The Role of Dispersion in the Non-Linear Wave-Particle Interaction in Magnetized Plasma

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Abstract

The present thesis investigates the interaction of a charged particle with a finite amplitude, dispersing electrostatic wave-packet in a uniformly magnetized plasma. We begin with the introduction of Hamiltonian perturbation theory which consists our main tool throughout this thesis in order to examine the various phenomena when a wave-particle interaction takes place. As a result, we continue with the application of Hamiltonian perturbation theory to various examples of wave-particle interactions in order for the reader to familiarize with the Hamiltonian formulation. The setup of our case is presented thoroughly in the fourth chapter where the large amplitude wave-packet decomposes to an ensemble of different amplitude electrostatic waves. From this point on, using both Hamiltonian and kinetic descriptions we are able to examine the interaction especially for the trapped particles regime which has never been treated analytically in the presence of a magnetic field. Finally we try to connect the perturbation theory formalism with the kinetic equations that describe the collective transport phenomena. Our formulation as well as our results can be applied to many other physical problems which fall into the same category, such as electrostatic turbulence and electromagnetic wave-particle interaction.

Extended Abstract in Greek

Η παρούσα διπλωματική εργασία εξετάζει την αλληλεπίδραση ενός φορτισμένου σωματιδίου με μεγάλου πλάτους ηλεκτροστατικό κυματοπακέτο σε ομογενες μαγνητισμένο πλάσμα. Με σκόπο η συγκεκριμένη εργασία να είναι αυτοσυνεπής, στο Πρώτο Κεφάλαιο παρουσιάζεται αναλυτικά η Κανονική Θεωρία Διαταραχών η οποία θα αποτελέσει βασικό εργαλείο στην Χαμιλτονιανή αντιμετώπιση των προβλημάτων που σχετίζονται με την αλληλεπίδραση σωματιδίων με κύματα σε γενικότερο πλαίσιο. Με την λογική αυτή, στο Δεύτερο Κεφάλαιο εφαρμόζουμε τις τεχνικές της θεωρίας διαταραχών σε διαφορετικά είδη αλληλεπίδρασης κυμάτων και σωματιδιών. Ο βασικός μας στόχος σε αυτό το στάδιο είναι να αναπαράγουμε και να επεκτείνουμε βασικές ερευνητικές εργασίες που έχουν πραγματοποιηθεί κατά το παρελθόν. Μέσα στα δύο πρώτα αυτά κεφάλαια ο αναγνώστης εισάγεται στον Χαμιλτονιανό φορμαλισμό και έρχεται σε επαφή με την μεθοδολογία αντιμετώπισης κλασσικών περιτπώσεων αλληλεπίδρασης σωματιδίου με κύμα. Με αυτόν τον τρόπο θέλουμε να δώσουμε στον αναγώστη την ικανότητα να διακρίνει και να κατανοήσει τις διαφορές και τις επεκτάσεις της δικής μας θεώρησης σε σχέση με τις ήδη υπάρχοντες. Στο Τρίτο Κεφάλαιο λοιπόν παρουσιάζεται διεξοδικά η θεώρηση και ο φορμαλισμός που ακολουθείται για την Χαμιλτονιανή κατασκευή της αλληλεπίδρασης του μεγάλου πλάτους κυματοπακέτου με το σωματίδιο υπό την παρουσία του μαγνητικού πεδίου. Η θεώρηση μας αποσκοπεί στην μελέτη της επίδραση της διασποράς του κυματοπακέτου στην αλληλεπίδραση του σωματιδίου με μια συλλογής κυμάτων με διαφορετικά πλάτη τα οποία συνθέτουν το κυματοπακέτο υπό την παρουσία του μαγνητικού πεδίου. Η επίδραση του μεγάλου πλάτους δημιουργεί την δυνατοτητα το σωματιδιο να είναι παγιδευμενο μεσα σε ενα δυναμικό μη γραμμικού ταλαντωτή και σε εκείνη την περιοχή του φασικού χώρου επικεντρώνουμε την προσοχή μας όπου υπάρχουν έκδηλα μη γραμμικά φαινόμενα τα οποία σχετίζονται άμεσα με την διασπορά. Στην συνέχεια προσπαθούμε να συνδέσουμε τον φορμαλισμό της Κανονικής Θεωρίας Διαταραχών με τις κινητικές εξισώσεις προσπαθώντας να μεταβούμε απο το ένα σωματίδιο σε εναν πληθυσμό σωματιδίων με σκοπό να πάρουμε αποτλέσματα για την συνάρτηση κατανομής των σωματιδίων που αλληλεπιδρούν πλέον με κύμα μέσα στο πλάσμα. Τέλος στο Πέμπτο Κεφάλαιο παρουσιάζουμε ενδεικτικά τα συμπεράσματα μας και περιγράφουμε τα σημεία της δουλειάς μας τα οποία χρείζουν περεταίρω γενίκευσης.

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Contents

Chapter 1

Introduction

The presence of coherent electromagnetic waves and their interaction with charged particles are ubiquitous phenomena in plasmas that are encountered in space as well as in laboratory fusion devices, dictating many aspects of energy transport. As an illustration, we can refer to tokamak plasma heating by Low Hybrid waves and RF waves, Alfvén waves heating the solar corona, modification of the distribution function of the charged particles and electromagnetic turbulence.
Subsequently, due to the significant aforementioned importance of the phenomenon, wave-particle interaction has been widely studied in the combined framework of Hamiltonian dynamical systems theory and plasma physics. The first works [1][2] considered a particle under the influence of an electrostatic potential perturbed from a single weak electrostatic wave in the absence of magnetic field in one direction and their goal was to investigate the stochastic instability of a perturbed non-linear oscillator (pendulum). Similar attempts [3][4] have been also proposed with the difference that they inducted adiabatic modulation in the amplitude of the one degree of freedom oscillator in order to examine the transient development of the stochastic instability. While the previous works are characterized by the absence of the magnetic field all four of them consider a finite amplitude potential capable of trapping particles. On the other hand, the classical works of Smith and Karney [5][6][7][8] introduced a uniform magnetic field in the Hamiltonian of a free particle as well as a perturbation that consisted of a small amplitude electrostatic wave which is treated in the framework of first order Hamiltonian perturbation theory in respect to the wave amplitude. Those works consentrate in applying the stochastic instability effect to examine and control the ion heating in fusion devices. The development of a chaotic domain in the phase space caused by resonance overlapping when the amplitude of the perturbing wave is no longer small, and the first order perturbation theory breaks down. Moreover the works of Smith and Karney investigate the dynamics of the system in the off-resonance case while Fukuyama [9] deals with the same problem in the on-resonance case.
More recent publications [10][11][12] have taken into consideration a more realistic scenario, investigating the interaction of a free particle in a uniform magnetic field with multiple electrostatic waves of small amplitude with the intention of extracting specific conditions for the optimal manipulation of the coherent ion heating in tokamaks. Now, second order Hamiltonian perturbation theory has been applied in order to extract non-linear conditions for the coherent acceleration of the ions while there is no trapping regions in the unperturbed system. Furthermore, wave-particle interactions have been extended for the case of particles interacting with a magnetically excited Alfvén wave in [13][14]. Alfvén waves are present in every astrophysical and fusion plasma configuration, thus play significant role in the energy transport. The final step towards the understanding of wave-particle interaction has been made in [15][16], where the interaction of a free particle with a coherent solitary electrostatic wave-packet has examined in the case of small modulation of the envelope of the wave packet.
Taking all the previous realizations into consideration we were motivated to combine all the im-

portant elements of the aforementioned publications and try to treat the most general problem. As a result, the present thesis regards the interaction of a charged particle with an electrostatic periodic wave packet which is consisted from multiple electrostatic waves with different amplitude magnitudes in a uniform magnetic field. In addition, we examine the role of dispersion of the wave packet in the interaction and the way it regulates the energy transport. As such, the structure of this thesis consists of three different parts. Firstly, we introduce the Hamiltonian formulation of perturbation theory and then represent and reproduce the main techniques and results that have been introduced by the classical works. Then, we proceed to the investigation of our case where a particle interacts with a dispersing periodic wave packet that decomposes with specific ordering into a finite amplitude wave and multiple electrostatic waves of smaller amplitude(perturbation strength). Finally we connect the results from the Hamiltonian formulation for one particle to the collective behaviour of a particle population in a collision-less plasma.
Specifically, in Chapter 1 we present the powerful methods of Canonical Perturbation Theory which consists the cornerstone of our analysis in the next chapters. Following Chapter 2, we introduce the reader to wave-particle interaction applying the techniques of Chapter 1. The treatment of our problem begins in Chapter 3 and concludes in Chapter 4 with the collective dynamics of particles in plasma. Final remarks and conclusion of our work has included in Chapter 5.

Chapter 2

Canonical Perturbation Theory

2.1 Classical Perturbation Theory

Most multidimensional systems are not integrable. However, for systems that differ slightly from integrable ones, one can attempt to obtain solutions to a specific degree of accuracy by an expansion of the generating function in powers of the small parameterϵ and then solve the Hamilton-Jacobi equation for each power. Throughout this Chapter we have used the same notation and formalism with [17].
We consider an autonomous Hamiltonian with N degrees of freedom of the form

$$H = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta}) \tag{2.1.1}$$

The unperturbed Hamiltonian H_0 is in action-angle form and its dynamics is governed by the following simple equations

$$\boldsymbol{J} = \boldsymbol{J_0} \tag{2.1.2}$$

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \boldsymbol{\omega}(t - t_0) \tag{2.1.3}$$

$$\boldsymbol{\omega} = \frac{\partial H_0}{\partial \boldsymbol{J}} \tag{2.1.4}$$

The symbol $\boldsymbol{\omega}$ just represents the pseudo-vector of the canonical frequencies defined from $\omega_i = \frac{\partial H_0}{\partial J_i}$.
We consider that the H_1 part is multiply periodic function of the angles, so that we can expand it in Fourier Series.

$$H_1 = \sum_{\boldsymbol{n}} H_{1\boldsymbol{n}}(\boldsymbol{J}) e^{i\boldsymbol{n}\cdot\boldsymbol{\theta}} \tag{2.1.5}$$

Where

$$\boldsymbol{n} \cdot \boldsymbol{\theta} = n_1\theta_1 + n_2\theta_2 + ... + n_N\theta_N$$

We seek a near identity transformation to new variables $\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}$ for which the new Hamiltonian K is a function only of $\bar{\boldsymbol{J}}$. In that way the new system is approximately integrable up to a certain order of accuracy. Using the mixed variable generating function $W(\bar{\boldsymbol{J}}, \boldsymbol{\theta})$ we expand W and K in power series of ϵ

$$W = \bar{\boldsymbol{J}}\boldsymbol{\theta} + \epsilon \sum_{\boldsymbol{n}} W_{1\boldsymbol{n}}(\bar{\boldsymbol{J}}) e^{i\boldsymbol{n}\cdot\boldsymbol{\theta}} + ... \tag{2.1.6}$$

$$K = K_0 + \epsilon K_1 + ... \tag{2.1.7}$$

The mixed relations between the new and the old generalized coordinates are given by

$$J_i = \frac{\partial W}{\partial \theta_i} = \bar{J}_i + \epsilon \frac{\partial W_1}{\partial \theta_i} + ... \tag{2.1.8}$$

$$\bar{\theta}_i = \frac{\partial W}{\partial \bar{J}_i} = \theta + \epsilon \frac{\partial W_1}{\partial \bar{J}_i} + ... \tag{2.1.9}$$

It is convenient to express the old variables in terms of the new variables which is relatively easy to first order

$$J_i = \bar{J}_i + \epsilon \frac{\partial W_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \bar{\theta}_i} + ... \tag{2.1.10}$$

$$\theta_i = \bar{\theta}_i - \epsilon \frac{\partial W_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \bar{J}_i} + ... \tag{2.1.11}$$

Therefore, from the Hamiltonian transformation theory we obtain

$$K(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}) = H(J_i(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}), \theta_i(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})) \tag{2.1.12}$$

We expand the right hand side of the equation (2.1.12) using the relations (2.1.10),(2.1.11), and equating the same order terms with equation (2.7) we obtain

$$K_0(\bar{\boldsymbol{J}}) = H_0(\bar{\boldsymbol{J}}) \tag{2.1.13}$$

$$K_1 = \boldsymbol{\omega}(\bar{\boldsymbol{J}}) \frac{\partial W_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} + H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}) \tag{2.1.14}$$

$$\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}} \tag{2.1.15}$$

The goal of this transformation is to suitably select the W_1 part of the generating function in (2.1.6) in order to eliminate the $\bar{\boldsymbol{\theta}}$ dependence from the K_1 part of the new Hamiltonian resulting to an approximately integrable K Hamiltonian.
We introduce the average part of H_1 as

$$\langle H_1 \rangle = \frac{1}{(2\pi)^N} \int H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})\, d^N \bar{\boldsymbol{\theta}} \tag{2.1.16}$$

In that way the oscillating part of the H_1 is described as

$$\{H_1\} = H_1 - \langle H_1 \rangle \tag{2.1.17}$$

As a result, from equations (2.14) and (2.16) we have

$$K_1 = \langle H_1 \rangle \tag{2.1.18}$$

$$\boldsymbol{\omega}(\bar{\boldsymbol{J}}) \frac{\partial W_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}} = -\{H_1\} \tag{2.1.19}$$

Consequently, the new integrable Hamiltonian K has the to first order the form

$$K = H_0(\bar{\boldsymbol{J}}) + \epsilon \langle H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}) \rangle \tag{2.1.20}$$

Equation (2.19) produces the W_1 part of the generating function. The solution for W_1 involves integration over the unperturbed orbits since

$$\frac{dW_1}{dt} = \frac{\partial W_1}{\partial t} + \frac{\partial W_1}{\partial \bar{\boldsymbol{\theta}}}\frac{d\bar{\boldsymbol{\theta}}}{dt} + \frac{\partial W_1}{\partial \bar{\boldsymbol{J}}}\frac{d\bar{\boldsymbol{J}}}{dt} \tag{2.1.21}$$

Moreover, since the K system is integrable up to this order, from Hamilton's equation we get

$$\frac{\mathrm{d}\bar{\boldsymbol{\theta}}}{\mathrm{d}t} = \frac{\partial K}{\partial \bar{\boldsymbol{J}}} = \bar{\boldsymbol{\omega}}(\bar{\boldsymbol{J}}) = \boldsymbol{\omega} + \epsilon\frac{\partial K_1}{\partial \bar{\boldsymbol{J}}} \tag{2.1.22}$$

$$\frac{\mathrm{d}\bar{\boldsymbol{J}}}{\mathrm{d}t} = -\frac{\partial K}{\partial \bar{\boldsymbol{\theta}}} = 0 \tag{2.1.23}$$

Combining equations (2.1.19) and (2.1.21)-(2.1.23) together with the fact that the system is autonomous we get to zero order

$$W_1 = -\int_t \{H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}(t'))\}\, dt' \tag{2.1.24}$$

An alternative way of finding the W_1 is from relations (2.1.5),(2.1.6) and (2.1.19) to integrate the Fourier series term by term in order to obtain that the generating function W_1 finally is

$$W = \bar{\boldsymbol{J}}\boldsymbol{\theta} + \epsilon i \sum_{n\neq 0} \frac{H_{1\boldsymbol{n}}(\bar{\boldsymbol{J}})}{\boldsymbol{n}\boldsymbol{\omega}(\bar{\boldsymbol{J}})} e^{i\boldsymbol{n}\boldsymbol{\theta}} \tag{2.1.25}$$

The form of the W generating function is an asymptotic expansion which as we said before has to be close in the identity transformation in order to suitably describe the slightly distorted KAM surfaces due to the effects of the perturbation. In that manner, the asymptotic form converges when

$$\left|H_{1\boldsymbol{n}}(\bar{\boldsymbol{J}})\right| \leq \left|\boldsymbol{n}\boldsymbol{\omega}(\bar{\boldsymbol{J}})\right| \tag{2.1.26}$$

Inequality (2.1.26) defines the action space where the invariants of the unperturbed system $H_0(\boldsymbol{J})$ are slightly distorted. On the other hand, when this condition is not satisfied then the phase space is strongly distorted and the invariants of the motion break down, giving rise to resonances.
In general the resonance condition is described by

$$\boldsymbol{n}\boldsymbol{\omega}(\boldsymbol{J}) = 0 \tag{2.1.27}$$

In perturbation theory the existence of small denominators-resonances represent a physical as well as a mathematical difficulty that has to bypassed by the techniques of the next section.
For the sake of complicity we note that when the Hamiltonian H is non autonomous the perturbation theory steps are in principle the same since in the extended phase space of $N+1$ degrees of freedom the pair $(-H, t)$ represents conjugate pair of canonical variables. As a result, the time dependent perturbation theory posses a time dependent Fourier component as well as an extra time derivative in equations (2.1.5),(2.1.6) and (2.1.14) respectively.
We conclude this section by computing the new invariants of the K Hamiltonian system. Those invariants to first order describe the phase space adequately when we are sufficiently far form the resonances.

$$\bar{J}_i = J_i - \epsilon i \sum_{n\neq 0} \frac{n_i H_{1\boldsymbol{n}}(\boldsymbol{J})}{\boldsymbol{n}\boldsymbol{\omega}(\boldsymbol{J})} e^{i\boldsymbol{n}\boldsymbol{\theta}} \tag{2.1.28}$$

2.2 Secular Perturbation Theory

Near a resonance in the unperturbed Hamiltonian a resonant denominator appears in the first order perturbation theory of the previous section. The resonant variables can be removed through a new canonical transformation. Then, a new invariant is discovered which in turn breaks down if the perturbation is large enough and secondary resonances appear. The way that secondary resonances destroy the modified invariant, mirroring that the original resonances destroy the invariant we have derived from the classical perturbation theory.
We start again from the Hamiltonian H of the form

$$H = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta}) \tag{2.2.1}$$

We restrict our investigation to two degrees of freedom for simplicity but the same techniques and results apply to higher dimensions. Again the $(\boldsymbol{J}, \boldsymbol{\theta})$ variables are the action-angle variables of the integrable system H_0 and H_1 is periodic in $\boldsymbol{\theta}$

$$H_1 = \sum_{l,m} H_{l,m}(\boldsymbol{J}) e^{i\boldsymbol{n}\boldsymbol{\theta}} \tag{2.2.2}$$

The integer vector $\boldsymbol{n}$ has the form $\boldsymbol{n} = (l, m)$ and $\boldsymbol{n}\boldsymbol{\theta} = l\theta_1 + m\theta_2$ as in the previous section.
As we have seen, the classical perturbation theory breaks down when resonances occur between the unperturbed frequencies. Specifically, if a resonance exists between the unperturbed frequencies then we can have

$$\frac{\omega_1}{\omega_2} = \frac{r}{s} \tag{2.2.3}$$

The r, s are integers and $\omega_1(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_1}$ and $\omega_2(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_2}$. Relation (2.2.3) can represent either a primary resonance of the system or a secondary resonance created be the harmonics of the oscillation island that is generated from the primary resonance.
In either case, we can apply a transformation that eliminates one of the original actions through the generating function

$$W = (r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2 \hat{J}_2 \tag{2.2.4}$$

The transformation relations from the old $(\boldsymbol{J}, \boldsymbol{\theta})$ to the new variables $(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}})$ are

$$J_1 = \frac{\partial W}{\partial \theta_1} = r\hat{J}_1 \tag{2.2.5}$$

$$J_2 = \frac{\partial W}{\partial \theta_2} = \hat{J}_2 - s\hat{J}_1 \tag{2.2.6}$$

$$\hat{\theta}_1 = \frac{\partial W}{\partial \hat{J}_1} = r\theta_1 - s\theta_2 \tag{2.2.7}$$

$$\hat{\theta}_2 = \frac{\partial W}{\partial \hat{J}_2} = \theta_2 \tag{2.2.8}$$

The new coordinates put the observer in a rotating frame. The form of the generating function allows us the freedom to choose which of the original variables to leave unchanged. The choice is considered by which of the original angles (θ_1, θ_2) is slower. Here we assume that the $\dot{\theta}_2$ is the slower of the two angles, hence we select the W generating function in order that angle to remain unchanged. Subsequently, through the equations (2.2.5)-(2.2.8) the original Hamiltonian is transformed to

$$\hat{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) \tag{2.2.9}$$

$$\hat{H}_1 = \sum_{l,m} H_{l,m}(\hat{\boldsymbol{J}}) \exp\left\{\frac{i}{r}[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]\right\} \tag{2.2.10}$$

Near a resonance from the relation (1.2.3) and (1.2.7) and the Hamilton equations we obtain to first order

$$\dot{\hat{\theta}}_1 = r\dot{\theta}_1 - s\dot{\theta}_2 = r\frac{\partial H}{\partial J_1} - s\frac{\partial H}{\partial J_2} = r\omega_1 - s\omega_2 + \mathcal{O}(\epsilon) \sim \mathcal{O}(\epsilon) \tag{2.2.11}$$

As a result, while for the original variables $\dot{\theta}_1 > \dot{\theta}_2$ in the rotating frame near the resonance we have $\dot{\hat{\theta}}_1 << \dot{\hat{\theta}}_2$. In that sense the fast $\hat{\theta}_2$ oscillations will not affect the motion and we can average the $\hat{H}$ system over $\hat{\theta}_2$ in order to obtain the transform Hamiltonian $\bar{H}$

$$\bar{H} = \bar{H}_0(\hat{\boldsymbol{J}}) + \epsilon \bar{H}_1(\hat{\boldsymbol{J}}, \hat{\theta}_1) \tag{2.2.12}$$

Where

$$\bar{H}_0 = \hat{H}_0(\hat{\boldsymbol{J}}) \tag{2.2.13}$$

$$\bar{H}_1 = \langle \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) \rangle_{\hat{\theta}_2} = \sum_{p=-\infty}^{\infty} H_{-pr,ps}(\hat{\boldsymbol{J}}) e^{-ip\hat{\theta}_1} \tag{2.2.14}$$

Averaging the equation (2.2.10) over $\hat{\theta}_2$ the only non zero term is defined for those l/m which it is valid that

$$ls + mr = 0 \tag{2.2.15}$$

In that way, we define a $p = \frac{m}{s}$ and from the previous (2.2.15) we have $l = -pr$ and $m = ps$. Since for the Hamiltonian $\bar{H}$ the angle $\hat{\theta}_2$ is cyclic we have that

$$\hat{J}_2 = constant \tag{2.2.16}$$

In fact, the $\hat{J}_2$ is an adiabatic invariant of the Hamiltonian in (2.2.9) and represents a combined invariant as can be seen in relation (2.2.6).

$$\hat{J}_2 = J_2 + \frac{s}{r} J_1 = constant \tag{2.2.17}$$

In the hat coordinates the invariant near the resonance is very different for the off-resonance case that we have obtained from classical perturbation theory. For secondary resonances, where $s >> r$ the modified invariant $\hat{J}_2$ is just a multiple of the unperturbed zero order invariant J_1.
The Hamiltonian $\bar{H}$ in (2.2.12) is integrable since it is autonomous and the dynamics can be examined in the $\hat{J}_1 - \hat{\theta}_1$ phase plane.
We proceed to find the stationary points $\hat{J}_{10}, \hat{\theta}_{10}$ in the $\hat{J}_1 - \hat{\theta}_1$ phase plane by

$$\left.\frac{\partial \bar{H}}{\partial \hat{J}_1}\right|_{\hat{J}_{10}} = 0 \tag{2.2.18}$$

$$\left.\frac{\partial \bar{H}}{\partial \hat{\theta}_1}\right|_{\hat{\theta}_{10}} = 0 \tag{2.2.19}$$

