Quantum Doubles and Anyonic Systems

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Abstract

This master's thesis is divided in three main parts. In the first part, we present the two fundamental transformations between anyons, Flux metamorphosis and Aharonov-Bohm effect. We introduce the braid group, its defining relations and its representation via the quantum double. Then, we construct the quantum double for the dihedral group D_5, specify the modular generators of the theory (S and T matrices) and give a simple example of anyon scattering.

In the second part, we focus on the quantum computational aspect of groups/models, with the goal to derive universal quantum computation. We define the encoding of a qubit in the fusion space and then we construct the generators of the braid group for various anyon models, including $\mathcal{D}(D_5)$. We establish general protocols of encoding and processing information with anyons, one using F and R-symbols as well as the fusion rules and another one with pairs of fluxes. We work with Fibonacci and Ising anyon models, where the former is considered as the ideal model for computing. We also point out the difficulty to construct known gates inside $\mathcal{D}(D_5)$.

In the third and final part, we give two examples of universal quantum computation, one with qutrit encoding in $\mathcal{D}(\mathcal{S}_3)$ and another one using pair of fluxes with simple perfect groups. We note in the conclusions that dihedral anyons have the potential of constructing a universal gate set but the way to illustrate this is beyond the purpose of this thesis.

Περίληψη

Η συγκεκριμένη διπλωματική εργασία χωρίζεται σε τρία κύρια μέρη. Στο πρώτο μέρος, παρουσιάζουμε τους δυο θεμελιώδεις μετασχηματισμούς μεταξύ ανυονίων, την μεταμόρφωση ροής και το φαινόμενο *Aharonov-Bohm*. Εισάγουμε το *braid group*, τις βασικές σχέσεις της ομάδας και την αναπαράσταση του μέσω του μοντέλου ανυονίων πεπερασμένης ομάδας. Έπειτα, κατασκευάζουμε το μοντέλο ανυονίων για την ομάδα D_5, προσδιορίζουμε τους γεννήτορες του *modular group* για αυτη την θεωρία (S και T πίνακες) και δίνουμε ένα απλό παράδειγμα σκέδασης ανυονίων.

Στο δεύτερο μέρος, εστιάζουμε στο υπολογιστικό κομμάτι των ομάδων/μοντέλων, με στόχο να εξάγουμε καθολική κβαντική υπολογιστική. Ορίζουμε την κωδικοποίηση ενός *qubit* στον χώρο σύντηξης και μετά κατασκευάζουμε τους γεννήτορες του *braid group* για διάφορα ανυονικά μοντέλα, συμπεριλαμβανομένου του $\mathcal{D}(D_5)$. Καθιερώνουμε γενικά πρωτόκολλα κωδικοποίησης και επεξεργασίας της πληροφορίας με ανυόνια, ένα χρησιμοποιώντας τα F και R σύμβολα καθώς και τους κανόνες σύντηξης και ένα άλλο με ζεύγη ροών. Εργαζόμαστε με τα *Fibonacci* και *Ising* μοντέλα ανυονίων, όπου τα πρώτο θεωρείται ως το ιδανικό μοντέλο για υπολογιστική. Επίσης, υποδεικνύουμε την δυσκολία κατασκευής γνωστών πυλών μέσα στο $\mathcal{D}(D_5)$.

Στο τρίτο και τελευταίο μέρος, δίνουμε δύο παραδείγματα καθολικής κβαντικής υπολογιστικής, ένα με *qutrit* κωδικοποίηση στο $\mathcal{D}(\mathcal{S}_3)$ και ένα άλλο χρησιμοποιώντας ζεύγη ροών με τέλειες απλές ομάδες. Σημειώνουμε στα συμπεράσματα ότι τα δίεδρα ανυόνια έχουν την δυνατότητα κατασκευής ενός καθολικού συνόλου πυλών αλλά ο τρόπος να υλοποιηθεί αυτό είναι πέραν του σκοπού αυτής της εργασίας.

Contents

1 Introduction

The concept of quantum computers was introduced by Richard Feynman in 1982 when he proposed that quantum many-body systems can be simulated exponential faster on a quantum computer that a classical one, by exploiting the basic principles of quantum mechanics. Later in 1994 Peter Shor followed, who created an innovative quantum algorithm for the factorization of a k digit number. The progress made was that information could be processed faster as well as the tolerance to the system size was enhanced. Quantum computation is based on three different procedures: initialization, unitary evolution and measurement. Suppose we have a system in hand, defined in a Hilbert space $\mathcal{H}$. Firstly, we initialize the state of the system to a desired and known state $|\psi_0\rangle$. Then under specific dynamics which we control (environment) the system evolves in time with a corresponding unitary operator $\hat{U}(t)$ to a final state $|\psi\rangle = \hat{U}(t)\,|\psi_0\rangle$ by the manner that Schrodinger's equation dictates. At last, we measure the state of the system at an instant of the evolution and read the outcome.

Nevertheless, we are dealing with a lot of problems when we conduct quantum computation in practice. The two predominant sources of error is decoherence and imperfect operations on the system. The former refers to the inability of sustaining large quantum systems bounded and not affecting their quantum properties (such as entanglement) due to their interactions with the outside environment (noise) while the latter alludes to the fact that quantum operations or as we call gates cannot be implemented with perfect accuracy so that small functional imperfections combined together result in a quantum computer that fails to operate. The need for a $fault-tolerant$ quantum computer still remains so obvious [1].

The idea imposed to fix such technical obstacles is topological quantum computing. Topological quantum computing proposes the use of 2D exotic quasiparticles called anyons for the purpose of processing and encoding information. Anyons are localized excitations that appear in 2D (artificial) materials which behave in a topological manner. They have highly entangled degenerate ground states, encode non locally information which can be revealed only when we bring them together (fusion) and we process this informational content when we encircle one another (braiding). They are indistinguishable by local operations and because of their topological nature, manipulating the information that they carry is so much more efficient because the path of their braids is topologically protected by small deformations of their trajectory [2]. So topology is so far the best tool to confront regular emerging errors.

In three spatial dimensions, the statistics of two identical particles is not as rich as in two dimensions. The reason for this argument is that any circular closed path that a particle follows in 3D can be continuously deformed (is equivalent) to the trivial path (no action at all or stay at the same place). This happens as we make the trajectory go above or below the particle that we encircle. This is why in 3D when we move a particle in a loop around its indistinguishable partner, the operation we do is trivial (act with the identity matrix) [2]:

$$\hat{R}^2\,|\psi_1\psi_2\rangle = \hat{I}\,|\psi_1\psi_2\rangle = |\psi_1\psi_2\rangle \Rightarrow \hat{R}\,|\psi_1\psi_2\rangle = \pm\,|\psi_2\psi_1\rangle\,,$$

where the exchange operator $\hat{R}$ acting on the eigenvector state $|\psi_1\psi_2\rangle$ of the two identical particles gives ± 1 as eigenvalues. So we have two kinds of particles: *bosons*, where the total wavefunction of the two particles is symmetric under the exchange, and $fermions$, where the total wavefunction is antisymmetric upon the exchange of the particles.

In two dimensions the scenery is far more intriguing. When we transport, in a loop, one of the two indistinguishable particle around the other, it is not necessary that we return to the initial state. The explanation is that when we attempt to deform continuously the loop to a single point we always cut through the other particle (we lack of an extra dimension here to slide over the particle). In general, when we exchange the two particles in 2D we get an arbitrary phase for *abelian* anyons and a unitary operator for *non abelian* anyons :

$$\hat{R}\left|\psi_1\psi_2\right\rangle = e^{i\phi}\left|\psi_2\psi_1\right\rangle \Rightarrow \hat{R}^2\left|\psi_1\psi_2\right\rangle = e^{2i\phi}\left|\psi_1\psi_2\right\rangle \qquad \textit{abelian anyons},$$

$$\hat{R}\left|\psi_1\psi_2\right\rangle = \hat{U}_{12}\left|\psi_2\psi_1\right\rangle \Rightarrow \hat{R}^2\left|\psi_1\psi_2\right\rangle = \hat{U}_{21}\hat{U}_{12}\left|\psi_1\psi_2\right\rangle \qquad \textit{non abelian anyons},$$

where $e^{2i\phi} \neq 1$ and $\hat{U}_{21}\hat{U}_{12} \neq \hat{I}$ for an arbitrary case. Only when $\phi = 0$ or $\phi = \pi$, we have *bosons* and $fermions$ respectively (the non abelian anyons, for which $\left[\hat{U}_{12}, \hat{U}_{21}\right] \neq \hat{I}$, are neither bosons nor fermions) [2].

Every distinct trajectory between a collection of n indistinguishable particles in 3D can be represented by some action of the elements of the permutation group S_n (because of the relation $\hat{R}^2 = \hat{I}$ in 3D for two particle systems). In 2D (because $\hat{R}^2 \neq \hat{I}$ in general) instead of the symmetric group S_n we have the braid group B_n, when we talk for indistinguishable particles. The one-dimensional representations of B_n represent the abelian anyons whereas higher-dimensional representations of the braid group describe non abelian anyons. When the particles are distinguishable, we have representations of the colored or pure braid group P_n, where the particles carry different colors to be distinguished and after any kind of braid each particle returns to its initial fixed position in the plane. Geometrically, an n-braid is a collection of n disjoint strings where the endpoints are fixed (evolving upwards in time). If we permit any kind of permutation between the endpoints we take B_n but if we prohibit it we have P_n. The groups S_n,B_n and P_n have infinite order meaning we can have infinite different combinations of their elements [7].

Finally, a physical phenomenon strongly connected with anyons is the $Aharonov-Bohm$ effect. It has been observed that when a particle with charge q is being transported adiabatically (given that the adiabatic condition is true) around a magnetic flux ϕ, then the wavefunction of the particle acquires a phase $e^{iq\phi}$, which has purely topological origin since it depends only on the winding number and not at the shape of the path [2]. We will later introduce this phenomenon in a more mathematical way, when we will define anyons as doublets of charges and fluxes (also have spin).

The most promising systems for the realization of anyons are the Fractional Quantum Hall (FQH) states in two dimensional electron (gas) systems at low temperatures and high magnetic fields, where several non abelian anyons occupy a degenerate ground state. When the highly degenerate Landau levels are filled up to a fraction ($\nu = p/q$ with $p, q \in \mathbb{Z}$), the elementary excitations carry fractional charge of magnitude $e^* = e/q$ and are candidates for realizations of all kinds of anyons.

2 Quantum double

Before we begin analyzing the quantum double model $\mathcal{D}(H)$, we have to introduce the tools that we will be working with and their transformation laws. We start with a Lagrangian governed by a *Chern-Simons* theory, which has it's gauge symmetry spontaneously broken down via Higgs mechanism so eventually we are left with a finite group H. This group can be abelian or non abelian. The elements of the group are the pure fluxes of this model while the irreducible representations of the group are our pure charges. The doublets of charges and fluxes will form our anyons [3]. We will discuss the basic topological interactions among these different flux/charge composites.

2.1 Flux metamorphosis

Suppose we are equipped with two fluxes (two infinite vertical strings referred to the magnetic vortices) $g, h \in H$. So the system is in the state :

$$|hg\rangle \equiv |h\rangle |g\rangle \tag{1}$$

When we interchange the h flux with the g flux counterclockwise, the stationary flux in the center becomes conjugate of g (global symmetry transformation) [3]:

$$\mathcal{R}|hg\rangle = \left|hgh^{-1}\right\rangle |h\rangle , \tag{2}$$

such that the total flux of the configuration $hg = hgh^{-1}h$ is conserved. The states of the two fluxes have been reversed after the interchange because of their position (from the left to right).

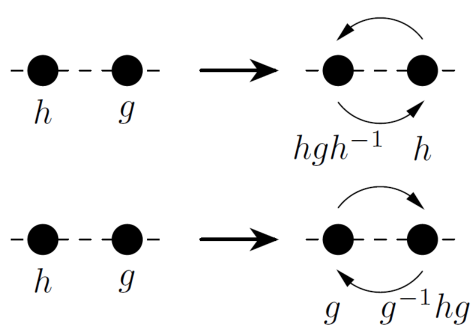


Figure 1: Interchange of two fluxes $h, g \in H$ counterclockwise (above) with the $\mathcal{R}$ operator and clockwise (below) with the $\mathcal{R}^{-1}$ operator.

When the two fluxes do not commute and at least one of the two carries non trivial charge, we have the non abelian $Aharonov - Bohm$ effect. Suppose we interchange again counterclockwise the two fluxes, now hgh^{-1} and h. We take :

$$\mathcal{R}^2 |hg\rangle = \mathcal{R}\left|hgh^{-1}\right\rangle |h\rangle = \left|(hgh^{-1})h(hgh^{-1})^{-1}\right\rangle \left|hgh^{-1}\right\rangle , \tag{3}$$

according to relation (2).

However, the group elements that we attach to the fluxes depend on our conventions. Suppose I am presented with k fluxons (particles that carry flux), and that I use my standard charges to measure the flux of each particle. I assign group elements $a_1, a_2, ..., a_k \in H$ to the k fluxons. You are then asked to measure the flux, to verify my assignments. But your standard charges differ from mine, because they have been surreptitiously transported around another flux (one that I would label with $h \in H$). Therefore you will assign the group elements $ha_1h^{-1}, ha_2h^{-1}, ..., ha_kh^{-1}$ to the k fluxons; our assignments differ by an overall conjugation by h. Because exactly that two fluxes transform with conjugation by the above interchange action, we will consider as different pure fluxons (anyons that carry flux and have trivial charge) those group elements that belong in different conjugacy classes of the finite group H [4].

2.2 Abelian Aharonov-Bohm effect

We will now present the transformation that comprises the mathematical analog of the abelian Aharonov-Bohm effect. When we include a mass term in the Lagrangian that was reported initially, then after the symmetry is spontaneously broken down, we also have (pure) charge particles in the form of any kind of the irreducible representations of the group H, except form the fluxes. Different types of charges correspond to all distinct irreducible representations of the group. Suppose we have a charge/flux pair in hand. When we encircle the charge counterclockwise around the flux we take the following transformation [3]:

$$\mathcal{R}^2 \left|h\right\rangle \left|v\right\rangle = \left|h\right\rangle \left|\Gamma(h)v\right\rangle, \tag{4}$$

where $\Gamma(h)$ is the matrix assigned to the group element h in the irep Γ, $|v\rangle$ is the internal charge state and $|h\rangle$ is the state of the flux ($h \in H$). Assuming the dimension of Γ is $|\Gamma|$ and defining an orthonormal basis for $|v\rangle$ (same dimension as Γ), with $i = 1, 2, ..., |\Gamma|$, the transformation of the charge basis vectors is written as [4]:

$$\left|v, i\right\rangle' = \sum_{j=1}^{|\Gamma|} \Gamma_{ij}(h) \left|v, j\right\rangle \tag{5}$$

The irreps of the group (and as a consequence the charge states) don't have to be one dimensional. When the irrep is 1D we have the classical and trivial abelian $Aharonov-Bohm$ effect, in which we get just a phase. When the irrep has a higher dimension, then we have the non-trivial (but still abelian) $Aharonov-Bohm$ effect in which we result with a unitary transformation (rotation) of the state $|v\rangle$. We have to specialize in this point which irreps count as different types of charges.

In principle, charge can be measured in an Aharonov-Bohm interference experiment [4]. We could hide the object whose charge is to be found behind a screen in between two slits, shoot a beam of carefully calibrated fluxons at the screen, and detect the fluxons on the other side. From the shift and visibility of the interference pattern revealed by the detected positions of the fluxons, we can determine $\Gamma(b)$ for each $b \in H$, and so deduce Γ.

However, there is a catch if the object being analyzed carries a nontrivial flux $a \in H$ as well as charge. Since carrying a flux b around the flux a changes a to bab^{-1}, the two possible paths followed by the b flux do not interfere, if a and b do not commute. After the b flux is detected, we could check whether the a flux has been modified, and determine whether the b flux passed through the slit on the left or the slit on the right. Since the flux (a or $N(bab^{-1})$) is correlated with the "which way" information (left or right slit), the interference is destroyed.

Therefore, the experiment reveals information about the charge only if a and b commute. Hence the charge attached to a flux a is not described as an irreducible representation of H; instead it is an irreducible representation of a subgroup of H, the centralizer $N(a)$ of a in H, which is defined as :

$$N(a) = \{b \in H | ab = ba\} \tag{6}$$

The centralizers $N(a)$ and $N(bab^{-1})$ are isomorphic, so we may associate the centralizer with a conjugacy class C_h of H rather than with a particular element $h \in H$, and denote it as $N(C_h)$. Thus, the list of the distinguishable charges is made by the ireps of the centralizers of the group H.

2.3 Superselection sectors

Collecting all the previous facts, each type of particle that can occur in our non abelian superconductor really has two labels : a conjugacy class C_h describing the flux, and the α-th irreducible representation Γ of $N(C_h)$ describing the charge [4]:

$$|C_h, {}^{\alpha}\Gamma\rangle$$

We say that C_h and ${}^{\alpha}\Gamma$ label the *superselection sectors* of the theory, as these are the properties of a localized object that must be conserved in all local physical processes. So, all the different anyon types that can be distinguished are labeled as $(C_h, {}^{\alpha}\Gamma)$ and their dimension is $d_{(C_h, {}^{\alpha}\Gamma)} = |C_h| \cdot |{}^{\alpha}\Gamma|$. The total dimension of the group is given by summing over all types of anyons :

$$\mathcal{D}^2 = \sum_{C_h} \sum_{{}^{\alpha}\Gamma} d^2_{(C_h, {}^{\alpha}\Gamma)} = \sum_{C_h} |C_h|^2 \sum_{{}^{\alpha}\Gamma} |{}^{\alpha}\Gamma|^2 \tag{7}$$

Accounting that the sum over the dimension squared for all irreducible representations of a finite group is the order of the group, and the order of the normalizer $N(C_h)$ is $|H|/|C_h|$, we obtain :

$$\mathcal{D}^2 = \sum_{C_h} |C_h|^2 \cdot \frac{|H|}{|C_h|} = |H| \sum_{C_h} |C_h| = |H|^2 \tag{8}$$

2.4 Algebraic structure of quantum double

There are two physical operations upon the particles of any discrete H gauge theory, explained previously. We can independently measure their magnetic flux and their electric charge through quantum interference experiments. Flux measurements then correspond to operators P_h projecting out a particular flux h, while the charge of a specific particle can be detected through its transformation properties under the residual global symmetry transformations $g \in N(h) \lhd H$ that commute with the flux h of the particle (the centralizer is a normal subgroup of H). The combination of global symmetry transformations followed by flux measurements :

$$\{P_h\, g\}_{h,g \in H}$$

generate the quantum double $\mathcal{D}(H) = \mathcal{F}(H) \otimes \mathbf{C}[H]$ [3]. If we have solely a flux measurement, this corresponds to trivial global symmetry transformation :

$$P_h \equiv P_h\, e \stackrel{\text{def}}{=} |h\rangle \langle h| \,, \tag{9}$$

which acts only in the flux space of a state :

$$P_h \,|h_i, {}^{\alpha}\Gamma\rangle = P_h \,|h_i\rangle \otimes |{}^{\alpha}\Gamma\rangle = \delta_{h,h_i} \,|h_i\rangle \otimes |{}^{\alpha}\Gamma\rangle = \delta_{h,h_i} \,|h_i, {}^{\alpha}\Gamma\rangle \tag{10}$$

The flux projecting operators P_h follow the algebra :

$$P_h P_{h'} = \delta_{h,h'} P_h \tag{11}$$

Proof. By making two successive flux measurements we get :

$$P_h P_{h'} = |h\rangle \langle h|h'\rangle \langle h'| = \delta_{h,h'} |h\rangle \langle h'|$$

But the right side above is non zero only for $h = h'$, so we can change $|h'\rangle$ to $|h\rangle$:

$$P_h P_{h'} = \delta_{h,h'} |h\rangle \langle h| = \delta_{h,h'} P_h \qquad \blacksquare$$

Flux projection operators and global symmetry transformations for a nonabelian finite gauge group H do not commute because global symmetry transformations $g \in H$ affect the fluxes through conjugation :

$$g P_h = P_{ghg^{-1}}\, g \tag{12}$$

Proof. Let $|h_i\rangle$ an arbitrary flux state (we omit the charge part $|^{\alpha}\Gamma\rangle$ of the state because flux measurements and global symmetry transformations act on the flux space). We have that :

$$g P_h |h_i\rangle = \delta_{h,h_i} g |h_i\rangle = \delta_{h,h_i} |g h_i g^{-1}\rangle$$

The action of the right side of the equation is :

$$P_{ghg^{-1}}\, g |h_i\rangle = P_{ghg^{-1}} |g h_i g^{-1}\rangle = \delta_{ghg^{-1}, gh_ig^{-1}} |g h_i g^{-1}\rangle$$

But $\delta_{ghg^{-1}, gh_ig^{-1}} = 1$ when $ghg^{-1} = gh_ig^{-1} \Rightarrow h = h_i$, so $\delta_{ghg^{-1}, gh_ig^{-1}} = \delta_{h,h_i}$ and :

$$P_{ghg^{-1}}\, g |h_i\rangle = \delta_{h,h_i} |g h_i g^{-1}\rangle = g P_h |h_i\rangle$$

Because $|h_i\rangle$ is arbitrary, the equation holds in general. $\blacksquare$

Equations (11) and (12) can be gathered in the next relation :

$$P_h\, g \cdot P_{h'}\, g' = \delta_{h, gh'g^{-1}}\, P_h\, gg' \tag{13}$$

Proof.

$$P_h\, g \cdot P_{h'}\, g' = P_h \left[P_{gh'g^{-1}}\, g \right] g' = P_h P_{gh'g^{-1}}\, gg' = \delta_{h, gh'g^{-1}} P_h\, gg' \qquad \blacksquare$$

The different particles $(C_h, {}^{\alpha}\Gamma)$, in which we concluded in subsection 2.3, for the theory constitute the complete set of inequivalent irreducible representations of the quantum double $D(H)$. To make explicit the irreducible action of the quantum double on these particles, we have to develop some further notation. To start with, we will label the group elements in the different conjugacy classes of H as :

$$C_h \equiv {}^{A}C = \left\{ {}^{A}h_1,\ {}^{A}h_2, \ldots,\ {}^{A}h_k \right\}$$

We take the centralizer of the first element ${}^{A}N = N({}^{A}h_1)$ (the centralizers of the other elements in the same conjugacy class are isomorphic to this). Since ${}^{A}N$ is a normal subgroup of H, we can form the quotient group $H/{}^{A}N$ which includes all the left cosets $h\,{}^{A}N$ with $h \in H$ (because ${}^{A}N$ is a normal subgroup of H it holds that $h\,{}^{A}N = {}^{A}N h$) :

$$H/{}^{A}N = \left\{ h\,{}^{A}N | h \in H \right\}$$

The same cosets that arise in the quotient group form an equivalence class. In number, there will be k equivalence classes (distinct cosets), same in number with the elements of the conjugacy class ${}^{A}C$. For each equivalence class, we choose a representative element ${}^{A}x_i$. Let $\{{}^{A}x_1,\ {}^{A}x_2, \ldots,\ {}^{A}x_k\}$ be a set of representatives for the equivalence classes of $H/{}^{A}N$, such that ${}^{A}h_i = {}^{A}x_i {}^{A}h_1 {}^{A}x_i^{-1}$. If the elements h_i, with $i = 1, 2, \ldots, k$, give different cosets $h_i {}^{A}N$, then the union of these cosets give the whole group, that is $h_1 {}^{A}N \cup h_2 {}^{A}N \cup \cdots \cup h_k {}^{A}N = H$. So all the elements of the total cosets are different. Suppose, that we pick as the first coset the one with the identity element inside, and take as its representative ${}^{A}x_1 = e$, for convenience. The basis vectors of the unitary irreducible representation ${}^{\alpha}\Gamma$ of the centralizer ${}^{A}N$ will be denoted by ${}^{\alpha}v_j$. With the previous conventions in mind, the internal Hilbert space V_α^A is spanned by the quantum states (we reduce the notation for the representations as ${}^{\alpha}\Gamma \equiv \alpha$) :

$$\left\{\left|{}^{A}h_i, {}^{\alpha}v_j\right\rangle\right\}_{i=1,\ldots,k}^{j=1,\ldots,|\alpha|}$$

The element $P_h\, g$ of the quantum double acting on the above eigenstates of V_α^A can be represented as (proof in [5]) :

$$\Pi_\alpha^A\left(P_h\, g\right)\left|{}^{A}h_i, {}^{\alpha}v_j\right\rangle = \delta_{h, g{}^{A}h_i g^{-1}}\left|g{}^{A}h_i g^{-1}, \alpha(\tilde{g})_{mj}{}^{\alpha}v_m\right\rangle, \tag{14}$$

with :

$$\tilde{g} := {}^{A}x_k^{-1} g {}^{A}x_i, \tag{15}$$

and ${}^{A}x_k$ defined through ${}^{A}h_k := g{}^{A}h_i g^{-1}$. It is easily verified that this element $\tilde{g}$ constructed from g and the flux ${}^{A}h_i$ indeed commutes with ${}^{A}h_1$ and therefore can be implemented on the centralizer charge.

In the case of a system with two particles $({}^{A}C, \alpha)$ and $({}^{B}C, \beta)$, the corresponding Hilbert space is the tensor product $V_\alpha^A \otimes V_\beta^B$ and the extension of the element $P_h\, g$ is the comultiplication :

$$\Delta\left(P_h\, g\right) = \sum_{h' \cdot h'' = h} P_{h'}\, g \otimes P_{h''}\, g \tag{16}$$

The braid operation is formally implemented by the universal R-matrix, which is an element of $\mathcal{D}(H) \otimes \mathcal{D}(H)$:

$$R = \sum_{h,g \in H} P_g \otimes P_h\, g = \sum_{h,g \in H} P_g\, e \otimes P_h\, g \tag{17}$$

The R matrix acts on a two particle state as a global symmetry transformation on the second particle by the flux of the first particle. The physical braid operator $\mathcal{R}$ that generates a counterclockwise interchange of the two particles is defined as the action of this R matrix followed by a transposition (permutation) map $\sigma : a \otimes b \to b \otimes a$ of the two particles :

$$\mathcal{R}_{\alpha\beta}^{AB} := \sigma \circ \left(\Pi_\alpha^A \otimes \Pi_\beta^B\right)(R) \tag{18}$$

The physical braid operator acting on the basis states $\left|{}^{A}h_i, {}^{\alpha}v_j\right\rangle\left|{}^{B}h_m, {}^{\beta}v_n\right\rangle$ of $V_\alpha^A \otimes V_\beta^B$ gives us the simple formula for it's representation/matrix :

$$\mathcal{R}_{\alpha\beta}^{AB}\left|{}^{A}h_i, {}^{\alpha}v_j\right\rangle\left|{}^{B}h_m, {}^{\beta}v_n\right\rangle = \left|{}^{A}h_i {}^{B}h_m {}^{A}h_i^{-1}, \beta({}^{A}\tilde{h}_i)_{ln}{}^{\beta}v_l\right\rangle\left|{}^{A}h_i, {}^{\alpha}v_j\right\rangle, \tag{19}$$

where the element ${}^{A}\tilde{h}_i$ is defined from equation (15) as ${}^{A}\tilde{h}_i = {}^{B}x_k^{-1} {}^{A}h_i {}^{B}x_m$ and ${}^{B}x_k$ is the representative of ${}^{B}h_k = {}^{A}h_i {}^{B}h_m {}^{A}h_i^{-1}$.

Proof.

$$
\begin{aligned}
\mathcal{R}^{AB}_{\alpha\beta}\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle &= \sigma\circ\left(\Pi^A_\alpha\otimes\Pi^B_\beta\right)(R)\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle \\
&= \sigma\circ\left(\Pi^A_\alpha\otimes\Pi^B_\beta\right)\left(\sum_{h,g} P_g\otimes P_h\, g\right)\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle \\
&= \sigma\circ\sum_{h,g}\Pi^A_\alpha\left(P_g\right)\otimes\Pi^B_\beta\left(P_h\, g\right)\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle \\
&= \sigma\circ\sum_{h,g}\Pi^A_\alpha\left(P_g\right)\left|{}^A h_i, {}^\alpha v_j\right\rangle\otimes\Pi^B_\beta\left(P_h\, g\right)\left|{}^B h_m, {}^\beta v_n\right\rangle
\end{aligned}
$$

and by using equations (10) and (14) we get :

$$
\begin{aligned}
\mathcal{R}^{AB}_{\alpha\beta}\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle &= \sigma\circ\sum_{h,g}\delta_{g,{}^A h_i}\delta_{h,g{}^B h_m g^{-1}}\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|g{}^B h_m g^{-1}, \beta(\tilde{g})_{ln}{}^\beta v_l\right\rangle \\
&= \sigma\circ\sum_{h}\delta_{h,{}^A h_i{}^B h_m{}^A h_i^{-1}}\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^A h_i{}^B h_m{}^A h_i^{-1}, \beta({}^A\tilde{h}_i)_{ln}{}^\beta v_l\right\rangle \\
&= \sigma\circ\left(\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^A h_i{}^B h_m{}^A h_i^{-1}, \beta({}^A\tilde{h}_i)_{ln}{}^\beta v_l\right\rangle\right) \\
&= \left|{}^A h_i{}^B h_m{}^A h_i^{-1}, \beta({}^A\tilde{h}_i)_{ln}{}^\beta v_l\right\rangle\left|{}^A h_i, {}^\alpha v_j\right\rangle \qquad \blacksquare
\end{aligned}
$$

2.5 Fusion, spin and braid groups

Let $(\Pi^A_\alpha, V^A_\alpha)$ and (Π^B_β, V^B_β) be two irreducible representations of the quantum double $\mathcal{D}(H)$. The tensor product representation $(\Pi^A_\alpha\otimes\Pi^B_\beta, V^A_\alpha\otimes V^B_\beta)$ constructed by means of the comultiplication (16) need not be irreducible. In general, it gives rise to a decomposition :

$$
\Pi^A_\alpha\otimes\Pi^B_\beta = \bigoplus_{C,\gamma} N^{AB\gamma}_{\alpha\beta C}\,\Pi^C_\gamma, \tag{20}
$$

where $N^{AB\gamma}_{\alpha\beta C}$ stands for the multiplicity of the irreducible representation $(\Pi^C_\gamma, V^C_\gamma)$. From the orthogonality relation for the characters of the irreducible representations of $\mathcal{D}(H)$, we infer (proof in [5]) :

$$
\begin{aligned}
N^{AB\gamma}_{\alpha\beta C} &= \frac{1}{|H|}\sum_{h,g} tr\left(\Pi^A_\alpha\otimes\Pi^B_\beta(\Delta(P_h\, g))\right)\, tr\left(\Pi^C_\gamma(P_h\, g)\right)^* \\
&= \frac{1}{|H|}\sum_{\substack{h'\cdot h''=h\\ h,g\in H}} tr\left(\Pi^A_\alpha(P_{h'}\, g)\otimes\Pi^B_\beta(P_{h''}\, g)\right)\, tr\left(\Pi^C_\gamma(P_h\, g)\right)^*,
\end{aligned} \tag{21}
$$

where the first trace is calculated in the basis $\left|{}^A h_i, {}^\alpha v_j\right\rangle\left|{}^B h_m, {}^\beta v_n\right\rangle$ and the second trace in the basis $\left|{}^C h_q, {}^\gamma v_r\right\rangle$. The fusion rule (21) now determines which particles $({}^C C, \gamma)$ can be formed in the composition of two given particles $({}^A C, \alpha)$ and $({}^B C, \beta)$, or if read backwards, gives the decay channels of the particle $({}^C C, \gamma)$.

The fusion algebra, spanned by the elements Π^A_α with multiplication rule (21), is commutative and associative and can therefore be diagonalized. The matrix implementing this diagonalization is the so-called modular S matrix [6] :

$$
\begin{aligned}
S^{AB}_{\alpha\beta} &:= \frac{1}{|H|} tr\mathcal{R}^{2\,AB}_{\ \ \alpha\beta} = \frac{1}{|H|} tr(\mathcal{R}^{BA}_{\beta\alpha}\mathcal{R}^{AB}_{\alpha\beta}) \\
&= \frac{1}{|H|} \sum_{\substack{{}^A h_i \in {}^A C, {}^B h_j \in {}^B C \\ [{}^A h_i, {}^B h_j] = e}} tr\left(\alpha({}^A x_i^{-1\,B} h_j {}^A x_i)\right)^* \ tr\big(\beta({}^B x_j^{-1\,A} h_i {}^B x_j)\big)^*,
\end{aligned} \tag{22}
$$

where the commutator between two group elements is defined as $[g, h] = ghg^{-1}h^{-1}$. The modular S matrix (22) contains all information concerning the fusion algebra defined in (20). In particular, the multiplicities (21) can be expressed in terms of the modular S matrix by means of Verlinde's formula [6] :

$$
N^{AB\gamma}_{\alpha\beta C} = \sum_{D,\delta} \frac{S^{AD}_{\alpha\delta} S^{BD}_{\beta\delta} (S^*)^{CD}_{\gamma\delta}}{S^{eD}_{1\delta}} \tag{23}
$$

The modular T matrix, which contains the spin factors assigned to the particles (anyons) of the theory is :

$$
T^{AB}_{\alpha\beta} := \delta_{\alpha,\beta}\delta^{A,B} \exp\big(2\pi i s_{(A,\alpha)}\big) = \delta_{\alpha,\beta}\delta^{A,B} \frac{1}{d_\alpha} tr\big(\alpha({}^A h_1)\big), \tag{24}
$$

where d_α stands for the dimension of the centralizer charge representation α of the particle $({}^A C, \alpha)$. The $S^{AB}_{\alpha\beta}$ and $T^{AB}_{\alpha\beta}$ matrices now realize a unitary representation of the special linear (modular) group $SL(2, \mathbb{Z})$, under matrix multiplication, over the integers :

$$
SL(2, \mathbb{Z}) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \Big\},
$$

with the following relations :

$$
\mathcal{C} = (ST)^3 = S^2, \tag{25}
$$

$$
S^* = \mathcal{C}S = S^{-1}, \qquad S^t = S, \tag{26}
$$

$$
T^* = T^{-1}, \qquad T^t = T. \tag{27}
$$

The matrices S and T are symmetric and unitary while the charge conjugation operator $\mathcal{C}$ assigns a unique anti-particle $\mathcal{C}({}^A C, \alpha) = ({}^{\bar{A}} C, \bar{\alpha})$ to each particle $({}^A C, \alpha)$, such that the vacuum channel occurs in the fusion rule (20) for the particle/anti-particle pairs.

