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Anyon model for the alternating group A_4

Master's Thesis

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Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών

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Διπλωματική Εργασία

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Διπλωματική Εργασία

Η εργασία αυτή εκπονήθηκε στα πλαίσια του μεταπτυχιακού προγράμματος ειδίκευσης ¨Φυσική¨.

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Abstract

In this master thesis the Quantum Double anyon model for the Alternating group A_4 is constructed. In the first part, we present an overview of the statistical properties of anyons, both abelian and non-abelian, through the Bohm-Aharonov effect as well as their use in quantum computation. In particular, we discuss the Braiding and Fusion rules for anyons as they are expressed through the R-matrix, the F-matrix and the fusion bases.

In the second part, we focus on the alternating group A_4 and by determining the superselection sectors through the conjugacy classes and their centralizers, we present the representations of each quantum double $D(A_4)$. We next compute the modular S-matrix and using the Verlinde formula we construct the fusion rules of the model. Also, we determine the modular T-matrix as well as the charge conjugation operator of the theory. Due to the high complexity of the model, we restrict the discussion to a small superselection sector whose fusion rules are closed and we calculate explicitly the R and F matrices. Finally, we show that the algebra $D(A_4)$ does not contain the necessary quantum gates for universal quantum computation.

Subject Area: Topological Quantum Computation
Keywords: Anyons, Anyon model, Quantum Double, Fusion Rules, Generalized Pentagon equation, Generalized Hexagon equation

Περίληψη

Στην διπλωματική εργασία αυτή, το μοντέλο ανυονίων για την ομάδα A_4 κατασκευάζεται. Στο πρώτο κομμάτι, πραγματοποιούμε μια ανασκόπηση των στατιστικών ιδιοτήτων των ανυονίων, αβελιανά και μη-αβελιανά, μέσω του φαινομένου $Bohm - Aharonov$ καθώς και την χρησιμότητά τους στους κβαντικούς υπολογιστές. Συγκεκριμένα, πραγματευόμαστε τους κανόνες πλεξίματος και συγχώνευσης των ανυονίων, όπως εκφράζονται μέσω των πινάκων R, F και τις συμβατικές βάσεις.

Στο δεύτερο κομμάτι, επικεντρωνόμαστε στην ομάδα A_4 και παραθέτουμε τις αναπαραστάσεις της κάθε κβαντικής διπλέτας $D(A_4)$ καθορίζοντας τους τομείς υπερεπιλογής μέσω των κλάσεων συζυγίας και των κεντροποιών τους ομάδων. Έπειτα, καθορίζουμε τον πίνακα S και μέσω της σχέσης $Verlinde$ κατασκευάζουμε τους κανόνες σύντηξης του μοντέλου. Επιπλέον, καθορίζουμε τον πίνακα T καθώς και τον τελεστή μετάθεσης φορτίου της θεωρίας. Λόγω της υψηλής πολυπλοκότητας του μοντέλου, περιορίζουμε την συζήτηση σε έναν μικρό τομέα υπερεπιλογής του οποίου οι κανόνες σύντηξης είναι κλειστοί και υπολογίζουμε ρητά τους πίνακες R και F. Τέλος, δείχνουμε οτι η άλγεβρα $D(A_4)$ δεν εμπεριέχει τις αναγκαίες κβαντικές πύλες για καθολική κβαντική υπολογιστική.

Θεματική Περιοχή: Τοπολογικοί Κβαντικοί Υπολογιστές
Λέξεις Κλειδιά: Ανυόνια, Ανυονικό Μοντέλο, Κβαντική Διπλέτα, Κανόνες Σύντηξης, Γενικευμένη εξίσωση Πενταγώνου, Γενικευμένη εξίσωση Εξαγώνου

Contents

1 Introduction

Since 1982 when Feynman first introduced the idea of a Quantum Computer [1], there have been many different attempts to realize Quantum Computers physically using various quantum systems, such as photons, which are best suited for transmitting data, or weakly interacting spins, that could serve as long lasting quantum memories [2]. Another approach that has recently shown up due to its robustness against decoherence is that of Topological Quantum Computation (TQC). TQC makes use of recently discovered anyonic excitations in materials with topological properties [3]. The anyonic [4] excitations are described by topological models which are fully determined by the fusion and braiding rules which correspond to the F and R matrices.

In sections **2** and **3** we give a short overview of the algebraic equations satisfied by the F and R matrices and we present a brief outline of their use in TQC in section **4**. In section **5** we work out in detail the quantum double algebra by determining the superselection sectors, the braiding and the fusion rules for the group A_4. Finally, in section **6** we calculate the R and F matrices for a specific small superselection sector of the model and we show that this sector is not enough to enable *Universal Quantum Computation*. It is known that the smallest group for non-abelian anyons enabling universal *classical* computation is A_5 [5].

2 Anyons

Suppose we have a pair of identical particles on the plane. By interchanging them once, in a counter-clockwise manner, the state of the system is multiplied by a phase factor $e^{i\theta}$ (often called exchange phase) as interchanging identical particles should not have any measurable difference on our state. Interchanging them once more returns us to the starting configuration. This double interchange is equivalent to a counter-clockwise rotation of one particle around the other and the resulting phase does not depend on the particular path of the rotation as long as there is no other particle around. If we now had access to the third spatial dimension we could make use of it and deform our path "upwards" or "downwards" so that it passes "above" or "below" the other particle and by doing so in a continuous fashion the result of the rotation should remain the shame. Now it is a simple matter to continuously deform that path to the trivial path that always leaves the initial particle where it is. As such in three dimensions, the state of the system $\Psi(\gamma_1)$ after a rotation of one particle on a path γ_1 around the other needs to be equal to the state $\Psi(\gamma_2)$ corresponding to the trivial path γ_2:

$$\Psi(\gamma_1) = \Psi(\gamma_2) \tag{1}$$

Given that a rotation is equivalent to two consecutive interchanges, the square of the phase factor has to equal 1 in order for (1) to hold

$$e^{2i\theta} = 1$$

which leads to $\theta = 0$ and $\theta = \pi$ that correspond to bosons and fermions respectively.

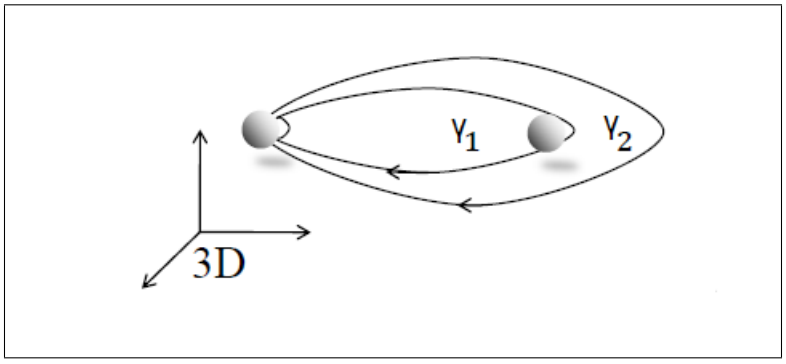


Figure 1: Exchanging a pair of particles in three dimensions.

If we don't have access to the third spatial dimension, to "bend" our path through, we cannot continuously deform a path γ_1, that revolves around the second particle, to the trivial path γ_2. As a result, if we are locked in two dimensions, (1) cannot be extrapolated. As a result, since the square of the phase factor $e^{i\theta}$ needs not equal 1, the variable θ does not have to equal 0 or π. Such two-dimensional particles that when interchanged do not multiply the state of the system by a simple $+1$ or -1 are called **Anyons**.

2.1 Abelian, non-abelian

Given a system of N identical anyons, interchanging pairs of them could potentially be more complicated if some states were degenerate. Suppose our system has $m \leq N$ degenerate states $|i\rangle\,,\ i = 0, 1, \ldots, m-1$ under a measurement A, meaning $\hat{A}\,|i\rangle = a\,|i\rangle \quad \forall\ i = 0, 1, \ldots, m-1$. Now interchanging a pair of anyons need not transform a state $|i\rangle$ into $e^{i\theta}\,|i\rangle$ but it could in theory give a superposition of all m degenerate states

$$|i'\rangle = \sum_{i=0}^{m-1} c_i\,|i\rangle$$

since we cannot differentiate between them by measuring A. In this case the state of the system transforms into $U\,|i\rangle$ where U is a *unitary matrix* instead of $e^{i\theta}\,|i\rangle\,, \quad e^{i\theta} \neq -1, 1$. Anyons that when interchanged give rise to simple complex phases are called **abelian** as consecutive interchanges of different anyons commute, since complex phases are numbers, whereas anyons that give rise to unitary matrices are called **non-abelian** since matrices in general do not commute.

2.2 Use of anyons in quantum computing

One of the main hurdles encountered in the realisation of Quantum Computers are errors accrued by the decoherence of our system, through its inevitable interaction with the environment. We can encode information non-locally in a system of anyons that can only be accessed when they are brought close together providing a decoherence resistant system as long as the anyons are kept far apart. Furthermore, those anyons can be used to efficiently simulate a quantum circuit allowing for fault-tolerant quantum computation. Although abelian anyons are easier to realise in practise, it is the non-abelian anyons that, given their greater computational power, are the most promising in constructing a Quantum Computer, were they to be realised experimentally [5].

2.3 Anyons through the Bohm-Aharonov effect

Consider the two dimensional plane and a magnetic field $\vec{B}$ confined in an area V perpendicular to the plane. It is known that if a charged particle moving on the plane, circulates the area V (counter-clockwise), ending up at the starting position, the state of the system acquires a phase

$$e^{iq\Phi} \tag{2}$$

where q is the charge of the particle and Φ is the magnetic flux passing through the area circulated by the particle. This is called the Aharonov-Bohm effect. It is important to note that the phase is not dependant on the exact path the particle follows around the area V but only on the number of rotations it performs, also called the winding number.

Let us examine a system of two composite particles each constituted of a charged particle and a magnetic flux. By interchanging these particles each charge is circulated half a turn around the other particle's flux altering the state of the system by the phase

$$e^{i(\frac{1}{2}q\Phi+\frac{1}{2}q\Phi)} = e^{iq\Phi}. \tag{3}$$

For $\theta \equiv q\Phi$ this phase matches the phase given when two anyons interchange.

Hence when needed anyons will be considered as composite particles made of a charge and a flux while long-range interaction between anyons will be limited only to the Aharonov-Bohm effect. Furthermore, we assume that the flux of all anyons in a model take values in a finite group G and the charges are unitary irreducible representations of said group.

It is possible to define an arbitrary orthonormal basis for the vector space that the irreducible representations act upon. It is important to note though, that the basis needs to be defined at a certain point since moving by parallel transport from one point to another will rotate the basis due to the Aharanov-Bohm effect. We can label a flux with a group element $g \in G$ by transporting a charge labeled R, with R being a unitary irreducible representation of G, around the flux g following a certain path and noting the action $U(g)$ that acts

on the basis of R. Since the action on the basis given a certain charge and flux is dependent on the path, we can label the path g as we did the flux.

3 Combining anyons

Anyons interact in three main ways

- by interchanging two anyons, or equivalently circulating one around the other, which is called **Braiding** because it is described by the Braid Groups instead of the Permutation Groups that describe exchange of fermions and bosons
- by combining two anyons to form a third, called **Fusion**, which can be written as
$$a \times b = \sum_c N_{ab}^c c \tag{4}$$
where N_{ab}^c called multiplicity is a non-negative integer that signifies the possible distinguishable ways an anyon with charge c can be obtained by two anyons with charge a and b.
- by splitting an anyon into two, called **Scattering**, which is the reverse process of fusing two anyons [6].

3.1 Braiding a pair of anyons

Consider two fluxes, prematurely labeled g_1 and g_2, and a standard basis for the charges at a starting point O as shown in the figure 2. By moving a charged

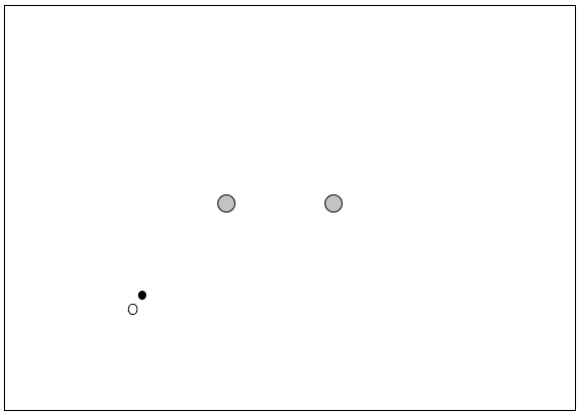


Figure 2: A pair of fluxes with labels g_1 and g_2 on the plane

particle by parallel transport counter-clockwise around the flux g_1 the basis is acted upon by $U(g_1)$, where U is a representation of the group $G \ni g_1$ where the fluxes' labels take values in, leading to a label of g_1 for both the flux it circulated

and the path it followed. Similarly, the second flux and the counter-clockwise path around it are labeled g_2. A path $g_1 g_2 g_1^{-1}$ can be constructed by the paths g_1, g_2 and g_1^{-1}, which is a clockwise rotation around g_1, where g_1^{-1} is traversed first then g_2 then g_1 as shown in figure 3. If the anyons are exchanged counter-clockwise while continuously deforming the paths in such a way as for them to never overlap, the path $g_1 g_2 g_1^{-1}$ is altered to the path g_1 and the path g_1 is altered to the path g_2. As such the action of a counter-clockwise interchange of a pair of anyons, represented by the braid operator R, on the fluxes is

$$R : (g_1, g_2) \to (g_1 g_2 g_1^{-1}, g_1) \tag{5}$$

which is also known as *flux metamorphosis*.

Clearly, if the group G were abelian braiding a pair of anyons would simply exchange the position of the two labels. More importantly though, if the arbitrary basis was chosen such that it had been transported around the flux g_1, when attempting to label the flux g_2 by transporting charges around it the effect on the new basis would not be $U(g_2)$ but rather $U(g_1 g_2 g_1^{-1})$. Consequently, although the labels assigned on a certain flux depend on the arbitrary selection of the basis they all belong on the same conjugacy class of the group. As such, the conjugacy class for a flux is invariant and all observers will note the same, which leads to anyons with equal charge whose flux belongs to the same conjugacy class to be considered as indistinguishable particles. Again, if the group G were abelian, different bases would not correspond to differences in the fluxes since each conjugacy class consists of only one element.