The Fourier amplitudes $H_{-pr,ps}(\hat{\boldsymbol{J}})$ in (2.2.24) as we will discuss in the next chapters fall rapidly as p increases, as such we are going to describe the integrable motion be keeping only the $p=-1,0,1$ terms with $H_{-r,s}=H_{r,-s}$. As a result the $\bar{H}$ Hamiltonian has the form

$$\bar{H}=\hat{H}_0(\hat{\boldsymbol{J}})+\epsilon H_{0,0}(\hat{\boldsymbol{J}})+2\epsilon H_{r,-s}(\hat{\boldsymbol{J}})\cos\hat{\theta}_1 \tag{2.2.20}$$

The combination of (2.2.19)-(2.2.20) provide us with the location of the fixed points

$$\frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}+\epsilon\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}+2\epsilon\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}\cos\hat{\theta}_{10}=0 \tag{2.2.21}$$

$$-2\epsilon H_{r,-s}\sin\hat{\theta}_{10}=0 \tag{2.2.22}$$

Equation (2.2.22) has the solutions $\hat{\theta}_{10}=0,\pi$. Therefore, the relation (2.2.21) with the resonance condition (2.2.3) becomes

$$\epsilon\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}\pm 2\epsilon\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}=0 \tag{2.2.23}$$

Now we distinct two different cases:

• If the resonance condition (2.2.3) of the unperturbed Hamiltonian H_0 is satisfied only for particular values of J_1, J_2 then the H_0 is accidentally degenerate and is transformed to rotating system to $\hat{H}_0$ witch is a function of both $\hat{J}_1,\hat{J}_2$

$$\hat{H}_0=\hat{H}_0(\hat{J}_1,\hat{J}_2) \tag{2.2.24}$$

• If the resonance condition (2.2.3) is satisfied for all values of J_1, J_2 then the Hamiltonian H_0 is intrinsically degenerate. The only way for the relation (2.2.3) to be met for all J_1, J_2 is the H_0 to has the form

$$H_0=H_0(sJ_1+rJ_2) \tag{2.2.25}$$

Then, from the rotating transformation relations (2.2.5) and (2.2.6) we obtain

$$\hat{H}_0=\hat{H}_O(\hat{J}_2) \tag{2.2.26}$$

For accidental degeneracy,using Hamilton's equations for $\bar{H}$ and (2.2.20) we obtain

$$\dot{\hat{J}}_1=-2\epsilon H_{r,-s}(\hat{\boldsymbol{J}})\sin\hat{\theta}_1=\mathcal{O}(\epsilon H_{r,-s}) \tag{2.2.27}$$

$$\dot{\hat{\theta}}_1=\mathcal{O}(1) \tag{2.2.28}$$

Hence, it is valid to expand Hamiltonian $\bar{H}$ in (2.2.20) around the stationary point $\hat{J}_{10}$ but not around $\hat{\theta}_{10}$

$$\hat{H}_0(\hat{\boldsymbol{J}})=\hat{H}_0(\hat{\boldsymbol{J}}_{10})+\frac{\partial\hat{H}_0}{\partial\hat{J}_{10}}\Delta\hat{J}_1+\frac{1}{2}\frac{\partial^2\hat{H}_0}{\partial\hat{J}_{10}^2}(\Delta\hat{J}_1)^2 \tag{2.2.29}$$

$$H_{0,0}(\hat{\boldsymbol{J}})=H_{0,0}(\hat{\boldsymbol{J}}_{10})+\frac{\partial H_{0,0}}{\partial\hat{J}_{10}}\Delta\hat{J}_1 \tag{2.2.30}$$

$$H_{r,-s}(\hat{\boldsymbol{J}})=H_{r,-s}(\hat{\boldsymbol{J}}_{10})+\frac{\partial H_{r,-s}}{\partial\hat{J}_{10}}\Delta\hat{J}_1 \tag{2.2.31}$$

Where

$$\Delta \hat{J}_1 = \hat{J}_1 - \hat{J}_{10} \tag{2.2.32}$$

Substituting relations (2.22.9)-(2.2.30) to (2.2.20) and using equation (2.2.21) we finally obtain to second order the Hamiltonian that describe the motion near a resonance

$$\Delta \bar{H} = \frac{1}{2} G(\Delta \hat{J}_1)^2 - F \cos \hat{\theta}_1 \tag{2.2.33}$$

$$G(\hat{\boldsymbol{J}}_{10}) = \frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} \tag{2.2.34}$$

$$F(\hat{\boldsymbol{J}}_{10}) = -2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}_{\mathbf{10}}) \tag{2.2.35}$$

This is a significant result, that proposes that the motion near a resonance is alike of the pendulum with the characteristic regions of libration, separatrix and rotation motion. Consequently the $\Delta \bar{H}$ Hamiltonian is usually referred as a the standard Hamiltonian. The frequency of the libration motion near the stable $\hat{\theta}_{10} = 0$ fixed point is slow and it is given when $FG > 0$ from

$$\hat{\omega}_1 = \sqrt{FG} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}] \tag{2.2.36}$$

The maximum excursion $\Delta \hat{J}_{1max}$ is given by half the separatrix width

$$\Delta \hat{J}_{1max} = 2\left(\frac{F}{G}\right)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}] \tag{2.2.37}$$

Thus for accidental degeneracy both the width of the resonance and the frequency of the libration motion are small of order $\mathcal{O}(\epsilon)$.
For intrinsic degeneracy we proceed analogously bu due to the fact that $\hat{H}_0$ has the form of (2.2.26) the relations (2.2.27) and (2.2.28) become

$$\dot{\hat{J}}_1 = \mathcal{O}(\epsilon H_{r,-s}) \tag{2.2.38}$$

$$\dot{\hat{\theta}}_1 = \mathcal{O}(\epsilon H_{0,0}, \epsilon H_{r,-s}) \tag{2.2.39}$$

In contrast with the accidental degeneracy, here $\dot{\hat{J}}_1$ and $\dot{\hat{\theta}}_1$ are of the same order, as such we must Taylor expand the modified equation (2.2.20) around both $\hat{J}_{10}$ and $\hat{\theta}_{10} = 0$ with $\Delta\hat{\theta}_1 = \hat{\theta}_1$

$$\hat{H}_0(\hat{\boldsymbol{J}}) = \hat{H}_0(\hat{J}_2) = constant \tag{2.2.40}$$

$$H_{0,0}(\hat{\boldsymbol{J}}) = H_{0,0}(\hat{\boldsymbol{J}}_{10}) + \frac{\partial H_{0,0}}{\partial \hat{J}_{10}} \Delta \hat{J}_1 + \frac{1}{2} \frac{\partial^2 \hat{H}_{0,0}}{\partial \hat{J}_{10}^2} (\Delta \hat{J}_1)^2 \tag{2.2.41}$$

$$H_{r,-s}(\hat{\boldsymbol{J}}) = H_{r,-s}(\hat{\boldsymbol{J}}_{10}) + \frac{\partial H_{r,-s}}{\partial \hat{J}_{10}} \Delta \hat{J}_1 + \frac{1}{2} \frac{\partial^2 \hat{H}_{r,-s}}{\partial \hat{J}_{10}^2} (\Delta \hat{J}_1)^2 \tag{2.2.42}$$

$$\cos \hat{\theta}_1 = 1 - \frac{1}{2}(\Delta \hat{\theta}_1)^2 \tag{2.2.43}$$

As a result, ignoring the constant terms the standard Hamiltonian in first order in ϵ has the form

$$\Delta\bar{H} = \frac{1}{2}G(\Delta\hat{J}_1)^2 - \frac{1}{2}(\Delta\hat{\theta}_1)^2 \tag{2.2.44}$$

$$G = \epsilon\frac{\partial^2\hat{H}_{0,0}}{\partial\hat{J}_{10}^2} + 2\epsilon\frac{\partial^2\hat{H}_{r,-s}}{\partial\hat{J}_{10}^2} \tag{2.2.45}$$

$$F = -2\epsilon H_{r,-s} \tag{2.2.46}$$

The frequency near the elliptic point $\hat{\theta}_{10} = 0$ now is

$$\hat{\omega}_1 = \sqrt{FG} = \mathcal{O}(\epsilon) \tag{2.2.47}$$

Hence, in the intrinsic degeneracy the libration frequency is very small compared to the accidental degeneracy. On the other hand the width of the resonance is large compared to the accidental case.

$$\Delta\hat{J}_{1max} = 2\left(\frac{F}{G}\right)^{1/2} = \mathcal{O}(1) \tag{2.2.48}$$

To conclude this section we will report that if ϵ is not sufficiently small then secondary resonances appear in the Hamiltonian of (2.2.9) that can destroy the adiabatic invariant $\hat{J}_2$. These resonances appear from the overlapping of the primary resonances in the $\hat{J}_1 - \hat{\theta}_1$ phase space and can be removed in analogous way with the primary resonances that we have just described.

2.3 Lie Perturbation Theory

In general, there are many physical problems where the previous techniques have to carried out to higher order. For instance, in the calculation of the ponderomotive force the first order calculation gives zero, so we have to calculate the next non-zero term which is of second order [17]. Carrying out higher order expansion using the classical procedures that we have described can be extremely difficult. That difficulty arises for the fact that in classical perturbation theory, the generating function from the old variables $(\boldsymbol{J}, \boldsymbol{\theta})$ to the new ones $(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})$, has a mixed variable form $W(\bar{\boldsymbol{J}}, \boldsymbol{\theta})$. As a consequence, the transformation appears in mixed form as well and that is the reason why expressions to second order and above are extremely lengthy to be derived.
In contrast to the classical perturbation theory, Lie perturbation theory is based on Lie transforms which act on functions as operators and not in variables.
Starting from an autonomous system let $\boldsymbol{x} = (\boldsymbol{p}, \boldsymbol{q})$ be a vector of generalized coordinates in the phase space. We consider a function $w(\bar{\boldsymbol{x}}, \epsilon)$ called Lie generating function that satisfies the equation

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}\epsilon} = [\bar{\boldsymbol{x}}, w] \tag{2.3.1}$$

Equation (2.3.1) describes Hamilton's equations in the form of a Poisson Bracket [ ] with the role of Hamiltonian the Lie generating function w and the parameter ϵ as time. In that way equation (2.3.1) generates a canonical transformation for any ϵ of the form

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon) \tag{2.3.2}$$

In addition, we introduce the evolution operator T which acts on any function g at the transformed point $\bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$ producing a new function f evaluated at the original point $\boldsymbol{x}$.

$$f = Tg \tag{2.3.3}$$

Then

$$f(\boldsymbol{x}) = g(\bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)) \tag{2.3.4}$$

As a result of (2.3.2),(2.3.3) and (2.3.4) when the g function is the identity function we obtain the representation of the transformation in terms of T

$$\bar{\boldsymbol{x}} = T\boldsymbol{x} \tag{2.3.5}$$

Again we have to highlight that the operator T is a functional operator in (2.3.3) and not the variable operator of (2.3.2). Assuming in a first case (2.3.2) we have deducted (2.3.5) as a result of the functional nature of T.
In order to evaluate the transformation T we introduce the Lie operator L

$$L = [w, \ \ ] \tag{2.3.6}$$

Combining relations (2.3.1),(2.3.5) and (2.3.6) we obtain the differential equation for T

$$\frac{\mathrm{d}T}{\mathrm{d}\epsilon} = -TL \tag{2.3.7}$$

The differential equation (2.3.7) has the solution

$$T = \exp\left\{-\int^{\epsilon} L(\epsilon')\, d\epsilon'\right\} \tag{2.3.8}$$

Thus, for any canonical transformation generated by the function w dictated by (2.3.1) the new Hamiltonian K has to be

$$K(\bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)) = H(\boldsymbol{x}) \tag{2.3.9}$$

As such, applying the definition (2.3.3) of the operator T we have

$$K = T^{-1}H \tag{2.3.10}$$

Now we want to generalise our results to time dependent Hamiltonian. For non autonomous systems w, T, L are explicit functions of time, then equations (2.3.1)-(2.3.8) are still valid. On the contrary, equation (2.3.10) is wrong and must be replaced [18] by the expression

$$K = T^{-1}H + T^{-1}\int_0^{\epsilon} T(\epsilon')\frac{\partial w(\epsilon')}{\partial t}\, d\epsilon' \tag{2.3.11}$$

Equations (2.3.6) and (2.3.8) provide complete description of the canonical transformations using Lie generating functions w.
Expanding w, L, T, H, K in powers of ϵ we derive perturbation series [19] that every order has to evaluated.

$$w = \sum_{n=0}^{\infty} \epsilon^n w_{n+1} \tag{2.3.12}$$

$$L = \sum_{n=0}^{\infty} \epsilon^n L_{n+1} \tag{2.3.13}$$

$$T = \sum_{n=0}^{\infty} \epsilon^n T_n \tag{2.3.14}$$

$$H = \sum_{n=0}^{\infty} \epsilon^n H_n \tag{2.3.15}$$

$$K = \sum_{n=0}^{\infty} \epsilon^n K_n \tag{2.3.16}$$

As a result from (2.3.6) we obtain

$$L_n = [w_n, \ ] \tag{2.3.17}$$

Combining relations (2.3.7),(2.3.13) and (2.3.14) and equating order by order we obtain the recursion relation for T_n

$$T_n = -\frac{1}{n} \sum_{m=0}^{n-1} T_m L_{n-m} \tag{2.3.18}$$

In fact, since $T_0 = I$ the identity operator we can have from (2.3.18) the expression of T_n in terms of all orders of L_n. Finally, for the inverse operator T^-1 with $TT^-1 = T^{-1}T = I \Rightarrow T_0^{-1} = I$ we have

$$\frac{\mathrm{d}T^-1}{\mathrm{d}\epsilon} = LT^{-1} \tag{2.3.19}$$

$$T_n^{-1} = -\frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_m^{-1} \tag{2.3.20}$$

Therefore,

$$T_1 = -L_1 \tag{2.3.21a}$$

$$T_2 = -\frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{2.3.21b}$$

$$T_3 = -\frac{1}{3} L_3 + \frac{1}{6} L_2 L_1 + \frac{1}{3} L_1 L_2 - \frac{1}{6} L_1^3 \tag{2.3.21c}$$

$$T_1^{-1} = L_1 \tag{2.3.22a}$$

$$T_2^{-1} = \frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{2.3.22b}$$

$$T_3^{-1} = \frac{1}{3} L + 3 + \frac{1}{6} L_1 L_2 + \frac{1}{3} L_2 L_1 = +\frac{1}{6} L_1^3 \tag{2.3.22c}$$

In order to obtain the equations for the w_n we multiply (2.3.11) with T and then we differentiate with respect to ϵ

$$\frac{\partial T}{\partial \epsilon} K + T \frac{\partial K}{\partial \epsilon} = \frac{\partial H}{\partial \epsilon} + T \frac{\partial w}{\partial \epsilon} \tag{2.3.23}$$

Using equation (1.3.7) with ($\frac{\mathrm{d}T}{\mathrm{d}\epsilon} \rightarrow \frac{\partial T}{\partial \epsilon}$) due to the explicit dependence and multiplying by T^{-1} we obtain

$$\frac{\partial w}{\partial t} = \frac{\partial K}{\partial \epsilon} - LK - T^{-1} \frac{\partial H}{\partial \epsilon} \tag{2.3.24}$$

Inserting in (2.3.24) the series expansions (2.3.12)-(2.3.16) and equating the same powers we obtain in nth order

$$\frac{\partial w_n}{\partial t} = nK_n - \sum_{m=0}^{n-1} Ln - mK_m - \sum_{m=1}^{n} mT_{n-m}^{-1} H_m \tag{2.3.25}$$

Manipulating equation (2.3.25), providing us with

$$D_0 w_n = n(K_n - H_n) - \sum_{m-1}^{n-1} (L_{n-m} K_m + m T_{n-m}^{-1} H_m) \tag{2.3.26}$$

The operator D_0 is the total time derivative along the unperturbed orbits of H_0 much like in the classical perturbation theory.

$$D_0 = \frac{\partial}{\partial t} + [\quad, H_0] \tag{2.3.27}$$

Thus, the equations for the w_n parts of the generating function are

$$D_0 w_1 = K_1 - H_1 \tag{2.3.28a}$$

$$D_0 w_2 = 2(K_2 - H_2) - L_1(K_1 + H_1) \tag{2.3.28b}$$

$$D_0 w_3 = 3(K_3 - H_3 - L_1(K_2 + 2H2) - L_2(K_1 + \frac{1}{2} H_1) - \frac{1}{2} L_1^2 H_1 \tag{2.3.28c}$$

Equations (2.3.28) characterized be the same set of variables since as we said before the operator T is a functional operator, as such the variables are dummy symbols in those equations. Solving equations (2.3.28) is extremely easier due to the dummy variables dependence. As in the classical theory the K Hamiltonian in every equation is selected properly in order to vanish any secular terms. In that way the new Hamiltonian K is expressed in the new coordinates just by changing $\boldsymbol{x} \to \bar{\boldsymbol{x}}$.

Finally, the generating Lie function w displays the same issues as far the resonances are concerned with the generating functions in classical perturbation theory.

Chapter 3

Wave-Particle Interactions

3.1 Hamiltonian Formulation

The aim of this chapter is to apply the techniques of perturbation theory in the framework of Hamiltonian formulation for the wave-particle interaction problem treated in [5]-[8].
The unperturbed system consists of a gyrating charged particle in a uniform magnetic field and is described by the unperturbed Hamiltonian

$$H_{0p} = \frac{1}{2M}\left|\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right|^2 \tag{3.1.1}$$

Where M is the mass of the particle,q is the charge and c the light velocity and the uniform magnetic field is $\boldsymbol{B}_0 = B_0\hat{z}$. The vector potential $\boldsymbol{A(r)}$ must satisfy the relation $\boldsymbol{B}_0 = \boldsymbol{\nabla} \times \boldsymbol{A}$, as such for uniform magnetic field $\boldsymbol{B}_0$ we have the freedom to select the vector potential as

$$\boldsymbol{A} = -B_0 y\hat{x} \tag{3.1.2}$$

The velocity of the particle is $\boldsymbol{v} = (v_x, v_y, v_z)$ whereas the canonical momentum with the aid of (3.1.2) is

$$\boldsymbol{p} = M\boldsymbol{v} + \frac{q}{c}\boldsymbol{A} = (Mv_x - M\Omega y, Mv_y, Mv_z) \tag{3.1.3}$$

Where Ω is the gyration/cyclotron frequency defined by

$$\Omega = \frac{qB_0}{Mc} \tag{3.1.4}$$

Thus, the Hamiltonian (3.1.1) is decomposed to

$$H_{0p} = \frac{p_z^2}{2M} + \frac{p_y^2}{2M} + \frac{1}{2M}(p_x + M\Omega y)^2 \tag{3.1.5}$$

The equilibrium Hamiltonian (3.1.5) is conserved and integrable, therefore the motion of the particle in completely tractable and characterized by a gyration perpendicular to $\boldsymbol{B}_0$ with frequency Ω and a uniform motion along $\boldsymbol{B}_0$.
Now that we have defined the unperturbed motion we want to investigate what happens when a wave interacts with the charged particle and the original motion is perturbed. For the rest of the chapter the interacting wave/s will be electrostatic, except the last section where we will treat the Alfvén wave interaction. As a result an electrostatic wave perturbation is defined by a potential

$$\Phi = \Phi_0 \sin\big(k_\parallel z + k_\perp y - \omega t\big) \tag{3.1.6}$$

Potential (3.1.6) corresponds to a elliptically polarized electrostatic wave in the $y-z$ plane of the form

$$\boldsymbol{E} = -\nabla\Phi = -\Phi_0[k_\parallel \cos\big(k_\parallel z + k_\perp y - \omega t\big)\hat{z} + k_\perp \cos\big(k_\parallel z + k_\perp y - \omega t\big)\hat{y}] \tag{3.1.7}$$

Without loss of generality we have assumed that the electrostatic wave propagates in the $y-z$ plane with $k_\perp > 0$ and obliquely to the magnetic field in an angle $\tan\theta = \frac{k_\perp}{k_\parallel}$.
As a result the perturbed motion is defined by

$$H = \frac{p_z^2}{2M} + \frac{p_y^2}{2M} + \frac{1}{2M}(p_x + M\Omega y)^2 + q\Phi_0 \sin\big(k_\parallel z + k_\perp y - \omega t\big) \tag{3.1.8}$$

A convenient canonical transformation is the guiding center transformation which describes the position of guiding center and the gyration of the particle about it, obtained from the generating function

$$W = M\Omega[\frac{1}{2}(y-Y)^2 \cot\phi - xY] \tag{3.1.9}$$

The transformation then yields

$$\tan\phi = \frac{v_x}{v_y} \tag{3.1.10a}$$

$$Y = y + \rho\sin\phi \tag{3.1.10b}$$

$$X = x - \rho\cos\phi \tag{3.1.10c}$$

$$p_x = -M\Omega Y \tag{3.1.10d}$$

$$p_y = -M\Omega\rho\cos\phi \tag{3.1.10e}$$

$$P_Y = M\omega X \tag{3.1.10f}$$

$$P_\phi = \frac{Mc}{q}\mu = \frac{1}{2}\frac{Mv_\perp^2}{\Omega} = \frac{1}{2}M\Omega\rho^2 \tag{3.1.10g}$$

As we have mention before Ω is the gyration frequency, μ is the magnetic moment,ϕ is the gyration angle, P_ϕ the angular momentum, ρ is the gyration radius and X, Y are the guiding center position. Then the Hamiltonian (3.1.8) transforms to

$$H = \frac{p_z^2}{2M} + \Omega P_\phi + q\Phi_0 \sin\big(k_\parallel z + k_\perp Y - k_\perp\rho\sin\phi - \omega t\big) \tag{3.1.11}$$

In Hamiltonian (3.1.11) we normalize time with the inverse of the gyro-frequency Ω of the ion, distances to the inverse of the parallel wave vector $k_\parallel$ and masses with particle's mass M.

$$h = \frac{p_z^2}{2} + P_\phi + \epsilon\sin(z + \alpha Y - \alpha\rho\sin\phi - \nu t) \tag{3.1.12}$$

Where we have defined

$$\epsilon = \frac{q\Phi_0 k_\parallel^2}{M\Omega^2} = \big(\frac{\omega_b}{\Omega}\big)^2 \tag{3.1.13}$$

$$\nu = \frac{\omega}{\Omega} = \frac{r}{s} \tag{3.1.14}$$

The ω_b frequency is the bounce frequency of the particle in the wave, ϵ is the perturbation strength, $\alpha = \frac{k_\perp}{k_\parallel}$ and the $\nu = \frac{r}{s}$ parameter with r, s integers, defines the relation between the wave frequency and the gyration frequency. Hamiltonian (3.1.12) is described by $X, Y = constant$, as such transforming to the wave frame using the generating function

$$W = (z + \alpha Y - \nu t)P_\psi + Y P'_Y \tag{3.1.15}$$

As a result we have the following transformation equations

$$\psi = z + \alpha Y - \nu t \tag{3.1.16a}$$

$$p_z = P_\psi \tag{3.1.16b}$$

$$Y' = Y \tag{3.1.16c}$$

$$P'_Y = P_Y \tag{3.1.16d}$$

Substituting (3.1.16a)-(3.1.16d) to (3.1.12) the Hamiltonian is transformed to

$$h = \frac{P_\psi^2}{2} - \nu P_\psi + P_\phi + \epsilon \sin(\psi - \alpha\rho \sin\phi) \tag{3.1.17}$$

Finally we make a final transformation using the generating function $W = (P_Z + \nu)Z$ which generates the relations

$$P_\psi = P_Z + \nu \tag{3.1.18a}$$

$$Z = \psi \tag{3.1.18b}$$

Therefore, dropping the constant terms the Hamiltonian (3.1.17) takes the final form

$$h = \frac{P_Z^2}{2} + P_\phi + \epsilon \sin(Z - \alpha\rho \sin\phi) \tag{3.1.19}$$

