathbf{u}_m}',v\rangle|\le\widetilde{\mathcal{K}},$$

hence

$$\|{\mathbf{u}_m}'\|_{L^\infty(\mathbb{R};H^{-1}(U))}=\|i\mathbf{u}_m'\|_{L^\infty(\mathbb{R};H^{-1}(U))}\le\widetilde{\mathcal{K}},\ \ \forall m\in\mathbb{N}. \tag{2.3.12}$$

Step 3α

We fix an arbitrary bounded J_0. From (2.3.11), (2.3.12), Point i) of Theorem 1.3.14 in [10][3] and Point 1. of Lemma 2.3.2, there exist a subsequence $\{\mathbf{u}_{m_l}\}_{l=1}^\infty\subseteq\{\mathbf{u}_m\}_{m=1}^\infty$ and a function

$$\mathbf{u}=\mathbf{u}_{J_0}\in L^\infty\big(J_0;H_0^1(U)\big)\cap W^{1,\infty}\big(J_0;H^{-1}(U)\big),$$

such that

$$\mathbf{u}_{m_l}\rightharpoonup\mathbf{u}\ \text{in } H_0^1(U)\ \text{everywhere in } \overline{J_0}\ \text{and also } \|\mathbf{u}\|_{L^\infty(J_0;H^1(U))}\le\widetilde{\mathcal{K}}. \tag{2.3.13}$$

Step 3β

$H^{-1}(U)$ is separable since $H_0^1(U)$ is separable, hence by the Dunford-Pettis theorem (see, e.g., Theorem 1, Section 3, Chaprter III in [13]) we have $L^\infty\big(J_0;H^{-1}(U)\big)\cong\big(L^1\big(J_0;H_0^1(U)\big)\big)^*$ (see, e.g., Theorem 1, Section 1, Chapter IV in [13]). In virtue of the above, from (2.3.12), the Banach-Alaoglu-Bourbaki theorem (see, e.g., Theorem 3.16 in [8]) and Point 2. of Lemma 2.3.2, there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^\infty$, which we still denote as such and a function

$$\mathbf{h}\in L^\infty\big(J_0;H^{-1}(U)\big),$$

such that

$${\mathbf{u}_{m_l}}'\overset{*}{\rightharpoonup}\mathbf{h}\ \text{in } L^\infty\big(J_0;H^{-1}(U)\big)\ \text{and also } \|\mathbf{h}\|_{L^\infty(J_0;H^{-1}(U))}\le\widetilde{\mathcal{K}}. \tag{2.3.14}$$

Let $\psi\in C_c^\infty(J_0^\circ)$ and $v\in H_0^1(U)$ be arbitrary. From

[3] We note that in [10], the normed space $\big(H^1_{0_\mathbb{R}}(U),\|\cdot\|_{H^1(U)}\big)$ is considered instead of $H_0^1(U)$. However, it becomes clear from its proof that the aforementioned result is also valid in our case.

1. the linearity of the functional,
2. the convergence in (2.3.14),
3. Lemma 1.1, Chapter III in [44],
4. the definition of the weak derivative,
5. Lemma 2.3.1,
6. the dominated convergence theorem and
7. the convergence in (2.3.13),

we obtain

$$\begin{aligned}\int_{J_0} \langle \mathbf{h}, v\rangle \psi dt &\overset{1.}{=} \int_{J_0} \langle \mathbf{h}, \psi v\rangle dt \overset{2.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{m_l}{}', \psi v\rangle dt \overset{1.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{m_l}{}', v\rangle \psi dt \overset{3.}{=} \\ &\overset{3.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{m_l}, v\rangle' \psi dt \overset{4.}{=} -\lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{m_l}, v\rangle \psi' dt \overset{5.}{=} -\lim_{l\nearrow\infty} \int_{J_0} \overline{(\mathbf{u}_{m_l}, v)} \psi' dt \overset{6.}{=} \\ &\overset{6.}{=} -\int_{J_0} \lim_{l\nearrow\infty} \overline{(\mathbf{u}_{m_l}, v)} \psi' dt \overset{7.}{=} -\int_{J_0} \overline{(\mathbf{u}, v)} \psi' dt \overset{5.}{=} -\int_{J_0} \langle \mathbf{u}, v\rangle \psi' dt \overset{4.}{=} \\ &\overset{4.}{=} \int_{J_0} \langle \mathbf{u}, v\rangle' \psi dt \overset{3.}{=} \int_{J_0} \langle \mathbf{u}', v\rangle \psi dt,\end{aligned}$$

hence $\mathbf{h} \equiv \mathbf{u}'$, since ψ and v are arbitrary.

Step 4

Since $H_0^1(U) \hookrightarrow\hookrightarrow L^2(U) \hookrightarrow H^{-1}(U)$, from (2.3.11), (2.3.12) and the Aubin-Lions-Simon lemma (see, e.g., Theorem II.5.16 in [7], or else Theorem 8.62 along with Exercise 8.63 in [35]), there exist a subsequence of $\{\mathbf{u}_{m_l}\}_{l=1}^{\infty}$, which we still denote as such, and a function $\mathbf{y} \in C(\overline{J_0}; L^2(U))$, such that

$$\mathbf{u}_{m_l} \to \mathbf{y} \text{ in } C(\overline{J_0}; L^2(U)). \tag{2.3.15}$$

From the convergence in (2.3.13), we deduce that $\mathbf{y} \equiv \mathbf{u}$. This fact has two direct consequences: First, $\mathbf{u}$ satisfies the initial condition, i.e.

$$\mathbf{u}(0) \equiv u_0,$$

which follows from (2.3.15) for $t = 0$ combined with $\mathbf{u}_m(0) \to u_0$ in $L^2(U)$ from Step 1β. Second, from (2.2.5) and (2.2.10), as well as (2.3.11), the bound in (2.3.13) and (2.3.15), we get

$$g_\lambda(\mathbf{u}_{m_l}) \to g_\lambda(\mathbf{u}) \text{ in } C(\overline{J_0}; Y_\alpha(U)). \tag{2.3.16}$$

and also

$$G_\lambda(\mathbf{u}_{m_l}) \to G_\lambda(\mathbf{u}) \text{ uniformly in } \overline{J_0}. \tag{2.3.17}$$

Step 5

We now show that $\mathbf{u}$ satisfies the variational equation in (2.2.6). Let now $\psi \in C_c^\infty(J_0^\circ)$ and fix $N \in \mathbb{N}$. We choose m_l such that $N \le m_l$ and $v \in \operatorname{span}\{w_k\}_{k=1}^N$, hence, by the linearity of the inner product, we get from (2.3.4) that

$$\int_{J_0} \langle i\mathbf{u}_{m_l}{}', \psi v\rangle + \langle \mathcal{L}_w(\mathbf{u}_{m_l} + \zeta), \psi v\rangle + \langle g_\lambda(\mathbf{u}_{m_l}), \psi v\rangle dt = 0.$$

From the convergence in (2.3.14) we get

$$\int_{J_0} \langle i\mathbf{u}_{m_l}{}', \psi v\rangle dt \to \int_{J_0} \langle i\mathbf{u}', \psi v\rangle dt.$$

From the convergence in (2.3.13) we have

$$\langle \mathcal{L}_w v, \mathbf{u}_{m_l}\rangle \to \langle \mathcal{L}_w v, \mathbf{u}\rangle, \text{ everywhere in } \overline{J_0},$$

since the functional $\langle \mathcal{L}_w v, \cdot \rangle : H_0^1(U) \to \mathbb{C}$ is linear and bounded, or else

$$\overline{\langle \mathcal{L}_w v, \mathbf{u}_{m_l} \rangle} \to \overline{\langle \mathcal{L}_w v, \mathbf{u} \rangle} \text{ everywhere in } \overline{J_0}.$$

From (5) (we deal as in Point 1. of Step 1 of the proof of Proposition 1.2.1), we then obtain

$$\langle \mathcal{L}_w \mathbf{u}_{m_l}, v \rangle = \overline{\langle \mathcal{L}_w v, \mathbf{u}_{m_l} \rangle} \to \overline{\langle \mathcal{L}_w v, \mathbf{u} \rangle} = \langle \mathcal{L}_w \mathbf{u}, v \rangle \text{ everywhere in } \overline{J_0},$$

and so

$$\langle \mathcal{L}_w (\mathbf{u}_{m_l} + \zeta), \psi v \rangle \to \langle \mathcal{L}_w (\mathbf{u} + \zeta), \psi v \rangle \text{ everywhere in } \overline{J_0}.$$

Hence, from the dominated convergence theorem, we get

$$\int_{J_0} \langle \mathcal{L}_w (\mathbf{u}_{m_l} + \zeta), \psi v \rangle dt \to \int_{J_0} \langle \mathcal{L}_w (\mathbf{u} + \zeta), \psi v \rangle dt.$$

From the Hólder inequality ($p_1 = p_2 = 2$) and (2.3.16), we also deduce

$$\int_{J_0} \langle g_\lambda (\mathbf{u}_{m_l}), \psi v \rangle dt \to \int_{J_0} \langle g_\lambda (\mathbf{u}), \psi v \rangle dt.$$

Since ψ is arbitrary, $\mathbf{u}$ satisfies the variational equation for every $v \in \operatorname{span}\{w_k\}_{k=1}^N$. We then get the desired result from the density argument, since N is arbitrary.

Step 6

As far as (2.3.3) is concerned, let $\epsilon > 0$ be arbitrary. From (2.3.9) and the equation in (2.3.10), we deduce that there exists $m_0 = m_0(\epsilon)$, such that

$$E_\lambda (\mathbf{u}_m) \le E_\lambda (u_0) + \epsilon \text{ everywhere in } \mathbb{R}, \text{ for every } m \ge m_0. \tag{2.3.18}$$

Now, it can be verified that the convergence in (2.3.13) implies that

$$\mathbf{u}_{m_l} \rightharpoonup \mathbf{u} \text{ in } H^1_{0_\mathbb{R}}(U) \text{ everywhere in } \overline{J_0}.$$

Indeed, let $f \in \left(H^1_{0_\mathbb{R}}(U) \right)^*$ be arbitrary. In view of the bijective isometry between the complex dual and the real dual (see also the proof of Proposition 1.2.1), there exists $h \in H^{-1}(U)$ such that $f = \operatorname{Re}(h)$. From (2.3.13) we deduce that

$$\langle h, \mathbf{u}_{m_l} \rangle \to \langle h, \mathbf{u} \rangle \text{ everywhere in } \overline{J_0},$$

thus

$$\operatorname{Re}(\langle h, \mathbf{u}_{m_l} \rangle) \to \operatorname{Re}(\langle h, \mathbf{u} \rangle) \text{ everywhere in } \overline{J_0},$$

that is

$$\langle f, \mathbf{u}_{m_l} \rangle \to \langle f, \mathbf{u} \rangle \text{ everywhere in } \overline{J_0}$$

and the desired result follows by the arbitrariness of f. This fact has two direct consequences: First, we get

$$\mathcal{L}[\mathbf{u}_{m_l}, \zeta] \to \mathcal{L}[\mathbf{u}, \zeta] \text{ everywhere in } \overline{J_0}, \tag{2.3.19}$$

since $\mathcal{L}[\cdot, \zeta] \in \left(H^1_{0_\mathbb{R}}(U) \right)^*$. Second, by the (sequentially) weak lower semi-continuity of the $(\mathcal{L}[\cdot,\cdot])^{\frac{1}{2}}$-norm, we obtain

$$(\mathcal{L}[\mathbf{u}, \mathbf{u}])^{\frac{1}{2}} \le \liminf_{l \nearrow \infty} (\mathcal{L}[\mathbf{u}_{m_l}, \mathbf{u}_{m_l}])^{\frac{1}{2}} \text{ everywhere in } \overline{J_0}. \tag{2.3.20}$$

Combining (2.3.17), (2.3.19) and (2.3.20) we deduce

$$E_\lambda (\mathbf{u}) \le \liminf_{l \nearrow \infty} E_\lambda (\mathbf{u}_{m_l}) \text{ everywhere in } \overline{J_0}. \tag{2.3.21}$$

From (2.3.21) and (2.3.18), we have

$$E_\lambda (\mathbf{u}) \le E_\lambda (u_0) + \epsilon \text{ everywhere in } \overline{J_0}$$

and we get the desired result from the arbitrariness of ϵ.

Step 7

Finally,

1. if ζ is real-valued, then $\overline{F_m}(z)=F_m(\bar{z})$, for every $z\in\mathbb{C}^m$ and
2. if u_0 is real-valued, then $d_m(0)\in\mathbb{R}^m$.

Hence, under these two assumptions, we apply Lemma 2.3.3 to get that $d_m(t)=\overline{d_m}(-t)$ and so $\overline{\mathbf{u}_m}(t)=\mathbf{u}_m(-t)$, for every $t\in\mathbb{R}$ and every $m\in\mathbb{N}$, which of course is equivalent to $\mathbf{u}_m(t)=\overline{\mathbf{u}_m}(-t)$, for every $t\in\mathbb{R}$ and every $m\in\mathbb{N}$. Now, the (conjugate) symmetry $\mathbf{u}(t)=\bar{\mathbf{u}}(-t)$, for every $t\in\overline{J_0}$ with $|t|\leq \operatorname{dist}(0,\partial J_0)$, follows straight from the aforementioned symmetry along with the convergence in (2.3.13) or (2.3.15).

□

Theorem 2.3.2. *If U is bounded and $u_0\in H_0^1(U)$, then for every bounded J_0 there exists a solution of (2.2.23), such that*

$$\|\mathbf{u}\|_{L^\infty(J_0;H^1(U))}+\|\mathbf{u}'\|_{L^\infty(J_0;H^{-1}(U))}\leq \\ \leq\mathcal{K}\Big(\|u_0\|_{H^1(U)},\|\zeta\|_{X^1(U)},\||\zeta|-\rho\|_{L^2(U)},\frac{1}{\theta},\|a\|_{L^\infty(U)},|J_0|\Big) \tag{2.3.22}$$

and also

$$\widetilde{E}_\lambda(\mathbf{u})\leq\widetilde{E}_\lambda(u_0)\ \textit{everywhere in}\ \overline{J_0}. \tag{2.3.23}$$

Moreover, if u_0 and ζ are real-valued, then the above solution satisfies $\mathbf{u}(t)=\bar{\mathbf{u}}(-t)$, for every $t\in\overline{J_0}$ with $|t|\leq\operatorname{dist}(0,\partial J_0)$.

Proof. Here, we use the notation

$$\widetilde{\mathcal{K}}:=\mathcal{K}\Big(\|u_0\|_{H^1(U)},\|\zeta\|_{X^1(U)},\||\zeta|-\rho\|_{L^2(U)},\frac{1}{\theta},\|a\|_{L^\infty(U)}\Big)$$

and

$$\widetilde{\mathcal{K}}_{J_0}:=\mathcal{K}\Big(\|u_0\|_{H^1(U)},\|\zeta\|_{X^1(U)},\||\zeta|-\rho\|_{L^2(U)},\frac{1}{\theta},\|a\|_{L^\infty(U)},|J_0|\Big).$$

Based on the proof of Theorem 2.3.1, it is essential to modify only its Steps 1β and 2.

Modified Step 1β

Employing (2.2.28) instead of (2.2.11), (2.3.7) gets the form

$$\widetilde{E}_\lambda(\mathbf{u}_m(0))\leq\widetilde{\mathcal{K}},\ \forall m\in\mathbb{N}. \tag{2.3.24}$$

Modified Step 2α

Instead of (2.3.7) we employ (2.3.24) to deduce

$$\widetilde{E}_\lambda(\mathbf{u}_m)\leq\widetilde{\mathcal{K}}\ \text{everywhere in}\ J_{0_m},\ \text{for every}\ m\in\mathbb{N} \tag{2.3.25}$$

and so, from (2.3.1) we deduce $\mathcal{L}[\mathbf{u}_m+\zeta,\mathbf{u}_m+\zeta]\leq\widetilde{\mathcal{K}}$, which implies $\|\nabla_w\mathbf{u}_m\|_{L^2(U)}\leq\widetilde{\mathcal{K}}$. Thus $J_{0_m}\equiv\mathbb{R}$ for every m, as well as

$$\|\nabla_w\mathbf{u}_m\|_{L^\infty(\mathbb{R};L^2(U))}\leq\widetilde{\mathcal{K}},\ \forall m\in\mathbb{N}. \tag{2.3.26}$$

In order to derive a bound for the L^2-norm which is independent of $|U|$, we have to find a different route instead of applying the Poincaré inequality. We note that

$$\widetilde{G}_\lambda(\mathbf{u}_m)\leq\widetilde{\mathcal{K}},\ \forall m\in\mathbb{N}, \tag{2.3.27}$$

which follows from (2.3.25) along with the fact that $\mathcal{L}[\mathbf{u}_m+\zeta,\mathbf{u}_m+\zeta]\geq 0$. Moreover, in view of (2.2.9), we fix some $C_1>\alpha+2$ and we have that

$$x^{2(\tau+1)}\leq C_1V(x),\ \text{for every}\ x\geq C_2,\ \text{for some}\ C_2>\rho^{\frac{1}{2}}. \tag{2.3.28}$$

Setting

$$\Omega(t) := \left\{x \in U \,\middle|\, |\mathbf{u}_m(t)+\zeta| \geq \max\{C_2, 1\}\right\} \subseteq U, \ \forall t \in \mathbb{R},$$

we get, from (2.3.28) and (2.3.27) (and the fact that $\lambda > 0$ of course), that

$$\int_{\Omega(t)} |\mathbf{u}_m+\zeta|^s dx \leq \widetilde{\mathcal{K}}, \ \forall m \in \mathbb{N}, \ \forall s \in (-\infty, 2(\tau+1)]. \tag{2.3.29}$$

Then, we multiply the variational equation (2.3.4) (for $\widetilde{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_\lambda$) by $d_m^k(t)$, sum for $k = 1, \ldots, m$ and take imaginary parts of both sides, and thus, in view of Lemma 2.3.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_m\|_{L^2(U)}^2 - \mathrm{Im}\int_U \nabla_w \zeta \cdot a \nabla_w \overline{\mathbf{u}_m} dx - \lambda \mathrm{Im}\Big(\left(|\mathbf{u}_m+\zeta|^{2\tau} - \rho^{2\tau}\right)(\mathbf{u}_m+\zeta), \mathbf{u}_m\Big) = 0.$$

For the middle term we apply the Hölder inequality ($p_1 = p_2 = 2$) and use the bound (2.3.26), while for the third term we expand in view of (2.2.22), to deduce that

$$\|\mathbf{u}_m\|_{L^2(U)}^2 \leq \widetilde{\mathcal{K}}\left(|t| + \left|\int_0^t \left(\int_U |\mathbf{u}_m|^{2\tau+1} dx\right) ds\right|\right), \ \forall t \in \mathbb{R}, \ \forall m \in \mathbb{N}. \tag{2.3.30}$$

In order to estimate the spatial integral, we write

$$\begin{aligned} \int_U |\mathbf{u}_m|^{2\tau+1} dx &= \int_{\Omega(t)^c \cap U} |\mathbf{u}_m|^2 |\mathbf{u}_m|^{2\tau-1} dx + \int_{\Omega(t)} |\mathbf{u}_m|^{2\tau+1} dx \leq \\ &\leq \int_{\left\{x \in U \,\middle|\, |\mathbf{u}_m| \leq \max\{C_2,1\} + \|\zeta\|_{L^\infty(U)}\right\}} |\mathbf{u}_m|^2 |\mathbf{u}_m|^{2\tau-1} dx + \\ &\quad + C \int_{\Omega(t)} |\zeta|^{2\tau+1} + |\mathbf{u}_m+\zeta|^{2\tau+1} dx \overset{(2.3.29)}{\leq} \\ &\overset{(2.3.29)}{\leq} \mathcal{K}\Big(\|\zeta\|_{L^\infty(U)}\Big)\|\mathbf{u}_m\|_{L^2(U)}^2 + \widetilde{\mathcal{K}} \leq \widetilde{\mathcal{K}}\left(1 + \|\mathbf{u}_m\|_{L^2(U)}^2\right). \end{aligned} \tag{2.3.31}$$

Let J_0 be arbitrary. From (2.3.30) and (2.3.31), we derive that

$$\|\mathbf{u}_m\|_{L^2(U)}^2 \leq \widetilde{\mathcal{K}}_{J_0}\left(1 + \left|\int_0^t \|\mathbf{u}_m\|_{L^2(U)}^2 ds\right|\right), \ \forall t \in \overline{J_0}, \ \forall m \in \mathbb{N}$$

and so, by the Grönwall inequality,

$$\|\mathbf{u}_m\|_{L^\infty(J_0; L^2(U))} \leq \widetilde{\mathcal{K}}_{J_0}, \text{ for every } m \in \mathbb{N}. \tag{2.3.32}$$

From (2.3.26) and (2.3.32) we conclude to

$$\|\mathbf{u}_m\|_{L^\infty(J_0; H^1(U))} \leq \widetilde{\mathcal{K}}_{J_0}, \ \forall m \in \mathbb{N}. \tag{2.3.33}$$

Modified Step 2β

We make use of (2.2.24) and (2.3.33), instead of (2.2.7) and (2.3.11), respectively, to get that

$$\|\mathbf{u}_m'\|_{L^\infty(J_0; H^{-1}(U))} \leq \widetilde{\mathcal{K}}_{J_0}, \ \forall m \in \mathbb{N}. \tag{2.3.34}$$

Modified Step 3

Instead of (2.3.11) and (2.3.12) we employ (2.3.33) and (2.3.34), respectively.