We introduce the *Artin* group representations for $\mathcal{B}_n$ and $\mathcal{P}_n$, that is the *braid group* and the *pure braid group*. The braid groups $\mathcal{B}_n$, $n \geq 2$, have finitely many generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ (the order of the elements and the group though are infinite) and defining relations [7] :

$$
\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{, if } |i - j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{, if } |i - j| = 1, \end{cases} \tag{28}
$$

where $i, j = 1, \ldots, n - 1$. The first equation says that braids between spatially remote strands commute and the second one is called the $Yang - Baxter$ relation. The *braid group* refers to a collection of identical particles. The generator σ_i produces a counterclockwise rotation ($\mathcal{R}$ matrix) between the ith and the $(i + 1)$th strand (Figure 2).

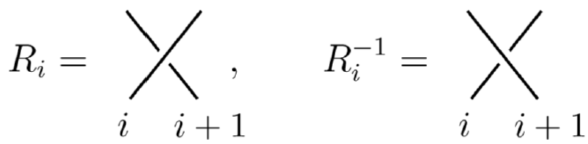


Figure 2: Counterclockwise rotation of two identical adjacent anyons in terms of R_i (σ_i) operator (left) and their clockwise rotation in terms of R_i^{-1} (σ_i^{-1}) operator (right).

On the other hand, the *pure braid group* admits generators [7] :

$$A_{ij} = \sigma_i \, \sigma_{i-1} \cdots \sigma_{j-2} \, \sigma_{j-1}^2 \, \sigma_{j-2}^{-1} \cdots \sigma_{i-1}^{-1} \, \sigma_i^{-1}, \tag{29}$$

which satisfy the relations :

$$A_{rs}^{-1} A_{ij} A_{rs} = \begin{cases} A_{ij} & , r < s < i < j \text{ or } i < r < s < j \,, \\ A_{rj} A_{ij} A_{rj}^{-1} & , r < s = i < j \,, \\ (A_{rj} A_{sj}) A_{ij} (A_{rj} A_{sj})^{-1} & , r = i < s < j \,, \\ (A_{rj} A_{sj} A_{rj}^{-1} A_{sj}^{-1}) A_{ij} (A_{sj} A_{rj} A_{sj}^{-1} A_{rj}^{-1}) & , r < i < s < j \,, \end{cases} \tag{30}$$

with $1 \leq i < j \leq n$. These generators correspond to starting at the ith strand, wrapping around (twist) the jth one, and returning on the same side of the intermediate strands. The ith strand crosses from the back the other strands, makes a loop around the jth strand in a clockwise manner and returns from the back to its initial position (the jth strand is separated from the others, as we see in Figure 3). The pure braid group corresponds to the process of braiding n distinguishable particles.

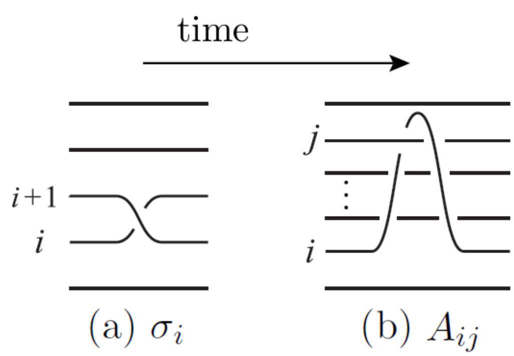


Figure 3: The geometrical braid diagrams for the generators σ_i and A_{ij} of the *braid group* and the *pure braid group* respectively.

The defining relations of the two groups (28) and (30) can easily be proved geometrically. The way we constructed the $\mathcal{R}$ matrices in the quantum double results for them in having finite order [3]:

$$\mathcal{R}^m = \mathbb{1}_{V_\alpha^A} \otimes \mathbb{1}_{V_\beta^B}, \tag{31}$$

where it denotes that the effect of m braidings in the space $V_\alpha^A \otimes V_\beta^B$ is trivial. So, given this restriction, the relations don't add up to the braid group but rather a finite order braid group (we cannot make arbitrary large and different braidings) which we call truncated braid group and symbolize it as $B(n,m)$. As the order of the $\mathcal{R}$ matrix or the particle number increases, the order of the truncated group becomes much higher (we have more distinct braiding possibilities). For n indistinguishable particles $({}^{A}C,\alpha)$, a representation is admitted for the braid group, in the tensor product space $(V_{\alpha}^{A})^{\otimes n}$, as :

$$\sigma_i \mapsto \mathcal{R}_i, \tag{32}$$

with

$$\mathcal{R}_i := \mathbb{1}^{\otimes(n-1)} \otimes \mathcal{R} \otimes \mathbb{1}^{\otimes(n-i-1)}, \tag{33}$$

where $\mathbb{1} \equiv \mathbb{1}_{V_{\alpha}^{A}}$ and $\mathcal{R} = \mathcal{R}_{\alpha\alpha}^{AA}$. In addition to equation (28), we have the extra relation of the truncated braid group (finite order) :

$$\sigma_i^m = e, \qquad\qquad i = 1, \dots, n-1 \tag{34}$$

If specifically $m = 2$, we get the permutation group on n strands :

$$B(n,2) \simeq S_n \tag{35}$$

Suppose now that we have n distinguishable anyons. The system realizes a representation of the pure braid group in the space $V_{\alpha_1}^{A_1} \otimes \cdots V_{\alpha_n}^{A_n}$ as :

$$A_{ij} \mapsto \mathcal{R}_i \cdots \mathcal{R}_{j-2}\mathcal{R}_{j-1}^2\mathcal{R}_{j-2}^{-1} \cdots \mathcal{R}_i^{-1}, \tag{36}$$

with $\mathcal{R}_i$ as defined in (33), with $\mathcal{R} \equiv \mathcal{R}_{\alpha_i\alpha_{i+1}}^{A_iA_{i+1}}$ the braid operator between the ith and $(i+1)$th (distinguishable) particles. Besides equation (29), because of (34), we also have that :

$$A_{ij}^{m/2} = e, \tag{37}$$

from which it is clear that the pure (colored) braid group $P(n,m)$ is only defined for even m.

3 Construction of $\mathcal{D}(D_5)$

We begin by introducing the group. The dihedral group D_5 is the group of symmetries of a regular pentagon (Figure 4).

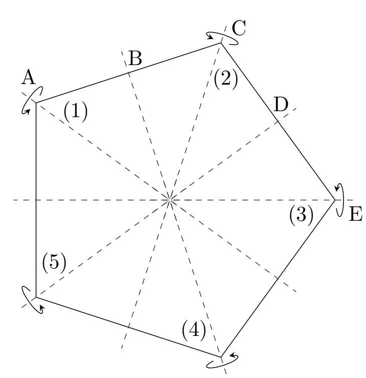


Figure 4: Geometrical axial symmetries of the D_5 group.

The group has two generators, rotation by $\frac{2\pi}{5} = 72^\circ$ denoted as r (arbitrary direction) and reflection across the x axis (line (3) in Figure 4) denoted as s. So we list the elements of the group :

$$\begin{aligned}
&R_0 \equiv e \rightarrow \text{identity element},\\
&R_1 \equiv r \rightarrow \text{rotate clockwise } 72^\circ,\\
&R_2 \equiv r^2 \rightarrow \text{rotate clockwise } 144^\circ,\\
&R_3 \equiv r^3 \rightarrow \text{rotate clockwise } 216^\circ,\\
&R_4 \equiv r^4 \rightarrow \text{rotate clockwise } 288^\circ,\\
&F_3 \equiv s \rightarrow \text{reflect across line } (3),\\
&F_5 \equiv sr = r^{-3}sr^3 \rightarrow \text{reflect across line } (5),\\
&F_2 \equiv sr^2 = r^{-1}sr^1 \rightarrow \text{reflect across line } (2),\\
&F_4 \equiv sr^3 = r^{-4}sr^4 \rightarrow \text{reflect across line } (4),\\
&F_1 \equiv sr^4 = r^{-2}sr^2 \rightarrow \text{reflect across line } (1).
\end{aligned}$$

So the group has the above 10 elements in total :

$$D_5 = \left\langle r, s | r^5 = s^2 = (sr)^2 = e \right\rangle = \left\{ e, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4 \right\}$$

3.1 Conjugacy classes

The conjugacy class of e is just ${}^0C = \{e\}$ because $heh^{-1} = e$ for every $h \in D_5$. To find the conjugacy class of r^n, note that when we conjugate r^n by r^m we get :

$$r^m r^n r^{-m} = r^n \quad \forall m,$$

and when we conjugate r^n by sr^m we get :

$$\begin{aligned}
sr^m r^n sr^m &= sr^m sr^{-n} r^m\\
&= ssr^{-m} r^{-n} r^m\\
&= r^{-n} \quad \forall m
\end{aligned}$$

(Note that since each reflection sr^m has order 2, $(sr^m)^{-1} = sr^m$). Thus the conjugacy class of each r^n contains r^n and r^{-n}. In the case of D_5, this gives us the classes :

$${}^1C = \left\{ r, r^4 \right\}, {}^2C = \left\{ r^2, r^3 \right\}$$

To find the conjugacy class of s, we notice that when we conjugate s by r^m we get :

$$r^m sr^{-m} = sr^{-2m} \quad \forall m,$$

and when we conjugate s by sr^m we get :

$$sr^m ssr^{-m} = s \quad \forall m$$

Since m can be $1, 2, 3$ or 4, $2m$ is either $2, 4, 1$ or 3 since we have to take $2m \pmod 5$. Thus all the reflections form a single conjugacy class :

$${}^3C = \left\{ s, sr, sr^2, sr^3, sr^4 \right\}$$

Since all of the reflections are in this conjugacy class, we don't need to compute the conjugacy class of any of the other reflections. Therefore, D_5 has the 4 conjugacy classes listed above.

3.2 Centralizers and their irreducible representations

For elements in the same conjugacy classes, their centralizers are isomorphic. So we find the centralizer for a single element of every conjugacy class, defined in equation (6). The centralizers are presented below :

Conjugacy class	Centralizer
${}^0C$	D_5
${}^1C$	$\mathbb{Z}_5$
${}^2C$	$\mathbb{Z}_5$
${}^3C$	$\mathbb{Z}_2$

Table 1: The four conjugacy classes of the group D_5 and the respective centralizers of their elements.

We will now derive the character tables for the representations of the centralizers $\mathbb{Z}_2$, $\mathbb{Z}_5$ and D_5.

3.2.1 Character table of $\mathbb{Z}_2$

The cyclic group of order 2 is :

$$\mathbb{Z}_2 = \langle h \rangle = \{e, h\},$$

with $h^2 = e$. The conjugacy classes of $\mathbb{Z}_2$ are $\{e\}$ and $\{h\}$. As the number of conjugacy classes is equal to the number of irreducible representations, the group has 2 irreps. By summing the squares of the dimensions of the irreps, we take the order of the group. That is :

$$|\mathbb{Z}_2| = 2 = 1^2 + 1^2,$$

so we have two 1D irreps. The character of the representation of the identity element e is always equal to the dimension of the irrep (the element e is represented with 1 for 1D irreps and with the identity matrix $\mathbb{1}_{n\times n}$ for nD irreps). Hence, the character table has the form :

$\mathbb{Z}_2$	e	h
$\chi({}^0\tilde{\Gamma})$	1	κ
$\chi({}^1\tilde{\Gamma})$	1	λ

where ${}^0\tilde{\Gamma}$ and ${}^1\tilde{\Gamma}$ are the two 1D irreps. By the character orthogonality :

$$\langle \chi_j, \chi_k \rangle = \frac{1}{|H|} \sum_{h \in H} \chi_j\left(h\right) \overline{\chi_k\left(h\right)} = \delta_{j,k}, \tag{38}$$

which indicates the row orthogonality in the character table, we have :

$$\text{For } j,k = 0 \rightarrow {}^0\tilde{\Gamma} \text{ as } \chi_i \equiv \chi({}^i\tilde{\Gamma}) \Rightarrow \frac{1}{|\mathbb{Z}_2|} \sum_{h \in \mathbb{Z}_2} \chi_0\left(h\right) \overline{\chi_0\left(h\right)} = 1 \Rightarrow$$
$$\frac{1}{2}\big(1 \cdot 1 + \kappa \cdot \kappa^*\big) = 1 \Rightarrow |\kappa|^2 + 1 = 2 \Rightarrow |\kappa|^2 = 1 \Rightarrow \kappa = \pm 1 \in \mathbb{R}$$

We choose $\kappa = 1$ so we have the *trivial* representation. Every group can be represented by the *trivial* representation (1D), so it always exists and it's irreducible (it satisfies every multiplication table).

$$\text{For } j = 0 \to {}^0\tilde{\Gamma} \text{ and } k = 1 \to {}^1\tilde{\Gamma} \Rightarrow \frac{1}{|\mathbb{Z}_2|} \sum_{h \in \mathbb{Z}_2} \chi_0(h) \overline{\chi_1(h)} = 1 \Rightarrow$$
$$\frac{1}{2}\left(1 \cdot 1 + 1 \cdot \lambda^*\right) = 0 \Rightarrow 1 + \lambda^* = 0 \Rightarrow \lambda^* = -1 = \lambda \in \mathbb{R}$$

and we get the *sign* representation. Thus, the character table of $\mathbb{Z}_2$ becomes :

$\mathbb{Z}_2$	e	h
$\chi({}^0\tilde{\Gamma})$	1	1
$\chi({}^1\tilde{\Gamma})$	1	-1

In the following step, we obtain the character table of $\mathbb{Z}_5$.

3.2.2 Character table of $\mathbb{Z}_5$

The cyclic group of order 5 is :

$$\mathbb{Z}_5 = \langle h \rangle = \left\{e, h, h^2, h^3, h^4\right\}$$

with $h^5 = e$. The conjugacy classes of $\mathbb{Z}_5$ are $\{e\}$, $\{h\}$, $\left\{h^2\right\}$, $\left\{h^3\right\}$ and $\left\{h^4\right\}$. So it has 5 irreducible representations. We see that the cyclic groups have exactly the same number of conjugacy classes and elements. To avoid the lengthy analytical process, we will prove a Theorem that reveals immediately the character table of $\mathbb{Z}_5$.

Theorem 3.1: Let V be a vector space over the field of the complex numbers $\mathbb{C}$. Let

$$\mathbb{Z}_n = \langle h | h^n = e \rangle = \left\{e, h, h^2, \ldots, h^{n-1}\right\}$$

be the cyclic group of order n, and let $\omega = e^{2\pi i/n} \in \mathbb{C}$.

$\boxed{1}$ For every $k = 0, 1, \ldots, n-1$, the pair $(\mathbb{C}_k, \Gamma_k)$ given by $\mathbb{C}_k = \mathbb{C}$, $\Gamma_k(h) = \left[\omega^k\right]$ is a one dimensional representation of $\mathbb{Z}_n$.

$\boxed{2}$ No two of these representations are isomorphic.

$\boxed{3}$ Let (V, Γ) be an irreducible representation of $\mathbb{Z}_n$. Then there exists exactly one $k \in \{0, 1, \ldots, n-1\}$ such that (V, Γ) is isomorphic to $(\mathbb{C}_k, \Gamma_k)$.

Proof. $\boxed{1}$ To prove that for every k, $(\mathbb{C}_k, \Gamma_k)$ is a representation of the group $\mathbb{Z}_n = \langle h | h^n = e \rangle$, it means that a group homomorphism $f : \mathbb{Z}_n \to H$ is determined by $f(h)$, as long as $f(h)^n = e_H$. For $\Gamma_k : \mathbb{Z}_n \to GL(\mathbb{C})$ given by $\Gamma_k(h) = \omega^k \mathrm{id}_{\mathbb{C}}$, we see that

$$\Gamma_k(h^n) = \omega^{kn} \mathrm{id}_{\mathbb{C}} = \mathrm{id}_{\mathbb{C}} = \Gamma_k(e).$$

$\boxed{2}$ To prove that different k give nonisomorphic representations, if $f : \mathbb{C}_k \to \mathbb{C}_l$ an isomorphism, then $f(x) = \alpha x$ for some $\alpha \neq 0 \in \mathbb{C}$. It occurs that

$$f(\alpha x) = \alpha f(x) \Rightarrow \omega^k \alpha x = \omega^l \alpha x \Rightarrow$$

$$\omega^k = \omega^l \Rightarrow k \equiv l \pmod{n} \Rightarrow k = l.$$

$\boxed{3}$ To prove that any irreducible (V, Γ) is isomorphic to one of these, we start by supposing that $\Gamma(h)$ has an eigenvector $v \in V$, with eigenvalue λ. Then, v is an eigenvector for $\Gamma\left(h^j\right)$ for all j. Let $\mathbb{C}_V$ a subrepresentation of V, then because V is irreducible $\Rightarrow V = \mathbb{C}_V$. So, $V = \mathbb{C}_V$ is one dimensional, which means $\Gamma(h) = \lambda \mathrm{id}$ and :

$$\mathrm{id} = \Gamma(e) = \Gamma(h^n) = \Gamma(h)^n = \lambda^n \mathrm{id} \Rightarrow$$

$$\lambda^n = 1 \Rightarrow \lambda = \omega^k \text{ for some } k$$

As a result, we get $(V, \Gamma) \cong (\mathbb{C}_k, \Gamma_k)$. ■

We can figure out the dimensions of the irreps ($d_i > 0$ for $i = 0, 1, 2, 3, 4$) from the fact that :

$$|\mathbb{Z}_5| = 5 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2,$$

which indicates that all the irreps are 1D (every cyclic groups has only 1D irreps). We found that the generator element $h \in \mathbb{Z}_5$ has 5 non equivalent representations :

$$\begin{aligned}
\Gamma_0(h) &\equiv {}^0\hat{\Gamma}(h) = \omega^0 = 1, \\
\Gamma_1(h) &\equiv {}^1\hat{\Gamma}(h) = \omega, \\
\Gamma_2(h) &\equiv {}^2\hat{\Gamma}(h) = \omega^2, \\
\Gamma_3(h) &\equiv {}^3\hat{\Gamma}(h) = \omega^3, \\
\Gamma_4(h) &\equiv {}^4\hat{\Gamma}(h) = \omega^4
\end{aligned}$$

According to the above irreps, we find the representations for the other elements of the group with number/scalar multiplication (all the irreps are 1D), following the multiplication rules of the group. The character table of $\mathbb{Z}_5$ is :

$\mathbb{Z}_5$	e	h	h^2	h^3	h^4
$\chi({}^0\hat{\Gamma})$	1	1	1	1	1
$\chi({}^1\hat{\Gamma})$	1	ω	ω^2	ω^3	ω^4
$\chi({}^2\hat{\Gamma})$	1	ω^2	ω^4	ω	ω^3
$\chi({}^3\hat{\Gamma})$	1	ω^3	ω	ω^4	ω^2
$\chi({}^4\hat{\Gamma})$	1	ω^4	ω^3	ω^2	ω

Next, we construct the character table for D_5.

3.2.3 Character table and representations of D_5

The group D_5, as we derived in subsection 3.1, has 4 conjugacy classes. The 4 irreps have dimensions $d_0 = d_1 = 1$ and $d_2 = d_3 = 2$.

Proof. The dimensions of the 4 irreps should satisfy the relation :

$$|D_5| = 10 = \sum_{i=0}^{4} d_i^2 = d_0^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2$$

We must examine all the possible cases.

Case 1: $d_i \geq 4$
We observe that $d_i < 4$ $(0 \leq i \leq 4)$ because if $d_i = 4$:

$$10 < 16 + d_j^2 + d_k^2 + d_l^2$$

Case 2: $d_i = 3$
Also $d_i \neq 3$ because in such case we would have :

$$10 = 9 + d_j^2 + d_k^2 + d_l^2 \Rightarrow d_j^2 + d_k^2 + d_l^2 = 1$$

but we know that :

$$d_j^2 + d_k^2 + d_l^2 \geq 1^2 + 1^2 + 1^2 = 3$$

Case 3: $d_i = 1 \quad \forall i$
We would have :

$$1^2 + 1^2 + 1^2 + 1^2 = 4 \neq 10$$

Case 4: $d_i = 2 \quad \forall i$
We would have :

$$2^2 + 2^2 + 2^2 + 2^2 = 16 \neq 10$$

Case 5: $d_i = d_j = d_k = 1$ and $d_l = 2$
We would have :

$$1^2 + 1^2 + 1^2 + 2^2 = 7 \neq 10$$

Case 6: $d_i = d_j = d_k = 2$ and $d_l = 1$
We would have :

$$2^2 + 2^2 + 2^2 + 1^2 = 13 \neq 10$$

So we are left only with the case when we have $d_i = d_j = 1$ and $d_k = d_l = 2$:

$$1^2 + 1^2 + 2^2 + 2^2 = 10 \qquad \blacksquare$$

We construct the character table of D_5 :

D_5	e	${}^1C$	${}^2C$	${}^3C$
$\chi({}^0\Gamma)$	1	1	1	1
$\chi({}^1\Gamma)$	1	c_1	c_2	c_3
$\chi({}^2\Gamma)$	2	c_4	c_5	c_6
$\chi({}^3\Gamma)$	2	c_7	c_8	c_9

The first irrep ${}^0\Gamma$ has to be the *trivial* representation (always $\exists$). Because elements in the same conjugacy class have the same character, we have that $\chi(r) = \chi(r^4)$ and $\chi(r^2) = \chi(r^3)$. Since $r^5 = e$, the second linear (1D) irrep ${}^1\Gamma$ must meet the relation :

$$\chi^5(r) = 1 \Rightarrow \chi(r) = c_1 = e^{2n\pi i/5}$$

But this solution has to satisfy also the multiplication rules of the group :

$$\chi^4(r) = \chi(r^4) = \chi(r) \Rightarrow c_1^4 = c_1 \Rightarrow c_1(c_1^3 - 1) = 0 \Rightarrow c_1 = 0, e^{2m\pi i/3}$$

The value $c_1 = 0$ doesn't satisfy the equation $c_1^5 = 1$ and we reject it. Furthermore, the character property $\chi(r^{-1}) = \overline{\chi(r)}$ gives us $\chi(r) = \chi(r^{-1}) = \overline{\chi(r)} \Rightarrow \chi(r) \in \mathbb{R}$. As a result, the complex numbers are rejected as solutions and we get $c_1 = 1$. By the same reasoning, we get $c_2 = 1$.

For $y \in {}^3C$ we must have :

$$\chi^2(y) = c_3^2 = 1 \Rightarrow c_3 = \pm 1$$

For $c_3 = 1$ we take again the *trivial* representation so we conclude that $c_3 = -1$ which leads to the *sign* representation. For the 2D irreps we use the character orthogonality :

$$\langle \chi({}^0\Gamma), \chi({}^2\Gamma) \rangle = 0 \Rightarrow \frac{1}{10}\big(2 + 2c_4^* + 2c_5^* + 5c_6^*\big) = 0,$$

and :

$$\langle \chi({}^1\Gamma), \chi({}^2\Gamma) \rangle = 0 \Rightarrow \frac{1}{10}\big(2 + 2c_4^* + 2c_5^* - 5c_6^*\big) = 0$$

Substracting the two above equations, we get $c_6^* = 0 = c_6$. With similar procedure, using the orthogonality relations $\langle \chi({}^0\Gamma), \chi({}^3\Gamma) \rangle = \langle \chi({}^1\Gamma), \chi({}^3\Gamma) \rangle = 0$ and then substracting the two, we get $c_9^* = 0 = c_9$. Moving on, the normalization of the 3rd row gives :

$$\langle \chi({}^2\Gamma), \chi({}^2\Gamma) \rangle = 1 \Rightarrow |c_4|^2 + |c_5|^2 = 3$$

But we earlier mentioned that $\chi({}^1C), \chi({}^2C) \in \mathbb{R}$ (for all the irreps), so we are left with the two following equations :

$$\left.\begin{aligned} c_4^2 + c_5^2 = 3 \\ 1 + c_4 + c_5 = 0 \end{aligned}\right\} \Rightarrow c_4^2 + (1 + c_4)^2 = 3 \Rightarrow c_4^2 + c_4 - 1 = 0$$

The solutions of the trinomial are $c_4 = \frac{-1\pm\sqrt{5}}{2}$. Suppose we accept $c_4 = \frac{-1+\sqrt{5}}{2} = 2\cos\frac{2\pi}{5}$. Then we get that $c_5 = \frac{-1-\sqrt{5}}{2} = 2\cos\frac{4\pi}{5}$. If we take the relations $\langle \chi({}^0\Gamma), \chi({}^3\Gamma) \rangle = 0$ and $\langle \chi({}^3\Gamma), \chi({}^3\Gamma) \rangle = 1$, we result with the same set (with the above) of equations for c_7 and c_8. In order not to have the same representation as ${}^2\Gamma$, we make the inverse selection of solutions, hence $c_7 = 2\cos\frac{4\pi}{5}$ and $c_8 = 2\cos\frac{2\pi}{5}$. Eventually, the character table of D_5 becomes :

D_5	e	${}^1C$	${}^2C$	${}^3C$
$\chi({}^0\Gamma)$	1	1	1	1
$\chi({}^1\Gamma)$	1	1	1	-1
$\chi({}^2\Gamma)$	2	$2\cos\frac{2\pi}{5}$	$2\cos\frac{4\pi}{5}$	0
$\chi({}^3\Gamma)$	2	$2\cos\frac{4\pi}{5}$	$2\cos\frac{2\pi}{5}$	0

As we have discovered the $1D$ representations of D_5 (equals to the character of the elements), we establish the 2D representations (in matrix form). We just have to find the matrices for the generator elements r and s. We work geometrically in Figure 4. For the irrep ${}^2\Gamma$, by transforming the pentagon edges $C \doteq \big(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}\big)$ to $E \doteq (1,0)$ with ${}^2\Gamma(r)$ and $A \doteq \big(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}\big)$ to $E \doteq (1,0)$ with ${}^2\Gamma(r^2) = {}^2\Gamma^2(r)$ in combination with the character relation $\chi({}^2\Gamma(r)) = \mathrm{tr}\{{}^2\Gamma(r)\} = 2\cos\frac{2\pi}{5}$, we result in the clockwise rotation by $\frac{2\pi}{5}$:

$${}^2\Gamma(r) = \begin{pmatrix} \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}$$

For the representation of the reflection s, using the relation ${}^2\Gamma(r){}^2\Gamma(s){}^2\Gamma(r) = {}^2\Gamma(s)$, the character relation $\chi({}^2\Gamma(s)) = \mathrm{tr}\{{}^2\Gamma(s)\} = 0$ and also transforming the edge $E \doteq (1,0)$ with ${}^2\Gamma(s)$ to itself (trivially), we get :

$$
{}^2\Gamma(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

For the irrep ${}^3\Gamma$, the procedure is similar here but the character relation for the r element $\chi({}^3\Gamma(r)) = \mathrm{tr}\{{}^3\Gamma(r)\} = 2\cos\frac{4\pi}{5}$ reveals that we are dealing with a clockwise rotation by $\frac{4\pi}{5}$:

$$
{}^3\Gamma(r) = \begin{pmatrix} \cos\frac{4\pi}{5} & \sin\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \end{pmatrix},
$$

which, for instance, transforms the edge $A \doteq \left(\cos\frac{4\pi}{5}, \sin\frac{4\pi}{5}\right)$ to $E \doteq (1,0)$. For ${}^3\Gamma(s)$, the respective relations as for ${}^2\Gamma(s)$ give the same matrix :

$$
{}^3\Gamma(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

3.3 Representations of $\mathcal{D}(D_5)$

The representations of the quantum double $\mathcal{D}(D_5)$ can be constructed using pairs of the conjugacy classes of the group and the irreducible representations of their centralizers (subsection 2.3). As a result, we have the following 16 inequivalent and irreducible representations of $\mathcal{D}(D_5)$:

$$
\begin{aligned}
&1 \equiv A \equiv \left|e, {}^0\Gamma\right\rangle, && 2 \equiv B \equiv \left|e, {}^1\Gamma\right\rangle, && 3 \equiv C \equiv \left|e, {}^2\Gamma\right\rangle, \\
&4 \equiv D \equiv \left|e, {}^3\Gamma\right\rangle, && 5 \equiv E \equiv \left|{}^1C, {}^0\hat{\Gamma}\right\rangle, && 6 \equiv F \equiv \left|{}^1C, {}^1\hat{\Gamma}\right\rangle, \\
&7 \equiv G \equiv \left|{}^1C, {}^2\hat{\Gamma}\right\rangle, && 8 \equiv H \equiv \left|{}^1C, {}^3\hat{\Gamma}\right\rangle, && 9 \equiv I \equiv \left|{}^1C, {}^4\hat{\Gamma}\right\rangle, \\
&10 \equiv J \equiv \left|{}^2C, {}^0\hat{\Gamma}\right\rangle, && 11 \equiv K \equiv \left|{}^2C, {}^1\hat{\Gamma}\right\rangle, && 12 \equiv L \equiv \left|{}^2C, {}^2\hat{\Gamma}\right\rangle, \\
&13 \equiv M \equiv \left|{}^2C, {}^3\hat{\Gamma}\right\rangle, && 14 \equiv N \equiv \left|{}^2C, {}^4\hat{\Gamma}\right\rangle, && 15 \equiv O \equiv \left|{}^3C, {}^0\tilde{\Gamma}\right\rangle, \\
&16 \equiv P \equiv \left|{}^3C, {}^1\tilde{\Gamma}\right\rangle
\end{aligned}
\tag{39}
$$

Among those, there are two one-dimensional anyons (1 and 2), twelve two-dimensional anyons $(3-14)$ and two five-dimensional anyons (15 and 16) that satisfy equations (7) and (8), that is $2 \cdot 1^2 + 12 \cdot 2^2 + 2 \cdot 5^2 = 100 = 10^2 = |D_5|^2 = |\mathcal{D}(D_5)|$, where $|\mathcal{D}(D_5)|$ is the order of the quantum double algebra.

The state $\left|e, {}^0\Gamma\right\rangle$ corresponds to the *trivial sector* (trivial electric and magnetic parts). The *purely magnetic flux sectors* are $\left|{}^1C, {}^0\hat{\Gamma}\right\rangle$,$\left|{}^2C, {}^0\hat{\Gamma}\right\rangle$ and $\left|{}^3C, {}^0\tilde{\Gamma}\right\rangle$. The *purely electric sectors* are the pairs of trivial magnetic flux and a non-trivial representation, which are $\left|e, {}^1\Gamma\right\rangle$,$\left|e, {}^2\Gamma\right\rangle$ and $\left|e, {}^3\Gamma\right\rangle$. The remaining sectors, being combinations of non-trivial fluxes and non-trivial representation of the centralizers, are *dyonic sectors*, meaning that they correspond to non-trivial magnetic fluxes and electric charges.

3.4 Modular matrices and charge conjugation

We will calculate the two generators of the modular group S and T, which encode information about the braiding properties ($S \to$ mutual statistics, $T \to$ self-statistics) and fusion rules of the anyons, and the charge conjugation operator $\mathcal{C}$ that determines the anti-particles of the anyons within the framework of the 16 particles of this theory.

3.4.1 S matrix

To begin with, we label the elements of the 4 conjugacy classes of D_5 :

$$\begin{aligned}
&{}^0C = \left\{{}^0h_1 = e\right\} \\
&{}^1C = \left\{{}^1h_1 = r, {}^1h_2 = r^4\right\} \\
&{}^2C = \left\{{}^2h_1 = r^2, {}^2h_2 = r^3\right\} \\
&{}^3C = \left\{{}^3h_1 = s, {}^3h_2 = sr, {}^3h_3 = sr^2, {}^3h_4 = sr^3, {}^3h_5 = sr^4\right\}
\end{aligned} \tag{40}$$

Then, we illustrate a choice of representatives for the equivalence classes of the quotient groups $H/{}^AN$ (subsection 2.4), which we need in order to calculate the S matrix from formula (22).
For the conjugacy class ${}^0C = \{e\}$ we have the centralizer ${}^0N = N\left({}^0C\right) = D_5$, so :

$$H/{}^0N = D_5/D_5 = \Big\{e\left\{e, r, \ldots, sr^4\right\}, r\left\{e, r, \ldots, sr^4\right\}, \ldots, sr^4\left\{e, r, \ldots, sr^4\right\}\Big\} = \{D_5\}$$

As we have 1 distinct coset (here the group D_5), the order of the quotient group is 1 :

$$\left|H/{}^0N\right| = \frac{|D_5|}{|D_5|} = 1$$

We select as a representative from this coset the identity element ${}^0x_1 = e$.
For the conjugacy class ${}^1C = \left\{r, r^4\right\}$ we have the centralizer ${}^1N = N\left({}^1C\right) = \mathbb{Z}_5$, so :

$$H/{}^1N = D_5/\mathbb{Z}_5 = \Big\{e\left\{e, r, \ldots, r^4\right\}, r\left\{e, r, \ldots, r^4\right\}, \ldots, sr^4\left\{e, r, \ldots, r^4\right\}\Big\} = \left\{\mathbb{Z}_5, {}^3C\right\}$$

As we have 2 distinct cosets, the group $\mathbb{Z}_5$ and the conjugacy class ${}^3C$, the order of the quotient group is 2 :

$$\left|H/{}^1N\right| = \frac{|D_5|}{|\mathbb{Z}_5|} = \frac{10}{5} = 2$$

We select as a representative from the coset $\mathbb{Z}_5$ the identity element ${}^1x_1 = e$ and from the coset ${}^3C$ the element ${}^1x_2 = s$. For the conjugacy class ${}^2C = \left\{r^2, r^3\right\}$ we have the centralizer ${}^2N = N\left({}^2C\right) = \mathbb{Z}_5$, so :

$$H/{}^2N = D_5/\mathbb{Z}_5 = \left\{e\mathbb{Z}_5, r\mathbb{Z}_5, \ldots, sr^4\mathbb{Z}_5\right\} = \left\{\mathbb{Z}_5, {}^3C\right\}$$

We have the same 2 distinct cosets and we choose again as representatives the elements ${}^2x_1 = e$ and ${}^2x_2 = s$. For the conjugacy class ${}^3C = \left\{s, sr, sr^2, sr^3, sr^4\right\}$ we have the centralizer ${}^3N = N\left({}^3C\right) = \mathbb{Z}_2$, so :

$$\begin{aligned}
H/{}^3N = D_5/\mathbb{Z}_2 = \Big\{e\{e, s\}, \ldots, sr^4\{e, s\}\Big\} &= \Big\{\{e, s\}, \left\{r^2, sr^3\right\}, \left\{r^4, sr\right\}, \left\{r, sr^4\right\}, \left\{r^3, sr^2\right\}\Big\} \\
&\equiv \{\mathbb{Z}_2, \mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}\}
\end{aligned}$$

As we have 5 distinct cosets, the group $\mathbb{Z}_2$ and the cosets $\mathbb{A}$,$\mathbb{B}$,$\mathbb{C}$ and $\mathbb{D}$, the order of the quotient group is 5 :

$$\left|H/{}^3N\right| = \frac{|D_5|}{|\mathbb{Z}_2|} = \frac{10}{2} = 5$$

We select as a representative from the coset $\mathbb{Z}_2$ the identity element ${}^3x_1 = e$, from the coset $\mathbb{A}$ the element ${}^3x_2 = r^2$, from the coset $\mathbb{B}$ the element ${}^3x_3 = r^4$, from the coset $\mathbb{C}$ the element ${}^3x_4 = r$ and from the coset $\mathbb{D}$ the element ${}^3x_5 = r^3$. The representatives are listed in section 6.

