

Topological Quantum Computation

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Chapter 1

Introduction

The idea of quantum computers go back in 1982 when Richard Feynman [1] thought that certain many-body quantum Hamiltonians could be simulated exponentially faster on a quantum computer than they could be on a classical computer. Simulations of large scale quantum many-body Hamiltonians are essentially hopeless on classical computers because of the exponentially-large size of the Hilbert space. Having the ability to manipulate an exponentially-large Hilbert space, may also enable progress in the solution of lattice gauge theory and quantum chromodynamics and help understand the strongly-interacting nuclear forces.

Another pioneer in quantum computation was Peter Shor [2] who lighted the fire in quantum computing research not only inside but also outside the physics community. In 1994, he invented an algorithm that could find the prime factors of an m digit number in a length of time $\sim m^2 \log m \log \log m$, much faster than the classic algorithm of a classical computer that would need $\sim \exp(m^{1/3})$ time. Since many encryption schemes depend on the difficulty of finding the solution to problems similar to finding the prime factors of a large number, there is an obvious application of a quantum computer which is of great and applied interest.

The computation is based on three steps: initialization, unitary evolution and measurement. We are equipped with a Hilbert space \mathcal{H} and assume that we can initialize the system in a known state $|\psi_0\rangle$. Then we unitary evolve the system in a final state $U(t)|\psi_0\rangle$ with $dU/dt = iH(t)U(t)/\hbar$ and $H(t)$ the Hamiltonian of the system. We require that we have control over this Hamiltonian so that $U(t)$ can be any unitary transformation we desire. Finally, we measure the state of the system at the end of the evolution and the whole process is called *Quantum Computation*. [3,35]

Although quantum computers have shown a great promise for solving efficiently exponentially hard problems, they are not easy to build as the biggest obstacle is posed by errors which would invariably happen during the computation. In a classical computer errors are typically corrected through redundancies i.e. by keeping multiple copies of information and checking against these copies.

In a quantum computer, however, the situation is much complex as every time we measure an intermediate quantum state during the computation to see if an error has occurred, we collapse the wavefunction and thus destroy the superposition and ruin the calculation

There are two major sources of error. First there is decoherence. These are random errors caused by the interaction between the quantum computer and the environment. Second, even if the system is protected from decoherence, it is almost certain that all operations performed on the quantum information during its processing will be imperfect. These errors will accumulate over the duration of the computation, eventually causing failure. Thus we need to build a quantum computer that is *fault-tolerant*.

Topological quantum computation is an approach of processing fault-tolerant quantum information using exotic quasiparticles, called anyons.[5-10,24,25] Due to their exotic statistical behaviour, they exhibit non-trivial quantum evolutions that are described by topology, i.e. the evolution depends solely if another anyon encircles them or not and not by the trajectory itself. While from the first sight it might look like an obscure way of manipulating quantum information it is linked to quantum error correction, the algorithmic means we have in dealing with errors during quantum computation. In a sense, anyonic quantum computers implement quantum error-correction at the hardware level, thus becoming resilient to control errors and erroneous perturbations. Suppose our system is characterized by the ideal conditions of zero temperature and infinite anyon separation, then the states in the fusion space have three very appealing properties.

- (i) All the states are perfectly degenerate.
- (ii) They are indistinguishable by local operations.
- (iii) They can coherently evolve by braiding anyons.

The system is immune to local perturbations because the operation performed depends only on the topology of the braid. The state of the system is encoded nonlocally and can only be measured by actually fusing the anyons and not by local interactions which makes the states resistant to decoherence. Errors could only occur under unlikely local perturbations to the hamiltonian that would create unwanted anyons and propagate them around the encoding ones.

Suppose we have a system of n identical particles in a three spatial and one time dimension world. The wavefunction that describes the multiparticle state has two possible symmetries. It is either symmetric under the exchange of two particles, and we call these particles *bosons*, or it is antisymmetric and we call the particles *fermions*. Under two consecutive exchanges the process is equivalent to the process where one particle transports around the other. In three dimensions transporting a particle around another is topologically equivalent to the process where no particle moves at all.

Suppose we consider all possible trajectories in $3 + 1$ dimensions which take the n particles from initial positions R_1, R_2, \dots, R_n at time t_i to final positions R_1, R_2, \dots, R_n at time t_f . If the particles are distinguishable then all trajectories can be deformed into the trajectory in which the particles do not move at all. If the particles are indistinguishable then the different trajectories fall into

topological classes corresponding to the elements of the permutation group S_n with each element of the group specifying how the initial positions are permuted to obtain the final positions. Fermions and bosons correspond to the only one dimensional irreducible representation of S_n .

In a two dimensional system though any particle loop that encircles another particle cannot be deformed to a point without cutting through the other particle. So when two particles are exchanged twice in a clockwise manner their trajectory involves a non trivial winding and the system does not necessarily come back to the same state.

Suppose we have two identical particles in two dimensions. Then when one particle is exchanged in a counterclockwise manner with the other, the wavefunction acquires a phase

$$\psi(r_1, r_2) \rightarrow e^{i\phi} \psi(r_1, r_2) \quad (1.1)$$

The phase is not ± 1 because after a second counterclockwise exchange where the particles return to their initial positions the final state is not the same as the initial.

$$\psi(r_1, r_2) \rightarrow e^{2i\phi} \psi(r_1, r_2) \quad (1.2)$$

The special cases $\theta = 0, \pi$ correspond to bosons and fermions. Particles with different values of statistical angle θ are called *anyons*. [11] The topological classes of trajectories which take these particles from initial positions R_1, R_2, \dots, R_n to final positions R_1, R_2, \dots, R_n are in one to one correspondence with the elements of the braid group B_n .

The braid group is infinite and thus it has an infinite number of irreducible representations. Indistinguishable particles that transform as a one-dimensional representation of the braid group are said to be *abelian anyons* and if they transform as nonabelian representations of higher dimension they are said to be *nonabelian anyons*.

Another important concept that will be useful in our analysis is the *Aharonov-Bohm effect*. That is, when an electric charge q is adiabatically transported counterclockwise around a flux Φ , the wavefunction of the charge acquires a *topological phase* $e^{iq\Phi}$. It is topological because the only thing that matters is the winding number i.e. the number of times the charge encircles the flux and not the type of its trajectory.

Plan of the Thesis

In chapter 1 we begin with a brief introduction on how anyons and their topological properties can be used for fault tolerant quantum computation. In chapter 2 we examine the symmetries of the hexagonal lattice, place spin 1/2 particles in each site, let them interact with a Kitaev's honeycomb lattice Hamiltonian and find that in a special case it allows the existence of abelian anyons. In chapter 3 we add a small perturbation that breaks the time reversal symmetry of the model creating an energy gap. This gap is sufficient for nonabelian anyons

whose fusion states can be used for creating the encoding computational states. Moreover, in chapter 4 we work on a continuous gauge theory broken down to the finite group S_3 , find the anyons of the model, construct the quantum double $D(S_3)$ and derive the fusion rules. Finally in chapter 5 and 6 we prove how to construct the fundamental gates with a nonabelian superconductor and with Fibonacci anyons respectively and how measurements can be done to achieve universal quantum computation.

Chapter 2

Eisestein Integers and Honeycomb Lattice Hamiltonian

Equipped with a hexagonal lattice, we place *spin*-1/2 particles in each vertex and let them interact with Kitaev's Hamiltonian [4]

$$H = -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y - J_z \sum_{z-links} \sigma_i^z \sigma_j^z \quad (2.1)$$

with $\sigma_i^\alpha, J_\alpha, \alpha = x, y, z$ denoting the Pauli matrices of the i -th particle and the strength of the interaction respectively. Two particles i, j connected with a specific link, for example x , will interact with the corresponding term in the Hamiltonian : $-J_x \sigma_i^x \sigma_j^x$. We want to examine the *symmetries* of our system. For that purpose we introduce the Eisenstein integers.

Eisestein integers are numbers of the form $Z(\omega) = m + n\omega$ with $m, n \in \mathbb{Z}$ and $\omega = e^{i2\pi/3}$ which can be written as $Z = (m, n)$. If $Z_1 = m_1 + n_1\omega$,

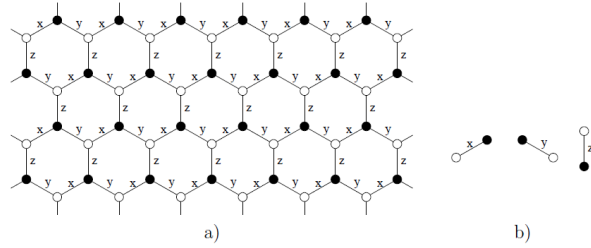


Figure 2.1

$Z_2 = m_2 + n_2\omega$ are two Eisenstein integers, then they satisfy the relations:

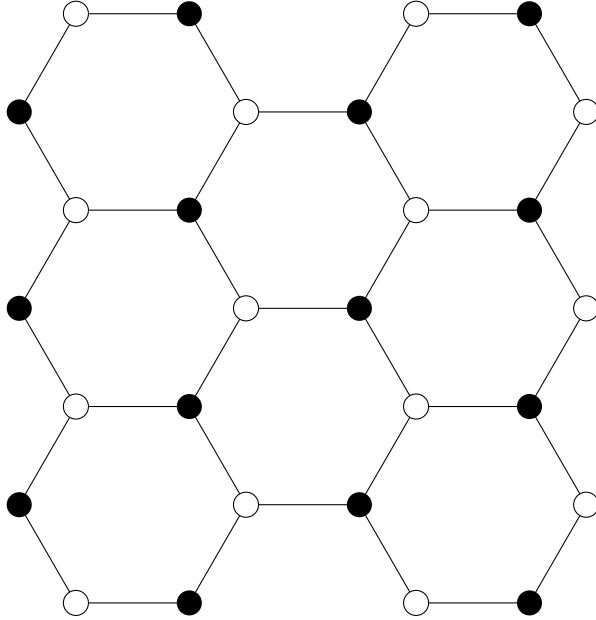
$$\begin{aligned} Z_1 + Z_2 &= (m_1 + m_2, n_1 + n_2) \\ Z_1 Z_2 &= (m_1 m_2 - n_1 n_2, m_1 n_2 + m_2 n_1 - n_1 n_2) \end{aligned} \quad (2.2)$$

It is easy to see that $\omega^3 = 1$ so we can define the norm of an Eisenstein integer as

$$\begin{aligned} |m + n\omega|^2 &= (m + n\omega)(m + n\bar{\omega}) \Rightarrow \\ |m + n\omega|^2 &= m^2 + n^2 - mn \end{aligned} \quad (2.3)$$

We can construct a hexagonal lattice by substracting all the Eisenstein integers which satisfy $(m + n - 2) \bmod 3 = 0$ and connect the lines between all the nearest neighbors. There are two type of vertices.

$$\begin{aligned} \text{Type 1, } (m + n) \bmod 3 = 0 &: \text{ White Circles} \\ \text{Type 2, } (m + n - 1) \bmod 3 = 0 &: \text{ Black Circles} \end{aligned} \quad (2.4)$$



In the infinite lattice above there are permutations that leave the lattice invariant. The allowed permutations are $2p + q\omega$, $p + 2q\omega$, $-p + q\omega$ with $p, q \in \mathbb{Z}$, but each one can be written as a linear combination of the other two. The permutations form a group with elements

$$T_{p,q} = p(2 + \omega) + q(-1 + \omega) = (2p - q) + \omega(p + q) \text{ with } p, q \in \mathbb{Z} \quad (2.5)$$

under the ordinary addition.

Theorem 2.0.1. *If $T_{p,q} = [(2p - q) + (p + q)\omega : p, q \in \mathbb{Z}]$ and $\Sigma_{m,n} = [m + n\omega : m, n \in \mathbb{Z}, (m + n) \bmod 3 = 0]$ then $T_{p,q} = \Sigma_{m,n}$.*

Proof. Suppose $z = (2p - q) + (p + q)\omega \in \mathbb{Z}$, then $2p - q + p + q = 3p = 0 \pmod{3}$ so $z \in \Sigma_{m,n}$.

If now $z \in \Sigma_{m,n}$, then $z = m + n\omega$ with $m + n = 0 \pmod{3} \Rightarrow m = 3p - n$. If $q = n - p$ then $m = 3p - n = 3p - q + p = 2p - q$ which means that z can be written as $z = (2p - q) + (p + q)\omega \in T_{p,q}$ so $T_{p,q} = \Sigma_{m,n}$. \square

If we rotate the lattice by an angle of $2\pi/3$ or by $4\pi/3$ then every element on the lattice is multiplied by ω and $\omega^2 = -\omega - 1$ respectively. It is easy to see that if an Eisenstein integer $z_1 = m_1 + n_1\omega$ belongs in the lattice, therefore $(m_1 + n_1 - 2) \pmod{3} \neq 0$, then after a rotation it will still belong in the lattice with $(m'_1 + n'_1 - 2) \pmod{3} \neq 0$.

Every element on the lattice is characterized by two integers (m, n) so we can define a basis on the two 2 dimension vector space. We can either work on *Eisenstein basis* with elements

$$\begin{bmatrix} m \\ n \end{bmatrix}$$

or work on *Standard basis* with elements

$$\begin{bmatrix} m - n/2 \\ \sqrt{3}n/2 \end{bmatrix}$$

The Matrix that transforms a vector from *Eisenstein basis* to *Standard basis* is

$$U = \begin{bmatrix} 1 & \sqrt{3}/3 \\ 0 & 2\sqrt{3}/3 \end{bmatrix}$$

and therefore the matrix that transforms an *Eisenstein vector* to a *Standard vector* is

$$U^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

The reflection through an axis with unit vector \hat{n} is defined as the matrix

$$P_{\hat{n}} = |\hat{n}\rangle \langle \hat{n}| - 1 = \begin{bmatrix} 2n_x n_x - 1 & 2n_x n_y \\ 2n_y n_x & 2n_y n_y - 1 \end{bmatrix} \quad (2.6)$$

that multiplies a standard vector. The reflection matrix on Eisenstein basis can be calculated by

$$P_{\hat{n}}^{Eis} = U P_{\hat{n}} U^{-1} \quad (2.7)$$

There are 3 reflections in the hexagonal lattice that leave it invariant as it can be seen above. On (a) axis the unit vector is $\hat{a} = 1/2\hat{x} + \sqrt{3}/3\hat{y}$ so

$$P_{\hat{a}} = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

On (b) axis the unit vector is $\hat{b} = \hat{x}$ so

$$P_{\hat{b}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and finally on (c) axis the unit vector is $\hat{c} = -1/2\hat{x} + \sqrt{3}/2\hat{y}$ so

$$P_{\hat{c}} = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

All three reflection matrices satisfy the relation $\det[P_{\hat{n}}] = -1$ and $P_{\hat{n}}^2 = 1$. The action of these matrices on the elements of the lattice is :

$$\begin{aligned} P_{\hat{a}} \begin{bmatrix} m \\ n \end{bmatrix} &= \begin{bmatrix} n - m/2 \\ \sqrt{3}m/2 \end{bmatrix} \\ P_{\hat{b}} \begin{bmatrix} m \\ n \end{bmatrix} &= \begin{bmatrix} m - n/2 \\ -\sqrt{3}n/2 \end{bmatrix} \\ P_{\hat{c}} \begin{bmatrix} m \\ n \end{bmatrix} &= \begin{bmatrix} -m/2 - n/2 \\ -\sqrt{3}m/2 + \sqrt{3}n \end{bmatrix} \end{aligned} \quad (2.8)$$

So putting all the transformations together and using Eisestein notation:

$$\begin{aligned} P_{\hat{a}} : m + n\omega &\rightarrow n + m\omega \\ P_{\hat{b}} : m + n\omega &\rightarrow (m - n) - n\omega \\ P_{\hat{c}} : m + n\omega &\rightarrow -m + (n - m)\omega \\ R_{\omega} : m + n\omega &\rightarrow -n + (m - n)\omega \\ R_{\omega^2} : m + n\omega &\rightarrow (n - m) - m\omega \end{aligned} \quad (2.9)$$

