Expanding Graphs and Balanced Separators

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ABSTRACT

A graph is an *expander* if it is sparse and has strong connectivity properties. Expanders are widely studied graphs, mainly due to their numerous applications in many different mathematical fields. The purpose of this thesis is to analyze the connections between expanders and other notions of graph theory, and study their substructures. Specifically, we will focus on the connection of balanced separators and expanders and provide an introduction on how the expansion of a graph is connected to the eigenvalues of its adjacency matrix. We will also study in detail the minors one can find in expanders.

ΣΥΝΟΨΗ

Ένα γράφημα ονομάζεται *εξαπλωτής* αν είναι αραιό αλλά ταυτόχρονα έχει ισχυρές ιδιότητες συνεκτικότητας. Οι εξαπλωτές είναι μία κατηγορία γραφημάτων η οποία, κυρίως λόγω των πολλών εφαρμογών τους σε διαφορετικά πεδία των μαθηματικών, έχουν μελετηθεί εκτενώς. Ο στόχος αυτής της εργασίας είναι να αναλύσουμε τη σύνδεση των εξαπλωτών με άλλες έννοιες της θεωρίας γραφημάτων, και να μελετήσουμε τις δομές που μπορούμε να βρούμε σε αυτούς. Συγκεκριμένα, θα επικεντρωθούμε στους ισορροπημένους διαχωριστές και πώς αυτοί συνδέονται με τους εξαπλωτές. Επιπλέον θα δούμε πιο σύντομα, πώς οι ιδιοτιμές του πίνακα γειτνίασης ενός γραφήματος συνδέονται με την εξάπλωσή του αλλά και με άλλες ιδιότητές του. Τέλος, θα ασχοληθούμε ιδιαίτερα με τα ελάσσονα γραφήματα ενός εξαπλωτή.

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CONTENTS

CHAPTER 1

INTRODUCTION

A graph G is said to have a balanced separator S, if there exists a vertex set $S \subseteq V(G)$ that disconnects the graph into two relatively equal connected components, namely each of size at most $2|V(G)|/3$. The problem of minimizing the size of a balanced separator in a graph, occurs naturally after studying the connectivity of graphs and its implications on the size and density of their subgraphs and minors. Although these two notions seem to be quite close, this is not the case. Notice that, adding an isolated vertex to a graph, affects its connectivity but not the existence of a small sized balanced separator, thus these two properties of graphs are essentially different.

Balanced separators have been widely studied, mainly due to their significance on applying the divide-and-conquer technique on developing algorithms that efficiently solve graph problems. One of the first results on this field was the $\sqrt{n}$-separator theorem which, Lipton and Tarjan proved [LT79] in 1979, and states that any planar n-vertex graph has a balanced separator of size $\mathcal{O}(\sqrt{n})$. In the same paper they provided a polynomial time algorithm that computes this separator, while one year later they issued a paper [LT80] that describes some applications of this theorem on known NP-complete graph problems, on planar graphs, such as the maximum independent set problem.

Later, in 1984, Gilbert, Hutchinson, Tarjan and Erde [GHT84] generalized this theorem for graphs that are embeddable on surfaces other than the plane. In particular, they showed that if an n-vertex graph can be drawn on a surface of genus g, then it has a balanced separator of size $\mathcal{O}(\sqrt{gn})$. Moreover they provided an algorithm that computes this balanced separator in linear time (in the number of edges of the graph), given an embedding of the graph in its genus surface. Alon, Seymour and Thomas [AST90b] proved a similar result for all non-planar graphs, namely, that every n-vertex graph either has a balanced separator of size $h^{3/2}\sqrt{n}$, or contains a clique of size h as a minor, and they also provided a $\mathcal{O}(h^{1/2}n^{1/2}m^{1/2})$ algorithm that realizes this theorem, where m is the number of edges in a graph.

Ideally, we would like also to have a polynomial time algorithm that computes a minimal balanced separator in a graph. However this problem has proven to be NP-complete (known as minimum bisection problem), so such an algorithm exists only if $P = NP$. However, due to the importance of this problem in complexity theory, much effort has been made to develop heuristics and approximation algorithms. The first heuristics for this problem were given by Kernighan and Lin [FM82] and subsequently improved in terms of running time by Fiduccia and Mattheyses [KL70]. Saran and Vazirani [SV95] provided the first non-trivial approximation algorithm for this problem, with approximation ratio $n/2$. Subsequently, the ratio was improved by Feige, Krauthgamer and Nissim [FKN00] to $\mathcal{O}(\sqrt{n} \cdot polylog(n))$. Feige and Krauthgamer [FK02] were able to improve the ratio to $\mathcal{O}(\log^2 n)$. The currently best known bound of an approximation algorithm for this problem is $\mathcal{O}(\log^{1/2} n)$ and was developed by Andreev and Räcke [AR06].

Sparse graphs that do not have small balanced separators are called expanders or expanding graphs. They were first defined by Bassalygo and Pinsker [BP73], and their existence was first proven by Pinsker [Pin73] in the early 70's. The original motivation for studying expanders was their applications on communication theory. Since then, in the past 40 years, substantial progress has been made on the properties and explicit constructions of expander graphs. That is mainly, due to the applications expanders have in numerous different fields, such as computer science, computational complexity and cryptography. In computer science, expanders have found extensive applications in designing algorithms, error correcting codes, extractors, pseudorandom generators, sorting networks [AKS83] and robust computer networks. In computational complexity, they have been used in proofs of many important results, such as $SL = L$ [Rei08] and the PCP theorem [Din07], while in cryptography they are used to construct hash functions.

Some of the many applications of expanders are due to their similarity, as far as edge distribution is concerned, to random and pseudorandom graphs. This similarity is expressed through the Expander Mixing Lemma and its converse. This lemma states that, for any two subsets S, T of a d-regular expander G, the number of edges between S and T is approximately what you would expect the number of edges between these two sets in a random d-regular graph to be. Although this lemma was observed by several researchers, it probably appeared in print first in [AC88]. The converse was proven later by Bilu and Linial [BL06].

Expanders are usually constructed using probabilistic, existential arguments (see [Pin73]), as this method is much easier than providing an explicit construction. Margulis [Mar73] gave the first construction of expander graphs which was later generalized in the theory of Ramanujan graphs [LPS88]. This construction is still among the most elegant and most easy to generate of all known constructions. The most recent explicit construction is, a combinatorial method for constructing expanders suggested by Reingold, Vadhan and Widgerson [RVW02] called the zig-zag product. They showed that the zig-zag product of two expanders is an expander as well, which lead to an iterative construction of an explicit family of expanders (i.e. expanders are closed under the zig-zag product operation).

Given the prominence of expanders and their usability it is also natural to study their substructures. As far as subgraphs of expanders is concerned, Krivelevich [Kri16] showed that every expander contains not only a path but also a cycle, of length that is affected only by its expansion parameter and its size. As it is natural, a lot more results occur when studying minors of expanders. They usually have the following form: if a graph G is sufficiently dense, or has sufficiently large average degree (plus possibly additional conditions imposed), then G contains a large minor. Kostochka [Kos82], [Kos84] and Thomasson [Tho84], independently showed, probably the most known result of this sort, that there exists an absolute constant c, such that every n-vertex graph G of average degree d contains a clique of size $cd/\sqrt{\log d}$ as a minor. The asymptotic value of c was later determined by Thomasson [Tho01].

The first results on finding minors in expanders arise through their connection to balanced separators. In particular, Plotkin, Rao and Smith [PRS94] proved that an expanding graph on n vertices , contains the complete graph $K_{c\sqrt{n/\log n}}$ as a minor, where c depends from the expansion of the graph. An even stronger result has been announced by Kawarabayashi and Reed [KR10a], who showed that an expander of size n and maximum degree bounded by d, contains the complete graph of size $\Omega(c\sqrt{n})$ as a minor. Here c depends from the expansion and d. Recently, Krivelevich and Nenadov [KN18] improved the dependence on the expansion α and the maximum degree d under a somewhat stronger definition of expansion.

Kleinberg and Rubinfield [KR96] also considered the same problem. Building on (the random walk-based) techniques of Broder, Frieze, and Upfal [BFU92], they showed that every expander G on n vertices contains every graph with $\mathcal{O}(n/\log^k n)$ vertices and edges as a minor. The exponent k depends on the expansion and the maximum degree of the graph. They also provided an efficient algorithm for finding a model of such a graph in G. While this result appears to be quite useful in finding large minors in sparse graphs, they used a rather weak definition of expansion, so it appears to be of limited value for the denser case and can not be used to show the existence of a clique minor of size larger than $\Omega(\sqrt{n/\log n})$. This result was later extended (as fas as the type of expansion is concerned and how it affects the constant of the clique minor) by Krievelevich and Sudakov [KS09], who improved the result of Kühn and Osthus [KO04].

As one can expect many of the results already stated arise from different fields of mathematics, such as finite geometry, spectral graph theory and extremal graph theory. Hence, in this thesis we will present only some of these results in detail. In particular, this thesis is organized as follows:

Chapter 2: This chapter is divided in 4 sections and its goal is to present important results on connectivity, minors and expanders, and to provide some basic tools about them that we will need further in this thesis. Specifically, in the first section we state some basic definitions from graph theory and some useful properties of the DFS algorithm. The second section is about connectivity, since the balanced separators come to answer questions that naturally occur from the study of connectivity in graphs and its implications. The next section, is focused on minors and especially on how the average degree of a graph can force a given minor, while in the last section we will introduce expanding graphs. Specifically, we will state some basic definitions, provide two examples of construction of expanders, and prove the existence of some substructures in expanders.

Chapter 3: In this chapter we will state and prove in detail some theorems about balanced separators, and is divided in three sections. In the first section, we provide some definitions about balanced separators that are essential about the proofs of the theorems that will follow. In the second and third section we will prove in detail some results on balanced separators by Lipton and Tarjan [LT79] and by Alon, Seymour and Thomas [AST90b] respectively.

Chapter 4: The last chapter of this thesis is about expanding graphs. There, first we will prove the connection balanced separators and expanders have, that is that expanders are graphs without small balanced separators. Next, we will provide a brief introduction in spectral graph theory, state the Expander Mixing Lemma and also state a connection between the combinatorial and the spectral definitions of expanders. We close this chapter by proving in detail the results about minors in expanders by Sudakov and Krivelevich [KS09].

CHAPTER 2

PRELIMINARIES

2.1 Definitions

A *simple graph* G, is a pair $(V(G), E(G))$ where $V(G)$ is a set of vertices and $E(G) \subseteq \binom{V(G)}{2}$ is a set of edges. For every edge $e = \{x, y\} \in E(G)$ we call $x, y \in V(G)$ *endpoints* of the edge e, and we usually denote e by xy. We also denote by $n(G)$ and $m(G)$ the number of vertices and the number of edges of G respectively. We call two vertices $x, y \in V(G)$ *adjacent*, if $xy \in E(G)$ and we say that an edge $e \in E(G)$ is *incident* to a vertex $x \in V(G)$ if x is an endpoint of e. If we allow $E(G)$ to be a multiset then G is called *multigraph*. Furthermore, when we refer to the *order* of a graph G, we mean the number of its vertices.

A graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of G. An alternative definition of the subgraph relationship between two graphs is the following: Let G and H be two graphs and $\phi : V(H) \to V(G)$ an injective mapping such that $\forall uv \in E(H)$, $\phi(u)\phi(v) \in E(G)$. We then say that H s a subgraph of G and denote it by $H \subseteq G$.

Given a graph G and a set $S \subseteq V(G)$ we will denote by $G \setminus S$ the induced subgraph $G[V(G) \setminus S]$ that we obtain after the removal of the vertices in U and their incident edges, from G. If $S = \{v\}$ is a singleton, we simply write $G \setminus v$. Similarly if F is a subset of $E(G)$ we denote by $G \setminus F$ the graph $G' = (V(G), E(G) \setminus F)$ that is obtained after the removal of the edges in F from G, and write $G \setminus e$ if $F = \{e\}$.

The number of edges incident to a vertex v is called *degree* of v in G and is denoted by $\mathbf{deg}_G(v)$. If it is clear to which graph we refer, we simply write $\mathbf{deg}(v)$. We can now define the maximum and the minimum degree of a graph G, which we will denote by $\Delta(G) = \max_{v \in V(G)} \mathbf{deg}(v)$ and $\delta(G) = \min_{v \in V(G)} \mathbf{deg}(v)$ respectively. We also define the *average degree* of a graph to be $\mathbf{d}(G) = \frac{\sum_{v \in V(G)} \mathbf{deg}(v)}{n(G)}$. The ratio $\frac{m(G)}{n(G)}$ is often denoted by $\epsilon(G)$. Each vertex adjacent to a vertex v is a neighbor of v while the set of those vertices is called *neighborhood* of v and is denoted by $N(v)$. Similarly, given a set $S \subseteq V(G)$ we call neighborhood of S the set $\{u \in V(G) | u \notin S \text{ and } N(u) \cap S \neq \emptyset\}$ and denote it by $N(S)$. If a vertex has degree 0 it is called *isolated*. Notice that if G is a simple graph and $v \in V(G)$, $\mathbf{deg}(v) = |N(v)|$. A graph is called *d-regular* if each of its vertices has degree exactly d.

A *path* P of length k is a graph with $k + 1$ vertices $v_0, \ldots, v_k$ and edges $e_i = v_i v_{i+1}$ for

$0 \leq i \leq k-1$. The vertices v_0 and v_k are called *endpoints* of the path P, whereas $v_1, \ldots, v_{k-1}$ are called *internal vertices* of P. Given a graph G we say that a path in G that connects the vertices $u, v \in V(G)$ is a subgraph of G which is a path with endpoints u, v. We will denote such a path by stating its vertices, for example $P = v_0, \ldots, v_k$ is the graph mentioned above. When we want to refer to a subpath of P we will write $P_{[v_i, v_j]}$, where v_i, v_j are its endpoints. We say that two paths that connect the same two vertices of a graph are *internally disjoint* if they have no common internal vertex and *edge disjoint* if they have no common edge.

The *distance* between two vertices $u, v \in V(G)$ is the length of the shortest path in G that connects them or infinity if there is no such a path, and is denoted by $\mathbf{dist}_G(u,v)$ (if it is obvious to which graph we refer to, we simply denote it by $\mathbf{dist}(u,v)$). The maximum distance that two vertices have on G is called *diameter* of G and is denoted by $\mathbf{diam}(G)$.

The graph $C_l = (\{v_1, \ldots, v_l\}, \{v_1v_2, \ldots, v_{l-1}v_l, v_lv_1\})$ is called *cycle* of length l. The length of the smallest cycle in a graph G, is called *girth* of the graph and is denoted by $\mathbf{g}(G)$. Furthermore,let $V_r = \{1, \ldots, r\}$, and consider the graph $(V_p \times V_q, \{(x_1,y_1)(x_2,y_2) \,|\, |x_1 - x_2| + |y_1 - y_2| = 1\})$. We call this graph $(p \times q)$*-grid*.

A *clique* of size $r \geq 0$ is the graph $(\{v_1, \ldots, v_r\}, \{v_iv_j | 1 \leq i < j \leq r\})$ and is denoted by K_r. An *independent set* of a graph G is a set $S \subseteq V(G)$ of pairwise non-adjacent vertices. A maximum independent set of G is an independent set of the largest possible size, and we will denote its size by $\alpha(G)$. Similarly we can define an independent set of edges, which is called a *matching*. We call a matching *perfect* if its cardinality is $\frac{n(G)}{2}$. A graph G is *bipartite* if we can partition $V(G)$ into to sets $X, Y \subseteq V(G)$ such that X and Y are independent sets of G and we say that a graph is complete bipartite and denote it by $K_{n,m}$ if X, Y are maximal independent sets and $|X| = n$, $|Y| = m$.

The *complement* of a given graph G is defined to be the graph $\bar{G} = (V(G), \{xy | xy \notin E(G)\})$, and its *line graph* to be $L(G) = (E(G), \{e_1e_2 | e_1, e_2 \in E(G) \text{ and } e_1 \cap e_2 \neq \emptyset\})$. We will also state here the definitions of some operations between graphs. Given two graphs G_1, G_2 and an integer $k \geq 0$ we define their *union* to be the graph $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and their *intersection* to be $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$.