With

$$\rho = \sqrt{2P_\phi} \tag{3.1.20}$$

Hamiltonian (3.1.19) is autonomous, thus the energy is conserved, but not integrable and represents a gyrating particle under the presence of a electrostatic wave. In this point it is important to highlight that the perturbation amplitude strength ϵ of the wave has to be considered small for Canonical perturbation theory to be applicable. Furthermore using the Jacobi-Anger relation we obtain the Fourier decomposition of the sinusoidal term

$$e^{-i\alpha\rho\sin\phi} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\alpha\rho) e^{-in\phi} \tag{3.1.21}$$

$$h = h_0 + \epsilon h_1 \tag{3.1.22}$$

$$h_0 = \frac{P_Z^2}{2} + P_\phi \tag{3.1.23}$$

$$h_1 = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\alpha\rho) \sin(Z - n\phi) \tag{3.1.24}$$

Equations (3.1.22)-(3.1.24) have the form of (2.2.1) and (2.2.2) with $H_{l,m} = h_{1,n} = \mathcal{J}_n(\alpha\rho)$ where $\mathcal{J}_n(\alpha\rho)$ is the Bessel function of integer n order. Consequently, the techniques and the results that we have derived in the previous chapter are applicable to the study of the Hamiltonian (3.1.22) due to the small amplitude perturbation amplitude ϵ. On the contrary, in Chapter 4 where our work is presented, the existence of a finite amplitude wave in the h_0 unperturbed Hamiltonian results to different classes of particles with different characteristics.
Hamiltonian h is already expressed in the action-angles variables $\boldsymbol{J} = (P_Z, P_\phi), \boldsymbol{\theta} = (Z, \phi)$ of the unperturbed h_0 that describes the gyro-motion of the ion. Whereas this formulation is very common in most textbooks and classical papers that investigate the wave-particle interaction, our framework posses the very important element of considering all the cases of the possible relationship between ω and Ω [20] for the general case of an obliquely propagating wave through the relation $\nu = \frac{\omega}{\Omega} = \frac{r}{s}$. In that way, we cover the whole frequency spectrum from very low frequency waves (VLF) and Alfvén waves in the magnetosphere to high frequency low hybrid waves and radio waves in fusion plasma. Last but not least our method includes cases where $\nu \notin \mathcal{Z}$. In principle equation (3.1.14) along with the respective resonance condition dictates the dynamics as well as the way that resonances arise.
The first order resonance condition of Hamiltonian h is

$$P_Z - n = 0 \tag{3.1.25}$$

With $\omega_Z = P_Z$ and $\omega_\phi = 1$ accordingly to equation (2.2.3). The resonance condition (3.1.25) in the original variables using (3.1.16b) and (3.1.18a) takes the well documented form, called ion-cyclotron resonance.

$$w_1 = -\int_{h_1} dt = \tag{3.1.26}$$

Or

$$p_z - \nu - n = 0 \Rightarrow k_\| v_z - \nu - n = 0 \tag{3.1.27}$$

Relation (3.1.27) is the general resonance condition that takes into consideration the nature of the perturbing electrostatic waves through the frequency spectrum. For instant for the case $k_\| = 0$ (intrinsic degeneracy) and super-harmonic wave $\omega > \Omega$ with $\nu \notin \mathcal{Z} \Rightarrow \frac{r}{s} \in \mathcal{Q}$, the resonance condition (3.1.27) can not be satisfied. As a result, for sufficiently small ϵ, Lie perturbation theory produce to first order smooth invariants of the perturbed motion. The gyro-motion is slightly distorted and no stochastic phenomena are present. On the other hand, when the perturbation strength increases, then second order Lie perturbation theory is needed to incorporate the non-linear dynamics, while new resonance conditions appear. Thus, the present formulation contains richer information about the nature of the interaction as well as about the transient shaping of the phase space as we increase the perturbation strength compared to other investigations.
As we have seen in the previous chapter the resonance condition (3.1.27) can be be divided in two different categories.
• For $k_\| \neq 0$ an accidental degeneracy occurs whenever resonance condition (3.1.27) is satisfied for a series of n integer values for particles with different z-momentum.
• For $k_\| = 0$ the propagation is transverse to the magnetic field $\boldsymbol{B}_0$ and the system is intrinsically degenerate, since whenever the resonance condition $\nu + n = 0$ is met for specific integer n it is valid for all values of p_z.
In the next sections we will apply the formalism of secular and Lie perturbation theory in order to derive important results about the nature of the dynamics for different frequency spectrum electrostatic waves.

3.2 Ion interaction with an Oblique Electrostatic Wave

In the present section we will investigate the interaction of oblique waves with ions. Due to the fact that for $k_\parallel \neq 0$ there is accidental degeneracy, resonances appear for any value of ν in both the super and sub harmonic cases. Oblique wave-ion interaction together with perpendicular wave propagation are well documented [5]-[9],[17][20] due to the strong resulting chaotic heating of the interacting ions which in turn has important application to the heating of the particles' distribution tail in fusion experiments.
We start from the autonomous normalized Hamiltonian, described in equations (3.1.22)-(3.1.24) with $k_\parallel \neq 0$ and we follow the same steps as in the Section 2.2 of secular theory. For a single resonance of l order in the rotating frame with weak perturbation $\epsilon << 1$ we have

$$W = (Z - l\phi)\hat{P}_Z + \phi\hat{P}_\phi \tag{3.2.1}$$

$$\hat{h} = \frac{\hat{P}_Z^2}{2} + (\hat{P}_\phi - l\hat{P}_Z) + \epsilon \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\alpha\hat{\rho}) \sin(\hat{Z} - (n-l)\hat{\phi}) \tag{3.2.2}$$

$$\hat{\rho} = \sqrt{2(\hat{P}_\phi - l\hat{P}_Z)} \tag{3.2.3}$$

Near a resonance variable $\hat{Z}$ is slowly varying so we can average over the fast angle $\hat{\phi}$ and we obtain

$$\bar{h} = \frac{\hat{P}_Z^2}{2} + (\hat{P}_\phi - l\hat{P}_Z) + \epsilon\mathcal{J}_l(\alpha\hat{\rho}) \sin\hat{Z} \tag{3.2.4}$$

With $\bar{h} = constant$ and $\hat{P}_\phi = P_\phi + lP_Z = constant$. The fixed points are given from analogous expressions to (1.2.22) and (1.2.23)

$$\hat{Z}_0 = \pm\frac{\pi}{2} \tag{3.2.5}$$

$$\hat{P}_Z - l \pm \epsilon\frac{\partial\mathcal{J}_l}{\partial\hat{P}_Z} = P_\psi - \nu - l \pm \epsilon\frac{\partial\mathcal{J}_l}{\partial P_\psi} = 0 \tag{3.2.6}$$

Expanding $\bar{h}$ around the stationary point $\hat{P}_{Z0}$ we obtain the standard Hamiltonian of the (2.2.33) form near the lth resonance.

$$\Delta\bar{h} = \frac{1}{2}(\Delta\hat{P}_Z)^2 + \epsilon\mathcal{J}_l(\alpha\hat{\rho}) \sin\hat{Z} \tag{3.2.7}$$

A a result, the maximum half width of the lth resonance according to (2.2.37) is

$$\Delta\hat{P}_{Zmax} = 2|\epsilon\mathcal{J}_l(\alpha\hat{\rho})|^{1/2} \tag{3.2.8}$$

Using the transformation relations (3.1.12)-(3.1.18), the width of the resonance in the velocity space is

$$\Delta v_{zmax} = 2\left|\frac{q\Phi_0\mathcal{J}_l(\alpha\rho)}{M}\right|^{1/2} \tag{3.2.9}$$

From equation (3.2.8) it is evident that increasing the perturbation strength ϵ the width of the resonance also increases. Hence there is a lower perturbation amplitude threshold ϵ_{th} where two neighboring resonances overlap and lead to stochastic motion of the ion. In that case our considerations breaks down and the invariant $\hat{P}_\phi = P_\phi + lP_Z$ is destroyed or modified. In that case we need to incorporate Lie perturbation theory to second order in order to include the non-linear

effects of the increased ϵ. As we will see soon we will derive a second order non-linear resonance condition associated with overlap of the primary resonances as well as an asymptotic expansion of the invariant of the system.
We first begin with the evaluation of the lower perturbation amplitude ϵ_{th} for stochasticity using the Chirikov criterion [21] which in fact states that when the separation between to adjacent resonances is smaller that the sum of their widths then the overlap process is taking place.
The separation δ of two adjacent resonances $l, l+1$ is given by the resonance condition (3.1.25)

$$\delta = \Delta P_Z = 1 \Rightarrow \delta = \Delta v_z = \frac{\Omega}{k_\parallel} \tag{3.2.10}$$

Thus, the criterion for the onset of stochasticity combining (3.2.10) and (3.2.9) yields

$$2\epsilon^{1/2}(|\mathcal{J}_l(\alpha\rho)|^{1/2} + |\mathcal{J}_{l+1}(\alpha\rho)|^{1/2}) = 1 \tag{3.2.11}$$

Considering that $|\mathcal{J}_l(\alpha\rho)| \sim |\mathcal{J}_{l+1}(\alpha\rho)|$ then stochasticity appears when

$$\epsilon > \epsilon_{th} = \frac{1}{16|\mathcal{J}_l(\alpha\rho)|} \tag{3.2.12}$$

Now we proceed to second order Lie perturbation methods that have been sketched in Chapter 2 for the Hamiltonian h from equations (3.1.22)-(3.1.24).

$$D_0 w_1 = K_1 - h_1 \tag{3.2.13}$$

We choose $K_1 = 0$ in order to examine the invariants of the motion away from the resonance $P_Z = l = n$. Therefore since $K_0 = h_0$ we have

$$K = K_0 = \frac{\bar{P}_Z}{2} + \bar{P}_\phi \tag{3.2.14}$$

Hamiltonian K is integrable and $\bar{P}_Z, \bar{P}_\phi = constants$. In order to evaluate the motion away of the resonance we need to find the generating function w_1 given from equation (3.2.13).

$$w_1 = -\int h_1\, dt = \sum_{n \neq l} \mathcal{J}_n(\alpha\rho) \frac{\cos(Z - n\phi)}{P_Z - n} \tag{3.2.15}$$

As such the integral of motion far from the resonance of lth order through the operator transformation of relation (2.3.21a) to first order are

$$\bar{P}_Z = TP_Z = P_Z - \epsilon\frac{\partial w_1}{\partial Z} = P_Z + \epsilon\sum_{n\neq l} \mathcal{J}_n(\alpha\rho)\frac{\sin(Z - n\phi)}{P_Z - n} \tag{3.2.16}$$

$$\bar{P}_\phi = TP_\phi = P_\phi - \epsilon\frac{\partial w_1}{\partial \phi} = P_\phi - \epsilon\sum_{n\neq l} n\mathcal{J}_n(\alpha\rho)\frac{\sin(Z - n\phi)}{P_Z - n} \tag{3.2.17}$$

Our results results can be interpreted as follows. For small perturbation ϵ, near the lth resonance the trapped particles in the resonance's pendulum potential characterized by the integral $\hat{P}_\phi = P_\phi + lP_Z = constant$. Away of the separatrix $\Delta\bar{h} = 0$ of the lth resonance, combining integrals (3.2.16) and (3.2.17) for $P_Z = l + l_0$ we have that the motion is characterized by the same integral $I = P_\phi + lP_Z = constant$ for $n = l$. As a result the is no significant acceleration of trapped particles near the separatrix of the resonance to the free region.

On the other hand, when the perturbation strength ϵ increases beyond the value ϵ_{th} a web is formed connecting the low energy particles to unbounded high energy states. Another way to depict these results is to consider the Hamiltonians in [3][4].

$$H_a = \frac{p^2}{2} - A(\epsilon t)\cos z \tag{3.2.18}$$

$$H_b = \frac{p^2}{2} + A(\epsilon z)\cos z \tag{3.2.19}$$

Hamiltonians H_a, H_b describe the development of the phase space near a resonance through the transition from small to large perturbation amplitude. Using adiabatic theory based on the elliptic functions of the pendulum followed by a Lie transformation results to the logarithmic divergence of the first order adiabatic invariant near the separatrix of the resonance due to the the stochastic region that has started to develop around the unstable separatrix. Although stochastic motion transports the low energetic ions near the resonance's separatrix to the unbound phase space it is described as a diffusion process, particles can become trapped and untapped from bouncing from one resonance to another. The important role to the stochastic web is to connect the coherent resonant islands with the chaotic region of the phase space where the ions can really accelerated efficiently.

We now want to examine the consequences of the non-linearity, when the perturbation strength ϵ in Hamiltonian h in (3.1.22)-(3.1.24) is large enough ($\epsilon \sim \epsilon_{th}$) and the first order perturbation theory breaks down, by extracting a non-linear resonance condition that corresponds to the overlap of the resonances. Hence we need to carry out Lie perturbation theory to order ϵ^2.

We proceed to find the non-linear resonance condition from the divergence of the w_2 Lie generating function setting $K_2 = 0$ in equation (2.3.28b). We want investigate the dynamics away from primary resonances in the overlapping phase space, as such equation (2.3.28.b) takes the form

$$D_0 w_2 = -L_1(K_1 + h_1) \tag{3.2.20a}$$

$$h_1 = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\alpha\rho)\sin(Z - n\phi) \tag{3.2.20b}$$

$$K_1 = \mathcal{J}_l(\alpha\rho)\sin(Z - l\phi) \tag{3.2.20c}$$

$$L_1 = [w_1, \ \ ] \tag{3.2.20d}$$

$$w_1 = \sum_{n\neq l} \mathcal{J}_n(\alpha\rho)\frac{\cos(Z - n\phi)}{P_Z - n} \tag{3.2.20e}$$

The resonance condition will be derived from the $-L_1 h_1$ term since term $-L_1 K_1$ has zero mean value for $\nu \neq l$. As a result we obtain

$$-L_1 h_1 = \sum_{\substack{n,m \\ n\neq l}} \{A_{nm}\cos(m-n)\phi + B_{nm}\cos(2Z - (m+n)\phi)\} \tag{3.2.21}$$

$$A_{nm} = -\frac{1}{2}\left[\frac{\mathcal{J}_m\mathcal{J}_n}{(P_Z - n)^2} + \frac{m\mathcal{J}_m\mathcal{J}_n'}{P_Z - n} + \frac{\mathcal{J}_m'\mathcal{J}_n}{P_Z - n}\right] \tag{3.2.22}$$

$$B_{nm} = -\frac{1}{2}\left[\frac{\mathcal{J}_m\mathcal{J}_n}{(P_Z - n)^2} + \frac{m\mathcal{J}_m\mathcal{J}_n'}{P_Z - n} - \frac{\mathcal{J}_m'\mathcal{J}_n}{P_Z - n}\right] \tag{3.2.23}$$

$$\mathcal{J}' = \frac{\partial\mathcal{J}(\alpha\rho)}{\partial P_\phi} = \frac{1}{\alpha\sqrt{2P_\phi}}\frac{\partial\mathcal{J}(\alpha\rho)}{\partial\rho} \tag{3.2.24}$$

As a result the non-linear resonance conditions are

$$2P_Z - (m+n) = 0 \Rightarrow 2(p_z - \nu) - (m+n) = 0 \Rightarrow k_{\|}v_z - \omega = \frac{m+n}{2}\Omega \tag{3.2.25}$$

And

$$m = n \tag{3.2.26}$$

For $m+n = 2\lambda$ resonance condition (2.2.25) is the same with the linear resonance (2.1.25). The separation δ between two adjacent secondary resonances from (3.2.25) is

$$\delta = \frac{1}{2} \tag{3.2.27}$$

Thus, the secondary resonances are going to interact for $\epsilon_{th,secondaries} << 2\epsilon_{th,primaries}$ and further contribute to the development of chaotic domain in phase space [17][22].

3.3 Ion Interaction with a Transverse Electrostatic Wave

We now head to the analysis for the transverse wave-particle interaction. Due to the intrinsic degeneracy the relation of the wave frequency ω to gyro-frequency Ω expressed by ν parameter, has important role in the nature of the interaction [7][8][17][20].
The normalized Hamiltonian h of the system is

$$h = P_\phi + \epsilon \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\rho) \sin(n\phi - \nu t) \tag{3.3.1}$$

Where we have eliminated the constant momentum term p_x using the generating functions $W = (y - p_{x0})p'_y$ and $W = (p'_x + p_{x0})x)$. Then the constant phase that is present in the sinusoidal term is time absorbed. Finally the generating function $W = \frac{y'^2}{2}\cot\phi$ has been used to obtain the guiding center variables.
Hamiltonian (3.3.1) reveals the resonance condition which is characterized by intrinsic degeneracy.

$$n - \nu = 0 \tag{3.3.2}$$

Thus a resonance can appear on specific value of ν but for successive values of P_ϕ since it does not depend from the action variable P_ϕ.
We will treat separately the case in which $\nu \notin \mathcal{Z}$ and no primary resonances appear for sufficiently small perturbation ϵ and the case where $\nu \in \mathcal{Z}$. The first case correspond to sub-harmonic and super-harmonic waves while the second only to super-harmonic.

3.3.1 $\nu \notin \mathcal{Z}$ Case

In this case the resonance condition (3.3.2) is not met. Thus we can choose $K_1 = 0$ the first order Lie perturbation theory.

$$K = K_0 = h_0 = \bar{P}_\phi \tag{3.3.3a}$$

$$D_0 w_1 = -h_1 \tag{3.3.3b}$$

$$w_1 = \sum_{n=-\infty}^{\infty} \frac{\mathcal{J}_n(\rho)}{n-\nu} \cos(n\phi - \nu t) \tag{3.3.3c}$$

$$\bar{P}_\phi = P_\phi + \epsilon \sum_{n=-\infty}^{\infty} \frac{n\mathcal{J}_n(\rho)}{n-\nu} \sin(n\phi - \nu t) = constant \tag{3.3.3d}$$

We proceed to second order calculation when ϵ increases to strength. Again we want to find the non-linear resonance condition first. In that manner we set $K_2 = 0$ in (2.3.28b) and we obtain

$$D_0 w_2 = -L_1 h_1 \tag{3.3.4}$$

Then

$$-L_1 h_1 = \sum_{n,m=-\infty}^{\infty} \left[A_{n,m} \cos\left((n+m)\phi - 2\nu t\right) + B_{n,m} \cos\left(n-m\right)\phi\right] \tag{3.3.5}$$

With

$$A_{n,m} = \frac{1}{2(m-\nu)} \left[n\mathcal{J}_n \mathcal{J}'_m - m\mathcal{J}_m \mathcal{J}'_n\right] \tag{3.3.6a}$$

$$b_{n,m} = \frac{1}{2(m-\nu)} \left[n\mathcal{J}_n \mathcal{J}'_m + m\mathcal{J}_m \mathcal{J}'_n\right] \tag{3.3.6b}$$

$$\mathcal{J}' = \frac{\partial \mathcal{J}(\rho)}{\partial P_\phi} = \frac{1}{\rho} \frac{\mathrm{d}\mathcal{J}(\rho)}{\mathrm{d}\rho} \tag{3.3.6c}$$

We can now evaluate the singular w_2 from (3.3.4) and (3.3.5) and obtaining the non-linear resonance conditions

$$w_2 = \sum_{n,m=-\infty}^{\infty} \left[\frac{A_{n,m}}{n+m-2\nu} \cos\left((n+m)\phi - 2\nu t\right) + \frac{B_{n,m}}{n-m} \cos\left(n-m\right)\phi\right] \tag{3.3.7}$$

The resonance conditions are

$$n + m = 2\nu \tag{3.3.8}$$

$$n = m \tag{3.3.9}$$

For sub-harmonic and super-harmonics waves with $s = 2,(\nu = r/s)$ the resonance condition can be satisfied while for waves with $s \neq 2$ there are no resonances except when $m = n$.

3.3.2 $\nu \in \mathcal{Z}$ Case

In contrast to the previous case, the first order resonance condition $n = \nu$ now can satisfied so we proceed with secular perturbation theory.
At first we evaluate the w_1 generating function by choosing an appropriate K_1 term in order to eliminate secularities in equation (1.3.28a) whereas the Hamiltonian h of the system is given from (3.3.1). The divergent form of w_1 is given in (3.3.3d) so we select K_1 as

$$K_1 = \mathcal{J}_\nu(\rho) \sin \nu(\phi - t) \tag{3.3.10}$$

Then the smooth w_1 is

$$w_1 = \sum_{n\neq\nu}^{\infty} \frac{\mathcal{J}_n(\rho)}{n-\nu} \cos\left(n\phi - \nu t\right) \tag{3.3.11}$$

The transformed Hamiltonian K is

$$K = \bar{P}_\phi + \epsilon \mathcal{J}_\nu(\bar{\rho}) \sin \nu(\bar{\phi} - t) \tag{3.3.12}$$

As we have mention in Section 2.3 Lie transforms concern functions and not variables, as such the bar-variables are dummy variables, plain symbols. So we will drop the bars from the variables in order to make the symbolism easier since we know that Hamiltonian K refers to the transformed variables.
We now proceed to secular perturbation theory. Using the generating function $W = (\phi - t)\nu \hat{P}_\phi$ we obtain

$$\hat{\phi} = \nu(\phi - t) \tag{3.3.13a}$$

$$P_\phi = \nu \hat{P}_\phi \tag{3.3.13b}$$

The Hamiltonian (2.3.12) the transforms to

$$K = \epsilon \mathcal{J}_\nu(\hat{\rho}) \sin \hat{\phi} \tag{3.3.14}$$

$$\hat{\rho} = \sqrt{2\nu \hat{P}_\phi} \tag{3.3.15}$$

Hamiltonian (3.3.14) is integrable since is autonomous with one degree of freedom. Therefore we search for the fixed points $(\hat{P}_{\phi 0}, \hat{\phi}_0)$ given by the analogous equations (2.2.21) and (2.2.22).

$$\frac{\epsilon \nu}{\hat{\rho}} \frac{\partial \mathcal{J}_\nu(\hat{\rho})}{\partial \hat{\rho}} \sin \phi = 0 \tag{3.3.16}$$

$$\epsilon \mathcal{J}_\nu(\hat{\rho}) \cos \hat{\phi} = 0 \tag{3.3.17}$$

Thus the fixed points are

$$\hat{\phi}_0 = \pm \frac{\pi}{2} \tag{3.3.18}$$

$$\left. \frac{\partial \mathcal{J}_\nu(\hat{\rho})}{\partial \hat{\rho}} \right|_{\hat{\rho}_0} = 0 \tag{3.3.19}$$

We expand (3.3.14) around both fixed points obtaining

$$\sin \hat{\phi} = 1 - \frac{1}{2}(\Delta \hat{\theta})^2 \tag{3.3.20}$$

$$\mathcal{J}_\nu(\hat{\rho}) = \mathcal{J}_\nu(\hat{\rho}_0) + \frac{1}{2} \frac{\partial^2 \mathcal{J}_\nu(\hat{\rho})}{\partial \hat{\rho}_0^2} (\Delta \hat{\rho})^2 \tag{3.3.21}$$

The first order term in (3.3.21) is zero due to relation (3.3.19). Now we manipulate equation (3.3.21) using the Bessel differential equation

$$\hat{\rho}_0^2 \frac{\partial^2 \mathcal{J}_\nu(\hat{\rho})}{\partial \hat{\rho}_0^2} + \hat{\rho}_0 \frac{\partial \mathcal{J}_\nu(\hat{\rho})}{\partial \hat{\rho}_0} + (\hat{\rho}_0^2 - \nu^2) \mathcal{J}_\nu(\hat{\rho}_0) = 0 \tag{3.3.22}$$

Again in (3.3.22) the middle term is zero due to (3.3.19). As a result combining (3.3.21) and (3.3.22) we produce

$$\mathcal{J}_\nu(\hat{\rho}) = \mathcal{J}_\nu(\hat{\rho}_0) + \mathcal{J}_\nu(\hat{\rho}_0) \frac{1}{2} \left[\left(\frac{\nu}{\hat{\rho}_0} \right)^2 - 1 \right] (\Delta \hat{\rho})^2 \tag{3.3.23}$$

Inserting relations (3.3.20) and (3.3.22) in the Hamiltonian in (3.3.14) we obtain the standard Hamiltonian near an intrinsically degenerate resonance like (2.2.44).

$$K = \frac{1}{2} G (\Delta \hat{\rho})^2 + \frac{1}{2} F (\Delta \hat{\phi})^2 \tag{3.3.24}$$

With

$$G = \epsilon \mathcal{J}_\nu(\hat{\rho}_0)\left[\left(\frac{\nu}{\hat{\rho}_0}\right)^2 - 1\right] \tag{3.3.25}$$

$$F = -\epsilon \mathcal{J}_\nu(\hat{\rho}_0) \tag{3.3.26}$$

The maximum half width of the intrinsic resonance is

$$\Delta\hat{\rho}_{max} = 2\left|\frac{F}{G}\right|^{1/2} = \frac{2}{\left|\left(\frac{\nu}{\hat{\rho}_0}\right)^2 - 1\right|^{1/2}} \tag{3.3.27}$$

Although the primary intrinsic resonances are independent of the perturbation strength ϵ, nonetheless when $\nu = \hat{\rho}_0$ then $\Delta\hat{\rho}_{max} \to \infty$. As a result even in low perturbation strength when $\nu = \hat{\rho}_0$ then the two primary resonances at $\hat{\phi}_0 = \pm\frac{\pi}{2}$ inevitably overlap, producing chaotic regions.

