HELLENIC REPUBLIC
National and Kapodistrian University of Athens
EST. 1837

Department of Mathematics

Topics of the Calculus of Variations

Vasilis Athanasakos

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics

February 2020

Supervisor: Professor Ioannis Stratis

Jury-Committee:

- Professor Ioannis Stratis
- Professor Gerassimos Barbatis
- Assistant Professor Panagiotis Gianniotis

Dedicated to Katerina, the joy of my life!

Contents

Περίληψη

Η εργασία πραγματεύεται θέματα από την περιοχή του Λογισμού Μεταβολών. Συγκεκριμένα αρχίζουμε την παρουσίασή μας με την εξαγωγή της βασικότατης εξίσωσης Euler-Lagrange, η οποία διαδραματίζει κυρίαρχο ρόλο σε όλη την εργασία, ως μια αναγκαία συνθήκη για ακρότατα. Θα συνεχίσουμε με γενικεύσεις και μια πλούσια παράθεση πολλών εφαρμογών και παραδειγμάτων, όπως χαρακτηριστικά αναφέρουμε το πρόβλημα του βραχυστοχρόνου, αυτό της ελάχιστης επιφάνειας, την αρχή του Dirichlet, καθώς επίσης και εφαρμογές στη κλασσική μηχανική, στην οπτική, στην οικονομία κλπ. Ολοκληρώνουμε το πρώτο κεφάλαιο με μια αναφορά στη δεύτερη μεταβολή, με κάποια σχόλια σχετικά με βασικά αποτελέσματα συμπάγειας, πολύ χρήσιμα στο τι θα επακολουθήσει, καθώς επίσης και μια σύντομη αναφορά σε υπολογιστικές μεθόδους, όπως τις μεθόδους των Ritz και Galerkin, οι οποίες μας βοηθούν συστηματικά στο να υλοποιούμε τις πρακτικές υπολογιστικές πτυχές των μεθόδων βελτιστοποίησης.

Στη συνέχεια θα ασχοληθούμε με το θεωρητικό πρόβλημα της ύπαρξης και της μοναδικότητας των ελαχιστοποιητών και υπό ποιες συνθήκες αυτά τα δυο ερωτήματα επιδέχονται καταφατική απάντηση, καθώς επίσης και με τις ασθενείς λύσεις της εξίσωσης E-L. Ύστερα θα συζητήσουμε θέματα σχετικά με την ομαλότητα τέτοιων λύσεων και αν μπορούμε να απαιτήσουμε περαιτέρω ομαλότητα. Ένα άκρως σημαντικό και ενδιαφέρον πλην δύσκολο ερώτημα.

Αφιερώνουμε το $3^{ο}$ κεφάλαιο στα προβλήματα που υπόκεινται σε περιορισμούς. Προς τούτο θα καταφύγουμε στη χρήση των Πολλαπλασιαστών Lagrange για να εξάγουμε μια ‘’βελτιωμένη’’ E-L, λαμβάνοντας υπόψη και τον περιορισμό και προσαρμόζοντας την ανάλυσή μας αναλόγως. Θα επικεντρωθούμε κυρίως σε ολοκληρωτικούς περιορισμούς, δίχως ωστόσο να παραλείψουμε την περίπτωση των αλγεβρικών. Για ακόμη μια φορά η παρουσίασή μας θα ακολουθήσει μια εύληπτη και διάφανη ροή, διανθισμένη με πολλές εφαρμογές και παραδείγματα. Επιλέγουμε να κλείσουμε το κεφάλαιο αυτό με τα λεγόμενα προβλήματα ελευθέρου συνόρου, όπου εδώ ελλείψει των συνοριακών τιμών, θα παρουσιάσουμε με ποιον τρόπο θα επιβάλουμε εμείς τις συνοριακές συνθήκες.

Στο $4^{ο}$ κεφάλαιο ασχολούμαστε με τη Χαμιλτονιανή θεώρηση και θα την αντιπαραβάλουμε με τη Νευτώνεια θεώρηση, εντοπίζοντας παράλληλα τα κοινά σημεία , αλλά και τα σημεία όπου οι δυο θεωρίες διαφοροποιούνται, παραθέτοντας τα υπέρ και κατά. Εν συνεχεία θα χρησιμοποιήσουμε την αρχή του Χάμιλτον για να

καταλήξουμε στη διατύπωση των κανονικών εξισώσεων, γνωστών και ως ''κανονικός φορμαλισμός''. Αυτές οι εξισώσεις είναι κεντρικής σημασίας στη Φυσική και θα επιδιώξουμε, μέσω μιας εκτενούς παράθεσης πολλών αναλυτικών παραδειγμάτων να αναδείξουμε τον κεντρικό αυτό ρόλο. Η ανάλυσή μας συμπληρώνεται με κάποια ενδιαφέροντα ιστορικά στοιχεία, όπως η αρχή ελάχιστης δράσης του Maupertuis, που αποσκοπούν στην καλύτερη συνοχή και συνεκτικότητα των παρουσιαζόμενων ιδεών. Κλείνουμε το κεφάλαιο με μια εκτενή αναφορά στα αντίστροφα προβλήματα, η σημασία των οποίων συνοψίζεται στο ερώτημα: ''αν μας δοθεί μια διαφορική εξίσωση, με ποιο τρόπο μπορούμε να προσδιορίσουμε τη Λαγκρανζιανή συνάρτηση;''.

Το $5^{ο}$ κεφάλαιο είναι αφιερωμένο στα κρίσιμα σημεία. Εξετάζουμε εδώ συναρτήσεις που ναι μεν καθιστούν κάποια δοσμένα συναρτησιακά στάσιμα, δίχως όμως να είναι ακρότατα αυτών. Εν συνεχεία παραθέτουμε ένα θεωρητικό αποτέλεσμα, το Θεώρημα Παραμορφώσεων, το οποίο θα χρησιμοποιήσουμε για να αποδείξουμε το Θεώρημα Ορεινής Διάβασης, το οποίο ουσιαστικά είναι μια αρχή minimax. Έπειτα μελετούμε την εφαρμογή αυτού του σημαντικού θεωρήματος στην απόδειξη ύπαρξης ασθενούς λύσης μιας ημιγραμμικής (quasilinear) ελλειπτικής μερικής διαφορικής εξίσωσης τύπου Poisson και κλείνουμε το κεφάλαιο αυτό με την εξαγωγή της ταυτότητας των Derrick-Pohozaev.

Τέλος στο $6^{ο}$ και τελευταίο κεφάλαιο ασχολούμαστε με τα αναλλοίωτα και το βασικότατο Θεώρημα της Noether, ένα κομμάτι μείζονος σημασίας στη Φυσική και ιδιαίτερα δε στην Μηχανική, απόλυτα συνυφασμένο με τις συμμετρίες των συστημάτων και τους νόμους διατήρησης. Για την ακρίβεια μελετούμε πως επηρεάζουν μετασχηματισμοί χωρίων και συναρτήσεων το πρόβλημά μας, μέσω της επίδρασής τους στη διαφορική εξίσωση Euler-Lagrange και καταλήγουμε σε πολλά χρήσιμα συμπεράσματα φυσικού περιεχομένου, μέσω εκτενών παραδειγμάτων και εφαρμογών.

Preface

Generally, the Calculus of Variations is concerned with the optimization (minimization or maximization) of variable quantities, called functional, over some admissible class of competing objects. Many of its methods, ideas and techniques were developed over 200 years ago by many great mathematicians like Euler, Lagrange, Bernoulli and many others. Its own development owes its existence to a constant and fruitful interaction between a strict mathematical theory and a continuous changing, and challenging as well physical consideration of the world. It continues to the present day to bring important techniques to many branches of physics, engineering, economics, optimal control theory, biological sciences, etc

We begin the presentation of this Master Thesis by referring to Euler-Lagrange equation and presenting how we can derive this equation. The techniques are based on the generalization of respective ideas from the calculus of real functions (perhaps in some cases slightly converted to fit with the basis where we shall develop our theory). This equation is of central significance throughout this thesis, since it provides us with a necessary condition for determining extrema. Of course we will accompany our presentation with plenty of examples which lead to a better/deeper comprehension of the ideas and with a wide variety of application as well, like the brachistochrone problem, the minimal surface problem, applications arising in economics and physics, like the plate equation or the equilibrium shape of a membrane overload, the Dirichlet's Principle and its connection to the minimal surface problem and many other interesting topics. We will close the first section by a short reference to the second variation, omitting however to further develop the related theory. Finally we will mention and analyze some results relevant to compactness. These will play a major role later in the presentation of the existence theory of minimizers. We also briefly present some computational methods like Ritz's and Galerkin's.

Next we devote a significant part of our work to the existence-uniqueness theory of minimizers for certain minimization problems and we investigate under which conditions the existence (and uniqueness) is possible. Here the material becomes a bit more ''advanced'' or abstract, since for our analysis we need to employ many techniques and results from Sobolev spaces. Moreover we shall deal with the weak solutions of the E-L equation, which could be regarded to be additional evidence which advocates in favor of using Sobolev spaces, being the suitable spaces for weak formulations. Afterwards we shall discuss the very important, but difficult too, topic of regularity and we will make some remarks regarding higher regularity. A reasonable question which may arise here is whether the theory is analogous to the one developed for linear, 2^{nd} order, elliptic pdes and under which conditions higher regularity can be achieved. Nonetheless we avoid further developing the theory as being beyond the scope of our thesis.

We continue by referring to minimization problems subject to constraints. This class of minimization problems is also known as isoperimetric problems. In this chapter, given the fact that we are dealing with constraints, we need to employ the Lagrange multiplier rule and to derive a slightly converted (or adjusted) E-L equation, by taking into consideration the constraints this time. We mostly dedicate our analysis to integral constraints which dominate the vast majority of our discussion, but we shall not neglect to refer to the algebraic constraints as well. Next we discuss topics related to unilateral constraints and variational inequalities. And eventually we choose to close this chapter by presenting the free boundaries. Once again, we accompany our presentation with lots of applications, like the shape of hanging rope, Schrödinger's equation from Quantum Mechanics, the classical isoperimetric problem from which the alternative name of this class of problems is derived and we close the chapter by referring to the Rayleigh Quotient and the eigenvalue minimization problem.

As mentioned earlier above, the theory that associates optimization with pdes is the calculus of variations. It can be used for both static and dynamic problems as well. Moreover the dynamical aspects of calculus of variations are based on Hamilton's principle and it is the central theme of our next chapter. We will derive Hamilton's canonical equations (what is known as Hamilton's Formalism) and we will apply Hamilton's principle for the wave propagation in elastic strings, membranes, vibrations of rods etc. We will also demonstrate how to handle Hamilton's formalism through many characteristic examples derived from classical mechanics. Next we will compare the Hamiltonian with the Newtonian approach and we shall present some supplementary material followed by interesting historical notes. We close the chapter with a discussion dedicated to the inverse problem, accompanied by many examples which illustrate a method of determining the Lagrangian from a given differential equation.

Regarding the two last sections now, one is devoted to the critical points of a functional. There by employing the results of the deformation theorem, we prove a very important theorem, the Mountain Pass Theorem, abbreviated as MPT, and we use it to prove the existence of a weak solution of a given, semi-linear Poisson b.v.p. In relation to the latter, we demonstrate extra material, the so called Derrick-Pohozaev identity.

Lastly, we have chosen to close this master thesis with an extensive report and analysis to Invariance and Noether's Theorem which plays a very crucial and centralized role in the conservation laws in Physics. This last part is enriched with lots of applications which highlight the great significance of Noether's theorem. We indicatively mention the scaling invariance, the monotonicity formulas, conservation of energy for non-linear wave equations and the conformal energy for the wave equations as well, accompanied by an application to local energy decay.

We have tried to cover an important range of topics relevant to the calculus of variations. This may have led to some partial repetition of the material (to some extent) in some parts of the text inevitably. We would like to apologize for this, although we recognize the benefits of the revision. On the other hand, there are some topics which unfortunately we omitted to present here, something we regret for. Nevertheless, given the restrictions in time (mostly), but in space (limitations in the thesis' extent) as well, something like that seems to be inevitable if someone does not wish to ''sacrifice'' the analyticity and the clarity of his presentation, without neglecting or omitting the crucial parts and making ''discounts''. Indicatively we mention some topics we wish to have covered like Legendre's and Jacobi's conditions regarding the 2^{nd} variation, the Hamilton-Jacobi equation and the Hopf-Lax resolving formula, the Legendre transform which highlights the duality of Lagrangian and Hamiltonian functions, the alternative derivation of Hamilton's canonical equations as the characteristics of the Hamilton-Jacobi equation as well as Pontryagin's Maximum Principle, which is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls.

As far for the notation, it is a nightmare. We have tried to simplify (if possible) the notation where it is necessary and we hope that we have achieved it at least a bit. In some cases, it was rather difficult given the wide range of different bibliography we have used.

Furthermore, although in some parts the presentation may seem a bit more technical at first sight, we have tried not to sacrifice the clarity and the simplicity of the ideas, focusing on the deep comprehension of the main results, and keep our approach very concrete, following a ''clear path''. Part of our philosophy is also to give sometimes a more practical-intuitive first approach which serves a more educational, say, character and shortly afterwards to profound to a deeper and more abstract level providing all the necessary details and ''machinery''. We hope that the work at hand serves faithfully this scope.

We wish and hope as well that each potential reader will find the current work interesting or at least a bit of ''something'' so as to turn his attention on it, at least for a while. Moreover, we would like to deeply thank the careful reader who may notice some errors in the text in advance. Errors are an inevitable part of human life. Needless to say that any suggestion targeting to improve this work is more than welcome!

Last but not least, I would like to thank all these people who significantly helped me during this journey till completing the work at hand. First I would like to deeply thank the *Alexandros Onassis Foundation* for the confidence and support to me all these two years of this Master and the very valuable scholarship it provided me with during the duration of my graduate studies. This made my entire life much easier these two years. I will always be grateful, recognizing at the same time this valuable source of funding my studies.

I would also like to thank my supervisor professor Ioannis Stratis for his help, his valuable remarks, suggestions and advices and all his guidance and supervision during this work at hand and for all the opportunities he has offered to me so far. Additionally I would also like to thank the other two members of my jury-committee, professor Gerassimos Barbatis and assistant professor Panagiotis Gianniotis for all the advice and suggestions targeting the improvement, when needed. Lastly a big thank to the secretary of the graduate studies in our department, Mrs Alkistis Ntai for all her help, her suggestions and her advice regarding bureaucratic and/or procedural issues and all her polite willingness to offer information relevant to my studies.

Next I would like to thank some close friends for their support all these years like Angelos Gikas, Giorgos Katopodis, Thodoris Giannopoulos and Giannis Arkoudis, four really good persons and good mathematicians I had the luck and the privilege meet and know during our postgraduate studies in the Department of Mathematics in the

National and Kapodistrian University of Athens.

As a young mathematician I owe a great thank to my teacher and mentor Vasilios Dougalis, who taught me, the most important of all according to my personal opinion in my mathematical life, how to think mathematically. I also feel a great and profound need to also thank him for his valuable advice, suggestions and remarks, as well as all these invaluable teaching moments during his lectures, moments of true inspiration, moments of true learning. This kind of moments which motivates you to get better, to try harder, this kind of moments which changes your life. As a person now, I owe another great thank to this great man, because he was present for me wherever and whenever I needed him. And one last thing. According to American author William Arthur Ward the mediocre teacher tells, the good teacher explains, the superior teacher demonstrates and the great teacher inspires!

Lastly among the persons I would like to thank and without them this work would never have come true, is actually a very important person in my life. My very best friend Nikos Housas, a devoted friend, a pleasant, encouraging and inspiring companion, willing to help and support in all occasions of my life. I owe him a great thank for all these inspiring and motivating discussions we have had and for his so much alive questioning spirit, full of peculiar curiosity and fruitful creativity.

Finally, the biggest thank of all is dedicated to the joy of my life, my wife Katerina for her support, her comprehension, her devotion, her faith in me, her patience with me, her insist on making me a better person, her tolerance, her encouragement, her enthusiasm and above all her love for me. Because she was, is and will be on my side in every challenge, in every difficulty, even in the darkest moments, believing in me and supporting me. Because she has offered the most precious and encouraging smiles to me, this kind of smiles which enriches you with optimism and hope for the future. This work would never have seen the light without Katerina, my heroine !

Athens
February 2020

1. The Euler-Lagrange Equation

1.1 The Euler-Lagrange Equation

Below we shall introduce the main ideas, methods and techniques in order to derive the Euler-Lagrange equation which plays a major and very central role in the calculus of variations and the optimization problems related to it. Our goal here is to make the reader familiar with the main concepts, to provide some further generalizations, accompanied by plenty of interesting examples and applications. As mentioned above, the core of this first section will be computational and our principal target will be to deal with the E-L equation and to use it extensively to applications.
To start with:

We consider the Lagrangian: $L = L(p,z,x) = L(p_1,...,p_n,z,x_1,...,x_n):\ \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$
(for $U \subset \mathbb{R}^n$ bounded, open with smooth ∂U.)
$p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U.$
p is the symbol of the variable for which we substitute $Dw(x)$ below, and z is the variable for which we substitute $w(x)$. We also set:

$D_p L = (L_{p_1},...,L_{p_n})$.

$D_z L = \frac{\partial L}{\partial z} = L_z$.

$D_x L = (L_{x_1},...L_{x_n})$..

Now for smooth functions $w : \bar{U} \to \mathbb{R}$ such that $u = g$ *on* ∂U, we define the functional

$$I[w] = \int_U L\left(x, \underbrace{w(x)}_{z}, \underbrace{Dw(x)}_{p} \right) dx.$$

If we suppose now that there is a smooth function u that is equal to g on the boundary which happens
to satisfy : $I[u] = \min_{w \in A} I[w]$, A is regarded to be a kind of functions' admissible set. Then, we will demonstrate that u is automatically the solution of a certain non-linear p.d.e , called E-L equation.

To confirm this statement, choose any smooth $u \in C_c^\infty(U)$ and consider the real-valued function $i(\tau) = I[u + \tau v]$, $\tau \in \mathbb{R}$.
Since u is a minimizer of $I[\cdot]$ and $u + \tau v = g + \tau \cdot 0 = g$ on ∂U, we notice that $i(\cdot)$ has a minimum at $\tau = 0$. Therefore by
Fermat's theorem $\Rightarrow$ $i'(0) = 0$. So we explicitly
compute this derivative (called the 1st variation).

$$i(\tau)=I[u+\tau v]=\int_U L(x,u+\tau v,Du+\tau Dv)dx \Rightarrow$$

$$i'(\tau)=\int_U\left[\sum_{i=1}^n\left(L_{p_i}\left(x,u+\tau v,Du+\tau Dv\right)v_{x_i}\right)+L_z\left(x,u+\tau v,Du+\tau Dv\right)v\right]dx.$$

Therefore we have for $\tau=0$

$$0=i'(0)=\int_U\left[\sum_{i=1}^n L_{p_i}\left(x,u,Du\right)v_{x_i}+L_z\left(x,u,Du\right)v\right]dx$$

and after an integration by parts

$$0=\int_U\left[-\sum_{i=1}^n\left(L_{p_i}\left(x,u,Du\right)\right)_{x_i}v+L_z\left(x,u,Du\right)v\right]dx+\oint_{\partial U}\sum_{i=1}^n L_{p_i}\left(x,u,Du\right)\cancel{v}^{\,0}n_i dS$$

since $v\in C_c^\infty(U),$ the last integral equals to zero.

$$0=\int_U\left\{-\sum_{i=1}^n\left(L_{p_i}\left(x,u,Du\right)\right)_{x_i}+L_z\left(x,u,Du\right)\right\}vdx\quad\forall v\in C_c^\infty\left(U\right)\Rightarrow$$

E-L equation associated to the energy functional $I[\cdot]$

$$\boxed{-\sum_{i=1}^n\left(L_{p_i}\left(x,u,Du\right)\right)_{x_i}+L_z\left(x,u,Du\right)=0\Leftrightarrow L_z-div\left(\nabla_p L\right)=0}$$

in U, wich is a non-linear PDE. Actually it is a quasilinear, 2nd order pde in divergence form.

In summary, any smooth minimizer of $I[\cdot]$ is a solution of E-L and thus conversly we can try to find a solution of E-L by searching for minimizers.

2.2 The Euler-Lagrange equation for systems

Systems:

Assume the smooth Lagrangian function $L:\mathrm{M}^{m\times n}\times\mathbb{R}^m\times\overline{U}\to\mathbb{R}$ where $\mathrm{M}^{m\times n}$ is the space of real $m\times n$ matrices. Hereafter we shall notate $L=L(P,z,x)=L(p_{11},p_{12},...p_{mn};z_1,...,z_m;x_1,...,x_n)$ for a matrix $P\in\mathrm{M}^{m\times n},\ z\in\mathbb{R},\ x\in U\subset\mathbb{R}^n,$ where $P=\begin{pmatrix}p_{11}&\cdots&p_{1n}\\ \vdots&\ddots&\vdots\\ p_{m1}&\cdots&p_{mn}\end{pmatrix}_{m\times n}$.

Consider now the functional

$I[w]=\int_U L\left(Dw(x),w(x),x\right)dx$ defined for smooth functions $w:\overline{U}\to\mathbb{R}^m$ with $w=(w_1,...,w_m)$

satisfying the boundary condition $w=g\ on\ \partial U$ for $g:\partial U\to\mathbb{R}^m$ being given. Here we denote

$Dw(x)=\begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \cdots & \frac{\partial w_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_m}{\partial x_1} & \cdots & \frac{\partial w_m}{\partial x_n} \end{pmatrix}_{m\times n}$ the gradient matrix of w at x. For simplicity we will write this matrix in a bit peculiar way as $Dw(x)=\begin{pmatrix} w^1_{x_1} & \cdots & w^1_{x_n} \\ \vdots & \ddots & \vdots \\ w^m_{x_1} & \cdots & w^m_{x_n} \end{pmatrix}_{m\times n}$. This will help us significantly with the calculations.

We will show now that any smooth minimizer $u=(u_1,...,u_m)$ of $I[\cdot]$, taken among functions equal to g on ∂U must solve a certain system of non linear pdes. We therefore fix a $v=(v_1,...,v_m)\in C_0^\infty(U;\mathbb{R}^m)$ and write as usual $i(\tau)=I[u+\tau v]$. As before we have that $i'(0)=0$, from which we may deduce (as above) the equality:

$0=i'(0)=\frac{d}{d\tau}I[u+\tau v]\Big|_{\tau=0}=\frac{d}{d\tau}\left[\int_U L\left(x,u+\tau v,Du+\tau Dv\right)dx\right]_{\tau=0}=\int_U\left\{\sum_{k=1}^m L_{z_k}(x,u,Du)v_k+\right.$ $\left.+\sum_{i=1}^n\sum_{k=1}^m L_{p_i^k}(x,u,Du)v^k_{x_i}\right\}dx,$ since $Du+\tau Dv=\left(u^k_{x_i}+\tau v^k_{x_i}\right)_{k,i}$. Now we notice that since this identity is valid for all choices of $v^1,v^2,...,v^m$, we conclude after integrating by parts that:

$0=\int_U\left\{\sum_{k=1}^m L_{z_k}(x,u,Du)v_k+\sum_{i=1}^n\sum_{k=1}^m L_{p_i^k}(x,u,Du)v^k_{x_i}\right\}dx=\int_U\left\{\sum_{k=1}^m L_{z_k}(x,u,Du)-\left(\sum_{i=1}^n\sum_{k=1}^m L_{p_i^k}(x,u,Du)\right)_{x_i}\right\}v_k dx$

$+\int_{\partial U}\cancelto{0}{\sum_{i=1}^n\sum_{k=1}^m L_{p_i^k}(x,u,Du)\,v_k\quad n_i dS}$, due to the compact support of v_k. Therefore, we obtain:

$\sum_{k=1}^m\left[L_{z_k}(x,u,Du)-\sum_{i=1}^n\left(L_{p_i^k}(x,u,Du)\right)_{x_i}\right]=0.$ Thus we have the system of E-L equations

$\boxed{L_{z_k}(x,u,Du)-\sum_{i=1}^n\left(L_{p_i^k}(x,u,Du)\right)_{x_i}}$ in U for $k=1,2,...,m$

1.3 The Null Lagrangian

Null Lagrangians:

Surprisingly, it turns out to be interesting to study certain systems of non linear pdes for which every smooth function is a solution. Before developing the idea of null Lagrangian let us first define what a null Lagrangian is:

Definition:

The function L is called a null Lagrangian if the system of E-L equations

$L_{z_k}(x,u,Du)-\sum_{i=1}^{n}\left(L_{p_i^k}(x,u,Du)\right)_{x_i}=0$ for $k=1,2,...,m$, in U is automatically solved by all smooth functions $u:U\to\mathbb{R}^m$.

The importance of null Lagrangians is that the corresponding energy: $I[w]=\int_U L(x,w,Dw)dx$ depends only on the boundary conditions. Specifically, we have the following result.

Theorem

Let L be a null Lagrangian. Assume $u,\tilde{u}$ are two functions $\mathrm{C}^2(\overline{U};\mathbb{R}^m)$ such that $u=\tilde{u}$ on ∂U. Then $I[u]=I[\tilde{u}]$.

Proof.

Define $i(\tau)=I[\tau u+(1-\tau)\tilde{u}]$, $0\le\tau\le1$. Then we have that

$i'(\tau)=\dfrac{d}{d\tau}I[\tau u+(1-\tau)\tilde{u}]=\dfrac{d}{d\tau}\int_U L\left(x\ ,\ \tau u+(1-\tau)\tilde{u}\ ,\ \tau Du+(1-\tau)D\tilde{u}\right)dx=$

$\int_U\left\{\sum_{k=1}^{m}L_{z_k}\left(x\ ,\ \tau u+(1-\tau)\tilde{u}\ ,\ \tau Du+(1-\tau)D\tilde{u}\right)\left(u_k-\tilde{u}_k\right)+\right.$

$\left.+\sum_{i=1}^{n}\sum_{k=1}^{m}L_{p_i^k}\left(x\ ,\ \tau u+(1-\tau)\tilde{u}\ ,\ \tau Du+(1-\tau)D\tilde{u}\right)\left(u_{x_i}^k-\tilde{u}_{x_i}^k\right)\right\}dx=$ after applying integration by parts and taking into consideration as well that $u=\tilde{u}$ on ∂U we obtain:

$=\sum_{k=1}^{m}\int_U\left\{L_{z_k}\left(x\ ,\ \tau u+(1-\tau)\tilde{u}\ ,\ \tau Du+(1-\tau)D\tilde{u}\right)-\right.$

$\left.-\sum_{i=1}^{n}\left(L_{p_i^k}\left(x\ ,\ \tau u+(1-\tau)\tilde{u}\ ,\ \tau Du+(1-\tau)D\tilde{u}\right)\right)_{x_i}\right\}\left(u_k-\tilde{u}_k\right)dx=0$. The last equality holds since the system of E-L equations is satisfied by the function $\tau u+(1-\tau)\tilde{u}$. Remember that L is assumed to be a null Lagrangian. In other words: $L_{z_k}(x,w,Dw)-\sum_{i=1}^{n}\left(L_{p_i^k}(x,w,Dw)\right)_{x_i}=0$ for $k=1,2,...,m$ in U, where $w=\tau u+(1-\tau)\tilde{u}$. Concluding we have shown that $i'(\tau)=0\ \Rightarrow\ i(\tau)=$ constant and therefore $i(\tau)=i(0)=i(1)$ where $i(0)=I[\tilde{u}]$ and $i(1)=I[u]$ which proves the required result □

1.4 Applications

Below we shall refer to some simplified expressions of the E-L equation, which we will use quite often later, when we will present a lot of applications. We will present the simplest E-L solving the simplest problem, some generalizations and what we call first integral which simplify significantly the E-L under certain cases.

First we will mention the simplest form of E-L. For this purpose we introduce the following definition

Simplified E-L, the simplest problem

Definition

Let $J : A \to \mathbb{R}$ be a functional on *A*, where $A \subseteq V$, and V a normed linear space. Let $y_0 \in A$ and $h \in V$ such that $y_0 + \varepsilon h \in A$ for all ε sufficiently small. Then the first variation (also called the *Gâteaux* derivative) of J *at* y_0 in the direction of h is defined by: $\boxed{\delta \mathrm{J}(y_0,h)=\frac{d}{d\varepsilon} J\left(y_0+\varepsilon h\right)\Big|_{\varepsilon=0}}$

provided the derivative exists. Such a direction h for which the derivative above exists is called an admissible variation at y_0.

Borrowing the idea from classic calculus, and to be more specific by Fermat's Theorem, we can proceed by introducing a necessary condition for minimizing a given functional which could be considered to be a direct analogue to the classic calculus. Therefore, we demand: $\boxed{\delta J(y_0,h)=\frac{d}{d\varepsilon} J(y_0+\varepsilon h)\Big|_{\varepsilon=0} = 0}$ for $y_0 \in \mathrm{A}$

and for all admissible variations *h*. This is a necessary minimizing condition.

Now we are ready to develope the principal and necessary for our needs theory of minimizing the functional $J[y]=\int_a^b L\left(x, y(x), y'(x)\right)dx$, where $y \in C^2[a,b]$, $y(a)=y_0$ and $y(b)=y_1$. L is a given, twice continuously differentiable function on $[a,\ b]\ \times\ \mathbb{R}^2$. Moreover, $h \in C_0^\infty[a,b]$ such that $h(a)=h(b)=0$ and the admissible set is considered to be the following: $\mathrm{A}=\left\{y \in C^\infty \,\middle|\, y(a)=y(b)=0\right\}$

Afterwards we will follow the usual procedure, by considering the first variation, which will lead us to derive the E-L equation.

$$J(y+\varepsilon h)=\int_a^b L\left(x, y+\varepsilon h, y'+\varepsilon h'\right)dx \ \Rightarrow\ \frac{d}{d\varepsilon}J(y+\varepsilon h)\Big|_{\varepsilon=0}=\int_a^b\left(L_y h+L_{y'}h'\right)dx=$$

by applying integration by parts, we obtain:

$$=\int_a^b\left(L_y-\frac{d}{dx}L_{y'}\right)h\ dx+\Big[\underbrace{L_{y'}\left(x,y,y'\right)\cancel{h(x)}}_{\text{0 due to the compact support}}\Big]_a^b \ \Rightarrow \int_a^b\left(L_y-\frac{d}{dx}L_{y'}\right)h(x)\ dx=0$$

$\forall h(x) \in C_0^\infty[a,b]$. Now we note that since this equality holds for every $h(x)$ with the properties mentioned above, we can conclude the E-L: $\boxed{\mathrm{L}_y-\frac{d}{dx}L_{y'}=0}$

Where in the last step we have used the following lemma:

$$\boxed{\int_a^b f(x)h(x)dx=0 \text{ with } f,h \text{ smooth and } h(a)=h(b)=0 \ \Rightarrow\ f(x)=0 \ in\ [a,b]}$$

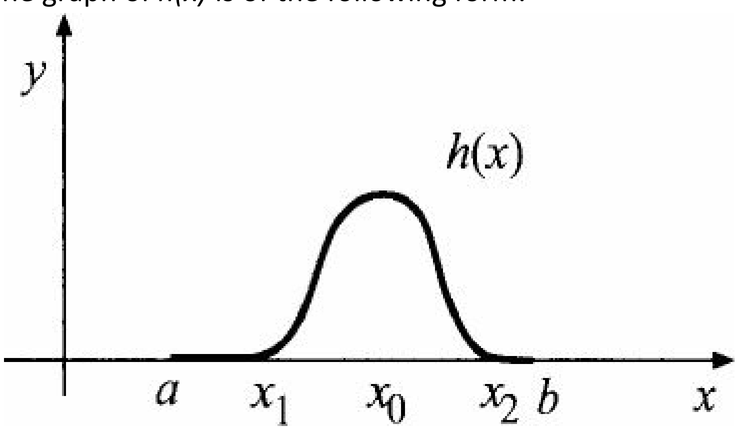
Hereafter we shall refer to this lemma as the fundamental lemma of calculus of variations. We will prove the general case of this lemma later in the chapter. But let us make a short remark for its proof. The important thing here is which $h(x)$ (satisfying the required property) to choose. If we choose it properly , we can use the method of contradiction to prove the lemma, by restricting f to a small interval, where without loss of generality, it will have been suppossed to be, say, positive. Then by considering, for example, the 6-spline,

which is $h(x)=\begin{cases}\left(x-x_1\right)^3\left(x_2-x\right)^3, & x \in [x_1,x_2] \subseteq [a,b] \\ 0, & \text{otherwise}\end{cases}$ we will be led to a contradiction.

We wish h(x) to be smooth at x_1 and x_2 so as to be C^2, with compact support as well. Then, it's straightforward to see that:

$0=\int_a^b f(x)h(x)dx = \int_{x_1}^{x_2} f(x)\left(x-x_1\right)^3\left(x_2-x\right)^3 dx > 0$ which is a contradiction if we suppose that f is not identically zero, but on the contrary, there exists a $x_0 \in (a,b)$ such that $f(x_0) \neq 0$. Why? Because then, by employing the continuity of f, we would be able to find an interval, say, $[x_1, x_2]$ where (wlg) f would be, say, positive.

The graph of *h(x)* is of the following form:



The E-L is a second order, non linear pde, provided that $\underline{L_{y'y'} \neq 0}$, because:

$$L_y - \frac{d}{dx}L_{y'}(x,y,y') = 0 \;\Rightarrow\; L_{y'x} + L_{y'y}y' + L_{y'y'}y'' - L_y = 0 \;\Rightarrow$$

$y''L_{y'y'} + y'L_{y'y} + L_{y'x} - L_y = 0 \qquad (L_{y'y'} \neq 0)$. Below we will discuss shortly about first integrals. In general, a first integral of a second order differential equation, say $F(x,y,y',y'') = 0$ is an expression of the form $g(x,y,y')$ involving only lower derivative, which is constant whenever y is the solution of the original equation $F(x,y,y',y'') = 0$. Hence $g(x,y,y') = c$ represents an integration of the second order equation. In Mechanics first integrals are called conservation laws.

One of the reasons for which we widely and quite often use first integrals is that they simplify significantly the E-L equation which otherwise would be quite complicated to be solved. They depend on the form of the Lagrangian, therefore we separate the following cases:

- $L = L(x, y') \Rightarrow L_{y'} = const.$
- $L = L(x, y) \Rightarrow L_y - \dfrac{d}{dx}\cancel{L_{y'}}^{0} = 0 \Rightarrow L_y = 0$
- $\boxed{L = L(y, y') \Rightarrow L - y'L_{y'} = const.}$ The proof is in fact quite simple:

$$\frac{d}{dx}\left(L - y'L_{y'}\right) = 0 \Rightarrow \cancel{L_x}^{0} + y'L_y + \cancel{y''L_{y'} - y''L_{y'}} - y'\left\{y'L_{y'y} + y''L_{y'y'}\right\} = 0 \Rightarrow$$

$$y'\left\{L_y - y'L_{y'y} - y''L_{y'y'}\right\} = 0 \Rightarrow y'\left\{\cancelto{\text{0 as the E-L equation}}{L_y - \frac{d}{dx}L_{y'}}\right\} = 0$$

Finally we would like to make a last remark regarding the E-L, by referring to an alternative way to represent the E-L: $\boxed{L_t - \dfrac{d}{dt}\left\{L - y'L_{y'}\right\} = 0}$ because:

$$L_t - \frac{d}{dt}\left\{L - y'L_{y'}\right\} = \cancel{L_t} - \cancel{L_t} - \cancel{y''L_{y'}} - y'L_y + \cancel{y''L_{y'}} + y'\left\{L_{y't} + L_{y'y}y' + L_{y'y'}y''\right\} =$$

$$-y'\left\{L_y - L_{y't} - L_{y'y}y' - L_{y'y'}y''\right\} = -y'\left\{\cancelto{\text{0 due to E-L equation}}{L_y - \frac{d}{dt}L_{y'}}\right\} = 0$$

Generalizations

- More variables

$L = L(x, y, u, u_x, u_y)$ for $u = u(x, y)$ a given smooth function

Let us now consider a function $h = h(x, y) \in C_0^2(U),$ where $U \subset \mathbb{R}^2$ an open and bounded set. Moreover, as usual, we 'll take the first variation to obtain:

$$J[u + \varepsilon h] = \int_U L(x, y, u + \varepsilon h, u_x + \varepsilon h_x, u_y + \varepsilon h_y)dxdy \Rightarrow \left.\frac{d}{d\varepsilon}J[u + \varepsilon h]\right|_{\varepsilon=0} = 0 \Rightarrow$$

$$\int_U \left(L_u h + L_{u_x}h_x + L_{u_y}h_y\right)dxdy = 0 \Rightarrow \int_U \left\{L_u h + \frac{\partial}{\partial x}\left(L_{u_x}h\right) - h\frac{\partial}{\partial x}L_{u_x} + \frac{\partial}{\partial y}\left(L_{u_y}h\right) - h\frac{\partial}{\partial y}L_{u_y}\right\}dxdy = 0$$

$$\Rightarrow \int_U \left\{L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial y}L_{u_y}\right\}h\ dxdy + \int_U \cancelto{0(*)}{\left\{\frac{\partial}{\partial x}\left(L_{u_x}h\right) + \frac{\partial}{\partial y}\left(L_{u_y}h\right)\right\}}dxdy = 0 \Rightarrow$$

$$\int_U \left\{L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial y}L_{u_y}\right\}h(x, y)dxdy = 0 \quad \forall\ h \in C_0^2(U).$$ From the Fundamental lemma $\Rightarrow$

$L_u - \frac{\partial}{\partial x} L_{u_x} - \frac{\partial}{\partial y} L_{u_y} = 0$ and in general: $\boxed{\mathrm{L}_u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} L_{u_{x_i}} = 0}$ E-L equation.

It remains to justify the $(*)$ deduction: By applying Green's Theorem for $Q(x,y) = L_{u_x} h$ and $P(x,y) = -L_{u_y} h$ we have that:

$$\int_U \left\{ \frac{\partial}{\partial x}\left(L_{u_x} h\right) + \frac{\partial}{\partial y}\left(L_{u_y} h\right) \right\} dxdy = \int_{\partial U} -L_{u_y} h dx + L_{u_x} h dy = \int_{\partial U} \left(-L_{u_y}, L_{u_x}\right) \cdot \overbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}^{\vec{T}} \cancelto{0 \text{ on} \partial \mathrm{U}}{h} \, ds = 0 \, .$$

- <u>Higher derivatives</u>

Once again we consider a function $h \in C_0^4[a,b]$ (at least), then

$J[y+\varepsilon h] = \int_a^b L(x, y+\varepsilon h, y'+\varepsilon h', y''+\varepsilon h'')dx \;\Rightarrow\; \frac{d}{d\varepsilon} J[y+\varepsilon h]\Big|_{\varepsilon=0} = 0$ the necessary condition for extrema $\Rightarrow \int_a^b \left(L_y h + L_{y'} h' + L_{y''} h''\right) dx = 0 \Rightarrow$ integration by parts $\Rightarrow$

$$\int_a^b \left\{ L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} \right\} h(x)dx + \cancelto{0}{\left[L_{y'} h \right]_a^b} + \cancelto{0}{\left[L_{y''} h' - \frac{d}{dx} L_{y''} h \right]_a^b} = 0$$ the last term

the last term equals zero since $h(a) = h'(a) = h(b) = h'(b) = 0 \;\Rightarrow$ Now by applying the fundamental lemma (which still holds true, it suffices to consider here the 10-spline, which is: $h(x) = \begin{cases} (x-x_1)^5 (x_2 - x)^5, & x \in [x_1, x_2] \\ 0, & \text{otherwise} \end{cases}$

as $h(x)$ and to be led to a contradiction as previously) to obtain:

$L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0$ and in general, (if it's C^m-differentiable) $\boxed{L_y + \sum_{k=1}^{m} (-1)^k \frac{d^k}{dx^k} L_{y^{(k)}} = 0}$

- More functions

$L = L(x, y_1, ..., y_n, y_1', ..., y_n')$ smooth as usual and we consider $h_1, ..., h_n$ smooth with compact support in $[a,b]$. Again, by taking the first variation we will be able to obtain a necessary condition for extrema, i.e. the E-L equation:

$$J[y_1 + \varepsilon h_1, ..., y_n + \varepsilon h_n] = \int_a^b L(x, y_1 + \varepsilon h_1, ..., y_n + \varepsilon h_n, y_1' + \varepsilon h_1', ..., y_n' + \varepsilon h_n')dx \Rightarrow \frac{d}{d\varepsilon} J[\vec{y} + \varepsilon \vec{h}]\Big|_{\varepsilon=0} = 0$$

$\int_a^b \left\{ L_{y_1} h_1 + ... + L_{y_n} h_n + L_{y_1'} h_1' + ... + L_{y_n'} h_n' \right\} dx = 0 \Rightarrow$ integration by parts to obtain:

$\int_a^b \left[\sum_{i=1}^{n} \left(L_{y_i} h_i - \frac{d}{dx} L_{y_i'} h_i \right) \right] dx = 0$ which is valid for every $h_i \Rightarrow \boxed{L_{y_i} - \frac{d}{dx} L_{y_i'} = 0}$ for $i = 1, 2, ..., n$

Remarks

Below we will examine some special cases of first integrals regarding the general cases of E-L as well as the necessary condition for the existence of extremals for the special case where L depends only on the first derivatives of the functions y and z, i.e. $L = L(y', z')$. Remember, since we have more than one functions, we are talking about a system of E-L equations, where an equivalent to the necessary condition $L_{y'y'} \neq 0$ for the existence of the extremals is : $L_{y'y'} L_{z'z'} - L_{y'z'}^2 \neq 0$.

In other words $\boxed{\det \begin{pmatrix} L_{y'y'} & L_{y'z'} \\ L_{z'y'} & L_{z'z'} \end{pmatrix} \neq 0}$. Now since $L = L(y', z')$ (depends only on y' and z'),

the E-L system has the following form:

$\begin{cases} L_y - \frac{d}{dt} L_{y'} = 0 \\ L_z - \frac{d}{dt} L_{z'} = 0 \end{cases} \Rightarrow \begin{cases} y'' L_{y'y'} + z'' L_{y'z'} = 0 \\ y'' L_{z'y'} + z'' L_{z'z'} = 0 \end{cases}$. Now we know from elementary Linear Algebra that this system can be uniquely solved iff $\det \neq 0$ and since it is homogenuous there exists only the null solution, i.e. $\begin{cases} y'' = 0 \\ z'' = 0 \end{cases} \Rightarrow \underline{y(t), z(t) \text{ are linear polynomials}}$

First integrals for the generalized cases:

$\bullet L = L(\vec{y},\vec{z})$ where $\vec{y} = (y_1, y_2, ..., y_n)$ and analogously $\vec{z} \in \mathbb{R}^n$.

Then a first integral is $\boxed{L - \sum_{i=1}^{n} y_i{}' L_{y_i'} = c}$ because:

$$\frac{d}{dt}\left(L - \sum_{i=1}^{n} y_i' L_{y_i'}\right) = \cancel{L_t}^{0} + \sum_{i=1}^{n} y_i' L_{y_i} + \cancel{\sum_{i=1}^{n} y_i'' L_{y_i'} - \sum_{i=1}^{n} y_i'' L_{y_i'}} - \sum_{i=1}^{n} y_i' \frac{d}{dt} L_{y_i'} = \sum_{i=1}^{n} y_i' \cancel{\left(L_{y_i} - \frac{d}{dt} L_{y_i'}\right)}^{0 \text{ as E-L}} = 0.$$

$\bullet L = L(x, y', y'') \Rightarrow L_y - \frac{d}{dx} L_{y'} + \frac{d^2}{dx^2} L_{y''} = 0 \Rightarrow \boxed{L_{y'} + \frac{d}{dx} L_{y''} = c}$.

$\bullet L = L(y, y', y'') \Rightarrow$ a first integral is $\boxed{L - y'\left(L_{y'} - \frac{d}{dx} L_{y''}\right) - y'' L_{y''} = c}$. In order to prove this result, we need to take thederivative with respect to x. This must be equal to zero. The required identity follows after some elementary computations.

Some examples:

$\bullet$ $J[u] = \iint_R \left(u_t^2 - c^2 u_x^2\right) dxdt \overset{E-L}{\Rightarrow} \cancel{L_u}^{0} - \frac{\partial}{\partial t} L_{u_t} - \frac{\partial}{\partial x} L_{u_x} = 0 \Rightarrow \frac{\partial}{\partial t}\left(\cancel{2}u_t\right) + \frac{\partial}{\partial x}\left(-\cancel{2}c^2 u_x\right) = 0 \Rightarrow$

$\boxed{u_{tt} - c^2 u_{xx} = 0}$ wave equation.

$\bullet$ $J[u] = \int_R \frac{1}{2}\left[u_t^2 - u_x^2 - u_y^2 - u_z^2 - m^2 u^2\right] dtdxdydz \overset{E-L}{\Rightarrow} L_u - \sum_{i=1}^{4} \frac{\partial}{\partial x_i} L_{u_{x_i}} = 0 \Rightarrow \overset{calculations}{..........} \Rightarrow$

$\boxed{u_{tt} - \Delta u = -m^2 u^2}$

$\bullet J[u] = \iint_R \left[\frac{1}{2}\left|\nabla u\right|^2 + p(x,y)u\right] dxdy \overset{E-L}{\Rightarrow} L_u - \frac{\partial}{\partial x} L_{u_x} - \frac{\partial}{\partial y} L_{u_y} = 0 \Rightarrow \begin{cases} L_u = p(x,y) \\ L_{u_x} = u_x \\ L_{u_y} = u_y \end{cases} \Rightarrow$

$p(x,y) = u_{xx} + u_{yy} \Rightarrow \boxed{\Delta u = p}$ Poisson's pde.

$$\bullet K[u]=\int_D \left\{ |\nabla u|^2 + \frac{gu^4}{2}\right\} dxdy \;\Rightarrow\; \left.\frac{d}{d\varepsilon}K[u+\varepsilon\psi]\right|_{\varepsilon=0} = \int_D \left(2\nabla u\cdot\nabla\psi + 2gu^3\psi\right)dxdy = 0 \Rightarrow$$

$0=\int_D\left(\nabla u\cdot\nabla\psi+gu^3\psi\right)dxdy=\int_{\partial D}\psi\frac{\partial u}{\partial\vec n}dS+\int_D\left(-\Delta u+gu^3\right)\psi\, dxdy=0 \quad \forall\psi\in H^1(D)$. For $\frac{\partial u}{\partial\vec n}=0$ on ∂D now, we conclude that: $\int_D\left(-\Delta u+gu^3\right)\psi\, dxdy=0 \quad \forall\psi\in H^1(D) \;\Rightarrow$

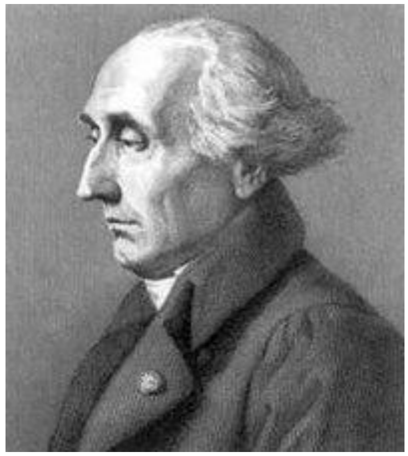
$\begin{cases}\Delta u-gu^3=0\ , \text{ in } D\\ \frac{\partial u}{\partial\vec n}=0 \text{ on } \partial D\end{cases}$ It remains to justify the first integral implication:

$$\left.\frac{d}{d\varepsilon}\left(|\nabla u+\varepsilon\nabla\psi|^2\right)\right|_{\varepsilon=0}=\left.\left(2|\nabla u+\varepsilon\nabla\psi|\frac{d}{d\varepsilon}\left(|\nabla u+\varepsilon\nabla\psi|\right)\right)\right|_{\varepsilon=0}=$$

$=2\cancel{|\nabla u+\varepsilon\nabla\psi|}\left.\frac{\left(\nabla u+\cancel{\varepsilon\nabla\psi}^{\,0}\right)\cdot\nabla\psi}{\cancel{|\nabla u+\varepsilon\nabla\psi|}}\right|_{\varepsilon=0}=2\nabla u\cdot\nabla\psi$. Otherwise we could obtain the same result by just expanding $|\nabla u+\varepsilon\nabla\psi|^2=|\nabla u|^2+2\varepsilon\nabla u\cdot\nabla\psi+\varepsilon^2|\nabla\psi|^2$ and operating finally the differentation with respect to ε.

Historical notes

Leonard Euler (1701–1783), a Swiss mathematician who spent much of his professional life in St. Petersburg, was perhaps one of the most prolific contributors to mathematics and science of all time. His collected works fill 92 volumes, more than anyone else in the field. Some may rank him at the top of all mathematicians. His name is attached to major results in nearly every area of study in mathematics. A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: "Read Euler, read Euler, he is the master of us all."



Joseph Louis Lagrange (1736–1813) was a Franco-Italian mathematician, physicist and astronomer, one of the great mathematicians in the 18th century, whose work had a deep influence on subsequent research. In 1766, on the recommendation of Swiss Leonhard Euler and French d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences. In 1787, at age 51, he moved from Berlin to Paris and became a member of the French Academy of Sciences. He remained in France until the end of his life. He was significantly involved in the decimalisation in Revolutionary France, became the first professor of analysis at the École Polytechnique upon its opening in 1794, was a founding member of the Bureau des Longitudes, and became Senator in 1799.

Dirichlet's Principle:

If we take $L(p,z,x)=\frac{1}{2}|p|^2$, then $L_{p_i}=p_i$ for $i=1,2,...,n$, $L_z=0$

and so the E-L associated with the functional $\boxed{I[w]=\frac{1}{2}\int|\nabla w|^2dx}$ is $\boxed{\Delta u=0}$,

i.e the solutions of Laplace equation (harmonic functions) minimize the energy's functional or Dirichlet's Integral.

Below we shall employ what is called the <u>energy method</u> to present a different but equivalent proof of the Dirichlet's Principle.

$\boxed{I\left[w\right]=\int_U \frac{1}{2}\left|\nabla w\right|^2 - wfdx,}$ where $w\in A:=\left\{w\in C^2\left(\overline{U}\right)\ \middle| w=g\ on\ \partial U\ \right\}$ admissible set.

$\boxed{Theorem}$

Dirichlet's principle now asserts that if $u\in C^2\left(\overline{U}\right)$ solves $\begin{cases}-\Delta u = f\text{, in U}\\ u=g\text{, on }\partial\text{U}\end{cases}$, then:

$(*)\ I[u]=\min_{w\in A} I[w]$

Conversely, if $u\in$ A satisfies the relation $(*)$, then u solves the *Poisson* b.v.p.

<u>Proof:</u>

Choose $w\in$ A, then the b.v.p $\begin{cases}-\Delta u = f\text{, in U}\\ u=g\text{, on }\partial\text{U}\end{cases}$ implies:

$$0=\int_U \cancel{\left(-\Delta u - f\right)}^0\left(u-w\right)dx=\int_U -\Delta u\left(u-w\right)dx+\int_U -f\left(u-w\right)dx=$$

$$\int_{\partial U} -\cancel{\left(u-w\right)}^0\frac{\partial u}{\partial \vec{n}}dS+\int_U \nabla u\cdot\nabla\left(u-w\right)dx-\int_U f\left(u-w\right)dx=\int_U\left[\nabla u\cdot\left(\nabla u-\nabla w\right)-f\left(u-w\right)\right]dx.$$

The surface integral equals to zero since $u=w=g$ on ∂U.

Hence: $\int_U \left|\nabla u\right|^2 - fudx=\int_U \nabla u\cdot\nabla w - fwdx$ which implies that

$$\int_U \left|\nabla u\right|^2 - fudx=\int_U \nabla u\cdot\nabla w - fwdx\overset{C-S}{\leq}\int_U\left|\nabla u\right|\left|\nabla w\right|-fwdx\overset{ab\leq\frac{a^2+b^2}{2}}{\leq}$$

$$\int_U\frac{\left|\nabla u\right|^2}{2}dx+\int_U\frac{\left|\nabla w\right|^2}{2}-wfdx\ \Rightarrow I\left[u\right]\leq I\left[w\right]$$

and $w\in A$ was chosen arbitrarily.

Since $u\in A$ and $I\left[u\right]\leq I\left[w\right]\ \ \forall w\in A\Rightarrow\ I\left[u\right]=\min_{w\in A} I\left[w\right]$.

Now let us suppose that $I\left[u\right]=\min_{w\in A} I\left[w\right]$ holds.

Then fix any $u\in\ C_c^{\infty}\left(U\right)$ and write $i\left(\tau\right)=I[u+\tau v]\ \ (\tau\in R)$

since $u+\tau v\in A$ because $u,v\in C^2\left(\bar{U}\right)$

in fact $v\in C_c^{\infty}\left(U\right)$ and $u+\tau v\big|_{\partial U}=g+\tau 0=g$

because $\text{supp}(v)=\text{compact}$

$\Rightarrow u+\tau v\in A$ indeed $\forall\,\tau\in R.$

Therefore,we conclude that the scalar function $i(\tau)$ has got a minimum

at $\tau=0$. Thus $i'(0)=0$. But $i(\tau)=I\left[u+\tau v\right]=\int_U \frac{1}{2}\left|\nabla u+\tau\nabla v\right|^2-(u+\tau v)\,fdx=$

$\int_U \frac{1}{2}\left|\nabla u\right|^2+\frac{1}{2}\tau^2\left|\nabla v\right|^2+\tau\nabla u\cdot\nabla v-\left(u+\tau v\right)fdx$

Consequently:

$i'(0)=0\Rightarrow 0=\int_U \nabla u\cdot\nabla v-vfdx=\int_{\partial U}\cancel{v}^{\,0}\,\frac{\partial u}{\partial\vec{n}}dS-\int_U v\Delta u+vfdx\ \Rightarrow$

$\int_U\left(-\Delta u-f\right)vdx=0\ \ \forall\ v\in C_c^{\infty}(U)\ \Rightarrow\ -\Delta u=f$ in U. □

Generalized Dirichlet's Principle

Consider $\boxed{L(p,z,x)=\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)p_ip_j-zf(x)}$ where $a^{ij}(x)=a^{ji}(x)$ (symmetric)

$(i,\,j=1,2,\ldots,n)$

Then $L_{p_i}\overset{\text{due to symmetry of } a^{ij}(x)}{=}\frac{\cancel{2}^{\,1}}{\cancel{2}}\sum_{j=1}^{n}a^{ij}(x)p_j$ and $L_z=-f(x)$. Hence the E-L equation

associated with the functional: $I[w]=\int_U\left(\frac{1}{2}\sum_{i,j=1}^{n}a^{ij}(x)w_{x_i}w_{x_j}-wf\right)$, where $w_{x_i}=p_i$ and

$w_{x_j}=p_j$, is the divergence-structure, linear, 2nd order pde:

$-\sum_{i,j=1}^{n}(a^{ij}u_{x_j})_{x_i}=f\Rightarrow\boxed{-div(A(x)\nabla u)=f}$ in U (note that in the simplest case

where the matrix $A(x)=I\Rightarrow -div(\nabla u)=-\nabla\cdot\nabla u=-\Delta u$ and we return back to the classical problem).We will see later that the uniform ellipticity condition on the $a^{ij}(x)$ is a natural further assumption, required to prove the existence of minimizers.

Consequently, from the non-linear viewpoint of calculus of variations, the divergence structure form of a linear, 2nd order elliptic pde is completely natural.

Remark: (Non-linear Poisson equation)
Given a smooth function $f:\mathbb{R}\to\mathbb{R}$ and its antiderivative

$F(z)=\int_0^z f(y)dy$ (such that $F'(z)=f$) $\Rightarrow$ E-L of the energy's functional

$I[w]=\int_U \frac{1}{2}|\nabla w|^2-F(w)dx$ is given by: $L_{p_i}(x,w,\nabla w)=p_i=w_{x_i}$

and $L_z=L_w=f(w)\Rightarrow -\sum_{i=1}^{n}(L_{p_i}(x,w,\nabla w)_{x_i}+L_z(x,w,\nabla w)=0\Rightarrow$

$-\sum_{i=1}^{n}\frac{\partial^2 w}{\partial x_i{}^2}+f(w)=0\Rightarrow \boxed{\Delta w=f(w)}$ in U, which is the non-linear *Poisson* equation.