We say that a given graph G is *connected* if $\forall x, y \in V(G)$ there is a path P connecting x and y in G. A maximal (to the number of vertices) connected subgraph of G is called a *connected component* of G. Notice that this is an equivalence relation on $V(G)$ with the connected components being its equivalence classes. A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G \setminus S$ is disconnected. Similarly an *edge cut* of a graph G is a set $F \subseteq E(G)$ such that $G \setminus F$ is disconnected. The *vertex connectivity* of a graph G is denoted by $\kappa(G)$ and equals to $\kappa(G) = \min\{|S| : S \text{ is a vertex cut}\}$. The edge connectivity is defined similarly and in this thesis we will refer to vertex connectivity simply as connectivity. Moreover, we define the connectivity of a graph to be 0 if it is disconnected, and $r-1$ if it is a clique of size r. Given a positive integer k, a graph G is called k-connected if for every subset $S \subseteq V(G)$ with cardinality $k-1$, the graph $G \setminus S$ is connected. For $S, T \subset V(G)$, we denote the set of edges of G from S to T by $E_G(S,T)$.

A graph T is called a *tree* if it is connected and has no cycle as a subgraph. Given a graph G, we say that T is a *spanning tree* of G, if $V(T) = V(G)$, $E(T) \subseteq E(G)$ and T is a tree. The *radius* of a graph G is denoted by $\mathbf{rad}(G)$ and equals to $\min\limits_{v \in V(G)} \max\limits_{u \in V(G)} \mathbf{dist}(u,v)$.

Given a graph G and an edge $e = xy \in E(G)$, the graph G/e is obtained from G by contracting the edge e. That means, that its endpoints x, y are replaced by a new vertex v_{xy} which is adjacent to

$N(x) \cup N(y) \setminus \{x, y\}$. A graph H obtained by a sequence of edge-contractions is called a *contraction* of G. Given two graphs G and H we say that H is a *minor* of G if it can be obtained from G via a sequence of edge and vertex removals, and edge contractions, and denote it by $H \leq_m G$. Notice also that this is equivalent to H being a contraction of some subgraph of G. We will now mention an equivalent definition of the minor relationship which will be useful later in this thesis.

A graph $H = (U, F)$ with a vertex set $U = \{u_1, \ldots, u_k\}$ is a minor of a graph $G = (V, E)$ if for every vertex $u_i \in U$ there is a connected subgraph G_{u_i} of G such that all subgraphs G_{u_i} are pairwise vertex disjoint, and G contains an edge between G_{u_i} and G_{u_j} whenever $u_i u_j \in F$.

We define the *density* of a simple graph G to be $\frac{2m(G)}{n(G)(n(G)-1)}$, which measures how close is G to the complete graph on $n(G)$ vertices. Notice that density of a graph is equal to 1 if $m(G) = \frac{n(G)(n(G)-1)}{2}$ which is the number of edges of the complete graph and 0 if G is composed just of isolated vertices. We say that G is *dense* if it has high density and *sparse* otherwise.

Let $f(n), g(n)$ be two function of n. We will write $f(n) = o(g(n))$, whenever $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$, $f(n) = \mathcal{O}(g(n))$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all n. Also, $f(n) = \Omega(g(n))$ if $g(n) = \mathcal{O}(f(n))$, and $f(n) = \Theta(g(n))$ if both $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$ are satisfied.

Depth First Search: *DFS* is a graph search algorithm that visits all vertices of a graph. The algorithm receives as an input a graph G; it is also assumed that an order σ on the vertices of G is given, and the algorithm prioritizes vertices according to σ. The algorithm maintains three sets of vertices: Let S be the set of vertices whose exploration is complete, T be the set of unvisited vertices and $U = V(G) \setminus (S \cup T)$, where the vertices of U are kept in a stack (the last in, first out data structure).

It initializes $S = U = \emptyset$ and $T = V$, and terminates once $U \cup T = \emptyset$. At each iteration, if the set U is non-empty, the algorithm queries T for neighbors of the last vertex v that has been added to U, scanning T according to σ. If v has a neighbor u in T, the algorithm deletes u from T and inserts it into U. Else, v is popped out of U and is moved to S. If U is empty, the algorithm chooses the first vertex of T according to σ, deletes it from T and pushes it into U. In order to complete the exploration of the graph, whenever the sets U and T have both become empty (at this stage, the connected component structure of G has already been revealed), we make the algorithm query all remaining pairs of vertices in $S = V$, not queried before.

Observe that the DFS algorithm starts revealing a connected component C of G at the moment the first vertex of C gets into (empty beforehand) U and completes discovering all of C when U becomes empty again. One can verify the following properties of DFS:

(P1) at each iteration one vertex moves, either from T to U, or from U to S,

(P2) at any stage of the algorithm, it has been revealed already that the graph G has no edges between the current set S and the current set T,

(P3) the set U always spans a path (indeed, when a vertex u is added to U, it happens because u is a neighbor of the last vertex v in U, thus u augments the path spanned by U, of which v is the last vertex).

2.2 Connectivity

The definition we gave in the previous section is somewhat unintuitive, as it only says what we need to do to disconnect the graph and does not give us any information about the "connections" vertices have between them. In this section we will refer to some results about connectivity, in order to understand better what a graph being k-connected means.

Proposition 2.1. *Let G be a graph of order $n \geq 2$. Then*

$$\kappa(G) \geq 1 \Rightarrow \exists v \in V(G) : \kappa(G[V(G) \setminus \{v\}]) \geq 1$$

Proof. Let, G be a connected graph of order n and P be a path that realizes the diameter of G, with w, u its end vertices. For each $v \in V(G)$ we denote by G_v the induced subgraph $G[V(G) \setminus v]$, of G. Consider now the graph G_w, and let $G_1, \ldots, G_l$ be its connected components. Without loss of generality, suppose that G_1 is the connected component such that $u \in V(G_1)$, and let $z \neq u$ be an arbitrary vertex of G_w, where $z \in V(G_i)$. Since $\kappa(G) \geq 1$ there is a path P' of length at most $|P| = \text{diam}(G)$ from u to z in G. Obviously $w \notin P'$, else $|P'| > |P|$. That means, that z and u are connected through P' also in G_w, so $i = 1$. Since z was an arbitrary vertex of G_w, G_w is connected. □

We can observe that the connectivity of a graph G is at most equal to its minimum degree, since in order to disconnect a vertex of minimum degree from the rest of the graph is suffices to delete its neighbors in G.

Lemma 2.2. *Let G be a connected graph of order n and $x \in V(G)$*

$$\kappa(G[V(G) \setminus x]) \geq \kappa(G) - 1$$

Proof. Let G be a connected graph of order n and $x \in V(G)$. Suppose that $\kappa(G \setminus x) \leq \kappa(G) - 2$. Then there exists a vertex set, S, of size $\kappa(G) - 2$ that separates G. However that would mean that the set $S \cup \{x\}$ separates the graph G and since its size is at most $\kappa(G) - 1$, we have a contradiction. □

Lemma 2.3. *Let G be a k-connected graph. Let also G' be the graph obtained from G by adding a new vertex x adjacent to at least k vertices of G. Then G' is also k-connected.*

Proof. Let S be a vertex set that disconnects G'. We will show that $|S| \geq k$. If $x \in S$, then $S \setminus x$ must disconnect G and since G is k-connected, then $|S \setminus x| \geq k$ so, $|S| \geq k + 1$. Suppose now that $x \notin S$. If $N(x) \subseteq S$ then $|S| \geq k$, since $|N_{G'}(x)| = k$. Else, if $N(x) \setminus S \neq \emptyset$, then $N(x) \setminus S$ belongs to a unique connected component of $G' \setminus S$ (the one that x also belongs). Hence, S disconnects G. Thus $|S| \geq k$ because G is k-connected. □

Notice that the same result holds for $G \setminus e$, where $e \in E(G)$ and can be proven using similar arguments. As one can expect the average degree of a graph can affect the connectivity of its subgraphs. This relationship is expressed by Mader's theorem as we will see below:

Theorem 2.4 (Mader 1972)**.** *Let k be a positive integer and G be a graph such that $d(G) \geq 4k$. Then there exists a $(k+1)$-connected subgraph $H \subseteq G$ such that $d(H) > \frac{d(G)}{2}$.*

Proof. Let k be a positive integer and G be a graph of average degree at least $4k$. Consider the family $\mathcal{G}$ of graphs such that for every $H \in \mathcal{G}$, $H \subseteq G$ and

$$|V(H)| \geq 2k \quad \text{and} \quad |E(H)| > \frac{d(G)}{2}(|V(H)| - k) \tag{2.1}$$

We can observe that since G satisfies these conditions $\mathcal{G} \neq \emptyset$. Let H be the graph in $\mathcal{G}$ that satisfies $\min_{G' \in \mathcal{G}} |V(G')|$.

Claim i. $|V(H)| \geq 2k+1$

Proof of Claim i. Suppose, for contradiction, that $|V(H)| = 2k$. Since H is in $\mathcal{G}$, we have that

$$|E(H)| > \frac{d(G)}{2}(V(H) - k) = \frac{d(G)}{2}k \geq 2k^2 > \binom{2k}{2} = \binom{|V(H)|}{2}$$

However that leads to a contradiction, since the complete graph on n vertices has $\binom{n}{2}$ edges. □

The minimality of $|V(H)|$ implies that $\delta(H) > \frac{d(G)}{2}$. If not, there exists a vertex $v \in V(H)$ such that $deg(v) < \frac{d(G)}{2}$. Consider the graph $G' = G[H \setminus v]$. We will show that G' satisfies (2.1). Due to Claim i, $|V(G')| \geq 2k$. Moreover, $|E(G')| > |E(H)| - \frac{d(G)}{2} > \frac{d(G)}{2}(|V(H)| - k) - \frac{d(G)}{2} > \frac{d(G)}{2}(|V(G')| - k)$, but that contradicts the minimality of $|V(H)|$. Hence $\delta(H) > \frac{d(G)}{2}$, and as a result $d(H) > \frac{d(G)}{2}$.

It is now left to show that $\kappa(H) \geq k+1$. Assume, for contradiction, that there exists a vertex set $S \subseteq V(H)$ of size at most k that disconnects H, and let H_1, H_2 be the two connected components of $G[H \setminus S]$. Since S disconnects H, $N_H(H_1) \subseteq V(H_1) \cup S$. Hence, $|H_1 \cup S| > 2k$, and as a result the subgraph of G that is induced by $V(H_1)$ has more than $2k$ vertices. Since H is minimal, and $|V(H_1) \cup S| \leq |V(H)|$, the number of edges spanned by $V(H_1)$ must be at most $\frac{d(G)}{2}(|V(H_1)| - k)$. Similarly, the number of edges spanned by $V(H_2)$ must be at most $\frac{d(G)}{2}(|V(H_2)| - k)$. Hence,

$$\begin{aligned}
|E(H)| &\leq \frac{d(G)}{2}(|V(H_1)| - k) + \frac{d(G)}{2}(|V(H_2)| - k) \\
&\leq \frac{d(G)}{2}(|V(H_1)| + |V(H_2)| - 2k) \\
&\leq \frac{d(G)}{2}(|V(H)| - k)
\end{aligned}$$

which contradicts the conditions (2.1) that H satisfies. Note that the last inequality holds because $|V(H_1) \cap V(H_2)| \leq k$. □

The following theorem is one of the cornerstones of graph theory.

Theorem 2.5 (Menger 1927)**.** *Let G be a graph and $A, B \subseteq V(G)$. Then the minimum numbers of vertices separating A from B in G is equal to the maximum number of pairwise vertex disjoint paths with one endpoint in A and the other in B.*

Proof. Given a graph G and two sets $A, B \subseteq V(G)$, we will denote by k the minimum number of vertices separating A from B in G. Clearly, G cannot contain more than k pairwise disjoint $A - B$ paths, so our goal is to show that k such paths exist. We will show that, by applying induction on $|E(G)|$.

If G has no edge, then $|A \cap B| = k$ and we have k trivial $A - B$ paths. So we may assume that there exists an edge $e = xy$ in G. Let G_e be the graph obtained by the contraction of e, and v_e be the vertex of G_e that occurs from this contraction. If one of x, y is in A then we count v_e also in A and similarly if one of x, y is in B then v_e is also counted in B. Note that, if G has no k pairwise disjoint $A - B$ paths then neither does G/e. By the induction hypothesis G_e contains a vertex set S that disconnects A from B of size less than k. If $v_e \notin S$, then S also disconnects A from B in G which contradicts our original assumption that k is the minimum number of vertices separating A from B in G. Hence $v_e \in S$, and as a result $S' = (S \setminus v_e) \cup \{x, y\})$, separates A from B in G and has exactly k vertices.

Consider now the graph $G' = G[G \setminus e]$. Since $x, y \in S'$, every set that separates A from S', also separates A from B in G, hence contains at least k vertices. So by induction there exist k pairwise disjoint $A - X$ paths in G'. Similarly there also exist k pairwise disjoint $X - B$ paths in G'. As X disconnects A from B, these two sets of paths do not meet outside S', and thus can be combined to k pairwise disjoint $A - B$ paths. $\square$

Theorem 2.6 (Global Version of Menger's Theorem)**.** *A graph is k-connected if and only if it contains k independent paths between any two vertices.*

Proof. If a graph G contains k independent paths between any two vertices, then $|G| > k$ and G cannot be separated by fewer than k vertices. Hence, $\kappa(G) \geq k$.

Conversely, suppose that G is k-connected (and, in particular, has more than k vertices) but contains two vertices $a, b \in V(G)$ that are not connected by k pairwise disjoint paths. If $ab \notin E(G)$, by applying Theorem 2.5 to the graph $G' = G[V(G) \setminus \{a, b\}]$ and the vertex sets $A = N_G(a)$ and $B = N_G(b)$ we have that the minimum number of vertices separating A from B is equal to the number of pairwise disjoint paths $A - B$ paths. Due to the selection of A and B, and because $ab \notin E(G)$ the number of pairwise disjoint paths in G that connect a to b, is equal to the minimum number of vertices needed to separate in G, a from b which is at least k since $\kappa(G) \geq k$.

If $ab \in E(G)$, let $G' = G \setminus ab$. Due to our original assumption G' contains at most $k-2$ pairwise disjoint paths that that connect a and b. Hence there exists a set $S \subseteq V(G')$ of size at most $k-2$, such that $\kappa(G[V(G)' \setminus S]) = 0$. As $|V(G)| > k$ there exists a vertex $v \in V(G) \setminus (S \cup \{a, b\})$.Then S separates v in G' either from a, or b, say from a. But then, $S \cup \{b\}$ is a set of at most $k - 1$ vertices that disconnects G, contradicting the k-connectedness of G. $\square$

2.3 Minors

Minors is one of the most central notions in modern graph theory. Thus, it is natural to expect the appearance of results that connects minors to the main subjects of this thesis, balanced separators and expanders. Indeed, as we will also see later, many progress has been made in combining these notions. Generally speaking the results of extremal minor theory can be stated as finding sufficient conditions for the existence of a minor from given family, or a concrete minor (say, a clique minor) in a given graph. For example, such a result is the following: Every graph G on n vertices with more than $3n - 6$ edges, contains the complete graph K_5 or the complete bipartite graph $K_{3,3}$ as a minor. Of course, this is nothing more than combining and rephrasing the following two theorems.

Theorem 2.7. *Any planar graph G of order $n \geq 3$ contains at most $3n - 6$ edges.*

Theorem 2.8 (Kuratowski-Wagner)**.** *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

When looking for large minors, one should remember that there is a limit to the size of the minor one can find in a graph. This limit occurs as a corollary of the following proposition.

Proposition 2.9. *If H, G are simple graphs the following holds*

$$H \leq_m G \Rightarrow |E(H)| \leq |E(G)|$$

Corollary 2.10. *A graph G of order n and average degree d cannot contain a graph H of average degree $k > \sqrt{nd}$ as a minor.*

As we will see by the results stated below the average degree of a graph affects its minors. We will focus on this relationship as the results that combine minors with expanders usually have the following form: If a graph G is sufficiently dense or has sufficiently large average degree, then G contains a large minor.

Proposition 2.11. *Every graph G contains a subgraph G' such that $d(G') \geq d(G)$ and $\delta(G') \geq \frac{d(G)}{2}$.*

Proof. We prove this proposition by induction on the number of vertices of G. For $|V(G)| = 1$ the assertion is trivial. Assume that this also holds for every graph on n vertices and let G be a graph such that $|V(G)| = n+1$. If $\delta(G) \geq \frac{d(G)}{2}$ then the proof is complete. Otherwise there exists a vertex v such that $deg(v) = k < \frac{d(G)}{2}$. Consider the graph $G' = G[G \setminus v]$. Since $deg(v) < \frac{d(G)}{2}$ we remove at most $\frac{d(G)}{2}$ degrees of the total degree of the graph. Notice now, that G' is an n-vertex graph with average degree

$$d(G') \geq \frac{\sum_{v \in V(G)} deg(v) - d(G)}{n} \geq d(G)$$

and as a result of the induction hypothesis G' has a subgraph of minimum degree at least $\frac{d(G')}{2}$ (hence the same is true for G). □

There is some interest in knowing the maximum size of graphs not having the complete graph K_r as a minor, not least because of the relationship between this extremal problem and the conjecture of Hadwiger [Had43], asserting that $K_r \leq_m G$ if $\chi(G) \geq r$. Wagner [Wag64] showed that a sufficiently large chromatic number (depending only on r) guarantees a K_r-minor, and Mader [Mad67] proved that a sufficiently large average degree will do.