3.4 Ion Interaction with Multiple Electrostatic Waves

In this section we will briefly examine the presence of multiple electrostatic waves interacting with a charged particle. This scenario is more realistic compared to the one wave case in both astrophysical and fusion plasmas.

The Hamiltonian of the interaction in guiding center variables for a finite spectrum of oblique electrostatic waves is

$$H = \frac{p_z^2}{2} + \Omega P_\phi + q\sum_{i=1}^{N}\Phi_0 \sin\left(k_\parallel^i z + k_\perp^i Y - k^i\rho\sin\phi - \omega^i t\right) \tag{3.4.1}$$

References [10]-[12] using second order Lie perturbation theory have proven that coherent acceleration of an ion takes place for $N = 2$ when the non-linear beating of the waves is of the form

$$\nu_1 - \nu_2 = n \ \ with \ \ n \in \mathcal{Z} \tag{3.4.2}$$

With $\nu_i = \frac{\omega^i}{\Omega}$ and Ω is the gyro-frequency.

3.5 Ion Interaction with an Alfvén Wave

In this section we are going to investigate the interaction between an ion and an Alfvén wave. Alfvén waves are magneto-hydrodynamic waves of fundamental nature that are present both in fusion and astrophysical plasmas. The presence of Alfvén waves is associated with solar corona heating and ion heating in tokamaks through chaotic heating in a similar way with the electrostatic waves. Since Alfvén waves are sub-harmonic with $\omega < \Omega$, sufficient ion heating can be produced when the amplitude of the wave is large, in other words if the magnetic perturbation is comparable to the ambient magnetic field. In that way the adiabatic invariant of magnetic moment μ breaks and the heating can be analogous to that of the oblique electrostatic wave case.

We begin by deriving the basic characteristics of Alfvén waves as magnetic excitations by perturbing an incompressible, static and infinite magnetic slab in the framework of Ideal Magneto-Hydrodynamics [23].

We consider a infinite,homogeneous, static, and incompressible plasma slab with density ρ_0, pressure P_0 and magnetic field $\boldsymbol{B}_0$. We proceed by perturbing the equilibrium state, described by the following equations

$$\boldsymbol{V} = \boldsymbol{v}_1 \tag{3.5.1}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}_1 \tag{3.5.2}$$

The subscript 0 refer to the equilibrium values while the subscript 1 describe the perturbation. The dynamics of the perturbed system are defined by the Ideal Magneto-Hydrodynamics equations which are consisted of continuity and momentum equations together with Maxwell equations for the magnetic field.

$$\rho_0 \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} \tag{3.5.3a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) \tag{3.5.3b}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) \tag{3.5.3c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0 \tag{3.5.3d}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{3.5.3e}$$

Equations (3.5.3a)-(3.5.3d) are expressed in c.g.s unit system. Using equations (3.5.1),(3.5.2) and considering that $|\boldsymbol{B}_0| >> |\boldsymbol{b}_1|$ expressions (3.5.3a)-(3.5.3d) linearized to first order

$$\rho_0 \frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \frac{(\boldsymbol{\nabla} \times \boldsymbol{b}_1) \times \boldsymbol{B}_0}{4\pi} \tag{3.5.4a}$$

$$\frac{\partial \boldsymbol{b}_1}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) \tag{3.5.4b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v}_1 = 0 \tag{3.5.4c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{b}_1 = 0 \tag{3.5.4d}$$

As a result, the solutions of (3.5.4a)-(3.5.4d) have the Fourier form

$$A_i e^{i(\boldsymbol{k}\boldsymbol{r} - \omega t)} \tag{3.5.5}$$

Where A_i are the respective constant amplitudes of each perturbed variable. Substituting the solutions (3.5.5) to (3.5.4a)-(3.5.4d) we obtain

$$-4\pi\rho_0\omega\boldsymbol{v}_1 = (\boldsymbol{k} \times \boldsymbol{b}_1) \times \boldsymbol{B}_0 = (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{b}_1 - (\boldsymbol{B}_0 \cdot \boldsymbol{b}_1)\boldsymbol{k} \tag{3.5.6a}$$

$$-\omega\boldsymbol{b}_1 = \boldsymbol{k} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) = (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{v}_1 - (\boldsymbol{k} \cdot \boldsymbol{v}_1)\boldsymbol{B}_0 \tag{3.5.6b}$$

$$\boldsymbol{k} \cdot \boldsymbol{v}_1 = 0 \tag{3.5.6c}$$

$$\boldsymbol{k} \cdot \boldsymbol{b}_1 = 0 \tag{3.5.6d}$$

Multiplying (3.5.6a) with $\boldsymbol{k}$ and using (3.5.6c), (3.5.6d) we obtain

$$\boldsymbol{B}_0 \cdot \boldsymbol{b}_1 = 0 \tag{3.5.7}$$

$$-\rho_0 \omega \boldsymbol{v}_1 = \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{b}_1}{4\pi} \tag{3.5.8}$$

$$-\omega \boldsymbol{b}_1 = (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{v}_1 \tag{3.5.9}$$

Equations (3.5.8) and (3.5.9) define that

$$\omega^2 = \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{4\pi\rho_0} \tag{3.5.10}$$

$$V_{phase} = \frac{\omega}{k} = V_A \cos\theta \tag{3.5.11}$$

Consequently, Alfvén waves are magnetic fluctuations, transverse to the equilibrium magnetic field $\boldsymbol{B}_0$ that propagate in an oblique direction ($\cos\theta = \hat{k} \cdot \hat{B}_0$) with phase velocity given by (3.5.11). The velocity V_A is called Alfvén velocity.

$$V_A = \frac{B_0}{\sqrt{4\pi\rho_0}} \tag{3.5.12}$$

We proceed now to the Hamiltonian formulation using the characteristics of Alfvén from equations (3.5.7) and (3.5.11).
Likewise the previous section we choose the equilibrium magnetic field $\boldsymbol{B}_0 = B_0\hat{z}$. Then from equation (3.5.7) and (3.5.11) an Alfvénic magnetic fluctuation $\boldsymbol{b}_1$ has the form

$$\boldsymbol{b}_1 = b_1 \hat{x} \sin(k_{\parallel} z + k_{\perp} y - \omega t) \tag{3.5.13}$$

$$\omega = k_{\parallel} V_A < \Omega \tag{3.5.14}$$

In addition, for the magnetic perturbation must be valid that $\boldsymbol{b}_1 = \boldsymbol{\nabla} \times \boldsymbol{A}_1$. As such we choose

$$\boldsymbol{A}_1 = \frac{b_1}{k_{\parallel}} \hat{y} \cos(k_{\parallel} z + k_{\perp} y - \omega t) \tag{3.5.15}$$

Again without loss of generality we confine our investigation in the $z - y$ plane for the wave propagation. The equilibrium Hamiltonian H_{0p} for the gyro-motion of the particle is given by (3.1.1) with

$$\boldsymbol{A}_0 = -B_0 y \hat{x} \tag{3.5.16}$$

Subsequently the Hamiltonian of the interaction is formulated as

$$H = \frac{1}{2M} \left| \boldsymbol{p} - \frac{q}{c}(\boldsymbol{A}_0 + \boldsymbol{A}_1) \right|^2 \tag{3.5.17}$$

Again we normalize distances with inverse of $k_{\parallel}$, times with inverse of Ω, masses with the ion mass M and magnetic fields with B_0. The obtained normalized Hamiltonian from (3.5.17) is

$$h = \frac{p_z}{2} + \frac{1}{2}(p_x + y)^2 + \frac{p_y}{2} - \frac{b_1 p_y}{B_0} \cos(z + \alpha y - \nu t) + \frac{1}{2}\left(\frac{b_1}{B_0}\right)^2 \cos^2(z + \alpha y - \nu t) \tag{3.5.18}$$

As before $\alpha = \frac{k_{\perp}}{k_{\parallel}}$. Furthermore from the condition $|\boldsymbol{B}_0| >> |\boldsymbol{b}_1|$ we set the amplitude ϵ as

$$\epsilon = \frac{b_1}{B_0} \tag{3.5.19}$$

The time dependence of the vector potential $\boldsymbol{A}_1$ introduces a perturbing electric field

$$\boldsymbol{E}_1 = -\frac{\partial \boldsymbol{A}_1}{\partial t} = -b_1 V_A \hat{y} \sin(k_{\parallel} z + k_{\perp} y - \omega t) \tag{3.5.20}$$

In order to eliminate the electric field we perform a transformation to the wave frame $z - V_A t$ followed by the standard guiding center transform.

$$W = \frac{1}{2}(y - Y)^2 \cot\phi - xY \tag{3.5.21a}$$

$$W = ((z + \alpha Y - \nu t)P_\psi + Y P'_Y \tag{3.5.21b}$$

As a result,

$$\psi = z + \alpha Y - \nu t \tag{3.5.22a}$$

$$p_z = P_\Psi \tag{3.5.22b}$$

$$p_y = -\rho\cos\phi \tag{3.5.22c}$$

$$y = Y - \rho\sin\phi \tag{3.5.22d}$$

Finally the Hamiltonian h takes the form

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 \tag{3.5.23}$$

$$h_0 = \frac{P_Z^2}{2} + P_\phi \tag{3.5.24}$$

$$h_1 = \sum_{n=-\infty}^{\infty} \rho \mathcal{J}_n(\alpha\rho)\cos\phi\cos(P_Z - n\phi) \tag{3.5.25}$$

$$h_2 = \frac{1}{4}\left[1 + \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\alpha\rho)\cos 2(P_Z - n\phi)\right] \tag{3.5.26}$$

We have used the transformation relations $P_\psi = P_Z + \nu$ and $Z = \psi$ from (3.1.18a) and (3.1.18b) along with the Jacobi-Anger expansion in (3.1.19). We can manipulate the h_1 part in (3.5.25) using

$$\cos\phi\cos(P_Z - n\phi) = \frac{1}{2}[\cos(P_Z - (n+1)\phi) + \cos(P_Z - (n-1)\phi)] \tag{3.5.27}$$

Then h_1 takes the form

$$h_1 = \frac{1}{2}\sum_{n=-\infty}^{\infty} \rho[\mathcal{J}_{n-1}(\alpha\rho) + \mathcal{J}_{n+1}(\alpha\rho)]\cos(P_Z - n\phi) \tag{3.5.28}$$

Finally, we use the Bessel identity

$$\mathcal{J}_{n-1}(\alpha\rho) + \mathcal{J}_{n+1}(\alpha\rho) = \frac{2n}{\alpha\rho}\mathcal{J}_n(\alpha\rho) \tag{3.5.29}$$

The h_1 has become

$$h_1 = \frac{1}{\alpha}\sum_{n=-\infty}^{\infty} n\mathcal{J}_n(\alpha\rho)\cos(P_Z - n\phi) \tag{3.5.30}$$

It is quite impressive that equation (3.5.30) is of the same form with (3.2.20b) which corresponds to the oblique electrostatic case. Therefore, even though the Alfvén wave is magnetically excited interacts with the charged particles to first order in an identical way with the electrostatic case in Section 3.1. The ion-cyclotron resonance $P_Z - n = 0$ appears as the primary resonance of the

interaction as well in the oblique electrostatic case.
Consequently it is possible for the sub-harmonic Alfvén waves with finite amplitude $\epsilon \sim \mathcal{O}(1)$ to accelerate cold ions in the solar corona through stochastic processes (overlap of resonances). Then the Maxwellian distribution tail will be heated up to a Lorentzian distribution characterized by super-thermal tail. That kind of distribution function for electrons has been proven to sufficiently accelerate the solar wind to the observable terminal velocities within the formulation of a kinetic exospheric model without a-priori assumption for the thermal energy in the base of the corona since heat energy is extracted from the model itself [24].

Chapter 4

Non-Linear Interactions of Ions with Electrostatic Wave Packets

4.1 Dispersive Wave Packets

In general, the waves either electromagnetic or electrostatic are not in the form of plane waves, rather they are wave packets that are localized in space and could also be of finite duration in time. This is commonly the case in fusion plasmas where the externally applied radio frequency waves have a finite spatial extent, as well as in space plasmas, where Low Hybrid solitary structures occur As a result, in the present chapter we investigate the realistic scenario of charged particle interaction with a finite amplitude wave packet and not with a monochromatic electrostatic wave.
A space and time dependent signal $f(\boldsymbol{r}, t)$ can described through its Fourier Transform representation as

$$f(\boldsymbol{r}, t) = \frac{1}{(2\pi)^4} \iint \tilde{f}(\boldsymbol{k}, \omega) e^{i(\boldsymbol{kr} - \omega t)} \, d\boldsymbol{k} d\omega \tag{4.1.1}$$

In most cases a non-vanishing $\tilde{f}(\boldsymbol{k}, \omega)$ exists only for specific values of ω related to the values of $\boldsymbol{k}$'s as $\omega = \omega_{disp}(\boldsymbol{k})$. In that way we can write

$$\tilde{f}(\boldsymbol{k}, \omega) = 2\pi \tilde{f}_0(\boldsymbol{k}) \delta(\omega - \omega_{disp}(\boldsymbol{k})) \tag{4.1.2}$$

As a result, equation (4.1.1) takes the form

$$f(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \int \tilde{f}_0(\boldsymbol{k}) e^{i(\boldsymbol{kr} - \omega_{disp}(\boldsymbol{k})t)} \, d\boldsymbol{k} \tag{4.1.3}$$

Generally, we can assume that wave vectors $\boldsymbol{k}$ lie around a main wave vector $\boldsymbol{k_0}$, then it is true that $\tilde{f}_0(\boldsymbol{k}) = 0$ when $|\boldsymbol{k} - \boldsymbol{k_0}| > \delta \boldsymbol{k}$. Expanding $\omega_{disp}(\boldsymbol{k})$ around $\boldsymbol{k}_0$ we obtain

$$\omega_{disp}(\boldsymbol{k}) = \omega_{disp}(\boldsymbol{k}_0) + \left. \frac{\partial \omega_{disp}(\boldsymbol{k})}{\partial \boldsymbol{k}} \right|_{\boldsymbol{k}_0} \boldsymbol{k}' + \frac{1}{2} \sum_{ij} \left. \frac{\partial^2 \omega_{disp}(\boldsymbol{k})}{\partial k_i \partial k_j} \right|_{\boldsymbol{k}_0} k_i' k_j' + ... \tag{4.1.4}$$

$$\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}_0 \tag{4.1.5}$$

We set $\omega_{disp}(\boldsymbol{k}_0) = \omega_0$ while second term in the expansion defines the group velocity

$$\boldsymbol{v}_g = \left. \frac{\partial \omega_{disp}(\boldsymbol{k})}{\partial \boldsymbol{k}} \right|_{\boldsymbol{k}_0} \tag{4.1.6}$$

Through expansion (4.1.4) equation (4.1.3) reads

$$f(\boldsymbol{r},t) = \frac{1}{(2\pi)^3}\int \tilde{g}(\boldsymbol{k}',t)e^{i[(\boldsymbol{k}'+\boldsymbol{k}_0)\boldsymbol{r}-(\omega_0+v_g\boldsymbol{k}')t]}\,d\boldsymbol{k}' \tag{4.1.7}$$

$$\tilde{g}(\boldsymbol{k}',t) = \tilde{f}_0(\boldsymbol{k}'+\boldsymbol{k}_0)\exp\left\{-\frac{i}{2}\sum_{ij}\left.\frac{\partial^2\omega_{disp}(\boldsymbol{k})}{\partial k_i\partial k_j}\right|_{\boldsymbol{k}_0}k_i'k_j't+...\right\} \tag{4.1.8}$$

Finally,

$$f(\boldsymbol{r},t) = e^{i(\boldsymbol{k}_0 r-\omega t)}\frac{1}{(2\pi)^3}\int \tilde{g}(\boldsymbol{k}',t)e^{i\boldsymbol{k}'(\boldsymbol{r}-\boldsymbol{v}_g t)}\,d\boldsymbol{k}' \tag{4.1.9}$$

The integral of relation (4.1.9) is the spatial Fourier Transform of $g(\boldsymbol{r}-\boldsymbol{v}_g t,t)$, therefore

$$f(\boldsymbol{r},t) = g(\boldsymbol{r}-\boldsymbol{v}_g t,t)e^{i(\boldsymbol{k}_0\boldsymbol{r}-\omega t)} \tag{4.1.10}$$

Equation (4.1.10) represents a plane wave with wave number $\boldsymbol{k}_0$ and frequency ω_0 whose amplitude is spatially and temporally modulated due to the envelope function $= g(\boldsymbol{r}-v_g t,t)$. The carrier wave travels with phase velocity v_{ph} and the envelope travels with group velocity v_g whose expressions are

$$\boldsymbol{v}_{ph} = \frac{\omega_0}{|\boldsymbol{k}_0|^2}\boldsymbol{k}_0 = \frac{\omega_{disp}(\boldsymbol{k}_0)}{|\boldsymbol{k}_0|^2}\boldsymbol{k}_0 \tag{4.1.11}$$

$$\boldsymbol{v}_g = \left.\frac{\partial\omega_{disp}(\boldsymbol{k})}{\partial\boldsymbol{k}}\right|_{\boldsymbol{k}_0} = \nabla_{\boldsymbol{k}}\omega_{disp}(\boldsymbol{k})|_{\boldsymbol{k}_0} \tag{4.1.12}$$

Thus, an interaction between a charged particle and a wave packet can be illustrated in the following Hamiltonian [15][16]

$$H = \frac{1}{2M}\left|\boldsymbol{p}-\frac{q}{c}\boldsymbol{A}\right|^2 + g(\boldsymbol{r}-v_g t,t)\sin(\boldsymbol{k}_0\boldsymbol{r}-\omega_0 t) \tag{4.1.13}$$

In this situation the particle can either interact with the envelope function non-resonantly and coherently change his momentum due to the ponderomotive force or with the carrier wave resonantly and accelerated through stochastic processes.
For short times provided by

$$t < \left[Max\left|\sum_{ij}\left.\frac{\partial^2\omega_{disp}(\boldsymbol{k})}{\partial k_i\partial k_j}\right|_{\boldsymbol{k}_0}\right|(\delta k)^2\right]^{-1} \tag{4.1.14}$$

the quadratic terms in (4.1.4) can be neglected. Then $\tilde{g}(\boldsymbol{k}',t) = \tilde{f}_0(\boldsymbol{k}'+\boldsymbol{k}_0)$, as such the envelope does not change shape (spreading) and the wave-packet is periodic. For larger times the higher order terms in (4.1.4) become important and the envelope function spreads in space as it propagates with $\boldsymbol{v}_g$, this is the case of solitary wave packet. Equation (4.1.14) suggests that when the spectrum of the wave packet is narrow or equivalently when the wave packet is broad in space then it can be characterized as periodic for all times.
We want to investigate those broad and large amplitude wave-packets by decompose them to discrete components-plain waves. We begin by expanding the time and space dependent signal $f(\boldsymbol{r},t)$ in Fourier Series. Since we have not specify the form of $f(\boldsymbol{r},t)$ this would be problematic for non-periodic signals because the Fourier Series would not converge to f function outside the periodicity range we have defined, even though $f(\boldsymbol{r},t)$ is an analytic function in the whole real regime. That difficulty can be bypassed by stating that in a tokamak, where toroidal symmetry exists, a particle encounters many wave-packets and we are only interested to examine the first passage. That

situation is perfectly reflected by the Fourier expansion where in the peridocity range the series converges to the suitably truncated version $f_c(\boldsymbol{r}, t)$ and outside of that range the series provides us with periodic wave-packets. The dynamics of a first passage provided by the Fourier expansion coincides with the dynamics of the wave-particle interaction as described in the beginning. In order to include all the important dynamics of the original wave-packet to the truncated one we have to select the periodicity range wide enough. A module of that range can be estimated from the width in the arguments of the wave packet. As a result by selecting a periodicity of $2L_i$ for the spatial dependencies and $2T$ for the time dependence of appropriate order verifies that we have included all the important attributes of the real wave packet in the truncated version.

$$f_c(\boldsymbol{r}, t) = \sum_{\boldsymbol{n},m}^{\infty} \tilde{f}_{\boldsymbol{n},m} e^{i(\boldsymbol{n}\boldsymbol{r}-mt)} \tag{4.1.15}$$

Equation (4.1.15) have derived by normalizing each spatial dependence with the corresponding inverse wave number $1/L_i$ and the time dependence with inverse frequency $1/T$. The complex Fourier coefficients $\tilde{f}_{\boldsymbol{n},m}$ are given from

$$\tilde{f}_{\boldsymbol{n},m} = \frac{1}{16L_1L_2L_3T} \int_{-L_1}^{L_1} \int_{-L_2}^{L_2} \int_{-L_3}^{L_3} \int_{-T}^{T} f_c(\boldsymbol{r}, t) e^{-i(\boldsymbol{n}\boldsymbol{r}-mt)} \, d\boldsymbol{r} \, dt \tag{4.1.16}$$

The Fourier coefficients in (4.1.15) are not equivalent [25] in magnitude and obey a specific ordering. The Fourier expansion converges to the analytic $f(\boldsymbol{r}, t)$ in the set $D = \{[-L_1, L_1] \times [-L_2, L_2] \times [-L_3, L_3] \times [-T, T]\}$, then we know that for $\boldsymbol{d} \in D$ the Fourier terms will exponentially decay as

$$\left| \tilde{f}_{\boldsymbol{n},m} \right| \leq A_{\boldsymbol{d}} e^{-(|n_1d_1|+|n_2d_2|+|n_3d_3|+|md_t|)} \tag{4.1.17}$$

The property of the exponential decay of Fourier coefficients provides us with a rule for grading the Fourier harmonics in groups of different order of smallness [25]. Specifically, by setting $(d_1, d_2, d_3, d_t) = (L_1, L_2, L_3, T)$ as an optimal value for any positive integer values of (K_1, K_2, K_3, K_T) we can define the following classes of smallness

$$terms\ of\ order\ 0 \leq |\boldsymbol{n}| + |m| \leq mean[K_i, K_T]\ smallness\ \mathcal{O}(A(e^{-(\boldsymbol{K}\boldsymbol{n})_i d_i})^0) \tag{4.1.18}$$

$$terms\ of\ order\ mean[K_i, K_T] \leq |\boldsymbol{n}| + |m| < 2mean[K_i, K_T]\ smallness\ \mathcal{O}(A(e^{-(\boldsymbol{K}\boldsymbol{n}_{js})_i d_i})^1) \tag{4.1.19}$$