Minimal Surface

Let $L(p,x,z)=(1+|p|^2)^{\frac{1}{2}}$ so that $I[w]=\int_U\left(1+|\nabla w|^2\right)^{\frac{1}{2}}dx$ is the area of the graph of the function $w:U\to\mathbb{R}$.

$$L_z=0 \text{ and } L_{p_i}=L_{u_{x_i}}=\frac{u_{x_i}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\Rightarrow L_z-\sum_{i=1}^{n}(L_{p_i})_{x_i}=0\Rightarrow\sum_{i=1}^{n}\left(\frac{u_{x_i}}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right)_{x_i}=0\Rightarrow$$

minimal surface equation: $\boxed{div\left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right)=0}$ in U.

Remark

the expression $div\left(\frac{\nabla u}{\left(1+|\nabla u|^2\right)^{\frac{1}{2}}}\right)$ is n times the mean curvature of the graph of u. Thus the minimal surface has zero mean curvature. A way to define the mean curvature of a function's graph is through the divergence of the unit outward normal vector , i.e. $H=-\frac{1}{n}\nabla\cdot\vec{n}=-\frac{1}{n}\nabla\cdot\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)$ where

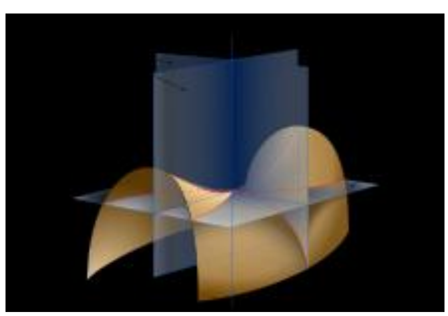
$$\varphi=z-u(x_1,...,x_n) \text{ and } \varphi=\varphi(\vec{x};z) \Rightarrow \begin{cases}\nabla\varphi=\left(-u_{x_1},...,-u_{x_n},1\right)=\left[-\nabla u \vdots 1\right]\\ |\nabla\varphi|=\left[1+|\nabla u|^2\right]^{\frac{1}{2}}\end{cases}$$

$$\text{Therefore}: \ H=-\frac{1}{n}div\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)=-\frac{1}{n}div\left(\frac{\left[-\nabla u \vdots 1\right]}{\sqrt{1+|\nabla u|^2}}\right)=$$

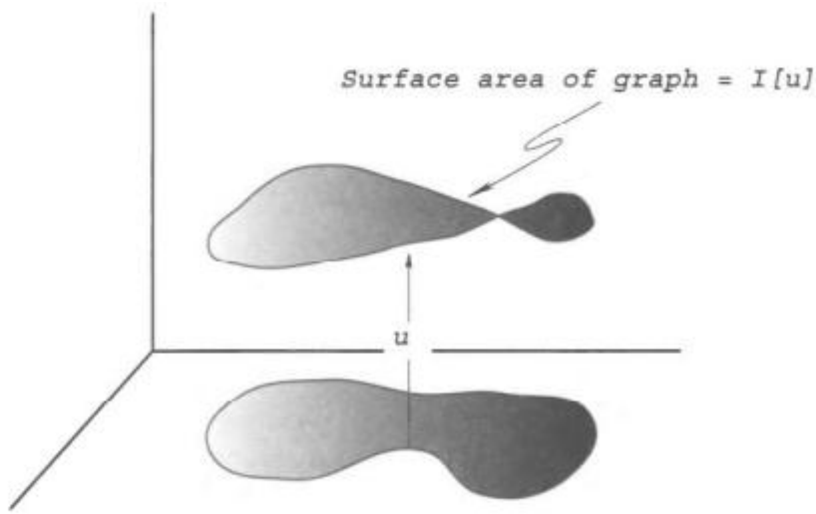
$$-\frac{1}{n}div\left(-\frac{\nabla u}{\left(1+|\nabla u|^2\right)^{\frac{1}{2}}}\right)+\frac{1}{n}\frac{\partial}{\partial z}\cancelto{0}{\left(\frac{1}{\left(1+|\nabla u|^2\right)^{\frac{1}{2}}}\right)} \quad \Rightarrow$$

$$\text{div}\left(\frac{\nabla u}{\left(1+|\nabla u|^2\right)^{\frac{1}{2}}}\right)=nH.$$

Additionally, there is a very interesting result: $\boxed{\text{a surface is a minimal surface} \Leftrightarrow \text{mean curvature } =0}$
We will omit the proof of the statement above, because this is beyond the scope of this work.



Minimal surface curvature planes. On a minimal surface, the curvature along the principal curvature planes are equal and opposite at every point. This makes the mean curvature zero.



A minimal surface

Some further remarks regarding the surface measure:

Let Γ be a simple, closed curve in $\mathbb{R}^3$. Now a surface whose boundary is Γ, is said to be spanned by Γ. Let us now consider a surface $S = S(u)$ characterized by a graph of function $u = u(x, y)$ defined over a region D in $\mathbb{R}^2$, such that the boundary ∂D is mapped by u to $\partial S = \Gamma$ (in particular $S(u)$ is spanned by Γ(figure1)). $E[u] = A\left[S[u]\right] = \int_D \sqrt{1+u_x^2+u_y^2}\,dxdy$ because:

$$Area\left[S[u]\right] = \int_{\partial D = S} 1dS = \iint_D \left\|T_x \times T_y\right\| dxdy = \iint_D \sqrt{1+u_x^2+u_y^2}\,dxdy =$$

$$\iint_D \sqrt{1+\left|\nabla u\right|^2}\,dxdy$$

For a given surface now which is the graph of a function, say u(x,y), i.e. $\varphi(x, y, z) = z - u(x, y)$, we have that $\nabla\varphi = \left(-u_x, -u_y, 1\right)$. Now we know that

$\left\|T_x \times T_y\right\| = \|\nabla\varphi\| = \sqrt{1+|\nabla u|^2}$. See figure below.

In general for a given, smooth surface parametrized by

φ(u,v)=$\big(x(u,v), y(u,v), z(u,v)\big): \mathbb{R}^2 \to \mathbb{R}^3$, we know that the following hold:

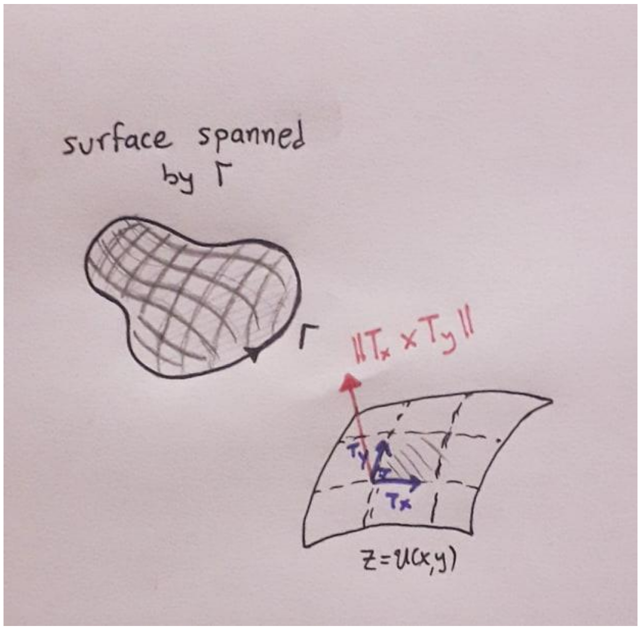
$\|T_u \times T_v\| = \sqrt{\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\dfrac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\dfrac{\partial(y,z)}{\partial(u,v)}\right|^2}$, where we denote :

$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = |\det(Jacobian)| = \left|\det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}\right|$. Analogously for the other.

Finally the admissible set: $\mathrm{A} = \{\psi \in C^1\big((D) \cap C(\overline{D})\big),\ \psi(x,y) = 0$

$\forall (x,y) \in \partial D\}$

In other words $\psi \in C_0^1(D)$. From the Calculus of Variations' point of view, we define $v = u + \varepsilon\psi$ and we try to minimize the functional E, i.e. to find a "u" such that $E[u] \le E[v] = E[v + \varepsilon\psi]$ for a small variation ε and for all $\psi \in \mathrm{A}$.



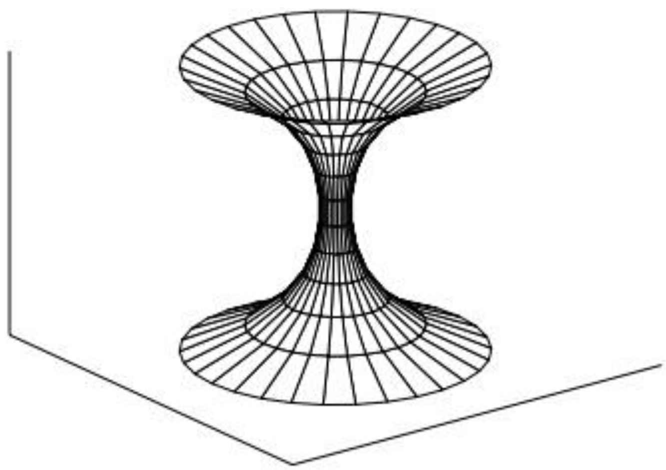
Comment :

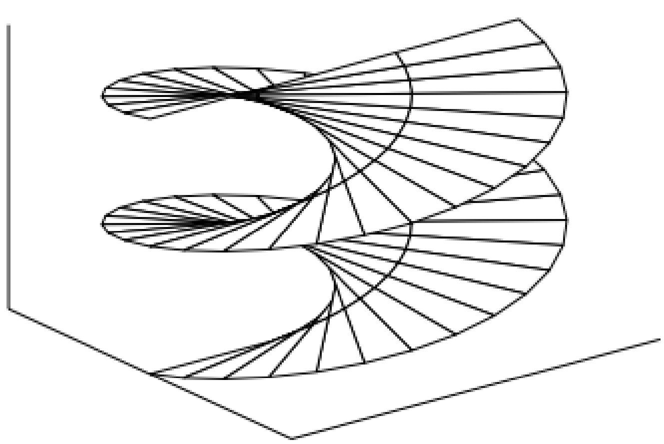
We should notice that both catenoid and helicoid surfaces cannot be represented as global graphs. Rather, they can be written explicitly in a parametric form.

$$Helicoid:\begin{cases} x=\rho\cos\theta, & 0\le\theta\le 2\pi \\ y=\rho\sin\theta, & a<\rho<b \\ z=d\rho \end{cases}$$

$$Catenoid:\begin{cases} x=d\cosh\dfrac{\rho}{d}\cos\theta \\ y=d\cosh\dfrac{\rho}{d}\sin\vartheta \\ z=\rho \end{cases}$$

We can see below how the **catenoid** and **helicoid** surfaces respectively look like:





Other examples of minimal surfaces taken by the great book of Hildebrandt-Giaquinta are the following:

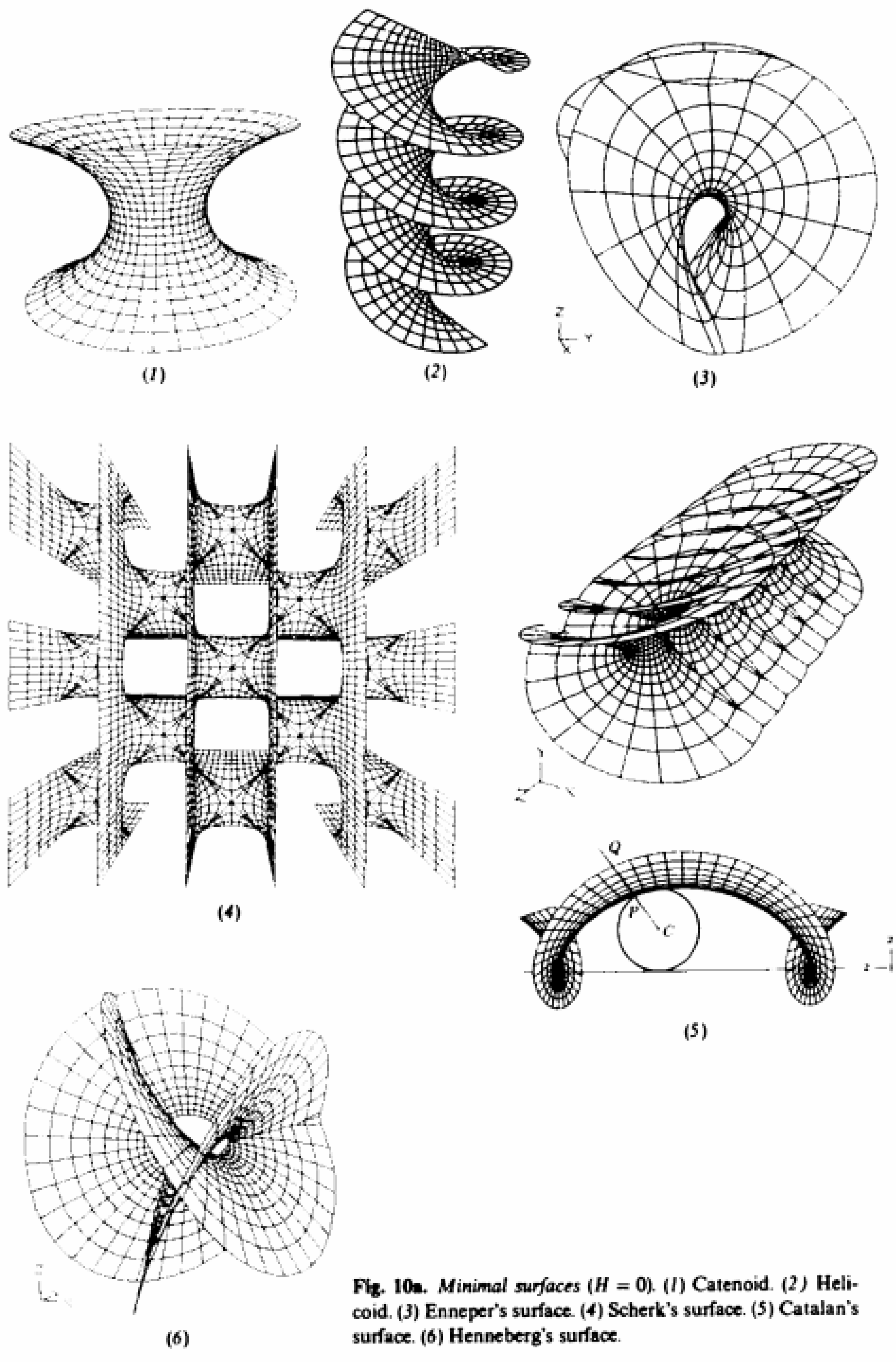
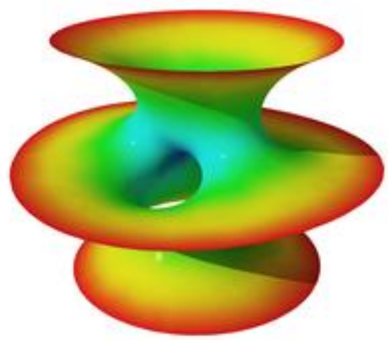


Fig. 10a. *Minimal surfaces* ($H = 0$). (*1*) Catenoid. (*2*) Helicoid. (*3*) Enneper's surface. (*4*) Scherk's surface. (*5*) Catalan's surface. (*6*) Henneberg's surface.

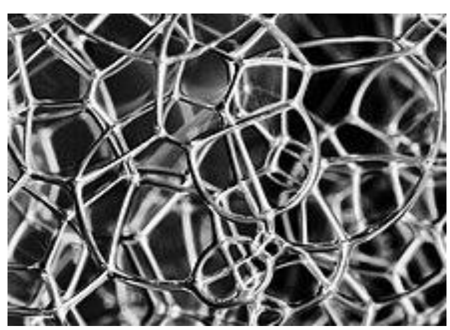
And a bit weird one, which is called Costa's minimal surface: Famous conjecture disproof. Described in 1982 by Celso Costa and later visualized by Jim Hoffman. Jim Hoffman, David Hoffman and William Meeks III then extended the definition to produce a family of surfaces with different rotational symmetries.



Joseph Antoine Plateau (1801-1883) was a Belgian mathematician and physicist with considerable contribution to optics as well as to the calculus of variation. He was also one of the first people to demonstrate the illusion of a moving image. To accomplish this, he used counter rotating disks with repeating drawn images in small increments of motion on one and regularly spaced slits in the other. He called this device of 1832 the *phenakistiscope*.

Additionally, he was widely known for the Physics of soap bubbles (Plateau's Laws) and Plateau's problem related to the minimal surface problem through the soap film experiments.

Incidentally, while such experiments are now frequently performed by children in science museums around the world, Plateau himself did not see a single minimal surface! He was blinded early in his scientific career as a result of looking directly at the sun while performing optical experiments.



Bubbles in a foam of soap. Soap films meet in threes at about 120° along Plateau borders and these borders meet at vertices at about the tetrahedral angle.

A boy blowing a soap bubble. Painting by the French artist Jean Baptiste Siméon Chardin (1699–1779), The Metropolitan Museum of Art, New York.

The connection between the minimal surface problem and the Dirichlet's principle

We shall follow a slightly different approach to our analysis below.
To be more specific, let us demonstrate our thought.
Assuming that the minimal surface u has "small derivatives", we could approximate the functional $E[u]$ by a siimpler functional, deducing parallelly the problem to a simpler one as well. Using the approximation (derived by using the Taylor's theorem)

$\sqrt{1+x} \simeq 1+\frac{x}{2}+...$ (Taylor around zero), we expand:

$E[u]=|D|+\frac{1}{2}\int_D \left(u_x^2+u_y^2\right)dxdy+...$ Neglecting now the terms of higher order, we replace now the problem of minimizing $E[u]$ with the problem of minimizing the functional: $G[u]=\frac{1}{2}\int_U |\nabla u|^2\, dxdy$ which is called the Dirichlet's energy functional. By regarding, as usual, the first variation of E at u, i.e. $\frac{d}{d\varepsilon}E\left[u+\varepsilon\psi\right]_{\varepsilon=0}=0$, which is a necessary minimization condition, we obtain the following:

$$G[u+\varepsilon\psi]=\frac{1}{2}\int_U |\nabla u+\varepsilon\nabla\psi|^2\, dxdy=\frac{1}{2}\int_U |\nabla u|^2\, dxdy+\frac{1}{2}\int_U \varepsilon^2\,|\nabla\psi|^2\, dxdy+$$

$$+\int_U \varepsilon \nabla u \cdot \nabla \psi dxdy \;\Rightarrow\; G[u+\varepsilon\psi] = G[u] + \varepsilon \int_U \nabla u \cdot \nabla \psi dxdy + \varepsilon^2 G[\psi] \;\Rightarrow$$

$$\frac{d}{d\varepsilon} G[u+\varepsilon\psi]\Big|_{\varepsilon=0} = \int_U \nabla u \cdot \nabla \psi dxdy + 2\varepsilon G[\psi]\Big|_{\varepsilon=0} = \int_D \nabla u \cdot \nabla \psi dxdy \;\Rightarrow$$

As denoted above, the necessary condition for u to be a local minimizer is to satisfy: $\int_U \nabla u \cdot \nabla \psi dxdy = 0 \;\forall \psi \in$ A, where A is the admissible set.

But we notice that:

$$0 = \int_U \nabla u \cdot \nabla \psi dxdy \overset{\substack{\text{integration} \\ \text{by parts}}}{=} \int_{\partial U} \psi \frac{\partial u}{\partial \vec{n}} dS - \int_U \psi \Delta u dxdy$$,where we observe that the surface integral equals 0 since ψ=0 on ∂U. Therefore we have that $\int_U \psi \Delta u dxdy = 0 \;\; \forall \psi \in$ A $\Rightarrow$ (if Δu is continuous) $\Delta u = 0$ *in D*, which is the Laplace linear, elliptic pde which coincides here with the E-L pde.

Moreover, by construction, u must satisfy the boundary condition $u(x,y) = g(x,y),\; (x,y) \in \partial D$. In other words, if a smooth function u minimizes the energy's functional, then u is a solution to the very well-known b.v.p. :

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = g \text{ on } \partial D \end{cases}$$. There is only one last thing which needs to be clarified.

Lemma:

Let $h(x,y,z)$ be a continuous function satisfying $\int_\Omega h(x,y,z)dV = 0 \;\forall$ domain $\Omega \;\Rightarrow\; h \equiv 0$

The proof is quite simple. For contradiction, let us assume that there exists a point $P = (x_0, y_0, z_0)$ where $h(P) \neq 0$ there. Without loss of generality, let's assume that $h(P) > 0$. Then, since h is continuous, there exists a domain (perhaps very small) D_0, containing P and ε>0 such that $h > \varepsilon > 0$ at each point of the domain D_0.

Therefore 0=$\int_\Omega h(x,y,z)dV = \int_{D_0} h(x,y,z)dV > \varepsilon Vol(D_0) > 0$,contradiction! QED

Reconstruction of a function from its gradient

Many applications in optics and other image analysis problems require a surface $u(x,y)$ to be computed from measurements of its gradient.

This procedure is extremely useful determining the phase of light or sound waves. If the measurements are precise, then the solution is straight forward. However, there is almost always an experimental error. Therefore the measurements can be considered at best as an approximation of the gradient. Denote the measure vector that approximates the gradient by $f^T = (f_1, f_2)$. Typically a given vector field is not the gradient of a scalar function. To be a gradient, f must satisfy the comppatibility condition $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$ because:

$$\vec{f}^T = (f_1, f_2) = \vec{\nabla} u \;\Rightarrow\; \left.\begin{aligned} f_1 &= \frac{\partial u}{\partial x} \\ f_2 &= \frac{\partial u}{\partial y} \end{aligned}\right\} \;\Rightarrow\; \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \;\Rightarrow\; \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}.$$

If this holds, then we can compute u by a single integral, indeed:

u(x,y)=$\int_{D_x} f_1(x,y)dx + \int_{D_y} f_2(x,y)dy.$ Nonetheless, very often,

the measurements' errors will corrupt the compatibility condition, we are obliged to seek other means for estimating the phase u. For example, the least square approximation: $\min K[u] = \int_D |\nabla u - f|^2 \, dxdy$ and we notice that

$$\left|\vec{\nabla} u - \vec{f}\right|^2 = |\nabla u|^2 + \left|\vec{f}\right|^2 - 2\nabla u \cdot \nabla f \;\Rightarrow$$

$$K[u + \varepsilon\psi] = \int_D \{|\nabla u + \varepsilon \nabla \psi|^2 + |f|^2 - 2\nabla u \cdot f - 2\varepsilon \nabla \psi \cdot f\} dxdy \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Rightarrow$$

$\delta K[u] = \int_D 2\nabla u \cdot \nabla \psi - 2\nabla \psi \cdot \vec{f} dxdy = 0,$ where with "δ" we denote the first variation of K, or the Gateaux Variation. The above leads to the following:

$$\int_D (\nabla u - f) \cdot \nabla \psi \, dxdy = 0 \overset{\substack{\text{integrating} \\ \text{by parts}}}{\Rightarrow} \int_{\partial D} \psi (\nabla u - f) \cdot \vec{n} dS - \int_D \psi \, div(\nabla u - f) \, dxdy = 0$$

where $\vec{n}$ is the outer unit vector to ∂U. Finally we have ended up with:

$\int_D \left(-\Delta u + div\vec{f}\right) \psi \, dxdy + \int_{\partial D} \psi \left(\frac{\partial u}{\partial \vec{n}} - \vec{f} \cdot \vec{n}\right) dS = 0.$ Now, since the first variation must vanish, in particular for functions ψ that are identically zero at ∂D, we have to equate the first integral to zero to obtain the E-L equation:

$\Delta u = div\vec{f} \quad \left(\textit{where we denote } div\vec{f} = \vec{\nabla} \cdot \vec{f}\right)$ for $(x, y) \in D.$

Then the surface integral reduces to

$\int_{\partial D}\left(\frac{\partial u}{\partial \vec{n}}-\vec{f}\cdot\vec{n}\right)dS=0,$ but this equation holds for every ψ that is non zero on ∂D as well, thus we obtain: $\frac{\partial u}{\partial \vec{n}}=\vec{f}\cdot\vec{n}$ for $(x,y)\in\partial D$. Such boundary conditions which are inherent to a variational problem (in contrast to being supplied from outside) are called natural boundary conditions. We shall refer to them more analytically later to the sequel.

<u>The shortest way to go from a point to another:</u> (straight line)

The arclength functional is $J[y]=\int_a^b\sqrt{1+(y')^2}\,dx=\int_C dS=\int_a^b 1\|T'(t)\|dt,$

$T'(t)=(1,y'(t))\Rightarrow \|T'(t)\|=\sqrt{1+(y'(t))^2}$ where curve's parametrization $T(t)=(t,y(t))$ here, the curve is a function's graph.

$L(x,y,y')=\sqrt{1+(y')^2}$

$Ly=0$ and $Ly'=\frac{y'}{\sqrt{1+(y')^2}}\overset{E-L}{=}\frac{y'}{\sqrt{1+(y')^2}}=c\Rightarrow (y')^2=c^2\left[1+(y')^2\right]\Rightarrow$

$(1-c^2)(y')^2=c^2\Rightarrow (y')^2=\frac{c^2}{1-c^2}\Rightarrow y'(t)=\text{constant}\Rightarrow$

$y(t)=kt+\lambda \quad k,\lambda\in R,$

after applying the boundary conditions, we end up with a straight line equation:

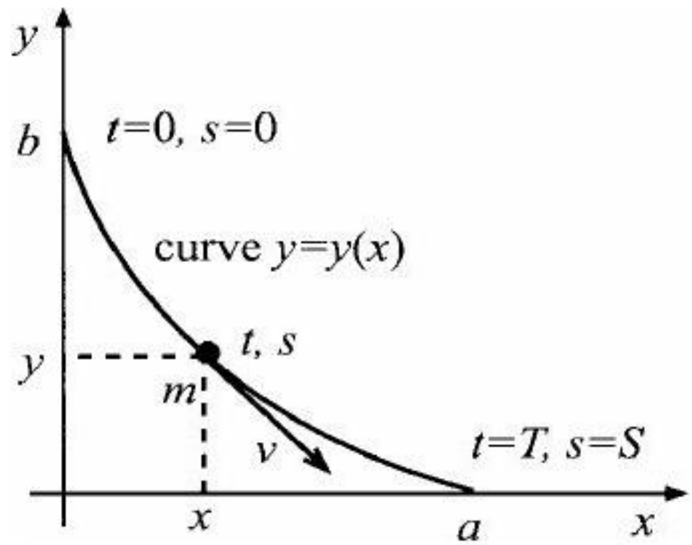
$y=\frac{y_1-y_0}{b-a}x+\frac{by_0-ay_1}{b-a}$ which in fact connects (a,y_0) with (b,y_1).

Brachistochrone

$$T = \int_0^T dt = \int_0^S \frac{dt}{ds} ds = \int_0^S \frac{1}{v} ds$$

S : total arclength of the curve

v : velocity



Since $ds = \sqrt{1+\left(y'(x)\right)^2}\,dx \;\Rightarrow\; T = \int_0^a \frac{\sqrt{1+\left(y'(x)\right)^2}}{v} dx$ and by using the energy conservation principle: $K_0 + U_0 = K_1 + U_1 \;\Rightarrow\; \frac{1}{2} mv^2 + mgy = 0 + mgb \;\Rightarrow$

$v = \sqrt{2g\left(b - y(x)\right)} \;\Rightarrow\; T[y] = \int_0^a \frac{\sqrt{1+\left(y'(x)\right)^2}}{\sqrt{2g\left(b-y(x)\right)}} dx$. Additionally here the set of admissible functions A is the following:

$A_{brachistochrone} = \{ y \in C^1[0,a] \;:\; y(0) = b \; and \; y(a) = 0$ and the conditions $y \le b$ and $\int_0^a \left(b-y\right)^{-\frac{1}{2}} dx < +\infty \}$ (the last holds in order to be able to guarantee that the functional $T(y)$ will be finite)

Now we observe that:

$$L=\frac{\sqrt{1+(y')^2}}{\sqrt{2g(b-y)}} \Rightarrow \text{(E=L) } L_y - y'L_{y'} = c \Rightarrow \frac{\sqrt{1+(y')^2}}{\sqrt{b-y}} - (y')^2 \frac{\left(1+(y')^2\right)^{-\frac{1}{2}}}{\sqrt{b-y}} = c$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-c^2(b-y)}{c^2(b-y)} \Rightarrow dx = -\frac{\sqrt{b-y}}{\sqrt{c_1-(b-y)}}dy, \text{(where } c_1 = \frac{1}{c^2} \text{ and}$$

sign"-" is because of $\frac{dy}{dx}<0$) $\Rightarrow$

$$\int dx = \int -\frac{\sqrt{b-y}}{\sqrt{c_1-(b-y)}}dy \overset{\substack{b-y=u \\ du=-dy}}{\Rightarrow} x+c = \int \frac{\sqrt{u}}{\sqrt{c_1-u}}du \Rightarrow$$

$$c_1(x+c) = \int \frac{\sqrt{u}}{\sqrt{1-u}}du \overset{\substack{u=\sin^2\varphi \\ du=\sin 2\varphi d\varphi}}{\Rightarrow} c_1(x+c) = \int \frac{\sin\varphi}{\cos\varphi} 2\cos\varphi \sin\varphi d\varphi \Rightarrow$$

$$c_1(x+c) = 2\int \sin^2\varphi d\varphi \Rightarrow c_1(x+c) = 2\int \frac{1-\cos 2\varphi}{2}d\varphi \Rightarrow$$

$$c_1(x+c) = \varphi - \frac{\sin^2\varphi}{2} \Rightarrow x = \frac{\tilde{c}}{2}(\varphi - \sin 2\varphi) + \hat{c} \text{ as well as}$$

$b-y=c_1\sin^2\frac{\varphi}{2}$ which are the parametric equations for the *cycloid curve.*

Here, in contrast to the problem of finding the curve of shortest length between two points, it's not clear that the cycloids just obtained actually minimize the given functional. Further calculations are required for confirmation.

Fermat's Principle in Geometric optics

$$T(y) = \int_P^Q \frac{ds}{c} = \int_{x_1}^{x_2} \eta(x,y)\sqrt{1+(y'(x))^2}dx \Rightarrow \text{ where } \eta(x,y) = 1/c \Rightarrow$$

$$\eta_y(x,y)\sqrt{1+(y'(x))^2} - \frac{d}{dx}\left(\frac{\eta(x,y)y'}{\sqrt{1+(y'(x))^2}}\right) = 0$$

In the limit of geometric optics the Principle of Fermat states that the time elapsed in the passage of light between two fixed points in the medium is the extremun with respect to all possible paths connecting the two points. For simplicity, we consider only light rays that lie xy plane. Let $c=c(x,y)$ be a positive, C^1

function representing the velocity of the light in the medium. It's reciprocal $\eta = c^{-1}$ is called the index of refraction of the medium. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two fixed points in the plane, then the time required for the light to travel along a given path $y = y(x)$ (connecting the two points) is :

$T(y) = \int_P^Q \frac{ds}{c} = \int_{x_1}^{x_2} \eta(x,y)\sqrt{1+(y'(x))^2}\,dx$ .Therefore the actual light path connecting P and Q is the one that extremizes the integral $T(y)$. E-L $\Rightarrow$

$$\eta_y(x,y)\sqrt{1+(y'(x))^2} - \frac{d}{dx}\left(\frac{\eta(x,y)y'}{\sqrt{1+(y'(x))^2}}\right) = 0 \Rightarrow \eta_y(x,y)\sqrt{1+(y'(x))^2}$$

$$-\eta_x(x,y)y'\frac{1}{\sqrt{1+(y'(x))^2}} - \frac{\eta_y(x,y)(y')^2}{\sqrt{1+(y'(x))^2}} - \eta(x,y)\left[\frac{y''}{\sqrt{1+(y'(x))^2}} - \frac{(y')^2 y''}{\left[1+(y'(x))^2\right]^{\frac{3}{2}}}\right] = 0 \Rightarrow$$

$$\eta_y\left(1+(y'(x))^2\right) - \eta_x y' - \eta_y (y')^2 + \frac{\eta y''}{1+(y'(x))^2} = 0 \Rightarrow \eta_y - y'\eta_x + \frac{y''\eta}{1+(y'(x))^2} = 0$$

which is the simplified E-L equation for the Fermat's Principle.

Minimal area of a surface of revolution (catenary)

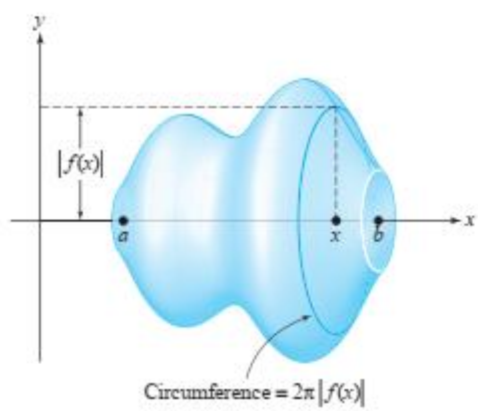


figure 7.4.3 The curve $y = f(x)$ rotated about the x axis.

$$J[y] = \int_a^b 2\pi y\sqrt{1+(y'(x))^2}\,dx \Rightarrow L = L(y,y') = 2\pi y\sqrt{1+(y'(x))^2}$$

independent of x $\Rightarrow$ 1st integral is $L - y'L_{y'} = c \Rightarrow 2\pi y\sqrt{1+(y'(x))^2}$

$$-\frac{2\pi y(y')^2}{\sqrt{1+(y'(x))^2}} = c \Rightarrow 2\pi y\left(1+(y'(x))^2\right) - 2\pi y(y')^2 = c\sqrt{1+(y'(x))^2} \Rightarrow$$

$$\frac{4\pi^2 y^2 - c^2}{c^2} = (y')^2$$ where $k = 2\pi / c$ and sign" + " due to $\frac{dy}{dx} > 0$ (without loss of generality). Therefore we obtain:

$$\int \frac{1}{\sqrt{k^2 y^2 - 1}} dy = x + c \underset{\substack{u=ky \\ du=kdy}}{\Rightarrow} \int \frac{1}{\sqrt{u^2 - 1}} du = kx + \tilde{c} \Rightarrow$$

$\cosh^{-1} u = kx + \tilde{c} \Rightarrow u(x) = \cosh(kx + \tilde{c}) \Rightarrow y(x) = \frac{\cosh(kx + \tilde{c})}{k}$ which is the *catenary*.

Remark: (analytic computation of the integral above)

By taking into consideration the following identities:

$\cosh^2 - \sinh^2 = 1$ as well as $\sinh' = \cosh$ and $\cosh' = \sinh$, we obtain:

$$\int \frac{1}{\sqrt{u^2 - 1}} du \underset{\substack{u=\cosh t \\ du=\sinh t dt \\ t=\cosh^{-1} u}}{=} \int \frac{1}{\sqrt{\cosh^2 t - 1}} \sinh t dt = \int \frac{\sinh t}{\sinh t} dt = t = \cosh^{-1} u$$

Economics

Let us consider $y = y(t)$ to be an individual's total capital at time t and let $r = r(t)$ be the rate that capital is spent. If $U = U(r)$ is the rate of enjoyment, then his total enjoyment over a lifetime with $0 \le t \le T$ is $E = \int_0^T e^{-\beta t} U\left(r(t)\right) dt$ where the exponential factor reflects the fact that future enjoyment is discounted over time. Initially his capital is Y, and he desires $y(T) = 0$. His capital gains interest at a rate $y' = ay - r(t)$. Furthermore, we assume (for simplicity's reason) that $a < 2\beta < 2a$. And we shall determine $r(t)$ and $y(t)$ for which the individual's total enjoyment is maximized, if his enjoyment function is $U(r) = 2\sqrt{r}$.

After some computations we will obtain:

$E(y) = 2\int_0^T e^{-\beta t}\sqrt{ay - y'}dt$. We shall employ once again the E-L equation in order to maximize this quantity. So we have:

$$L_y - \frac{d}{dt}L_{y'} = 0 \Rightarrow \frac{e^{-\beta t}a}{\sqrt{ay-y'}} + \frac{d}{dt}\left(\frac{e^{-\beta t}}{\sqrt{ay-y'}}\right) = 0 \Rightarrow$$

$$\frac{e^{-\beta t}a}{\sqrt{ay-y'}} - \frac{\beta e^{-\beta t}}{\sqrt{ay-y'}} + e^{-\beta t}\frac{d}{dt}\left(\frac{1}{\sqrt{ay-y'}}\right) = 0 \Rightarrow$$

$$\frac{a-\beta}{\sqrt{ay-y'}} + \frac{d}{dt}\left(\frac{1}{\sqrt{ay-y'}}\right) = 0 \Rightarrow$$

$$\frac{a-\beta}{\sqrt{ay-y'}} - \frac{1}{2}\frac{ay'-y''}{(ay-y')^{\frac{3}{2}}} = 0 \Rightarrow$$

$y'' + (a+2\beta-2)y' + 2a(1-\beta)y = 0$ which is a linear equation with constant coefficients. By taking into consideration the characteristic polynomial of this linear equation, which is : $r^2 + (a+2\beta-2)r + 2a(1-\beta) = 0$, where r_1 and r_2 are the roots, we have that: $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$. This is the solution given that $r(t) = ay(t) - y'(t)$.

Equilibrium shape of a membrane underload

Physical systems in equilibrium are often characterized by a function that is a local minimum of the potential energy of the system. This is one of the reasons for the great value of variational methods. Below we consider two classical problems from the theory of elasticity.

- Consider a thin membrane occupying a domain $D \subset \mathbb{R}^2$ being at a horizontal rest position and denote its vertical displacement by u(x,y). Assume also that the membrane is subject to a trasverse force (called in elasticity load) ℓ(x,y) and constrained to satisfy $u(x,y) = g(x,y)$ *for* $(x,y) \in \partial D$.

Since the membrane is assumed to be in equilibrium, its potential energy must be at a minimum. The potential energy consists of the energy stored in the stretching of the membrane and the work done by membrane against the load ℓ. The local stretching of the membrane from its horizontalrest shape is given by $d(\sqrt{1+u_x^2+u_y^2}-1)$, where d is the elasticity constant of the membrane. Assuming that the membrane's slopes are small, we approximate the local stretching by $\frac{1}{2}d\left(u_x^2+u_y^2\right)$.

The work against the load is $-\ell u$. Therefore, we have to minimize:

$Q[u]=\int_D\left[\frac{d}{2}\left(u_x^2+u_y^2\right)-\ell u\right]dxdy$. Consequently, as usual, regarding the *Gateaux Variation*, we obtain:

$$\mathrm{Q}[\mathrm{u}+\varepsilon\psi]=\int_D\left\{\frac{d}{2}\left[\left(u_x+\varepsilon\psi_x\right)^2+\left(u_y+\varepsilon\psi_y\right)^2\right]-\ell\left(u+\varepsilon\psi\right)\right\}dxdy \overset{\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}}{\Rightarrow}$$

$$\delta Q[u]=\int_D\left\{d\left[\left(u_x+\cancel{\varepsilon}^{0}\psi_x\right)\psi_x+\left(u_y+\cancel{\varepsilon}^{0}\psi_y\right)\psi_y\right]-\ell\,\psi\right\}dxdy \Rightarrow$$

$$\delta Q[u]=\int_D\left(d\nabla u\cdot\nabla\psi-\ell\,\psi\right)dxdy \Rightarrow \int_{\partial D}d\psi\,\cancel{\frac{\partial u}{\partial\vec{n}}}^{0}\,dS-\int_D\left(d\Delta u+\ell\right)\psi\,dxdy=0$$

but ψ=0 on ∂D $\Rightarrow \int_D\left(d\Delta u+\ell\right)\psi\,dxdy=0 \overset{\text{Fundamental lemma of cov}}{\Rightarrow}$

$$\begin{cases}\Delta u=-\dfrac{1}{d}\ell(x,y)\ ,\ (x,y)\in D\\ u(x,y)=g(x,y)\ ,\ (x,y)\in\partial D\end{cases}$$ which is the E-L for the membrane. In fact it is the Poisson equation. An alternative way to obtain the above result is by directly considering the E-L equation, i.e. $L_u-\frac{\partial}{\partial x}L_{u_x}-\frac{\partial}{\partial y}L_{u_y}=0 \Rightarrow$

$$-\ell-\frac{\partial}{\partial x}\left(du_x\right)-\frac{\partial}{\partial y}\left(du_y\right)=0 \Rightarrow \ell(x,y)+d\left[u_{xx}+u_{yy}\right]=0 \Rightarrow \Delta u=-\frac{1}{d}\ell(x,y)$$

The Plate equation

• Now let us consider a thin plate under a load ℓ whose amplitude with respect to a planar domain D is given by $u(x,y)$. Integration of the *elasticity equation* gives the following expression for the plate's energy.

$P[u]=\int_D \left\{ \frac{d}{2}\left[(\Delta u)^2 + 2(1-\lambda)^{-1}\left(u_{xy}^2 - u_{xx}u_{yy}\right)\right] - \ell u \right\} dxdy$ where λ is called the Poisson ratio and d is called the flexural rigidity of the plate.

The Poisson ratio is a characteristic of the medium composing the plate.
It measures the transversal compression of an element of the plate when it is stretched longitudinally. For example, $\lambda \simeq 0.5$ for rubber, and $\lambda \simeq 0.27$ for steel. The parameter d depends not only on the material constituting the plate, but also on its thickness.

To find the Euler–Lagrange equations for the plate we compute the first variation of P. To simplify the calculations we assume that the plate is clamped, i.e. both u and $\frac{\partial u}{\partial \vec{n}}$ are given on ∂D. This means that the boundary conditions imply that ψ and $\frac{\partial \psi}{\partial \vec{n}} = 0$ on ∂D. By employing the usual procedure

$$P[u+\varepsilon\psi]=\int_D \left\{ \frac{d}{2}\left[\left(u_{xx}+\varepsilon\psi_{xx}\right) + \left(u_{yy}+\varepsilon\psi_{yy}\right)\right]^2 + \frac{d}{1-\lambda}\left[\left(u_{xy}+\varepsilon\psi_{xy}\right)^2 - \right.\right.$$

$$\left.\left. -\left(u_{xx}+\varepsilon\psi_{xx}\right)\left(u_{yy}+\varepsilon\psi_{yy}\right)\right] - \ell u - \varepsilon \ell \psi \right\} dxdy \overset{\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}}{\Rightarrow}$$

$$\int_D d\left[\left(u_{xx}+\cancel{\varepsilon}^0\psi_{xx}\right) + \left(u_{yy}+\cancel{\varepsilon}^0\psi_{yy}\right)\right]\left(\psi_{xx}+\psi_{yy}\right) +$$

$$\frac{d}{1-\lambda}\left[2\left(u_{xy}+\cancel{\varepsilon}^0\psi_{xy}\right)\psi_{xy} - \psi_{xx}\left(u_{yy}+\cancel{\varepsilon}^0\psi_{yy}\right) - \psi_{yy}\left(u_{xx}+\cancel{\varepsilon}^0\psi_{xx}\right)\right] -$$

$-\ell\psi\, dxdy = 0$. Consequently we obtain:

$$\int_D \left\{ d\Delta u \Delta \psi + \frac{d}{1-\lambda}\left[2u_{xy}\psi_{xy} - \psi_{xx}u_{yy} - \psi_{yy}u_{xx}\right] - \ell\psi \right\} dxdy = 0 \;\Rightarrow$$

$$\overbrace{\int_D \{d\Delta u\Delta\psi - \ell\psi\}\, dxdy}^{I_1} + \overbrace{\int_D \frac{d}{1-\lambda}\left[2u_{xy}\psi_{xy} - \psi_{xx}u_{yy} - \psi_{yy}u_{xx}\right]dxdy}^{I_2} = 0 \Rightarrow$$

$I_1 + I_2 = 0$ Now we will calculate each integral separately, thus:

$$I_1 = -\int_D d\nabla(\Delta u)\cdot\nabla\psi\, dxdy + \int_{\partial D} d\Delta u \cancel{\frac{\partial\psi}{\partial\vec{n}}}^{0} dS = -\int_{\partial D} d\, \cancel{\psi(x,y)}^{0}\vec{\nabla}(\Delta u)\cdot\vec{n}\ dS +$$

$+\int_D d\psi(x,y)\ \Delta(\Delta u)\, dxdy$, since $div(\nabla f) = \nabla\cdot\nabla f = \Delta f$. Finally we obtain

$\boxed{I_1 = \int_D \left(d\ \Delta^2 u - \ell(x,y)\right)\psi(x,y)dxdy}$ and we proceed to compute the second integral below.

where $I_2 = \int_D \frac{d}{1-\lambda}\left\{2u_{xy}\psi_{xy} - \psi_{xx}u_{yy} - \psi_{yy}u_{xx}\right\} dxdy$. For this purpose we need to take into consideration the following useful identity:

$$2u_{xy}\psi_{xy} - u_{xx}\psi_{yy} - \psi_{xx}u_{yy} = \frac{\partial}{\partial x}\left(\overbrace{u_{xy}\psi_y - u_{yy}\psi_x}^{P}\right) + \frac{\partial}{\partial y}\left(\overbrace{u_{xy}\psi_x - u_{xx}\psi_y}^{Q}\right) = div\vec{F}$$

for the vector field $\vec{F} = (P,Q)$. Thanks to this identity and the divergence theorem of Gauss, we can convert the integral I_2 into a boundary integral. Hence:

$$I_2 = \frac{d}{1-\lambda}\int_D div\left(u_{xy}\psi_y - u_{yy}\psi_x\ ,\ u_{xy}\psi_x - u_{xx}\psi_y\right)dxdy \overset{Green}{=}$$

$$= \frac{d}{1-\lambda}\int_{\partial D}\left(u_{xy}\cancel{\psi_y}^{0} - u_{yy}\cancel{\psi_x}^{0}\ ,\ u_{xy}\cancel{\psi_x}^{0} - u_{xx}\cancel{\psi_y}^{0}\right)\cdot\overbrace{\left(\frac{dy}{dt}, -\frac{dx}{dt}\right)}^{\vec{n}} dS = 0$$

However the boundary integral involves the first derivatives of ψ. But since $\frac{\partial\psi}{\partial\vec{n}} = \nabla\psi\cdot\vec{n} = \psi_x\frac{dy}{dt} - \psi_y\frac{dx}{dt} = 0$ and ψ=0 also at the boundary, then both the normal and the tangential derivatives of ψ vanish there. The tangential derivative of ψ is $\nabla\psi\cdot\vec{T} = (\psi_x,\psi_y)\cdot\left(\frac{dx}{dt},\frac{dy}{dt}\right) = \psi_x\frac{dx}{dt} + \psi_y\frac{dy}{dt} = 0$. Therefore we obtain the following system:

$$\begin{cases} \psi_x \dfrac{dy}{dt} - \psi_y \dfrac{dx}{dt} = 0 \\ \psi_x \dfrac{dx}{dt} + \psi_y \dfrac{dy}{dt} = 0 \end{cases}$$ So we have that the of this linear system is

$\det\begin{pmatrix} \dot{y} & -\dot{x} \\ \dot{x} & \dot{y} \end{pmatrix} = (\dot{y})^2 + (\dot{x})^2 \neq 0,$ which means that it's solvable, so

$\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \psi}{\partial y} = 0$ on the boundary ∂D. This explains why the boundary integral is identically zero. At last we are able to have a representation expression for $\delta P[u] = \int_D \left(d\ \Delta^2 u - \ell\right)\psi(x,y)dxdy$ which implies that the E-L for this plate

is given by : $\boxed{\Delta^2 u = \dfrac{\ell(x,y)}{d}}$,where we denote by Δ^2 the biharmonic

operator, which is : $\Delta^2 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$.

Alternatively we can get the same results by employing directly the E-L equation

i.e. $L_u - \cancelto{0}{\sum_{i=1}^{2} \dfrac{\partial}{\partial x_i} L_{u_{x_i}}} + \sum_{i=1}^{2} \dfrac{\partial^2}{\partial x_i^2} L_{u_{x_i x_i}} + \dfrac{\partial^2}{\partial x \partial y} L_{u_{xy}} = 0$, where $x_1 = x$ and $x_2 = y$ $\Rightarrow$

$$-\ell + \frac{\partial^2}{\partial x^2}\left[d\left(u_{xx}+u_{yy}\right) - \frac{d}{1-\lambda}u_{yy}\right] + \frac{\partial^2}{\partial y^2}\left[d\left(u_{xx}+u_{yy}\right) - \frac{d}{1-\lambda}u_{xx}\right] + \frac{\partial^2}{\partial x \partial y}\left[\frac{2d}{1-\lambda}u_{xy}\right] = 0$$

$$\Rightarrow\ -\ell(x,y) + du_{xxxx} - \frac{d}{1-\lambda}u_{xxyy} + du_{yyyy} - \frac{d}{1-\lambda}u_{xxyy} + \frac{2d}{1-\lambda}u_{xxyy} + 2du_{xxyy} = 0 \Rightarrow$$

$2\dfrac{\partial^4 u}{\partial x^2 \partial y^2} + \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^4 u}{\partial y^4} = \dfrac{\ell(x,y)}{d} \Rightarrow \Delta^2 u = \dfrac{\ell(x,y)}{d}$ which is the required equation.

Another way to see that the plate's expression middle term does not contribute to the E–L equation is to observe that the corresponding integrand is in fact the Hessian $u_{xx}u_{yy} - u^2{}_{xy}$, which actually is the divergence of a vector field; i.e. it equals $\nabla \cdot (u_x u_{yy}, -u_x u_{xy}) \Rightarrow \int_D \left(u_{xx}u_{yy} - u_{xy}^2\right)dxdy = \int_D div(u_x u_{yy}, -u_x u_{xy})dxdy \overset{Gauss}{=}$

$$\int_{\partial D} (u_x u_{yy}, -u_x u_{xy}) \cdot \vec{n}\ dS = \int_{\partial D} \left(u_x u_{yy}\frac{dy}{dt} - u_x u_{xy}\frac{dx}{dt}\right)dt\ .$$

What we think that it's really interesting here is the fact that the Poisson ratio does not play any role to the final equation! In order to avoid a misunderstanding though, let us clarify that this does not mean that clamped rubber plates and clamped steel plates bend in the same way under the same load because the coefficient *d* does depend on the material (in addition of course to its dependence on the plate's thickness). However, we *could* conclude the surprising fact that for any given steel plate, there is a rubber plate that bends in exactly the same way. And one last comment. We just derived a fourth order equation above. As it turns out, the fourth-order equations are relatively rare in applications. Among the exceptions are the plate equation, the equation for the vibrations of rods that we shall derive later, and certain equations in lens design.

1.5 Second Variation

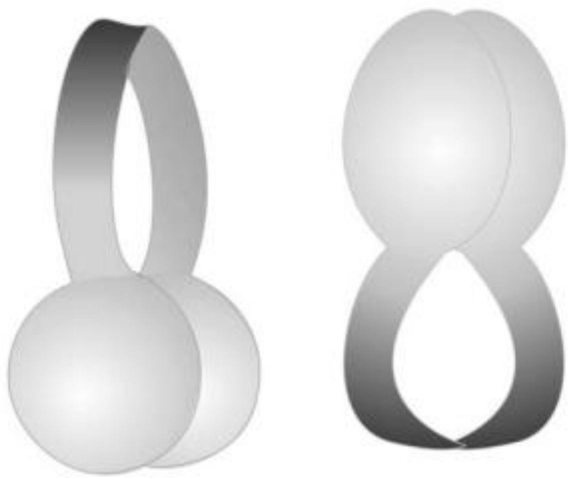
It is well known that equating the first derivative of a real (scalar) function $f(x)$ to zero only provides a necessary condition for potential minimizers of f. To determine whether a stationary point x_0 (where $f'(x_0) = 0$) is indeed a local minimizer, we have to examine higher derivatives of f . For example, if $f''(x_0) > 0$, we can conclude that indeed x_0 is a local minimizer.

Similarly, to verify that a function u is a local minimum of some functional, we must compute the second variation of the functional, and evaluate it at u. When considering a general functional $Q(u)$, the first variation was defined as: $\boxed{\delta Q[u] = \frac{d}{d\varepsilon} Q[u+\varepsilon\psi]\Big|_{\varepsilon=0} = 0}$ for ψ in an appropriate function space. Similarly, if the first variation of Q at u is zero, we define the second variation of Q there through: $\boxed{\delta^2 Q[u] = \frac{d^2}{d\varepsilon^2} Q[u+\varepsilon\psi]\Big|_{\varepsilon=0} = 0}$. Just like the case of the first variation, the second variation is a functional of ψ that depends on u.

A functional Q such that $\delta^2 Q(u)(\psi) > 0$ for all appropriate u and ψ is called (strictly) <u>*convex*</u>. Such functionals are particularly useful to identify since they have a unique minimizer as we will see later in the existence theory of minimizers. However, a very reasonable question one may ask is if there always exists a unique minimum. We shall try to answer this particularly interesting question later.

This question has far reaching implications in many branches of science and technology. In fact, it is also raised in unexpected disciplines such as philosophy and even theology. In contrast to the ethical monotheism of the Prophets of Israel, the Hellenic monotheism was based on logical arguments, basically claiming that since God is the best, i.e. optimal, and since the best must be unique, then there is only one god. This argument did not convince the ancient Greeks (were they aware of the possibility of many local extrema?), who stuck to their belief in a plurality of gods.

Indeed one of the intriguing questions raised by Plateau and many mathematicians after him was whether the minimal surface problem has a unique solution for any given spanning curve . The answer is no! In the figure below we depict an example of a spanning curve for which there exist more than one minimal surfaces.



Next we will try to derive a necessary condition for deciding if a critical point is indeed an extremum or not.

With respect to the discussion above, since $I[\cdot]$ admits a minimum at the function u, i.e. $i(\tau)$ admits a minimum at $\tau=0 \Rightarrow i'(0)=0,\ i''(0)\geq 0$. Now in view of $i(\tau)=\int_U L(x,u+\tau v,Du+\tau Dv)dx$, we calculate that $\left('=\dfrac{d}{d\tau} \text{ and analogously } ''=\dfrac{d^2}{d\tau^2}\right)$

$i'(\tau)=\int_U \left\{ \sum_{i=1}^{n} L_{p_i}(x,u+\tau v,Du+\tau Dv)v_{x_i} + L_z(x,u+\tau v,Du+\tau Dv)v \right\} dx$. Analogously:

$$i''(\tau)=\int_U \left\{ \sum_{i,j=1}^{n} L_{p_i p_j}(x,u+\tau v,Du+\tau Dv)v_{x_i}v_{x_j} + 2\sum_{i=1}^{n} L_{p_i z}(x,u+\tau v,Du+\tau Dv)v_{x_i}v + \right.$$

$\left. + L_{zz}(x,u+\tau v,Du+\tau Dv)v^2 \right\} dx$. By setting now $\tau=0$, we obtain the following expression:

$$0\leq i''(0)=\int_U \left\{ \sum_{i,j=1}^{n} L_{p_i p_j}(x,u,Du)v_{x_i}v_{x_j} + 2\sum_{i=1}^{n} L_{p_i z}(x,u,Du)v_{x_i}v + L_{zz}(x,u,Du)v^2 \right\} dx \ \ \forall v\in C_0^\infty(U). \ (*)$$

We can extract useful information from the inequality above. First note, after a routine approximation argument, that estimate $(*)$ is valid for any Lipschitz continuous function v vanishing on ∂U. We then fix $\xi\in\mathbb{R}^n$ and by borrowing the idea from the mollifiers as an inspiration here, we define:

$\boxed{v(x)=\varepsilon\rho\left(\dfrac{x\cdot\xi}{\varepsilon}\right)\zeta(x)}$ $(x\in U)$ where $\zeta\in C_0^\infty(U)$ and $\rho:\mathbb{R}\to\mathbb{R}$ is the periodic

"zig-zag" function, defined by $\rho(x)=\begin{cases} x, & 0\leq x\leq 1/2 \\ 1-x, & 1/2\leq x\leq 1 \end{cases}$ ("1-periodic", i.e. $\rho(x+1)=\rho(\mathrm{x})\ \forall x\in\mathbb{R}$) .