Modified Step 4

Instead of (2.3.11) and (2.3.12) we employ (2.3.33) and (2.3.34), respectively. Moreover, instead of (2.2.5) and (2.2.10) we employ (2.2.13), (2.2.15), (2.2.17) and (2.2.25), (2.2.27), respectively.

□

Theorem 2.3.3. *Theorem 2.3.2 is also valid for every unbounded U.*

Proof. In Step 1 we construct an approximation sequence for the initial datum and in Step 2 we consider an approximation sequence of problems as well as their solutions. These problems are considered in an expanding sequence of bounded sets that eventually cover the whole unbounded set. In Step 3 we take the limit of the aforementioned approximation sequence of solutions, and the verification that this limit is indeed a solution of the variational equation takes places in Step 4. The crux for the latter step is the application of Proposition 1.2.4. Lastly, in Step 5 we verify the initial condition, the energy estimate and the symmetry of the solution. Now, as we do in the proof of Theorem 2.3.2, we write

$$\widetilde{\mathcal{K}}_{J_0} := \mathcal{K}\left(\|u_0\|_{H^1(U)}, \|\zeta\|_{X^1(U)}, \||\zeta|-\rho\|_{L^2(U)}, \frac{1}{\theta}, \|a\|_{L^\infty(U)}, |J_0| \right).$$

Step 1α

We fix an arbitrary $x_0 \in U$ and we set[4]

$$B_k := B(x_0, k), \ \forall k \in \mathbb{N}.$$

In view of Proposition 1.2.11, we consider a sequence $\{\phi_k\}_k \subset C_c^\infty(\mathbb{R}^n; [0,1])$, such that

1. $\operatorname{supp}(\phi_k) \subseteq \overline{B_{k+1}}$, for every k,
2. $\phi_k \equiv 1$ in $\overline{B_k}$, for every k, and
3. $\|\nabla \phi_k\|_{L^\infty(\mathbb{R}^n)} \leq C$, uniformly for every k.

We then set

$$\varphi_k := (\mathcal{R}(\mathbb{R}^n, B_{k+2}))\, \phi_k, \ \forall k.$$

Evidently, $\varphi_k \in C_c^\infty(B_{k+2}; [0,1])$ for every k, with

1. $\operatorname{supp}(\varphi_k) \subseteq \overline{B_{k+1}}$, for every k,
2. $\varphi_k \equiv 1$ in $\overline{B_k}$, for every k, and
3. $\|\nabla \varphi_k\|_{L^\infty(B_{k+2})} \leq C$, uniformly for every k.

Moreover, we set

$$U_k := B_k \cap U, \ v_k := (\mathcal{R}(B_{k+2}, U_{k+2}))\, \varphi_k, \ v_{0_k} := (\mathcal{R}(U, U_{k+2}))\, u_0 \text{ and}$$
$$u_{0_k} := v_k v_{0_k}, \text{ for every } k.$$

In view of Lemma 2.3.4, we have that

$$u_{0_k} \in H_0^1(U_{k+2}), \text{ with } \|u_{0_k}\|_{H^1(U_{k+2})} \leq C \|u_0\|_{H^1(U)}, \text{ uniformly for every } k. \tag{2.3.35}$$

Step 1β

We set

$$u_{00_k} := (\mathcal{E}_0(U_{k+2}, U))\, u_{0_k}, \ \forall k.$$

In virtue of Proposition 1.2.3 along with (2.3.35), we deduce that

$$u_{00_k} \in H_0^1(U), \text{ with } \|u_{00_k}\|_{H^1(U)} = \|u_{0_k}\|_{H^1(U_{k+2})} \leq C \|u_0\|_{H^1(U)}, \text{ uniformly for every } k. \tag{2.3.36}$$

Now, we claim that

$$u_{00_k} \to u_0 \text{ in } H^1(U). \tag{2.3.37}$$

Indeed, from

1. $(\mathcal{R}(U, U_k))\, u_{00_k} \equiv (\mathcal{R}(U, U_k))\, u_0$, for every k,

[4] For convenience, in this proof we abuse the notation U_δ for $\delta > 0$ (see Remark 1.2.6). When a natural number appears as a subscript of a set, it indicates that this set is an element of a sequence.

2. $(\mathcal{R}(U,U_k^{\mathsf{c}}\cap U_{k+1}))\,u_{00_k} \equiv ((\mathcal{R}(B_{k+2},U_k^{\mathsf{c}}\cap U_{k+1}))\,\varphi_k)\,((\mathcal{R}(U,U_k^{\mathsf{c}}\cap U_{k+1}))\,u_0)$, for every k,
3. $(\mathcal{R}(U,U_{k+1}^{\mathsf{c}}\cap U))\,u_{00_k} \equiv 0$, for every k and
4. (1.2.9),

we have that

$$\begin{aligned}\|u_{00_k}-u_0\|_{H^1(U)} &\overset{1.}{=} \|(\mathcal{R}(U,U_k^{\mathsf{c}}\cap U))\,(u_{00_k}-u_0)\|_{H^1(U_k^{\mathsf{c}}\cap U)} \overset{2.}{\underset{3.}{=}}\\ \overset{2.}{\underset{3.}{=}} \|(((\mathcal{R}(B_{k+2},U_k^{\mathsf{c}}\cap U_{k+1}))\,\varphi_k)-1)\,((\mathcal{R}(U,U_k^{\mathsf{c}}\cap U_{k+1}))\,u_0)\|_{H^1(U_k^{\mathsf{c}}\cap U_{k+1})}+\\ +\|(\mathcal{R}(U,U_{k+1}^{\mathsf{c}}\cap U))\,u_0\|_{H^1(U_{k+1}^{\mathsf{c}}\cap U)} \overset{4.}{\leq}\\ \overset{4.}{\leq} C\|(\mathcal{R}(U,U_k^{\mathsf{c}}\cap U_{k+1}))\,u_0\|_{H^1(U_k^{\mathsf{c}}\cap U_{k+1})} + \|(\mathcal{R}(U,U_{k+1}^{\mathsf{c}}\cap U))\,u_0\|_{H^1(U_{k+1}^{\mathsf{c}}\cap U)} \leq\\ \leq C\|(\mathcal{R}(U,U_{k+1}^{\mathsf{c}}\cap U))\,u_0\|_{H^1(U_{k+1}^{\mathsf{c}}\cap U)} \to 0.\end{aligned}$$

Step 1γ

We show that

$$\widetilde{E}_\lambda(u_{00_k}) \to \widetilde{E}_\lambda(u_0). \tag{2.3.38}$$

Indeed, from (2.2.25) and (2.2.27), along with (2.3.37), we obtain

$$\widetilde{G}_\lambda(u_{00_k}) \to \widetilde{G}_\lambda(u_0).$$

Additionally, we expand as

$$\mathcal{L}[u_{00_k}+\zeta,u_{00_k}+\zeta]=\mathcal{L}[u_{00_k},u_{00_k}]+2\mathcal{L}[u_{00_k},\zeta]+\mathcal{L}[\zeta,\zeta].$$

The second term converges to $2\mathcal{L}[u_0,\zeta]$, since

$$\begin{aligned}\mathcal{L}[u_{00_k},\zeta]-\mathcal{L}[u_0,\zeta]=\mathcal{L}[u_{00_k}-u_0,\zeta]\leq\\ \leq\|a\|_{L^\infty(U)}\|\nabla_w(u_{00_k}-u_0)\|_{L^2(U)}\|\nabla_w\zeta\|_{L^2(U)}\to 0,\end{aligned}$$

from the Hölder inequality ($p_1=p_2=2$) and (2.3.37). As for the first term, we have that it converges to $\mathcal{L}[u_0,u_0]$, since

$$\begin{aligned}\mathcal{L}[u_{00_k},u_{00_k}]-\mathcal{L}[u_0,u_0]=\mathcal{L}[u_{00_k}-u_0,u_{00_k}]+\mathcal{L}[u_0,u_{00_k}-u_0]\leq\\ \leq\mathcal{K}\Big(\|a\|_{L^\infty(U)},\|u_0\|_{H^1(U)}\Big)\|\nabla_w(u_{00_k}-u_0)\|_{L^2(U)}\to 0,\end{aligned}$$

by dealing as before and also employing the bound in (2.3.36). Hence

$$\mathcal{L}[u_{00_k}+\zeta,u_{00_k}+\zeta]\to\mathcal{L}[u_0+\zeta,u_0+\zeta]$$

and (2.3.38) follows.

Step 2α

Let J_0 be arbitrary and bounded. For every k, we consider (2.2.23) in U_{k+2} instead of U, where we take u_{0_k} as our initial datum instead of u_0, and we set

$$\mathbf{u}_k\in L^\infty\big(J_0;H_0^1(U_{k+2})\big)\cap W^{1,\infty}\big(J_0;H^{-1}(U_{k+2})\big)$$

to be a solution that Theorem 2.3.2 provides. For every $\mathbf{u}_k$ we have that

$$\begin{aligned}\|\mathbf{u}_k\|_{L^\infty(J_0;H^1(U_{k+2}))}+\|\mathbf{u}_k{}'\|_{L^\infty(J_0;H^{-1}(U_{k+2}))}\leq\\ \leq\mathcal{K}\Big(\|u_{0_k}\|_{H^1(U_{k+2})},\|\zeta_k\|_{X^1(U_{k+2})},\|(|\zeta|-\rho)_k\|_{L^2(U_{k+2})},\frac{1}{\theta},\|a_k\|_{L^\infty(U_{k+2})},|J_0|\Big),\end{aligned}\tag{2.3.39}$$

where

$$\begin{aligned}\zeta_k\coloneqq(\mathcal{R}(U,U_{k+2}))\,\zeta,\ \ (|\zeta|-\rho)_k\coloneqq(\mathcal{R}(U,U_{k+2}))\,(|\zeta|-\rho)\text{ and}\\ a_k\coloneqq(\mathcal{R}(U,U_{k+2}))\,a,\ \text{ for every }k\end{aligned}$$

and

$$\widetilde{E}_\lambda(\mathbf{u}_k)\leq\widetilde{E}_\lambda(u_{0_k})\leq\widetilde{E}_\lambda(u_{00_k}) \text{ everywhere in } \overline{J_0}, \tag{2.3.40}$$

as well as $\mathbf{u}_k(t)=\overline{\mathbf{u}_k}(-t)$, for every $t\in\overline{J_0}$ with $|t|\leq \operatorname{dist}(0,\partial J_0)$, if u_0 (hence if u_{0_k}) and ζ (hence ζ_k) are real-valued. From the bound in (2.3.35), along with the increasing property of $\mathcal{K}$ and the fact that the bound in (2.3.39) is independent of U, (2.3.39) gets the form

$$\|\mathbf{u}_k\|_{L^\infty(J_0;H^1(U_{k+2}))}+\|\mathbf{u}_k{}'\|_{L^\infty(J_0;H^{-1}(U_{k+2}))}\leq\widetilde{\mathcal{K}}_{J_0}, \text{ uniformly for every } k. \tag{2.3.41}$$

Step 2β

In view of Lemma 2.3.4 and (2.3.41) we have that

$$(v_k\mathbf{u}_k)\in H_0^1(U_{k+2}), \text{ with } \|v_k\mathbf{u}_k\|_{H^1(U_{k+2})}\leq C\|\mathbf{u}_k\|_{H^1(U_{k+2})}\leq\widetilde{\mathcal{K}}_{J_0}, \text{ uniformly for every } k,$$

where v_k is as in Step 1α. Hence, in view of Proposition 1.2.3, we define

$$\begin{gathered}\mathbf{v}_k\in L^\infty\big(J_0;H_0^1(U)\big) \text{ as } \mathbf{v}_k\coloneqq(\mathcal{E}_0(U_{k+2},U))\,(v_k\mathbf{u}_k), \text{ for every } k,\\ \text{with } \|\mathbf{v}_k\|_{L^\infty(J_0;H^1(U))}\leq\widetilde{\mathcal{K}}_{J_0}, \text{ uniformly for every } k.\end{gathered} \tag{2.3.42}$$

Moreover, in view of Lemma 2.3.4 we have that

$$\begin{gathered}(v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right))\in H_0^1(U_{k+2}), \text{ with}\\ \|v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right)\|_{H^1(U_{k+2})}\leq C\|v\|_{H^1(U)}, \text{ for every } v\in H_0^1(U),\\ \text{uniformly for every } k.\end{gathered}$$

Hence, employing (2.3.41), for every k we define

$$\begin{gathered}\mathbf{f}_k\in L^\infty\big(J_0;H^{-1}(U)\big) \text{ by } \langle\mathbf{f}_k,v\rangle\coloneqq\langle\mathbf{u}_k{}',v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right)\rangle,\\ \text{for every } v\in H_0^1(U), \text{ for every } k, \text{ with } \|\mathbf{f}_k\|_{L^\infty(H^{-1}(U))}\leq\widetilde{\mathcal{K}}_{J_0},\\ \text{uniformly for every } k.\end{gathered} \tag{2.3.43}$$

We now claim that

$$\mathbf{v}_k\in L^\infty\big(J_0;H_0^1(U)\big)\cap L^\infty\big(J_0;H^{-1}(U)\big), \text{ with } \mathbf{v}_k{}'\equiv\mathbf{f}_k, \text{ for every } k. \tag{2.3.44}$$

Indeed, let $v\in H_0^1(U)$ be arbitrary. Employing

1. Lemma 1.1, Chapter III in [44],
2. Lemma 2.3.1 and
3. the fact that v_k is real-valued for every k,

we derive

$$\begin{gathered}\langle\mathbf{f}_k,v\rangle=\langle\mathbf{u}_k{}',v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right)\rangle\overset{1.}{=}\langle\mathbf{u}_k,v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right)\rangle'\overset{2.}{=}\\ \overset{2.}{=}\overline{\left(\mathbf{u}_k,v_k\left((\mathcal{R}(U,U_{k+2}))\,v\right)\right)'}\overset{3.}{=}\overline{\left(\int_{U_{k+2}}\mathbf{u}_k\,v_k\left((\mathcal{R}(U,U_{k+2}))\,\bar{v}\right)dx\right)'}=\\ =\overline{\left(\int_U\left((\mathcal{E}_0(U_{k+2},U))\,(v_k\mathbf{u}_k)\right)\bar{v}dx\right)'}=\overline{\left((\mathcal{E}_0(U_{k+2},U))\,(v_k\mathbf{u}_k),v\right)'}\overset{2.}{=}\\ \overset{2.}{=}\langle(\mathcal{E}_0(U_{k+2},U))\,(v_k\mathbf{u}_k),v\rangle'.\end{gathered}$$

Therefore, from the arbitrariness of v along with Lemma 1.1, Chapter III in [44], we get (2.3.44).

Step 2γ

For every open and bounded $V\subset U$, there exists $k_V\in\mathbb{N}$, such that $V\subseteq U_{k+2}$ for every $k\geq k_V$. Now, for every fixed such V, we define[5]

$$\begin{gathered}\mathbf{v}_{V,k}\coloneqq((\mathcal{R}(U_{k+2},V))\,\mathbf{u}_k)\in L^\infty\big(J_0;H^1(V)\big), \text{ for every } k\geq k_V,\\ \text{with } \|\mathbf{v}_{V,k}\|_{L^\infty(J_0;H^1(V))}\leq\widetilde{\mathcal{K}}_{J_0}, \text{ uniformly for every such } k.\end{gathered} \tag{2.3.45}$$

[5] We highlight that we do not claim that $\mathbf{v}_{V,k}\in L^\infty\big(J_0;H_0^1(V)\big)$ for every $k\geq k_V$.

where the bound above follows directly from the bound in (2.3.41). Moreover, in view of Definition 1.2.5 and the bound in (2.3.41), we claim that

$$\begin{aligned}&\mathbf{v}_{V,k} \in L^\infty\big(J_0; H^1(V)\big) \cap L^\infty\big(J_0; H^{-1}(V)\big), \text{ with } \mathbf{v}_{V,k}{}' \equiv (\mathcal{R}(U_{k+2}, V))\,(\mathbf{u}_k{}'),\\ &\text{for every } k \geq k_V, \text{ thus } \|\mathbf{v}_{V,k}{}'\|_{L^\infty(J_0; H^{-1}(V))}, \text{ uniformly for every such } k.\end{aligned} \tag{2.3.46}$$

Indeed, let $v \in H_0^1(V)$ be arbitrary. From

1. Definition 1.2.5,
2. Lemma 1.1, Chapter III in [44] and
3. Lemma 2.3.1,

we derive, for every $k \geq k_V$, that

$$\begin{aligned}\langle (\mathcal{R}(U_{k+2}, V))\,(\mathbf{u}_k{}'), v\rangle &\overset{1.}{=} \langle \mathbf{u}_k{}', (\mathcal{E}_0(V, U_{k+2}))\,v\rangle \overset{2.}{=} \langle \mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))\,v\rangle' \overset{3.}{=}\\ &\overset{3.}{=} \overline{(\mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))\,v)'} = \overline{\left(\int_{U_{k+2}} \mathbf{u}_k\ ((\mathcal{E}_0(V, U_{k+2}))\,\bar{v})dx\right)'} =\\ &= \overline{\left(\int_V ((\mathcal{R}(U_{k+2}, V))\,\mathbf{u}_k)\ \bar{v}dx\right)'} = \overline{((\mathcal{R}(U_{k+2}, V))\,\mathbf{u}_k, v)'} \overset{3.}{=} \langle (\mathcal{R}(U_{k+2}, V))\,\mathbf{u}_k, v\rangle',\end{aligned}$$

thus, (2.3.46) follows from the arbitrariness of v along with Lemma 1.1, Chapter III in [44]. Additionally, we have

$$(\mathcal{L}_w(a_V, \theta))\,\mathbf{v}_{V,k} \equiv ((\mathcal{R}(U_{k+2}, V)) \circ \mathcal{L}_w(a_k, \theta))\,\mathbf{u}_k,\ \forall k \geq k_V, \tag{2.3.47}$$

where

$$a_V := (\mathcal{R}(U, V))\,a.$$

Indeed, we consider an arbitrary $v \in H_0^1(V)$. From

1. Definition 1.2.5,
2. (1.2.6),
3. (1.2.5) and
4. the definition in (2.3.45),

we get, for every $k \geq k_V$, that

$$\begin{aligned}\langle ((\mathcal{R}(U_{k+2}, V)) \circ \mathcal{L}_w(a_k, \theta))\,\mathbf{u}_k, v\rangle &\overset{1.}{=} \langle (\mathcal{L}_w(a_k, \theta))\,\mathbf{u}_k, (\mathcal{E}_0(V, U_{k+2}))\,v\rangle =\\ &= \int_{U_{k+2}} ((\nabla_w \circ (\mathcal{E}_0(V, U_{k+2})))\,v) \cdot a_k \nabla_w \overline{\mathbf{u}_k} dx \overset{2.}{=}\\ &\overset{2.}{=} \int_{U_{k+2}} (((\mathcal{E}_0(V, U_{k+2})) \circ \nabla_w)\,v) \cdot a_k \nabla_w \overline{\mathbf{u}_k} dx =\\ &= \int_V \nabla_w v \cdot (\mathcal{R}(U_{k+2}, V))\,(a_k \nabla_w \overline{\mathbf{u}_k})dx =\\ &= \int_V \nabla_w v \cdot a_V\,(((\mathcal{R}(U_{k+2}, V)) \circ \nabla_w)\,\overline{\mathbf{u}_k})dx \overset{3.}{=}\\ &\overset{3.}{=} \int_V \nabla_w v \cdot a_V\,((\nabla_w \circ (\mathcal{R}(U_{k+2}, V)))\,\overline{\mathbf{u}_k})dx \overset{4.}{=} \int_V \nabla_w v \cdot a_V \nabla_w \overline{\mathbf{v}_{V,k}} dx =\\ &= \langle (\mathcal{L}_w(a_V, \theta))\,\mathbf{v}_{V,k}, v\rangle\end{aligned}$$

and the desired result follows from the arbitrariness of v. Finally, in view of (2.2.29) and the definition in (2.3.45), $\widetilde{g}_\lambda(\mathbf{v}_{V,k})$ is well defined for every $k \geq k_V$, hence we directly get

$$\widetilde{g}_\lambda(\mathbf{v}_{V,k}) \equiv ((\mathcal{R}(U_{k+2}, V)) \circ (\widetilde{g}_\lambda))\,\mathbf{u}_k,\ \forall k \geq k_V. \tag{2.3.48}$$