If the lattice is not infinite then there are no permutations that leave the lattice invariant but only rotations and reflections. The symmetry transformations form a Group $G = [e, P_a, P_b, P_c, R_{\omega}, R_{\omega^2}]$ where e means that we apply no transformation. We can calculate the multiplication table of the Group:

G	e	R_{ω}	R_{ω^2}	$P_{\hat{a}}$	$P_{\hat{b}}$	$P_{\hat{c}}$
e	e	R_{ω}	R_{ω^2}	$P_{\hat{a}}$	$P_{\hat{b}}$	$P_{\hat{c}}$
R_{ω}	R_{ω}	R_{ω^2}	e	$P_{\hat{b}}$	$P_{\hat{c}}$	$P_{\hat{a}}$
R_{ω^2}	R_{ω^2}	e	R_{ω}	$P_{\hat{c}}$	$P_{\hat{a}}$	$P_{\hat{b}}$
$P_{\hat{a}}$	$P_{\hat{a}}$	$P_{\hat{c}}$	$P_{\hat{b}}$	e	R_{ω^2}	R_{ω}
$P_{\hat{b}}$	$P_{\hat{b}}$	$P_{\hat{a}}$	$P_{\hat{c}}$	R_{ω}	e	R_{ω^2}
$P_{\hat{c}}$	$P_{\hat{c}}$	$P_{\hat{b}}$	$P_{\hat{a}}$	R_{ω^2}	R_{ω}	e

Moreover the subgroups of G are :

$$\begin{aligned} H_1 &= [e, P_{\hat{a}}], H_2 = [e, P_{\hat{b}}] \\ H_3 &= [e, P_{\hat{c}}], H_4 = [e, R_{\omega}, R_{\omega^2}] \end{aligned} \quad (2.10)$$

with H_4 being the only *normal subgroup* $\Rightarrow gH_4 = H_4g, \forall g \in G$. The conjugacy classes are:

$$\begin{aligned} C_1 &= [e], C_2 = [P_{\hat{a}}, P_{\hat{b}}, P_{\hat{c}}] \\ C_3 &= [R_{\omega}, R_{\omega^2}] \end{aligned} \quad (2.11)$$

The group G is isomorphic to the group S_3 , the group of permutations of three objects, $G \cong S_3$. To be precise:

$$\begin{aligned} e &\rightarrow e, R_\omega \rightarrow (123), R_{\omega^2} \rightarrow (132) \\ P_{\hat{a}} &\rightarrow (12), P_{\hat{b}} \rightarrow (23), P_{\hat{c}} \rightarrow (13) \end{aligned} \quad (2.12)$$

It is known that if d_k is the dimension of an irreducible representation and $|G|$ is the order of the group, then

$$\sum_{k=1}^n d_k^2 = |G| \quad (2.13)$$

There is always the one dimensional irreducible representation, Γ_1 , where every element is mapped to unity, so knowing that the number of irreducible representations is equal to the number of the conjugacy classes of the group then the solution in the above equation is:

$$d_1 = 1, d_2 = 1, d_3 = 2 \quad (2.14)$$

The matrices $P_{\hat{a}}, P_{\hat{b}}, P_{\hat{c}}$ represent reflections so their determinant is equal to -1 but on the contrary, R_ω and R_{ω^2} represent rotations and thus have determinant 1. So the other 1 dimensional irreducible representation, Γ_2 , is:

$$\begin{aligned} [e, R_\omega, R_{\omega^2}] &\rightarrow [1] \\ [P_{\hat{a}}, P_{\hat{b}}, P_{\hat{c}}] &\rightarrow [-1] \end{aligned} \quad (2.15)$$

To determine the two dimensional irreducible representation, Γ_3 , we use the fact that $G \cong S_3$ and S_3 being the group describing the symmetries of an equilateral triangle. The rotation matrix in the standard (x, y) basis is:

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

By setting $\theta = 2\pi/3$ and $\theta = 4\pi/3$ we can calculate $R(2\pi/3) = R[(123)]$ and $R(4\pi/3) = R[(132)]$ respectively.

$$\begin{aligned} R[(123)] &= \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \\ R[(132)] &= \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} \end{aligned} \quad (2.16)$$

Moreover, $(12), (23), (13)$ represent reflections. Setting

$$R[(23)] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

along with conditions:

$$\begin{aligned} R[(23)] \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} &= \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \\ R[(23)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \quad (2.17)$$

we find that

$$R[(23)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (2.18)$$

But representations respect the group multiplications. So, if $(123)(23) = (13)$ then $R[(12)]R[(23)] = R[(13)]$.

$$\begin{aligned} R[(13)] &= R[(123)]R[(23)] = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \\ R[(12)] &= R[(132)]R[(23)] = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \end{aligned} \quad (2.19)$$

which is exactly the form of the reflection matrices we calculated above. So the 2 dimensional irreducible representation of G is:

$$\begin{aligned} e &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_\omega = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \quad R_{\omega^2} = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} \\ P_{\hat{a}} &= \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \quad P_{\hat{b}} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad P_{\hat{c}} = \frac{1}{2} \begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \end{aligned} \quad (2.20)$$

To complete our analysis we can construct the character table of G

	C_1	C_2	C_3
Γ_1	1	1	1
Γ_2	1	-1	1
Γ_3	2	0	-2

with the columns and lines of the table beeing orthogonal.

The Hamiltonian respects the symmetry group G along with time reversal symmetry. We define the operator

$$K_{jk}^\alpha = \sigma_j^\alpha \sigma_k^\alpha \text{ with } \alpha \text{ the type of link that connects } jk \quad (2.21)$$

and the plaquette operator

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \quad (2.22)$$

with the enumeration done as in (2.2). It is easy to see that

$$\begin{aligned} K_{12}K_{23}K_{34}K_{45}K_{56}K_{61} &= \sigma_1^z \sigma_2^z \sigma_2^x \sigma_3^x \sigma_3^y \sigma_4^y \sigma_4^z \sigma_5^z \sigma_5^x \sigma_6^x \sigma_6^y \sigma_1^y \\ \Rightarrow W_p &= K_{12}K_{23}K_{34}K_{45}K_{56}K_{61} \end{aligned} \quad (2.23)$$

Squarring the plaquette operator: $W_p W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = \mathbb{I}$, so the operator W_p has eigevalues ± 1 with high degeneracy. Precisely, the operators W_p are 64×64 matrices with 32 eigenvalues $+1$ and 32 eigevalues -1 . The plaquette operators acting on different plaquettes p, p' clearly commute since they act on different spaces.

$$[W_p, W_{p'}] = 0 \quad \forall \quad p, p' \quad (2.24)$$

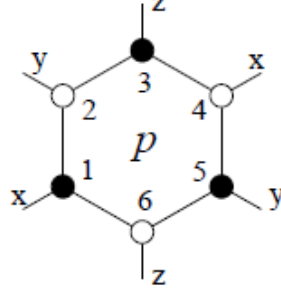


Figure 2.2

The Hamiltonian can be expressed with K_{jk}^α operators as:

$$H = -J_x \sum_{x-links} K_{jk}^x - J_y \sum_{y-links} K_{jk}^y - J_z \sum_{z-links} K_{jk}^z \quad (2.25)$$

and using fact that $[K_{jk}^\alpha, W_p] = 0 \forall K_{jk}^\alpha$ we conclude that the plaquette operators commute with the Hamiltonian

$$[H, W_p] = 0 \forall p \quad (2.26)$$

For an infinite hexagonal lattice every vertex belongs in three plaquettes and every plaquette has 6 vertices. So we actually have 2 vertices per plaquette. We can divide the total Hilbert space \mathcal{L} into sectors:

$$\mathcal{L} = \mathcal{L}_{W_1} \oplus \mathcal{L}_{W_2} \oplus \dots \mathcal{L}_{W_m} \quad (2.27)$$

So if n is the number of vertices, the Hilbert space can be divided in $n/2$ sectors of dimension $n/2$ where in each sector the W_p operators have eigenvalues ± 1 . A system with n fermionic modes can be described by the creation and annihilation operators $\alpha_k^\dagger, \alpha_k$ with $k = 1, \dots, n$. We introduce two fermionic modes $a_{1,i}$ and $a_{2,i}$ for every spin-1/2 particle in each site. By taking linear combinations of creation and annihilation operators we define new operators:

$$\begin{aligned} c_{2k+1} &= \alpha_k + \alpha_k^\dagger \\ c_{2k} &= \frac{\alpha_k - \alpha_k^\dagger}{i} \end{aligned} \quad (2.28)$$

The operators above satisfy $c_{2k}^\dagger = c_{2k}$, $c_{2k+1}^\dagger = c_{2k+1}$, $c_j c_l = -c_l c_j$ if $j \neq l$, i.e. the particles are their own antiarticles and are called *Majorana Operators*. We

define:

$$\begin{aligned}
b^x &\equiv c_1 \Rightarrow b^x = a_1 + a_1^\dagger \\
b^y &\equiv c_2 \Rightarrow b^y = i(a_1^\dagger - a_1) \\
b^z &\equiv c_3 \Rightarrow b^z = a_2 + a_2^\dagger \\
c &\equiv c_4 \Rightarrow c = i(a_2^\dagger - a_2)
\end{aligned} \tag{2.29}$$

with the above operators anticommuting. The following step is to represent Spin operators with Majorana operators. Every *spin*-1/2 particle has its own two dimensional space, so the 4 Majorana fermions have a four dimensional space. For the correspondance to be consise we ought to remone 2 fermionic states. The operators σ_i act on the *Physical Space* \mathcal{M}_i of each particle, on contrary to Majorana operators which act on $\tilde{\mathcal{M}}$, the *Extended space*. We make the choice that when a fermion is the $|\uparrow\rangle$ state with $\sigma_z = 1/2$ then the modes $a_{1,i}$, $a_{2,i}$ are empty and when the fermion is in $|\downarrow\rangle$ state with $\sigma_z = -1/2$ then the modes are full. So

$$\begin{aligned}
|\uparrow\rangle &= |00\rangle \quad |\downarrow\rangle = |11\rangle \\
\text{with } a_{1,i} |00\rangle &= a_{2,i} |00\rangle = 0 \text{ and } |11\rangle = a_{1,i}^\dagger a_{2,i}^\dagger |00\rangle
\end{aligned} \tag{2.30}$$

We "throw out" the states with just one of the fermionic modes full: $|01\rangle$, $|10\rangle$. We need an operator that will act on states that belong in the extended space and restrict us on the physical space. For that, we define the operator:

$$D_i = b_i^x b_i^y b_i^z c_i \tag{2.31}$$

The operator can be brought in the form:

$$\begin{aligned}
b_i^x b_i^y b_i^z c_i &= i(a_{1,i}^\dagger - a_{1,i})(a_{2,i} + a_{2,i}^\dagger)i(a_{1,i} + a_{1,i}^\dagger) \\
&\Rightarrow b_i^x b_i^y b_i^z c_i = (1 - 2a_{1,i}^\dagger a_{1,i})(1 - 2a_{2,i}^\dagger a_{2,i}) \\
&\Rightarrow D_i = (1 - 2a_{1,i}^\dagger a_{1,i})(1 - 2a_{2,i}^\dagger a_{2,i})
\end{aligned} \tag{2.32}$$

If $|\Psi\rangle = |00\rangle$ then $D_i |00\rangle = |00\rangle$ and if $|\Psi\rangle = |11\rangle$ then $D_i |11\rangle = |11\rangle$. But if the operator D_i act on the $|10\rangle$ and $|01\rangle$ states, then $D_i |\Psi\rangle = -|\Psi\rangle$. For the representation to be faithfull:

$$|\Psi\rangle \in \mathcal{L} \text{ if and only if } D_i |\Psi\rangle = |\Psi\rangle \text{ for every } i \tag{2.33}$$

The next step is to represent Pauli Matrices σ^x , σ^y , σ^z on the Physical space with new $\tilde{\sigma}^x$, $\tilde{\sigma}^y$, $\tilde{\sigma}^z$ on the extended space $\tilde{\mathcal{L}}$ with the condition that when the new Pauli operators $\tilde{\sigma}^x$, $\tilde{\sigma}^y$, $\tilde{\sigma}^z$ are restricted on the Physical subspace they will satisfy the same algrbraic relations with σ^x , σ^y , σ^z . We make the identification:

$$\tilde{\sigma}_i^a = i b_i^a c_i \text{ for } a = x, y, z \tag{2.34}$$

The Pauli matrices commute with the operator D_i , $[D_i, \tilde{\sigma}_i^a] = 0$. If $a = x$ then

$$\begin{aligned}
[D_i, \sigma_i^x] &= [b_i^x b_i^y b_i^z c_i, i b_i^x c_i] = \\
b_i^x b_i^y b_i^z c_i i b_i^x c_i - i b_i^x c_i b_i^x b_i^y b_i^z c_i &= 0
\end{aligned} \tag{2.35}$$

If for example a state $|\xi\rangle \in \mathcal{L}$ then under the action of $\tilde{\sigma}^a$ it will still belong in \mathcal{L} . So the action of $\tilde{\sigma}^x, \tilde{\sigma}^y, \tilde{\sigma}^z$ preserves the subspace \mathcal{L} . Moreover $i\tilde{\sigma}_i^x \tilde{\sigma}_i^y \tilde{\sigma}_i^z = i b_i^x c_i i b_i^y c_i i b_i^z c_i = b_i^x b_i^y b_i^z c_i = D_i$. But on the physical subspace, $D_i = \mathbb{I}$, so the algebra of the Pauli matrices is satisfied:

$$i\tilde{\sigma}_i^x \tilde{\sigma}_i^y \tilde{\sigma}_i^z = \mathbb{I} \quad (2.36)$$

We need to express the new Hamiltonian using the above operators. We evaluate the product $\tilde{K}_{ij}^a = \tilde{\sigma}_i^a \tilde{\sigma}_j^a$:

$$\tilde{\sigma}_i^a \tilde{\sigma}_j^a = i b_i^a c_i i b_j^a c_j = -i (i b_i^a b_j^a) c_i c_j = -i \hat{u}_{ij}^a c_i c_j \quad (2.37)$$

where

$$\hat{u}_{ij}^a = i b_i^a b_j^a \quad (2.38)$$

are link operators because they depend on the type of link that connects the two different sites i, j . They are antisymmetric $\hat{u}_{ij}^a = -\hat{u}_{ji}^a$, hermitian and square to unity $(\hat{u}_{ij}^a)^2 = (b_i^a)^2 (b_j^a)^2 = \mathbb{I}$ since $(\sigma^a)^2 = (b_i^a)^2 = \mathbb{I}$. The Hamiltonian thus can be expressed as

$$\begin{aligned} \tilde{H} &= -J_x \sum_{x-links} (-i) \hat{u}_{ij}^x c_i c_j - J_y \sum_{y-links} (-i) \hat{u}_{ij}^y c_i c_j - J_z \sum_{z-links} (-i) \hat{u}_{ij}^z c_i c_j \\ \tilde{H} &= \frac{i}{2} \sum_{i,j} J_{a_{ij}} \hat{u}_{ij}^{a_{ij}} c_i c_j \end{aligned} \quad (2.39)$$

or

$$\tilde{H} = \frac{i}{4} \sum_{i,j} \hat{A}_{ij} c_i c_j \quad (2.40)$$

where

$$\hat{A}_{ij} = \begin{cases} 2J_{a_{ij}} \hat{u}_{ij}^{a_{ij}} & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases} \quad (2.41)$$