Thus $|\rho'|=1$ a.e. Observe further that

$$v_{x_i}=\cancel{\varepsilon}\rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\frac{\xi_i}{\cancel{\varepsilon}}\zeta(x)+\varepsilon\rho\left(\frac{x\cdot\xi}{\varepsilon}\right)\zeta_{x_i} \ \Rightarrow\ v_{x_i}=\rho'\left(\frac{x\cdot\xi}{\varepsilon}\right)\xi_i\zeta(x)+O(\varepsilon) \ \text{ as } \ \varepsilon\to 0.$$

and so our substitution of the new, well-defined function $v(x)$ into the inequality $(*)$ yields:

$0\leq \int_U \sum_{i,j=1}^{n} L_{p_i p_j}(x,u,Du)\left(\rho'\right)^2 \xi_i\xi_j\zeta^2 dx+O(\varepsilon)$ (all the other remaining terms involve at least the first power of "ε" , which justifies the existence of $\mathrm{O}(\varepsilon)$). At this point we recall that $|\rho'|=1$ a.e. and send $\varepsilon\to 0$, thereby obtaining the following inequality:

$0\leq \int_U \sum_{i,j=1}^{n} L_{p_i p_j}(x,u,Du)\xi_i\xi_j\zeta^2 dx$. Now, since this estimate holds true for all $\zeta\in \mathrm{C}_0^\infty(U),$ we deduce

$\boxed{\sum_{i,j=1}^{n} L_{p_i p_j}(x,u,Du)\xi_i\xi_j\geq 0}$ $\left(\xi\in\mathbb{R}^n, x\in U\right)$. Actually this necessary condition contains a clue as to the basic convexity assumption on the Lagrangian L required for the existence theory which we shall examine analytically later.

1.6 *A few words and remarks regarding compactness*

When we studied in calculus the problem of minimizing real valued functions, we had at our disposal a theorem that guaranteed that a continuous function in a closed bounded set K must achieve its maximum and minimum in K. Establishing a priori the existence of a minimizer for a functional is much harder. To understand the difficulty involved, let us recall from calculus that if A is a set of real numbers bounded from below, then it has a well-defined infimum. Moreover, there exists at least one sequence $a_n \subset A$ that converges to the infimum. Consider now, for example, the Dirichlet integral $G(u)$ defined over the functions in:

$$\mathrm{B} = \left\{ u \in C^1(D) \cap C(\bar{D}),\ u = g \text{ for } x \in \partial D \right\}$$

for some domain D. Clearly G is bounded from below by 0. Therefore, there exists a sequence:

$\{u_k\}$ such that $\boxed{\lim_{k\to\infty} G(u_k) = \inf_{u\in B} G(u)}$. Such a $\{u_k\}$ is called a <u>minimizing sequence</u>.

The trouble is that a priori it is not clear that the infimum is achieved, and in fact, it is not even clear that the minimizing sequence u_k has convergent subsequences in B. Achieving the infimum is not always possible even for a sequence of numbers (for example if they are defined over an open interval), but we do like to retain some sort of convergence. In $\mathbb{R}^n$ we know that any bounded sequence has at least one convergent subsequence. This is the compactness property of bounded sets in $\mathbb{R}^n$. Is it also true for the space B? The answer is no. There are examples in which a Fourier series converges strongly to a discontinuous function. This is a case in which a sequence of functions in B– the partial sums of the Fourier series –does not have any subsequence converging to a function in B.
It turns out that, if we consider infinite bounded sequences of functions in Hilbert spaces, we can still maintain to some extent the property of compactness. Unfortunately we have to weaken the meaning of convergence.

Definition:

A sequence of functions $\{f_n\}$ in a Hilbert space H is said to converge strongly to a function f in H, if $\lim_{n\to\infty}\|f_n - f\| = 0$

A sequence of functions $\{f_n\}$ in a Hilbert space H is said to converge weakly to a function f in H, if $\boxed{\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle}$ $\forall g \in H.$

Note that by the *Riemann – Lebesgue lemma* , any (infinite) orthonormal sequence in a given infinite-dimensional inner product space converges weakly to 0. The following theorem explains why we call the property presented above, *weak convergence*, and also provides the *fundamental compactness property* of Hilbert spaces.

Theorem

Let H be a Hilbert space. Then the following statements hold:

(i) Every strongly convergent sequence $\{u_n\}_{n\in\mathbb{N}}$ in H also converges weakly. But the converse is not necessarily true. Moreover the weak limit is unique.

(ii) If $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to u, then $\|u\| \le \liminf_{n\to\infty}\|u_n\|$. The equality can be achieved iff $u_n \xrightarrow[n\to\infty]{strongly} u$ in H.

(iii) Every sequence $\{u_n\}_{n\in\mathbb{N}}$ in H that is bounded, in the sense that $\|u_n\| \le C\ \forall n \in \mathbb{N}$, has at least one convergent subsequence.

Note: Normally H is supposed to also be *separable* (i.e. it has a *dense* and *countable* subset). This statement (iii) is a special case of the *Banach - Alaoglu* theorem.

(iv) Every weakly convergent subsequence in H is bounded.

Proof:

(i) We need to show that if $\|u_n - u\| \xrightarrow[n\to\infty]{} 0$ in H, then $u_n - u \xrightarrow{weakly} 0$. For this purpose we write for an arbitrary function $f \in H$: $\left|\langle u_n, f\rangle - \langle u, f\rangle\right| \overset{\substack{\text{linearity of}\\ \text{inner product}}}{=} \left|\langle u_n - u, f\rangle\right| \overset{C-S}{\le} \|u_n - u\|\|f\| \xrightarrow[n\to\infty]{} 0 \Rightarrow$ $\langle u_n, f\rangle \xrightarrow[n\to\infty]{} \langle u, f\rangle\ \forall f \in H \Rightarrow u_n \xrightarrow{weakly} u.$

We shall prove now that the converse is not always true. For doing so, we use a counterexample. Let us consider the sequence $\{\sin nx\}_{n\in\mathbb{N}} \subset L^2\left(\left[0,2\pi\right]\right)$. Then by using the *Riemann - Lebesque* lemma we get that $\int_a^b \sin nx\, f(x)dx = \langle \sin nx,\, f(x)\rangle \xrightarrow[n\to\infty]{} 0\ \forall f \in H$. Hence we obtain $\sin nx \xrightarrow[n\to\infty]{weakly} 0,$ while $\|\sin nx\| = \sqrt{\pi/2}$ and therefore $\{\sin nx\}$ does not converge strongly to 0, otherwise we would have had $\sqrt{\pi/2} = 0,$ by the limit's uniqueness, which is of course a contradiction if we select $f(x) = \sin nx.$

However let us note here that the following holds true: $\begin{cases} u_n \xrightarrow{weakly} u \\ \|u_n\| \to \|u\| \end{cases} \Rightarrow\ u_n \xrightarrow{strongly} u$, because:

$\|u_n - u\|^2 = \langle u_n - u, u_n - u\rangle = \langle u,u\rangle + \langle u_n,u_n\rangle - 2\langle u_n,u\rangle \xrightarrow[n\to\infty]{} 0$, since $\langle u_n,u\rangle \to \langle u,u\rangle$ and also $\langle u_n,u_n\rangle = \|u_n\|^2 \to \|u\|^2 = \langle u,u\rangle.$

(ii) Supposing now that $u_n \xrightarrow{weakly} u$, then in particular $\langle u_n,u\rangle \to \|u\|^2$, since by the definition of weak convergence $\langle u_n, f\rangle \to \langle u, f\rangle\ \forall f \in H,$ so after selecting $f = u$, we get the result above. Now by employing the *Cauchy – Schwarz* inequality we get the required as following:

$$\|u\|^2 = \langle u,u\rangle = \lim_{n\to\infty}\langle u_n,u\rangle \overset{\substack{monotonicity \\ of\ the\ \lim}}{\leq} \|u\| \liminf_{n\to\infty} \|u_n\|.$$

(iii) Let $(h_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Morover let $H_0 = \overline{span\langle h_1, h_2, ...\rangle}$. Then H_0 is separable as the set of all finite linear combinations of points in (h_n) with rational coefficients is a *countable* and *dense* subset of H_0. For each $n \in \mathbb{N}$ let $f_n : H_0 \to \mathbb{R}$ be defined for all $h \in H_0$ by $\boxed{f_n(h) = \langle h_n, h\rangle}$.

Afterwards, we observe that each f_n is a linear functional on H_0, something that it follows from the linearity of the inner product. Furthermore, f_n is bounded with $\|f_n\| \leq \|h_n\|$, since for each $h \in H,$ we have that $|f_n(h)| = |\langle h_n, h\rangle| \overset{C-S}{\leq} \|h_n\|\|h\| \ \Rightarrow\ \dfrac{|f_n(h)|}{\|h\|} \leq \|h_n\| \ \Rightarrow\ \sup\limits_{\substack{h\neq 0 \\ h\in H}} \dfrac{|f_n(h)|}{\|h\|} \leq \|h_n\| \ \Rightarrow\ \|f_n\| \leq \|h_n\|.$

Consequently, $\{f_n\}_{n\in\mathbb{N}}$ is a bounded sequence of linear functionals on the separable space H_0. By employing now *Helly's theorem* , we get that $\{f_n\}_{n\in\mathbb{N}}$ has a weak*-convergent subsequence, say $\{f_{n_k}\}_{k\in\mathbb{N}}$, which weak*-converges to $f_0 \in H_0$. The *Riesz representation theorem* for Hilbert spaces now states that there exists an $h_0 \in H_0$ such that $f_0(h) = \langle h, h_0\rangle\ \forall h \in H_0$. So $\{f_{n_k}\}_{k\in\mathbb{N}}$ weak*-converges to $f_0(h)$ and so for every $h \in H \ \Rightarrow\ \lim\limits_{k\to\infty} f_{n_k}(h) = \langle h, h_0\rangle$ and also $\lim\limits_{k\to\infty}\langle h, h_{n_k}\rangle = \langle h, h_0\rangle$

The reader can find more about the Helly's Theorem, the Riemann-Lebesgue Lemma and the Riesz representation Theorem in [B1].

Finally let P be the orthogonal projection of H onto H_0. Then for each $k \in \mathbb{N}$, we have that: $\left\langle (I-P)(h), h_{n_k} \right\rangle = \langle I-P(h), h_0 \rangle$. Hence for each $h \in H$ we have that $\lim_{k\to\infty} \left\langle h_{n_k}, h \right\rangle = \langle h, h_0 \rangle$. At the end, from the characteriziation of weak convergence in Hilbert spaces, we obtain that $h_{n_k} \xrightarrow[k\to\infty]{weakly} h_0 \in H$. So every bounded sequence in H has a weakly convergent subsequnce.

(iv) we omit the proof of the last statement, since we need to use the *principle of uniform boundness* which is beyond the scope of the work at hand. □

Remark

If a Hilbert space is finite-dimensional, e.g. an Euclidean space, then the concepts of weak and strong convergence are the same!

Part "c" actually states that we do have compactness in Hilbert spaces with strong convergence replaced by weak.

An alternative way to prove "c" would be by using a "*diagonal* argument". We will shortly sketch this alternative proof. Employ the C-S inequality to obtain:

$\left|\langle u_j, v\rangle\right| \le \|u_j\| \|v\|$ and by our assumptions we get $\|u_j\| \le c \ \forall j \in \mathbb{N},$ since $\{u_n\}$ is bounded. Hence $\left(\langle u_j, v\rangle\right)_{j\in\mathbb{N}}$ is a bounded sequence of real numbers for each fixed $v \in H$. The "trick" now is to apply a *diagonal* argument, by first constructing subsequences for $v_1, v_2, ...$ and then take the diagonal sequence. So first we note that $\left\{\langle u_j, v_1\rangle\right\}_{j\in\mathbb{N}}$ is a bounded sequence of real numbers ($\mathbb{R}$ is a complete space), so it has had a convergent subsequence $\left\{\langle u_{j_{1,k}}, v_1\rangle\right\}_{k\in\mathbb{N}}$ with, say, $\left|\langle u_{j_{1,k}}, v_1\rangle - a_1\right| \le 2^{-k} L_0$ and $k \le j_{1,k} \to \infty$. And we continue in this way. Inductively, with exactly the same argument,we get for $m = 2, 3, ...$ subsequences $\left\{u_{j_{m,k}}\right\}$ with $j_{m-1,k} \le j_{m,k} \to \infty$ as $k \to \infty$ and $\left|\langle u_{j_{m,k}}, v_n\rangle - a_n\right| \le 2^{-k} L_0$ for $n = 1, 2, ..., m$. Finally we take the diagonal sequence $\left\{u_{j_{m,m}}\right\}$. Then as usual $\left|\langle u_{j_{m,m}}, v_n\rangle - a_n\right| \le 2^{-m} L_0$ for $n = 1, 2, ..., m$. Hence the subsequence $\langle u_{j_{m,m}}, v_n\rangle$ is convergent for each *n*.

Moreover all the discussion above (especially the theorem) can also be regarded as an introductory section or a first approach-touch to the next section dedicated to the theory of existence of minimizers where we shall extensively deal with such matters and ideas.

1.7 A short reference to computational methods (Ritz-Galerkin)

Below we shall refer to two very important methods. We begin with a brief presentation of the main ideas and concept of what we call the Ritz method, avoiding a technical presentation, but instead prefering a more intuitive approach.

Ritz method

Consider the problem of minimizing a functional $G(u)$, where u is taken from some Hilbert space H. The Ritz method is based on selecting a basis B (preferably orthonormal) for H, and expressing the unknown minimizer u in terms of the elements φ_n of B:

$\boxed{u = \sum_{n=1}^{\infty} a_n \varphi_n}$ .The functional minimization problem has been transformed to an algebraic (infinite-dimensional) minimization problem in the unknown coefficients a_n. This process is similar to our discussion after the introduction of the Rayleigh quotient later in the context. Practically, we can use the fact that since the series expansion for u is convergent, we expect the coefficients to decay as $n \to \infty$. We can therefore truncate the expansion at some finite term N and write: $\boxed{u \approx \sum_{n=1}^{N} a_n \varphi_n}$. (see the Rayleigh Quotient, developed later below, for more details regarding the convergence of the generalized Fourier series and the conditions under which we can achieve it. Briefly we mention that if we consider an orthonormal and complete sequence of eigenfunctions as a basis, and u to be continuous and piecewise differentiable, this can guarantee uniform convergence of the generalized Fourier series to the function. Under our assumptions, many interesting qualitative properties hold, like the Parseval equality and the Riemann-Lebesque lemma). This approximation leads to a finite-dimensional algebraic system that can be handled by a variety of numerical tools, like the *Finite Elements Method* which is regarded to be among the most important methods.

In general, an interesting question is: what would be an optimal basis?
It is clear that some bases are superior to others. For example, the series above might converge much faster in one basis than in another basis. In fact, the series might even be finite if we are fortunate (or clever). For instance, suppose that we happened to choose a basis that contains the minimizing function *u* itself. Then the series expansion would consist of just one term!
On the other hand, we might face the problem of not having *any* obvious candidate for a basis. This would happen when we consider a Hilbert space of functions defined over a general domain that has no symmetries.

To better demonstrate the Ritz method (and illustrate our comment above), we return to the problem of reconstruction of a function from its gradient (or you may have seen it as phase reconstruction problem). In typical applications D is the unit disk. We shall seek the minimizer of $K(u)$ in the space $H^1(D)$. What would be a good basis for this space? The first candidate that comes normally to mind, or at least what we would expect at first sight is: (in polar coordinates)

$\boxed{\left\{ J_n\left(\frac{a_{n,m}}{\alpha}r\right)\cos n\vartheta \right\} \cup \left\{ J_n\left(\frac{a_{n,m}}{\alpha}r\right)\cos n\vartheta \right\}}$, which are the eigenfunctions of the Dirichlet problem for the eigenvalue problem of Laplacian operator in the disk. The corresponding eigenvalues are given by:

$\boxed{\lambda_{n,m} = \left(\frac{a_{n,m}}{\alpha}\right)^2}$ for $n = 0,1,2,....$ and $m = 1,2,....$ where each eigenvalue is of multiplicity 2, except for the case $n = 0$. J_n now is the $Bessel's$ function of $1st\ kind$ in honor of the German mathematician and astronomer Friedrich Wilhelm Bessel $(1784–1846)$ who was among the first to study these functions. Below we will briefly refer to the Bessel functions and some basic properties they have as a remind.

So, we know that the sequence of eigenfunctions, form a complete and orthonormal system for the space of continuous functions in the disk of radius α with respect to the inner product (with weight r):

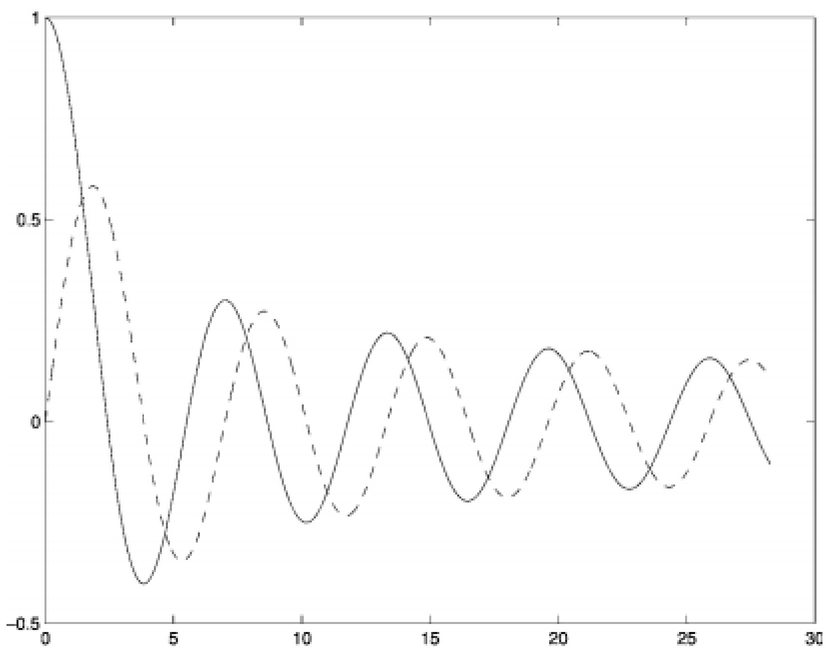
$\langle f, g \rangle = \int_0^{2\pi}\int_0^{\alpha} f(r,\vartheta)g(r,\vartheta)r\ drd\vartheta$ and the generalized Fourier-Bessel expansion for a function, say $h(r,\vartheta)$, over that disk is given by: $\boxed{h(r,\vartheta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{a_{n,m}}{\alpha}r\right)\left(A_{n,m}\cos n\vartheta + B_{n,m}\sin n\vartheta\right)}$

and the Fourier-Bessel coefficients are given by:

$$\boxed{A_{n,m} = \frac{2}{\pi\alpha^2 J_{n+1}\left(a_{n,m}\right)}\int_0^{2\pi}\int_0^{\alpha} h(r,\vartheta)J_n\left(\frac{a_{n,m}}{\alpha}r\right)\cos n\vartheta\ rdrd\vartheta},$$

$$\boxed{B_{n,m} = \frac{2}{\pi\alpha^2 J_{n+1}\left(a_{n,m}\right)}\int_0^{2\pi}\int_0^{\alpha} h(r,\vartheta)J_n\left(\frac{a_{n,m}}{\alpha}r\right)\sin n\vartheta\ rdrd\vartheta}$$

The careful reader may wonder why we obtain this representation of the Fourier-Bessel coefficients. It may seem to be weird at first sight, but it makes sense after taking into consideration the result below, whose proof we omit here (it is neither technical nor difficult, just for space's economy):
For $n = 0,1,2,\dots$, $m = 1,2,\dots$ and $a_{n,m}$ be the zeros of the Bessel's function of first kind J_n, we have that $\boxed{\int_0^\alpha r J_n^2\left(\frac{a_{n,m}}{\alpha} r\right) dr = \frac{\alpha^2}{2} J_{n+1}^2(a_{n,m})}$. Another interesting fact is that for every nonnegative integer n, the *zeros* of J_n form a sequence of real positive numbers $a_{n,m}$ that diverge to ∞ as $m \to \infty$.
Furthermore, the difference between two consecutive zeros converges to π in the limit $m \to \infty$.
We close this brief reference to the Bessel functions and some of their properties by the observation that the following recursive formula holds: $\boxed{sJ_{n+1}(s) = nJ_n(s) - sJ_n'(s)}$. Notice that according to this formula it is enough to compute J_0, and then use this function to evaluate J_n for $n > 0$. The Bessel functions J_0 and J_1 are depicted in the figure below:



The Bessel functions J_0 (solid line) and J_1 (dashed line)

After this short break in order to present some facts about Bessel functions, let us return to our initial investigation about a suitable basis. Let us also recall that we have considered this candidate as a basis:

$$\left\{ J_n\left(\frac{a_{n,m}}{\alpha} r\right) \cos n\vartheta \right\} \cup \left\{ J_n\left(\frac{a_{n,m}}{\alpha} r\right) \cos n\vartheta \right\}$$

While this basis would certainly do the work, it turns out that in practice physicists use another basis. Phase reconstruction is an important step in a process called adaptive optics, in which astronomers correct images obtained by telescopes. These images are corrupted by atmospheric turbulence (this is similar to scintillation of stars when they are observed by a naked eye). Thus astronomers measure the phase and use these measurements to adjust flexible mirrors to correct the image. The Dutch physicist Frits Zernike (1888–1966) proposed in 1934 to expand the phase in a basis in which he replaced the Bessel functions above by radial functions that are polynomials in *r*.

The *Zernike basis* for the space L_2 over the unit disk consists of functions that have the same angular form as the Bessel basis above. The radial Bessel functions, though, are replaced by orthogonal polynomials. Using complex number notation, we write the Zernike functions as

$\boxed{Z_n^m(r,\vartheta)=R_n^m(r)e^{im\vartheta}}$ where the polynomials R_n^m are orthogonal over the interval (0,1) with respect to the inner product $\langle f(r),g(r)\rangle=\int_0^1 f(r)g(r)rdr$. For some reason Zernike did not choose the polynomials to be orthogonal, but rather set $\langle R_n^m,R_{n'}^m\rangle=\left[1/2(n+1)\right]\delta_{n,n'}$. In fact one can write the polynomials explicitly $\big($they are only defined for $n\geq|m|\geq 0\big)$:

$$\boxed{R_n^m(r)=\begin{cases}\sum_{l=0}^{(n-|m|)/2}\dfrac{(-1)^l(n-l)!}{l!\left(\frac{1}{2}(n+|m|)-l\right)!\left(\frac{1}{2}(n-|m|)-l\right)!}r^{n-2l} & \text{, for } n-|m| \text{ even}\\ 0 & \text{, for } n-|m| \text{ odd.}\end{cases}}$$

The phase is expanded in the form: $\boxed{u(r,\vartheta)=\sum_{n,m}a_{n,m}Z_n^m(r,\vartheta)}$. We then substitute this expansion into our minimization problem to obtain an infinite dimensional quadratic minimization problem for the unkown coefficients $\{a_{n,m}\}$. In practice the series is truncated at some finite term, and then, since the functional is quadratic in the unknown coefficients, the minimization problem is reduced to solving a system of linear algebraic equations. Notice that this method has a fundamental practical flaw: since the functional involves derivatives of *u*, and the derivatives of the Zernike functions are not orthogonal, we need to evaluate all the inner products of these derivatives. Moreover, this implies that the matrix associated with the linear algebraic system we mentioned above is generically full; in contrast if we select a clever basis, we can obtain linear algebraic systems that are associated with sparse matrices, whose solution can be computed much faster.

Galerkin method - weak solutions

Let us consider the following minimization problem:

$\boxed{\min Y[u]=\int_D \left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}u^2+fu\right)dx}$ where D is a bounded domain in $\mathbb{R}^n$ and f is a given continuous function, satisfying without loss of generality $|f|\le 1$ in D. Next we shall compute the first variation $Y[u+\varepsilon\psi]=\int_D \left(\frac{1}{2}|\nabla u+\varepsilon\nabla\psi|^2+\frac{1}{2}(u+\varepsilon\psi)^2+fu+\varepsilon f\psi\right)dx \Rightarrow \left.\frac{d}{d\varepsilon}Y[u+\varepsilon\psi]\right|_{\varepsilon=0}$

$\Rightarrow$ and taking into account the identity: $|\nabla u|^2+\varepsilon^2|\nabla\psi|^2+2\varepsilon\nabla u\cdot\nabla\psi=|\nabla u+\varepsilon\nabla\psi|^2 \;\Rightarrow$

$$\delta Y[u]=\int_D\left(\cancel{\varepsilon}^0|\nabla\psi|^2+\nabla u\cdot\nabla\psi+\left(u+\cancel{\varepsilon\psi}^0\right)\psi+f\psi\right)dx=\int_D\left(\nabla u\cdot\nabla\psi+(u+f)\psi\right)dx$$

We seek a minimizer in the space $H^1(D)$ and take the variation ψ also to belong to this space as well. So the problem takes the following variational form: $(*)\ \int_D\left(\nabla u\cdot\nabla\psi+u\psi+f\psi\right)dx=0\ \forall\psi\in H^1(D)$.

Additionally we assume that the minimizer $u\in C^2\left(\bar{D}\right)$ and that ∂D is also smooth. Consequently we get:

$$0=\int_D\left(\nabla u\cdot\nabla\psi+(u+f)\psi\right)dx=\int_{\partial D}\psi\,\cancel{\frac{\partial u}{\partial\vec{n}}}^0\,dS+\int_D\left\{-\psi\overbrace{\nabla\cdot(\nabla u)}^{div(\nabla u)=\Delta u}+(u+f)\psi\right\}dx=\int_D\left\{-\Delta u+u+f\right\}\psi\,dx\ \Rightarrow$$

$$\int_D\left\{-\Delta u+u+f\right\}\psi\,dx=0\ ,\ \forall\psi\in H^1(D)\ \overset{\substack{Fundamental\\ lemma}}{\Rightarrow}\ (\#)\ \boxed{-\Delta u+u=-f}\ \text{ for } x\in D \text{ and } \frac{\partial u}{\partial\vec{n}}=0 \text{ on } \partial D.$$

Equation $(*)$ however, is more general than (#), since it also holds under the weaker assumption that $u\in C^1(D)\cap C\left(\bar{D}\right)$ and not $u\in C^2\left(\bar{D}\right)$ as demanded previously, or at least is a suitable limit of functions in $C^1(D)$. Therefore, we call $(*)$ the weak formulation of (#).

Boris Grigoryevich Galerkin (4 March 1871 – 12 July 1945), born in Polotsk, Vitebsk Governorate, Russian Empire, was a Soviet mathematician and an engineer.

$\boxed{\textit{Theorem}}$

The weak formulation $(*)$ has a unique solution u^* which is a minimizer of the minimization problem we presented above.

<u>Proof</u>:

Since $|f| \le 1$ in D, $f^2 \le 1 \Rightarrow -\dfrac{f^2}{2} \ge -\dfrac{1}{2}$ and $\dfrac{u^2+f^2}{2} + uf \ge 0$ $\left(\text{since}\,(u+f)^2 \ge 0\right)$. Therefore $\dfrac{u^2}{2} + uf \ge -\dfrac{f^2}{2} \ge -\dfrac{1}{2}$ $\forall x \in D$. Consequently $Y[u] = \displaystyle\int_D \left(\frac{|\nabla u|^2}{2} + \frac{u^2}{2} + fu \right) dx \ge \int_D \left(\frac{|\nabla u|^2}{2} - \frac{f^2}{2} \right) dx \ge$

$\ge \displaystyle\int_D \left(\frac{|\nabla u|^2}{2} - \frac{1}{2} \right) dx \ge \int_D -\frac{1}{2}\, dx = -\frac{|D|}{2}$ $\Rightarrow$ $Y[u] \ge -\dfrac{|D|}{2}$ and thus the functional is bounded from below.

Now let $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence, i.e. $\lim\limits_{n\to\infty} Y[u_n] = I = \inf\limits_{u\in H^1(D)} Y[u]$. The C-S inequality implies:

$$\left|\int_D fu\ dx\right| \le \int_D |f||u|\,dx \overset{|f|\le 1}{\le} \int_D |u|\,dx \overset{C-S}{\le} \left(\int_D 1^2 dx\right)^{1/2} \left(\int_D |u|^2\, dx\right)^{1/2} = \sqrt{|D|}\,\|u\|_{L^2(D)} \ \Rightarrow\ \boxed{\left|\int_D fu\ dx\right| \le |D|^{1/2}\,\|u\|_{L^2(D)}}$$

Now we note that it suffices to consider u_n such that $Y[u_n] < Y[0] = 0$. That's because $Y[u] \ge -\dfrac{|D|}{2}$ (which is a negative quantity) $\forall u \in$ Admissible set and $\exists$ inf since there exists a lower bound. So we can choose u_n such that the minimizing sequense is negative. Now it follows that:

$\dfrac{1}{2}\|u_n\|^{\cancel{2}}_{L^2(D)} \le \dfrac{1}{2}\|u_n\|^2_{H^1(D)} \le \left|\displaystyle\int_D fu_n dx\right| \le |D|^{1/2}\,\cancel{\|u_n\|_{L^2(D)}}$ $\Rightarrow$ $\boxed{\|u_n\|_{L^2(D)} \le C}$,bounded where $C = 2\sqrt{D}$. Let us now illustrate how we made the transit in the first and the second inequality. Regarding the first:

$\|u\|_{H^1(D)} = \left[\|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)}\right]^{1/2} \Rightarrow$ $\|u\|^2_{L^2(D)} \le \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)} = \|u\|^2_{H^1(D)}$. As far for the second

$$Y[u_n] = \int_D \left(\frac{|\nabla u_n|^2}{2} + \frac{u_n^2}{2} + fu_n \right) dx \overset{assumption}{<} Y[0] = 0 \ \Rightarrow\ \frac{1}{2}\int_D \left(|\nabla u_n|^2 + |u_n|^2\right) dx \le -\int_D fu_n dx \le \left|\int_D fu_n dx\right| \ \Rightarrow$$

$$\frac{1}{2}\|u_n\|^2_{H^1(D)} \le \left|\int_D fu_n dx\right|.$$

But we also discover (again from the inequalities above) that $\frac{1}{2}\|u_n\|^2_{H^1(D)} \le |D|^{1/2}\|u_n\|_{L^2(D)} <$

$< |D|^{1/2}\overbrace{2|D|^{1/2}}^{C} \Rightarrow \|u_n\|^2_{H^1(D)} \le 4|D| \Rightarrow \boxed{\|u_n\|_{H^1(D)} \le C}$ as well for the same constant C. Now we know that every bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H^1(D)$, in the sense that $\|u_n\|_{H^1(D)} \le C \ \forall n \in \mathbb{N}$, has got at least one weakly convergent subsequence. Let us denote this subsequence by $\{u_{n_k}\}_{k\in\mathbb{N}} \subset H^1(D)$. Moreover we shall denote its weak limit by u^*. But since $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to u^*, i.e. $\langle u_{n_k}, v\rangle \xrightarrow[k\to\infty]{} \langle u^*, v\rangle \ \forall v \in H^1(D)$, we also know that $\|u^*\|_{H^1(D)} \le \liminf_{k\to\infty}\|u_{n_k}\|_{H^1(D)}$, furthermore the fact that weak convergence in $H^1(D)$ implies weak convergence in $L^2(D)$, (something that can be verified very easy (it is almost obvious), because $\langle u,v\rangle_{H^1(D)} = \int_D (uv + \nabla u\cdot\nabla v)\,dx = \langle u,v\rangle_{L^2(D)} +$ $+\langle \nabla u, \nabla v\rangle_{L^2(D)}$, the continuity of inner product and the uniqueness of the limit) it follows that:

(Below we denote the subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ as $\{u_n\}_{n\in\mathbb{N}}$,this notation simplifies the computations a lot)

$$Y[u^*] = \frac{1}{2}\|u^*\|^2_{H^1(D)} + \int_D fu^* dx \le \frac{1}{2}\liminf_{n\to\infty}\|u_n\|^2_{H^1(D)} + \overbrace{\lim_{n\to\infty}\int_D fu_n dx}^{\lim_{n\to\infty}\langle u_n, f\rangle_{L^2(D)}} \le \lim_{n\to\infty} Y[u_n] \overset{\substack{\text{minimizing}\\ \text{sequence}}}{=} I = \inf_{u\in H^1(D)} Y[u] \le Y[u^*]$$

since $u^* \in H^1(D)$. At this point, in order to fully justify the inequalities above, let us note that:

$$\lim_{n\to\infty} Y[u_n] = \lim_{n\to\infty}\left[\int_D \left\{\overbrace{\left(\frac{|\nabla u_n|^2 + |u_n|^2}{2}\right)}^{(1)} + fu_n\right\}dx\right] \overset{(1)}{=} \lim_{n\to\infty}\frac{\|u_n\|^2_{H^1(D)}}{2} + \lim_{n\to\infty}\int_D fu_n dx$$

, where we observe that:

$\lim_{n\to\infty}\frac{\|u_n\|^2_{H^1(D)}}{2} \ge \lim_{n\to\infty}\frac{\|u_n\|^2_{L^2(D)}}{2} \ge \frac{1}{2}\liminf_{n\to\infty}\|u_n\|^2_{L^2(D)}$. Thus indeed: $\lim_{n\to\infty} Y[u_n] \ge \frac{1}{2}\liminf_{n\to\infty}\|u_n\|^2_{L^2(D)} + \lim_{n\to\infty}\int_D fu_n dx$.

These computations lead to $Y[u^*] \le I \le Y[u^*] \Rightarrow Y[u^*] = I = \inf\limits_{u\in H^1(D)} Y[u]$. Hence u^* is indeed a minimizer of the problem.

Uniqueness

Now we fix $\psi \in H^1(D)$ and we observe that

$g(\varepsilon) = Y[u^* + \varepsilon\psi] = Y[u^*] + \dfrac{\varepsilon^2}{2}\|\psi\|^2_{H^1(D)} + \varepsilon\left\langle u^*,\psi\right\rangle_{H^1(D)} + \varepsilon\int\limits_D f\psi dx$. We saw previously that $g(\varepsilon)$ has got a minimum at $\varepsilon = 0$. Thus $g'(0) = 0$. As a consequence we get

$$g'(\varepsilon) = \varepsilon\|\psi\|^2_{H^1(D)} + \left\langle u^*,\psi\right\rangle_{H^1(D)} + \int\limits_D f\psi dx \overset{g'(0)=0}{\Rightarrow} \left\langle u^*,\psi\right\rangle_{H^1(D)} + \int\limits_D f\psi dx = 0 \Rightarrow \boxed{\left\langle u^*,\psi\right\rangle_{H^1(D)} = -\int\limits_D f\psi dx}$$

$\forall\, \psi \in H^1(D)$, which is the *weak formulation.*

Because this relation holds for every $\psi \in H^1(D)$, we have established the existence of a weak solution of the weak formulation above. Now let us for contradiction assume that there exists two solutions u_1^*, u_2^* . Then we form the difference $v^* = u_1^* - u_2^*$ and obtain for v^* that $\left\langle v^*,\psi\right\rangle_{H^1(D)} = -\int\limits_D f\psi dx + \int\limits_D f\psi dx = 0$. Thus $\left\langle v^*,\psi\right\rangle_{H^1(D)}$ $\forall\, \psi \in H^1(D)$. In particular we choose $\psi = v^*$, since $v^* \in H^1(D)$, so $\|v^*\|_{H^1(D)} = 0 \Rightarrow v^* = 0$. □

Remarks:

- If we are able to prove that $u^* \in C^2(D)\cap C^1(\bar{D})$, then by the previous problem, we have the existence of a classical solution to the elliptic b.v.p. $\begin{cases} -\Delta u + u = -f, \text{ in D} \\ \dfrac{\partial u}{\partial \vec{n}} = 0 \text{ , on } \partial\text{D} \end{cases}$.
- *Disadvantage and cure*:

The proof was not constructive. The limit u^* was identified as a limit of an as yet unknown sequence. We therefore introduce now a practical method for computing the solution. The idea is to construct a chain of subspaces $H^{(1)}, H^{(2)}, ..., H^{(k)}, ...$ with the property that $H^{(k)} \subset H^{(k+1)}$ and $\dim H^{(k)} = k$, such that their union exhausts the full $H^1(D)$, i.e. there exists a basis $\{\varphi_k\}_{k=1}^{\infty}$ of $H^1(D)$ with $\varphi_k \in H^{(k)}$. In each subspace $H^{(k)}$ now, we select a basis $\{\varphi_1^k, \varphi_2^k, ..., \varphi_k^k\}$ and we write the weak formulation in $H^{(k)}$ as $\boxed{\left\langle v^k,\varphi_i^k\right\rangle_{H^1(D)} = -\int\limits_D f\varphi_i^k dx}$, for $i = 1, 2, ..., k$.

If we further express the unknown function v^k in terms of the basis φ^k, i.e. $\boxed{v^k = \sum\limits_{j=1}^{k} a_j^k \varphi_j^k}$ we obtain for the unknown coefficient vector $\vec{a}^k$ the following algebraic (system) for $i = 1, 2, ..., k$:

$$\left\langle v^k, \varphi_i^k \right\rangle_{H^1(D)} = -\int_D f \varphi_i^k dx \;\Rightarrow\; \left\langle \sum_{j=1}^{k} a_j^k \varphi_j^k, \varphi_i^k \right\rangle_{H^1(D)} = -\int_D f \varphi_i^k dx \;\Rightarrow$$

$$\sum_{j=1}^{k} a_j^k \left\langle \varphi_i^k, \varphi_j^k \right\rangle_{H^1(D)} = -\int_D f \varphi_i^k dx \text{ , for } i = 1, 2, ..., k \;\Rightarrow\; \begin{cases} K_{ij}^k = \left\langle \varphi_i^k, \varphi_j^k \right\rangle_{H^1(D)} \\ d_i = -\int_D f \varphi_i^k dx \end{cases} \Rightarrow \boxed{\sum_{j=1}^{k} K_{ij}^k a_j^k = d_i} \;\; (i = 1, ..., k)$$

K is called *stiffness matrix* and the vector $\vec{d}$ is called the *force vector*.

The algebraic system above has got a unique solution for all k and the sequence v^k converges strongly to u^*(for more about Galerkin and Finite Elements Methods see [B3] and/or [L3]). This practical method, which we just presented above, is called the Galerkin method after the Russian mathematician and engineer Boris Galerkin (1871-1945). Actually, for the minimization problem at hand, Galerkin and Ritz methods happen to be identical. Sometimes, these two methods are confused (or perhaps just fused) with each other and go together under the title Galerkin-Ritz method. We point out however that the Galerkin method is more general than the Ritz method in the sense that it is not limited to problems where the weak formulation is derived from a variation of a functional. In fact, given any pde of the abstract from $L[u] = f$, where L is a linear or even a non linear operator,we are able to apply the Galerkin method by writing the equation to the form $\langle L[u] - f, \psi \rangle = 0 \;\forall \psi \in H$, for a suitbale Hilbert space H. Afterwards, by integrating by parts, we will be able to throw some derivatives of u to ψ and thus obtain a formulation which requires less regularity, for its solution.

This is the central idea and the hard core of the weak formulation into the well known Sobolev spaces.
The last but not the least is an important question regarding how we will choose the subspaces $H^{(k)}$.
A very important class of such subspaces forms a numerical method called *finite elements*. A very useful choice for the subspace's basis is the *splines* (piecewise continuous, or continuous, differentiable functions) like linear, cubic, hermitian splines etc.

2. Existence of minimizers

In the current section we shall identify some conditions on the Lagrangian *L* which ensure the functional I does indeed have a minimizer, at least within an appropriate Sobolev space.

2.1 Coercivity, lower semi-continuity

- **Coercivity**

Before proceeding to stating the "strict" defintions, let us make a short "intuitive" remark, to better introduce the reader to the subject of coercivity and the reason why it is necessary for our job here. Let us note that even a smooth function $f:\mathbb{R}\to\mathbb{R}$ and bounded below need not attain its infimum. Take for example e^x. This example (intuitive approach) suggests that we in general will need some hypothesis controlling $I[w]$ for "large" functions w. Certainly the most effective way to ensure this will be to assume that $I[w]$ "grows rapidly as $|w|\to\infty$".
More specifically, let us assume $1<p<\infty$ is fixed. We will then suppose that there exist constants $a>0,\ \beta\geq 0$ such that $\boxed{L(x,z,p)\geq a|p|^q-\beta}$ for all $p\in\mathbb{R}^n, z\in\mathbb{R}, x\in U$. Therefore:

$$I[w]=\int_U L(x,w,\nabla w)dx\geq\int_U\left\{a|p|^q-\beta\right\}dx\overset{p=\nabla w}{=}\int_U\left[a|\nabla w|^q-\beta\right]dx=a\int_U|\nabla w|^q\,dx-\beta\int_U dx=$$

$$=a\|\nabla w\|^q_{L^q(U)}-\beta|U|\ \Rightarrow\ \boxed{I[w]\geq a\|\nabla w\|^q_{L^q(U)}-\beta|U|}\text{ for } a>0,\ \beta\geq 0.$$

Thus $I[w]\to\infty$ as $\|\nabla w\|_{L^q(U)}\to\infty$. It is customary to call this estimate *coercivity condition* on $I[\cdot]$. Furthermore, this inequality leads us to assume that it is quite reasonable to define $I[w]$ not only for smooth functions u, but also for functions w belonging to the Sobolev space $W^{1,q}(U)$, where $W^{1,q}(U)=\left\{u\in L^q(U)\mid\ \exists\nabla u\text{ in the weak sense and }\nabla u\in L^q(U)\right\}$, that satisfy the boundary condition $w=g$ on ∂U in the *trace* sense. After all, the wider the class of functions w for which $I[w]$ is defined, the more candidates we will have for a minimizer. We will henceforth write:
$\boxed{A=\left\{w\in W^{1,q}(U)\mid w=g\text{ on }\partial U\text{ in the \textit{trace} sense}\right\}}$ in order to denote the class of admissible functions w. Note here in view of inequality $L(x,z,p)\geq a|p|^q-\beta$ that $I[w]$ is defined (but may equal $+\infty$) for each $w\in A$. For a detailed presentation of the Traces in Sobolev Spaces see [B1].

- **Lower semi-continuity**

Next, let us observe that although a continuous functions $\mathbb{R}\to\mathbb{R}$ satisfying a coercivity condition does indeed attain its infimum, our integral function $I[\cdot]$ in general will not. To better understand the problem, set $\boxed{m=\inf_{w\in A}I[w]}$ and choose functions $u_k\in A\ \ (k=1,2,\ldots)$ so that $I[u_k]\xrightarrow[k\to\infty]{}m$. We call $\{u_k\}_{k=1}^\infty$ a <u>minimizing sequence</u>. We would like to show now that some subsequence of $\{u_k\}_{k=1}^\infty$

converges to an actual minimizer. For this purpose, however, we need some kind of compactness and this is defintely a problem, since $W^{1,q}(U)$ is an infinite dimensional space. Indeed, if we utilize the coercivity inequality above, it turns out (we will see that shortly afterwards) that we can only conclude that the minimizing subsequence lies in a bounded subset of $W^{1,q}(U)$. But this does not imply that there exists any subsequence which converges in $W^{1,q}(U)$!

We therefore turn our attention to the weak topology. Since we are assuming $1<q<\infty$ so that $L^q(U)$ is *reflexive*, we conclude that there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(U)$ such that: $\begin{cases} u_{k_j} \xrightarrow{weakly} u \text{ in } L^q(U) \\ \nabla u_{k_j} \xrightarrow{weakly} \nabla u \text{ in } L^q(U;\mathbb{R}^n) \end{cases}$. Hereafter we will abbreviate it by saying that $u_{k_j} \xrightarrow{weakly} u$ in $W^{1,q}(U)$. Furthermore, $u = g$ on ∂U in the *trace* sense and so $u \in A$.

Consequently by shifting to the weak topology we have recovered enough compactness from the coercivity inequality to deduce that $u_{k_j} \xrightarrow{weakly} u$ in $W^{1,q}(U)$ for an appropriate subsequence. But now another difficulty arises, for in essentially all cases of interest the functional $I[\cdot]$ is not continuous with respect to weak convergence! In other words, we cannot deduce from $I[u_k] \xrightarrow[k\to\infty]{} m$ and $u_{k_j} \xrightarrow{weakly} u$ in $W^{1,q}(U)$ that $(\not\Rightarrow)$ $I[u] = \lim_{j\to\infty} I[u_{k_j}]$ and thus that u is a minimizer. The problem in fact here is that $\nabla u_{k_j} \xrightarrow{weakly} \nabla u$ does not imply that $\nabla u_{k_j} \longrightarrow \nabla u$ a.e. It is quite possible for instance that the gradients ∇u_{k_j}, although bounded in $L^q(U)$, are oscillating more and more widely as $k \to \infty$. What saves us is the final, key observation that we don't really need the full strength of $I[u] = \lim_{j\to\infty} I[u_{k_j}]$, but it would suffice instead to know only $\boxed{I[u] \le \liminf_{j\to\infty} I[u_{k_j}]}$. Then we could deduce that $I[u] \le m$, but owing to $m = \inf_{w\in A} I[w]$, $m \le I[u] \Rightarrow u$ is indeed a minimizer.

Definition:
$I[\cdot]$ is (sequentially) *weakly lower semicontinuous* on $W^{1,q}(U)$, provided $I[u] \le \liminf_{k\to\infty} I[u_k]$, whenever $u_k \xrightarrow{weakly} u$ in $W^{1,q}(U)$.

Convexity

Let us look instantly back to our second variation analysis and recall we derived there the inequality $\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} L_{p_i p_j}\left(x,u(x),\nabla u(x)\right)\xi_i\xi_j \geq 0}$ for $\xi \in \mathbb{R}^n, x \in U$ holding as a necessary condition, whenever u is a smooth minimizer. In fact this inequality strongly suggests that it is very reasonable to assume that L is *convex* in its third argument.

$\boxed{\textit{Theorem}}$ (Weak lower semi-continuity)
Assume that L is smooth, bounded below and in addition the mapping $p \mapsto L(x,z,p)$ is *convex* with respect to p for each $z \in \mathbb{R}, x \in U$. Then $I[\cdot]$ is *weakly lower semicontinuous* on $W^{1,q}(U)$, *i.e.* $I[u] \leq \liminf_{k\to\infty} I[u_k]$
For the proof the reader can see [B5] and [B8].

2.2 Existence and uniqueness

Now we have developed all the necessary ideas and tools in order to be able to continue by presenting the basic theorems regarding the existence and uniqueness of minimizers

$\boxed{\textit{Theorem}}$ (Existence of minimizers)
Assume $I[\cdot]$ satisfies the coercivity condition $L(x,z,p) \geq a\left|p\right|^q - \beta$ for constants $a>0, \beta \geq 0$ and $\forall p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U$ and is also *convex* in the variable p. Suppose also that $A \neq \varnothing$. Then there exists at least one function $u \in A$ solving $I[u] = \min_{w\in A} I[w]$

Proof.
$\boxed{1.}$ Set $m = \inf_{w\in A} I[w]$. If $m = +\infty$ we are done. Therefore let us assume that m is finite. Select a minimizing sequence $\left\{u_k\right\}_{k=1}^{\infty}$. Then $I[u_k] \xrightarrow[k\to\infty]{} m.$
$\boxed{2.}$ We may take $\beta = 0$ in the convexity condition inequality, since (as previously above) we could otherwise just consider $\tilde{L} = L + \beta$. Thus $L \geq a\left|p\right|^q$ and so $I[w] \geq a\int_U \left|\nabla w\right|^q dx$. Now since $m < +\infty$, we conclude (in combination with $I[u_k] \xrightarrow[k\to\infty]{} m$) that: $\sup_k \left\|\nabla u_k\right\|_{L^q(U)} < +\infty$ because of the fact $a\left\|\nabla u_k\right\|_{L^q(U)}^q = a\int_U \left|\nabla u_k\right|^q dx \leq I[u_k] \xrightarrow[k\to\infty]{} m < +\infty \overset{a>0}{\Rightarrow}$ we obtain clearly that $\boxed{\sup_k \left\|\nabla u_k\right\|_{L^q(U)} < +\infty}$ (1).

$\boxed{3.}$ Now fix any function $w \in A$. Since w, u_k both equal g on ∂U in the *trace* sense, we have $u_k - w \in W_0^{1,q}(U)$ $\big($by the definition of the set A it follows that $w, u_k \in W^{1,q}(U)\big)$. Therefore we are able to apply the *Poincaré* 's inequality, i.e. $\|v\|_{L^q(U)} \le c\|\nabla v\|_{L^q(U)}$ $\forall v \in W_0^{1,q}(U)$, so in the case at hand:

$$\|u_k\|_{L^q(U)} \overset{\substack{\text{triangular}\\ \text{inequality}}}{\le} \|u_k - w\|_{L^q(U)} + \|w\|_{L^q(U)} \overset{\text{Poincaré}}{\le} c\|\nabla u_k - \nabla w\|_{L^q(U)} + c\|\nabla w\|_{L^q(U)} \overset{\substack{\text{triangular}\\ \text{inequality}}}{\le} 2c\|\nabla w\|_{L^q(U)} + c\|\nabla u_k\|_{L^q(U)} \le$$

$$\le 2c\|\nabla w\|_{L^q(U)} + c\sup_k \|\nabla u_k\|_{L^q(U)} \overset{\sup_k \|\nabla u_k\|_{L^q(U)} < +\infty}{\le} C\,,$$ since $\|\nabla w\|_{L^q(U)} \le \|\nabla w\|_{W^{1,q}(U)} < \infty$ because $w \in W^{1,q}(U)$.

Consequently we conclude that: $\boxed{\sup_k \|u_k\|_{L^q(U)} < +\infty}\,(2)$. Estimates (1) and (2) now imply that $\{u_k\}_{k=1}^{\infty}$ is bounded in $W^{1,q}(U)$.

$\boxed{4.}$ As a consequence there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W^{1,q}(U)$ such that $u_{k_j} \xrightarrow{weakly} u$ in $W^{1,q}(U)$. We assert next that $u \in A$. To see that, note that for $w \in A$ as above, $u_k - w \in W_0^{1,q}(U)$. Now $W_0^{1,q}(U)$ is closed, linear subspace of $W^{1,q}(U)$ and so, by *Mazur's* theorem we conclude that it is weakly closed. At this point let us remind the statement of Mazur, i.e. a *convex* and *closed* subset of *reflexive Banach space*, say X, is *weakly closed*.

Hence $u - w \in W_0^{1,q}(U)$. Consequently the trace of u on ∂U is g. In view of the previous theorem now $\big($remember that by assumption $L = L(x, z, p)$ is convex with respect to the variable p and obviously bounded below by the convexity condition$\big) \Rightarrow$ $I[\cdot]$ is lower semi-continuous, i.e. $I[u] \le \liminf_{j\to\infty} I[u_{k_j}] = m$. But since $u \in A$, it follows that: $I[u] = m = \min_{w\in A} I[w]$. □

We turn our attention now to the problem of uniqueness. In general there can be many minimizers, so we need to require further assumptions in order to ensure uniqueness. Suppose for instance $L = L(x, p)$ (does not depend on z) and there exists $\vartheta>0$ such that $\underbrace{\boxed{\sum_{i,j=1}^{n} L_{p_i p_j}(p, x)\xi_i\xi_j \ge \vartheta|\xi|^2}}_{\text{uniformly convex condition}}$ for $p, \xi \in \mathbb{R}^n$ and $x \in U$. This condition says that the mapping $p \mapsto L(x, p)$ is uniformly convex $\forall x \in U$.

$\boxed{\textit{Theorem}}$ (Uniqueness of minimizers)

Suppose that $L = L(x, p)$ and also that the uniformly convex condition, i.e. $\sum_{i,j=1}^{n} L_{p_i p_j}(p, x)\xi_i\xi_j \ge \vartheta|\xi|^2$ hold. Then a minimizer $u \in A$ of the functional $I[\cdot]$ is unique.

Proof:

$\boxed{1.}$ Assume $u,\tilde{u} \in A$ are both minimizers of $I[\cdot]$ over A. Then $v = \dfrac{u+\tilde{u}}{2} \in A$ as well and we claim:

$I[v] \le \dfrac{I[u]+I[\tilde{u}]}{2}$ with strict inequality unless $u = \tilde{u}$ a.e.

$\boxed{2.}$ To see this, note from the uniform convexity assumption that we have

$$L(p,x) = L\big(q+(p-q),x\big) \overset{Taylor}{=} L(q,x) + \nabla_p L(q,x)\cdot(p-q) + \frac{1}{2}\big(p-q\big)^T D_p^2 L(\tilde{q},x)(p-q) \overset{\substack{uniform\\convexity}}{\ge}$$

$\ge L(q,x) + \nabla_p L(q,x)\cdot(p-q) + \dfrac{\vartheta}{2}|p-q|^2$ for $x \in U$ and $p,q \in \mathbb{R}^n$. Here by $D_p^2 L(\tilde{q},x)$ we denote the *Hessian* matrix of the second partial derivatives, $\tilde{q} \in B\big(q,x\big)$ and we justify the last inequality by noting that: $\dfrac{1}{2}\big(p-q\big)^T D_p^2 L(\tilde{q},x)(p-q) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{n} L_{p_i p_j}(\tilde{q},x)(p_i-q_i)(p_j-q_j) \ge \dfrac{\vartheta}{2}|p-q|^2$.

Set now $q = \dfrac{\nabla u + \nabla\tilde{u}}{2}$, $p = \nabla u$ in this inequality and integrate over U to get:

$I[u] \ge I[v] + \displaystyle\int_U \nabla_p L\left(\frac{\nabla u+\nabla\tilde{u}}{2},x\right)\cdot\left(\frac{\nabla u-\nabla\tilde{u}}{2}\right)dx + \frac{\vartheta}{8}\int_U |\nabla u - \nabla\tilde{u}|^2\,dx$. Similarly by setting again $q = \dfrac{\nabla u+\nabla\tilde{u}}{2}$, but $p = \nabla\tilde{u}$ this time, we obtain:

$I[\tilde{u}] \ge I[v] - \displaystyle\int_U \nabla_p L\left(\frac{\nabla u+\nabla\tilde{u}}{2},x\right)\cdot\left(\frac{\nabla u-\nabla\tilde{u}}{2}\right)dx + \frac{\vartheta}{8}\int_U |\nabla u - \nabla\tilde{u}|^2\,dx$. Finally, if we add them and divide by 2, we deduce: $\boxed{\dfrac{I[u]+I[\tilde{u}]}{2} \ge I[v] + \dfrac{\vartheta}{8}\displaystyle\int_U |\nabla u-\nabla\tilde{u}|^2\,dx \ge I[v]}\,(*)$ $\;\Rightarrow I[v] \le \dfrac{I[u]+I[\tilde{u}]}{2}$

$\boxed{3.}$ As $I[u] = I[\tilde{u}] = \min\limits_{w\in A} I[w] \le I[v]$ $\Rightarrow$ $\dfrac{I[u]+I[\tilde{u}]}{2} \le I[v]$ $\Rightarrow$ $\dfrac{I[u]+I[\tilde{u}]}{2} = I[v]$ $\Rightarrow (*)$ and $\vartheta>0$

$\displaystyle\int_U |\nabla u-\nabla\tilde{u}|^2\,dx = 0$ $\Rightarrow$ $|\nabla u - \nabla\tilde{u}| = 0$ $\Rightarrow$ $\nabla u = \nabla\tilde{u}$ in U a.e. Finally. since $u = \tilde{u} = g$ on ∂U in the trace sense, it follows that $u = \tilde{u}$ a.e. in U. □

2.3 Weak solutions of E-L equations

We wish next to demonstrate that any minimizer $u \in A$ of $I[\cdot]$ solves the E-L equation in some suitable sense. We will need some growth conditions on L and its derivatives. Let us hereafter suppose the *growth* conditions:

$\boxed{|L(x,z,p)| \le c\left(|p|^q + |z|^q + 1\right)}$ and also $\boxed{|\nabla_i L(x,z,p)| \le c\left(|p|^{q-1} + |z|^{q-1} + 1\right)}$ $(i = z$ and $p)$,for all $p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U$. Let us now refer to the motivation for the definition of weak solution. Consider the b.v.p. for the E-L pde, associated with our functional $L,$ which for a smooth minimizer u reads: $(*)\begin{cases} -\sum_{i=1}^{n}\left(L_{p_i}(x,u,\nabla u)\right)_{x_i} + L_z(x,u,\nabla u) = 0 \text{ in } U \\ u = g \text{ on } \partial U \end{cases}$

If we multiply now this b.v.p. by a test function $v \in C_0^\infty(U)$ and next integrate by parts, we arrive at the equality $\int_U \left\{\sum_{i=1}^{n} L_{p_i}(x,u,\nabla u)v_{x_i} + L_z(x,u,\nabla u)v\right\}dx = 0$ (due to the compact support of v).