It is worth mentioning that the inverse of the braiding operator R^{-1}, which corresponds to a clockwise interchange of anyons, acts as

$$R^{-1} : (g_1, g_2) \to (g_2, g_2^{-1} g_1 g_2). \tag{6}$$

Two successive counter-clockwise exchanges, which correspond to the so called monodromy operator R^2, affect the fluxes as:

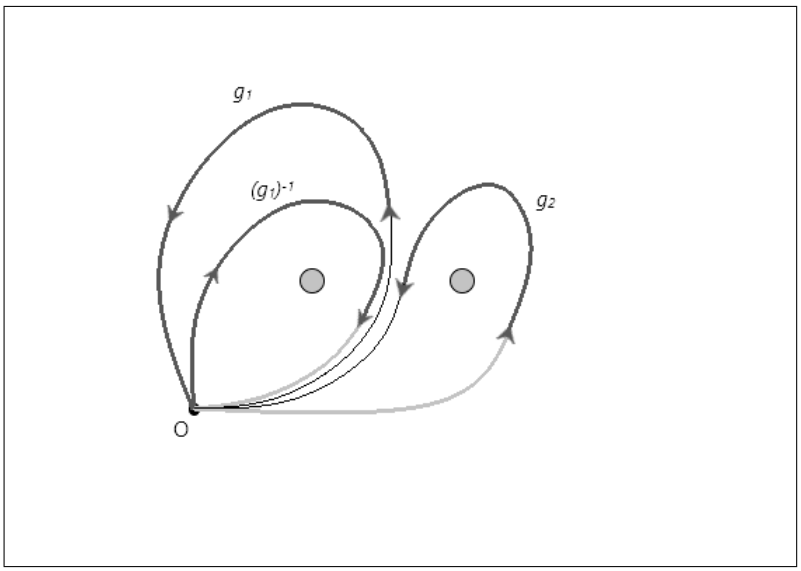
$$R^2 : (g_1, g_2) \to [(g_1 g_2) g_1 (g_1 g_2)^{-1}, (g_1 g_2) g_2 (g_1 g_2)^{-1}] \tag{7}$$

Suppose the fluxes of the pair of anyons are g_1 and g_1^{-1}. This case is special because although the total flux of the pair is zero the total charge need not be. We can measure that charge by circulating a third flux g_2 around the pair. If the action $U(g_2) = I$ for all $g_2 \in G$ then the charge of the pair equals zero. Carrying the flux g_2 around the pair though, alters the fluxes as

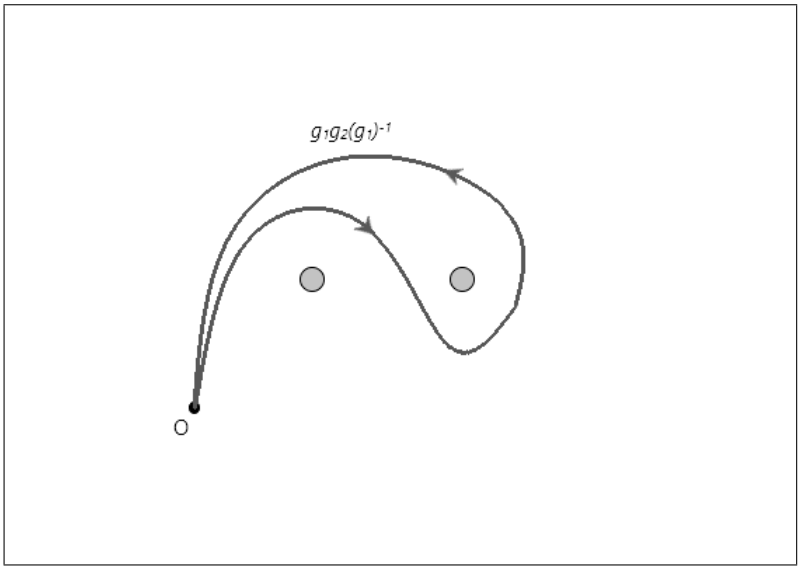
$$(g_1, g_1^{-1}) \to (g_2 g_1 g_2^{-1}, g_2 g_1^{-1} g_2^{-1}) \tag{8}$$

since a rotation of one anyon around another corresponds to two successive interchanges, which is the monodromy operator. This action is generally non-trivial for a non-abelian group G. Interestingly, this new state also has zero total flux without necessarily having zero charge. In order for a state $|0, I\rangle$ with trivial charge to be constructed, a uniform superposition of all class elements is needed

$$|0, I\rangle = \frac{1}{\sqrt{|C_g|}} \sum_{g \in C_g} |g g^{-1}, \Gamma\rangle \tag{9}$$



(a) Paths g_1^{-1}, g_2 and g_1



(b) Path $g_1 g_2 g_1^{-1}$

Figure 3: (b) By combining the paths g_1^{-1}, g_2 and g_1 in that order we construct the path $g_1 g_2 g_1^{-1}$. (a) Note that the pale parts of the paths g_1^{-1} and g_2 cancel each other out since the rotation inflicted on the basis by parallel transport along the first is undone along the second. Similarly, the effect on the basis along the thin parts of g_2 and g_1 cancel each other out.

where C_g is the conjugacy class the element g belongs to and Γ is a representation of the group G. In order for a pair of anyons with zero flux to annihilate when brought together they need to be in the aforementioned state in order for charge conservation to be observed.

3.1.1 Superselection Sectors

Consider an anyon with a non-trivial flux $\beta \in G$, where G is a non-abelian Group. In a sector with non-trivial flux, global transformations can only be implemented if they belong to the centralizer $N(\beta)$ of the group, defined as

$$N(\beta) = \{g \in G | \beta g = g\beta\} \tag{10}$$

and as such the charge is an *irreducible representation* of the *centralizer* of the group. Furthermore, the centralizers of elements belonging to the same conjugacy class are isomorphic to each other, meaning that an anyons flux is described by a conjugacy class C_g of G and its charge is described by the α-th representation Γ of the centralizer $N(g)$

$$|C_g, {}^{\alpha}\Gamma\rangle$$

which also labels the *superselection sectors* of the theory as these need to be conserved in any physical process.

3.2 Fusion/Scattering of anyons

Another property of anyons that needs addressing is what happens when two or more anyons are combined, also known as fusion, which is to say what happens to their charges and fluxes. Scattering of anyons is simply the inverse process where an anyon splits into more. The flux of the anyon after a fusion can belong to any of the conjugacy classes obtained by the product of the fluxes of the constituents. The possible values of the charge are given by the decomposition of the product of the representations of the centralizers of the components as a direct sum of representations

$${}^{\alpha}\Gamma_i \otimes {}^{\beta}\Gamma_j = \bigoplus_{\gamma} {}^{\gamma}\Gamma_k \tag{11}$$

where ${}^{\alpha}\Gamma_i$ signifies the α representation of the centralizer of the group element h that labels the flux of the i anyon.

By labelling each sector and its conserved charge with $i \in \{a, b, \dots\}$ fusion of anyons can more generally be described by the *fusion rules* that describe the possible distinguishable ways two anyons with charges a and b respectively can combine to give an anyon with charge c

$$a \times b = \sum_c N_{ab}^c c \tag{12}$$

where N_{ab}^c is a non-negative integer called multiplicity. Since combining an anyon with charge a with an anyon with charge b is the same as combining an anyon with charge b with an anyon with charge a in both equations (11) and (12) the products are commutative. If $N_{ab}^c = 0$ then an anyon with charge c cannot be obtained when combining anyons with charge a and b, if $N_{ab}^c = 1$ then the charge c can be obtained in a unique way and if $N_{ab}^c \geq 2$ then the charge c can be obtained in N_{ab}^c distinguishable ways. For abelian anyons $\sum_c N_{ab}^c$ is always equal to one whereas for non-abelian anyons $\sum_c N_{ab}^c \geq 2$.

The N_{ab}^c distinguishable ways c can arise from the fusion of a and b, can be considered a basis of an N_{ab}^c-th dimensional Hilbert space, called fusion space. The basis elements can be symbolized as:

$$\{|a, b \to c; \mu\rangle, \mu = 1, 2, \ldots, N_{ab}^c\} \tag{13}$$

For an anyon $|i\rangle \equiv |C_g, {}^{\alpha}\Gamma\rangle$ we can define the *quantum dimension* $d_i \equiv |C_g||{}^{\alpha}\Gamma|$.

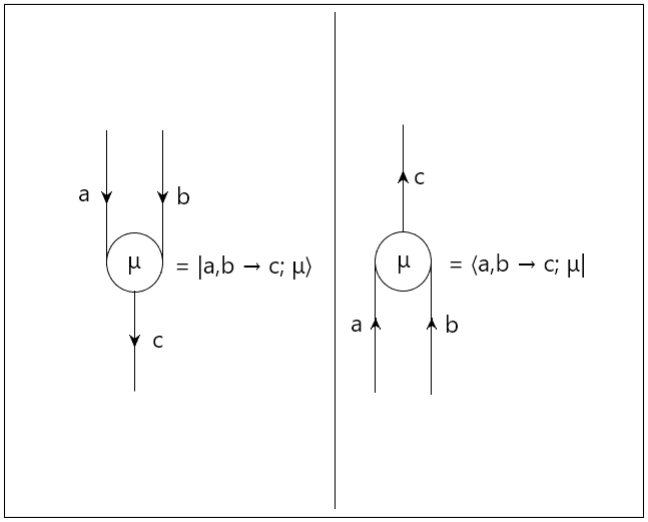


Figure 4: The fusion basis of $\mathcal{M}_{ab}^c$ diagrammatically.

Abelian anyons always have a quantum dimension of one whereas non-abelian anyons have $d_i > 1$, not necessarily integer. For anyons with fusion rules (12), the quantum dimensions satisfy:

$$d_a d_b = \sum_c N_{ab}^c d_c \tag{14}$$

Finally, a quantum dimension of an anyon model can be defined as

$$D \equiv \sqrt{\sum_i d_i^2} \tag{15}$$

where the sum runs over all anyon types in a model.

3.3 R-matrix, F-matrix, n anyon basis

Although, the braiding and fusion processes are described by the R and F operators accordingly, during calculations they are usually represented by matrices, namely the R-matrix and F-matrix accordingly, that act on a fusion state.

3.3.1 R-matrix

Given the state $|b, a \to c; \mu\rangle$ in the Hilbert space $\mathcal{M}_{ba}^{c}$ and the canonical basis $\{|a, b \to c; \mu'\rangle, \mu' = 1, 2, \ldots, N_{ba}^{c}\}$ of $\mathcal{M}_{ab}^{c}$, the state $|b, a \to c; \nu\rangle$ in the Hilbert space $\mathcal{M}_{ab}^{c}$ can be attained by braiding a and b, which corresponds to a counterclockwise exchange of the two anyons. The braid operator can in this case be described by the unitary matrix

$$R : |b, a \to c; \mu\rangle \to \sum_{\mu'} (R_{ab}^{c})_{\mu}^{\mu'} \, |a, b \to c; \mu'\rangle \tag{16}$$

where the sum runs over all the basis states $\{|a, b \to c; \mu'\rangle, \mu' = 1, 2, \ldots, N_{ba}^{c}\}$.

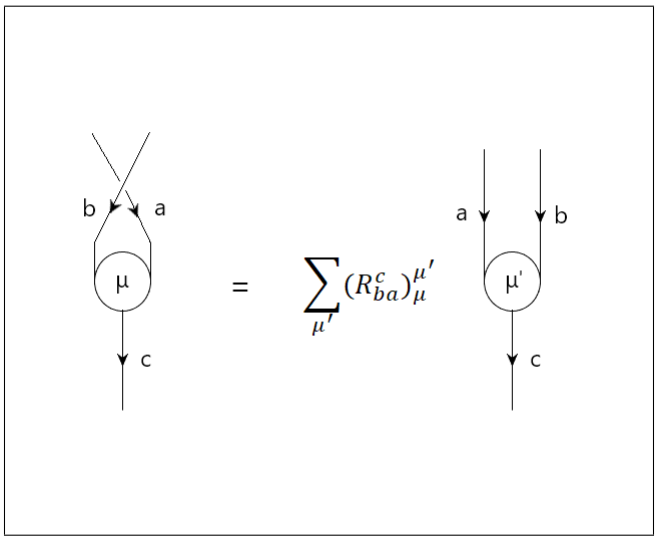


Figure 5: The R-matrix diagrammatically.

The braid operator obeys the same rules as the generators τ_i of the braid group B_n

$$\tau_i \tau_j = \tau_j \tau_i \ , \quad |i - j| \geq 2 \tag{17}$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \ , \quad i = 1, 2, \ldots, n - 2 \tag{18}$$

where (17) states that braiding of strands commute as long as different strands are involved and (18) is called the *Yang-Baxter* equation.

3.3.2 F-matrix

Suppose we have a system of three anyons with general labels a, b and c that fuse giving an anyon d

$$a \times b \times c = d. \tag{19}$$

Fusion of anyons, as well as scattering, are always considered pair processes, meaning that exactly two anyons combine to give a third in a single fusion and an anyon will always split into two in a single scattering. As such, a fusion of three anyons is simply a fusion of two of them followed by a fusion of the resulting anyon with the third. Similarly, a scattering of an anyon into more than two products is achieved by successive scatterings of the products of each previous scattering.

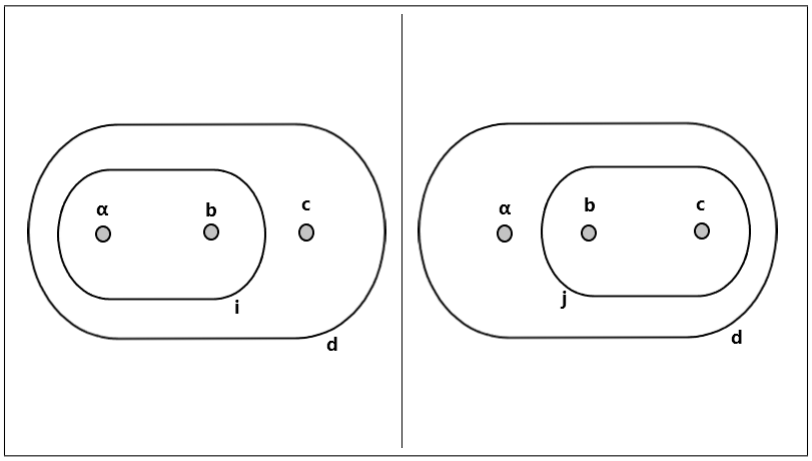


Figure 6: Another representation of the two bases shown in figure 7, ignoring multiplicities [7]. It is obvious that fusing two anyons, that is labeling them with a single label that signifies their total charge, need not require them to be brought close together only that they need to be aligned side by side along the line.