The Hamiltonian commutes with the operator D_i :

$$\begin{aligned} [\tilde{H}, D_i] &= \left[\frac{i}{4} \sum_{jk} \hat{A}_{jk} c_j c_k, i b_i^x b_i^y b_i^z c_i \right] = \\ &= \frac{i}{4} \sum_{j,k} \hat{A}_{jk} c_j c_k i b_i^x b_i^y b_i^z c_i - i b_i^x b_i^y b_i^z c_i \sum_{j,k} \hat{A}_{jk} c_j c_k = 0 \end{aligned} \quad (2.42)$$

so we can diagonalise the Hamiltonian and restrict on the states where $D_i |\Psi\rangle = |\Psi\rangle$. Furthermore,

$$[\tilde{H}, \hat{u}_{ij}^a] = 0 \quad (2.43)$$

Just like we did before we can divide \mathcal{L} in subspaces of \hat{u}_{ij}^a of fixed eigenvalues $u_{ij}^a = \pm 1$, $\tilde{\mathcal{L}} = \oplus_u \tilde{\mathcal{L}}_u$ with summation over all fixed values of u_{jk}^a . Thus,

the operators \hat{A}_{jk} can be replaced by numbers and the hamiltonian becomes $\tilde{H}_u = \frac{i}{4} \sum_{i,j} A_{ij} c_i c_j$, a more familiar one. It describes free fermions. But there is a catch. The operators \hat{u}_{ij}^a do not commute with the operators D_i , but rather they anticommute:

$$\begin{aligned} \{D_i, \hat{u}_{ij}^a\} &= \{b_i^x b_i^y b_i^z c_i, i b_i^a b_j^a\} = \\ i b_i^x b_i^y b_i^z c_i i b_i^a b_j^a + i b_i^x b_i^y i b_i^a b_j^z c_i i b_i^x b_i^y &= 0 \end{aligned} \quad (2.44)$$

The \tilde{L}_u subspace is not gauge invariant since acting with the gauge operator D_i on a vertex i , changes the sign of the eigenvalues of the link operators that connect the i vertex with the other 3 j vertices. So the states $|\tilde{\Psi}_u\rangle$ do not belong to the physical space. We can construct the gauge invariant state $|\Psi_w\rangle \in \mathcal{L}$

$$|\Psi_w\rangle = \prod_i \left(\frac{1 + D_i}{2} \right) |\tilde{\Psi}_u\rangle \in \mathcal{L} \quad (2.45)$$

The plaquette operators can be written as

$$\hat{W}_p = \tilde{\sigma}_1^x \tilde{\sigma}_2^y \tilde{\sigma}_3^z \tilde{\sigma}_4^x \tilde{\sigma}_5^y \tilde{\sigma}_6^z = \prod_{(i,j) \in p} \hat{u}_{ij}^a \quad (2.46)$$

So the eigenvalues of the plaquette operators are the product of the eigenvalues of the link operators that span the boundary of the plaquette p .

$$W_p = \prod_{(j,k) \in \partial P} u_{jk}^a \quad (2.47)$$

But since $[\tilde{W}_p, \tilde{H}] = 0$ and $[\tilde{W}_p, D_i] = 0$ we can still express the Hamiltonian as $\tilde{H} = \frac{i}{4} \sum_{i,j} A_{ij} c_i c_j$. Acting with D_i still changes the sign of u_{ij}^a but in such way that their product, and thus the eigenvalues W_p , remain constant. To be precise, acting with D_i on a specific vertex i changes the sign of the three links connected to i . But only the two of them belong in the same plaquette. So the action D_i leave the W_p 's invariant. We will interpret the eigenvalues of the plaquette operators as the magnetic flux. If $W_p = -1$ then the plaquette will carry a *vortex*. We define the *fermionic path operator*:

$$W(j_0, \dots, j_n) = K_{j_n, j_{n-1}} \dots K_{j_1, j_0} = \left(\prod_{s=1}^n -i \hat{u}_{j_s, j_{s-1}}^a \right) c_n c_0 \quad (2.48)$$

We can see that the fermionic path operator for the sites belonging on the boundary of a plaquette is the plaquette operator W_p . If the path forms a closed loop then it is called *Wilson loop*.

We will work on the special case where $|J_x|, |J_y| \ll |J_z|$ and $J_z > 0$. We write the Hamiltonian as

$$H = H_0 + V \quad (2.49)$$

with

$$\begin{aligned} H_0 &= -J_z \sum_{z-links} \sigma_i^z \sigma_j^z \\ V &= -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y \end{aligned} \quad (2.50)$$

and treat V as a perturbation. The ground state $|\Psi_0\rangle$ of the unperturbed Hamiltonian is:

$$H_0 |\Psi_0\rangle = E_0 |\Psi_0\rangle \Rightarrow -J_z \sum_{z-links} \sigma_i^z \sigma_j^z |\Psi_0\rangle = E_0 |\Psi_0\rangle \quad (2.51)$$

so $E_0 = -NJ_z$ with N , the number of unit cells. The ground state is highly degenerate, with degeneracy 2^N . Each particles in a unit cell can be either $|\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\rangle$. We define $P_0 : \mathcal{L}_{eff} \rightarrow \mathcal{L}$ the projector operator that projects the perturbed states in the space \mathcal{L} where the ground states reside. We will denote the states that span the groundspace of H_0 as

$$\left\{ |m^{(0)}\rangle \right\} \text{ with } m = 1, 2, \dots, 2^N \quad (2.52)$$

so the projector operator is

$$P_0 = \sum_{m \in \mathcal{L}_0} |m^{(0)}\rangle \langle m^{(0)}| \quad (2.53)$$

The projector operator that projects us out of the \mathcal{L}_0 is

$$P_1 = \mathcal{I} - P_0 = \sum_{h \notin \mathcal{L}_0} |h\rangle \langle h| \quad (2.54)$$

With the projection operators satisfying:

$$\begin{aligned} P_0^2 &= P_0, \quad P_1^2 = P_1, \\ P_0 P_1 &= P_1 P_0 = 0, \quad P_0 + P_1 = 1 \end{aligned} \quad (2.55)$$

We will use *Bloch's Perturbation theory* to find an effective Hamiltonian such that, for $|a\rangle, |b\rangle \in \mathcal{L}_0$

$$\langle a | H_0 | b \rangle = \lambda \langle a | H_{eff}^{(1)} | b \rangle + \lambda^2 \langle a | H_{eff}^{(2)} | b \rangle + \lambda^3 \langle a | H_{eff}^{(3)} | b \rangle + \lambda^4 \langle a | H_{eff}^{(4)} | b \rangle + \dots \quad (2.56)$$

On first term on perturbation theory:

$$\langle a | H_{eff}^{(1)} | b \rangle = \langle a | P_0 V P_0 | b \rangle = 0 \quad (2.57)$$

because V is not diagonal on the Hamiltonian basis, so when V acts on a state $|b\rangle \in \mathcal{L}_0$ it excites it in another state $|n\rangle \notin \mathcal{L}_0$. On second term:

$$\begin{aligned} \langle a | H_{eff}^{(2)} | b \rangle &= \langle a | P_0 V P_1 (E_0 - H_0)^{-1} P_0 | b \rangle \\ \langle a | V \sum_{|h\rangle \notin \mathcal{L}_0} |h\rangle \langle h| (E_0 - H_0)^{-1} \sum_{|h\rangle \notin \mathcal{L}_0} |h\rangle \langle h| V | b \rangle &= \\ \sum_{|h\rangle \notin \mathcal{L}_0} \frac{\langle a | V | h \rangle \langle h | V | b \rangle}{E_0 - E_h} & \end{aligned} \quad (2.58)$$

We can see that an excited state achieved by acting on a ground state once will have an energy difference $E_0 - E_h = 4J_z$ so on second term

$$\sum_{|h\rangle \notin \mathcal{L}_0} \frac{\langle a|V|h\rangle \langle h|V|b\rangle}{E_o - E_h} = - \sum_{x-links} \frac{J_x^2}{4J_z} - \sum_{y-links} \frac{J_y^2}{4J_z} = -N \frac{J_x^2 + J_y^2}{4J_z} \quad (2.59)$$

and so

$$\langle a|H_{eff}^{(2)}|b\rangle = \langle a|\mathcal{I}\left(-N \frac{J_x^2 + J_y^2}{4J_z}\right)|b\rangle = -N \frac{J_x^2 + J_y^2}{4J_z} \delta_{ab} \quad (2.60)$$

which means that initializing the system on a ground state, then the interaction V , on second term cannot transform the initial state in a different state of the groundspace of the Hamiltonian. It will only add an energy difference. Continuing on the third term of the effective Hamiltonian:

$$H_{eff}^{(3)} = P_0 V P_1 (E_0 - H_0)^{-1} P_1 V P_1 (E_0 - H_0)^{-1} P_1 V P_0 \quad (2.61)$$

But acting with the interaction on a ground state three consecutive times cannot return the system on a ground state of the groundspace. So on the third term $H_{eff}^{(3)} = 0$ and we can generalise this result that odd terms of the effective hamiltonian have zero contribution i.e. $H_{eff}^{(n)} = 0$ for $n = \text{odd}$. The most important term in the effective Hamiltonian is the fourth term as we will show below. Calculating the fourth term gives us:

$$H_{eff}^{(4)} = P_0 V P_1 (E_0 - H_0)^{-1} P_1 V P_1 (E_0 - H_0)^{-1} P_1 V P_1 (E_0 - H_0)^{-1} P_1 V P_0 \quad (2.62)$$

and the matrix element is

$$\langle a|H_{eff}^{(4)}|b\rangle = \sum_{h_1 \notin \mathcal{L}_0} \sum_{h_2 \notin \mathcal{L}_0} \sum_{h_3 \notin \mathcal{L}_0} \frac{\langle a|V|h_1\rangle \langle h_1|V|h_2\rangle \langle h_2|V|h_3\rangle \langle h_3|V|b\rangle}{(E_o - E_{h_1})(E_o - E_{h_2})(E_o - E_{h_3})} \quad (2.63)$$

Although for the full calculation we need to calculate many terms we are only interested on the case where we start on an initial ground state but end up on a different ground state. Suppose we have this part of the lattice. The interaction can be written for this part as

$$V = v_1 + v_2 + v_3 + v_4 = -J_x \sigma_1^x \sigma_2^x - J_y \sigma_2^y \sigma_3^y - J_x \sigma_4^x \sigma_5^x - J_y \sigma_5^y \sigma_6^y \quad (2.64)$$

Let

$$\begin{aligned} |a\rangle &= |\uparrow_1 \downarrow_6 \uparrow_2 \uparrow_7 \downarrow_3 \downarrow_4 \uparrow_5 \uparrow_8\rangle \\ |b\rangle &= |\uparrow_1 \uparrow_6 \uparrow_2 \uparrow_7 \uparrow_3 \uparrow_4 \uparrow_5 \uparrow_8\rangle \end{aligned} \quad (2.65)$$

We know that

$$\begin{aligned} \sigma^x |\uparrow\rangle &= |\downarrow\rangle \quad \sigma^x |\downarrow\rangle = |\uparrow\rangle \\ \sigma^y |\uparrow\rangle &= i |\downarrow\rangle \quad \sigma^y |\downarrow\rangle = -i |\uparrow\rangle \end{aligned} \quad (2.66)$$

We will calculate

$$\langle a | H_{eff}^{(4)} | b \rangle = \sum_{h_1 \notin \mathcal{L}_0} \sum_{h_2 \notin \mathcal{L}_0} \sum_{h_3 \notin \mathcal{L}_0} \frac{\langle a | V | h_1 \rangle \langle h_1 | V | h_2 \rangle \langle h_2 | V | h_3 \rangle \langle h_3 | V | b \rangle}{(E_o - E_{h_1})(E_o - E_{h_2})(E_o - E_{h_3})} \quad (2.67)$$

First, we will calculate the terms where v_i appears only once. For a cyclic translation of v_i we have

$$\begin{aligned} |h_1\rangle &= |\uparrow_1 \downarrow_6 \downarrow_2 \uparrow_7 \downarrow_3 \downarrow_4 \uparrow_5 \uparrow_8\rangle, \quad E_o - E_{h_1} = -4J_z \\ |h_2\rangle &= |\uparrow_1 \downarrow_6 \uparrow_2 \uparrow_7 \downarrow_3 \downarrow_4 \uparrow_5 \uparrow_8\rangle, \quad E_o - E_{h_2} = -4J_z \\ |h_3\rangle &= |\uparrow_1 \downarrow_6 \uparrow_2 \uparrow_7 \uparrow_3 \downarrow_4 \downarrow_5 \uparrow_8\rangle, \quad E_o - E_{h_3} = -4J_z \end{aligned} \quad (2.68)$$

Computing the matrix elements:

$$\begin{aligned} \langle a | v_1 | h_1 \rangle &= \langle \downarrow_1 \uparrow_2 | (-J_x \sigma_1^x \sigma_2^x) | \uparrow_1 \downarrow_2 \rangle = -J_x \\ \langle h_1 | v_2 | h_2 \rangle &= \langle \downarrow_2 \downarrow_3 | (-J_y \sigma_2^y \sigma_3^y) | \uparrow_2 \uparrow_3 \rangle = J_y \\ \langle h_2 | v_3 | h_3 \rangle &= \langle \downarrow_4 \uparrow_5 | (-J_x \sigma_4^x \sigma_5^x) | \uparrow_4 \downarrow_5 \rangle = -J_x \\ \langle h_3 | v_4 | b \rangle &= \langle \downarrow_5 \downarrow_6 | (-J_y \sigma_5^y \sigma_6^y) | \uparrow_5 \uparrow_6 \rangle = J_y \end{aligned} \quad (2.69)$$

So

$$\sum_{h_1 \notin \mathcal{L}_0} \sum_{h_2 \notin \mathcal{L}_0} \sum_{h_3 \notin \mathcal{L}_0} \frac{\langle a | V | h_1 \rangle \langle h_1 | V | h_2 \rangle \langle h_2 | V | h_3 \rangle \langle h_3 | V | b \rangle}{(E_o - E_{h_1})(E_o - E_{h_2})(E_o - E_{h_3})} = -\frac{J_x^2 J_y^2}{64 J_z^3} \quad (2.70)$$

Due to the symmetry of the hexagonal lattice all the above terms have the same contribution

$$\begin{aligned} &v_1 v_2 v_3 v_4, \quad v_4 v_3 v_2 v_1 \\ &v_2 v_3 v_4 v_1, \quad v_3 v_2 v_1 v_4 \\ &v_3 v_4 v_1 v_2, \quad v_2 v_1 v_4 v_3 \\ &v_4 v_1 v_2 v_3, \quad v_1 v_4 v_3 v_2 \end{aligned} \quad (2.71)$$

So the total contribution of these 8 terms is $8(-\frac{J_x^2 J_y^2}{64 J_z^3})$. If we make the same calculations and use the symmetries of the hexagonal lattice, the total contribution of the 24 terms is

$$-8 \frac{J_x^2 J_y^2}{64 J_z^3} + 8 \frac{J_x^2 J_y^2}{64 J_z^3} - 8 \frac{J_x^2 J_y^2}{128 J_z^3} = -\frac{J_x^2 J_y^2}{16 J_z^3} \quad (2.72)$$

The above arguments can easily be executed in the case $J_z > 0$. We have

$$\begin{aligned} \sigma_1^x \sigma_6^y | \uparrow_1 \uparrow_6 \rangle &= i | \downarrow_1 \downarrow_6 \rangle \\ \sigma_2^x \sigma_2^y | \uparrow_2 \uparrow_7 \rangle &= i | \uparrow_2 \uparrow_7 \rangle \\ \sigma_3^y \sigma_4^x | \uparrow_3 \uparrow_4 \rangle &= i | \downarrow_3 \downarrow_4 \rangle \\ \sigma_5^y \sigma_5^x | \uparrow_5 \uparrow_8 \rangle &= -i | \uparrow_5 \uparrow_8 \rangle \end{aligned} \quad (2.73)$$