Of course this is the identity we first obtained in our derivation of this equality in the introductory chapter. Now assume $u \in W^{1,q}(U)$. Then using one of the growth conditions, to be more specific $|\nabla_i L(x,z,p)| \le c\left(|p|^{q-1} + |z|^{q-1} + 1\right)$for $i = z$ or $p,$ so $\left|\nabla_p L(x,u,\nabla u)\right| \le c\left(|u|^{q-1} + |\nabla u|^{q-1} + 1\right) \in L^{q'}(U)$ for $q' = \dfrac{q}{q-1}$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Similarly we obtain $|\nabla_z L(x,u,\nabla u)| \le c\left(|u|^{q-1} + |\nabla u|^{q-1} + 1\right) \in L^{q'}(U)$

Consequently, we see using a standard approximation argument that the equality $\int_U \left\{\sum_{i=1}^{n} L_{p_i}(x,u,\nabla u)v_{x_i} + L_z(x,u,\nabla u)v\right\}dx = 0$ is valid for any $v \in W_0^{1,q}(U)$. Actually we get this result because of the fact that $C_0^\infty(U)$ is *dense* to $W_0^{1,q}(U)$.

Definition:

We say $u \in A$ is a *weak solution* of the b.v.p. $(*)$ for the E-L equation provided

$\boxed{\int_U \left\{\sum_{i=1}^{n} L_{p_i}(x,u,\nabla u)v_{x_i} + L_z(x,u,\nabla u)v\right\}dx = 0}$ for all $v \in W_0^{1,q}(U)$

$\boxed{\textit{Theorem}}$ (Solution of E-L equation)

Assume L verifies the growth conditions above and $u \in A$ satisfies $I[u] = \min_{w\in A} I[w]$. Then u is a *weak solution* of the b.v.p. $(*)$.

<u>Proof</u>

Fix $v \in W_0^{1,q}(U)$ and set $i(\tau) = I[u+\tau v]$ for $\tau \in \mathbb{R}$. In view of the first growth condition, i.e. $|L(x,z,p)| \le c\left(|p|^q + |z|^q + 1\right)$, we see that $i(\tau)$ is finite for all τ, due to the fact that $|i(\tau)| \le \int_U \left|L\left(x, u+\tau v, \nabla u + \tau\nabla v\right)\right| dx \le \int_U c\left(|u+\tau v|^q + |\nabla u + \tau \nabla v|^q + 1\right) dx$ and since $u, v \in W^{1,q}(U) \Rightarrow |i(\tau)| < +\infty$. Let now $\tau \ne 0$ and write the difference quotient:

$\dfrac{i(\tau)-i(0)}{\tau} = \displaystyle\int_U \frac{L\left(x,u+\tau v,\nabla u+\tau\nabla v\right) - L\left(x,u,\nabla u\right)}{\tau} dx$ and in order to simplify a bit the notation we denote the difference quotient: $L^\tau(x) = \dfrac{L\left(x,u+\tau v,\nabla u+\tau\nabla v\right) - L\left(x,u,\nabla u\right)}{\tau}$ for a.e. $x \in U$. Clearly

$L^\tau(x) \xrightarrow[\tau\to 0]{} \displaystyle\sum_{i=1}^n L_{p_i}\left(x,u,\nabla u\right) v_{x_i} + L_z\left(x,u,\nabla u\right) v$ a.e. and this because of:

$L = L\left(x,u+\tau v,\nabla u+\tau\nabla v\right) \Rightarrow \left.\dfrac{dL}{d\tau}\right|_{\tau=0} = L_z\left(x,u,\nabla u\right)v + \nabla_p L\left(x,u,\nabla u\right)\cdot \nabla v$ and also (by definition)

$\left.\dfrac{dL}{d\tau}\right|_{\tau=0} = \lim_{\tau\to 0} L^\tau(x) = \lim_{\tau\to 0} \dfrac{L\left(x,u+\tau v,\nabla u+\tau\nabla v\right) - L\left(x,u,\nabla u\right)}{\tau}$. Furthermore we observe that

$L^\tau(x) = \dfrac{1}{\tau}\displaystyle\int_0^\tau \frac{d}{ds} L\left(x,u+sv,\nabla u+s\nabla v\right) ds =$ (which is the fundamental theorem of calculus)

$= \dfrac{1}{\tau}\displaystyle\int_0^\tau \left\{ \sum_{i=1}^n L_{p_i}\left(x,u+sv,\nabla u+s\nabla v\right) v_{x_i} + L_z\left(x,u+sv,\nabla u+s\nabla v\right) v \right\} ds$.

Now we will use Young's identity $ab \le \dfrac{a^q}{q} + \dfrac{b^{q'}}{q'}$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Then, since $u, v \in W^{1,q}(U)$ *growth* inequalities and *Young's* inequality imply after some elementary calculations that: $|L^\tau(x)| \le c\left(1 + |u|^q + |v|^q + |\nabla u|^q + |\nabla v|^q\right) \in L^1(U)$ for each $\tau \ne 0$. Consequently, we may invoke Lebesque's <u>dominated convergence theorem</u> to conclude that $i'(0)$ exists and equals

$\displaystyle\int_U \left\{ \sum_{i=1}^n L_{p_i}\left(x,u,\nabla u\right) v_{x_i} + L_z\left(x,u,\nabla u\right) v \right\} dx = 0$. But then since $i(\cdot)$ has a minimum at $\tau = 0$, we know that $i'(0) = 0$ and thus u is a *weak solution* according to our definition above. □

Remarks:

• In general E-L equation will have other solutions which do not correspond to minima of $I[\cdot]$ (we shall discuss it analytically later, in the section of "critical points"). However, in the special case that the joint mapping $(p,z)\mapsto L(x,z,p)$ is *convex* for each x, then each weak solution is in fact a minimizer. To see this, suppose $u\in A$ solves $\begin{cases} -\sum\limits_{i=1}^{n}\left(L_{p_i}\left(x,u,\nabla u\right)\right)_{x_i}+L_z\left(x,u,\nabla u\right)=0, & U \\ u=g, & \partial U \end{cases}$

in the weak sense and select any $w\in A$. Utilizing now the convexity of the mapping $(p,z)\mapsto L(x,z,p)$ we have $L(x,z,p)+\nabla_p L(x,z,p)\cdot(q-p)+\nabla_z L(x,z,p)\cdot(w-z)\le L(x,w,q)$ (convexity's identity) and let $p=\nabla u(x),\ q=\nabla w(x),\ \ z=u(x)$ and $w=w(x)$ and integrate over U to find that:
$I[u]+\int\limits_U\left\{\nabla_p L(x,u,\nabla u)\cdot(\nabla w-\nabla u)+\nabla_z L(x,u,\nabla u)\cdot(w-u)\right\}dx\le I[w]$. Now in view of the E-L equation the integral is zero and therefore $I[u]\le I[v]$ for each $w\in A$. Let us now clarify why the integral equals zero. $\left(\text{remember here the notation: } div\left(\vec{\nabla}_p L\right)=\nabla\cdot\left(\nabla_p L\right)=\sum\limits_{i=1}^{n}\left(L_{p_i}\right)_{x_i}\right)$

$\int\limits_U\left\{\nabla_p L(x,u,\nabla u)\cdot(\nabla w-\nabla u)\right\}dx=\int\limits_{\partial U}\cancel{(w-u)}^{0}\,\vec{\nabla}_p L(x,u,\nabla u)\cdot\vec{n}dS-\int\limits_U div\left(\vec{\nabla}_p L\right)(w-u)dx$, so we get:

$$\int\limits_U\left\{\nabla_p L(x,u,\nabla u)\cdot(\nabla w-\nabla u)+\nabla_z L(x,u,\nabla u)\cdot(w-u)\right\}dx=\int\limits_U\underbrace{\left\{-div\left(\vec{\nabla}_p L\right)+\nabla_z L\right\}}_{E-L}\left(w-u\right)dx=0.$$

• In the case of systems, we have exactly the same results as here. Also the *growth*, the *convexity* and the *coercivity* conditions are the same, as well as the assumptions in existence, uniqueness and the weak solution theorems. Needless to say that the results are the same too :)

2.4 Regularity

In the current section we shall discuss the smoothness of minimizers to our energy functionals. This is generally a quite difficult topic and so we shall make a number of strong simplifying assumptions, most notably that $\underline{L \text{ depends only on } p}$. Thus we henceforth assume that our functional $I[\cdot]$ has got the following form. $I[w]=\int\limits_U\left\{L\left(\nabla w\right)-wf\right\}dx$ for $f\in L^2\left(U\right)$. We also take $q=2$ and suppose as well the growth condition $\left|\nabla_p L(p)\right|\le c\left(\left|p\right|+1\right)$ for $p\in\mathbb{R}^n$. Then, any minimizer $u\in A$ is a weak solution of E-L equation $-\sum\limits_{i=1}^{n}\left(L_{p_i}(\nabla u)\right)_{x_i}=f$ in U,

$\begin{pmatrix} \text{due to } \tilde{L}_z = -f, \\ \text{here } \tilde{L} = L(\nabla w) - wf \end{pmatrix} \Rightarrow -div\left(\nabla_p L(\nabla u)\right) = f$ in U that is (in weak formulation): we employ simply integration by parts to obtain

$$\int_U \sum_{i=1}^n L_{p_i}(\nabla u) v_{x_i} dx = \int_U fv\ dx \Rightarrow \int_U \nabla_p L(\nabla u) \cdot \nabla v\ dx = \int_U fv\ dx\ \forall v \in H_0^1(U), \text{ where } H_0^1(U) \equiv W_0^{1,2}(U)$$

Second derivative estimates

We now intend to show that if $u \in H^1(U)$ is a weak solution of the non-linear pde (E-L equation) $-\sum_{i=1}^n \left(L_{p_i}(\nabla u)\right)_{x_i} = f$ in U , then $u \in H^2_{loc}(U)$! This is the main regularity's result. But in order to establish this, we need to strengthen our *growth* conditions on *L*. Let us first of all assume that $\boxed{\left|D^2 L(p)\right| \le c}$,where here $D^2 L(p)$ is the Hessian matrix of second derivatives, for $p \in \mathbb{R}^n$. In addition, we also suppose that L is *uniformly convex*, i.e. there exists a constant ϑ>0 such that $\boxed{\sum_{i,j=1}^n L_{p_i p_j}(p)\xi_i \xi_j \ge \vartheta|\xi|^2}$ for all $p, \xi \in \mathbb{R}^n$. Clearly this is some sort of non-linear analogue of the *uniform ellipticity* condition for linear pdes, where we have had $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \ge \vartheta|\xi|^2$ for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$ for the partial differential operator L such that (in divergence form):

$\mathrm{L}u = -\sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x)u,$ because we observe that :

$$div\left(A\nabla u\right) = div\left(\sum_{j=1}^n a^{ij}(x) u_{x_j}\right) = \sum_{i,j=1}^n \left(a^{ij}(x) u_{x_j}\right)_{x_i} = \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i,j=1}^n u_{x_j}\left(a^{ij}(x)\right)_{x_i} = \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} +$$

$+\sum_{j=1}^n u_{x_j} \underbrace{\left(\sum_{i=1}^n \frac{\partial}{\partial x_i} a^{ij}(x)\right)}_{b^j(x)}$, so after a change of the index from j to i in the last some, we justify why the above expression is indeed in divergence form.

$\boxed{Theorem}$ (Second derivatives for minimizers)

(i) Let $u \in H^1(U)$ be a *weak solution* of the E-L $-\sum_{i=1}^{n}\left(L_{p_i}(\nabla u)\right)_{x_i} = f$ in U, where L satisfies the *growth* condition $\left|D^2L(p)\right| \le c$ and the *uniform convexity* condition $\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \ge \vartheta\left|\xi\right|^2$.

Then: $\boxed{u \in H^2_{loc}(U)}$

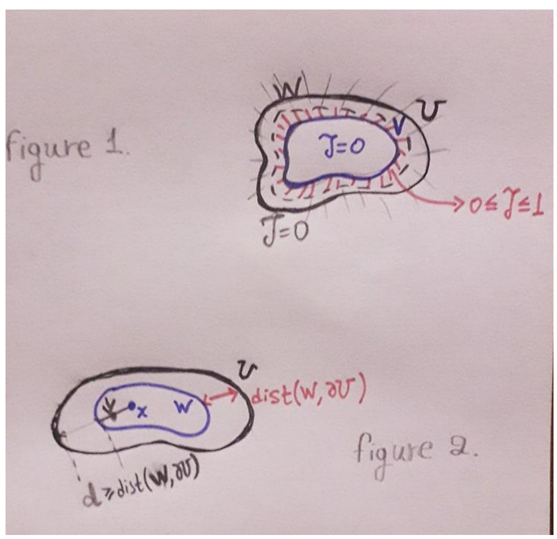
(ii) If in addition $u \in H^1_0(U)$ and ∂U is C^2, then: $\boxed{u \in H^2}$ with the estimate $\boxed{\left\|u\right\|_{H^2(U)} \le c\left\|f\right\|_{L^2(U)}}$

Proof:

$\boxed{1.}$ Fix any open set $V \subset\subset U$ and choose then an open set W so that $V \subset\subset W \subset\subset U$.

Select a smooth cut off function ζ satisfying $\begin{cases} \zeta \equiv 1, \text{ on } V \\ \zeta \equiv 0, \text{ in } \mathbb{R}^n \setminus W \\ 0 \le \zeta \le 1, \text{ inbetween} \end{cases}$

(for the cut off function, see figure 1 below)



- Figure 1 illustrates how we define our ''cut off'' function over these three sets, the two of them are compactly contained to the third.
- Figure 2 illustrates one assertion we will make below, in our attempt to prove a significant for our proof statement, according to which a suitable selected change of variables will still belong to the superset *U.*

The purpose of a cut off function in the subsequent calculations will be to restrict all expressions to the subset W which is a positive distance away from ∂U. This is necessary, as we have no information concerning the behaviour of u near ∂U.

Let now $|h|>0$ be small, choose $k\in\{1,2,...,n\}$ and substitute $\boxed{v=-D_k^{-h}\left(\zeta^2 D_k^h u\right)}$ into the weak formulation of E-L equation, i.e. $\int_U \sum_{i=1}^n L_{p_i}(\nabla u)v_{x_i}\,dx=\int_U fv\,dx\,,\forall v\in H_0^1(U)$. We are employing here the notation for the difference quotient $\boxed{D_k^h u(x)=\dfrac{u(x+he_k)-u(x)}{h}}$ for $x\in W$. As a result $D_k^{-h}u(x)=\dfrac{u(x-he_k)-u(x)}{-h}$. Moreover the following identity $\int_U uD_k^{-h}v dx=-\int_U vD_k^h u dx$ holds, since after applying the change of variables $x-he_k=y$ so $dx=dy$, taking into consideration that the domain U does not change since we have taken $|h|>0$ sufficiently small and selecting k appropriately so as to still be inside the domain U we get that: $\int_U uD_k^{-h}v dx=-\int_U u(x)\dfrac{v(x-he_k)-v(x)}{h}dx=$

$=-\int_U u(y+he_k)\dfrac{v(y)-v(y+he_k)}{h}dy=-\int_U \dfrac{u(x+he_k)v(x)}{h}dx+\int_U \dfrac{u(x+he_k)v(x+he_k)}{h}dx\overset{y=x+he_k}{=}$

$=-\int_U \dfrac{u(x+he_k)v(x)}{h}dx+\int_U \dfrac{u(x)v(x)}{h}dx=-\int_U vD_k^h u dx$. As a consequence we deduce that:

$\int_U \sum_{i=1}^n L_{p_i}(\nabla u)\left(-D_k^{-h}\left(\zeta^2 D_k^h u\right)\right)_{x_i}dx=\int_U f\left(-D_k^{-h}\left(\zeta^2 D_k^h u\right)\right)dx \Rightarrow \left(\text{obviously } \left(D_k^h f\right)_{x_i}=D_k^h\left(f_{x_i}\right)\right)\Rightarrow$

$\sum_{i=1}^n\int_U D_k^h\left(L_{p_i}(\nabla u)\right)\left(\zeta^2 D_k^h u\right)_{x_i}dx=-\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)dx$. Now let us observe that

$D_k^h\left(L_{p_i}(\nabla u)\right)=\dfrac{L_{p_i}\left(\nabla u(x+he_k)\right)-L_{p_i}\left(\nabla u(x)\right)}{h}\overset{\substack{\text{Fundamental}\\ \text{theorem}}}{=}\dfrac{1}{h}\int_0^1\dfrac{d}{ds}L_{p_i}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)ds=$

$\dfrac{1}{h}\int_0^1\sum_{j=1}^n L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)\left(u_{x_j}(x+he_k)-u_{x_j}(x)\right)ds=\boxed{\sum_{j=1}^n a_{ij}^h(x)D_k^h u_{x_j}(x)}$ for

$\boxed{a_{ij}^h(x)=\int_0^1 L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)ds}$ $(i,\,j=1,2,...,n)$. Now we must substitute this suitable expression of $D_k^h\left(L_{p_i}(\nabla u)\right)$ above and perform simple calculations, to arrive at the identity:

$$\sum_{i=1}^{n}\int_U D_k^h\left(L_{p_i}(\nabla u)\right)\left(\zeta^2 D_k^h u\right)_{x_i} dx = -\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)dx \;\Rightarrow$$

$$\sum_{i,j=1}^{n}\int_U a_{ij}^h(x)D_k^h u_{x_j}(x)\left(\zeta^2 D_k^h u\right)_{x_i} dx = -\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)dx \;\Rightarrow$$

$$\sum_{i,j=1}^{n}\int_U a_{ij}^h(x)D_k^h u_{x_j}\zeta^2 D_k^h u_{x_i}dx + \sum_{i,j=1}^{n}\int_U a_{ij}^h(x)D_k^h u_{x_j}2\zeta\zeta_{x_i}D_k^h u\; dx = -\int_U fD_k^{-h}\left(\zeta^2 D_k^h u\right)dx$$, where

we denote the last equality's terms as A_1, A_2 and B respectively. So the equality above can be abbreviated as $A_1 + A_2 = B$. Now the uniform convexity condition $\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \geq \vartheta|\xi|^2$ (actually this is equivalent to $\xi^T D^2 L(p)\xi \geq \vartheta|\xi|^2 \overset{\xi\neq 0}{\Leftrightarrow}$(by definition) $D^2L(p) > 0 \Leftrightarrow$ the Hessian matrix is positive definite $\Leftrightarrow$ convex) implies that:

$$A_1 = \int_U\int_0^1\sum_{i,j=1}^{n}\zeta^2 L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)\overbrace{D_k^h u_{x_i}}^{\xi_i}\,\overbrace{D_k^h u_{x_j}}^{\xi_j}dsdx \geq \int_U \zeta^2\vartheta\left|D_k^h(\nabla u)\right|^2 dx \;\Rightarrow$$

$\boxed{A_1 \geq \vartheta\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx}$. Furthermore, by employing the growth condition $\left|D^2L(p)\right| \leq c,$

$A_2 = \int_U\int_0^1\sum_{i,j=1}^{n} L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)D_k^h u_{x_j}D_k^h u 2\zeta\zeta_{x_i}dsdx$, therefore we have that:

(remember that $V \subset\subset W \subset\subset U$ and the fact that $\zeta = \begin{cases}1, \text{on } V\\ 0\leq\zeta\leq 1\\ 0, \text{in } \mathbb{R}^n\setminus W\end{cases}$ is a *smooth* cut off function)

$|A_2| \leq \int_U\int_0^1\left|\sum_{i,j=1}^{n} L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)D_k^h u_{x_j}D_k^h u 2\zeta\zeta_{x_i}\right|dsdx$, but we observe that

$$\sum_{i,j=1}^{n} L_{p_ip_j}\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)D_k^h u_{x_j}\zeta_{x_i} = \left(\nabla\zeta\right)^T D^2L\left(s\nabla u(x+he_k)+(1-s)\nabla u(x)\right)D_k^h(\nabla u) \Rightarrow$$

$$\left|\sum_{i,j=1}^{n} L_{p_ip_j}D_k^h\left(u_{x_j}\right)D_k^h u 2\zeta\zeta_{x_i}\right| = \left|\left(\nabla\zeta\right)^T D^2L(p)D_k^h(\nabla u)D_k^h u 2\zeta\right| \overset{\substack{|D^2L(p)|\leq c\\ and\ \zeta\geq 0}}{\leq} c|\nabla\zeta|\left|D_k^h(\nabla u)\right|2\zeta\left|D_k^h u\right| \leq$$

$\leq c\tilde{c}\left|D_k^h(\nabla u)\right|2\zeta\left|D_k^h u\right|$, where the last inequality holds since ζ is smooth over a closed and bounded domain $\subset\mathbb{R}^n$ and $\dim\mathbb{R}^n = n < +\infty$ (finite dimensional) $\Rightarrow$ *compact* $\Rightarrow \exists\tilde{c} > 0$ such that $|\nabla\zeta| \leq \tilde{c},$ in other words, a continuous function over a compact domain is bounded. As a consequence, we obtain the following estimate for the integral A_2 : (note first that since ζ=0 in $\mathbb{R}^n\setminus W,$ ζ=0 outside $W,$ that's why we choose to restrict our analysis below inside W)

$|A_2| \leq \int_W c2\zeta\left|D_k^h(\nabla u)\right|\left|D_k^h u\right|dx \overset{\substack{cauchy\\ with\ \varepsilon}}{\leq} \varepsilon\int_W \zeta^2\left|D_k^h(\nabla u)\right|^2 dx + \frac{c}{\varepsilon}\int_W\left|D_k^h u\right|^2 dx$, where we have employed the

Cauchy inequality with ε, i.e. $ab \le \varepsilon a^2 + \dfrac{b^2}{4\varepsilon}$ $(a,b>0$ and $\varepsilon>0)$. At last we obtain:

$\boxed{|A_2| \le \varepsilon \int_W \zeta^2 \left|D_k^h(\nabla u)\right|^2 dx + \frac{C}{\varepsilon}\int_W \left|D_k^h u\right|^2 dx}$. Finally we shall find an estimation for the quantity B as well. To be more specific, we will try to show that the following estimate holds true:

$|B| \le \varepsilon \int_U \zeta^2 \left|D_k^h(\nabla u)\right|^2 dx + \frac{C}{\varepsilon}\int_U f^2 + |\nabla u|^2 dx$. For this purpose we first need to establish that the following inequality is indeed valid: $\boxed{\int_W \left|D_k^h u\right|^2 dx \le \int_U |\nabla u|^2 dx}$

The procedure we will follow below is quite similar with the proof of Poincar*é* 's inequality.

Observe first that $\dfrac{d}{dt}u(x+the_k) = u_{x_k}(x+the_k)he_k$, therefore it follows that:

$$u(x+he_k)-u(x) \overset{\substack{\text{Fundamental}\\ \text{theorem}}}{=} \int_0^1 \frac{d}{dt}u(x+the_k)dt = \int_0^1 u_{x_k}(x+the_k)he_k dt \Rightarrow$$

$$|u(x+he_k)-u(x)| \overset{|e_k|=1}{\le} h\int_0^1 \left|u_{x_k}(x+the_k)\right| dt \overset{\left|\frac{\partial u}{\partial x_k}\right| \le |\nabla u|}{\le} h\int_0^1 |\nabla u(x+the_k)| dt \Rightarrow$$

$$\frac{|u(x+he_k)-u(x)|}{h} \le \int_0^1 |\nabla u(x+the_k)| dt \Rightarrow \left|D_k^h u\right| \le \int_0^1 1|\nabla u(x+the_k)| dt \overset{C-S}{\le}$$

$$\le \cancel{\left(\int_0^1 1^2 dt\right)^{1/2}}^{\,1} \left(\int_0^1 |\nabla u(x+the_k)|^2 dt\right)^{1/2} \Rightarrow \left|D_k^h u\right|^2 \le \int_0^1 |\nabla u(x+the_k)|^2 dt.$$ Therefore

$\int_W \left|D_k^h u\right|^2 dx \le \int_W \int_0^1 |\nabla u(x+the_k)|^2 dtdx \overset{Fubini}{=} \int_0^1 \int_W |\nabla u(x+the_k)|^2 dxdt$ and now we will proceed by making the following change of variables $y = x+the_k$, so $dy = dx$. Moreover we note that since $x \in W$, $y \in U$ because of the choice of h. Remember that we have selected $|h|>0$ small enough. A convenient selection would be $0<|h|<\frac{1}{2}\operatorname{dist}(W,\partial U)$. In that way, we would be able to a-priori guarantee that for a given $x \in W$, the change of variables $y = x+hte_k$ would still belong to U, since $|y-x| = |hte_k| \overset{t\in[0,1]}{\le} |h| < \frac{1}{2}\operatorname{dist}(W,\partial U) \Rightarrow y \in U$ (in the worst case).

(see figure 2 above)

Consequently $\int\limits_W \left|D_k^h u\right|^2 dx \le \int\limits_0^1 \int\limits_W \left|\nabla u\left(x+the_k\right)\right|^2 dxdt \le \int\limits_0^1 \int\limits_U \left|\nabla u(y)\right|^2 dydt = \int\limits_U \left|\nabla u(x)\right|^2 dx$, where the last step holds since we integrate a non-negative function, hence the extention of the domain of integration does not affect in any way the inequality (we have assumed $V \subset\subset W \subset\subset U$). Thus we have proved the required estimation, i.e. $\boxed{\int\limits_W \left|D_k^h u\right|^2 dx \le \int\limits_U \left|\nabla u(x)\right|^2 dx}\,(*)$. This inequality will be proved very useful in our computation below. Now that we have established this result, we are ready to continue by proving the estimate for *B*. And last but not the least, let us note that it still holds true for D_k^{-h} and $W \equiv U$, because first D_k^{-h} apparently does not change anything at all and secondly we can consider $W \equiv U$ (in the sense that we extend it), since ζ=0 in $\mathbb{R}^n \setminus W$.

$\left|B\right| = \left|\int\limits_U fv dx\right| \le \int\limits_U |f||v| dx \overset{\substack{Cauchy\\ with\ \varepsilon}}{\le} \varepsilon \int\limits_U |v|^2 dx + \frac{1}{4\varepsilon}\int\limits_U |f|^2 dx$. At this point we need to further estimate the $\int\limits_U |v|^2 dx$, therefore we observe that: $\int\limits_U |v|^2 dx = \int\limits_U \left|D_k^{-h}\left(\zeta^2 D_k^h u\right)\right|^2 dx \overset{(*)}{\le} \int\limits_U \left|\nabla\left(\zeta^2 D_k^h u\right)\right|^2 dx \le$

$\le c\int\limits_W \left(\left|D_k^h u\right|^2 + \zeta^2\left|D_k^h\left(\nabla u\right)\right|^2\right) dx \overset{(*)}{\le} c\int\limits_U \left(\left|\nabla u\right|^2 + \zeta^2\left|D_k^h\left(\nabla u\right)\right|^2\right) dx$. Now the pre-last inequality does indeed hold true since: $\nabla\left(\zeta^2 D_k^h u\right) = 2\zeta\nabla\zeta D_k^h u + \zeta^2\nabla\left(D_k^h u\right) = 2\zeta\nabla\zeta D_k^h u + \zeta^2 D_k^h(\nabla u)$, because the difference quotient operator is linear and as such it can be converted with ∇ during the derivation.

$\left|\nabla\left(\zeta^2 D_k^h u\right)\right| \overset{\substack{triangular\\ inequality}}{\le} 2\zeta\left|\nabla\zeta\right|\left|D_k^h u\right| + \zeta^2\left|D_k^h(\nabla u)\right| \le \zeta\left(2c^*\left|D_k^h u\right| + \zeta\left|D_k^h(\nabla u)\right|\right)$, where $\left|\nabla\zeta\right| \le c^*$, since ζ is smooth, so ∇ζ is continuous in a bounded domain $\Rightarrow \left|\nabla\zeta\right|$ is bounded. Now we get:

$\left|\nabla\left(\zeta^2 D_k^h u\right)\right|^2 \le \zeta^2\left(2c^*\left|D_k^h u\right| + \zeta\left|D_k^h(\nabla u)\right|\right)^2 \overset{0\le\zeta\le 1}{\le} \left(2c^*\left|D_k^h u\right| + \zeta\left|D_k^h(\nabla u)\right|\right)^2 \overset{(a+b)^2 \le 2\left(a^2+b^2\right)}{\le}$

$\le \tilde{c}\left|D_k^h u\right|^2 + 2\zeta^2\left|D_k^h(\nabla u)\right|^2 \le c\left(\left|D_k^h u\right|^2 + \zeta^2\left|D_k^h(\nabla u)\right|^2\right)$, for $c = \max\left\{\tilde{c}, 2\right\}$. Therefore we have proved the required: $\left|\nabla\left(\zeta^2 D_k^h u\right)\right|^2 \le c\left(\left|D_k^h u\right|^2 + \zeta^2\left|D_k^h(\nabla u)\right|^2\right)$, which verifies that indeed the following $\int\limits_U |v|^2 dx \le c\int\limits_U \left(\left|\nabla u\right|^2 + \zeta^2\left|D_k^h\left(\nabla u\right)\right|^2\right) dx$ holds true. Applying now this estimate in the estimate for *B*, we obtain: $|B| \le \varepsilon \int_U |v|^2\,dx + \frac{1}{4\varepsilon}\int_U |f|^2\,dx \le c\varepsilon\int_U |\nabla u|^2\,dx + c\varepsilon\int_U \zeta^2 \left|D_k^h(\nabla u)\right|^2 dx + \frac{1}{4\varepsilon}\int_U |f|^2\,dx.$

We conclude $|B| \le \varepsilon\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx + \frac{c}{\varepsilon}\int_U |f|^2\,dx + c\varepsilon\int_U |\nabla u|^2\,dx \le \varepsilon\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx +$

$+\frac{\hat{c}}{\varepsilon}\int_U \left(f^2 + |\nabla u|^2\right)dx$, for $\hat{c} = \max\{\varepsilon^2, 1\}\,c$, which was the required estimation for the quantity *B*.

$\boxed{|B| \le \varepsilon\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx + \frac{c}{\varepsilon}\int_U \left(f^2 + |\nabla u|^2\right)dx}$. We select now $\varepsilon = \frac{\vartheta}{4},$ to deduce from the foregoing

bounds on A_1, A_2, B the estimate:

$\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx \le c\int_W \left(f^2 + \left|D_k^h u\right|^2\right)dx \le c\int_U \left(f^2 + |\nabla u|^2\right)dx$, the last inequality holds by the application of $(*)$ and the fact that for given non-negative functions, we can extend the domain of integration.

$\boxed{2.}$ Now from our estimate above, i.e. $\int_U \zeta^2\left|D_k^h(\nabla u)\right|^2 dx \le c\int_U \left(f^2 + |\nabla u|^2\right)dx$ and from the fact that $\zeta = 1$ on $V,$ we find that: $\boxed{\int_V \left|D_k^h(\nabla u)\right|^2 dx \le c\int_U \left(f^2 + |\nabla u|^2\right)dx}$ for $k = 1, 2, \dots, n$ and all sufficiently small $|h| > 0$. Consequently, by applying the following theorem from the Sobolev spaces, we can deduce that $\nabla u \in H^1(V)$ and so $u \in H^2(V)$. This is true for each $V \subset\subset U,$ thus $u \in H^2_{loc}(U)$.

Theorem

(i) Suppose that $1 \le p < \infty$ and $W^{1,p}(U)$. Then for each $V \subset\subset U \Rightarrow \boxed{\left\|D^h u\right\|_{L^p(V)} \le c\left\|\nabla u\right\|_{L^p(U)}}$ for some constant $c > 0$ and for all $0 < |h| < \frac{1}{2}dist(V, \partial U)$. Here we denote $D^h u = \left(D_1^h u, D_2^h u, \dots, D_n^h u\right)$ where $D_k^h u = \frac{u(x + he_k) - u(x)}{h}$ the difference quotient for $0 < |h| < \frac{1}{2}dist(V, \partial U)$ and $k = 1, 2, \dots, n.$

(ii) Assume now $1 < p < \infty$ and $u \in L^p(V)$ and also that there exists a constant $c > 0$ such that $\left\|D^h u\right\|_{L^p(V)} \le c$ for all $0 < |h| < \frac{1}{2}dist(V, \partial U)$. Then $\boxed{u \in W^{1,p}(V) \text{ with } \left\|\nabla u\right\|_{L^p(V)} \le c}$.

$\boxed{3.}$ If now $u \in H_0^1(U)$ is a weak solution of E-L equation $-\sum_{i=1}^{n}\left(L_{p_i}(\nabla u)\right)_{x_i} = f$ in U and the boundary ∂U is C^2, we can then prove that $\underline{u \in H^2(U)}$ with estimate: $\boxed{\|u\|_{H^2(U)} \le c\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right)}$. However we shall omit the proof of this statement here as it is rather longish and technical as well. Now let us observe that from the condition $\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \ge \vartheta|\xi|^2$ follows the inequality $\left(\nabla L(p) - \nabla L(0)\right)\cdot p \ge \vartheta|p|^2$ for $p \in \mathbb{R}^n$. To see that let us note that it is a straightforward result from the Mean Value Theorem, because

$\nabla L(p) - \nabla L(0) = D^2 L(\tilde{p})p$ for $\tilde{p} \in B(0,p) \Rightarrow \left(\nabla L(p) - \nabla L(0)\right)\cdot p = p^T D^2 L(\tilde{p})p \ge \vartheta|p|^2$

If we then put $v = u$ in the variational formulation of the E-L equation, i.e. $\int_U \sum_{i=1}^{n} L_{p_i}(\nabla u)v_{x_i}\,dx = \int_U fv dx$ for all $v \in H_0^1(U)$, we can employ the estimate above to derive the bound $\|u\|_{H^1(U)} \le c\|f\|_{L^2(U)}$ and so finish the proof of the theorem, i.e. $\|u\|_{H^2(U)} \le c\|f\|_{L^2(U)}$, because apparently we have $\|u\|_{H^2(U)} \le c\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right) \le c\left(\|f\|_{L^2(U)} + \tilde{c}\|f\|_{L^2(U)}\right) \le \hat{c}\|f\|_{L^2(U)}$ for $\hat{c} = \max\{1,\tilde{c}\}c$. Finally, in order to complete the proof, let us justify the bound at $\|u\|_{H^1(U)} \le c\|f\|_{L^2(U)}$.

$\int_U \sum_{i=1}^{n} L_{p_i}(\nabla u)u_{x_i}\,dx = \int_U fudx \Leftrightarrow \int_U fudx = \int_U \nabla L(p)\cdot pdx \ge \int_U \left(\vartheta|p|^2 + \nabla L(0)\cdot p\right)dx \overset{p=\nabla u}{=} \int_U \vartheta|\nabla u|^2\,dx +$ $+\int_U \nabla L(0)\cdot\nabla u\;dx$. Consequently we get $\int_U \left(\vartheta|\nabla u|^2 + \nabla L(0)\cdot\nabla u\right)dx \le \int_U fudx \Rightarrow$

$\vartheta\|\nabla u\|_{L^2(U)}^2 \le \int_U \left(fu - \nabla L(0)\cdot\nabla u\right)dx \le \int_U |fu - \nabla L(0)\cdot\nabla u|\,dx \le \int_U \left(|fu| + |\nabla L(0)\cdot\nabla u|\right)dx \overset{Cauchy}{\le}$

$\int_U \left(|f||u| + |\nabla L(0)||\nabla u|\right)dx \overset{C-S}{\le} \|f\|_{L^2(U)}\|u\|_{L^2(U)} + c\|\nabla u\|_{L^2(U)}$, where the constant c appears because L is smooth and thus ∇L is continuous in a bounded domain $U \subset \mathbb{R}^n$, so $\exists\, c > 0$ such that $|\nabla L| \le c$. Now remember the assuumption in the third step of the proof, i.e. $u \in H_0^1$. So we are able to apply Poincaré

inequality $\|u\|_{L^2(U)} \le \overbrace{c(U)}^{\hat{c}}\|\nabla u\|_{L^2(U)}$. Therefore $\|f\|_{L^2(U)}\|u\|_{L^2(U)} \le \hat{c}\|f\|_{L^2(U)}\|\nabla u\|_{L^2(U)}$. As a result, we get:

$\vartheta\|\nabla u\|_{L^2(U)}^{\cancel{2}} \le \|f\|_{L^2(U)}\|u\|_{L^2(U)} + c\|\nabla u\|_{L^2(U)} \le \hat{c}\|f\|_{L^2(U)}\|\nabla u\|_{L^2(U)} + c\|\nabla u\|_{L^2(U)} \overset{\hat{c}=\max\{c,\hat{c}\}}{\le} \hat{c}\cancel{\|\nabla u\|_{L^2(U)}}\left(\|f\|_{L^2(U)} + 1\right)$

$\Rightarrow \|\nabla u\|_{L^2(U)} \le \frac{\hat{c}}{\vartheta}\left(\|f\|_{L^2(U)} + 1\right) \overset{\substack{archimedean\\ property}}{\le} c\|f\|_{L^2(U)}$, archimedean property actually asserts that $\exists\,\bar{c}$ such that

$\hat{c} \leq \bar{c} \| f \|_{L^2(U)}$. Now we shall apply Poincaré 's inequality once again and in combination with what we just showed, we get: $\|u\|_{L^2(U)} \leq c(U)\|\nabla u\|_{L^2(U)} \leq c(U)c \ \| f \|_{L^2(U)} \Rightarrow$

$\|u\|_{H^1(U)} = \left(\|u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)}\right)^{1/2} \leq c\| f \|_{L^2(U)}$, which proves the required. QED □

Remarks on higher regularity

We would like to show that if L is infinitely differentiable, then so is u. By analogy with the regularity theory for second order linear elliptic pdes, it may seem natural to extend the H^2_{loc} estimate from the previous section to obtain further estimates in higher Sobolev spaces $H^k_{loc}(U)$ for $k = 3,4,5,...$ Unfortunately, this method will not work :(for the non-linear E-L pde! The reason is this. For linear equations, we could, roughly speaking, differentiate the equation many times and still obtain a linear pde of the same general form. Whereas if we differentiate a non-linear pde many times, the resulting increasingly complicated expression quickly becomes impossible to handle! In general, much deeper ideas and techniques are called for, the full development of which is beyond the scope of the work at hand. We will nevertheless at least outline the basic plan.

To start with, choose a test function $w \in C_0^\infty(U)$, select $k \in \{1,2,...,n\}$ and set $v = -w_{x_k}$ in the identity $\int_U \sum_{i=1}^n L_{p_i}(\nabla u) v_{x_i} dx = \int_U fv dx$, where for simplicity we now take $f = 0$. So we get that

$$\int_U \sum_{i=1}^n L_{p_i}(\nabla u)\left(-w_{x_k}\right)_{x_i} dx = \int_U \sum_{i=1}^n L_{p_i}(\nabla u)\left(-w_{x_i}\right)_{x_k} dx = \int_U -\cancel{f}^{\,0} w_{x_k} dx = 0 \ \Rightarrow$$

$\int_U \sum_{i=1}^n L_{p_i}(\nabla u)\left(-w_{x_i}\right)_{x_k} dx = 0$. Since we know that $u \in H^2_{loc}(U)$, we can integrate by parts to find:

$$\Rightarrow -\int_{\partial U} \sum_{i=1}^n L_{p_i}(\nabla u) \cancel{w_{x_i}}^{\,0} n_k dS + \int_U w_{x_i}\left(\sum_{i=1}^n L_{p_i}(\nabla u)\right)_{x_k} dx \ \Rightarrow \ \int_U w_{x_i}\left(\sum_{i=1}^n L_{p_i}(\nabla u)\right)_{x_k} dx = 0$$

, because the surface integral above is zero, since $w \in C_0^\infty(U)$ and so $w_{x_i} \in C_0^\infty(U)$. But now we note that

$$\frac{\partial}{\partial x_k}\left[\sum_{i=1}^n L_{p_i}(\nabla u)\right] = \sum_{i=1}^n\left[\frac{\partial}{\partial x_k} L_{p_i}(u_{x_1},...,u_{x_n})\right] = \sum_{i=1}^n\left[\sum_{j=1}^n L_{p_i p_j}(\nabla u) u_{x_k x_j}\right] = \sum_{i,j=1}^n L_{p_i p_j}(\nabla u) u_{x_k x_j}.$$

Therefore

$(a)\ \boxed{\int_U \sum_{i,j=1}^n L_{p_i p_j}(\nabla u) u_{x_k x_j} w_{x_i} dx = 0}$. Next we write $(b)\ \boxed{\tilde{u} = u_{x_k}}$ and $(c)\ \boxed{a^{ij} = L_{p_i p_j}(\nabla u)}$ $(i.j = 1,2,...,n)$

Fix also any $V \subset\subset U$. Then after approximating we find from (a), (b) and (c) that:

$\boxed{\boxed{\int_U \sum_{i,j=1}^n a^{ij}(x)\tilde{u}_{x_j} w_{x_i} dx = 0 \text{ for all } w \in H_0^1(V)}}$ and this is true because $C_0^\infty(V)$ is *dense* to $H_0^1(V)$.

This is to say that $\tilde{u} \in H^1(V)$ is a weak solution of the linear, second order, elliptic pde:

$-\sum_{i,j=1}^n \left(a^{ij}(x)\tilde{u}_{x_j}\right)_{x_i} = 0 \text{ in V} \Rightarrow \boxed{\boxed{div\left(A\nabla\tilde{u}\right) = 0 \text{ in } V}}$, for $A = \left(a^{ij}(x)\right)_{ij} = \left(L_{p_i p_j}\right)_{ij} = D^2 L = Hess(L)$.

But pay attention to the fact that we cannot just apply the regularity theory developed for linear, 2nd order, elliptic pdes to conclude from the equation $div\left(A\nabla\tilde{u}\right) = 0$ in V that $\tilde{u}$ is smooth! The reason being that we can deduce from the growth condition $\left|D^2 L(p)\right| \le c$ and the $a^{ij} = L_{p_i p_j}(\nabla u)$ only (!)

that $\left|a^{ij}(x)\right| \le c$, i.e. that $a^{ij} \in L^\infty(V)$ for $\left(i, j = 1, 2, ..., n\right)$, whereas according to the regularity theory for linear, 2nd order, elliptic pdes a^{ij} needs at least be C^1 if we wish to achieve H^2-regularity, while we must demand $a^{ij} \in C^{m+1}$ in the case we would like to attain higher regularity. Finally in order to obtain infinite differentiability, a^{ij} needs apparently be C^∞.

However, a deep theorem, due independently to *De Giorgi* and *Nash*, asserts that <u>any weak solution of $div\left(A\nabla\tilde{u}\right)$=0 in V must in fact be *locally Hölder continuous* for some exponent γ.</u> (see [B11], chapter 8, for more details and results related to this theorem). Thus if $W \subset\subset V$, we have $\tilde{u} \in C^{0,\gamma}(W)$ and so $u \in C_{loc}^{1,\gamma}(U)$. Return to the definition $a^{ij} = L_{p_i p_j}(\nabla u)$ for $i, j = 1, 2, ..., n$. If L is smooth, we now know that $a^{ij} \in C_{loc}^{0,\gamma}(U)$. Then the relation $-\sum_{i=1}^n \left(L_{p_i}(\nabla u)\right)_{x_i} = f$ in U and the *Schauder's* fixed point theorem (see again [B11] , chapters 4 and 6) assert in fact that $u \in C_{loc}^{2,\gamma}(U)$. But then $a^{ij} \in C_{loc}^{1,\gamma}(U)$ and so another version of Schauder's estimate implies that $u \in C_{loc}^{3,\gamma}(U)$. We can continue this so-called "bootstrap" argument, eventually to deduce that $u \in C_{loc}^{k,\gamma}(U)$ for $k = 1, 2, 3, ...$ and consequently $\boxed{u \in C^\infty}$.

For even more information and a deeper analysis regarding topics of regularity in the calculus of variations see [B10]. Finally we shall close this section by presenting, as a reminder, the definition of the *Hölder* space.

The *Hölder* space $C^{k,\gamma}\left(\bar{U}\right)$ consists of all functions $u \in C^{k}\left(\bar{U}\right)$ for which the norm $\|u\|_{C^{k,\gamma}(\bar{U})} < +\infty$, where $\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|a|\le k} \|D^{a}u\|_{C(\bar{U})} + \sum_{|a|=k} \left[D^{a}u \right]_{C^{0,\gamma}(\bar{U})}$. Now $\|u\|_{C(\bar{U})} = \sup_{x\in U} |u(x)|$, the $\gamma^{th} - \textit{Hölder}$ semi-norm is $[u]_{C^{0,\gamma}(\bar{U})} = \sup_{\substack{x,y\in U \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{\gamma}} \right\}$ and the $\gamma^{th} - \textit{Hölder}$ norm is $\|u\|_{C^{0,\gamma}(\bar{U})} = \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$.

So the space $C^{k,\gamma}\left(\bar{U}\right)$ consists of those functions u that are $k-$times continuous differentiable and whose k^{th}-partial derivatives are *Hölder* continuous with exponent γ.

Such functions are well behaved and furthermore the space $C^{k,\gamma}\left(\bar{U}\right)$ itself possesses a good mathematical structure. Finally we would like to mention that $C^{k,\gamma}\left(\bar{U}\right)$ is a *Banach* space.

3. Constraints

3.1 Non linear eigenvalue problems

We first investigate problems subject to integral constraints. To be more specific let us look at the following problem of minimizing the energy functional $I[w] = \frac{1}{2}\int_{U} |\nabla w|^{2}\, dx$ over all functions w with, say, $w = 0$ on $\partial U,$ but subject now also to the side condition that $J[w] = \int_{U} G(w)dx = 0,$ where $G : \mathbb{R} \to \mathbb{R}$ is a given, smooth function . We will henceforth write $g = G'$. Assume that $|g(z)| \le c\left(1+|z|\right)$ and so $|G(z)| \le c\left(1+|z|^{2}\right),\ z\in\mathbb{R},$ for some constant c. Finally we introduce as well an appropriate admissible class $\mathrm{A} = \left\{ w\in H_{0}^{1}(U) \ \mid\ J[w]=0 \right\},$ where U is an open, bounded and simply connected set with $\partial U \in C^{1},$ i.e. it has a smooth boundary.

$\boxed{Theorem}$ (existence of constrained minimizers)

Assuming that $\mathrm{A} \neq \varnothing$. Then there exists $u \in \mathrm{A}$ satisfying $I[u] = \min\limits_{w \in \mathrm{A}} I[w]$

Proof.

Choose, as usual, a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset \mathrm{A}$ with the property $I[u_k] \to m = \inf\limits_{w \in \mathrm{A}} I[w]$.

Then we know that we can extract a subsequence $u_{k_j} \xrightarrow{weakly} u$ in $H_0^1(U)$ with $I[u] \leq m$.

We will be done once we show that $J[u] = 0$ so that $u \in \mathrm{A}$. Then if $u \in \mathrm{A}$ and $I[u] \leq m = \inf\limits_{w \in \mathrm{A}} I[w] \;\Rightarrow\; I[u] = m$ and since there is an element of the class A that attains the infimum, this means that it is indeed a minimum. Utilizing the Rellich-Kondrachov compactness Theorem, we deduce from the fact that $u_{k_j} \xrightarrow{weakly} u$ in $H_0^1(U)$ that $u_{k_j} \longrightarrow u$ in L^2.

Consequently:

$$\left|J[u]\right| = \Big| J[u] - \cancel{J[u_k]}^{\substack{0 \text{ since} \\ u_k \in \mathrm{A}}} \Big| \leq \int_U \left|G(u) - G(u_k)\right| dx \overset{MVT}{\leq} \int_U \left|G'(\xi)\right| \left|u - u_k\right| dx = \;(G \in C^1 \text{ and}$$

$$\xi \text{ is between } u \text{ and } u_k) = \int_U \left|g(\xi)\right| \left|u - u_k\right| dx \leq \;\Big(because\;\; \left|g(u)\right| \leq c\big(1 + |u|\big)\Big) \leq$$

$$\int_U c\left|u - u_k\right| \big(|\xi| + 1\big) dx \overset{|\xi| \leq \max\{|u|, |u_k|\}}{\leq} c\int_U \left|u - u_k\right| \big(|u| + |u_k| + 1\big) dx \xrightarrow[k \to \infty]{} 0 \;\Rightarrow\; J[u] = 0 \;\square$$

$\boxed{Theorem}$ (E-L equation through Lagrange-multiplier)

Let $u \in \mathrm{A}$ satisfy $I[u] = \min\limits_{w \in \mathrm{A}} I[w]$. Then there exists a real number λ such that

$$\boxed{\int_U \nabla u \cdot \nabla v \; dx = \lambda \int_U g(u) v \; dx} \quad \forall v \in H_0^1(U).$$

Proof.

$\boxed{1.}$ Fix any function $v \in H_0^1(U)$. Assume first $g(u) \neq 0$ a.e. within U. Choose then any function $w \in H_0^1(U)$ with $\int_U g(u) w \; dx \neq 0$. This is possible because of our assumption $g(u) \neq 0$ a.e. in U. Now write

$j(\tau, \sigma) = J[u + \tau v + \sigma w] = \int_U G(u + \tau v + \sigma w) dx \;\; (\tau, \sigma \in \mathbb{R})$. Clearly

$j(0,0) = J[u] = \int_U G(u) dx = 0.$ In addition, j is C^1 as a composition of a C^1 function G and the linear $u + \tau v + \sigma w$. Moreover we compute that

$$\begin{cases} \dfrac{\partial j}{\partial \tau}(\tau, \sigma) = \displaystyle\int_U g(u + \tau v + \sigma w) v \; dx \\ \dfrac{\partial j}{\partial \sigma}(\tau, \sigma) = \displaystyle\int_U g(u + \tau v + \sigma w) w \; dx \end{cases}$$

Consequently $\int_U g(u) w \; dx \neq 0$ implies that $\boxed{\dfrac{\partial j}{\partial \sigma}(0,0) \neq 0}$

As a result we can apply the Implicit Function Theorem . Therefore, there exists a C^1-function $\phi:\mathbb{R}\to\mathbb{R}$ (we write $\sigma=\phi(\tau)$) such that $\phi(0)=0$ and $j(\tau,\phi(\tau))=0$ for all sufficiently small τ, say $|\tau|\le\tau_0$. Differentiating implicitly now, we discover that:

$\frac{d}{d\tau}j(\tau,\phi(\tau))=\frac{\partial j}{\partial\tau}(\tau,\phi(\tau))+\frac{\partial j}{\partial\sigma}(\tau,\phi(\tau))\ \phi'(\tau)=0.$ So for $\tau=0$ we have that $\phi(0)=0$ and whence $\phi'(0)=-\dfrac{\frac{\partial j}{\partial\tau}(0,0)}{\frac{\partial j}{\partial\sigma}(0,0)}\ \Rightarrow \boxed{\phi'(0)=-\dfrac{\int_U g(u)v\,dx}{\int_U g(u)w\,dx}}$.

$\boxed{2.}$ Now set $w(\tau)=\tau v+\phi(\tau)w,\ \left(|\tau|\le\tau_0\right)$

and write $i(\tau)=I[u+w(\tau)].$ Since $j(\tau,\varphi(\tau))=0\ \forall|\tau|\le\tau_0$ implies that $J[u+w(\tau)]=\int_U G(u+w(\tau))dx=0$ (by the definition of j) we see that $u+w(\tau)\in$ A.

So the C^1 function $i(\cdot)$ has a minimum at 0. Thus we have that:

$$0=i'(0)=\frac{d}{d\tau}I[u+w(\tau)]\Big|_{\tau=0}=\frac{d}{d\tau}I[u+\tau v+w\varphi(\tau)]\Big|_{\tau=0}=\frac{d}{d\tau}\frac{1}{2}\int_U\left|\nabla u+\tau\nabla v+\varphi(\tau)\nabla w\right|^2dx=$$

$$=\frac{1}{\cancel{2}}\int_U\cancel{2}\left(\nabla u+\cancel{\tau\nabla v}^{\,0}+\cancel{\varphi(\tau)\nabla w}^{\,0}\right)\cdot\left(\nabla v+\varphi'(\tau)\nabla w\right)dx\Big|_{\tau=0}\overset{\varphi(0)=0}{=}\int_U\nabla u\cdot\left(\nabla v+\varphi'(0)\nabla w\right)dx$$

At this point recall the previous calculation, where we have found that

$\varphi'(0)=-\dfrac{\int_U g(u)v\,dx}{\int_U g(u)w\,dx}$ and define $\boxed{\lambda=\dfrac{\int_U\nabla u\cdot\nabla w\,dx}{\int_U g(u)w\,dx}}$ to deduce the desired equality which is

$\int_U\nabla u\cdot\nabla v\,dx=\lambda\int_U g(u)v\,dx\quad\forall v\in H_0^1(U).$

$\boxed{3.}$ Suppose now that $g(u)=0$ a.e. in U.

Approximating g by bounded functions, we deduce that $\nabla G(u)=g(u)\nabla u=0$ a.e.

Hence, since U is simply connected, $G(u)$ is constant a.e. It follows then that $G(u)=0$ a.e. because $J[u]=\int_U G(w)dx=0.$ As $u=0$ on ∂U in the trace sense, it follows that $G(0)=0.$

But then $u=0$ a.e. as otherwise $I[u]>I[0]=0.$ Since $g(u)=0$ a.e. , the identity $\int_U\nabla u\cdot\nabla v\,dx=\lambda\int_U g(u)v\,dx\quad\forall v\in H_0^1(U)$ is trivially valids in the case at hand, for any λ □

Remark

According to the identity above, u is a weak solution of the non linear b.v.p.

$\begin{cases} -\Delta u = \lambda g(u), \text{ in } U \\ u = 0, \text{ on } \partial U \end{cases}$ where λ is the Lagrange multiplier corresponding to the integral constraint $J[u] = 0$. A problem of the form above for the pair of the unknowns (u, λ), with $u \neq 0$ is a non linear eigenvalue problem.

3.2 Unilateral constraints, variational inequalities

We study now calculus of variations problems with certain pointwise, one-sided constraints on the values of $u(x)$ for each $x \in U$. For definiteness, let us consider the problem of minimizing, say, the energy functional: $I[w] = \int_U \frac{1}{2}|\nabla w|^2 - fw \, dx$ among all functions w belonging to the "admissible" set: $A = \{ w \in H_0^1(U) | w \geq h \text{ } a.e. \text{ in } U \}$ for $h : \bar{U} \to \mathbb{R}$ is a given smooth function, called the *obstacle.*

The convex admissible set A thus comprises those functions $w \in H_0^1(U)$ satisfying the one-side, or unilateral, constraint that $w \geq h$. We suppose as well that f is a given, smooth function.

$\boxed{Theorem}$ (Existence of minimizers)

Assume that $A \neq \varnothing$. Then there exists a unique function $u \in A$ such that $I[u] = \min_{w \in A} I[w]$.

Proof :

$\boxed{1.}$ The existence of a minimizer follows very easily from the general ideas discussed so far. We need only note explicitly that if $\{u_{k_j}\}_{j=1}^{\infty} \subset A$ is a minimizing sequence with $u_{k_j} \xrightarrow{weakly} u$ in $H_0^1(U)$, then by employing (as usual) compactness, we have that $u_{k_j} \xrightarrow{strongly} u$ in $L^2(U)$. Since $u_{k_j} \geq h$ a.e. , it follows that $u \geq h$ a.e. Therefore $u \in A$.