The proof of the following early bound on the average degree needed to force a K_r minor contains a key idea from which all the later results were developed.

Proposition 2.12. *Every graph of average degree at least 2^{r-2} has a K_r minor.*

Proof. We apply induction on r. For $r = 2$ the result holds, since graphs of average degree at least 2^0 must have an edge (so they contain K_2 as a minor). For the induction step let $r \geq 3$ and let G be an arbitrary graph such that $d(G) \geq 2^{r-2}$. Then $\epsilon(G) = \frac{m(G)}{n(G)} = \frac{\sum_{v \in V(G)} deg(v)}{2n(G)} \geq 2^{r-3}$. Let H be a minimal minor of G with $\epsilon(H) \geq 2^{r-3}$, and let x be an arbitrary vertex of H. By the minimality of H, it is connected, so x is not isolated. Moreover, each of its neighbors has at least 2^{r-3} common neighbors with x. Suppose, for contradiction, that there exists a vertex $y \in N_H(x)$ such that $N_H(x) \cap N_H(y) < 2^{r-3}$. Consider the graph $H' = G/xy$ and notice that by the contraction of the edge xy $|E(H)| - |E(H')| \leq 2^{r-3}$, so $\epsilon(H') \geq 2^{r-3}$, which contradicts the minimality of H.

Hence, the subgraph induced by the neighbors of x in H, has minimum degree, and as a result also average degree, at least 2^{r-3}. By the induction hypothesis, this graph contains K_{r-1} as a minor. Together with x this yields the desired K_r minor of G. □

Kostochka [Kos82], [Kos84] and Thomasson [Tho84] independently proved the following Theorem, which, as we will see in the fourth chapter, Sudakov and Krivelevich [KS09] used to find complete minors in expanding graphs.

Theorem 2.13 (Kostochka, Thomasson 1982)**.** *There exists a constant $c \in \mathbb{R}$ such that, for every $r \in \mathbb{N}$, every graph G of average degree $d(G) \geq cr\sqrt{\log r}$ contains K_r as a minor.*

The correct value of the average degree needed to force a K_r minor is known almost precisely. Thomasson, in 2001 [Tho01] determined, asymptotically, the smallest constant c that makes Theorem 2.13 true. It can be written as $c = \alpha + o(1)$, where $o(1)$ stands for a function of r tending to zero as $r \to \infty$ and $\alpha = 0.319\ldots$ is an explicit constant. Later, in 2005, Kühn and Osthus [KO04] proved that, if G is a locally sparse graph, in the sense that it does not contain a fixed complete bipartite graph $K_{s,s}$ as a subgraph , then G has a K_p minor where p is asymptotically much larger than the average degree of G. Formally they proved the following theorem, which was later improved by Krivelevich and Sudakov [KS09] as we will see in more detail in chapter 4.

Theorem 2.14 (Kühn, Osthus 2004). *For every integer $s \geq 2$ there exists an r_s such that every $K_{s,s}$-free graph of average degree at least $r \geq r_s$, contains a K_p minor for all*

$$p \leq \frac{r^{1+\frac{1}{2(s-1)}}}{(\log r)^3}$$

2.4 Expanding Graphs

The basic concept of this thesis is expanding graphs. Informally a graph is said to be an expanding graph, or an expander, if every subset X of $V(G)$ has relatively many neighbors outside X. This is what is usually called *vertex expansion*. Sometimes an alternative notion of *edge expansion* is used, where every set $X \subseteq V(G)$ is required to be incident to many edges crossing between X and its complement in G. Of course a formal definition is required, firstly to measure the expansion quantitatively, and secondly to distinguish between the expansion of small and large sets (note that a set $X \subseteq V(G)$ containing half the vertices of G cannot have more than $|X|$ outside neighbors in G, while a much smaller set X can expand by a much larger factor). There are several definitions of expanders in common use, capturing sometimes rather different expansion properties. In this section we will provide two definitions of vertex and edge expansion and we will later see some alternative algebraic definitions specifically for edge expansion.

Definition 2.15 (Vertex Expansion). Let $t > 0$, $0 < \alpha < 1$. A graph $G = (V, E)$ is (t, α)-*expanding* if

$$\forall X \subseteq V(G) : |X| \leq \frac{\alpha |V(G)|}{t} \Rightarrow |N_G(X)| \geq t|X|$$

that is, that every set X of size $|X| \leq \frac{\alpha|V|}{t}$ expands by a factor of at least t.

Definition 2.16 (Edge Expansion). A cut in G is a bipartition $(S, \bar{S})$ of its vertices, that is , $S \cup \bar{S} = V$, $S \cap \bar{S} = \emptyset$ and $S, \bar{S} \neq \emptyset$. The *sparsity* of the cut $(S, \bar{S})$ is $e_G(S, S')/\min\{|S|, |S'|\}$. The *edge expansion* of a graph G, is denoted by $\phi(G)$, is the minimum sparsity of any cut in G.

Given a parameter $\alpha > 0$, we say that a graph G is an α-*edge-expander* if and only if $\phi(G) \geq \alpha$. Equivalently, for every subset S of at most $|V(G)|/2$ vertices of G, $E_G(S, \bar{S}) \geq \alpha|S|$.

As one can easily notice, the complete graph is an expander with respect to both edge end vertex expansion. However, when we talk about expansion we usually want the graph we refer to, to have as few edges as possible. Informally we say that a graph G is a good expander if it has low degree and high expansion properties. That means that good expanders are sparse graphs which can't be separated into two large components.

Now we will provide two examples of explicit constructions of good expanders. Usually we are interested in the additional property that these graphs are regular (although we allow parallel edges and self-loops) and have a fixed constant degree independent of n. Ideally, we would like to have a construction with n vertices for every n, however usually the constructions work only for some subsets of integers n. The two main approaches in constructing expanders are the algebraic approach and the combinatorial one. The first example that we will provide is Margulis construction [Mar73], which arises from the algebraic approach, and is the following:

For every n, we construct the graph, G, of order n^2, and we think $V(G)$ as the group of pairs $\mathbb{Z}_n \times \mathbb{Z}_n$. A vertex (x, y) is connected to the vertices

$$(x \pm 2y, y), (x \pm (2y+1), y), (x, y \pm 2x), (x, y \pm (2x+1))$$

so that G is 8-regular (notice that all operations are modulo n). The analysis of this construction is due to Gabber and Galil [GG81].

Another example that its proof depends on a deep result of number theory, the Selberg's 3/16 theorem [Sel65], and is a construction of a family of 3-regular graphs, of order p, where p is a prime number. Each vertex x of this graph is adjacent to the following vertices

$$x+1, x-1, x^{-1}.$$

Notice that all operations are modulo p and we define the inverse of 0 to be 0.

Next, we will see some elementary results about the substructures one can find in an expanding graph. The following statement was proved by Krivelevich [Kri16], from which it follows that if G is an α-edge-expander on n vertices, G contains a path of length at least $\frac{\alpha n}{2}$ (set $k = \frac{n}{2}$ and $l = \frac{\alpha n}{2}$ in the following Proposition).

Proposition 2.17. *Let k, l be positive integers. Assume that G is a graph on more than k vertices such that*

$$\forall S \subseteq V(G) : |S| = k \Rightarrow |N_G(S)| \geq l$$

Then G contains a path of length l.

Proof. Run the DFS algorithm on G, with σ being an arbitrary ordering of V. Consider the moment during the algorithm execution when the size of the set S of already processed vertices becomes exactly k (there is such an instance due to Property **(P1)** of DFS, as the vertices of G move into S one by one, till eventually they all move there). By Property **(P2)**, the current set S has no neighbors in the current set T, and thus $N_G(S) \subseteq U$ implying $|U| \geq l$. During the last iteration a vertex from U is moved to S, so before this move U is one vertex larger. The set U always spans a path in G, by Property **(P3)**, hence G contains a path of length l. □

Krivelevich [Kri17] also proved the following theorem (still based on the DFS algorithm and its properties), from which he deduced that for every α, a (k, α)-expander contains a cycle of length at least $\frac{\alpha k}{2}$. The linear dependence on α is optimal in the range $0 < \alpha < 1$, as shown by the example of the complete bipartite graph with parts of size k and $\lceil \alpha k \rceil$, where the longest cycle has length $2\lceil \alpha k \rceil$.

Theorem 2.18 (Krivelevich 2017). *Let $k > 0$, $t \geq 2$ be integers. Let G be a graph of order at least k, satisfying*

$$|N(S)| \geq t \quad \textit{for every} \quad S \subseteq V(G) : \frac{k}{2} \leq |S| \leq k$$

Then G contains a cycle of length at least $t+1$.

The following results are about vertex expansion and will be useful to the proof of Theorem 4.21.

Proposition 2.19. *Let G be a (t, α)-expanding graph of order n. Then*

$$\forall X \subseteq V(G) : \frac{\alpha n}{t} \leq |X| \leq \frac{\alpha n}{2} \Rightarrow |N(X)| \geq \frac{\alpha n}{2}$$

Proof. We observe that $|X| \geq \frac{\alpha n}{t} \Rightarrow \exists Y \subseteq X : |Y| = \frac{an}{t}$. Since $Y \subseteq X$, its neighbors will either be in $X \setminus Y$ or in $N(X)$. Moreover, G is (t, α)-expanding and $|Y| = \frac{\alpha n}{t}$, so $|N(Y)| \geq \alpha n$. We can now deduce that

$$|N(X)| \geq |N(Y)| - |X| \geq t|Y| - |X| = \alpha n - |X| \geq \frac{\alpha n}{2}$$

□

Lemma 2.20. *Let G be a connected (t, α)-expanding graph of order n. Then*

$$\mathbf{diam}(G) \leq \frac{3}{\alpha} \frac{\log n}{\log t}$$

Proof. Let G be a connected (t, α)-expanding graph of order n.
We will first prove by induction on q, that for every $v \in V(G)$ and every $q \in \mathbb{N}$ there exist at least $\min\{t^q, \alpha n\}$ vertices of G in distance at most q, from v.
Let v be an arbitrary vertex of G and denote by Y_i the set of vertices of G, in distance at most i from v. We observe that $Y_i = Y_{i-1} \cup N(Y_{i-1})$, $\forall i \geq 2$. Since G is (t, α)-expanding, we have that $|Y_1| = |N_G(v)| \geq t$ so the above condition holds for $i = 1$. We suppose that $|Y_i| \geq \min\{t^i, \alpha n\}$, $\forall i \leq q - 1$.

- If $|Y_{q-1}| \geq \alpha n$, since $Y_{q-1} \subseteq Y_q$, we have $|Y_q| \geq \alpha n$.
- If $t^{q-1} \leq |Y_{q-1}| \leq \frac{\alpha n}{t}$, since G is (t, α)-expanding we have $|Y_q| \geq |N(Y_{q-1})| \geq t|Y_{q-1}| \geq t^q$.
- If $|Y_{q-1}| \geq \frac{\alpha n}{t}$, using Proposition 2.1 as many times as needed, we can find a connected subgraph of $G[Y_{q-1}]$, W, such that $|V(W)| = \frac{\alpha n}{t}$. Since G is (t, α)-expanding we have that $|N(W)| \geq t|V(W)| = \alpha n$, and because every $v \in V(W)$ is either in Y_{q-1} or in $N(Y_{q-1})$ we have that $|Y_q| \geq \alpha n$.

Let now, $q = \left\lceil \frac{\log n}{\log t} \right\rceil$, so for every $v \in V(G)$, $|Y_q| \geq \min\{n, \alpha n\}$. Suppose that

$$\exists u, w \in V(G) : d(u, w) \geq \frac{3}{\alpha} \frac{\log n}{\log t}$$

and let $P = \{u = v_1, \ldots, v_l = w\}$ be a path that realizes this distance. That means, that there are $k < \frac{1}{\alpha}$ vertices in $V(P)$, $v_1 = u, \ldots, v_k = w$, such that the distance between each pair of them is at least $\frac{2 \log n}{\log t}$. We now observe that for each of these vertices, the corresponding sets $Y_q(v_i)$ are pairwise disjoint and each of size at least αn. Thus, $|\bigcup_{i=1}^{k} V_i| \geq kan > n$ which lead us to a contradiction, because we have supposed that the order of G is n. □

Proposition 2.21. *Let G be an α-edge-expander of order n. Then*

$$\mathbf{diam}(G) \leq \left\lceil \frac{2(\log n - 1)}{\log(1 + \alpha)} \right\rceil + 1$$

Proof Sketch. We can prove this proposition using similar arguments as in the proof of Lemma 2.20. The basic difference is that in order to apply the basic argument described in detail in that proof, we first have to prove by induction that the number of vertices at distance at most d in an α-edge-expander is at least $\min\{\frac{n}{2}, (1 + \alpha)^t\}$. □

In the next chapters we will see more results on substructures of expanding graphs, especially on minors. However, first we will see in detail some results about balanced separators and how they are connected to expanders.

CHAPTER 3

BALANCED SEPARATORS

"Divide and conquer" is one of the classic and most widely used techniques for designing efficient algorithms. Divide-and-conquer algorithms partition their inputs into two or more independent subproblems, solve those subproblems recursively, and then combine the solutions to those subproblems to obtain the final output. In order for this technique to succeed the following three conditions must be satisfied. (*i*) the subproblems must be of the same type as the original and independent of each other (in a suitable sense), (*ii*) the cost of solving the original problem given the solutions of the subproblems must be small, and (*iii*) the subproblems must be significantly smaller than the original. Notice, that this strategy can be successfully applied to several graph problems, provided we can quickly separate the graph into roughly equal subgraphs (so that the previous conditions are satisfied).

Balanced separators serve to measure quantitatively the connectivity of large vertex sets in graphs. The fact that all balanced separators of a graph G are large, indicates that it is costly to break G into large pieces not connected by any edge and sometimes, if G is well connected finding a small sized separator might be impossible. Balanced separators came into prominence with the celebrated result of Lipton and Tarjan [LT79], to which we refer to, with more details in the section "Planar Graphs" of this chapter, asserting that every planar graph on n vertices has a balanced separator of size $\mathcal{O}(\sqrt{n})$. Later, Alon, Seymour and Thomas [AST90b] proved than a graph will either have a small balanced separator or a large minor which we will see in more detail in the section "Non-planar graphs" of this chapter, while Kawarabayashi and Reed [KR10b], and Plotkin, Rao and Smith [PRS94], also addressed that issue.

As we will see in the next chapter expanders and balanced separators are closely related, since expanders are graphs that do not have small balanced separators.

3.1 Definitions

Definition 3.1. Let G be a graph of order n. We say that the vertex set $S \subseteq V(G)$ is a *balanced separator* of G if $V(G)$ can be partitioned into the sets A, B, S such that $|E_G(A,B)| = 0$ and $|A| \leq \frac{2n}{3}$, $|B| \leq \frac{2n}{3}$.

Definition 3.2. A *finite element graph* G is any graph formed from a planar embedding of a planar graph by adding all possible diagonals to each face (the finite element graph has a clique corresponding to each face of the embedded planar graph). The embedded planar graph is called the *skeleton* of the finite element graph and each of its faces is an *element* of the finite element graph.

Definition 3.3. Let G be a graph of order n and a vertex set $X \subseteq V(G)$. An X-*flap* is the vertex set of some component of $G \setminus X$.

Definition 3.4. Let G be a graph of order n. By a *haven of order* k in G we mean a function β, which assigns to each subset $X \subseteq V(G)$, of size at most k, an X-flap $\beta(X)$, in such a way that if $X \subseteq Y$ and $|Y| \leq k$, then $\beta(Y) \subseteq \beta(X)$.

Definition 3.5. Let G be a graph of order n. A *covey* in G is a set $\mathcal{C}$ of (non-null) trees of G, mutually vertex-disjoint, such that for all distinct $C_1, C_2 \in \mathcal{C}$, there is an edge with one endpoint in C_1 and one in C_2.