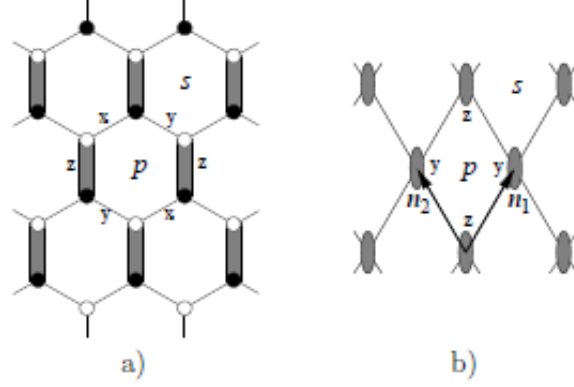


Figure 2.3

so if we visualize the spins of a z -bond as one, as seen on figure (2.3), then we can easily make the correspondance

$$\begin{aligned}
 \sigma_1^x \sigma_6^y &= \sigma_{16}^y \equiv \sigma_{left}^y \\
 \sigma_2^x \sigma_2^y &= i\sigma_{27}^z \equiv i\sigma_{up}^z \\
 \sigma_3^y \sigma_4^x &= \sigma_{34}^y \equiv \sigma_{right}^y \\
 \sigma_5^y \sigma_5^x &= -i\sigma_{58}^z \equiv -i\sigma_{down}^z
 \end{aligned} \tag{2.74}$$

So the effective hamiltonian on 4th term is

$$\langle a | H_{eff}^{(4)} | b \rangle = \langle a | \left(-\frac{J_x^2 J_y^2}{16 J_z^3} \sum_p \sigma_{left}^y \sigma_{right}^y \sigma_{up}^z \sigma_{down}^z \right) | b \rangle \tag{2.75}$$

We define a new operator

$$Q_p = \sigma_{left(p)}^y \sigma_{right(p)}^y \sigma_{up(p)}^z \sigma_{down(p)}^z \tag{2.76}$$

So the lattice has been transformed as seen in (2.3). We construct a new square lattice, Λ' and the Hamiltonian becomes

$$H_{eff} = -\frac{J_x^2 J_y^2}{16 |J_z|^3} \left(\sum_{vertices} Q_s + \sum_{plaquettes} Q_p \right) \tag{2.77}$$

Clearly the translational symmetry of our original model is now lost. We make a transformation U on the Hamiltonian

$$U = \prod_{horizontal\ links} X_j \prod_{vertical\ links} Y_k \tag{2.78}$$

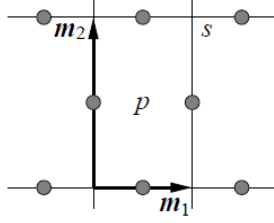


Figure 2.4

and bring it in a more familiar form:

$$\begin{aligned}
 H'_{eff} &= U H_{eff} U^\dagger = -\frac{J_x^2 J_y^2}{16|J_z|^3} \left(\sum_{vertices} A_s + \sum_{plaquettes} B_p \right) \\
 H_{eff} &= -\frac{J_x^2 J_y^2}{16|J_z|^3} \left(\sum_{vertices} A_s + \sum_{plaquettes} B_p \right)
 \end{aligned} \tag{2.79}$$

where

$$\begin{aligned}
 A_s &= \sigma_{s,1}^x \sigma_{s,2}^x \sigma_{s,3}^x \sigma_{s,4}^x = \prod_{j \in star(s)} \sigma_{s,j}^x \\
 B_p &= \sigma_{p,1}^z \sigma_{p,2}^z \sigma_{p,3}^z \sigma_{p,4}^z = \prod_{j \in plaquette(p)} \sigma_{p,j}^z
 \end{aligned} \tag{2.80}$$

The operators A_s and B_p commute, $[A_s, B_p] = 0$ because a star (s) and a plaquette (p) can either have two or none common particles. Hence they commute with the Hamiltonian

$$[H, A_s] = [H, B_p] = 0 \quad \forall s, p \tag{2.81}$$

It's clear that the operators A_s, B_p have eigenvalues ± 1 . We define the ground state $|\Psi\rangle$

$$|\Psi\rangle = \prod_s \frac{1}{\sqrt{2}} (I + A_s) |\uparrow\uparrow \dots \uparrow\rangle \tag{2.82}$$

with $A_s |\Psi\rangle = |\Psi\rangle$ and $B_p |\Psi\rangle = |\Psi\rangle$. The excited states arise when one of the operators has an eigenvalue -1 . These will be the positions of the *anyons*. We want to find out how can we generate anyons. We can use the anticommutation relations:

$$\{A_s, \sigma_{s,j}^z\} = 0, \quad \{B_p, \sigma_{p,j}^x\} = 0 \tag{2.83}$$

If $|\xi\rangle = \sigma_{s,i}^z |\Psi\rangle$ then $A_s |\xi\rangle = -\sigma_{s,i}^z |\Psi\rangle = -|\xi\rangle$. The quasiparticles that created by the action of the operator A_s with eigenvalue -1 are e -type anyons and we will refer to them as *electric charges*.

$$|e_s, e_k\rangle = \sigma_i^z |\Psi\rangle \tag{2.84}$$

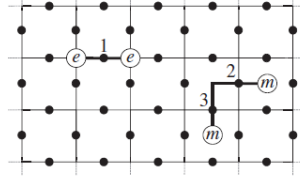


Figure 2.5

The indices s, k point the position of the quasiparticles. They "live" on the different vertices s, k . Similarly acting with an operator B_p we can generate m -type anyons and we will refer to them as *magnetic vortices* i.e. quasiparticles with eigenvalue $B_p = -1$.

$$|m_p, m_k\rangle = \sigma_i^x |\Psi\rangle \quad (2.85)$$

The magnetic vortices lie inside the different plaquettes p, k . We can visualize it as living inside the dual lattice.

Acting with both of the operators gives as a new state $|\xi'\rangle = \sigma_{p,i}^x \sigma_{s,i}^z |\Psi\rangle$ with eigenvalues $A_s = -1$ and $B_p = -1$. We interpret this as a third type anyon, ϵ -anyon, composite of the two others.

$$|\epsilon, \epsilon\rangle = \sigma_{p,i}^x \sigma_{s,i}^z |\Psi\rangle \quad (2.86)$$

These three along with the vacuum, (1), are the *superselection sectors* of our model. Acting twice with the same operator gives eigenvalue +1 so the fusion rules put together are:

$$\begin{aligned} m \times m &= e \times e = \epsilon \times \epsilon = 1 \\ m \times e &= \epsilon, m \times \epsilon = e, e \times \epsilon = m \end{aligned} \quad (2.87)$$

The anyon model is nonabelian as there is only one specific outcome for the fusion of any two different anyons. We would like to see the statistical behaviour of our model. We can transport the anyons on the lattice with consecutive actions of the operators σ^x, σ^z . Every loop of σ^z actions can be written as a product of B_p 's and every loop of σ^x as a product of A_s operators.

In order to exchange two e anyons we need to act with σ^z operators and the operators must form a loop.

So if $|\Psi_i\rangle$ is the initial state, the final state will be $|\Psi_f\rangle = f(B_p) |\Psi_i\rangle$ with f being some function of the operators B_p . But the B_p 's act on plaquettes without an m anyon. So the final state is identical to the initial state and thus e are *bosons* with respect to themselves. The same stands for the magnetic vortices. We want to rotate an e particle around an m . In order to have an m anyon in the lattice we must have acted with some operators C_{σ^x} on the ground state $|\Psi\rangle$. The same stands for e . So the initial state is $|\Psi_i\rangle = C_{\sigma^x} C_{\sigma^z} |\Psi\rangle$. In order to transport the e anyon and bring it to the original position we must act with a series of σ_z operators which form a loop and call the product of them as L_{σ^z} . But L_{σ^z} and C_{σ^x} have just one spin in common so $L_{\sigma^z} C_{\sigma^x} = -C_{\sigma^x} L_{\sigma^z}$.

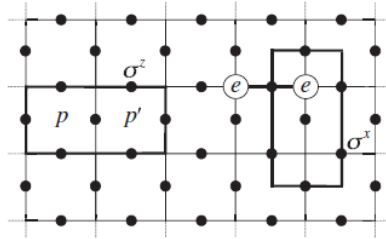


Figure 2.6

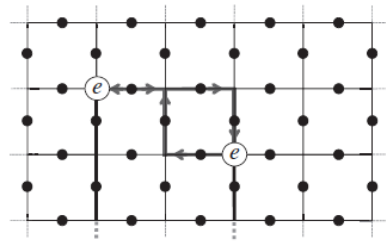


Figure 2.7

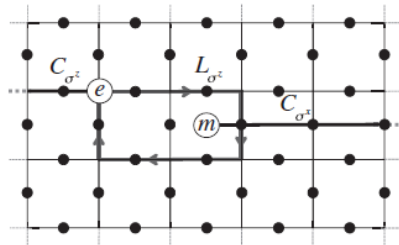


Figure 2.8

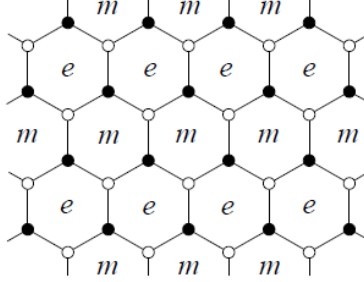


Figure 2.9

Thus $|\Psi_f\rangle = L_{\sigma^z} |\Psi_i\rangle = -|\Psi_i\rangle$ and thus rotating an e around an m gives a -1 . Continuing with the statistics of ϵ particles we want to exchange two ϵ particles. Each ϵ particle consists of one e and one m . Each e anyon must transport by π around the m of the other ϵ . This is topologically equivalent with rotating just the one e around m by 2π . So the exchange of two ϵ 's gives a factor -1 so the ϵ 's are *fermions* with respect to themselves. Tracing back we see that the e 's and m 's live on different rows of the hexagonal lattice. We see that anyons can exist in the exotic phase where J_z is much bigger than J_x, J_y . But there is a problem. The fusion space of an abelian anyon model is one dimensional and thus we cannot construct the universal gates and achieve universal quantum computation with abelian anyons. This can only be done if the model is nonabelian and the fusion space V_{ab}^c has dimension $\dim V_{ab}^c = N_{ab}^c > 1$ and so possible fusion states can serve as qubits.

Chapter 3

Gapped and Gapless phases

But, which is the ground state of our model? There is a beautiful theorem by proved Lieb [30] that states that the minimum energy is achieved by the vortex free configuration and that is for $w_p = 1$. So we may assume that $u_{jk}^a = 1$. According to the first chapter we can "move" on the lattice using 3 Eisenstein integers, $s_1 = 1$, $s_2 = \omega$, $s_3 = -\omega - 1$ or written as vectors in the standard (x, y) basis:

$$\begin{aligned}\vec{s}_1 &= (1, 0) \\ \vec{s}_2 &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \vec{s}_3 &= \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\end{aligned}\tag{3.1}$$

We employ a Fourier transform on the operators c_i and c_j :

$$\begin{aligned}c_i &= c_{\vec{r}} = \frac{1}{2N} \sum_{\vec{p}} e^{-i\vec{p}\vec{r}} c_{\vec{p}} \\ c_j &= c_{\vec{r}} = \frac{1}{2N} \sum_{\vec{p}} e^{-i\vec{p}\vec{r}} c_{\vec{p}}\end{aligned}\tag{3.2}$$

with the restriction that $\vec{r} = \vec{r} + \vec{s}_a$, $a = 1, 2, 3$. We drop tildes and write the Hamiltonian as:

$$\begin{aligned}H &= \frac{i}{4} \sum_{\vec{r}} 2c_{\vec{r}} (J_z c_{\vec{r}+\vec{s}_1} + J_x c_{\vec{r}+\vec{s}_2} + J_y c_{\vec{r}+\vec{s}_3}) + \\ &\frac{i}{4} \sum_{\vec{r}} (-2)c_{\vec{r}} (J_z c_{\vec{r}-\vec{s}_1} + J_x c_{\vec{r}-\vec{s}_2} + J_y c_{\vec{r}-\vec{s}_3}) \\ &= \frac{i}{4} \sum_{\vec{p}} \sum_{a=1,2,3} 2J_a e^{-i\vec{p}\vec{s}_a} c_{-\vec{p}} c_{\vec{p}} + h.c\end{aligned}\tag{3.3}$$

By definition, $c_{-\vec{p}} = c_{\vec{p}}^\dagger$ and if:

$$\begin{aligned}\tilde{c}_{\vec{p}} &= e^{-i\pi/4} c_{\vec{p}} \\ \tilde{c}'_{\vec{p}} &= e^{i\pi/4} c'_{\vec{p}} \\ f(\vec{p}) &= \sum_{a=1,2,3} = 2J_a e^{-i\vec{p}\vec{s}_a}\end{aligned}\tag{3.4}$$

then the Hamiltonian is written

$$H = \frac{1}{4} \sum_{\vec{p}} \begin{bmatrix} \tilde{c}_{\vec{p}} & \tilde{c}'_{\vec{p}} \end{bmatrix} \begin{bmatrix} 0 & f(\vec{p}) \\ f^*(\vec{p}) & 0 \end{bmatrix} \begin{bmatrix} \tilde{c}_{\vec{p}} \\ \tilde{c}'_{\vec{p}} \end{bmatrix}\tag{3.5}$$

We define the *1-particle Hamiltonian* as:

$$H(\vec{p}) = \begin{bmatrix} 0 & f(\vec{p}) \\ f^*(\vec{p}) & 0 \end{bmatrix}\tag{3.6}$$

The spectrum of the Hamiltonian is

$$\begin{aligned}\epsilon(\vec{p}) &= \pm |f(\vec{p})| \\ \epsilon(\vec{p}) &= \pm 2|J_x e^{-i\vec{p}\vec{s}_2} + J_y e^{-i\vec{p}\vec{s}_3} + J_z e^{-i\vec{p}\vec{s}_1}|\end{aligned}\tag{3.7}$$

We want to examine if the spectrum allows *fermi points* i.e. points (p_x, p_y) with eigenvalues $\epsilon = 0$. Thus, there must be $\vec{p} = (p_x, p_y)$ that satisfy:

$$J_x e^{-i\vec{p}\vec{s}_2} + J_y e^{-i\vec{p}\vec{s}_3} + J_z e^{-i\vec{p}\vec{s}_1} = 0\tag{3.8}$$

This equation represents three vectors in the complex plane. In order for the sum of three vectors to be zero, they must form a triangle and so the length of it's sides must satisfy the triangular inequalities:

$$|J_x| \leq |J_y| + |J_z|, |J_y| \leq |J_x| + |J_z|, |J_z| \leq |J_x| + |J_y|\tag{3.9}$$

Going back to (3.7), we have to find (p_x, p_y) that satisfy:

$$\begin{aligned}J_x \cos\left(-\frac{p_x}{2} + \frac{\sqrt{3}p_y}{2}\right) + J_y \cos\left(-\frac{p_x}{2} - \frac{\sqrt{3}p_y}{2}\right) + J_z \cos p_x &= 0 \\ J_x \sin\left(-\frac{p_x}{2} + \frac{\sqrt{3}p_y}{2}\right) + J_y \sin\left(-\frac{p_x}{2} - \frac{\sqrt{3}p_y}{2}\right) + J_z \sin p_x &= 0\end{aligned}\tag{3.10}$$

After calculations the fermi points are:

$$\begin{aligned}p_y &= \pm \frac{\sqrt{3}}{2} \arccos\left(\frac{J_z^2 - J_x^2 - J_y^2}{2J_x J_y}\right) \\ p_x &= \pm \frac{2}{3} \arccos\left(\frac{J_z^2 - J_y^2 - J_x^2}{2J_x J_y}\right)\end{aligned}\tag{3.11}$$

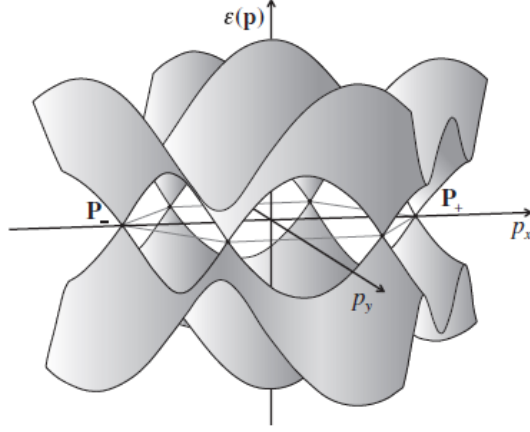


Figure 3.1

In the special case where $J_x = J_y = J_z = J$, the energy eigenvalues become

$$\epsilon(\vec{p}) = \pm \frac{J}{2} \sqrt{1 + 4\cos^2 \frac{\sqrt{3}p_y}{2} + 4\cos \frac{3p_x}{2} \cos \frac{\sqrt{3}p_y}{2}} \quad (3.12)$$

The figure is shown above. There are infinite fermi points and spectrum is gapless.