$\boxed{2.}$ We now prove uniqueness. Assume u and $\tilde{u} \in A$ are two minimizers with $u \neq \tilde{u}$. Then $w = \dfrac{u+\tilde{u}}{2} \in A$ and $I[w] = \int_U \frac{1}{2}\left|\frac{\nabla u + \nabla \tilde{u}}{2}\right|^2 - f\left(\frac{u+\tilde{u}}{2}\right) dx = \int_U \frac{1}{8}\left(|\nabla u|^2 + 2\nabla u \cdot \nabla \tilde{u} + |\nabla \tilde{u}|^2\right) -$ $- f\left(\dfrac{u+\tilde{u}}{2}\right) dx$. Now by employing the equality: $2a \cdot b = |a|^2 + |b|^2 - |a-b|^2$, we obtain

$I[w] = \int_U \frac{1}{8}\left\{2|\nabla u|^2 + 2|\nabla \tilde{u}|^2 - |\nabla(u-\tilde{u})|^2\right\} - f\left(\frac{u+\tilde{u}}{2}\right) dx < \frac{1}{2}\int_U \frac{1}{2}|\nabla u|^2 - fu\ dx +$ $+ \frac{1}{2}\int_U \frac{1}{2}|\nabla \tilde{u}|^2 - f\tilde{u}\ dx \Rightarrow \boxed{I[w] < \frac{I[u]+I[\tilde{u}]}{2}}$, the "strict" inequality holds since $u \neq \tilde{u}$. However this is a *contradiction* since u and $\tilde{u}$ have been both considered to be minimizers □

Now our target is to compute the analogue of E-L equation, which for the case at hand turns out to be an inequality.

$\boxed{Theorem}$ (Variational characterization of minimizers)

Let $u \in A$ be the unique solution of $I[u] = \min_{w \in A} I[w]$. Then we have the variational inequality:

$$\boxed{\int_U \nabla u \cdot \nabla(w-u)\ dx \geq \int_U f(w-u)\ dx} \quad \forall w \in A.$$

Proof:

$\boxed{1.}$ Fix any element $w \in A$. Then for each $0 \leq \tau \leq 1$, we consider the convex combination $u + \tau(w-u) = (1-\tau)u + \tau w$ which belongs apparently to A, since A is a convex set. Thus if we set $i(\tau) = I[u + \tau(w-u)]$, we see that $i(0) \leq i(\tau)\ \forall\ 0 \leq \tau \leq 1$ and $i(\tau)$ is also smooth. Therefore $i'(0) \geq 0$.

$\boxed{2.}$ Now if $0<\tau\le 1$, then we've got that:

$$\frac{i(\tau)-i(0)}{\tau}=\frac{1}{\tau}\int_U\left\{\frac{\left|\nabla u+\tau\nabla\left(w-u\right)\right|^2-\left|\nabla u\right|^2}{2}-f\left(\cancel{u}+\tau\left(w-u\right)-\cancel{u}\right)\right\}dx=$$

$$=\int_U\left\{\nabla u\cdot\nabla\left(w-u\right)+\frac{\tau\left|\nabla\left(w-u\right)\right|^2}{2}-f\left(w-u\right)\right\}dx$$. Thus $i'(0)\ge 0$ implies (by taking "$\lim_{\tau\to 0}$")

$0\le i'(0)=\int_U\left\{\nabla u\cdot\nabla\left(w-u\right)-f\left(w-u\right)\right\}dx$, which proves the theorem's assertion □

Interpretation of the variational inequality

To gain some insight into the above variational inequality, let us quote without proof a regularity assertion, which states $u\in W^{2,\infty}\left(U\right)$, provided ∂U is smooth. Hence the set $O=\left\{x\in U\middle|u(x)>h(x)\right\}$ is open and $C=\left\{x\in U\middle|u(x)=h(x)\right\}$ is relatively close. We claim that in fact $u\in C^\infty\left(O\right)$ and $-\Delta u=f$ in O.

To see this, fix any test function $v\in C_0^\infty\left(O\right)$. Then if $\left|\tau\right|$ is sufficiently small, $w=u+\tau v\ge h$, and so $u\in$ A. Thus the variational inequality

$\int_U\nabla u\cdot\nabla\left(w-u\right)\,dx\ge\int_U f\left(w-u\right)\,dx$, $\forall w\in$ A implies that $\tau\int_O\left\{\nabla u\cdot\nabla v-fv\right\}dx\ge 0.$

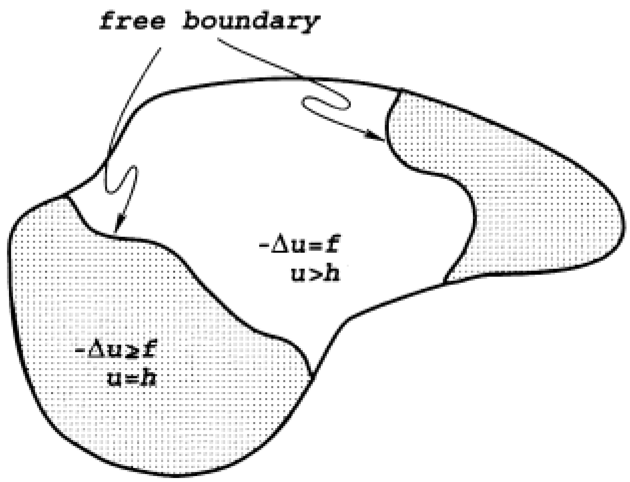
This inequality is valid for all sufficiently small τ, both positive and negative, and so in fact $\int_O\left\{\nabla u\cdot\nabla v-fv\right\}dx=0\ \ \forall\ v\in C_0^\infty\left(O\right)$. Hence u is a weak solution $-\Delta u=f$ in O, whence linear regularity theory shows that $u\in C^\infty(O)$.

Now if $v\in C_0^\infty\left(U\right)$ satisfies $v\ge 0$ and if $0<\tau\le 1$, then $w=u+\tau v\in$ A, whence $\int_U\left(\nabla u\cdot\nabla v-fv\right)dx\ge 0$. But since $u\in W^{2,\infty}\left(U\right)$, we can integrate by parts to deduce that $\int_U\left(-\Delta u-f\right)v\,dx\ge 0$ for all nonnegative functions $v\in C_0^\infty\left(U\right)$. Thus $-\Delta u\ge f$ a.e. in U.

We summarize our conclusions by observing from $-\Delta u=f$ in O, $-\Delta u\ge f$ in U that

$$\begin{cases}u\ge h,\ -\Delta u\ge f \text{ a.e. in } U\\ -\Delta u=f \text{ on } U\cap\left\{u>h\right\}\end{cases}\quad(\#)$$

3.3 Free boundaries



The free boundary for the obstacle problem

The set $\boxed{F = \partial O \bigcap U}$ is called the *free boundary*. Many interesting problems in applied mathematics involve pdes with free boundaries. Such of these problems as can be recast as variational inequalities become relatively easy to study, especially since there is no explicit mention of the free boundary in the inequalities (#). Applications arise in stopping time optimal control problems for Brownian motion, in groundwater hydrology, in plasticity theory etc. For more about this topic see the great book of Kinderlehrer-Stampacchia.

Now we shall present some applications regarding both the constraints problem and the free boundary problems. What follows is based on our discussion above. In the most cases we have chosen to deal with simple cases for simplicity in our calculations. In the vast majority of such problems the main idea as well as the basic methods used to solve the problems are quite the same with what we have already presented.

Integral constraints, the simplest case

As mentioned above, in this part of the chapter we are basically dealing with the procedure of minimizing a given functional in which the ''competing'' functions are required to conform to certain integral restrictions, in addition to the normal endpoint-boundary conditions. We shall present how to handle the case of the simplest problem below by using the Lagrange multipliers method.

$$\begin{cases} (1)\ J[y]=\int_a^b L(x,y,y')dx \\ (2)\ \underbrace{\boxed{W[y]=\int_a^b G(x,y,y')dx=k}}_{\text{isoperimetric constraint}} \end{cases}$$ where $L,G\in C^2[a,b]$ and $y\in C^2[a,b]$ with $y(a)=y_0$ and $y(b)=y_1$, k fixed constant. An one-parameter family $y(x)+\varepsilon h(x)$ is not a suitable choice here, because some curves may not maintain the constancy of W. Therefore we will introduce a two--parameter family $z(x)=y(x)+\varepsilon_1 h_1(x)+\varepsilon_2 h_2(x)$, where $h_1,h_2\in C^2[a,b]$ and $h_1(a)=h_1(b)=h_2(a)=h_2(b)=0$. And $\varepsilon_1,\varepsilon_2\in\mathbb{R}$. Additionally, we assume that W does not have an extremum at y. Then for any choice of h_1,h_2 there will be values of $\varepsilon_1,\varepsilon_2$ in the neighborhood of (0,0) for which $W(z)=k$. Evaluating now J and W at $z=y+\varepsilon_1 h_1+\varepsilon_2 h_2$ gives:

$$\Im(\varepsilon_1,\varepsilon_2)=J[y+\varepsilon_1 h_1+\varepsilon_2 h_2]=J[z]=\int_a^b L(x,z,z')dx$$

$$\Re(\varepsilon_1,\varepsilon_2)=W[y+\varepsilon_1 h_1+\varepsilon_2 h_2]=W[z]=\int_a^b G(x,z,z')dx=k$$

Now we know that y is a local minimum (by our assumption) for (1) subject to the constraint (2). Thus the point $(\varepsilon_1,\varepsilon_2)=(0,0)$ must be a local minimum for $\Im(\varepsilon_1,\varepsilon_2)$ subject to the constraint $\Re(\varepsilon_1,\varepsilon_2)=k$. Hence, by applying the lagrange multiplier rule, we have that:

$\left.\dfrac{\partial\Im^*}{\partial\varepsilon_1}\right|_{(0,0)}=\left.\dfrac{\partial\Im^*}{\partial\varepsilon_2}\right|_{(0,0)}=0$ where $\Im^*=\Im+\lambda\Re=\int_a^b L^*(x,z,z')dx$ and $L^*=L+\lambda G$ (we denote here by λ the Lagrange multiplier) $\Rightarrow$ $\left.\dfrac{\partial\Im^*}{\partial\varepsilon_i}\right|_{(0,0)}=\int_a^b\left\{L_y^*(x,y,y')h_i+L_{y'}^*(x,y,y')h_i'\right\}dx$ (for $i=1,2$) $\overset{\substack{\text{integration}\\ \text{by parts}}}{\Rightarrow}$

$\dfrac{\partial\Im^*}{\partial\varepsilon_i}(0,0)=\int_a^b\left\{L_y^*-\dfrac{d}{dx}L_{y'}^*\right\}h_i(x)\,dx=0$ for $i=1,2$. $\overset{\substack{\text{Fundamental}\\ \text{lemma}}}{\Rightarrow}$ $\boxed{L_y^*-\dfrac{d}{dx}L_{y'}^*=0}$ which is the necessary condition for extremals. The last implication holds since h_1,h_2 are arbitrary.

At this point let us remark that in order to be able to evaluate both λ and the solution of the problem, we need the two boundary conditions in combination with the substitution of *y(x)* into the isoperimetric constraint.

3.4 Applications and examples

Shape of a hanging rope

A rope of length l with constant density ρ hangs from two fixed points (a, y_0) and $(b,\ y_1)$ in the plane. Let $y(x)$ be an arbitrary configuration of the rope with the y axis adjusted so that $y(x)\ >\ 0$. A small element of length ds at $(x,\ y)$ has mass ρds and potential energy ρgy ds relative to $y = 0$. Therefore, the total potential energy of the rope hanging in the arbitrary configuration $y = y(x)$ is given by the functional:

$J[y]=\int_0^1 \rho gyds=\int_a^b \rho gy\sqrt{1+\left(y'\right)^2}dx$. We also know that the actual configuration minimizes the potential energy. Therefore, the constraint will be :

$W[y]=\int_a^b \sqrt{1+\left(y'\right)^2}dx=\ell$. Hence we obtain: $L^*=L+\lambda G=\rho gy\sqrt{1+\left(y'\right)^2}+\lambda\sqrt{1+\left(y'\right)^2}$

(L^* does not depend explicitly on x, thus we can use a first integral despite the direct E-L)

$$L^*-y'L^*_{y'}=c\ \Rightarrow\ \left(\rho gy+\lambda\right)\left(\sqrt{1+\left(y'\right)^2}-\frac{\left(y'\right)^2}{\sqrt{1+\left(y'\right)^2}}\right)=c\ \Rightarrow\ \frac{dy}{\sqrt{\left(\rho gy+\lambda\right)^2+c^2}}=\frac{dx}{c}\Rightarrow$$

change of variable $\begin{cases}u=\rho gy+\lambda\\ du=\rho gdy\end{cases}\Rightarrow\ \int\frac{du}{\sqrt{u^2+c^2}}=\frac{x}{c}+c_1\ \Rightarrow$

$\frac{1}{\rho g}\cosh^{-1}\left(\frac{u}{c}\right)=\frac{x}{c}+c_1$. Finally this leads to: $\boxed{y=-\frac{\lambda}{\rho g}+\frac{c}{\rho g}\cosh\left(\frac{\rho gx}{c}+c_2\right)}$ <u>*catenary*</u> .

Only one thing remains to be clarified. The computation of the left-hand integral above:

$$\frac{1}{c}\int\frac{du}{\sqrt{\left(\frac{u}{c}\right)^2-1}}\overset{\substack{\frac{u}{c}=t\\ \frac{du}{c}=dt}}{=}\int\frac{dt}{\sqrt{t^2-1}}\overset{\substack{t=\cosh y\\ dt=\sinh y}}{=}\int\frac{\sinh y}{\underbrace{\sqrt{\cosh^2 y-1}}_{\sinh y}}dy=\int dy=y=\cosh^{-1}t=\cosh^{-1}\left(\frac{u}{c}\right)$$

The constants c, c_2 and λ may be determined from the isoperimetric constraint and the endpoint conditions $y(a)\ =y_0$ and $y(b)\ =y_1$. In practice this calculation is difficult, and there may not be a smooth solution for large values of ℓ .

Schrödinger's Equation

$$\begin{cases} J[\psi] = \int\limits_{-\infty}^{+\infty} \left(\frac{\hbar^2}{2m} (\psi')^2 + V(x)\psi^2 \right) dx \\ W[\psi] = \int\limits_{-\infty}^{+\infty} \psi^2(x) dx = 1 \end{cases} \quad \text{where} \begin{cases} \psi^2: \text{ probability density function} \\ \psi\text{: wave function} \\ m\text{: mass of the particle} \\ V\text{: potential} \\ W[\psi]=1\text{: normalization condition for the} \\ \text{wave function } \psi \end{cases}$$

By employing the E-L equation subject to integral constraints we obtain:

$L^* = \frac{\hbar^2}{2m}(\psi')^2 + V(x)\psi^2 - E\psi^2(x)$, where $-E$ is the Lagrange multiplier $\Rightarrow$

$L^*_\psi - \frac{d}{dx} L^*_{\psi'} = 0 \Rightarrow \boxed{-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi}$ which is *Schrödinger's equation* for the wave function in quantum mechanics for a particle of mass m under the influence of a potential V. In general, solutions of Schrödinger's equation will exist only for discrete values of the multiplier E, which are identified with the possible energy levels of the particle and are the eigenvalues.

Generalizing the above to $\mathbb{R}^3$ we get:

$J[\psi] = \int\limits_D \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + V\psi^2 \right) dxdydz$, subject to integral constraint (probability density function must be integrated to 1) $W[\psi] = \int\limits_D \psi^2 dxdydz = 1$. Then, as usual, by employing the Lagrange multiplier rule: $L^* = L + \lambda G = \frac{\hbar^2}{2m}|\nabla \psi|^2 + V\psi^2 + \lambda\psi^2 \Rightarrow L^*_\psi - \frac{\partial}{\partial x} L^*_{\psi_x} - \frac{\partial}{\partial y} L^*_{\psi_y} - \frac{\partial}{\partial z} L^*_{\psi_z} = 0 \Rightarrow$

$2(V+\lambda)\psi - \frac{\hbar^2}{m}\left[\psi_{xx} + \psi_{yy} + \psi_{zz}\right] = 0 \Rightarrow \boxed{-\frac{\hbar^2}{2m}\Delta\psi + V\psi = -\lambda\psi}$ *Schödinger's equation.*

The Laplace multiplier is the eigenvalue of *Schödinger's operator*: $\boxed{\mathrm{K} = \frac{\hbar^2}{2m}\Delta - V}$, i.e.

$K\psi = \lambda\psi$ and represents the particle's energy level.

Connections between Isoperimetric and Sturm-Liouville problem

Below we will examine the connection between the integral constraint and Sturm-Liouville problem. For this purpose let us consider, as usual, the functional and the integral constraint:

$$\begin{cases} J[y]=\int_a^b \left\{ p(x)\left(y'\right)^2+q(x)y^2 \right\} dx \\ W[y]=\int_a^b r(x)y^2 dx=1 \end{cases} \quad \text{Then for } \begin{cases} L_y^*=2qy+2\lambda ry \\ L_{y'}^*=2py' \end{cases} \Rightarrow L^*=L+\lambda G=$$

$$=p\left(y'\right)^2+qy^2+\lambda ry^2 \overset{L_y^*-\frac{d}{dt}L_{y'}^*=0}{\Rightarrow} 2qy+2\lambda ry-2\left(py'\right)'=0 \Rightarrow \begin{cases} -\left(py'\right)'+\left(q+\lambda r\right)y=0 \\ y(a)=y(b)=0 \end{cases}$$

which is a typical S-L b.v.p.

The classical Isoperimetric Problem

Given a vector field $F:\mathbb{R}^2\to\mathbb{R}^2$ such that $F(x,y)=\left(P(x,y)\ ,\ Q(x,y)\right)$ with $\begin{cases} P(x,y)=-y \\ Q(x,y)=x \end{cases}$

and considering the functional $J[x,y]=\frac{1}{2}\int_{t_0}^{t_1}\left(xy'-yx'\right)dt,$ we 'll show that it represents the area of a domain in $\mathbb{R}^2$, say D. Indeed that is true because:

$$\int_{t_0}^{t_1}\left(xy'-yx'\right)dt=\int_{t_0}^{t_1}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt=\oint_{C=\partial D}\vec{F}ds \overset{Green}{=} \int_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=$$

$$\int_D 1-(-1)dxdy=2\int_D dxdy=2Area(D) \Rightarrow Area(D)=\frac{1}{2}\int_{t_0}^{t_1}\left(xy'-yx'\right)dt=J[x,y]\ .$$

Moreover we know that the length of a given smooth curve C is given by:

$\oint_C ds=\int_{t_0}^{t_1}\left\|\sigma'(t)\right\|dt=\int_{t_0}^{t_1}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}dt$, where $\sigma(t)=\left(x(t),y(t)\right):\mathbb{R}\to\mathbb{R}^2$. Thus

the problem $\begin{cases} J[x,y]=\frac{1}{2}\int_{t_0}^{t_1}\left(xy'-yx'\right)dt \\ W[x,y]=\int_t^{t_1}\sqrt{\left(x'\right)^2+\left(y'\right)^2}dt=\ell \end{cases}$ is in fact the problem of determining which curve of a specified perimeter encloses the maximum area. The term *Isoperimetric* , meaning same perimeter, originated in this context. Hence by employing the same techninques we have

$L^* = L + \lambda G = xy' - yx' + \lambda\sqrt{(x')^2 + (y')^2} = L^*(x, y, x', y')$ which is independent of t and depends on two functions $x = x(t)$ and $y = y(t)$ at the same time. As a consequence a first integral is: $\boxed{L^* - x'L^*_{x'} - y'L^*_{y'} = c}$, however after some elementary calculations we observe that this equality holds as an identity. Therefore we prefer to do the computations by using the system of E-L equations instead. So:

$$(\#)\begin{cases} L^*_x - \dfrac{d}{dt}L^*_{x'} = 0 \\ L^*_y - \dfrac{d}{dt}L^*_{y'} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{y'}{2} = \dfrac{d}{dt}\left(-\dfrac{y}{2} + \dfrac{\lambda x'}{\sqrt{(x')^2 + (y')^2}}\right) \\ -\dfrac{x'}{2} = \dfrac{d}{dt}\left(\dfrac{x}{2} + \dfrac{\lambda y'}{\sqrt{(x')^2 + (y')^2}}\right) \end{cases} \Rightarrow \begin{cases} y(t) - \dfrac{\lambda x'}{\sqrt{(x')^2 + (y')^2}} = d \\ x(t) + \dfrac{\lambda y'}{\sqrt{(x')^2 + (y')^2}} = c \end{cases} \Rightarrow$$

$$\begin{cases} (y - d)^2 = \dfrac{\lambda^2 (x')^2}{(x')^2 + (y')^2} \\ (x - c)^2 = \dfrac{\lambda^2 (y')^2}{(x')^2 + (y')^2} \end{cases} \overset{(+)}{\Rightarrow} (x - c)^2 + (y - d)^2 = \lambda^2,$$ which are circles! Finally we only need to show that the system of E-L (#) is indeed a necessary condition for extrema regarding the minimization problem subject to the integral constraint above.

For this purpose let us consider the following problem:

$$\begin{cases} J[y_1,y_2]=\int\limits_a^b L\left(x,y_1,y_2,y_1{}',y_2{}'\right)dx \text{ subject to} \\ W[y_1,y_2]=\int\limits_a^b G\left(x,y_1,y_2,y_1{}',y_2{}'\right)dx=c \end{cases} \quad \text{for } \begin{cases} y_1(a)=A_1,\, y_1(b)=B_1 \\ y_2(a)=A_2,\, y_2(b)=B_2 \end{cases}$$

. As before we define a two-parameter family of "competing" functions, which are;

$$\begin{cases} z_1(t)=y_1(t)+\varepsilon_1 h_1(t)+\varepsilon_2 h_2(t) \\ z_2(t)=y_2(t)+\varepsilon_1 h_3(t)+\varepsilon_2 h_4(t) \end{cases}$$

and all $h_1,h_2,h_3,h_4\in C_0^2[a,b]$. As usual, we proceed like in the simplest case demonstrated above earlier, by defining $L^*=L+\lambda G,$ where λ is the Lagrange multiplier (the classical calculus technique for minimization subject to certain constraint). So

$$\frac{\partial\mathfrak{I}^*}{\partial\varepsilon_1}(0,0)=\frac{\partial}{\partial\varepsilon_1}\bigg|_{(\varepsilon_1,\varepsilon_2)=\vec{0}}\int\limits_a^b L^*\Bigg(t,\overbrace{y_1+\varepsilon_1 h_1+\varepsilon_2 h_2}^{z_1},\overbrace{y_2+\varepsilon_1 h_3+\varepsilon_2 h_4}^{z_2},\overbrace{y_1{}'+\varepsilon_1 h_1{}'+\varepsilon_2 h_2{}'}^{z_1'},$$

$$,\overbrace{y_2{}'+\varepsilon_1 h_3{}'+\varepsilon_2 h_4{}'}^{z_2{}'}\Bigg)dt=\int\limits_a^b\left(L^*_{y_1}h_1+L^*_{y_2}h_3+L^*_{y_1'}h_1'+L^*_{y_2'}h_3'\right)dt\overset{\substack{\text{integration}\\ \text{by parts}}}{=}\int\limits_a^b\left\{\left[L^*_{y_1}-\frac{d}{dt}L^*_{y_1'}\right]h_1+\right.$$

$$\left.+\left[L^*_{y_2}-\frac{d}{dt}L^*_{y_2'}\right]h_3\right\}dt=0$$

. Similarly we obtain that:

$$\frac{\partial\mathfrak{I}^*}{\partial\varepsilon_2}(0,0)=\frac{\partial}{\partial\varepsilon_2}\bigg|_{(\varepsilon_1,\varepsilon_2)=\vec{0}}\int\limits_a^b\ldots\ldots\overset{\substack{\text{exactly the}\\ \text{same calculations}\\ \text{as above}}}{=}\int\limits_a^b\left\{\left[L^*_{y_1}-\frac{d}{dt}L^*_{y_1'}\right]h_2+\left[L^*_{y_2}-\frac{d}{dt}L^*_{y_2'}\right]h_4\right\}dt=0$$

Finally by employing the Fundamental Lemma $\Rightarrow$ $\boxed{\begin{cases} L^*_{y_1}-\dfrac{d}{dt}L^*_{y_1'}=0 \\ L^*_{y_2}-\dfrac{d}{dt}L^*_{y_2'}=0 \end{cases}}$

Minimization subject to integral constraint that leads to the eigenvalue problem for the Laplacian operator.

$$\begin{cases} J[u] = \int_D |\nabla u|^2 \, dxdy \text{ for } U \subset \mathbb{R}^2 \text{ with } C^1 - boundary. \\ W[u] = \int_D u^2 dxdy = 1 \text{ and } u = 0 \text{ on } \partial D \end{cases}$$

We shall introduce the Lagrange's multiplier λ and we shall solve the following minimization problem: $\min \left\{ \int_D |\nabla u|^2 \, dxdy \; + \lambda \left(1 - \int_D u^2 dxdy \right) \right\}$ for all u that vanish on ∂D.

Afterwards, we 'll equate the first variation to zero to find the E-L equation. The careful reader may have already noticed that this is an equal expression of the tactic (Lagrange's multiplier) we used above to approach the minimization problem subject to constraints.

Alternatively, we shall consider $L^* = L + \lambda G$ and the functional $\Im(\varepsilon_1, \varepsilon_2) = J[u + \varepsilon_1 \psi_1 + \varepsilon_2 \psi_2]$ where the quantity inside the branches is the two-parametrized variational family as usual.

Moreover, $\Im(\varepsilon_1, \varepsilon_2) = \int_D L^* \left(\vec{x}, \vec{z}, \dot{\vec{z}} \right) d\vec{x}$, for $\vec{z} = u + \varepsilon_1 \psi_1 + \varepsilon_2 \psi_2$. Then, by Lagrange's multipliers Theorem we obtain that: $\left. \dfrac{\partial \Im}{\partial \varepsilon_i} \left(\varepsilon_1, \varepsilon_2 \right) \right|_{(\varepsilon_1, \varepsilon_2) = \vec{0}} = 0$ for $i = 1, 2 \Rightarrow \boxed{L_u^* - \sum_{i=1}^{2} \dfrac{\partial}{\partial x_i} L_{u_{x_i}}^* = 0} \Rightarrow$

$$L^* = |\nabla u|^2 + \lambda u^2 \Rightarrow L^* - \frac{\partial}{\partial x_1} L_{u_{x_1}}^* - \frac{\partial}{\partial x_2} L_{u_{x_2}}^* = 0 \Rightarrow 2\lambda u - 2u_{xx} - 2u_{yy} = 0 \Rightarrow$$

$\begin{cases} \Delta u = \lambda u, \text{ in } D \\ u = 0, \text{ on } \partial D \end{cases}$ which is actually the eigenvalue problem for the Laplacian operator. We shall notice the similarities with the *Rayleigh quatient* in the sequel.

Algebraic constraints

We have only seen integral constraints where the Lagrange's multiplier is a real number so far. A reasonable question which may rise here is that the only possible constraints are of integral form? The answer is no. They may as well be of algebraic form. But the multiplier this time will be a function rather than a real number. Let us specialize what we are discussing here. For this purpose we consider the minimization problem:

$$\begin{cases} J[y,z]=\int_a^b L\left(t,y,z,y',z'\right)dt \\ G(t,y,z)=0 \;\text{ with }\left(\dfrac{\partial G}{\partial z}\neq 0\right) \\ y(a)=y(b)=z(a)=z(b)=0 \end{cases}$$

, where $y(t), z(t)$ sufficiently smooth and provide a local minimum. Then the above implies that $\exists\ \lambda(t)$ such that $\begin{cases} L_y-\dfrac{d}{dt}L_{y'}=\lambda(t)G_y \\ L_z-\dfrac{d}{dt}L_{z'}=\lambda(t)G_z \end{cases}$.

The proof is quite simple and actually we will take advantage of the IFT, since $\dfrac{\partial G}{\partial z}\neq 0$ (Implicit Function Theorem). Thus we are able to solve the constraint equation to obtain: $z=g(t,y)$. Substituting now this into the $J[y,z]$ expression to see:

$W[y]=\int_a^b F(t,y,y')dt=\int_a^b L\left(t,y,g(t,y),y',g_t+g_yy'\right)dt$ and as usual the E-L is

$$F_y-\frac{d}{dt}F_{y'}=0 \;\Rightarrow\; L_y+L_zg_y+L_{z'}\left(g_{ty}+g_{yy}y'\right)-\frac{d}{dt}\left(L_{y'}+L_{z'}g_y\right)=0 \;\Rightarrow$$

$$L_y-\frac{d}{dt}L_{y'}+g_y\left(L_z-\frac{d}{dt}L_{z'}\right)+L_{z'}\left(\cancel{g_{ty}+g_{yy}y'}\right)-L_{z'}\cancel{\frac{d}{dt}g_y}=0 \;\Rightarrow$$

$\left(L_y-\dfrac{d}{dt}L_{y'}\right)+g_y\left(L_z-\dfrac{d}{dt}L_{z'}\right)=0$. Now let us note that by the IFT we may also obtain

$g_y=-\dfrac{G_y}{G_z}$ and this is valid, since if we derivate the equation $G\left(t,y,\overbrace{g(t,y)}^{z}\right)=0$ with respect to y, i.e. $0=G_y+G_zg_y$, we obtain the required. Thus by substituting this to the last equality above:

$\dfrac{L_y-\frac{d}{dt}L_{y'}}{G_y}=\dfrac{L_z-\frac{d}{dt}L_{z'}}{G_z}$ and these two expressions must be equal to the same function of $t,$ that is

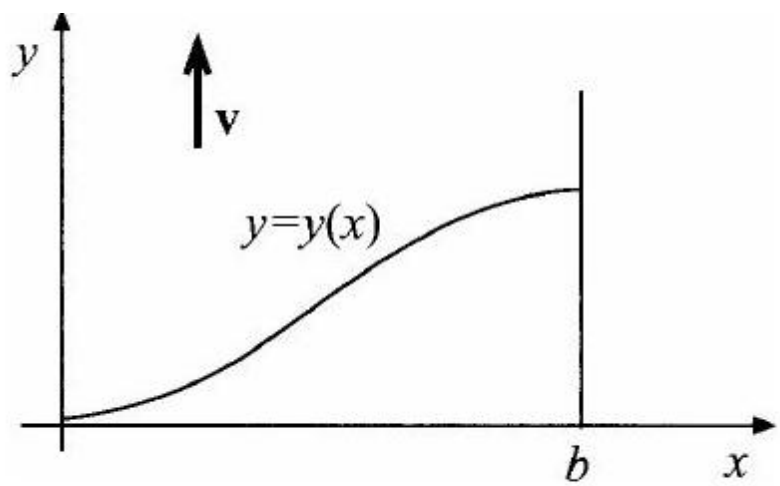
$\dfrac{L_y-\frac{d}{dt}L_{y'}}{G_y}=\dfrac{L_z-\frac{d}{dt}L_{z'}}{G_z}=\lambda(t)$. Therefore we get: $\begin{cases} L_y-\dfrac{d}{dt}L_{y'}=\lambda(t)G_y \\ L_z-\dfrac{d}{dt}L_{z'}=\lambda(t)G_z \end{cases}$

3.5 Natural boundary condition (supplementary remarks)

Let us consider the following problem: A river with parallel straight banks b units apart has stream velocity given by $v(x,\ y)\ = v(x)\,j$, where j is the unit vector in the y direction (see Figure below). Assuming that one of the banks is the y axis and that the point $(0,0)$ is the point of departure, what route should a boat take to reach the opposite bank in the shortest possible time? Assume that the speed of the boat in still water is c, where $c > v$.

This problem differs from those in earlier sections in that the right-hand endpoint, the point of arrival on the line $x = b$, is not specified; it must be determined as part of the solution to the problem.

Such a problem is called a free endpoint problem, and if $y(x)$ is an extremal, then a certain condition must hold at $x = b$. Conditions of these types, called *natural boundary conditions*, are the subject of this section. Just as common are problems where both endpoints are unspecified.



A classical natural boundary condition problem

Consider the functional $J[y]=\int_a^b L(x,y,y')dx$ and let $y\in C^2[a,b]$ with $y(a)=y_0,\ y(b)$ free, be a local minimum. Then the variation h must be C^2 and must also satisfy the single condition $h(a)=0$. But pay attention to the fact that no condition on h at $x=b$ is required due to the fact that the admissible functions here are not specified at the right endpoint. So:

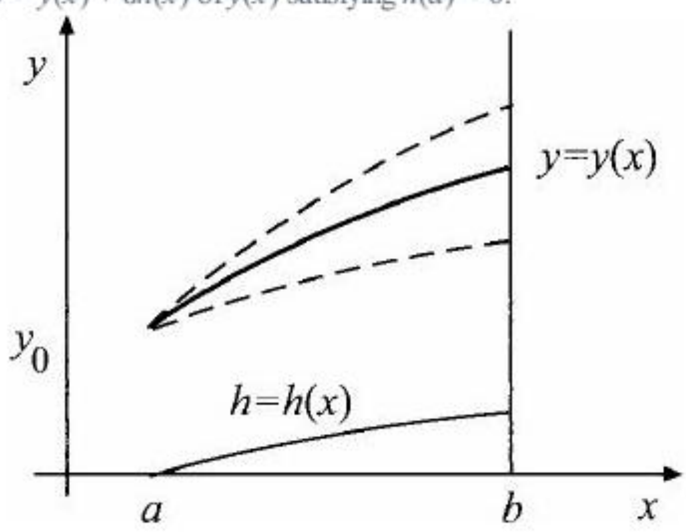
$$\frac{d}{d\varepsilon}J[y+\varepsilon h]\bigg|_{\varepsilon=0}=\int_a^b\left(L_y h+L_{y'}h'\right)dx\overset{\substack{\text{integration}\\ \text{by parts}}}{=}\int_a^b\left(L_y-\frac{d}{dx}L_{y'}\right)h(x)dx+h(b)L_{y'}\left(b,y(b),y'(b)\right)=0$$

$\forall h\in C^2[a,b]$ with additionally $h(a)=0$. Therefore, it also holds for those "h" which satisfy the condition $h(b)=0$ and by the Fundamental lemma we get: $L_y-\frac{d}{dx}L_{y'}=0$ and finally by substituting the E-L above, we obtain: $L_{y'}\left(b,y(b),y'(b)\right)h(b)=0,$ valid for any choice of $h(b)$.

Thus $\boxed{L_{y'}\left(b,y(b),y'(b)\right)=0}$

which is a condition on the extremal y at x=b. This is called a *natural boundary condition.* The E-L equation, the fixed boundary condition $y(a)\ =y_0$,and the natural boundary condition are enough to determine the extremal for the variational we presented earlier above. By similar arguments if the left endpoint $y(a)$ is unspecified, then the natural boundary condition on an extremal y at $x=a$ is: $\boxed{L_{y'}\left(a,y(a),y'(a)\right)=0}$

Variations $y=y(x)+\varepsilon h(x)$ of $y(x)$ satisfying $h(a)=0$.



Below we shall present a few examples of natural (or free) boundary condition problems:

Sturm-Liouville equation as a condition for extremals

$J[y]=\int_0^1\left[p(x)\left(y'\right)^2-q(x)y^2\right]dx$ with boundary conditions $\begin{cases}y(0)=0\\ y(1)\ \ free\end{cases}$

We shall compute the E-L equation for the functional above. For this purpose we compute: $L_y=-2q(x)y$ and $L_{y'}=2p(x)y'\ \Rightarrow$

$-2q(x)y+2p'(x)y'+2p(x)y'(x)=0$ with natural boundary condition:

$2p(1)y'(1)=0$ and because $p(x)>0\ \Rightarrow\ \frac{d}{dx}\left(p(x)y'\right)+q(x)y=0$ with

$y(0)=0\ and\ y'(1)=0.$

Ramsey's Growth Model

One version of Ramsey's Growth model in Economics involves minimizing the total product $J[M]=\int_0^T(aM-M'-b)^2dt$ for a,b>0 , over a fixed planning period $[0,T]$, where $M=M(t)$ is the capital at time t and $M(0)=M_0$ is the initial capital. If $M(t)$ minimizes J, the capital $M(t)$ at the end of the planning period can be obtained as : $L_M-\frac{d}{dt}L_{M'}=0\ \Rightarrow$

$2a(aM-M'-b)+\frac{d}{dt}2\left(aM-M'-b\right)=0\ \Rightarrow\ M''(t)-a^2M(t)+ab=0\ \Rightarrow$

$M(t)=Ae^{at}+Be^{-at}+\frac{b}{a}$ with $M(0)=M_0$ and the natural boundary condition $L_{M'}(T,M(T),M'(T))=0,$ or $aM(T)-M'(T)=b\ \Rightarrow$ Therefore, the required result can be obtained by simple computations which will be omitted here.

Supplements to the theory (more variables)

$J[u]=\iint_D L\left(x,y,u,u_x,u_y\right)dxdy$, where $u\in C^2(D),$ but undefined on $\partial D.$ For this purpose, we define the test functions with compact support $h\in C_0^2(D)$ which means that $h\in C^2(D)$ and additionally $h=0$ on $\partial D.$

$J[u+\varepsilon h]=\iint_D L\left(x,y,u+\varepsilon h,u_x+\varepsilon h_x,u_y+\varepsilon h_y\right)dxdy\ \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}^{\text{first variation}}\ \Rightarrow$

$\iint_D\left[L_uh+L_{u_x}h_x+L_{u_y}h_y\right]dxdy=0$ and by employing the identity $\frac{\partial}{\partial x}\left(L_{u_x}h\right)=L_{u_x}h_x+h\frac{\partial}{\partial x}L_{u_x}$

(similarly for $L_{u_y}h_y$) $\Rightarrow \iint_D \left\{ L_u - \frac{\partial}{\partial x} L_{u_x} - \frac{\partial}{\partial y} L_{u_y} \right\} h(x,y)dxdy + \iint_D \left[\frac{\partial}{\partial x} \overbrace{\left(L_{u_x}h\right)}^{Q} + \frac{\partial}{\partial y} \overbrace{\left(L_{u_y}h\right)}^{-P} \right] dxdy$

Now we are ready to apply *Green's* Theorem for the vector field $\vec{F}(x,y) = \left(P(x,y), Q(x,y)\right)$:

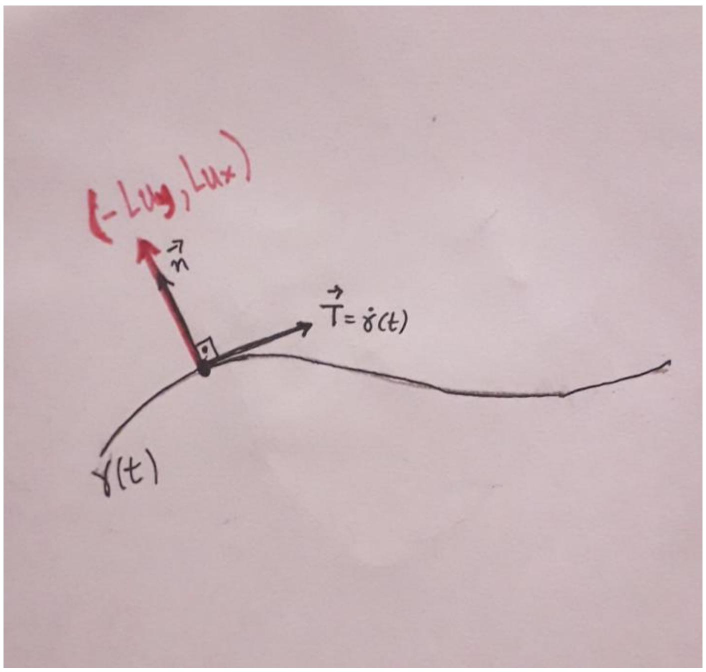
$\iint_D \left[\frac{\partial}{\partial x}\left(L_{u_x}h\right) + \frac{\partial}{\partial y}\left(L_{u_y}h\right) \right] dxdy = \oint_{\partial D} \left[-L_{u_y}\frac{dx}{dt} + L_{u_x}\frac{dy}{dt} \right] h\ dt \overset{\substack{\text{line}\\\text{integral}}}{=} \oint_{\partial D} \vec{F}\cdot\vec{T}\ h(x,y)ds$, where

$\vec{T} = \dot{\gamma}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. For the smooth curve $\gamma(t) = \left(x(t), y(t)\right)$ and the vector field

$\vec{F} = \left(-L_{u_y}, L_{u_x}\right)$. Now the typical argument can be applied. Therefore, since the equality above is valid for every $h \in C_0^2(D)$, then it is also valid for those $h = 0$ on ∂D. Consequently it is reduced to the following equality: $\iint_D \left[\frac{\partial}{\partial x} L_{u_x} + \frac{\partial}{\partial y} L_{u_y} - L_u \right] hdxdy = 0$

where by applying, as usual, the fundamental lemma, we obtain the typical E-L equation. Finally, by substituting the E-L above, we get the *natural boundary condition.*

$\vec{F}\cdot\vec{T} = 0 \ \Rightarrow \ \boxed{\left(-L_{u_y}, L_{u_x}\right) \perp \vec{T} = \left(\dot{x}(t), \dot{y}(t)\right) \text{ or equivalently } \left(-L_{u_y}, L_{u_x}\right) \| \vec{n} = \left(\dot{y}(t), -\dot{x}(t)\right)}$



This figure depicts the orthogonality of tangential and normal vectors and justifies our underlined assertion *example for a given Lagrangian:*

$J[u]=\iint\limits_D \left\{a\left(u_x^2+u_y^2\right)-bu^2\right\}dxdy$ where $a=a(x,y)$ and $b=b(x,y)$ smooth.

$J[u+\varepsilon h]=\iint\limits_D \left\{a\left[\left(u_x+\varepsilon h_x\right)^2+\left(u_y+\varepsilon h_y\right)^2\right]-b\left(u+\varepsilon h\right)^2\right\}dxdy \Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\overset{\substack{\text{necessary}\\ \text{condition}}}{=0} \Rightarrow$

$\left.\iint\limits_D \left\{a(x,y)\left[\cancel{2}\left(u_x+\cancel{\varepsilon} h_x\right)h_x+\cancel{2}\left(u_y+\cancel{\varepsilon} h_y\right)h_y\right]-\cancel{2}b(x,y)\left(u+\cancel{\varepsilon} h\right)h\right\}dxdy\right|_{\varepsilon=0}=0 \Rightarrow$

$\iint\limits_D \left\{au_xh_x+au_yh_y-buh\right\}dxdy=0.$ Now we can either proceed exactly as above, in other words by writing $au_xh_x=\frac{\partial}{\partial x}\left(au_xh\right)-\frac{\partial}{\partial x}\left(au_x\right)h,$ similarly for au_yh_y, and apply $Green's$ Theorem, or we could equivalently integrate by parts. We shall demonstrate both ways here.

a)

$$\iint\limits_D \left\{\frac{\partial}{\partial x}\left(au_xh\right)-\frac{\partial}{\partial x}\left(au_x\right)h+\frac{\partial}{\partial y}\left(au_yh\right)-\frac{\partial}{\partial y}\left(au_y\right)h-buh\right\}dxdy=0 \quad \Rightarrow$$

$$\iint\limits_D -\left\{b(x,y)u+\frac{\partial}{\partial x}\left(au_x\right)+\frac{\partial}{\partial y}\left(au_y\right)\right\}hdxdy+\iint\limits_D \frac{\partial}{\partial x}\overbrace{\left(au_xh\right)}^{Q}+\frac{\partial}{\partial y}\overbrace{\left(au_yh\right)}^{-P}dxdy$$

Now the E-L equation is $L_u-\frac{\partial}{\partial x}L_{u_x}-\frac{\partial}{\partial y}L_{u_y}=0$ with $L=a\left(u_x^2+u_y^2\right)-bu^2,$ therefore

$\begin{cases} L_u=-2bu \\ L_{u_x}=2au_x \\ L_{u_y}=2au_y \end{cases}$ and apply Green's theorem for the second double integral $\Rightarrow$

(the first double integral is zero due to E-L)

$$\oint\limits_{\partial D}\left\{-a(x,y)u_yh\frac{dx}{dt}+a(x,y)u_xh\frac{dy}{dt}\right\}dt \overset{\substack{\text{line}\\ \text{integral}}}{=} \oint\limits_{\partial D} h(x,y)\vec{F}\cdot\vec{T}ds \text{ , where } \vec{F}=\left(-au_y,au_x\right)$$

Consequently the *natural boundary condition* would be $\vec{F}\cdot\vec{T}=0,$ since the line integral must be zero for every $h\in C_0^2(D)$ and therefore the condition above is justified. But

$\vec{T}\perp\vec{n}\Rightarrow\vec{F}\parallel\vec{n}\Rightarrow\ \left(-au_y,au_x\right)\perp\left(au_x,au_y\right)\overset{\substack{\text{because the}\\ \text{inner product}\\ \text{equals to zero}}}{=} a\nabla u.$ So since $a\nabla u\perp\vec{F}$ which is parallel to outward unit normal $\vec{n},$ we get that $a\nabla u\perp\vec{n},$ i.e. $\boxed{a\frac{\partial u}{\partial\vec{n}}=0}$.

b) integration by parts

$$\begin{cases} \displaystyle\iint_D \left[a(x,y)u_x h_x\right]dxdy = \int_{\partial D} ahu_x n_1 dS - \iint_D \left[h(x,y)\frac{\partial}{\partial x}\left(au_x\right)\right]dxdy \\ \displaystyle\iint_D \left[a(x,y)u_y h_y\right]dxdy = \int_{\partial D} ahu_y n_2 dS - \iint_D \left[h(x,y)\frac{\partial}{\partial y}\left(au_y\right)\right]dxdy \end{cases} \quad \text{this implies:}$$

$$\frac{d}{d\varepsilon}J[u+\varepsilon h]\Big|_{\varepsilon=0} = \iint_D \cancelto{0}{\left\{-a_x u_x - au_{xx} - a_y u_y - au_{yy} - bu\right\}} h(x,y)dxdy + \int_{\partial D} ha\left(u_x n_1 + u_y n_2\right)dS = 0$$

the first integral eqauls to zero due to E-L and the second in fact equals:

$$\int_{\partial D} ha(x,y)\left(\nabla u\cdot\vec{n}\right)dS = 0 \quad (h \text{ is arbitrary}) \Rightarrow a\left(\nabla u\cdot\vec{n}\right) = 0 \Rightarrow \boxed{a(x,y)\frac{\partial u}{\partial\vec{n}} = 0}^{\text{natural boundary condition}}$$

3.6 Rayleigh Quotient

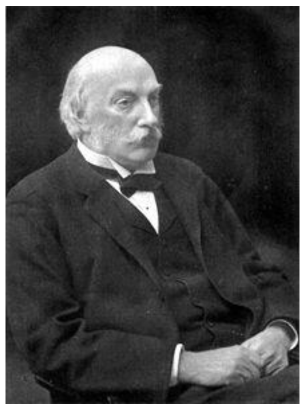
In this section we will define the Rayleigh Quotient and we will present some properties of the eigenvalues and the eigenfunctions of the Dirichlet's problem for the eigenvalue problem of Laplacian operator (the 1-dimensional case of which is the S-L b.v.p.) as well as the connection with the calculus of variations.

An important problem that arises frequently in chemistry and physics is how to compute the spectrum of a quantum system. The system is modeled by a Schrödinger operator. In the one-dimensional case such operators are of the Sturm-Liouville type. For instance, the information from the spectrum of the Schrödinger operator enables us to determine the discrete frequencies of the radiation from excited atoms. In addition, using the information from the spectrum, one can understand the stability of atoms and molecules. We do not present here a precise definition of the spectrum of a given linear operator, but roughly speaking, the (point) *spectrum* of a quantum system is given by the eigenvalues of the corresponding Schrödinger operator. It is particularly important to find the first (minimal) eigenvalue, or at least a good approximation of it. The minimal eigenvalue of the eigenvalue problem for the Laplacian operator, which we shall examine blow, is called the *principal eigenvalue* (or the ground state energy), and the corresponding eigenfunction is called the *principal eigenfunction* (or the ground state).

The British scientist John William Strutt (Lord Rayleigh) (1842–1919) observed that the expression:

$\boxed{R[u] = -\dfrac{\int\limits_{\Omega} u\Delta u\ dx}{\int\limits_{\Omega} u^2 dx}}$ $\left(Rayleigh\ Quotient\right)$ plays a very imprtant role in this context.

It is also equal to this expression: $\dfrac{\int\limits_{\Omega} |\nabla u|^2\ dx}{\int\limits_{\Omega} u^2 dx}$ which we shall use below. We will see that actually are the same (by integrating by parts) in the following sections.



John William Strutt (3rd Baron of Rayleigh) (1842–1919) was a British scientist who made extensive contributions to both theoretical and experimental Physics. He spent all of his academic career at the University of Cambridge. Among many honors, he received the 1904 Nobel Prize in Physics for his investigations of the densities of the most important gases and for his discovery of Argon, in connection to his studies. He served as president of the Royal Society from 1905-1908 and as a Chancellor of Cambridge from 1908-1919.

Properties of the eigenvalues and the eigenfunctions :

- Symmetry

$$\boxed{\int\limits_{\Omega} v\Delta u\ dx = \int\limits_{\Omega} u\Delta v\ dx}$$

$\int\limits_{\Omega} v\Delta u\ dx = -\int\limits_{\Omega} \nabla v \cdot \nabla u\ dx = \int\limits_{\Omega} u\Delta v\ dx\ symmetry$ of Laplace operator, because:

$\int\limits_{\Omega} v\Delta u\ dx = \int\limits_{\Omega} v div\left(\nabla u\right)\ dx = \int\limits_{\Omega} v\nabla\cdot\left(\nabla u\right)\ dx = \int\limits_{\partial\Omega} v\cancel{\dfrac{\partial u}{\partial \vec{n}}}^{0}\ dS - \int\limits_{\Omega} \nabla v \cdot \nabla u\ dx \overset{\substack{\text{similar}\\ \text{argument}}}{=} \int\limits_{\Omega} u\Delta v\ dx$, where the surface integral is zero because under our assumptions $u, v = 0$ on $\partial\Omega$.

- Orthogonality

eigenfunctions associated to different eigenvalues are orthogonal to each other.

$$\left(\lambda_n \neq \lambda_m\right) \text{ and consider } \begin{cases} \Delta u_n = -\lambda_n u_n \\ \Delta u_m = -\lambda_m u_m \end{cases} \Rightarrow \begin{cases} u_m \Delta u_n = -\lambda_n u_m u_n \\ u_n \Delta u_m = -\lambda_m u_n u_m \end{cases} \overset{\int_\Omega}{\Rightarrow} \begin{cases} \int_\Omega u_m \Delta u_n \; dx = -\lambda_n \int_\Omega u_m u_n \; dx \\ \int_\Omega u_n \Delta u_m \; dx = -\lambda_m \int_\Omega u_n u_m \; dx \end{cases}$$

by subtracting by parts and using the symmetry property now we get:

$$\left(\lambda_n - \lambda_m\right) \int_\Omega u_n u_m \; dx = 0 \overset{\lambda_n \neq \lambda_m}{\Rightarrow} \left\langle u_n, u_m \right\rangle = 0 \;\Rightarrow\; u_n \perp u_m$$

- The eigenvalues are real

Let us assume for contradiction that $\lambda \in \mathbb{C}$.

$$\begin{cases} \Delta u = -\lambda u, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases} \Rightarrow \begin{cases} \Delta u + \lambda u = 0, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases} \Rightarrow \begin{cases} \overline{\Delta u + \lambda u} = 0, \text{ in } \Omega \\ \bar{u} = 0, \text{ on } \partial\Omega \end{cases} \Rightarrow \begin{cases} \Delta \bar{u} = -\bar{\lambda} \bar{u}, \text{ in } \Omega \\ \bar{u} = 0, \text{ on } \partial\Omega \end{cases} \Rightarrow$$

$\left(\bar{u}, \bar{\lambda}\right)$ is an eigenfunction-eigenvalue pair $\Rightarrow$ $\left(\text{assuming } \lambda \neq \bar{\lambda}\right) \Rightarrow$

$$0 = \left\langle u, \bar{u} \right\rangle = \int_\Omega u \bar{u} dx = \int_\Omega |u|^2 \, dx > 0, \text{ since } u \neq 0, \text{ which is a contradiction.}$$

- The eigenfunctions are real

The proof is quite simple. It is based on the separation into the real and the imaginary part and the observation that both the real and the imaginary part of the eigenfunction solve the differential equation and satisfy the boundary condition as well. Since at least one of these two functions are not zero, it follows that at least one of them is an eigenfunction. If now λ is simple we have a real eigenfunction. On the contrary if it is of higher multiplicity, say 2, we can consider the real and imaginary parts of two linearly independent eigenfunctions. By a simple dimensional consideration, it follows that out of these four real functions, one can extract at least one pair of linearly independent functions.

- Multiplicity of the eigenvalues

One of the main differences between the 1-dimensional S-L problem and its multi-dimensional generalization, which is Dirichlet's b.v.p. for the eigenvalue problem of Laplace operator, involves multiplicity. In the multidimensional case at hand here,

$$(\#)\begin{cases} -\Delta u = \lambda u, \; \vec{x} \in \Omega \\ u = 0, \; \vec{x} \in \partial\Omega \end{cases} \text{ the multiplicity might be larger than one (but always finite!).}$$

This is a fact of great physical significance.

- there exists a sequence of eigenvalues diverging to infinity

Proposition

(a) The set of the eigenvalues for (#) consists of a monotone, non-decreasing sequence that diverges to $+\infty$, i.e. $0<\lambda_0 \le \lambda_1 \le \ldots \le \lambda_n \le \lambda_{n+1} \le \ldots \xrightarrow[n\to\infty]{} +\infty$.

(b) The eigenvalues are all positive and they have also finite multiplicity.

We shall only show the positivity (i.e. $\forall n \in \mathbb{N},\ \lambda_n>0$) here. The other results are derived by the spectral theorem for the Laplace eigenvalue problem.

$-\Delta u = \lambda u \Rightarrow -u\Delta u = \lambda u^2 \Rightarrow \int_\Omega \lambda u^2 dx = -\int_\Omega u\Delta u\ dx,$ but we observe that:

$$-\int_\Omega u\Delta u\ dx = -\int_{\partial\Omega} \cancel{u}^{0} \frac{\partial u}{\partial \vec{n}} dS + \int_\Omega \nabla u \cdot \nabla u\ dx = \int_\Omega |\nabla u|^2 dx \ \Rightarrow \ \boxed{\lambda = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2\ dx}} > 0 \text{ for } u \ne 0$$

Since u is constant is not an eigenfunction, it follows that $\lambda>0$ □

Jean-Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) was a French mathematician and physicist born in Auxerre and best known for initiating the investigation of Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations. The Fourier transform and Fourier's law of conduction are also named in his honor. Fourier is also generally credited with the discovery of the greenhouse effect.

- Generalized Fourier Series

We know that the infinite orthonormal set of the eigenfunctions is complete, thus we can formally expand smooth functions defined in Ω into a generalized Fourier series, converging on the average (due to completeness), i.e.

$v(x) = \sum_{n=0}^{\infty} a_n u_n(x)$, where $a_n = \langle u_n(x), v(x) \rangle$ are the generalized Fourier coefficients and $\{u_n\}_{n=0}^{\infty}$ the orthonormal basis.

- Asymptotic behaviour of the eigenvalues when n tends to infinity

It can be shown that for $\Omega \subset \mathbb{R}^j$ the n^{th} eigenvalue associated with the b.v.p

$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$ has the following asymptotic behaviour in the limit $n \to \infty$

$\boxed{\lambda_n \sim 4\pi^2 \left(\dfrac{n}{\omega_j |\Omega|} \right)^{\frac{2}{j}}}$ for $j = 1, 2, 3, \ldots$ This is called *Weyl's asymptotic formula*.

Here ω_j denotes the volume of the unit ball in $\mathbb{R}^j$. For example $\begin{cases} \omega_1 = 2 \\ \omega_2 = \pi \\ \omega_3 = \dfrac{4\pi}{3} \end{cases}$ etc.

An optimization problem for the first eigenfunction.

We have already developed above an integral formulation for the eigenvalues,

i.e. $\boxed{\lambda = \dfrac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 dx}}$. Let us now denote the smallest (minimal) eigenvalue, which hereafter

we shall call *principal*, by λ_0. Then we will show that $\boxed{\lambda_0 = \inf_{u \in V} \dfrac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 dx}}$, where

$V = \left\{ u \in C^2(\Omega) \cap C(\bar{\Omega}) \middle| u \neq 0 \text{ and } u|_{\partial\Omega} = 0 \right\}$. Moreover λ_0 is a simple eigenvalue and the infimum is only achieved for the associated eigenfunction.