A fusion of anyons corresponds to a labeling of that subsystem with a single label that signifies their total charge; no interaction needs to occur between those anyons, the only requirement is for the subsystem to be able to be isolated from the rest of the system, as can be clearly seen in figure 6. Consequently, fusing the anyons a and b to a product i and that with c or b and c to a product j and that with a, will both yield the same final charge d, since the total charge of a system is independent of our unintrusive selection of fusion order

$$(a \times b) \times c = a \times (b \times c). \tag{20}$$

The selection of fusion order corresponds to a selection of a basis in the fusion

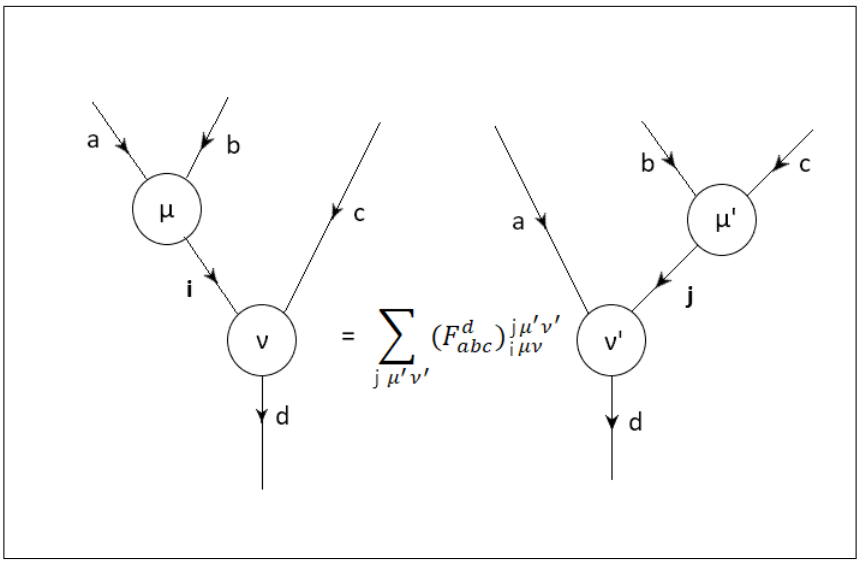


Figure 7: The F-matrix diagrammatically.

space:

$$|(ab), c \to i, c \to d; \mu, \nu\rangle \equiv |(ab), c \to d; i, \mu, \nu\rangle \equiv |a, b \to i; \mu\rangle \otimes |i, c \to d; \nu\rangle$$
$$|a, (bc) \to j, c \to d; \mu', \nu'\rangle \equiv |a, (bc) \to d; j, \mu', \nu'\rangle \equiv |a, b \to j; \mu'\rangle \otimes |j, c \to d; \nu'\rangle$$

The two bases are related by the unitary matrix F

$$|(ab), c \to d; i, \mu, \nu\rangle = \sum_{j,\mu',\nu'} (F^d_{abc})^{i,\mu,\nu}_{j,\mu',\nu'} \, |a, (bc) \to d; j, \mu', \nu'\rangle \tag{21}$$

which is simply called the *F-matrix*.

3.3.3 Standard basis

During the actual implementation of a topological quantum computer a sizeable number of anyons need to be controlled, making the generalization of anyon combination processes necessary for any number N of anyons. From now on, all anyons of a system will be considered to have been arranged in a line.

Consider a system of N anyons carrying labels $a_1, a_2, \ldots, a_N$ where the two first anyons fuse giving a label i_1, then i_1 is fused with a_3 giving i_2 and so on until i_{N-2} is fused with the last anyon a_N giving a total charge b. This selection of fusion order corresponds to a fusion basis, also called the *standard basis*,

$$\{|a_1, a_2; i_1, \mu_1\rangle \otimes |i_1, a_3; i_2, \mu_2\rangle \otimes \ldots \otimes |i_{N-2}, a_N; b, \mu_{N-1}\rangle\} \tag{22}$$

of the topological Hilbert space $\mathcal{M}^b_{a_1,a_2,\ldots,a_N}$, with $i_1, i_2, \ldots i_{N-2}$ symbolising the intermediate labels and $\{\mu_j\}$ being the basis of the fusion space $\mathcal{M}^{i_j}_{i_{j-1},a_{j+1}}$.

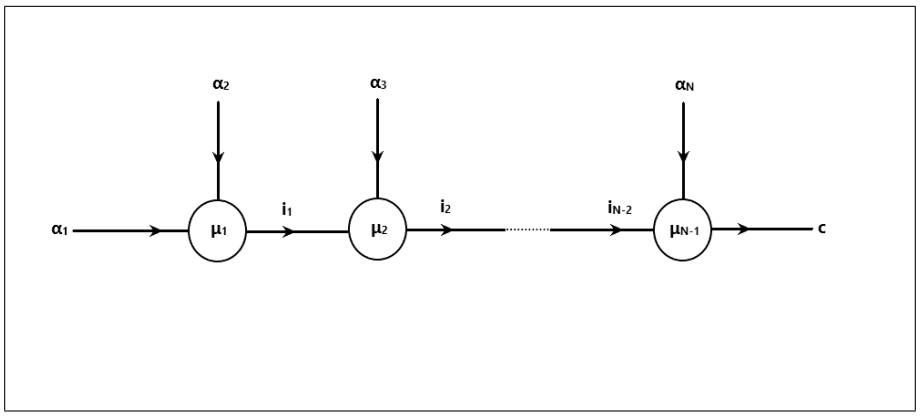


Figure 8: The standard basis diagrammatically. The N anyons to be fused to total charge c are labeled with labels a_j, the intermediate charges are labeled with i_j and the multiplicities are labeled with μ_j.

In a system with more than two anyons, braiding by use of the operator R, can only be accomplished between anyons that can be isolated into a subsystem with a certain total charge, which for a state in the standard basis, corresponds to them being in the same fusion channel. In order to braid two anyons of a system, that belongs to the standard basis, when they are not in the same fusion channel, enough F moves (Fusion operator acting on the state) need to be applied on the state to bring the two anyons under the same channel, then an R move (Braiding operator acting on the state) to perform the actual braid and finally the inverse of the F moves (by applying the F^{-1}-matrices on the state) to return to the standard basis state. This operation on the standard basis is performed by the so called *B-matrix*, which is composed by the needed F-matrices followed by an R-matrix followed by the inverse F^{-1}-matrices.

The B-matrices that braid pairs of anyons of a system of n anyons form representation of the *Braid group* on n strands. In the special case where the braiding operation is performed on the first pair of anyons the B-matrix is simply the R-matrix for that pair.

3.3.4 Pentagon and Hexagon equations

For a set of fusion rules with fusion coefficients $N_{ab}^{c} \leq 1$, there are two consistency conditions that need to be observed. Firstly, if two different fusion states of a system can be connected through different F moves, then those sets of moves need to correspond to the same process. By applying this condition on four anyons with the labels 1,2,3,4 and total charge 5, as seen in figure 9, it gives rise to the so called *Pentagon equation*, which can be used to calculate the F-matrices.

Secondly, if two different fusion states of a system can be connected through different F and R moves, then again those sets of moves need to correspond to

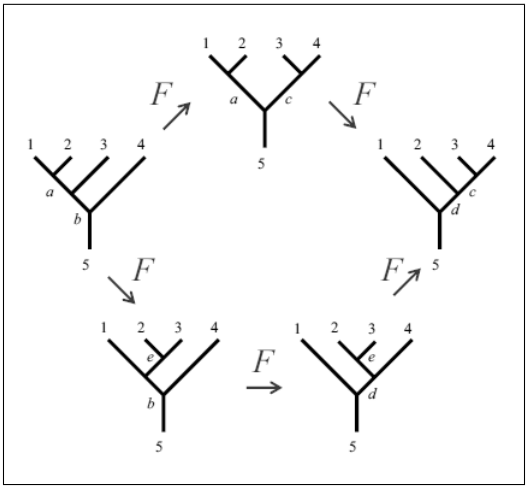


Figure 9: The Pentagon equation diagrammatically [5]. The label 5 corresponds to some set total charge of the system whereas the labels a,b,c,d and e take all possible allowed values.

the same process. The resulting equation when the aforementioned condition is imposed on three anyons, labeled 1,2,3 and total charge 4, as seen in figure 10, is called the *Hexagon equation*, which gives a way of calculating the R-matrices given the F-matrices.

A specific anyon model can finally be defined, in an unambiguous way, given a certain set of *fusion rules* and the corresponding *F-matrices* and *R-matrices*.

4 Quantum computation with anyons

Through the operations of Fusion/Scattering and Braiding, a *topological quantum computation* with anyons is performed in three general phases:

- *Initialization*: A system of anyons in a known fusion state is constructed. Perhaps the simplest way to do so is by scattering the vacuum into pairs of anyons anti-anyons. Since the topological charge of each pair is zero the total charge of the system must also be zero. It is worth noting that depending on the fusion rules of an anyon model, the pair of anyon anti-anyon could potentially be a pair of the same particle, as seen for example in the Fibonacci anyon model.

- *Quantum Gates*: Anyons are braided, while being kept far apart to avoid decoherence, in a pattern that replicates the effect of certain quantum gates. If for a specific anyon model those quantum gates are universal then it can be used to perform universal quantum computation. For anyons in the standard basis, the braiding is accomplished by combination of R and F-moves.

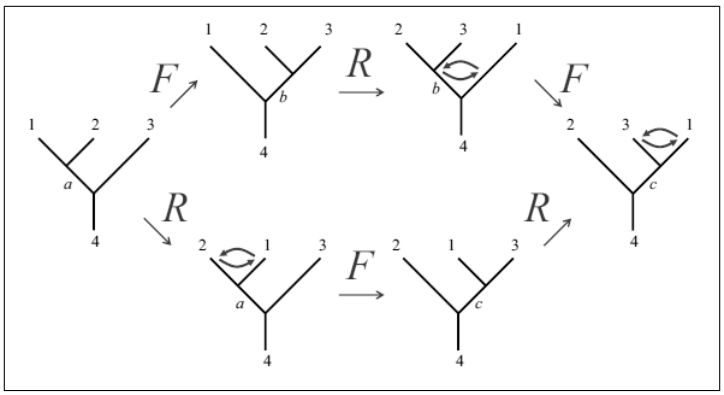


Figure 10: The Hexagon equation diagrammatically [5]. The label 4 corresponds to some set total charge of the system whereas a,b and c take all possible allowed values.

- *Measurement*: Finally, the pairs are brought together again and it is observed whether they annihilate completely, that is to say whether or not their charge has remained trivial.

In the case where N pairs of anyons anti-anyons are prepared in the standard basis, the Hilbert space of the system is

$$\mathcal{M}^1_{a_1,a_2,\ldots,a_n} = \bigoplus_{i_j} \mathcal{M}^{i_1}_{a_1 a_2} \otimes \mathcal{M}^{i_2}_{i_1 a_3} \otimes \ldots \otimes \mathcal{M}^1_{i_{n-2} a_n} \tag{23}$$

where a_j are the labels of the starting anyons, i_j are the intermediate anyons and $n = 2N$ is the total number of anyons created. The dimension of the space is

$$d(\mathcal{M}^1_{a_1,a_2,\ldots,a_n}) = \sum_{i_1,i_2,\ldots,i_n-3} N^{i_1}_{a_1 a_2} N^{i_2}_{i_1 a_3} \ldots N^1_{i_{n-2} a_n} \tag{24}$$

where N^c_{ab} are the multiplicities. In the case of non-abelian anyons the dimension of the space $d(\mathcal{M}) \propto d_a^n$ increases exponentially for each anyon a added to a system of n anyons, which makes this topological Hilbert space suitable for quantum information processing.

4.1 A qubit made of anyons

In order to perform a quantum computation, qubits need to be encoded in anyons. Since a qubit is a two-level quantum mechanical system, its Hilbert space dimension is equal to two. As such, in order to encode k qubits, the dimension of the fusion space must be

$$d(\mathcal{M}^b_{a_1,a_2,\ldots,a_n}) = \sum_{i_1,i_2,\ldots,i_n-3} N^{i_1}_{a_1 a_2} N^{i_2}_{i_1 a_3} \ldots N^b_{i_{n-3} a_n} = 2^k \tag{25}$$

where b labels the total charge of the system. The above mentioned dimension, is a function not only of the number of anyons n but also the types of anyons a_i and the end result b.

5 Constructing an anyon model of the group A_4

In this section, the first step of constructing an anyon model, that is finding the fusion rules, for the alternating group on four elements A_4 will be covered. First, a brief overview of the structure of the group is given followed by the character tables of all centralizers of the group. The characters will then be used in calculating the modular S-matrix [8] and finally the multiplicities will be evaluated through the Verlinde formula [9].

5.1 Alternating group on four elements

The alternating group on four elements A_4 consists of all even permutations on four elements and as such has $4!/2 = 12$ elements. Those elements are the identity, eight 3-cycles and three double-transpositions

$$\begin{aligned} A_4 = \{&e, (123), (134), (124), (234), (132), (143), (142), (243), \\ &(12)(34), (13)(24), (14)(23)\} \end{aligned}$$

using cyclic notation. There are four conjugacy classes:

$$\begin{aligned} e &= \{e\}, \\ {}^1C &= \{(12)(34), (13)(24), (14)(23)\}, \\ {}^2C &= \{(123), (134), (142), (243)\}, \\ {}^3C &= \{(124), (132), (143), (234)\} \end{aligned}$$

5.1.1 Centralizers

The *centralizer* $N(g)$ of a single element g in a group G, consists of all group elements that commute with g:

$$N(g) = \{a \in G | ag = ga\} \tag{26}$$

For a single element the centralizer and the normalizer are equivalent. Moreover, the centralizers of elements that belong to the same conjugacy class correspond to groups that are isomorphic to one another.

In the case of the group A_4, the centralizer of the identity element ${}^0C$ is obviously the whole group since every element of the group commutes with the identity element. The elements in ${}^1C$ give rise to groups that are isomorphic to the Klein four group K_4, which is an abelian group with elements a, b, c and the identity, where $ab = c$ and each element is its own inverse. The elements of the other two conjugacy classes ${}^2C$ and ${}^3C$ all have centralizers that are isomorphic to the cyclic group of order three Z_3. In the table 1 the conjugacy classes and the centralizers of their elements are collected.