Equations (4.1.18) and (4.1.19) reveal that the $(\boldsymbol{n}, m)$ values are not independent in a way that we can define $m = m(\boldsymbol{n})$ that obeys the ordering of magnitude of the Fourier coefficients. Choosing parameter K_i, K_T in a way to only one harmonic $(\boldsymbol{n}_0, m_0)$ be of zero Fourier order we have

$$f_c(\boldsymbol{r}, t) = \tilde{f}_{\boldsymbol{n}_0} e^{i(\boldsymbol{n}_0\boldsymbol{r}-m_0(\boldsymbol{n}_0)t)} + \sum_{(\boldsymbol{n},m)\in\mathcal{E}} \tilde{f}_{\boldsymbol{n}} e^{i(\boldsymbol{n}\boldsymbol{r}-m(\boldsymbol{n})t)} \tag{4.1.20}$$

Unlike the Fourier expansion of (4.1.15), relation (4.1.20) posses finite harmonic terms in set $\mathcal{E}$ defined by equation (4.1.19) as well as ordered properly. We can write (4.1.20) as

$$f_c(\boldsymbol{r}, t) = \left[\tilde{f}_{\boldsymbol{n}_0} + \sum_{(\boldsymbol{n},m)\in\mathcal{E}} \tilde{f}_{\boldsymbol{n}} e^{i((\boldsymbol{n}-\boldsymbol{n}_0)\boldsymbol{r}-(m(\boldsymbol{n})-m_0(\boldsymbol{n}_0))t)} \right] e^{i(\boldsymbol{n}_0\boldsymbol{r}-m_0(\boldsymbol{n}_0)t)} \tag{4.1.21}$$

Equation (4.1.21) is in principle the discrete counterpart to equation (4.1.9) by whiting

$$m(\boldsymbol{n}) = m_0(\boldsymbol{n}_0) + \boldsymbol{v}_g(\boldsymbol{n})(\boldsymbol{n} - \boldsymbol{n}_0) \tag{4.1.22}$$

Where bold symbols are integer valued vectors in $\mathcal{E}$ and $\boldsymbol{v}_g(\boldsymbol{n})$ is the group velocity defined as finite difference. As a result we obtain

$$f_c(\boldsymbol{r}, t) = \left[\tilde{f}_{\boldsymbol{n}_0} + \sum_{\boldsymbol{n}', \in \mathcal{E}} \tilde{f}_{\boldsymbol{n}'} e^{i[\boldsymbol{n}'(\boldsymbol{r} - \boldsymbol{v}_g(\boldsymbol{n}')t)]}\right] e^{i(\boldsymbol{n}_0 \boldsymbol{r} - m_0(\boldsymbol{n}_0)t)} \tag{4.1.23}$$

The series in the bracket represent a truncated Fourier Series version of the envelope function $g_c(\boldsymbol{r} - \boldsymbol{v}_g t)$. In the following sections we are interested to explore the interaction of a charged particle with a discrete spectrum of planes waves with different and ordered amplitudes that form a wave packet. In that way, we can naturally introduce the interaction of an ensemble of large amplitude monochromatic wave accompanied by a finite number of narrow spectrum of lower amplitude waves that form a broad wave packet with a charged particle. This representation of the interaction coincides with the evidences of large-amplitude coherent electromagnetic and electrostatics waves in the near-Earth space. Unlike the broad spectrum low amplitude waves occupying large coordinate and time domains, these large amplitude waves represent various solitary structures such as electrostatic solitons, Langmuir waves, Alfvén pulses, whistler wave packets, and Low-Hybrid wave bursts. Similar large amplitude wave bursts were reproduced in fusion laboratory experiments [26].

4.2 Theoretical Hamiltonian Formulation

According to the previous realizations the Hamiltonian that we are going to investigate is

$$H = \frac{1}{2M}\left|\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right|^2 + q\Phi_0 \sin(\boldsymbol{k}\boldsymbol{r} - \omega t) + \sum_{n=1}^{N} q\Phi_n \sin(\boldsymbol{k}_n \boldsymbol{r} - \omega_n t + \psi_n) \tag{4.2.1}$$

In order to make our calculations easier we will assume that $\boldsymbol{k} = k_\parallel \hat{z}$, while $\boldsymbol{k}_n = k_{n\parallel}\hat{z} + k_{n\perp}\hat{y}$. The magnetic field is uniform $\boldsymbol{B} = B_0 \hat{z}$ and the particle charge is q.
We normalize distances with inverse of $k_\parallel$, time with inverse of Ω and masses with ion mass M. Finally we use the following transformations similarly to Section 3.1.

$$W = (z - \nu t + \frac{\pi}{2})P_\psi \tag{4.2.2a}$$

$$W = (P_Z + \nu)Z \tag{4.2.2b}$$

$$W = \frac{1}{2}(y - Y)^2 \cot\phi - xY \tag{4.2.2c}$$

Therefore the Hamiltonian in (4.2.1) takes the form

$$h = \frac{P_Z^2}{2} + P_\phi - A\cos Z + A\sum_{n=1}^{N}\sum_{b=-\infty}^{\infty} \epsilon_n \mathcal{J}_m(\alpha_{n\perp}\rho)\sin(\alpha_{n\parallel}Z - b\phi - (\nu_n - \alpha_{n\parallel}\nu)t + \psi) \tag{4.2.3}$$

Where we have set

$$A = \frac{q\Phi_0 k_\parallel^2}{M\Omega^2} = \left(\frac{\omega_b}{\Omega}\right)^2 \tag{4.2.4a}$$

$$\epsilon_n = \frac{\Phi_n}{\Phi_0} \tag{4.2.4b}$$

$$\nu_n = \frac{\omega_n}{\Omega} \tag{4.2.4c}$$

$$\alpha_{n\parallel} = \frac{k_{n\parallel}}{k_{\parallel}} \tag{4.2.4d}$$

$$\alpha_{n\perp} = \frac{k_{n\perp}}{k_{\parallel}} \tag{4.2.4e}$$

Equation (4.2.4b) is derived due to the ordering of the Fourier coefficients in (4.1.20) as such we can set $\epsilon_n = \epsilon$ and the standard Jacobi-Anger expansion for Bessel functions have been used. Equations (4.2.4a) and (4.2.4b) constitute the cornerstone of our analysis and the reason why stands out from the previous frameworks. The ordering ϵ in the third term of (4.2.3) is fixed due to the Fourier ordering of (4.1.20) in respect to the amplitude A. In other words, while we can vary amplitude A from large to small values, parameter ϵ guarantee us that the rest of the waves are small amplitude compared to A. For instance, when $A = 1/\epsilon$ then $\epsilon A = 1$, compared to the smallness parameter ϵ both perturbing terms are in the non linear regime but due to the fact that they will always retain the same ordering between them, the application of perturbation theory of first order is valid. In the present framework we want the trapping amplitude A to be large, so trapping can occurred in the unperturbed motion. The second important aspect of our formulation reflects on the $\nu_n - \alpha_{n\parallel}\nu$ term which expresses the magnitude of dispersion in the wave packet. It is obvious from equation (4.2.3) that the dispersion term defines how the resonances are formed as well as the dynamics of the system. Constant terms have been either dropped out or absorbed in the phase ψ_n. Hamiltonian (4.2.3) in contrast to the previous Chapter, represents a gyrating particle inside a stationary electrostatic potential while interacting with perturbing electrostatic waves with varying magnitude frequencies. As such, this Hamiltonian not only describes perfectly the electrostatic turbulence but also determines the kind of turbulence through the magnitude of the dispersive term $\nu_n - \alpha_{n\parallel}\nu$. Hence low frequency turbulence which is responsible for anomalous transport can be obtained when $\nu_n - \alpha_{n\parallel}\nu = \mathcal{O}(\epsilon)$.

In order to exploit the techniques of perturbation theory we have to express the integrable h_0 part of (4.2.3) in action angles variables $(\boldsymbol{J}, \boldsymbol{\theta})$.

$$h_0 = \frac{P_Z^2}{2} + P_\phi - A\cos Z \tag{4.2.5}$$

The h_0 Hamiltonian resembles that of a pendulum and the P_ϕ momentum is already the action of the system. This is maybe the first time that Hamiltonian (4.2.5) is treated in the literature and once more we have to point out that the trapping of the gyrating particles leads to development of important non-linear phenomena [26]. The parameter which defines whether the particles gyrates around the magnetic field freely or between two reflective points in z-axis is

$$m = \frac{2A}{h_0 - P_\phi + A} \tag{4.2.6}$$

The first scenario described from $m < 1$ while the second one for $m > 1$ and we will consider each one of them separately.

- For the free particle case $m < 1$ with $-\pi \leq Z \leq \pi$ we have

$$J_r = \frac{4}{\pi}\left(\frac{A}{m}\right)^{1/2} E(m) \tag{4.2.7}$$

$$\omega_r = \frac{\partial h_0}{\partial J_r} = \frac{\partial h_0}{\partial m}\frac{\partial m}{\partial J_r} = -\frac{2A}{m^2}\frac{\partial m}{\partial J_r} = -\frac{2A}{m^2}\left(\frac{\partial J_r}{\partial m}\right)^{-1} = \left(\frac{A}{m}\right)^{1/2}\frac{\pi}{K(m)} \tag{4.2.8}$$

- For the trapped particle case $m > 1$ with $-Z_{3c} \leq Z \leq Z_{3c}$ we have

$$Z_{3c} = 2\arcsin\Big(\frac{1}{\sqrt{m}}\Big) \tag{4.2.9}$$

$$J_l = \frac{8}{\pi}A^{1/2}\Big[E(1/m) + \Big(\frac{1-m}{m}\Big)K(1/m)\Big] \tag{4.2.10}$$

$$\omega_l = \frac{\partial h_0}{\partial J_l} = 2A\frac{\partial(1/m)}{\partial J_l} = 2A\Big(\frac{\partial J_l}{\partial(1/m)}\Big)^{-1} = \frac{\pi A^{1/2}}{2K(1/m)} \tag{4.2.11}$$

From the definition of m in equation (4.2.6) we can already express h_0 in action variables without even find the appropriate generating function for that transformation. As a result we obtain

$$h_0 = \Big[\frac{2}{m(J_i)} - 1\Big]A + P_\phi \; with \;\; i = r, l \tag{4.2.12}$$

The generating function and the angle variables relations for free gyration $m < 1$ are

$$W_r(J_r, J_\phi, Z, \phi) = 4\Big(\frac{A}{m}\Big)^{1/2} E(s|m) + J_\phi\phi \tag{4.2.13a}$$

$$s = \frac{Z}{2} \tag{4.2.13b}$$

$$\theta_r = \frac{\pi u}{K(m)} \tag{4.2.13c}$$

$$u = F(s|m) \tag{4.2.13d}$$

While for trapped gyration case $m > 1$

$$W_l(J_l, J_\phi, Z, \phi) = 4A^{1/2}\Big[E(\tilde{s}|1/m) + \Big(\frac{1-m}{m}\Big)K(\tilde{s} \mid 1/m)\Big] + J_\phi\phi \tag{4.2.14a}$$

$$\sin\tilde{s} = \sqrt{m}\sin s \tag{4.2.14b}$$

$$\theta_l = \frac{\pi\tilde{u}}{2K(1/m)} \tag{4.2.14c}$$

$$\tilde{u} = F(\tilde{s}|1/m) \tag{4.2.14d}$$

Since we succeeded in expressing the integrable part in action variables now we have to express the perturbation part h_1 in (4.2.3) in the action-angle variables of the integrable system. This process will be treated separately for each case.

4.2.1 $m < 1$ Case

In this case, the particle is not trapped in the large amplitude wave potential,due to the fact that his total energy is greater than potential energy. The particle gyrates around the magnetic field interacting with the rest of the small amplitude waves. Thus it is expected for his unperturbed motion to be non-periodic. Our goal is to express the variable Z as $Z = Z(\theta_r, J_r)$. Using the Jacobi Elliptic Functions and their properties we obtain

$$Z = \theta_r + 4\sum_{s=1}^{\infty} x_s \frac{\sin(s\theta_r)}{s} \tag{4.2.15}$$

$$x_s = \frac{q^s}{1+q^{2s}} \tag{4.2.16}$$

Substituting (4.2.15) in the h_1 part of (4.2.3) we obtain a term that needs special treatment due to its complexity.

$$\exp\left\{ i4\alpha_{n\parallel} \sum_{s=1}^{\infty} x_s \frac{\sin(s\theta_r)}{s} \right\} = \exp\left\{ i \sum_{s=1}^{\infty} a_s(n)\sin(s\theta_r) \right\} \tag{4.2.17}$$

$$a_s(n) = \frac{4\alpha_{n\parallel}}{s} x_s \tag{4.2.18}$$

In order to exploit this term we will use a generalization of Bessel Functions, the Infinite variable Bessel Functions of one index of integer order $\mathcal{J}_n(\{a_m\}_1^\infty)$. The convergence of the $\sum_{s=1}^{\infty} s a_s(n)$ verifies the existence of such functions which admit the following expansion of the Jacobi-Anger type

$$\exp\left\{ i \sum_{s=1}^{\infty} a_s(n)\sin(s\theta_r) \right\} = \sum_{l=-\infty}^{\infty} \mathcal{J}_l(\{a_s(n)\}_1^\infty) e^{il\theta_r} \tag{4.2.19}$$

As a result the h Hamiltonian in action angle variables for $m < 1$ takes the form

$$h = h_0 + h_1 \tag{4.2.20}$$

$$h_0 = \left[\frac{2}{m(J_r)} - 1 \right] A + J_\phi \tag{4.2.21}$$

$$h_1 = A\epsilon \sum_{n=1}^{N} \sum_{b,l} \mathcal{J}_b(\alpha_{n\perp}\rho)\mathcal{J}_l(\{a_s(n)\}_1^\infty) \sin\left((\alpha_{n\parallel} + l)\theta_r - b\phi - (\nu_n - \alpha_{n\parallel}\nu)t + \psi\right) \tag{4.2.22}$$

For $\epsilon << 1$ Lie perturbation theory produces the singular first order w_1 generating function for $K_1 = 0$

$$w_{1r} = A\epsilon \sum_{n=1}^{N} \sum_{b,l} \mathcal{J}_b(\alpha_{n\perp}\rho)\mathcal{J}_l(\{a_s(n)\}_1^\infty) \frac{\cos\left((\alpha_{n\parallel} + l)\theta_r - b\phi - (\nu_n - \alpha_{n\parallel}\nu)t + \psi\right)}{R} \tag{4.2.23}$$

Where R is the resonant denominator

$$R = (\alpha_{n\parallel} + l)\omega_r - b - (\nu_n - \alpha_{n\parallel}\nu) \tag{4.2.24}$$

Thus, the K Hamiltonian is integrable with

$$K = h_0 = \left[\frac{2}{m(\bar{J}_r)} - 1 \right] A + \bar{J}_\phi \tag{4.2.25}$$

With

$$\bar{J}_r = J_r - \epsilon \frac{\partial w_{1r}}{\partial \theta_r} \tag{4.2.26}$$

$$\bar{J}_\phi = J_\phi - \epsilon \frac{\partial w_{1r}}{\partial \phi} \tag{4.2.27}$$

Invariants (4.2.26) and (4.2.27) diverge not only in the resonant points but also as we close out the separatrix ($m \to 1$) since

$$\lim_{m \to 1} K(m) = \frac{1}{2} \ln \left(\frac{16}{1-m} \right) \tag{4.2.28}$$

The logarithmic divergence of the invariants near the seperatrix indicates a diffusion process where particles cross this boundary multiple times changing their status from trapped to free and vice versa. Let now consider the simple case where $n = 1$ and the particles are far from the separatrix $m << 1$ then we have

$$h_0 = \frac{2A}{m(J_r)} \tag{4.2.29}$$

Moreover, only the fist term $s = 1$ has significant impact in equation (4.2.15) while the Nome q can be written in the $m << 1$ limit as

$$q = \frac{m}{16} \tag{4.2.30}$$

Then equation (4.2.15) transforms to

$$Z = \theta_r + \frac{m}{4} \sin \theta_r \tag{4.2.31}$$

Finally from relation (4.2.7) we obtain that

$$J_r = 2 \left(\frac{A}{m} \right)^{1/2} \tag{4.2.32}$$

As a result, the Hamiltonian h in the limit $m << 1$ takes the form

$$h = \frac{J_r^2}{2} + P_\phi + \epsilon A \sum_{b,l} \mathcal{J}_b(\alpha_{1\perp}\rho) \mathcal{J}_l(\alpha_{1\parallel} \frac{m}{4}) \sin\left((\alpha_{1\parallel} + l)\theta - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi\right) \tag{4.2.33}$$

Hamiltonian h in (4.2.33) represents a free particle gyration perturbed by a wave packet. When there is no dispersion $(\nu_1 - \alpha_{1\parallel}\nu) = 0$ we obtain a modified the ion-cyclotron resonance condition

$$(\alpha_{1\parallel} + l)J_r - b = 0 \tag{4.2.34}$$

In the following we will highlight the important attributes of resonance condition

$$(\alpha_{1\parallel} + l)J_r - b - (\nu_1 - \alpha_{1\parallel}\nu) = 0 \tag{4.2.35}$$

In general resonance condition (4.2.35) suggests accidental degeneracy in the system which in fact depends from the magnitude of the dispersion term $(\nu_1 - \alpha_{1\parallel}\nu)$. From equation (4.2.32) for $m << 1 \sim \mathcal{O}(\epsilon)$ then $J_r \sim \mathcal{O}(\frac{1}{\epsilon^{1/2}})$. For instance, for large dispersion the resonance condition (4.2.35) can be satisfied only for large integer values of w. Moreover for $\alpha_{1\parallel} + l = 0$ the the system develops a transition from accidental degeneracy to intrinsic degeneracy. Resonances in the high energy region of phase space suggest an unbounded chaotic domain available for the particle to accelerated due to overlap condition.

Hamiltonian h in (4.2.33) contains rich information about the dynamics of the $m << 1$ system that must also exist in the general case.
For $\alpha_{1\parallel} = 0 \Rightarrow k_{1\parallel} = 0$ then the Hamiltonian (4.2.33) takes the form

$$h = \frac{J_r^2}{2} + P_\phi - \epsilon A \sum_b \mathcal{J}_b(\alpha_{1\perp}\rho) \sin(b\phi + \nu_1 t - \psi) \tag{4.2.36}$$

Hamiltonian (4.2.36) is essential the Hamiltonian (3.3.1) of a transverse electrostatic wave interacting with a free gyrating particle as we have studied in Section 2.3, with the amplitude parameter being exactly the ϵA of (4.2.36).
For $\alpha_{1\parallel} \neq 0$ since $m << 1$ we can use the expansion of the Bessel function

$$\mathcal{J}_l(\alpha_{1\parallel}\frac{m}{4}) = \frac{1}{\Gamma(l+1)}\left(\frac{\alpha_{1\parallel}m}{8}\right)^l \; for \; l > 0 \tag{4.2.37}$$

$$\mathcal{J}_{-l}(\alpha_{1\parallel}\frac{m}{4}) = \frac{(-1)^l}{\Gamma(l+1)}\left(\frac{\alpha_{1\parallel}m}{8}\right)^l \; for \; l > 0 \tag{4.2.38}$$

As a result for $m << 1$ the most important terms in the h_1 part of (4.2.33) are given for $l = -1, 0, 1$ and the Hamiltonian takes the form

$$h = h_0 + h_1 \tag{4.2.39a}$$

$$h_0 = \frac{J_r^2}{2} + P_\phi \tag{4.2.39b}$$

$$h_1 = \epsilon A \sum_{\substack{b \\ l=-1,0,1}} \mathcal{J}_b(\alpha_{1\perp}\rho)\mathcal{J}_l(\alpha_{1\parallel}\frac{m}{4}) \sin((\alpha_{1\parallel} + l)\theta - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi) \tag{4.2.39c}$$

Finally from the relation $J_r = J_r(m)$ in (4.2.32), (4.2.39c) and (4.2.37)-(4.2.38) we obtain that

$$h_1 = h_{1a} + h_{1\beta} \tag{4.2.40}$$

$$h_{1a} = \epsilon A \sum_b \mathcal{J}_b(\alpha_{1\perp}\rho) \sin(\alpha_{1\parallel}\theta - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi) \tag{4.2.41}$$

$$h_{1\beta} = \frac{\epsilon\alpha_{1\parallel}A^2}{2J_r^2} \sum_{\substack{b \\ l=-1,1}} l\mathcal{J}_b(\alpha_{1\perp}\rho) \sin((\alpha_{1\parallel} + l)\theta - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi) \tag{4.2.42}$$

Equation (4.2.42) reveals the ordering of a second order force compared to the interaction proposed from h_{1a} in (4.2.41). Specifically, the combined amplitude ϵA is exactly the same with that of the plane wave-ion interaction in Chapter 3.
As a result the interaction of a particle with a plane oblique electrostatic wave corresponds to a sub-case $m << 1$ for the non-linear interaction of a particle with an electrostatic wave packet to first order in trapping strength A. Setting $A = 1$, the h_{1a} part indicates the interaction with a plane electrostatic wave which is of order ϵ. On the other hand, the magnitude of the $h_{1\beta}$ term is varying due to the dependence from the action J_r, since this expression is valid for $m << 1$ considering $m \sim \epsilon$ the interaction proposed from $h_{1\beta}$ is of order ϵ^2. This is a significant result that reveals that for a free particle the main interaction includes the carrier wave of the wave packet

while the interaction proposed by $h_{1\beta}$ is of second order. Subsequently, it is reasonable to proceed to the examination of the dominant way of interaction between the free particle and the wave packet which is through the resonant interaction with the carrier wave of the wave packet.
Therefore, if the resonance condition $\alpha_{1\parallel}J_r - b - (\nu_1 - \alpha_{1\parallel}\nu) = 0$ defined by relation (4.2.41) is satisfied for the integer b_0 then from first order Lie perturbation theory we choose K_1 in order to eliminate the secularity in the w_1 generating function

$$K_1 = \epsilon A \mathcal{J}_{w_0}(\alpha_{1\perp}\rho) \sin(\alpha_{1\parallel}\theta_r - w_0\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi) \tag{4.2.43}$$

The new transformed Hamiltonian K is

$$K = K_0 + K_1 \tag{4.2.44}$$

$$K_0 = \frac{J_r^2}{2} + P_\phi \tag{4.2.45}$$

$$K_1 = \epsilon A \mathcal{J}_{b_0}(\alpha_{1\perp}\rho) \sin(\alpha_{1\parallel}\theta_r - b_0\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi) \tag{4.2.46}$$

We transform to the rotating frame in order to derive the standard Hamiltonian near the accidental resonance.

$$W = (\alpha_{1\parallel}\theta_r - b_0\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi)\hat{J}_r + \phi\hat{P}_\phi \tag{4.2.47a}$$

$$\hat{\theta}_r = \alpha_{1\parallel}\theta_r - b_0\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi \tag{4.2.47b}$$

$$J_r = \alpha_{1\parallel}\hat{J}_r \tag{4.2.47c}$$

$$P_\phi = -b_0\hat{J}_r + \hat{P}_\phi \tag{4.2.47d}$$

$$\hat{\rho} = \sqrt{2(\hat{P}_\phi + -b_0\hat{J}_r)} \tag{4.2.47e}$$

Using transformations (4.2.47a)-(4.2.47e) to (4.2.45) and (4.2.46) we obtain

$$K_0 = \alpha_{1\parallel}^2\frac{\hat{J}_r^2}{2} - (b_0 + \nu_1 - \alpha_{1\parallel}\nu)\hat{J}_r + \hat{P}_\phi \tag{4.2.48}$$

$$K_1 = \epsilon A \mathcal{J}_{b_0}(\alpha_{1\perp}\hat{\rho}) \sin\hat{\theta}_r \tag{4.2.49}$$

The result of the removal of the resonance is that Hamiltonian $K = K_0 + K_1$ from equations (4.2.45) and (4.2.46) is integrable. The fixed points in the $\hat{J}_r - \hat{\theta}_r$ plane are

$$\left.\frac{\partial K}{\partial \hat{J}_r}\right|_{\hat{J}_{r0}} = 0 \tag{4.2.50}$$

$$\left.\frac{\partial K}{\partial \hat{\theta}_r}\right|_{\hat{\theta}_{r0}} = 0 \Rightarrow \hat{\theta}_{r0} = \pm\frac{\pi}{2} \tag{4.2.51}$$

Expanding Hamiltonian K around the $\hat{J}_{r0}$, together with the fixed point equation (4.2.48) and dropping constant terms we produce to first order the standard Hamiltonian near the resonance

$$K = \alpha_{1\parallel}^2(\Delta\hat{J}_r)^2 + \epsilon A \mathcal{J}_{w_0}(\alpha_{1\perp}\hat{\rho}) \sin\hat{\theta}_r \tag{4.2.52}$$

Thus the maximum half width of the resonance is

$$\Delta \hat{J}_{rmax} = 2\left|\frac{\epsilon A \mathcal{J}_{b_0}(\alpha_{1\perp}\hat{\rho})}{\alpha_{1\parallel}^2}\right|^{1/2} \tag{4.2.53}$$

Using the Chirikov criterion [21] in an analogous way like (3.2.9) and (3.2.10) together with the resonance condition (4.2.35) for $l = 0$ we obtain the trapping amplitude A_{th}, above which resonance overlapping occurs.

$$A_{th} = \frac{1}{16\epsilon|\mathcal{J}_{b_0}(\alpha_{1\perp}\hat{\rho}_0|)} \tag{4.2.54}$$

Equation (4.2.55) defines the magnitude of the non linearity in order for a chaotic web to be formed into the upper energy domain of phase space as $A_{th} \sim 1/\epsilon$. In particular for $m << 1$ the phase space exhibits distant resonant islands due to the accidental degeneracy condition of (4.2.35), when satisfied for $l \neq (-1, 0, 1)$. Specifically when the resonance happens for $l = 0$ then the combined ϵA_{th} threshold is exactly the same with the threshold amplitude in equation (3.2.12) for the one wave. Therefore, around the resonance region characterized by $l_0 = 0$ a stochastic web could be evident while the rest of the high energy phase space exhibits distant islands. On the other hand when $\alpha_{1\parallel} + l = 0$, then intrinsic degeneracy takes place and the resonance condition is

$$b + (\nu_1 - \alpha_{1\parallel}\nu) = 0 \tag{4.2.55}$$

Due to the intrinsic degeneracy it is possible the phase space between the distance accidental resonant islands to be altered. Finally if $\alpha_{1\parallel} \notin \mathcal{Z}$ then only accidental resonances occur in the system.