Proof:

Let $\{\lambda_n\}_{n=1}^{\infty}$ be the non-decreasing sequence of the real eigenvalues (not all of them necessarily simple) , such that $0<\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \lambda_{n+1} \le \ldots$, $\lim\limits_{n\to\infty}\lambda_n = +\infty$. Moreover let $\{\varphi_n\}_{n=1}^{\infty}$ be the orthonormal and complete sequence of the corresponding eigenfunctions, such that $\begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n, \text{ in } \Omega \\ \varphi_n = 0, \text{ on } \partial\Omega \end{cases}$ Then (see the remarks below) we know that we are able to expand $u \in V = \left\{ u \in C^2(\Omega)\bigcap C(\overline{\Omega}) \;\middle|\; u \ne 0 \text{ and } u\big|_{\partial\Omega} = 0 \right\}$ as a generalized Fourier series, i.e. $\boxed{u = \sum_{n=0}^{\infty} a_n\varphi_n \text{ where } a_n = \langle u, \varphi_n\rangle}$ which converges uniformly

Therefore:

$$R[u] = \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2 dx} \underset{\substack{\text{see}\\\text{below}}}{=} \frac{\sum_{n=0}^{\infty} a_n^2\lambda_n}{\sum_{n=0}^{\infty} a_n^2} \ge \lambda_0 \cancel{\frac{\sum_{n=0}^{\infty} a_n^2}{\sum_{n=0}^{\infty} a_n^2}} = \lambda_0 \;\Rightarrow\; R[u] \ge \lambda_0 \;\; \forall\; u \in V \text{ and thus } \inf_{u\in v} R[u] \ge \lambda_0.$$

Calculations:

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega \langle \nabla u, \nabla u\rangle\,dx = \int_\Omega \left\langle \sum_{n=0}^{\infty} a_n \nabla\varphi_n, \sum_{m=0}^{\infty} a_m\nabla\varphi_m \right\rangle dx = \sum_{n,m=0}^{\infty} a_n a_m \int_\Omega \langle \nabla\varphi_n, \nabla\varphi_m\rangle\,dx \text{ , but}$$

$$\int_\Omega \langle \nabla\varphi_n, \nabla\varphi_m\rangle\,dx = \int_{\partial\Omega} \cancelto{0}{\varphi_m \frac{\partial\varphi_n}{\partial\vec{n}}}\,dS - \int_\Omega \varphi_m\Delta\varphi_n\;dx = \int_\Omega \lambda_n\varphi_m\varphi_n dx = \lambda_n \langle\varphi_m,\varphi_n\rangle_{Hilbert}. \text{ Therefore}$$

$$\sum_{n,m=0}^{\infty} a_n a_m \int_\Omega \nabla\varphi_n \cdot \nabla\varphi_m dx = \sum_{n,m=0}^{\infty} a_n a_m\lambda_n \int_\Omega \varphi_n\varphi_m dx = \sum_{n=0}^{\infty} a_n^2\lambda_n \ge \sum_{n=0}^{\infty} a_n^2\lambda_0$$ due to the non decreasing sequence of the eigenvalues and also $\langle\varphi_n,\varphi_m\rangle_{Hilbert} = \delta_{nm}$ since they are orthonormal.

Regarding the denominator now, since we have a complete and orthonormal system of eigenfunctions which forms a basis, the *Bessel* inequality is in fact equality here (*Parseval*).

So we get: $\int_\Omega u^2 dx = \|u\|_{L^2(\Omega)}^2 \overset{Parseval}{=} \sum_{n=0}^{\infty} |\langle u, \phi_n\rangle|^2 = \sum_{n=0}^{\infty} a_n^2$.

Now it can be easily verified that equality holds iff $u = c\varphi_0$, because then we can achieve that $R[u] = \lambda_0$ indeed, and by this way we are done. To be more specific we have that: $|\nabla u|^2 = c^2 |\nabla \varphi_0|^2$ and $u^2 = c^2\varphi_0^2$ and $\|\varphi_0\|^2_{L^2(\Omega)} = \int_\Omega \varphi_0^2 dx = 1$. So we get that

$$R[u] = \frac{\int_\Omega \cancel{c^2} |\nabla \varphi_0|^2 dx}{\int_\Omega \cancel{c^2} \varphi_0^2 dx} = \frac{\int_\Omega |\nabla \varphi_0|^2 dx}{1} = \int_\Omega \nabla \varphi_0 \cdot \nabla \varphi_0 dx = -\int_\Omega \varphi_0 \Delta \varphi_0 dx = \int_\Omega \lambda_0 \varphi_0^2 dx = \lambda_0$$

where we have used integration by parts and the fact that $\varphi_0 = 0$ on $\partial\Omega$ □

Remark in completeness:

In general if we have an orthonormal sequence, say $\{u_n\}_{n=1}^\infty$, finite or infinite, then we know *Bessel's inequality*, i.e. $\sum_{n=1}^\infty |\langle u_n, u\rangle|^2 \leq \|u\|^2$, and the *Riemann-Lebesgue lemma*, i.e. $\lim_{n\to\infty} \langle u_n, u\rangle = 0$ hold true. Now if it is also *complete* then the Bessel holds as an equality called *Parseval*.

Remark in convergence:

We shall make two comments regarding the convergence of the eigenfunction expansions, which we referred to previously above.

If u is *piecewise differentiable*, then the eigenfunction expansion converges to the average, i.e. $\frac{u_+ + u_-}{2}$, which is the Dirichlet's theorem.

If now u is *piecewise differentiable* and *continuous*, then the eigenfunction expansion converges uniformly to u.

That's why the smooth u we have considered above converges uniformly, as mentioned.

Connections to the calculus of variations

We shall notice now the similarity of our approach above with the following minimization problem from the calculus of variations (subject to an integral constraint). For this consider

$$\begin{cases} J[u] = \int_{\Omega} |\nabla u|^2 dxdy \\ W[u] = \int_{\Omega} u^2 dxdy = 1 \\ u = 0 \text{ on } \partial\Omega \text{ as well} \end{cases}$$

. By using the Lagrange multiplier rule we see that for

$L^* = L + \lambda G,$ where $L = L(x, y, u, u_x, u_y) = |\nabla u|^2$ and $G(x, y, u, u_x, u_y) = u^2$ the E-L is

$$L^*_u - \frac{\partial}{\partial x} L^*_{u_x} - \frac{\partial}{\partial y} L^*_{u_y} = 0 \Rightarrow 2\lambda u - \frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) = 0 \Rightarrow \begin{cases} \Delta u = \lambda u, \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

which apparently is the eigenvalue problem for the Laplacian operator. Additionally let us note that here the Lagrange multipliers are in fact the eigenvalues. Regarding the Rayleigh quotient for this problem, it takes the following form:

$$R[u] = -\frac{\int_{\Omega} u\Delta u \, dxdy}{\int_{\Omega} u^2 dxdy} = \frac{-\int_{\partial\Omega} \cancel{u}^{\,0} \frac{\partial u}{\partial \vec{n}} dS + \int_{\Omega} \nabla u \cdot \nabla u \, dxdy}{\int_{\Omega} u^2 dxdy} = \frac{\int_{\Omega} |\nabla u|^2 dxdy}{\int_{\Omega} u^2 dxdy}$$

which is obviously similar to what we were dealing with previously! As a consequence the Rayleigh quotient method $\left(\lambda_0 = \inf_{u \in V} \mathrm{R[u]}\right)$ is an alternative method of finding the minimal "principal" eigenvalue, in comparison with the variational method of minimization, being discussed above. In general, both methods are of equal usefulness and perhaps the Rayleigh quotient (seen as a minimization method) is even more useful in some cases, especially from the numerical point of view. Nowadays, the vast majority of numerical methods for computing the eigenvalues is based on what we call Rayleigh-Quotient iteration. We use this method to obtain an eigenvalue approximation from an eigenfunction initial approximation.

4. Hamilton's Principle-Canonical form

According to the doctrine of classical dynamics, one associates with the system being described a set of quantities or dynamical variables, each of which has a well-defined value at each instant of time and which defines the state of the dynamical system at that instant. Further, it is assumed that the time evolution of the system is completely determined if its state is known at some given instant. Analytically this doctrine is expressed by the fact that the dynamical variables satisfy a set of differential equations (the equations of motion of the system) as functions of time, along with initial conditions.

The program of classical dynamics consists of listing the dynamical variables and formulating the equations of motion that predict the system's evolution in time. Newton's second law of motion describes the dynamics of a mechanical system.

Another method of obtaining the equations of motion is from a variational principle. This method is based on the idea that a system should evolve along a path of least resistance.

Principles of this sort have a long history in physical theories dating back to antiquity when Hero of Alexandria stated a minimum principle concerning the path of reflected light rays. In the 17th century, Fermat's principle, that light rays travel along the path of shortest time, was put forth. For mechanical systems, Maupertuis's principle of least action stated that a system should evolve from one state to another in such a way that the *action* (a vaguely defined term with the energy × time) is smallest. Lagrange and Gauss were advocates of similar principles. In the early part of the 19th century, however, W. R. Hamilton (1805–1865) stated what has become an encompassing, aesthetic principle that can be generalized to embrace many areas of physics. Hamilton's principle states that *the time evolution of a mechanical system occurs in such a manner that the integral of the difference between kinetic and potential energy is stationary*. To be more precise, let

$y_1,\dots,y_n$ denote a set of *generalized coordinates* of a given dynamical system. That is, regarded as functions of time, we assume that $y_1,\dots,y_n$ completely specify the state of the system at any instant. Further, we assume that there are no relations among the y_i, so that they may be regarded as independent. In general, the y_i may be lengths, angles, or whatever. The time derivatives $\dot{y}_1,\dots,\dot{y}_n$ are called the *generalized velocities*. The *kinetic energy* T is, in the most general case, a quadratic form in the y_i, that is,

$\boxed{T=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\left(y_1,...,y_n\right)\dot{y}_i\dot{y}_j}$, where the a_{ij} are known functions of the coordinates

$y_1,\dots,y_n$. The *potential energy* V is a scalar function $\boxed{V=V\left(t,y_1,...,y_n\right)}$. We define now the Lagrangian of the system by $L=L\left(t,y_1,...,y_n,\dot{y}_1,...,\dot{y}_n\right)=T-V,$ so $L=T-V$.

4.1 Hamilton's Principle

Hamilton's principle for these systems may then be stated as follows: Consider a mechanical system described by *generalized coordinates* $y_1,\ldots,y_n$ with Lagrangian as above. Then the motion of the system from time t_0 to t_1 is such that the functional: (*action* integral)

$\boxed{J[y_1,\ldots,y_n]=\int_{t_0}^{t_1} L\left(t,y_1,\ldots,y_n,\dot{y}_1,\ldots,\dot{y}_n\right)dt}$ is stationary for the functions $y_1(t),\ldots,y_n(t)$, which describe the actual time evolution of the system. If we regard the set of coordinates $y_1,...,y_n$ as coordinates in n-dimensional space, then the equations $y_i = y_i(t),\ i=1,2,...,n\ (t_0 \le t \le t_1)$ are parametric equations of a curve C that joins two states $S_0:\ \left(y_1(t_0),\ldots,y_n(t_0)\right)$ and S_1: $\left(y_1(t_1),\ldots,y_n(t_n)\right)$. Hamilton's principle then states that among all paths in configuration space connecting the initial state S_0 to the final state S_1, the actual motion takes place along the path that affords an extreme value to the integral presented above. The actual path is an extremal. In physics and engineering, Hamilton's principle is often stated concisely as:

$$\delta\int_{t_0}^{t_1} L\left(t,y_1,\ldots,y_n,\dot{y}_1,\ldots,\dot{y}_n\right)dt = 0\ .$$

Because the curve $y_i = y_i(t)$, $i=1,\ldots,n$, along which the motion occurs, makes the functional J stationary, it follows from the calculus of variations that the $y_i(t)$ must satisfy the Euler equations, i.e. $\boxed{L_{y_i}-\frac{d}{dt}L_{\dot{y}_i}=0}$ for $i=1,\ldots,n$. In Mechanics, the Euler equations are called Lagrange's equations. They form the equations of motion, or governing equations, for the system. We say that the governing equations follow from a variational principle if we can find an L such that $\delta\int Ldt = 0$ gives those governing equations as necessary conditions for an extremum. If the Lagrangian L is independent of time t, that is, $L_t = 0$, or equivalently $L = L\left(\vec{y},\vec{y}'\right)$, then a first integral is given, as we know, by: $\boxed{L-\sum_{i=1}^{n}\dot{y}_i L_{\dot{y}_i}=c}$. This equation is a *conservation law*. The quantity $\boxed{-L+\sum_{i=1}^{n}\dot{y}_i L_{\dot{y}_i}}$ is called the *Hamiltonian* of the system, and it frequently represents the *total energy*. Thus, if L is independent of time, then energy is conserved.

Sir William Rowan Hamilton (4 August 1805 – 2 September 1865) was an Irish mathematician, Andrews Professor of Astronomy at Trinity College Dublin, and Royal Astronomer of Ireland. He worked in both pure mathematics and mathematics for physics. He made important contributions to optics, classical mechanics and algebra. Although Hamilton was not a physicist–he regarded himself as a pure mathematician–his work was of major importance to physics, particularly his reformulation of Newtonian mechanics, now called Hamiltonian mechanics. This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics. In pure mathematics, he is best known as the inventor of quaternions.

3 basic examples

- **Harmonic Oscillator**

Consider the restoring force (Hooke's Law) $F=-ky,\ k>0,$ where k is the spring's constant.

$$\begin{cases} T=\dfrac{1}{2}m\dot{y}^2 \\ V=-\displaystyle\int F dy=\dfrac{1}{2}ky^2 \end{cases} \Rightarrow L=T-V=\frac{1}{2}m\dot{y}^2-\frac{1}{2}ky^2 \ \Rightarrow\ L_t=0, \text{ thus the energy is conserved.}$$

$J[y]=\displaystyle\int_{t_0}^{t_1}\left[\frac{1}{2}m\dot{y}^2-\frac{1}{2}ky^2\right]dt$ is stationary, hence by employing the E-L: $L_y-\dfrac{d}{dt}L_{\dot{y}}=0\Rightarrow$

$$-ky-\frac{d}{dt}\left(m\dot{y}\right)=0 \ \Rightarrow\ \underbrace{\ddot{y}+\frac{k}{m}y=0}_{Newton's\ second\ law} \ \Rightarrow\ \boxed{y(t)=c_1\cos\sqrt{\frac{k}{m}}t+c_2\sin\sqrt{\frac{k}{m}}t}$$ harmonic oscillator equation.

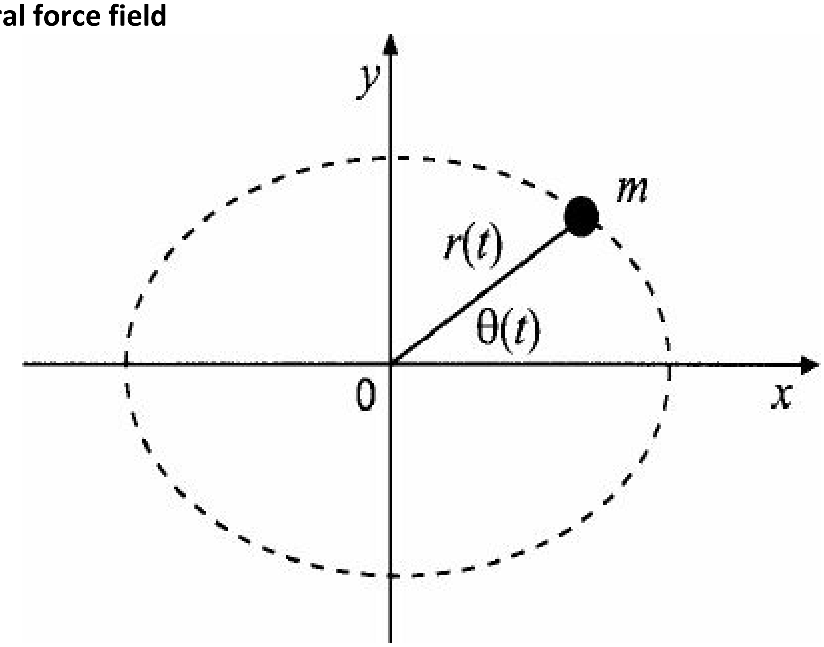
- **Pendulum**

$$\begin{cases} \ell = \text{length} \\ m = \text{mass} \\ \theta = \text{angle} \\ \text{frictionless support} \\ t = \text{time} \\ s = \ell\theta \end{cases}$$ Considering now the kinetic and potential energy of the pendulum

$K = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$ and $V = mg(\ell - \ell\cos\theta)$. Now we have seen that the Lagrangian is given by $L = T - V$, therefore we consider: $J[\theta] = \int_{t_0}^{t_1}\left\{\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)\right\}dt \overset{E-L}{\Rightarrow}$

$L_\theta - \frac{d}{dt}L_{\dot{\theta}} = 0 \;\Rightarrow\; -mg\ell\sin\theta - \frac{d}{dt}\left(m\ell^2\dot{\theta}\right) = 0 \;\Rightarrow\; \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$. For small displacements $\left(\sin\theta \simeq \theta\right)$, we obtain: $\ddot{\theta} + \frac{g}{\ell}\theta = 0 \;\Rightarrow\; \boxed{\theta(t) = c_1\cos\sqrt{\frac{g}{\ell}}t + c_2\sin\sqrt{\frac{g}{\ell}}t}$ simple harmonic motion.

- **Central force field**



Consider the planar motion of a mass m that is attracted to the origin with a force inversely proportional to the square of the distance from the origin (see figure above). For generalized coordinates we take the polar coordinates r and θ of the position of the mass. The kinetic energy is: $T=\frac{1}{2}mv^2=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}m\left\{\left[\frac{d}{dt}\left(r\cos\theta\right)\right]^2+\left[\frac{d}{dt}\left(r\sin\theta\right)\right]^2\right\}=$

$=\frac{1}{2}m\left\{\left(\dot{r}\cos\theta-r\sin\theta\dot{\theta}\right)^2+\left(\dot{r}\sin\theta+r\cos\theta\dot{\theta}\right)^2\right\}=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$. As a consequence we get

$\boxed{T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)}$. Moreover we compute the potential energy as $V=-\int Fdr=-\int-\frac{k}{r^2}dr\Rightarrow$

$\boxed{V=-\frac{k}{r}}$. Thus the E-L gives: $L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}$ and the corresponding functional

$J[r,\theta]=\int_{t_0}^{t_1}\left\{\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}\right\}dt \;\Rightarrow\; \begin{cases}\cancel{L_\theta}-\frac{d}{dt}L_{\dot{\theta}}=0\\ L_r-\frac{d}{dt}L_{\dot{r}}=0\end{cases} \Rightarrow \begin{cases}mr^2\theta=c\\ m\ddot{r}-mr\dot{\theta}^2+\frac{k}{r^2}=0\end{cases}$ which is a coupled system of odes, the solution of which gives the required equations of motion.

4.2 Hamilton vs Newton

In summary, Hamilton's principle gives us a procedure for finding the equations of motion of a system if we can write down the kinetic and potential energies. This offers an alternative approach to writing down Newton's second law for a system, which requires that we know the forces. Because Hamilton's principle only results in writing the equations of motion, why not just directly determine the governing equations and forgo the variational principle altogether? Actually, this may be a legitimate objection, particularly in view of the fact that the variational principle is usually derived *a posteriori*, that is, from the known equations of motion and not conversely, as would be relevant from the point of view of the calculus of variations. Moreover, if a variational principle is given as the basic principle for the system, then there are complicated sufficient conditions for extrema that must be considered, and they seem to have little or no role in physical problems. Finally, although variational principles do to some extent represent a unifying concept for physical theories, the extent is by no means universal; it is impossible to state such principles for some systems with constraints or dissipative forces.

On the other hand, aside from the aesthetic view, the *ab initio* formulation of the governing law by a variational principle has arguments on its side. The action integral plays a fundamental role in the development of numerical methods for solving differential equations (Rayleigh–Ritz method, Galerkin methods, etc. which we shall present below); it also plays a decisive role in the definition of Hamilton's characteristic function, the basis for the Hamilton–Jacobi theory. Furthermore, many variational problems occur in geometry and other areas apart from physics; in these problems the action or fundamental integral *is* an *a priori* notion. In summary, the calculus of variations provides a general context in which to study wide classes of problems of interest in many areas of science, engineering, and mathematics.

Carl Gustav Jacob Jacobi (10 December 1804 – 18 February 1851) was a German mathematician who made fundamental contributions to elliptic functions, dynamics, differential equations, determinants, and number theory.

4.3 Hamilton's equations

The E-L equations for the variational problem $J[y_1,...,y_n]=\int_{t_0}^{t_1} L\left(t, y_1,..., y_n, \dot{y}_1,..., \dot{y}_n\right)dt$ form form a system of n second-order odes. We now introduce a canonical method for reducing these equations to a system of $2n$ first-order equations. For simplicity we examine the case $n=1$, i.e. $J[y]=\int_{t_0}^{t_1} L(t, y, \dot{y})dt$ and the E-L equation is $L_y-\frac{d}{dt}L_{\dot{y}}=0$. Furthermore we introduce a new variable p, called the *canonical momentum* by $\boxed{p=L_{\dot{y}}(t, y, \dot{y})}$. If $L_{\dot{y}\dot{y}}\neq 0,$ then the IFT guarantees that the equation $p=L_{\dot{y}}(t, y, \dot{y})$ can be solved for $\dot{y}$ in terms of $t,$ y and p to get $\boxed{\dot{y}=\varphi\left(t, y, p\right)}$. Now we define the *Hamiltonian* H by: $\boxed{H(t, y, p)=-L\left(t, y, \varphi\left(t, y, p\right)\right)+\varphi\left(t, y, p\right)p}$, which actually is the expression $-L+\dot{y}L_{\dot{y}}$ we defined earlier. In many systems H is the total energy.

Now we are ready to derive the Hamilton's canonical equations by derivating H partially with respect first to p and to y. In other words we have:

$$\begin{cases} \dfrac{\partial H}{\partial p} = \cancel{-L_{\dot{y}}\dfrac{\partial \varphi}{\partial p}} + \cancel{p\dfrac{\partial \varphi}{\partial p}} + \varphi = \varphi(t,y,p) = \dot{y} \text{ , because } p = L_{\dot{y}}(t,y,\dot{y}) \\ \dfrac{\partial H}{\partial y} = -L_y - \cancel{L_{\dot{y}}\dfrac{\partial \varphi}{\partial y}} + \cancel{\dfrac{\partial \varphi}{\partial y}p} = -L_y \overset{E-L}{=} -\dfrac{d}{dt}L_{\dot{y}} = -\dfrac{d}{dt}p = -\dot{p} \end{cases} \Rightarrow \boxed{\begin{cases} \dot{y} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial y} \end{cases}}$$

These equations are called Hamilton's equations or *canonical equations* and they form a system of first order differential equations for y and p.

An alternative way to derive the Hamilton's equations is the following: $\big($keep in mind what we have defined so far, i.e. $H(t,y,p) = -L + \dot{y}L_{\dot{y}}$ where $p = L_{\dot{y}}(t,y,\dot{y})\big)$

$J[y] = \int_{t_0}^{t_1} L(t,y,\dot{y})dt \overset{Hamiltonian}{=} \int_{t_0}^{t_1} \overbrace{\{\dot{y}p - H(t,y,p)\}}^{F} dt = \int_{t_0}^{t_1} F(t,y,p,\dot{y})dt$ and we shall treat y and p as independent and will find the E-L equations like previously, i.e. $\begin{cases} F_y - \dfrac{d}{dt}F_{\dot{y}} = 0 \\ F_p - \cancel{\dfrac{d}{dt}F_{\dot{p}}}^{0} = 0 \end{cases} \Rightarrow$

$$\begin{cases} -\dfrac{\partial H}{\partial y} - \dfrac{dp}{dt} = 0 \\ \dot{y} - \dfrac{\partial H}{\partial p} = 0 \end{cases} \Rightarrow \begin{cases} \dot{y} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial y} \end{cases}$$ which is the required canonical form.

Example (Harmonic oscillator)

Consider the harmonic oscillator whose Lagrangian is: $L(t,y,\dot{y}) = \dfrac{m}{2}\dot{y}^2 - \dfrac{k}{2}y^2$. Then the canonical momentum is $p = L_{\dot{y}} = m\dot{y} \Rightarrow \dot{y} = \dfrac{p}{m}$. Consequently $H = -L + \dot{y}L_{\dot{y}} = -\dfrac{1}{2}m\dot{y}^2 +$

$+\dfrac{1}{2}ky^2 + \dfrac{p^2}{m} = \dfrac{1}{2}ky^2 + \dfrac{p^2}{2m}$, therefore $\begin{cases} \dfrac{\partial H}{\partial p} = \dot{y} = \dfrac{p}{m} \\ \dfrac{\partial H}{\partial y} = -\dot{p} = -L_y = -ky \end{cases} \Rightarrow \begin{cases} \dot{y} = \dfrac{p}{m} \\ \dot{p} = -ky \end{cases} \Rightarrow \dfrac{dp}{dy} = -\dfrac{ky}{p/m}$

$\Rightarrow \dfrac{p}{m}dp = -kydy \Rightarrow \dfrac{p^2}{2m} = -\dfrac{ky^2}{2} + c \Rightarrow p^2 + kmy^2 = \tilde{c}$ which is a family of ellipses in the yp (phase) plane.

Generalization

Once again consider the functional: $J=\int_{t_0}^{t_1} L(t,y_1,...,y_n,\dot{y}_1,...,\dot{y}_n)dt$ and as previously define the generalized momenta: $p_i=L_{\dot{y}_i}(t,y_1,...,y_n,\dot{y}_1,...,\dot{y}_n)$ for $i=1,2,...,n.$ Assume also that $\boxed{\det\left(L_{\dot{y}_i\dot{y}_j}\right)\neq 0}$ in order to be able to employ the IFT and solve the system of generalized momenta for $y_1,...,y_n$ and get (as previously): $\dot{y}_i=\varphi_i(t,y_1,...,y_n,\dot{y}_1,...,\dot{y}_n)$ for $i=1,2,...,n.$

Then $H(t,y_1,...,y_n,p_1,...,p_n)=-L+\sum_{i=1}^{n}\dot{y}_iL_{\dot{y}_i}=-L(t,y_1,...,y_n,\varphi_1,...,\varphi_n)+$

$+\sum_{i=1}^{n}\varphi_i\left(t,y_1,...,y_n,\varphi_1,...,\varphi_n\right)p_i$. Finally using an argument exactly like the one above, we get $\begin{cases}\dot{y}_i=\dfrac{\partial H}{\partial p_i}\\ \dot{p}_i=-\dfrac{\partial H}{\partial y_i}\end{cases}$ for $i=1,2,...,n.$ These are the Hamilton's equations. They represent $2n$ first order odes for the $2n$ functions $y_1,...,y_n,p_1,...,p_n$. More about the role of canonical formalism in the Calculus of Variations, Geometry, and Physics one could find in [B22].

Applications:

A canonical physical model concerns a system whose kinetic and potential energy are given by: $T=\frac{1}{2}\int_D u_t^2dx$ and $V=\int_D\left\{\frac{|\nabla u|^2}{2}+V(u)\right\}dx$. Here $u(x,t)$ is a function which characterizes the system, $D\subseteq\mathbb{R}^3$ and V a known, smooth function. Therefore, as usual, we have that:

$L=T-V=\frac{u_t^2}{2}-\frac{|\nabla u|^2}{2}-V(u)$ and of course $J[u]=\int_{t_0}^{t_1}\int_D\left\{\frac{u_t^2}{2}-\frac{u_{x_1}^2+u_{x_2}^2+u_{x_3}^3}{2}-V(u)\right\}dxdt\overset{E-L}{\Rightarrow}$

$L_u-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}L_{u_{x_i}}=0\Rightarrow\frac{dV}{du}-\frac{\partial}{\partial t}(u_t)-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(u_{x_i}\right)=0\Rightarrow\left('=\frac{d}{du}\right)V'(u)-u_{tt}-\Delta u=0\Rightarrow$

$\boxed{u_{tt}+\Delta u=V'\left(u\right)}$ which is called *Klein-Gordon* equation.

(N-particles problem)

Consider a system of n-particles where m_i is the mass of the *ith* particle and (x_i, y_i, z_i) is its position in space. Then the kinetic energy is $T = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$ and the potential energy is $V = V(t, x_1, ..., x_n, y_1, ..., y_n, z_1, ..., z_n)$ such that the force acting on the *ith* particle has components: $F_i = -\frac{\partial V}{\partial x_i}$, $G_i = -\frac{\partial V}{\partial y_i}$, $H_i = -\frac{\partial V}{\partial z_i}$. Then the *Hamilton's principle* applied to this system yields:

$$\begin{cases} L_{x_i} - \frac{d}{dt}L_{\dot{x}_i} = 0, \\ L_{y_i} - \frac{d}{dt}L_{\dot{y}_i} = 0 \\ L_{z_i} - \frac{d}{dt}L_{\dot{z}_i} = 0 \end{cases} \Rightarrow \begin{cases} -\frac{\partial V}{\partial x_i} - \frac{d}{dt}\left(m_i\dot{x}_i\right) = 0 \\ -\frac{\partial V}{\partial y_i} - \frac{d}{dt}\left(m_i\dot{y}_i\right) = 0 \\ -\frac{\partial V}{\partial z_i} - \frac{d}{dt}\left(m_i\dot{z}_i\right) = 0 \end{cases} \Rightarrow \begin{cases} m_i\ddot{x}_i = F_i \\ m_i\ddot{y}_i = G_i \\ m_i\ddot{z}_i = H_i \end{cases} \text{ for } i = 1, 2, ..., n.$$

These apparently are the Newton's equations for a system of n-particles.

4.4 Supplementary material accompanied by some historical comments

Newton founded his theory of Mechanics in the second part of the seventeenth Century. The theory was based upon three laws postulated by him. The laws provided a set of tools for computing the motion of bodies, given their initial positions and initial velocities, by calculating the forces they exert on each other, and relating these forces to the acceleration of the bodies. Motivated by the introduction of steam machines towards the end of the eighteenth century and the beginning of the nineteenth century, scientists developed the theory of Thermodynamics, and with it the important concept of energy. Then, in 1824 Hamilton started his systematic derivation of an axiomatic geometric theory of light. He realized that his theory is equivalent to a variational principle, called the *Fermat principle*, (mentioned earlier in the context) which states that light propagates so as to travel between two arbitrary points in minimal time. During his Optics research, Hamilton observed that apparently different notions such as optical travel time and energy are in fact related by another physical object called *action*. Moreover, he showed that the entire theory of Newtonian mechanics can be formulated in terms of actions and energies, instead of in terms of forces and acceleration. Hamilton’s new theory, now called *Hamilton’s principle*, enabled the use of variational methods to study not just static equilibria, but also dynamical problems.

Below we will re-demonstrate Hamilton's principle by applying it to the problem of *n interacting particles*, a standard problem in classical mechanics. We have already handled this particular problem earlier and inevitably this will lead to a sort of material's cover.

Our goal here is to explain better (more profoundly) why we consider the difference of kinetic and potential energy, to present Maupertuis' least action principle (mentioned also earlier to the context) and lastly to demonstrate Hamilton's principle to the elastic string and extract the equation for the vibrations of rods.

$$\begin{cases} E_k = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} m_i \dot{x}_i^2 \\ E_p = E_p(x_1,...,x_n) \end{cases} \overset{\substack{Newton's \\ second\ law}}{\Rightarrow} \boxed{\frac{d}{dt}\left(m_i \frac{dx_i}{dt}\right) = F = -\nabla_{x_i}\left(E_p\right)} \text{ for } i = 1,2,...,n$$

Now the energy of the system (if it is conserved, it is also called Hamiltonian) is defined by $E = E_k + E_p$ as we know from the theory of classical mechanics. We additionally define the Lagrangian of the system as $L = E_k - E_p$ and we shall explain later in the context why we have made this choice. Finally we shall also define the action in Hamilton's formalism as:

$J = \int_{t_1}^{t_2} L dt$. By taking into consideration the above discussion, we are ready to (re)introduce the Hamilton's principle. Hamilton postulated that a mechanical system evolves such that $\delta J = 0$, where the variation is taken with respect to all orbits $\left(y_1(t), y_2(t),..., y_n(t)\right)$, such that $y_i(t_1) = x_i(t_1)$, $y_i(t_2) = x_i(t_2)$ for $i = 1,2,...,n$.

$L = L\left(t, x_1,..., x_n, \dot{x}_1,..., \dot{x}_n\right)$. If $L = L\left(\vec{x}, \dfrac{d\vec{x}}{dt}\right)$ (i.e. L is independent of time) energy is conserved, which implies that $E_{total} = H$.

$$\delta J = \frac{d}{d\varepsilon} J[\vec{x} + \varepsilon\vec{\varphi}]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{t_0}^{t_1} \overbrace{\left[\frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{x}_i + \varepsilon\dot{\varphi}_i\right)^2 - E_p\left(x_1 + \varepsilon\varphi_1,..., x_n + \varepsilon\varphi_n\right)\right]}^{L} dt\Bigg|_{\varepsilon=0} =$$

$$\int_{t_0}^{t_1}\left[\sum_{i=1}^{n}\left(m_i \frac{dx_i}{dt}\frac{d\varphi_i}{dt} - \frac{\partial E_p}{\partial x_i}\varphi_i\right)\right]dt \overset{\substack{\text{integration} \\ \text{by parts}}}{=} \int_{t_0}^{t_1}\left[-\sum_{i=1}^{n}\left(m_i \frac{d^2x_i}{dt^2} - \frac{\partial E_p}{\partial x_i}\right)\varphi_i\right]dt = 0 \text{ , where } \varphi_i \in C_0^1[t_0, t_1]$$

is the variation with respect to the particle x_i. Conclusion: $\displaystyle\int_{t_0}^{t_1}\left[-\sum_{i=1}^{n}\left(m_i \frac{d^2x_i}{dt^2} - \frac{\partial E_p}{\partial x_i}\right)\varphi_i(t)\right]dt = 0 \Rightarrow$

Fundamental lemma $\Rightarrow$ $\dfrac{\partial E_p}{\partial x_i} = \boxed{F_i = m_i \dfrac{d^2x_i}{dt^2}}$ for $i = 1,2,...,n$, which is the Newton's 2nd law.

But why should we consider the difference of kinetic and potential energy as *L*?

The concept of the Lagrangian seems a bit odd at first sight. The sum of the kinetic and potential energies is the total energy, which is an intuitively natural physical object. But why should we consider their difference? To give an intuitive meaning to the difference of the energies, it is useful to look a bit closer at the historical development of Mechanics. Although Newton wrote clear laws for the dynamics of bodies, he and many other scientists looked for metaphysical principles behind them.

As the mainstream philosophy of the eighteenth century was based on the idea of a single God, it was natural to assume that such a God would create a world that is 'perfect' in some sense. This prompted the French scientist Pierre de Maupertuis (1698–1759) to define the notion of action of a moving body. According to Maupertuis, the action of a body moving from *a* to *b* is:

$A=\int_a^b p\,dx$, where p is the particle's momentum. He then formulated his principle of least action, stating that the world is such that action is always minimized.

Converting this definition of action to energy-related terms we write:

$A=\int_a^b p\,dx=\int_a^b m\frac{dx}{dt}dx=\int_{t_1}^{t_2} m\left(\frac{dx}{dt}\right)^2 dt=2\int_{t_1}^{t_2} E_k dt$. The difficulty with this approach is in fact that it only includes the kinetic energy, while the combination (sum) of both kinetic and potential energy determines the motion. Therefore, Lagrange used the identity

$2E_k=E_{total}+L \;\Rightarrow\; A=\int_{t_1}^{t_2}\left(E_{total}+L\right)dt$. But since the energy is a constant of the motion (since conserved) , minimizing A is actually the same as minimizing $\int_{t_1}^{t_2} L\,dt$.

Pierre Louis Moreau de Maupertuis (1698 – 27 July 1759) was a French mathematician, philosopher and man of letters. He became the Director of the Académie des Sciences, and the first President of the Prussian Academy of Science, at the invitation of Frederick the Great.

We proceed now to demonstrate Hamilton's principle for the *elastic string*.

Elastic String

We set $u_a = u(a,t)$ and analogously $u_b = u(b,t)$. Moreover we define $u(x,t)$ to be the string's deviation from the horizontal rest position, ℓ(x,t) be the load on the string, $d(x,t)$ be the string's elasticity and $p(x,t)$ be the mass density. As we know the line element is given by $ds = \sqrt{1+u_x^2}\,dx$ and $E_k = \dfrac{1}{2}\int_a^b p u_t^2 ds$. The potential energy now consists of the sum of the energy due to the stretching of the string and the work done against a load $\ell(x,t)$:

$E_p = \int_a^b \left\{ d\left(\sqrt{1+u_x^2}-1\right) - \ell u\sqrt{1+u_x^2} \right\} dx$. Notice that we allow the density, the elastic coefficient, and the load to depend on x and t. The action is thus given by:

$$J = \int_{t_0}^{t_1}\int_a^b \left[\frac{1}{2}\sqrt{1+u_x^2}\, p u_t^2 - d\left(\sqrt{1+u_x^2}-1\right) + \ell u\sqrt{1+u_x^2} \right] dxdt$$

Consider variations $u+\varepsilon\psi$ such that ψ vanishes at the string's endpoints a and b,and also at the initial and terminal timepoints t_0 and t_1. Neglecting the term that is cubic in the derivatives u_x, u_t we get for the first variation:

$\delta J = \int_{t_0}^{t_1}\int_a^b \left\{ \sqrt{1+u_x^2}\, p u_t \psi_t - d\left(1+u_x^2\right)^{-\frac{1}{2}} u_x\psi_x + \ell\sqrt{1+u_x^2}\,\psi \right\} dxdt$. Integrating by parts, the boundary conditions imply that the boundary terms (both the spatial and the temporal) vanish. Therefore equating the first variation to zero and integrating by parts:

$\int_{t_0}^{t_1}\int_a^b \left\{ -\dfrac{\partial}{\partial t}\left(\dfrac{p u_t}{\sqrt{1+u_x^2}} \right) + \dfrac{\partial}{\partial x}\left(\dfrac{p u_x}{\sqrt{1+u_x^2}} \right) + \ell\sqrt{1+u_x^2} \right\} \psi\, dxdt = 0$. This implies the dynamical equation for the string's vibration: $\boxed{\left(p u_t\right)_t - \dfrac{\left(d\left(1+u_x^2\right)^{-\frac{1}{2}} u_x \right)_x}{\sqrt{1+u_x^2}} - \ell = 0}$. Moreover, if we assume p and d to be constant and also use the *small slope approximation* $|u_x| \ll 1$, we obtain the 1-dimensional wave equation: $\mathrm{pu}_{tt} - \dfrac{d\left(\dfrac{u_x}{\cancel{\sqrt{1+u_x^2}}^{\,1}} \right)_x}{\cancel{\sqrt{1+u_x^2}}^{\,1}} = \ell(x,t) \;\Rightarrow\; \boxed{u_{tt} - \dfrac{d}{p} u_{xx} = \dfrac{\ell(x,t)}{p}}$ non homogenuous wave eq.

The potential energy of the string is stored in its stretching i.e. a string resists being stretched. We define a rod as an elastic body that also resists being bent. This means that we have to add to the elastic energy of the string a term that penalizes bending. The amount of bending of a curve $f(x)$ is measured by its *curvature*: $\boxed{k(x) = \dfrac{f_{xx}}{\left(1+f_x^2\right)^{3/2}}}$

Therefore, the Lagrangian for a rod under a load ℓ can be written as:

$L = \int_a^b \left\{ \frac{1}{2}\sqrt{1+u_x^2}\, pu_t^2 - \frac{d_1}{2}\frac{u_{xx}^2}{\left(1+u_x^2\right)^2} - d_2\left(\sqrt{1+u_x^2}-1\right) + \ell u\sqrt{1+u_x^2} \right\} dx$. To simplify now the calculations, we introduce the *small slopes* assumption at the outset , i.e. $|u_x| \ll 1$, therefore $\sqrt{1+u_x^2} \simeq 1$. Thus $J = \int_{t_0}^{t_1}\int_a^b \left\{ \frac{1}{2}pu_t^2 - \frac{d_1}{2}u_{xx}^2 - \frac{d_2}{2}u_x^2 + \ell u \right\} dxdt$. Computing the first variation now, we find that: $\delta J = \int_{t_0}^{t_1}\int_a^b \left\{ pu_t\psi_t - d_1 u_{xx}\psi_{xx} - d_2 u_x \psi_x + \ell\psi \right\} dxdt.$

In order to obtain the Euler–Lagrange equation we need to integrate the last integral by parts. Just as in the case of the plate, we assume that the rod is clamped, i.e. we specify u and u_x at the end points a and b. Therefore, the variation ψ vanishes at the spatial and temporal endpoints, and in addition, ψ_x vanishes at a and b. We thus obtain that the vibrations of rods are determined by the equation: $\boxed{\left(pu_t\right)_t - \left(d_2 u_x\right)_x + \left(d_1 u_{xx}\right)_{xx} - \ell = 0}$

4.5 Inverse Problem

How can we determine the Lagrangian L if we know the equations of motion? This problem is known as the *inverse* problem of the calculus of variations. Next we shall formulate the inverse problem (keeping our approach in an elementary level) by considering the case n=1 for simplicity.

Given a second order ode: $(*)\ y'' = F(t, y, y')$ we will try to find the corresponding Lagrangian $L = L(t, y, y')$ such that $(*)$ is the E-L equation $L_y - \frac{d}{dt} L_{y'} = 0$. Generally the problem has infinitely many solutions. In order to determine L , we write the E-L: $L_y - L_{y't} - y'L_{y'y} - y''L_{y'y'} = 0$ or equivalently: $L_y - L_{y't} - y'L_{y'y} - FL_{y'y'} = 0$. We assume that L is three times continuous differentiable in order to be able to change the partial derivatives. Differentiating with respect to $\dot{y}$:

$\frac{d}{d\dot{y}}\left[-L_y + L_{\dot{y}t} + \dot{y}L_{\dot{y}y} + FL_{\dot{y}\dot{y}}\right] = -\cancel{L_{y\dot{y}}} + L_{t\dot{y}\dot{y}} + \cancel{L_{\dot{y}y}} + \dot{y}L_{y\dot{y}\dot{y}} + FL_{\dot{y}\dot{y}\dot{y}} + F_{\dot{y}}L_{\dot{y}\dot{y}} = 0$

(vanishing because $L_{y\dot{y}} = L_{\dot{y}y}$) $\Rightarrow\ L_{\dot{y}\dot{y}t} + \dot{y}L_{\dot{y}\dot{y}y} + FL_{\dot{y}\dot{y}\dot{y}} + L_{\dot{y}\dot{y}}F_{\dot{y}} = 0$. Setting now $u(t, y, \dot{y}) = L_{\dot{y}\dot{y}}(t, y, \dot{y})$ we get the first order (with respect to u) pde : $\boxed{u_t + \dot{y}u_y + Fu_{\dot{y}} + uF_{\dot{y}} = 0}$. However it is often much simpler to proceed directly by matching terms in the E-L equation to terms in the given equation.

A representative example

Next let us consider a simple example in order to clarify a bit more the whole procedure. *Motion's equation*: $my'' + ky + ay' = 0$ which is a damped harmonic oscillator. This system is not conservative due to the damping and there is not a scalar potential. We cannot apply *Hamilton's principle* directly here. We seek a Lagrangian such that $my'' + ay' + ky = 0$ is an E-L equation. Multiplying by a nonnegative function $f(t)$, we get (we additionally demand the equation to coincide with the typical E-L): $\begin{cases} mf(t)\ddot{y} + af(t)\dot{y} + kf(t)y = 0 \\ L_{\dot{y}\dot{y}}\ddot{y} + L_{\dot{y}y}\dot{y} + \left(L_{\dot{y}t} - L_y\right) = 0 \end{cases} \Rightarrow$

$L_{\dot{y}\dot{y}} = mf(t) \ \Rightarrow\ L_{\dot{y}} = mf(t)\dot{y} + M(t, y) \ \Rightarrow\ L = \frac{mf(t)}{2}\dot{y}^2 + M(t, y)\dot{y} + N(t, y)$ where M, N are arbitrary functions. Then it follows that $-L_y + L_{\dot{y}t} + L_{\dot{y}y}\dot{y} = af(t)\dot{y} + kf(t)y \ \Rightarrow$

$-\left(\cancel{M_y\dot{y}} + N_y\right) + mf'(t)\dot{y} + M_t + \cancel{M_y\dot{y}} = af(t)\dot{y} + kf(t)y$. Thus we have that:

$mf'(t) = af(t) \ \Rightarrow\ f(t) = e^{\frac{at}{m}}$ and $-N_y + M_t = kye^{\frac{at}{m}}$, but since M, N are arbitrary, we are able to select $M = 0$ and get that: $N(t, y) = -\left(\frac{k}{2}\right)y^2e^{\frac{at}{m}}$ and consequently a Lagrangian for the damped harmonic oscillator is: $\boxed{L(t, y, \dot{y}) = \left\{\frac{m}{2}\dot{y}^2 - \frac{k}{2}y^2\right\}e^{\frac{at}{m}}}$. Finally let us make a comment.

This expression is the Lagrangian for the undamped oscillator times a time-dependent factor $e^{\frac{at}{m}}$.

Next we will discover for which Lagrangian functions the E-L equation is satisfied identically.

$$L_y - \frac{d}{dt}L_{\dot y} = 0 \;\Rightarrow\; \ddot y L_{\dot y\dot y} + \dot y L_{\dot y y} + L_{\dot y t} - L_y = 0 \;\Rightarrow\; L_{\dot y\dot y} = 0 \;\Rightarrow\; L_{\dot y} = M(t,y) \;\Rightarrow$$

$L = M(t,y)\dot y + N(t,y)$, but $L_{\dot y y} = M_y$ and $L_{\dot y t} = M_t \;\Rightarrow\; \cancel{\dot y M_y} + M_t - \cancel{\dot y M_y} - N_y = 0 \;\Rightarrow$

$\boxed{\dfrac{\partial M}{\partial t} = \dfrac{\partial N}{\partial y}}$ compatibility condition. Thus this leads to $\boxed{L = M(t,y)\dot y + N(t,y)}$ with M, N satisfying the compatibility condition. For these Lagrangians, which are linear in $\dot y$, with additionally coeffiecients which satisfy the compatibility condition, the E-L is identity!

Now we continue by presenting below some further examples-applications to Physics, where we shall practice analytically the method of finding the Lagrangian corresponding to a given differential equation developed above.

Emden-Fowler equation:

In this example we shall consider the *Emden - fowler* equation, i.e. $\ddot y + \frac{2}{t}\dot y + y^5 = 0,$ and we will attempt to obtain the Lagrangian (inverse problem) by imitating the procedure above. Hence multiplying by a function $\varphi(t) \ge 0 \Rightarrow \begin{cases} \varphi(t)\ddot y + \frac{2}{t}\varphi(t)\dot y + \varphi(t) y^5 = 0 \\ L_{\dot y\dot y}\ddot y + L_{\dot y y}\dot y + L_{\dot y t} - L_y = 0 \end{cases} \Rightarrow$ *equalizing term by term*

$L_{\dot y\dot y} = \varphi(t) \Rightarrow L_{\dot y} = \varphi(t)\dot y + M(t,y) \Rightarrow L = \frac{1}{2}\varphi(t)\dot y^2 + M(t,y)\dot y + N(t,y)$. Equalize now the remaining parts to obtain: $L_{\dot y y}\dot y + L_{\dot y t} - L_y = \frac{2}{t}\varphi(t)\dot y + \varphi(t)y^5 \;\Rightarrow\; \cancel{\dot y M_y} + \varphi'(t)\dot y + M_t - \cancel{\dot y M_y} -$

$-N_y = \frac{2\varphi(t)}{t}\dot y + \varphi(t)y^5 \;\Rightarrow\;$ (equalizing the coefficients of $\dot y$) $\Rightarrow \;\frac{d\varphi}{dt} = \frac{2\varphi}{t} \Rightarrow \varphi(t) = t^2,$

substitute now the value of φ to the equalizing condition above and take into consideration that M is arbitrarily chosen (so we can demand $M = 0$) to get: $-N_y = \varphi(t)y^5 = y^5 t^2 \;\Rightarrow$

$N(t,y) = -\frac{y^6}{6}t^2 + \tilde\varphi(t) \;\Rightarrow\; \boxed{L = \frac{t^2}{2}\dot y^2 - \frac{t^2 y^6}{6} + \tilde\varphi(t)}$ infinitely many Lagrangians.

An even faster way to determine the Lagrangian is by using the "integrating factor". This simplifies a lot the procedure. Below we shall demonstrate this equivalent method through some examples .

Example (i)

A particle's motion in a constant external force field with frictional force proportional to its velocity:

Consider the linear equation $\ddot{y}+a\dot{y}+b=0$ $(a,b>0)$. The integrating factor is e^{at}, thus after multiplying the equation with it, we get: $\underbrace{\left(e^{at}y'\right)'}_{-L_{y'}}+\underbrace{e^{at}b}_{L_y}=0 \Rightarrow \begin{cases} L_y = e^{at}b \\ L_{y'} = -e^{at}y' \end{cases} \Rightarrow$

$$\begin{cases} L = e^{at}yb + N(t,y') \\ L = -e^{at}\dfrac{(y')^2}{2} + M(t,y) \end{cases} \Rightarrow \boxed{L = e^{at}\left[by - \frac{(y')^2}{2}\right]}$$

Example (ii)

Consider the non linear equation $\ddot{y}+p(t)\dot{y}+f(y)=0,$ where p and f are continuous functions with anti-derivatives $P(t)$ and $F(t)$ respectively. Next we will compute a Lagrangian $L=L(t,y,\dot{y})$ such that the E-L equation associated with the functional $J[y]=\int_a^b L(t,y,\dot{y})dt$ is equivalent to the given non linear equation above. Here the integrating factor is $e^{P(t)}$, therefore multiplying the equation by $e^{P(t)}$ we get: $\frac{d}{dt}\underbrace{\left(e^{P(t)}\dot{y}\right)}_{-L_{\dot{y}}}+\underbrace{e^{P(t)}f(y)}_{L_y}=0 \Rightarrow \begin{cases} L_y = e^{P(t)}f(y) \\ L_{\dot{y}} = -e^{P(t)}\dot{y} \end{cases} \Rightarrow$

$\begin{cases} L = e^{P(t)}F(y) + M(t,\dot{y})_{M,N \text{ arbitrary smooth functions}} \\ L = -e^{P(t)}\dfrac{\dot{y}^2}{2} + N(t,y) \end{cases} \Rightarrow$ Now we can either compare the two equations or equivalently plug the first into the second. So $L_{\dot{y}} = M_{\dot{y}} = -e^{P(t)}\dot{y} \Rightarrow M(t,\dot{y}) = \dfrac{-e^{P(t)}\dot{y}^2}{2} + \tilde{\varphi}(t)$

Hence we obtain the Lagrangian: $\boxed{L = e^{P(t)}\left[F(y) - \frac{\dot{y}^2}{2}\right] + \tilde{\varphi}(t)}$

Example (iii)

Consider an electrical circuit RLC , which satisfies the following differential equation:

$L\ddot{y}+R\dot{y}+\dfrac{y}{C}=0$ where $y(t)$ is the electrical current, R is the resistance, C the capacitance, L is the inductance. By multiplying with the integrating factor $e^{\frac{R}{L}t}$ we get that:

$$\left(y'e^{\frac{R}{L}t}\right)'+\left(y\frac{e^{\frac{R}{L}t}}{LC}\right)=0 \Rightarrow \begin{cases} L_{\dot{y}}=-e^{\frac{R}{L}t}\dot{y} \\ L_y=y\dfrac{e^{\frac{R}{L}t}}{LC}\end{cases} \Rightarrow \begin{cases} L=-e^{\frac{R}{L}t}\dfrac{\dot{y}^2}{2}+M(t,y) \\ L=e^{\frac{R}{L}t}\dfrac{y^2}{2LC}+N(t,\dot{y})\end{cases} \Rightarrow \text{ as usual we obtain:}$$

$$\boxed{L(t,y,\dot{y})=\frac{e^{\frac{R}{L}t}}{2}\left[\dot{y}^2-\frac{y^2}{LC}\right]}$$

5. Critical Points and the Mountain Pass Theorem

We have concentrated our study so far on the problem of locating minimizers of various energy functionals, subject to constraints (perhaps) and of discovering the appropriate E-L equation they satisfy. For this chapter, we shall turn our attention to the problem of finding additional solutions of the E-L pde, by looking for other critical points. These critical points will not in general be minimizers, but rather ''saddle points'' of the functional I. We shall develop next some ''machinery'' that ensures that an abstract functional I has a critical point.

5.1 Critical points, deformations

Hereafter H denotes a real Hilbert space, with norm $\|\ \|$ and inner product $\langle\cdot,\cdot\rangle$ and let $I:H\to\mathbb{R}$ be a non-linear functional on H.

Definition:

$\triangleright$ We say I is differentiable at $u\in H$ if there exists $v\in H$ such that

$\boxed{I[w]=I[u]+\langle v,w-u\rangle+o\left(\|w-u\|\right)}$ for $w\in H$. The element v, if it exists, is unique.

We then write $\boxed{I'[u]=v}$

Definition:

▹ We say that I belongs to $C^1(H;\mathbb{R})$, if $I'[u]$ exists for each $u \in H$ and the mapping $I': H \to H$ is continuous.

The theory we develop below holds if $I \in C^1(H;\mathbb{R})$, but the proofs will be greatly streamlined provided we additionally assume that the mapping $(*)$ $I': H \to H$ is *Lipschitz continuous* on bounded subsets of H.

Notation:

(i) We denote by C the collection of functions $I \in C^1(H;\mathbb{R})$ satisfying $(*)$.

(ii) If $c \in \mathbb{R}$, we write $\boxed{A_c = \{u \in H \mid I[u] \le c\}}$ and $\boxed{K_c = \{u \in H \mid I[u] = c \ , \ I'[u] = 0\}}$

Definition

▹ We say $u \in H$ is a critical point if $I'[u] = 0$

▹ The number $c \in \mathbb{R}$ is a critical value if $K_c \neq \emptyset$

We now wish to prove that if c is not a critical value, we can "nicely" deform the set $A_{c+\varepsilon} = \{u \in H \mid I[u] \le c + \varepsilon\}$ into $A_{c-\varepsilon}$ for some $\varepsilon > 0$. The idea will be to solve an appropriate ode in H and to follow the resulting flow "downhill". As H is generally infinite dimensional, we will need some kind of compactness condition.

Definition:

▹ A functional $I \in C^1(H;\mathbb{R})$ satisfies the *Palais - Smale* compactness condition if each sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that

(i) $\{I[u_k]\}_{k=1}^{\infty}$ is bounded

(ii) $I'[u_k] \to 0$ in H

is *precompact* in H.

$\boxed{\textit{Deformation Theorem}}$

Assume $I \in \mathrm{C}$ satisfies the *Palais – Smale* condition. Suppose also $K_c = \varnothing$.
Then for each sufficiently small $\varepsilon > 0$, there exists a constant $0 < \delta < \varepsilon$ and a function $n \in C\left([0,1] \times H; H\right)$ such that the mappings $n_t(u) = n(t,u) \quad \left(0 \le t \le 1\ ,\ u \in H\right)$ satisfy:

(i) $n_0(u) = u \quad (u \in H)$

(ii) $n_1(u) = u \quad (u \notin I^{-1}[c-\varepsilon, c+\varepsilon])$

(iii) $I[n_t(u)] \le I[u] \quad (u \in H, 0 \le t \le 1)$

(iv) $n_1(A_{c+\delta}) \subset A_{c-\delta}$

The proof of the theorem can be found in [B21]

5.2 Mountain Pass Theorem

Next we present an interesting ''min-max'' technique and we use the deformation *n* built above to the deformation theorem to deduce the existence of a critical point.

$\boxed{\textit{Theorem}}$ (Mountain Pass Theorem)

Assume $I \in \mathrm{C}$ satisfies the *Palais – Smale* condition. Suppose also :

(i) $I[0] = 0$

(ii) There exist constants $r, a > 0$ such that $I[u] \ge a$ if $\|u\| = r$ and

(iii) There exists an element $v \in H$ with $\|v\| > r,\ I[v] \le 0$.