We add an extra term in the hamiltonian

$$H_2 = -K \sum_{i,j,k} \sigma_i^x \sigma_j^y \sigma_k^z \quad (3.13)$$

where K is the effective magnetic field. Every plaquette contributes 6 terms

$$\sum_{\{i,j,k\} \in p} \sigma_i^x \sigma_j^y \sigma_k^z = \sigma_1^x \sigma_2^z \sigma_3^y + \sigma_2^y \sigma_3^x \sigma_4^z + \sigma_3^z \sigma_4^y \sigma_5^x + \sigma_4^x \sigma_5^z \sigma_6^y + \sigma_5^y \sigma_6^x \sigma_1^z + \sigma_6^z \sigma_1^y \sigma_2^x \quad (3.14)$$

We added this term because it breaks the time-reversal symmetry and preserves the exact solvability of the model. We have

$$\begin{aligned} \sigma_i^x \sigma_j^y \sigma_k^z &= ib_i^x c_i ib_j^y c_j ib_k^z c_k = ib_i^x b_k^x ib_j^y b_k^y ib_k^x b_k^y b_k^z c_k \implies \\ \sigma_i^x \sigma_j^y \sigma_k^z &= -i \hat{u}_{ik}^x \hat{u}_{jk}^y D_k c_i c_j \end{aligned} \quad (3.15)$$

But $D_k = 1$ on the physical subspace and thus

$$\sigma_i^x \sigma_j^y \sigma_k^z = -i \hat{u}_{ik}^x \hat{u}_{jk}^y c_i c_j \quad (3.16)$$

The hamiltonian is now written

$$H_{eff} = \frac{i}{4} \sum_{\{i,j\}} \left(2J_{ij} \hat{u}_{ij} c_i c_j + \frac{2ki}{4} \sum_k \hat{u}_{ik} \hat{u}_{jk} c_i c_j \right) = \frac{i}{4} \sum_{\{i,j\}} A_{ij} c_i c_j \quad (3.17)$$

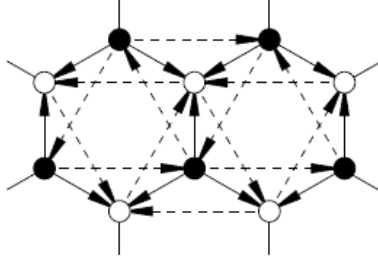


Figure 3.2

with

$$A_{ij} = 2J_{ij}\hat{u}_{ij} + 2K \sum_k \hat{u}_{ik}\hat{u}_{jk} \quad (3.18)$$

Thus the effective hamiltonian is shown in figure (3.2) The arrow \rightarrow is just a notation for the matrix \hat{u}_{jk} which is $+1$ if it points from j to k and -1 otherwise. We define $--\rightarrow$ in the same way. If we employ again the Fourier transform in (3.2) then the second part of the hamiltonian becomes

$$\begin{aligned} H_2 &= \frac{i}{4} \sum_{\vec{r}} \sum_{\vec{p}} \sum_{\vec{p}'} \frac{2K}{2N} \left(e^{-i\vec{p}'\vec{n}_1} - e^{-i\vec{p}'\vec{n}_2} + e^{i\vec{p}'(\vec{n}_1 - \vec{n}_2)} + e^{-i\vec{p}'\vec{n}_1} + e^{i\vec{p}'\vec{n}_2} - e^{-i\vec{p}'(\vec{n}_1 - \vec{n}_2)} \right) \\ &= \frac{iK}{2} \sum_{\vec{p}} \frac{1}{2i} [-\sin(\vec{p}\vec{n}_1) + \sin(\vec{p}\vec{n}_2) + \sin \vec{p}(\vec{n}_1 - \vec{n}_2)] \end{aligned} \quad (3.19)$$

The vectors are

$$\begin{aligned} \vec{n}_1 &= \vec{s}_2 - \vec{s}_3 = \left(0, \sqrt{3}\right) \\ \vec{n}_2 &= -\vec{s}_1 + \vec{s}_3 = \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) \\ \vec{n}_1 - \vec{n}_2 &= \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned} \quad (3.20)$$

and we define

$$\Delta(\vec{p}) = 4K (-\sin(\vec{p}\vec{n}_1) + \sin(\vec{p}\vec{n}_2) + \sin \vec{p}(\vec{n}_1 - \vec{n}_2)) \quad (3.21)$$

If we use the relation (3.4) then

$$H_2 = \frac{i}{4} \sum_{\vec{p}} \Delta(\vec{p}) \left(\tilde{c}_{\vec{p}}^\dagger \tilde{c}_{\vec{p}} - \tilde{c}_{\vec{p}}^\dagger \tilde{c}_{-\vec{p}} \right) \quad (3.22)$$

So the effective hamiltonian is now written

$$H = \frac{1}{4} \sum_{\vec{p}} \begin{bmatrix} \tilde{c}_{\vec{p}} & \tilde{c}_{-\vec{p}} \end{bmatrix} \begin{bmatrix} \Delta(\vec{p}) & f(\vec{p}) \\ f^*(\vec{p}) & -\Delta(\vec{p}) \end{bmatrix} \begin{bmatrix} \tilde{c}_{\vec{p}} \\ \tilde{c}_{-\vec{p}} \end{bmatrix} \quad (3.23)$$

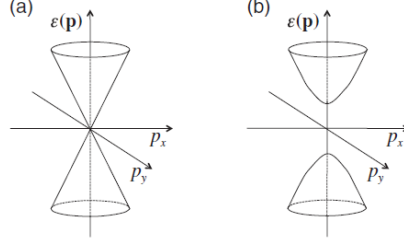


Figure 3.3

This new extra term modifies the energy spectrum. To be precise it creates an energy gap

$$\Delta = 6\sqrt{3}K \quad (3.24)$$

around the Fermi points between the valence and conductance bands.

The starting point is this: *if ν (the Chern number) is odd then each vortex carries an unpaired majorana mode.* In our model the Chern number is equal to 1. We consider a Majorana operator γ_i at a specific position. Consider $2n$ spatially well-seperated Majoranas $\gamma_1, \dots, \gamma_{2n}$. Since a Majorana degree of freedom is half a fermionic degree of freedom, one can combine them to give rise to full fermions

$$a_j = \frac{1}{2} (\gamma_{2j-1} + i\gamma_{2j}) \quad (3.25)$$

The majorana operator satisfies

$$\begin{aligned} \gamma_i &= \gamma_i^\dagger \\ \gamma_i^2 &= 1 \end{aligned} \quad (3.26)$$

Let us take 4 localised majorana fermions $\gamma_1, \dots, \gamma_4$. We can combine these 4 Majorana fermions in two different ways as seen in the figure (3.4).

$$\begin{aligned} z_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \quad z_2 = \frac{1}{2}(\gamma_3 + i\gamma_4) \\ w_1 &= \frac{1}{2}(\gamma_1 + i\gamma_3), \quad w_2 = \frac{1}{2}(\gamma_2 + i\gamma_4) \end{aligned} \quad (3.27)$$

They satisfy the anticommutation relations

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= 2\delta_{ij} \\ \{z_i, z_i^\dagger\} &= 1, \quad \{w_i, w_i^\dagger\} = 1 \end{aligned} \quad (3.28)$$

It does not make sense to talk about Majorana number since it is identically one i.e. $\gamma_i^\dagger \gamma_i = 1$. Similarly, $\gamma_i \gamma_i^\dagger = 1$. Thus in the traditional sense, the Majorana mode is empty and full at the same time. However is possible to speak of the number of states $|n_i\rangle$, which are the eigenstates of the number operator

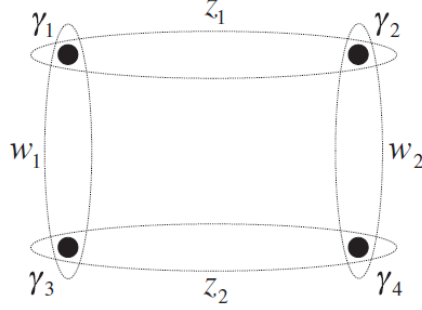


Figure 3.4

$n_i = a_i^\dagger a_i$, $i = 1, \dots, n$. In terms of Majorana operators, these number operators are given by

$$n_i = a_i^\dagger a_i = \frac{1}{2} (1 + i\gamma_{2j-1}\gamma_{2j-1}) \quad (3.29)$$

So in general for spatially separated Majorana fermions, the way to re-write them in terms of traditional fermions is non-unique. These 2 different ways of grouping the Majorana fermions are not independent as

$$\{z_1, w_1\} = \frac{1}{2} \quad (3.30)$$

The population states are given by $|ij\rangle = |i\rangle_{z_1} \otimes |j\rangle_{z_2}$ with $i, j = 0, 1$ and i is the population of the z_1 and j the population of z_2 . The operators $z_i^\dagger z_i$, $w_i^\dagger w_i$ project on the zero population states while $z_i z_i^\dagger$, $w_i w_i^\dagger$ project on the states with populated modes. We are interested in the $|00\rangle_{z/w}$, $|11\rangle_{z/w}$ states. The states $|00\rangle_z$, $|11\rangle_z$ satisfy

$$\begin{aligned} z_1^\dagger z_1 |00\rangle_z &= 0, \quad z_1 z_1^\dagger |00\rangle_z = |00\rangle_z \\ z_1^\dagger z_1 |11\rangle_z &= |11\rangle_z, \quad z_1 z_1^\dagger |11\rangle_z = 0 \end{aligned} \quad (3.31)$$

As the operators w_1 and z_1 do not anticommute we can find what is the z population of the state $(2w_1^\dagger w_1 - 1) |00\rangle_z$.

$$z_1^\dagger z_1 (2w_1^\dagger w_1 - 1) |00\rangle_z = (2w_1^\dagger w_1 - 1) |00\rangle_z \quad (3.32)$$

and so

$$|11\rangle_z = (2w_1^\dagger w_1 - 1) |00\rangle_z \quad (3.33)$$

Rewriting the above equation

$$|11\rangle_w = \sqrt{2} w_1^\dagger w_1 |00\rangle_z = \frac{1}{\sqrt{2}} (|00\rangle_z + |11\rangle_z) \quad (3.34)$$

and with the same way

$$|00\rangle_w = \sqrt{2} w_1 w_1^\dagger |11\rangle_z = -\frac{1}{\sqrt{2}} (|00\rangle_z - |11\rangle_z) \quad (3.35)$$

and so z and w bases are related by the transformation matrix

$$F = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad (3.36)$$

This matrix is the F matrix of the Ising model. We can continue our calculation and find that the phase factor $\theta = \pi/8$, that of the Ising model. So we are led to the conclusion that a vortex carrying an unpaired majorana mode is just one of the superselection sectors, denoted by σ . The others are ϵ (fermions) and 1 (vacuum). If two vortices fuse, then they either annihilate completely or leave a fermion behind

$$\sigma \times \sigma = 1 + \epsilon \quad (3.37)$$

The complete set of the fusion rules is

$$\begin{aligned} \sigma \times \epsilon &= \epsilon \\ \epsilon \times \epsilon &= 1 \\ \sigma \times \sigma &= 1 + \epsilon \end{aligned} \quad (3.38)$$

This model of anyons is called *Ising anyons* and it clearly nonabelian. Thus it can be used for quantum computation. The basis vectors that can serve as our computational basis are $|\psi_1^{\sigma\sigma}\rangle$ and $|\psi_\epsilon^{\sigma\sigma}\rangle$.

Chapter 4

Discrete gauge theories and Quantum double

Let us consider a continuous gauge theory that is spontaneously broken down to a finite nonabelian group G in two spatial dimensions. This *nonabelian superconductor* contains particles that carry magnetic flux, called *fluxons*, that take values in a nonabelian finite group G and *electric charges* that are the unitary irreducible representations of the group G [12,13]. Particles that carry flux and charge are called *dyons* and they are labeled by a group element of G and an irreducible representation of the normalizer group of the group element.

Let a be an irreducible representation of G whose dimension is denoted by $|a|$. The representation acts on a vector space spanned by vectors

$$|a, i\rangle, \quad i = 1, \dots, |a| \quad (4.1)$$

When a charge a is transported around a closed path which encircles a magnetic flux $b \in G$ then the basis vectors transform according to

$$|a, i\rangle' = \sum_j^{|a|} D_{ij}^a(b) |a, j\rangle \quad (4.2)$$

So given a number of charges we can attach labels to all fluxes as long as the representation $|a|$ is faithful. Suppose that I use my standard charges to measure the flux of each particle and that I assign the group elements b_1, b_2, \dots, b_n in each particle. But what if someone else asked to verify my assignments given the same charges but had transported them around another flux $g \in G$. Then he would assign the group elements $gb_1g^{-1}, gb_2g^{-1}, \dots, gb_kg^{-1}$. We would agree only on the thing that each particle will be assigned a group element which belongs in the same conjugacy class. So the fluxons belonging in the same conjugacy class are regarded as indistinguishable particles. Given a number of fluxons we will measure the total flux from right to left, so given a pair $|a, b\rangle$, its total flux is

ab . We define the R -matrix which acts on the space of two fluxons, exchanges them counterclockwise and conserves the total flux of the pair as

$$R|a, b\rangle = |aba^{-1}, a\rangle \quad (4.3)$$

The clockwise exchange is defined as

$$R^{-1}|a, b\rangle = |b, b^{-1}ab\rangle \quad (4.4)$$

Two consecutive counterclockwise exchanges is the *monodromy operator*

$$R^2|a, b\rangle = |(ab)a(ab)^{-1}, (ab)b(ab)^{-1}\rangle \quad (4.5)$$

We can see that both of the fluxes are conjugated by (ab) . An important observation is taking R to an arbitrary power n is:

$$R^n|a, b\rangle = \begin{cases} |(ab)^{\frac{n}{2}}a(ab)^{-\frac{n}{2}}, (ab)^{\frac{n}{2}}b(ab)^{-\frac{n}{2}}\rangle & n \text{ even} \\ |(ab)^{\frac{n-1}{2}}aba^{-1}(ab)^{-\frac{n-1}{2}}, (ab)^{\frac{n-1}{2}}a(ab)^{-\frac{n-1}{2}}\rangle & n \text{ odd} \end{cases} \quad (4.6)$$

Theorem 4.0.1. *Let fluxons take values on a finite group G . Then the R -matrix to the power of the least common multiple of the order of the group elements times two equals the identity.*

This is an important observation due to the fact that the system of two fluxons does not keep on braiding but gets unbraided after a finite number of exchanges. R acts a representation of the truncated colored braid group $P(n, m)$. We will return to that briefly.

Let the fluxes take values on the permutation group S_3 . The permutation group is of order $|S_3| = 6$ and consists of the elements

$$S_3 = \{e, (123), (132), (12), (23), (13)\} \quad (4.7)$$

The conjugacy classes are

$$\begin{aligned} C_1 &= \{e\} \\ C_2 &= \{(123), (132)\} \text{ 3-cycles} \\ C_3 &= \{(12), (23), (13)\} \text{ 2-cycles} \end{aligned} \quad (4.8)$$

The R matrix is shown in the figure below.