Conjugacy class	Centralizer
$\{e\}$	A_4
${}^1C$	K_4
${}^2C$	Z_3
${}^3C$	Z_3

Table 1: The four conjugacy classes of the group A_4 and the respective centralizers of their elements.

5.1.2 Character table of Z_3

The group $Z_3 = \{e, g, g^2\}$ has three conjugacy classes $\{e\}, \{g\}$ and $\{g^2\}$ and as such must have three irreducible representations $\tilde{\Gamma}^0, \tilde{\Gamma}^1$ and $\tilde{\Gamma}^2$. Since $|Z_3| = 3 = 1^2 + 1^2 + 1^2$, the three irreducible representations are one dimensional and must equal their characters; this in turn leads to the characters having to obey group multiplication. The yet to be filled character table is:

Z_3	e	g	g^2
$\tilde{\Gamma}^0$	1	κ_1	κ_2
$\tilde{\Gamma}^1$	1	κ_3	κ_4
$\tilde{\Gamma}^2$	1	κ_5	κ_6

Given that $g^3 = e$ the characters (as they obey group multiplication) must belong to the 3rd roots of unity, meaning $\kappa_i \in \{1, \exp\frac{2\pi i}{3}, \exp\frac{4\pi i}{3}\}\ \forall\ i = 1, \ldots, 6$ and $\kappa_j = \kappa_{j-1}^2\ \forall\ j = 2, 4, 6$. If $\tilde{\Gamma}^0$ is the trivial representation then $\kappa_1 = 1$ and $\kappa_2 = 1$. Due to character orthogonality we have $\kappa_3 = \exp\frac{2\pi i}{3} \equiv \omega$ and $\kappa_5 = \exp\frac{4\pi i}{3}$, which gives the filled character table:

Z_3	e	g	g^2
$\tilde{\Gamma}^0$	1	1	1
$\tilde{\Gamma}^1$	1	ω	ω^2
$\tilde{\Gamma}^2$	1	ω^2	ω

5.1.3 Character table of K_4

The group K_4 is of order four and since it is abelian has four conjugacy classes, one for each element $e, a \equiv (12)(34), b \equiv (13)(24), c \equiv (14)(23)$. As a result is has four one-dimensional irreducible representations $\hat{\Gamma}^0, \hat{\Gamma}^1, \hat{\Gamma}^2$ and $\hat{\Gamma}^3$. The character table is of the form

Z_3	e	a	b	c
$\hat{\Gamma}^0$	1	λ_1	λ_2	λ_3
$\hat{\Gamma}^1$	1	λ_4	λ_5	λ_6
$\hat{\Gamma}^2$	1	λ_7	λ_8	λ_9
$\hat{\Gamma}^3$	1	λ_{10}	λ_{11}	λ_{12}

with λ_i's to be determined.

Similarly to the group Z_3, since all representations are one-dimensional their characters must obey the rules of group multiplication and given that the square of any element is equal to the identity element, the square of each character must in turn equal one. If we take $\hat{\Gamma}^0$ to be the trivial representation, $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Since the rows need to be orthogonal to each other the character table takes the form:

Z_3	e	(12)(34)	(13)(24)	(14)(23)
$\hat{\Gamma}^0$	1	1	1	1
$\hat{\Gamma}^1$	1	1	-1	-1
$\hat{\Gamma}^2$	1	-1	1	-1
$\hat{\Gamma}^3$	1	-1	-1	1

5.1.4 Character table of A_4

The group A_4 has four conjugacy classes $e, {}^1C, {}^2C$ and ${}^3C$ meaning that it must have four irreducible representations $\Gamma^0, \Gamma^1, \Gamma^2$ and Γ^3. Since the group is of order twelve, three of the representations must be one-dimensional and one must be three-dimensional, in order for the relation $dim(\Gamma^0)^2 + dim(\Gamma^1)^2 + dim(\Gamma^2)^2 + dim(\Gamma^3)^2 = 12 = |A_4|$. The character table is of the form

A_4	e	${}^1C$	${}^2C$	${}^3C$
Γ^0	1	μ_1	μ_2	μ_3
Γ^1	1	μ_4	μ_5	μ_6
Γ^2	1	μ_7	μ_8	μ_9
Γ^3	3	μ_{10}	μ_{11}	μ_{12}

where Γ^3 is assumed to be the three-dimensional one.

First of, if Γ^0 is to be the trivial representation then $\mu_1 = \mu_2 = \mu_3 = 1$. Since $\mu_i,\ i = 4, 5, \ldots, 9$ are characters of one-dimensional representations they must obey group multiplication given by the multiplication table of the group (Table 10). As such the following relations need to be obeyed:

$$\begin{aligned}
&\mu_4^2 = \mu_4 \quad and \quad \mu_4^2 = 1 \\
&\mu_7^2 = \mu_7 \quad and \quad \mu_7^2 = 1 \\
&\mu_5^2 = \mu_6, \quad \mu_5^3 = \mu_4 \quad and \quad \mu_5^3 = 1 \\
&\mu_8^2 = \mu_9, \quad \mu_8^3 = \mu_7 \quad and \quad \mu_8^3 = 1
\end{aligned}$$

Given the above equation and since rows need to be orthogonal to one another we get $\mu_4 = \mu_7 = 1$, $\mu_5 = \mu_9 = \exp\frac{2\pi i}{3} \equiv \omega$ and $\mu_6 = \mu_8 = \omega^2$. The characters μ_{10}, μ_{11} and μ_{12} can be obtained by the orthogonality of columns. Finally, the character table is:

A_4	e	${}^1C$	${}^2C$	${}^3C$
Γ^0	1	1	1	1
Γ^1	1	1	ω	ω^2
Γ^2	1	1	ω^2	ω
Γ^3	3	-1	0	0

5.2 Constructing the representations of the algebra $D(A_4)$

The representation of the group algebra $D(A_4)$ can be constructed using the conjugacy classes of the group and the irreducible representations of the centralizers. As a result we have the following 14 inequivalent irreducible representations of $D(A_4)$:

$$1 = |e, \Gamma^0\rangle\,, \qquad 2 = |e, \Gamma^1\rangle\,, \qquad 3 = |e, \Gamma^2\rangle\,, \tag{27}$$

$$4 = |e, \Gamma^3\rangle\,, \qquad 5 = |{}^1C, \hat{\Gamma}^0\rangle\,, \quad 6 = |{}^1C, \hat{\Gamma}^1\rangle\,, \quad 7 = |{}^1C, \hat{\Gamma}^2\rangle\,, \quad 8 = |{}^1C, \hat{\Gamma}^3\rangle\,, \tag{28}$$

$$9 = |{}^2C, \tilde{\Gamma}^0\rangle\,, \quad 10 = |{}^2C, \tilde{\Gamma}^1\rangle\,, \quad 11 = |{}^2C, \tilde{\Gamma}^2\rangle\,, \tag{29}$$

$$12 = |{}^3C, \tilde{\Gamma}^0\rangle\,, \quad 13 = |{}^3C, \tilde{\Gamma}^1\rangle\,, \quad 14 = |{}^3C, \tilde{\Gamma}^2\rangle \tag{30}$$

Among those, there are three one-dimensional (27), five three-dimensional (28) and six four-dimensional ones (29)-(30) that satisfy $3 \cdot 1^2 + 5 \cdot 3^2 + 6 \cdot 4^2 = 144 = 12^2 = |D(A_4)|$, where $|D(A_4)| = |A_4|^2$ is the order of the algebra.

The state $1 = |e, \Gamma^0\rangle$ corresponds to the trivial sector. The purely magnetic flux sectors are $|{}^1C, \Gamma^0\rangle\,, |{}^2C, \Gamma^0\rangle$ and $|{}^3C, \Gamma^0\rangle$. The purely electric sectors are the pairs of trivial magnetic flux and a non-trivial representation, which are $|e, \Gamma^1\rangle\,, |e, \Gamma^2\rangle$ and $|e, \Gamma^3\rangle$. The remaining sectors, being combinations of non-trivial fluxes and non-trivial representation of the centralizers, are dyonic sectors, that is to say they correspond to non-trivial magnetic fluxes and electric charges.

5.3 The S-matrix

In order to calculate the fusion rules of the anyon model we must first compute the modular S-matrix. This S-matrix is one of two generators of the modular group, the other being the T-matrix which contains the spin factors of the anyons in the model. For two given representations of $D(A_4)$, $|{}^AC, \Gamma^\alpha\rangle$ and $|{}^BC, \Gamma^\beta\rangle$ it takes the form [6]

$${}^A_a S^B_b = \frac{1}{|A_4|} \sum_{\substack{{}^Ag_i \in {}^AC,\ {}^Bg_j \in {}^BC \\ [{}^Ag_i, {}^Bg_j] = e}} \chi^*_a({}^Ax_i^{-1\,B}g_j\,{}^Ax_i)\chi^*_b({}^Bx_j^{-1\,A}g_i\,{}^Bx_j) \tag{31}$$

where $A, B \in \{0, 1, 2, 3\}$, $\Gamma \in \{\Gamma, \hat{\Gamma}, \tilde{\Gamma}\}$ and $a = a(A),\ b = b(B)$. The function χ^*_a signifies the complex conjugate of the character of the argument for the representation with label a. It is important to note that although the characters

have only one label they also depend on the conjugacy class A, since a is a function of A. The group elements ${}^{A}x_i$ are representatives of the equivalence classes of the quotient group $A_4/{}^{A}N$, with ${}^{A}N$ being the centralizer of the first element of the conjugacy class ${}^{A}C$.

The representatives are chosen as follows

1. Set an arbitrary ordering of the elements in each conjugacy class, ${}^{A}C = \{{}^{A}g_1, {}^{A}g_2, \dots\}$.
2. For a certain conjugacy class ${}^{A}C$ calculate the centralizer of the first element $N({}^{A}g_1)$.
3. Determine the quotient group $A_4/N({}^{A}g_1) \equiv A_4/{}^{A}N$.
4. Finally, compute the representatives ${}^{A}x_i$ through the relation

$$ {}^{A}g_i = {}^{A}x_i g_1 {}^{A}x_i^{-1} \tag{32} $$

   where it is always assumed that ${}^{A}x_1 = e$.

We demonstrate with the conjugacy class of the double transpositions:

$$ {}^{1}C = \{{}^{1}g_1 = (12)(34), {}^{1}g_2 = (13)(24), {}^{1}g_3 = (14)(23)\} $$

The centralizer of the first element ${}^{1}g_1$, is the conjugacy class itself:

$$ {}^{1}N \equiv {}^{1}N[{}^{1}g_1 = (12)(34)] = {}^{1}C $$

The equivalence classes of the quotient group are:

$$ \{\{{}^{A}N\}, \{(123){}^{A}N\}, \{(124){}^{A}N\}\} = \{\{{}^{1}C\}, \{{}^{2}C\}, \{{}^{3}C\}\} $$

Through (32) we find the representatives:

$$ \{{}^{1}x_1 = e, {}^{1}x_2 = (124), {}^{1}x_3 = (123)\} $$

Similarly, for

$$ {}^{2}C = \{{}^{2}g_1 = (123), {}^{2}g_2 = (134), {}^{2}g_3 = (142), {}^{2}g_4 = (243)\} $$
$$ {}^{3}C = \{{}^{3}g_1 = (124), {}^{3}g_2 = (132), {}^{3}g_3 = (143), {}^{3}g_4 = (234)\} $$

we find:

$$ \{{}^{2}x_1 = e, {}^{2}x_2 = (234), {}^{2}x_3 = (143), {}^{2}x_4 = (124)\} $$
$$ \{{}^{3}x_1 = e, {}^{3}x_2 = (134), {}^{3}x_3 = (243), {}^{3}x_4 = (123)\} $$

Obviously, for the trivial conjugacy class ${}^{0}C = \{{}^{0}g_1 = e\}$, there is only one representative ${}^{0}x_1 = e$.

To summarize, the elements of the four conjugacy classes are ordered as follows

$$
\begin{aligned}
{}^0C &= \{{}^0g_1 = e\} \\
{}^1C &= \{{}^1g_1 = (12)(34), {}^1g_2 = (13)(24), {}^1g_3 = (14)(23)\} \\
{}^2C &= \{{}^2g_1 = (123), {}^2g_2 = (134), {}^2g_3 = (142), {}^2g_4 = (243)\} \\
{}^3C &= \{{}^3g_1 = (124), {}^3g_2 = (132), {}^3g_3 = (143), {}^3g_4 = (234)\}
\end{aligned}
$$

and the representatives are collected in the table 9. Using these and the values of all the characters of the representations of the centralizers, the S-matrix can now, through (31), be determined.

5.3.1 Computing the S-matrix

Before we begin the express calculations, it is worth noting which of the group elements commute, since only those are summed in (31). Other than the identity element, that obviously commutes with any element of the group, the elements of the conjugacy class ${}^1C$ commute amongst themselves whereas each element in ${}^2C$ or ${}^3C$ commutes only with itself and its inverse in the other conjugacy class.