Step 3α

In virtue of the bounds in (2.3.42) and (2.3.43) (along with (2.3.44)), we argue exactly as in

Step 3 of the proof of Theorem 2.3.1 in order to derive that there exist $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}\subset\{\mathbf{u}_k\}_{k=1}^{\infty}$ and a function

$$\mathbf{u}=\mathbf{u}_{J_0}\in L^{\infty}\big(J_0;H_0^1(U)\big)\cap W^{1,\infty}\big(J_0;H^{-1}(U)\big),$$

such that

$$\begin{aligned}\mathbf{v}_{k_l}=(\mathcal{E}_0(U_{k_l+2},U))\,(v_{k_l}\mathbf{u}_{k_l})\rightharpoonup\mathbf{u}\text{ in }H_0^1(U)\text{ everywhere in }\overline{J_0}\\ \text{and also }\|\mathbf{u}\|_{L^{\infty}(J_0;H^1(U))}\leq\widetilde{\mathcal{K}}_{J_0},\end{aligned}\tag{2.3.49}$$

as well as

$$\mathbf{v}_{k_l}{}'\overset{*}{\rightharpoonup}\mathbf{u}'\text{ in }L^{\infty}\big(J_0;H^{-1}(U)\big)\text{ and also }\|\mathbf{u}'\|_{L^{\infty}(J_0;H^{-1}(U))}\leq\widetilde{\mathcal{K}}_{J_0}.\tag{2.3.50}$$

Step 3β

Let $V\subset U$ be a fixed, arbitrary, open and bounded set. In virtue of the bounds in (2.3.45) and (2.3.46), again we deal exactly as in Step 3 of the proof of Theorem 2.3.1, but with one exception, that is we employ a slightly modified Point i) of Theorem 1.3.14 in [10] where we consider $H^1(V)$ instead of $H_0^1(V)$. We note that it is direct to check the validity of this modification just by a straightforward adaptation of the proof of the aforementioned result. Hence, we get a subsequence of $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$, which we still denote as such (we assume that $k_l\geq k_V$, for every $l\in\mathbb{N}$, where k_V is as in Step 2γ), and a function

$$\mathbf{u}_V=\mathbf{u}_{V,J_0}\in L^{\infty}\big(J_0;H^1(V)\big)\cap W^{1,\infty}\big(J_0;H^{-1}(V)\big),$$

such that

$$\mathbf{v}_{V,k_l}=(\mathcal{R}(U_{k_l+2},V))\,\mathbf{u}_{k_l}\rightharpoonup\mathbf{u}_V\text{ in }H^1(V)\text{ everywhere in }\overline{J_0},\tag{2.3.51}$$

as well as

$$\mathbf{v}_{V,k_l}{}'\overset{*}{\rightharpoonup}\mathbf{u}_V{}'\text{ in }L^{\infty}\big(J_0;H^{-1}(V)\big).\tag{2.3.52}$$

Step 3γ

We claim that

$$\begin{aligned}(\mathcal{R}(U,V))\,\mathbf{u}\equiv\mathbf{u}_V\text{ and }(\mathcal{R}(U,V))\,\mathbf{u}'\equiv\mathbf{u}_V{}',\\ \text{for every open and bounded }V\subset U.\end{aligned}\tag{2.3.53}$$

Indeed, first of all, for every V as above there exists $l_V\in\mathbb{N}$, such that $V\subseteq U_{k_l}$ for every $l\geq l_V$. Now, for the first equivalence we consider an arbitrary $\phi\in C_c^{\infty}(V)$. We then use

1. the convergence in (2.3.49),
2. $V\subseteq U_{k_l+2}$ for every $l\in\mathbb{N}$ by the definition of the sequence $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$,
3. $(\mathcal{R}(U_{k_l+2},V))\,v_{k_l}\equiv1$ for every $l\geq l_V$, since $(\mathcal{R}(U_{k_l+2},U_{k_l}))\,v_{k_l}\equiv1$ by the definition of v_{k_l} for every $l\in\mathbb{N}$, as well as $V\subseteq U_{k_l}$ for every $l\geq l_V$, and
4. (2.3.51),

to deduce that

$$\begin{aligned}\int_V((\mathcal{R}(U,V))\,\mathbf{u})\,\phi dx&=\int_U\mathbf{u}\,((\mathcal{E}_0(V,U))\,\phi)dx\overset{1.}{=}\lim_{l\nearrow\infty}\int_U\mathbf{v}_{k_l}\,((\mathcal{E}_0(V,U))\,\phi)dx=\\&=\lim_{l\nearrow\infty}\int_V((\mathcal{R}(U,V))\,\mathbf{v}_{k_l})\,\phi dx=\\&=\lim_{l\nearrow\infty}\int_V(((\mathcal{R}(U,V))\circ(\mathcal{E}_0(U_{k_l+2},U)))\,(v_{k_l}\mathbf{u}_{k_l}))\,\phi dx=\\&=\lim_{\substack{l\nearrow\infty\\ l\geq l_V}}\int_V(((\mathcal{R}(U,V))\circ(\mathcal{E}_0(U_{k_l+2},U)))\,(v_{k_l}\mathbf{u}_{k_l}))\,\phi dx\overset{2.}{=}\\&\overset{2.}{=}\lim_{\substack{l\nearrow\infty\\ l\geq l_V}}\int_V((\mathcal{R}(U_{k_l+2},V))\,(v_{k_l}\mathbf{u}_{k_l}))\,\phi dx\overset{3.}{=}\lim_{\substack{l\nearrow\infty\\ l\geq l_V}}\int_V((\mathcal{R}(U_{k_l+2},V))\,\mathbf{u}_{k_l})\,\phi dx=\\&=\lim_{\substack{l\nearrow\infty\\ l\geq l_V}}\int_V\mathbf{v}_{V,k_l}\,\phi dx\overset{4.}{=}\int_V\mathbf{u}_V\,\phi dx\end{aligned}$$

everywhere in $\overline{J_0}$, and the result follows from the arbitrariness of ϕ. As for the second equivalence, let $\psi\in C_c^{\infty}(J_0^{\circ})$ and $v\in H_0^1(V)$ be arbitrary. From

1. Definition 1.2.5,
2. the linearity of the functional,
3. the convergence in (2.3.50),
4. Lemma 1.1, Chapter III in [44],
5. the definition of the weak derivative,
6. Lemma 2.3.1,
7. $V \subseteq U_{k_l+2}$ for every $l \in \mathbb{N}$ by the definition of the sequence $\{\mathbf{u}_{k_l}\}_{l=1}^{\infty}$,
8. $(\mathcal{R}(U_{k_l+2}, V))\, v_{k_l} \equiv 1$ for every $l \geq l_V$, and
9. (2.3.52),

we have

$$\begin{aligned}
&\int_{J_0} \langle (\mathcal{R}(U,V))\,\mathbf{u}', v\rangle\,\psi dt \overset{1.}{=} \int_{J_0} \langle \mathbf{u}', (\mathcal{E}_0(V,U))\,v\rangle\,\psi dt \overset{2.}{=} \\
&\overset{2.}{=} \int_{J_0} \langle \mathbf{u}', \psi\,((\mathcal{E}_0(V,U))\,v)\rangle dt \overset{3.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle {\mathbf{v}_{k_l}}', \psi\,((\mathcal{E}_0(V,U))\,v)\rangle dt \overset{2.}{=} \\
&\overset{2.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle {\mathbf{v}_{k_l}}', (\mathcal{E}_0(V,U))\,v\rangle\,\psi dt = \\
&= \lim_{l\nearrow\infty} \int_{J_0} \langle {\mathbf{u}_{k_l}}', v_{k_l}\,((\mathcal{R}(U,U_{k_l+2}))\circ(\mathcal{E}_0(V,U)))\,v\rangle\,\psi dt \overset{4.}{=} \\
&\overset{4.}{=} \lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{k_l}, v_{k_l}\,((\mathcal{R}(U,U_{k_l+2}))\circ(\mathcal{E}_0(V,U)))\,v\rangle'\psi dt \overset{5.}{=} \\
&\overset{5.}{=} -\lim_{l\nearrow\infty} \int_{J_0} \langle \mathbf{u}_{k_l}, v_{k_l}\,((\mathcal{R}(U,U_{k_l+2}))\circ(\mathcal{E}_0(V,U)))\,v\rangle\,\psi' dt \overset{6.}{=} \\
&\overset{6.}{=} -\lim_{l\nearrow\infty} \int_{J_0} \overline{(\mathbf{u}_{k_l}, v_{k_l}\,((\mathcal{R}(U,U_{k_l+2}))\circ(\mathcal{E}_0(V,U)))\,v)}\psi' dt = \\
&= -\lim_{l\nearrow\infty} \int_{J_0} \overline{\left(\int_{U_{k_l+2}} \mathbf{u}_{k_l}\, v_{k_l}\,(((\mathcal{R}(U,U_{k_l+2}))\circ(\mathcal{E}_0(V,U)))\,\bar{v})dx\right)}\psi' dt \overset{7.}{=} \\
&\overset{7.}{=} -\lim_{l\nearrow\infty} \int_{J_0} \overline{\left(\int_{V} ((\mathcal{R}(U_{k_l+2},V))\,(v_{k_l}\mathbf{u}_{k_l}))\,\bar{v}dx\right)}\psi' dt = \\
&\quad -\lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \overline{\left(\int_{V} ((\mathcal{R}(U_{k_l+2},V))\,(v_{k_l}\mathbf{u}_{k_l}))\,\bar{v}dx\right)}\psi' dt \overset{8.}{=} \\
&\overset{8.}{=} -\lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \overline{\left(\int_{V} ((\mathcal{R}(U_{k_l+2},V))\,\mathbf{u}_{k_l})\,\bar{v}dx\right)}\psi' dt = -\lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \overline{\left(\int_{V} \mathbf{v}_{V,k_l}\,\bar{v}dx\right)}\psi' dt = \\
&= -\lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \overline{(\mathbf{v}_{V,k_l}, v)}\psi' dt \overset{6.}{=} -\lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle\,\psi' dt \overset{5.}{=} \lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \langle \mathbf{v}_{V,k_l}, v\rangle'\psi dt \overset{4.}{=} \\
&\overset{4.}{=} \lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \langle {\mathbf{v}_{V,k_l}}', v\rangle\,\psi dt \overset{2.}{=} \lim_{\substack{l\nearrow\infty\\ l\geq l_V}} \int_{J_0} \langle {\mathbf{v}_{V,k_l}}', \psi v\rangle dt \overset{9.}{=} \int_{J_0} \langle {\mathbf{u}_V}', \psi v\rangle dt \overset{2.}{=} \\
&\overset{2.}{=} \int_{J_0} \langle {\mathbf{u}_V}', v\rangle\,\psi dt,
\end{aligned}$$

thus, the desired result follows since ψ and v are arbitrary. We also claim that

$$((\mathcal{R}(U,V))\circ\mathcal{L}_w)\,\mathbf{u} \equiv (\mathcal{L}_w(a_V,\theta))\,\mathbf{u}_V, \text{ for every open and bounded } V \subset U. \tag{2.3.54}$$

Indeed, let $v \in H_0^1(V)$ be arbitrary. From

1. Definition 1.2.5,
2. (1.2.6),
3. (1.2.5) and
4. the first equivalence in (2.3.53),

we get

$$\langle ((\mathcal{R}(U,V))\circ\mathcal{L}_w)\,\mathbf{u},v\rangle \overset{1.}{=} \langle \mathcal{L}_w\mathbf{u},(\mathcal{E}_0(V,U))\,v\rangle =$$
$$= \int_U ((\nabla_w\circ(\mathcal{E}_0(V,U)))\,v)\cdot a\nabla_w\bar{\mathbf{u}}dx \overset{2.}{=}$$
$$\overset{2.}{=} \int_U (((\mathcal{E}_0(V,U))\circ\nabla_w)\,v)\cdot a\nabla_w\bar{\mathbf{u}}dx = \int_V \nabla_w v\cdot(\mathcal{R}(U,V))\,(a\nabla_w\bar{\mathbf{u}})dx =$$
$$= \int_V \nabla_w v\cdot a_V\,(((\mathcal{R}(U,V))\circ\nabla_w)\,\bar{\mathbf{u}})dx \overset{3.}{=} \int_V \nabla_w v\cdot a_V\,((\nabla_w\circ(\mathcal{R}(U,V)))\,\bar{\mathbf{u}})dx \overset{4.}{=}$$
$$\overset{4.}{=} \int_V \nabla_w v\cdot a_V\nabla_w\overline{\mathbf{u}_V}dx = \langle(\mathcal{L}_w(a_V,\theta))\,\mathbf{u}_V,v\rangle$$

and the desired result follows from the arbitrariness of v. Finally, we have

$$((\mathcal{R}(U,V))\circ\widetilde{g}_\lambda)\,\mathbf{u}\equiv\widetilde{g}_\lambda(\mathbf{u}_V),\ \text{for every open and bounded } V\subset U. \qquad (2.3.55)$$

For the above equivalence, we only notice that, in view of the first equivalence in (2.3.53) along with (2.2.29), $\widetilde{g}_\lambda(\mathbf{u}_V)$ is well defined.

Step 4α

Since every $\mathbf{u}_{k_l}$ satisfies the variational equation in U_{k_l+2}, we have that

$$\langle i\mathbf{u}_{k_l}{}' + (\mathcal{L}_w(a_k,\theta))\,(\mathbf{u}_{k_l}+\zeta_{k_l}) + \widetilde{g}_\lambda(\mathbf{u}_{k_l}),v_{k_l}\rangle = 0,\ \forall v_{k_l}\in H_0^1(U_{k_l+2}),\ \forall l\in\mathbb{N}.$$

Hence, for every open and bounded $V\subset U$ we have

$$\langle(\mathcal{R}(U_{k_l+2},V))\,(i\mathbf{u}_{k_l}{}' + (\mathcal{L}_w(a_k,\theta))\,(\mathbf{u}_{k_l}+\zeta_{k_l}) + \widetilde{g}_\lambda(\mathbf{u}_{k_l})),v\rangle = 0,\ \forall v\in H_0^1(V),\ \forall l\in\mathbb{N}.$$

In virtue of the equivalence in (2.3.46), as well as the equivalences (2.3.47) and (2.3.48) (along with the definition of the sequence $\{\mathbf{u}_{k_l}\}_{l=1}^\infty$), the above equation becomes

$$\langle i\mathbf{v}_{V,k_l}{}' + (\mathcal{L}_w(a_V,\theta))\,(\mathbf{v}_{V,k_l}+\zeta_V) + \widetilde{g}_\lambda(\mathbf{v}_{V,k_l}),v\rangle = 0,\ \forall v\in H_0^1(V),\ \forall l\in\mathbb{N}, \qquad (2.3.56)$$

where

$$\zeta_V \coloneqq (\mathcal{R}(U,V))\,\zeta.$$

Step 4β

Directly from (2.3.52) we have

$$\int_{J_0}\langle i\mathbf{v}_{V,k_l}{}',\psi v\rangle dt \to \int_{J_0}\langle i\mathbf{u}_V{}',\psi v\rangle dt,\ \forall\psi\in C_c^\infty(J_0^\circ),\ \forall v\in H_0^1(V). \qquad (2.3.57)$$

Moreover, in view of (2.3.51), we argue exactly as in Step 5 of the proof of Theorem 2.3.1 to obtain

$$\int_{J_0}\langle(\mathcal{L}_w(a_V,\theta))\,(\mathbf{v}_{V,k_l}+\zeta_V),\psi v\rangle dt \to \int_{J_0}\langle(\mathcal{L}_w(a_V,\theta))\,(\mathbf{u}_V+\zeta_V),\psi v\rangle dt, \qquad (2.3.58)$$

for every ψ and v as above. Additionally, in virtue of the bound (2.3.42) for k_l instead of k, along with the scaling invariant compact embeddings (see Proposition 1.2.5)

$$H_0^1(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\left(L^2(U)\right) \text{ and } H_0^1(U)\hookrightarrow\hookrightarrow(\mathcal{R}(U,V))\left(L^4(U)\right)\ (n=1,2,3),$$

we deduce that there exists a subsequence of $\{\mathbf{v}_{k_l}\}_{l=1}^\infty$, which we still denote as such, and a function $\mathbf{z}\in\mathcal{F}\big(J_0;L^4(V)\big)$, such that

$$(\mathcal{R}(U,V))\,(\mathbf{v}_{k_l}(t))\to\mathbf{z}(t) \text{ in } L^2(V) \text{ and } (\mathcal{R}(U,V))\,(\mathbf{v}_{k_l}(t))\to\mathbf{z}(t) \text{ in } L^4(V), \qquad (2.3.59)$$

for every $t\in J_0$. Since

$$\mathbf{v}_{V,k_l}\equiv(\mathcal{R}(U,V))\,\mathbf{v}_{k_l},\ \forall l\ge l_V, \qquad (2.3.60)$$

where l_V is as in Step 3γ, we deduce, from (2.3.51), that

$$\mathbf{z}\equiv\mathbf{u}_V. \qquad (2.3.61)$$

In virtue of (2.3.59), (2.3.60), (2.3.61), along with (2.2.31), (2.2.32) and (2.2.33), we derive that

$$\widetilde{g}_\lambda(\mathbf{v}_{V,k_l}) \to \widetilde{g}_\lambda(\mathbf{u}_V) \text{ in } \begin{cases} L^2(V), & \text{if } n=1,2 \\ Y_2(V), & \text{if } n=3, \end{cases} \text{ everywhere in } J_0.$$

Hence, from the dominated convergence theorem we get

$$\int_{J_0} \langle \widetilde{g}_\lambda(\mathbf{v}_{V,k_l}), \psi v\rangle dt \to \int_{J_0} \langle \widetilde{g}_\lambda(\mathbf{u}_V), \psi v\rangle dt, \ \forall \psi \in C_c^\infty(J_0^\circ), \ \forall v \in H_0^1(V). \tag{2.3.62}$$

Gathering (2.3.57), (2.3.58) and (2.3.62), we get from (2.3.56) that

$$\langle i\mathbf{u}_V{}' + (\mathcal{L}_w(a_V, \theta))(\mathbf{u}_V + \zeta_V) + \widetilde{g}_\lambda(\mathbf{u}_V), v\rangle = 0, \ \forall v \in H_0^1(V). \tag{2.3.63}$$

Step 4γ

In virtue of the second equivalence in (2.3.53), as well as the equivalences (2.3.54) and (2.3.55), we get from (2.3.63) that

$$(\mathcal{R}(U,V))(i\mathbf{u}' + \mathcal{L}_w(\mathbf{u}+\zeta) + \widetilde{g}_\lambda(\mathbf{u})) \overset{H^{-1}(V)}{\equiv} 0.$$

Since $V \subset U$ is arbitrary open and bounded, we deduce from Proposition 1.2.4 that $\mathbf{u}$ satisfies the variational equation in U.

Step 5α

As far as the initial condition is concerned, we first note that

$$\mathbf{v}_k(0) \to u_0 \text{ in } H^1(U). \tag{2.3.64}$$

Indeed, we have

$$\mathbf{v}_k(0) = (\mathcal{E}_0(U_{k+2}, U))(v_k u_{0_k}), \ \forall k,$$

thus, we get (2.3.64) by dealing exactly as in Step 1β. Hence, combining (2.3.64) with the convergence in (2.3.49) for $t=0$, we obtain $\mathbf{u}(0) \equiv u_0$.