We want to know what happens when a particle carries a flux labeled by a group element but also carries an internal charge. Let the particles whose charge we want to measure is in the flux/charge eigenstate

$$|flux, charge\rangle = |b, u\rangle \quad (4.9)$$

Let the initial state of the 2-particle system be $|a, b\rangle$. Then if we transport the flux b around a counterclockwise then

$$R^2|b, a\rangle = |(ab)a(ab)^{-1}, (ab)b(ab)^{-1}\rangle = |(ab)a(ab)^{-1}, (aba^{-1})\rangle \quad (4.10)$$

groups. There are 3 conjugacy classes so we have to find three normalizer groups.

$$\begin{aligned}
N(e) &= S_3 \\
N(123) &= \{e, (123), (132)\} \cong Z_3 \\
N(12) &= \{e, (12)\} \cong N(23) \cong N(13) \cong Z_2
\end{aligned} \tag{4.14}$$

Z_3 has 3 conjugacy classes and thus 3 irreducible representations of dimension 1. We can construct the character table:

	$C_1^{Z_3}$	$C_2^{Z_3}$	$C_3^{Z_3}$
ρ_1	1	1	1
ρ_2	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$
ρ_3	1	$e^{4\pi i/3}$	$e^{2\pi i/3}$

Z_2 has 2 irreducible representations of dimension 1.

	C_1	C_2
$\tilde{\rho}_1$	1	1
$\tilde{\rho}_2$	1	-1

And S_3 has three irreducible representations. Two with dimension one, Γ_1 and Γ_2 which correspond to the trivial and the non-trivial and one with dimension two, Γ_3 . So for the case $G = S_3$, there are 8 particle types

<i>Type</i>	<i>Flux</i>	<i>Charge</i>	<i>Dimension</i>
<i>A</i>	<i>e</i>	Γ_1	1
<i>B</i>	<i>e</i>	Γ_2	1
<i>C</i>	<i>e</i>	Γ_3	2
<i>D</i>	(123)	$\tilde{\rho}_1$	3
<i>E</i>	(123)	$\tilde{\rho}_2$	3
<i>F</i>	(12)	ρ_1	2
<i>G</i>	(12)	ρ_2	2
<i>H</i>	(12)	ρ_3	2

When two different particles are fused together the composite object can be of various types and the *fusion rules* specify which types are possible. The decomposition theorem can be used to decompose reducible representations of a group into its irreducible components. But in our case every particle is labeled by a conjugacy class and a representation of a specific normalizer group, not the same for all particles. For this case, we are led to the construction of the quantum double[14,15].

Quantum double or *Drinfeld double*, denoted by $D(G)$ is a Hopf algebra:

$$D(G) = F(G) \otimes \mathbb{C}[G] \tag{4.15}$$

where G is a finite group, $\mathbb{C}[G]$ its group algebra, i.e. the complex vector space with basis $(e_g)_{g \in G}$ equipped with the product $e_g e_h = e_{gh}$ and $F(G)$ the abelian algebra of complex functions defined on G .

The magnetic flux of a particle is given by the element $g \in G$ and the charges are the irreducible representations of the centralizer gN of the particle's flux $g \in G$. We can measure the flux by projecting it out with an operator P_g and the charge with global transformations $h \in {}^gN \subset G$ that commute with the flux g of the particle.

The elements spanning the quantum group $D(G)$ are $(P_h g)_{g,h \in G}$. The projection operators satisfy the algebra:

$$P_h P_{h'} = \delta_{h,h'} P_h \text{ with } h, h' \in G \quad (4.16)$$

The elements P_h, g do not commute:

$$g P_h = P_{ghg^{-1}} g \quad (4.17)$$

Combining the above relations we conclude that the algebra of the elements of the quantum group is

$$P_h g P_{h'} g' = \delta_{h,gh'g^{-1}} P_h (gg') \quad (4.18)$$

Let $\{C_A\}$, $A = 1, \dots, C$ be the conjugacy classes of the group G with $\{C_1\} = \{e\}$. In every C_A we choose an element ${}^A h_1$ and define the normalizer of the element ${}^A h_1$:

$$N_{A h_1} = \{h \in G | h {}^A h_1 = {}^A h_1 h\} \quad (4.19)$$

The normalizers of the elements in the same conjugacy class are isomorphic so we will denote N_A the group $N_{A h_1}$. Let ${}^A x_1, \dots, {}^A x_q$ a set of equivalence classes of G/N_A so that ${}^A h_i = {}^A x_1^A h_1^A x_i^{-1}$ i.e.

$$\begin{aligned} {}^A h_1 &= {}^A x_1^A h_1^A x_1^{-1} \\ {}^A h_2 &= {}^A x_2^A h_1^A x_2^{-1} \\ &\dots \\ {}^A h_q &= {}^A x_q^A h_1^A x_q^{-1} \end{aligned} \quad (4.20)$$

We choose for convenience ${}^A x_1 = e$. So the conjugacy class C_A can now be written as

$$C_A = \{{}^A h_1, \dots, {}^A h_q\} = \{{}^A x_1^A h_1^A x_1^{-1}, \dots, {}^A x_q^A h_1^A x_q^{-1}\} \quad (4.21)$$

Let a be an irreducible representation of N_A acting on a subspace V_A with basis vectors $|{}^a v_j\rangle_{j=1, \dots, \dim a}$. We define the Hilbert space V_a^A with basis vectors

$$|{}^A h_i, {}^a v_j\rangle_{j=1, \dots, \dim a}^{i=1, \dots, q} \quad (4.22)$$

We are not interested in all the elements of the quantum double, but only on the subalgebra B_A of $D(G)$ with elements

$$(P_h g)_{h \in G, g \in N_A} \quad (4.23)$$

Let Π_a^A the mapping from B_A to V_a^A such that

$$\Pi_a^A(P_h g) |^A h_i, {}^a v_j\rangle = \delta_{h, g^A h_i g^{-1}} |g^A h_i g^{-1}, a(\tilde{g})_{mj} {}^a v_m\rangle \quad (4.24)$$

with $\tilde{g} = {}^A x_k^{-1} g {}^A x_i \in N_A$. x_k is defined in such way the \tilde{g} is always in N_A .

$$\tilde{g}^A h_1 = {}^A h_1 \tilde{g} \Rightarrow {}^A h_k = g^A h_i g^{-1} \quad (4.25)$$

The map Π_a^A satisfies the relation

$$\Pi_a^A(P_h g) \Pi_a^A(P_{h'} g') = \Pi_a^A(P_h g P_{h'} g') \quad (4.26)$$

so it is a representation of the quantum double. The representations satisfy the *orthogonality relation*

$$\frac{1}{|G|} \sum_{h \in G, g \in N_A} \text{tr}(\Pi_a^A(P_h g)) \text{tr}(\Pi_\beta^B(P_h g))^* = \delta_{A,B} \delta_{a,\beta} \quad (4.27)$$

with

$$\text{tr}(\Pi_a^A(P_h g)) = \begin{cases} 0 & \text{if } h \notin C_A \text{ or } hg \neq gh \\ \text{tr}(a(\tilde{g})) & \text{where } \tilde{g} = {}^A x_k^{-1} g {}^A x_i \in N_A \end{cases} \quad (4.28)$$

Let a be an irreducible representation of N_{A_h} , if h is in C_A we will define a representation a^h of N_h by

$$a^h(\tilde{g}) = a(x^{-1} \tilde{g} x) \text{ where } h = x^A h x^{-1}, \tilde{g} \in N_h \quad (4.29)$$

We will use the notation $\rho_a^h(\tilde{g}) = \text{tr}(a^h(\tilde{g}))$. As a result

$$\text{tr}(\Pi_a^A(P_h g)) = \begin{cases} 0, & \text{if } g \notin C_A \text{ or } hg \neq gh \\ \rho_a^h(\tilde{g}) & \end{cases} \quad (4.30)$$

The superselection sectors are the couples (C_A, a) where C_A is the conjugacy class and a is the irreducible representation of the normalizer of the element that belongs in the conjugacy class C_A . We will denote them as $[\phi_a^A]$ and call them *chiral sectors*. We will show that the vectors that span the Hilbert space V_a^A have the same *topological spin*. We define the central element as $\sum_h P_h h$. Then

$$\Pi_a^A \left(\sum_h P_h h \right) |^A h_i, {}^a v_j\rangle = |^A h_i, a({}^A h_1)_{mj} {}^a v_m\rangle \quad (4.31)$$

Since ${}^A h_1$ from its definition commutes with all the elements of N_A then from Schur's lemma it will be proportional to the identity matrix of the V_a subspace

$$a({}^A h_1) = e^{2\pi i s(A,a)} I_a \quad (4.32)$$

so every sector (C_A, a) has its own different value of topological spin.

The Hilbert space that describes two sectors (C_A, a) and (C_B, β) is $V_a^A \otimes V_\beta^B$. The extension of the action of the quantum double from the 1-particle states in the 2-particles states is given by the *comultiplication*

$$\Delta(P_h g) = \sum_{h' h'' = h} P_{h'} g \otimes P_{h''} g \quad (4.33)$$

The representation of the tensor product is

$$\sum_{h' h'' = h} \Pi_a^A(P_{h'} g) \otimes \Pi_\beta^B(P_{h''} g) = \Pi_a^A \otimes \Pi_\beta^B(\Delta(P_h g)) \quad (4.34)$$

and can be decomposed on a direct sum of representations $(\Pi_\gamma^C, V_\gamma^C)$:

$$\Pi_a^A \otimes \Pi_\beta^B = \bigoplus_{C, \gamma} N_{a\beta C}^{AB\gamma} \Pi_\gamma^C \quad (4.35)$$

The coefficients $N_{a\beta C}^{AB\gamma}$ are called *fusion multiplicities*.

An important aspect of the comultiplication is that it is coassociative :

$$(id \oplus \Delta)\Delta(P_h g) = (\Delta \oplus id)\Delta(P_h g) = \sum_{h' h'' h''' = h} P_{h'} g \otimes P_{h''} g \otimes P_{h'''} g \quad (4.36)$$

We define the R-matrix that acts on the Hilbert space of the tensor product of fluxes that span the group G as

$$R = \sum_{h, g \in G} P_g e \otimes P_h g \quad (4.37)$$

It's action on a state $|a\rangle |b\rangle$ is

$$R |a\rangle |b\rangle = \sum_{h, g \in G} \delta_{g, a} |a\rangle \otimes \delta_{h, gbg^{-1}} |gbg^{-1}\rangle = |a\rangle |aba^{-1}\rangle \quad (4.38)$$

But we have already mentioned that the R-matrix action on a state $|a\rangle |b\rangle$ is

$$R |a\rangle |b\rangle = |aba^{-1}\rangle |a\rangle \quad (4.39)$$

So we can redefine the R-matrix as

$$R = \sigma \circ \sum_{h, g \in G} P_g e \otimes P_h g \quad (4.40)$$

with the notation σ meaning $\sigma \circ |a\rangle |b\rangle = |b\rangle |a\rangle$.

The physical braid operator is defined so that it acts on the states belonging in the Hilbert space $V_a^A \otimes V_\beta^B$

$$R_{a\beta}^{AB} = \sigma \circ (\Pi_a^A \otimes \Pi_\beta^B) R = \sigma \circ \sum_{h, g \in G} \Pi_a^A(P_g e) \otimes \Pi_\beta^B(P_h g) \quad (4.41)$$

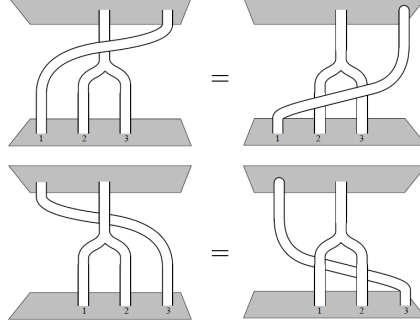


Figure 4.2

The action of $R_{a\beta}^{AB}$ is

$$R_{a\beta}^{AB} |^A h_i, {}^a v_j\rangle |^B h_m, {}^\beta v_n\rangle = \left| {}^A h_i^B h_m^A h_i^{-1}, \beta({}^A \tilde{h}_i)_{ln} {}^\beta v_n \right\rangle |^A h_i, {}^a v_j\rangle \quad (4.42)$$

The physical braid operator satisfies the quasitriangularity conditions:

$$\begin{aligned} R\Delta(P_h g) &= \Delta(P_h g)R \\ (id \otimes \Delta)R &= R_2 R_1 \\ (\Delta \otimes id)R &= R_1 R_2 \end{aligned} \quad (4.43)$$

with R_1 and R_2 satisfying the *Yang-Baxter equation*

$$R_1 R_2 R_1 = R_2 R_1 R_2 \quad (4.44)$$

It is clear now that R is a homomorphism of the Braid group. The R operators satisfy the equations of the generators of the Braid group:

$$\begin{aligned} R_i R_{i+1} R_i &= R_{i+1} R_i R_{i+1}, \quad i = 1, \dots, n-2 \\ R_i R_j &= R_j R_i, \quad |i-j| \geq 2 \end{aligned} \quad (4.45)$$

along with the condition

$$R^m = 1 \quad (4.46)$$

where m is the least common multiple of the order of the elements of the group times two. The groups satisfying the above equations are called *Truncated Braid Groups* and it is clear that the system unbraids after a finite number of knots. They are denoted as $B(n, m)$ where n is the number of particles and m the order of the generators.

Let (Π_a^A, V_a^A) and (Π_β^B, V_β^B) two irreducible representations of the quantum double $D(G)$. The representation of the tensor product $(\Pi_a^A \oplus \Pi_\beta^B, V_a^A \otimes V_\beta^B)$ is reducible

$$\Pi_a^A \otimes \Pi_\beta^B = \bigoplus_{C, \gamma} N_{a\beta C}^{AB\gamma} \Pi_\gamma^C \quad (4.47)$$

with

$$N_{a\beta C}^{AB\gamma} = \frac{1}{G} \sum_{h,g \in G} \text{tr}[\Pi_a^A \otimes \Pi_\beta^B(\Delta(P_h g))] \text{tr}[\Pi_\gamma^C(P_h g)]^* \quad (4.48)$$

These are the fusion rules. They determine which particles (C_C, γ) can be formulated by two particles $(C_A, a), (C_B, \beta)$. We can define the modular S-matrix:

$$S_{a\beta}^{AB} = \frac{1}{|G|} \text{tr}(R_{a\beta}^{AB})^{-2} \\ = \frac{1}{|G|} \sum_{\substack{A h_i \in C_A, B h_j \in C_B \\ [A h_i, B h_j] = e}} \text{tr}(a(A x_i^{-1B} h_j^A x_i))^* \text{tr}(\beta(B x_j^{-1A} h_i^B x_j))^* \quad (4.49)$$

For the case that $G = S_3$ the S-Matrix is:

$$S = \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{bmatrix} \quad (4.50)$$

Verlinde found a formula which produces the non-negative fusion multiplicities from S-matrix [16,27-29].

$$N_{a\beta C}^{AB\gamma} = \sum_{D, \delta} \frac{S_{a\delta}^{AD} S_{\beta\delta}^{BD} (S_{\gamma\delta}^{CD})^*}{S_{0\delta}^{eD}} \quad (4.51)$$

So the fusion rules for our model is:

Fusion rules								
	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	A⊕B⊕C	D⊕E	D⊕E	G⊕H	F⊕H	F⊕G
D	D	E	D⊕E	A⊕C⊕F⊕G⊕H	B⊕C⊕F⊕G⊕H	D⊕E	D⊕E	D⊕E
E	E	D	D⊕E	B⊕C⊕F⊕G⊕H	A⊕C⊕F⊕G⊕H	D⊕E	D⊕E	D⊕E
F	F	F	G⊕H	D⊕E	D⊕E	A⊕B⊕F	H⊕C	G⊕C
G	G	G	F⊕H	D⊕E	D⊕E	H⊕C	A⊕B⊕G	F⊕C
H	H	H	F⊕G	D⊕E	D⊕E	G⊕C	F⊕C	A⊕B⊕H

Let Π be a finite dimension representation of $D(G)$ acting on a vector space W . So

$$\Pi = \bigoplus_{A,a} n_{A,a} \Pi_a^A \quad (4.52)$$

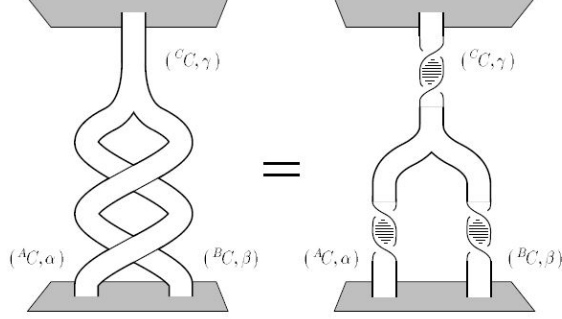


Figure 4.3

and there is a unique decomposition of W such that

$$W = \bigoplus_{A,a} W_a^A \quad (4.53)$$

where

$$W_a^A = \bigoplus_{i=1}^{n_{A,a}} W_i \text{ with } W_i \cong V_a^A \quad (4.54)$$

W_a^A are called *isotypic components*. We construct the projection operator P_a^A , which projects in W_a^A using the character's orthogonality.

$$P_a^A = \frac{\dim a}{N_A} \sum_{g,h \in G} \text{tr}(\Pi_a^A(P_g h))^* \Pi(P_g h) \quad (4.55)$$

Let $K_{a\beta\gamma}^{ABC}$ be the projection of the isotypic component W_γ^C on $V_a^A \otimes V_\beta^B$

$$K_{a\beta\gamma}^{ABC} = \frac{\dim \gamma}{N_C} \sum_{gx=xg} \rho_\gamma^{gh}(x)^* \Pi_a^A(P_g x) \otimes \Pi_\beta^B(P_h x) \quad (4.56)$$

It is proven that

$$K_{a\beta\gamma}^{ABC} R_{\beta a}^{BA} R_{a\beta}^{AB} = e^{2\pi i(s(C,\gamma) - s(A,a) - s(B,\beta))} K_{a\beta\gamma}^{ABC} \quad (4.57)$$

So acting with the monodromy operator and then fusing the two particles is equivalent with fusing the two particles and then rotating the 2 particles clockwise by 2π and the composite object counterclockwise by 2π [26] as seen in figure (4.3).