Starting off, for an initial state $|{}^0C, \Gamma^a\rangle$ and any final state, the elements of the S-matrix that connect them are

$$
{}^0_aS^B_b = \frac{1}{12} \sum_{{}^Bg_j \in {}^BC} \chi^*_a(e\,{}^Bg_j\,e)\chi^*_b({}^Bx_j^{-1}\,e\,{}^Bx_j) \tag{33}
$$

$$
= \frac{1}{12}\chi^*_b(e) \sum_{{}^Bg_j \in {}^BC} \chi^*_a({}^Bg_j) \quad \forall\ B, a, b(B) \tag{34}
$$

since ${}^0C = \{e\}$. In this case $B = 0, 1, 2, 3$, $a = 0, 1, 2, 3$ and $[b(0) = 0, 1, 2, 3\ ,$ $b(1) = 0, 1, 2, 3\ ,\ b(2) = 0, 1, 2\ ,\ b(3) = 0, 1, 2]$. Depending on the combination of indices (B, b) we have two possible values of $\chi^*_b(e)$:

$$
\chi^*_b(e) = \begin{cases} 3 & for\ (B = 0, b = 3) \\ 1 & else \end{cases}
$$

Summing over the characters of the representations ${}^a\Gamma$ for all ${}^Bg_j \in {}^BC$ we get the S-matrix elements for $|{}^0C, \Gamma^a\rangle \to |{}^BC, \Gamma^b\rangle\,, \quad \Gamma^b \in \{\Gamma^b, \hat{\Gamma}^b, \tilde{\Gamma}^b\}$ shown in Table 2 below:

Moving on, for an initial state $|{}^1C, \hat{\Gamma}^a\rangle$ the S-matrix can have non-zero elements only for final states with indices B=0,1 since no element of ${}^1C$ commutes with any element of ${}^2C$ or ${}^3C$. For final states $|{}^0C, \Gamma^b\rangle$ the S-matrix elements

${}^0_aS^B_b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	3	3	3	3	3	4	4	4	4	4	4
2	1	1	1	3	3	3	3	3	$4\omega^2$	$4\omega^2$	$4\omega^2$	4ω	4ω	4ω
3	1	1	1	3	3	3	3	3	4ω	4ω	4ω	$4\omega^2$	$4\omega^2$	$4\omega^2$
4	3	3	3	9	-3	-3	-3	-3	0	0	0	0	0	0

Table 2: The modular S-matrix of $D(A_4)$ for initial states $|^0C, \Gamma^a\rangle$ up to an overall factor $\frac{1}{12}$

are

$$\begin{aligned} {}^1_aS^0_b =& \frac{1}{12} \sum_{{}^1g_i \in {}^1C} \chi^*_a(e)\chi^*_b({}^1g_i) \\ =& \frac{1}{12}\chi^*_a(e) \sum_{{}^1g_i \in {}^1C} \chi^*_b({}^1g_i) \\ =& \frac{1}{12} \sum_{{}^1g_i \in {}^1C} \chi^*_b({}^1g_i) \quad \forall\ a = 0, 1, 2, 3 \end{aligned} \tag{35}$$

since $\chi_a(e) = 1$ for any pair of indices $(A, a) \neq (0, 3)$. For final states $|^1C, \Gamma^b\rangle$ the elements are of the form:

$$\begin{aligned} {}^1_aS^1_b =& \frac{1}{12} \sum_{{}^1g_i \in {}^1C,\ {}^1g_j \in {}^1C} \chi^*_a(({}^1x_i^{-1}){}^1g_j{}^1x_i)\chi^*_b(({}^1x_j^{-1}){}^1g_i{}^1x_j) \\ =& \frac{1}{12} 3[\chi^*_a((12)(34))\chi^*_b((12)(34)) + \chi^*_a((14)(23))\chi^*_b((13)(24)) + \\ & \chi^*_a((13)(24))\chi^*_b((14)(23))] \end{aligned} \tag{36}$$

The results of both (35) and (36) are collected in Table 3.

${}^1_aS^B_b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
5	3	3	3	-3	9	-3	-3	-3	0	0	0	0	0	0
6	3	3	3	-3	-3	9	-3	-3	0	0	0	0	0	0
7	3	3	3	-3	-3	-3	-3	9	0	0	0	0	0	0
8	3	3	3	-3	-3	-3	9	-3	0	0	0	0	0	0

Table 3: The modular S-matrix of $D(A_4)$ for initial states $|^1C, \Gamma^a\rangle$ up to an overall factor $\frac{1}{12}$.

In the case where the initial state is one of $\{|^2C, \tilde{\Gamma}^a\rangle\}$, the S-matrix elements will be zero for a final state in the set $\{|^1C, \hat{\Gamma}^a\rangle\}$ since the elements of these

conjugacy classes do not commute. For a final state $|^0C,\Gamma\rangle^a$ the S-matrix is

$$\begin{aligned}{}^2_aS^0_b =&\frac{1}{12}\sum_{{}^2g_i\in{}^2C}\chi^*_a(e)\chi^*_b({}^2g_i)\\ =&\frac{1}{12}[\chi^*_b(123)+\chi^*_b(134)+\chi^*_b(142)+\chi^*_b(243)]\end{aligned} \tag{37}$$

given that $\chi_a(e)=1\ \forall\ (A,a)\neq(0,3)$. For a final state among $\{|^2,C,\tilde{\Gamma}^b\rangle\}$ the S-matrix elements are

$$\begin{aligned}{}^2_aS^2_b =&\frac{1}{12}\sum_{{}^2g_i\in{}^2C,\ {}^2g_j\in{}^2C}\chi^*_a(({}^2x_i^{-1})^2g_j{}^2x_i)\chi^*_b(({}^2x_j^{-1})^2g_i{}^2x_j)\\ =&\frac{1}{12}\sum_{{}^2g_i\in{}^2C,\ {}^2g_j\in{}^2C}\chi^*_a(({}^2x_i^{-1})^2g_i{}^2x_i)\chi^*_b(({}^2x_i^{-1})^2g_i{}^2x_i)\\ =&\frac{1}{12}4[\chi^*_a((123))\chi^*_b((123))]\end{aligned} \tag{38}$$

since ${}^2g_j=\sum_i{}^2g_i\delta^i_j$, as within the conjugacy class ${}^2C$, elements commute only with themselves. Finally, for a final state $|^3C,\tilde{\Gamma}^b\rangle$ the elements of the S-matrix take the form

$$\begin{aligned}{}^2_aS^3_b =&\frac{1}{12}\sum_{{}^2g_i\in{}^2C,\ {}^3g_j\in{}^3C}\chi^*_a(({}^2x_i^{-1})^3g_j{}^2x_i)\chi^*_b(({}^3x_j^{-1})^2g_i{}^3x_j)\\ =&\frac{1}{12}4[\chi^*_a((132))\chi^*_b((142))]\end{aligned} \tag{39}$$

where the only element ${}^3g_j\in{}^3C$ that commutes with ${}^2g_i\in{}^2C$ is its inverse. These results are collected in Table 4.

${}^2_aS^B_b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
9	4	$4\omega^2$	4ω	0	0	0	0	0	4	$4\omega^2$	4ω	4	$4\omega^2$	4ω
10	4	$4\omega^2$	4ω	0	0	0	0	0	$4\omega^2$	4ω	4	4ω	4	$4\omega^2$
11	4	$4\omega^2$	4ω	0	0	0	0	0	4ω	4	$4\omega^2$	$4\omega^2$	4ω	4

Table 4: The modular S-matrix of $D(A_4)$ for initial states $|^2C,\Gamma^a\rangle$ up to an overall factor $\frac{1}{12}$.

Similarly to $|^2C,\tilde{\Gamma}^a\rangle$, for an initial state $|^3C,\tilde{\Gamma}^a\rangle$, depending on the final state, we have the elements of the S-matrix

$$\begin{aligned}{}^3_aS^0_b =&\frac{1}{12}\sum_{{}^3g_i\in{}^3C}\chi^*_a(e)\chi^*_b({}^3g_i)\\ =&\frac{1}{12}[\chi^*_b(124)+\chi^*_b(132)+\chi^*_b(143)+\chi^*_b(234)]\end{aligned} \tag{40}$$

$$
\begin{aligned}
{}^3_aS^2_b =& \frac{1}{12}\sum_{{}^3g_i\in{}^3C,\ {}^2g_j\in{}^2C}\chi^*_a(({}^3x_i^{-1})^2g_j{}^3x_i)\chi^*_b(({}^2x_j^{-1})^3g_i{}^2x_j) \\
=& \frac{1}{12}4[\chi^*_a((132))\chi^*_b((142))]
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
{}^3_aS^3_b =& \frac{1}{12}\sum_{{}^3g_i\in{}^3C,\ {}^3g_j\in{}^3C}\chi^*_a(({}^3x_i^{-1})^3g_j{}^3x_i)\chi^*_b(({}^3x_j^{-1})^3g_i{}^3x_j) \\
=& \frac{1}{12}\sum_{{}^3g_i\in{}^3C,\ {}^3g_j\in{}^3C}\chi^*_a(({}^3x_i^{-1})^3g_i{}^3x_i)\chi^*_b(({}^3x_i^{-1})^3g_i{}^3x_i) \\
=& \frac{1}{12}4[\chi^*_a((124))\chi^*_b((124))]
\end{aligned}
\tag{42}
$$

that are collected in the Table 5.

${}^3_aS^B_b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
12	4	4ω	$4\omega^2$	0	0	0	0	0	4	4ω	$4\omega^2$	4	4ω	$4\omega^2$
13	4	4ω	$4\omega^2$	0	0	0	0	0	$4\omega^2$	4	4ω	4ω	$4\omega^2$	4
14	4	4ω	$4\omega^2$	0	0	0	0	0	4ω	$4\omega^2$	4	$4\omega^2$	4	4ω

Table 5: The modular S-matrix of $D(A_4)$ for initial states $|{}^3C,\Gamma^a\rangle$ up to an overall factor $\frac{1}{12}$.

Combining all the previous results the whole S-matrix can be constructed (Table 6).

5.4 Determining the Fusion Rules

Now that we have evaluated the modular S-matrix of $D(A_4)$, we can input it in the Verlinde formula [9] to determine the fusion rules of the model:

$$
N_{ab^Cc}^{{}^AC^BCc} \equiv N_{abC}^{ABc} = \sum_{D,d}\frac{[{}^A_aS^D_d][{}^B_bS^D_d][{}^C_c(S^*)^D_d]}{{}^0_0S^D_d} \tag{43}
$$

Specifically, since the conjugacy classes ${}^0C$ and ${}^1C$ have three representations and the conjugacy classes ${}^2C$ and ${}^3C$ have two, the sums take the form:

$$
\begin{aligned}
N_{abC}^{ABc} = &\sum_{d=0}^{3}\left\{\frac{[{}^A_aS^0_d][{}^B_bS^0_d][{}^C_c(S^*)^0_d]}{{}^0_0S^0_d}+\frac{[{}^A_aS^1_d][{}^B_bS^1_d][{}^C_c(S^*)^1_d]}{{}^0_0S^1_d}\right\} \\
&+\sum_{d=0}^{2}\left\{\frac{[{}^A_aS^2_d][{}^B_bS^2_d][{}^C_c(S^*)^2_d]}{{}^0_0S^2_d}+\frac{[{}^A_aS^0_d][{}^B_bS^0_d][{}^C_c(S^*)^0_d]}{{}^0_0S^0_d}\right\}
\end{aligned}
\tag{44}
$$

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	3	3	3	3	3	4	4	4	4	4	4
2	1	1	1	3	3	3	3	3	$4\omega^2$	$4\omega^2$	$4\omega^2$	4ω	4ω	4ω
3	1	1	1	3	3	3	3	3	4ω	4ω	4ω	$4\omega^2$	$4\omega^2$	$4\omega^2$
4	3	3	3	9	-3	-3	-3	-3	0	0	0	0	0	0
5	3	3	3	-3	9	-3	-3	-3	0	0	0	0	0	0
6	3	3	3	-3	-3	9	-3	-3	0	0	0	0	0	0
7	3	3	3	-3	-3	-3	-3	9	0	0	0	0	0	0
8	3	3	3	-3	-3	-3	9	-3	0	0	0	0	0	0
9	4	$4\omega^2$	4ω	0	0	0	0	0	4	$4\omega^2$	4ω	4	$4\omega^2$	4ω
10	4	$4\omega^2$	4ω	0	0	0	0	0	$4\omega^2$	4ω	4	4ω	4	$4\omega^2$
11	4	$4\omega^2$	4ω	0	0	0	0	0	4ω	4	$4\omega^2$	$4\omega^2$	4ω	4
12	4	4ω	$4\omega^2$	0	0	0	0	0	4	4ω	$4\omega^2$	4	4ω	$4\omega^2$
13	4	4ω	$4\omega^2$	0	0	0	0	0	$4\omega^2$	4	4ω	4ω	$4\omega^2$	4
14	4	4ω	$4\omega^2$	0	0	0	0	0	4ω	$4\omega^2$	4	$4\omega^2$	4	4ω

Table 6: The modular S-matrix of $D(A_4)$ up to an overall factor $\frac{1}{12}$.

In order to better present the fusion rules we redefine the representations of $D(A_4)$ as follows:

$$1 \equiv |e, \Gamma^0\rangle \equiv \Phi_0 \tag{45}$$

$$\Phi_a \equiv |^0C, \Gamma^a\rangle\,, \quad a = 1, 2 \tag{46}$$

$$\Psi_i \equiv \begin{cases} |^0C, \Gamma^3\rangle & for\ i = 0 \\ |^1C, \hat{\Gamma}^{i-1}\rangle & for\ i = 1, 2, 3, 4 \end{cases} \tag{47}$$

$$\Lambda_l^+ \equiv |^2C, \tilde{\Gamma}^l\rangle\,, \quad l = 0, 1, 2 \tag{48}$$

$$\Lambda_l^- \equiv |^3C, \tilde{\Gamma}^l\rangle\,, \quad l = 0, 1, 2 \tag{49}$$

First of all, the fusion of any anyon in the model with the trivial 1, leaves the anyon unchanged. Also, the fusion of anyons as a process is symmetric, that is to say $a \times b = b \times a$, which leads to $N_c^{ab} = N_c^{ba}$ for the multiplicities.

That being said, the fusion rules for at least one initial anyon Φ_a are

$$\Phi_a \times \Phi_a = \Phi_b, \qquad \Phi_a \times \Phi_b = 1 \tag{50}$$

$$\Phi_a \times \Lambda_l^\sigma = \Lambda^\sigma_{(a+l)mod3} \qquad \Phi_a \times \Psi_i = \Psi_i \tag{51}$$

where $a, b = 1, 2 \;\; a \neq b, \;\; l = 0, 1, 2$ and $\sigma \in \{+, -\}$.

For at least one initial anyon Ψ_i they are

$$\Psi_i \times \Psi_i = 1 + \sum_a \Phi_a + 2\Psi_i \tag{52}$$

$$\Psi_i \times \Psi_j = \sum_{k \neq i,j} \Psi_k \tag{53}$$

$$\Psi_i \times \Lambda_l^\sigma = \sum_m \Lambda_m^\sigma \tag{54}$$

where $i, j, k = 0, 1, 2, 3, 4,\ \ i \neq j,\ \ a = 1, 2,\ \ l, m = 0, 1, 2$ and $\sigma \in \{+, -\}$.

The fusion rules for the Λ_l^σ are

$$\Lambda_l^+ \times \Lambda_m^+ = \sum_n \Lambda_n^- + \Lambda_{(l+m)mod3}^- \tag{55}$$

$$\Lambda_l^- \times \Lambda_m^- = \sum_n \Lambda_n^+ + \Lambda_{(l+m)mod3}^+ \tag{56}$$

$$\Lambda_l^+ \times \Lambda_m^- = \Phi_{(l+m)mod3} + \sum_{i=0}^{4} \Psi_i \tag{57}$$

where $l, m, n = 0, 1, 2$ and $\Phi_0 \equiv 1$.