Step 5β

We show that

$$\widetilde{E}_\lambda(\mathbf{u}) \le \widetilde{E}_\lambda(u_0) \text{ everywhere in } \overline{J_0}.$$

Indeed, we have from Theorem 2.3.2 that

$$\widetilde{E}_\lambda(\mathbf{u}_k) \le \widetilde{E}_\lambda(u_{0_k}) \text{ everywhere in } \overline{J_0},$$

hence, employing (2.3.40) and (2.3.1), we deduce

$$\widetilde{E}_\lambda(\mathbf{v}_{V,k}) \le \widetilde{E}_\lambda(u_{00_k}) \text{ everywhere in } \overline{J_0}, \text{ for every open } V \subseteq U_{k+2}, \text{ for every } k.$$

Let $\epsilon > 0$ be arbitrary. In virtue of (2.3.38), we have that there exists $k_0 = k_0(\epsilon)$, such that

$$\widetilde{E}_\lambda(\mathbf{v}_{V,k}) \le \widetilde{E}_\lambda(u_0) + \epsilon \text{ everywhere in } \overline{J_0}, \text{ for every open } V \subseteq U_{k+2}, \text{ for every } k \ge k_0. \tag{2.3.65}$$

From (2.3.59), (2.3.60), (2.3.61), along with (2.2.34) and (2.2.35), we derive that

$$\widetilde{G}_\lambda(\mathbf{v}_{V,k_l}) \to \widetilde{G}_\lambda(\mathbf{u}_V), \text{ for every open and bounded } V \subset U. \tag{2.3.66}$$

Moreover, dealing as in Step 6 of the proof of Theorem 2.3.1, we can verify that (2.3.51) implies

$$\mathbf{v}_{V,k_l} \rightharpoonup \mathbf{u}_V \text{ in } H^1_{\mathbb{R}}(V) \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U, \tag{2.3.67}$$

from which we directly get that

$$\mathcal{L}[\mathbf{v}_{V,k_l},\zeta] \to \mathcal{L}[\mathbf{u}_V,\zeta] \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U. \quad (2.3.68)$$

Additionally, considering $H^1_{\mathbb{R}}(V)$ equipped with the norm introduced in Lemma 2.3.5, we obtain from (2.3.67) along with the (sequentially) weak lower semi-continuity of the respective norm, that

$$\left(\|\mathbf{u}_V\|^2_{L^2(V)}+\mathcal{L}[\mathbf{u}_V,\mathbf{u}_V]\right)^{\frac{1}{2}} \le \liminf_{l\nearrow\infty}\left(\|\mathbf{v}_{Vk_l}\|^2_{L^2(V)}+\mathcal{L}[\mathbf{v}_{V,k_l},\mathbf{v}_{V,k_l}]\right)^{\frac{1}{2}}, \\ \text{everywhere in } \overline{J_0},$$

for every V as above. In virtue of the first convergence in (2.3.59) (along with (2.3.60) and (2.3.61)), the above inequality reads

$$\mathcal{L}[\mathbf{u}_V,\mathbf{u}_V] \le \liminf_{l\nearrow\infty}\mathcal{L}[\mathbf{v}_{V,k_l},\mathbf{v}_{V,k_l}] \text{ everywhere in } \overline{J_0}, \text{ for every } V \text{ as above.} \quad (2.3.69)$$

We now consider k_l for $l \le l_V$ instead of k in (2.3.65). From (2.3.66), (2.3.68) and (2.3.69), we get

$$\widetilde{E}_\lambda(\mathbf{u}_V) \le \widetilde{E}_\lambda(u_0)+\epsilon \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U,$$

or else

$$\widetilde{E}_\lambda(\mathbf{u}_V) \le \widetilde{E}_\lambda(u_0) \text{ everywhere in } \overline{J_0}, \text{ for every open and bounded } V \subset U, \quad (2.3.70)$$

since ϵ is arbitrary. In virtue of the first equivalence in (2.3.53), it is only left for us to consider in (2.3.70) an increasing sequence $\{V_k \subset U\}_k$ of open and bounded sets with $V_k \nearrow U$, e.g. $V_k = U_k$ for every k, and to let $k \nearrow \infty$, in order to obtain the desired result.

Step 5γ

The (conjugate) symmetry around $t=0$ follows directly from the convergence in (2.3.49) along with the fact that every $\mathbf{v}_{k_l}$ satisfies the same symmetry.

□

Remark 2.3.1. *Concerning the estimate*

$$\widetilde{E}_\lambda(\mathbf{u}) \le \widetilde{E}_\lambda(u_0) \text{ everywhere in } \overline{J_0},$$

we notice that for its proof in Theorem 2.3.1 and Theorem 2.3.2 we do not use (2.3.1), while in Theorem 2.3.3 we do. Skipping the details, we note that for the classic case $\zeta,\rho \equiv 0$, where the "charge", i.e. $\|\mathbf{u}\|_{L^2(U)}$, is conserved, there is no need for the use of (2.3.1), not even in Theorem 2.3.3.

2.3.2 Uniqueness and globality

It is obvious that the uniqueness of the local solutions of Subsection 2.3.1 implies the "globality" of those solutions. Before we proceed, we note that an upper bound for the constant in the following version of the Gagliardo-Nirenberg interpolation inequality (2.2.4)

$$\|u\|_{L^{2\tau}(U)} \le C\|\nabla_w u\|^{1-\frac{1}{\tau}}_{L^2(U)}\|u\|^{\frac{1}{\tau}}_{L^2(U)}, \ \forall u \in H^1_0(U), \ \forall \tau \in [1,\infty), \ n=2, \quad (2.3.71)$$

is

$$C\tau^{\frac{1}{2}}, \quad (2.3.72)$$

for an elegant proof of which we refer to Lemma 2 in [40] and the references therein.

Proposition 2.3.1. *Let* $\mathbf{u}$ *be as in Theorem 2.3.1 with $\zeta \in L^\infty(U)$, as in Theorem 2.3.2, or as in Theorem 2.3.3. If*

i. $n=1$,

ii. $n=2$ *and* $\alpha\in(0,2]$ *(i.e.* $\tau=1$ *for the case of the last two theorems), or*

iii. $U=\mathbb{R}^n$ $(n=1,2,3)$ *and* a *is as in (3),*

then $\mathbf{u}$ *is unique everywhere in* $\overline{J_0}$.

Proof. Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two solutions of the same problem, that the aforementioned theorems provide. Setting $\mathbf{w}:=\mathbf{u}_1-\mathbf{u}_2$, we have

$$i\mathbf{w}'+\mathcal{L}_w\mathbf{w}-(f(\mathbf{u}_1)-f(\mathbf{u}_2))\overset{H^{-1}(U)}{=}0,\text{ a.e. in } J_0, \tag{2.3.73}$$

where f stands for either g_λ or $\widetilde{g}_\lambda$, depending on the problem which we consider. We apply the functional of (2.3.73) on $\mathbf{w}(t)$, for arbitrary $t\in J_0$ and take the imaginary parts of both parts to get

$$\|\mathbf{w}\|^2_{L^2(U)}\le C\left|\int_0^t|\langle f(\mathbf{u}_1)-f(\mathbf{u}_2),\mathbf{w}\rangle|ds\right|,\ \forall t\in\overline{J_0}. \tag{2.3.74}$$

i. Since $H^1_0(U)\hookrightarrow L^\infty(U)$, from (2.2.13) (since $\zeta\in L^\infty(U)$) we deduce that

$$\|\mathbf{w}\|^2_{L^2(U)}\le C\left|\int_0^t\|\mathbf{w}\|^2_{L^2(U)}ds\right|,$$

hence, from the Grönwall inequality, $\mathbf{w}\equiv0$ everywhere in $\overline{J_0}$ and uniqueness follows.

ii. From (2.3.74) and the fact that $\zeta\in L^\infty(U)$, we get

$$\|\mathbf{w}\|^2_{L^2(U)}\le C\left|\int_0^t|\langle f(\mathbf{u}_1)-f(\mathbf{u}_2),\mathbf{w}\rangle|ds\right|\le$$
$$\le C\left|\int_0^t\left(\|\mathbf{w}\|^2_{L^2(U)}+\int_U\left(|\mathbf{u}_1|^2+|\mathbf{u}_2|^2\right)|\mathbf{w}|^2dx\right)ds\right|,\ \ \forall t\in\overline{J_0}.$$

In order to estimate the spatial integral, let $\tau>2$. Then, by the use of the Hölder inequality for $p_1=\frac{\tau}{2}$ and $p_2=\frac{\tau}{\tau-2}$, we get

$$\begin{aligned}\int_U|\mathbf{u}_1|^2|\mathbf{w}|^2dx&=\int_U\left(|\mathbf{u}_1|^\tau|\mathbf{w}|^2\right)^{\frac{2}{\tau}}|\mathbf{w}|^{\frac{2(\tau-2)}{\tau}}dx\le\\ \le\left(\int_U|\mathbf{u}_1|^\tau|\mathbf{w}|^2dx\right)^{\frac{2}{\tau}}\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}&\le\left(\int_U|\mathbf{u}_1|^{2\tau}dx\right)^{\frac{1}{\tau}}\|\mathbf{w}\|^{\frac{4}{\tau}}_{L^4(U)}\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}.\end{aligned} \tag{2.3.75}$$

Since $H^1_0(U)\hookrightarrow L^4(U)$, we apply (2.3.71) and (2.3.72) to get

$$\int_U|\mathbf{u}_1|^2|\mathbf{w}|^2dx\le C\tau\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}.$$

By repeating the above argument for the second term inside the parenthesis, we deduce, for τ sufficiently large such that $\|\mathbf{w}\|^2_{L^2(U)}\le\tau\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}$, that

$$\|\mathbf{w}\|^2_{L^2(U)}\le C\tau\left|\int_0^t\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}ds\right|,\ \ \forall t\in\overline{J_0}.$$

Therefore,

$$\left|\int_0^t\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}ds\right|\le(C\,|t|)^{\frac{\tau}{2}},\ \ \forall t\in\overline{J_0}.$$

Choosing $t_0\in\overline{J_0}$ such that $|t_0|$ is sufficiently small so that $C\,|t_0|<1$, we have that

$$\liminf_{\tau\nearrow\infty}\left|\int_0^{t_0}\|\mathbf{w}\|^{\frac{2(\tau-2)}{\tau}}_{L^2(U)}ds\right|\le0,$$

which, in view of the Fatou lemma, implies that $\mathbf{w}\equiv0$ everywhere in $[-|t_0|,|t_0|]$. By repeating the above argument as many times as needed in order to cover $\overline{J_0}$, we get the uniqueness.

iii. We first note that we already have a stronger result, i.e. not nessesarily $a \equiv \text{id}$, for $n=1$ and for a subcase of $n=2$. We then show the result only for $n=2,3$. We recall that $\mathbf{w}$ takes the form (see, e.g., Proposition 3.1.3 in [10]),

$$\mathbf{w} = i\int_0^t \mathcal{T}(t-s)\left(\widetilde{g}_\lambda(\mathbf{u}_1)-\widetilde{g}_\lambda(\mathbf{u}_2)\right)ds, \text{ for a.e. } t\in J_0, \tag{2.3.76}$$

where $\mathcal{T}(t)$ stands for the linear and bounded operator from $L^{\frac{p}{p-1}}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for every $p\in[2,\infty]$ and every $t\neq 0$, with

$$\mathcal{T}(t)u = \left(\frac{1}{4\pi i t}\right)^{\frac{n}{2}} e^{\frac{i|\cdot|^2}{4t}} * u, \ \forall u\in L^{\frac{p}{p-1}}(\mathbb{R}^n), \ \forall t\neq 0 \text{ and}$$

$$\|\mathcal{T}(t)\|_{\mathcal{L}\left(L^{\frac{p}{p-1}}(\mathbb{R}^n);L^p(\mathbb{R}^n)\right)} \leq \left(\frac{1}{4\pi|t|}\right)^{n\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

We have that the pairs $(\infty,2)$ and

$$\begin{cases}(4,4), & \text{if } n=2\\ \left(\frac{8}{3},4\right), & \text{if } n=3\end{cases}$$

are admissible[6]. From (2.3.76), (2.2.14), (2.2.18) for $p_1=\frac{8}{5}$ and $p_2=1$, as well as the Strichartz estimate (see, e.g., Theorem 2.3.3 in [10], or Theorem 2.3 in [43]), we have

$$\begin{cases}\|\mathbf{w}\|_{L^\infty(J_0;L^2(U))}+\|\mathbf{w}\|_{L^4(J_0;L^4(U))} \leq C\left(\|\mathbf{w}\|_{L^1(J_0;L^2(U))}+\|\mathbf{w}\|_{L^1(J_0;L^2(U))}\right), & \text{if } n=2\\ \|\mathbf{w}\|_{L^\infty(J_0;L^2(U))}+\|\mathbf{w}\|_{L^{\frac{8}{3}}(J_0;L^4(U))} \leq C\left(\|\mathbf{w}\|_{L^1(J_0;L^2(U))}+\|\mathbf{w}\|_{L^{\frac{8}{5}}(J_0;L^2(U))}\right), & \text{if } n=3,\end{cases}$$

hence, from Lemma 4.2.2 in [10], $\mathbf{w}\equiv 0$ everywhere in $\overline{J_0}$ and uniqueness follows.

□

2.3.3 Conservation of energy and well posedness

Here, we utilize the existence backwards in time as well as the uniqueness of a solution, in order to complete the puzzle of the well posedness of the problem, for certain cases. First, we show the following result, concerning the conservation of the energy of a solution.

Proposition 2.3.2. *The energy of a unique solution* $\mathbf{u}$ *of Theorem 2.3.1, Theorem 2.3.2 or Theorem 2.3.3 is conserved, that is*

$$F(\mathbf{u}) = F(u_0) \text{ everywhere in } \overline{J_0}, \tag{2.3.77}$$

where F *stands for either* E_λ *or* $\widetilde{E}_\lambda$, *depending on the problem which we consider.*

Proof. Let $t_1,t_2\in\overline{J_0}$ with $t_1<t_2$ be arbitrary. We consider two solutions

$$\mathbf{w},\mathbf{z}\in L^\infty\big((t_1-t_2,t_2-t_1);H_0^1(U)\big)\cap W^{1,\infty}\big((t_1-t_2,t_2-t_1);H^{-1}(U)\big),$$

of the respective problems

$$\begin{cases}\langle i\mathbf{w}',v\rangle+\mathcal{M}[\mathbf{w},v]=0, \text{ for every } v\in H_0^1(U), \text{ a.e. in } (t_1-t_2,t_2-t_1)\\ \mathbf{w}(0)=\mathbf{u}(t_1)\end{cases} \tag{2.3.78}$$

and

$$\begin{cases}\langle i\mathbf{z}',v\rangle+\mathcal{M}[\mathbf{z},v]=0, \text{ for every } v\in H_0^1(U), \text{ a.e. in } (t_1-t_2,t_2-t_1)\\ \mathbf{z}(0)=\mathbf{u}(t_2)\end{cases}$$

[6] A pair $(p,q)\in[2,\infty]^2$ is called admissible if $\frac{2}{p}+\frac{n}{q}=\frac{n}{2}$ and $(p,q,n)\neq(2,\infty,2)$.

that the aforementioned theorems provide us, where $\mathcal{M}$ stands for either $\mathcal{N}_\lambda$ or $\widetilde{N}_\lambda$, depending on the problem which we consider. We then have

$$F(\mathbf{w}(t_2-t_1)) \le F(\mathbf{u}(t_1)) \text{ and } F(\mathbf{z}(t_1-t_2)) \le F(\mathbf{u}(t_2)). \tag{2.3.79}$$

Moreover, we claim that

$$\mathbf{w}(t-t_1) = \mathbf{u}(t) = \mathbf{z}(t-t_2), \ \forall t \in [t_1, t_2]. \tag{2.3.80}$$

We show only the first equation and in analogous manner we derive the second one. From the uniqueness of the solution in the time-interval $[t_1, t_2]$, it suffices to show that

$$\mathbf{w}(\cdot - t_1) \in L^\infty\big((t_1,t_2);H_0^1(U)\big) \cap W^{1,\infty}\big((t_1,t_2);H^{-1}(U)\big)$$

with

$$\begin{cases} \big\langle i(\mathbf{w}(\cdot-t_1))', v\big\rangle + \mathcal{M}[\mathbf{w}(\cdot-t_1), v] = 0, \text{ for every } \in H_0^1(U), \text{ a.e. in } (t_1,t_2) \\ \mathbf{w}(0) = \mathbf{u}(t_1). \end{cases}$$

For the first result, we only note that $(\mathbf{w}(\cdot-t_1))' \in L^\infty\big((t_1,t_2);H^{-1}(U)\big)$ and in particular

$$(\mathbf{w}(t-t_1))' \equiv \mathbf{w}'(t-t_1), \text{ for a.e. } t \in (t_1,t_2),$$

by the use of the common chain rule for the normed-space-valued functions (see, e.g., Theorem 3.59 in [35]). As for the second result, in view of (2.3.78), it suffices to show only the variational equation. Indeed, let $\psi \in C_c^\infty((t_1,t_2))$ be arbitrary. If we set $\phi(t) := \psi(t+t_1)$ for every $t \in (0, t_2-t_1)$, then $\phi \in C_c^\infty((0,t_2-t_1))$. From

1. Lemma 1.1, Chapter III in [44],
2. the definition of the weak derivative,
3. Lemma 2.3.1 and
4. the change of variables formula,

we have

$$\begin{aligned} 0 &= \int_0^{t_2-t_1} (\langle i\mathbf{w}', v\rangle + \mathcal{M}[\mathbf{w},v])\,\phi dt \overset{1.}{=} \int_0^{t_2-t_1} \big(\langle i\mathbf{w}, v\rangle' + \mathcal{M}[\mathbf{w},v]\big)\,\phi dt \overset{2.}{=} \\ &\overset{2.}{=} \int_0^{t_2-t_1} -\langle i\mathbf{w}, v\rangle\,\phi' + \mathcal{M}[\mathbf{w},v]\phi dt \overset{3.}{=} \int_0^{t_2-t_1} -\overline{(i\mathbf{w}, v)}\phi' + \mathcal{M}[\mathbf{w},v]\phi dt \overset{4.}{=} \\ &\overset{4.}{=} \int_{t_1}^{t_2} -\overline{(i\mathbf{w}(\cdot-t_1), v)}\psi' + \mathcal{M}[\mathbf{w}(\cdot-t_1),v]\psi dt \overset{3.}{=} \\ &\overset{3.}{=} \int_{t_1}^{t_2} -\langle i\mathbf{w}(\cdot-t_1), v\rangle\,\psi' + \mathcal{M}[\mathbf{w}(\cdot-t_1),v]\psi dt \overset{2.}{=} \\ &\overset{2.}{=} \int_{t_1}^{t_2} \big(\langle i\mathbf{w}(\cdot-t_1), v\rangle' + \mathcal{M}[\mathbf{w}(\cdot-t_1),v]\big)\,\psi dt \overset{1.}{=} \\ &\overset{1.}{=} \int_{t_1}^{t_2} (\langle i\mathbf{w}'(\cdot-t_1), v\rangle + \mathcal{M}[\mathbf{w}(\cdot-t_1),v])\,\psi dt, \end{aligned}$$

thereby follows the desired equality, since ψ is arbitrary. Now, (2.3.80) implies that

$$\mathbf{w}(t_2-t_1) = \mathbf{u}(t_2) \text{ and } \mathbf{z}(t_1-t_2) = \mathbf{u}(t_1). \tag{2.3.81}$$

Combining (2.3.79) and (2.3.81), we get

$$F(\mathbf{u}(t_1)) = F(\mathbf{u}(t_2)),$$

thus, (2.3.77) follows from the arbitrariness of t_1 and t_2. □

Corollary 2.3.1. *If a is as in* (3) *and $\zeta \equiv$ const, then a unique solution* $\mathbf{u}$ *of Theorem 2.3.1, Theorem 2.3.2 or Theorem 2.3.3 is strong H_0^1-solution in J_0, i.e.*

$$\mathbf{u} \in C\big(\overline{J_0}; H_0^1(U)\big) \cap C^1\big(\overline{J_0}; H^{-1}(U)\big).$$

If, in addition, U is bounded, then $\mathbf{u}$ *is also continuously dependent on the initial datum.*

Proof. For the regularity, since $\mathbf{u}\in C\big(\overline{J_0};L^2(U)\big)$, we deduce that

$$\|\mathbf{u}\|_{L^2(U)}\in C\big(\overline{J_0}\big) \tag{2.3.82}$$

by the triangle inequality, as well as that

$$P(\mathbf{u})\in C\big(\overline{J_0}\big) \tag{2.3.83}$$

by (2.2.10), (2.2.25) and (2.2.27), where P stands for either G_λ or $\widetilde{G}_\lambda$, depending on the the problem which we consider. From (2.3.77), (2.3.83), as well as the facts that $a\equiv\mathrm{id}$ and $\zeta\equiv\mathrm{const}$, we get that

$$\|\mathbf{u}\|_{L^2(U)}\in C\big(\overline{J_0}\big). \tag{2.3.84}$$

Therefore, from (2.3.82) and (2.3.84), we derive

$$\|\mathbf{u}\|_{H^1(U)}\in C\big(\overline{J_0}\big),$$

which implies that $\mathbf{u}\in C\big(\overline{J_0};H_0^1(U)\big)$ and so, by the variational equation, we get $\mathbf{u}'\in C\big(\overline{J_0};H^{-1}(U)\big)$.