4.2.2 $m > 1$ Case

In contrast to the previous case the particle now is trapped in the potential while gyrates around the magnetic field. The $Z = Z(\theta_l, J_l)$ relation is given from

$$Z = 4\sum_{s+\frac{1}{2}}^{\infty} x_s \frac{\sin(2s\theta_l)}{s} \tag{4.2.56}$$

Thus using the previous relations we obtain

$$h = h_0 + h_1 \tag{4.2.57}$$

$$h_0 = \left[\frac{2}{m(J_l)} - 1\right] A + J_\phi \tag{4.2.58}$$

$$h_1 = A\epsilon \sum_{n=1}^{N}\sum_{b,l} \mathcal{J}_b(\alpha_{n\perp}\rho)\mathcal{J}_l(\{a_s(n)\}_{s+\frac{1}{2}}^{\infty}) \sin\left(l\theta_l - b\phi - (\nu_n - \alpha_{n\parallel}\nu)t + \psi\right) \tag{4.2.59}$$

Similarly with the previous, the first order generating function which provide us with the position of the resonances is

$$w_{1l} = A\epsilon \sum_{n=1}^{N}\sum_{b,l} \mathcal{J}_w(\alpha_{n\perp}\rho)\mathcal{J}_l(\{a_s(n)\}_{s+\frac{1}{2}}^{\infty}) \frac{\sin\left(l\theta_l - b\phi - (\nu_n - \alpha_{n\parallel}\nu)t + \psi\right)}{R} \tag{4.2.60}$$

Again the resonance condition is given from

$$R = l\omega_l - b - (\nu_n - \alpha_{n\parallel}\nu) = 0 \tag{4.2.61}$$

The transformed integrable new Hamiltonian K has the form

$$K = h_0 = \left[\frac{2}{m(\bar{J}_l)} - 1\right] A + \bar{J}_\phi \tag{4.2.62}$$

As we close out the separatrix $m \to 1$ the invariants $\bar{J}_l$ and $\bar{J}_\phi$ logarithmically diverge as in the $m < 1$ case. It is of great interest to examine if strongly trapped particles $m >> 1$ can find their way to higher energies. As before we select only one $n = 1$ linear wave.
When $m >> 1 \Rightarrow \frac{1}{m} << 1$ thus the h_0 part in (4.2.58) becomes

$$h_0 = -A + P_\phi \tag{4.2.63}$$

Equation (4.2.63) reveals that for strongly trapped particles the bounce motion is negligible compared to the gyro-motion, but from equation (4.2.11) $\omega_l = A^{1/2}$. As such, we must first express $J_l(m)$ from (4.2.10) and then insert it into the h_0 term dropping the constant $-A$ term from the Hamiltonian.

$$h_0 = A^{1/2} J_l + P_\phi \tag{4.2.64}$$

Furthermore since $\frac{1}{m} << 1$ the first important term in equation (4.2.56) is for $s = \frac{1}{2}$

$$q = \frac{1}{16m} \tag{4.2.65}$$

$$Z = \frac{2}{\sqrt{m}} \sin\theta_l \tag{4.2.66}$$

Hence, the h_1 part in (4.2.59) can be written as

$$h_1 = A\epsilon \sum_{b,l} \mathcal{J}_b(\alpha_{1\perp}\rho)\mathcal{J}_l(\frac{2\alpha_{1\parallel}}{\sqrt{m}}) \sin\left(l\theta_l - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi\right) \tag{4.2.67}$$

From equation (4.2.10) for J_l expanding $E(1/m)$ and $K(1/m)$ to first order for the limiting case $\frac{1}{m} << 1$ we obtain

$$J_l = \frac{2A^{1/2}}{m} \tag{4.2.68}$$

Thus, h_1 has the form

$$h_1 = A\epsilon \sum_{b,l} \mathcal{J}_b(\alpha_{1\perp}\rho)\mathcal{J}_l\Big(\frac{2\alpha_{1\parallel} J_l^{1/2}}{(2A^{1/2})^{1/2}}\Big) \sin\left(l\theta_l - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi\right) \tag{4.2.69}$$

The resonance condition for the system takes the form

$$lA^{1/2} - b - (\nu_n - \alpha_{1\parallel}\nu) = 0 \tag{4.2.70}$$

The resonance condition in (4.2.70) for the strongly trapped particle, proposes intrinsic degeneracy for the system while includes the trapping strength A as a primary element of the resonance existence. Additionally, the dispersion magnitude plays significant role in the way the resonances formed along with A. The resonance condition (4.2.70) can be fulfilled in both cases where $(\nu_n - \alpha_{1\parallel}\nu) \notin \mathcal{Z}$ and $(\nu_n - \alpha_{1\parallel}\nu) \in \mathcal{Z}$ according to the values of $A^{1/2}$. The strong trapping of the particles is reflected in the $\mathcal{J}_l$ Bessel term which can be expanded as before

$$\mathcal{J}_{l>0}\Big(\frac{2}{\sqrt{m}}\Big) = \frac{1}{\Gamma(l+1)}\Big(\frac{1}{\sqrt{m}}\Big)^l \tag{4.2.71}$$

$$\mathcal{J}_{-l}\Big(\frac{2}{\sqrt{m}}\Big) = (-1)^l \mathcal{J}_{l>0}\Big(\frac{1}{\sqrt{m}}\Big)^l \tag{4.2.72}$$

In that way only the $l = -1, 0, 1$ terms contribute to the sum of in (4.2.69) and the h_1 part can be written

$$h_1 = h_{1a} + h_{1b} \tag{4.2.73}$$

$$h_{1a} = -\epsilon A \alpha_{1\parallel} \sum_b \mathcal{J}_b(\alpha_{1\perp}\rho) \sin\left(b\phi + (\nu_1 - \alpha_{1\parallel}\nu)t - \psi\right) \tag{4.2.74}$$

$$h_{1\beta} = \epsilon A^{3/4} \alpha_{1\parallel} \left(\frac{J_l}{2}\right)^{1/2} \sum_{\substack{b \\ l=-1,1}} l \mathcal{J}_b(\alpha_{1\perp}\rho) \sin\left(l\theta_l - b\phi - (\nu_1 - \alpha_{1\parallel}\nu)t + \psi\right) \tag{4.2.75}$$

Equation (4.2.74) proposes that the interaction of a deeply trapped particle with the wave packet can be described with the interaction with a plane wave to order ϵ. The same result have been obtained from equation (4.2.41) for the free particle. The important difference between the two cases, reflects on equation (4.2.42). For $A = 1$ and $\frac{1}{m} \sim \epsilon$ the interaction defined by $h_{1\beta}$ is of order $\epsilon^{3/2}$ in contrast to equation (4.2.42) where it was of order ϵ^2. As a result in order examine possible energy gain or loss for the trapped particle we need to take account both parts $h_{1a}, h_{1\beta}$ in equations (4.2.75) and (4.2.76). As we have expected the qualitative differences of our work with the rest of the aforementioned references are pointed out in the apparent non-linear interaction between the particle and the wave-packet in the trapping phase space.

Chapter 5

Collective Particle Dynamics and Transport

5.1 Averaged Action Variations

Aside from the stochastic acceleration a particle can also be accelerated or decelerated in the phase space without the existence of a threshold in perturbation strength. The Lie perturbation theory provides us with the ability to systematically evaluate the variation in the action variables due to coherent dynamics.

The evolution of the a well behaved function $f(\boldsymbol{z})$ expressed in the action angle phase space $\boldsymbol{z} = (\boldsymbol{J}, \boldsymbol{\theta}, t)$ is given by

$$f(\boldsymbol{z}) = S_H(t) f(\boldsymbol{z}_0) \tag{5.1.1}$$

Where $S_H(t)$ is the time evolution operator defined from the solutions of the equations of motion from the Hamiltonian $H = H_0(\boldsymbol{J}) + H_1(\boldsymbol{z})$ of the system. In general the time evolution operator is very difficult to computed especially for the non integrable systems that we have been examining. On the other hand, the Lie perturbation theory provides us with the capability to simplify the new Hamiltonian of the system K through equations (2.3.28a)-(2.3.28c) by selecting properly $K_1, K_2, ... = 0$ in order to obtain an integrable system that depends only from the new variables $\bar{\boldsymbol{J}}$. Then the new variables $\bar{\boldsymbol{J}}$ are the new actions of the system and the equation of motion as well as the new time evolution operator $S_K(t)$ is easily evaluated.

$$H = H_0(\boldsymbol{J}) + H_1(\boldsymbol{z}) \xrightarrow{Lie\ transform} K(\bar{\boldsymbol{J}}) \tag{5.1.2}$$

$$\bar{\boldsymbol{J}} = \bar{\boldsymbol{J}}_0 = constant \tag{5.1.3}$$

$$\bar{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}}_0 + \boldsymbol{\omega}_K(\bar{\boldsymbol{J}}_0)(t - t_0) \tag{5.1.4}$$

$$\bar{f}(\bar{\boldsymbol{z}}) = S_K(t) f(\bar{\boldsymbol{z}}_0) = f[\bar{\boldsymbol{J}}_0, \bar{\boldsymbol{\theta}}_0 + \boldsymbol{\omega}_K(\bar{\boldsymbol{J}}_0)(t - t_0), t] \tag{5.1.5}$$

As we have mentioned before $\boldsymbol{\omega}_K(\bar{\boldsymbol{J}}_0)$ symbolizes the frequencies for each $\bar{J}_i$ with $\boldsymbol{\omega}_K(\bar{\boldsymbol{J}}_0) = \nabla_{\bar{\boldsymbol{J}}} K|_{\bar{\boldsymbol{J}}_0}$. In order to transform back to the original variables we exploit the Lie operator definition (2.3.3) and equation (5.1.5)

$$f(\boldsymbol{z}) = T f(\bar{\boldsymbol{z}}) = T(\boldsymbol{z}_0) S_K(t) f(\bar{\boldsymbol{z}}_0) = T(\boldsymbol{z}_0, t_0) S_K(t) T^{-1}(\boldsymbol{z}_0) f(\boldsymbol{z}_0) \tag{5.1.6}$$

It is important to mention that the Lie evolution operators have been evaluated in the finite time interval $[t_0, t]$ where $w_i(\boldsymbol{z}_0) = 0$ so that $T(\boldsymbol{z}_0) = I$. The S_K operator now acts on $T^{-1}(\boldsymbol{z}_0)$ Lie operator as defined form equation (5.1.5). As a result the evolution of $f(\boldsymbol{z}_0)$ from equation (5.1.6) in the time interval $[t_1, t_2]$ is

$$f(\boldsymbol{z})_{t_2} = T^{-1}[\boldsymbol{J}_{t_1}, \boldsymbol{\theta}_{t_1} + \boldsymbol{\omega}_K(\boldsymbol{J}_{t_1})(t_2 - t_1), t_2] f(\boldsymbol{z})_{t_1} \tag{5.1.7}$$

Where we have defined $f(\boldsymbol{z})_t = f(\boldsymbol{z(t)})$.
So far, our formulation was conducted for the motion of one particle in the presence of a perturbation defined by the Hamiltonian $H = H_0(\boldsymbol{J}) + H_1(\boldsymbol{z})$. In order to realistically describe the dynamics in a plasma we need to expand our results to many particles case.
According to equation (5.1.2) the new Hamiltonian of the system through Lie perturbation method,for one particle, is $K = K(\bar{\boldsymbol{J}})$. Assuming that the plasma is collision-less which is a valid assumption for both fusion and space plasma physics then the distribution function $\bar{f}(\bar{\boldsymbol{z}})$ for the one particle is the same with the distribution function for a collection of particles in a collision-less plasma which has to obey Vlasov equation

$$\frac{\partial \bar{f}(\bar{\boldsymbol{z}})}{\partial t} + [\bar{f}(\bar{\boldsymbol{z}}), K(\bar{\boldsymbol{J}})] = 0 \tag{5.1.8}$$

The form of Vlasov equation (5.1.8) suggests that any function $\bar{f}(\bar{\boldsymbol{z}})$ that depends from the integrals of Hamiltonian K is a solution, thus from equation (5.1.3) any function $\bar{f}(\bar{\boldsymbol{J}})$ is a solution to the Vlasov equation that dictates the collective particle behaviour in a collision-less plasma. As a result, relation (5.1.6) through the Lie operator, transfers us to the collective particle dynamics of the original system H when we choose $f(\boldsymbol{z}) = \boldsymbol{J}$. Then, we can use (5.1.7) to evaluate the variation in the actions for the time interval $[t_1, t_2]$ to second order as

$$\Delta J_n = J_n(t_2) - J_n(t_1) = (L_1 + \frac{1}{2}L_2 + \frac{1}{2}L_1^2) J_n(t_1) \tag{5.1.9}$$

Relations (2.3.22a) and (2.3.22b) have been used for the expansion of T^{-1} operator and the index n cores ponds to the conjugate angle variable, thus $n = \{(r,l), \phi\}$. We define the average of any dynamical variable ζ of the system as in equation (2.1.16)

$$\langle \zeta \rangle = \frac{1}{(2\pi)^2} \iint \zeta \, d\theta d\phi \tag{5.1.10}$$

Averaging equation (5.1.9) we obtain the averaged variation of the original action of the system

$$\langle \Delta J_n \rangle = \langle [w_1, J_n] \rangle + \frac{1}{2} \langle [w_2, J_n] \rangle + \frac{1}{2} \langle [w_1 [w_1, J_n]] \rangle \tag{5.1.11}$$

We are interested to employ equation (5.1.11) in evaluating the coherent dynamics of the strongly trapped particle which described in equations (4.2.63),(4.2.74) and (4.2.75). Due to the fact that both $h_{1\alpha}$ and h_{1b} are periodic in θ_l, ϕ variables in equations (4.2.74) and (4.2.75) then the generating Lie function w_1 would be periodic as well. As a result we obtain

$$\langle [w_1, J_n] \rangle = \langle [w_2, J_n] \rangle = 0 \tag{5.1.12}$$

Finally integrating by parts the third non-zero term in relation (5.1.11) we get

$$\langle \Delta J_n \rangle = \frac{1}{2} \frac{\partial}{\partial J_n} \left\langle \left(\frac{\partial w_1}{\partial n} \right)^2 \right\rangle \tag{5.1.13}$$

The averaged variation of the actions for a particle collection due to a complete interaction with the wave packet is obtained in the limit $w_1(t_1 \to -\infty, t_2 \to \infty)$. Equation (5.1.13) is very powerful since it evaluates the averaged variations to second order in ϵ with the aid only of the first order generating function w_1.

5.2 Coherent Dynamics of Strongly Trapped Particles

Using the results from Section 5.1 we want to examine the possible transport of the strongly trapped particles interacting with the wave packet through Hamiltonian terms $h_{1\alpha}$ and h_{1b} in equations (4.2.74) and (4.2.75). We choose to examine the strongly trapped particles in order to highlight the non-linear interaction and the finite amplitude wave component that leads to trapping. Moreover it is of great interest to examine the dispersion condition for the strong trapped particles to gain energy in order to switch on the unbounded phase space plane where they can stochastically accelerated. As a result of equation (5.1.13) using (4.2.74) and (4.2.75) results to

$$\langle \Delta J_l \rangle = \frac{\epsilon^2 A^{3/2}}{8} \alpha_{1\parallel}^2 \sum_b \mathcal{J}_b^2(\alpha_{1\perp}\rho) \left[\frac{1}{(A^{1/2} - b - p)^2} + \frac{1}{(A^{1/2} + b + p)^2} \right] \tag{5.2.1}$$

In equation (5.2.1) we have set $p = \nu_1 - \alpha_{1\parallel}\nu$ and we have dropped periodically time fluctuating terms since we want a complete interaction.
In order to examine the coherent acceleration in the trapped region we have to stay sufficiently far from the resonance conditions $A^{1/2} \pm b \pm p = 0$. In that way we distinct two different cases.
• When $A^{1/2} \in \mathcal{Z}$ then it always exists $b_0 \in \mathcal{Z}$ such that $A^{1/2} \pm b_0 = 0$. As a result, the condition for coherent acceleration to take place is

$$\frac{\epsilon^2 A^{3/2} \alpha_{1\parallel}^2 \mathcal{J}_{b_0}^2(\alpha_{1\perp}\rho)}{8p^2} \geq 1 \tag{5.2.2}$$

Therefore, the dispersion magnitude p has to be

$$0 < p \leq \left| \frac{\epsilon A^{3/4} \alpha_{1\parallel} \mathcal{J}_{b_0}(\alpha_{1\perp}\rho)}{2\sqrt{2}} \right| \leq \frac{\epsilon A^{3/4} \alpha_{1\parallel}}{2\sqrt{2}} \tag{5.2.3}$$

The second inequality is a very crude estimation since Bessel functions fall off rapidly, and is accurate only for small values of b_0. Equation (5.2.3) has two important consequences. Firstly when $A \to 0$ then we have only one, small amplitude wave and coherent acceleration can not be achieved, particles can only gain energy stochastically as it has examined in Chapter 2. Secondly, the stronger the non-linear trapping amplitude A is the larger the values of b_0 satisfy the relation $A^{1/2} \pm b \pm p = 0$, but in the same time the Bessel function $\mathcal{J}_{b_0}$ has fallen extremely rapidly, according to the asymptotic form [27]

$$\mathcal{J}_{b_0}(\alpha_{1\perp}\rho) \sim \frac{1}{\sqrt{2\pi b_0}} \left(\frac{e\alpha_{1\perp}\rho}{2b_0} \right)^{b_0} \tag{5.2.4}$$

Hence coherent acceleration can not be achieved in the very large $A^{1/2}$ trapping amplitude limit. Only through resonant acceleration the particles can find their way to the unbounded phase space. In general, the upper limit in relation (5.2.3) is smaller than ϵ provide us with the information that only for small dispersion $p \leq \epsilon$ it is possible for the particles to coherently gain energy.
• When $A^{1/2} \notin \mathcal{Z}$ then we can express it as $A^{1/2} = I + D$ where I and D is the integer and decimal part of $A^{1/2}$. Hence, there is always exists $b_0 \in \mathcal{Z}$ such that $I \pm b_0 = 0$. As a result, the inequality that defines the magnitude of dispersion capable to observe coherent acceleration is

$$0 < |D \pm p| \leq \left| \frac{\epsilon A^{3/4} \alpha_{1\parallel} \mathcal{J}_{b_0}(\alpha_{1\perp}\rho)}{2\sqrt{2}} \right| \leq \frac{\epsilon A^{3/4} \alpha_{1\parallel}}{2\sqrt{2}} \tag{5.2.5}$$

Equation (5.2.5) reveals that coherent acceleration can be proposed only when the magnitude of dispersion p is near the decimal value D so that $0.1 \leq p < 1$. Finally, since strongly trapped particles characterized from $\frac{1}{m} << 1$ then from equation (4.2.68) we obtain that the J_l action of those particles must obey

$$J_l << 2A^{1/2} \tag{5.2.6}$$

Stronger amplitude A translates to larger phase space available for trapping. Next, we proceed to the visualization of our results for the different cases. For $A = 1$ from equation (5.2.6) the domain

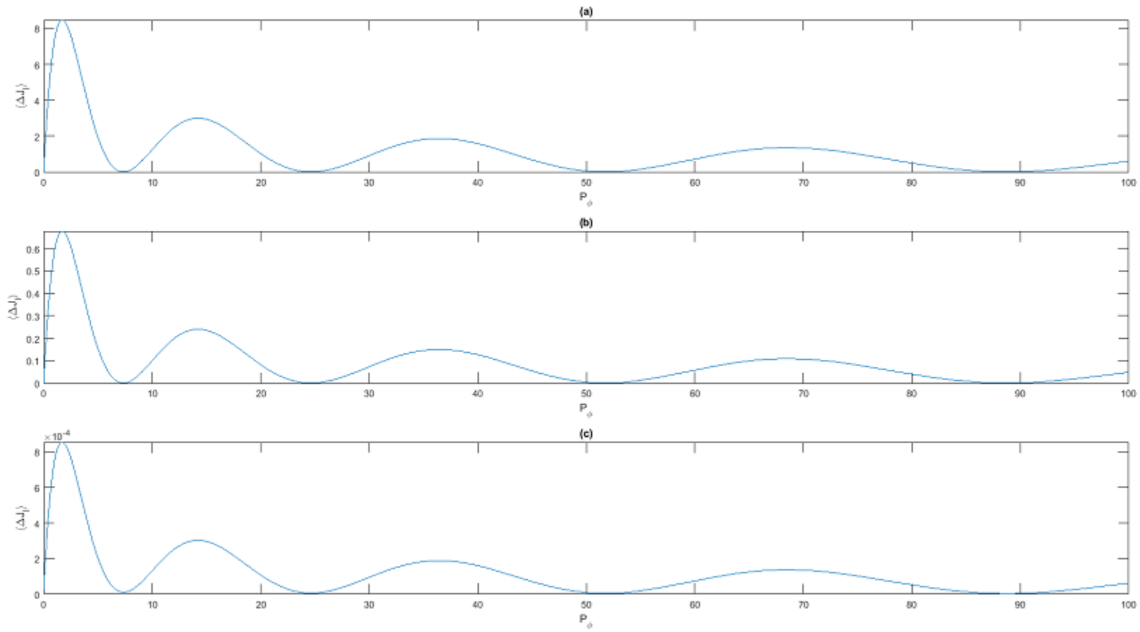


Figure 5.1: Averaged action variation $\langle \Delta J_l \rangle$ over P_ϕ for trapping amplitude strength $A = 1$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p compared to the dispersion threshold value p_{th} provided in equation (5.2.3). (a) $p = 0.001$, (b) $p = 0.0035$, (c) $p = 0.1$.