When two indistinguishable particles are exchanged then they are transformed as a representation of the truncated braid group $B(2, m)$. If the particles though are distinguishable, they transform as a representation of the truncated colored braid group $P(2, m)$ with $m/2 \in \mathbb{Z}$ the order of the monodromy operator R^2 of this particular system. So a system with n indistinguishable particles

transform as the truncated colored braid group $P(n, m)$ and if the particles are distinguishable transform as the truncated braid group $B(n, m)$. If the system is composed of both distinguishable and indistinguishable then they transform as the Partially Truncated Braid group.

Chapter 5

Quantum computation using anyons

In the case of a nonabelian superconductor, the fluxes take values in a nonabelian finite group G . In a system of anyons there are 3 physical processes [17,18].

Pair creation and identification. We can create pair of particles from the vacuum where every particle is characterized by it's conjugacy class and the irreducible representation of the normalizer group of the flux (C_A, a) . Particles that either belong in a different conjugacy class or in a different representation of the normalizer group of the flux are distinguishable.

Pair annihilation. We can bring any two particles together and see if they annihilate, and thus have trivial total flux and charge. We will call this process *destructive measurement*.

Braiding. We can move the anyons in specific trajectories of our choice and exchange particles.

The first thing we need to do in a system of anyons is to calibrate the fluxes. Suppose we are given two pair of fluxes $|a, a^{-1}\rangle$ and $|b, b^{-1}\rangle$ and we want to test if the pairs match. Consider a pair of particles each of which carries charge but not flux such that the total charge of the pair is trivial. We will call them *chargeon-antichargeon* and if the chargeon transforms as a unitary irreducible representation R of the group G acting on a vector space V with dimension $\dim V = |R|$ then the antichargeon transforms as the conjugate irreducible representation \bar{R} that combined with R gives the trivial representation. If $\{|R, i\rangle\}$ is a basis of R then $\{|\bar{R}, i\rangle\}$ can be chosen for \bar{R} so that the pair can be described by the state

$$|0; R\rangle = \frac{1}{|R|} \sum_{i=1}^{|R|} |R, i\rangle \otimes |\bar{R}, i\rangle \quad (5.1)$$

Imagine we create a pair of fluxes $|a, a^{-1}\rangle$ and a pair of chargeon-antichargeon. We wind the chargeon around the first member of the flux pair, so due to

Bohm-Aharonov effect it's state transforms as

$$|R, i\rangle' = \sum_i^{|R|} D_{ij}^R(a) |R, j\rangle \quad (5.2)$$

so the chargeon-antichargeon state becomes

$$|0; R\rangle' = \frac{1}{|R|} \sum_{i=1}^{|R|} |R, i\rangle' \otimes |\bar{R}, i\rangle = \frac{1}{|R|} \sum_{i=1}^{|R|} D_{ij}^R(a) |R, j\rangle \otimes |\bar{R}, i\rangle \quad (5.3)$$

We bring the members of the pair together and see if they annihilate. The amplitude of the process is

$$\langle 0; R | 0; R \rangle' = \frac{\sum_{i=1}^{|R|} D_{ii}^R(a)}{\dim R} = \frac{\chi^R(a)}{|R|} \quad (5.4)$$

so the probability they will annihilate is

$$Prob(0) = \left| \frac{\chi^R(a)}{|R|} \right|^2 \quad (5.5)$$

So back to our case, and the two pair of fluxes. We wind the chargeon around the first member of the first pair and then around the second member of the second pair. The probability they will annihilate after the two consecutive exchanges is

$$Prob(0) = \left| \frac{\chi^R(b^{-1}a)}{|R|} \right|^2 \quad (5.6)$$

which clearly is less than zero if $b \neq a$. After a number of repetitions we can say with high statistical confidence if the fluxes are different. We can sort in that way all the different fluxes on separate "bins". The next step is to label the fluxes so that they match the group composition rules because the chance of getting it right on the first random labeling is $1/(|G|!)$. Suppose we take 3 pairs from 3 different bins, $|a, a^{-1}\rangle$, $|b, b^{-1}\rangle$ and $|c, c^{-1}\rangle$ and we want to check whether $c = ab$. We create again a chargeon-antichargeon pair and wind it first around a , then b and then c^{-1} and then reunite the chargeon-antichargeon to see if it annihilates. If $ab = c$ then every time we bring the chargeon-antichargeon pair together it will annihilate. We construct in that way a *flux bureau of standards* and label the fluxes according to the group G that they take values. So given an unknown pair of fluxes $|d, d^{-1}\rangle$ we can use any of the labeled fluxes, suppose $|a, a^{-1}\rangle$, repeat the above process and determine the flux d . We will call this process *projective flux measurement*.

Suppose we have two pairs of fluxes $|a, a^{-1}\rangle$ and $|b, b^{-1}\rangle$. If we transport the pair of fluxes $|a, a^{-1}\rangle$ counterclockwise around the first member of the second pair, then the state transforms as

$$|a, a^{-1}\rangle \rightarrow |bab^{-1}, ba^{-1}b^{-1}\rangle \quad (5.7)$$

This is the basic gate we can use and we will call it *conjugation gate*. Recapitulating, the 3 basic processes are *Projective flux measurement*, *Destructive measurement* and *Conjugation gate*.

Next step is to choose the computational basis. We make the choice

$$|n\rangle = |a^n b a^{-n}\rangle \otimes |a^n b^{-1} a^{-n}\rangle \quad 0 \leq n < d \quad (5.8)$$

with d prime. Let the fluxes take values on a simple perfect group. Every simple perfect group has even order so we can always find a group element a such that $a^2 = 1$ and thus work with *qubits* [19]. We choose two noncommuting elements a, b of G such that $a^2 = 1$ and we define the computational basis

$$\begin{aligned} |0\rangle &= |b, b^{-1}\rangle \\ |1\rangle &= |aba^{-1}, ab^{-1}a^{-1}\rangle \end{aligned} \quad (5.9)$$

The matrix that executes the projective measurement is the Z -Pauli matrix

$$Z |0\rangle = |0\rangle, \quad Z |1\rangle = -|1\rangle \quad (5.10)$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (5.11)$$

The X -Pauli is defined as

$$X |0\rangle = |1\rangle, \quad X |1\rangle = |0\rangle \quad (5.12)$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (5.13)$$

The eigenvectors of X in the computational basis are

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} (|b, b^{-1}\rangle + |aba^{-1}, ab^{-1}a^{-1}\rangle) \\ |\tilde{1}\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} (|b, b^{-1}\rangle - |aba^{-1}, ab^{-1}a^{-1}\rangle) \end{aligned} \quad (5.14)$$

Suppose we create a pair of fluxes from the vacuum. What is the state of the pair? Suppose that the members of the pair belong in the conjugacy class C_A . They should not be carrying any conserved charges and so the state that describes the pair is

$$|0; C_A\rangle = \frac{1}{\sqrt{|C_A|}} \sum_{g \in C_A} |g, g^{-1}\rangle \quad (5.15)$$

That means that if we bring two members of a flux pair $|g, g^{-1}\rangle$ together the chance that they will annihilate is not 1 but $1/|C_A|$. If the state of a pair is $|\tilde{1}\rangle$ then the amplitude of the components of the pair to fuse in the vacuum is

$$\langle \tilde{1} | 0; C_A \rangle = 0 \quad (5.16)$$

thus they will never fuse into the vacuum. On the other side taking the amplitude

$$\langle \tilde{0} | 0; C_A \rangle = \sqrt{\frac{2}{|C_A|}} \quad (5.17)$$

and thus there is finite probability that they will fuse in the vacuum.

We can construct the states $|\tilde{0}\rangle$. We begin by creating a pair in the state $|0; C_A\rangle$. If C_A has just two elements then we are done. But if it has more than two, we bring the pair near a calibrated pair $|c, c^{-1}\rangle$ where $c \in C_A$ and $c \neq b, aba^1$. If it doesn't match $|c, c^{-1}\rangle$ for every $c \in C_A$ then it must be $|\tilde{0}\rangle$.

It will be more convenient to denote the pairs $|a, a^{-1}\rangle$ as $|a\rangle$. Suppose we have the 3 pair state $|x, y, z\rangle$. We can wind the third pair clockwise or counterclockwise around the first pair and execute the gate

$$|x, y, z\rangle \rightarrow |x, y, xzx^{-1}\rangle, |x, y, z\rangle \rightarrow |x, y, x^{-1}zx\rangle \quad (5.18)$$

or do the same with the second pair

$$|x, y, z\rangle \rightarrow |x, y, yzy^{-1}\rangle, |x, y, z\rangle \rightarrow |x, y, y^{-1}zy\rangle \quad (5.19)$$

or even borrow a pair of fluxes $|c\rangle$ from the bureau of standards

$$|x, y, z\rangle \rightarrow |x, y, czc^{-1}\rangle \quad (5.20)$$

for every $c \in G$. So we can execute any gate of the form

$$|x, y, z\rangle \rightarrow |x, y, f(x, y)zf(x, y)^{-1}\rangle \quad (5.21)$$

where $f(x, y)$ is a function with a product form.

Theorem 5.0.1. *If G is a simple and perfect finite group then every function $f(g_1, g_2, \dots, g_n)$ can be expressed as a product of inputs $\{g_i\}$ their inverses $\{g_i^{-1}\}$ and fixed elements of G any of which may appear multiple times in the product*

The smallest simple and perfect group is A_5 the group of even permutations of five objects. Gottesman [20] has proved in his paper that for d prime being able to apply products of X 's and Z 's plus a Toffoli is universal for quantum computation.

The Toffoli gate is defined to act on a 3 qubit state as

$$|x, y, z\rangle \rightarrow |x, y, z \oplus xy\rangle \quad (5.22)$$

so in our case it can be written in a more convenient way as

$$|a^i ba^{-i}, a^j ba^{-j}, a^k ba^{-k}\rangle \rightarrow |a^i ba^{-i}, a^j ba^{-j}, a^{ij+k} ba^{-ij-k}\rangle \quad (5.23)$$

So, the Toffoli gate conjugates the third qubit by the function

$$f(a^i ba^{-1}, a^j ba^{-j}) = a^{ij} \quad (5.24)$$

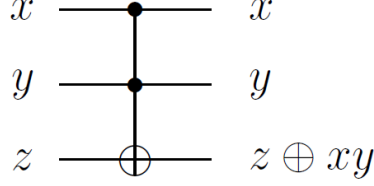


Figure 5.1

Suppose we have 2 qubits with fluxes g_1 and g_2 with $g_i \in \{b, aba^{-1}\}$. We define new variables $g'_i = g_i b^{-1} \in \{1, c\}$ with c is defined as the commutator of a and b

$$c \equiv [a, b] = aba^{-1}b^{-1} \quad (5.25)$$

We choose an element d such that it doesn't commute with c and define $f \equiv [c, d]$. If we find two functions that can be written in a product form such that

$$\begin{aligned} h_1(c) &= d, \quad h_1(e) = 1 \\ h_2(f) &= a, \quad h_2(e) = 1 \end{aligned} \quad (5.26)$$

then the Toffoli function can be written as

$$f(g_1, g_2) = h_2([g'_1, h_1(g'_2)]) \quad (5.27)$$

Let the fluxes take values on A_5 . According to the above analysis we need an element a such that $a^2 = e$ in order to work with qubits. We choose $a = (12)(34)$. Next we choose an element b that doesn't commute with a , and d that doesn't commute with $c \equiv [a, b]$. We make the choice $b = (345)$ and $d = (234)$. The computational basis is

$$\begin{aligned} |0\rangle &= |b\rangle = |(345)\rangle \\ |1\rangle &= |aba^{-1}\rangle = |(435)\rangle \end{aligned} \quad (5.28)$$

So

$$\begin{aligned} h_1(c) &= d, \quad h_1(e) = 1 \\ h_2(f) &= a, \quad h_2(e) = 1 \end{aligned} \quad (5.29)$$

It's easy to see that two functions with these properties are

$$h_1(g) = h_2(g) = (521)g(125) \quad (5.30)$$

So the Toffoli function is

$$\begin{aligned} f(g_1, g_2) &= \{(521)[g_1(435), (521)g_2(435)(125)](125)\} \Rightarrow \\ f(g_1, g_2) &= \{(521)g_1(14325)g_2(124)g_1^{-1}(15432)g_1^{-1}(521)\} \end{aligned} \quad (5.31)$$

So given 3 qubits $|g_1, g_2, g_3\rangle$, if we want to execute the Toffoli gate for the group A_5 all we have to do is to conjugate the third qubit by the above function

$f(g_1, g_2)$. This is the "recipe" for the function $f(g_1, g_2)$ that can be used for any simple perfect group and its exact form can be found using the functions h_i .

Now that we constructed the Toffoli gate we can move to the next step. We need to be able to apply products of X 's and Z 's i.e. $X^a Z^b$. An important observation is that

$$(X^a Z^b)^d = \omega^{abd(d-1)/2} X^{ad} Z^{bd} = \begin{cases} -I^{ab}, & \text{if } d = 2 \\ I, & \text{if } d = \text{odd} \end{cases} \quad (5.32)$$

with $\omega = e^{2\pi i/d}$. We work with qubits so $\omega = e^{2\pi i/2}$. The eigenvalues of $ZX = iY$ are

$$\begin{aligned} |0_Y\rangle &= \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}} (|b\rangle + i|aba^{-1}\rangle) \\ |1_Y\rangle &= \frac{1}{\sqrt{2}} (|0\rangle - i|1\rangle) = \frac{1}{\sqrt{2}} (|b\rangle - i|aba^{-1}\rangle) \end{aligned} \quad (5.33)$$

The $d = 2$ case is invariant under complex conjugation and thus there is no way of distinguishing the two eigenstates of ZX . Before we continue we need to refer *Kitaev's phase estimation technique* that refers that if we have an operator with eigenvalues the d th roots of unity and if we apply a Controlled- U and measure in the X basis is equivalent with measuring the operator U .