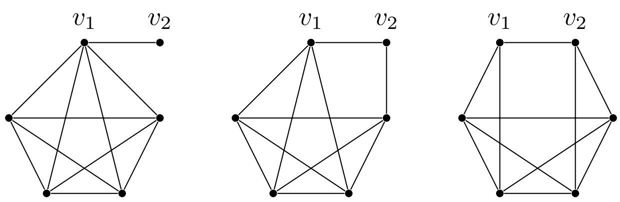
3.2 Planar graphs

The basic results stated in this section were proved by Richard J.Lipton and Robert Endre Tarjan [LT79]. They proved that any planar graph on n vertices has a balanced separator of size at most $2\sqrt{2}\sqrt{n}$, and also provided a polynomial-time algorithm that computes this balanced separator. Their motivation was to apply the divide and conquer technique to solve efficiently a number of problems defined on graphs. Since in some applications it is useful to have a result more general, as we will se in Lemma 3.8, first planar graphs with non-negative costs on their vertices are considered and the desired balanced separator theorem occurs as a special case, where the assigned costs on the vertices is equal. Previously known balanced separator theorems include the following: (i) Any binary tree of order n, can be separated into two subtrees, each with at most $2n/3$ vertices, by removing a single edge, (ii) Any n-vertex tree has a balanced separator S, such that $|S| = 1$, and (iii) Any grid of order n has a balanced separator of size $\sqrt{n}$, hence, a $\sqrt{n}$-separator theorem holds for the class of grid graphs.

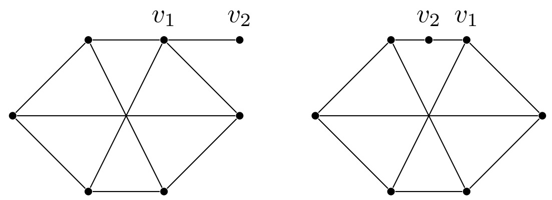
Theorem 3.6 (Jordan curve theorem [WG55])**.** *Let C be any closed curve in the plane. The removal of C divides the plane into exactly two connected regions, the "inside"and the "outside" of C.*

Lemma 3.7. *Let G be any planar graph. Contracting any edge of G to a single vertex preserves planarity. This implies that contracting any connected subgraph of G to a single vertex, preserves planarity.*

Proof. Let G be a planar graph. Suppose, for contradiction, that there exists an edge $e = v_1v_2 \in E(G)$ such that, the resulting graph, $G^* = G/e$, is not planar. Let, also, v be the resulting vertex from the contraction of e, in G^*. From Theorem 2.8, G^* contains either K_5 or $K_{3,3}$ as a minor, and let H be the induced subgraph of G^* that contains exactly the vertices which contribute in the resulting K_5 or $K_{3,3}$. Obviously $x \in E(H)$, or else $H \subseteq G$ and as a result G would not be planar. Suppose that $K_5 \leq_m G^*$. Then x is either a vertex of K_5 or has "disappeared" from the contractions on H. If the second case holds, then $K_5 \leq_m G$ which leads to a contradiction. In the first case K_5 can occur from one of the following three graphs,



all of which lead to G containing K_5 as a minor, which contradicts the original hypothesis that G is planar. Suppose now that $K_{3,3} \leq_m G^*$ and consider the graph H as we did in the previous case. Using similar arguments as in the case of K_5, one can see that both of the following possible graphs lead to a contradiction to the planarity of G,



which completes our proof for the case where one edge is contracted. The case where a connected subgraph is contracted is immediate, by applying induction on the number of vertices in the subgraph to be contracted. □

As mentioned in the beginning of this section, in order to have a more general result, we will fist prove that any graph, to which non-negative costs are assigned to its vertices, can be separated into two parts, each with cost at most 2/3 of the total cost, by removing $\mathcal{O}(\sqrt{n})$ of its vertices. The following two Lemmata will be useful for the proof of Theorem 3.10, of which the $\sqrt{n}$-balanced-separator theorem is a corollary.

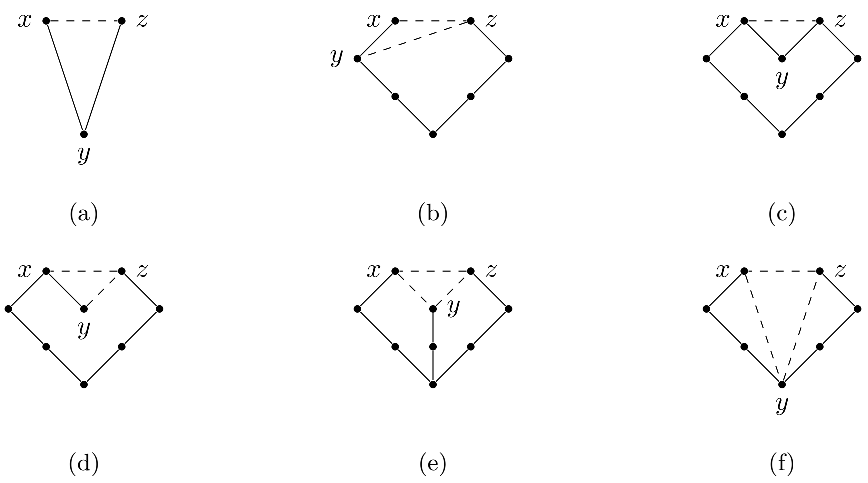
Lemma 3.8. *Let G be any planar graph of order n and $f : V(G) \to [0, 1)$ a cost assignment to the vertices of G, such that $f(V(G)) \leq 1$. Let also T be a spanning tree of G such that $\mathbf{rad}(T) = r$ and root t. Then there exists a set $S \subseteq V(G)$ such that $t \in S$ and $|S| \leq 2r + 1$, that disconnects G into the sets A, B such that $f(A) \leq \frac{2}{3}$ and $f(B) \leq \frac{2}{3}$*

Proof. Let G be a planar graph and f, T, the function and the spanning tree described in the lemma, respectively. Without loss of generality, we can suppose that $\forall v \in V(G)$, $f(v) \leq \frac{1}{3}$, since otherwise the lemma would be true. Now, we embed G on the plane and we make each "internal" face a triangle by adding a suitable number of edges to G. Let E^+ be the set of added edges to obtain this triangulation of G and G^* the resulting graph. Notice that any edge $e \in E(G^*) \setminus (E^+ \cup E(T))$ forms a cycle with some of the tree edges (otherwise it would be an edge of T). Since $\mathbf{rad}(T) = r$ each of these cycles is of length at most $2r + 1$ if it contains t and at most $2r - 1$ otherwise. Due to Theorem 3.6 each of these cycles divides the plane into two parts. Now it suffices to prove that there exists a cycle such that neither the inside nor the outside contains vertices whose total cost exceeds $\frac{2}{3}$ (due to the triangulation we know that there exists at least one cycle). Let C be a cycle in G^*. We will denote by $IN(C)$ the set of vertices that lie inside C, and by $OUT(C)$ the set of vertices outside C.

Claim i. *There exists a cycle $C \subseteq G^*$, such that the $f(IN(C)) \leq \frac{2}{3}$ and $f(OUT(C)) \leq \frac{2}{3}$*

Proof of Claim i. Let xz be an edge in $E(G^*) \setminus (E^+ \cup E(T))$ such that its corresponding cycle C minimizes the maximum cost either inside or outside the cycle. Break ties by choosing the nontree edge whose cycle has the smallest number of faces on the same side as the maximum cost and suppose that, that side is the inside of C. Suppose, without loss of generality, that $f(IN(C)) \geq f(OUT(C))$, therefore if $f(IN(C)) \leq \frac{2}{3}$ the proof of the Claim is complete. Suppose that $f(IN(C)) \geq \frac{2}{3}$.

Notice that the face that has xz on its boundary forms a triangle in G^* and let y be its third vertex. By examining each of the 6 possible cases which are illustrated below and by using the



(a) (b) (c) (d) (e) (f)

way we selected xz, one can show that it is impossible for the total cost of the vertices inside the cycle to exceed $\frac{2}{3}$, and that proves the claim.

The first case leads to a contradiction due to our assumption that $f(IN(C)) \geq \frac{2}{3}$, hence $IN(C) \neq \emptyset$. In the second case we suppose that $xy \in E(C)$ (alternatively $yz \in E(C)$ which can lead to a contradiction using similar arguments). Then $yz \notin E(T)$ and it defines a cycle C', such that $IN(C') = IN(C)$, and has one less face than C which contradicts to the selection of xz. The third case is not possible since T cannot contain a cycle by the definition of the spanning tree. For the next case, we suppose that $xy \in E(T)$ (alternatively $yz \in E(T)$ which can lead to a contradiction using similar arguments). Let C'' be the cycle in G^* defined by yz. Notice that $IN(C'') = IN(C) \setminus \{y\}$, hence $f(IN(C'')) \leq f(IN(C))$. Moreover C'' contains one less face than C, thus if $f(IN(C'')) \geq f(OUT(C''))$, yz would have been chosen instead of xz. Else, if $f(IN(C'')) \leq f(OUT(C''))$, then $f(OUT(C'') = f(OUT(C)) + f(y)$. However, since $f(OUT(C)) \leq \frac{1}{3}$ and $f(y) \leq \frac{1}{3}$, then $f(OUT(C'')) \leq \frac{2}{3}$ which would lead to the selection of yz instead of xz. If neither xy nor yz are in $E(T)$, one of the two last cases would occur. Then, as we see in case (e), each of xy and yz defines a cycle, we denote them by C_1 and C_2 respectively. In that case, $IN(C) = IN(C_1) \cup IN(C_2) \cup (E(C_1) \cap E(C_2))$. Suppose, without loss of generality, that $f(IN(C_1)) \geq f(IN(C_2))$. Then $f(IN(C_1)) \leq f(IN(C))$ and C_1 contains less faces than C. Thus, if $f(IN(C_1)) \geq f(OUT(C_1))$, xy would have been chosen in place of xz. On the other hand, suppose that $f(IN(C_1)) \leq f(OUT(C_1))$. Then, since $f(IN(C)) \geq \frac{2}{3}$ and $f(IN(C_1)) \geq f(IN(C_2))$, $f(C_1) + f(IN(C_1)) \geq \frac{1}{3}$ and $f(OUT(C_1)) \leq \frac{2}{3}$, which would lead to the selection of xy instead of xz. The same arguments apply, in case (f), and lead to a contradiction. Thus, all cases are impossible and C satisfies Claim i. □

Consider now the cycle C, whose existence we proved in Claim i. Since $|C| \leq 2r + 1$, and $f(IN(C)) \leq \frac{2}{3}$, $f(OUT(C)) \leq \frac{2}{3}$, C is the desired vertex set, and that completes our proof. □

Lemma 3.9. *Let G be a connected planar graph of order n and $f : V(G) \to [0, 1)$ be a cost assignment to the vertices of G such that $f(V(G)) \leq 1$. Suppose that the vertices of G are partitioned into levels according to their distance from some vertex $v \in V(G)$, and that $L(l)$ denotes the number of vertices on level l. If r is the maximum distance of any vertex from v, let $r + 1$ be an additional level containing no vertices. Given any two levels l_1, l_2 such that levels 0 through $l_1 - 1$ have total cost at most $\frac{2}{3}$ and levels $l_2 + 1$ through $r + 1$ have total cost at most $\frac{2}{3}$, it is possible to find a* *vertex set $S \subseteq V(G)$ of size at most $L(l_1) + L(l_2) + \max\{0, 2(l_2 - l_1 - 1)\}$ that separates G into two sets A, B such that $f(A) \leq \frac{2}{3}$, $f(B) \leq \frac{2}{3}$.*

Proof. If $l_1 \geq l_2$ we can set S to be all vertices on level l_1. That separates G into two sets A, B, which contain all the vertices on levels 0 through $l_1 - 1$ and through the levels $l_1 + 1$ through r respectively. Since all the vertices on levels $l_2 + 1$ through r have total cost at most $\frac{2}{3}$ and $l_1 \geq l_2$ that holds also for the vertices on levels $l_1 + 1$ through r, so the lemma is true.

If $l_1 < l_2$ consider the vertex sets of G that occur after the removal of the vertices of both of those levels. Those are, the vertices on levels 0 through $l_1 - 1$, l_1 through l_2 and $l_2 + 1$ through $r + 1$. The only part of those which can have total cost exceeding $\frac{2}{3}$ is the vertices on level l_1 through l_2. If it does not, the lemma is true. Else, consider the graph G^* that occurs after the removal of the vertices on levels l_2 through r and the contraction of all vertices on levels 0 through l_1 to a single vertex v^* and set $f(v^*)$ to be 0. Due to Lemma 3.7, G^* is planar and also remains connected due to the separation of the vertices into levels according to their distance from a vertex v. G^* has a spanning tree with root the vertex v^* of radius $l_2 - l_1 - 1$.

After applying Lemma 3.8 to G^*, we can find a set $S \subseteq V(G^*)$ such that $v^* \in S$ of size at most $2(l_2 - l_1 - 1)$ that disconnects G^* into sets A, B such that $f(A) \leq \frac{2}{3}$ and $f(B) \leq \frac{2}{3}$ and suppose without loss of generality that $f(A) \geq f(B)$. Consider now the following partition of $V(G)$: $S' = S \cup V(l_1) \cup V(l_2) \setminus v^*$, $A' = A$ and $B' = V(G) \setminus (S' \cup A')$. By Lemma 3.8 $f(A') \leq \frac{2}{3}$. Moreover since $f(A' \cup S) \geq \frac{1}{3}$ (since $f(B) \leq \frac{2}{3}$) we have that $f(B') \leq \frac{2}{3}$. After the observation that $|S'| \leq L(l_1) + L(l_2) + 2(l_2 - l_1 - 1)$, we can conclude that the lemma is true. □

Now, we will present the proof of the main theorem of this section, on which, the polynomial-time algorithm for finding a balanced separator in any planar graph G, was based.

Theorem 3.10. *Let G be a planar graph of order n and $f : V(G) \to [0, 1)$ be a cost assignment to the vertices of G such that $f(V(G)) \leq 1$. Then there exists a vertex set $S \subseteq V(G)$ of size at most $2\sqrt{2}\sqrt{n}$ that separates G into two sets A, B such that $f(A) \leq \frac{2}{3}$, $f(B) \leq \frac{2}{3}$.*

Proof. Let G be a connected graph and v an arbitrary vertex of G. Using BFS we can partition the vertices of G into levels according to their distance from v. Suppose that the furthest vertex from v lies in the level r. We will denote by $L(l)$ the number of vertices on the level l (as we did in Lemma 3.9) and we will add two additional levels, -1 and $r + 1$, each containing 0 vertices.

Consider a level l_1, such that the total cost of the vertices from the level 0 through the level $l_1 - 1$ is less than $1/2$, but the total cost of the vertices from the level 0 through l_1 is at least $1/2$. We can suppose that such a level exist, otherwise the total cost of all vertices in G is less than $1/2$, so the Theorem is satisfied for $B = S = \emptyset$. Let k be the number of vertices on levels 0 through l_1. Now, we want to find a level l_0, such that $l_0 \leq l_1$ and

$$L(l_0) \leq 2\sqrt{k} - 2(l_1 - l_0)$$

and a level l_2 such that $l_1 + 1 \leq l_2$ and

$$L(l_2) \leq 2\sqrt{n - k} - 2(l_2 - l_1 - 1)$$

Suppose that two such levels exist. Notice that, due to the selection of the level l_1, the total cost of the vertices on levels 0 through $(l_0 - 1)$, and l_2 through $r + 1$ is not exceeding $2/3$. Then by applying Lemma 3.9, there exist a vertex set $S \subseteq V(G)$ that separates G into two vertex sets

A, B, each with total cost at most $2/3$ and

$$\begin{aligned}|S| &\leq L(l_0) + L(l_2) + \max\{0, 2(l_2 - l_0 - 1)\} \\ &\leq 2\sqrt{k} - 2(l_1 - l_0) + 2\sqrt{n-k} - 2(l_2 - l_1 - 1) + 2(l_2 - l_0 - 1) \\ &\leq 2(\sqrt{k} + \sqrt{n-k})\end{aligned}$$

But $2(\sqrt{k} + \sqrt{n-k}) \leq 2(\sqrt{n/2} + \sqrt{n/2}) = 2\sqrt{2}\sqrt{n}$, hence, the theorem holds if suitable levels l_0 and l_2 exist.

Suppose, now, that a suitable level l_0 does not exist. Then $\forall i \leq l_1$, $L(i) \geq 2\sqrt{k} - 2(l_1 - i)$. Since $L(0) = 1$, this means that $1 \geq 2\sqrt{k} - 2l_1 \Rightarrow l_1 + \frac{1}{2} \geq \sqrt{k}$. Thus, $l_1 = \lfloor l_1 + \frac{1}{2} \rfloor \geq \lfloor \sqrt{k} \rfloor$, and

$$\begin{aligned}k = \sum_{i=0}^{l_1} L(i) &\geq \sum_{i=l_1 - \lfloor\sqrt{k}\rfloor}^{l_1} 2\sqrt{k} - 2(l_1 - i) \\ &\geq (\lfloor\sqrt{k}\rfloor + 1)2\sqrt{k} - 2\sum_{i=0}^{\lfloor\sqrt{k}\rfloor} i \\ &\geq (\lfloor\sqrt{k}\rfloor + 1)2\sqrt{k} - \lfloor\sqrt{k}\rfloor(\lfloor\sqrt{k}\rfloor + 1) \\ &\geq (\lfloor\sqrt{k}\rfloor + 1)(2\sqrt{k} - \lfloor\sqrt{k}\rfloor) \\ &\geq \sqrt{k}(\lfloor\sqrt{k}\rfloor + 1) > k\end{aligned}$$

which is a contradiction. A similar contradiction occurs if a suitable level l_2 does not exist, and this complete the proof in the case where G is connected.