It is worth noting that, for the pure electric states, the fusion process does correspond to a tensor product of the two representations since there is no flux metamorphosis involved. As such, the fusion rules for the pure electric states can be obtained by reducing the tensor product into a direct sum of irreducible representations.

Also, we see that fusion, being a decomposition of products of representation, can only generate representations of dimension equal or lower than the maximum dimension of the representations being fused.

5.4.1 Anyon model dimension

Having calculated all the different multiplicities, we can now calculate the the quantum dimensions of the anyons in the model through (14) by taking $a = b = 1, 2, \ldots, 14$

$$d_1 = 1 \tag{58}$$

$$d_2^2 = d_3 \ and \ d_3^2 = d_2 \implies d_2 = d_3 = 1 \tag{59}$$

$$\begin{aligned} &d_4^2 = d_1 + d_2 + d_3 + 2d_4 \implies d_4^2 - 2d_4 - 3 = 0 \implies \\ &d_4 = 3 = d_5 = d_6 = d_7 = d_8 \end{aligned} \tag{60}$$

$$d_9^2 = d_1 + \sum_{i=4}^{8} d_i \implies d_9 = 4 = d_{10} = d_{11} \tag{61}$$

$$d_{12}^2 = 2d_9 + d_{10} + d_{11} \implies d_{12} = 4 = d_{13} = d_{14} \tag{62}$$

since $d_i \in \mathbb{R}^+$. Given the individual quantum dimensions the dimension of the anyon model is determined through (15) to be:

$$D = \sqrt{\sum_{i=1}^{14} d_i^2} = 12 \tag{63}$$

5.5 The T-matrix

The modular T-matrix is the second generator of the modular group. It is a diagonal matrix with the spin factors of the different anyons in the model on

the diagonal:

$$T_{ab}^{AB} \equiv \delta^{A,B}\delta_{a,b}\exp\left(2\pi i s_{(A,a)}\right) \tag{64}$$

This expression can be simplified by noting the following. The element ${}^{A}g_1 \in {}^{A}C$ commutes by definition with all elements in its centralizer ${}^{A}N$ and as such in the representation a its proportional to the unit matrix

$$a({}^{A}g_1) = \exp\left(2\pi i s_{(A,a)}\right)\mathbb{1}_{d_a \times d_a} \tag{65}$$

where d_a is the dimension of the representation a. Since anyons in the same superselection sector $({}^{A}C, a)$ are described by the same spin factor, that is to say they gain the same phase under a counter-clockwise rotation of 2π, (65) describes all of them. Given the above, the T-matrix can take the form:

$$T_{ab}^{AB} = \delta^{A,B}\delta_{a,b}\frac{1}{d_a}tr(a({}^{A}g_1)) = \delta^{A,B}\delta_{a,b}\frac{1}{d_a}\chi_a({}^{A}g_1) \tag{66}$$

Using the character tables in section **6.1** the matrix can now be easily calculated and the results are collected in Table 7.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	-1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	ω	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	ω^2	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	ω	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	ω^2

Table 7: The modular T-matrix of $D(A_4)$.

5.5.1 Charge conjugation operator

The S and T matrices, being generators of the modular group, satisfy the following relations:

$$(ST)^3 = S^2 \tag{67}$$

$$S^4 = I \tag{68}$$

They are also unitary and symmetric:

$$S^* = S^{-1} \qquad S^t = S \tag{69}$$

$$T^* = T^{-1} \qquad T^t = T \tag{70}$$

Through the relation (67), the matrix $\mathcal{C}$ that represents the charge conjugation operator can be defined as:

$$\mathcal{C} \equiv (ST)^3 \tag{71}$$

Charge conjugation gives the anti-particle $\mathcal{C}({}^AC, a) = ({}^{\overline{A}}C, \overline{a})$ of each particle $({}^AC, a)$. The matrix $\mathcal{C}$ is given in Table 8. The following particle - anti-particle

$\mathcal{C}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	1	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	1
11	0	0	0	0	0	0	0	0	0	0	0	0	1	0
12	0	0	0	0	0	0	0	0	1	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	0	0	1	0	0	0	0

Table 8: The charge conjugation matrix of $D(A_4)$.

relations can be read off the matrix $\mathcal{C}$ immediately

$$({}^{\overline{0}}C, \overline{1}) = ({}^0C, 2) \tag{72}$$

$$({}^{\overline{2}}C, \overline{0}) = ({}^3C, 0) \tag{73}$$

$$({}^{\overline{2}}C, \overline{1}) = ({}^3C, 1) \tag{74}$$

$$({}^{\overline{2}}C, \overline{2}) = ({}^3C, 2) \tag{75}$$

and in the other cases each particle is its own anti-particle.

6 F-matrix, R-matrix, quantum gates

In this final section, we will demonstrate the calculation of the R and F matrices, for a specific subset of particles of the $D(A_4)$ theory whose fusion rules are closed, meaning that no possible fusion of those can result in one of the other particles

of the theory. Finally, we will attempt to construct specific quantum gates through those matrices and see whether or not they are sufficient to support universal quantum computation.

6.1 Generalized Pentagon and Hexagon equations

Now that we have all the fusion rules of the model in our hands we can determine, up to a choice of gauge, all the possible R and F matrices. One way to do so is by solving all the allowed Pentagon and Hexagon equations. Since in our model we have a set of fusion rules (52) with multiplicities $N_{\Psi_i \Psi_i}^{\Psi_i} = 2$, we will have to construct the Pentagon and Hexagon equations for any multiplicity $N_{ab}^{c} \in \mathbb{N}$ which we will call the generalized Pentagon and Hexagon equations.

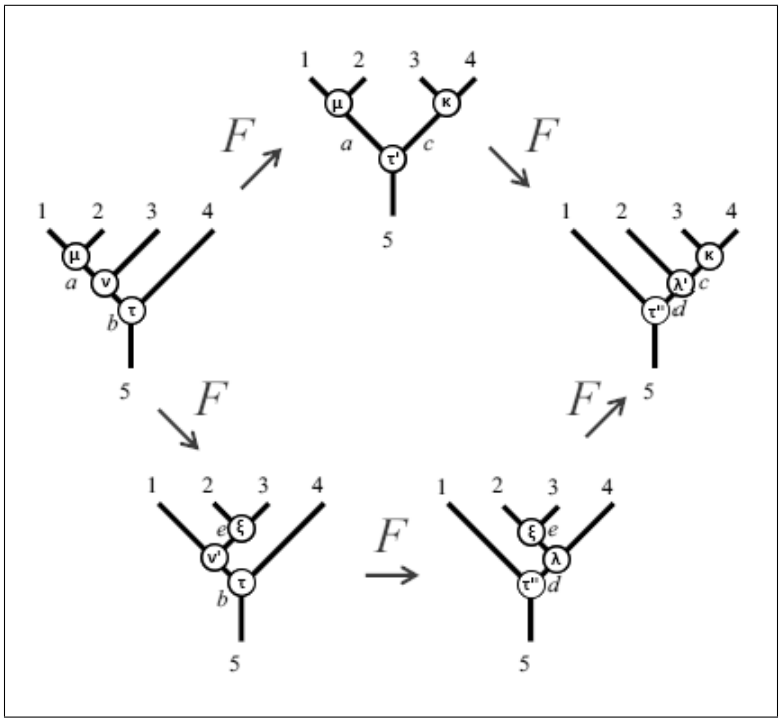


Figure 11: The generalized Pentagon equation diagrammatically for any N_{ab}^{c}. The label 5 corresponds to some set total charge of the system whereas the labels a,b,c,d and e take all possible allowed values.

Let us start off with the generalized Pentagon equation. The basis shown furthest to the left in Figure 11 is $\{|left; a, b, \mu, \nu, \tau\rangle\}$, where anyons 1 and 2 fuse first in the μ-th distinguishable way to give a, which is fused with anyon 3 giving anyon b in the ν-th distinguishable way and finally b is fused with 4 giving total charge 5 in the τ-th distinguishable way. Similarly, the right most basis in the figure is $\{|right; c, d, \kappa, \lambda', \tau''\rangle\}$, where in this case anyons labeled 3

and 4 fuse in the κ-th distinguishable way to give c, which is fused with anyon 2 giving d in the λ'-th distinguishable way and then anyons d and 1 are fused in the τ''-th distinguishable way giving the total charge 5 of the system. These two bases are connected by two F-moves across the top and three F-moves across the bottom giving in each case accordingly

$$|left; a, b, \mu, \nu, \tau\rangle = \sum_{c,d,\kappa,\lambda,\tau',\tau''} (F^5_{a34})^{c\kappa\tau'}_{b\nu\tau} (F^5_{12c})^{d\lambda\tau''}_{a\mu\tau'} \, |right; c, d, \kappa, \lambda', \tau''\rangle \quad (76)$$

and

$$|left; a, b, \mu, \nu, \tau\rangle = \sum_{\substack{e,d,c,\xi,\lambda,\\ \kappa,\nu',\tau'',\lambda'}} (F^b_{123})^{e\xi\nu'}_{a\mu\nu} (F^5_{1e4})^{d\lambda\tau''}_{b\nu'\tau} (F^d_{234})^{c\kappa\lambda'}_{e\xi\lambda} \, |right; c, d, \kappa, \lambda', \tau''\rangle \, . \quad (77)$$

As we have discussed in section **4.3.4** these set of moves need to correspond to the same process and as such by equating (76) and (77) we get

$$\sum_{\tau'} (F^5_{a34})^{c\kappa\tau'}_{b\nu\tau} (F^5_{12c})^{d\lambda\tau''}_{a\mu\tau'} = \sum_{e,\xi,\nu',\lambda'} (F^b_{123})^{e\xi\nu'}_{a\mu\nu} (F^5_{1e4})^{d\lambda\tau''}_{b\nu'\tau} (F^d_{234})^{c\kappa\lambda'}_{e\xi\lambda} \quad (78)$$

since the basis elements of a set are orthogonal amongst themselves. This is the *Generalized Pentagon equation* (GPE).

In a similar manner the generalized Hexagon equation can also be calculated. Once again, we have the left most basis in Figure 12 $\{|left; a.\mu, \nu\rangle\}$, where

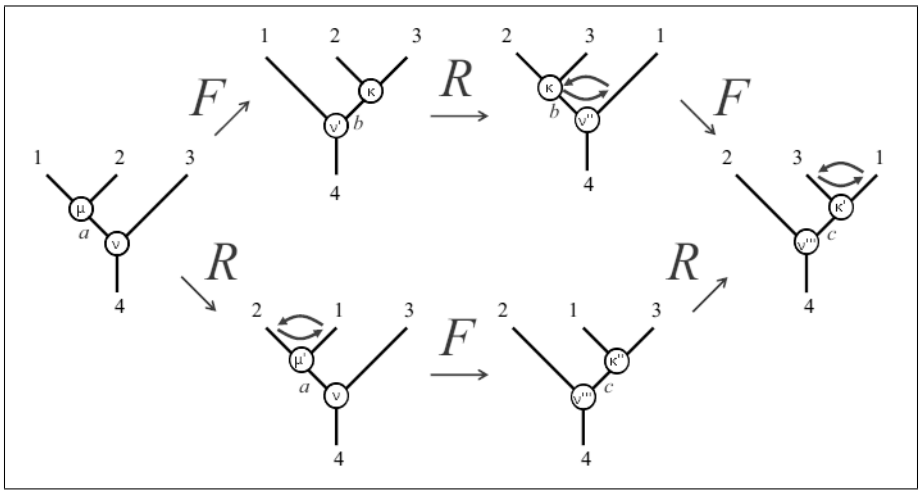


Figure 12: The generalized Pentagon equation diagrammatically for any N^c_{ab}. The label 5 corresponds to some set total charge of the system whereas the labels a,b,c,d and e take all possible allowed values.

anyons labeled 1 and 2 fuse to a in the μ-th distinguishable way and then it is fused with 3 in the ν-th distinguishable way giving the total charge 4. The right most basis in the figure is $\{|right; c, \kappa', \nu'''\rangle$, where anyons 3 and 1 fuse to c in the κ'-th distinguishable way and then it is fused with anyon 2 in the ν'''-th distinguishable way giving the total charge 4 of the system. These two bases are connected by two F-moves and an R-move across the top and two R-moves and an F-move across the bottom, giving

$$|left; a, \mu, \nu\rangle = \sum_{\substack{b,c,\kappa,\kappa',\\ \nu',\nu'',\nu'''}} (F^4_{231})^{b\kappa\nu'}_{a\mu\nu} (R^4_{1b})^{\nu''}_{\nu'} (F^4_{123})^{c\kappa'\nu'''}_{b\kappa\nu''} \, |right; c, \kappa', \nu'''\rangle \qquad (79)$$

and

$$|left; a, \mu, \nu\rangle = \sum_{c,\mu',\kappa'',\kappa',\nu'''} (R^a_{12})^{\mu'}_{\mu} (F^4_{213})^{c\kappa''\nu'''}_{a\mu'\nu} (R^c_{13})^{\kappa'}_{\kappa''} \, |right; c, \kappa', \nu'''\rangle \qquad (80)$$

accordingly. By equating the two expressions (79) and (80) for the same reason as before we get

$$\sum_{\mu',\kappa''} (R^a_{12})^{\mu'}_{\mu} (F^4_{213})^{c\kappa''\nu'''}_{a\mu'\nu} (R^c_{13})^{\kappa'}_{\kappa''} = \sum_{b,\kappa,\nu',\nu''} (F^4_{231})^{b\kappa\nu'}_{a\mu\nu} (R^4_{1b})^{\nu''}_{\nu'} (F^4_{123})^{c\kappa'\nu'''}_{b\kappa\nu''} \qquad (81)$$

which is the *Generalized Hexagon equation* (GHE).

At this point, it is worthwhile to mention some of the characteristics of these two consistency conditions. First off, each arrangement of starting anyons and total charge (which in figures 9 and 10 we have labeled with numbers) both in different particles used and permutations of the same particles, defines a different system of equations, for each consistency condition, compromised by a number of equations equal the square of the dimension of the topological Hilbert space ($dim^2(\mathcal{M}^5_{1234})$ for the GPE and $dim^2(\mathcal{M}^4_{123})$ for the GHE) since we have one equation for each combination of basis elements from the left and right most bases. These equations are constructed by choosing different **allowed** combinations of the non-summed over labels in (78) and (81) for the GPE and GHE accordingly (labels $\{c, d, b, a, \kappa, \lambda, \nu, \mu, \tau'', \tau'\}$ in the GPE and $\{a, c, \kappa', \mu, \nu, \nu'''\}$ in the GHE).