So suppose someone provides with a state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + \omega|1\rangle) \quad (5.34)$$

It could be any of the eigenvectors of ZX . We label it as the $+i$ eigenstate. We use Kitaev's phase estimation technique and act with control- ZX on the state $|\tilde{0}\rangle \otimes |\Psi\rangle$ with $|\tilde{0}\rangle$ as the control ancilla and $|\Psi\rangle$ as the target. So the circuit performs the transformation

$$|\tilde{0}\rangle \otimes |\Psi\rangle \rightarrow |\Psi\rangle \otimes |\Psi\rangle \quad (5.35)$$

So the copy of the second state occurs either way. This is quite helpful as we can execute gates on the first qubit without employing a destructive measurement on the second. We can always act with Z on $|\Psi\rangle$ and construct the orthogonal state $|\Phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle - \omega|\Psi\rangle)$. Now that we have copied our state we can construct a new controlled- ZX with the ancilla $|\Psi\rangle$ as the control qubit and another $|\Psi\rangle$ as the target and then measure the first qubit in the X basis:

$$|\Psi\rangle \otimes |\Psi\rangle \rightarrow |\tilde{1}\rangle \otimes |\Psi\rangle \quad (5.36)$$

Doing the same thing with $|\Phi\rangle$ as the ancilla and then measuring again in the X basis:

$$|\Phi\rangle \otimes |\Psi\rangle \rightarrow |\tilde{0}\rangle \otimes |\Psi\rangle \quad (5.37)$$

so as long as we are consistent in using the same ancilla $|\Psi\rangle$ we will have broken the conjugation symmetry and have found a new way of labeling creating and

measuring eigenstates of ZX . But how we create the first state $|\Psi\rangle$ ourselves? If we want to prepare an eigestate of $ZX = iY$ at random, it is equivalent for the density matrix to be proportional to the identity

$$\rho = \frac{1}{2}I = \frac{1}{2} |0_Y\rangle \langle 0_Y| + \frac{1}{2} |1_Y\rangle \langle 1_Y| \quad (5.38)$$

By using a controlled-NOT gate from a $|\tilde{0}\rangle$ ancilla to $|0\rangle$ ancilla we produce the state

$$CNOT(|\tilde{0}\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}} (|0_Y 1_Y\rangle + |1_Y 0_Y\rangle) \quad (5.39)$$

So by tracing out one qubit of the bell state we have produced the desired $|\Psi\rangle$. Therefore we proved how to construct eigestates of ZX , X , Z and execute the Toffoli gate and thus achieve universal quantum computation.

Chapter 6

Fibonacci Anyons

Suppose we have a model of two different anyons. The first is the *vacuum sector* labeled 0 and the second is the nontrivial label 1 with $\bar{1} = 1$. There is only one fusion rule

$$1 \times 1 = 0 + 1 \quad (6.1)$$

The model is nonabelian as the nontrivial label 1 can fuse with itself in two different ways. The topological Hilbert space that describes the fusion of n different 1 anyons is denoted $V_{1^n}^b$ with $b \in \{0, 1\}$. When we create anyons from the vacuum, their total charge is trivial. So the Hilbert space of anyons created from the vacuum is $V_{1^n}^0$. We denote the dimension of the Hilbert space $\dim V_{1^n}^0 = N_{1^n}^0 = N_n^0$. The number N_n^0 , i.e. the dimension of the topological Hilbert space, describes the different ways the anyons can fuse to the vacuum. By creating the trees for each n we notice that every 0 must always be followed by an 1. So if the first two anyons give us trivial total charge then the remaining anyons can fuse with N_{n-2}^0 different ways and if the fuse to 1 then the can fuse with N_{n-1}^0 different ways. So the numbers N_n^0 satisfy the recursion relation

$$N_n^0 = N_{n-1}^0 + N_{n-2}^0 \quad (6.2)$$

The dimensions of the Hilbert space follow the Fibonacci sequence and that's why the model is called *Fibonacci model* [21-23]. One important thing we observe is that V_{1111}^0 is two dimensional and thus work with qubits. The two different vectors that span V_{1111}^0 are seen below.

As we know fusion is associative. So if V_{abc}^d is a Hilbert space then it can be decomposed in two different ways

$$V_{abc}^d = \oplus_e V_{ab}^e \otimes V_{ec}^d = \oplus_{e'} V_{ae'}^d \otimes V_{bc}^{e'} \quad (6.3)$$

What connects these two different decompositions is the F -matrix. Schemati-

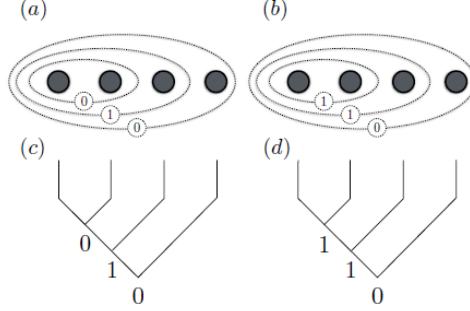


Figure 6.1

$$\begin{array}{c} a \quad b \quad c \\ \swarrow \quad \downarrow \quad \searrow \\ \mu \quad e \quad \nu \\ \downarrow \\ d \end{array} = \sum_{e' \mu' \nu'} \left(F_{abc}^d \right)_{e \mu \nu}^{e' \mu' \nu'} \begin{array}{c} a \quad b \quad c \\ \swarrow \quad \downarrow \quad \searrow \\ \mu' \quad e' \quad \nu' \\ \downarrow \\ d \end{array}$$

Figure 6.2

cally: We can compute all the elements of the different F-matrices.

$$\begin{aligned}
 (abcd) = (1101) &\Rightarrow (F_{110}^1)_1^1 = 1, (F_{110}^1)_0^1 = 0 \\
 (abcd) = (0000) &\Rightarrow (F_{000}^0)_0^0 = 1, (F_{000}^0)_1^0 = 0 \\
 (abcd) = (1100) &\Rightarrow (F_{110}^0)_1^0 = 1, (F_{110}^0)_0^0 = 0 \\
 (abcd) = (1110) &\Rightarrow (F_{111}^0)_1^1 = 1, (F_{111}^0)_0^1 = 0 \\
 (abcd) = (1010) &\Rightarrow (F_{111}^0)_1^1 = 1, (F_{101}^0)_0^1 = 0 \\
 (abcd) = (0111) &\Rightarrow (F_{011}^1)_1^1 = 1, (F_{011}^1)_0^1 = 0 \\
 (abcd) = (0110) &\Rightarrow (F_{011}^0)_0^1 = 1, (F_{011}^0)_1^1 = 0 \\
 (abcd) = (1011) &\Rightarrow (F_{101}^1)_1^1 = 1, (F_{101}^1)_0^1 = 0
 \end{aligned} \tag{6.4}$$

The case $(abcd) = (1111)$ is not trivial. We need to calculate all the elements of F_{111}^1 defined as

$$F_{111}^1 = \begin{bmatrix} (F_{111}^1)_0^0 & (F_{111}^1)_0^1 \\ (F_{111}^1)_1^0 & (F_{111}^1)_1^1 \end{bmatrix} \tag{6.5}$$

In order to find the matrix elements, we need the *pentagon equation* as seen 6.3. Written as an equation

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = \sum_e (F_{234}^d)_e^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \tag{6.6}$$

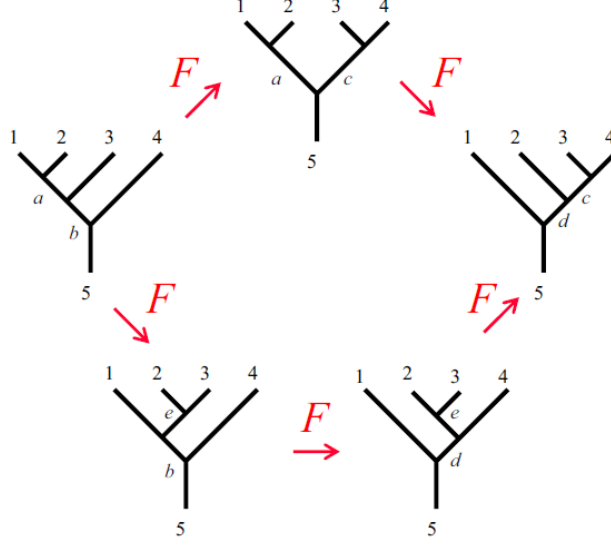


Figure 6.3

Taking $a = b = c = d = 1$

$$[(F_{111}^1)_1^1]^2 = (F_{111}^1)_0^1 (F_{111}^1)_1^0 + [(F_{111}^1)_1^1]^3 \quad (6.7)$$

Then setting $a = 0, b = c = d = 1$

$$[(F_{111}^1)_1^1]^2 + (F_{111}^1)_0^0 = 1 \quad (6.8)$$

Moreover, if $b = c = 1, a = d = 0$:

$$(F_{111}^1)_0^0 = (F_{111}^1)_1^0 (F_{111}^1)_0^1 \quad (6.9)$$

If $\Phi = \frac{1+\sqrt{5}}{2}$ is the *golden mean* then the matrix F_{111}^1 is

$$F_{111}^1 = \begin{bmatrix} \frac{1}{\Phi} & \frac{1}{\sqrt{\Phi}} \\ \frac{1}{\sqrt{\Phi}} & -\frac{1}{\Phi} \end{bmatrix} \quad (6.10)$$

In the next step I will calculate R-matrix which is a map from $V_{ba}^c \longrightarrow V_{ab}^c$ and represents a counterclockwise exchange of two adjacent anyons. It is clear that if the exchange includes an anyon which is the vacuum sector, then it will have no physical consequence in the system. That is the phases below are all equal to one.

$$\begin{aligned} R_1^{10} &= 1, \quad R_1^{01} = 1 \\ R_0^{00} &= 1 \end{aligned} \quad (6.11)$$

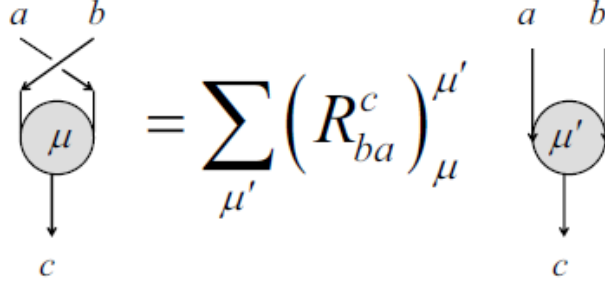


Figure 6.4

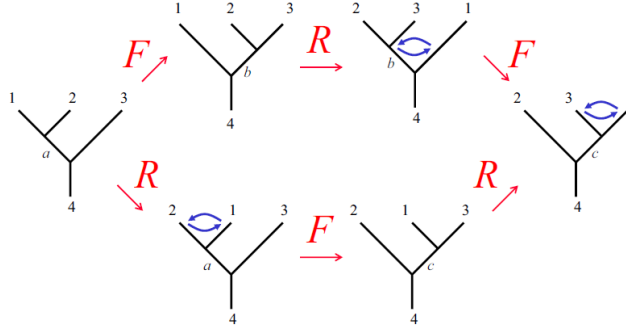


Figure 6.5

The matrix R is clearly diagonal because an exchange of two anyons cannot change the outcome of the fusion, that is R is of the form

$$R^{11} = \begin{bmatrix} R_0^{11} & 0 \\ 0 & R_1^{11} \end{bmatrix} \quad (6.12)$$

Just as in the case of F matrix in order to calculate the matrix elements we need something analogous to the pentagon equation. There exists an equation that includes the R and F matrices and is called *hexagon equation*.

$$R_c^{13} (F_{213}^4)_a^c R_a^{12} = \sum_b (F_{231}^4)_b^c R_4^{1b} (F_{123}^4)_a^b \quad (6.13)$$

We set $c = a = 0$

$$(R_0^{11})^2 = \frac{1}{\Phi} + R_1^{11} \quad (6.14)$$

and $a = 0, c = 1$

$$R_1^{11} R_0^{11} \frac{1}{\sqrt{\Phi}} = \frac{1}{\Phi} + R_1^{11} \frac{1}{\Phi^{3/2}} \quad (6.15)$$

The R matrix is

$$R^{11} = \begin{bmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{bmatrix} \quad (6.16)$$

If we want to work with qubits then we will denote $|0\rangle$ the state (a) in figure (3.1) and $|1\rangle$ the state (b). Let σ_1 then counterclockwise exchange of the first and second anyon and σ_2 the counterclockwise exchange of the second and third.

$$\begin{aligned}\sigma_1 |0\rangle &= R_0^{11} |0\rangle = e^{-4\pi i/5} |0\rangle \\ \sigma_1 |1\rangle &= R_1^{11} |1\rangle = e^{3\pi i/5} |1\rangle\end{aligned}\tag{6.17}$$

$$\begin{aligned}\sigma_2 |0\rangle &= (F_{111}^1)_0^0 R_0^{11} (F_{111}^1)_0^0 |0\rangle + (F_{111}^1)_1^0 R_1^{11} (F_{111}^1)_0^1 |0\rangle \\ &\quad + (F_{111}^1)_0^0 R_0^{11} (F_{111}^1)_1^0 |1\rangle + (F_{111}^1)_1^0 R_1^{11} (F_{111}^1)_1^1 |1\rangle \\ \sigma_2 |1\rangle &= (F_{111}^1)_0^1 R_0^{11} (F_{111}^1)_0^0 |0\rangle + (F_{111}^1)_1^1 R_1^{11} (F_{111}^1)_0^1 |0\rangle \\ &\quad + (F_{111}^1)_1^1 R_0^{11} (F_{111}^1)_1^0 |1\rangle + (F_{111}^1)_1^1 R_1^{11} (F_{111}^1)_1^1 |1\rangle\end{aligned}\tag{6.18}$$

So the 2 σ_i matrices are

$$\sigma_1 = \begin{bmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \Phi^{-1} e^{4\pi i/5} & \Phi^{-1/2} e^{-3\pi i/5} \\ \Phi^{-1/2} e^{-3\pi i/5} & -\Phi^{-1} \end{bmatrix}\tag{6.19}$$

These matrices are a representation of the braid group B_3 whose image is dense in $SU(2)$. Therefore each braid can be simulated with arbitrary accuracy by some finite braid. We can also use the inverse of each of the σ_i 's as the opposite braid can be applied. From our analysis on previous chapters we know that these matrices are actually representations of the truncated braid group $B(3, m)$. In the case of the Fibonacci, the σ_i 's are representations of the truncated braid group $B(3, 10)$. Moreover $\sigma_i^6 = \sigma_i^{-4}$ so the number of braids is significantly reduced. If we want to construct two qubit gates then we need 8 Fibonacci anyons whose fusion space is 13-dimensional and whose image is dense in $SU(13)$ which of course includes the $SU(4)$ that acts on the two encoded qubits. Therefore, we can apply the Solovay-Kitaev theorem to conclude that the universal gates of the circuit model can be simulated to accuracy ϵ with braids of length $\text{poly}(\log(1/\epsilon))$ and so a universal quantum computer can be simulated efficiently using Fibonacci anyons.

Conclusion

Putting everything together, we saw that topological quantum computation is of high interest because the way the information is manipulated is fault tolerant. We discussed two different ways to achieve universality. One with anyons from a discrete nonsolvable finite group and one with Fibonacci anyons. The difference between these 2 models is that the first one needs intermediate measurements in contrast with the other where the measurements are done after the braids are completed. We also discussed Kitaev's honeycomb lattice model where we introduced a perturbation that doesn't preserve the time reversal symmetry, creating an energy gap which allows the existence of nonabelian anyons. But have anyons been observed? Experimental evidence for nonabelian anyons is weak. The only promising lead seems to be the possible observation of Majorana zero modes trapped at the ends of superconducting wires, as described by the theoretical model of the Kitaev chain. Recent works show that the quantum Hall states at filling factors $5/2$ and $7/2$ are expected to have Abelian charge $e/2$ quasiparticles and non-Abelian charge $e/4$ quasiparticles.

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