Let G be a graph that is not connected, and let $G_1, G_2, \ldots, G_k$ be its connected components, with vertex sets $V_1, V_2, \ldots, V_k$ respectively. Suppose that $f(V_i) \leq \frac{1}{3}, \forall i \in [k]$. Let i be the minimum index such that $\sum_{j=1}^{i} f(V_i) \geq \frac{1}{3}$, and also let $A = \cup_{j=1}^{i} V_j$, $B = \cup_{z=i+1}^{k} V_z$. Since the total cost of V_i does not exceed $1/3$ and i is the minimum index that satisfies the previous condition, the total cost of both A and B does not exceed $2/3$. Thus, the theorem holds for $S = \emptyset$.

If for some connected component, say G_i, $\frac{1}{3} \leq f(V_i) \leq \frac{2}{3}$, the theorem holds for $A = V_i$, $B = \cup_{j=1}^{k} V_j \setminus V_i$ and $S = \emptyset$.

Finally, if for some connected component, say G_i, $f(V_i) \geq \frac{2}{3}$, consider the partition A^*, B^*, S^*, that occurs after applying the same arguments as we did in the case where G is connected, to the component G_i. Now let A be the set among A^*, B^* with the greater cost, say A^*, $S = S^*$ and B the remaining vertices of G. Notice that $f(A) \leq \frac{2}{3}$ and $f(B) \leq \frac{2}{3}$ (because $f(\cup_{j=1}^{k} V_j \setminus V_i) \leq \frac{1}{3}$ and $f(B^*) \leq \frac{1}{3}$ since $f(A^* \cup B^* \cup S^*) \geq \frac{2}{3}$ and A^* is the vertex set with the maximum total cost), so S is a balanced separator of the desired size.

Now, the proof is complete, since we showed that, in all cases, a planar graph G has a balanced separator which is either empty, or connected in only one connected component of G. □

The following Theorem is a corollary of the Theorem 3.10 if we set the cost function to be $\frac{1}{|V(G)|}$ for each vertex of a planar graph G.

Theorem 3.11 ($\sqrt{n}$-separator theorem)**.** *Let G be a planar graph of order n. Then G has a balanced separator of size at most $2\sqrt{2}\sqrt{n}$.*

Richard J.Lipton and Robert Endre Tarjan [LT79] also proved that the total cost of each of the sets A, B of Theorem 3.10 can be reduced to at most $1/2$, if we allow the balanced separator to have size at most $\frac{2\sqrt{2}\sqrt{n}}{1-\sqrt{2/3}}$. It is natural to ask whether a similar theorem is true for non-planar graphs. Richard J.Lipton and Robert Endre Tarjan [LT79] showed that to be the case for "almost" planar graphs, referring to finite element graphs, a result which was later “extended” by Noga Alon, Paul Seymour and Robin Thomas [AST90a] for nonplanar graphs with a fixed excluded minor, as we will see in more detail in the next section.

3.3 Non-planar Graphs

The results stated in this section were proved by Alon, Seymour and Thomas[AST90a], who also provided an algorithm that runs in time $\mathcal{O}(h^{1/2}n^{1/2}m)$, which given a graph G, computes either a K_h minor of G or a balanced separator of G, of size at most $h^{3/2}\sqrt{n}$. This result was later improved (for $h >> \sqrt{\log n}$) by Plotkin, Rao and Smith [PRS94], who proved that any graph that excludes K_h as a minor, has a balanced separator S of size $\mathcal{O}(h\sqrt{n\log n})$. In 2010, Kawarabayashi and Reed [KR10a] proved, what was earlier conjectured by Alon, Seymour and Thomas, that for each t, there is a balanced separator of size $\mathcal{O}(t\sqrt{n})$ in any graph G, of order n, with no K_t minor and they also provided an $\mathcal{O}(n^2)$ time algorithm to obtain such a balanced separator. This bound is the best possible, since every 3-regular expander graph G, of order n, is a graph with no K_t minor for $t = c\sqrt{n}$ and with no balanced separator of size dn for appropriately chosen positive constants c, d. Moreover, this result generalized the result of Gilbert, Hutchinson and Tarjan [GHT84],that every graph on n vertices and genus g has a balanced separator of size $\mathcal{O}(\sqrt{g}\sqrt{n})$, as K_h has genus at least $\Omega(h^2)$.

The results proved in the previous section also hold if we extend the cost function f to $\mathbb{R}^+ = (0, +\infty)$, and use $\frac{2f(V(G))}{3}$ as an upper bound on the total cost of each set of the partition. Therefore, Theorem 3.10 can be expressed in the following form:

Let G be a planar graph of order n and $f : V(G) \to \mathbb{R}^+$ be a function that assigns costs to each vertex of G. Then there exists a vertex set $S \subseteq V(G)$ of size at most $2\sqrt{2}\sqrt{n}$ that separates G into two sets A, B such that $f(A) \leq \frac{2f(V(G))}{3}$, $f(B) \leq \frac{2f(V(G))}{3}$.

Lemma 3.12. *Let G be a graph of order n, $A_1, \dots, A_k$ subsets of $V(G)$ and $r \geq 1$ a real number. Then one of the following holds:*

(i) Either there exists a tree T in G of size at most r such that $V(T) \cap A_i \neq \emptyset$ for $i \in [k]$,

(ii) or, there exists a vertex set $Z \subseteq V(G)$ of size at most $\frac{(k-1)n}{r}$, such that no Z-flap intersects all of $A_1, \dots, A_k$.

Proof. Let G be a graph of order n, $A_1, \dots, A_k$ subsets of $V(G)$ and $r \geq 1$ a real number. We may assume that $k \geq 2$. Let $G^1, \dots, G^{k-1}$ be isomorphic copies of G, mutually disjoint. For each vertex $v \in V(G)$ and $1 \leq i \leq k-1$, let v^i be the corresponding vertex of G^i. Now consider the graph G' obtained from $\bigcup_{i=1}^{k-1} G^i$ by adding for $2 \leq i \leq k-1$ and all $v \in A_i$ the edge $v^{i-1}v^i$, and let $X = \{v^1 : v \in A_1\}$, $Y = \{v^{k-1} : v \in A_k\}$. Let $d(u)$ be the number of vertices in the shortest path in G' that connects X to any vertex u (or ∞ if no such path exists). Now we will examine the following two cases:

There exists a vertex $u \in Y$, such that $d(u) \leq r$. Let P be a path that realizes that $d(u)$ for this vertex $u \in Y$. Let $S = \{v \in V(G) : v^i \in V(P) \text{ for some } i, 1 \leq i \leq k-1\}$. Since each vertex of S corresponds to at least one vertex in P, we have that $|S| \leq |V(P)| \leq r$. Moreover, because each G^i is isomorphic to G and P is a path, $G[S]$ is a tree in G. We can think of G' as a graph that has k levels (the copies of G and G), in which the only way to go from the i^{th} level to the $(i-1)^{th}$ level is through a vertex in A_i, and the only way to go from the i^{th} level to the $(i-k)^{th}$ level is to pass through all the levels between them. After that observation, we see that in order to connect a vertex in Y to a vertex in X we have to pass from each level, so $V(S) \cap A_i \neq \emptyset$ for $i \in [k]$. That means that case (i) is satisfied.

The next case to examine is, the case where $d(u) > r$, $\forall u \in Y$. Let $t = \lceil r \rceil$ and define for all $j \in [t]$ the sets $Z_j = \{u \in V(G') : d(u) = j\}$, which are mutually disjoint (by the definition of $d(u)$ for a vertex u). Moreover, since $|V(G')| = (k-1)n$, at least one of the sets $Z_1, \ldots, Z_t$, say Z_j has cardinality at most $\frac{(k-1)n}{t}$. That means that $|Z_j| \leq \frac{(k-1)n}{t} \leq \frac{(k-1)n}{r}$. Since $d(u) > r\ \forall u \in Y$, every path between X and Y has a vertex in Z_j. Let $Z = \{v \in V(G) : v^i \in Z_j$ for some $i, 1 \leq i \leq k-1\}$ and notice that $|Z| \leq |Z_j| \leq \frac{(k-1)n}{r}$. Suppose, for contradiction, that F is a Z-flap of G which intersects all of $A_1, \ldots A_k$, and let $a_i \in F \cap A_i$, $\forall i \in [k]$. Since $G[F]$ is a connected component of $G \setminus Z$ there exist a path, P_i, that connects a_i to a_{i+1} for all $i \in [k-1]$. Let P^i be the path of G^i corresponding to P_i and consider the vertex set $V(P^1) \cup \ldots \cup V(P^{k-1})$. Due to the observation of the graph G' we did in the previous case, that vertex set includes a path of G' between X and Y, which as we assumed has length at least r. However, since F is a Z-flap, that path should be disjoint from $Z = Z_j$, which lead us to a contradiction. Thus, there exists no Z-flap that intersects all of $A_1, \ldots, A_k$ so case (ii) is satisfied.

The proof of this lemma is now complete since for any graph and any real number r we either found a tree that satisfies the first condition or a vertex set that satisfies (ii). $\square$

Theorem 3.13. *Let $h \geq 1$ be an integer, and G be a graph of order n with a haven of order $h^{3/2}n^{1/2}$. Then G has a K_h-minor.*

Proof. Let G be a graph of order n and β be a haven in G of order $h^{3/2}n^{1/2}$. We can choose a vertex set $X \subseteq V(G)$ and a covey $\mathcal{C}$ with $|\mathcal{C}| \leq h$ such that

1. $X \subseteq \cup_{C \in \mathcal{C}} V(C)$
2. $|X \cap V(C)| \leq h^{1/2}n^{1/2}$ for each $C \in \mathcal{C}$
3. $V(C) \cap \beta(X) = \emptyset$ for each $C \in \mathcal{C}$
4. subject to 1, 2 and 3, $|\mathcal{C}| + 3|\beta(X)| + |X|$ is minimum.

Such a set X and covey $\mathcal{C}$ exists, since all the above conditions are satisfied for $X = \mathcal{C} = \emptyset$. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ and suppose, for a contradiction, that $k < h$. Let also, for $1 \leq i \leq k$, A_i to be the set of all vertices $v \in \beta(X)$ adjacent to a vertex in C_i and $G' = G[\beta(X)]$. By applying Lemma 3.12 to G' with $r = h^{1/2}n^{1/2}$ one of the following cases holds.

(i) There exists a tree T in G' of size at most $h^{1/2}n^{1/2}$, such that $V(T) \cap A_i \neq \emptyset$, $\forall i \in [k]$. In this case we can replace $\mathcal{C}$ by $\mathcal{C}' = \mathcal{C} \cup \{T\}$ and X by $X' = X \cup V(T)$. Since $X \subseteq \cup_{C \in \mathcal{C}} V(C)$, we have that $X' \subseteq \cup_{C \in \mathcal{C}'} V(C)$. Moreover, because $|V(T)| \leq h^{1/2}n^{1/2}$ and $T \cap X = \emptyset$ condition 2 also holds for X' and $\mathcal{C}'$. Furthermore, $X' \subseteq X$ and $|X'| \leq h^{3/2}n^{1/2}$, so $\beta(X') \subseteq \beta(X)$. Moreover, since $V(T) \subseteq X'$, no vertex of T is in $\beta(X')$, so $\beta(X') \subseteq \beta(X) \setminus T$. As a result, $\mathcal{C}'$ is a covey and for each $C \in \mathcal{C}'$,

$$V(C) \cap \beta(X') \subseteq V(C) \cap (\beta(X) \setminus V(T)) = \emptyset$$

so condition 3 is also satisfied. Notice that $T \subseteq G'$ so $|\beta(X')| \leq |\beta(X) \setminus T| = |\beta(X)| - |V(T)|$, and that $|X'| \leq |X \cup V(T)|$, but $X \cap T = \emptyset$, so $|X'| \leq |X| + |V(T)|$. Thus,

$$|\mathcal{C}'| + 3|\beta(X')| + |X'| \leq |\mathcal{C}| + 1 + 3|\beta(X)| - 3|V(T)| + |X| + V(T) \leq |\mathcal{C}| + 3|\beta(X)| + |X|$$

which contradicts to the minimality of the fourth condition.

(ii) There exists a vertex set $Z \subseteq \beta(X)$ of size at most $\frac{(k-1)|\beta(X)|}{h^{1/2}n^{1/2}} \leq h^{1/2}n^{1/2}$ such that no Z-flap of G' intersects all of $A_1, \ldots, A_k$. Let $Y = X \cup Z$ be a vertex set. From condition 2 that X satisfies, $|X| \leq n^{1/2}h^{1/2}k$ and because of our assumption that $k \leq h-1$ we have that $|X| \leq (h-1)h^{1/2}n^{1/2}$. Because $|Z| \leq h^{1/2}n^{1/2}$, we have that $|Y| \leq h^{3/2}n^{1/2}$. That means that $\beta(Y)$ exists and $\beta(Y) \subseteq \beta(X)$. Since $\beta(Y)$ is a Z-flap of G' there exists an index $i \in [k]$ such that $\beta(Y) \cap A_i = \emptyset$. We now extend C_i to be a maximal tree C_i' of G that is disjoint from $\beta(Y)$ and from each C_j, $(j \neq i)$. We also define $Z' = V(C_i') \cap Z$, $X' = Z' \cup (X - V(C_i))$ and $W = V(C_i') \cup (V(G) - \beta(X))$. We will show that $\beta(X') \cap W = \emptyset$.

Suppose for a contradiction that $\beta(X') \cap W \neq \emptyset$. Notice that $X' \subseteq Y$, so $\beta(Y) \subseteq \beta(X')$ and hence there exists a path P that connects W to $\beta(Y)$, in $G[\beta(X')]$ (so it is disjoint from X'). Due to our assumption that $\beta(X') \cap W \neq \emptyset$ and the fact that $\beta(Y) \subseteq \beta(X')$, $\beta(Y) \cap W = \emptyset$. Hence, there exist two consecutive vertices $u, v \in P$, such that $u \in W$ and $v \in V(G) \setminus W \subseteq \beta(X)$. Since $uv \in E(G)$ and $v \in \beta(X)$, $u \in X \cup \beta(X)$ and because also $u \in W$ and $P \cap X' = \emptyset$,

$$u \in (X \cup \beta(X)) \cap (W \setminus X') \subseteq V(C_i')$$

Since $v \notin W$, it follows by the maximality of C_i' that $v \in \beta(Y)$ (else since $u \in W$ and $uv \in E(G)$ we can extend C_i'). Moreover, since $u \in V(C_i')$ and $u \notin \beta(Y)$ we deduce that $u \in Y$, and as a result

$$u \in Y \cap (V(C_i') \setminus X') \subseteq V(C_i)$$

However, by the definition of A_i, $v \in A_i$, which contradicts our assumption that $A_i \cap \beta(Y) = \emptyset$, so $\beta(X') \cap W = \emptyset$.

Hence, $\beta(X') \subseteq \beta(X)$. Let $\mathcal{C}' = (\mathcal{C} \setminus \{C_i\}) \cap \{C_i'\}$, which is a covey (since C_i' is disjoint from all elements of $\mathcal{C} \setminus \{C_i\}$). We observe that

1. $X' \subseteq \cup(V(C) : C \in \mathcal{C}')$, for $Z' \subseteq V(C_i')$
2. $|X' \cap V(C)| \leq h^{1/2}n^{1/2}$ for each $C \in \mathcal{C}'$, if $C \neq C_i'$ then $X' \cap V(C) = X \cap V(C)$, and $X' \cap V(C_i') = Z'$
3. $V(C) \cap \beta(x') = \emptyset$ for each $C \in \mathcal{C}'$ (since $\beta(X') \cap W = \emptyset$).

By the minimality stated in 4, the following should hold

$$|\mathcal{C}'| + 2(|\beta(X')| + |X' \cup \beta(X')|) \geq |\mathcal{C}| + 2(|\beta(X)| + |X \cup \beta(X)|)$$

However, since $|\mathcal{C}'| = |\mathcal{C}|$, $\beta(X') \subseteq \beta(X) \subseteq (X \cup \beta(X)) \setminus (X \cap V(C_i))$, and because $X \cap V(C_i) = \emptyset$, we have that $|\mathcal{C}| + 2(|\beta(X)| + |X \cup \beta(X)|) \geq |\mathcal{C}'| + 2(|\beta(X')| + |X' \cup \beta(X')|) \Rightarrow |\mathcal{C}| + 2(|\beta(X)| + |X \cup \beta(X)|) = |\mathcal{C}'| + 2(|\beta(X')| + |X' \cup \beta(X')|)$. Hence, $\mathcal{C} \setminus \{C_i\}, X$ satisfy the conditions 1, 2, 3 and contradict 4.