Now define $\Gamma = \left\{g \in C\left([0,1]; H\right) \ \mid\ g(0) = 0\ ,\ g(1) = v\right\}$. Then:

$\boxed{c = \inf_{g \in \Gamma} \max_{0 \le t \le 1} I[g(t)]}$ is a *critical value* of $I[\cdot]$.

Proof:

Clearly $c \geq a$. (why?) Because by applying the IVT (intermediate value theorem), something we are allowed to do because the real value function g is continuous in [0,1] and $0 = g(0) \neq g(1) = v$, since $\|v\| > r > 0$, we get that $\exists\, t^* \in (0,1)$ such that $g(t^*) = r$, so we have $g(t^*) \in \partial B(0,r)$, since $\|g(t^*)\| = \|r\| = r$, which according to the theorem's assumptions means that $a \leq I[g(t^*)]$. Therefore $a \leq I[g(t^*)] \leq \max_{0 \leq t \leq 1} I[g(t)]$, which is valid $\forall g \in \Gamma$.

So obviously $a \leq \inf_{g\in\Gamma} \max_{0 \leq t \leq 1} I[g(t)] = c$.

Assume now that c is not a critical value of $I[\cdot]$, so that $K_c = \varnothing$. Choose then any sufficiently small number $0 < \varepsilon < \dfrac{a}{2}$. According to the *deformation theorem* now, there exists a constant $0 < \delta < \varepsilon$ and a homeomorphism $n : H \to H$ with $n\left(A_{c+\delta}\right) \subset A_{c-\delta}$ and $n(u) = u$ if $u \notin I^{-1}[c-\varepsilon, c+\varepsilon]$. Now select $g \in \Gamma$ satisfying $\max_{0 \leq t \leq 1} I[g(t)] \leq c + \delta$, something we are allowed to do because of the δ-characterization of infimum, i.e. $\forall\ \delta > 0\ \exists\ g \in \Gamma$ such that $\max_{0 \leq t \leq 1} I[g(t)] \leq \inf_{g\in\Gamma} \max_{0 \leq t \leq 1} I[g(t)] + \delta = c + \delta$. Then we observe that if we define $\hat{g} = n \circ g$, this also belongs to Γ, since $n\left(g(0)\right) = n(0) = 0$ and $n\left(g(1)\right) = n(v) = v$, according to $n(u) = u$ if $u \notin I^{-1}[c-\varepsilon, c+\varepsilon]$.

To be more specific, from our assumptions we have that $I[0] = 0$ as well as $I[v] \leq 0$, while we note that $\varepsilon < \dfrac{a}{2} < a \leq c \ \Rightarrow\ c - \varepsilon > 0$. Hence neither v nor $0 \in I^{-1}[c-\varepsilon, c+\varepsilon]$. Consequently we are ok :)

Needless to say that it is continuous as a composition of continuous functions.

But then $\max_{0 \leq t \leq 1} I[g(t)] \leq c + \delta$ implies that $\max_{0 \leq t \leq 1} I[\hat{g}(t)] \leq c - \delta$, because of the fact that $n\left(A_{c+\delta}\right) \subset A_{c-\delta}$ and $\hat{g} = n(g)$. Again, in order to be more specific, let us justify this assertment.

Fix an arbitrary $t \in [0,1]$. Then $g(t) \leq \max_{0 \leq t \leq 1} I[g(t)] \leq c + \delta \Rightarrow g(t) \in A_{c+\delta}$. But now we observe that

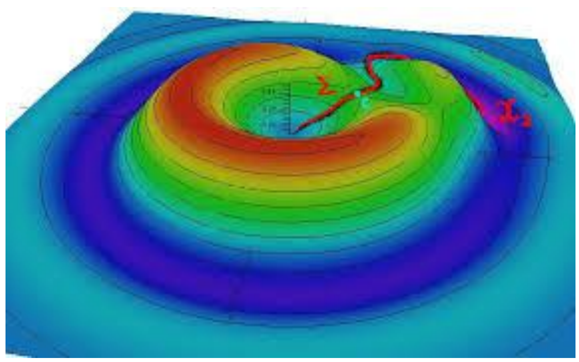
$\hat{g}(t) = n\left(g(t)\right) \overset{n\left(A_{c+\delta}\right) \subset A_{c-\delta}}{\in} A_{c-\delta}$, i.e. $I[\hat{g}(t)] \leq c - \delta$ and since t was arbitrarily chosen, we get that $\max_{0 \leq t \leq 1} I[\hat{g}(t)] \leq c - \delta$. Finally, since $\hat{g} \in \Gamma$, we have that $c = \inf_{g\in\Gamma} \max_{0 \leq t \leq 1} I[g(t)] \leq \max_{0 \leq t \leq 1} I[\hat{g}(t)] \leq c - \delta$. So:

$c = \inf_{g\in\Gamma} \max_{0 \leq t \leq 1} I[g(t)] \leq c - \delta$, which is obviously a contradiction, because $0 \leq -\delta$. □

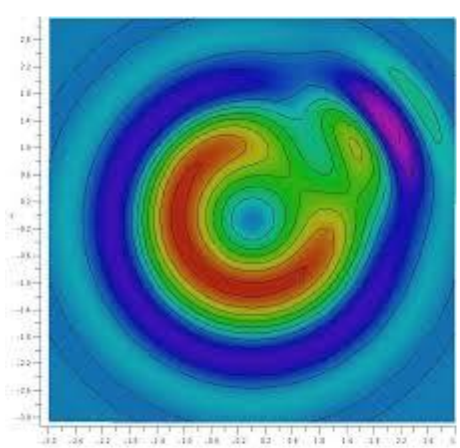
Remarks:

▹ The Mountain Pass Theorem (MPT) is an existence theorem. Given certain conditions on a function the theorem demonstrates the existence of a saddle point. The theorem is unusual in that there are many other theorems regarding the existence of extrema, but few regarding saddle points.

▹ Think of the graph of $I[\cdot]$ as a landscape with a low spot at zero, surrounded by a ring of mountains. Beyond these mountains lies another low spot at v. The idea is to look for a "*path*" g connecting 0 to v, which passes through a mountain pass, that is a saddle point for $I[\cdot]$. But note carefully that we are only asserting the existence of a critical point at the "*energy level*" c, which may not necessarily correspond to a true saddle point.



This is a photo which depicts the main idea and illustrates the main assertion of our remark above, however, since it has been taken from the net, the notation does not match. The same situation seen from above:



5.3 Application to semilinear elliptic pde

Next, we shall illustrate the utility of MTP. For this purpose let us investigate the following semilinear b.v.p.

(#) $\begin{cases} -\Delta u = f(u), \text{ in } U \\ u = 0, \text{ on } \partial U \end{cases}$ f is smooth and for some $1 < p < \dfrac{n+2}{n-2}$ we have that $|f(z)| \le c\left(1+|z|^p\right)$, $|f'(z)| \le c\left(1+|z|^{p-1}\right)$, $z \in \mathbb{R}$ and c is a constant. We will also suppose that $0 \le F(z) \le \gamma f(z)z$ for some constant $\gamma < \dfrac{1}{2}$, where $F(z) = \int_0^z f(s)ds$. We hypothesize finally that for constants $0 < a \le A$ $a|z|^{p+1} \le |F(z)| \le A|z|^{p+1}$ $(z \in \mathbb{R})$. Now this implies $f(0) = 0$, because by applying the mean value theorem, we obtain $a|z|^p \le \left|\dfrac{F(z) - \cancel{F(0)}^{0}}{z-0}\right| \le A|z|^p \overset{MVT}{\Rightarrow} a|z|^p \le f(\xi) \le A|z|^p$, for some ξ between 0 and z. And by letting $z \to 0$, we obtain that $f(0) = 0$. Obviously then, $u \equiv 0$ is a trivial solution of the b.v.p. (#). We wish to find another. Observe that the pde $-\Delta u = u|u|^{p-1}$, where $f(u) = u|u|^{p-1}$, falls under the hypotheses above.

$\boxed{\textit{Theorem}}$ (Existence)

$\begin{cases} -\Delta u = f(u), \text{ in } U \\ u = 0, \text{ on } \partial U \end{cases}$ has at least one weak solution $u \neq 0$.

Proof:

$\boxed{1.}$ Define the *energy functional*: $I[u] = \int_U \left(\frac{1}{2}|\nabla u|^2 - F(u)\right)dx$ for $u \in H_0^1(U)$. We intend to apply the MPT to $I[\cdot]$. We will also simplify a bit the notation by setting $H = H_0^1(U)$, with norm $\|u\| = \left(\int_U |\nabla u|^2\, dx\right)^{1/2}$ and inner product $\langle u, v\rangle = \int_U \nabla u \cdot \nabla v\, dx$. At this point let us note that due to Poincaré's inequality (we are able to use it since $u \in H_0^1(U)$) the norms $\|\nabla u\|_{L^2(U)}$ and $\|u\|_{H^1(U)}$ are equivalent, i.e. $\|\nabla u\|_{L^2(U)} \sim \|u\|_{H^1(U)}$, so we can consider the norm above instead of the typical norm. To check the equivalence is quite easy, if not directly obvious, since $\|\nabla u\|^2_{L^2(U)} \le \|u\|^2_{H^1(U)} =$ $= \left(\|u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)}\right) \overset{\text{Poincaré}}{\le} \|\nabla u\|^2_{L^2(U)}\left(c(U)+1\right) \Rightarrow \|\nabla u\|^2_{L^2(U)} \le \|u\|^2_{H^1(U)} \le c\|\nabla u\|^2_{L^2(U)}$.

Then we have that: $\boxed{I[u] = \frac{1}{2}\|u\|^2 - \int_U F(u)dx = I_1[u] - I_2[u]}$

$\boxed{2.}$ We first claim that I belongs to class C. To see that, first note that for each $u, w \in H$, we have:

$$I_1[\mathrm{w}] = \frac{1}{2}\|w\|^2 = \frac{1}{2}\|u+w-u\|^2 = \frac{1}{2}\|u\|^2 + \langle u, w-u\rangle + \frac{1}{2}\|w-u\|^2 \quad \Rightarrow$$

$I_1[w] = I_1[u] + \langle u, w-u\rangle + o(\|w-u\|)$. Hence, according to the definiton given earlier, we conclude that I_1 is differentiable at u, with $\boxed{I_1'[u]=u}$. Consequently $I_1 \in$ C.

$\boxed{3.}$ Now we need to examine the term I_2. According to the Lax-Milgram theorem we have that for each element $v^* \in H^{-1}(U)$ (the dual space of H), the problem $\begin{cases} -\Delta v = v^*, \text{ in } U \\ v = 0, \text{ on } \partial U \end{cases}$ has a unique solution $v \in H_0^1(U) \equiv H$. Now we shall justify analytically the assertion above.

Weak formulatin of b.v.p. $\begin{cases} -\Delta u = u^*, \text{ in } U \\ u = 0, \text{ on } \partial U \end{cases} \Rightarrow$ we multiply now by a test function $\varphi \in C_0^\infty(U) \Rightarrow$

$\int_U -\Delta u\varphi dx = -\int_{\partial U} \cancel{\varphi}^0 \frac{\partial u}{\partial \vec{n}} dS + \int_U \nabla u \cdot \nabla \varphi dx = \int_U u^* \varphi dx,$ where the surface integral equals zero, since φ is zero at the boundary due to the compact support. Therefore $\int_U \nabla u \cdot \nabla \varphi dx = \int_U u^* \varphi dx \ \forall \varphi \in H_0^1(U)$, because $C_0^\infty(U)$ is *dense* to $H_0^1(U)$. Now by defining the bilinear form: $B[u,v] = \int_U \nabla u \cdot \nabla v dx$, we get:

$\boxed{B[u,\varphi] = \langle u^*, \varphi\rangle \ \forall \varphi \in H_0^1(U)}$ *weak formulation*

We also define $f(u) = u$ which is (obviously linear) continuous $\Leftrightarrow$ bounded since $u \in H^{-1}(U)$, which is the dual space. Now the bilinear form $B[u,v]$ is: (i) <u>bounded</u>, since

$$|B[u,v]| \leq \int_U |\nabla u \cdot \nabla v| dx = \int_U |\nabla u||\nabla v| dx \overset{C-S}{\leq} \left(\int_U |\nabla u|^2 dx\right)^{1/2} \left(\int_U |\nabla v|^2 dx\right)^{1/2} = \|\nabla u\|_{L^2(U)} \|\nabla v\|_{L^2(U)} = \|u\|\|v\|$$

and moreover (ii) <u>coercive</u>, since $B[u,u] = \int_U |\nabla u|^2 dx = \|\nabla u\|^2_{L^2(U)} = \|u\|^2$. Therefore the assumptions of Lax-Milgram theorem are satisfied and consequently we can proceed by applying it. We now write $\boxed{v = Kv^*}$, so that $K : H^{-1}(U) \to H_0^1(U)$ is an *isometry*. Note in particular that if $w \in L^{\frac{2n}{n+2}}(U)$, then the linear functional w^* defined by $\boxed{\langle w^*, u\rangle = \int_U wudx}$, $u \in H_0^1(U)$ belonds to $H^{-1}(U)$. (we will misuse notation and say "$w \in H^{-1}(U)$"). In order to see this, it suffices to show that it is bounded and this implies that it is continuous (as linear) as well, thus it belongs to the dual space. Therefore we observe

$\left|\langle w^*, u\rangle\right| \leq \int_U |wu| dx \overset{\text{Hölder}}{\leq} \|w\|_{L^{\frac{2n}{n+2}}(U)} \|u\|_{L^{2^*}(U)} \overset{\substack{\text{remarks below} \\ \text{after the proof}}}{\leq} \|w\|_{L^{\frac{2n}{n+2}}(U)} \|u\|$, from the theorems (Sobolev type inequalities) presented shortly afterwards the end of this proof and the fact that $\frac{2n}{n+2}$ and 2^* are conjugate. Consequently: $\|w^*\| \le \|w\|_{L^{\frac{2n}{n+2}}(U)} < \infty,$ something that verifies the assertion above.

Observe next that $p\left(\frac{2n}{n+2}\right) < \frac{n+2}{n-2} \cdot \frac{2n}{n+2} = 2^*$ and so we have that $f(u) \in L^{\frac{2n}{n+2}}(U) \subset H^{-1}(U)$,

if $u \in H_0^1(U)$. Indeed $\| f(u)\|_{L^{\frac{2n}{n+2}}(U)}^{\frac{2n}{n+2}} \le \int_U | f(u)|^{\frac{2n}{n+2}} dx \le \int_U c\left(1+|u|^p\right)^{\frac{2n}{n+2}} dx < \infty$, because $\|u\|_{L^{2^*}} < \infty,$ since if we see again the Sobolev's estimates presented below, $u \in H_0^1(U) \subset H^1(U) \subset L^{2^*}$.

We now demonstrate that if $u \in H_0^1(U)$, then $\boxed{I_2'[u] = K\left[f(u)\right]}$.

To see this, note first (we expand through Taylor theorem with integral residual,

i.e. $F(a+b) \overset{Taylor}{=} F(a) + \overbrace{f(a)}^{F'(a)} b + b^2 \int_0^1 (1-s)\, f'\left(a+sb\right) ds$). Thus for each $w \in H_0^1(U),$ we get that

$F(\overbrace{u}^{a} + \overbrace{w-u}^{b}) = F(u) + f(u)\left(w-u\right) + \tilde{R}$, where $\tilde{R}$ is the integral residual. As a result this leads to:

$$I_2[w] = \int_U F(w)dx = \int_U F(u+w-u)dx \overset{R=\int_U \tilde{R}dx}{=} \int_U \left\{F(u) + f(u)\left(w-u\right)\right\} dx + R = \left(I_2[u] = \int_U F(u)dx\right) =$$

$= I_2[u] + \int_U \nabla K\left[f(u)\right] \cdot \nabla\left(u-w\right) dx + R$, where the last equality follows from the way we have defined K and the result of Lax-Milgram theorem. Now the remainder term R satisfies,

according to $|f(z)| \le c\left(1+|z|^p\right),\ |f'(z)| \le c\left(1+|z|^{p-1}\right),$

$$|R| = \left|\int_U (w-u)^2 \int_0^1 (1-s)\, f'\left(u+s(w-u)\right) ds dx\right| \le \int_U \left\{|w-u|^2 \int_0^1 (1-s)\left| f'\left(u+s(w-u)\right)\right| ds\right\} dx \le$$

$$\overset{(1)}{\le} c\int_U \left(1+|u|^{p-1}+|w-u|^{p-1}\right)|w-u|^2\, dx \overset{(2)}{\le} c\int_U \left(|w-u|^2+|w-u|^{p+1}\right) dx + c\left(\int_U |u|^{p+1}\, dx\right)^{\frac{p-1}{p+1}} \left(\int_U |w-u|^{p+1}\, dx\right)^{\frac{2}{p+1}}$$

(1) $\left| f'\left(u+s(w-u)\right)\right| \le c\left(1+\left|u+s(w-u)\right|^{p-1}\right)$, where $\left|u+s(w-u)\right| \overset{\substack{triangular \\ inequality}}{\le} |u| + s|w-u| \overset{0\le s\le 1}{\le} |u| + |w-u| \le$

$2\max\left\{|u|, |w-u|\right\}$, consequently: $\frac{\left|u+s(w-u)\right|^{p-1}}{2^{p-1}} \le \left(\max\left\{|u|, |w-u|\right\}\right)^{p-1} \le |u|^{p-1} + |w-u|^{p-1},$ and thus:

$$c\left(1+|u+s(w-u)|^{p-1}\right)=\overbrace{2^{p-1}c}^{\widehat{c}}\left(\frac{1}{2^{p-1}}+\frac{|u+s(w-u)|^{p-1}}{2^{p-1}}\right)\leq\widehat{c}\left(\frac{1}{2^{p-1}}+|u|^{p-1}+|w-u|^{p-1}\right)\leq \quad \left(\frac{1}{2^{p-1}}\leq 1\right)\leq$$

$\leq\widehat{c}\left(1+|u|^{p-1}+|w-u|^{p-1}\right)$, so we have established that $\left|f'(u+s(w-u))\right|\leq\widehat{c}\left(1+|u|^{p-1}+|w-u|^{p-1}\right)$.

(2) We obtain this by applying the $H\ddot{o}lder's$ inequality, to be more specific we have:

$$\int_U|u|^{p-1}|w-u|^2\,dx\overset{\text{Hölder}}{\leq}\left\|u^{p-1}\right\|_{L^{\frac{p+1}{p-1}}(U)}\left\|(w-u)^2\right\|_{L^{\frac{p+1}{2}}(U)}\overset{i.e.}{=}\left(\int_U u^{\cancel{p-1}\frac{p+1}{\cancel{p-1}}}dx\right)^{\frac{p-1}{p+1}}\left(\int_U|w-u|^{\cancel{2}\frac{p+1}{\cancel{2}}}dx\right)^{\frac{2}{p+1}}$$, because of

the conjugate exponents: $\dfrac{p-1}{p+1}+\dfrac{2}{p+1}=1.$

Now since 2 and $p+1<2^*$, because $p+1<\dfrac{n+2}{n-2}+1=\dfrac{2n}{n-2}=2^*$ and $2<2^*$since $2n-4<2n$,

the Sobolev inequalities show that $R=o\left(\|w-u\|\right)$. Again by the Sobolev's inequality:

if $u\in H_0^1(U)$ then $\|u\|_{L^q(U)}\leq c\|\nabla u\|_{L^2(U)}$ $\forall\ 1\leq q\leq 2^*$ (for more about it see the remark below)

and the fact that $\|\nabla u\|_{L^2(U)}\sim\|u\|$ as we have already noted in the beginning of the proof, we have that:

$\boxed{\|u\|_{L^2},\|u\|_{L^{p+1}}\leq c\|u\|}$. So now from the relation: $|R|\leq\int_U\left(|w-u|^2+|w-u|^{p+1}\right)dx+\left\|u^{p-1}\right\|_{L^{\frac{p+1}{p-1}}}\left\|(w-u)^2\right\|_{L^{\frac{p+1}{2}}}$

we deduce that $R=o\left(\|w-u\|\right)$,because (a)$\left\|(w-u)^2\right\|_{L^{\frac{p+1}{2}}}=\left(\int_U|w-u|^{\cancel{2}\frac{p+1}{\cancel{2}}}dx\right)^{\frac{2}{p+1}}=\|w-u\|_{L^{p+1}}^2\leq c\|w-u\|^2$

and (b)$\int_U\left(|w-u|^2+|w-u|^{p+1}\right)dx=\|w-u\|_{L^2}^2+\|w-u\|_{L^{p+1}}^{p+1}\leq c\left(\|w-u\|^2+\|w-u\|^{p+1}\right)$

Thus we see that

$I_2[w]=I_2[u]+\left\langle K\left[f(u)\right],w-u\right\rangle+o\left(\|w-u\|\right)$ as required , since we have shown earlier above

$I_2[w]=I_2[u]+\int_U\nabla K\left[f(u)\right]\cdot\nabla\left(u-w\right)dx+R$ and $K:H^{-1}(U)\to H_0^1(U)$ is an isometry. In other

words we have shown that $\boxed{I_2'[u]=K\left[f(u)\right]}$. Finally we note that if $u,\tilde{u}\in H_0^1(U)$ with $\|u\|,\|\tilde{u}\|\leq L$

then: $\|I_2'[u]-I_2'[\tilde{u}]\|=\left\|K\left[f(u)\right]-K\left[f(\tilde{u})\right]\right\|_{H_0^1(U)}=\|f(u)-f(\tilde{u})\|_{H^{-1}(U)}\leq\|f(u)-f(\tilde{u})\|_{L^{\frac{2n}{n+2}}(U)}$ and this

due to the way we have defined $K,$ i.e. $K\left[f(u)\right]=K[u^*]=u,$ and here $f(u)=u,$

hence $K\left[f(u)\right]=f(u)$. Moreover the embedding $f(u)\in L^{\frac{2n}{n+2}}(U)\subset H^{-1}(U)$, if $u\in H_0^1(U)$ justifies the last inequality. But:

$\|f(u)-f(\tilde{u})\|_{L^{2n/n+2}(U)}=\left(\int_U|f(u)-f(\tilde{u})|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}}$ and we remind ourselves that $\left|f'(z)\right|\leq c\left(1+|z|^{p-1}\right)$.

Then by MVT $\left|f(u)-f(\tilde{u})\right| = \left|f'(\xi)\right|\left|u-\tilde{u}\right| \le c\left(1+\left|\xi\right|^{p-1}\right)\left|u-\tilde{u}\right| \le c\left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)\left|u-\tilde{u}\right|$, where the last inequality is justified by: $\left|\xi\right| \le \max\left\{\left|u\right|,\left|\tilde{u}\right|\right\} \Rightarrow 1+\left|\xi\right|^{p-1} \le 1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}$. Taking into account now this note, we return to our calculation and observe that:

$$\left\| f(u)-f(\tilde{u})\right\|_{L^{2n/n+2}(U)} = \left(\int_U \left|f(u)-f(\tilde{u})\right|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}} \le c\left(\int_U \left\{\left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)\left|u-\tilde{u}\right|\right\}^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{2n}} \overset{\text{Hölder}}{\le}$$

$$\left(\text{keep in mind that } 2^* = \frac{2n}{n-2}\right) \le c\left(\int_U \left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)^{\frac{2n}{\cancel{n+2}}\frac{\cancel{n+2}}{4}}dx\right)^{\frac{\cancel{4}\,2}{\cancel{2}n}} \left\|u-\tilde{u}\right\|_{L^{2^*}(U)} \overset{\substack{\text{in other}\\ \text{words}}}{=}$$

$$c\left(\int_U \left[\left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)^{\frac{2n}{n+2}}\right]^{\frac{n+2}{4}}dx\right)^{\frac{4}{n+2}\frac{n+2}{2n}} \left(\int_U \left(\left|u-\tilde{u}\right|^{\frac{2n}{\cancel{n+2}}}\right)^{\frac{\cancel{n+2}}{n-2}=2^*}dx\right)^{\frac{n-2}{\cancel{n+2}}\frac{\cancel{n+2}}{2n}=\frac{1}{2^*}} =$$

$$=\left(\left\|\left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)^{\frac{2n}{n+2}}\right\|_{L^{\frac{n+2}{4}}(U)}\right)^{\frac{n+2}{2n}} \left(\left\|\left|u-\tilde{u}\right|^{\frac{2n}{n+2}}\right\|_{L^{\frac{n+2}{n-2}}(U)}\right)^{\frac{n+2}{2n}}, \text{where } \left\|u-\tilde{u}\right\|_{L^{2^*}(U)} = \left(\left\|\left|u-\tilde{u}\right|^{\frac{2n}{n+2}}\right\|_{L^{\frac{n+2}{n-2}}(U)}\right)^{\frac{n+2}{2n}}.$$

We continue the computation: $c\left(\int_U \left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)^{\frac{2n}{\cancel{n+2}}\frac{\cancel{n+2}}{4}}dx\right)^{\frac{\cancel{4}\,2}{\cancel{2}n}} \left\|u-\tilde{u}\right\|_{L^{2^*}(U)} \le C(L)\left\|u-\tilde{u}\right\|_{L^{2^*}(U)} \le$

$\le C(L)\left\|u-\tilde{u}\right\|$, again by Sobolev's inequality and since $\left\|u\right\|,\left\|\tilde{u}\right\| \le L$ because $u,\tilde{u} \in H_0^1(U),$ we easily explain the bound: $c\left(\int_U \left(1+\left|u\right|^{p-1}+\left|\tilde{u}\right|^{p-1}\right)^{\frac{n}{2}}dx\right)^{\frac{2}{n}} \le C(L).$ Thus we have established the estimate:

$\left\| f(u)-f(\tilde{u})\right\|_{L^{2n/n+2}(U)} \le C(L)\left\|u-\tilde{u}\right\|$. Eventually, this leads to the following estimate

$\left\|I_2'[u]-I_2'[\tilde{u}]\right\| \le \left\| f(u)-f(\tilde{u})\right\|_{L^{2n/n+2}(U)} \le C(L)\left\|u-\tilde{u}\right\| \;\Rightarrow\; \boxed{\left\|I_2'[u]-I_2'[\tilde{u}]\right\| \le C(L)\left\|u-\tilde{u}\right\|}$. Hence, after all this longish analysis, we conclude that $I_2' : H_0^1(U) \to H_0^1(U)$ is <u>*Lipschitz continuous*</u> on bounded sets. As a consequence we get that $\boxed{I_2 \in \mathrm{C}}$ and at last we have established the assertion that

$I = I_1[u]-I_2[u]$ belongs to class C, since both $I_1, I_2 \in \mathrm{C}$.

$\boxed{4.}$ Now we need to verify the *Palais - Smale condition.* For this purpose, suppose $\left\{u_k\right\}_{k=1}^{\infty} \subset H_0^1(U)$ with $\left\{I[u_k]\right\}_{k=1}^{\infty}$ bounded and $I'[u_k] \to 0$ in $H_0^1(U)$. According to the foregoing now,

$u_k - K[f(u_k)] \to 0$ in $H_0^1(U)$, because $I'[u_k] = I_1'[u_k]-I_2'[u_k] = u_k - K[f(u_k)] \to 0.$

Thus for each $\varepsilon > 0$ we have $\left|\langle I'[u_k],v\rangle\right| = \left|\langle u_k - K[f(u_k)],v\rangle\right| = \left|\langle u_k,v\rangle - \langle K[f(u_k)],v\rangle\right| =$ (see the definition of the inner product given in the first step of the proof and remember the way we defined the isometry $K,$ so that $K[f(u)] = f(u)$) $= \left|\int_U \nabla u_k \cdot \nabla v - f(u_k)v\ dx\right|$. Another way to see why this equality is true is to observe that $\left|\langle u_k - K[f(u_k)],v\rangle\right| \overset{def.}{=} \left|\int_U \left(\nabla u_k \cdot \nabla v - \nabla\left(K\left[f(u_k)\right]\right)\cdot \nabla v\right)\ dx\right| = \quad \left(K[f] = f\right) \quad =$

$= \left|\int_U \nabla u_k \cdot \nabla v - f(u_k)v\ dx\right|$ and this because $\int_U \nabla f(u_k)\cdot\nabla v\ dx = \cancel{\int_{\partial U} v\nabla f(u_k)\cdot \vec{n} dS}^{\ 0 \text{ since } v\in H_0^1(U)} - \int_U v\Delta f(u_k)dx =$

$= \left[-\Delta f(u_k) \overset{f(u)=u}{=} -\Delta u_k \overset{b.v.p.}{=} f(u_k)\right] = \int_U vf(u_k)dx$. We also note that by C-S inequality:

$\left|\langle I'[u_k],v\rangle\right| = \left|\int_U \nabla u_k \cdot \nabla v - f(u_k)v\ dx\right| = \left|\langle u_k - K[f(u_k)],v\rangle\right| \overset{C-S}{\leq} \left\|u_k - K[f(u_k)]\right\|\left\|v\right\| < \varepsilon\left\|v\right\|$ for $v\in H_0^1(U),$ since $u_k - K[f(u_k)] \to 0$ in $H_0^1(U),$ i.e. $\left\|u_k - K[f(u_k)]\right\| < \varepsilon,$ for k sufficiently large. Now let $v = u_k$ above to find $\left|\int_U \left|\nabla u_k\right|^2 - f(u_k)u_k dx\right| \leq \varepsilon\left\|u_k\right\|$ for each $\varepsilon > 0$ and for all k sufficiently large. For $\varepsilon = 1$ in particular we see that:

$-\int_U \left|\nabla u_k\right|^2 dx + \int_U f(u_k)u_k dx \leq \left|\int_U \left|\nabla u_k\right|^2 dx - \int_U f(u_k)u_k dx\right| \leq \left\|u_k\right\|$, however according to our definition for the inner product which was $\left\|u_k\right\|^2 = \langle u_k,u_k\rangle = \int_U \left|\nabla u_k\right|^2 dx,$ the above yields:

$(*)\ \boxed{\int_U f(u_k)u_k dx \leq \left\|u_k\right\| + \left\|u_k\right\|^2}$ for all k large enough. But since $\left\{I[u_k]\right\}_{k=1}^\infty$ is bounded (1st Palais-Smale assumption) we get that $\left|I[u_k]\right| \leq c$ for all k and some constant c. So:

$\left|\int_U \frac{1}{2}\left|\nabla u_k\right|^2 - F(u_k)dx\right| \leq c \ \Rightarrow\ \left|\frac{\left\|u_k\right\|^2}{2} - \int_U F(u_k)dx\right| \leq c \ \Rightarrow\ \frac{\left\|u_k\right\|^2}{2} - \int_U F(u_k)dx \leq c < +\infty.$

Finally we deduce that $\left\|u_k\right\|^2 \leq c + 2\int_U F(u_k)dx \leq c + 2\int_U \gamma f(u_k)u_k dx \overset{(*)}{\leq} c + 2\gamma\left(\left\|u_k\right\| + \left\|u_k\right\|^2\right)$. Now since $2\gamma < 1,$ we discover that the sequence $\left\{u_k\right\}_{k=1}^\infty \subset H_0^1(U)$ is <u>bounded</u> . Hence there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^\infty$ and $u\in H_0^1(U)$ such that $u_{k_j} \xrightarrow{weakly} u$ in $H_0^1(U)$ and so $u_{k_j} \xrightarrow{strongly} u$

in $L^{p+1}(U)$, the latter assertion holding since $p+1<2^*$ and therefore it follows from the Rellich-Kondrachov compactness theorem, which in our case states that $H_0^1(U)\subset\subset L^{p+1}(U)$.
But then $f(u_k)\to f(u)$ in $H^{-1}(U)$, because f is *continuous*,
since it belongs to the dual space, remember: $f\in L^{2n/n+2}\subset H^{-1}(U)$. Whence
$K[f(u_k)]\to K[f(u)]$ in $H_0^1(U)$, because we have shown in (3) that $K[f(u)]$ is Lipschitz continuous.
As a consequence the relation: $u_k-K[f(u_k)]\to 0$ in $H_0^1(U)$ implies in fact that $u_{k_j}-K[f(u_{k_j})]\to 0$ in $H_0^1(U)$, but $K[f(u_{k_j})]\to K[f(u)]=f(u)=u$ in $H_0^1(U)$. Therefore we get that $\boxed{u_{k_j}\to u \text{ in } H_0^1(U)}$, deduced by the uniqueness of the limit.

$\boxed{5.}$ We finally verify the remaining hypothesis of the MPT. Clearly $I[0]=0$. Suppose now that $u\in H_0^1(U)$, with $\|u\|=r$, for $r>0$ to be selected below. Then: $I[u]=I_1[u]-I_2[u]=\frac{r^2}{2}-I_2[u]$.
Now hypothesis $a|z|^{p+1}\le|F(z)|\le A|z|^{p+1}$ for $0<a\le A$ and $z\in\mathbb{R}$ implies, since $p+1<2^*$, that $|I_2[u]|=\left|\int_U F(u)dx\right|\le\int_U c|u|^{p+1}dx=c\|u\|_{L^{p+1}(U)}^{p+1}\le c\|u\|_{L^{2^*}(U)}^{p+1}\le c\|u\|^{p+1}=cr^{p+1}$, where the first inequality is because $p+1<2^*$ and consequently the embedding $L^{2^*}\subset L^{p+1}$ implies that $\|\ \|_{L^{p+1}}\le\|\ \|_{L^{2^*}}$ and the second inequality is due to Gagliardo-Nirenberg-Sobolev's inequality.
Now in view of $I[u]=\frac{r^2}{2}-I_2[u]$ we note that $I[u]\ge\frac{r^2}{2}-cr^{p+1}\ge\frac{r^2}{4}=a>0$, provided $r>0$ is small enough (archimedean property guarantees that we can find such a small $r>0$ to satisfy the inequality $\frac{r^2}{2}-cr^{p+1}\ge\frac{r^2}{4}$, since $p+1>2$). Thus, given this selection of r, we select $a=\frac{r^2}{4}$. Now fix some element $u\in H$, $u\neq 0$. Write $\boxed{v=tu}$ for $t>0$ to be selected. Then $I[v]=I_1[v]-I_2[v]=I_1[tu]-I_2[tu]=$
$=t^2I_1[u]-\int_U F(tu)dx\overset{at^{p+1}|u|^{p+1}\le|F(tu)|}{\le}t^2I_1[u]-at^{p+1}\int_U|u|^{p+1}dx<0$ (we demand it in order to satisfy the 3rd assumption of the MPT) , for $t>0$ large enough. On the other hand, we need to take into account that according to the 3rd assumption, it must be $\|v\|>r$, i.e. $r<\|v\|=\|tu\|\Rightarrow t>\frac{r}{\|u\|}$.These lead to our choice.

$\boxed{6.}$ We have at last checked all the hypotheses of the MPT. Therefore it provides us with a critical value, so there must consequently exist a function $u\in H_0^1(U)$, $u\neq 0$, with $I'[u]=u-K[f(u)]=0$. In particular for each $v\in H_0^1$ we have that $0=\langle I'[u],v\rangle=\langle u-K[f(u)],v\rangle=\int_U(\nabla u\cdot\nabla v-f(u)v)dx$,as we have proved previously in step 4. So: $\int_U\nabla u\cdot\nabla v\,dx=\int_U f(u)vdx$ and therefore u is a weak solution of b.v.p. □

Remark:

Because we have referred to the so called Sobolev's inequality many times on the foregoing proof of the theorem above, we shall shortly present the inequalities of Gagliardo-Nirenberg- Sobolev type and its different versions as well as the emdeddings implied by them and we will conclude to the basic compactness result, which is the Rellich- Kondrachov Compactness theorem.

Definition:

If $1 \le p < n,$ the Sobolev conjugate of p is $\boxed{p^* = \dfrac{np}{n-p}}$. Note that $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$, $p^* > p.$

$\boxed{\textit{Theorem}}$ (Gagliardo-Nirenberg-Sobolev)

Assume $1 \le p < n.$ Then $\exists\, c > 0$ (depending only on p and n) such that $\boxed{\|u\|_{L^{p^*}(\mathbb{R}^n)} \le c\|\nabla u\|_{L^p(\mathbb{R}^n)}}$ $\forall u \in C_0^1(\mathbb{R}^n).$

$\boxed{\textit{Theorem}}$ (Estimates for $W^{1,p}, 1 \le p < n$)

$U \subset \mathbb{R}^n,$ open and bounded and ∂U is C^1. Assume $u \in W^{1,p}(U)$ and $1 \le p < n.$ Then we have that: $u \in L^{p^*}(U)$ with the estimate $\boxed{\|u\|_{L^{p^*}(U)} \le c\|u\|_{W^{1,p}(U)}}$ with c depends only on $p, n, U.$

$\boxed{\textit{Theorem}}$ (Estimates for $W_0^{1,p}, 1 \le p < n$)

$U \subset \mathbb{R}^n,$ open and bounded and $u \in W_0^{1,p}(U)$ for some $1 \le p < n,$ then $\boxed{\|u\|_{L^q(U)} \le c\|\nabla u\|_{L^p(U)}}$ $\forall q \in [1, p^*]$ with c depends only on $p, q, n, U.$

Let us now note two things:

▹ The last estimate is sometime also called Poincaré 's inequality. The difference with the pre-last estimate above is that only ∇u appears on the right hand side of the inequality.

▹ In view of this theorem, the norm $\|\nabla u\|_{L^p(U)}$ is equivalent to $\|u\|_{W^{1,p}(U)}$ on $W_0^{1,p}(U)$, if U is bounded of course.

As we have already seen above, the Gagliardo-Nirenberg-Sobolev inequality implies the embedding of $W^{1,p}(U)$ into $L^{p^*}(U)$ for $1 \le p < n$, $p^* = \dfrac{np}{n-p}$. We will now demonstrate that $W^{1,p}(U)$ is in fact <u>compactly embedded</u> in $L^q(U)$, for $1 \le q < p^*$.

$\boxed{\textit{Theorem}}$ (Rellich-Kondrachov compactness theorem)

$U \subset \mathbb{R}^n$ open and bounded and ∂U is C^1. For $1 \leq p < n \Rightarrow W^{1,p}(U) \subset\subset L^q(U)$ for $1 \leq q < p^*$.

Finally let us remark that since $p^* > p$ and $p^* \xrightarrow[p \to n]{} \infty$, we have in particular $W^{1,p}(U) \subset\subset L^p(U)$ for $1 \leq p \leq \infty$. (notice that if $n < p \leq \infty$, this follows from Morrey's inequality and the Ascoli-Arzela compactness criterion). Note finally that $W_0^{1,p}(U) \subset\subset L^p(U)$ even if we do not assume ∂U to be C^1.

5.4 Complementary material [Derrick-Pohozaev identity]

Closing this section dedicated to Mountain Pass theorem and its applications, we would like to present the *Derrick-Pohozaev* identity as an ''additional material'' topic, relevant to our discussion regarding the application of MPT to the semilinear elliptic pdes like the non linear Poisson's equations of the following form:

b.v.p.

$(\#)\begin{cases} -\Delta u = u|u|^{p-1}, \text{ in } U \\ u = 0, \text{ on } \partial U \end{cases}$. We know from the theory of existence developed as an application of MPT earlier that there exists a non-trivial solution $(u \neq 0)$ of the b.v.p. (#), provided that $1 < p < \frac{n+2}{n-2}$. Let us now instead suppose that $\boxed{\frac{n+2}{n-2} < p < \infty}$ $(*)$. Our goal here is to demonstrate under a certain geometric condition on U that $(*)$ implies $u = 0$ is the only smooth solution of b.v.p. (#) ! We see therefore that the restriction to the condition above, i.e. $1 < p < \frac{n+2}{n-2}$, was in some sense natural and consequently say $p = \frac{n+2}{n-2}$ is a critical exponent.

$\underline{Lemma}$ [*normals to a star-shaped region*]

Assume ∂U is C^1 and U is a star-shaped set with respect to O (origin). Then $\boxed{\vec{x}\cdot\vec{n}(\vec{x})\geq 0}$ $\forall \vec{x}\in\partial U$.

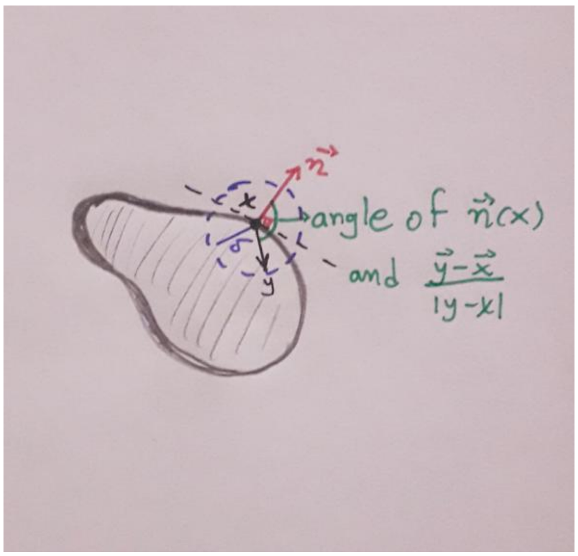
Proof:

Since ∂U is C^1 , if $x\in\partial U$ then for each ε>0, there exists δ>0 such that $|y-x|<\delta$ and $y\in\bar{U}$ imply $\vec{n}(x)\cdot\frac{(y-x)}{|y-x|}\leq\varepsilon$. In particular $\limsup\limits_{\substack{y\to x\\ y\in\bar{U}}}\left(\vec{n}(x)\cdot\frac{(y-x)}{|y-x|}\right)\leq 0$. (why?) Because:

Due to the fact that ∂U is C^1, i.e. $\vec{n}(x)$ is well defined, we have from C^1 definition of ∂U that for given $x\in\partial U$: $\forall\varepsilon$>0 $\exists\delta$>0 such that, if $y\in\bar{U}$ and $y\in B(x,\delta)$, then

$\vec{n}\cdot\frac{(y-x)}{|y-x|}=\cancel{|\vec{n}|}^{1}\left|\cancel{\frac{y-x}{|y-x|}}\right|^{1}\cos\left(\widehat{\vec{n},\frac{(y-x)}{|y-x|}}\right)\leq\varepsilon$,because this angle is larger than $\frac{\pi}{2}$(see figure)

i.e. that for given $x\in\partial U$ and an arbitrarily small number ε, we are always able to find a number (radius) δ>0 , so as to be able to select a y ($y\in\bar{U}\cap B(x,\delta)$) such that the angle of the vectors $\vec{n}$ and $\frac{y-x}{|y-x|}$ to be larger than $\frac{\pi}{2}rad$, i.e. $\cos\leq 0\Rightarrow \limsup\limits_{\substack{y\to x\\ y\in\bar{U}}}\left(\vec{n}(x)\cdot\frac{(y-x)}{|y-x|}\right)\leq 0$. Finally let $y=\lambda x$ for $0<\lambda<1$. Then, apparently $y\in\bar{U}$, since U is a star-shaped set and each line segment lies in it. Hence $\vec{n}(x)\cdot\frac{x}{|x|}\overset{y=\lambda x}{=}-\lim\limits_{\lambda\to 1^-}\vec{n}(x)\cdot\frac{(\lambda x-x)}{|\lambda x-x|}\geq 0$ □



In this figure we illustrate our assertion in the Lemma's proof above that the angle between these two vectors is indeed obtuse

We next prove that there can exist no trivial solution to the b.v.p. (#) for for supercritical growth, provided U is star-shaped. The proof is a remarkable calculation initiated by multiplying the pde $-\Delta u = |u|^{p-1} u$ by $x \cdot \nabla u$ and continually integrating by parts.

$\boxed{Theorem}$ [*Nonexistence of non-trivial solutions*]

Assume that $u \in C^2(\bar{U})$ is a solution of $\begin{cases} -\Delta u = u|u|^{p-1}, \text{in } U \\ u = 0, \text{ on } \partial U \end{cases}$ and the exponent p satisfies $\frac{n+2}{n-2} < p < \infty$. Suppose further that U is a star-shaped set with respect to the origin and the boundary ∂U is C^1. Then $u \equiv 0$ within U.

Proof:

$\boxed{1.}$ We multiply the pde by $x \cdot \nabla u$ and integrate over U, to find $\left(f = |u|^{p-1} u \right)$:

$$\int_U -\Delta u (x \cdot \nabla u) dx = \int_U f (x \cdot \nabla u) dx \Rightarrow \underbrace{\int_U |u|^{p-1} u (x \cdot \nabla u) dx}_{B} = -\int_{\partial U} \underbrace{(x \cdot \nabla u) \frac{\partial u}{\partial \vec{n}} dS}_{A_2} + \underbrace{\int_U \nabla u \cdot \nabla (x \cdot \nabla u) dx}_{A_1}$$

where $A = A_1 + A_2$. At this point let us note the followings:

$x \cdot \nabla u = \sum_{j=1}^n x_j u_{x_j} \Rightarrow \frac{\partial}{\partial x_i}(x \cdot \nabla u) = \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n x_j u_{x_j} \right) = \sum_{j=1}^n \left(x_j u_{x_j} \right)_{x_i} = \sum_{j=1}^n \left[\delta_{ij} u_{x_j} + x_j u_{x_i x_j} \right]$. Consequently $\nabla u \cdot \nabla (x \cdot \nabla u) = \sum_{i=1}^n u_{x_i} (x \cdot \nabla u)_{x_i} = \sum_{i=1}^n u_{x_i} \left(\sum_{j=1}^n \left[\delta_{ij} u_{x_j} + x_j u_{x_i x_j} \right] \right) = \sum_{i,j=1}^n \left\{ \delta_{ij} u_{x_i} u_{x_j} + x_j u_{x_i} u_{x_i x_j} \right\}$. Therefore these calculations lead to:

$$A_1 = \int_U \nabla u \cdot \nabla (x \cdot \nabla u) dx = \int_U \sum_{i,j=1}^n \left\{ \delta_{ij} u_{x_i} u_{x_j} + x_j u_{x_i} u_{x_i x_j} \right\} dx \Rightarrow \boxed{A_1 = \sum_{i,j=1}^n \int_U \left\{ \delta_{ij} u_{x_i} u_{x_j} + x_j u_{x_i} u_{x_i x_j} \right\} dx}$$

Analogously, we have after similar computation, a representation expression for A_2 quantity as well.

$$A_2 = -\int_{\partial U} (x \cdot \nabla u) \frac{\partial u}{\partial \vec{n}} dS = -\int_{\partial U} \left(\sum_{j=1}^n x_j u_{x_j} \right) \left(\sum_{i=1}^n u_{x_i} n_i \right) dS = -\int_{\partial U} \sum_{i,j=1}^n u_{x_i} n_i x_j u_{x_j} dS \Rightarrow \boxed{A_2 = \sum_{i,j=1}^n \int_{\partial U} u_{x_i} n_i x_j u_{x_j} dS}$$

$\boxed{2.}$ Now we shall proceed by computing a little better representation forms for the quantities A_1, A_2, B. To start with:

$$A_1=\sum_{i,j=1}^{n}\int_U\left\{\delta_{ij}u_{x_i}u_{x_j}+x_ju_{x_i}u_{x_ix_j}\right\}dx=\int_U|\nabla u|^2+\sum_{j=1}^{n}\left[\left(\frac{|\nabla u|^2}{2}\right)_{x_j}x_j\right]dx$$, because of the fact that:

$$\bullet\sum_{i,j=1}^{n}\left(\delta_{ij}u_{x_i}u_{x_j}\right)=\sum_{i=1}^{n}\left(u_{x_i}\sum_{j=1}^{n}\delta_{ij}u_{x_j}\right)=\sum_{i=1}^{n}\left(u_{x_i}u_{x_i}\right)=\sum_{i=1}^{n}\left(u_{x_i}\right)^2=\nabla u\cdot\nabla u=|\nabla u|^2,\text{ where }\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

$$\bullet\sum_{i,j=1}^{n}\left(x_ju_{x_i}u_{x_ix_j}\right)=\left(\text{observe that }\frac{\partial}{\partial x_j}\left(\frac{u_{x_i}^2}{2}\right)=\frac{\cancel{2}u_{x_i}u_{x_ix_j}}{\cancel{2}}=u_{x_i}u_{x_ix_j}\right)=\sum_{j=1}^{n}\left[x_j\sum_{i=1}^{n}\left(u_{x_i}u_{x_ix_j}\right)\right]=$$

$$=\sum_{j=1}^{n}\left(x_j\sum_{i=1}^{n}\left(\frac{u_{x_i}^2}{2}\right)_{x_j}\right)=\sum_{j=1}^{n}\left(x_j\left(\sum_{i=1}^{n}\frac{u_{x_i}^2}{2}\right)_{x_j}\right)=\sum_{j=1}^{n}\left(x_j\left(\frac{|\nabla u|^2}{2}\right)_{x_j}\right)$$. Finally we continue by noticing that

$$\int_U|\nabla u|^2+\sum_{j=1}^{n}\left[\left(\frac{|\nabla u|^2}{2}\right)_{x_j}x_j\right]dx=\int_U|\nabla u|^2\,dx+\int_{\partial U}\underbrace{\sum_{j=1}^{n}\left(\frac{|\nabla u|^2}{2}x_jn_j\right)}_{\frac{|\nabla u|^2}{2}(\vec{x}\cdot\vec{n})}dS-\sum_{j=1}^{n}\int_U\frac{|\nabla u|^2}{2}dx=\int_{\partial U}\frac{|\nabla u|^2}{2}\left(\vec{x}\cdot\vec{n}\right)dS+$$

$+\left(1-\frac{n}{2}\right)\int_U|\nabla u|^2\,dx$. As a consequence: $\boxed{A_1=\left(1-\frac{n}{2}\right)\int_U|\nabla u|^2\,dx+\int_{\partial U}\frac{|\nabla u|^2}{2}\left(\vec{x}\cdot\vec{n}\right)dS}$

On the other hand now, since $u=0$ on ∂U, $\nabla u(x)$ is parallel to the outward normal $\vec{n}(x)$ at each $x\in\partial U$. In other words this expresses what we know from multivariable calculus, i.e. $\vec{n}=\pm\frac{\nabla u}{|\nabla u|}\Rightarrow\ \vec{\nabla}u(x)=\pm\left|\vec{\nabla}u(x)\right|\vec{n}(x)$. Using this equality in the A_2 above, we calculate:

$$A_2=-\int_{\partial U}\left(x\cdot\nabla u\right)\frac{\partial u}{\partial\vec{n}}dS=-\sum_{i,j=1}^{n}\int_{\partial U}u_{x_i}n_ix_ju_{x_j}dS=-\sum_{i,j=1}^{n}\int_{\partial U}\pm|\nabla u|n_i^2x_j\left(\pm|\nabla u|\right)n_jdS=$$

$$=-\sum_{i,j=1}^{n}\int_{\partial U}|\nabla u|^2n_i^2x_jn_jdS=-\int_{\partial U}|\nabla u|^2\left(\sum_{i=1}^{n}|n_i|^2\right)\left(\sum_{j=1}^{n}x_jn_j\right)dS=-\int_{\partial U}|\nabla u|^2\cancel{|\vec{n}|^2}^{\,1}\left(\vec{x}\cdot\vec{n}\right)dS\Rightarrow$$

$$\boxed{A_2=-\int_{\partial U}|\nabla u|^2\left(\vec{x}\cdot\vec{n}\right)dS}\overset{A=A_1+A_2}{\Rightarrow}A=\left(1-\frac{n}{2}\right)\int_U|\nabla u|^2\,dx+\int_{\partial U}\frac{|\nabla u|^2}{2}\left(\vec{x}\cdot\vec{n}\right)dS-\int_{\partial U}|\nabla u|^2\left(\vec{x}\cdot\vec{n}\right)dS\Rightarrow$$

$$\boxed{A=\frac{2-n}{2}\int_U|\nabla u|^2\,dx-\frac{1}{2}\int_{\partial U}|\nabla u|^2\left(\vec{x}\cdot\vec{n}\right)dS}(I)$$

$\boxed{3.}$ Returning to the initial relation now, obtained after multiplying the pde by $x \cdot \nabla u$ and integrating over U, i.e. $\underbrace{\int_U -\Delta u\left(\vec{x}\cdot\vec{\nabla}u\right)dx}_{A=A_1+A_2} = \underbrace{\int_U |u|^{p-1}u\left(\vec{x}\cdot\vec{\nabla}u\right)dx}_{B}$, we compute that:

$$B=\int_U |u|^{p-1}u\left(\vec{x}\cdot\vec{\nabla}u\right)dx=\sum_{j=1}^{n}\int_U u|u|^{p-1}x_j u_{x_j}\,dx=\sum_{j=1}^{n}\int_U\left(\frac{|u|^{p+1}}{p+1}\right)_{x_j}x_j dx=\sum_{j=1}^{n}\int_{\partial U}\frac{|u|^{p+1}}{p+1}x_j n_j dS-$$

$$-\sum_{j=1}^{n}\int_U\frac{|u|^{p+1}}{p+1}dx=\int_{\partial U}\frac{|u|^{p+1}}{\cancelto{0}{p+1}\left(\vec{x}\cdot\vec{n}\right)}\,dS-\frac{n}{p+1}\int_U |u|^{p+1}dx$$, where the surface integral vanishes because $u=0$ on $\partial U \Rightarrow\ |u|^{p+1}=0$ on ∂U. Finally all the above lead to: $\boxed{B=-\frac{n}{p+1}\int_U |u|^{p+1}dx}$ (II)

An alternative way to get this result would be the following:

$$\sum_{j=1}^{n}\int_U\left(\frac{|u|^{p+1}}{p+1}\right)_{x_j}x_j dx=\int_U\vec{\nabla}_x\left(\frac{|u|^{p+1}}{p+1}\right)\cdot\vec{x}\ dx=\int_{\partial U}\cancelto{0}{\frac{|u|^{p+1}}{p+1}}\left(\vec{x}\cdot\vec{n}\right)dS-\int_U\frac{|u|^{p+1}}{p+1}div\left(\vec{x}\right)dx=-\frac{n}{p+1}\int_U |u|^{p+1}dx$$

since $div\left(\vec{x}\right)=\sum_{i=1}^{n}\frac{\partial x_i}{\partial x_i}=1+1...+1=n.$ Moreover let us also justify the previous-step derivation:

$$\left(\frac{|u|^{p+1}}{p+1}\right)_{x_j}=|u|^p\frac{\partial}{\partial x_j}\left(|u|\right)=|u|^p\operatorname{sgn}(u)\frac{\partial u}{\partial x_j}=|u|^p\overbrace{\frac{u}{|u|}}^{\operatorname{sgn}(u)}u_{x_j}=u|u|^{p-1}u_{x_j}$$, where we have made use of the following known properties of the function "sgn": $\frac{\partial}{\partial x}\left(|x|\right)=\operatorname{sgn}x$ and $\operatorname{sgn}x=\frac{|x|}{x}=\frac{x}{|x|}$, $x\neq 0$

$\boxed{4.}$ This calculation and the initial relation $\int_U -\Delta u\left(\vec{x}\cdot\vec{\nabla}u\right)dx=\int_U |u|^{p-1}u\left(\vec{x}\cdot\vec{\nabla}u\right)dx$ yield:

$$\boxed{\boxed{\frac{n-2}{2}\int_U |\nabla u|^2 dx+\frac{1}{2}\int_{\partial U}|\nabla u|^2\left(\vec{x}\cdot\vec{\nabla}u\right)dS=\frac{n}{p+1}\int_U |u|^{p+1}dx}}$$ *Derrick - Pohozaev identity.*

In view of the lemma presented above, i.e. $\vec{x}\cdot\vec{n}\left(\vec{x}\right)\geq 0\ \forall\vec{x}\in\partial U$, given that ∂U is C^1 and U is a star-shaped set with respect to the origin, we then obtain the inequality:

$\frac{n-2}{2}\int_U |\nabla u|^2\, dx \le \frac{n}{p+1}\int_U |u|^{p+1}\, dx$. But once we multiply the pde $-\Delta u = u|u|^{p-1}$ by u and integrate by parts, we produce the equality: $\int_U -u\Delta u dx = \int_U u^2 |u|^{p-1}\, dx \Rightarrow -\int_{\partial U} u \cancel{\frac{\partial u}{\partial \vec{n}}}^{0}\, dS + \int_U \nabla u \cdot \nabla u dx =$

$= \int_U |u|^2 |u|^{p-1}\, dx \Rightarrow \boxed{\int_U |\nabla u|^2\, dx = \int_U |u|^{p+1}\, dx}$ and substituting this above, we conclude that:

$\frac{n-2}{2}\int_U |u|^{p+1}\, dx = \frac{n-2}{2}\int_U |\nabla u|^2\, dx \le \frac{n}{p+1}\int_U |u|^{p+1}\, dx \Rightarrow \left[\frac{n-2}{2} - \frac{n}{p+1}\right]\int_U |u|^{p+1}\, dx \le 0$. Hence if $u \ne 0$, it follows that $\frac{n-2}{2} - \frac{n}{p+1} \le 0 \Rightarrow p \le \frac{n+2}{n-2}$ which is a contradiction, therefore $u = 0$ in U □

6. Invariance - Noether's theorem

Next we study variational integrands that are invariant under appropriate domain and function variations and show that solutions of the corresponding E-L equations then automatically solve also certain divergence structure conservation laws.