We will evaluate a single element of the S matrix just to show how the matrix was calculated. The notation for an anyon state, out of the total 16, will be $|i\rangle = |{}^{A}C, {}^{\alpha}\Gamma\rangle$ with $A \in \{0,1,2,3\}$, $\Gamma \in \left\{\Gamma, \hat{\Gamma}, \tilde{\Gamma}\right\}$ and $\alpha = \alpha(A)$. For two given representations of $\mathcal{D}(D_5)$, $|{}^{A}C, {}^{\alpha}\Gamma\rangle = \left|{}^{1}C, {}^{0}\hat{\Gamma}\right\rangle$ and $\left|{}^{B}C, {}^{\beta}\Gamma\right\rangle = \left|{}^{1}C, {}^{0}\hat{\Gamma}\right\rangle$, we have :

$$S_{55} \equiv \langle 5| S |5\rangle \equiv \left\langle {}^{1}C, {}^{0}\hat{\Gamma}\middle|S\middle|{}^{1}C, {}^{0}\hat{\Gamma}\right\rangle = \frac{1}{10} \sum_{\substack{{}^{1}h_i \in {}^{1}C, {}^{1}h_j \in {}^{1}C \\ [{}^{1}h_i, {}^{1}h_j] = e}} tr\left({}^{0}\hat{\Gamma}({}^{1}x_i^{-1}{}^{1}h_j{}^{1}x_i)\right)^* \, tr\left({}^{0}\hat{\Gamma}({}^{1}x_j^{-1}{}^{1}h_i{}^{1}x_j)\right)^*$$

We examine for which elements we make the summation. For $i = j \to [{}^{1}h_i, {}^{1}h_i] = e$ for all ${}^{1}h_i \in {}^{1}C$. For $i \neq j \to [{}^{1}h_i, {}^{1}h_j] \neq e$ with $i, j = 1, 2$. So we get :

$$S_{55} = \frac{1}{10}\left[2\chi^*_{{}^{0}\hat{\Gamma}}\left({}^{1}h_1\right)\chi^*_{{}^{0}\hat{\Gamma}}\left({}^{1}h_1\right) + 2\chi^*_{{}^{0}\hat{\Gamma}}\left({}^{1}h_2\right)\chi^*_{{}^{0}\hat{\Gamma}}\left({}^{1}h_2\right)\right],$$

and reading the value for $\chi_{{}^{0}\hat{\Gamma}}\left({}^{1}C\right)$ from the character table of $\mathbb{Z}_5$, we find :

$$S_{55} = \frac{1}{10}\left[2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 1\right] = \frac{4}{10}$$

The complete S matrix is :

$$S = \frac{1}{10}\begin{pmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 5 \\
1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -5 & -5 \\
2 & 2 & 4 & 4 & x & x & x & x & x & y & y & y & y & y & 0 & 0 \\
2 & 2 & 4 & 4 & y & y & y & y & y & x & x & x & x & x & 0 & 0 \\
2 & 2 & x & y & 4 & x & y & y & x & 4 & x & y & y & x & 0 & 0 \\
2 & 2 & x & y & x & y & y & x & 4 & y & y & x & 4 & x & 0 & 0 \\
2 & 2 & x & y & y & y & x & 4 & x & x & 4 & x & y & y & 0 & 0 \\
2 & 2 & x & y & y & x & 4 & x & y & x & y & y & x & 4 & 0 & 0 \\
2 & 2 & x & y & x & 4 & x & y & y & y & x & 4 & x & y & 0 & 0 \\
2 & 2 & y & x & 4 & y & x & x & y & 4 & y & x & x & y & 0 & 0 \\
2 & 2 & y & x & x & y & 4 & y & x & y & x & x & y & 4 & 0 & 0 \\
2 & 2 & y & x & y & x & x & y & 4 & x & x & y & 4 & y & 0 & 0 \\
2 & 2 & y & x & y & 4 & y & x & x & x & y & 4 & y & x & 0 & 0 \\
2 & 2 & y & x & x & x & y & 4 & y & y & 4 & y & x & x & 0 & 0 \\
5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -5 \\
5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 5
\end{pmatrix}, \tag{41}$$

with $x = \sqrt{5} - 1 = 4\cos\frac{2\pi}{5}$ and $y = -(\sqrt{5}+1) = 4\cos\frac{4\pi}{5}$. From subsection 3.2.2, for D_5 we have $\omega = e^{2\pi i/5}$ and we can alternatively have the expressions $x = 2(\omega + \overline{\omega})$ and $y = 2(\omega^2 + \overline{\omega}^2)$.

3.4.2 T matrix

We use the equation (24) :

$$T_{11} \equiv T^{0,0}_{{}^{0}\Gamma,{}^{0}\Gamma} = \frac{\chi_{{}^{0}\Gamma}\left({}^{0}h_1\right)}{d_{{}^{0}\Gamma}} = \frac{1}{1} = 1, \qquad T_{22} \equiv T^{0,0}_{{}^{1}\Gamma,{}^{1}\Gamma} = \frac{\chi_{{}^{1}\Gamma}\left({}^{0}h_1\right)}{d_{{}^{1}\Gamma}} = \frac{1}{1} = 1,$$

$$T_{33} \equiv T^{0,0}_{{}^{2}\Gamma,{}^{2}\Gamma} = \frac{\chi_{{}^{2}\Gamma}\left({}^{0}h_1\right)}{d_{{}^{2}\Gamma}} = \frac{2}{2} = 1, \qquad T_{44} \equiv T^{0,0}_{{}^{3}\Gamma,{}^{3}\Gamma} = \frac{\chi_{{}^{3}\Gamma}\left({}^{0}h_1\right)}{d_{{}^{3}\Gamma}} = \frac{2}{2} = 1,$$

$$
\begin{aligned}
T_{55} &\equiv T^{1,1}_{0_{\hat\Gamma},0_{\hat\Gamma}} = \frac{\chi_{0_{\hat\Gamma}}\left({}^1h_1\right)}{d_{0_{\hat\Gamma}}} = \frac{1}{1} = 1, & T_{66} &\equiv T^{1,1}_{1_{\hat\Gamma},1_{\hat\Gamma}} = \frac{\chi_{1_{\hat\Gamma}}\left({}^1h_1\right)}{d_{1_{\hat\Gamma}}} = \frac{\omega}{1} = \omega,\\
T_{77} &\equiv T^{1,1}_{2_{\hat\Gamma},2_{\hat\Gamma}} = \frac{\chi_{2_{\hat\Gamma}}\left({}^1h_1\right)}{d_{2_{\hat\Gamma}}} = \frac{\omega^2}{1} = \omega^2, & T_{88} &\equiv T^{1,1}_{3_{\hat\Gamma},3_{\hat\Gamma}} = \frac{\chi_{3_{\hat\Gamma}}\left({}^1h_1\right)}{d_{3_{\hat\Gamma}}} = \frac{\omega^3}{1} = \omega^3,\\
T_{99} &\equiv T^{1,1}_{4_{\hat\Gamma},4_{\hat\Gamma}} = \frac{\chi_{4_{\hat\Gamma}}\left({}^1h_1\right)}{d_{4_{\hat\Gamma}}} = \frac{\omega^4}{1} = \omega^4, & T_{1010} &\equiv T^{2,2}_{0_{\hat\Gamma},0_{\hat\Gamma}} = \frac{\chi_{0_{\hat\Gamma}}\left({}^2h_1\right)}{d_{0_{\hat\Gamma}}} = \frac{1}{1} = 1,\\
T_{1111} &\equiv T^{2,2}_{1_{\hat\Gamma},1_{\hat\Gamma}} = \frac{\chi_{1_{\hat\Gamma}}\left({}^2h_1\right)}{d_{1_{\hat\Gamma}}} = \frac{\omega^2}{1} = \omega^2, & T_{1212} &\equiv T^{2,2}_{2_{\hat\Gamma},2_{\hat\Gamma}} = \frac{\chi_{2_{\hat\Gamma}}\left({}^2h_1\right)}{d_{2_{\hat\Gamma}}} = \frac{\omega^4}{1} = \omega^4,\\
T_{1313} &\equiv T^{2,2}_{3_{\hat\Gamma},3_{\hat\Gamma}} = \frac{\chi_{3_{\hat\Gamma}}\left({}^2h_1\right)}{d_{3_{\hat\Gamma}}} = \frac{\omega}{1} = \omega, & T_{1414} &\equiv T^{2,2}_{4_{\hat\Gamma},4_{\hat\Gamma}} = \frac{\chi_{4_{\hat\Gamma}}\left({}^2h_1\right)}{d_{4_{\hat\Gamma}}} = \frac{\omega^3}{1} = \omega^3,\\
T_{1515} &\equiv T^{3,3}_{0_{\hat\Gamma},0_{\hat\Gamma}} = \frac{\chi_{0_{\hat\Gamma}}\left({}^3h_1\right)}{d_{0_{\hat\Gamma}}} = \frac{1}{1} = 1, & T_{1616} &\equiv T^{3,3}_{1_{\tilde\Gamma},1_{\hat\Gamma}} = \frac{\chi_{1_{\hat\Gamma}}\left({}^3h_1\right)}{d_{1_{\hat\Gamma}}} = \frac{-1}{1} = -1
\end{aligned}
$$

Collecting the elements of the T matrix, we get :

$$
T = diag\big(1,1,1,1,1,\omega,\omega^2,\omega^3,\omega^4,1,\omega^2,\omega^4,\omega,\omega^3,1,-1\big) \tag{42}
$$

All the non-diagonal elements of the T matrix are zero. The spin factors of the anyons are (equation (24)) :

$$
\begin{aligned}
T_{11} &= \exp\big(2\pi i s_1\big) = 1 \Rightarrow s_1 = 0,\\
T_{22} &= \exp\big(2\pi i s_2\big) = 1 \Rightarrow s_2 = 0,\\
T_{33} &= \exp\big(2\pi i s_3\big) = 1 \Rightarrow s_3 = 0,\\
T_{44} &= \exp\big(2\pi i s_4\big) = 1 \Rightarrow s_4 = 0,\\
T_{55} &= \exp\big(2\pi i s_5\big) = 1 \Rightarrow s_5 = 0,\\
T_{66} &= \exp\big(2\pi i s_6\big) = \omega = \exp\big(2\pi i/5\big) \Rightarrow s_6 = 1/5,\\
T_{77} &= \exp\big(2\pi i s_7\big) = \omega^2 = \exp\big(4\pi i/5\big) \Rightarrow s_7 = 2/5,\\
T_{88} &= \exp\big(2\pi i s_8\big) = \omega^3 = \exp\big(6\pi i/5\big) \Rightarrow s_8 = 3/5,\\
T_{99} &= \exp\big(2\pi i s_9\big) = \omega^4 = \exp\big(8\pi i/5\big) \Rightarrow s_9 = 4/5,\\
T_{1010} &= \exp\big(2\pi i s_{10}\big) = 1 \Rightarrow s_{10} = 0,\\
T_{1111} &= \exp\big(2\pi i s_{11}\big) = \omega^2 = \exp\big(4\pi i/5\big) \Rightarrow s_{11} = 2/5,\\
T_{1212} &= \exp\big(2\pi i s_{12}\big) = \omega^4 = \exp\big(8\pi i/5\big) \Rightarrow s_{12} = 4/5,\\
T_{1313} &= \exp\big(2\pi i s_{13}\big) = \omega = \exp\big(2\pi i/5\big) \Rightarrow s_{13} = 1/5,\\
T_{1414} &= \exp\big(2\pi i s_{14}\big) = \omega^3 = \exp\big(6\pi i/5\big) \Rightarrow s_{14} = 3/5,\\
T_{1515} &= \exp\big(2\pi i s_{15}\big) = 1 \Rightarrow s_{15} = 0,\\
T_{1616} &= \exp\big(2\pi i s_{16}\big) = -1 = \exp\big(\pi i\big) \Rightarrow s_{16} = 1/2
\end{aligned} \tag{43}
$$

We characterize the abelian anyons $1, 2$ ($d_1 = d_2 = 1$) as "*bosons*" due to value of their spin factor (this theory has no "*fermions*" i.e. 1D anyons with $\phi = \pi \rightarrow s = 1/2$).

3.4.3 $\mathcal{C}$ matrix

From the relation (25), using the S matrix of D_5 (equation (41)) :

$$\mathcal{C} = S^2 = diag\big(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\big) = \mathbb{1}_{16\times 16} \tag{44}$$

So we see that every particle has itself as antiparticle.

3.5 Fusion rules

We determine the fusion rules by utilizing the Verlinde's formula (equation (23)), where $S^* = S$ because all the elements of the S matrix are real. For our theory, the multiplicities are $N^{AB\gamma}_{\alpha\beta C} \equiv N^{ab}_{c} = 0, 1$, so two anyons $a \equiv ({}^{A}C, \alpha)$ and $b \equiv ({}^{B}C, \beta)$ can either fuse to an anyon $c \equiv ({}^{C}C, \gamma)$ by a single channel or not at all. Because the fusion of anyons is an associative process, that is to say $a \times b = b \times a$, we can save time in the computation by calculating half of the elements of the fusion rules. It is worth noting that for the pure electric states, the fusion process does correspond to a tensor product of the two representations since there is no flux metamorphosis involved. As such, the fusion rules for the pure electric states can be obtained by reducing the tensor product into a direct sum of irreducible representations. Also, we see that fusion, being a decomposition of products of representation, can only generate representations of dimension equal or lower than the maximum dimension of the representations being fused. The fusion rules are included in section 6.

3.6 Anyon scattering

Here, we look at a simple example of abelian *Aharonov − Bohm* scattering, namely for two anyons with magnetic fluxes that commute [3]. For $({}^{A}C, \alpha) = ({}^{1}C, {}^{0}\hat{\Gamma}) \equiv 5$ and $({}^{B}C, \beta) = ({}^{2}C, {}^{1}\hat{\Gamma}) \equiv 11$, we find their monodromy matrix $\mathcal{R}^2 = \mathcal{R}_{115}\mathcal{R}_{511}$ and use it to find the cross section of the scattering process with the anyon 5 being the projectile and the anyon 11 being the scatterer (Figure 5).

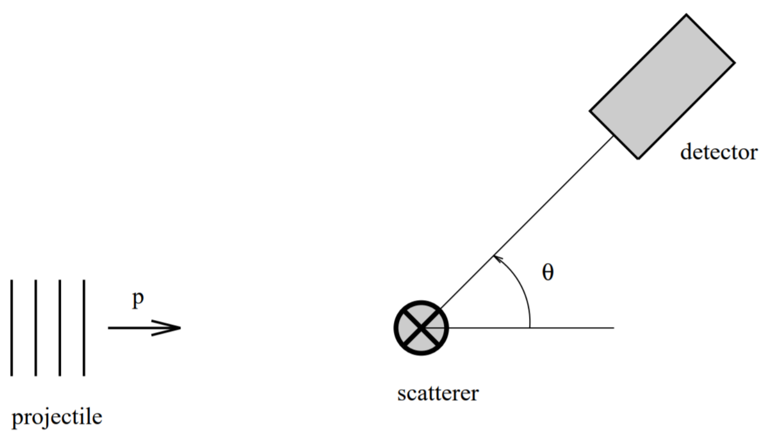


Figure 5: The geometry of the *Aharonov − Bohm* elastic scattering experiment. The projectile anyon (plane wave with momentum p) never enters the nearby region of the scatterer, who has it's position fixed at the origin. The cross section for the scattered projectile is measured by a detector placed at a scattering angle θ.

We see that for arbitrary ${}^1h_i \in {}^1C$ and ${}^2h_j \in {}^2C$ we have $\left[{}^1h_i, {}^2h_j\right] = r^k r^l r^{-k} r^{-l} = e$ for $k = 1, 4$ and $l = 2, 3$, thus we are dealing with the abelian case of the phenomenon. To calculate $\mathcal{R}_{511}$, we use equation (19) :

$$\begin{aligned}
\mathcal{R}_{511}\left|{}^1h_1, {}^0\hat{v}\right\rangle\left|{}^2h_1, {}^1\hat{v}\right\rangle &= \omega\left|{}^2h_1, {}^1\hat{v}\right\rangle\left|{}^1h_1, {}^0\hat{v}\right\rangle \Rightarrow \mathcal{R}_{511}\left|1\right\rangle = \omega\left|1\right\rangle,\\
\mathcal{R}_{511}\left|{}^1h_1, {}^0\hat{v}\right\rangle\left|{}^2h_2, {}^1\hat{v}\right\rangle &= \overline{\omega}\left|{}^2h_2, {}^1\hat{v}\right\rangle\left|{}^1h_1, {}^0\hat{v}\right\rangle \Rightarrow \mathcal{R}_{511}\left|2\right\rangle = \overline{\omega}\left|3\right\rangle,\\
\mathcal{R}_{511}\left|{}^1h_2, {}^0\hat{v}\right\rangle\left|{}^2h_1, {}^1\hat{v}\right\rangle &= \overline{\omega}\left|{}^2h_1, {}^1\hat{v}\right\rangle\left|{}^1h_2, {}^0\hat{v}\right\rangle \Rightarrow \mathcal{R}_{511}\left|3\right\rangle = \overline{\omega}\left|2\right\rangle,\\
\mathcal{R}_{511}\left|{}^1h_2, {}^0\hat{v}\right\rangle\left|{}^2h_2, {}^1\hat{v}\right\rangle &= \omega\left|{}^2h_2, {}^1\hat{v}\right\rangle\left|{}^1h_2, {}^0\hat{v}\right\rangle \Rightarrow \mathcal{R}_{511}\left|4\right\rangle = \omega\left|4\right\rangle,
\end{aligned}$$

so the braid matrix $\mathcal{R}_{511}$, in the basis of the two anyons $\left|i\right\rangle$ with $i = 1, 2, 3, 4$, is :

$$\mathcal{R}_{511} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

In the same basis, we similarly find :

$$\mathcal{R}_{115} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The monodromy matrix of the anyons 5 and 11 is :

$$\mathcal{R}^2 = \mathcal{R}_{115}\mathcal{R}_{511} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

We see that the monodromy matrix is diagonal with phases $e^{2\pi i\alpha}$ as elements, as we expected for the abelian case. The differential cross section for this particular scattering experiment turns out [3] :

$$\frac{d\sigma}{d\theta} = \frac{\sin^2\left(\pi\alpha\right)}{2\pi p \sin^2\left(\theta/2\right)} = \frac{\sin^2\left(\pi/5\right)}{2\pi p \sin^2\left(\theta/2\right)}, \tag{45}$$

because $\omega, \overline{\omega} = e^{\pm 2\pi i/5} = e^{2\pi i\alpha} \Rightarrow \alpha = \pm 1/5$. In every abelian case of $\mathcal{D}(D_5)$, the resulting phases are always ω^n and ω^{-n} with $n \in \{0, 1, 2, 3, 4\}$, so $\alpha = \pm n/5$ but we have the same cross section for both signs of a given α value as the sinus is squared in the above formula. For the non-abelian case, the particles involved may exchange internal flux/charge quantum numbers which corresponds to non-diagonal monodromy matrices $\mathcal{R}^2$. There are two formulas for $d\sigma/d\theta$ in the non-abelian case, with their difference being in whether the detector can distinguish between the different internal appearances of the projectile or not [3],[8]. In addition, we note that all the $\mathcal{R}$ matrices are included in section 6.

4 Quantum computation with anyon models

In this section, we provide the basic tools of how to use anyon models for encoding and processing quantum information. The general idea is to encode a qudit in the dimension of the fusion space. Then, we introduce two main techniques to construct the braiding generators, the first being with F and R symbols and the second one via the representations of the quantum double. We illustrate these methods on both Fibonacci and Ising anyons and the quantum double models $\mathcal{D}(S_3)$ and $\mathcal{D}(D_5)$. Finally, we point out how the Fibonacci anyons can perfom universal quantum computation and also introduce two other ideas, in relevance with our work, that can solve this major problem of universality, that is constructing the whole spectrum of quantum gates.

4.1 Measurements with pairs of fluxes

As pure fluxes we consider the anyon types that carry trivial representation of the centralizer of the conjugacy class in which the flux element belongs :

$$\big|h\big\rangle \equiv \big|h, {}^{0}\Gamma\big\rangle$$

We look separately each conjugacy class. In D_5, for the conjugacy classes ${}^0C$, ${}^1C$ and ${}^2C$ we have only one pair of different fluxes, each one having trivial total flux $hh' = e$ with $h, h' \in {}^AC$:

$$\begin{aligned}
{}^0C &\rightarrow \big|h, h'\big\rangle = \big|e, e\big\rangle = \big|0; {}^0C\big\rangle, \\
{}^1C &\rightarrow \big|h, h'\big\rangle = \big|r, r^{-1}\big\rangle = \big|r, r^4\big\rangle = \big|0; {}^1C\big\rangle, \\
{}^2C &\rightarrow \big|h, h'\big\rangle = \big|r^2, r^{-2}\big\rangle = \big|r^2, r^3\big\rangle = \big|0; {}^2C\big\rangle
\end{aligned}$$

The braiding of the two fluxes is also trivial (h and $h' = h^{-1}$ commute) :

$$R\big|h, h'\big\rangle = \big|hh'h^{-1}, h\big\rangle = \big|h', h\big\rangle \Rightarrow R^2\big|h, h'\big\rangle = \big|h, h'\big\rangle \Rightarrow R^2 = I,$$

where we used the flux metamorphosis rule (2). When flux metamorphosis happens between pure fluxes, the action of the braiding matrix (19) gives a trivial scalar factor, as their representation is trivial. But for $h, h' \in {}^3C$ ($h \neq h'$), we have non trivial total flux and the braid operator R has orbits of length five. If we choose $\big|h, h'\big\rangle = \big|s, sr\big\rangle$ as a pair of fluxes, we have [4]:

$$R : \big|s, sr\big\rangle \rightarrow \big|sr^4, s\big\rangle \rightarrow \big|sr^3, sr^4\big\rangle \rightarrow \big|sr^2, sr^3\big\rangle \rightarrow \big|sr, sr^2\big\rangle \rightarrow \big|s, sr\big\rangle \Rightarrow R^5 = I,$$

which holds for every arbitrary pair $\big|h, h'\big\rangle$. Thus, if the two fluxons are exchanged five times, they swap positions (the number of exchanges is odd), yet the labeling of the state is unmodified. This observation means that there can be quantum interference between the "direct" and "exchange" scattering of two fluxons that carry distinct labels in the same conjugacy class, reinforcing the notion that fluxes carrying conjugate labels ought to be regarded as indistinguishable particles.
Since the braid operator acting on pairs of distinct fluxes in ${}^3C$ satisfies $R^5 = I$, its eigenvalues are fifth roots of unity. For example, by taking linear combinations of the five states with total flux

$hh' = r$, we obtain the R eigenstates :

$$\begin{aligned}
R = 1 &: \ \big|s, sr\big\rangle + \big|sr^4, s\big\rangle + \big|sr^3, sr^4\big\rangle + \big|sr^2, sr^3\big\rangle + \big|sr, sr^2\big\rangle, \\
R = \omega &: \ \big|s, sr\big\rangle + \overline{\omega}\big|sr^4, s\big\rangle + \overline{\omega}^2\big|sr^3, sr^4\big\rangle + \omega^2\big|sr^2, sr^3\big\rangle + \omega\big|sr, sr^2\big\rangle, \\
R = \omega^2 &: \ \big|s, sr\big\rangle + \overline{\omega}^2\big|sr^4, s\big\rangle + \omega\big|sr^3, sr^4\big\rangle + \overline{\omega}\big|sr^2, sr^3\big\rangle + \omega^2\big|sr, sr^2\big\rangle, \\
R = \overline{\omega}^2 &: \ \big|s, sr\big\rangle + \omega^2\big|sr^4, s\big\rangle + \overline{\omega}\big|sr^3, sr^4\big\rangle + \omega\big|sr^2, sr^3\big\rangle + \overline{\omega}^2\big|sr, sr^2\big\rangle, \\
R = \overline{\omega} &: \ \big|s, sr\big\rangle + \omega\big|sr^4, s\big\rangle + \omega^2\big|sr^3, sr^4\big\rangle + \overline{\omega}^2\big|sr^2, sr^3\big\rangle + \overline{\omega}\big|sr, sr^2\big\rangle,
\end{aligned}$$

where $\omega = e^{2\pi i/5}$.
Although a pair of fluxes $\big|h, h^{-1}\big\rangle$ with trivial total flux has trivial braiding properties, it is interesting for another reason — it carries *charge*. The way to detect the charge of an object is to carry a flux g around the object (counterclockwise); this modifies the object by the action of ${}^{\alpha}\Gamma\big(g\big)$ for some representation α of H. If the charge is zero then the representation is trivial — $\Gamma\big(g\big) = I$ for all $g \in H$. But if we carry a flux g counterclockwise around the state $\big|h, h^{-1}\big\rangle$, the state transforms as (due to equation (2)) :

$$\mathcal{R}\big|g\big\rangle\big|h, h^{-1}\big\rangle \to \big|ghg^{-1}, gh^{-1}g^{-1}\big\rangle\big|g\big\rangle,$$

a nontrivial action (for at least some g) if h belongs to a conjugacy class with more than one element. In fact, for each conjugacy class C_h, there is a unique state $\big|0; C_h\big\rangle$ with zero charge, the uniform superposition of the class representatives :

$$\big|0; C_h\big\rangle = \frac{1}{\sqrt{|C_h|}} \sum_{h \in C_h} \big|h, h^{-1}\big\rangle,$$

where $\big|C_h\big|$ denotes the order of C_h. A pair of fluxons in the class C_h, that can be created in a local process, must not carry any conserved charges and therefore must be in the state $\big|0; C_h\big\rangle$ (we create them from the vaccum because they have trivial flux and charge). Other linear combinations orthogonal to $\big|0; C_h\big\rangle$ carry nonzero charge. This charge carried by a pair of fluxons can be detected by other fluxons, yet oddly the charge cannot be localized on the core of either particle in the pair. Rather it is a collective property of the pair. If two fluxons with a nonzero total charge are brought together, complete annihilation of the pair will be forbidden by charge conservation, even though the total flux is zero.
In the case of a pair of fluxons from the conjugacy class ${}^{3}C$ of $H = D_5$, for example, there is a two-dimensional subspace with trivial total flux and nontrivial charge, for which we may choose (not uniquely) the basis :

$$\begin{aligned}
\big|0\big\rangle &= \big|s, s\big\rangle + \overline{\omega}\big|sr, sr\big\rangle + \overline{\omega}^2\big|sr^2, sr^2\big\rangle + \omega^2\big|sr^3, sr^3\big\rangle + \omega\big|sr^4, sr^4\big\rangle, \\
\big|1\big\rangle &= \big|s, s\big\rangle + \omega\big|sr, sr\big\rangle + \omega^2\big|sr^2, sr^2\big\rangle + \overline{\omega}^2\big|sr^3, sr^3\big\rangle + \overline{\omega}\big|sr^4, sr^4\big\rangle,
\end{aligned}$$

with evident orthogonality $\to \big\langle 0\big|1\big\rangle = \big\langle 1\big|0\big\rangle = 0$ and not orthonormality $\to \big\langle 0\big|0\big\rangle = \big\langle 1\big|1\big\rangle = 5 \neq 1$.
If the generator flux r of D_5 is carried around these two basis states consisting of fluxon pairs, we take :

$$\mathcal{R}\big|r\big\rangle\big|0\big\rangle = \overline{\omega}^2\big|0\big\rangle\big|r\big\rangle$$
$$\mathcal{R}\big|r\big\rangle\big|1\big\rangle = \omega^2\big|1\big\rangle\big|r\big\rangle$$

Therefore, the action (by conjugation) of $r \in D_5$ on these states is :

$$\Gamma(r) = \begin{pmatrix} \overline{\omega}^2 & 0 \\ 0 & \omega^2 \end{pmatrix},$$

which is the complex extension of the real matrix ${}^3\Gamma(r)$ we found in subsection 3.2.3. It represents a unitary clockwise rotation by $4\pi/5$ on the complex plane (acts on the complex numbers $x + iy = (x\ \ y)^T$) whereas ${}^3\Gamma(r)$ represents a clockwise rotation on the real plane by the same angle.
When the second generator of the group $s \in D_5$ is carried around the basis states, we get :

$$\mathcal{R}|s\rangle|0\rangle = |1\rangle|s\rangle,$$
$$\mathcal{R}|s\rangle|1\rangle = |0\rangle|s\rangle,$$

so the action (by conjugation) of s on the basis states is :

$$\Gamma(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is the representation of the reflection element in both 2D irreps of $D_5 \to {}^2\Gamma(s) = {}^3\Gamma(s)$. We conclude that the action by conjugation is just the two-dimensional irreducible representation ${}^3\Gamma$ of D_5, so the charge of each pair of fluxons $|h, h^{-1}\rangle$ with $h \in {}^3C$ is ${}^3\Gamma$.
Furthermore, under braiding, this charge carried by a pair of fluxons can be transferred to other particles. For example, consider a pair of particles, each of which carries charge but no flux (we refer to such particles as *chargeons*), such that the total charge of the pair is trivial. If one of the chargeons transforms as the unitary irreducible representation Γ of H, there is a unique conjugate (dual) representation $\overline{\Gamma}$ that can be combined with Γ to give the trivial representation; if $\{|v, i\rangle\}$ is a basis for Γ, then a basis $\{|\overline{v}, i\rangle\}$ can be chosen for $\overline{\Gamma}$, such that the *chargeon-antichargeon* pair with trivial charge can be expressed as :

$$|0; \Gamma\rangle = \frac{1}{\sqrt{|\Gamma|}} \sum_{i=1}^{|\Gamma|} |v, i\rangle \otimes |\overline{v}, i\rangle$$

Imagine that we create a pair of fluxons in the state $|0; C_h\rangle = |h, h^{-1}\rangle$ and also create a pair of chargeons in the state $|0; \Gamma\rangle$. Then we wind the chargeon with charge Γ counterclockwise around the first member of the pair of fluxons, so due to *Bohm-Aharonov* effect it's state transforms exactly as in equation (5). The chargeon-antichargeon state becomes :

$$|0; \Gamma\rangle' = \frac{1}{\sqrt{|\Gamma|}} \sum_{i=1}^{|\Gamma|} |v, i\rangle' \otimes |\overline{v}, i\rangle = \frac{1}{\sqrt{|\Gamma|}} \sum_{i=1}^{|\Gamma|} \Gamma_{ij}(h)|v, j\rangle \otimes |\overline{v}, i\rangle$$

We bring the members of the pair together and see if they annihilate (the total charge of the pair of chargeons has remained zero after the winding). The amplitude of the process is :

$$\langle 0; \Gamma | 0; \Gamma\rangle' = \frac{1}{|\Gamma|} \sum_{i=1}^{|\Gamma|} \Gamma_{ij}(h)\langle v, i|v, j\rangle\langle \overline{v}, i|\overline{v}, i\rangle = \frac{1}{|\Gamma|} \sum_{i=1}^{|\Gamma|} \Gamma_{ij}(h)\delta_{ij} = \frac{\sum_{i=1}^{|\Gamma|} \Gamma_{ii}(h)}{|\Gamma|} = \frac{\chi(\Gamma(h))}{|\Gamma|},$$

so the probability they will annihilate is :

$$Prob(0) = \left| \frac{\chi\big(\Gamma(h)\big)}{|\Gamma|} \right|^2$$

Since the total charge of all four particles is zero and charge is conserved, after the winding the two pairs have opposite charges — if the pair of chargeons has total charge Γ', then the pair of fluxons must have total charge $\overline{\Gamma}'$, combined with Γ' to give trivial total charge. This probability is less than one, provided that the representation of Γ is not one dimensional and the class C_h is not represented trivially $({}^{\alpha}\Gamma\big(C_h\big) \neq {}^{0}\Gamma\big(C_h\big) = 1)$.
In the case where $C_h = {}^3C$ of $H = D_5$ and $\Gamma = {}^3\Gamma$ which came up in the analysis earlier, we see that $\chi\big({}^3\Gamma\big({}^3C\big)\big) = 0$. Therefore, charge is transfered with certainty; after the winding, both the fluxon pair and the chargeon pair transform as $\Gamma' = {}^3\Gamma$.
We want to show now how we can use the chargeons to calibrate the fluxons and assemble a flux bureau of standards. Suppose that we are presented with two pairs of fluxons in the states $\big|a, a^{-1}\big\rangle$ and $\big|b, b^{-1}\big\rangle$, and we wish to determine whether the fluxes a and b match or not. We are also equipped with a *chargeon-antichargeon* pair. We wind the chargeon around the first member of the first fluxon pair and then around the second member of the second fluxon pair. The state of the chargeons is transformed to :

$$\big|0;\Gamma\big\rangle' = \frac{1}{\sqrt{|\Gamma|}} \sum_{i=1}^{|\Gamma|} \big|v,i\big\rangle' \otimes \big|\overline{v},i\big\rangle = \frac{1}{\sqrt{|\Gamma|}} \sum_{i=1}^{|\Gamma|} \Gamma_{ij}\big(b^{-1}a\big)\big|v,j\big\rangle \otimes \big|\overline{v},i\big\rangle$$

The probability that the chargeon pair will annihilate after the two consecutive exchanges is :

$$Prob(0) = \left| \frac{\chi\big(\Gamma\big(b^{-1}a\big)\big)}{|\Gamma|} \right|^2,$$

which clearly is less than zero if $b \neq a$ (assuming that the representation Γ is not one-dimensional and represents $b^{-1}a$ nontrivially). After a number of repetitions we can say with high statistical confidence if the fluxes are different. We can sort in that way all the different fluxes on seperate "bins".
The next step is to label the fluxes so that they match the group composition rules, because the chance of getting it right on the first random labeling is $1/\big(|H|!\big)$. Suppose we take 3 pairs from 3 different bins, $\big|a, a^{-1}\big\rangle$, $\big|b, b^{-1}\big\rangle$ and $\big|c, c^{-1}\big\rangle$, and we want to check whether $c = ba$. We create again a chargeon-antichargeon pair and wind the chargeon first around a (carry the chargeon around a closed path that encloses a), then around b and finally around c^{-1}. The probability whether the reunited chargeon pair annihilates is :

$$Prob(0) = \left| \frac{\chi\big(\Gamma\big(c^{-1}ba\big)\big)}{|\Gamma|} \right|^2$$

Every time the chargeon-antichargeon pair annihilates, when we bring it together, it means that $ba = c$. We construct in that way a flux bureau of standards and label the fluxes according to the values that they take in group H. So given an unkown pair of fluxes $\big|d, d^{-1}\big\rangle$, we can use any of the labeled fluxes, suppose $\big|a, a^{-1}\big\rangle$, repeat the above process and determine the flux d. We will call this process *projective flux measurement*.