Secondly, as the GPE involves many different F-matrices, the system of equations that it generates is in general indefinite and in order for the F-matrices to be determined, all equations for every allowed combination and permutation of the numbered labels (1,2,3,4 and 5) will need to be combined to construct a solvable system of equations. Similarly, for the GHE different R-matrices appear in a single generated system of equations but since only three initial anyons are used to determine this consistency condition the only F-matrices that are involved are F^4_{213}, F^4_{231} and F^4_{123} which is a detail that we will make use of later on.

Based on the above, it is straightforward to see that as the number of different anyons in our model or their dimension increases, the number of equations

that are generated by the GPE and GHE and need to be solved greatly increases, which is why this method of determining the R and F matrices of the model is usually only applied for the two simplest non-abelian models, those being the Fibonnaci and Ising anyon models.

In the case of our $D(A_4)$ theory, the number non-trivial systems of equations that would need to be solved together only for the GPE are between 1.001 and 24.024 depending on how many arrangements of the different labels are allowed by the fusion rules of our model.

6.2 Determining the F and R matrices for a subset of the model

As the number of equations needed to be solved is great and since we are ultimately interested in simulating a quantum computer we will limit ourselves to a certain process involving the subset of the spectrum of the theory, containing the representations $1, \Phi_1, \Phi_2$ and one of the Ψ_i, whose fusion is closed. As can be seen in fusion rules (50) , (51) and (52) by fusing multiple Ψ_i anyons together the result can only ever be the vacuum 1, the abelian anyons Φ_1 and Φ_2 or Ψ_i itself since the abelian anyons are absorbed into Ψ_i and no other non-abelian anyons can be created. Even by limiting ourselves to these particles, in order to calculate the F-matrices involved we would have to solve the GPE for every allowed arrangement of five particles in the subset which would lead to a system with a number of equations of order 10^2. In order to avoid solving these, we will deduce the R-matrices involved, by taking advantage of the fact that the only non-abelian fusion in the subset, is of two same particles Ψ_i. Afterwards, we will compute the needed F-matrices through the GHE.

6.2.1 Hilbert space dimension

As a topological quantum computation is performed by creating pairs of anyons - anti-anyons out of the vacuum, we seek to fuse enough anyons with a total charge of 1 whose topological Hilbert space dimension is enough to encode qubits. Out of the particles in the subset, the abelian anyons Φ_1 and Φ_2 which form a pair of particle - anti-particle cannot be used to encode a qubit since no matter how many abelian anyons are fused the dimension of the space is equal to one as the result is deterministic. That leaves the non-abelian anyon Ψ_i, which forms a particle - anti-particle pair with itself. For a system of Ψ_i anyons we have the following dimensions of the Hilbert spaces:

$$dim(\mathcal{M}^1_{\Psi_i\Psi_i}) = N^1_{\Psi_i\Psi_i} = 1 \tag{82}$$

$$dim(\mathcal{M}^1_{\Psi_i\Psi_i\Psi_i}) = \sum_{j_1} N^{j_1}_{\Psi_i\Psi_i} N^1_{j_1\Psi_i} = N^{\Psi_i}_{\Psi_i\Psi_i} N^1_{\Psi_i\Psi_i} = 2 \cdot 1 = 2 \tag{83}$$

$$
\begin{aligned}
dim(\mathcal{M}^1_{\Psi_i\Psi_i\Psi_i\Psi_i}) =& \sum_{j_1,j_2} N^{j_1}_{\Psi_i\Psi_i} N^{j_2}_{j_1\Psi_i} N^1_{j_2\Psi_i} = \sum_{j_1} N^{j_1}_{\Psi_i\Psi_i} N^{\Psi_i}_{j_1\Psi_i} N^1_{\Psi_i\Psi_i} \\
=& N^1_{\Psi_i\Psi_i} N^{\Psi_i}_{1\Psi_i} N^1_{\Psi_i\Psi_i} + N^{\Phi_1}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_1\Psi_i} N^1_{\Psi_i\Psi_i} \\
&+ N^{\Phi_2}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_2\Psi_i} N^1_{\Psi_i\Psi_i} + N^{\Psi_i}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Psi_i\Psi_i} N^1_{\Psi_i\Psi_i} \\
=& 1+1+1+2\cdot 2 = 7
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
dim(\mathcal{M}^1_{\Psi_i\Psi_i\Psi_i\Psi_i\Psi_i}) =& \sum_{j_1,j_2,j_3} N^{j_1}_{\Psi_i\Psi_i} N^{j_2}_{j_1\Psi_i} N^{j_3}_{j_2\Psi_i} N^1_{j_3\Psi_i} \\
=& \sum_{j_1,j_2} N^{j_1}_{\Psi_i\Psi_i} N^{j_2}_{j_1\Psi_i} N^{\Psi_i}_{j_2\Psi_i} N^1_{\Psi_i\Psi_i} \\
=& \sum_{j_2} \Big[N^1_{\Psi_i\Psi_i} N^{\Psi_i}_{1\Psi_i} N^{\Psi_i}_{\Psi_i\Psi_i} + N^{\Phi_1}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_1\Psi_i} N^{\Psi_i}_{\Psi_i\Psi_i} \\
&+ N^{\Phi_2}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_2\Psi_i} N^{\Psi_i}_{\Psi_i\Psi_i} + N^{\Psi_i}_{\Psi_i\Psi_i} N^{j_2}_{\Psi_i\Psi_i} N^{\Psi_i}_{j_2\Psi_i} \Big] N^1_{\Psi_i\Psi_i} \\
=& 6 + 2\Big(N^1_{\Psi_i\Psi_i} N^{\Psi_i}_{1\Psi_i} + N^{\Phi_1}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_1\Psi_i} \\
&+ N^{\Phi_2}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Phi_2\Psi_i} + N^{\Psi_i}_{\Psi_i\Psi_i} N^{\Psi_i}_{\Psi_i\Psi_i} \Big) = 6 + 2\cdot 7 = 20
\end{aligned}
\tag{85}
$$

In order to encode a single qubit we need a system with two states that span a 2D subspace of a certain energy, which in our case of anyonic systems, corresponds to equal total charge. As such, we see that we can encode a qubit onto a system with three Ψ_i anyons with trivial total charge but we require at least five Ψ_i with trivial total charge in order to encode three qubits.

6.2.2 Encoding a single qubit

As we showed above, a single qubit can be encoded in the fusion states of a system of three Ψ_i particles with trivial total charge. The standard basis for this process is $\{|(\Psi_i\Psi_i)\Psi_i \to 1; \Psi_i, \mu\rangle\}$ where the two basis elements $|0\rangle \equiv |(\Psi_i\Psi_i)\Psi_i \to 1; \Psi_i, 1\rangle$ and $|1\rangle \equiv |(\Psi_i\Psi_i)\Psi_i \to 1; \Psi_i, 2\rangle$ correspond to the two distinguishable ways the first two Ψ_i particles can fuse to another Ψ_i.

In order to construct quantum gates we will need the matrices $R^{\Psi_i}_{\Psi_i\Psi_i}$, $R^1_{\Psi_i\Psi_i}$ and $F^1_{\Psi_i\Psi_i\Psi_i}$. By making use of the GHE for anyons $1, 2, 3 = \Psi_i$ and $4 = 1$ we get, from (81), the system

$$
\sum_{\mu',\kappa''} (R^a_{\Psi_i\Psi_i})^{\mu'}_{\mu} (F^1)^{c\kappa''\nu'''}_{a\mu'\nu} (R^c_{\Psi_i\Psi_i})^{\kappa'}_{\kappa''} = \sum_{b,\kappa,\nu',\nu''} (F^1)^{b\kappa\nu'}_{a\mu\nu} (R^1_{\Psi_i b})^{\nu''}_{\nu'} (F^1)^{c\kappa'\nu'''}_{b\kappa\nu''} \tag{86}
$$

where we have defined $F^1_{\Psi_i\Psi_i\Psi_i} \equiv F^1$ for simplicity's sake. Given that the only fusion process that can happen in more than one distinguishable way is $\Psi_i \times \Psi_i = 2\Psi_i$, the labels ν, ν', ν'', ν''' can only take one value $\nu = \nu' = \nu'' = \nu''' = 1$. Also the labels a, b, c need to be Ψ_i in order for the total charge to be

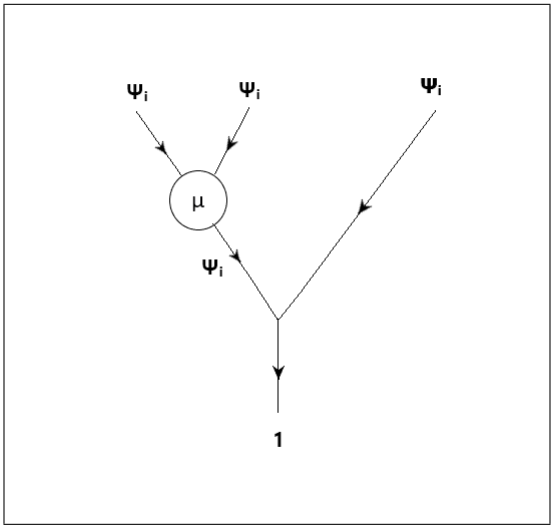


Figure 13: The standard basis diagrammatically for three Ψ_i anyons fusing to 1. The different basis elements correspond to the two distinguishable ways the first two Ψ_i particles fuse to Ψ_i

trivial. These simplify (86) to:

$$\sum_{\mu',\kappa''} (R^{\Psi_i}_{\Psi_i\Psi_i})^{\mu'}_{\mu} (F^1)^{\kappa''}_{\mu'} (R^{\Psi_i}_{\Psi_i\Psi_i})^{\kappa'}_{\kappa''} = \sum_{\kappa} (F^1)^{\kappa}_{\mu} (R^1_{\Psi_i\Psi_i}) (F^1)^{\kappa'}_{\kappa} \tag{87}$$

that gives:

$$\begin{aligned} &(R^{\Psi_i}_{\Psi_i\Psi_i})^1_\mu (F^1)^1_{\mu'} (R^{\Psi_i}_{\Psi_i\Psi_i})^{\kappa'}_1 + (R^{\Psi_i}_{\Psi_i\Psi_i})^2_\mu (F^1)^1_2 (R^{\Psi_i}_{\Psi_i\Psi_i})^{\kappa'}_1 \\ &\quad + (R^{\Psi_i}_{\Psi_i\Psi_i})^1_\mu (F^1)^2_1 (R^{\Psi_i}_{\Psi_i\Psi_i})^{\kappa'}_2 + (R^{\Psi_i}_{\Psi_i\Psi_i})^2_\mu (F^1)^2_2 (R^{\Psi_i}_{\Psi_i\Psi_i})^{\kappa'}_2 \\ &\qquad = (F^1)^1_\mu (R^1_{\Psi_i\Psi_i}) (F^1)^{\kappa'}_1 + (F^1)^2_\mu (R^1_{\Psi_i\Psi_i}) (F^1)^{\kappa'}_2 \end{aligned} \tag{88}$$

We see that knowing the matrices $R^1_{\Psi_i\Psi_i}$ and $R^{\Psi_i}_{\Psi_i\Psi_i}$, suffices to determine the matrix $F^{\Psi_i}_{\Psi_i\Psi_i\Psi_i}$. The action of the braid operator acting on particle - anti-particle pairs with trivial total charge is straightforward and it is

$$R^1_{a\bar{a}} = e^{-i\theta_a} = T^{AA}_{aa} \tag{89}$$

since creating a pair, exchanging and then annihilating them is equivalent to rotating the particle a by -2π. By consulting the T-matrix in section **6.5** which contains the phase values, we can immediately read off the following:

$$R^1_{\Psi_0\Psi_0} = R^1_{\Psi_1\Psi_1} = R^1_{\Psi_2\Psi_2} = 1 \qquad R^1_{\Psi_3\Psi_3} = R^1_{\Psi_4\Psi_4} = -1 \tag{90}$$

Computing the 2×2 $R^{\Psi_i}_{\Psi_i\Psi_i}$ matrix is slightly more convoluted since the total charge is non-trivial. We will do so by using arguments that only apply

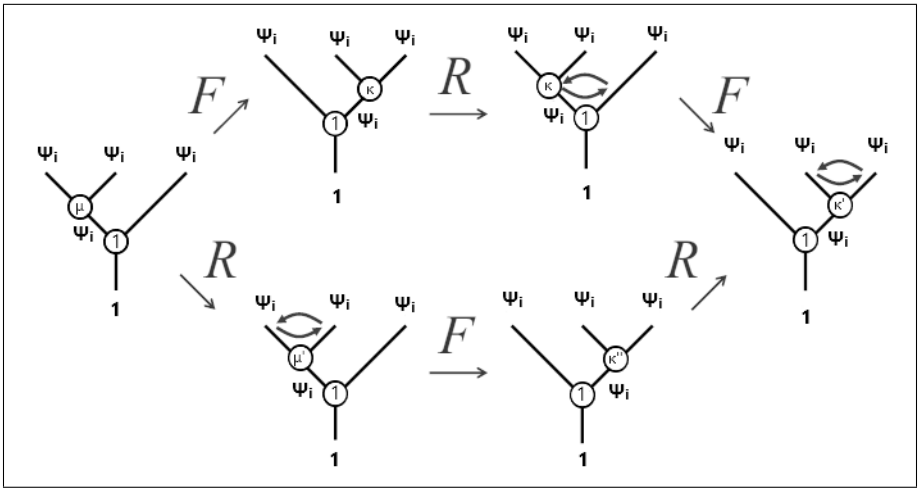


Figure 14: The Generalized Hexagon equation for the process $\Psi_i\Psi_i\Psi_i \to 1$.