of the strongly trapped particles is characterized by $J_l << 2$. As a result, Figure 5.1 proposes that there is significant J_l momentum transport in case (a) which represents the domain $p < p_{th} \simeq 0.0035$. Specifically, the J_l momentum variation exhibits a peak for low angular momentum values $P_\phi < 10$ while the acceleration is still persistent in higher P_ϕ values as it gradually wears off in strength compared to the first peak. In case (b) where $p = p_{th}$ we can barely consider important variation around the first peak. Finally for $p > p_{th}$ strongly trapped particles cannot accelerated coherently. Similar results are proposed when $A = 10$ but now the dispersion threshold p_{th} is defined from equation (5.2.5) and the domain of strongly trapped particles is $J_l << 2\sqrt{10}$. Moreover, according to Figure 5.2 when $A = 10$ the P_ϕ domain that strong coherent acceleration appears, has shifted to higher P_ϕ values compared to the $A = 1$ case. Figure 5.3(a) reveals that coherent momentum J_l transfer is achieved only for large values of P_ϕ momentum since the first accelerating peak is met around $P_\phi = 70$ in contrast to cases $A = 1, 10$. The momentum transfer still exhibits a persistent "periodic" nature as in the previous. In case (b) we cannot consider any acceleration because we have overestimated the true value of p_{th} from equation (5.2.3). Finally, in Figure 5.4 we verify that not only the primary J_l transport peak has shifted to even greater gyro-radius than before but also that the width of the peak broadens as we move from low to greater trapping amplitudes A.
This particular behaviour of the coherent J_l acceleration can be explained through the formalism of the previous chapter. The unperturbed Hamiltonian h_0 for the strongly trapped particles is given from equation (4.2.64) as $h_0 = A^{1/2} J_l + P_\phi$. As a result, the angular momentum must have the same ordering with the bounce momentum J_l.

$$P_\phi \sim A^{1/2} J_l \tag{5.2.7}$$

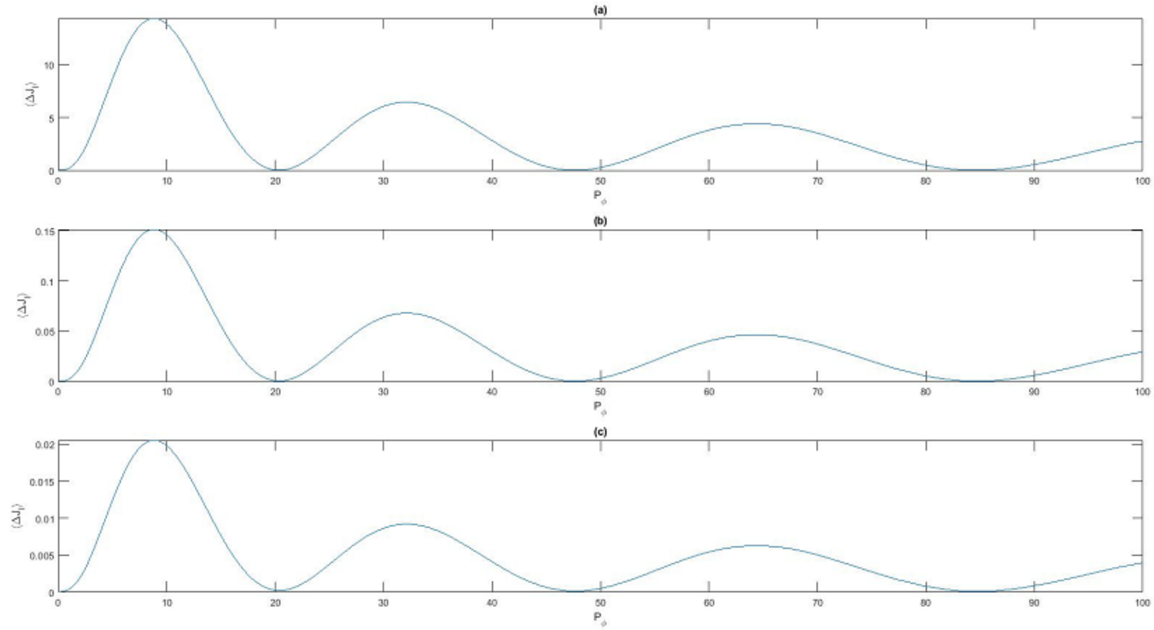


Figure 5.2: Averaged action variation $\langle \Delta J_l \rangle$ over P_ϕ for trapping amplitude strength $A = 10$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p compared to the dispersion threshold value p_{th} provided in equation (5.2.5). (a) $p = 0.16$, (b) $p = 0.14$, (c) $p = 0.1$.

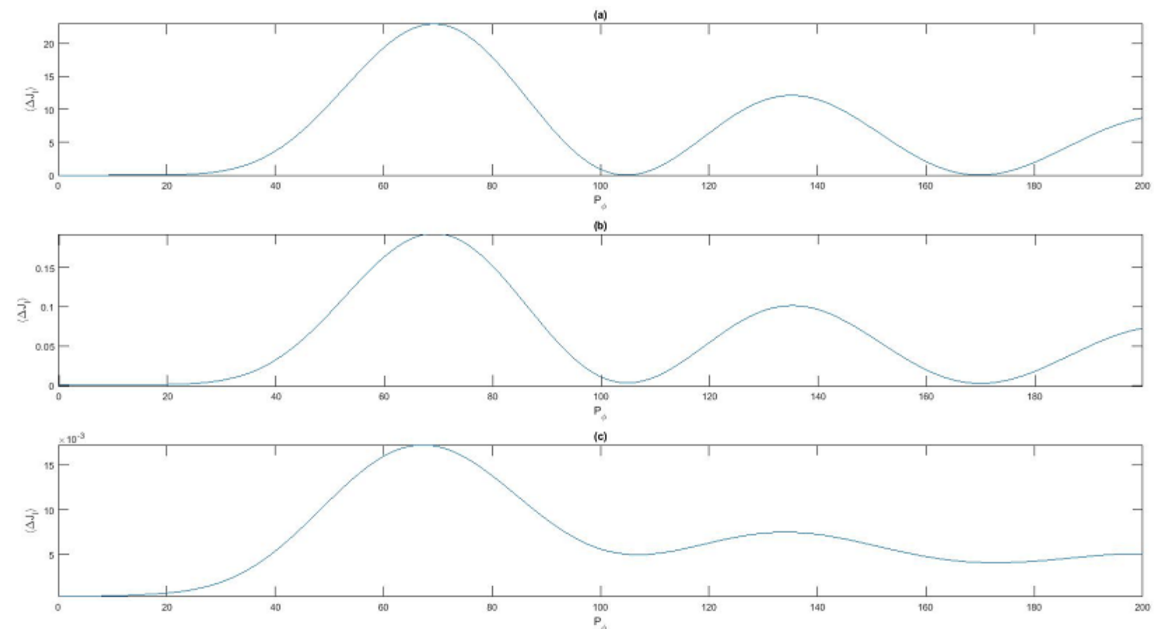


Figure 5.3: Averaged action variation $\langle \Delta J_l \rangle$ over P_ϕ for trapping amplitude strength $A = 100$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p compared to the dispersion threshold value p_{th} provided in equation (5.2.3). (a) $p = 0.01$, (b) $p = 0.11$, (c) $p = 0.8$.

In addition the J_l momentum of the strongly trapped particles is given in equation (5.2.6) $J_l << 2A^{1/2}$. Therefore equation (5.2.7) reads

$$A^{1/2} J_l \sim P_\phi << 2A \tag{5.2.8}$$

Equation (5.2.8) explains the behaviour of $\langle \Delta J_l \rangle$ variation in respect with P_ϕ momentum. As we

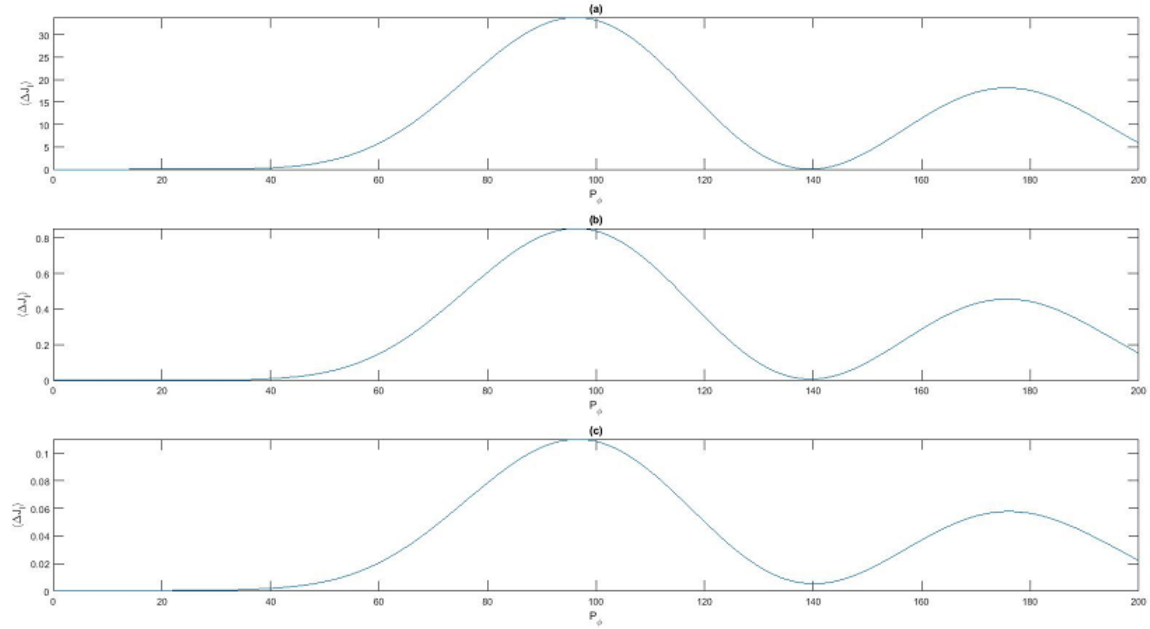


Figure 5.4: Averaged action variation $\langle \Delta J_l \rangle$ over P_ϕ for trapping amplitude strength $A = 150$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p compared to the dispersion threshold value p_{th} provided in equation (5.2.5). (a) $p = 0.2$, (b) $p = 0.1$, (c) $p = 1$.

increase the trapping strength A the P_ϕ momentum has to increase as well in order to be comparable to $A^{1/2} J_l$. In the same time the J_l domain of trapped particles has expanded due to the increased trapping strength A and that leads to wider accepted values of P_ϕ momentum. For instance when $A = 100$ for a trapped particle with $J_l = 1$ equation (5.2.8) proposes that $10 \sim P_\phi << 200$ while from Figure 5.2 the peak of the strongest acceleration is $P_\phi \simeq 70$ and the width of the strongest accelerating curve is characterized by $40 \leq P_\phi \leq 90$. Consequently our simplified analytical insight explains adequately the J_l acceleration.

The importance of our results concerning the mean variation $\langle \Delta J_l \rangle$ is of great significance because by adjusting the amplitude A we can select the range of gyro-radius of the particles that we want to enhance their J_l momentum.

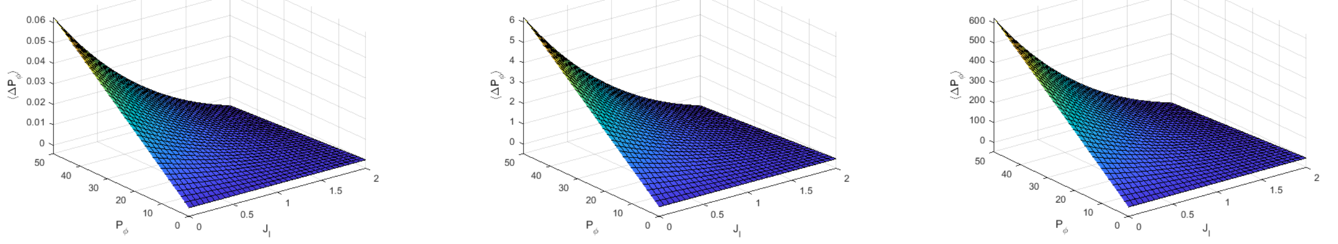
In the same manner as before for the P_ϕ variation we obtain

$$\langle \Delta P_\phi \rangle = \frac{(\epsilon A \alpha_{1\parallel})^2}{2\rho} \sum_b b^2 \mathcal{J}_b(\alpha_{1\perp}\rho) \frac{\partial \mathcal{J}_b}{\partial \rho} \left[\frac{1}{(b+p)^2} + \frac{J_l}{2A^{1/2}(A^{1/2} - b - p)^2} + \frac{J_l}{2A^{1/2}(A^{1/2} + b + p)^2} \right] \tag{5.2.9}$$

One significant difference between equations (5.2.9) and (5.2.1) is that in the h_{1a} term in (4.2.74) participates in the mean variation of the P_ϕ momentum. In addition relation (5.2.9) depends from both J_l, P_ϕ actions of the system. Analytical estimation of the behavior of $\langle \Delta P_\phi \rangle$ is difficult due to the dependence from the derivative of the Bessel function $\frac{\partial \mathcal{J}_b}{\partial \rho}$ which can only evaluated from the recurrence relation [27]

$$\frac{\partial \mathcal{J}_b}{\partial \rho} = \frac{b}{\alpha_{1\perp}\rho} \mathcal{J}_b(\alpha_{1\perp}\rho) - \mathcal{J}_{b+1}(\alpha_{1\perp}\rho) \tag{5.2.10}$$

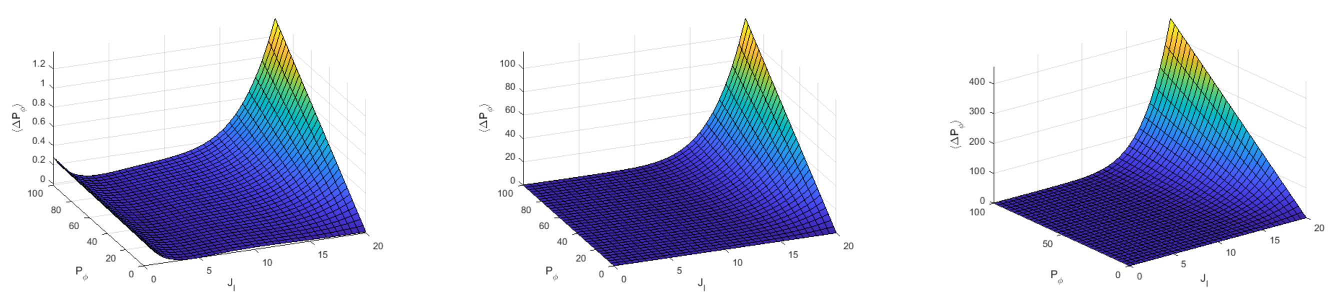
As a result, we will examine possible acceleration with various combinations of A, p through the $\langle \Delta P_\phi \rangle = \langle \Delta P_\phi \rangle (J_l, P_\phi)$ surface.



(a) p=0.01 (b) p=0.01 (c) p=0.001

Figure 5.5: Averaged action variation $\langle \Delta P_\phi \rangle$ over P_ϕ and J_l for trapping amplitude strength $A = 1$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p. The range of J_l has selected from equation (5.2.6).

Figures 5.5a-5.5c reveal that for $A = 1$, with the trapped particles characterized from $J_l << 2$ and small dispersion $p \simeq 0.001$ leads to significant gyro-acceleration of the trapped particles even for small P_ϕ values. In that way the magnetic moment breaks down as an invariant and the ensemble of the trapped particles is heated.
On the other hand the behaviour of $\langle \Delta P_\phi \rangle$ is very different from that of $\langle \Delta J_l \rangle$ in larger trapping amplitudes. In particular, Figures 5.6a-5.6c propose that P_ϕ acceleration is only possible around the $J_l = 20$ even for small initial values of P_ϕ. But, for $A = 100$ the allowed domain of J_l for the strongly trapped particles is $J_l << 20$. As a result strongly trapped particles cannot gyro-accelerated coherently for large trapping strength A and only the "weakly" trapped particles can feel the acceleration. This an important realization since the coherent acceleration of the weakly trapped particles can synergies with the stochastic area around the separatrix of the perturbed pendulum Hamiltonian. Strickly speaking in order to examine the exact acceleration of the weakly trapped particles we must refine our formulation in Chapter 4 by keeping more terms in the $1/m$ expansions in order to adequately describe larger portion of the trapped phase space.



(a) p=0.01 (b) p=0.01 (c) p=0.005

Figure 5.6: Averaged action variation $\langle \Delta P_\phi \rangle$ over P_ϕ and J_l for trapping amplitude strength $A = 100$, $\epsilon = 0.01$ and $\alpha_{1\perp} = \alpha_{1\parallel} = 1$ for different values of dispersion p. The range of J_l has selected from equation (5.2.6).

In conclusion, it is obvious that both mean angle variations $\langle \Delta P_\phi \rangle$ and $\langle \Delta J_l \rangle$ propose a selective nature of momentum transport for $p < 1$ of the trapped particles in plasma. The transferred momentum J_l in the strongly trapped particles depends from a low p_{th} dispersion threshold given from relations (2.5.3) and (2.5.5). The selective nature of J_l accelerations reflects on the fact that for stronger trapping amplitude A the allowed domain of the initial P_ϕ values for which the acceleration occurs is increased as it shifts to higher values. On the contrary the P_ϕ transferred momentum is

also constrained from a dispersion threshold but as the amplitude A increases it is present only to the weakly trapped particles.
These transport properties enables us to manipulate the properties of the waves suitably in order to heat up specific particle populations in a fusion plasma and possibly heal any detected instabilities.

5.3 Hierarchy of Kinetic Equations

5.3.1 Approximate Solutions of Vlasov Equation

In this Section we are going to exploit the Lie perturbation theory in order evaluate approximate expansion for the evaluation of the distribution function of a population of particles in plasma.
The general formalism of the Hamiltonian that describes the interaction between an ion and electromagnetic waves as we have already encountered is

$$H = H_0(\boldsymbol{J}) + H_1(\boldsymbol{J}, \boldsymbol{\theta}, t) \tag{5.3.1}$$

Where H_1 is periodic over some angles. From the Lie transformation theory we have

$$\bar{\boldsymbol{x}} = T\boldsymbol{x} \tag{5.3.2}$$

The T operator to first order has the form

$$T = T_0 + T_1 = I - [w_1, \quad] \tag{5.3.3}$$

Then the transformation from the old action-angle variables $\boldsymbol{z} = (\boldsymbol{J}, \boldsymbol{\theta})$ to the new variables $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})$ is given from

$$\bar{\boldsymbol{J}} = \boldsymbol{J} - \frac{\partial w_1}{\partial \boldsymbol{\theta}} \tag{5.3.4}$$

$$\bar{\boldsymbol{\theta}} = \boldsymbol{\theta} - \frac{\partial w_1}{\partial \boldsymbol{J}} \tag{5.3.5}$$

In order to evaluate the new variables in (5.3.4) and (5.3.5) we need the Lie generating function w_1 which we obtain by substituting $K_1 = 0$ in equation (2.3.28a). In that way, the new integrable to first order system has the form $K = K_0 = H_0(\bar{\boldsymbol{J}})$.
The Vlassov equation for the K Hamiltonian is

$$\frac{\partial \bar{f}}{\partial t} + [\bar{f}, K] = 0 \tag{5.3.6}$$

Since the plasma is collision-less as we have already mentioned the distribution function in the phase space of one charged particle corresponds to the distribution function of the whole population. We will return to this statement later in order to highlight the significance of it.
The Hamiltonian K is time independent, so the solution of the Vlassov must have the form $\bar{f} = \bar{f}(\bar{\boldsymbol{J}})$. For instance it can be Maxwellian

$$\bar{f}(\bar{\boldsymbol{J}}) = \Big(\frac{M}{2\pi k_B T}\Big) \exp\left\{ -\frac{M\bar{\boldsymbol{J}}^2}{2k_B T} \right\} \tag{5.3.7}$$

Due to the fact that the T operator acts on functions using relation (5.3.3) we obtain the first order approximation in the distribution function $f(\boldsymbol{J}, \boldsymbol{\theta}, t)$ in the $(\boldsymbol{J}, \boldsymbol{\theta})$ space as

$$f(\boldsymbol{z}) = T\bar{f}(\bar{\boldsymbol{J}}) = \bar{f}(\boldsymbol{J}) - \frac{\partial w_1}{\partial \boldsymbol{\theta}} \frac{\partial \bar{f}(\boldsymbol{J})}{\partial \boldsymbol{J}} = f_0(\boldsymbol{J}) + f_1(\boldsymbol{z}) \tag{5.3.8}$$

Where we have set $\boldsymbol{z} = (\boldsymbol{J}, \boldsymbol{\theta}, t)$.
The distribution function $f(\boldsymbol{z}) = f_0(\boldsymbol{J}) + f_1(\boldsymbol{z})$ satisfies to first order the Vlassov equation of the original system

$$\frac{\partial f}{\partial t} + [f, H] = 0 \tag{5.3.9}$$

Equation (5.3.7) is very important since we have derived the first order correction of distribution function $f(\boldsymbol{z})$ in a self-contained manner in action angle variables without any linearazation procedures of the electromagnetic fields as in [28]. All the important dynamics of Hamiltonian H that have been embed in w_1 they have also transferred in the f_1 part of the distribution function. As a result the distribution function in (5.3.7) retains all the characteristics of the initial Hamiltonian $H = H_0 + H_1$. Moreover, due to the collision-less nature of the plasma the transition to the collective distribution function has achieved without any Markovian statistical assumption. The Markovian assumption is contrary to the dynamical behavior of particles interacting with coherent waves. The particles phase space is a mix of chaotic and coherent motions with islands of coherent motion embedded within chaotic regions. Also, the phase space is bounded and near the boundaries, or near islands, particles can get stuck and undergo coherent, correlated, motion for times very much longer than an interaction time. Furthermore, in practice, particles do not continuously interact with the same spectrum of waves, either because the waves evolve in time or because the waves are spatially confined. Particles undergoing multiple transits are likely to drift away from the location where the previous interaction took place. This occurs in tokamaks where the radio frequency waves used for heating and current drive are localized over part of the plasma [29].
In general we can approximate the solution of the Vlassov equation to the same order with the perturbation order of the operator T.