Suppose we have two pairs of fluxes $|a, a^{-1}\rangle$ and $|b, b^{-1}\rangle$. We want to realize the effect on the states if we transport the pair of fluxes $|a, a^{-1}\rangle$ counterclockwise around the first member of the second fluxon pair. Since the $|a, a^{-1}\rangle$ pair has trivial total flux, the $|b, b^{-1}\rangle$ pair is unaffected by this procedure. But since in effect the flux b travels counterclockwise about both members of the pair whose initial state was $|a, a^{-1}\rangle$, this pair is transformed as :

$$\mathcal{R}|b\rangle|a, a^{-1}\rangle = |bab^{-1}, ba^{-1}b^{-1}\rangle|b\rangle$$

We will refer to this operation as the *conjugation gate* acting on the fluxon pair. Recapitulating, the 3 basic processes with fluxon pairs are *Projective flux measurement*, *Destructive measurement* and *Conjugation gate*.

4.2 Encode a qudit

Let us now try to give a bit more precise mathematical meaning to idea of fusion and how we can encode a qudit in it's space. We start with a theory $\mathcal{A}$ (anyon model) with a finite collection of superselection sector labels $\mathcal{C}_{\mathcal{A}} = \{a, b, c, \dots\}$ called topological or anyonic charges. The process of combining two of these anyons is also called fusion while the rules describing the allowed fusion outcomes of two anyons are called fusion rules. A convenient way of writing the fusion rules is given by [9],[10]:

$$a \times b = \sum_{c \in \mathcal{C}_{\mathcal{A}}} N_c^{ab} c,$$

where the fusion multiplicities N_c^{ab} are non-negative integers which indicate the different ways that the two anyons a and b can be combined to produce the anyon c (these different ways are called the different fusion channels). The fusion algebra is both commutative and associative :

$$\begin{aligned} a \times b = b \times a \ &\Leftrightarrow \ N_c^{ab} = N_c^{ba}, \\ \big(a \times b\big) \times d = a \times \big(b \times d\big) \ &\Leftrightarrow \ \sum_x N_x^{ab} N_c^{xd} = \sum_x N_c^{ax} N_x^{bd} \end{aligned} \tag{46}$$

To each fusion product we assign a fusion vector space V_c^{ab} (a Hilbert space), with $dim\big(V_c^{ab}\big) = N_c^{ab}$. The vector space V_c^{ab} is spanned by so called *fusion states*, which form the basis (diagrammatically in Figure 6) :

$$\big\{|a, b; c, \mu\rangle | \mu = 1, 2, \dots, N_c^{ab}\big\} \tag{47}$$

The dual space (hermitian conjugate) of V_c^{ab}, formed by the bras, is denoted by V_{ab}^c and it's basis vectors are (Figure 6):

$$\big\{\langle a, b; c, \mu| | \mu = 1, 2, \dots, N_{ab}^c = N_c^{ab}\big\} \tag{48}$$

It is called *splitting space* and is described by the states that arise when the anyon c splits into the anyons a and b, while it is a dual basis in the sense that the arrows are reversed. The full Hilbert space for the fusion of the anyons a and b is given by $\oplus_c V_c^{ab}$. It's basis states are orthonormal to the corresponding dual basis states :

$$\langle a, b; c', \mu' | a, b; c, \mu\rangle = \delta_{cc'} \delta_{\mu,\mu'}, \tag{49}$$

and the completeness relation of the basis is expressed as :

$$\sum_{c,\mu} \big|a, b; c, \mu\big\rangle\big\langle a, b; c, \mu\big| = I^{ab}, \tag{50}$$

where I^{ab} is the projector onto the full Hilbert space $\oplus_c V_c^{ab}$.

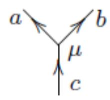
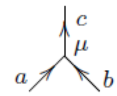
 $= |a, b; c, \mu\rangle \in V_c^{ab},$  $= \langle a, b; c, \mu| \in V_{ab}^c$

Figure 6: Graphical notation for the fusion states of V_c^{ab} showing the fusion of a and b into c (left). The graphical notation for the basis states of the dual space V_{ab}^c emphasizes the splitting of c into a and b (right).

One can as well consider more general fusion spaces V^{ab}, carried by particles a and b, where the fusion outcome is not fixed. The structure of such spaces is given by the direct sum over all the subspaces indexed by the possible fusion outcomes c :

$$V^{ab} = \bigoplus_c V_c^{ab}, \qquad dim\big(V^{ab}\big) = \sum_c N_c^{ab} \tag{51}$$

Since for each c there is a proper subspace, the orthonormal basis in V^{ab} is given by :

$$\Big\{\big|a, b; c, \mu\big\rangle\big|c, \mu = 1, \ldots, N_c^{ab}\Big\}, \;\; \big\langle a, b; c, \mu\big|a, b; c', \mu'\big\rangle = \delta_{c,c'}\delta_{\mu,\mu'} \tag{52}$$

From the definition (51), one can see that $dim\big(V^{ab}\big) > 1$ only for non-abelian models. In an abelian model there would be no topological degeneracy and the outcome of every fusion would always be unique. The topological Hilbert space would coincide with the only subspace labeled by a single c, $V^{ab} \simeq V_c^{ab}$ and thus $dim\big(V^{ab}\big) = N_c^{ab} = 1$ for all a and b. Since one wants to consider the fusion spaces as an arena for quantum computation, this reinforces the notion that quantum computation with anyons is only possible for a non-abelian model.

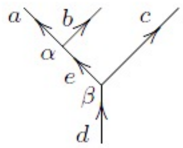
The two-particle fusion spaces serve as simple examples of what are sometimes called *topological Hilbert spaces*. However, they are hardly of particular interest, because unless there is fusion degeneracy, i.e. $N_c^{ab} \geq 2$, V_c^{ab} can not be used to encode quantum information. Consequently, the fusion spaces V^{ab} are directly out of the question, because one cannot form superpositions of states belonging to different superselection sectors. To overcome these restrictions, one must consider the more general fusion spaces $V_c^{a_1,\ldots,a_N}$ carried by some N-particles, whose total charge has been restricted to c. To study their structure, one needs to decompose them in terms of the elementary fusion spaces V_c^{ab}.

When three particles with charges a, b and c are fused to yield a total charge of d, there are two natural ways to decompose the Hilbert space V_d^{abc} (Figure 7) :

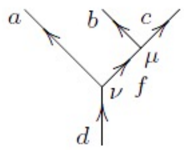
$$V_d^{abc} \cong \bigoplus_e V_e^{ab} \otimes V_d^{ec} \cong \bigoplus_f V_d^{af} \otimes V_f^{bc} \tag{53}$$

These isomorphisms (called F-moves) are written as :

$$\big|a, b; e, \alpha\big\rangle\big|e, c; d, \beta\big\rangle = \sum_{f,\mu,\nu} \big[F_d^{abc}\big]_{\left(e,\alpha,\beta\right)\left(f,\mu,\nu\right)} \big|a, f; d, \nu\big\rangle\big|b, c; f, \mu\big\rangle \tag{54}$$



$$= |a,b;e,\alpha\rangle \otimes |e,c;d,\beta\rangle \in V_e^{ab} \otimes V_d^{ec} \in V_d^{abc}$$



$$= |a,f;d,\nu\rangle \otimes |b,c;f,\mu\rangle \in V_d^{af} \otimes V_f^{bc} \in V_d^{abc}$$

Figure 7: Two distinct decompositions of the fusion space V_d^{abc} into tensor product of two anyon splitting spaces.

and are unitary for anyon models (we give the description in the following subsection).
The decomposition of multi-particle fusion spaces $V_c^{a_1,\dots,a_N}$ is :

$$V_c^{a_1,\dots,a_N} \simeq \bigoplus_{b_1,b_2,\dots,b_{N-2}} V_{b_1}^{a_1a_2} \otimes V_{b_2}^{b_1a_3} \otimes \dots \otimes V_c^{b_{N-2}a_N}, \tag{55}$$

where $b_1, b_2, \dots, b_N$ are particles which may occur during intermediate stages of fusing all particles together. From this expression, one can immediately read off the dimension of $V_c^{a_1,\dots,a_N}$:

$$dim\big(V_c^{a_1\dots a_N}\big) = N_c^{a_1\dots a_N} = \sum_{b_1,b_2,\dots,b_{N-2}} N_{b_1}^{a_1a_2} N_{b_2}^{b_1a_3} \cdots N_c^{b_{N-2}a_N} \tag{56}$$

Rather than the tensor product of the subspace bases, we adopt a more compact notation for the basis states of $V_c^{a_1\dots a_N}$ as :

$$\big\{\big|a_1a_2\cdots a_N; c,\mu\big\rangle\big|\mu = 1,\dots,N_c^{a_1a_2\dots a_N}\big\}, \quad \big\langle a_1a_2\cdots a_N; c,\mu\big|a_1a_2\cdots a_N; c,\mu'\big\rangle = \delta_{\mu,\mu'} \tag{57}$$

When the end charge c is not fixed, we have the Hilbert space $V^{a_1\dots a_N}$ with :

$$dim\big(V^{a_1\dots a_N}\big) = \sum_c V_c^{a_1\dots a_N} = \sum_{c,b_1,b_2,\dots,b_{N-2}} N_{b_1}^{a_1a_2} N_{b_2}^{b_1a_3} \cdots N_c^{b_{N-2}a_N} \tag{58}$$

In particular, we can bring N anyons of the same type a together. The asymptotic dimension of the resulting Hilbert space $V^{aa\dots a} = V^{a^{\otimes N}}$ is written as :

$$dim\big(V^{aa\dots a}\big) \to \big(d_a\big)^N \quad \text{as } N \to \infty$$

Here d_a is called the *quantum dimension* of the anyon. They obey $d_a \geq 1$. The vacuum anyon 1 always has $d_1 = 1$. Very roughly speaking, the quantum dimension should be thought of as the number of degrees of freedom carried by in a single anyon. However, as we'll see, these numbers are typically non-integers, reflecting the fact that we can't really think of the information as being stored on an individual anyon.
From (58) and using the fact that $N_c^{ab} = N_c^{ba}$, we can write the dimension of $V^{aa\dots a}$ as :

$$dim\big(V^{a^{\otimes N}}\big) = \sum_{c,b_1,\dots,b_{N-2}} N_{b_1}^{aa} N_{b_2}^{ab_1} \cdots N_c^{ab_{N-2}} = \sum_{c,b_1,\dots,b_{N-2}} \big[N_a\big]_{ab_1} \big[N_a\big]_{b_1b_2} \cdots \big[N_a\big]_{b_{N-2}c} = \sum_c \big(\big[N_a\big]_{ac}\big)^N, \tag{59}$$

where N_a is the matrix with components $\left[N_a\right]_{bc} = N_c^{ab}$, which is raised to the N^{th} power in the expression above. But, in the limit $N \to \infty$, such a product is dominated by the largest eigenvalue of the matrix N_a. This eigenvalue is the quantum dimension d_a. There is therefore an eigenvector $\mathbf{e}_a = \left(e_1, \ldots, e_N\right)^T$ satisfying :

$$N_a \mathbf{e}_a = d_a \mathbf{e}_a \Rightarrow N_c^{ab} e_c = d_a e_b$$

For what it's worth, the *Perron-Frobenius* theorem in mathematics deals with eigenvalue equations of this type. Among other things, it states that all the components of $\mathbf{e}_a$ are strictly positive. In fact, in the present case the symmetry of $N_c^{ab} = N_c^{ba}$ tells us what they must be. For the right-hand-side to be symmetric, we must have $e_b = d_b$. So the quantum dimensions obey :

$$d_a d_b = \sum_c N_c^{ab} d_c \tag{60}$$

The process when two anyons fuse to a given anyon c via a specific channel represents a single basis vector so it corresponds to a certain measurement outcome, as we know that the total wavefunction collapses after the measurement from the superposition of the basis to an eigenstate of the physical quantity we observe. Some examples of fusion spaces (trees) are given in Figure 8.

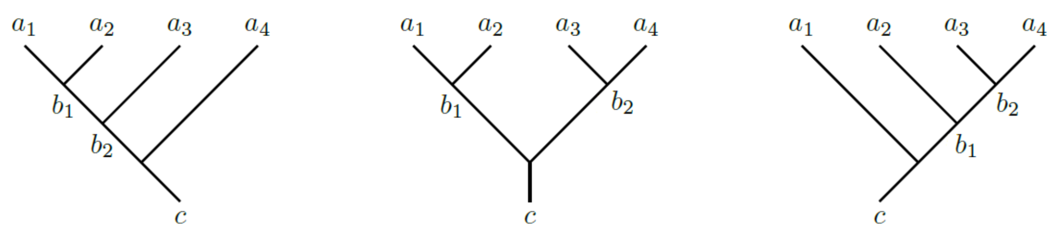


Figure 8: Three splitting tree bases for the space $V_c^{a_1 \ldots a_4}$.

4.3 F and R symbols

Let us consider the case of three anyons a, b and c that fuse together to form an anyon d. One can describe the state of these three particles in two different ways. We can interpret the fusion process by describing how a fuses with b (left of Figure 9), or by starting with the fusion of b and c anyons (right of Figure 9). The e and f are the possible intermediate anyons of each fusion and μ and ν denote the channels (ways) from which the process took place. However, in the two different cases these states are described in different bases. We define the change of basis (order of the fusion) as a set of unitary matrices called F-symbols or F-moves, as shown in Figure 9 [9],[10].

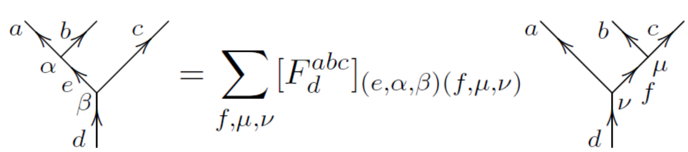
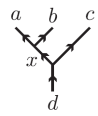
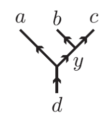


Figure 9: The F symbols as the amplitudes in the change of fusion basis.

With the below definitions :

$|x\rangle \equiv$ 

, $|y\rangle \equiv$ 

,

where $x, y = 0, 1, \ldots, n$, we construct two different orthonormal bases (the same dimension because we are talking about the same space V_d^{abc}), so :

$$|x\rangle = \sum_y c_y |y\rangle = \sum_y \left(F_d^{abc}\right)_{xy} |y\rangle$$

We verified that we are delving with a change of basis, with $c_y = \left(F_d^{abc}\right)_{xy}$ being the fusion probability amplitude and as such $P\big(y\big|x\big) = |c_y|^2 = \left|\left(F_d^{abc}\right)_{xy}\right|^2$ is the conditional probability to measure the anyon y in the second basis given that we have prepared the anyons a and b to fuse exactly to the anyon x (in the $|x\rangle$ basis). We want the action of the F-symbols to preserve the norm (probability) :

$$\sum_y |c_y|^2 = \sum_y \left|\left(F_d^{abc}\right)_{xy}\right|^2 = 1,$$

so we demand F to be unitary. Unitarity of the model amounts to :

$$\left[\left(F_d^{abc}\right)^\dagger\right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[F_d^{abc}\right]^*_{(e,\alpha,\beta)(f,\mu\nu)} = \left[\left(F_d^{abc}\right)^{-1}\right]_{(f,\mu,\nu)(e,\alpha,\beta)}, \tag{61}$$

and diagrammatically refers to the inverse procedure in the change between the two basis (Figure 10). If we require the two above basis for the splitting trees to be orthonormal and the transformation between them to be unitary, then the F-symbols (as well as the R-symbols) can always be represented by unitary matrices.

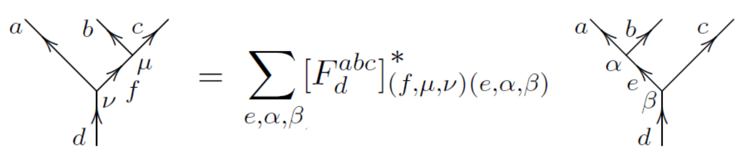


Figure 10: The action of the inverse (unitary) F symbol.

There is a certain freedom present in any anyon model. This corresponds to the choice of bases in the V_c^{ab} space. We can apply unitary transformations in V_c^{ab} without changing the theory, denoted by u_c^{ab}. This gives new basis states $\big|a, b; c\big\rangle'$ (Figure 11).

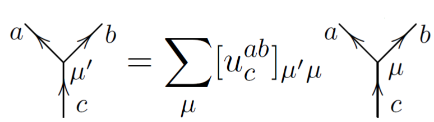


Figure 11: Unitary gauge transformation of a vertex.

When there are no multiplicities ($N_c^{ab} = 0, 1$), the transformations u_c^{ab} are just complex phases. In this case, the transformation on the basis vectors is $\big|a, b; c\big\rangle' = u_c^{ab}\big|a, b; c\big\rangle$ and we have the freedom to redefine the F symbols as it follows :

$$\left[F_d^{abc}\right]'_{ef} = \frac{u_f^{bc} u_d^{af}}{u_e^{ab} u_d^{ec}} \left[F_d^{abc}\right]_{ef} \tag{62}$$

We can always use this freedom to switch to a more convenient set of F symbols. When the state of the system with anyons a, b and c is transformed by a global $U(1)$ phase factor, we get the same physical state meaning that we cannot measure this phase so we have the same expectation values for every observable :

$$|\psi\rangle \sim |\psi\rangle' = e^{i\phi}|\psi\rangle = e^{i\phi}\sum_x c_x|x\rangle = e^{i\phi}\sum_{x,y} c_x\left(F_d^{abc}\right)_{xy}|y\rangle = \sum_{x,y} c_x\left(F_d^{abc}\right)'_{xy}|y\rangle,$$

where we absorbed the phase in the F-symbols due the gauge redundancy that we mentioned previously. By the fact that F-symbols correspond to probability amplitudes, so we want their absolute square (probability) to be fixed, we have the arbitrariness of a global phase when we calculate the total F and R matrices in any space we work. When it comes to the R-symbols or R-moves, their effect is a counterclockwise rotation of two anyons given that the fuse to an anyon c (measurement). The diagrammatic relation of the R-symbol is the following (Figure 12).

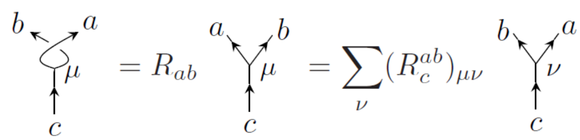


Figure 12: The phase accumulated by exchanging two particles a and b that fuse to c in terms of the R_c^{ab} symbol.

The same arguments that we analyzed for the F-symbols hold for the R-symbols. The unitary braiding operator $\mathcal{R}_{ab}$ upon a pair of anyons a and b, acting on the basis vector of their fusion gives :

$$\mathcal{R}_{ab}|a, b; c, \mu\rangle = \sum_\nu \left[R_c^{ab}\right]_{\mu\nu}|b, a; c, \nu\rangle, \tag{63}$$

which is the mathematical description of Figure 12. Similar equations hold also for the inverse R-moves. Unitarity of the R-symbols amounts to :

$$\left[\left(R_c^{ab}\right)^{-1}\right]_{\mu\nu} = \left[\left(R_c^{ab}\right)^{\dagger}\right]_{\mu\nu} = \left[R_c^{ab}\right]^*_{\nu\mu} \tag{64}$$

Under a gauge transformation, the R-symbols become :

$$\left[R_c^{ab}\right]' = \frac{u_c^{ba}}{u_c^{ab}} R_c^{ab} \tag{65}$$

The inverse $\left(R_c^{ab}\right)^{-1}$ symbol corresponds to a phase when a and b are rotated in a clockwise manner, given that they fuse to c (Figure 13). When we have no multiplicities (one channel at most for every fusion), the sum in (63) simplifies to just one phase. We will only get involved with such practical anyon models.
For the F-moves in a tree diagram, if we result with value 1 or because of the gauge freedom a

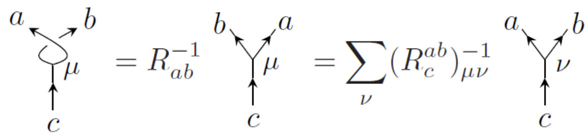


Figure 13: The phase accumulated by exchanging two particles a and b clockwise that fuse to c in terms of the $\left(R_c^{ab}\right)^{-1}$ symbol.

phase $e^{i\phi}$ (norm equal to 1), we know that despite the change of basis, we are dealing with the same diagram on the right hand side of the equation. If a diagram on the right is not permitted by the fusion rules, we put the corresponding F-symbol to zero. We will later do calculations with such symbols.
For the R moves, we introduce the crucial spin-statistics (which relates the spin of the anyons with the action of the braid/monodromy operator) equation [3]. Consider that we have two anyons a and b that fuse to an anyons c. If we exchange, with R_c^{ab} symbol, counterclockwise k times the two anyons then we can continuously transform this braiding process to the one that a and b twist their own wordlines (ribbons) k times by and angle π and also the c anyon twists by itself k times counterclockwise by an angle π (Figure 14). The spin-statistics theorem dictates that the amplitudes of these two processes have to be equal. Each twist is directly linked with the topological spin of each particle. If an anyon, with spin s, twists counterclockwise by an angle ϕ then we get the phase $e^{i\phi s}$ and when this happens clockwise we get $e^{-i\phi s}$. We result with the formula [2]:

$$\left(R_c^{ab}\right)^k = e^{-ik\pi s_a} e^{-ik\pi s_b} e^{ik\pi s_c} \tag{66}$$

We repeatedly use the above relation to calculate the R-symbols.

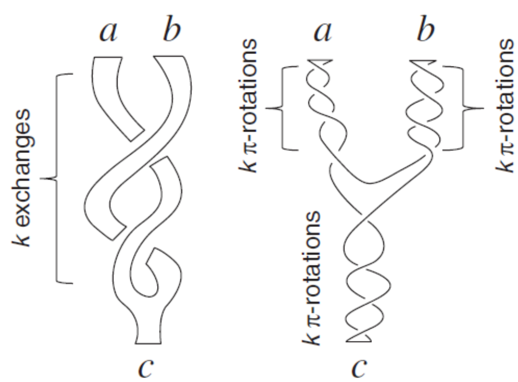


Figure 14: The two topologically equivalent configurations described by the spin-statistics theorem.

4.3.1 Pentagon and hexagon equation

We will consider a multiplicity-free theory ($N_c^{ab} = 0, 1$). If we have higher multiplicities than 1 for the fusion between anyons, then in every F or R-move we have to sum for the multiplicities of every participating vertex. The reason we restrict ourselves is for the notation to be clear. We present the two consistency equations that F and R-moves should satisfy. We start by the *pentagon equation* (Figure 15) [12].

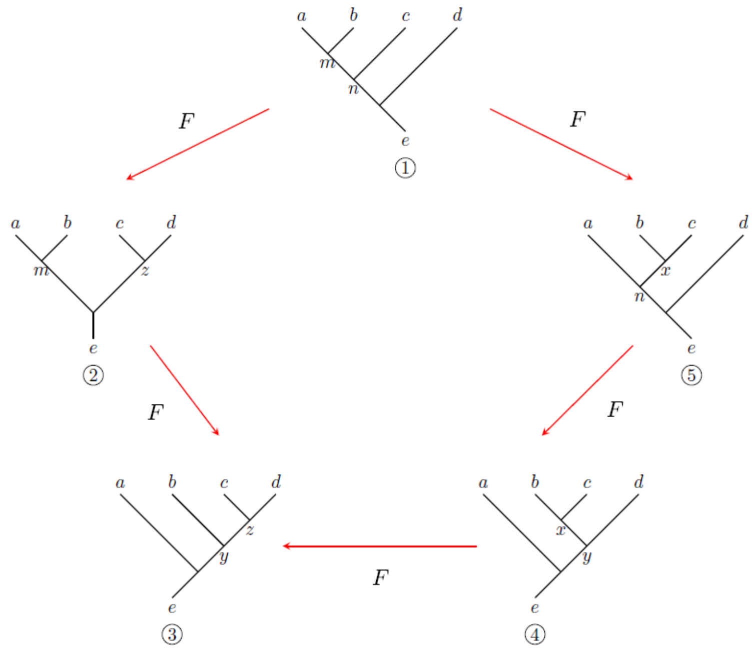


Figure 15: Consistency condition for F-moves.

We see that the geometrical figure constructed is a *pentagon*. The space of the trees is V_e^{abcd}. There are five splitting tree bases in total. We can convert one to another by a sequence of F-moves. But there are different sequences of F-moves achieving this purpose. It is natural to require that different sequences of F-moves give the same transformation. More precisely, in the space V_e^{abcd}, to convert one basis to another, there are exactly two sequences of F-moves. To convert the splitting tree basis labeled by ① to the basis labeled by ③, one can follow either the path ① $\rightarrow$ ② $\rightarrow$ ③ or the path ① $\rightarrow$ ⑤ $\rightarrow$ ④ $\rightarrow$ ③. Denote by $\big|ⓘ;\alpha,\beta\big\rangle$ the random basis state in the basis labeled by ⓘ. Then following the path from ① to the basis labeled by ③, we have :

$$
\begin{aligned}
\big|①;m,n\big\rangle &= \sum_z \left(F_e^{mcd}\right)_{nz}\big|②;m,z\big\rangle \\
&= \sum_{z,y} \left(F_e^{mcd}\right)_{nz}\left(F_e^{abz}\right)_{my}\big|③;z,y\big\rangle
\end{aligned}
$$

And following the path ① $\rightarrow$ ⑤ $\rightarrow$ ④ $\rightarrow$ ③, we have :

$$
\begin{aligned}
\big|①;m,n\big\rangle &= \sum_x \left(F_n^{abc}\right)_{mx}\big|⑤;x,n\big\rangle \\
&= \sum_{x,y} \left(F_n^{abc}\right)_{mx}\left(F_e^{axd}\right)_{ny}\big|④;x,y\big\rangle \\
&= \sum_{x,y,z} \left(F_n^{abc}\right)_{mx}\left(F_e^{axd}\right)_{ny}\left(F_y^{bcd}\right)_{xz}\big|③;z,y\big\rangle
\end{aligned}
$$

By requiring these two sequences to give the same transformation, we arrive at the following equation, known as *pentagon equation* :

$$\left(F_e^{mcd}\right)_{nz}\left(F_e^{abz}\right)_{my} = \sum_x \left(F_n^{abc}\right)_{mx}\left(F_e^{axd}\right)_{ny}\left(F_y^{bcd}\right)_{xz}, \quad \forall a,b,c,d,e,m,n,y,z. \tag{67}$$

Now we exhibit the second geometrical consistency relation called the *hexagon equation* (Figure 16). To convert the basis labeled by ① to the basis labeled by ④, there are two sequences of F/R-moves : ① → ② → ③ → ④ and ① → ⑥ → ⑤ → ④. Again, it is natural to require these two sequences of moves produce the same transformation. For the sequence ① → ② → ③ → ④, the transformation is given by :

$$\begin{aligned}
\big|①; m\big\rangle &= \sum_x \left(F_d^{abc}\right)_{mx}\big|②; x\big\rangle \\
&= \sum_x \left(F_d^{abc}\right)_{mx} R_d^{xa}\big|③; x\big\rangle \\
&= \sum_{x,n} \left(F_d^{abc}\right)_{mx} R_d^{xa}\left(F_d^{bca}\right)_{xn}\big|④; n\big\rangle
\end{aligned}$$

For the sequence ① → ⑥ → ⑤ → ④, we have :

$$\begin{aligned}
\big|①; m\big\rangle &= R_m^{ba}\big|⑥; m\big\rangle \\
&= \sum_n R_m^{ba}\left(F_d^{bac}\right)_{mn}\big|⑤; n\big\rangle \\
&= \sum_n R_m^{ba}\left(F_d^{bac}\right)_{mn} R_n^{ca}\big|④; n\big\rangle
\end{aligned}$$

Hence we arrive at the *hexagon equation* :

$$R_m^{ba}\left(F_d^{bac}\right)_{mn} R_n^{ca} = \sum_x \left(F_d^{abc}\right)_{mx} R_d^{xa}\left(F_d^{bca}\right)_{xn} \tag{68}$$

In general, it is a very hard task to solve the *pentagon* and *hexagon equations*. The pentagon equation makes the F-moves consistent not only for the case of four anyons a, b, c and d but also for the case of arbitrary n anyons. In general we have infinitely many solutions from the *pentagon* and *hexagon equations* (also non unitary solutions can exist). If we fix the gauge (equations (62) and (65)) and also with the constraints that F and R matrices have to be unitary, we end up with finite (in number) solutions from these two relations. If there are no solutions, it means that the anyon model we chose (fusion rules and braiding) does not exist (in the sense that it can not be realized by a gapped, local Hamiltonian).

4.3.2 Quantum double method

Throughout this section, it will be very convenient to condense the notation in the following manner. We will change the particle labels from $({}^AC, {}^\alpha\Gamma), ({}^BC, {}^\beta\Gamma), ({}^CC, {}^\gamma\Gamma), \ldots$ to $a, b, c, \ldots$. As a result, we replace the internal spaces $V_\alpha^A, V_\beta^B, \ldots$ to $V^a, V^b, \ldots$ and the representation maps are denoted as $\Pi^a = \Pi_\alpha^A$, etcetera. We can label the basis states of a by an index $i = 1, \ldots, d_a$ with d_a the dimension of the anyon a. We denote the basis states of a therefore as :

$$\left\{\big|a, i\big\rangle\right\}, \quad i = 1, \ldots, d_a,$$

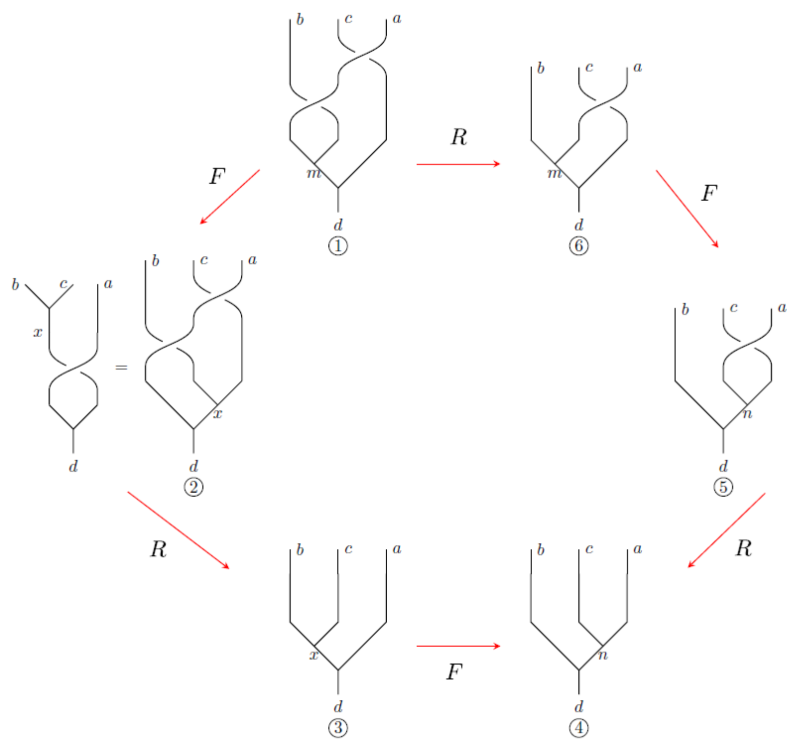


Figure 16: Consistency condition for R-moves.

leaving the particle label explicit, instead of $\left|{}^{A}h_j, {}^{\alpha}v_k\right\rangle$ as we did before. The coordinate functions of the representation matrices are denoted as Π^{a}_{ij}. The fusion product $a \times b$ means the tensor product representation.

4.3.2.1 Computing the F-symbols

The F-symbols are conveniently calculated using *Clebsch-Gordan* coefficients, which describe how the irreps are precisely embedded in tensor product representations (Figure 17). When there are no fusion multiplicities ($N^{ab}_c = 0, 1$) we can calculate the *Clebsch-Gordan* symbols using a generalization of the projection operator technique (a technique well known from the theory of group representations).

In a multiplicity-free theory, there is a unique basis [10]:

$$\left\{\left|c, k\right\rangle\right\}, \quad N^{ab}_c = 1,\ k = 1, \dots, d_c$$

in $a \times b$ corresponding to the decomposition in irreps. Of course, there is also the standard inner product basis :

$$\left|a, i\right\rangle\left|b, j\right\rangle, \quad i = 1, \dots, d_a, j = 1, \dots, d_b$$

The *Clebsch-Gordan* coefficients, sometimes also called $3j$-symbols, precisely give the relation between these two. They are defined by [11]:

$$\left|c, k\right\rangle = \sum_{i,j} C^{abc}_{ijk}\left|a, i\right\rangle\left|b, j\right\rangle = \sum_{i,j} \left(\begin{array}{cc|c} a & b & c \\ i & j & k \end{array}\right)\left|a, i\right\rangle\left|b, j\right\rangle \tag{69}$$

$$\left(\begin{array}{cc|c} a & b & c \\ i & j & k \end{array}\right) =$$

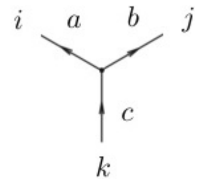


Figure 17: The irrep c embedded in the tensor product of irreps $a \times b$.