in the special case where the two particles braided are the same but which are not quite enough to unambiguously determine the solution. As we have talked about in section **4.3.1**, the action of the braid operator on the basis state $|ba \to c;\mu\rangle$ in $\mathcal{M}_{ba}^{c}$ produces a state $|ab \to c;\mu\rangle_R$ in $\mathcal{M}_{ab}^{c}$, that is in general a linear superposition of the basis elements $\{|ab \to c;\mu'\rangle\}$ of that space. Suppose now, we braid the anyons a and b twice by applying two consecutive braid operators which together are called the monodromy operator R^2, then as the particles are returned to their initial setup the final state must be the initial state multiplied by a phase. That phase can be shown to be [5]:

$$(R_{ab}^{c})^2 = \frac{e^{-i\theta_a - i\theta_b}}{e^{-i\theta_c}} \tag{91}$$

As such we have the following

$$\begin{aligned} \frac{e^{-i\theta_a - i\theta_b}}{e^{-i\theta_c}} |ab \to c;\mu\rangle &= |ab \to c;\mu\rangle_{R^2} \\ &= \sum_{\mu''} (R_{ab}^{c})_{\mu}^{\mu''} \, |ba \to c;\mu''\rangle_R \\ &= \sum_{\mu'',\mu'} (R_{ab}^{c})_{\mu}^{\mu''} (R_{ba}^{c})_{\mu''}^{\mu'} \, |ab \to c;\mu'\rangle \end{aligned} \tag{92}$$

and by taking the inner product

$$\begin{aligned}\frac{e^{-i\theta_a-i\theta_b}}{e^{-i\theta_c}}\langle ab\to c';\nu|ab\to c;\mu\rangle &= \sum_{\mu'',\mu'}(R^c_{ab})^{\mu''}_{\mu}(R^c_{ba})^{\mu'}_{\mu''}\langle ab\to c';\nu|ab\to c;\mu'\rangle \\ \Leftrightarrow \frac{e^{-i\theta_a-i\theta_b}}{e^{-i\theta_c}}\delta^{c'}_{c}\delta^{\nu}_{\mu} &= \sum_{\mu''}(R^c_{ab})^{\mu''}_{\mu}(R^c_{ba})^{\nu}_{\mu''}\delta^{c'}_{c} \\ \Leftrightarrow \frac{e^{-i\theta_a-i\theta_b}}{e^{-i\theta_c}}\delta^{\nu}_{\mu} &= \sum_{\mu''}(R^c_{ab})^{\mu''}_{\mu}(R^c_{ba})^{\nu}_{\mu''}\end{aligned} \tag{93}$$

where we have used the orthogonality relation of the fusion basis:

$$\langle ab\to c';\nu|ab\to c;\mu\rangle = \delta^{c'}_{c}\delta^{\nu}_{\mu}$$

Supposing $\mu,\nu,\mu''=1,2$ we get the system

$$\begin{aligned}(R^c_{ab})^1_1(R^c_{ba})^1_1+(R^c_{ab})^2_1(R^c_{ba})^1_2 &= \frac{e^{-i\theta_a-i\theta_b}}{e^{-i\theta_c}} \\ (R^c_{ab})^1_2(R^c_{ba})^2_1+(R^c_{ab})^2_2(R^c_{ba})^2_2 &= \frac{e^{-i\theta_a-i\theta_b}}{e^{-i\theta_c}} \\ (R^c_{ab})^1_2(R^c_{ba})^1_1+(R^c_{ab})^2_2(R^c_{ba})^1_2 &= 0 \\ (R^c_{ab})^1_1(R^c_{ba})^2_1+(R^c_{ab})^2_1(R^c_{ba})^2_2 &= 0\end{aligned} \tag{94}$$

which for $a=b=c=\Psi_i$ becomes

$$\begin{aligned}(R_{11})^2+(R_{12})(R_{21}) &= exp(-i\theta_{\Psi_i}) \\ (R_{21})(R_{12})+(R_{22})^2 &= exp(-i\theta_{\Psi_i}) \\ (R_{21})(R_{11})+(R_{22})(R_{21}) &= 0 \\ (R_{11})(R_{12})+(R_{12})(R_{22}) &= 0\end{aligned} \tag{95}$$

where we have defined $(R^{\Psi_i}_{\Psi_i\Psi_i})^j_k = R_{kj}$ to simplify the symbolism.

Taking Ψ_2 as the anyon of choice, the system (95) becomes

$$\begin{aligned}(R_{11})^2+(R_{12})(R_{21}) &= 1 \\ (R_{21})(R_{12})+(R_{22})^2 &= 1 \\ (R_{21})(R_{11})+(R_{22})(R_{21}) &= 0 \\ (R_{11})(R_{12})+(R_{12})(R_{22}) &= 0\end{aligned} \tag{96}$$

since $exp(-i\theta_{\Psi_2}) = T^{11}_{11} = 1$, as seen in table 7. These equations together with the condition that the R-matrix needs to be unitary, accept the solutions

$$R^{\Psi_2}_{\Psi_2\Psi_2} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ (R^{\Psi_2}_{\Psi_2\Psi_2})' = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\ (R^{\Psi_2}_{\Psi_2\Psi_2})'' = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{97}$$

up to a total phase. The trivial solution also gives a trivial $F^{\Psi_i}_{\Psi_i\Psi_i\Psi_i}$-matrix whereas the other to solutions lead to the same non-trivial F-matrix. Out of these three possible solutions we accept the one signified as R', since it corresponds to a two dimensional representation of the Braid group on three strands [10] being of the form

$$\begin{pmatrix} -z & 0 \\ 0 & 1 \end{pmatrix}$$

with $z \neq -1, 0$ and as we know the B-matrices form irreducible representations of the Braid group [6].

Having both of our needed R-matrices, $R^1_{\Psi_2\Psi_2}$ and $R^{\Psi_2}_{\Psi_2\Psi_2}$, we can now go ahead and input them in (88) for $\Psi_i = \Psi_2$ which gives the system of equations

$$\begin{aligned} F_{11} &= (F_{11})^2 + F_{12}F_{21} \\ -F_{12} &= F_{11}F_{12} + F_{12}F_{22} \\ -F_{21} &= F_{21}F_{11} + F_{22}F_{21} \\ F_{11} &= F_{21}F_{12} + (F_{22})^2 \end{aligned} \tag{98}$$

where we have again defined $F_{kj} = (F^{\Psi_2}_{\Psi_2\Psi_2\Psi_2})^j_k$. The solution to the system is the F-matrix:

$$F^{\Psi_2}_{\Psi_2\Psi_2\Psi_2} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \tag{99}$$

We can also construct the B-matrix that braids the second and third Ψ_2 anyons, which as they are not in the same fusion channel in the standard basis will not be in Block-diagonal form:

$$B^{\Psi_2}_{\Psi_2\Psi_2\Psi_2} = F^{-1}RF = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{100}$$

This matrix as expected corresponds to the second two-dimensional representation of the braid group on three strands [10] since it is of the form

$$\begin{pmatrix} \frac{1}{z+1} & f \\ g & -\frac{z^2}{z+1} \end{pmatrix}$$

for $z = 1$ and $fg = \frac{z(z^2+z+1)}{(z+1)^2} = \frac{3}{4}$.

Obviously, these two matrices, $R^{\Psi_i}_{\Psi_i\Psi_i}$ and $B^{\Psi_2}_{\Psi_2\Psi_2\Psi_2}$, do not span the whole $SU(2)$ and as such by using only Ψ_2 anyons, or any Ψ_i anyons for that matter, universal quantum computation cannot be achieved. In truth, even if the whole spectrum of particles in the theory were to be used, it would still not be enough as the non-abelian simple group of minimal order for universal classical computation is the alternating group on five elements A_5 [5].

7 Conclusion

Putting everything together, we see that although the Quantum Double anyon model for the alternating group on four elements $D(A_4)$ is described by an anyonic particle spectrum on a fourteen dimensional Hilbert space, it is still lacking in computational power. Given that the higher dimensional superselection sectors Ψ_i and $\Lambda_l^{\pm}$ have dimension three (Ψ_i) or five ($\Lambda_l^{\pm}$) accordingly, the dimension grows much faster than 2^n for every particle added.

Moreover, were we to compare this model with two of the most established ones, the Fibonacci and Ising anyon models, we see that it lacks the computational power of the Fibonacci model while being much more complex than both of them, as thousands of calculations are required in order to compute all of the F and R matrices either through the Pentagon and Hexagon equations or straight from the action of the algebra representations on the anyon states [6].

Finally, it is unlikely that in the near future anyons in the $D(A_4)$ theory will be experimentally realized, as that has yet to be accomplished even for the seemingly much simpler Fibonacci anyons. The full complexity and the power for universal quantum computation presents an interesting set of problems for future research.

A Tables

	Representatives
${}^0C$	${}^0x_1 = e$
${}^1C$	${}^1x_1 = e, {}^1x_2 = (124), {}^1x_3 = (123)$
${}^2C$	${}^2x_1 = e, {}^2x_2 = (234), {}^2x_3 = (143), {}^2x_4 = (124)$
${}^3C$	${}^3x_1 = e, {}^3x_2 = (134), {}^3x_3 = (243), {}^3x_4 = (123)$

Table 9: The representatives of the equivalence classes of the quotient groups $A_4/A_4, A_4/{}^1N, A_4/{}^2N$ and $A_4/{}^3N$.

A_4	e	(123)	(124)	(132)	(134)	(142)	(143)	(234)	(243)	(12)(34)	(13)(24)	(14)(23)
e	e	(123)	(124)	(132)	(134)	(142)	(143)	(234)	(243)	(12)(34)	(13)(24)	(14)(23)
(123)	(123)	(132)	(13)(24)	e	(234)	(143)	(14)(23)	(12)(34)	(124)	(134)	(243)	(142)
(124)	(124)	(14)(23)	(142)	(134)	(13)(24)	e	(243)	(123)	(12)(34)	(143)	(132)	(234)
(132)	(132)	e	(243)	(123)	(12)(34)	(14)(23)	(142)	(134)	(13)(24)	(234)	(124)	(143)
(134)	(134)	(124)	(12)(34)	(14)(23)	(143)	(234)	e	(13)(24)	(132)	(123)	(142)	(243)
(142)	(142)	(234)	e	(13)(24)	(132)	(124)	(12)(34)	(14)(23)	(143)	(243)	(134)	(123)
(143)	(143)	(12)(34)	(123)	(243)	e	(13)(24)	(134)	(142)	(14)(23)	(124)	(234)	(132)
(234)	(234)	(13)(24)	(134)	(142)	(14)(23)	(12)(34)	(123)	(243)	e	(132)	(143)	(124)
(243)	(243)	(143)	(14)(23)	(12)(34)	(124)	(132)	(13)(24)	e	(234)	(142)	(123)	(134)
(12)(34)	(12)(34)	(243)	(234)	(143)	(142)	(134)	(132)	(124)	(123)	e	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(142)	(143)	(234)	(243)	(123)	(124)	(132)	(134)	(14)(23)	e	(12)(34)
(14)(23)	(14)(23)	(134)	(132)	(124)	(123)	(243)	(234)	(143)	(142)	(13)(24)	(12)(34)	e

Table 10: The multiplication table of the group A_4 using cyclic notation.

References

[1] R.P. Feynman. "Simulating physics with computers". Int. J. Theor. Phys. 21 (1982), pp. 467.

[2] G. Kurizki et al. "Quantum technologies with hybrid quantum systems". PNAS 112 (2015), pp. 3866-3873. DOI: 10.1073/pnas.1419326112.

[3] S. Chang. "Topological states of quantum matter". APS Physics 1, 6 (2008). DOI: 10.1103/Physics.1.6, url: http://link.aps.org/doi/10.1103/Physics.1.6

[4] F. Wilczek. "Quantum Mechanics of Fractional-Spin Particles". Phys. Rev. Lett. 49 (1982) pp. 957-959. DOI: 10.1103/PhysRevLett.49.957.

[5] J. Preskill. Lecture Notes for Physics 219: Quantum Computation. `http://www.theory.caltech.edu/~preskill/ph219/topological.pdf`. Accessed on 2020/01/15. June 2004.

[6] M. de Wild Propitius and F.A. Bais, *Discrete gauge theories*. 1995. `arXiv:hep-th/9511201v2` [hep-th].

[7] C. Carnahan, D. Zeuch and N.E. Bonesteel. "Systematically generated two-qubit anyon braids". Phys. Rev. A93 (2016).

[8] P. van Driel, F.A. Bais and M. de Wild Propitius. "Quantum symmetries in discrete gauge theories". Phys.Lett. B280 (1992), pp. 63–70.

[9] E. P. Verlinde. "Fusion Rules and Modular Transformations in 2D Conformal Field Theory" 1988 Nucl. Phys. B300 pp. 360-376.

[10] M. Kirdar. "Two Dimensional Representations of the Braid Group B3 and its Burau Representation Squared". arXiv:1905.08452 [math.RT]

[11] J. K. Pachos, *Introduction to Topological Quantum Computation*, Cambridge University Press (2012), doi:10.1017/CBO9780511792908.

[12] V. Lahtinen and J. K. Pachos, "A Short Introduction to Topological Quantum Computation", SciPost Phys. 3, 021 (2017). doi:10.21468/SciPostPhys.3.3.021.

[13] S. Sternberg. *Group theory and physics*. Cambridge University Press (1994). ISBN 0 521 55885 9.

[14] S. Trebst et al. "A Short Introduction to Fibonacci Anyon Models". Progress of Theoretical Physics Supplement 176 (2008), pp. 384–407. issn: 0375-9687. doi:10.1143/ptps.176.384. url: `http://dx.doi.org/10.1143/PTPS.176.384`.

[15] A. Kitaev. *Anyons in an exactly solved model*. Annals of Physics 321.1 (Jan. 2006), pp. 2–111. issn: 0003-4916. doi:10.1016/j.aop.2005.10.005. url: `http://dx.doi.org/10.1016/j.aop.2005.10.005`.

[16] F. A. Bais and J. C. Romers. “The modular S-matrix as order parameter for topological phase transitions”. New Journal of Physics 14.3 (Mar. 2012), p. 035024. issn: 1367-2630. doi:10.1088/1367-2630/14/3/035024. url: http://dx.doi.org/10.1088/1367-2630/14/3/035024.

[17] R. Dijkgraaf, V. Pasquier and P. Roche. "Quasi hopf algebras, group cohomology and orbifold models". Nucl. Phys. B (Proc. Suppl.) 18B (1991), pp. 60-72.

[18] F.A. Bais. "Flux Metamorphosis". Nucl. Phys. B 170 (1980) pp. 32-43.

[19] M. Freedman, A. Kitaev, M. Larsen and Z. Wang. "Topological Qunatum Computation". Bulletin of the American Mathematical Society 40 (2002) pp. 31-38.

[20] C. Nayak et al. "Non-Abelian Anyons and Topological Quantum Computation". Rev. Mod. Phys. 80 (2008) pp. 1083, doi: 10.1103/RevModPhys.80.1083, arXiv:0707.1889 [cond-mat.str-el].