Since both cases resulted in a contradiction, $k \geq h$. Hence, there exist h vertex disjoint trees such that any two of them are connected through an edge. By contracting each such a tree to a vertex, we can conclude that $K_h \leq_m G$ as required. ☐

Proposition 3.14. *Let $h \geq 1$ be an integer and G be a graph of order n with no K_h-minor. Let also, $f : V(G) \to \mathbb{R}^+$ be a cost assignment to the vertices of G. Then there exists a set $X \subseteq V(G)$ of size at most $h^{3/2}n^{1/2}$ such that $f(F) \leq \frac{1}{2}f(V(G))$ for every X-flap F.*

Sketch of Proof. Suppose for a contradiction, that for each set $X \subseteq V(G)$ of size at most $h^{3/2}n^{1/2}$, there exists an X-flap F, such that $f(F) > \frac{1}{2}f(V(G))$. Notice that if we define for each X of that size the function $\beta(X) = F$ (where F is the vertex set with the properties that we described before), then β is a haven of order $h^{3/2}n^{1/2}$ and thus, by Theorem 3.13, G should have K_h as a minor, which leads to a contradiction. $\square$

Note that since the complete graph K_h contains every simple graph of order h as a subgraph,the above results hold for every H of order h, so the following Theorem is a corollary of them.

Theorem 3.15. *Let G, H be graphs of order n and h respectively and $f : V(G) \to \mathbb{R}^+$ a cost assignment to the vertices of G. Then exactly one of the following is true:*

(i) Either $H \leq_m G$,

(ii) or, there exist a vertex set $S \subseteq V(G)$ of size at most $h^{3/2}\sqrt{n}$ that separates G into two sets A, B such that $f(A) \leq \frac{2f(V(G))}{3}$, $f(B) \leq \frac{2f(V(G))}{3}$.

As we did in the previous section, by assigning the same cost, $\frac{1}{|V(G)|}$, to each vertex of a graph G, we can obtain the following result.

Corollary 3.16. *Let G, H be graphs of order n and h respectively. Then exactly one of the following is true:*

(i) Either $H \leq_m G$,

(ii) or, there exist a balanced separator $S \subseteq V(G)$, of size at most $h^{3/2}n^{1/2}$.

Alon, Seymour and Thomas [AST90a] also provided an algorithm that realizes Corollary 3.16 in time $\mathcal{O}(h^{1/2}n^{1/2}m)$ by converting the proofs of Lemma 3.12 and of Theorem 3.13.

CHAPTER 4

EXPANDERS

4.1 Expanders and Balanced Separators

In this section we will prove what we have previously mentioned, that expanders are graphs that do not have small balanced separators. This simple yet powerful connection between two central graph theoretic notions (expanders and separation), usually perceived as belonging to quite different worlds (extremal graph theory and structural graph theory, respectively) can be quite fruitful and illuminating.

Proposition 4.1. *Let G be an α-edge-expander of order n and $S \subseteq V(G)$ be a balanced separator of G. Then $|S| \geq \frac{\alpha n}{3(1+\alpha)}$.*

Proof. Let G be an α-edge-expander of order n and $S \subseteq V(G)$ be a balanced separator that separates G into the vertex sets A, B. By the definition of a balanced separator of a graph $|A|, |B| \leq \frac{2n}{3}$, and suppose without loss of generality that, $|A| \leq |B|$. Since $|B| \leq \frac{2n}{3}$, $A \cup B \cup S = V(G)$ and A, B, S are pairwise disjoint subsets of $V(G)$,

$$|A| + |S| \geq \frac{n}{3} \tag{4.1}$$

Moreover, because $|S|$ is a balanced separator, we have that $N_G(A) \subseteq S$. Now, notice that by the definition of an α-edge-expander and because $|A| \leq \frac{n}{2}$, we know that $|E_G(A, \bar{A})| = |E_G(A, S)| \geq \alpha|A|$ from which we conclude that

$$\begin{aligned}
|S| - \alpha|A| &\geq 0 \Leftrightarrow \\
\frac{1}{\alpha}(|S| - \alpha|A|) &\geq 0 \Leftrightarrow \\
\frac{1}{\alpha}(|S| - \alpha|A|) + |A| + |S| &\geq \frac{n}{3} \Leftrightarrow \\
|S|\left(\frac{1}{\alpha} + 1\right) &\geq \frac{n}{3} - |A|\left(1 - \frac{1}{\alpha}\right) \Leftrightarrow \\
|S| &\geq \frac{\frac{n}{3} - |A|\left(1 - \frac{1}{\alpha}\right)}{\left(\frac{1}{\alpha} + 1\right)} \Leftrightarrow \\
|S| &\geq \frac{\alpha n}{3(1+\alpha)}
\end{aligned}$$

Notice that the second inequality holds since $\alpha > 0$ while the third one occurs after adding the inequality (4.1) to the second one. Since S was an arbitrarily chosen balanced separator of G, our proof is now complete. $\square$

The following Proposition shows that the opposite implication is also true, in the sense that graphs without small balanced separators contain large induced edge-expanders.

Proposition 4.2. *Let $\alpha > 0$ be a constant and G a graph of order n. If all balanced separators of G are of size at least αn, then G contains an induced $\left(\frac{3\alpha}{2}\right)$-edge-expander of at least $\frac{2n}{3}$ vertices.*

Proof. Let G be a graph and $\alpha > 0$ be a constant as described above. If every vertex subset of size at most $\frac{n}{2}$ has edge expansion at least $\left(\frac{3\alpha}{2}\right)$, then, by the definition of edge expansion, G is an $\left(\frac{3\alpha}{2}\right)$-edge-expander so our proof is complete.

Else, there is a subset $V_1 \subseteq V(G)$ of size at most $\frac{n}{2}$ such that $|E_G(V_1, \bar{V_1})| < \left(\frac{3\alpha|V_1|}{2}\right)$. Consider now the graph $G_1 = G[G \setminus V_1]$ and observe that since all balanced separators of G have size at least αn, G_1 is connected. If G_1 is still not an $\left(\frac{3\alpha}{2}\right)$-edge-expander we can construct the graph G_2 by deleting a subset of it vertices as we did while constructing G_1. Consider Z to be the union of the sets of vertices that are deleted at each iteration. Suppose that k^{th} is the first iteration after which $|Z| \geq \frac{n}{3}$ and let V_k be the set that is deleted from G_{k-1} in order to obtain G_k. Due to the selection of V_k we have that $|V_k| \leq \frac{n-|Z \setminus V_k|}{2}$ and $|Z \setminus V_k| < \frac{n}{3}$. Combining these two inequalities, we have

$$|Z| \leq \frac{2n}{3} \tag{4.2}$$

The set $N_G(Z)$ forms a balanced separator in G (separating Z and $V(G) \setminus (Z \cup N_G(Z))$), hence its size is at least αn. However, due to inequality (4.2), we have that $\alpha n \geq \frac{3\alpha|Z|}{2}$, which leads to a contradiction (because at each iteration we select the set V_i such that $N_G(V_i) < \frac{3\alpha|V_i|}{2}$).

Hence, the removal process stops with $|Z| < \frac{n}{3}$ and the final graph of this process is a $\left(\frac{3\alpha}{2}\right)$-edge-expander on at least $\frac{2n}{3}$ as required. $\square$

Thus, when aiming to prove results about graphs without sublinear balanced separators, we can choose instead to operate on expanders.

4.2 Expanders and eigenvalues

There are various matrices that are naturally associated with a graph, such as the adjacency matrix, the incidence matrix and the Laplacian. One of the main problems of algebraic graph theory is to determine precisely how or whether, properties of graphs are reflected in the algebraic properties of such matrices. Hence, apart from balanced separators, another scope from which we can operate when studying expanders, is the properties of the matrices related to them. In order to do so, we will need some basic definitions and properties from linear algebra (mainly for symmetric matrices).

Definition 4.3. The *trace* of a square matrix A is the sum of its diagonal entries and is denoted by $\mathbf{tr}(A)$.

Definition 4.4. Given a matrix A, a vector x is defined to be an eigenvector of A if and only if there exists a λ such that $Ax = \lambda x$. In that case λ is called an eigenvalue of A.

Proposition 4.5. *Let A be a square matrix. Then the sum of its eigenvalues is equal to $\mathbf{tr}(A)$.*

Lemma 4.6. *Let A be a symmetric matrix with real entries. If u and v are eigenvectors of A with different eigenvalues, then u and v are orthogonal.*

Lemma 4.7. *The eigenvalues of a real symmetric matrix A are real numbers.*

Definition 4.8. Let G be a graph of order n.

- The *edge boundary* of a set $S \subseteq V(G)$, denoted by ∂S, is $\partial S = E(S, \bar{S})$.
- The (edge) *expansion ratio* of G, denoted $h(G)$, is defined as

$$h(G) = \min_{\{S \subseteq V(G) \mid |S| \leq \frac{n}{2}\}} \frac{|\partial S|}{|S|}$$

Definition 4.9. A sequence of d-regular graphs $\{G_i\}_{i \in \mathbb{N}}$ of size increasing with i, is a *family of expander graphs* if there exists $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ for all i.

Definition 4.10 (Algebraic definition of expansion)**.** The *Adjacency Matrix* of an n-vertex, simple graph G, denoted $A = A(G)$, is an $n \times n$ matrix whose (u, v) entry is 1 if $uv \in E(G)$ and 0 otherwise. Notice that since A is a symmetric matrix with integer values, it has n real eigenvalues which we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We refer to those eigenvalues as the *spectrum* of the graph G.

Observation 4.11. *If G is a d-regular simple graph, then the eigenvalues of its adjacency matrix satisfy the equality $\sum_{i \in [n]} \lambda_i = 0$ (since $\sum_{i \in [n]} \lambda_i = \mathbf{tr}(A)$ for any matrix, and G is a simple graph, so the elements of its diagonal are 0).*

As we can see from the following proposition, the spectrum of a graph encodes a lot of information about it.

Proposition 4.12. *Let G be a d-regular graph of order n and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its spectrum. Then,*

(i) $\lambda_1 = d$, and the corresponding eigenvector is $v_1 = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

(ii) $\lambda_1 = \max_{\|x\|=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{\|x\|^2}$

(iii) G is connected if and only if $\lambda_1 > \lambda_2$.

(iv) G is bipartite if and only if $\lambda_1 = -\lambda_n$.

Proof. (i) Let $x = (x_v)_{v \in V(G)} \neq 0$ be an eigenvector corresponding to the largest eigenvalue,λ_1, and x_u be the entry of x with maximum absolute value. By the definition of an eigenvector we have that $\lambda_1 \cdot x = A(G) \cdot x$, and since A is the adjacency matrix of G, we have that,

$$\lambda_1 x_u = \sum_{v \in N(u)} x_u,$$

so, due to the selection of x_u and the fact that G is d-regular,

$$|\lambda_1 \cdot x_u| = \left| \sum_{v \in N(u)} x_v \right| \leq \sum_{v \in N(u)} |x_v| \leq \sum_{v \in N(u)} |x_u| = d|x_u|.$$

Now, it is easy to verify that d is indeed an eigenvalue of $A(G)$, and that its corresponding vector is $v_1 = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$, using the fact that G is d-regular.

(ii) Let $x \in \mathbb{R}^n$, such that $\|x\| = 1$. If x is an eigenvector of an eigenvalue λ of A then

$$Ax = \lambda x \Rightarrow x^T A x = x^T \lambda x \Rightarrow x^T A x = \lambda \leq \lambda_1$$

If x is not an eigenvector of A. Let $x_1, \ldots, x_n$ be the orthonormal basis of the eigenvectors of A (notice that if two eigenvectors of an adjacency matrix correspond to different eigenvalues, they are orthogonal to each other). Then there exist $k_1, \ldots, k_n \in \mathbb{R}$ such that $x = k_1x_1 + \ldots + k_nx_n$, and because $\|x\| = 1$, $1 = \|x\|^2 = k_1^2 + \ldots + k_n^2$. Hence

$$\begin{aligned} xTAx = (k_1x_1^T + \ldots + k_nx_n^T)A(k_1x_1 + \ldots + k_nx_n) = \\ k_1^2\lambda_1 + \ldots + k_n^2\lambda_n \leq (k_1^2 + \ldots + k_n^2)\lambda_1 = \lambda_1 \end{aligned} \tag{4.3}$$

As a result $\lambda_1 \geq \max_{\|x\|=1} x^TAx$, and sunce for the corresponding normalized eigenvector of λ_1, x_1, $\lambda_1 = x_1^TAx_1$, we have that

$$\lambda_1 = \max_{\|x\|=1} x^TAx$$

Moreover if $x \in \mathbb{R}^n$, we know that for $x' = \frac{x}{\|x\|}$, $\{x'\} = 1$. Thus

$$\lambda_1 = \max_{\|x\|=1} x^TAx = \max_{x\in\mathbb{R}^n} \frac{x^TAx}{\|x\|\|x\|}$$

(iii) Suppose that G is a disconnected graph and let C be one of its connected components. Consider the vectors $x = (x_1, \ldots, x_n)$, where $x_i = 1$ if the corresponding vertex is in C and 0 otherwise and and $y = (1, \ldots, 1)$. Notice now that since each connected component of G is d-regular, $d \cdot x = A(G) \cdot x$ but also $d \cdot y = A(G) \cdot y$, so $\lambda_1 = \lambda_2$.

Now, supose that G is a d-regular connected graph and that there exists an eigenvector that correspond to the eigenvalue d other than, $v = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$. Let that vector be $x = (x_1, \ldots, x_n)$, and let also v_m be the vertex that corresponds to the largest x_m. As we have seen before, $d \cdot x_m = \sum_{v\in N(v_m)} x_v$, and because $x_i \leq x_m$ for all i, $x_i = x_m$, for all $v_i \in N(v_m)$. By repeating this, for the vertices that are not neighbors of v_m,because G is connected, we have that $x_i = x_m$ for all i, which contradicts our assumption that $x \neq v$.

(iv) ($\Rightarrow$) Let G be a d-regular bipartite graph. Since G is bipartite we can re-index its vertices such that

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

We proved that d is an eigenvalue of $A(G)$ and let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be its eigenvector. Then we have

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d \begin{bmatrix} x \\ y \end{bmatrix}$$

and as a result $By = dx$ and $B^Tx = dy$. Hence

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} -By \\ B^Tx \end{bmatrix} = \begin{bmatrix} -dx \\ dy \end{bmatrix} = -d \begin{bmatrix} x \\ -y \end{bmatrix}$$

So, $-d$ is an eigenvalue of $A(G)$ and its corresponding eigenvector is $\begin{bmatrix} x \\ -y \end{bmatrix}$.

($\Leftarrow$)Now suppose that G is a connected graph such that $\lambda_1 = -\lambda_n$. Let also x_n be the eigenvector corresponding to the eigenvalue λ_n and y be the vector that its i^{th} entry is

$|x_n(i)|$. We now have,

$$\begin{aligned} |\lambda_n| = |-d| = |x_n^T A x_n| &\leq \sum_{i,j} A_{ij}|x_n(i)||x_n(j)| \\ &= \sum_{i,j} A_{ij} y(i) y(j) \\ &= y^T A y \\ &\leq \lambda_1 \end{aligned}$$

The assumption that $\lambda_n = -\lambda_1$ implies that all the above inequalities are equalities, so y is an eigenvector corresponding to λ_1. Because $y \geq 0$ and G is connected (so we know the eigenvector that corresponds to λ_1), we have that $y > 0$ and as a result $x_n(i) \neq 0$ for all $i \in [n]$.

Since all inequalities are equalities we have $\sum_{i,j} A_{ij}|x_n(i)||x_n(j)| = |\sum_{i,j} A_{ij} x_n(i) x_n(j)|$, so $x_n(i)x_n(j)$ has the same sign whenever A_{ij} is positive. Since $\lambda_n = -d = x_n^T A x_n < 0$, all of these products must be negative. This implies that for any $v_i v_j \in E(G)$, either $x_n(i) > 0$ and $x_n(j) < 0$, or $x_n(j) > 0$ and $x_n(i) < 0$. This induces the bipartition

$$V = \{i : x_n(i) < 0\}$$
$$W = \{i : x_n(i) > 0\}$$

which shows that G is bipartite ($v_1, v_2 \in W$ and $v_1 v_2 \in E(G)$, imply that $x_n(1), x_n(2) > 0$ and $x_n(1)x_n(2) < 0$ respectively which is a contradiction).