As far as the continuous dependence for bounded U is concerned, we fix an arbitrary $u_0\in H_0^1(U)$. Let $\{u_{0,m}\}\subset H_0^1(U)$ be such that $u_{0,m}\to u_0$ in $H^1(U)$, and so

$$\|u_{0,m}\|_{H^1(U)}\le\mathcal{K}\Big(\|u_0\|_{H^1(U)}\Big).$$

We write as $\mathbf{u}$ and $\mathbf{u}_m$, the unique corresponding solutions of the problem (2.2.6) or (2.2.23). We deduce that $\{\mathbf{u}\}\cup\{\mathbf{u}_m\}\subset C\big(\overline{J_0};H_0^1(U)\big)$ from the previous regularity result. From (2.3.2) and (2.3.22), the above estimate, as well as the increasing property of $\mathcal{K}$ we have

$$\|\mathbf{u}_m\|_{L^\infty(J;H^1(U))}+\|\mathbf{u}_m{}'\|_{L^\infty(J;H^{-1}(U))}\le\widetilde{\mathcal{K}},$$

where $\widetilde{\mathcal{K}}$ is as in the proof of the corresponding theorems. Hence, by dealing as in the proof of Theorem 2.3.1 from Step 3 to Step 5, there exist a subsequence $\{\mathbf{u}_{m_l}\}\subseteq\{\mathbf{u}_m\}$ and a function $\mathbf{y}\in L^\infty\big(J_0;H_0^1(U)\big)\cap W^{1,\infty}\big(J_0;H^{-1}(\mathrm{U})\big)$, such that $\mathbf{y}$ solves the problem (2.2.6) or (2.2.23) and also

$$\mathbf{u}_{m_l}\to\mathbf{y}\ \text{in}\ C\big(\overline{J_0};L^2(U)\big),\ \text{as well as}\ P(\mathbf{u}_{m_l})\to P(\mathbf{y}).$$

From the uniqueness we deduce that $\mathbf{y}\equiv\mathbf{u}$. Moreover, from the above convergences along with (2.3.77), we obtain that

$$\|\mathbf{u}_{m_l}\|_{H^1(U)}\to\|\mathbf{u}\|_{H^1(U)},\ \text{uniformly in}\ \overline{J_0}.$$

Hence, from Point iii) of Proposition 3.1.14 in [10] we get that $\mathbf{u}_{m_l}\to\mathbf{u}$ in $C\big(\overline{J_0};H_0^1(U)\big)$. Since $\{u_{0,m}\}$ is arbitrary we deduce that for every $\{u_{0,m}\}\subset H_0^1(U)$ such that $u_{0,m}\to u_0$ in $H^1(U)$, there exists a subsequence $\{u_{0,m_l}\}\subset\{u_{0,m}\}$ such that $\mathbf{u}_{m_l}\to\mathbf{u}$ in $C\big(\overline{J_0};H_0^1(U)\big)$. Hence, $\mathbf{u}_m\to\mathbf{u}$ in $C\big(\overline{J_0};H_0^1(U)\big)$ also, and since $\{u_{0,m}\}$ and u_0 are arbitrary we conclude that the map $u_0\mapsto\mathbf{u}$ is continuous. ☐

2.4 Regularity of solutions

Here, we study the regularity of solutions of Section 2.3. In particular, we consider the problem (2.2.23) only for the cases where τ is as in (2.1.2). We recall that a solution of such a problem possesses certain fine properties, such as uniqueness, globality and conservation of energy. We see here that it also can be infinitely regular, if the initial datum is regular enough.

Before we proceed to the statement and proof of the main results, we provide some preliminary ones. First, we derive an estimate with the use of the following Gagliardo-Nirenberg interpolation inequality

$$\big\|\nabla^j u\big\|_{L^{\frac{2m}{j}}(\mathbb{R}^n)}\le C\|\nabla^m u\|_{L^2(\mathbb{R}^n)}^{\frac{j}{m}}\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{m}},\ \ \forall\, j=0,1,\dots,m,\ \ \forall u\in C_c^\infty(\mathbb{R}^n), \tag{2.4.1}$$

which allows us to handle certain non-linearities such as ours.

Proposition 2.4.1. *Let* $m\in\mathbb{N}$ *and* $f\in C^m([0,\infty);\mathbb{R})$*. Then, for every* $u\in C_c^\infty(\mathbb{R}^n)$*,*

$$\sum_{k=1}^{m}\left\|\nabla^k\left(f\left(|u|^2\right)u\right)\right\|_{L^2(\mathbb{R}^n)}\leq C\left(\sum_{k=1}^{m}\left\|\nabla^k u\right\|_{L^2(\mathbb{R}^n)}\right)\left(\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)}\|u\|^{2k}_{L^\infty(\mathbb{R}^n)}\right). \quad (2.4.2)$$

Proof. Let $u\in C_c^\infty(\mathbb{R}^n)$ and $\alpha\in\mathbb{N}_0^n$ with $1\leq|\alpha|\leq m$ be arbitrary. From the Leibniz rule and the already known multivariate Faá di Bruno formula in Theorem 1.2.3, we have that[7]

$$D^\alpha\left(f\left(|u|^2\right)u\right)=f\left(|u|^2\right)D^\alpha u+\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|,\\|\alpha_1|\geq 1}}D^{\alpha_1}\left(f\left(|u|^2\right)\right)D^{\alpha_2}u=$$

$$=f\left(|u|^2\right)D^\alpha u+\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|,\\|\alpha_1|\geq 1}}\sum_{1\leq|\beta|\leq|\alpha_1|}M_{\alpha_1,|\beta|}\left(|u|^2\right)D^{|\beta|}f\left(|u|^2\right)D^{\alpha_2}u\coloneqq I_1+I_2,$$

where

$$M_{\alpha_1,|\beta|}\left(|u|^2\right)\coloneqq\alpha_1!\sum_{s=1}^{|\alpha_1|}\sum_{p_s(\alpha_1,|\beta|)}\prod_{j=1}^{s}\frac{1}{\gamma_j!(\delta_j!)^{|\gamma_j|}}\left(D^{\delta_j}|u|^2\right)^{\gamma_j},$$

with $\gamma_j\in\mathbb{N}$, $\delta_j\in\mathbb{N}_0^n$,

$$p_s(\alpha_1,|\beta|)\coloneqq\left\{(\gamma_1,\ldots,\gamma_s,\delta_1,\ldots,\delta_s)\;\middle|\;0\prec\delta_1\prec\ldots\prec\delta_s,\;\sum_{j=1}^{s}\gamma_j=|\beta|,\;\sum_{j=1}^{s}\gamma_j\delta_j=\alpha_1\right\}$$

and $\mu\prec\nu$ for $\mu,\nu\in\mathbb{N}_0^n$ as in the aforementioned theorem.

I_1 is easily estimated. Indeed,

$$\|I_1\|_{L^2(\mathbb{R}^n)}\leq\|D^\alpha u\|_{L^2(\mathbb{R}^n)}\|f\|_{L^\infty\left(\left(0,\|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)}.$$

As far as I_2 is concerned, we have

$$\|I_2\|_{L^2(\mathbb{R}^n)}\leq C\sum_{\substack{|\alpha_1|+|\alpha_2|=|\alpha|,\\|a_1|\geq 1}}\sum_{l=1}^{|\alpha_1|}\|f^{(l)}\|_{L^\infty\left(\left(0,\|u\|^2_{L^\infty(\mathbb{R}^n)}\right)\right)}\sum_{s=1}^{|\alpha_1|}\sum_{p_s(\alpha_1,l)}I_2',$$

where

$$I_2'\coloneqq\left\|\prod_{j=1}^{s}\left(D^{\delta_j}|u|^2\right)^{\gamma_j}D^{\alpha_2}u\right\|_{L^2(\mathbb{R}^n)}=\left\|\prod_{i_1=1}^{\gamma_1}\left(D^{\delta_1}|u|^2\right)\cdots\prod_{i_s=1}^{\gamma_s}\left(D^{\delta_s}|u|^2\right)D^{\alpha_2}u\right\|_{L^2(\mathbb{R}^n)}.$$

From the Hölder inequality for $p_{j,i_j}=\frac{|\alpha|}{|\delta_j|}$, for $i_j=1,\ldots,\gamma_j$, $j=1,\ldots,s$ and $p_{s+1}=\frac{|\alpha|}{|\alpha_2|}$, we get

$$I_2'\leq\prod_{i_1=1}^{\gamma_1}\left\|D^{\delta_1}|u|^2\right\|_{L^{\frac{2|\alpha|}{|\delta_1|}}(\mathbb{R}^n)}\cdots\prod_{i_s=1}^{\gamma_s}\left\|D^{\delta_s}|u|^2\right\|_{L^{\frac{2|\alpha|}{|\delta_s|}}(\mathbb{R}^n)}\|D^{\alpha_2}u\|_{L^{\frac{2|\alpha|}{|\alpha_2|}}(\mathbb{R}^n)}.$$

From the Leibniz rule we have

$$D^{\delta_j}|u|^2=\sum_{|\delta_{1,j}|+|\delta_{2,j}|=|\delta_j|}D^{\delta_{1,j}}u\,D^{\delta_{2,j}}\overline{u},$$

thus, once again from the Hölder inequality for $p_1=\frac{|\delta_j|}{|\delta_{1,j}|}$ and $p_2=\frac{|\delta_j|}{|\delta_{2,j}|}$, we get

$$\left\|D^{\delta_j}|u|^2\right\|_{L^{\frac{2|\alpha|}{|\delta_j|}}(\mathbb{R}^n)}\leq\sum_{|\delta_{1,j}|+|\delta_{2,j}|=|\delta_j|}\left\|D^{\delta_{1,j}}u\right\|_{L^{\frac{2|\alpha|}{|\delta_{1,j}|}}(\mathbb{R}^n)}\left\|D^{\delta_{2,j}}u\right\|_{L^{\frac{2|\alpha|}{|\delta_{2,j}|}}(\mathbb{R}^n)}.$$

hence, applying (2.4.1), we deduce that

$$\left\|D^{\delta_j}|u|^2\right\|_{L^{\frac{2|\alpha|}{|\delta_j|}}(\mathbb{R}^n)}\leq C\left\|\nabla^{|\alpha|}u\right\|^{\frac{|\delta_j|}{|\alpha|}}_{L^2(\mathbb{R}^n)}\|u\|^{2-\frac{|\delta_j|}{|\alpha|}}_{L^\infty(\mathbb{R}^n)}.$$

[7] If $n=1$, then $D^\beta=D^{|\beta|}$, for every multi-index β.

Again from (2.4.1), we get

$$\|D^{\alpha_2}u\|_{L^{\frac{2|\alpha|}{|\alpha_2|}}(\mathbb{R}^n)} \leq C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)}^{\frac{|\alpha_2|}{|\alpha|}}\|u\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\alpha_2|}{|\alpha|}}.$$

Therefore,

$$I_2' \leq C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)}^{2l}$$

and so

$$\|I_2\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla^{|\alpha|}u\|_{L^2(\mathbb{R}^n)}\sum_{1\leq|\beta|\leq|\alpha|}\sum_{l=1}^{|\beta|}\|f^{(l)}\|_{L^\infty\left(\left(0,\|u\|_{L^\infty(\mathbb{R}^n)}^2\right)\right)}\|u\|_{L^\infty(\mathbb{R}^n)}^{2l}.$$

□

If we further assume that $f \neq \text{const}$ and $f(0)=0$, then the above estimate becomes

$$\sum_{k=1}^{m}\left\|\nabla_w^k\left(f\left(|u|^2\right)u\right)\right\|_{L^2(U)} \leq C\left(\sum_{k=1}^{m}\|\nabla_w^k u\|_{L^2(U)}\right)\left(\sum_{k=1}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\|u\|_{L^\infty(\mathbb{R}^n)}^2\right)\right)}\|u\|_{L^\infty(U)}^{2k}\right),$$

for every $u \in C_c^\infty(U)$, along with the obvious generalization for $f^{(k)}(0)=0$, with $k=1,\ldots,m-1$. This fact, however, does not make any difference for us here. Moreover, if $n<2m$, we directly deduce that the above results holds for every $u \in H_0^m(U)$ and every arbitrary U, by employing the $\mathcal{E}_0(U,\mathbb{R}^n)$ operator and the scaling-invariant Sobolev embedding $H_0^m(U) \hookrightarrow L^\infty(U)$.

Now, in virtue of Theorem 1.2.1, we extend Proposition 2.4.1 for functions in non-zero-trace Sobolev spaces.

Corollary 2.4.1. *Let U with $\partial U \in \mathrm{Lip}(\varepsilon,K,L)$, $m \in \mathbb{N}$ with $n<2m$, $f \in C^m([0,\infty);\mathbb{R})$ and $u \in H^m(U)$. Then $\left(f\left(|u|^2\right)u\right) \in H^m(U)$ also, with*

$$\sum_{k=1}^{m}\left\|\nabla_w^k\left(f\left(|u|^2\right)u\right)\right\|_{L^2(U)} \leq \mathcal{K}(K,L)\times$$
$$\times\left(\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k u\|_{L^2(U)}\right)\left(\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\mathcal{K}(K)\|u\|_{L^\infty(U)}^2\right)\right)}\|u\|_{L^\infty(U)}^{2k}\right). \tag{2.4.3}$$

Proof. Considering the extended function, (2.4.2) gets the form

$$\sum_{k=1}^{m}\left\|\nabla^k\left(f\left(|u|^2\right)u\right)\right\|_{L^2(U)} \leq \sum_{k=1}^{m}\left\|\nabla^k\left(f\left(|(\mathcal{E}(U,\mathbb{R}^n))\,u|^2\right)(\mathcal{E}(U,\mathbb{R}^n))\,u\right)\right\|_{L^2(\mathbb{R}^n)} \leq$$
$$\leq C\left(\sum_{k=1}^{m}\left\|\left(\nabla^k\circ(\mathcal{E}(U,\mathbb{R}^n))\right)u\right\|_{L^2(\mathbb{R}^n)}\right)\times$$
$$\times\left(\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\|(\mathcal{E}(U,\mathbb{R}^n))u\|_{L^\infty(\mathbb{R}^n)}^2\right)\right)}\|(\mathcal{E}(U,\mathbb{R}^n))\,u\|_{L^\infty(\mathbb{R}^n)}^{2k}\right).$$

From the bounds in Theorem 1.2.1 we obtain

$$\sum_{k=1}^{m}\left\|\left(\nabla^k\circ(\mathcal{E}(U,\mathbb{R}^n))\right)u\right\|_{L^2(\mathbb{R}^n)} \leq \mathcal{K}(K,L)\left(\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\|\nabla^k u\|_{L^2(U)}\right).$$

Moreover, in view of Corollary 1.2.2, we have that $u \in L^\infty(U)$, therefore, again from the aforementioned bounds we get

$$\|(\mathcal{E}(U,\mathbb{R}^n))\,u\|_{L^\infty(\mathbb{R}^n)} \leq \mathcal{K}(K)\|u\|_{L^\infty(U)},$$

thereby follows the desired result. □

For the next result we notice that if $U \subseteq \mathbb{R}$ with $\partial U \in \mathrm{Lip}^m(\varepsilon,K,L)$ for some $m \in \mathbb{N}$, then in fact $\partial U \in \mathrm{Lip}(\varepsilon,K,0)$, and vice versa.

Corollary 2.4.2. *Let* $U\subset\mathbb{R}$ *with* $|U|<\infty$ *as well as* $\partial U\in\operatorname{Lip}(\varepsilon,K,0)$, $m\in\mathbb{N}\setminus\{1\}$, $f\in C^m([0,\infty);\mathbb{R})$, $u\in H^m(U)$ *and* $\zeta\in X^m(U)$. *Then* $\left(f\left(|u+\zeta|^2\right)(u+\zeta)\right)\in H^m(U)$ *also, with*

$$\sum_{k=1}^{m}\left\|\nabla_w^k\left(f\left(|u+\zeta|^2\right)(u+\zeta)\right)\right\|_{L^2(U)}\leq\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\times\times\left(1+\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}\right).\tag{2.4.4}$$

If, in addition, $\mathcal{L}_w(a,\theta)$ *is such that* $a\in W^{m-1,\infty}(U)$, $u\in H^m(U)\cap H_0^1(U)$, *as well as*

$$\left(L_w^j u\right)\in H_0^1(U),\ \forall j=0,\ldots,\left\lfloor\frac{m}{2}\right\rfloor-1,$$

then we have

$$\sum_{k=1}^{m}\left\|\nabla_w^k\left(f\left(|u+\zeta|^2\right)(u+\zeta)\right)\right\|_{L^2(U)}\leq\leq\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)}\right)\times\times\left(1+\sum_{\substack{j\in\mathbb{N},\\2j+1\leq m}}\left\|\left(\nabla_w\circ L_w^j\right)u\right\|_{L^2(U)}+\sum_{\substack{j\in\mathbb{N},\\2j\leq m}}\left\|L_w^j u\right\|_{L^2(U)}\right).\tag{2.4.5}$$

Proof. We have that $\zeta\in H^m(U)$, since $|U|<\infty$, hence $(u+\zeta)\in H^m(U)$. Employing (2.4.3), we get

$$\sum_{k=1}^{m}\left\|\nabla_w^k\left(f\left(|u+\zeta|^2\right)(u+\zeta)\right)\right\|_{L^2(U)}\leq\mathcal{K}(K)\times\times\left(\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\left\|\nabla_w^k(u+\zeta)\right\|_{L^2(U)}\right)\left(\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\mathcal{K}(K)\|u+\zeta\|^2_{L^\infty(U)}\right)\right)}\|u+\zeta\|^{2k}_{L^\infty(U)}\right).$$

For the term inside the first parenthesis we have

$$\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\left\|\nabla_w^k(u+\zeta)\right\|_{L^2(U)}\leq\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\left\|\nabla_w^k u\right\|_{L^2(U)}+\sum_{k=0}^{m}\frac{1}{\varepsilon^{m-k}}\left\|\nabla_w^k\zeta\right\|_{L^2(U)}\leq$$
$$\leq\max\left\{1,\frac{1}{\varepsilon^m}\right\}\sum_{k=0}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+\frac{1}{\varepsilon^m}\|\zeta\|_{L^2(U)}+\max\left\{1,\frac{1}{\varepsilon^m}\right\}\sum_{k=1}^{m}\left\|\nabla_w^k\zeta\right\|_{L^2(U)}\leq$$
$$\leq C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\left(\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+\|u\|_{H^1(U)}\right)+\frac{1}{\varepsilon^m}|U|^{\frac{1}{2}}\|\zeta\|_{X^m(U)}+$$
$$+C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\|\zeta\|_{X^m(U)}\leq C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\left(\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+\|u\|_{H^1(U)}\right)+$$
$$+\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\|\zeta\|_{X^m(U)}+C\max\left\{1,\frac{1}{\varepsilon^m}\right\}\|\zeta\|_{X^m(U)}\leq$$
$$\leq C\max\left\{1,\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\right\}\left(\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+\|u\|_{H^1(U)}+\|\zeta\|_{X^m(U)}\right)\leq$$
$$\leq C\max\left\{1,\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\}\right\}\max\left\{1,\|u\|_{H^1(U)}+\|\zeta\|_{X^m(U)}\right\}\left(\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+1\right)=$$
$$=\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(\sum_{k=2}^{m}\left\|\nabla_w^k u\right\|_{L^2(U)}+1\right).$$

As for the term inside the second parenthesis, we have that

$$\|u\|_{L^\infty(U)}\leq\mathcal{K}\left(\frac{1}{\varepsilon},K\right)\|u\|_{H^1(U)},$$

from the scaling dependent embedding $H^1(U)\hookrightarrow L^\infty(U)$ (see Corollary 1.2.2), which implies

$$\sum_{k=0}^{m}\|f^{(k)}\|_{L^\infty\left(\left(0,\mathcal{K}(K)\|u+\zeta\|^2_{L^\infty(U)}\right)\right)}\|u+\zeta\|^{2k}_{L^\infty(U)}\leq\mathcal{K}\left(\frac{1}{\varepsilon},K,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right).$$

Directly from (2.4.4) and the bound in Proposition 1.2.14, we get (2.4.5). □

Lastly, the following version of the Brezis-Gallouët-Wainger inequality

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le \mathcal{K}\Big(\|u\|_{H^1(\mathbb{R}^2)}\Big)\left(1+\left(\ln\left(1+\|\nabla^2 u\|_{L^2(\mathbb{R}^2)}\right)\right)^{\frac{1}{2}}\right),\ \ \forall u\in C_c^\infty(\mathbb{R}^2), \tag{2.4.6}$$

which is a straightforward adaptation of Lemma 2 in [9], is essential for the following useful result. In fact, we need a consequence of the above estimate.