Amalie Emmy Noether (23 March 1882 – 14 April 1935) was a German mathematician who made important contributions to abstract algebra and theoretical physics. She invariably used the name "Emmy Noether" in her life and publications. She was described by Pavel Alexandrov, Albert Einstein, Jean Dieudonné, Hermann Weyl and Norbert Wiener as the most important woman in the history of mathematics. As one of the leading mathematicians of her time, she developed the theories of rings, fields, and algebras. In physics, Noether's theorem explains the connection between symmetry and conservation laws.

6.1 Invariant variational problems – Noether's Theorem

We again turn our attention to the functional $I[w] = \int_U L(x, w, \nabla w) dx$ where $U \subset \mathbb{R}^n$ and $w : U \to \mathbb{R}$. We as usual write $L = L(x, z, p)$ $(z \in \mathbb{R})$.

Notation:

(i) Let $x : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{\chi} = \boldsymbol{\chi}(x, \tau)$, be a smooth family of vector fields satisfying $\boldsymbol{\chi}(x, 0) = x$ for all $x \in \mathbb{R}^n$. Then for small $|\tau|$, the mapping $x \mapsto \boldsymbol{\chi}(x, \tau)$ is a smooth diffeomorphism. We call the mapping $x \mapsto \boldsymbol{\chi}(x, \tau)$ a <u>domain variation</u>. Define also $\boxed{\mathbf{v}(x) = \boldsymbol{\chi}_\tau(x, 0)}$ and $\boxed{U(\tau) = \boldsymbol{\chi}(U, \tau)}$.

(ii) Next, given a smooth $u : \mathbb{R}^n \to \mathbb{R}$, we consider a smooth family of <u>function variation</u> $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $w(x, \tau) = w$ such that $\boxed{w(x, 0) = u(x)}$. Write $\boxed{m(x) = w_\tau(x, 0)}$. For reasons that will be clear shortly, we call m a *multiplier*.

Task:

Given a functional $I[\cdot]$ of the upper form (i.e. $I[w] = \int_U L(x, z, p)$ for $z = w$ and $p = \nabla w$), we ask if we can find domain and function variations that are compatible with the Lagrangian L , in the sense that $I[\cdot]$ is unchanged under these variations.

Definition:

We say that a functional $I[\cdot]$ is invariant under both the <u>domain variation</u> $\boldsymbol{\chi}$ and the <u>function variation</u> w provided (#) $\boxed{\int_U L\left(x, w(x, \tau), \nabla w(x, \tau)\right) dx = \int_{U(\tau) = \boldsymbol{\chi}(U, \tau)} L(x, u, \nabla u) dx}$ for all small $|\tau|$ and all open sets $U \subset \mathbb{R}^n$. $\left(\text{Here we write } \nabla w = \nabla_x w\right)$

The idea behind this definition is that given a domain variation $\boldsymbol{\chi}$ and a function u , we will look for w as some expression involving $u(\boldsymbol{\chi}(x,\tau))$. We will try to check (#) by changing variables in the integral term on the left side, after which the integration will be over the region $U(\tau)$. Below we will show that *invariance of the functional implies that the corresponding E-L equation can be transformed into divergence form!*

$\boxed{Noether's\ theorem}$

Suppose that the functional $I[\cdot]$ is invariant under the domain variation $\boldsymbol{\chi}$ and the function variation w corresponding to a smooth function $u,$ i.e. $\int\limits_U L\left(x,w(x,\tau),\nabla w(x,\tau)\right)dx = \int\limits_{U(\tau)} L\left(x,u,\nabla u\right)dx$

Then:

(i) $\sum_{i=1}^{n}\left[mL_{p_i}(x,u,\nabla u)-L(x,u,\nabla u)v_i\right]_{x_i} = m\left[\sum_{i=1}^{n}\left(L_{p_i}(x,u,\nabla u)\right)_{x_i} - L_z(x,u,\nabla u)\right]$,which has the vector form: $\boxed{div\left[m\vec{\nabla}_p L - L\vec{v}\right] = m\left[div\left(\vec{\nabla}_p L\right)-L_z\right]}$ $(*)$, where $\vec{v}=(v_1,v_2,...,v_n)$ is defined by $\vec{v}=\mathbf{v}=\boldsymbol{\chi}_\tau(x,0)$ and the multiplier m by $m(x)=w_\tau\left(x,0\right)$.

(ii) In particular, if u is a critical point of $I[\cdot]$ and so solves the E-L equation, i.e. $-div\left(\nabla_p L\right)+L_z=0,$ we have the divergence identity $\sum_{i=1}^{n}\left(mL_{p_i}(x,u,\nabla u)-L(x,u,\nabla u)v_i\right)_{x_i}=0$, which has the following vector form: $\boxed{div\left[m\vec{\nabla}_p L - L\vec{v}\right]=0}$, known as Noether's divergence identity.

Proof:

Differentiating the invariance identity $\int\limits_U L\left(x,w(x,\tau),\nabla w(x,\tau)\right)dx = \int\limits_{U(\tau)=\boldsymbol{\chi}(U,\tau)} L\left(x,u,\nabla u\right)dx$ with respect to τ and then setting $\tau=0$ yields the identity:

(remember our notation according to which: $z=w(x,\tau)$, $p=\nabla w(x,\tau)$ and $U(\tau)=\boldsymbol{\chi}(U,\tau)$)

$$\int\limits_U L\left(x,w(x,\tau),\nabla w(x,\tau)\right)dx = \int\limits_{U(\tau)} L\left(x,u,\nabla u\right)dx \overset{\frac{d}{d\tau}}{\Rightarrow}$$

$\int\limits_U\left\{L_z\left(x,w(x,\tau),\nabla w(x,\tau)\right)w_\tau(x,\tau)+\sum_{i=1}^{n}L_{p_i}\left(x,w(x,\tau),\nabla w(x,\tau)\right)\left(w_\tau(x,\tau)\right)_{x_i}\right\}dx =$ (we can obtain this result, because we can change the derivatives, i.e. $\left(w_{x_i}(x,\tau)\right)_\tau=\left(w_\tau(x,\tau)\right)_{x_i}$) $=$

$\int\limits_{\partial U(\tau)} L(x,u,\nabla u)\ \vec{v}(x)\cdot\vec{n}\ dS + \int\limits_{U(\tau)} \cancel{L_\tau(x,u,\nabla u)}^{0}\,dx$, where here we applied the differentiation formula for moving regions, i.e. $\boxed{\frac{d}{d\tau}\int\limits_{U(\tau)} f(x,\tau)dx = \int\limits_{U(\tau)} f_\tau(x,\tau)dx + \int\limits_{\partial U(\tau)} f\ \vec{v}\cdot\vec{n}dS}$ where $\begin{cases}\vec{n}:\ \text{outward normal}\\ \vec{v}:\ \text{velocity of the moving boundary of } \partial U(\tau)\end{cases}$

By setting now $\tau=0$ and by taking into consideration that $w(x,0)=u(x),\ m(x)=w_\tau\left(x,0\right),$ $\vec{v}=\mathbf{v}(x)=\boldsymbol{\chi}_\tau(x,0)$ and $\boldsymbol{\chi}(U,\tau)=U\left(\tau\right)\ \Rightarrow\ \boldsymbol{\chi}(U,0)=U$, we get that:

$$\int\limits_U\left\{L_z(x,u,\nabla u)m(x)+\sum_{i=1}^{n}L_{p_i}(x,u,\nabla u)\left(m(x)\right)_{x_i}\right\}dx = \int\limits_{\partial U(0)=\partial U} L(x,u,\nabla u)\vec{v}\cdot\vec{n}\ dS \ \Rightarrow$$

$\boxed{\int_U \left[L_z m + \vec{\nabla}_p L \cdot \vec{\nabla} m\right] dx = \int_{\partial U} L\vec{v} \cdot \vec{n}\ dS}$ Now by applying integration by parts and the Gauss-Green theorem $\Rightarrow \int_{\partial U} m(x)\left(\vec{\nabla}_p L \cdot \vec{n}\right) dS + \int_U \left[L_z m - \left(\nabla \cdot \nabla_p L\right) m\right] dx = \int_{\partial U} L\vec{v} \cdot \vec{n}\ dS \ \Rightarrow$

$\int_U \left[L_z - div\left(\nabla_p L\right)\right] m(x) dx = \int_{\partial U} \left(-m\vec{\nabla}_p L + L\vec{v}\right) \cdot \vec{n}\ dS \overset{Gauss}{=} \int_U div\left(-m\vec{\nabla}_p L + L\vec{v}\right) dx$. This identity is valid for all regions U, and so the identity $(*)$ in (i) follows. Therefore:

$div\left[m\vec{\nabla}_p L - L\vec{v}\right] = m\left[div\left(\vec{\nabla}_p L\right) - L_z\right]$ which leads to (i) and (ii) respectively:

$\sum_{i=1}^{n}\left[mL_{p_i}(x,u,\nabla u) - L(x,u,\nabla u) v_i\right]_{x_i} = m\left[\sum_{i=1}^{n}\left(L_{p_i}(x,u,\nabla u)\right)_{x_i} - L_z(x,u,\nabla u)\right]$ where apparently if u is a critical point, i.e. the E-L equation $div\left(\vec{\nabla}_p L\right) - L_z = 0$ is satisfied, then we have that:

$div\left[m\vec{\nabla}_p L - L\vec{v}\right] = 0 \ \Rightarrow \ \sum_{i=1}^{n}\left[mL_{p_i}(x,u,\nabla u) - L(x,u,\nabla u) v_i\right]_{x_i} = 0$ which is the required. □

As noted earlier, we can sometimes first guess a domain variation **χ** and then look for a corresponding function variation *w* as some formula involving *u*(**χ**(*x,τ*)). Then we will be able to compute the multiplier *m* in terms of *u* and its partial derivatives. Next we shall present numerous ''examples-applications'' illustrating this procedure.

6.2 Examples- Applications

(I) Lagrangian independent of *χ*

If $L = L(p,z)$ does not depend upon the independent variable x, then the functional $I[w] = \int_U L(w,\nabla w) dx$ is invariant under translations in space. To be more specific, let us select $k \in \{1,2,...,n\}$ and define $\boxed{\boldsymbol{\chi}(x,\tau) = x + \tau e_k}$, $\boxed{w(x,\tau) = u\left(x + \tau e_k\right)}$. Then $\mathbf{v}(x) = \boldsymbol{\chi}_\tau(x,0) = e_k \ \Rightarrow \ \boxed{\mathbf{v} = e_k}$ and similarly $m(x) = w_\tau(x,0) \ \Rightarrow \ \boxed{m(x) = u_{x_k}}$ since $u\left(x + \tau e_k\right) = u\left(x_1,...,x_k + \tau,...,x_n\right)$. As a consequence, if u is a critical point, the Noether's Theorem provides us with the identity: $div\left(m\vec{\nabla}_p L - L\vec{v}\right) = 0 \ \Rightarrow \ \sum_{i=1}^{n}\left[u_{x_k} L_{p_i} - L\delta_{ik}\right]_{x_i} = 0$ for $k = 1,2,...,n.$ As we have seen, these formulas follow directly from the E-L equation (confirmed also by the result (ii) of Noether's theorem as well) by simple calculation and they are known as "first integral".

Let us remind the relation $L-\sum_{i=1}^{n}\dot{y}_i L_{\dot{y}_i}$ = constant. The point here is that Noether's theorem provides us with a systematic procedure for finding first integrals and in general such identities.

(II) Scaling Invariance

The functional $I[w]=\int_U |\nabla w|^p\, dx$, smooth minimizers of which solve the p-Laplacian equation

$\boxed{div\left(|\nabla u|^{p-2}\nabla u\right)=0}$ is invariant under the *scaling* transformation $x\mapsto \lambda x$ and $\boxed{u\mapsto \lambda^{\frac{n-p}{p}}u(\lambda x)}$.

First let us note (in order to justify the statement above) the following:

E-L $\Rightarrow$ $L_w-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}L_{w_{x_i}}=0 \Rightarrow L_w=div\left(L_{w_{x_1}},...,L_{w_{x_n}}\right)$, where we note

that $L_w=0$ and $L_{w_{x_i}}=w_{x_i}\,p|\nabla w|^{p-2} \Rightarrow div\left(p|\nabla w|^{p-2}\nabla w\right)=0 \overset{\substack{p \text{ is a}\\ \text{constant}}}{\Rightarrow} div\left(|\nabla u|^{p-2}\nabla u\right)=0.$

Now in order to be consistent with previous notation, we put $\boxed{\lambda=\mathrm{e}^{\tau}}$ and define: $\boldsymbol{\chi}(x,\tau)=e^{\tau}x$ as well as $w(x,\tau)=e^{\tau\frac{n-p}{p}}u(e^{\tau}x)$. Then as a result we obtain: $\mathbf{v}(x)=x$ and $m=\nabla u\cdot\vec{x}+\dfrac{n-p}{p}u.$

The corresponding divergence identity $div\left(m\vec{\nabla}_pL-L\vec{v}\right)=0$ derived by the Noether's theorem takes the following form (for a minimizer which is in fact a solution of the p-Laplacian equation

$div\left(|\nabla u|^{p-2}\nabla u\right)=0$): $div\left\{\left[\nabla u\cdot\vec{x}+\dfrac{n-p}{p}u\right]\vec{\nabla}_pL-|\nabla u|^p\,\vec{x}\right\}=0 \Rightarrow \begin{cases}L_{p_i}=u_{x_i}\,p|\nabla u|^{p-2}\ so: \\ \vec{\nabla}_pL=p|\nabla u|^{p-2}\nabla u\end{cases}\Rightarrow$

$div\left\{\left[\nabla u\cdot\vec{x}+\dfrac{n-p}{p}u\right]p|\nabla u|^{p-2}\nabla u-|\nabla u|^p\,\vec{x}\right\}=0.$ Therefore this leads to

$\boxed{\sum_{i=1}^{n}\left[\left[\nabla u\cdot\vec{x}+\dfrac{n-p}{p}u\right]p|\nabla u|^{p-2}u_{x_i}-|\nabla u|^p x_i\right]_{x_i}=0}$. It is again straightforward to check this identity by a direct calculation from the E-L equation.

<u>Application: Monotonicity formulas</u>

Assume that u is a smooth solution f the p-Laplacian $div\left(|\nabla u|^{p-2}\nabla u\right)=0\Leftrightarrow\Delta\left(|\nabla u|^p\right)=0$ within some region U and that the ball $B(0,r)$ lies within U. If we integrate the divergence identity we found above, i.e. $\sum_{i=1}^{n}\left[\left[\nabla u\cdot\vec{x}+\frac{n-p}{p}u\right]p\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial x_i}-\left|\nabla u\right|^{p}x_i\right]_{x_i}=0$ over the ball $B(0,r)$ and simplify using Gauss theorem, we discover that:

$\int\limits_{B(0,r)} div\left\{\left(\nabla u\cdot\vec{x}+\frac{n-p}{p}u\right)p\left|\nabla u\right|^{p-2}\vec{\nabla}u-\left|\nabla u\right|^{p}\vec{x}\right\}dx=0\Rightarrow$ Gauss theorem on the ball's surface $\partial B(0,r)$ where if $\vec{x}\in\partial B(0,r)$ then $\vec{n}=\dfrac{\vec{x}}{r}$ is apparently the outward unit normal and moreover we set:

$u_r=\dfrac{\partial u}{\partial\vec{n}}=\vec{\nabla}u\cdot\vec{n}=\dfrac{\nabla u\cdot x}{|x|}$, since $|\vec{x}|=r$ for $\vec{x}\in\partial B(0,r).\Rightarrow$

$$\int\limits_{\partial B(0,r)}\left\{\left(\nabla u\cdot\vec{x}+\frac{n-p}{p}u\right)p\left|\nabla u\right|^{p-2}\vec{\nabla}u-\left|\nabla u\right|^{p}\vec{x}\right\}\cdot\vec{n}\;dS=0\Rightarrow$$

$$\int\limits_{\partial B(0,r)}\left(\nabla u\cdot\vec{x}+\frac{n-p}{p}u\right)p\left|\nabla u\right|^{p-2}\vec{\nabla}u\cdot\vec{n}\;dS=\int\limits_{\partial B(0,r)}\left|\nabla u\right|^{p}\vec{x}\cdot\vec{n}\;dS\Rightarrow$$

$$\int\limits_{\partial B(0,r)}\frac{(\nabla u\cdot x)}{\|x\|^2}p\left|\nabla u\right|^{p-2}\left(\vec{\nabla}u\cdot\vec{n}\right)\|x\|^2\,dS+\int\limits_{\partial B(0,r)}(n-p)u\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial\vec{n}}dS=\int\limits_{\partial B(0,r)}\left|\nabla u\right|^{p}\vec{x}\cdot\vec{n}\;dS\Rightarrow$$

$$\int\limits_{\partial B(0,r)}\left\{\left|\nabla u\right|^{p}\vec{x}\overbrace{\frac{\vec{x}}{r}}^{\vec{n}}-\frac{\overbrace{(\nabla u\cdot x)^2}^{u_r^2}}{\|x\|^2}p\left|\nabla u\right|^{p-2}\vec{x}\cdot\vec{n}\right\}dS=\int\limits_{\partial B(0,r)}(n-p)u\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial\vec{n}}dS\Rightarrow\;\left(\|x\|=r\text{ for }x\in\partial B(0,r)\right)$$

$$\int\limits_{\partial B(0,r)}\left\{\left|\nabla u\right|^{p}\frac{r^{\cancel{2}}}{\cancel{r}}-u_r^2\,p\left|\nabla u\right|^{p-2}\frac{r^{\cancel{2}}}{\cancel{r}}\right\}dS=(n-p)\int\limits_{\partial B(0,r)}u\left|\nabla u\right|^{p-2}\frac{\partial u}{\partial\vec{n}}dS\Rightarrow$$

$$r\int\limits_{\partial B(0,r)}\left(\left|\nabla u\right|^{p}-p\left|\nabla u\right|^{p-2}u_r^2\right)dS=(n-p)\int\limits_{\partial B(0,r)}u\left|\nabla u\right|^{p-2}\nabla u\cdot\vec{n}\;dS\Rightarrow$$

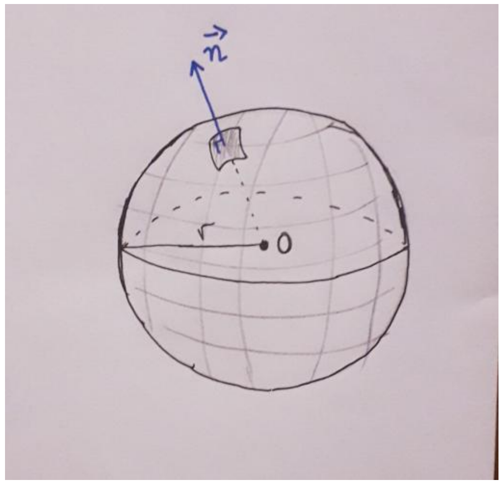
$$\boxed{r\int\limits_{\partial B(0,r)}\left(\left|\nabla u\right|^{p}-p\left|\nabla u\right|^{p-2}u_r^2\right)dS=(n-p)\int\limits_{B(0,r)}\left|\nabla u\right|^{p}dx}\quad(\#)$$

At this point we need to prove that $(n-p)\int\limits_{B(0,r)}\left|\nabla u\right|^{p}dx=(n-p)\int\limits_{\partial B(0,r)}u\left|\nabla u\right|^{p-2}u_r\;dS.$

For this purpose, we observe that the E-L equation in divergence form can be written

$\nabla\left(\left|\nabla u\right|^{p}\right)=p\left|\nabla u\right|^{p-1}\nabla\left(\left|\nabla u\right|\right)=p\left|\nabla u\right|^{p-1}\dfrac{\nabla u}{\left|\nabla u\right|}=p\left|\nabla u\right|^{p-2}\nabla u$, because of the fact that

$\dfrac{\partial}{\partial x_i}\left[\left|\nabla u\right|^{p}\right]=p\left|\nabla u\right|^{p-1}\dfrac{u_{x_i}}{\left|\nabla u\right|}=p\left|\nabla u\right|^{p-2}u_{x_i}$. Therefore we can rewrite the quantity $\left|\nabla u\right|^{p-2}\nabla u$ inside



This figure depicts the outward unit normal

the "div" operator in the E-L equation $div\left(|\nabla u|^{p-2}\,\nabla u\right)=0$ as following: (p is constant)

$0=div\left(|\nabla u|^{p-2}\,\nabla u\right)=div\left(p|\nabla u|^{p-2}\,\nabla u\right)=div\left(\nabla\left(|\nabla u|^{p}\right)\right)=\nabla\cdot\nabla\left(|\nabla u|^{p}\right)=\Delta\left(|\nabla u|^{p}\right).$ So we have

$\boxed{div\left(|\nabla u|^{p-2}\,\nabla u\right)=0 \;\Leftrightarrow\; \Delta\left(|\nabla u|^{p}\right)=0}$. Now we are ready to obtain the required by employing the Green's identity:

$0=\int_{B(0,r)} u\,\overset{=0}{\cancel{\Delta\left(|\nabla u|^{p}\right)}}\,dx=\int_{\partial B(0,r)} u\vec{\nabla}\left(|\nabla u|^{p}\right)\cdot\vec{n}\;dS-\int_{B(0,r)}\vec{\nabla}\left(|\nabla u|^{p}\right)\cdot\vec{\nabla}u\;dx$, but we have shown that $\nabla\left(|\nabla u|^{p}\right)=p|\nabla u|^{p-2}\,\vec{\nabla}u,$ therefore we get:

$\int_{\partial B(0,r)} \cancel{p}u|\nabla u|^{p-2}\,\overbrace{\vec{\nabla}u\cdot\vec{n}}^{u_r}\,dS=\int_{B(0,r)} \cancel{p}\,|\nabla u|^{p-2}\overbrace{\left(\vec{\nabla}u\cdot\vec{\nabla}u\right)}^{|\nabla u|^2}dx$. Consequently we have shown the required which is: $\int_{B(0,r)}|\nabla u|^{p}\,dx=\int_{\partial B(0,r)}u|\nabla u|^{p-2}\,u_r\;dS$. As a result from all the calculations above, after differentiating with respect to $r,$ we obtain:

$\frac{d}{dr}\left(\frac{1}{r^{n-p}}\int_{B(0,r)}|\nabla u|^{p}\,dx\right)=\frac{p-n}{r^{n-p+1}}\int_{B(0,r)}|\nabla u|^{p}\,dx+\frac{1}{r^{n-p}}\int_{\partial B(0,r)}|\nabla u|^{p}\,dS$, where we have used the integration formula in polar coordinates: $\frac{d}{dr}\int_{B(r)} f dx=\frac{d}{dr}\int_0^r\int_{\partial B(p)} f\;dSdp=\int_{\partial B(r)} fdS$. Now we would also like to note

$\frac{p-n}{r^{n-p+1}}\int_{B(0,r)}|\nabla u|^{p}\,dx\overset{(\#)}{=}-r^{p-n}\int_{\partial B(0,r)}\left\{|\nabla u|^{p}-p|\nabla u|^{p-2}\,u_r^2\right\}dS$, (due to the relation (#) above). Consequently:

$\frac{d}{dr}\left(\frac{1}{r^{n-p}}\int_{B(0,r)}|\nabla u|^{p}\,dx\right)=\frac{p-n}{r^{n-p+1}}\int_{B(0,r)}|\nabla u|^{p}\,dx+\frac{1}{r^{n-p}}\int_{\partial B(0,r)}|\nabla u|^{p}\,dS\overset{(\#)}{=}-r^{p-n}\int_{\partial B(0,r)}\left\{\cancel{|\nabla u|^{p}}-p|\nabla u|^{p-2}\,u_r^2\right\}dS+$

$+\frac{1}{r^{p-n}}\cancel{\int_{\partial B(0,r)}|\nabla u|^{p}\,dS}=\frac{p}{r^{n-p}}\int_{\partial B(0,r)}|\nabla u|^{p-2}\,u_r^2dS\;\geq 0.$ Therefore we have proved the following monotonicity formula: $\boxed{\dfrac{d}{dr}\left(\dfrac{1}{r^{n-p}}\displaystyle\int_{B(0,r)}|\nabla u|^p\,dx\right)=\dfrac{p}{r^{n-p}}\displaystyle\int_{\partial B(0,r)}|\nabla u|^{p-2}u_r^2 dS\ \geq 0}$, implying that $r\mapsto\dfrac{1}{r^{n-p}}\displaystyle\int_{B(0,r)}|\nabla u|^p\,dx$ is non decreasing.

6.3 Time dependent problems

If one of the dependent variables is identified with time, then we can interpret the equation in the second leg of Noether's Theorem $div\left[m\vec{\nabla}_p L-L\vec{v}\right]=0$ as a *conservation law* resulting from the invariance of our variational integral.

Conservation of energy for non-linear wave equations:

Consider the integral expression $\boxed{I[w]=\displaystyle\int_0^T\int_{\mathbb{R}^n}\left\{\frac{w_t^2}{2}-\left(\frac{1}{2}|\nabla w|^2+F(w)\right)\right\}dxdt}$ $(*)$ defined for functions $w=w(x,t)$ with, say, compact support. As usual, we write $\nabla w=\nabla_x w$. We can interpret the Lagrangian as representing the kinetic energy minus the potential energy ($L=T-V$). The corresponding E-L equation is the semi-linear wave equation: $\boxed{u_{tt}-\Delta u+f(u)=0}$ where $f=F'$.
The integrand of $(*)$ does not depend on the time variable t and is consequently invariant under shifts on this variable. Noether's theorem implies that this invariance forces a conservation law, in the case at hand *conservation of energy*. More precisely we define: $\boldsymbol{\chi}(x,t,\tau)=(x,t+\tau)$, $w(x,t,\tau)=u(x,t+\tau)$ so that $\mathbf{v}=e_{n+1}$ because $\mathbf{v}(x)=\boldsymbol{\chi}_\tau(x,0)$ and $\boldsymbol{\chi}(x,t,\tau)=(x,t+\tau)\in\mathbb{R}^n\times\mathbb{R}\Rightarrow$ $\boldsymbol{\chi}_\tau(x,0)=(0,1)\in\mathbb{R}^n\times\mathbb{R}\Rightarrow\mathbf{v}(x)=e_{n+1}$. Finally, again as usual, we find that $m=u_t$. Remember now that the Lagrangian is $L=\dfrac{u_t^2}{2}-\left(\dfrac{1}{2}|\nabla u|^2+F(u)\right)$. Moreover $L=L(x,u,\overbrace{u_t,\nabla_x u}^{p})=\vec{\nabla}_p L=$

$=\nabla_{u_x}L+e_{n+1}L_{u_t}=-\nabla u+e_{n+1}u_t$, where $\nabla_{u_x}L=-\nabla u$, since $\dfrac{\partial L}{\partial u_{x_i}}=-u_{x_i}$.Here by writing ∇u we mean $\nabla u=\left(u_{x_1},u_{x_2},\ldots,u_{x_n},0\right)$. Then $div\left[m\vec{\nabla}_p L-L\vec{v}\right]=0$ implies that:

$$div\left[u_t\left\{-\nabla u+e_{n+1}u_t\right\}-e_{n+1}\left\{\frac{u_t^2}{2}-\left(\frac{1}{2}|\nabla u|^2+F(u)\right)\right\}\right]=0\ \Rightarrow$$

$$div\left[\left(-u_t\frac{\partial u}{\partial x_1},\ldots,--u_t\frac{\partial u}{\partial x_n},u_t^2-\frac{u_t^2}{2}+\left(\frac{1}{2}|\nabla u|^2+F(u)\right)\right)\right]=0.\text{ Consequently }\left(div=\sum_{i=1}^n\frac{\partial}{\partial x_i}\right)\ \Rightarrow$$

$\sum_{i=1}^{n}\left(-u_{x_i}u_t\right)_{x_i}+\left(u_t^2-\frac{1}{2}\left[u_t^2-|\nabla u|^2\right]+F(u)\right)_t=0$. Of course this can be rewritten as:

$\left[\frac{u_t^2}{2}+\frac{1}{2}|\nabla u|^2+F(u)\right]_t-\underbrace{\sum_{i=1}^{n}\left(u_t u_{x_i}\right)_{x_i}}_{div(u_t\nabla u)}=0 \Rightarrow \boxed{e_t-div\left(u_t\vec{\nabla}u\right)=0}(**)$ considering the *energy density*:

$e=\frac{u_t^2}{2}+\frac{1}{2}|\nabla u|^2+F(u).$

The divergence operator in relation $(**)$ now acts in the $\vec{x}$ variables only if u has compact support in space at each time, it follows that the total energy is conserved in time, i.e.

$\frac{d}{dt}\int_{\mathbb{R}^n} e\,dx=\frac{d}{dt}\int_{\mathbb{R}^n}\frac{1}{2}\left[u_t^2+|\nabla u|^2\right]+F(u)\,dx=\int_{\mathbb{R}^n} e_t dx\overset{(**)}{=}\int_{\mathbb{R}^n} div\left[u_t\nabla u\right]dx\overset{Gauss}{=}\int_{\partial U}u_t\frac{\partial u}{\partial\vec{n}}dS\overset{\substack{\text{compact}\\\text{support}}}{=}0,$ where $U=\operatorname{supp}(u)$. Consequently, we have conservation of energy!

Scaling invariance for the wave equation

Consider the linear wave equation $\Box u=u_{tt}-\Delta u=0$ which corresponds to the action functional $\boxed{I[w]=\frac{1}{2}\int_0^T\int_{\mathbb{R}^n} w_t^2-|\nabla w|^2\,dxdt}$. Similarly to the case above, this functional is invariant under the scaling transformation $(x,t)\mapsto(\lambda x,\lambda t)$ and $u\mapsto\lambda^{\frac{n-1}{2}}u(\lambda x,\lambda t)$. As before we put $\boxed{\lambda=e^\tau}$ and define $\boldsymbol{\chi}(x,t,\tau)=\left(e^\tau x,e^\tau t\right)$ as well as $w(x,t,\tau)=e^{\tau\frac{n-1}{2}}u\left(e^\tau x,e^\tau t\right)$. Therefore, taking into consideration that $\mathbf{v}(x)=\boldsymbol{\chi}_\tau(x,0)$ and $\boldsymbol{\chi}_\tau(x,0)=(x,t)$ we obtain obviously that $\mathbf{v}=(x,t)$ and $m(x)=w_\tau(x,0),$ so we have:

$w_\tau(x,t,\tau)=e^{\tau\frac{n-1}{2}}\frac{n-1}{2}u(e^\tau x,e^\tau t)+e^{\tau\frac{n-1}{2}}\vec{\nabla}u(e^\tau x,e^\tau t)\cdot\vec{x}e^\tau+e^{\tau\frac{n-1}{2}}u_t(e^\tau x,e^\tau t)te^\tau$ and for $\tau=0\Rightarrow$

$\boxed{m=tu_t+x\cdot\nabla u+\frac{n-1}{2}u}$. The conservation law now provided by $div\left[m\vec{\nabla}_pL-L\vec{v}\right]=0$ asserts that:

$$div\left\{\left(tu_t+x\cdot\nabla u+\frac{n-1}{2}u\right)\left(-\vec{\nabla}u,u_t\right)+\left(-\frac{u_t^2}{2}+\frac{|\nabla u|^2}{2}\right)(\vec{x},t)\right\}=0\Rightarrow$$

$$\sum_{i=1}^{n}\left\{-\left(tu_t+x\cdot\nabla u+\frac{n-1}{2}u\right)u_{x_i}-\frac{1}{2}\left(u_t^2-|\nabla u|^2\right)x_i\right\}_{x_i}+\left\{\left(tu_t+x\cdot\nabla u+\frac{n-1}{2}u\right)u_t+\left(-\frac{u_t^2}{2}+\frac{|\nabla u|^2}{2}\right)t\right\}_t=0$$

$$\Rightarrow \boxed{p_t-div\vec{q}=0}\text{, where }\begin{cases}\vec{q}=\left(tu_t+x\cdot\nabla u+\frac{n-1}{2}u\right)\vec{\nabla}u+\frac{1}{2}\left(u_t^2-|\nabla u|^2\right)\vec{x}\\ p=\frac{t}{2}\left(u_t^2+|\nabla u|^2\right)+\left(\vec{x}\cdot\vec{\nabla}u\right)u_t+\frac{n-1}{2}uu_t\end{cases}$$

Conformal energy for wave equation

The following much more sophisticated example illustrates how Noether's theorem, even when not directly applicable, can sometimes help us identify useful multipliers. The mapping $(x,t)\mapsto(\bar{x},\bar{t})=\left(\frac{x}{|x|^2-t^2},\frac{t}{|x|^2-t^2}\right)$, where $|x|^2\neq t^2$, is called *hyperbolic inversion*. It is related to the *hyperbolic Kelvin transform* $\boxed{Ku=\bar{u}}$ defined by $\bar{u}(x,t)=u(\bar{x},\bar{t})\left||\bar{x}|^2-\bar{t}^2\right|^{\frac{n-1}{2}}=$

$=u\left(\frac{x}{|x|^2-t^2},\frac{t}{|x|^2-t^2}\right)\frac{1}{\left||x|^2-t^2\right|^{\frac{n-1}{2}}}$. Moreover, after a lot of long calculations (see [B5]) we can derive that: if $\square u=u_{tt}-\Delta u=0\ \Rightarrow\ \square\bar{u}=0$. Our intention here is to use hyperbolic inversion and hyperbolic Kelvin transform to help us design variations in (x,t) and in u. For the first let us map (x,t) to $(\bar{x},\bar{t})$, then add τe_{n+1} and lastly apply hyperbolic inversion again. A rather long calculation (omitted here :( ) shows that the result is the mapping $\boldsymbol{\chi}(x,t,\tau)=\gamma\left(x,t+\tau\left(|x|^2-t^2\right)\right)$ for $\gamma=\frac{|x|^2-t^2}{|x|^2-\left(t+\tau\left(|x|^2-t^2\right)\right)^2}$. We next employ a similar procedure to build variations of u. In fact we apply the Kelvin hyperbolic transform to compute $Ku=\bar{u}$, then add (as above) τe_{n+1} within the argument of $\bar{u}$ and lastly apply once again Kelvin transform. Another (quite similar) calculation reveals the resulting function variation to be $w(x,t,\tau)=\gamma^{\frac{n-1}{2}}u(\boldsymbol{\chi}(x,t,\tau))$. We next compute the corresponding multiplier dy differentiating with respect to τ and then setting $\tau=0$ to obtain: $\mathbf{v}(x)=\left(2xt,|x|^2+t^2\right)$, $m(x)=\left(|x|^2+t^2\right)u_t+2t\vec{x}\cdot\vec{\nabla}u+(n-1)tu$. Now we don't assert (!) that the energy functional $I[u]=\int_0^T\int_{\mathbb{R}^n}\left\{\frac{u_t^2}{2}-\frac{|\nabla u|^2}{2}\right\}dxdt$ is invariant under these domain and function variations.

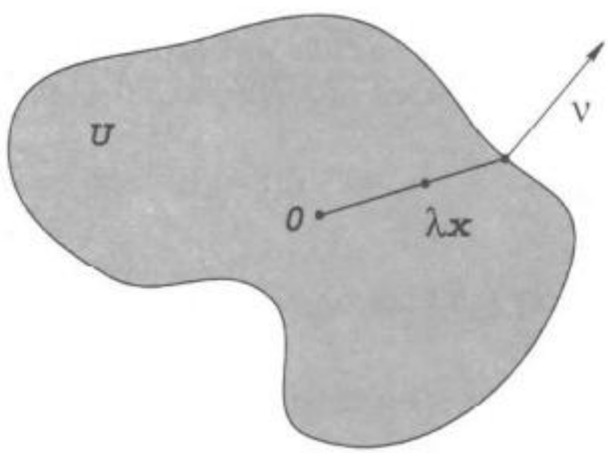
Rather, we guess that since the hyperbolic Kelvin transformation preserves solutions of the wave equation, then it might be useful to multiply the wave equation $\square u=0$ by the multiplier m we have computed above. This turns out to be so, and after a longish calculation we derive the *Morawetz*'s identity: $\boxed{c_t-div\vec{r}=0}$, where "c" is the density of the so-called *conformal energy* which is: $\boxed{c=\frac{1}{2}\left(|x|^2+t^2\right)\left(u_t^2+|\nabla u|^2\right)+2t\left(\vec{x}\cdot\vec{\nabla}u\right)u_t+(n-1)tuu_t-\frac{n-1}{2}u^2}$ and $\vec{r}$ is given by: $\boxed{\vec{r}=\left\{\left(|x|^2+t^2\right)u_t+2t\left(\vec{x}\cdot\vec{\nabla}u\right)+(n-1)tu\right\}\vec{\nabla}u+t\left(u_t^2-|\nabla u|^2\right)\vec{x}}$, Morawetz's identity is important since the conformal energy density c can be written for $n \neq 2$ as a sum of non-negative terms plus a divergence in the $\vec{x}$-variables, which is precisely the following: $\left(\text{we denote: } u_r = \dfrac{\vec{\nabla} u \cdot \vec{x}}{|x|} = \dfrac{\partial u}{\partial \vec{n}} \right)$

$$c = \overbrace{\frac{(t+|x|)^2}{4}\left(u_t + u_r + \frac{n-1}{2|x|}u\right)^2}^{A_1} + \overbrace{\frac{(t-|x|)^2}{4}\left(u_t - u_r - \frac{n-1}{2|x|}u\right)^2}^{A_2} + \overbrace{\frac{|x|^2+t^2}{2}\left(|\nabla u|^2 - u_r^2 + \frac{(n-3)(n-1)}{4|x|^2}u^2\right)}^{A_3} -$$

$$-\frac{n-1}{4} div\left(\frac{|x|^2+t^2}{|x|^2}u\vec{x}\right)$$

. This is the conformal energy density. (We shall use below A_1, A_2, A_3 notation)

Application: Local energy decay



A star-shaped domain

$\underline{Definition: \ (Star-shaped)}$

An open set U is called $Star-shaped$ set with respect to O provided for each $x \in \bar{U}$, the line segment $\{\lambda x \mid 0 \le \lambda \le 1\}$ lies in $\bar{U}$. For a graphical representation of such sets, see the figure above.

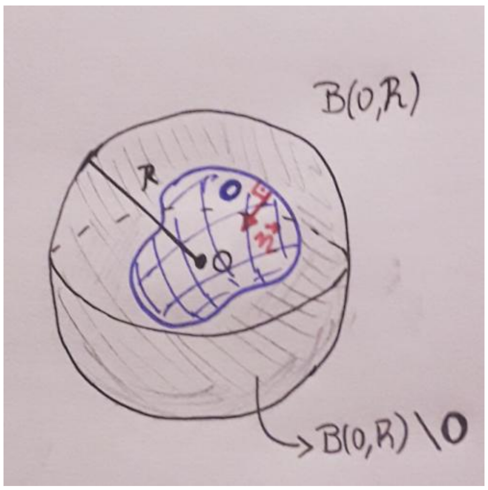
Remark:

Clearly if U is convex and $\text{O} \in U$, then U is star-shaped with respect to O. But a general star-shaped region needn't be convex.

Suppose that $O \subset \mathbb{R}^n$ denotes a bounded, smooth, open subset of $\mathbb{R}^n$ that is star-shaped with respect to the origin. Define the exterior region $U = \mathbb{R}^n \setminus \bar{O}$. Assume in addition that u is a smooth solution of this i.b.v.p. for the wave equation , outside of the "obstacle" O: $(\#)\begin{cases} u_{tt} - \Delta u = 0, \text{ in } U \times (0,+\infty) \\ u = 0, \text{ on } \partial U \times \{t=0\} \\ u = g \ , u_t = h, \text{ on } U \times \{t=0\} \end{cases}$

for which the initial data h, g have compact support. We assert that if $n \ge 3$ and if $O \subset B(0,R)$, there exists a constant c such that $\boxed{\int\limits_{B(0,R)\setminus O} \left\{u_t^2 + |\nabla u|^2\right\} dx \le \frac{c}{t^2}}$ for $t \ge 2R$. Consequently *the energy within any bounded region decays to zero as* $t \to \infty$ *, although the total energy is conserved*. This statement is the local energy decay!

In order to prove that, we first observe from the conservation law $c_t - div\vec{r} = 0$ (Morawetz's identity) that $\frac{d}{dt}\int\limits_U c dx = \int\limits_U c_t dx \overset{Morawetz}{=} \int\limits_{U=\mathbb{R}^n \setminus \bar{O}} div\vec{r} \ dx \overset{Gauss}{=} \int\limits_{\partial U = \partial O} \vec{r} \cdot \vec{n} \ dS$, where $\vec{n}$ denotes the inward unit normal to ∂O.



A figure which demonstrates how to consider the outward unit normal in the interior boundary.

Now (from the compact support of the initial data), $u = u_t = 0$ on ∂O and hence we are able to compute from the representation formula of $\vec{r}$, which is the following (computed earlier above)

$\vec{r} = \left\{ \left(|x|^2 + t^2 \right) \cancel{u_t}^{0} + 2t\left(\vec{x} \cdot \vec{\nabla} u \right) + (n-1) \cancel{tu}^{0} \right\} \vec{\nabla} u + t\left(u_t^2 - |\nabla u|^2 \right) \vec{x}$, these terms vanish on $\partial U \equiv \partial O$, that $\vec{r} \cdot \vec{n} = 2t\left(\vec{x} \cdot \vec{\nabla} u \right)\left(\vec{\nabla} u \cdot \vec{n} \right) - t|\nabla u|^2 \left(\vec{x} \cdot \vec{n} \right)$ along ∂O. Since $u = 0$ on ∂O, we have that $\nabla u = \left(\vec{\nabla} u \cdot \vec{n} \right) \vec{n}$ there. Using this observation in the formula above leads to: $\boxed{\vec{r} \cdot \vec{n} = t|\nabla u|^2 \left(\vec{x} \cdot \vec{n} \right) \le 0}$ since O is a star-shaped set with respect to the origin and $\vec{n}$ is the inward pointing unit normal. From the representation formula for the *conformal energy density* and our relation (seen above) $\frac{d}{dt}\int_U c\,dx = \int_{\partial O} \vec{r} \cdot \vec{n}\,dS = \int_{\partial O} t|\nabla u|^2 \left(\vec{x} \cdot \vec{n} \right) dS \le 0$ we have that: $\int_U c\,dx \le const \Rightarrow$

$$I = \int_{B(0,R)\backslash O} \left\{ \overbrace{\frac{\left(t + |x| \right)^2}{4}\left(u_t + u_r + \frac{n-1}{2|x|} u \right)^2}^{A_1} + \overbrace{\frac{\left(t - |x| \right)^2}{4}\left(u_t - u_r - \frac{n-1}{2|x|} u \right)^2}^{A_2} + \frac{|x|^2 + t^2}{2}\left(|\nabla u|^2 - u_r^2 \right) \right\} dx \le const.$$

for each $t \ge 0$.

Let us now fully justify the implication to the integral I. Therefore:

$\int_{U = \mathbb{R}^n \backslash \bar{O}} c\,dx \le const \Rightarrow \int_U \left(A_1 + A_2 + A_3 \right) dx - \int_U \frac{n-1}{4} div\left(\frac{|x|^2 + t^2}{|x|^2} u\vec{x} \right) dx \le const.$, but we observe that $\int_{U = \mathbb{R}^n \backslash \bar{O}} div\left(\frac{|x|^2 + t^2}{|x|^2} u\vec{x} \right) dx \overset{Gauss}{=} \int_{\partial O} \frac{|x|^2 + t^2}{|x|^2} \cancel{u}^{0} (\vec{x} \cdot \vec{n}) dS = 0$, because $u = 0$ on ∂O, due to the compact support (here $\vec{n}$ denotes the inward normal). Thus we get that:

$$\int_U \overbrace{\left(A_1 + A_2 + A_3 \right)}^{\ge 0} dx \le const.$$

Now because $A_1, A_2, A_3 \ge 0$ we get: $\int_{B(0,R)\backslash O} \left[A_1 + A_2 + \frac{\left(|x|^2 + t^2 \right)\left(|\nabla u|^2 - u_r^2 \right)}{2} \right] dx \le$ (by monotonicity of the integral) $\le \int_{B(0,R)\backslash O} \left[A_1 + A_2 + A_3 \right] dx \le$ (since $A_1 + A_2 + A_3 \ge 0$, we expand the integration's domain) $\int_U \left(A_1 + A_2 + A_3 \right) dx$, because of the fact that $\frac{\left(|x|^2 + t^2 \right)\left(|\nabla u|^2 - u_r^2 \right)}{2} \le A_3$ and $B(0,R) \backslash O \subset U = \mathbb{R}^n \backslash \bar{O}$ and the required implication follows.

Taking now $t \geq 2R$ and making some simple estimates, we derive from this the estimates:

(1) $\boxed{\int\limits_{B(0,R)\backslash O} \left(|\nabla u|^2 - u_r^2\right) dx \leq \frac{\tilde{c}}{t^2}}$, (where here c is the constant and not the conformal energy density)

because $|x| \leq R$ in the region $B(0,R)\backslash O$ and t is taken to be greater than or equal to $2R$. So $t > |x| \Rightarrow$

$\int\limits_{B(0,R)\backslash O} \frac{t^2}{2}\left(|\nabla u|^2 - u_r^2\right) dx \leq I \leq c \;\Rightarrow\; \int\limits_{B(0,R)\backslash O} \left(|\nabla u|^2 - u_r^2\right) dx \leq \frac{\tilde{c}}{t^2}$ (where $\tilde{c} = 2c$) and this due to the fact

that $|\nabla u|^2 - u_r^2 = |\nabla u|^2 - \left(\frac{\partial u}{\partial \vec{n}}\right)^2 = |\nabla u|^2 - \left(\vec{\nabla} u \cdot \vec{n}\right)^2 = |\nabla u|^2 - \left(|\nabla u|\, \cancel{|\vec{n}|}^{1} \cos\vartheta\right)^2 = |\nabla u|^2\left(1 - \cos^2\vartheta\right) \geq 0.$

And we have also: (2) $\boxed{\int\limits_{B(0,R)\backslash O} \left\{u_t^2 + u_r^2 + \frac{n-1}{|x|} u u_r + \frac{(n-1)^2}{2|x|^2} u^2\right\} dx \leq \frac{\tilde{c}}{t^2}}$. (this estimate follows by simple algebraic inequalities regarding the squares of sums).

But now let us note that : $\frac{u u_r}{|x|} = div\left(\frac{u^2 \vec{x}}{2|x|^2}\right) - \frac{(n-2)}{2}\frac{u^2}{|x|^2}$ because of the fact that :

$div\left(\frac{u^2\vec{x}}{2|x|^2}\right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{u^2 x_i}{2|x|^2}\right] = \sum_{i=1}^{n} \frac{u u_{x_i} x_i}{|x|^2} + \sum_{i=1}^{n} \frac{u^2}{2|x|^2} - \sum_{i=1}^{n} \frac{u^2 x_i^2}{|x|^4} = \frac{u}{|x|^2}\left(\vec{\nabla} u \cdot \vec{x}\right) + \frac{u^2}{2|x|^2} n - \frac{u^2}{|x|^2}$ and

$u_r = \frac{\vec{\nabla} u \cdot \vec{x}}{|x|}$. As a consequence, we notice: (here we have chosen to denote $\Omega = B(0,R)\backslash O$)

$\int\limits_{\Omega} (n-1) div\left(\frac{u^2\vec{x}}{2|x|^2}\right) dx = \int\limits_{\partial\Omega = \partial O \cap \partial B(0,R)} (n-1)\frac{u^2}{2|x|^2}\left(\vec{x}\cdot\vec{n}\right) dS = \int\limits_{\partial O} (n-1)\cancel{\frac{u^2}{2|x|^2}}^{0}\left(\vec{x}\cdot\vec{n}_1\right) dS +$

$+ \int\limits_{\partial B(0,R)} (n-1)\frac{u^2}{2|x|^2}\left(\vec{x}\cdot\vec{n}_2\right) dS \geq 0$, the first surface integral vanishes since $u = 0$ on ∂O and moreover the second surface integral is non-negative since $\vec{x} \parallel \vec{n}_2 \;\Rightarrow\; \cos\left(\widehat{\vec{x},\vec{n}_2}\right) \geq 0 \;\Rightarrow\; \vec{x}\cdot\vec{n}_2 \geq 0$ (see figure)

Therefore we are able to conclude that:

$\int\limits_{\Omega} \frac{n-1}{|x|} u u_r dx = \int\limits_{\Omega} (n-1) div\left(\frac{u^2\vec{x}}{2|x|^2}\right) dx - \frac{(n-1)(n-2)}{2}\int\limits_{\Omega} \frac{u^2}{|x|^2} dx =$(from our computation with surface integrals above)$= \underbrace{\int\limits_{\partial B(0,R)} (n-1)\frac{u^2}{2|x|^2}\left(\vec{x}\cdot\vec{n}_2\right) dS}_{\geq 0} - \frac{(n-1)(n-2)}{2}\int\limits_{B(0,R)\backslash O} \frac{u^2}{|x|^2} dx$. As a consequence:

$$\int\limits_{\Omega}\left\{\frac{n-1}{|x|} u u_r + \frac{(n-1)^2}{2|x|^2}u^2\right\} dx = \underbrace{\int\limits_{\partial B(0,R)} (n-1)\frac{u^2}{2|x|^2}\left(\vec{x}\cdot\vec{n}_2\right) dS}_{\geq 0} + \int\limits_{\Omega} \frac{u^2}{2|x|^2}\left\{\overbrace{(n-1)^2 - (n-1)(n-2)}^{=(n-1)(\cancel{n}-1-\cancel{n}+2)=n-1\geq 0}\right\} dx \geq 0$$

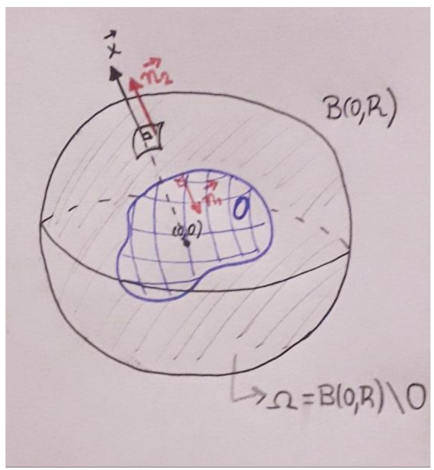
As a conclusion we have that

$$\int_\Omega \left\{u_t^2 + u_r^2\right\} dx \le \int_\Omega \left\{ u_t^2 + u_r^2 + \overbrace{\frac{n-1}{|x|} u u_r + \frac{(n-1)^2}{2|x|^2} u^2}^{\ge 0} \right\} \le \frac{\tilde{c}}{t^2}.$$

This result in combination with the first estimate (estimate (1)) leads to:

$$\left\langle \begin{array}{l} \displaystyle\int_\Omega \left\{u_t^2 + \cancel{u_r^2}\right\} dx \le \frac{\tilde{c}}{t^2} \\ \displaystyle\int_\Omega \left\{|\nabla u|^2 - \cancel{u_r^2}\right\} dx \le \frac{\tilde{c}}{t^2} \end{array} \right. \overset{(+)}{\Rightarrow} \int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx \le \frac{2\tilde{c}}{t^2} \text{ for } t \ge 2R$$

,which is the energy decay estimate. □



In this figure we demonstrate how to consider the inward and outward normals we used in the computations of the surface integrals above.

Fin

Bibliography

Books:

1. Brezis H., Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011
2. Caratheodory C., The calculus of Variations and Partial Differential Equations of first order, Chelsea, 1982
3. Ciarlet P., The Finite Element Method for Elliptic Problems, Classics in Applied Mathematics 40, SIAM, 2^{nd} edition, 2002
4. Courant R. and Hilbert D., *Methods of Mathematical Physics,* Vols. I,II, New York, NY: John Wiley & Sons, 1996.
5. Evans L.C., Partial Differential Equations, 2^{nd} edition, Graduate Studies in Mathematics Vol.19, AMS, 2010
6. Friedman A., Variational Principles and Free Boundary Problems, Wiley-Interscience, 1982
7. Gelfand, I. M. & Fomin, S. V. 1963. *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, NJ (reprinted by Dover Publications, Inc., 2000).
8. Giaquinta M. and Hildebrandt S., Calculus of Variations, Vol. I-II, Springer, 1996
9. Giaquinta M., Hildebrandt S. and Buttazzo M., An Introduction to One Dimensional Variational Problems, Oxford Lecture Series in Mathematics and Its Applications 15, 1998
10. Giaquinta M., Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton University Press, 1983
11. Gilbarg D. and Trudinger N., Elliptic Partial Differential Equations of Second Order (2^{nd} edition), Springer, 1983
12. Kinderlehrer D. and Stampacchia G., An Introduction to Variational Inequalities and Their Applications, Academic Press, 1980
13. Kot M., A First Course in the Calculus of Variations, Student Mathematical Library Vol. 72, AMS, 2014
14. Logan, J. D. 1977. *Invariant Variational Principles*, Academic Press, New York
15. Logan, J. D. 2013. *Applied Mathematics 4^{th} edition* , John Wiley & Sons, New Jersey
16. Marsden J. and Tromba A., Internet Supplement for Vector Calculus, 5^{th} edition, version October 2003, W. H. Freeman and Company, New York
17. Marsden J. and Tromba A., Vector Calculus, 6^{th} edition, W. H. Freeman and Company, New York, 2012
18. Moser J., Selected Chapters in the Calculus of Variations, Lecture Notes by Oliver Knill, ETH Zurich, Birkhauser, 2003
19. Neuenschwander, D. E. 2010. *Emmy Noether's Wonderful Theorem*, Johns Hopkins University Press.
20. Pinchover Y. and Rubinstein J., An Introduction to Partial Differential Equations, Cambridge University Press, 2005

21. Rabinowitz P., Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conference Series in Mathematics 65, AMS, 1986
22. Rund, H. 1966. *The Hamilton-Jacobi Theory in the Calculus of Variations*, Van Nostrand, London.
23. Sagan H., An Introduction to the Calculus of Variations, Dover Publications Inc, New York, 1992
24. Troutman J.L., *Variational Calculus and Optimal Control,* second edition. Undergraduate Texts in Mathematics. New York, NY: Springer-Verlag, 1996.
25. Zeidler E., Nonlinear Functional Analysis and Its Applications, Vol. III, Variational Methods and Optimization, Springer-Verlag New York Inc. ,1985

Lecture notes:

1. Arthurs A., Calculus of Variations
2. Danielli D. and Garofalo N., Calculus of Variations, Properties of entire solutions of non-uniformly elliptic equations arising in geometry and in phase transitions, 2002
3. Dougalis V., Finite Elements, Graduate notes, University of Athens, 2013
4. Emery J., The Calculus of Variations, University of Pennsylvania, 1999
5. Figueroa J. and O'Farrill, Brief Notes on the Calculus of Variations
6. Fonseca I. and Leoni G., Calculus of Variations, Carnegie Mellon University, 2018
7. Miersemann E., Calculus of Variations, Department of Mathematics Leipzig University, October 2012
8. Olver P., Introduction to the Calculus of Variations, University of Minnesota, 2019
9. Rindler F., Introduction to the Modern Calculus of Variations, MA4G6 Lecture Notes, University of Warwick, Spring Term 2015
10. Strang G., Calculus of Variations, MIT, 2006