5.3.2 Diffusion Equation

The main problem with solution (5.3.7) for the distribution function is that they are not smooth, they have singular points that represent the resonant points in the phase space. As a result, the charge density as well as the current that dictate the Maxwell equations exhibit singular behavior. In order to solve the situation we use a coarse-grain form of $f(\boldsymbol{z})$ by averaging over the angles θ

$$F(\boldsymbol{J}, t) = \langle f(\boldsymbol{z}) \rangle_{\boldsymbol{\theta}} \tag{5.3.10}$$

Due to periodic dependence of w_1 from at least one angle the averaging of the distribution function in (5.3.7) produces

$$F(\boldsymbol{J}, t) = f_0(\boldsymbol{J}) \tag{5.3.11}$$

Since the f_0 is a Maxwellian given in relation (5.3.6) according to the previous discussion the averaging method does not produce a satisfactory solution. In that way, we need to expand the Lie operator T to second order as in Section 5.1 in order to have a non-zero contribution in f_0 from the averaging. Similarly to equations (5.1.11)-(5.1.13) we obtain

$$F(\boldsymbol{J}, t) = f_0(\boldsymbol{J}) - \frac{1}{2}\left\langle \left(\frac{\partial w_1}{\partial \boldsymbol{\theta}}\right)^2 \right\rangle \frac{\partial^2 \bar{f}(\boldsymbol{J})}{\partial \boldsymbol{J}^2} \tag{5.3.12}$$

The distribution function $F(\boldsymbol{J}, t)$ is not a solution of the Vlassov equation. Therefore, such a smooth, angle averaged distribution function cannot satisfy Vlassov but a different Fokker-Planck type equation
In order to evaluate the equation for $F(\boldsymbol{J}, t)$ we use equation (5.1.7) in the $[t, t + \Delta t]$ time interval [29]

$$f(\boldsymbol{z})_{t+\Delta t} = T^{-1}[\boldsymbol{J}_t, \boldsymbol{\theta}_t + \boldsymbol{\omega}_K(\boldsymbol{J}_t)\Delta t, t + \Delta t] f(\boldsymbol{z})_t \tag{5.3.13}$$

As a result we can write

$$f(\boldsymbol{z})_{t+\Delta t} - f(\boldsymbol{z})_t = [T^{-1} - I][\boldsymbol{J}_t, \boldsymbol{\theta}_t + \boldsymbol{\omega}_K(\boldsymbol{J}_t)\Delta t, t + \Delta t] f(\boldsymbol{z})_t \tag{5.3.14}$$

Dividing equation (5.3.13) with $\Delta t \to 0$ we obtain

$$\frac{\partial f(\boldsymbol{z})}{\partial t} = \frac{\partial [T^{-1} - I](\boldsymbol{z})}{\partial t} f(\boldsymbol{z}) \tag{5.3.15}$$

Equation (5.3.14) is an approximation to Vlassov equation to the same order of the T expansion. Averaging equation (5.3.14) we get the evolution equation for $F(\boldsymbol{J}, t)$ since the average operator commutes with the partial time derivative operator

$$\frac{\partial F(\boldsymbol{J}, t)}{\partial t} = \frac{\partial \langle [T^{-1} - I](\boldsymbol{z}) \rangle_{\boldsymbol{\theta}}}{\partial t} F(\boldsymbol{J}, t) \tag{5.3.16}$$

Expanding operator T to second order we have the analogous results with those of equations (5.1.11)-(5.1.13)

$$\langle L_n F(\boldsymbol{J}, t) \rangle_{\boldsymbol{\theta}} = \langle [w_n(\boldsymbol{z}), F(\boldsymbol{J}, t)] \rangle_{\boldsymbol{\theta}} = \langle \nabla_{\boldsymbol{\theta}} w_n \rangle \nabla_{\boldsymbol{J}} F = 0 \ for \ n = 1, 2 \tag{5.3.17}$$

On the other hand,

$$\langle L_1^2 F(\boldsymbol{J}, t) \rangle_{\boldsymbol{\theta}} = \nabla_{\boldsymbol{J}} \cdot [\langle \nabla_{\boldsymbol{\theta}} w_1 \nabla_{\boldsymbol{\theta}} w_1 \rangle_{\boldsymbol{\theta}} \cdot \nabla_{\boldsymbol{J}} F(\boldsymbol{J}, t)] \tag{5.3.18}$$

Again our results are accurate to second order using only the first order generating function w_1. Then, the evolution equation (5.3.15) takes the form

$$\frac{\partial F(\boldsymbol{J}, t)}{\partial t} = \nabla_{\boldsymbol{J}} \cdot [\boldsymbol{D}(\boldsymbol{J}, t) \cdot \nabla_{\boldsymbol{J}} F(\boldsymbol{J}, t)] \tag{5.3.19}$$

Where $\boldsymbol{D}(\boldsymbol{J}, t)$ is the generalized quasi-linear tensor [29] which has the form

$$\boldsymbol{D}(\boldsymbol{J}, t) = \frac{1}{2} \frac{\partial \langle \nabla_{\boldsymbol{\theta}} w_1 \nabla_{\boldsymbol{\theta}} w_1 \rangle_{\boldsymbol{\theta}}}{\partial t} \tag{5.3.20}$$

Setting $F = \boldsymbol{J}$ we obtain the results of Section 5.1. The diffusion tensor $\boldsymbol{D}(\boldsymbol{J}, t)$ in time dependent and non-singular since is derived from finite time interval perturbation theory. In fact the diffusion tensor exhibits functional dependence of the form

$$R(\Omega, t, t_0) = \frac{e^{i\Omega t} - e^{i\Omega t_0}}{i\Omega} = \int_{t_0}^{t} e^{i\Omega t} \, dt \tag{5.3.21}$$

Function R is smooth and localized around the resonances $\Omega = 0$. Again, in the derivation of the diffusion tensor we have included any Markovian assumption for the statistics of the particles. The standard dependence of a singular diffusion tensor under Markovian statistical assumption is obtained time interval $\Delta t \to \infty$. Then $R = 2\pi\delta(\Omega)$ and the particles orbits are completely decorrelated. But, as we have already figure, the phase space is non-homogeneous and regular orbits characterized as quasi-periodic around resonant islands must be strongly correlated. As such, a Markovian assumption cannot describe globally the phase space.

Finally, the methods we have described in Section 5.3 cannot globally applied to our problem of charged particle interaction with electrostatic wave-packet since different action variables J_l, J_r exist for different regions of phase space. Hence, we can only apply our conclusions for the kinetic equations for each region (trapped, free) separately.

Chapter 6

Conclusions and Future Research

In this final part we are going to sum up the results that have been produced over the Chapters 4 and 5 as well as to point out possible generalizations of our work as references for future research. The most important elements of our research are the trapping amplitude A and the dispersion magnitude p, both of them been introduced naturally with the assumption of an electrostatic, finite amplitude, periodic wave packet. As we have mentioned this type of interaction not only generalizes the electrostatic cases of Chapter 2 but also reflects the nature of the interactions in both fusion and astrophysical plasmas.

As a result of our formulation the unperturbed gyro-motion of the particle can be constrained due to the introduced trapping potential. On the other hand, if the particle is energetic enough then his motion can be described to first order by free gyro-motion under the presence of a low amplitude wave and falls on the cases that we have investigated in Chapter 2. Subsequently, it is of great interest to investigate the dynamics of the trapped gyrating particle since it represents the low energy spectrum of distribution function.

Since the plasma is collision-less our results for the dynamics of the one particle can be applied for a collective behaviour of particles in plasma. As such we are in the position to examine the transport properties of the trapped gyrating particles under different conditions of A and p.

Large values of dispersion $p > 1$ are associated with the resonant particle acceleration. On the other hand, small values of $p < 1$ are contributing to coherent acceleration of the particles that connects with anomalous transport. This particular realization comes from the sensitive and selective nature of the mean action variations from the values of A, p.

Finally, we use the formalism of Lie perturbation theory in order to find approximate distribution solutions of Vlassov equation as well as to extract a new Fokker-Planck type equation for the system. Our results are not constrained from any statistical assumption and they are derived from the dynamics of the system itself without any linearization.

In the end, possible topics and generalizations of our work as references for future research may include:

- Non-uniform magnetic field for astrophysical plasmas, toroidal geometry magnetic field for fusion plasmas.
- The presence of electromagnetic waves.
- The interaction with a finite amplitude solitary electrostatic wave packet.
- The co-existence of small amplitude electrostatic wave-packets.
- Advanced interpretation of the stochastic nature of the system.
- Global phase space description of diffusion equation and solutions of Vlassov equation.

Appendix A

Jacobi Elliptic Functions in Pendulum

The Hamiltonian h_0 in (3.2.5) resembles to that of a pendulum and has to be formulated in action-angle variables suitable for perturbation theory. This procedure incorporates the Jacobi Elliptic Functions and it will be presented in the following.
The Hamiltonian in (3.2.5) is

$$h_0 = \frac{P_Z^2}{2} + P_\phi - A\cos Z \tag{A.1}$$

The action integral is given from

$$J = \frac{1}{2\pi}\oint P_Z\, dZ \tag{A.2}$$

With

$$P_Z = \pm\left\{\frac{4A}{m}\left[1 - m\sin^2\frac{Z}{2}\right]\right\}^{1/2} \tag{A.3}$$

$$m = \frac{2A}{E - P_\phi + A} \tag{A.4}$$

Parameter m defines the motion of the particle in the potential.
• For $m < 1$, the total energy of the particle $E = h_0$ in larger than the maximum potential amplitude $A + P_\phi$. Thus the particle "rotates" freely either in the upper half plane or in the down half plane of $P_Z - Z$ space according to the sign of (A.3).
• For $m > 1$, the total energy of the particle $E = h_0$ in lesser than the maximum potential amplitude $A + P_\phi$. Thus the particle "librates" trapped in the potential, oscillating between $|Z| \leq Z_c$.
• For $m = 1$, the total energy of the particle $E = h_0$ equals the maximum potential amplitude $A + P_\phi$ and the solutions represent the separatrices which connect the unstable equilibrium points $Z = \pm\pi$. and separates the rotation from the libration motion.
We will examine each motion separately.

A.1 Rotation

For the untapped particle case $m < 1$ with $-\pi \leq Z \leq \pi$ for $P_Z > 0$ we introduce the Complete Elliptic integral of Second Kind $E(m)$

$$E(m) = \int_0^{\frac{\pi}{2}} (1 - m\sin^2\xi)^{1/2}\, d\xi \tag{A.1.1}$$

Thus the action combining (A.2) and (A.3) is

$$J_r = \frac{4}{\pi}\left(\frac{A}{m}\right)^{1/2} E(m) \tag{A.1.2}$$

In order to evaluate the conjugate angle variable of J_r we need to compute the generating function $W_r = W_r(J_r, Z)$ since P_ϕ is already and action for h_0.

$$P_Z = \frac{\partial W_r}{\partial Z} \tag{A.1.3}$$

$$W_r = \int_0^Z P_Z(m, Z')\, dZ' \tag{A.1.4}$$

Now we introduce the Incomplete Integral of Second Kind $E(s|m)$

$$E(s|m) = \int_0^s (1 - m\sin^2\xi)^{1/2}\, d\xi \tag{A.1.5}$$

$$s = \frac{Z}{2} \tag{A.1.6}$$

$$E(-s|m) = -E(s|m) \tag{A.1.7}$$

Thus, the generating function W_r takes the form

$$W_r = \pm 4\left(\frac{A}{m}\right)^{1/2} E(s|m) \tag{A.1.8}$$

The conjugate angle variable θ_r is derived from

$$\theta_r = \frac{\partial W_r}{\partial J_r} = \frac{\partial W_r}{\partial m}\frac{\partial m}{\partial J_r} = \frac{\partial W_r}{\partial m}\left(\frac{\partial J_r}{\partial m}\right)^{-1} \tag{A.1.9}$$

In order to evaluate (A.1.9) we will exploit the following derivatives in respect with the parameter x

$$\frac{\mathrm{d}E(s|x}{\mathrm{d}x} = \frac{1}{2x}[E(s|x) - F(s|x)] \tag{A.1.10a}$$

$$\frac{\mathrm{d}E(x)}{\mathrm{d}x} = \frac{1}{2x}[E(x) - K(x)] \tag{A.1.10b}$$

$$\frac{\mathrm{d}F(s|x}{\mathrm{d}x} = \frac{E(s|x)}{2x(1-x)} - \frac{F(s|x)}{2x} \tag{A.1.10c}$$

$$\frac{\mathrm{d}K(x)}{\mathrm{d}x} = \frac{E(x)}{2x(1-x)} - \frac{K(x)}{2x} \tag{A.1.10d}$$

In the previous relations we have produced the Complete Elliptic Integral of First Kind $K(m)$ and the Incomplete Elliptic Integral of First Kind $F(s|m)$ defined as follows

$$F(s|m) = \int_0^s \frac{d\xi}{(1 - m\sin^2\xi)^{1/2}} \tag{A.1.11}$$

$$K(m) = F(s = \frac{\pi}{2}|m) \tag{A.1.12}$$

Thus, equation (A.1.9) provides us with

$$\theta_r = \pm \frac{\pi u}{K(m)} \tag{A.1.13}$$

With

$$u = F(s|m) \tag{A.1.14}$$

Through the integral u we can define the Jacobi Elliptic Function $pq(u|m)$ as

$$s = am(u|m) \tag{A.1.15}$$

$$\sin s = sn(u|m) \tag{A.1.16}$$

$$\cos s = cn(u|m) \tag{A.1.17}$$

$$dn(u|m) = \frac{\mathrm{d}E(s|m)}{\mathrm{d}m} = (1 - m\sin^2 s)^{1/2} \tag{A.1.18}$$

The letters p, q in the Jacobi Elliptic Functions can be any two pf the letters s, c, d, n having a simple zero at p and a simple pole in q. Similarly to expressions (A.1.15)-(A.1.18) we can define all the $pq(u|m)$ functions. The Hamiltonian h_0 in the J_r, P_ϕ, θ_r action- angle variables is

$$h_0 = \left[\frac{2}{m(J_r)} - 1\right] A + P_\phi \tag{A.1.19}$$

Now we seek the relation $Z = Z(\theta_r, J_r)$. From equation (A.1.16) we obtain that

$$\sin s = sn(u|m) \Rightarrow s = \arcsin\left(sn(u|m)\right) \tag{A.1.20}$$

Furthermore,

$$\int_0^u dn(u|m)\, du = \arcsin\left(sn(u|m)\right) \tag{A.1.21}$$

We continue by expanding the $dn(u|m)$ in Lamden Series in order to evaluate the integral in (A.1.21)

$$dn(u|m) = \frac{\pi}{2K(m)} + \frac{2\pi}{K(m)} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos\left(\frac{n\pi u}{K(m)}\right) \tag{A.1.22}$$

The Nome $q(m)$ is defined as

$$q = q(m) = \exp\left[\frac{-\pi K'(m)}{K(m)}\right] \tag{A.1.23a}$$

$$K'(m) = \int_0^{\frac{\pi}{2}} (1 - m_1 \sin^2\xi)^{-1/2}\, d\xi \tag{A.1.23b}$$

$$m + m_1 = 1 \tag{A.1.23c}$$

While m is the parameter m_1 is the complementary parameter of the elliptic functions. The integration of (A.1.22) combined with (A.1.21), (A.1.6) and (A.1.13) gives

$$Z = \pm\left[\theta_r + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \frac{\sin(n\theta_r)}{n}\right] \tag{A.1.24}$$

A.2 Libration

For the trapped particle case $m > 1$ with $-Z_c \leq Z \leq Z_c$ we have first to find the bounce points for $P_Z = 0$ using the equation in (A.3), as such

$$Z_c = 2\arcsin\left(\frac{1}{\sqrt{m}}\right) \tag{A.2.1}$$

Then the action integral in (A.2) takes the form

$$J_l = \frac{2}{\pi}\left(\frac{4A}{m}\right)^{1/2}\int_{-Z_c}^{Z_c}\left[1 - m\sin^2\frac{Z}{2}\right]^{1/2} dZ \tag{A.2.2}$$

In contrast to previous case, now $m > 1$ and we cannot use the $E(m)$ integral. In order to bypass this difficulty we use the reciprocal transformation of the parameter m

$$\sin\frac{Z}{2} = \frac{1}{\sqrt{m}}\sin\xi \tag{A.2.3}$$

Then (A.2.2) transforms to

$$J_l = \frac{4A^{1/2}}{\pi m}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\xi\left(1 - \frac{1}{m}\sin^2\xi\right)^{-1/2} d\xi \tag{A.2.4}$$

The parameter of the Elliptic Integrals and Jacobi functions is $\frac{1}{m} < 1$ for $m > 1$. As a result the action (A.2.4) is

$$J_l = \frac{8A^{1/2}}{\pi}\left[E(1/m)) + \left(\frac{1-m}{m}\right)K(1/m)\right] \tag{A.2.5}$$

The generating function W_l of the transformation again from (A.1.4)

$$W_l = 2\left(\frac{A}{m}\right)^{1/2}\int_0^Z\left[1 - m\sin^2\frac{Z'}{2}\right]^{1/2} dZ' \tag{A.2.6}$$

Again with the use of the reciprocal transformation we have

$$\sin\frac{Z'}{2} = \sin s' = \frac{1}{\sqrt{m}}\sin\xi \tag{A.2.7}$$

$$W_l = \frac{4A^{1/2}}{m}\int_0^{\tilde{s}}\cos^2\xi\left(1 - \frac{1}{m}\sin^2\xi\right)^{-1/2} d\xi \tag{A.2.8}$$

$$\sin\tilde{s} = \sqrt{m}\sin s \tag{A.2.9}$$

Then

$$W_l = 4A^{1/2}\left[E(\tilde{s}|1/m) + \left(\frac{1-m}{m}\right)K(\tilde{s}|1/m)\right] \tag{A.2.10}$$

It is now possible to extract the θ_l variable

$$\theta_l = \frac{\partial W_l}{\partial J_l} = \frac{\partial W_l}{\partial(1/m)}\frac{\partial(1/m)}{\partial J_l} = \frac{\partial W_l}{\partial(1/m)}\left(\frac{\partial J_l}{\partial(1/m)}\right)^{-1} \tag{A.2.11}$$

$$\theta_l = \frac{\pi\tilde{u}}{2K(1/m)} \tag{A.2.12}$$

$$\tilde{u} = F(\tilde{s}|1/m) \tag{A.2.13}$$

Again the Hamiltonian h_0 has the same form as in the rotation case but the action now is J_l

$$h_0 = \left[\frac{2}{m(J_l)} - 1\right] A + P_\phi \tag{A.2.14}$$

In order to complete the transformation we need the relation between Z and θ_l. We need to highlight that the parameter of the functions now is $1/m$ and the amplitudes from the $\tilde{s}$. As a result

$$\sin \tilde{s} = sn(\tilde{u}|1/m) \tag{A.2.15}$$

$$\cos \tilde{s} = cn(\tilde{u}|1/m) \tag{A.2.16}$$

Moreover, it is valid that

$$dn(\tilde{u}|1/m) = \left(1 - \frac{1}{m}\sin^2 \tilde{s}\right)^{1/2} \tag{A.2.17}$$

$$dn(\tilde{u}|1/m) = 1 - \frac{1}{m} + \frac{1}{m} cn(\tilde{u}|1/m) \tag{A.2.18}$$

Combining (A.2.15)-(A.2.18) we produce

$$\cos s = (1 - \sin s)^{1/2} = \left(1 - \frac{1}{m} sn(\tilde{u}|1/m)\right)^{1/2} = dn(\tilde{u}|1/m) \Rightarrow s = \arccos(dn(\tilde{u}|1/m)) \tag{A.2.19}$$

In addition

$$\int_0^{\tilde{u}} cn(\tilde{u}|1/m)\, d\tilde{u} = m^{1/2} \arccos\left(dn(\tilde{u}|1/m)\right) \tag{A.2.20}$$

Like the previous case we have to expand $cn(\tilde{u}|1/m)$ in order to solve for $s = z/2$.

$$cn(\tilde{u}|1/m) = \frac{2\pi m^{1/2}}{K(1/m)} \sum_{n+\frac{1}{2}}^{\infty} \frac{q^n}{1+q^{2n}} \cos\left(\frac{n\pi\tilde{u}}{K(1/m)}\right) \tag{A.2.21}$$

The Nome q is characterized by the dependence $q = q(1/m)$ in contrast to the rotation case and can defined as

$$q = q(1/m) = \exp\left[\frac{-\pi K'(1/m)}{K(1/m)}\right] \tag{A.2.22a}$$

$$K'(1/m) = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{m_1}\sin^2 \xi\right)^{-1/2} d\xi \tag{A.2.22b}$$

$$\frac{1}{m} + \frac{1}{m_1} = 1 \tag{A.2.22c}$$

Solving the integral (A.2.20) with the aid of (A.2.21) together with (A.2.19) and (A.2.12) we obtain

$$Z = 4 \sum_{n+\frac{1}{2}}^{\infty} \frac{q^n}{1+q^{2n}} \frac{\sin(2n\theta_l)}{n} \tag{A.2.23}$$

Appendix B

Generalized Bessel Functions

Infinite variable Bessel Functions are the generalized extension of finite dimensional Bessel Functions through the integral definition [30]

$$\mathcal{J}_n(\{\beta_m\}) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - \beta_1 \sin\theta - \beta_2 \sin 2\theta - ... - \beta_m \sin m\theta - ...)\, d\theta \tag{B.1}$$

Where $\{\beta_m\}$ are real and satisfying that the following series is convergent

$$\sum_{m=1}^{\infty} m|\beta_m| < \infty \tag{B.2}$$

Under those two assumptions the integral in (B.1) converges and ensures the existence of such functions.
From Fourier Series we can expand a function $f(\theta)$ continuous, piece-wise smooth and periodic in $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ as

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} = \frac{\alpha}{2} + \sum_{n=1}^{\infty} \alpha_n \cos n\theta + \sum_{n=1}^{\infty} \beta_n \sin n\theta \tag{B.3}$$

If we consider functions with higher degree of smoothness then the following theorem holds for the Fourier expansion.

Theorem 1 *If a real function, $f(\theta)$ and its derivative, $f^{(1)}(\theta)$ are continuous in $[-\pi, \pi]$ with*

$$f^{(l)}(-\pi) = f^{(l)}(\pi) \ \ for \ \ l = 0, 15 \tag{B.4}$$

and the second derivative $f^{(2)}$ exists and is piece-wise continuous in $[-\pi, \pi]$ then the series

$$\sum_{m=1}^{\infty} m^l(|\alpha_m| + |\beta_m|) \ \ l = 0, 1 \tag{B.5}$$

is convergent with $\{\alpha_m\}$ and $\{\beta_m\}$ the Fourier coefficients of (B.3).

Finally if the function $f(\theta)$ is odd then $f(\pm\pi) = 0$ and $\{\alpha_m\} = 0$. As such, the Fourier sine coefficients, $\{\beta_m\}$, of an arbitrary function of Theorem 1, which is odd, satisfy the existence conditions for the infinite-dimensional Bessel functions, $\mathcal{J}_n(\{\beta_m\})$
We are now interested to prove the Jacobi-Anger expansion of the form

$$\exp\left\{ i \sum_{m=1}^{\infty} \beta_m \sin(m\theta) \right\} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(\{\beta_m\}_1^{\infty}) e^{in\theta} \tag{B.6}$$

We expand the r.h.s of (B.6) in Fourier Series

$$\exp\left\{ i \sum_{m=1}^{\infty} \beta_m \sin(m\theta) \right\} = \sum_{n=\infty}^{\infty} e^{in\theta} c_n(\{\beta_m\}) \tag{B.7}$$

The Fourier coefficients are given from

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ i \sum_{m=1}^{\infty} \beta_m \sin(m\theta) \right\} e^{-in\theta}\, d\theta \tag{B.8}$$

Taking into consideration the Euler formula and the parities of the sinusoidal functions we obtain

$$c_n = \frac{1}{\pi} \int_{0}^{\pi} \cos\left(n\theta - \sum_{m=1}^{\infty} \beta_m \sin(m\theta) \right) d\theta \tag{B.9}$$

But according to definition (B.1) we get that

$$c_n = \mathcal{J}_n(\{\beta_m\}) \tag{B.10}$$

Subsequently, inserting (B.10) to equation (B.7) yields the Jacobi-Anger expansion of (B.6).

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