The inverse of this relation is written as :

$$\big|a, i\big\rangle\big|b, j\big\rangle = \sum_{c,k} \left(\begin{array}{c|cc} c & a & b \\ k & i & j \end{array}\right) \big|c, k\big\rangle \tag{70}$$

To calculate the actual coefficients, we will use projectors $\mathcal{P}^a_{ij}$ defined by :

$$\mathcal{P}^a_{ij} = \frac{d_a}{|H|} \sum_{h,g\in H} \Pi^{a\,*}_{ij}\big(P_h g\big)\Pi\big(P_h g\big), \tag{71}$$

where the last $\Pi\big(P_h g\big)$ stands for the appropriate representation of the $P_h g$ element of the quantum double, depending on what it is acting on. These projectors, which can be applied in any representation, act as :

$$\mathcal{P}^a_{ij}\big|b, k\big\rangle = \delta_{a,b}\delta_{j,k}\big|a, i\big\rangle \tag{72}$$

Proof.

$$\big\langle b, l\big|\mathcal{P}^a_{ij}\big|b, k\big\rangle = \frac{d_a}{|H|} \sum_{h,g\in H} \Pi^{a\,*}_{ij}\big(P_h g\big)\big\langle b, l\big|\Pi\big(P_h g\big)\big|b, k\big\rangle = \frac{d_a}{|H|} \sum_{h,g\in H} \Pi^{a\,*}_{ij}\big(P_h g\big)\Pi^b_{lk}\big(P_h g\big)$$

The orthogonality relation for the coordinate functions Π^a_{ij} is :

$$\frac{d_a}{|H|} \sum_{h,g\in H} \Pi^{a\,*}_{ij}\big(P_h g\big)\Pi^b_{lk}\big(P_h g\big) = \delta_{a,b}\delta_{i,l}\delta_{j,k}, \tag{73}$$

so we have :

$$\big\langle b, l\big|\mathcal{P}^a_{ij}\big|b, k\big\rangle = \delta_{a,b}\delta_{i,l}\delta_{j,k}$$

From the formula we want to show :

$$\mathcal{P}^a_{ij}\big|b, k\big\rangle = \delta_{a,b}\delta_{j,k}\big|a, i\big\rangle = \delta_{a,b}\delta_{j,k}\big|b, i\big\rangle \Rightarrow \big\langle b, l\big|\mathcal{P}^a_{ij}\big|b, k\big\rangle = \delta_{a,b}\delta_{j,k}\big\langle b, l\big|b, i\big\rangle = \delta_{a,b}\delta_{i,l}\delta_{j,k},$$

which agrees with the previous formula. ∎

Applying the projector to a direct product of two states (70) and using equation (69), gives :

$$\begin{aligned} \mathcal{P}^c_{lk}\big|a, i\big\rangle\big|b, j\big\rangle &= \left(\begin{array}{c|cc} c & a & b \\ k & i & j \end{array}\right) \big|c, l\big\rangle \\ &= \sum_{i',j'} \left(\begin{array}{c|cc} c & a & b \\ k & i & j \end{array}\right) \left(\begin{array}{cc|c} a & b & c \\ i' & j' & l \end{array}\right) \big|a, i'\big\rangle\big|b, j'\big\rangle \end{aligned} \tag{74}$$

By using the definition (71), this is seen to be equal to :

$$\begin{aligned}\mathcal{P}^c_{lk}\big|a,i\big\rangle\big|b,j\big\rangle &= \frac{d_c}{|H|}\sum_{h,b\in H}\Pi^c_{lk}{}^*\big(P_h g\big)\Pi\big(P_h g\big)\big|a,i\big\rangle\big|b,j\big\rangle \\ &= \frac{d_c}{|H|}\sum_{h,g\in H}\Pi^c_{lk}{}^*\big(P_h g\big)\sum_{\substack{h'h''=h \\ i',j'}}\Pi^a_{i'i}\big(P_{h'}g\big)\Pi^b_{j'j}\big(P_{h''}g\big)\big|a,i'\big\rangle\big|b,j'\big\rangle,\end{aligned} \tag{75}$$

where we inserted the definition of the comultiplication (16), which acts on the product state. Equating expressions (74) and (75), we obtain :

$$\left(\begin{matrix} c \\ k \end{matrix}\,\middle|\,\begin{matrix} a & b \\ i & j \end{matrix}\right)\left(\begin{matrix} a & b \\ i' & j' \end{matrix}\,\middle|\,\begin{matrix} c \\ l \end{matrix}\right) = \frac{d_c}{|H|}\sum_{h,g\in H}\Pi^c_{lk}{}^*\big(P_h g\big)\sum_{h'h''=h}\Pi^a_{i'i}\big(P_{h'}g\big)\Pi^b_{j'j}\big(P_{h''}g\big) \tag{76}$$

Unitarity amounts to :

$$\left(\begin{matrix} c \\ k \end{matrix}\,\middle|\,\begin{matrix} a & b \\ i & j \end{matrix}\right) = \left(\begin{matrix} a & b \\ i & j \end{matrix}\,\middle|\,\begin{matrix} c \\ k \end{matrix}\right)^* \tag{77}$$

Now we pick some triple (i,j,k) such that :

$$\frac{d_c}{|H|}\sum_{h,g\in H}\Pi^c_{kk}{}^*\big(P_h g\big)\sum_{h'h''=h}\Pi^a_{ii}\big(P_{h'}g\big)\Pi^b_{jj}\big(P_{h''}g\big) \tag{78}$$

is non-zero. From equation (76) with $i = i'$, $j = j'$ and $k = k'$ and from the unitarity condition (77), it follows that this number is real and positive. This fixes one of the *Clebsch-Gordan* symbols, by :

$$\left(\begin{matrix} a & b \\ i & j \end{matrix}\,\middle|\,\begin{matrix} c \\ k \end{matrix}\right) = \left(\frac{d_c}{|H|}\sum_{h,g\in H}\sum_{h'h''=h}\Pi^c_{kk}{}^*\big(P_h g\big)\Pi^a_{ii}\big(P_{h'}g\big)\Pi^b_{jj}\big(P_{h''}g\big)\right)^{1/2} \tag{79}$$

Since we have demanded the unitarity condition, once we compute a positive (non-zero) F-symbol (element of the F matrix) we know that the relative phase among all the elements of that matrix is constant, let's say $e^{i\phi}$. We mentioned earlier that these matrices can be can be identified with the arbitrariness of a total complex phase (in our example this $e^{i\phi}$ phase). We use equation (76) to calculate all others, which results in :

$$\left(\begin{matrix} a & b \\ i' & j' \end{matrix}\,\middle|\,\begin{matrix} c \\ k' \end{matrix}\right) = \sqrt{\frac{d_c}{|H|}}\,\frac{\sum_{h,g\in H}\sum_{h'h''=h}\Pi^c_{k'k}{}^*\big(P_h g\big)\Pi^a_{i'i}\big(P_{h'}g\big)\Pi^b_{j'j}\big(P_{h''}g\big)}{\Big(\sum_{h,g\in H}\sum_{h'h''=h}\Pi^c_{kk}{}^*\big(P_h g\big)\Pi^a_{ii}\big(P_{h'}g\big)\Pi^b_{jj}\big(P_{h''}g\big)\Big)^{1/2}} \tag{80}$$

From the *Clebsch-Gordan* symbols it is straightforward to calculate the F-symbols. We use the definitions of F and *Clebsch-Gordan* symbols (Figures 9 and 17), when there are no multiplicities meaning $\alpha,\beta,\mu,\nu = 0,1$. We sum in the internal states of the intermediate anyon $e \to \big|e,m\big\rangle$ (we have specified the result of the fusion of the anyons a and b but it's state is unknown so we have to include contributions from all of them). The states of the anyons $a \to \big|a,i\big\rangle$, $b \to \big|b,j\big\rangle$ and $c \to \big|c,k\big\rangle$ are given (we prepare them in some desired states) and we let free the state of the final anyon $d \to \big|d,l\big\rangle$ in the end of the fusion diagram. The emerging relation is :

$$\sum_{f,n}\big(F^{abc}_d\big)_{ef}\left(\begin{matrix} b & c \\ j & k \end{matrix}\,\middle|\,\begin{matrix} f \\ n \end{matrix}\right)\left(\begin{matrix} a & f \\ i & n \end{matrix}\,\middle|\,\begin{matrix} d \\ l \end{matrix}\right) = \sum_m \left(\begin{matrix} a & b \\ i & j \end{matrix}\,\middle|\,\begin{matrix} e \\ m \end{matrix}\right)\left(\begin{matrix} e & c \\ m & k \end{matrix}\,\middle|\,\begin{matrix} d \\ l \end{matrix}\right) \tag{81}$$

Using the orthogonality of the *Clebsch-Gordan* symbols :

$$\sum_{i,j} \left(\begin{array}{cc|c} a & b & c \\ i & j & k \end{array}\right) \left(\begin{array}{cc|c} a & b & c' \\ i & j & k' \end{array}\right) = \delta_{c,c'}\delta_{k,k'}, \tag{82}$$

we finally have :

$$\left(F_d^{abc}\right)_{ef} = \sum_{i,j,k,n,m} \left(\begin{array}{cc|c} a & b & e \\ i & j & m \end{array}\right) \left(\begin{array}{cc|c} e & c & d \\ m & k & l \end{array}\right) \left(\begin{array}{cc|c} b & c & f \\ j & k & n \end{array}\right)^* \left(\begin{array}{cc|c} a & f & d \\ i & n & l \end{array}\right)^* \tag{83}$$

4.3.2.2 Computing the R-symbols

We look at Figure 12. As we use the quantum double here, $R_{ab} \equiv \mathcal{R}^{ab}$ defined in (18). After the winding of the two particles a and b, with the action of the braiding matrix $\mathcal{R}_{ab}$, we don't know in which exactly states these particles end up so we have to sum in all possible states they can exist. For the simple case where there are no multiplicities of c, similarly as we did for F-symbols, here we have :

$$R_c^{ab} \left(\begin{array}{cc|c} b & a & c \\ j' & i' & k \end{array}\right) = \sum_{i,j} \mathcal{R}^{ab}_{(i,i'),(j,j')} \left(\begin{array}{cc|c} a & b & c \\ i & j & k \end{array}\right), \tag{84}$$

where $\mathcal{R}^{ab}_{(i,i'),(j,j')}$ are the representation elements of the action of $\mathcal{R}^{ab}$ on the basis states $|a,i\rangle|b,j\rangle$:

$$\begin{aligned} \mathcal{R}^{ab}_{(i,i'),(j,j')} \equiv \langle a,i'|\langle b,j'|\mathcal{R}^{ab}|a,i\rangle|b,j\rangle &= \langle a,i'|\langle b,j'|\sigma \circ \left(\Pi^a \otimes \Pi^b\right)(R)|a,i\rangle|b,j\rangle \\ &= \sum_{h,g\in H} \langle a,i'|\langle b,j'|\Big(\Pi^b(P_h g) \otimes \Pi^a(P_h)\Big)|b,j\rangle|a,i\rangle \end{aligned} \tag{85}$$

Using equation (82), we get :

$$R_c^{ab} = \sum_{i,i'=1}^{d_a} \sum_{j,j'=1}^{d_b} \left(\begin{array}{cc|c} a & b & c \\ i & j & k \end{array}\right) \left(\begin{array}{cc|c} b & a & c \\ j' & i' & k \end{array}\right)^* \mathcal{R}^{ab}_{(i,i'),(j,j')}, \tag{86}$$

where again k (internal state of the fusion resulting anyon c) can be chosen freely.
If multiplicities of c do occur, we can still obtain the $\left(R_c^{ab}\right)_{\mu\nu}$ by first including the μ^{th} copy of c and next project onto the ν^{th} copy. This can be done by introducing more general *Clebsch-Gordan* coefficients that also carry and index μ that keeps track of the copy c. The formula then becomes :

$$\sum_{\nu} \left(R_c^{ab}\right)_{\mu\nu} \left(\begin{array}{cc|c} b & a & c,\nu \\ j' & i' & k \end{array}\right) = \sum_{i,j} \mathcal{R}^{ab}_{(i,i'),(j,j')} \left(\begin{array}{cc|c} a & b & c,\mu \\ i & j & k \end{array}\right) \Rightarrow$$

$$\sum_{\nu,i',j'} \left(R_c^{ab}\right)_{\mu\nu} \left(\begin{array}{cc|c} b & a & c,\nu \\ j' & i' & k \end{array}\right) \left(\begin{array}{cc|c} b & a & c',\nu' \\ j' & i' & k' \end{array}\right)^* = \sum_{i,i',j,j'} \mathcal{R}^{ab}_{(i,i'),(j,j')} \left(\begin{array}{cc|c} a & b & c,\mu \\ i & j & k \end{array}\right) \left(\begin{array}{cc|c} b & a & c',\nu' \\ j' & i' & k' \end{array}\right)^* \Rightarrow$$

$$\sum_{\nu} \left(R_c^{ab}\right)_{\mu\nu} \delta_{c,c'}\delta_{\nu,\nu'}\delta_{k,k'} = \sum_{i,i',j,j'} \mathcal{R}^{ab}_{(i,i'),(j,j')} \left(\begin{array}{cc|c} a & b & c,\mu \\ i & j & k \end{array}\right) \left(\begin{array}{cc|c} b & a & c',\nu' \\ j' & i' & k' \end{array}\right)^* \Rightarrow$$

$$\left(R_c^{ab}\right)_{\mu\nu} = \sum_{i,i'=1}^{d_a} \sum_{j,j'=1}^{d_b} \left(\begin{array}{cc|c} a & b & c,\mu \\ i & j & k \end{array}\right) \left(\begin{array}{cc|c} b & a & c,\nu \\ j' & i' & k \end{array}\right)^* \mathcal{R}^{ab}_{(i,i'),(j,j')} \tag{87}$$

4.4 Fibonacci anyons

Here we have a model of two different anyons. We have the abelian trivial sector (vacuum) which we label as $\mathbf{1}$ and a non-trivial charge (anyon) which we label as τ with $\overline{\tau} = \tau$, meaning that it is it's own antiparticle (self-dual like $\mathbf{1}$). The fusion rules of this model are [2],[13],[14]:

$$\begin{aligned} &\mathbf{1} \otimes \mathbf{1} = \mathbf{1} \\ &\tau \otimes \mathbf{1} = \mathbf{1} \otimes \tau = \tau \\ &\tau \otimes \tau = \mathbf{1} \oplus \tau \end{aligned} \tag{88}$$

The first two fusion rules give an abelian model as the fusion of $\mathbf{1}$ with any other anyon $x = \{\mathbf{1}, \tau\}$ gives back the same anyon x (because $\mathbf{1}$ is abelian). The interesting non-trivial fusion rule is the third one with the fusion of the two non-abelian anyons τ, which indeed makes the model non-abelian because we result with more than one type of anyons after the fusion. The topological Hilbert space that describes the fusion of N different τ anyons is denoted $V_c^{\tau^{\otimes N}}$ with $c \in \{\mathbf{1}, \tau\}$. We usually create N anyons from the vacuum (the reverse process of fusion called splitting), so in these cases the overall charge is trivial and the Hilbert space they live is $V_{\mathbf{1}}^{\tau^{\otimes N}}$. The dimension of this fusion space is $dim\big(V_{\mathbf{1}}^{\tau^{\otimes N}}\big) = N_{\mathbf{1}}^{\tau^{\otimes N}} = \big[D_{\mathbf{1}}\big]_N$. The number $\big[D_{\mathbf{1}}\big]_N$, i.e. the dimension of the topological Hilbert space, describes the different ways the N anyons can fuse to the vacuum. By creating the trees for each N we notice that every $\mathbf{1}$ must always be followed by a τ because $\mathbf{1} \otimes \tau = \tau$. So if the first two anyons fuse to trivial total charge $\mathbf{1}$ then, because this trivial charge with the third τ give back τ, then the remaining $\big(N-2\big)\tau$ anyons can fuse with $\big[D_{\mathbf{1}}\big]_{N-2}$ different ways to end up with topological charge $\mathbf{1}$. If the first two anyons fuse to τ then we are left with $\big(N-1\big)\tau$ anyons that can fuse with $\big[D_{\mathbf{1}}\big]_{N-1}$ different ways to $\mathbf{1}$. So the numbers $\big[D_{\mathbf{1}}\big]_N$ satisfy the recursion relation :

$$\big[D_{\mathbf{1}}\big]_N = \big[D_{\mathbf{1}}\big]_{N-1} + \big[D_{\mathbf{1}}\big]_{N-2} \tag{89}$$

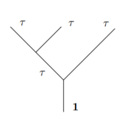
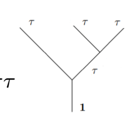
The dimensions of the Hilbert space follow the Fibonacci sequence and that's the reason why the model is called *Fibonacci model*. The matrix N_τ, with components $N_c^{\tau b}$, can be extracted from the fusion rules :

$$N_\tau = \begin{pmatrix} N_{\mathbf{1}}^{\tau\mathbf{1}} & N_{\tau}^{\tau\mathbf{1}} \\ N_{\mathbf{1}}^{\tau\tau} & N_{\tau}^{\tau\tau} \end{pmatrix} = \begin{pmatrix} \big(N_\tau\big)_{\mathbf{1}\mathbf{1}} & \big(N_\tau\big)_{\mathbf{1}\tau} \\ \big(N_\tau\big)_{\tau\mathbf{1}} & \big(N_\tau\big)_{\tau\tau} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{90}$$

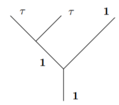
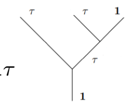
The space $V_{\mathbf{1}}^{\tau\tau\tau}$ of three τ anyons that fuse to the vaccum $\mathbf{1}$ is one-dimensional :

$$dim\big(V_{\mathbf{1}}^{\tau\tau\tau}\big) = \sum_x N_x^{\tau\tau} N_{\mathbf{1}}^{x\tau} = N_\tau^{\tau\tau} N_{\mathbf{1}}^{\tau\tau} = 1$$

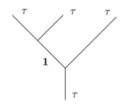
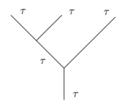
The change of basis transformation F in $V_{\mathbf{1}}^{\tau\tau\tau}$ is trivial ($\mu = 1$) :

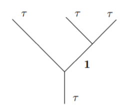
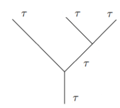
$\big|\tau\big\rangle_1 \equiv$  $= \big(F_{\mathbf{1}}^{\tau\tau\tau}\big)_{\tau\tau}$  $\equiv \big(F_{\mathbf{1}}^{\tau\tau\tau}\big)_{\tau\tau}\big|\tau\big\rangle_2,$

where with the state $\big|\tau\big\rangle$ we note the product anyon τ of the first fusion (the numbers 1 and 2 on the states are referring to the basis). Without loss of generality we can take $F_{\mathbf{1}}^{\tau\tau\tau} = 1$ due to gauge freedom. The diagram at the right above is identical with the left one if we rotate it around the main/central vertical vertex by π angle (the scalar coefficient F can be a phase at most as we have the same course of fusions). Similarly for the space $V_{\mathbf{1}}^{\tau\tau\mathbf{1}}$:

$\big|\mathbf{1}\big\rangle_1 \equiv$  $= \big(F_{\mathbf{1}}^{\tau\tau\mathbf{1}}\big)_{\mathbf{1}\tau}$  $\equiv \big(F_{\mathbf{1}}^{\tau\tau\mathbf{1}}\big)_{\mathbf{1}\tau}\big|\tau\big\rangle_2,$

so we deal again with an 1D space and we choose $\big(F_{\mathbf{1}}^{\tau\tau\mathbf{1}}\big)_{\mathbf{1}\tau} = 1$. In general, we know that the total probability to measure the $\big|\tau\big\rangle$ state in the right basis given that we prepared the system in the state $\big|\mathbf{1}\big\rangle$ in the left basis is $P\big(\mathbf{1}|\tau\big) = \big|\big(F_{\mathbf{1}}^{\tau\tau\mathbf{1}}\big)_{\mathbf{1}\tau}\big|^2 = 1 \to \big(F_{\mathbf{1}}^{\tau\tau\mathbf{1}}\big)_{\mathbf{1}\tau} = e^{i\theta}$ and we select $\theta = 0$. We cannot encode a qubit in these two trivial spaces. The first non-trivial space to encode a qubit is $V_{\tau}^{\tau\tau\tau}$, with states :

$\big|\mathbf{1}\big\rangle_1 \equiv$  , $\big|\tau\big\rangle_1 \equiv$  ,

$\big|\mathbf{1}\big\rangle_2 \equiv$  , $\big|\tau\big\rangle_2 \equiv$ 

The general unitary transformation F between the two bases has the form :

$$F \equiv F_{\tau}^{\tau\tau\tau} = \begin{pmatrix} \big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\mathbf{1}} & \big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\tau} \\ \big(F_{\tau}^{\tau\tau\tau}\big)_{\tau\mathbf{1}} & \big(F_{\tau}^{\tau\tau\tau}\big)_{\tau\tau} \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & -a^* \end{pmatrix},$$

with $\sum_y \big|\big(F_{\tau}^{\tau\tau\tau}\big)_{xy}\big|^2 = \big|a\big|^2 + \big|b\big|^2 = 1$. Without any multiplicities in the theory, the pentagon equation (67) for $a = b = c = d = e = m = y = \tau$ and $n, z = \mathbf{1}$ gives :

$$\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\mathbf{1}}\big(F_{\tau}^{\tau\tau\mathbf{1}}\big)_{\tau\tau} = \sum_x \big(F_{\mathbf{1}}^{\tau\tau\tau}\big)_{\tau x}\big(F_{\tau}^{\tau x\tau}\big)_{\mathbf{1}\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{x\mathbf{1}} = \big(F_{\mathbf{1}}^{\tau\tau\tau}\big)_{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{\tau\mathbf{1}} \Rightarrow$$
$$\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\mathbf{1}} = \big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{\tau\mathbf{1}} \Rightarrow a = |b|^2,$$

so $|a|^2 + a - 1 = 0 \Rightarrow a_{1,2} = \frac{-1\pm\sqrt{5}}{2}$. We pick $a = \frac{-1+\sqrt{5}}{2} = \phi^{-1}$ with $\phi = \frac{1+\sqrt{5}}{2}$ the *golden mean*. We then have to solve for $b \to |b|^2 = a = \phi^{-1} \Rightarrow b = e^{i\theta}\sqrt{\phi^{-1}}$. The phase $e^{i\theta}$ can be set to unity with a gauge transformation (in each space $V_c^{\tau^{\otimes N}}$ separately we have a gauge freedom or else for every different F matrix). The F matrix becomes :

$$F = \begin{pmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{\phi^{-1}} & -\phi^{-1} \end{pmatrix} \tag{91}$$

All pentagon relations are now indeed satisfied as we fixed the gauge. Next, we calculate the R matrix. It is clear that if the exchange includes an anyon which is the vacuum sector, this action will have no physical consequence in the system. Since the vacuum braids trivially, the phases below, which are allowed by the fusion rules, are equal to 1 :

$$R_{\tau}^{\tau\mathbf{1}} = R_{\tau}^{\mathbf{1}\tau} = R_{\mathbf{1}}^{\mathbf{1}\mathbf{1}} = 1$$

The R matrix is clearly diagonal because an exchange of two anyons cannot change the outcome of the fusion, so it has the form :

$$R = R_c^{\tau\tau} = \begin{pmatrix} R_{\mathbf{1}}^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix}$$

Having found the F matrix, we use the hexagon equation (68) to find the components of the R matrix. For $a=b=c=d=\tau$ and $m=n=\mathbf{1}$ we take :

$$R_{\mathbf{1}}^{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{11}}R_{\mathbf{1}}^{\tau\tau}=\sum_{x}\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}x}R_{\tau}^{x\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{x\mathbf{1}}=\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{11}}^{2}+\big(F_{\tau}^{\tau\tau\tau}\big)_{\mathbf{1}\tau}R_{\tau}^{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{\tau\mathbf{1}}\Rightarrow$$
$$\phi^{-1}\big(R_{\mathbf{1}}^{\tau\tau}\big)^{2}=\phi^{-2}+\phi^{-1}R_{\tau}^{\tau\tau}\Rightarrow\big(R_{\mathbf{1}}^{\tau\tau}\big)^{2}=\phi^{-1}+R_{\tau}^{\tau\tau}$$

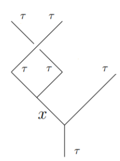
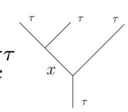
For $a=b=c=d=m=\tau$ and $n=\mathbf{1}$ we get :

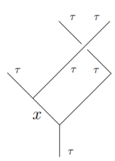
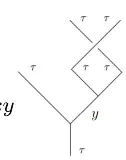
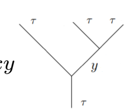
$$R_{\tau}^{\tau\tau}R_{\mathbf{1}}^{\tau\tau}\sqrt{\phi^{-1}}=\phi^{-3/2}-R_{\tau}^{\tau\tau}\phi^{-3/2}\Rightarrow R_{\tau}^{\tau\tau}R_{\mathbf{1}}^{\tau\tau}=\phi^{-1}-R_{\tau}^{\tau\tau}\phi^{-1}$$

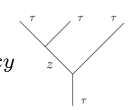
The solutions from the two equations we derived above are $R_{\mathbf{1}}^{\tau\tau}=e^{-4\pi i/5}$ and $R_{\tau}^{\tau\tau}=e^{3\pi i/5}$, so we result with the following R matrix :

$$R=\begin{pmatrix}e^{-4\pi i/5} & 0\\ 0 & e^{3\pi i/5}\end{pmatrix}\tag{92}$$

We work in the qubit basis $\big|x\big\rangle$ with $x\in\big\{\mathbf{1},\tau\big\}$ that we established previously. The generators of the braid group $\mathcal{B}_3$ among three identical τ particles can be determined by the F and R matrices that we found :

$\sigma_1\big|x\big\rangle\equiv$  $=R_{x}^{\tau\tau}$  $=R_{x}^{\tau\tau}\big|x\big\rangle,$

$\sigma_2\big|x\big\rangle\equiv$  $=\sum_{y}\big(F_{\tau}^{\tau\tau\tau}\big)_{xy}$  $=\sum_{y}R_{y}^{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{xy}$ 

$=\sum_{y,z}\big(F_{\tau}^{\tau\tau\tau}\big)_{yz}^{-1}R_{y}^{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{xy}$  $=\sum_{y,z}\big(F_{\tau}^{\tau\tau\tau}\big)_{yz}^{-1}R_{y}^{\tau\tau}\big(F_{\tau}^{\tau\tau\tau}\big)_{xy}\big|z\big\rangle$

So the unitary generators σ_i are ($F^{-1}=F^{\dagger}=F$ because the matrix F is hermitian) :

$$\sigma_1=R=\begin{pmatrix}e^{-4\pi i/5} & 0\\ 0 & e^{3\pi i/5}\end{pmatrix},\quad \sigma_2=F^{-1}RF=\begin{pmatrix}\phi^{-1}e^{4\pi i/5} & \sqrt{\phi^{-1}}e^{-3\pi i/5}\\ \sqrt{\phi^{-1}}e^{-3\pi i/5} & -\phi^{-1}\end{pmatrix}\tag{93}$$

These matrices comprise a representation of the braid group $\mathcal{B}_3$ with an image that is dense in $SU\big(V_{\tau}^{\tau\tau\tau}\big)=SU\big(2\big)$. Therefore, each braid/weave can be simulated with arbitrary accuracy by some finite braid. We can also use the inverse of each of the σ_i's as the opposite braid can be applied. The image of these two generators is the truncated braid group $B\big(3,10\big)$ which has infinite order (three identical τ anyons where the order of the generators is $10\to\sigma_i^{10}=\mathbb{1}_{2\times2}$). Because of this finite order of the generators, we result with $\sigma_i^{n}=\sigma_i^{n-10}$ where $n=n$ (mod10), so the number of braids we need to approximate or even construct analytically a unitary gate is significantly reduced [15].
Even though the determinants of the generators are not 1, we can redefine them with a freedom of a common total phase $e^{\pi i/5}$ so that they belong in $SU\big(2\big)$. Otherwise, we approximate unitary

gates with braids up to an overall phase factor. Since in quantum mechanics the overall phase of a system cannot be measured, this overall phase is generally of no consequence. In order to create a two-qubit entangling gate, we need 6 Fibonacci anyons with a 5D fusion space and whose image is dense in $SU\left(V_{\mathbf{1}}^{\tau^{\otimes 6}}\right) = SU(5)$ (Figure 18) or 8 anyons that form a dense set in $SU\left(V_{\mathbf{c}}^{\tau^{\otimes 8}}\right) = SU(13)$. Both spaces of course include the $SU(4)$ as 4×4 computational blocks in their representations that act on the two encoded qubits. The representations of the generators for the 6 and 8 anyon cases have a trivial and a 9×9 non-computational blocks respectively (there is leakage which leads to errors between the computational and non-computational states whenever the representation has non diagonal elements) [14],[15].
Universality refers to the existence of a universal set of quantum gates, the elements of which can perform any unitary evolution in $SU(n)$ with arbitrary accuracy. It requires a minimum set of quantum gates, such as the set of single qubit braids (rotation gates that can span $SU(2)$ or geometrically the *Bloch sphere*) and a two-qubit entangling gate such as the controlled-NOT (CNOT) gate. Alternatively, universality can be achieved only with the *Hadamard* gate and *CCNOT* or *Toffoli* gate (3-qubit gate). The weaves (sequences of successive braids) of some universal gates like Hadamard, NOT and CNOT with Fibonacci anyons are included (with diagrams) in [15].
Not all anyon models, such as Ising anyons, provide universality by braiding alone. Such computers need to be supplemented by non-topological operations (like measurement) in order to achieve universality. The application of the *Solovay-Kitaev* theorem concludes that the universal gates of the circuit model can be simulated to accuracy ϵ with braidwords of length $poly\left[\log(1/\epsilon)\right]$ and so a universal quantum computer can be simulated efficiently using Fibonacci anyons [15].

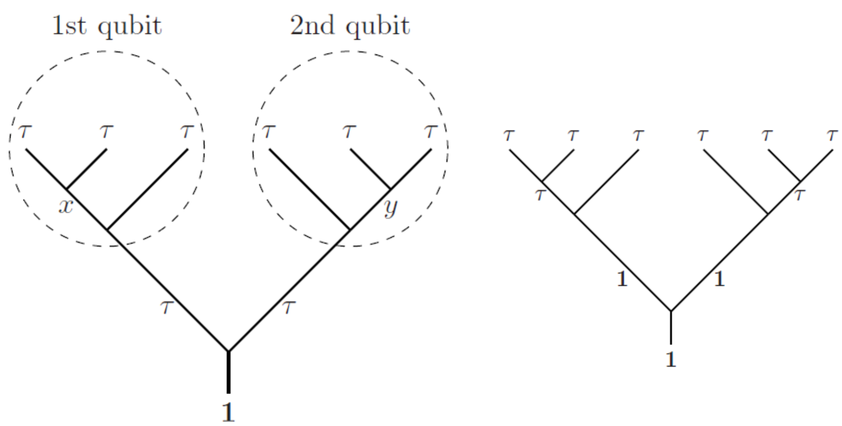


Figure 18: The 6τ anyon encoding of two qubits where $x, y = \mathbf{1}, \tau$ (left) and the trivial non-computational basis element (right).