□

Given a d-regular graph G of order n, we denote $\lambda = \lambda(G) = \max\{|\lambda_2|, |\lambda_n|\}$ (that means that λ is the largest absolute value of an eigenvalue other than $\lambda_1 = d$). As we will see below, the graph's second eigenvalue is closely related to its expansion parameter.

Theorem 4.13. *Let G be a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\frac{d-\lambda}{2} \leq h(G) \leq \sqrt{2d(d-\lambda)}$$

This theorem is due to Cheeger [Che69], and Buser [Bus82] in the continuous case. In the discrete case, it was proved by Dodziuk [Dod84] and independently by Alon-Milan [AM85], and by Alon [Alo86]. More concretely we see that $d - \lambda$, also known as the *Spectral Gap*, provides an estimate on the expansion of a graph. In particular, a d-regular graph has an expansion ratio $h(G)$ bounded away from zero if and only if its spectral gap is bounded away from zero.

The following lemma shows that a small second eigenvalue in a graph implies that its edges are "spread out", a hallmark of random graphs. This useful bound probably appeared in print first by Alon and Chung [AC88].

Lemma 4.14 (Expander Mixing Lemma)**.** *Let G be a d-regular graph of order n and set $\lambda = \lambda(G)$. Then for all $S, T \subseteq V(G)$:*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}$$

The left-hand side of the above inequality measures the deviation between two quantities: one is $|E(S,T)|$, the number of edges between the two sets and the other, the expected number of edges between S and T in a random graph of edge density $\frac{d}{n}$, namely $\frac{d|S||T|}{n}$. A small λ implies that this deviation is small, so the graph is nearly random in this sense.

When the spectral gap of G is much smaller than d, the upper and the lower bounds of theorem 4.13 differ substantially. This is expressed through the converse of the expander lemma, which was proven by Bilu and Linial [BL06].

Lemma 4.15. *(Converse of the Expander Mixing Lemma) Let G be a d regular graph and suppose that there exists a positive number ρ, such that the following inequality holds for every two subsets S, T, of $V(G)$.*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \rho\sqrt{|S||T|}$$

Then, $\lambda \leq \mathcal{O}(\rho(1+\log(\frac{d}{\rho}))$. This bound is tight.

Although many results occur through the algebraic notion of spectral expansion, in some contexts it is more convenient to use the definition of vertex expansion, such as when constructing expander codes, as Sipser and Spielman did in [SS94]. As one can imagine there is a connection between these two definitions. This relationship was discovered in a series of works by Alon, Milman and Tanner ([Alo86] [AM85], [Tan84]), and is expressed through the following two theorems.

Theorem 4.16. *Let G be a graph of order n and let, its second largest eigenvalue be λ. Then G is also a $(\alpha n, \frac{1}{(1-\alpha)\lambda^2+\alpha})$-vertex expander.*

Notice that the relationship expressed by this theorem behaves as we would expect, that means, the smaller the λ, the greater the vertex expansion of the graph. The next theorem gives the converse relationship.

Theorem 4.17. *Let G be a d regular, $(1+\alpha)$-vertex expander, of order n. Then, there exists a real number $\gamma = \Omega(\alpha^2/d)$ such that the second largest eigenvalue, λ, of G is equal to $(1-\gamma)$.*

In the next section we will see how through the combinatorial definition of vertex expansion we can obtain results about minors in expanders.

4.3 Minors in Expanders

Given the prominence of theory of minors it is only natural to expect to see meaningful research connecting expanding graphs and minors. Indeed, there have been several papers addressing this subject directly or indirectly. The theorems we are going to see in detail in this section are about vertex-expanders, were proved by Krivelevich and Sudakov [KS09] and are an extension of results of Alon, Seymour and Thomas [AST90b], Plotkin, Rao and Smith [PRS94] and of Kleinberg and Rubinfeld [KR96], who cover basically the case of expansion by a constant factor $t = \Theta(1)$. The main theorem of this section, states that

If G is a (t,α)-expanding graph of order n and $t \geq 10$, then G contains a minor with average degree at least

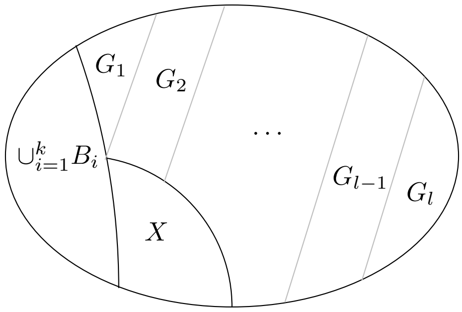
$$c\alpha^3 \frac{\sqrt{nt\log t}}{\sqrt{\log n}}$$

where $c > 0$ is some absolute constant independent of α.

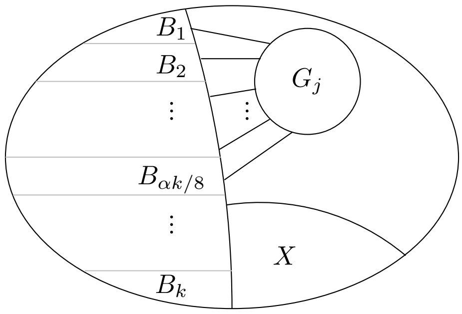
The idea of the proof of this theorem is to repeat the following iterative procedure p times. In the beginning of iteration $k+1$ we will have k pairwise disjoint sets of vertices of G, $B_1, \ldots, B_k$

each of size $|B_i| = q$, such that all induced subgraphs $G[B_i]$ are connected. We will construct a new subset B_{k+1}, also of size q, such that the induced subgraph $G[B_{k+1}]$ is connected and there are at least $\frac{\alpha k}{8}$ indices $1 \leq i \leq k$ such that there is an edge from B_i to B_{k+1}. In the end of this algorithm if we contract all subsets B_i, and choose the values of p and q carefully, we will obtain a graph of the desired average degree .The construction and the proof of the properties of the set B_{k+1}, will be completed in two stages

Stage 1: We will first prove that there exists a subset $X \subseteq V(G) \setminus B$ such that, every connected component, G_i, of the resulting graph $G'' = G[V(G) \setminus (B \cup X)]$ is a $(\frac{t}{2}, \frac{\alpha}{2})$-expander.



Stage 2: Using probabilistic arguments we will show that there exists one connected component G_j that is connected with at least $\frac{\alpha k}{8}$ of the sets B_i through an edge.



In order to prove the existence of a set X that is described in Stage 1, which will also be useful for the proof of Proposition 4.24 we will need the following lemmata:

Lemma 4.18. *Let G be a (t, α)-expanding graph of order n, and $t \geq 10$. Let also, $B \subseteq V(G)$ be a vertex set of G, of size at most $0.06\alpha n$. Then, for the graph $G' = G[V(G) \setminus B]$, the following holds:*

$$\forall X \subseteq V(G') : \frac{2|B|}{t} \leq |X| \leq \frac{an}{t} \Rightarrow |N_{G'}(X)| \geq \frac{t|X|}{2}$$

Proof. Suppose, for contradiction, that,

$$\exists X \subseteq V(G') : \frac{2|B|}{t} \leq |X| \leq \frac{an}{t} \Rightarrow |N_{G'}(X)| < \frac{t|X|}{2}$$

Hence,

$$|N_G(X)| \leq |N_{G'}(X)| + |B| < \frac{t|X|}{2} + |B| \leq t|X| \tag{4.4}$$

The first inequality of (4.4) holds due to the fact that $G' = G \backslash B$, the second one as $|N_{G'}(X)| < \frac{t|X|}{2}$, while the last one comes as a result of the size of $|X|$. However, since G is (t, α)-expanding and X is a vertex set of G of size, $|X| \leq \frac{\alpha n}{t}$, by the definition of (t, α)-expanders we have that $|N_G(X)| \geq t|X|$, which contradicts inequality (4.4). $\square$

Using now Lemma 4.18, we will prove the following.

Lemma 4.19. *Let G be a (t,α)-expanding graph of order n, and $t \geq 10$. Let also, $B \subseteq V(G)$ be a vertex set of G, of size at most $0.06\alpha n$, and denote by C the vertex set $V(G) \setminus B$. Then, for the graph $G' = G[C]$, the following holds:*

$$\exists X \subseteq C\big(|X| < \frac{2|B|}{t} \wedge \forall Q \subset C \setminus X(|Q| \leq \frac{\alpha n}{t} \Rightarrow |N_{G[C\setminus X]}(Q)| \geq \frac{t|Q|}{2})\big)$$

Proof. During the proof of this lemma we will denote $D_X = C \setminus X$, $G''_X = G[D_X]$. Suppose, for contradiction, that

$$\forall X \subseteq C\big(|X| < \frac{2|B|}{t} \wedge \exists Q \subseteq D_X \ (|Q| \leq \frac{\alpha n}{t} \Rightarrow |N_{G''_X}(Q)| < \frac{t|Q|}{2})\big). \tag{4.5}$$

Let $X = \emptyset$. From the above assumption, $\exists Q_\emptyset \subseteq D_\emptyset : |Q_\emptyset| \leq \frac{\alpha n}{t} \wedge |N_{G''_\emptyset}(Q_\emptyset)| < \frac{t|Q_\emptyset|}{2}$. Let $Q_\emptyset$ be such a maximal set. Since $G' = G''_\emptyset$, we can apply Lemma 4.18 to $G''_\emptyset$ and deduce that $|Q_\emptyset| < \frac{2|B|}{t}$ (otherwise, $N_{G''_\emptyset} \geq \frac{t|Q_\emptyset|}{2}$).

Since (4.5) is true for every subset of C, of size less than $\frac{2|B|}{t}$, it will also be true for $Q_\emptyset$. For the rest of this proof, we will denote the set $Q_\emptyset$ by $\bar{X}$. Due to the choice of $\bar{X}$, and because G is (t,α)-expanding we conclude respectively that

$$|N_{G'}(\bar{X})| = |N_{G''_\emptyset}(\bar{X})| \quad < \quad \frac{t|\bar{X}|}{2} \tag{4.6}$$

$$|N_G(\bar{X})| \quad \geq \quad t|\bar{X}| \tag{4.7}$$

Let $Q_{\bar{X}} \subseteq D_{\bar{X}}$ be a vertex set, such that $|Q_{\bar{X}}| \leq \frac{\alpha n}{t}$ and

$$|N_{G''_{\bar{X}}}(Q_{\bar{X}})| < \frac{t|Q_{\bar{X}}|}{2} \tag{4.8}$$

We now observe that $Q_{\bar{X}} \neq \emptyset$, in order for inequality (4.8) to hold, and that $\bar{X}, Q_{\bar{X}}$ are disjoint subsets of C. Thus, one of the following cases should be true:

(i) $|\bar{X} \cup Q_{\bar{X}}| < \frac{2|B|}{t}$
This case, since $Q_{\bar{X}} \neq \emptyset$, contradicts the choice of $\bar{X}$ as a maximal set with the required properties.

(ii) $\frac{2|B|}{t} \leq |\bar{X} \cup Q_{\bar{X}}| \leq \frac{\alpha n}{t}$
In this case we have that

$$\begin{aligned} |N_{G''_{\bar{X}}}(Q_{\bar{X}})| + |N_{G'}(\bar{X})| &\geq |N_{G'}(\bar{X} \cup Q_{\bar{X}})| \\ &\geq \frac{t(|\bar{X}| + |Q_{\bar{X}}|)}{2} \\ &\geq |N_{G'}(\bar{X})| + \frac{t|Q_{\bar{X}}|}{2} \end{aligned}$$

The second inequality is a result of Lemma 4.18, while the third, of the inequality (4.7). However the fact that $|N_{G''_{\bar{X}}}(Q_{\bar{X}})| \geq \frac{t|Q_{\bar{X}}|}{2}$ contradicts the inequality (4.8).

(iii) $|\bar{X} \cup Q_{\bar{X}}| > \frac{\alpha n}{t}$.
In this case, since $|\bar{X}| \leq \frac{2|B|}{t} - 1$, we have that

$$\frac{\alpha n - 2|B|}{t} < |Q_{\bar{X}}| \leq \frac{\alpha n}{t}$$

and because G is (t,α)-expanding, $|N_G(Q_{\bar{X}})| \geq t|Q_{\bar{X}}| > \alpha n - 2|B|$. Moreover because $|\bar{X}| \leq \frac{2|B|}{t} - 1$ we know that $N_{G[B]}(Q_{\bar{X}}) \leq |B|$ and $N_{G[\bar{X}]}(Q_{\bar{X}}) \leq \frac{2|B|}{t} - 1$. Observe now that $N_{G''_{\bar{X}}}(Q_{\bar{X}}) = N_G(Q_{\bar{X}}) - N_{G[B]}(Q_{\bar{X}}) - N_{G[\bar{X}]}(Q_{\bar{X}})$ and $|B| \leq 0.06an$ so

$$\begin{aligned} N_{G''_{\bar{X}}}(Q_{\bar{X}}) &\geq \alpha n - 2|B| + 1 - |B| - \frac{2|B|}{t} + 1 \geq \alpha n - 0.18\alpha n - \frac{0.12\alpha n}{t} + 2 \\ &= 0.82\alpha n - \frac{0.12}{t} + 2 \geq 0.82\alpha n \geq \frac{\alpha n}{2} \geq \frac{t|Q_{\bar{X}}|}{2} \end{aligned}$$

which contradicts inequality (4.8)

Since all three possible cases resulted to a contradiction, $\bar{X}$ is a vertex set of size at most $\frac{2|B|}{t}$ that does not satisfy the relationship (4.5) so we obtain the desired contradiction and the proof of this lemma is now complete. □

Lemma 4.20. *Let G be a (t,α)-expanding graph of order n, and $t \geq 10$. Let also, $B \subseteq V(G)$ be a vertex set of G, of size at most $0.06\alpha n$, and denote by C the vertex set $V(G) \setminus B$. Then there exists a vertex set $X \subseteq C$, such that every connected component of $G'' = G[C \setminus X]$ has size at least $\frac{\alpha n}{2}$.*

Proof. We will show that the vertex sets X that satisfy Lemma 4.19 also satisfy this lemma. Let X be such a vertex set and denote by D the set $C \setminus X$ and by G'' the induced subgraph $G[D]$. Let also, v be an arbitrary vertex of G'', and V_1 its connected component. Denote by Y_i the set of vertices in V_1, in distance at most i from v. Obviously $Y_i \subseteq Y_{i+1}$. Also from Lemma 4.19 we have that $|Y_1| = |N_{G''}(v)| \geq \frac{t}{2}$. We will now repeat the following procedure until we find an index j for which $|Y_j| \geq \frac{\alpha n}{2}$.

1. We set $Z = Y_1$.
2. If $|Z| \leq \frac{\alpha n}{t}$ we notice that from Lemma 4.19, and because $t \geq 10$

$$|Y_{j+1}| \geq \frac{t|Z|}{2} > |Y_j|$$

   and we repeat this step for $Z = Y_{j+1}$.
3. If $|Z| \geq \frac{\alpha n}{t}$, we terminate the procedure.

This procedure will terminate since each time we repeat step 2 we increase the size of Z. Moreover, once we reach step 3, from Proposition 2.1, we can repeatedly remove as many vertices as needed from $G[Z]$ in order to obtain a connected subgraph of V_1, H, of size exactly $\frac{\alpha n}{t}$. Now, from Lemma 4.19, we have that

$$|N(H)| \geq \frac{t|H|}{2} \Rightarrow |N(H)| \geq \frac{\alpha n}{2}$$

Since v is arbitrary, and because we will eventually reach step 2, every component of G'' has size at least $\frac{\alpha n}{2}$. □

Now we are ready to use these lemmata to prove the following theorem:

Theorem 4.21. *Let G be a (t,α)-expanding graph of order n and let $t \geq 10$. Then G contains a minor with average degree at least*

$$c\alpha^3 \frac{\sqrt{nt\log t}}{\sqrt{\log n}}$$

where $c > 0$ is some absolute constant independent of α.

Proof. Let

$$p = \frac{\alpha^2}{100}\frac{\sqrt{nt\log t}}{\sqrt{\log n}}, \quad q = \frac{6}{\alpha}\frac{\sqrt{n\log n}}{\sqrt{t\log t}}$$

Consider being in the $k+1$ iteration, hence we already have constructed the k subsets of $V(G)$ with the properties that are described above. Denote

$$B = \bigcup_{i=1}^{k} B_i \text{ and } b = |B|$$

Since k is at most p, $b = kq \leq pq = 0.06\alpha n$. We will denote $C = V(G) \setminus B$ and by G' the induced graph $G[C]$. Hence, we can apply Lemma 4.18, to G, B, and have as a result that $\forall X \subseteq C : \frac{2b}{t} \leq |X| \leq \frac{an}{t} \Rightarrow |N_{G'}(X)| \geq \frac{t|X|}{2}$.