Lemma 2.4.1. *Let $U\subseteq\mathbb{R}^2$ with $\partial U\in \mathrm{Lip}(\varepsilon,K,L)$ and $m\in\mathbb{N}\smallsetminus\{1\}$. Then*

$$\|u\|_{L^\infty(U)} \le \mathcal{K}\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)}\right)\left(1+\left(\ln\left(1+\sum_{k=2}^m \|\nabla_w^k u\|_{L^2(U)}\right)\right)^{\frac{1}{2}}\right),\ \ \forall u\in H^m(U). \tag{2.4.7}$$

Proof. Let $u\in H^m(U)$ be arbitrary. Since $m\ge 2$, then $u\in L^\infty(U)$. Considering the extended function, (2.4.6) gets the form

$$\|u\|_{L^\infty(U)} \le \mathcal{K}\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)}\right)\left(1+\left(\ln\left(1+\|\nabla_w^2 u\|_{L^2(U)}\right)\right)^{\frac{1}{2}}\right),$$

thereby follows (2.4.7). □

Corollary 2.4.3. *Let $U\subset\mathbb{R}^2$ with $|U|<\infty$ as well as $\partial U\in\mathrm{Lip}(\varepsilon,K,L)$, $m\in\mathbb{N}\smallsetminus\{1\}$, $u\in H^m(U)$ and $\zeta\in X^m(U)$. Then $\left(|u+\zeta|^2(u+\zeta)\right)\in H^m(U)$ also, with*

$$\begin{aligned}\sum_{k=1}^m \left\|\nabla_w^k\left(|u+\zeta|^2(u+\zeta)\right)\right\|_{L^2(U)} &\le \mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\times\\ &\times\left(1+\sum_{k=2}^m\|\nabla_w^k u\|_{L^2(U)}\right)\left(1+\ln\left(1+\left(\sum_{k=2}^m\|\nabla_w^k u\|_{L^2(U)}\right)^2\right)\right).\end{aligned} \tag{2.4.8}$$

If, in addition, $\partial U\in\mathrm{Lip}^m(\varepsilon,K,L)$, $\mathcal{L}_w(a,\theta)$ is such that $a\in W^{m-1,\infty}(U)$, $u\in H^m(U)\cap H_0^1(U)$, as well as

$$\left(L_w^j u\right)\in H_0^1(U),\ \ \forall j=0,\dots,\left\lfloor\frac{m}{2}\right\rfloor-1,$$

then we have

$$\begin{aligned}&\sum_{k=1}^m \left\|\nabla_w^k\left(|u+\zeta|^2(u+\zeta)\right)\right\|_{L^2(U)}\le\\ &\le\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)}\right)\times\\ &\quad\times\left(1+\sum_{\substack{j\in\mathbb{N},\\2j+1\le m}}\left\|\left(\nabla_w\circ L_w^j\right)u\right\|_{L^2(U)}+\sum_{\substack{j\in\mathbb{N},\\2j\le m}}\left\|L_w^j u\right\|_{L^2(U)}\right)\times\\ &\quad\times\left(1+\ln\left(1+\sum_{\substack{j\in\mathbb{N},\\2j+1\le m}}\left\|\left(\nabla_w\circ L_w^j\right)u\right\|_{L^2(U)}^2+\sum_{\substack{j\in\mathbb{N},\\2j\le m}}\left\|L_w^j u\right\|_{L^2(U)}^2\right)\right).\end{aligned} \tag{2.4.9}$$

Proof. We have that $\zeta\in H^m(U)$, since $|U|<\infty$, hence $(u+\zeta)\in H^m(U)$. Employing (2.4.3), we get

$$\sum_{k=1}^m \left\|\nabla_w^k\left(|u+\zeta|^2(u+\zeta)\right)\right\|_{L^2(U)}\le\mathcal{K}(K,L)\left(\sum_{k=0}^m\frac{1}{\varepsilon^{m-k}}\|\nabla_w^k(u+\zeta)\|_{L^2(U)}\right)\|u+\zeta\|_{L^\infty(U)}^2.$$

In order to estimate the term inside the parenthesis, we deal exactly as in Corollary 2.4.2 and we deduce that

$$\begin{aligned}\sum_{k=1}^m \left\|\nabla_w^k\left(|u+\zeta|^2(u+\zeta)\right)\right\|_{L^2(U)}&\le\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\times\\ &\times\left(1+\sum_{k=2}^m\|\nabla_w^k u\|_{L^2(U)}\right)\|u+\zeta\|_{L^\infty(U)}^2.\end{aligned}$$

For the last term, we employ (2.4.7) to get

$$\begin{aligned}\|u+\zeta\|^2_{L^\infty(U)}&\le C\left(\|u\|^2_{L^\infty(U)}+\|\zeta\|^2_{L^\infty(U)}\right)\le\mathcal{K}\Big(\|\zeta\|_{X^m(U)}\Big)\left(1+\|u\|^2_{L^\infty(U)}\right)\le\\ &\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(1+\ln\left(1+\sum_{k=2}^m\left\|\nabla^k_w u\right\|_{L^2(U)}\right)\right)\le\\ &\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L,\|u\|_{H^1(U)},\|\zeta\|_{X^m(U)}\right)\left(1+\ln\left(1+\left(\sum_{k=2}^m\left\|\nabla^k_w u\right\|_{L^2(U)}\right)^2\right)\right).\end{aligned}$$

Now, directly from (2.4.8) and the bound in Proposition 1.2.14, we get (2.4.9). □

We are ready to proceed to the statement and the proof of the main results of this section.

Theorem 2.4.1. *Let $n=1,2$, U be bounded, τ be as in (2.1.2), $u_0\in H^1_0(U)$ and $\mathbf{u}$ be the solution of (2.2.23) that Theorem 2.3.2 provides. If*

1. $\partial U\in\bigcap_{m=1}^\infty \operatorname{Lip}^m(\varepsilon,K,L_m)$,
2. $a\in\bigcap_{m=1}^\infty W^{m-1,\infty}(U)$,
3. $\zeta\in\bigcap_{m=1}^\infty X^m(U)$ *and*
4. $u_0\in\bigcap_{m=2}^\infty H^m(U)\cap H^1_0(U)$, *with* $\left(L^j u_0\right)\in H^1_0(U)$ *for every* $j\in\mathbb{N}_0$,

then $\mathbf{u}\in L^\infty_{\mathrm{loc}}\left(\mathbb{R};\bigcap_{m=2}^\infty H^m(U)\cap H^1_0(U)\right)\cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R};\bigcap_{m=0}^\infty H^m(U))$, *with*

$$\begin{aligned}&\|\mathbf{u}\|_{L^\infty(J_0;H^m(U))}+\|\mathbf{u}'\|_{L^\infty(J_0;H^{m-2}(U))}\le\\ &\le\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac12}\right\},K,L_m,\|u_0\|_{H^m(U)},\|\zeta\|_{X^{m+2}(U)},\||\zeta|-\rho\|_{L^2(U)},\right.\\ &\left.\qquad,\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)},|J_0|\right),\end{aligned}\tag{2.4.10}$$

for every $m\in\mathbb{N}\setminus\{1\}$ and every J_0.

Proof. It suffices to show (2.4.10). Let $m\in\mathbb{N}\setminus\{1\}$ and J_0 be arbitrary and we set

$$\begin{aligned}\widetilde{\mathcal{K}}&:=\mathcal{K}\left(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac12}\right\},K,L_m,\|u_0\|_{H^m(U)},\|\zeta\|_{X^{m+2}(U)},\||\zeta|-\rho\|_{L^2(U)},\right.\\ &\left.\qquad,\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)},|J_0|\right).\end{aligned}$$

Step 1

Let $\{\mathbf{u}_k\}_{k=1}^\infty$ be the Faedo-Galerkin approximations, as in the proof of Theorem 2.3.2. We recall that for every w_l there exists $\lambda_l>0$, such that $\mathcal{L}_w w_l=\lambda_l w_l$ in $H^{-1}(U)$. In virtue of Proposition 1.2.13, $L_w w_l=\lambda_l w_l$ everywhere in U (and not just almost everywhere). Therefore, $L^j_w w_l=\lambda_l^j w_l$ everywhere in U, for every $j\in\mathbb{N}$, that is $L^j_w(\mathbf{u}_k(0))\in\operatorname{span}\{w_l\}_{l=1}^k$, for every $j\in\mathbb{N}_0$, and so

$$\{\mathbf{u}_k\}_{k=1}^\infty\subset C^\infty\left(\mathbb{R},\bigcap_{m=2}^\infty H^m(U)\cap H^1_0(U)\right),$$

as well as

$$\left(L^j(\mathbf{u}_k(0)),\mathbf{u}_k(0)\right)=\left(L^j(\mathbf{u}_k(0)),u_0\right),\ \ \forall j\in\mathbb{N}_0.\tag{2.4.11}$$

Moreover, we have

$$\left(L^i(\mathbf{u}_k(0)),L^j(\mathbf{u}_k(0))\right)=\left(\mathbf{u}_k(0),L^{i+j}(\mathbf{u}_k(0))\right),\ \ \forall i,j\in\mathbb{N}_0.\tag{2.4.12}$$

Indeed, from

1. the common integration by parts formula and
2. (5),

we have, for every $i\in\mathbb{N}$ and every $j\in\mathbb{N}_0$, that

$$\begin{aligned}\int_U L^i(\mathbf{u}_k(0))\,\overline{L^j(\mathbf{u}_k(0))}dx &= \int_U \operatorname{div}\left(a^{\mathrm{T}}\left(\nabla\circ L^{i-1}\right)(\mathbf{u}_k(0))\right)\overline{L^j(\mathbf{u}_k(0))}dx \overset{1.}{=}\\ &\overset{1.}{=} -\int_U a^{\mathrm{T}}\left(\nabla\circ L^{i-1}\right)(\mathbf{u}_k(0))\cdot\overline{\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))}dx =\\ &= -\int_U\left(\nabla\circ L^{i-1}\right)(\mathbf{u}_k(0))\cdot a\overline{\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))}dx \overset{2.}{=}\\ &\overset{2.}{=} -\int_U\left(\nabla\circ L^{i-1}\right)(\mathbf{u}_k(0))\cdot\overline{a^{\mathrm{T}}\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))}dx \overset{1.}{=}\\ &\overset{1.}{=} \int_U L^{i-1}(\mathbf{u}_k(0))\,\overline{L^{j+1}(\mathbf{u}_k(0))}dx\end{aligned}$$

and thus, (2.4.12) follows easily by induction. Now, we claim that

$$\begin{aligned}&\sum_{\substack{j\in\mathbb{N},\\2j+1\le m}}\left\|\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right\|_{L^2(U)}+\sum_{\substack{j\in\mathbb{N},\\2j\le m}}\left\|L^j(\mathbf{u}_k(0))\right\|_{L^2(U)}\le\\ &\le\mathcal{K}\left(\frac{1}{\varepsilon},K,L_m,\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)}\right)\|u_0\|_{H^m(U)}.\end{aligned}$$

In view of Proposition 1.2.14, it suffices to show

$$\begin{aligned}&\sum_{\substack{j\in\mathbb{N},\\2j+1\le m}}\left\|\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right\|_{L^2(U)}+\sum_{\substack{j\in\mathbb{N},\\2j\le m}}\left\|L^j(\mathbf{u}_k(0))\right\|_{L^2(U)}\le\mathcal{K}\left(\frac{1}{\theta},\|a\|_{L^\infty(U)}\right)\times\\ &\times\left(\sum_{\substack{j\in\mathbb{N}_0,\\2j+1\le m}}\left\|\left(\nabla\circ L^j\right)u_0\right\|_{L^2(U)}+\sum_{\substack{j\in\mathbb{N},\\2j\le m}}\left\|L^j u_0\right\|_{L^2(U)}\right).\end{aligned}$$

Indeed, from

1. (2.4.11),
2. (2.4.12),
3. (4) and
4. the common integration by parts formula along with (5),

we obtain, for every $j\in\mathbb{N}$, that

$$\begin{aligned}&\left\|L^j(\mathbf{u}_k(0))\right\|^2_{L^2(U)}=\left(L^j(\mathbf{u}_k(0)),L^j(\mathbf{u}_k(0))\right)\overset{1.}{=}\left(\mathbf{u}_k(0),L^{2j}(\mathbf{u}_k(0))\right)\overset{2.}{=}\\ &\overset{2.}{=}\left(u_0,L^{2j}(\mathbf{u}_k(0))\right)\overset{1.}{=}\left(L^j u_0,L^j(\mathbf{u}_k(0))\right)\le\frac{1}{2}\left\|L^j(\mathbf{u}_k(0))\right\|^2_{L^2(U)}+\frac{1}{2}\left\|L^j u_0\right\|^2_{L^2(U)},\end{aligned}$$

as well as

$$\begin{aligned}&\left\|\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right\|^2_{L^2(U)}\overset{3.}{\le}\frac{1}{\theta^2}L\left[\left(\nabla\circ L^j\right)(\mathbf{u}_k(0)),\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right]\overset{4.}{=}\\ &\overset{4.}{=}-\frac{1}{\theta^2}\left(L^j(\mathbf{u}_k(0)),L^{j+1}(\mathbf{u}_k(0))\right)\overset{1.}{=}-\frac{1}{\theta^2}\left(\mathbf{u}_k(0),L^{2j+1}(\mathbf{u}_k(0))\right)\overset{2.}{=}\\ &\overset{2.}{=}-\frac{1}{\theta^2}\left(u_0,L^{2j+1}(\mathbf{u}_k(0))\right)\underset{4.}{\overset{1.}{=}}\frac{1}{\theta^2}L\left[\left(\nabla\circ L^j\right)u_0,\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right]\le\\ &\le\mathcal{K}\left(\frac{1}{\theta},\|a\|_{L^\infty(U)}\right)\left(\left|\left(\nabla\circ L^j\right)u_0\right|,\left|\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right|\right)\le\\ &\le\frac{1}{2}\left\|\left(\nabla\circ L^j\right)(\mathbf{u}_k(0))\right\|^2_{L^2(U)}+\mathcal{K}\left(\frac{1}{\theta},\|a\|_{L^\infty(U)}\right)\left\|\left(\nabla\circ L^j\right)u_0\right\|^2_{L^2(U)}.\end{aligned}$$

Step 2

We multiply the variational equation (2.3.4) (for $\widetilde{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_\lambda$) by

$$\begin{cases}d_k^l(t)\lambda_l^{2j}, & \text{for every } j\in\mathbb{N} \text{ such that } 2j\le m\\ -d_k^l(t)\lambda_l^{2j+1}, & \text{for every } j\in\mathbb{N} \text{ such that } 2j+1\le m,\end{cases}$$

sum for $l=1,\dots,k$, integrate by parts and take imaginary parts of both sides to find

$$\frac{1}{2}\frac{d}{dt}\left\|L^j\mathbf{u}_k\right\|^2_{L^2(U)} - \operatorname{Im}\left(L^{j+1}\zeta, L^j\mathbf{u}_k\right) - \operatorname{Im}\left(L^j\left(\left(|\mathbf{u}_k+\zeta|^{2\tau}\right)(\mathbf{u}_k+\zeta)\right), L^j\mathbf{u}_k\right) = 0,$$

for every $j\in\mathbb{N}$ with $2j\le m$, and

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}\left[L^j\mathbf{u}_k, L^j\mathbf{u}_k\right] - \operatorname{Im}\int_U\left(\nabla\circ L^{j+1}\right)\zeta\cdot a\overline{\left(\nabla\circ L^j\right)\mathbf{u}_k}dx-$$
$$-\operatorname{Im}\int_U\left(\nabla\circ L^j\right)\left(\left(|\mathbf{u}_k+\zeta|^{2\tau}\right)(\mathbf{u}_k+\zeta)\right)\cdot a\overline{\left(\nabla\circ L^j\right)\mathbf{u}_k}dx = 0,$$

for every $j\in\mathbb{N}$ with $2j+1\le m$. We sum the above equations for every j, integrate with respect to t, employ the Young and the Hölder inequality, as well as (2.4.5) and (2.4.9) along with the estimate for the H^1-norm of each $\mathbf{u}_k$ from the proof of Theorem 2.3.2, to obtain the estimates

$$A\le\widetilde{\mathcal{K}}\left(1+\left|\int_0^t A ds\right|\right),\ \text{for every } t\in J_0,\ \text{if } n=1$$

and also

$$A\le\widetilde{\mathcal{K}}\left(1+\left|\int_0^t A\left(1+\ln\left(1+A\right)\right)ds\right|\right),\ \text{for every } t\in J_0,\ \text{if } n=2,$$

where

$$A := \sum_{\substack{j\in\mathbb{N},\\ 2j+1\le m}}\left\|\left(\nabla\circ L^j\right)\mathbf{u}_k\right\|^2_{L^2(U)} + \sum_{\substack{j\in\mathbb{N},\\ 2j\le m}}\left\|L^j\mathbf{u}_k\right\|^2_{L^2(U)}.$$

Consequently, $A\le\widetilde{\mathcal{K}}$ everywhere in J_0, which, combined with the estimate for the H^1-norm of each $\mathbf{u}_k$ from the proof of Theorem 2.3.2, gives us

$$\|\mathbf{u}_k\|_{L^\infty(J_0;H^m(U))}\le\widetilde{\mathcal{K}},\ \ \forall k\in\mathbb{N},$$

since every $\mathbf{u}_k$ satisfies the necessary compatibility conditions for the validity of Proposition 1.2.14. Now, dealing in an analogous manner as in Step 3β of the proof of Theorem 2.3.1, we deduce that $\mathbf{u}\in L^\infty(J_0;H^m(U))$ with

$$\|\mathbf{u}\|_{L^\infty(J_0;H^m(U))}\le\widetilde{\mathcal{K}}.$$

Moreover, directly from the differential equation, we deduce that $\mathbf{u}'\in L^\infty\left(J_0;H^{m-2}(U)\right)$ with

$$\|\mathbf{u}'\|_{L^\infty(J_0;H^{m-2}(U))}\le\widetilde{\mathcal{K}}.$$

□

Employing the same argument as in the proof of Theorem 2.4.1, after the differentiation of the approximating equations with respect to the temporal variable[8], we can show by induction the following generalization of the aforementioned result, the proof of which is omitted.

Corollary 2.4.4. *Let $n=1,2$, U be bounded, τ be as in (2.1.2), $u_0\in H^1_0(U)$ and $\mathbf{u}$ be the solution of (2.2.23) that Theorem 2.3.2 provides. If*

1. $\partial U\in\bigcap_{m=1}^\infty \operatorname{Lip}^m(\varepsilon,K,L_m)$,
2. $a\in\bigcap_{m=1}^\infty W^{m-1,\infty}(U)$,
3. $\zeta\in\bigcap_{m=1}^\infty X^m(U)$ *and*
4. $u_0\in\bigcap_{m=2}^\infty H^m(U)\cap H^1_0(U)$, *with* $\left(L^j u_0\right)\in H^1_0(U)$ *for every* $j\in\mathbb{N}_0$,

[8] As we have already notice in Step 1α of the proof of Theorem 2.3.1, the Faedo-Galerkin approximations are infinitely smooth with respect to t.

then $\mathbf{u}\in\bigcap_{j=0}^{\infty} W_{\mathrm{loc}}^{j,\infty}(\mathbb{R};\bigcap_{m=2}^{\infty} H^m(U))$, *with*

$$\left\|\mathbf{u}^{(j)}\right\|_{L^\infty(J_0;H^m(U))}\leq\mathcal{K}\Bigg(\frac{1}{\varepsilon^m}\max\left\{1,|U|^{\frac{1}{2}}\right\},K,L_m,\|u_0\|_{H^m(U)},\|\zeta\|_{X^{m+2}(U)},\\ ,\||\zeta|-\rho\|_{L^2(U)},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(U)},|J_0|\Bigg), \tag{2.4.13}$$

for every $j\in\mathbb{N}_0$, *every* $m\in\mathbb{N}\smallsetminus\{1\}$ *and every* J_0.

Now, we show the analogous regularity result for the case where $U=\mathbb{R}^n$ ($n=1,2$ of course).

Theorem 2.4.2. *Let* $n=1,2$, τ *be as in (2.1.2),* $u_0\in H^1(\mathbb{R}^n)$ *and* $\mathbf{u}$ *be the solution of (2.2.23) that Theorem 2.3.3 provides. If*

1. $a\in\bigcap_{m=1}^{\infty} W^{m-1,\infty}(\mathbb{R}^n)$,
2. $\zeta\in\bigcap_{m=1}^{\infty} X^m(\mathbb{R}^n)$ *and*
3. $u_0\in\bigcap_{m=2}^{\infty} H^m(\mathbb{R}^n)$,

then $\mathbf{u}\in\bigcap_{j=0}^{\infty} W_{\mathrm{loc}}^{j,\infty}(\mathbb{R};\bigcap_{m=1}^{\infty} H^m(\mathbb{R}^n))$, *with*

$$\left\|\mathbf{u}^{(j)}\right\|_{L^\infty(J_0;H^m(\mathbb{R}^n))}\leq\mathcal{K}\Bigg(\|u_0\|_{H^m(\mathbb{R}^n)},\|\zeta\|_{X^{m+2}(\mathbb{R}^n)},\||\zeta|-\rho\|_{L^2(\mathbb{R}^n)},\\ ,\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(\mathbb{R}^n)},|J_0|\Bigg), \tag{2.4.14}$$

for every $j\in\mathbb{N}_0$, *every* $m\in\mathbb{N}\smallsetminus\{1\}$ *and every* J_0.