4.5 Ising anyons

There are three elements in the Ising anyon model. A fermion denoted as ψ, an anyon denoted as σ and the vacuum $\mathbf{1}$, obeying the following fusion rules [2],[16]:

$$\begin{aligned} \mathbf{1} \otimes \mathbf{1} = \mathbf{1},\ \ \mathbf{1} \otimes \psi = \psi,\ \ \mathbf{1} \otimes \sigma = \sigma, \\ \psi \otimes \psi = \mathbf{1},\ \ \sigma \otimes \psi = \sigma, \\ \sigma \otimes \sigma = \mathbf{1} \oplus \psi. \end{aligned} \tag{94}$$

The fusion rule for two σ particles states that this anyon is indeed a non-abelian one, because there exist two possible fusion outcomes. Since all other fusion rules determine the outcomes uniquely, higher dimensional fusion spaces are always carried by σ particles. Using the last fusion rule successively, we get the fusion rules for $N\sigma$ particles :

$$\begin{aligned}
\sigma \otimes \sigma \otimes \sigma &= 2\sigma, \\
\sigma \otimes \sigma \otimes \sigma \otimes \sigma &= 2\mathbf{1} \oplus 2\psi, \\
\sigma \otimes \sigma \otimes \sigma \otimes \sigma \otimes \sigma &= 4\sigma, \\
&\dots \\
\big(\sigma\big)^{\otimes N} &= \begin{cases} 2^{\frac{N-2}{2}}\mathbf{1} \oplus 2^{\frac{N-2}{2}}\psi, & N \text{ is even,} \\ 2^{\frac{N-1}{2}}\sigma, & N \text{ is odd} \end{cases}
\end{aligned} \tag{95}$$

From these, we can detect the smallest non-trivial fusion spaces :

$$\begin{aligned}
V_{\sigma}^{\sigma^{\otimes 3}} &\equiv V_{\sigma}^{\sigma\sigma\sigma}, \quad dim\big(V_{\sigma}^{\sigma\sigma\sigma}\big) = N_{\sigma}^{\sigma^{\otimes 3}} = 2, \\
V_{\mathbf{1}}^{\sigma^{\otimes 4}} &\equiv V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma}, \quad dim\big(V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma}\big) = N_{\mathbf{1}}^{\sigma^{\otimes 4}} = 2, \\
V_{\psi}^{\sigma^{\otimes 4}} &\equiv V_{\psi}^{\sigma\sigma\sigma\sigma}, \quad dim\big(V_{\psi}^{\sigma\sigma\sigma\sigma}\big) = N_{\psi}^{\sigma^{\otimes 4}} = 2
\end{aligned} \tag{96}$$

Since one anticipates that the computational space should belong to the vacuum sector, the interest lies particularly in the space $V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma}$ because they could be used to encode a qubit as the space is two-dimensional. To encode m qubits, one needs a fusion space carried by $\big(2m+2\big)\sigma$ particles. In general, the dimension of the fusion space carried by N particles can be determined by using the last fusion rule from (94) successively :

N	1	2	3	4	5	6	7	8	9	10	11	12	...
$N_{\mathbf{1}}^{\sigma^{\otimes N}}$	0	1	0	2	0	4	0	8	0	16	0	32	...

$$\Rightarrow dim\big(V_{\mathbf{1}}^{\sigma^{\otimes N}}\big) = 2^{\frac{N-2}{2}}, \quad N \text{ is Even},$$

which means that the topological Hilbert space grows exponentially with N. Since the fusion multiplicities are zero for all odd N, one can restrict to consider only spaces carried by an even number of particles. This is in line with the anticipated initialization of the quantum computer, where one draws some number of particles-anti-particle pairs out of the vacuum, which implies that one always ends up with an even number of particles. The space $V_{\mathbf{1}}^{\sigma\sigma\sigma}$ is zero-dimensional (forbidden by the fusion rules). By inspection with the tree diagrams, we collect the values of the F-symbols for the one-dimensional spaces $V_{\mathbf{1}}^{\sigma\sigma x}$, $V_{\mathbf{1}}^{x\sigma\sigma}$ and $V_{\mathbf{1}}^{\sigma x\sigma}$ with $x \in \big\{\mathbf{1}, \psi\big\}$:

$$\big(F_{\mathbf{1}}^{\sigma\sigma y}\big)_{x\sigma} = a_y\delta_{x,y}, \quad \big(F_{\mathbf{1}}^{x\sigma\sigma}\big)_{\sigma y} = b_y\delta_{x,y}, \quad F_{\mathbf{1}}^{\sigma y\sigma} = c_y, \quad |a_y|^2 = |b_y|^2 = |c_y|^2 = 1,$$

for some $a_y, b_y, c_y \in \mathbb{C}$, meaning that these F-moves introduce only overall phases which are non-physical and can be set to unity $\to a_i = b_i = c_i = 1$. The zero values, because of the Kronecker delta, correspond to the forbidden F transformations. However, the space $V_{\sigma}^{\sigma\sigma\sigma}$ is 2D and so a potential computational space for a single qubit. So $F_{\sigma}^{\sigma\sigma\sigma}$ is a genuine 2×2 unitary matrix because it is the only one acting in a non-trivial fusion space :

$$F \equiv F_{\sigma}^{\sigma\sigma\sigma} = \begin{pmatrix} \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\mathbf{11}} & \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\mathbf{1}\psi} \\ \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\psi\mathbf{1}} & \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\psi\psi} \end{pmatrix} = \begin{pmatrix} F_{\mathbf{11}} & F_{\mathbf{1}\psi} \\ F_{\psi\mathbf{1}} & F_{\psi\psi} \end{pmatrix},$$

where the components have to satisfy the constraints following from unitarity :

$$\begin{cases} |F_{\mathbf{11}}|^2 + |F_{\mathbf{1}\psi}|^2 = 1, \\ |F_{\psi\mathbf{1}}|^2 + |F_{\psi\psi}|^2 = 1, \\ F_{\mathbf{11}}\big(F_{\psi\mathbf{1}}\big)^* + F_{\mathbf{1}\psi}\big(F_{\psi\psi}\big)^* = 0 \end{cases}$$

From the pentagon equation (67), for $a = b = c = d = n = y = \sigma$ and $e = \mathbf{1}$ we take :

$$\big(F_{\mathbf{1}}^{m\sigma\sigma}\big)_{\sigma z}\big(F_{\mathbf{1}}^{\sigma\sigma z}\big)_{m\sigma} = \sum_{x=\mathbf{1},\psi} \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{mx}\big(F_{\mathbf{1}}^{\sigma x\sigma}\big)_{\sigma\sigma}\big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{xz},$$

from which we take two equations for $m = z = \mathbf{1}, \psi$:

$$\begin{aligned} \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\mathbf{11}}^2 + \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\mathbf{1}\psi}\big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\psi\mathbf{1}} = 1 \Rightarrow F_{\mathbf{11}}^2 + F_{\mathbf{1}\psi}F_{\psi\mathbf{1}} = 1, \\ \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\psi\psi}^2 + \big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\mathbf{1}\psi}\big(F_{\sigma}^{\sigma\sigma\sigma}\big)_{\psi\mathbf{1}} = 1 \Rightarrow F_{\psi\psi}^2 + F_{\mathbf{1}\psi}F_{\psi\mathbf{1}} = 1, \end{aligned}$$

where we used the F symbols equal to 1 that we derived earlier. We can pick $F_{\mathbf{11}} \in \mathbb{R}$. We have this freedom to select an element from the whole matrix to be real and then the fixed gauge will determine properly the other 3 elements. From the above two equations we get $F_{\mathbf{11}}^2 = F_{\psi\psi}^2 \Rightarrow F_{\psi\psi} = \pm F_{\mathbf{11}} \in \mathbb{R}$. We then conclude that :

$$F_{\mathbf{1}\psi}F_{\psi\mathbf{1}} = |F_{\mathbf{1}\psi}|^2 = |F_{\psi\mathbf{1}}|^2 \Rightarrow F_{\mathbf{1}\psi} = F_{\psi\mathbf{1}}^*$$

The third equation of the unitarity constraint becomes :

$$F_{\mathbf{11}}F_{\mathbf{1}\psi} + F_{\mathbf{1}\psi}F_{\psi\psi} = 0$$

Adding by members this equation to the pentagon relations, we see that the F matrix elements are determined as solutions to the polynomial equations :

$$\begin{cases} 1 = F_{\mathbf{11}}\big(F_{\mathbf{11}} + F_{\mathbf{1}\psi}\big) + F_{\mathbf{1}\psi}\big(F_{\psi\mathbf{1}} + F_{\psi\psi}\big), \\ 1 = F_{\psi\mathbf{1}}\big(F_{\mathbf{11}} + F_{\mathbf{1}\psi}\big) + F_{\psi\psi}\big(F_{\psi\mathbf{1}} + F_{\psi\psi}\big) \end{cases}$$

The set of equations has four types of general solutions :

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \quad \text{and} \quad \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & -1 \end{pmatrix},$$

where $\phi = \big[0, 2\pi\big]$ is an undetermined arbitrary parameter (different from the gauge freedom of a common total phase which we fixed when we demanded $F_{\mathbf{11}} \in \mathbb{R}$). Of these matrices, the first three are trivial in the sense that they only redefine the basis up to some overall phase. Fixing the arbitrary phase by setting $\phi = 0$ and choosing the solution with an overall '+' sign, the matrix implementing the non-trivial F-move in the standard basis of the model is :

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{97}$$

This solution is of particular interest because it is the Hadamard gate, which was already encountered as one of the gates in one particular universal gate set. To find how the braid group acts in the fusion space of the model, one should find the unitary matrices representing the R-moves as solutions of the hexagon equation (68). For $a = b = c = d = \sigma$ we take :

$$R^{\sigma\sigma}_m \left(F^{\sigma\sigma\sigma}_\sigma\right)_{mn} R^{\sigma\sigma}_n = \sum_{x=\mathbf{1},\psi} \left(F^{\sigma\sigma\sigma}_\sigma\right)_{mx} R^{x\sigma}_\sigma \left(F^{\sigma\sigma\sigma}_\sigma\right)_{xn}$$

This time all the $R^{y\sigma}_\sigma$, with $y \in \{\mathbf{1}, \psi\}$, are complex constants with unit norm. This is because the spaces $V^{y\sigma}_\sigma$ are 1D for a given y, which implies that braiding can only contribute non-physical overall phases that can again be set to unity. From the definition of the R-move as a map $R : V^{ab}_c \to V^{ba}_c$, also $R^{\sigma\sigma}_y$ with $y \in \{\mathbf{1}, \psi\}$ are phases, because there are no fusion degeneracies. However, this is a 2D fusion space to encode a single qubit, as the action of braiding depends whether one braids particles which fuse to yield either $\mathbf{1}$ or ψ. Therefore, these phases are physical and correspond to the eigenvalues of a matrix implementing an R-move in $V^{\sigma\sigma\sigma}_\sigma$ (or $V^{\sigma\sigma\sigma\sigma}_\mathbf{1}$) :

$$R \equiv \begin{pmatrix} R^{\sigma\sigma}_\mathbf{1} & 0 \\ 0 & R^{\sigma\sigma}_\psi \end{pmatrix}$$

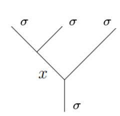
From the hexagon equation for $m = n = \mathbf{1}, \psi$, we get $\left(R^{\sigma\sigma}_\mathbf{1}\right)^2 \equiv \left(R_\mathbf{1}\right)^2 = \sqrt{2}$ and $\left(R^{\sigma\sigma}_\psi\right)^2 \equiv \left(R_\psi\right)^2 = -\sqrt{2}$ so combining the two results we have :

$$\left(R^{\sigma\sigma}_\psi\right)^2 = e^{i\pi}\left(R^{\sigma\sigma}_\mathbf{1}\right)^2 \Rightarrow R^{\sigma\sigma}_\psi = e^{i\pi/2} R^{\sigma\sigma}_\mathbf{1}$$

We want the R matrix to be unitary so $|R_\mathbf{1}|^2 = |R_\psi|^2 = 1$, which restricts that it's elements must have unit norm (complex phases). Since all the solutions give a different representation of the same model, the simplest one will be to choose $R_\mathbf{1} = 1$ and represent the R matrix as :

$$R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{98}$$

This particular matrix appears also in the theory of quantum computation, where it is known as the phase gate S. We can now find a representation of the braid group $\mathcal{B}_3$, the braid group on three identical (same color) strands. Just like we did with Fibonacci anyons, with the basis states $|x\rangle$, $x \in \{\mathbf{1}, \psi\}$, now :

$|x\rangle \equiv$  ,

we find the generators of $\mathcal{B}_3$ in $V^{\sigma\sigma\sigma}_\sigma$:

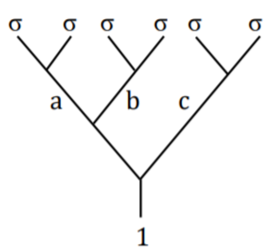
$$\sigma_1 = R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_2 = F^{-1} R F = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{99}$$

We verify that the generators σ_i indeed form a representation of the braid group $\mathcal{B}_3$, i.e. that they satisfy the Yang-Baxter equation :

$$\sigma_1 \sigma_2 \sigma_1 = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\pi/4} F = \sigma_2 \sigma_1 \sigma_2$$

As it happens that both sides are proportional to F, this also demonstrates that the F matrix is a physically meaningful transformation (Hadamard gate) which can be implemented by braiding

anyons.
The generators are unitary operations $\rightarrow \sigma_i \sigma_i^\dagger = \mathbb{1}_{2\times 2}$. Another thing to be noticed is the order 4 of the generators $\rightarrow \sigma_1^4 = \sigma_2^4 = \mathbb{1}_{2\times 2}$, which means that we are dealing with the truncated braid group $B\big(3,4\big)$ rather than the braid group $\mathcal{B}_3$ that has infinite number of elements. The truncated braid group has a finite number of elements and this sets a limit on the number of different braidings, which could be implemented. Since braiding is the only tool to perform unitary transformations in the fusion space, dealing with truncated braid groups implies that there is also only a limited number of unitary transformations available. However, models that produce truncated braid groups are not automatically invalid for universal quantum computation since some may generate subgroups which are dense in the unitary group $U\big(n\big)$ (in our example with Ising anyons $det\big(\sigma_i\big) = i \neq 1 \rightarrow$ dense in $U\big(2\big)$ and not $SU\big(2\big)\big)$. For instance, even though single qubit unitary transformations are limited to the elements in $B\big(3,4\big)$ here, even this relatively simple group is of order 96 [3] and it is far from obvious whether it admits universal quantum computation.
Unfortunately, such a model is not universal for quantum computation. Even though the Hadamard gate H can be realized, instead of the $\frac{\pi}{8}$-phase gate T, one can only produce the phase gate $S = \sigma_1 = T^2$. Because the physical braid group generators σ_i are the most elementary unitary transformations that can be implemented on the system, there can not exist a transformation $T \in B\big(3,4\big)$ because then σ_1 could be decomposed as two successive even more elementary operations T, which should satisfy the pentagon and hexagon equations. However, no such solutions were obtained and thus even without considering if any entangling gates arise though braiding anyons, it can be concluded that the Ising anyon model does not admit universal quantum computation.
Two qubit gates can be constructed in the fusion space of 6 Ising anyons as shown in Figure 19 [16]. The braidwords and their diagrams of constructing 1-qubit and also 2-qubits gates like CNOT can be found in [17]. In bibliography, the σ_1 generator is defined with an overall phase $e^{-i\pi/8}$ as it emerges from the hexagon equation. We omitted this phase because it cannot be measured.

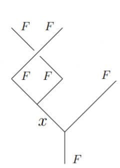
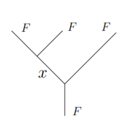


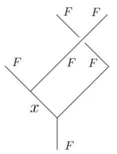
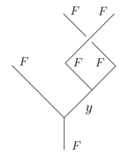
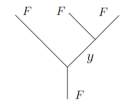
	a	b	c
$\|00\rangle$	1	1	1
$\|10\rangle$	ψ	ψ	1
$\|01\rangle$	1	ψ	ψ
$\|11\rangle$	ψ	1	ψ

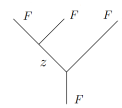
Figure 19: The 6σ anyon encoding of two qubits where the table shows the identification of the fusion channels with the computational basis.

4.6 $\mathcal{D}(D_5)$ anyons

We are working with 3 identical anyons (F type) so we have to find the two generators of the braid group $\mathcal{B}_3$ via the F and R symbols. In the qutrit basis we selected, we have :

$\sigma_1 \big|x\big\rangle \equiv$  $= R_x^{FF}$  $= R_x^{FF}\big|x\big\rangle,$

$$\sigma_2|x\rangle \equiv$$  $$= \sum_y \left(F_F^{FFF}\right)_{xy}$$  $$= \sum_y R_y^{FF}\left(F_F^{FFF}\right)_{xy}$$ 

$$= \sum_{y,z} \left(F_F^{FFF}\right)^{-1}_{yz} R_y^{FF} \left(F_F^{FFF}\right)_{xy}$$  $$= \sum_{y,z} \left(F_F^{FFF}\right)^{-1}_{yz} R_y^{FF} \left(F_F^{FFF}\right)_{xy} |z\rangle,$$

where $x, y, z \in \{A, B, L\}$ and because $\left(F_F^{FFF}\right)$ is unitary we have $\left(F_F^{FFF}\right)^{-1}_{yz} = \left(F_F^{FFF}\right)^{*}_{zy}$. So :

$$\sigma_1 = R\ ,\ \ \sigma_2 = F^{-1}RF = F^{\dagger}RF, \tag{100}$$

where :

$$R \equiv \begin{pmatrix} R_A^{FF} & 0 & 0 \\ 0 & R_B^{FF} & 0 \\ 0 & 0 & R_L^{FF} \end{pmatrix}\ ,\ \ F \equiv F_F^{FFF} \equiv \begin{pmatrix} \left(F_F^{FFF}\right)_{AA} & \left(F_F^{FFF}\right)_{AB} & \left(F_F^{FFF}\right)_{AL} \\ \left(F_F^{FFF}\right)_{BA} & \left(F_F^{FFF}\right)_{BB} & \left(F_F^{FFF}\right)_{BL} \\ \left(F_F^{FFF}\right)_{LA} & \left(F_F^{FFF}\right)_{LB} & \left(F_F^{FFF}\right)_{LL} \end{pmatrix} \tag{101}$$

We calculate the F-symbols with the quantum double method (*Clebsch-Gordan* symbols). We will calculate the first element $\left(F_F^{FFF}\right)_{AA}$ as an example. We have from (83) :

$$\left(F_F^{FFF}\right)_{FF} = \sum_{i,j,k,n,m} \left(\begin{array}{cc|c} F & F & A \\ i & j & 1 \end{array}\right) \left(\begin{array}{cc|c} A & F & F \\ 1 & k & 1 \end{array}\right) \left(\begin{array}{cc|c} F & F & A \\ j & k & 1 \end{array}\right)^{*} \left(\begin{array}{cc|c} F & A & F \\ i & 1 & 1 \end{array}\right)^{*},$$

where $m, n = 1$ because the anyon A (vaccum) is one-dimensional and we also chose $l = 1$ for the two-dimensional F anyon. To find a non-zero coefficient, we start with (79) in which we calculate diagonal elements of the representations Π^A and Π^F and because of (14), the relation $h'h'' = h$ in the sum of (79) defines which coefficients are non zero. For instance :

$$\left(\begin{array}{cc|c} F & F & A \\ 1 & 1 & 1 \end{array}\right) = 0 \ \text{ because } {}^1h_1{}^1h_1 = r^2 \neq e = {}^0h_1,$$

$$\left(\begin{array}{cc|c} F & F & A \\ 2 & 2 & 1 \end{array}\right) = 0 \ \text{ because } {}^1h_2{}^1h_2 = r^3 \neq e = {}^0h_1$$

So the two non-zero coefficients for this selection of anyons ($a = b = F$ and $c = A$) are :

$$\left(\begin{array}{cc|c} F & F & A \\ 1 & 2 & 1 \end{array}\right) \text{ and } \left(\begin{array}{cc|c} F & F & A \\ 1 & 2 & 1 \end{array}\right) \quad \text{because } {}^1h_1{}^1h_2 = {}^1h_2{}^1h_1 = e = {}^0h_1$$

We first find from (79) that :

$$\left(\begin{array}{cc|c} F & F & A \\ 1 & 2 & 1 \end{array}\right) = \left(\frac{1}{10}\sum_{g\in\mathbb{Z}_5} {}^1\hat{\Gamma}(g){}^1\hat{\Gamma}(sgs)\right)^{1/2} = \left(\frac{1}{10}\left(1 + \omega\cdot\omega^4 + \omega^2\cdot\omega^3 + \omega^3\cdot\omega^2 + \omega^4\cdot\omega\right)\right)^{1/2} = \frac{1}{\sqrt{2}},$$

where $\mathbb{Z}_5 = \{e, r, r^2, r^3, r^4\}$. Then we find the other non-zero CG coefficient from (80) :

$$\left(\begin{array}{cc|c} F & F & A \\ 2 & 1 & 1 \end{array}\right) = \frac{1}{\sqrt{10}} \frac{\sum_{g\in{}^3C} {}^1\hat{\Gamma}(sg){}^1\hat{\Gamma}(gs)}{\sqrt{5}} = \frac{1}{\sqrt{2}},$$

where we summed in the elements of the conjugacy class ${}^3C$. The other non-zero symbols that we need are calculated as above :

$$\begin{pmatrix} A & F & \big| & F \\ 1 & 1 & \big| & 1 \end{pmatrix} = \begin{pmatrix} F & A & \big| & F \\ 1 & 1 & \big| & 1 \end{pmatrix} = 1$$

Gathering what we computed, the sum for $\left(F_F^{FFF}\right)_{AA}$ reduces to one term :

$$\left(F_F^{FFF}\right)_{AA} = \begin{pmatrix} F & F & \big| & A \\ 1 & 2 & \big| & 1 \end{pmatrix} \begin{pmatrix} A & F & \big| & F \\ 1 & 1 & \big| & 1 \end{pmatrix} \begin{pmatrix} F & F & \big| & A \\ 2 & 1 & \big| & 1 \end{pmatrix}^* \begin{pmatrix} F & A & \big| & F \\ 1 & 1 & \big| & 1 \end{pmatrix}^* = \frac{1}{2}$$

The total F matrix results :

$$F \equiv F_F^{FFF} \equiv \begin{pmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ -1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \tag{102}$$

We reassure that $\sum_{j=1}^{3} \left|\left(F_F^{FFF}\right)_{ij}\right|^2 = 1$ for $i = 1, 2, 3$.
For the R_x^{FF} symbols we use the simple relation (66) for $k = 1$:

$$\begin{aligned} R_A^{FF} &= e^{-i\pi s_F} e^{-i\pi s_F} e^{i\pi s_A} = e^{-2\pi i/5} = \omega^{-1} = \overline{\omega}, \\ R_B^{FF} &= e^{-i\pi s_F} e^{-i\pi s_F} e^{i\pi s_B} = \overline{\omega}, \\ R_L^{FF} &= e^{-i\pi s_F} e^{-i\pi s_F} e^{i\pi s_L} = \omega, \end{aligned}$$

where we used the anyon spins from (43). The total R matrix is :

$$R \equiv \begin{pmatrix} \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix} \tag{103}$$

Alternatively, we could have used (71) with :

$$\mathcal{R}^{FF} \equiv \mathcal{R}_{66} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

and the non-zero $3j$-symbols (CG symbols) :

$$\begin{aligned} \begin{pmatrix} F & F & \big| & A \\ 1 & 2 & \big| & 1 \end{pmatrix} &= \begin{pmatrix} F & F & \big| & A \\ 1 & 2 & \big| & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}, \\ \begin{pmatrix} F & F & \big| & B \\ 1 & 2 & \big| & 1 \end{pmatrix} &= \begin{pmatrix} F & F & \big| & B \\ 2 & 1 & \big| & 1 \end{pmatrix} = -\frac{1}{\sqrt{2}}, \\ \begin{pmatrix} F & F & \big| & L \\ 1 & 1 & \big| & 1 \end{pmatrix} &= \begin{pmatrix} F & F & \big| & L \\ 2 & 2 & \big| & 2 \end{pmatrix} = 1 \end{aligned}$$

and take the same result for the R matrix as in (103).
The generators of $\mathcal{B}_3$ are ($F^\dagger = F$, as F is symmetric with real elements) from (100) :

$$\sigma_1 = \begin{pmatrix} \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix} \ , \ \sigma_2 = FRF = \begin{pmatrix} \frac{\omega+\overline{\omega}}{2} & -\frac{(\omega+\overline{\omega})}{2} & \frac{\overline{\omega}}{\sqrt{2}} \\ -\frac{(\omega+\overline{\omega})}{2} & \frac{\omega+\overline{\omega}}{2} & -\frac{\overline{\omega}}{\sqrt{2}} \\ \frac{\overline{\omega}}{\sqrt{2}} & -\frac{\overline{\omega}}{\sqrt{2}} & \overline{\omega} \end{pmatrix}, \tag{104}$$

where $\omega + \overline{\omega} = 2\cos 2\pi/5$. The generators act on a general qutrit state :

$$\big|\psi\big\rangle = a\big|0\big\rangle + b\big|1\big\rangle + c\big|2\big\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{105}$$

which represents the initial state of the three F anyons we braid. We can approximate or even find exactly a qutrit gate by a sequence of actions of the generators σ_1 and σ_2 on the state $\big|\psi\big\rangle$. Working with the group D_5, such a task is very hard because among the elements of the generators we have the 5^{th} roots of unity.
Without loss of generality, we may multiply the $\sigma_i{}'$s by a common factor (we explained this freedom earlier) so that they all have determinant equal to 1 (note that all the $\sigma_i{}'$s are conjugate to each other). This is because the images of the representations we take are subgroups of $SU(V_z^{mm\cdots})$ (square matrices with determinant = 1 that have the dimension of the space $V_z^{mm\cdots}$). For our case, we have 3 anyons $m = F$ that fuse to $z = F$ and $dim(V_F^{FFF}) = 3$. The new matrices with determinant = 1 are $\tilde{\sigma_i} = \tau\sigma_i$ with $\tau = e^{2\pi/15}$. The matrix σ_1 (as well as $\tilde{\sigma}_1$) is unitary but σ_2 (and $\tilde{\sigma}_2$) is not. The order of the first generator is finite $\rightarrow \sigma_1^5 \equiv \Gamma^5\big(\sigma_1\big) = \mathbb{1}_{3\times 3}$ but the second generator $\sigma_2 \equiv \Gamma\big(\sigma_2\big)$ has infinite order $\rightarrow \sigma_2^m \neq \mathbb{1}_{3\times 3}\ \forall m \in \mathbb{N}$.
The quantum double $\mathcal{D}(D_5)$, as well as others like $\mathcal{D}(S_3)$, have a lot of options for qutrit encoding so we can get as images of the representations of the braid group generators some interesting subgroups of $SU(3)$ which are rarely the *braid group* $\mathcal{B}_3$ or the *truncated braid group* $B(3, m)$ (the matrices of the generators don't obey the *Yang-Baxter* equation but the multiplication table of the image subgroup of $SU(3)$ which can have finite or infinite order). Nevertheless, those matrices indeed correspond to representations of the braiding generators as we showed geometrically with F and R moves even though they don't actually produce the *braid group*. For quantum computation reasons, we want the braiding transformations $\sigma_i{}'$s to be unitary in order for the process to conserve the total probability. Generators that are not in this category still represent braiding transformations between anyons but are not ideal for building braiding quantum circuits. Our aim always is to build circuits that lead to a universal gate set.
We want to examine if the representation of $\mathcal{B}_3$ that we found is irreducible and identify the images on each irreducible summand (we also call them *sectors* but we shouldn't confuse the word with the anyonic sectors/states). To find a subspace invariant under the matrices $\sigma_1 = \Gamma(\sigma_1)$ and $\sigma_2 = \Gamma(\sigma_2)$, we carry out the following procedure: First, we create new matrices from σ_1 and σ_2 via addition and multiplication. When we find a matrix that has a nontrivial null space, we take a basis for that null space, and apply the matrices σ_1 and σ_2 to the vectors in that basis. We record the resulting vectors, apply σ_1 and σ_2 to them, and continue in this fashion until applying σ_1 and σ_2 ceases to produce vectors that are linearly independent from the ones we have already recorded. After some testing, the first combination of the two generators that has non-trivial null space is

their product :

$$\sigma_1\sigma_2 = \begin{pmatrix} \frac{1+\overline{\omega}^2}{2} & -\frac{(1+\overline{\omega}^2)}{2} & \frac{\overline{\omega}^2}{\sqrt{2}} \\ -\frac{(1+\overline{\omega}^2)}{2} & \frac{1+\overline{\omega}^2}{2} & -\frac{\overline{\omega}^2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

We transform this matrix into row echelon form :

$$\sigma_1\sigma_2 \to \begin{pmatrix} 1 & -1 & \sqrt{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so it's null space is spanned by a single vector :

$$\begin{pmatrix} 1 & -1 & \sqrt{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Any linear combination of addition and multiplication of σ_1 and σ_2 we act on the above vector gives the same vector multiplied by a complex scalar or the trivial null vector which implies that the null space is 1-dimensional. We name this irreducible subspace as W. We normalize the vector to norm 1 :

$$\overrightarrow{w} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}(|A\rangle + |B\rangle),$$

where $|A\rangle \equiv |0\rangle$ and $|B\rangle \equiv |1\rangle$ are the two basis states of the qutrit space. The representations of the two generators in this 1D basis are $\sigma_1 = \overline{\omega}$ and $\sigma_2 = 0$. We will now find the irreducible 2D orthogonal complement of $W \to W^{\perp}$ ($dim(V_F^{FFF}) = 3 = dim(W) + dim(W^{\perp})$). We want to find two linearly independent vectors that are orthogonal to the vector $\overrightarrow{w}$. Those two real vectors must satisfy :

$$\frac{1}{\sqrt{2}} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow x + y = 0.$$

For $x = 1 \to y = -1$, we have the normalized vector :

$$\overrightarrow{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}}(|A\rangle - |B\rangle)$$

We can pick as the second vector of $W^{\perp}$ the unit basis vector in the z direction ($\overrightarrow{w}$ and $\overrightarrow{w}_1$ are in the xy plane) :

$$\overrightarrow{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to |L\rangle \equiv |2\rangle$$

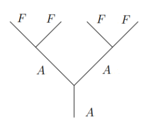
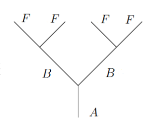
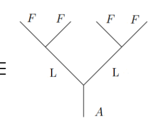
The three vectors that we found are linearly independent :

$$\frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = b = c = 0.$$

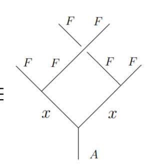
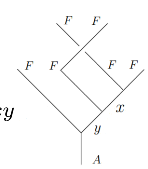
The representations of the generators in the basis $\big\{\big(\big|A\big\rangle - \big|B\big\rangle\big)/\sqrt{2}, \big|L\big\rangle\big\}$ of $W^{\perp}$ are :

$$
\begin{aligned}
\sigma_1 &= \begin{pmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{pmatrix} = \overline{\omega}\begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} \omega + \overline{\omega} & \overline{\omega} \\ \overline{\omega} & \overline{\omega} \end{pmatrix} = \overline{\omega}\begin{pmatrix} 1+\omega^2 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1+\omega^2 & 1 \\ 1 & 1 \end{pmatrix},
\end{aligned}
\tag{106}
$$

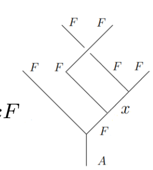
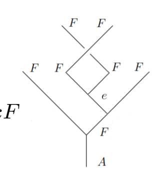
if we reduce a common phase $\overline{\omega}$ ($\sigma_1 = \overline{\omega} \sim 1$). The 2 dimensional sector satisfies obviously the same multiplication relations as the 3D reducible representation we first derived and also carries qutrit information even though it is 2-dimensional (the basis of $W^{\perp}$ includes all of the fusion basis states $\big|A\big\rangle$, $\big|B\big\rangle$ and $\big|L\big\rangle$). So, we still encode a qutrit in the 2D basis and process it's information with the above 2D matrices σ_1 and σ_2. We proved that the reducible representations of the generators in V_F^{FFF} decompose into the irreducible representations in the sectors W and $W^{\perp}$.
We could have worked with 4 identical F type anyons (this type choice is arbitrary) but we need more F symbols to calculate in order to find the 3 generators of the $\mathcal{B}_4$ group. We analyze the case where the space is described by four F anyons that fuse into the vaccum A. The basis states are :

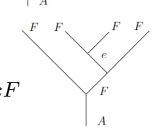
$\big|0\big\rangle \equiv \big|A\big\rangle \equiv$  , $\big|1\big\rangle \equiv \big|B\big\rangle \equiv$  , $\big|2\big\rangle \equiv \big|L\big\rangle \equiv$  ,

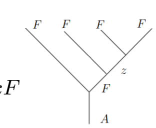
where we symbolize the states $\big|x, x\big\rangle \equiv \big|x\big\rangle$ because the two anyons are the same kind. For the generator σ_2 :

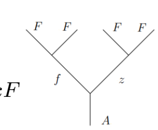
$\sigma_2\big|x\big\rangle \equiv$  $= \sum_y \big(F_A^{FFx}\big)_{xy}$ 

For $x = A, B$ we have $y = F$ from the fusion rules. If $x = L$ then $y = F, L$. But for $y = L \to F \oplus L \neq A$. So we accept only $y = F$. We continue :

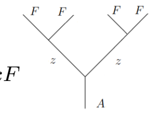
$\sigma_2\big|x\big\rangle = \big(F_A^{FFx}\big)_{xF}$  $= \sum_e \big(F_F^{FFF}\big)^{-1}_{xe}\big(F_A^{FFx}\big)_{xF}$ 

$= \sum_e R_e^{FF}\big(F_F^{FFF}\big)^{-1}_{xe}\big(F_A^{FFx}\big)_{xF}$ 

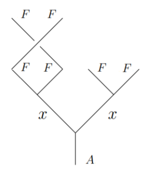
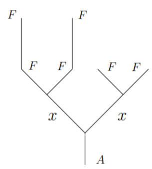
$= \sum_{e,z} \big(F_F^{FFF}\big)_{ez} R_e^{FF}\big(F_F^{FFF}\big)^{-1}_{xe}\big(F_A^{FFx}\big)_{xF}$ 

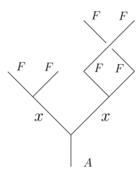
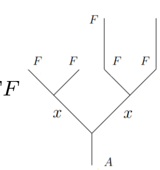
$= \sum_{e,z,f} \big(F_A^{FFz}\big)^{-1}_{Ff}\big(F_F^{FFF}\big)_{ez} R_e^{FF}\big(F_F^{FFF}\big)^{-1}_{xe}\big(F_A^{FFx}\big)_{xF}$ 

But because $F \times F = A \oplus B \oplus L$, we should have $f = z$ in order to eventually fuse to the vacuum A, so :

$$\sigma_2|x\rangle = \sum_{e,z} \left(F_A^{FFz}\right)^{-1}_{Fz} \left(F_F^{FFF}\right)_{ez} R_e^{FF} \left(F_F^{FFF}\right)^{-1}_{xe} \left(F_A^{FFx}\right)_{xF}$$ 

We see that it requires more effort than the case with three F anyons as here we have to calculate not only the matrix F_F^{FFF} that we calculated but also the symbols F_A^{FFA}, F_A^{FFB} and F_A^{FFL}. The generators σ_1 and σ_3 are identified straightforward :

$\sigma_1|x\rangle \equiv$  $= R_x^{FF}$  ,

$\sigma_3|x\rangle \equiv$  $= R_x^{FF}$  ,

so $\sigma_1 = \sigma_3 = R$ from (103).
We will now construct representations of the braiding generators with the quantum double. The case of indistinguishable particles is very easy to be examined and we can verify that the generators satisfy the *Yang-Baxter* equation (28). Suppose we have one anyon type a, we only need the braiding matrix $\mathcal{R}^{aa}$ (from equation (19)) to construct them with equation (33). They all have the same order and so we result with representations of the *truncated braid group* $B(n, m)$.
We will do the calculations for the case of distinguishable particles. Suppose we have three anyons of types E, F and G (or else 5, 6 and 7). In order to take a representation for the *pure truncated braid* group, all the $\mathcal{R}$ matrices involved must have the same even order. The braid matrices we need are :

$$\mathcal{R}_{65} = \mathcal{R}_{75} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathcal{R}_{56} = \mathcal{R}_{76} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \ \mathcal{R}_{57} = \mathcal{R}_{67} = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega}^2 & 0 \\ 0 & \overline{\omega}^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix}$$