Now, let X be a subset of C that satisfies Lemma 4.19, which we have proved that also satisfies Lemma 4.20. For the rest of this proof we will denote $D = C \setminus X$ and by G'' the induced subgraph $G[D]$. We will also denote the connected components of G'' by $G_1, \ldots, G_l$, where $l \leq \frac{2}{\alpha}$ due to Lemma 4.20.

Let G_k be a connected component of G'', and $Y \subseteq V(G_k) : |Y| \leq \frac{\alpha|V(G_k)|}{t}$. Obviously $|Y| \leq \frac{\alpha n}{t}$, since $V(G_k) \leq n$, so from Lemma 4.19 we have that $|N_{G''}(Y)| = |N_{G_k}(Y)| \geq \frac{t|Y|}{2}$. Since $\frac{\alpha}{2} \leq 1$ and G_k is arbitrary, each connected component of G'' is $(\frac{t}{2}, \frac{\alpha}{2})$-expanding. From Lemma 2.20 we have that

$$\mathbf{diam}(G_i) \leq \frac{7\log n}{\alpha \log t}, 1 \leq i \leq l \tag{4.9}$$

We will now proceed into proving the second stage of this proof.

Claim i. *There exists an index i, such that there are at least $r = \frac{k}{2l}$ sets B_j, each having at least $\frac{t|B_j|}{2l}$ neighbors in G_i.*

Proof of Claim i. Suppose that for every G_i there exist at most $r-1$ sets B_j, which have at least $\frac{t|B_j|}{2l}$ neighbors in it. Thus, there exist $k - l\frac{k}{2l} = \frac{k}{2}$ sets B_j, each having at most $l\frac{t|B_j|}{2l} = \frac{tq}{2}$ neighbors in $G'' = \bigcup_{i=1}^{l} G_i$ (since each has at most $\frac{tq}{2l}$ neighbors in each G_i). We know that each B_j is of size q, and that they are pairwise disjoint, so the union of these $\frac{k}{2}$ sets, B', has size $\frac{kq}{2} = \frac{b}{2}$. Now, considering the size of B' we will examine the following cases:

(i) If $|B'| \leq \frac{\alpha n}{t}$, since G is a (t, α)-expanding graph

$$|N_G(B')| \geq t|B'| = \frac{tb}{2} \tag{4.10}$$

Moreover, due to the above assumption, B' has at most $\frac{k}{2}\frac{tq}{2} = \frac{tb}{4}$ neighbors in G''. So the rest $\frac{tb}{4}$ neighbors of B' in G must be in $X \cup B$. However,

$$|X \cup B| = |X| + |B| \leq \frac{2b}{t} - 1 + b < \frac{2b}{t} + b = \frac{b(2+t)}{t}$$

We also observe that for $t \geq 10$, $\frac{b(2+t)}{t} < \frac{tb}{4}$, because $(t-2)^2 - 12 > 0 \Rightarrow t^2 - 4t - 8 > 0 \Rightarrow \frac{2+t}{t} < \frac{t}{4}$ and $b > 0$. So B' can not have $\frac{tb}{4}$ neighbors in $X \cup B$, which contradicts inequality (4.10).

(ii) If $|B'| \geq \frac{\alpha n}{t}$, we can select $\lfloor \frac{\alpha n}{tq} \rfloor$ subsets of B', B_j, whose union has size at most $\frac{\alpha n}{t}$ (this is possible since $\frac{\alpha n}{tq} \leq \frac{k}{2}$ in this case). Because G is (t, α)-expanding, it has at least $\frac{tb}{2}$ neighbors in G, which lead us to a contradiction, using the same argument as in the first case.

□

Due to Claim i, without loss of generality, we suppose that each of $B_1, \dots, B_r$ has at least $\frac{tq}{2l}$ neighbors inside G_1. Denote these sets of neighbors by $U_1, \dots, U_r$, respectively.
Pick uniformly at random with repetition $\frac{|G_1|}{\frac{tq}{2l}}$ vertices of G_1 and denote this set by W. Let also, Ω be the set of all the possible sets W after the end of the selection that is described above and define the random variables

$$\begin{aligned} A(W) &:= \#\{i \in [r] : \quad U_i \cap W \neq \emptyset\} \\ A_i(W) &:= \begin{cases} 0, \text{ if } W \cap U_i = \emptyset \\ 1, \text{ if } W \cap U_i \neq \emptyset \end{cases} \end{aligned}$$

Claim ii. $\mathbb{E}[A] > \frac{r}{2}$

Proof of Claim ii. First, we will prove that

$$\mathbb{P}[W \cap U_i = \emptyset] \leq \left(1 - \frac{|U_i|}{|G_1|}\right)^{|W|} \leq \frac{1}{e}.$$

Let $v \in G_1$ be a vertex uniformly at random selected. Notice that the probability that $v \notin U_i$ is equal to $\frac{|G_1| - |U_i|}{|G_1|} = 1 - \frac{|U_i|}{G_1}$, so for every U_i , $\mathbb{P}[W \cap U_i = \emptyset] = \left(1 - \frac{|U_i|}{|G_1|}\right)^{|W|}$. Moreover, since $|U_i| \geq \frac{tq}{2l}$ and $|W| = \frac{|G_1|}{\frac{tq}{2l}}$ we have that

$$(1 - \frac{|U_i|}{|G_1|})^{|W|} \leq (1 - \frac{\frac{tq}{2l}}{|G_1|})^{|W|} = (1 - \frac{1}{|W|})^{|W|} \leq \frac{1}{e}$$

We also know that $\mathbb{E}[A] = \mathbb{E}\left[\sum_{i=1}^{r} A_i\right] = \sum_{i=1}^{r} \mathbb{E}[A_i]$ and

$$\mathbb{E}[A_i] = \sum_{W \subseteq G_1} A_i(W)\mathbb{P}[W] = \sum_{\substack{W \subseteq V(G_1) \\ W \cap U_i \neq \emptyset}} \mathbb{P}[W] = \mathbb{P}[W \cap U_i \neq \emptyset] \geq 1 - \frac{1}{e}$$

so, $\mathbb{E}[A] \geq r\left(1 - \frac{1}{e}\right) > \frac{r}{2}$. □

Due to Claim ii there is a set W, such that

$$\sum_{\substack{U_i \cap W \neq \emptyset \\ i=1}}^{r} 1 \geq \frac{r}{2} \geq \frac{k}{4l} \geq \frac{\alpha k}{8} \tag{4.11}$$

with the last inequality being true because $l \leq \frac{2}{\alpha}$. Let w_0 be a vertex of W and $\mathcal{P}_{w_0}$ be a collection of paths that realize the distance from w_0 to any other vertex of W. We now have

$$\begin{aligned} |V(\mathcal{P}_{w_0})| &\leq \frac{7|W|\log n}{a \log t} = \frac{14l|G_1|\log n}{tqa \log t} \leq \frac{7ln \log n}{tq \log t} \\ &= \frac{7\alpha l\sqrt{n \log n}}{6\sqrt{t \log t}} \leq \frac{14\sqrt{n \log n}}{6\sqrt{t \log t}} < q \end{aligned}$$

The above inequalities hold due to equation (4.9), $|W| = \frac{2l|G_1|}{tq}$, $|G_1| \geq \frac{\alpha n}{2}$, $q = \frac{6}{\alpha}\frac{\sqrt{n \log n}}{\sqrt{t \log t}}$ and $l \leq \frac{2}{\alpha}$ respectively. Obviously the induced graph $G[\mathcal{P}_{w_0}]$ is connected. By adding in $\mathcal{P}_{w_0}$ as much vertices with their corresponding edges, as needed, we can obtain a connected subgraph of G_1, of size q, which contains W. We denote this set by B_{k+1} and due to equation (4.11) we have that this set is connected with at least $\frac{\alpha k}{8}$ sets B_i, $1 \leq i \leq k$.

After p repetitions of this procedure, and by the contraction of the sets $B_1, \ldots, B_p$ we obtain the vertices $b_1, \ldots, b_p$ respectively, each of a degree at least $\frac{\alpha(i-1)}{8}$. We denote the induced graph of these vertices by G_m and we notice that the average degree of this graph is

$$\begin{aligned} d(G_m) &= \frac{\sum_{i=1}^{p} deg(b_i)}{p} \geq \frac{\sum_{i=1}^{p} \frac{2\alpha(i-1)}{8}}{p} = \frac{\alpha}{4p} \sum_{i=0}^{p} i \\ &= \frac{\alpha}{4p} \frac{p(p-1)}{2} = \frac{\alpha(p-1)}{8} \\ &= \frac{\alpha}{8} \left(\frac{\alpha^2}{100} \frac{\sqrt{nt \log t}}{\sqrt{\log n}} - 1 \right) \end{aligned}$$

which completes the proof of Theorem 4.21. $\square$

Corollary 4.22. *Let G be a (t, α)-expanding graph of order n, and let $t \geq 10$. Then G has a clique of size*

$$c\alpha^3 \frac{\sqrt{nt \log t}}{\log n}$$

as a minor, where c is an absolute constant, independent of α.

Proof. Let G be a (t, α)-expanding graph of order n, and also let $t \geq 10$. Due to Theorems 4.21 and 2.13 we deduce that G has a clique of size

$$\Omega \left(\frac{c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}}}{\sqrt{\log c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}}}} \right)$$

as a minor. Notice that $t \leq n$, $\alpha \leq 1$ and $c \leq 1$ so

$$\begin{aligned} c_1 \left(\frac{c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}}}{\sqrt{\log c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n}}}} \right) &\geq c_1 \left(c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n} \sqrt{\log c \frac{\sqrt{n^2 \log n}}{\sqrt{\log n}}}} \right) \\ &\geq c_1 \left(c\alpha^3 \frac{\sqrt{nt \log t}}{\sqrt{\log n} \sqrt{\log n}} \right) \\ &\geq c_1 \left(c\alpha^3 \frac{\sqrt{nt \log t}}{\log n} \right) \end{aligned}$$

$\square$

Lemma 4.23. *Let G be a (t, α)-expanding graph of order n and $A \subseteq V(G)$ a subset of size at most $\frac{\alpha n}{8}$. Then $G' = G[V(G) \setminus A]$ has a connected component of size at least $\frac{\alpha n}{4}$*

Proof. Let G' be a graph as described above and suppose that all of its connected components have size at most $\frac{\alpha n}{4}$. Then, by taking the union of some of those components we obtain a subset of $V(G)$, A' such that $\frac{\alpha n}{4} \leq |A'| \leq \frac{\alpha n}{2}$. Since A' is a union of connected components of G', $N(A') \cap G' = \emptyset$, so the neighbors of A' is only inside the set A. However from Proposition 2.19 we have that $N_G(A') \geq \frac{\alpha n}{2}$, which contradicts to the size of A. $\square$

Proposition 4.24. *Let G be a (t, α)-expanding graph of order n and $t \geq 10$. Then G contains a clique minor of size*

$$\Omega \left(\alpha^2 \sqrt{\frac{n \log t}{\log n}} \right)$$

Proof. Let G be a (t,α)-expanding graph of order n, and

$$p = \frac{\alpha}{100}\sqrt{\frac{n\log t}{\log n}}, \quad q = \sqrt{\frac{n\log n}{\log t}}$$

We will now construct p pairwise disjoint subsets of $V(G)$, $B_1,\dots,B_p$ each of size q, such that, the induced subgraphs $G[B_i]$ are connected, doing the following procedure:

1. Choose B_1 to be an arbitrary subset of $V(G)$ such that the induced subgraph $G[B_1]$ is connected and $|B_1| = q$.

2. In order to choose the sets B_i, $2 \le i \le p$, we denote $A = \bigcup_{l=1}^{i-1} B_l$. Since $pq = \frac{\alpha n}{100}$, we have $|A| \le \frac{\alpha n}{8}$. Due to Lemma 4.23, there is a connected component of $G[V(G)\setminus A]$, G_j, of size at least $\frac{\alpha n}{4}$. Now, due to Proposition 2.1, we can choose B_i to be the set of vertices of a connected subgraph of G_j, such that $|B_i| = q$.

Denote $B' = \bigcup_{i=1}^{p} B_i$ and let B'' be a subset of $V(G)$ of size at most $\frac{|B'|}{10}$, such that $B' \cap B'' = \emptyset$. Also denote $B = B' \cup B''$.
Using the same arguments as in the proof of Theorem 4.21 and the fact that $t \ge 10$, we can prove that

$$\exists X \subseteq V(G)\setminus B, |X| \le \frac{5|B|}{t} \le \frac{|B|}{2}:$$

- The graph $G' = G[V(G)\setminus (X\cup B)]$ is $(\frac{t}{2},\alpha)$-expanding, with at most $l = \frac{2}{\alpha}$ connected components $G_1,\dots,G_l$, each of diameter at most $\frac{7\log n}{\alpha\log t}$.

- There exists a connected component G_i, $1 \le i \le l$ such that at least $\frac{p}{2l} \ge \frac{\alpha p}{4}$ sets B_j have neighbors in it.

We will now select B'' in a way that it contains vertices of pairwise disjoint paths (that are not in B'), each of them connecting a different pair of the sets B_j,$1 \le j \le p$ we have already constructed. We will prove that, through this selection we can find $\frac{\alpha p}{4}$ sets B_j that are pairwise connected by disjoint paths. Notice that, if two sets B_i, B_j are connected through a path P, then $|P| \le \frac{7\log n}{\alpha\log t}+2$, since they should either be connected through one edge, or through a connected component G_i, which, as we mentioned above, has diameter at most $\frac{7\log n}{\alpha\log t}$. Also

$$\begin{aligned}\binom{p}{2}\frac{7\log n}{\alpha\log t} &\le \frac{p^2}{2} = \binom{p}{2}\frac{7\log n}{\alpha\log t} = \frac{\alpha^2}{10^4}\frac{n\log t}{\log n}\frac{7\log n}{\alpha\log t} \\ &= \frac{7\alpha n}{10^4} \le \frac{\alpha n}{10^3} = \frac{pq}{10} = \frac{|B'|}{10}\end{aligned} \tag{4.12}$$

so even if all sets in B' were pairwise connected, the size of B'' would still be at most $\frac{|B'|}{10}$ (because the endpoints of each path will obviously be in B').
Consider now the following iterative procedure, which, in each iteration, adds in B'', the vertices of a path that connects two sets on B' who were not connected before:

1. Let G' be the graph that is constructed, using B, as described above. Then there exists a connected component G_i which has neighbors in at least $\frac{\alpha p}{4}$ of the sets in B'.

2. If those $\frac{\alpha p}{4}$ sets are already pairwise connected, stop this procedure.

3. Else, there exists a pair B_l, B_k of them that is not connected through a path in B''. Since both have neighbors in G_i, and G_i is connected, we can find a path P, through G_i that connects them. Select this path to be a minimal one, so it has at most $\frac{7\log n}{\alpha\log t}$ vertices in G_i. Add these vertices to B'' and repeat step 1.

The above procedure will be repeated at most $\binom{p}{2}$ times, so due to equation (4.12) there will always be a component G_i with the required properties in G', in order to add a path in B''.
After the end of this procedure, we will have $\frac{\alpha p}{4}$ sets B_j that are pairwise connected through pairwise disjoint paths. The contraction of these sets and of the corresponding paths results in a clique of size $\frac{\alpha p}{4} = \frac{1}{400}\alpha^2\sqrt{\frac{n \log t}{\log n}}$, which completes our proof. $\square$

One other well-known result in this area, due to Kawarbayashi and Reed [KR10b], shows that every α-expander G with n vertices and maximum vertex degree bounded by d contains a clique with $\Omega(\alpha\frac{\sqrt{n}}{d})$ vertices as a minor. Recently, Krivelevich and Nenadov [KN18] improved the dependence on the expansion α and the maximum vertex degree d under a somewhat stronger definition of expansion.

CHAPTER 5

CONCLUSION

In this thesis we proved in detail some basic theorems on balanced separators, and how they are connected to expanding graphs. Moreover, we provided a brief introduction on spectral graph theory, that is, how the eigenvalues of a graph are connected to its expansion. We also stated the Expander mixing lemma and a correlation between the algebraic and the combinatorial definition of expansion. As far as the substructures in expanding graphs is concerned we studied in detail the minors one can find in them.

Finding large minors in expanders has been studied by several researchers, with the most recent result being that of Chuzhoy and Nimavat [CN19], who proved the following, which also achieves a tight dependence on n: There exists a universal constant c, such that every α-expander, G, of order n and maximum degree at most d, contains every graph with at most $\frac{n}{c \log n} \cdot \left(\frac{\alpha}{d}\right)^c$ vertices and edges as a minor. They also provided a randomized algorithm with time $\mathrm{poly}(n, d/\alpha)$ that realizes this theorem.

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