Proof. We set

$$\widetilde{\mathcal{K}}:=\mathcal{K}\Bigg(\|u_0\|_{H^m(\mathbb{R}^n)},\|\zeta\|_{X^{m+2}(\mathbb{R}^n)},\||\zeta|-\rho\|_{L^2(\mathbb{R}^n)},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(\mathbb{R}^n)},|J_0|\Bigg).$$

Let $\{\mathbf{u}_k\}_{k=1}^{\infty}$ be the sequence of solutions, as in the proof of Theorem 2.3.3. Since

$$B_1\in\bigcap_{m=1}^{\infty}\mathrm{Lip}^m(\varepsilon,K,L_m),$$

then, in view of Proposition 1.2.7, we deduce that

$$U_k\equiv B_k\in\bigcap_{m=1}^{\infty}\mathrm{Lip}^m(k\,\varepsilon,K,L_m),\ \ \forall k\in\mathbb{N}.$$

Hence, $\{\mathbf{u}_k\}_{k=1}^{\infty}\subset\bigcap_{k=0}^{\infty} W_{\mathrm{loc}}^{k,\infty}\big(\mathbb{R};\bigcap_{m=1}^{\infty} H^m(U_{k+2})\cap H_0^1(U_{k+2})\big)$, with

$$\left\|\mathbf{u}_k^{(j)}\right\|_{L^\infty(J_0;H^m(U_{k+2}))}\leq\mathcal{K}\Bigg(\frac{1}{((k+2)\,\varepsilon)^m}\max\left\{1,|U_{k+2}|^{\frac{1}{2}}\right\},K,L_m,\|u_0\|_{H^m(\mathbb{R}^n)},\\ ,\|\zeta\|_{X^{m+2}(\mathbb{R}^n)},\||\zeta|-\rho\|_{L^2(\mathbb{R}^n)},\frac{1}{\theta},\|a\|_{W^{m-1,\infty}(\mathbb{R}^n)},|J_0|\Bigg),$$

for every $j\in\mathbb{N}_0$, every $m\in\mathbb{N}\smallsetminus\{1\}$ and every J_0. Since

$$\frac{1}{((k+2)\,\varepsilon)^m}\max\left\{1,|U_{k+2}|^{\frac{1}{2}}\right\}=\frac{|U_{k+2}|^{\frac{1}{2}}}{((k+2)\,\varepsilon)^m}\leq C(k+2)^{\frac{n}{2}-m}\leq C \text{ uniformly for every } k\in\mathbb{N},$$

therefore

$$\left\|\mathbf{u}_k^{(j)}\right\|_{L^\infty(J_0;H^m(U_{k+2}))}\leq\widetilde{\mathcal{K}},$$

for every j, m and J_0 as above, and the same is true for the respective norms of $\mathbf{v}_k^{(j)}$. Now, dealing as in Step 3β of the proof of Theorem 2.3.1, we deduce that $\mathbf{u}^{(j)}\in L^\infty(J_0;H^m(\mathbb{R}^n))$ with

$$\left\|\mathbf{u}^{(j)}\right\|_{L^\infty(J_0;H^m(\mathbb{R}^n))}\leq\widetilde{\mathcal{K}}.$$

□

Remark 2.4.1. *The usual regularity results for unbounded sets appeared in the bibliography (see, e.g., Chapter 10 in [8]) also concern sets with bounded boundaries, such as exterior domains, and not only the whole Euclidean space. Such results can be obtained for the classic version of our problem, i.e. for* $\zeta, \rho \equiv 0$*, by the use of the technique we present here. However, it is not possible to consider* $\varepsilon_k = k\varepsilon \nearrow \infty$ *in Theorem 2.4.2 for the case of bounded (and non-empty) boundary.*

Remark 2.4.2. *We can also deal with the regular problem in the semi-line, simply by considering the odd or the even extension of both* u_0 *and* ζ*, depending on the behaviour of these functions at the boundary.*

Chapter 3

The inviscid limit of the linearly damped and driven NLSE

3.1 Introduction

The goal in this chapter is to show, under certain conditions, that the linearly damped and driven NLSE can be considered as a perturbation of the respective NLSE.

NLSE models with gain and loss effects have found applications to many physical fields such as non-linear optics and fluid mechanics (see [3] and the references therein). The use of damping and forcing effects for the NLSE is not a novelty for physicists (see, e.g., [5] and [39]). On the other hand, some cases of the linearly damped and driven NLSE have already been studied, concerning the solvability and the long time behavior of solutions and their attractors of Cauchy problems (see, e.g., [21], [46], [34], [27], [2], [32], [28] and [29]). Comparisons between the two equations have also been made (see, e.g., [16] about some blow-up issues). Even though these two equations seem quite similar, they exhibit important differences. In particular, many of the symmetries of the NLSE do not hold for the respective linearly damped and driven equation, such as the known scaling symmetry, the Galilean invariance and the time reversal symmetry (see, e.g., [43]). To the author's best knowledge, some questions of "inviscid limit" type for these equations still remain unasked. In [4], the respective linearly damped and driven NLSE arises from a perturbation study of the sine-Gordon equation and in [48] it is shown that the NLSE is the inviscid limit of complex Ginzburg-Landau equation. However, it is natural for us to expect that the linearly damped and driven NLSE could be a perturbation of the NLSE and this viewpoint is the scope here.

In particular, we extract a sufficient relation between f and γ of the form $\|f\|=O(\gamma)$, as $\gamma \searrow 0$ (see (3.4.1)), in order to get two approximation results in Section 3.4. First (see Proposition 3.4.1 and Corollary 3.4.1), we approximate a solution (or the solution in case of uniqueness) v of the problem (7) by a sequence $\{u_m\}_m$ of solutions of the problems (8), as $\gamma_m \searrow 0$, $f_m \to 0$ and $u_{0m} \to v_0$. Second (see Proposition 3.4.2), we estimate the rate of this approximation for $n=1$.

In proving the above results, we first show, in Section 3.3, the existence of a bounded solution of (8), which satisfies a certain estimate (see Theorem 3.3.1 and Theorem 3.3.2). The aforementioned sufficient condition $\|f\|=O(\gamma)$, as $\gamma \searrow 0$, comes naturally from that estimate.

We note that, since our main interest lies in inviscid limit results, we deal with the defocusing and the subcritical focusing case, as well as the critical focusing case with sufficiently small initial datum (see (3.3.4)), where the analysis for the extraction of energy estimates is not that extended in comparison with the supercritical focusing case for sufficiently small initial datum. Hence, we exclude this case, not because of inefficiency of our approach, but in order to keep the work as compact as possible and stay focused on our main result.

3.2 Formulation of the problem

We deal exactly as in the previous chapter in order to define, for every $\lambda \in \mathbb{R}^*$ and every α as in (2.1.1), the operator $g_\lambda : H_0^1(U) \to L^{\frac{\alpha+2}{\alpha+1}}(U) \hookrightarrow H^{-1}(U)$ to be the non-linear and bounded operator such that

$$g_\lambda(u;\alpha) := \lambda |u|^\alpha \bar{u}, \ \forall u \in H_0^1(U),$$

or else

$$\langle g_\lambda(u), v\rangle := \lambda \int_U |u|^\alpha \bar{u}\, v dx,\ \ \forall u, v \in H_0^1(U),$$

with

$$\|g_\lambda(u)-g_\lambda(v)\|_{L^{\frac{\alpha+2}{\alpha+1}}(U)} \le \mathcal{K}\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}\Big)\|u-v\|_{L^{\alpha+2}(U)},\ \ \forall u, v \in H_0^1(U) \tag{3.2.1}$$

and

$$\|g_\lambda(u)-g_\lambda(v)\|_{L^{\frac{\alpha+2}{\alpha+1}}(U)} \le \mathcal{K}\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}\Big)\|u-v\|_{L^2(U)}^{1-\frac{n\alpha}{2(\alpha+2)}},\ \ \forall u, v \in H_0^1(U). \tag{3.2.2}$$

Moreover, for every $\gamma \in [0,\infty)$, we define $\mathcal{N}_{\lambda,\gamma}[*,\star] : H_0^1(U)^2 \to \mathbb{C}$ to be the form which is associated with the operator $\mathcal{L}_w(a,\theta)+g_\lambda+i\gamma\mathrm{id}$, such that $\mathcal{N}_{\lambda,\gamma}[u,v] := \langle\mathcal{L}_w u, v\rangle + \langle g(u), v\rangle + i\gamma\langle u, v\rangle$, for every $u, v \in H_0^1(U)$, satisfying the estimate

$$|\mathcal{N}_{\lambda,\gamma}[u,v]| \le \mathcal{K}\Big(\|u\|_{H^1(U)}, \|v\|_{H^1(U)}\Big),\ \ \forall u, v \in H_0^1(U). \tag{3.2.3}$$

We then restate the problems (7) and (8): for every $v_0, u_0 \in H_0^1(U)$, we seek solutions

$$\mathbf{v}, \mathbf{u} \in L^\infty\big(J_0; H_0^1(U)\big) \cap W^{1,\infty}\big(J_0; H^{-1}(U)\big)$$

of

$$\begin{cases} \langle i\mathbf{v}', w\rangle + \mathcal{N}_{\lambda,0}[\mathbf{v}, w] = 0,\ \ \forall w \in H_0^1(U),\ \text{a.e. in}\ (0,T) \\ \mathbf{v}(0) = v_0, \end{cases} \tag{3.2.4}$$

and of

$$\begin{cases} \langle i\mathbf{u}', w\rangle + \mathcal{N}_{\lambda,\gamma}[\mathbf{u}, w] = \langle \mathbf{f}, w\rangle,\ \ \forall w \in H_0^1(U),\ \text{a.e. in}\ (0,T) \\ \mathbf{u}(0) = u_0, \end{cases} \tag{3.2.5}$$

respectively.

3.3 Weak existence results

Before we proceed with the main results of this section, we need some preliminary lemmata.

Lemma 3.3.1. *Let $\alpha \in \big(0, \frac{4}{n}\big)$, $\epsilon > 0$ and $u \in H_0^1(U)$. Then*

$$\|u\|_{L^{\alpha+2}(U)}^{\alpha+2} \le \epsilon\|\nabla_w u\|_{L^2(U)}^2 + C\|u\|_{L^2(U)}^{2-\frac{4\alpha}{4-n\alpha}}. \tag{3.3.1}$$

Proof. Direct application of the Young inequality with constant $\epsilon > 0$ for $p = \frac{4}{n\alpha}$ and $q = \frac{4}{4-n\alpha}$ into (2.2.4). □

Below follows a straightforward adaptation of a well-known result from [47] (see also Definition 8.1.13, as well as Theorems 8.1.4, 8.1.5 and 8.1.6 in [10], and Chapter B, Appendix in [43]).

Lemma 3.3.2. *Let $\alpha = \frac{4}{n}$ and $R \in H^1(\mathbb{R}^n)$ be the spherically symmetric, positive ground state of the elliptic equation $-\Delta R + R = |R|^\alpha R$, in $H^{-1}(\mathbb{R}^n)$. Then, the best constant C in*

$$\|u\|_{L^{\alpha+2}(U)}^{\alpha+2} \le C\|\nabla_w u\|_{L^2(U)}^2\|u\|_{L^2(U)}^\alpha,\ \ \text{for every}\ u \in H_0^1(U),\ \text{for any open}\ U \subseteq \mathbb{R}^n \tag{3.3.2}$$

is

$$C = C_{cr} := \frac{\alpha+2}{2\|R\|_{L^2(\mathbb{R}^n)}^\alpha}. \tag{3.3.3}$$

Theorem 3.3.1. *If* U *is bounded,* $T,\theta\in(0,\infty)$*,* $\mathbf{f}\in W^{1,\infty}\big((0,T);L^2(U)\big)$*,* $u_0\in H_0^1(U)$ *and also*

$$\begin{aligned} &i.\ \lambda>0,\ \text{or}\\ &ii.\ \lambda<0 \text{ and } \alpha\in\left(0,\frac{4}{n}\right),\ \text{or}\\ &iii.\ \lambda<0,\ \alpha=\frac{4}{n} \text{ and } \max\left\{\|u_0\|_{L^2(U)},C\right\}<\left(-\frac{\theta}{\lambda}\right)^{\frac{1}{\alpha}}\|R\|_{L^2(\mathbb{R}^n)},\\ &\text{where } C=\begin{cases}0, & \text{if we consider the problem (3.2.4)}\\ \frac{1}{\gamma}\|\mathbf{f}\|_{L^\infty((0,T);L^2(U))}, & \text{if we consider the problem (3.2.5)},\end{cases}\end{aligned} \tag{3.3.4}$$

where R *is as in Theorem 3.3.2, then there exist solutions of (3.2.4) and (3.2.5), such that*

$$\|\mathbf{v}\|_{L^\infty((0,T);H^1(U))}+\|\mathbf{v}'\|_{L^\infty((0,T);H^{-1}(U))}\le\mathcal{K}\left(\|u_0\|_{H^1(U)},\frac{1}{\theta},\|a\|_{L^\infty(U)}\right) \tag{3.3.5}$$

and

$$\begin{aligned}&\|\mathbf{u}\|_{L^\infty((0,T);H^1(U))}+\|\mathbf{u}'\|_{L^\infty((0,T);H^{-1}(U))}\le\\ &\le\mathcal{K}\left(\|u_0\|_{H^1(U)},\max\left\{1,\frac{1}{\gamma}\right\}\|\mathbf{f}\|_{W^{1,\infty}((0,T);L^2(U))},\frac{1}{\theta},\|a\|_{L^\infty(U)}\right),\end{aligned} \tag{3.3.6}$$

respectively.

Proof. We only show the result for the problem (3.2.5), since the respective result for the simpler problem (3.2.4) follows analogously. We set

$$\widetilde{\mathcal{K}}:=\mathcal{K}\left(\|u_0\|_{H^1(U)},\max\left\{1,\frac{1}{\gamma}\right\}\|\mathbf{f}\|_{W^{1,\infty}((0,T);L^2(U))},\frac{1}{\theta},\|a\|_{L^\infty(U)}\right).$$

As in Theorem 2.3.1, we make use of the standard Faedo-Galerkin method, by considering the complete set of eigenfunctions for the operator $\mathcal{L}_w$ restricted to $H_0^1(U;\mathbb{R})$, which we denote as $\{w_k\}_{k=1}^\infty$. This set is an orthogonal basis of both $H_0^1(U;\mathbb{C})$ and $L^2(U;\mathbb{C})$. We also assume that $\{w_k\}_{k=1}^\infty$ is appropriately normalized so that it is an orthonormal basis of $L^2(U;\mathbb{C})$.

Step 1

For every $m\in\mathbb{N}$ we define $d_m\in C^1((0,T_0);\mathbb{C}^m)$, with $T_0\le T$ and $d_m(t):=\big(d_m^k(t)\big)_{k=1}^m$, to be the unique maximal solution of the initial-value problem

$$\begin{cases} d_m{}'(t)=F_m(t,d_m(t)),\ \forall t\in(0,T_0)\\ d_m(0)=((w_k,u_0))_{k=1}^m\ (=(\langle u_0,w_k\rangle)_{k=1}^m, \text{ in view of Lemma 2.3.1}),\end{cases}$$

where $F_m\in C\Big([0,T]^{2m+1};\mathbb{C}^m\Big)$ (we note that $W^{1,\infty}\big((0,T);L^2(U)\big)\hookrightarrow C\big([0,T];L^2(U)\big)$) with

$$F_m^k(t,d_m(t)):=i\mathcal{N}_{\lambda,\gamma}[\sum_{l=1}^m d_m^l(t)w_l,w_k]-i\,(w_k,\mathbf{f}(t)),\ \forall k=1,\dots,m.$$

Now, we define $\mathbf{u}_m\in C^1\big((0,T_0);H_0^1(U;\mathbb{C})\big)$, with

$$\mathbf{u}_m(t):=\sum_{k=1}^m\overline{d_m^k}(t)w_k.$$

In view of Lemma 2.3.1, it is direct to verify that

$$\langle i\mathbf{u}_m{}',w_k\rangle+\mathcal{N}_{\lambda,\gamma}[\mathbf{u}_m,w_k]=\langle\mathbf{f},w_k\rangle \text{ everywhere in } (0,T_0)\,, \text{ for every } k=1,\dots,m. \tag{3.3.7}$$

We can also deal as in Step 1β of the proof of Theorem 2.3.1 to deduce that

$$\|\mathbf{u}_m(0)\|_{L^2(U)}\le\|u_0\|_{L^2(U)} \text{ and } \|\nabla_w\mathbf{u}_m(0)\|_{L^2(U)}\le\mathcal{K}\left(\frac{1}{\theta},\|a\|_{L^\infty(U)}\right)\|\nabla_w u_0\|_{L^2(U)}.$$

as well as

$$\mathbf{u}_m(0)\to u_0 \text{ in } L^2(U).$$

Step 2

We multiply the variational equation (3.3.7) by $d_m^k(t)$, sum for $k=1,\dots,m$ and take imaginary parts of both sides to find

$$\frac{d}{dt}\|\mathbf{u}_m\|_{L^2(U)}^2+2\gamma\|\mathbf{u}_m\|_{L^2(U)}^2\le 2\left|(\mathbf{f},\mathbf{u}_m)\right|,$$

hence, from the Young inequality for $\epsilon=\frac{\gamma}{2}$ $(p=q=2)$, we obtain

$$\frac{d}{dt}\|\mathbf{u}_m\|_{L^2(U)}^2+\gamma\|\mathbf{u}_m\|_{L^2(U)}^2\le\frac{1}{\gamma}\|\mathbf{f}\|_{W^{1,\infty}((0,T);L^2(U))}^2,$$

which implies that $T_0=T$ for every $m\in\mathbb{N}$, as well as the estimate

$$\|\mathbf{u}_m\|_{L^\infty((0,T),L^2(U))}\le\max\left\{\|u_0\|_{L^2(U)},\frac{1}{\gamma}\|\mathbf{f}\|_{W^{1,\infty}((0,T);L^2(U))}\right\},\ \forall m\in\mathbb{N}. \tag{3.3.8}$$

Step 3

We multiply the variational equation (3.3.7) by ${d_m^k}'(t)+\gamma d_m^k(t)$, sum for $k=1,\dots,m$ and take real parts of both sides to find

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\gamma\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\frac{\gamma}{2}\mathcal{L}[\mathbf{u}_m,\mathbf{u}_m]+\frac{\gamma\lambda\,(\alpha+1)}{\alpha+2}\|\mathbf{u}_m\|_{L^{\alpha+2}(U)}^{\alpha+2}=-\left(\mathbf{f}',\mathbf{u}_m\right), \tag{3.3.9}$$

where

$$\mathcal{J}[v,g]:=\frac{1}{2}\mathcal{L}[\mathbf{u}_m,\mathbf{u}_m]+\frac{\lambda}{\alpha+2}\|v\|_{L^{\alpha+2}(U)}^{\alpha+2}-\operatorname{Re}\left(g,v\right),\ \forall v\in H_0^1(U),\ \forall g\in L^2(U).$$

In view of the estimate in Step 1, along with the scaling invariant embedding $H_0^1(U)\hookrightarrow L^{\alpha+2}(U)$, we have that

$$\mathcal{J}[\mathbf{u}_m(0),\mathbf{f}(0)]\le\mathcal{K}\left(\|u_0\|_{H^1(U)},\|\mathbf{f}\|_{L^\infty((0,T);L^2(U))},\frac{1}{\theta},\|a\|_{L^\infty(U)}\right).$$

In order to show

$$\|\nabla_w\mathbf{u}_m\|_{L^2(U)}\le\widetilde{\mathcal{K}},\ \text{uniformly for every } m\in\mathbb{N}, \tag{3.3.10}$$

we consider the three cases of (3.3.4).

i. Since

$$\frac{\gamma}{2}\mathcal{L}[\mathbf{u}_m,\mathbf{u}_m]+\frac{\gamma\lambda\,(\alpha+1)}{\alpha+2}\|\mathbf{u}_m\|_{L^{\alpha+2}(U)}^{\alpha+2}\ge 0,$$

from the Hölder inequality $(p=q=2)$ and (3.3.8) we get

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\gamma\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\le\|\mathbf{u}_m\|_{L^2(U)}\|\mathbf{f}'\|_{L^2(U)}\le\widetilde{\mathcal{K}}\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))},$$

which implies

$$\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\le\max\left\{\mathcal{J}[\mathbf{u}_m(0),\mathbf{f}(0)],\widetilde{\mathcal{K}}\frac{1}{\gamma}\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right\}.$$

Hence

$$\frac{1}{2}\mathcal{L}[\mathbf{u}_m,\mathbf{u}_m]\le$$
$$\le\widetilde{\mathcal{K}}\|\mathbf{f}\|_{L^\infty((0,T);L^2(U))}+\max\left\{\mathcal{J}[\mathbf{u}_m(0),\mathbf{f}(0)],\widetilde{\mathcal{K}}\frac{1}{\gamma}\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right\},$$

thereby we get (3.3.10).

ii. Employing (3.3.1) for

$$\epsilon = -\frac{\alpha+2}{2\lambda(\alpha+1)}$$

to estimate the last term on the left-hand side of (3.3.9), we have

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\gamma\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\leq\widetilde{\mathcal{K}}\left(\gamma+\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right),$$

which implies

$$\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\leq\max\left\{\mathcal{J}[\mathbf{u}_m(0),\mathbf{f}],\widetilde{\mathcal{K}}\left(1+\frac{1}{\gamma}\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right)\right\}.$$

Therefore, applying again (3.3.1) for

$$\epsilon = \frac{\delta(\alpha+2)}{\lambda}, \text{ for some } \delta\in\left(0,\frac{1}{2}\right),$$

we get

$$\frac{1}{2}\mathcal{L}[\mathbf{u}_m,\mathbf{u}_m] \leq \widetilde{\mathcal{K}}\left(1+\|\mathbf{f}\|_{L^\infty((0,T);L^2(U))}\right)+$$
$$+\max\left\{\mathcal{J}[\mathbf{u}_m(0),\mathbf{f}],\widetilde{\mathcal{K}}\left(1+\frac{1}{\gamma}\widetilde{\mathcal{K}}\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right)\right\},$$

hence (3.3.10) follows.

iii. Employing (3.3.2) for $C=C_{cr}$ as in (3.3.3) to estimate the last term on the left-hand side of (3.3.9), as well as (3.3.8), we obtain

$$\frac{d}{dt}\mathcal{J}[\mathbf{u}_m,\mathbf{f}]+\gamma\mathcal{J}[\mathbf{u}_m,\mathbf{f}]\leq\widetilde{\mathcal{K}}\left(\gamma+\|\mathbf{f}'\|_{L^\infty((0,T);L^2(U))}\right),$$

since

$$\frac{1}{2}+\frac{\lambda C_{cr}}{\theta(\alpha+2)}\left(\max\left\{|u_0|_{0,2,U},\frac{1}{\gamma}\|\mathbf{f}\|_{L^\infty((0,T);L^2(U))}\right\}\right)^\alpha>0.$$

(3.3.10) then follows.