From equations (33) and (36), we define the 3 generators of the *pure truncated braid* group (all the above matrices have order $6 \to \mathcal{R}^{10} = \mathbb{1}_{4\times4}$) :

$$\begin{aligned} A_{12} &= \mathcal{R}_1^2 = \mathcal{R}_{65}\mathcal{R}_{56} \otimes \mathbb{1}_{2\times2}, \\ A_{13} &= \mathcal{R}_1\mathcal{R}_2^2\,\mathcal{R}_1^{-1} = \big(\mathcal{R}_{65} \otimes \mathbb{1}_{2\times2}\big)\big(\mathbb{1}_{2\times2} \otimes \mathcal{R}_{75}\mathcal{R}_{57}\big)\big(\mathcal{R}_{56}^{-1} \otimes \mathbb{1}_{2\times2}\big), \\ A_{23} &= \mathcal{R}_2^2 = \mathbb{1}_{2\times2} \otimes \mathcal{R}_{76}\mathcal{R}_{67}, \end{aligned} \tag{107}$$

where $\mathcal{R}_i$, $\mathcal{R}_i^{-1}\,'$s correspond to single braiding operators while $\mathcal{R}_i^2\,'$s correspond to monodromy operators. The matrices end up being :

$$\begin{aligned} A_{12} &= diag\big(\omega,\omega,\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\omega,\omega\big),\\ A_{13} &= diag\big(\omega,\omega^2,\overline{\omega}^2,\overline{\omega},\overline{\omega},\overline{\omega}^2,\omega^2,\omega\big),\\ A_{23} &= diag\big(\overline{\omega}^2,\omega^2,\omega^2,\overline{\omega}^2,\overline{\omega}^2,\omega^2,\omega^2,\overline{\omega}^2\big), \end{aligned} \tag{108}$$

where all the generators have order $5 \to A_{ij}^{10/2} = A_{ij}^5 = \mathbb{1}_{8\times 8}$ (equation (37)) and form the *pure truncated group* $P(3,10)$ (3 distinguishable particles with all the generators having order $\frac{10}{2}=5$). We verify with an example that the generators we found satisfy the group relations (30) :

$$\begin{aligned} A_{12}^{-1} &= diag\big(\overline{\omega},\overline{\omega},\omega,\omega,\omega,\omega,\overline{\omega},\overline{\omega}\big),\\ A_{13}^{-1} &= diag\big(\overline{\omega},\overline{\omega}^2,\omega^2,\omega,\omega,\omega^2,\overline{\omega}^2,\overline{\omega}\big), \end{aligned}$$

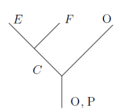
so we have :

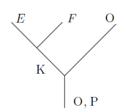
$$A_{12}^{-1}A_{23}A_{12} = A_{13}A_{23}A_{13}^{-1} = diag\big(\overline{\omega}^2,\omega^2,\omega^2,\overline{\omega}^2,\overline{\omega}^2,\omega^2,\omega^2,\overline{\omega}^2\big)$$

The braiding generators A_{ij} act on the general initialized state :

$$\big|\psi\big\rangle = \big(a\big|E,1\big\rangle + b\big|E,2\big\rangle\big)\otimes\big(c\big|F,1\big\rangle + d\big|F,2\big\rangle\big)\otimes\big(e\big|G,1\big\rangle + f\big|G,2\big\rangle\big), \tag{109}$$

where we use the notation of the internal states of the 3 types of anyons (the $\mathcal{R}$ matrices and consequently the braiding generators in the quantum double are represented in the internal states of the anyons). It holds that $|a|^2+|b|^2=|c|^2+|d|^2=|e|^2+|f|^2=1$. We cannot encode any qudit in the fusion space of E, F and G because it is one dimensional as $E\times F = C\oplus K$, $C\times G = F\oplus H$ and $K\times G = I\oplus L$ so we get only 1 time each anyon in the end of the fusion tree.
There are a few options in $\mathcal{D}(D_5)$ in which we can encode a qubit in the fusion options between 3 distinguishable anyons. If we select the anyons E, F and O, we result with the following fusion space basis :

$\big|0\big\rangle \equiv$ 

and $\big|1\big\rangle \equiv$ 

,

so we can encode a qubit in this $2D$ space with state $\big|\psi\big\rangle = a\big|0\big\rangle + b\big|1\big\rangle$. We can define the braiding generators between the 3 anyons just like in equation (107) by substituting the indices of the anyon 7 with the anyon 15 in the $\mathcal{R}$ matrices (also now the rightmost anyon lives in a 5-dimensional space so in the tensor product when we don't act on the 3rd anyon we put $\mathbb{1}_{5\times5}$ instead of $\mathbb{1}_{2\times2}$). The additional $\mathcal{R}$ matrices we need :

$$\mathcal{R}_{155} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix},\ \mathcal{R}_{156} = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&\overline{\omega}&0&0&0&0&0&0\\ 0&0&0&0&0&\overline{\omega}^2&0&0&0&0\\ 0&0&0&0&0&0&0&\omega^2&0&0\\ 0&0&0&0&0&0&0&0&0&\omega\\ 1&0&0&0&0&0&0&0&0&0\\ 0&0&\omega&0&0&0&0&0&0&0\\ 0&0&0&0&\omega^2&0&0&0&0&0\\ 0&0&0&0&0&0&\overline{\omega}^2&0&0&0\\ 0&0&0&0&0&0&0&0&\overline{\omega}&0 \end{pmatrix},$$

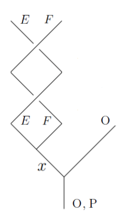
with order 10 $\rightarrow \mathcal{R}_{155}^{10} = \mathcal{R}_{156}^{10} = \mathbb{1}_{10\times 10}$. So now not all $\mathcal{R}$ matrices involved have the same order. The generators that arise are :

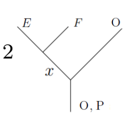
$$A_{12} = diag\big(\omega,\omega,\omega,\omega,\omega,\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\overline{\omega},\omega,\omega,\omega,\omega,\omega\big),$$
$$A_{13} = \left(\begin{array}{c|c} \mathbb{O}_{10\times 10} & \begin{matrix} \omega\mathbb{A} & \mathbb{O}_{5\times 5} \\ \mathbb{O}_{5\times 5} & \overline{\omega}\mathbb{A} \end{matrix} \\ \hline \begin{matrix} \overline{\omega}\mathbb{B} & \mathbb{O}_{5\times 5} \\ \mathbb{O}_{5\times 5} & \omega\mathbb{B} \end{matrix} & \mathbb{O}_{10\times 10} \end{array}\right),\ A_{23} = \left(\begin{array}{c|c} \begin{matrix} \mathbb{O}_{5\times 5} & \mathbb{C} \\ \mathbb{D} & \mathbb{O}_{5\times 5} \end{matrix} & \mathbb{O}_{10\times 10} \\ \hline \mathbb{O}_{10\times 10} & \begin{matrix} \mathbb{O}_{5\times 5} & \mathbb{C} \\ \mathbb{D} & \mathbb{O}_{5\times 5} \end{matrix} \end{array}\right), \tag{110}$$

where with $\mathbb{O}$ we denote the zero/null matrices and :

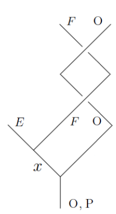
$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},\ \mathbb{B} = \mathbb{A}^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathbb{C} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \overline{\omega} \\ \overline{\omega}^2 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \end{pmatrix},\ \mathbb{D} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \\ \overline{\omega}^2 & 0 & 0 & 0 & 0 \\ 0 & \overline{\omega} & 0 & 0 & 0 \end{pmatrix} \tag{111}$$

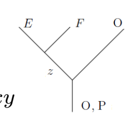
The generators have order 5, 2 and 10 $\rightarrow A_{12}^5 = A_{13}^2 = A_{23}^{10} = \mathbb{1}_{20\times 20}$, a consequence of the different orders of the $\mathcal{R}$ matrices. The image group of this representation, formed by elements that are all the possible different results for any action of the generators, now is a different finite group (because all the generators have finite order) than the *pure truncated braid* group. It is obvious to confirm that the relations of the *pure braid* group (30) are not satisfied here.
In the attempt to define the generators in the fusion space rather than the internal states of the anyons (quantum double), using the F and R symbols, we can construct the generators A_{12} and A_{23} as :

$A_{12}|x\rangle \equiv$ 

$= \big(R_x^{EF}\big)^2$ 

$= \big(R_x^{EF}\big)^2|x\rangle,$

$A_{23}|x\rangle \equiv$ 

$= \sum_{y,z} \big(F_{O,P}^{EFO}\big)^{-1}_{yz}\big(R_y^{FO}\big)^2\big(F_{O,P}^{EFO}\big)_{xy}$ 

$$= \sum_{y,z} \big(F_{O,P}^{EFO}\big)^{-1}_{yz}\big(R_y^{FO}\big)^2\big(F_{O,P}^{EFO}\big)_{xy}|z\rangle,$$

where we define geometrically the generators A_{ij} as in Figure 3 (the particles are distinguishable) by using the monodromy symbols $\big(R_x^{EF}\big)^2$ and $\big(R_y^{FO}\big)^2$ from equation (66) (for $k=2$). We cannot construct the generator A_{13} with F and R symbols because there is no way to make the leftmost anyon E and rightmost anyon O fuse while braiding separately from anyon F at some point in the process, in order to use for their braiding the monodromy symbol $\left(R_c^{EO}\right)^2$. That's why this method is applicable only when we work with a collection of indistinguishable particles.

4.7 Universal quantum computation

In this final section, we inspect two fundamental cases of universal quantum computation with certain anyon models. There are also other interesting suggestions that propose an efficient way to achieve universality, such as metaplectic anyons or more generally the $SU(2)_k$ anyon theories from which we compared the Ising ($k = 2$) anyon and Fibonacci ($k = 3$) anyon models, motivated by their potential for future realizations based on $Majorana\ fermion\ quasiparticles$ or *exotic* $fractional$ *quantum-Hall states*, respectively.

4.7.1 Simple perfect groups

Instead of dealing with the general case of non-solvable groups, we will deal with the smaller set of groups that are both simple and perfect. Non-solvable groups are those that contain a perfect subgroup; and a perfect group is any non-trivial group, whose commutator subgroup equals the full group: $\left[H, H\right] = H$. The property of simplicity means that the group has exactly two subgroups that are invariant under conjugation: the trivial group and the whole group. Because the commutator subgroup is invariant under conjugation, it should be clear that any simple non-abelian group is perfect. However, we shall refer to these groups as simple and perfect to remind the reader that we are dealing with a subcase of the general non-solvable case.
The set of simple perfect groups, which includes the alternating groups A_n for $n > 4$, is powerful for computing because in some sense we can get from one non-trivial element to any other using operations that fix the identity. The simplest example of that category of groups is A_5 which will be referred during the process.
We will work with a computational basis of qudits of trivial net flux [18]:

$$\big|n\big\rangle = \big|a^n b a^{-n}\big\rangle \otimes \big|a^n b^{-1} a^{-n}\big\rangle,\ \ 0 \leq n \leq d, \tag{112}$$

with d the smallest prime number such that $a^d b a^{-d} = b$. The pure fluxes, as we defined them in subsection 4.1, here take values in a simple perfect group. Every simple perfect group has even order, so we can always find a group element a such that $a^2 = e$ and thus work with qubits (e is the trivial element). We choose two non-commuting elements $a, b \in H$ such that $a^2 = e$ and we define the basis states :

$$\begin{aligned} \big|0\big\rangle &\equiv \big|b, b^{-1}\big\rangle, \\ \big|1\big\rangle &\equiv \big|aba^{-1}, ab^{-1}a^{-1}\big\rangle \end{aligned} \tag{113}$$

The matrix that executes the projective measurement is the *Z-Pauli* matrix :

$$\begin{gathered} Z\big|0\big\rangle = \big|0\big\rangle,\ \ Z\big|1\big\rangle = -\big|1\big\rangle, \\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{gathered} \tag{114}$$

The X-$Pauli$ is defined as :

$$\begin{gathered} X\big|0\big\rangle = \big|1\big\rangle, \;\; Z\big|1\big\rangle = \big|0\big\rangle, \\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{gathered} \tag{115}$$

The eigenvectors of X in the computational basis are :

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}\big(\big|0\big\rangle + \big|1\big\rangle\big) = \frac{1}{\sqrt{2}}\big(\big|b, b^{-1}\big\rangle + \big|aba^{-1}, ab^{-1}a^{-1}\big\rangle\big), \\ |-\rangle &= \frac{1}{\sqrt{2}}\big(\big|0\big\rangle - \big|1\big\rangle\big) = \frac{1}{\sqrt{2}}\big(\big|b, b^{-1}\big\rangle - \big|aba^{-1}, ab^{-1}a^{-1}\big\rangle\big) \end{aligned} \tag{116}$$

Suppose we create a pair of fluxes from the vacuum. The state, as we explained in subsection 4.1, should be $\big|0; C_h\big\rangle$, with $b, aba^{-1} \in C_h$, as we cannot have any conserved charges. This means that if we bring two members of a flux pair $\big|h, h^{-1}\big\rangle$ together, the chance (probability) that they will annihilate is not 1 but :

$$P = \big|\big\langle 0; C_h\big|h, h^{-1}\big\rangle\big|^2 = 1/|C_h|$$

If the state of a pair is $|-\rangle$, then the amplitude of the pair components to fuse into the vacuum is :

$$\big\langle 0; C_h \big|-\big\rangle = \frac{1}{\sqrt{2|C_h|}}\big(\big\langle 0\big| + \big\langle 1\big| + \dots\big)\big(\big|0\big\rangle - \big|1\big\rangle\big) = 0,$$

thus they will never fuse to the vacuum. On the other side, if the pair is in the state $|+\rangle$, we have :

$$\big\langle 0; C_h \big|+\big\rangle = \frac{1}{\sqrt{2|C_h|}}\big(\big\langle 0\big| + \big\langle 1\big| + \dots\big)\big(\big|0\big\rangle + \big|1\big\rangle\big) = \sqrt{\frac{2}{|C_h|}},$$

and thus there is a finite probability that they will fuse to the vacuum.
We can construct the state $|+\rangle$. We begin by creating a pair in the vacuum state $\big|0; C_h\big\rangle$. If C_h has just two elements, then $\big|0; C_h\big\rangle = |+\rangle$ and we are done. But if the specific conjugacy class has more than two elements, we bring the pair near a calibrated pair $\big|c, c^{-1}\big\rangle$, where $c \in C_h$ and $c \neq b, aba^{-1}$. If it doesn't match $\big|c, c^{-1}\big\rangle \rightarrow \big\langle c, c^{-1}\big|0; C_h\big\rangle = 0$ for every $c \in C_h$, then the state of the pair must be $|+\rangle$.
We denote for convenience the pair states as $\big|a\big\rangle \equiv \big|a, a^{-1}\big\rangle$. Suppose we have the 3 pair state $\big|x, y, z\big\rangle$. We can wind the third pair clockwise or counterclockwise around the first pair and execute the gate :

$$\mathcal{R}_{xz}\big|x, y, z\big\rangle = \big|x, y, xzx^{-1}\big\rangle, \;\; \mathcal{R}_{xz}^{-1}\big|x, y, z\big\rangle = \big|x, y, x^{-1}zx\big\rangle,$$

or do the same with the second pair :

$$\mathcal{R}_{yz}\big|x, y, z\big\rangle = \big|x, y, yzy^{-1}\big\rangle, \;\; \mathcal{R}_{yz}^{-1}\big|x, y, z\big\rangle = \big|x, y, y^{-1}zy\big\rangle,$$

or even borrow a pair of fluxes $\big|c\big\rangle$ from the bureau of standards and wind $\big|z\big\rangle$ around it :

$$\big|x, y\big\rangle \otimes \mathcal{R}\big|c\big\rangle\big|z\big\rangle = \big|x, y, czc^{-1}\big\rangle\big|c\big\rangle,$$

for every $c \in H$. So we can execute any gate of the form :

$$\prod_i \mathcal{R}_i\big|x, y, z\big\rangle = \big|x, y, f\big(x, y\big)zf\big(x, y\big)^{-1}\big\rangle,$$

with $\mathcal{R}_i = \mathcal{R}_{xz}^{\pm}, \mathcal{R}_{yz}^{\pm}$ and $f\big(x, y\big)$ is a function with a product form of x, y factors and their inverses.

Theorem 4.1: If H is a simple and perfect finite group, then any function $f(h_1,\dots,h_n) : H^n \to H$ can be expressed as a product of the inputs $\{h_i\}$, their inverses $\{h_i^{-1}\}$ and fixed elements of H, any of which may appear multiple times in the product.

The smallest simple and perfect group is A_5, the group of even permutations of five objects. Gottesman has proved in one of his papers that for d prime, being able to apply products of X's and Z's plus a Toffoli gate is universal for quantum computation.
The Toffoli gate is defined to act on a 3-qubit state as :

$$T|x,y,z\rangle = |x,y,z\oplus xy\rangle, \tag{117}$$

so in our case, where a pair of fluxes can be written in the form $|h\rangle = |a^m b a^{-m}\rangle$, we can write the action of the Toffoli gate in a more convenient way as :

$$T|a^i b a^{-i}, a^j b a^{-j}, a^k b a^{-k}\rangle = |a^i b a^{-i}, a^j b a^{-j}, a^{ij+k} b a^{-ij-k}\rangle \tag{118}$$

Thus, the Toffoli gate conjugates the third qubit by the function :

$$f(x,y) = f(a^i b a^{-i}, a^j b a^{-j}) = a^{ij}$$

Suppose we have 2 qubits with fluxes h_1 and h_2 with $h_i \in \{b, aba^{-1}\}$. We define new variables $h_i' = h_i b^{-1} \in \{e, c\}$, with c defined as the commutator of a and b :

$$c \equiv [a,b] = aba^{-1}b^{-1}$$

We choose an element g such that it doesn't commute with c and define $l \equiv [c,g]$. If we find two functions $p_{1,2}$ that can be written in a product form, such that :

$$\begin{aligned} p_1(c) = g,\ \ p_1(e) = e,\\ p_2(l) = a,\ \ p_2(e) = e, \end{aligned} \tag{119}$$

then the Toffoli function can be written as :

$$f(h_1,h_2) = p_2\Big([h_1', p_1(h_2')]\Big)$$

Let the fluxes take values in A_5. According to the above analysis, we need an element a such that $a^2 = e$ in order to work with qubits. We choose $a = (12)(34)$. Next, we choose an element b that doesn't commute with a, and an element g that doesn't commute with $c \equiv [a,b]$. A little trial and error yield $b = (345)$ and $g = (234)$. The computational basis is now defined as :

$$\begin{aligned} |0\rangle = |b\rangle = |(345)\rangle,\\ |1\rangle = |aba^{-1}\rangle = |(435)\rangle, \end{aligned} \tag{120}$$

and the remaining group elements are fixed as :

$$\begin{aligned} c = (aba^{-1})b^{-1} = (435)(435) = (345),\\ l = (cgc^{-1})g^{-1} = (245)(324) = (25)(34) \end{aligned} \tag{121}$$

It's easy to see that two functions with the properties in (119) are :

$$p_1(h) = p_2(h) = (521)h(125) \tag{122}$$

Putting all the steps together, we get the Toffoli function :

$$f(h_1, h_2) = \Big\{(521)h_1(14352)h_2(124)h_1^{-1}(15342)h_2^{-1}(521)\Big\} \tag{123}$$

So given 3 qubits $|h_1, h_2, h_3\rangle$, if we want to execute the Toffoli gate for the group A_5 all we have to do is conjugate the third qubit by the above function $f(h_1, h_2)$. This "recipe" of finding the Toffoli function $f(h_1, h_2)$ can be used for any simple perfect group and it's exact form can be found using the functions p_i.

Now that we constructed the Toffoli gate, we move on to the next step. We need to be able to apply and measure products of X's and Z's, i.e. X^aZ^b. An important observation is that :

$$\big(X^aZ^b\big)^d = \omega^{abd(d-1)/2}X^{ad}Z^{bd} = \begin{cases} \;\; I \;\;, & \text{if } d = \text{odd}, \\ -I^{ab}, & \text{if } d = 2, \end{cases} \tag{124}$$

with $\omega = e^{2\pi i/d}$. We work with qubits so $\omega = e^{2\pi i/2} = e^{\pi i}$. The eigenvectors of $ZX = iY$ are :

$$\begin{aligned} |0_Y\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big) = \frac{1}{\sqrt{2}}\big(|b\rangle + i|aba^{-1}\rangle\big), \\ |1_Y\rangle &= \frac{1}{\sqrt{2}}\big(|0\rangle - i|1\rangle\big) = \frac{1}{\sqrt{2}}\big(|b\rangle - i|aba^{-1}\rangle\big) \end{aligned} \tag{125}$$

In order to construct the method for measuring operators of the form X^aZ^b for every possible value of d, we introduce a vital trick, that is $Kitaev's$ *phase estimation technique*. Assuming that we are working in a system with qudits, and we have an operator U with eigenvalues that are d^{th} roots of unity. If we are able to apply a controlled-U, and measure in the X basis, it is equivalent to being able to measure the operator U.

The general case for d an odd prime number is easy because the eigenvalues of X^aZ^b are the d^{th} roots of unity just like those of X and Z. Assuming we can make measurements in the X basis (which includes preparation of X eigenstates), all that remains is to construct the controlled-X^aZ^b. That is, we need to be able to apply the gate :

$$\begin{aligned} CX^aZ^b\big(|n\rangle \otimes |\psi\rangle\big) &= \sum_{s=0}^{d_c-1} P_s \otimes \big(X^aZ^b\big)^s\big(|n\rangle \otimes |\psi\rangle\big) = |n\rangle \otimes \big(X^aZ^b\big)^n|\psi\rangle \\ &= |n\rangle \otimes X^{an}Z^{bn}\omega^{abn(n-1)/2}|\psi\rangle, \end{aligned}$$

composed of the total phase :

$$CX^aZ^b|n, m\rangle = \omega^{bnm+abn(n-1)/2}|n, m\rangle,$$

as $Z^{bn}|m\rangle = \omega^{bnm}|m\rangle$, followed by controlled-sums (a in number) which illustrate the X^{an} gate above. The controlled-sum is just a Toffoli with an input (qubit) fixed to one, so in practice it acts always as X^n for the third qudit (one control qubit, one control qudit and one target qudit) :

$$\big(Csum\big)^a|1\rangle \otimes |n\rangle \otimes |\psi\rangle = T^a|1\rangle \otimes |n\rangle \otimes |\psi\rangle = |1\rangle \otimes |n\rangle \otimes X^{an}|\psi\rangle,$$

where we use an extra qubit ancilla in the state $|1\rangle$. As for the phase, because we have a Toffoli, we have universal classical computation. We can thus compute the computational basis vector indexed by $q = bnm + abn(n-1)/2$ in an ancilla by the action of T^q in $|1,1,0\rangle$ (two control qubits and one target qudit) :

$$T^q|1,1,0\rangle = |1,1\rangle \otimes X^q|0\rangle = |1,1,q\rangle,$$

apply a Z to this ancilla in order to create the phase, and then erase the computation. We used for the previous the generalized hybrid Toffoli gate :

$$T = \sum_{r=0}^{d_c-1}\sum_{s=0}^{d'_c-1} P_r \otimes P_s \otimes X^{rs} = \sum_{r=0}^{d_c-1}\sum_{s=0}^{d'_c-1} |r\rangle\langle r| \otimes |s\rangle\langle s| \otimes X^{rs},$$

with d_c,d'_c the dimensions of the two control qudits. We showed how to compute X^aZ^b operators for the qudit case.

The $d = 2$ qubit case is invariant under complex conjugation and thus there is no way of distinguishing the two eigenstates of ZX. Suppose someone provides us with the state :

$$|\Psi\rangle = |0_Y\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + \omega|1\rangle\big)$$

It could be any of the two eigenvectors of ZX. We label it as the $\omega = +i$ eigenstate. We use again Kitaev's phase estimation technique and act with control-ZX on the state $|+\rangle \otimes |\Psi\rangle$, with $|+\rangle$ as the control ancilla (X eigenstate) and $|\Psi\rangle$ as the target. So the circuit performs the transformation :

$$CZX\big(|+\rangle \otimes |\Psi\rangle\big) = \frac{1}{2}\big(|00\rangle + \omega|01\rangle - |11\rangle + \omega|10\rangle\big) = |\Psi\rangle \otimes |\Psi\rangle$$

So the copy of the second state occurs either way. This is quite helpful as we can execute gates on the first qubit without employing a destructive measurement on the second. We can always act with Z on $|\Psi\rangle$ and construct the orthogonal state $|\Phi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - \omega|1\rangle\big)$. Now that we have copied our state, we can construct a new controlled-ZX gate with the ancilla $|\Psi\rangle$ as the control qubit and another $|\Psi\rangle$ as the target, and then measure the first qubit in the X basis :

$$(X \otimes I)CZX\big(|\Psi\rangle \otimes |\Psi\rangle\big) = \frac{1}{2}\big(|10\rangle + \omega|11\rangle - \omega|01\rangle - |00\rangle\big) = -|-\rangle \otimes |\Psi\rangle,$$

where the phase $e^{i\pi} = -1$ is irrelevant as it is not observable. Doing the same thing with $|\Phi\rangle$ as the ancilla, and then measuring again in the X basis :

$$(X \otimes I)CZX\big(|\Phi\rangle \otimes |\Psi\rangle\big) = \frac{1}{2}\big(|10\rangle + \omega|11\rangle + \omega|01\rangle + |00\rangle\big) = |+\rangle \otimes |\Psi\rangle,$$

so as long as we are consistent in using the same ancilla $|\Psi\rangle$, we will have broken the conjugation symmetry and have found a new way of labeling, creating and measuring eigenstates of ZX.

All that remains to be explained is how to create the first copy of $|\Psi\rangle$. Because a state with a density matrix proportional to the identity can be written as :

$$\rho = \frac{1}{2}I = \frac{1}{2}|0_Y\rangle\langle 0_Y| + \frac{1}{2}|1_Y\rangle\langle 1_Y|,$$

it is equivalent to having prepared an eigenstate of $ZX = iY$ chosen at random. By using a controlled-NOT gate from a $|+\rangle$ ancilla to a $|0\rangle$ ancilla, we produce the state :

$$CNOT\big(|+\rangle \otimes |0\rangle\big) = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \frac{1}{\sqrt{2}}\big(|0_Y 1_Y\rangle + |1_Y 0_Y\rangle\big)$$

So by discarding one qubit of the above bell state (trace out it's degrees of freedom), we have produced the desired $|\Psi\rangle$. Therefore, we proved also for the qubit case, which is of main interest, how to construct eigenstates of ZX, X and Z as well as the way to execute the Toffoli gate. Summing up what we showed, we have the power to achieve universal quantum computation

4.7.2 Qutrit encoding in $\mathcal{D}(\mathcal{S}_3)$ model

The superselection sectors of $\mathcal{D}(\mathcal{S}_3)$ (representations of the quantum double model $\mathcal{D}(\mathcal{S}_3)$) are [19]:

$$\begin{aligned} &1 \equiv A \equiv |e, {}^0\Gamma\rangle\,, && 2 \equiv B \equiv |e, {}^1\Gamma\rangle\,, && 3 \equiv C \equiv |e, {}^2\Gamma\rangle\,, \\ &4 \equiv D \equiv |{}^1C, {}^0\hat{\Gamma}\rangle, && 5 \equiv E \equiv |{}^1C, {}^1\hat{\Gamma}\rangle, && 6 \equiv F \equiv |{}^2C, {}^0\tilde{\Gamma}\rangle, \\ &7 \equiv G \equiv |{}^2C, {}^1\tilde{\Gamma}\rangle, && 8 \equiv H \equiv |{}^2C, {}^2\tilde{\Gamma}\rangle \end{aligned} \tag{126}$$

The fusion rules of $\mathcal{D}(\mathcal{S}_3)$ are included in the last section of *Tables and* $\mathcal{R}$ *matrices*. The T matrix of the model is :

$$T = diag\big(1, 1, 1, 1, -1, 1, \omega, \omega^2\big), \tag{127}$$

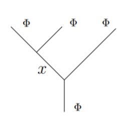
with $\omega = e^{2\pi i/3}$, from which we find the spin factors $s_1 = s_2 = s_3 = s_4 = s_6 = 0$, $s_5 = 1/2$, $s_7 = 1/3$ and $s_8 = 2/3$. We can use the spins to find the R symbols from (66).
Now, the fusion rules of the whole $\mathcal{D}(S_3)$ anyon model are too complicated to serve as an illustrative model. Hence, the simplest fusion subalgebra M we can distinguish is [20]:

$$M = \{1, \Lambda, \Phi\}, \tag{128}$$

with $1 \equiv A$, $\Lambda \equiv B$ and we have 4 options for $\Phi \equiv C, F, G, H$. These choices of particles give closed fusion rules :

$$\Phi \otimes \Phi = 1 \oplus \Lambda \oplus \Phi, \;\; \Lambda \otimes \Lambda = 1, \;\; \Phi \otimes \Lambda = \Phi,$$

where Φ is the anyon with the non-abelian identity. We elaborate two cases for the $\Lambda - \Phi$ submodel, one in the space where three Φ particles fuse to Φ and the other with four anyons of the same type that fuse to the vacuum. We choose $\Phi = C$. The basis states in $V_\Phi^{\Phi\Phi\Phi}$ are :

$|x\rangle \equiv$ 

,

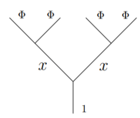
with $x \in \{1, \Lambda, \Phi\}$. We calculate the unitary F and R matrices :

$$\begin{aligned} F \equiv F_\Phi^{\Phi\Phi\Phi} &= \begin{pmatrix} F_{11} & F_{1\Lambda} & F_{1\Phi} \\ F_{\Lambda 1} & F_{\Lambda\Lambda} & F_{\Lambda\Phi} \\ F_{\Phi 1} & F_{\Phi\Lambda} & F_{\Phi\Phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \\ R &= \begin{pmatrix} R_1^{\Phi\Phi} & 0 & 0 \\ 0 & R_\Lambda^{\Phi\Phi} & 0 \\ 0 & 0 & R_\Phi^{\Phi\Phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{129}$$

The unitary generators of $\mathcal{B}_3$ are $(F^{-1} = F^\dagger = F)$ [19]:

$$\sigma_1 = e^{i\pi}R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = e^{i\pi}FRF = \frac{1}{2}\begin{pmatrix} -1 & 1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \tag{130}$$

where we multiplied both σ_1 and σ_2 with a total phase $e^{i\pi} = -1$ so that the have determinant equal to 1. The image of this representation of the generators σ_i is the symmetric group of permutations on 3 objects $\mathcal{S}_3$, as $\sigma_1^2 = \sigma_2^2 = \mathbb{1}_{3\times 3}$ and also the Yang-Baxter relation is satisfied.
When we deal with the fusion space $V_1^{\Phi\Phi\Phi\Phi}$ $(\Phi = C)$, we have the computational qutrit basis :

$$\big|x\big\rangle \equiv \big|x, x\big\rangle \equiv$$ 

,

with $x \in \{1, \Lambda, \Phi\}$. The generators of $\mathcal{B}_4$ are identified in the same manner as we did in $\mathcal{D}(D_5)$:

$$\sigma_1\big|x\big\rangle = \sigma_3\big|x\big\rangle = R_x^{\Phi\Phi}\big|x\big\rangle,$$

$$\sigma_2\big|x\big\rangle = \sum_{e,z}\big(F_1^{\Phi\Phi z}\big)^{-1}_{\Phi z}\big(F_\Phi^{\Phi\Phi\Phi}\big)_{ez}R_e^{\Phi\Phi}\big(F_\Phi^{\Phi\Phi\Phi}\big)^{-1}_{xe}\big(F_1^{\Phi\Phi x}\big)_{x\Phi}\big|z\big\rangle$$

The extra one-dimensional F-symbols we need, in order to determine σ_2, are $\big(F_1^{\Phi\Phi 1}\big)^\pm = \big(F_1^{\Phi\Phi\Lambda}\big)^\pm = \big(F_1^{\Phi\Phi\Phi}\big)^\pm = 1$. We result with the expressions :

$$\sigma_1 = \sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = -\frac{1}{2}\begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}, \tag{131}$$

where we added again a common factor $e^{i\pi} = -1$ so that $det\big(\sigma_i\big) = 1$. This reducible representation splits into two sectors S_1 and S_2, where S_1 is a 1-dim irrep mapping σ_i to 1 and S_2 is a 2-dim irrep spanned by the basis $\big\{\big|\Lambda, \Lambda\big\rangle, \frac{\sqrt{3}}{3}\big|1, 1\big\rangle - \frac{\sqrt{6}}{3}\big|\Phi, \Phi\big\rangle\big\} = \big\{\big|B, B\big\rangle, \frac{\sqrt{3}}{3}\big|A, A\big\rangle - \frac{\sqrt{6}}{3}\big|C, C\big\rangle\big\}$. The matrices of the σ_i's under the basis of S_2 are given by :

$$\sigma_1 = \sigma_3 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \frac{i}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{132}$$

They generate a group which is isomorphic to the non-abelian semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ [19], which has order 12. Thus, combining the action of the generators in any possible way, we derive only 12 different matrices that form a representation for this specific group (they satisfy it's multiplication table). This group is generated by the elements $\rho = \big(\sigma_1\sigma_2\big)^2$ and $\tau = \sigma_2$, with the relations $\rho^3 = \mathbb{1}_{3\times 3} = \tau^4$ and $\tau\rho = \rho^{-1}\tau$. Note that ρ and τ^2 commute, so $\rho\tau^2$ has order six. We verify that the group is non abelian because $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. The three generators of $\mathcal{B}_4$ have the same order $\sigma_i^4 = \mathbb{1}_{2\times 2}$ and satisfy the Yang-Baxter equation. Nevertheless, there is no finite truncated braid group $B\big(4, 4\big)$. The only finite truncated braid group with order $m = 4$ is $B\big(3, 4\big)$.
If we choose $\Phi = G$, the qutrit basis in the fusion space $V_1^{\Phi\Phi\Phi\Phi} \equiv V_A^{GGGG}$ is $\big\{\big|A, A\big\rangle, \big|B, B\big\rangle, \big|G, G\big\rangle\big\}$, while we have :

$$F \equiv F_\Phi^{\Phi\Phi\Phi} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}, \quad R = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{133}$$

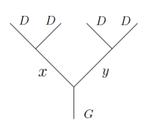
with $\omega = e^{2\pi i/3}$. All the other F symbols we use in the formula of σ_2 are 1D and equal to unity. We find the representation of $\mathcal{B}_4$:

$$\sigma_1 = \sigma_3 = \tau \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & -\omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \text{and} \quad \sigma_2 = \frac{\tau}{2} \begin{pmatrix} \omega & -\omega & \sqrt{2}\omega^2 \\ -\omega & \omega & \sqrt{2}\omega^2 \\ \sqrt{2}\omega^2 & \sqrt{2}\omega^2 & 0 \end{pmatrix}, \tag{134}$$

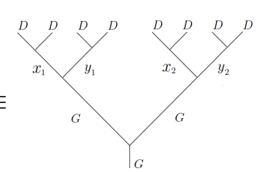
where the phase $\tau = e^{-\frac{\pi i}{9}}$ fixes the determinant of the generators to 1. This representation is irreducible and the group generated by σ_i's has a structure of the semidirect product $\big(\mathbb{Z}_9 \times \mathbb{Z}_3\big) \rtimes \mathcal{S}_3$ with order 162, which is isomorphic to the group $D\big(9,1,1;2,1,1\big)$ [19],[21]. This group has three generators :

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F\big(9,1,1\big) = \begin{pmatrix} e^{\frac{2\pi i}{9}} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{9}} & 0 \\ 0 & 0 & e^{-\frac{4\pi i}{9}} \end{pmatrix}, \quad G\big(2,1,1\big) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

So the group is defined as $D\big(9,1,1;2,1,1\big) :=< E, F\big(9,1,1\big), G\big(2,1,1\big) >$. Actually one can show that the group generated by the σ_i's is isomorphic to $D\big(9,1,1;2,1,1\big)$ via a conjugation by some unitary matrix. Obviously, the σ_i's belong in that group and can be produced by the 3 matrices above.
Now, let's present a model inside $\mathcal{D}\big(\mathcal{S}_3\big)$ that succeeds universality, and this is by using D type anyons. We revert to the fusion tree formation of 4 anyons that was used throughout the thesis. The natural choice will be to encode a qubit in V_A^{DDDD}. Unfortunately, we did not succeed in finding a model that could be made universal even with measurements and ancillary states. Therefore, we turn back to V_G^{DDDD} based on our knowledge of the braid group representations :

$\big|x, y\big\rangle \equiv$ 

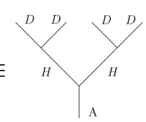
The space V_G^{DDDD} is nine dimensional with a basis $\big\{\big|G,G\big\rangle, \big|A,G\big\rangle, \big|G,A\big\rangle, \big|F,C\big\rangle, \big|C,F\big\rangle, \big|F,H\big\rangle, \big|H,F\big\rangle, \big|C,H\big\rangle, \big|H,C\big\rangle\big\}$. Let $U = span\big\{\big|G,G\big\rangle, \big|A,G\big\rangle, \big|G,A\big\rangle\big\}$, $V = span\big\{\frac{1}{\sqrt{2}}\big(\big|F,C\big\rangle + \big|C,F\big\rangle\big), \frac{1}{\sqrt{2}}\big(\big|F,H\big\rangle + \big|C,H\big\rangle\big), \frac{1}{\sqrt{2}}\big(\big|H,F\big\rangle + \big|H,C\big\rangle\big)\big\}$ and $W = span\big\{\frac{1}{\sqrt{2}}\big(\big|F,C\big\rangle - \big|C,F\big\rangle\big), \frac{1}{\sqrt{2}}\big(\big|C,H\big\rangle - \big|F,H\big\rangle\big), \frac{1}{\sqrt{2}}\big(\big|H,F\big\rangle - \big|H,C\big\rangle\big)\big\}$. To remind ourselves that these bases are used as computational bases, we also write them as $\big\{\big|0\big\rangle_x, \big|1\big\rangle_x, \big|2\big\rangle_x\big\}, x = U, V, W$, where the subscript x indicates which subspace we are referring to, e.g. $\big|0\big\rangle_U = \big|G,G\big\rangle$. The representation of $\mathcal{B}_4$ splits into the direct sum of a 6-dim irreducible summand $U \oplus V$ and a 3-dim irreducible summand W.
To encode 2-qutrits, we consider the following fusion tree :

$\big|x_1, y_1; x_2, y_2\big\rangle = \big|x_1, y_1\big\rangle \otimes \big|x_2, y_2\big\rangle \equiv$ 

The 2-qutrits are the tensor product of the two qutrits on the two branches. This encoding of 2-qutrits is called the *sparse encoding* because encoding with fewer anyons, called *dense encoding*,

is also possible. To encode n-qutrits, we simply use the tensor product of n such branches, so there are totally $4n$ anyons. We will refer to the three qutrit models that encode 1-qutrit in the subspaces U, V and W respectively, with the computational bases above as the *qutrit* $U - model$, $V - model$ and $W - model$ respectively.
For 1-qutrit braiding circuits, we need to know the representation matrices of $\mathcal{B}_4$ and for 2-qutrit braiding circuits, the representation matrices of $\mathcal{B}_8$. Since both collections of matrices are finite, they are not sufficient to simulate the standard qutrit circuit model. To gain extra power, we consider measurement and ancilla. In anyon theory, there are two kinds of measurements to determine the total charge of a collection of anyons: projective and interferometric. Both types of measurements always lead to some decoherence in the model. Therefore, ideally we should only use them at the end of the computation. Since we cannot avoid using them for weakly integral anyons, we will allow ourselves to determine whether or not the total charge of two anyons is trivial in the middle of the computation. Then based on the outcome, we choose how to continue our computation. For this reason, we call such models *adaptive*.
Measurement 1. Let $\mathcal{M}_A = \{P_A, P_{A'}\}$ be the projective measurement onto the total charge $= A$ sector and it's complement. Then $\mathcal{M}_A$ allows us to distinguish between the anyon A and the other anyons; namely, check whether an anyon is trivial or not. Moreover, the state after measurement for each outcome is still coherent.
The next measurement that we use is problematic, but it is unavoidable due to our choices of computational subspaces. It allows us to project states back to the computational subspaces.
Measurement 2. Let S be a subspace of an anyonic space and $S^{\perp}$ be it's orthonormal complement. Then $\mathcal{M}_S = \{P_S, P_{S^{\perp}}\}$ be the projective measurement that projects a state to S or $S^{\perp}$.
For example, applying $\mathcal{M}_S$ to $S = U$ in V_G^{DDDD}, we obtain the orthogonal projection to $U = span\{|G,G\rangle, |A,G\rangle, |G,A\rangle\}$ and it's orthogonal complement $V \oplus W = span\{|F,C\rangle, |C,F\rangle, |F,H\rangle, |H,F\rangle, |C,H\rangle, |H,C\rangle\}$.
The main idea is that braiding supplemented by measurements $\mathcal{M}_A$ and $\mathcal{M}_U$ leads to a universal gate set for the U-model and V-model. To make the qutrit W-model universal, we need to use the extra ancillary state :

$$|H\rangle_A \equiv |H,H\rangle_A \equiv$$



Then the second result from this analysis is that braiding supplemented by measurements $\mathcal{M}_A$ and $\mathcal{M}_U$ and the ancilla state $|H\rangle_A$ leads to a universal gate set for the W-model.
Measurement 3. Let $\mathcal{M}_{|0\rangle} = \{P_{|0\rangle}, P_{|0\rangle^{\perp}}\}$ be the projective measurements that is the orthogonal projection to $span\{|0\rangle\}$ and it's orthogonal complement $span\{|1\rangle, |2\rangle\}$ in a qutrit.
The generalized Hadamard gate for qutrit is the following :

$$h = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \text{where } \omega = e^{\frac{2\pi i}{3}}. \tag{135}$$

The SUM gate for qudits is a generalized version of $CNOT$, which maps the basis element $|i,j\rangle$ to $|i, i + j \bmod 3\rangle$.
We define the qutrit gate $FLIP_2$ by the map: $FLIP_2|0\rangle = |0\rangle$, $FLIP_2|1\rangle = |1\rangle$, $FLIP_2|2\rangle = -|2\rangle$.

Theorem 4.2: The 1-qutrit classical gates, generalized Hadamard gate, SUM gate, and Measurement 3 form a universal gate set for the standard qutrit quantum circuit model.

Note that with 1-qutrit gates, the generalized Hadamard gate h and Measurement 3, we can easily construct the following ancilla and measurements :

1. $|i\rangle$, $i = 0, 1, 2$.
2. $\widetilde{|i\rangle} = \sum_{j=0}^{2} \omega^{ij} |j\rangle = h|i\rangle$, $i = 0, 1, 2$.
3. Projection of a 1-qutrit state to any computational state, preserving the coherence of the orthogonal complement. For example, projection to $span\{|0\rangle, |1\rangle\}$ and it's complement $span\{|2\rangle\}$.
4. Measurement of a qutrit in the standard computational basis.
5. Projection to $span\{\widetilde{|1\rangle}, \widetilde{|2\rangle}\}$ and it's complement $span\{\widetilde{|0\rangle}\}$.
6. Measurement of a qubit in the standard basis if we take $\{|0\rangle, |1\rangle\}$ as the computational basis. This follows from 4.

From the set of operations given in Theorem 4.2, it can be shown that we can construct the qutrit (qubit) gates (measurements) stated in Lemmas 4.1 – 4.3.
Lemma 4.1: The gate $FLIP_2$ can be constructed.
Lemma 4.2: The 3-qubit gate $\bigwedge^2(Z)$ which maps $|i, j, k\rangle$ to $(-1)^{ijk}|i, j, k\rangle$ can be constructed. In particular, $\bigwedge^2(Z)$ and Z can be constructed since we have the ancilla $|1\rangle$.
We note that the state $|+\rangle$ can be obtained by projecting the state $\widetilde{|0\rangle}$ to the space $span\{|0\rangle, |1\rangle\}$.
Lemma 4.3: Measurement of X can be constructed on a qubit.
Lemma 4.4: The following set of qubit operations are universal for quantum computation :

1. Create the state $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, $|0\rangle$ and $|1\rangle$.
2. Measure Z.
3. Measure X.
4. The Toffoli gate $T = \bigwedge^2(X)$.

Lemma 4.5: The following set of qubit operations are universal for quantum computation :

1. Create the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
2. Measure Z.
3. Measure X.
4. The Toffoli gate $T = \bigwedge^2(Z)$.

By the lemmas above in this subsection, all the operations in Lemma 4.5 can be created from the operations given in Theorem 4.2 if we pick a qubit from the qutrit space . Thus, Lemma 4.5 implies Theorem 4.2.
The U-model, V-model and W-model can be made universal provided measurement and ancilla are allowed besides braiding. Our main theorems are :

Theorem 4.3: Braiding quantum gates and Measurements 1 and 2 provide a universal gate set for the qutrit U-model and V-model.

Theorem 4.4: A universal gate set for the W-model can be constructed from braidings and Measurements 1 and 2 when the ancillary state $\big|H\big\rangle_A$ is used.

Universality for U- and V-models. $U \oplus V$ is a 6-dim irreducible representation of $\mathcal{B}_4$. Under the basis $span\{|0\rangle_U, |1\rangle_U, |2\rangle_U, |0\rangle_V, |1\rangle_V, |2\rangle_V\}$, the generators σ_i's have the following matrices :

$$\sigma_1 = \begin{pmatrix} \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}, \quad \sigma_2 = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega & \sqrt{2}\omega^2 & \sqrt{2} & \sqrt{2} \\ \omega & 1 & \omega & \sqrt{2} & \sqrt{2} & \sqrt{2}\omega^2 \\ \omega & \omega & 1 & \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2} \\ \sqrt{2}\omega^2 & \sqrt{2} & \sqrt{2} & -\omega & -\omega^2 & -\omega^2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2}\omega^2 & -\omega^2 & -\omega & -\omega^2 \\ \sqrt{2} & \sqrt{2}\omega^2 & \sqrt{2} & -\omega^2 & -\omega^2 & -\omega \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $p = \sigma_1\sigma_2\sigma_1$ and $q = \sigma_3\sigma_2\sigma_3$. Then :

$$p^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad q^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad p^2q^2p^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Therefore, when restricted to the subspace U or V, p^2 and q^2 generate all the classical gates on 1 qutrit and $p^2q^2p^2$ is equal to h^2, where h is the generalized Hadamard gate defined previously :

$$p^2q^2p^2 = \begin{pmatrix} h^2 & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & h^2 \end{pmatrix} = \mathbb{1}_{2\times 2} \otimes h^2$$

Let $h' = q^2pq^2$. Then (in block form) :

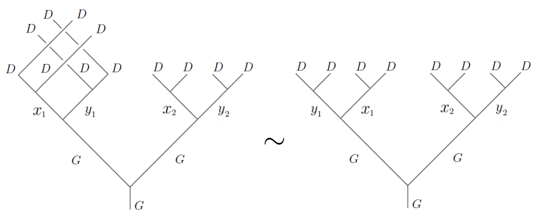
$$h' = \frac{1}{\sqrt{3}} \begin{pmatrix} h & \sqrt{2}h^{-1} \\ \sqrt{2}h^{-1} & -h \end{pmatrix}, \quad h'^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} h^{-1} & \sqrt{2}h \\ \sqrt{2}h & -h^{-1} \end{pmatrix}$$

We define a unitary transformation between the basis states of the two subspaces U and V as $\gamma : U \to V$, $\gamma\big|j\big\rangle_U = \big|j\big\rangle_V$, $j = 0, 1, 2$.

Lemma 4.6: By alternating use of h' (or h'^{-1}) and Measurement 2, one can eventually obtain the generalized Hadamard gate on both U and V, as well as the transformation γ and γ^{-1}. Moreover, the probability to successfully construct these transformations approaches to 1 exponentially fast in the number of measurements and the gate h'.

The following lemma shows Measurement 3 can be constructed in both U and V.
Lemma 4.7: Using Measurement 1, 2 and braiding, one can perform Measurement 3 in both the space U and V.
Up to now, we only considered gates and operations on one qutrit. Next, we want to construct a 2-qutrit gate, the Controlled-Z gate $\bigwedge(Z)$ which maps $|i,j\rangle$ to $\omega^{ij}|i,j\rangle$.
We use the fusion tree to encode 2-qutrits that we mentioned before. Let $s_1 = \sigma_2\sigma_1\sigma_3\sigma_2$, namely, s_1 is the braiding of the first pair with the second pair. Similarly let $s_2 = \sigma_4\sigma_3\sigma_5\sigma_4$, $s_3 = \sigma_6\sigma_5\sigma_7\sigma_6$. Clearly s_1 exchanges x_1 with y_1 with a phase in the 2-qudit splitting tree, namely it maps $|x_1, y_1; x_2, y_2\rangle$ to $|y_1, x_1; x_2, y_2\rangle$ up to a phase :

$s_1|x_1, y_1; x_2, y_2\rangle \equiv$



$\equiv |y_1, x_1; x_2, y_2\rangle$

Similarly, s_3 exchanges x_2 with y_2. The gate s_2 is much more complicated since it involves F-moves. Let $CZ = s_1^{-1}s_2^2s_1s_3^{-1}s_2^2s_3$. Through direct calculations, we find that CZ is a diagonal matrix. Moreover, when restricted to the space U, CZ is exactly the Controlled-Z gate $\bigwedge(Z)$. Again, via the transformation γ, one also obtains the Controlled-Z gate in the space V.
The SUM gate maps $|i,j\rangle$ to $|i, i+j\rangle$ and can be obtained by conjugating $\bigwedge(Z)$ via the Hadamard gate. Explicitly :

$$SUM = \big(Id \otimes h\big) \bigwedge(Z)^{-1}\big(Id \otimes h^{-1}\big)$$

So we can also construct the SUM gate in the space U and V.
To sum up, with Measurement 1, 2 and braiding, we can construct all the 1-qutrit classical gates, generalized Hadamard gate, SUM gate and Measurement 3 im both spaces U and V.
Finally, Theorem 2 follows from Theorem 1 and the arguments analyzed in Universality for U- and V-models.
Universality for W-model. At this point, we examine the representation on W. Under the basis of W given by $\{|0\rangle_W, |1\rangle_W, |2\rangle_W\}$, the σ_i's have the matrices :

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{6} & -\frac{1}{2} + \frac{\sqrt{3}i}{6} & \frac{1}{2} + \frac{\sqrt{3}i}{6} \end{pmatrix}$$

The same as in the U, V-models, we define $p = \sigma_1\sigma_2\sigma_1$, $q = \sigma_3\sigma_2\sigma_3$. Then :

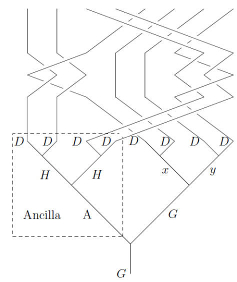
$$p^2 = -\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q^2 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

So p^2 and q^2 generate all the 1-qutrit classical gates in W. Also from σ_1 and σ_3, we obtain the generalized Z-gate and Phase gate P :

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

where Z maps $|i\rangle$ to $\omega^i|i\rangle$ and P maps $|i\rangle$ to $\omega^{\frac{i^2-i}{2}}|i\rangle$. Moreover, let $h' = q^2pq^2$, then $h' = e^{-\frac{i\pi}{2}}h = i^{-1}h$, which is exactly the generalized Hadamard gate up to a phase.
Therefore, in the space W, we obtained the classical 1-qutrit gates, generalized Z-gate, the Phase gate and the generalized Hadamard gate by braiding.
Now we turn to constructing the 2-qutrit gate $\bigwedge(Z)$. One may try the same braiding method as we did for the space U. But it turns out that braiding doesn't work for W. Instead, we try to construct a transformation similar to γ. Consider the following picture of braiding :

$$R|H\rangle_A|x,y\rangle = P^{-1}QP|H\rangle_A|x,y\rangle \equiv$$



Let $P = \sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4$, $Q = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_6\sigma_7\sigma_7\sigma_6$ and let $R = P^{-1}QP$. Then the braiding in the picture is given by R.
We denote the state in the picture before braiding by $|H\rangle_A|x,y\rangle$. Then the braiding R gives the following transformation :

$$R|H\rangle_A|i\rangle_W = \frac{1}{2}\big(-|H\rangle_A|i\rangle_W + |H\rangle_B|i\rangle_V - \sqrt{2}|H\rangle_B|-i\rangle_U\big)$$

and :

$$R|H\rangle_B|i\rangle_U = \frac{1}{\sqrt{2}}\big(|H\rangle_A|-i\rangle_W + |H\rangle_B|-i\rangle_V\big),$$

where $i = 0, 1, 2$ and $-i$ is taken to be modulo 3.
We define a unitary transformation $\beta : |H\rangle_A \otimes W \to |H\rangle_B \otimes U$ with $\beta\big(|H\rangle_A|i\rangle_W\big) = |H\rangle_B|i\rangle_U$. Here $|H\rangle_A$ is the ancilla.
Lemma 4.8: With braiding, Measurement 1, 2 and Ancilla $|H\rangle_A$, the transformation β and β^{-1} can be constructed with probability approaching to 1 exponentially fast in the number of measurements and the gates applied.
By going back and forth between W and U via β and β^{-1}, any operation in the space U can be performed in W accordingly. In particular, the Controlled-Z gate and Measurement 3 can be constructed in W. Collecting all the results in Universality for W-model, we finish the proof of Theorem 4.4.
The whole process can be found analytically in [19], which also contains proofs for the Lemmas and Theorems that were claimed in order to derive Universality for U, V and W-models.

5 Conclusions

The quantum mechanical description of quantum hall systems focuses on how the landau levels are filled and as we change that filling, our system will cycle through a variety of some very interesting quantum hall phases that can be quite extraordinary exotic and differ in the kind of anyonic excitations they allow us to generate. Both abelian and non abelian anyons can be realized in FQHE systems [22]. For example, abelian anyons with $e^* = e/3$ are realized in $\nu = 7/3$ FQH state whereas the $\nu = 5/2$ FQH state can harbor non abelian anyons, known as Ising anyons, with $e^* = e/4$ [23],[24]. In relevance with the latter case, it was discoved that the non abelian physics of this non trivial and highly correlated phase of matter can be distilled into a much simpler weakly correlated platform, namely the $2D$ "spinless" $p + ip$ superconductor, where superconducting vortices form the non abelian Majorana zero modes, meaning that each pair of quasiparticles contain a neutral fermion orbital which can be occupied or unoccupied and hence can act as a qubit [25].

If we want to build an inherently fault tolerant topological quantum computer, we have to find topological phases of matter that harbor non abelian anyons as collective excitations. These emergent particles have zero energy deegres of freedom, in the way that when they populate a certain exotic topological phase, they generate a degenerate ground state (they are locally indistinguishable). We want their unitary braiding to commute with the Hamiltonian of the system so that our state transforms between these degenerate ground states. Also, there has to be a sufficient energy gap between the ground states and their corresponding first excited states, a crucial requirement as we want our system to be immune in local operations and external noise. In that way, we can encode information non locally meaning that it is protected by damage that can be caused by decoherence through topology. Non abelian anyons can be implemented with an engineering approach (we can design them). We take materials which individually we understand perfectly well (find them in laboratory) and then combine those materials in a very precise way to really force non abelian anyons in to our system, even if individually we will have no chance finding something so exotic by looking at the individual components. Read and Rezayi proposed a phase state where there are correlations among triplets of electrons ($3e = e+e+e$) and Fibonacci anyons emerge. It is possible to mimic this multi-particle clustering physics by using the "simple" abelian quantum hall state and deposit $2D$ arrays of superconducting islands on top of this quantum Hall phase (charged $2e = 3 \times \big(2e/3\big)$ Cooper pairs are forced into a quantum Hall fluid that supports fractionalized excitations) and find Fibonacci anyons in practice [26].

When it comes to their computational capabilities, we pointed out some interesting conclusions on how the Fibonacci are capable of producing a universal gate set while the Ising anyons are not. With their discrete braiding operations we cannot reach the entire Bloch sphere of a qubit. So some quantum gates required for universal quantum computation will be missing from the set that we can do with braiding. These additional gates can be supplemented by topologically not protected operations on the qubit, but without absolute gate fidelity. There are also suggestions that abelian anyons, provided that we support them with non topological operations like measurements, can potentially achieve universality [27],[28] (see also Kitaev's work in [29]). We presented that A_5 anyons can be universal and also we can find submodels inside $\mathcal{D}\big(\mathcal{S}_3\big)$ that give a universal qutrit gate set. The dihedral anyons in $\mathcal{D}\big(D_5\big)$ are not preferred for computational purposes, as they produce limited gates and also due to their fusion rules, there in no way to encode qubits between indistinguishable anyons. However, there is a *Yang-Baxterixation* process which leads to universal quantum computation with dihedral anyons [30]. This method is worth mentioning but exceeds the logic and the scope of this thesis.

The field of research aims in finding topological phases of matter that can host non abelian anyons in order to built a fault tolerant quantum computer. There has been progress in the field of Majorana zero modes, which can be found in the boundary edges of $1D$ "spinless" p-wave superconducting nanowires, where by adding an extra nanowire and exploiting the second dimension we can construct a T-junction and perform quantum gates with them [31]. They can also be spotted as vortices in the defects of $2D$ superconductors or quantum spin liquids (i.e. when we have a chiral Majorana edge state in the boundary of a Kitaev spin liquid which yields quantized values of thermal Hall conductance and ψ, σ Ising anyons appear) [32]. In the near future, it is promising that the hardware will be created so that non abelian anyons, such as Fibonacci, to be detected, then prototypes of topological qubits to be constructed and finally quantum simulations to happen with anyons that give eventually universal operations to execute a variety of gates and algorithms.

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6 Tables and $\mathcal{R}$ matrices

D_5	e	r	r^2	r^3	r^4	s	sr	sr^2	sr^3	sr^4
e	e	r	r^2	r^3	r^4	s	sr	sr^2	sr^3	sr^4
r	r	r^2	r^3	r^4	e	sr^4	s	sr	sr^2	sr^3
r^2	r^2	r^3	r^4	e	r	sr^3	sr^4	s	sr	sr^2
r^3	r^3	r^4	e	r	r^2	sr^2	sr^3	sr^4	s	sr
r^4	r^4	e	r	r^2	r^3	sr	sr^2	sr^3	sr^4	s
s	s	sr	sr^2	sr^3	sr^4	e	r	r^2	r^3	r^4
sr	sr	sr^2	sr^3	sr^4	s	r^4	e	r	r^2	r^3
sr^2	sr^2	sr^3	sr^4	s	sr	r^3	r^4	e	r	r^2
sr^3	sr^3	sr^4	s	sr	sr^2	r^2	r^3	r^4	e	r
sr^4	sr^4	s	sr	sr^2	sr^3	r	r^2	r^3	r^4	e

Table 2: Multiplication table of D_5.

D_5	Representatives
${}^0C$	${}^0x_1 = e$
${}^1C$	${}^1x_1 = e, {}^1x_2 = s$
${}^2C$	${}^2x_1 = e, {}^2x_2 = s$
${}^3C$	${}^3x_1 = e, {}^3x_2 = r^2, {}^3x_3 = r^4, {}^3x_4 = r, {}^3x_5 = r^3$

Table 3: Representatives for the equivalence classes of $D_5/{}^AN$.

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

Table 4: Fusion rules of $\mathcal{D}(\mathcal{S}_3)$.

$\otimes$	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
B	B	A	C	D	E	F	G	H	I	J	K	L	M	N	P	O
C	C	C	$A \oplus B \oplus D$	$C \oplus D$	$F \oplus I$	$E \oplus G$	$F \oplus H$	$G \oplus I$	$E \oplus H$	$K \oplus N$	$J \oplus L$	$K \oplus M$	$L \oplus N$	$J \oplus M$	$O \oplus P$	$O \oplus P$
D	D	D	$C \oplus D$	$A \oplus B \oplus C$	$G \oplus H$	$H \oplus I$	$E \oplus I$	$E \oplus F$	$F \oplus G$	$L \oplus M$	$M \oplus N$	$J \oplus N$	$J \oplus K$	$K \oplus L$	$O \oplus P$	$O \oplus P$
E	E	E	$F \oplus I$	$G \oplus H$	$A \oplus B \oplus J$	$C \oplus K$	$D \oplus L$	$D \oplus M$	$C \oplus N$	$E \oplus J$	$F \oplus N$	$G \oplus M$	$H \oplus L$	$I \oplus K$	$O \oplus P$	$O \oplus P$
F	F	F	$E \oplus G$	$H \oplus I$	$C \oplus K$	$A \oplus B \oplus L$	$C \oplus M$	$D \oplus N$	$D \oplus J$	$I \oplus N$	$E \oplus M$	$F \oplus L$	$G \oplus K$	$H \oplus J$	$O \oplus P$	$O \oplus P$
G	G	G	$F \oplus H$	$E \oplus I$	$D \oplus L$	$C \oplus M$	$A \oplus B \oplus N$	$C \oplus J$	$D \oplus K$	$H \oplus M$	$I \oplus L$	$E \oplus K$	$F \oplus J$	$G \oplus N$	$O \oplus P$	$O \oplus P$
H	H	H	$G \oplus I$	$E \oplus F$	$D \oplus M$	$D \oplus N$	$C \oplus J$	$A \oplus B \oplus K$	$C \oplus L$	$G \oplus L$	$H \oplus K$	$I \oplus J$	$E \oplus N$	$F \oplus M$	$O \oplus P$	$O \oplus P$
I	I	I	$E \oplus H$	$F \oplus G$	$C \oplus N$	$D \oplus J$	$D \oplus K$	$C \oplus L$	$A \oplus B \oplus M$	$F \oplus K$	$G \oplus J$	$H \oplus N$	$I \oplus M$	$E \oplus L$	$O \oplus P$	$O \oplus P$
J	J	J	$K \oplus N$	$L \oplus M$	$E \oplus J$	$I \oplus N$	$H \oplus M$	$G \oplus L$	$F \oplus K$	$A \oplus B \oplus E$	$C \oplus I$	$D \oplus H$	$D \oplus G$	$C \oplus F$	$O \oplus P$	$O \oplus P$
K	K	K	$J \oplus L$	$M \oplus N$	$F \oplus N$	$E \oplus M$	$I \oplus L$	$H \oplus K$	$G \oplus J$	$C \oplus I$	$A \oplus B \oplus H$	$C \oplus G$	$D \oplus F$	$D \oplus E$	$O \oplus P$	$O \oplus P$
L	L	L	$K \oplus M$	$J \oplus N$	$G \oplus M$	$F \oplus L$	$E \oplus K$	$I \oplus J$	$H \oplus N$	$D \oplus H$	$C \oplus G$	$A \oplus B \oplus F$	$C \oplus E$	$D \oplus I$	$O \oplus P$	$O \oplus P$
M	M	M	$L \oplus N$	$J \oplus K$	$H \oplus L$	$G \oplus K$	$F \oplus J$	$E \oplus N$	$I \oplus M$	$D \oplus G$	$D \oplus F$	$C \oplus E$	$A \oplus B \oplus I$	$C \oplus H$	$O \oplus P$	$O \oplus P$
N	N	N	$J \oplus M$	$K \oplus L$	$I \oplus K$	$H \oplus J$	$G \oplus N$	$F \oplus M$	$E \oplus L$	$C \oplus F$	$D \oplus E$	$D \oplus I$	$C \oplus H$	$A \oplus B \oplus G$	$O \oplus P$	$O \oplus P$
O	O	P	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$A \oplus C \oplus D \oplus \cdots \oplus N$	$B \oplus C \oplus D \oplus \cdots \oplus N$
P	P	O	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$O \oplus P$	$B \oplus C \oplus D \oplus \cdots \oplus N$	$A \oplus C \oplus D \oplus \cdots \oplus N$

Table 5: Fusion rules of $\mathcal{D}(D_5)$.

The $\mathcal{R}$ matrices of $\mathcal{D}\big(D_5\big)$ are :

1. $\mathcal{R}_{(1)} \equiv \mathcal{R}_{11} = \mathcal{R}_{12} = \mathcal{R}_{22} = 1$
2. $\mathcal{R}_{(2)} \equiv \mathcal{R}_{13} = \mathcal{R}_{14} = \mathcal{R}_{15} = \mathcal{R}_{16} = \mathcal{R}_{17} = \mathcal{R}_{18} = \mathcal{R}_{19} = \mathcal{R}_{110} = \mathcal{R}_{111} = \mathcal{R}_{112} = \mathcal{R}_{113} = \mathcal{R}_{114} = \mathcal{R}_{23} = \mathcal{R}_{24} = \mathcal{R}_{25} = \mathcal{R}_{26} = \mathcal{R}_{27} = \mathcal{R}_{28} = \mathcal{R}_{29} = \mathcal{R}_{210} = \mathcal{R}_{211} = \mathcal{R}_{212} = \mathcal{R}_{213} = \mathcal{R}_{214} = \mathbb{1}_{2\times 2}$
3. $\mathcal{R}_{(3)} \equiv \mathcal{R}_{115} = \mathcal{R}_{116} = \mathcal{R}_{215} = \mathcal{R}_{216} = \mathbb{1}_{5\times 5}$
4. $\mathcal{R}_{(4)} \equiv \mathcal{R}_{33} = \mathcal{R}_{34} = \mathcal{R}_{35} = \mathcal{R}_{36} = \mathcal{R}_{37} = \mathcal{R}_{38} = \mathcal{R}_{39} = \mathcal{R}_{310} = \mathcal{R}_{311} = \mathcal{R}_{312} = \mathcal{R}_{313} = \mathcal{R}_{314} = \mathcal{R}_{44} = \mathcal{R}_{45} = \mathcal{R}_{46} = \mathcal{R}_{47} = \mathcal{R}_{48} = \mathcal{R}_{49} = \mathcal{R}_{410} = \mathcal{R}_{411} = \mathcal{R}_{412} = \mathcal{R}_{413} = \mathcal{R}_{414} = \mathcal{R}_{55} = \mathcal{R}_{510} = \mathcal{R}_{610} = \mathcal{R}_{710} = \mathcal{R}_{810} = \mathcal{R}_{910} = \mathcal{R}_{1010}$
5. $\mathcal{R}_{(5)} \equiv \mathcal{R}_{56} = \mathcal{R}_{511} = \mathcal{R}_{66} = \mathcal{R}_{711} = \mathcal{R}_{811} = \mathcal{R}_{911} = \mathcal{R}_{1013} = \mathcal{R}_{1113} = \mathcal{R}_{1213} = \mathcal{R}_{1313}$
6. $\mathcal{R}_{(6)} \equiv \mathcal{R}_{57} = \mathcal{R}_{512} = \mathcal{R}_{67} = \mathcal{R}_{77} = \mathcal{R}_{712} = \mathcal{R}_{812} = \mathcal{R}_{912} = \mathcal{R}_{1011} = \mathcal{R}_{1111}$
7. $\mathcal{R}_{(7)} \equiv \mathcal{R}_{58} = \mathcal{R}_{513} = \mathcal{R}_{68} = \mathcal{R}_{78} = \mathcal{R}_{713} = \mathcal{R}_{88} = \mathcal{R}_{813} = \mathcal{R}_{913} = \mathcal{R}_{1014} = \mathcal{R}_{1114} = \mathcal{R}_{1214} = \mathcal{R}_{1314} = \mathcal{R}_{1414}$
8. $\mathcal{R}_{(8)} \equiv \mathcal{R}_{59} = \mathcal{R}_{514} = \mathcal{R}_{69} = \mathcal{R}_{79} = \mathcal{R}_{714} = \mathcal{R}_{89} = \mathcal{R}_{814} = \mathcal{R}_{99} = \mathcal{R}_{914} = \mathcal{R}_{1012} = \mathcal{R}_{1112} = \mathcal{R}_{1212}$
9. $\mathcal{R}_{(9)} \equiv \mathcal{R}_{315} = \mathcal{R}_{316} = \mathcal{R}_{415} = \mathcal{R}_{416}$
10. $\mathcal{R}_{(10)} \equiv \mathcal{R}_{515} = \mathcal{R}_{516} = \mathcal{R}_{615} = \mathcal{R}_{616} = \mathcal{R}_{715} = \mathcal{R}_{716} = \mathcal{R}_{815} = \mathcal{R}_{816} = \mathcal{R}_{915} = \mathcal{R}_{916}$
11. $\mathcal{R}_{(11)} \equiv \mathcal{R}_{1015} = \mathcal{R}_{1016} = \mathcal{R}_{1115} = \mathcal{R}_{1116} = \mathcal{R}_{1215} = \mathcal{R}_{1216} = \mathcal{R}_{1315} = \mathcal{R}_{1316} = \mathcal{R}_{1415} = \mathcal{R}_{1416}$
12. $\mathcal{R}_{(12)} \equiv \mathcal{R}_{1515} = -\mathcal{R}_{1516} = -\mathcal{R}_{1616}$

with :

$$\mathcal{R}_{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{R}_{(5)} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad \mathcal{R}_{(6)} = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 0 & \overline{\omega}^2 & 0 \\ 0 & \overline{\omega}^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix},$$

$$\mathcal{R}_{(7)} = \begin{pmatrix} \overline{\omega}^2 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \overline{\omega}^2 \end{pmatrix}, \quad \mathcal{R}_{(8)} = \begin{pmatrix} \overline{\omega} & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & \overline{\omega} \end{pmatrix}, \quad \mathcal{R}_{(9)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$
\mathcal{R}_{(10)} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad
\mathcal{R}_{(11)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix},
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\mathcal{R}_{(12)} = \begin{pmatrix}
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\end{pmatrix}
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