From (3.3.8) and (3.3.10) we conclude that

$$\|\mathbf{u}_m\|_{L^\infty((0,T);H^1(U))}\leq\widetilde{\mathcal{K}}, \text{ uniformly for every } m\in\mathbb{N}. \tag{3.3.11}$$

Step 4

We can deal as in Step 2β of the proof of Theorem 2.3.1, minding to employ (3.2.3) instead, we derive that

$$\|\mathbf{u}_m{}'\|_{L^\infty((0,T);H^{-1}(U))}\leq\widetilde{\mathcal{K}}, \text{ uniformly for every } m\in\mathbb{N}. \tag{3.3.12}$$

The rest of the proof follows from (3.3.11) and (3.3.12), in analogous manner as in the aforementioned proof.

□

Since the estimates (3.3.5) and (3.3.6) are independent of U, we can deal as in the proof of Theorem 2.3.3 to show the following result, the proof of which is omitted.

Theorem 3.3.2. *Theorem 3.3.1 is also valid for every unbounded U.*

3.4 NLSE as limit case of linearly damped and driven NLSE

Here, we consider $\{u_{0_m}\}_m \cup \{v_0\} \subset H_0^1(U)$, $\{\mathbf{f}_m\}_m \subset W^{1,\infty}\big((0,T)\,;L^2(U)\big)$ and $\{\gamma_m\}_m \subset (0,\infty)$, such that

$$\begin{gathered} \gamma_m \searrow 0, \\ \mathbf{f}_m \to 0 \text{ in } W^{1,\infty}\big((0,T)\,;L^2(U)\big), \text{ with } \|\mathbf{f}_m\|_{W^{1,\infty}((0,T);L^2(U))} = O(\gamma_m) \text{ and} \\ u_{0_m} \to v_0 \text{ in } H_0^1(U), \end{gathered} \tag{3.4.1}$$

as $m \nearrow \infty$.

Proposition 3.4.1. *For every v_0 and $\{(u_{0_m},\mathbf{f}_m,\gamma_m)\}_m$ as in (3.4.1), as well as every corresponding sequence $\{\mathbf{u}_m\}_m$ of solutions of (3.2.5), which Theorem 3.3.1 or 3.3.2 provides, there exist a subsequence $\{\mathbf{u}_{m_l}\}_l \subseteq \{\mathbf{u}_m\}_m$ and a solution $\mathbf{v}$ of (3.2.4), such that*

$$\begin{gathered} \mathbf{u}_{m_l} \rightharpoonup \mathbf{v} \text{ in } H_0^1(U), \text{ everywhere in } [0,T], \\ \mathbf{u}_{m_l}' \overset{*}{\rightharpoonup} \mathbf{v}' \text{ in } L^\infty\big((0,T)\,;H^{-1}(U)\big). \end{gathered}$$

Proof. In view of the former proofs, it is sufficient to show that

$$\Big\{\|\mathbf{u}_m\|_{L^\infty((0,T);H^1(U))} + \|\mathbf{u}_m'\|_{L^\infty((0,T);H^{-1}(U))}\Big\}_m$$

is uniformly bounded. Indeed, it is direct from the combination of (3.3.6) with (3.4.1), that

$$\|\mathbf{u}_m\|_{L^\infty((0,T);H^1(U))} + \|\mathbf{u}_m'\|_{L^\infty((0,T);H^{-1}(U))} \le \mathcal{K}\bigg(\|v_0\|_{H^1(U)}, \frac{1}{\theta}, \|a\|_{L^\infty(U)}\bigg),$$

uniformly for every m. □

Before we proceed to the next result, we note that it is easy to check that Proposition 2.3.1 also holds for the solutions of (3.2.4) and (3.2.5).

Corollary 3.4.1. *If the solutions of (3.2.4) and (3.2.5) are unique, then, for every v_0 and $\{(u_{0_m},\mathbf{f}_m,\gamma_m)\}_m$ as in (3.4.1), the corresponding sequence $\{\mathbf{u}_m\}_m$ of solutions of (3.2.5) converges to the corresponding solution $\mathbf{v}$ of (3.2.4), in the sense that*

$$\begin{gathered} \mathbf{u}_m \rightharpoonup \mathbf{v} \text{ in } H_0^1(U), \text{ everywhere in } [0,T], \\ \mathbf{u}_m' \overset{*}{\rightharpoonup} \mathbf{v}' \text{ in } L^\infty\big((0,T)\,;H^{-1}(U)\big). \end{gathered}$$

Proof. From Proposition 3.4.1 and uniqueness, we have, for every such v_0 and $\{(u_{0_m},\mathbf{f}_m,\gamma_m)\}_m$, that there exists a subsequence $\{\mathbf{u}_{m_l}\}_l \subseteq \{\mathbf{u}_m\}_m$ such that

$$\begin{gathered} \mathbf{u}_{m_l} \rightharpoonup \mathbf{v} \text{ in } H_0^1(U), \text{ everywhere in } [0,T], \\ \mathbf{u}_{m_l}' \overset{*}{\rightharpoonup} \mathbf{v}' \text{ in } L^\infty\big((0,T)\,;H^{-1}(U)\big). \end{gathered} \tag{3.4.2}$$

Now, seeking a contradiction, we assume that a sequence $\{\mathbf{u}_m\}_m$ does not converge to $\mathbf{v}$ in the above first sense, i.e. there exists $t_0 \in [0,T]$ such that

$$\mathbf{u}_m(t_0) \not\rightharpoonup \mathbf{v}(t_0) \text{ in } H_0^1(U).$$

Then there exist $\epsilon > 0$, $w \in H_0^1(U)$ and a subsequence of $\{\mathbf{u}_m\}_m$, which we still denote as such, for which we have

$$\Big|(\mathbf{u}_m(t_0),w)_{H^1(U)} - (\mathbf{v}(t_0),w)_{H^1(U)}\Big| \ge \epsilon, \ \forall m,$$

which is a contradiction to the first convergence of (3.4.2). The proof of the second convergence is similar. □

In fact, if $n=1$, then $\mathbf{u}_m \to \mathbf{v}$ in $C\big([0,T]\,;L^2(U)\big)$ also, as we show below.

Proposition 3.4.2. *If $n=1$, then for every convergent sequence $\{\mathbf{u}_m\}_m$ of solutions of (3.2.5) to a solution $\mathbf{v}$ of (3.2.4), as in Proposition 3.4.1 or Corollary 3.4.1, there exist*

$$C_1 = C_1\Big(\|v_0\|_{H^1(U)}\Big) \text{ and } C_2 = C_2\Big(\|v_0\|_{H^1(U)}, \|\mathbf{f}_m\|_{W^{1,\infty}((0,T);L^2(U))}, \gamma_m\Big) \text{ with}$$
$$C_2 = O\big(\gamma_m^2\big) \text{ as } m \nearrow \infty,$$

such that

$$\|\mathbf{u}_m - \mathbf{v}\|^2_{L^2(U)} \le \|u_{0_m} - v_0\|^2_{L^2(U)} e^{C_1 t} + C_2\left(e^{C_1 t} - 1\right), \;\; \forall t \in [0,T], \;\; \forall m. \tag{3.4.3}$$

In particular, if

$$\|u_{0m} - v_0\|_{L^2(U)} = O(\gamma_m) \text{ as } m \nearrow \infty,$$

then

$$\|\mathbf{u}_m - \mathbf{v}\|_{L^\infty((0,T);L^2(U))} = O(\gamma_m) \text{ as } m \nearrow \infty.$$

Proof. Let m be arbitrary and set $\mathbf{w}_m \coloneqq \mathbf{u}_m - \mathbf{v}$. Then

$$i\mathbf{w}_m{}' + \mathcal{L}_w \mathbf{w}_m + g_\lambda(\mathbf{u}_m) - g_\lambda(\mathbf{v}) + i\gamma_m \mathbf{u}_m \overset{H^{-1}(U)}{=} \mathbf{f}_m, \text{ a.e. in } (0,T).$$

Applying (2.2.3) and dealing as usual we get

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}_m\|^2_{L^2(U)} \le C \int_U |\mathbf{w}_m|^2 \left(|\mathbf{u}_m|^\alpha + |\mathbf{v}|^\alpha\right) dx + C\|\mathbf{w}_m\|^2_{L^2(U)} + C\gamma_m^2 \|\mathbf{u}_m\|^2_{L^2(U)} + \\ + C\|\mathbf{f}_m\|^2_{W^{1,\infty}((0,T);L^2(U))},\end{aligned} \tag{3.4.4}$$

a.e. in $(0,T)$. From (3.4.4) and the embedding $H_0^1(U) \hookrightarrow L^\infty(U)$ we obtain (3.4.3) with

$$C_1 = \mathcal{K}\Big(\|v_0\|_{H^1(U)}\Big) \text{ and } C_2 = \frac{1}{C_1}\left(\mathcal{K}\Big(\|v_0\|_{H^1(U)}\Big)\gamma_m^2 + \|\mathbf{f}_m\|^2_{W^{1,\infty}((0,T);L^2(U))}\right).$$

□

Bibliography

[1] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces, 2nd edition*, volume 140 of *Pure and Applied Mathematics*. Academic Press, Oxford, UK, 2003.

[2] Naïma Akroune. Regularity of the attractor for a weakly damped nonlinear Schrödinger equation on $\mathbb{R}$. *Applied Mathematics Letters*, 12(3):45–48, 1999.

[3] Zacharias A. Anastassi, Georgios Fotopoulos, Dimitri J. Frantzeskakis, Theodoros P. Horikis, Nikolaos I. Karachalios, Panayotis G. Kevrekidis, Ioannis G. Stratis, and Konstantinos Vetas. Spatiotemporal algebraically localized waveforms for a nonlinear Schrödinger model with gain and loss. *Physica D: Nonlinear Phenomena*, 355:24–33, 2017.

[4] Alan R. Bishop, Randy Flesch, Gregory M. Forest, David W. McLaughlin, and Edward A Overman II. Correlations between chaos in a perturbed sine-Gordon equation and a truncated model system. *SIAM Journal on Mathematical Analysis*, 21(6):1511–1536, 1990.

[5] Keith J. Blow and Nick J. Doran. Global and local chaos in the pumped nonlinear Schrödinger equation. *Physical Review Letters*, 52(7):526–529, 1984.

[6] Jean Bourgain. *Global Solutions of Nonlinear Schrödinger Equations*, volume 46 of *Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, USA, 1999.

[7] Franck Boyer and Pierre Fabrie. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, New York, USA, 2013.

[8] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, New York, USA, 2011.

[9] Haim Brezis and Thierry Gallouët. Nonlinear Schrödinger evolution equations. *Nonlinear Analysis: Theory, Methods & Applications*, 4(4):677–681, 1980.

[10] Thierry Cazenave. *Semilinear Schrödinger Equations*, volume 10 of *Lecture Notes*. American Mathematical Society, Providence, Rhode Island, USA, 2003.

[11] Thierry Cazenave and Alain Haraux. *An Introduction to Semilinear Evolution Equations, revised edition*, volume 13 of *Oxford Lectrure Series in Mathematics and its Applications*. Clarendon Press, Oxford, UK, 1998.

[12] Gregory M. Constantine and Thomas H. Savits. A multivariate Faà di Bruno formula with applications. *Transactions of the American Mathematical Society*, 348(2):503–520, 1996.

[13] Joseph Diestel and Jerry J. Uhl jr. *Vector Measures*, volume 15 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, Rhode Island, USA, 1977.

[14] Burak M. Erdoğan and Nikolaos Tzirakis. *Dispersive Partial Differential Equations*, volume 86 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, UK, 2016.

[15] Lawrence C. Evans. *Partial Differential Equations, 2nd edition*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, USA, 2010.

[16] Gadi Fibich. Self-focusing in the damped nonlinear Schrödinger equation. *SIAM Journal on Applied Mathematics*, 61(5):1680–1705, 2001.

[17] Gadi Fibich. *The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse*, volume 192 of *Applied Mathematical Sciences*. Springer, Cham, Switzerland, 2015.

[18] Clément Gallo. Schrödinger group on Zhidkov spaces. *Advances in Differential Equations*, 9(5-6):509–538, 2004.

[19] Clément Gallo. The Cauchy problem for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity. *Communications in Partial Differential Equations*, 33(5):729–771, 2008.

[20] Patrick Gérard. The Cauchy problem for the Gross-Pitaevskii equation. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, 23(5):765–779, 2006.

[21] Jean-Michel Ghidaglia. Finite dimensional behavior for weakly damped driven Schrödinger equations. *Annales de l'IHP Analyse Non Linéaire*, 5(4):365–405, 1988.

[22] Nikolaos Gialelis. The inviscid limit of the linearly damped and driven nonlinear Schrödinger equation. *Submitted*.

[23] Nikolaos Gialelis, Nikolaos I. Karachalios, and Ioannis G. Stratis. Regularity of non-vanishing-at-infinity solutions of the defocusing NLS. *In preparation*.

[24] Nikolaos Gialelis and Ioannis G. Stratis. Nonvanishing at spatial extremity solutions of the defocusing nonlinear Schrödinger equation. *Mathematical Methods in the Applied Sciences*, 2018, In press.

[25] Nikolaos Gialelis and Ioannis G. Stratis. On the 1-dim defocusing NLS equation with non-vanishing initial data at infinity. *Modern Methods, Problems and Applications of Operator Theory and Harmonic Analysis, Alexey Karapetyants, Vladislav Kravchenko and Elijah Liflyand (eds), Springer Proceedings in Mathematics & Statistics*, 2019, In press.

[26] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag Berlin Heidelberg, Germany, 2001.

[27] Olivier Goubet. Regularity of the attractor for a weakly damped nonlinear Schrödinger equation. *Applicable Analysis*, 60(1-2):99–119, 1996.

[28] Olivier Goubet and Luc Molinet. Global attractor for weakly damped nonlinear Schrödinger equations in $L^2(\mathbb{R})$. *Nonlinear Analysis: Theory, Methods & Applications*, 71(1-2):317–320, 2009.

[29] Olivier Goubet and Ezzeddine Zahrouni. Finite dimensional global attractor for a fractional nonlinear Schrödinger equation. *NoDEA: Nonlinear Differential Equations and Applications*, 24(5):59, 2017.

[30] Pierre Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 69 of *Classics in Applied Mathematics*. SIAM, Philadelphia, USA, 2011.

[31] Luis Hernández Encinas and Jaime Muñoz Masqué. A short proof of the generalized Faà di Bruno's formula. *Applied Mathematics Letters*, 16(6):975–979, 2003.

[32] Nikolaos I. Karachalios and Nikolaos M. Stavrakakis. Global attractor for the weakly damped driven Schrödinger equation in $H^2(\mathbb{R})$. *NoDEA: Nonlinear Differential Equations and Applications*, 9(3):347–360, 2002.

[33] Panayotis G. Kevrekidis, Dimitri J. Frantzeskakis, and Ricardo Carretero González. *The Defocusing Nonlinear Schrödinger Equation*. SIAM, Philadelphia, Pennsylvania, USA, 2015.

[34] Philippe Laurençot. Long-time behaviour for weakly damped driven nonlinear Schrödinger equations in $\mathbb{R}^N$, $N \leq 3$. *NoDEA: Nonlinear Differential Equations and Applications*, 2(3):357–369, 1995.

[35] Giovanni Leoni. *A First Course in Sobolev Spaces, 2nd edition*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, USA, 2017.

[36] Felipe Linares and Gustavo Ponce. *Introduction to Nonlinear Dispersive Equations, 2nd edition.* Springer, New York, New York, USA, 2015.

[37] Vladimir Maz'ya. *Sobolev Spaces with Applications to Elliptic Partial Differential Equations, 2nd revised & augmented edition*, volume 342 of *Die Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag Berlin Heidelberg, Germany, 2011.

[38] William McLean. *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, New York, New York, USA, 2000.

[39] Kazuhiro Nozaki and Naoaki Bekki. Low-dimensional chaos in a driven damped nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena*, 21(2):381–393, 1986.

[40] Takayoshi Ogawa. A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations. *Nonlinear Analysis: Theory, Methods & Applications*, 14(9):765–769, 1990.

[41] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*, volume 30 of *Princeton Mathematical Series.* Princeton University Press, Princeton, New Jersey, USA, 1970.

[42] Catherine Sulem and Pierre-Louis Sulem. *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, volume 139 of *Applied Mathematical Sciences.* Springer, New York, New York, USA, 1999.

[43] Terence Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*, volume 106 of *Regional Conference Series in Mathematics.* American Mathematical Society, Providence, Rhode Island, USA, 2006.

[44] Roger Temam. *Navier-Stokes Equations, revised edition*, volume 2 of *Studies in Mathematics and its Applications.* North-Holland, New York, New York, USA, 1979.

[45] Toshio Tsuzuki. Nonlinear waves in the Pitaevskii-Gross equation. *Journal of Low Temperature Physics*, 4(4):441–457, 1971.

[46] Xiaoming Wang. An energy equation for the weakly damped driven nonlinear Schrödinger equations and its application to their attractors. *Physica D: Nonlinear Phenomena*, 88(3):167–175, 1995.

[47] Michael I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Communications in Mathematical Physics*, 87(4):567–576, 1983.

[48] Jiahong Wu. The inviscid limit of the complex Ginzburg-Landau equation. *Journal of Differential Equations*, 142(2):413–433, 1998.

[49] Vladimir E. Zakharov and Aleksei B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34(1):62–69, 1972.

[50] Vladimir E. Zakharov and Aleksei B. Shabat. Interaction between solitons in a stable medium. *Soviet Physics JETP*, 37(5):823–828, 1973.

[51] Peter E. Zhidkov. The Cauchy problem for a nonlinear Schrödinger equation (in Russian). *JINR Communications, Dubna*, R5-87-373, 1987.

[52] Peter E. Zhidkov. *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory*, volume 1756 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, Heidelberg, Germany, 2001.