National and Kapodistrian University of Athens
School of Sciences
Department of Physics
Section of Astrophysics, Astronomy and Mechanics

Self-Similar Relativistic Magnetized Flows with Finite Conductivity

MSc Thesis

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Abstract

Relativistic magnetohydrodynamic flows of a finite resistivity are thought to possess a crucial role in high energy astrophysical phenomena such as, magnetic reconnection, accretion disks, and particle acceleration in magnetospheric charge gaps. This thesis presents a detailed modelling of resistive relativistic magnetized flows. The equations of steady state and axisymmetric, resistive, relativistic Magnetohydrodynamics are derived, followed by the derivation of the system of first order ordinary differential equations which governs steady-state and axisymmetric, radially self-similar flows of finite resistivity. Semianalytic solutions of this system of equations are presented, enabling the exploration of the dynamics of such flows, as well as of the geometric configuration of their velocity and electromagnetic fields. The efficiency of the emergent electric field component parallel to the flow's poloidal magnetic field as a particle acceleration mechanism is also examined.

Key words: Relativistic Magnetohydrodynamics, self-similarity, resistivity, particle acceleration

Περίληψη

Οι σχετικιστικές μαγνητισμένες ροές με πεπερασμένη αγωγιμότητα βρίσκουν εφαρμογή σε μία πληθώρα αστροφυσικών φαινομένων υψηλών ενεργειών, όπως, η μαγνητική επανασύνδεση, οι δίσκοι προσαύξησης, και η επιτάχυνση σωματιδίων σε υψηλές ενέργειες στις μαγνητόσφαιρες αστέρων νετρονίων. Στην παρούσα διπλωματική εργασία παρατίθεται μία λεπτομερής μελέτη ροών τέτοιου τύπου. Αρχικά παρουσιάζεται το σύστημα των εξισώσεων το οποίο περιγράφει χρονοανεξάρτητες, αξισυμμετρικές και ακτινικά αυτοόμοιες σχετικιστικές ροές πεπερασμένης αγωγιμότητας. Στην συνέχεια, μέσω ημιαναλυτικών λύσεων αυτού του συστήματος εξισώσεων εξερευνάται η επίδραση διαφόρων προφίλ και τιμών της ειδικής αντίστασης στην δυναμική τέτοιων ροών, καθώς και στην γεωμετρική διαμόρφωση των πεδίων τους. Τέλος, εξετάζεται η αποδοτικότητα της επιτάχυνσης σωματιδίων από την παράλληλη στο πολοειδές μαγνητικό πεδίο συνιστώσα του ηλεκτρικού πεδίου της ροής, η οποία εμφανίζεται λόγω της πεπερασμένης αγωγιμότητάς της.

Λέξεις κλειδια: Σχετικιστική Μαγνητοϋδροδυναμική, αυτοομοιότητα, αντίσταση, επιτάχυνση σωματιδίων

Ευχαριστίες

Θα ήθελα καταρχάς να ευχαριστήσω θερμά τον επιβλέποντα αυτής της εργασίας και σύμβουλο καθηγητή μου κ. Νεκτάριο Βλαχάκη, για την πολύτιμη καθοδήγησή του σε όλα τα θέματα τα οποία προέκυψαν κατά την εκπόνηση της παρούσας εργασίας. Επίσης, τα μέλη της τριμελούς εξεταστικής επιτροπής μου, κ. Απόστολο Μαστιχιάδη και κα. Μαρία Πετροπούλου, για τον χρόνο που αφιέρωσαν στην ανάγνωση αυτής της εργασίας και στην αξιολόγησή της. Θέλω ακόμα να ευχαριστήσω τους Γρηγόρη Κατσουλάκο και Βασίλη Μπισκετζή, για τις συζητήσεις και τις ενδιαφέρουσες ιδέες που μοιράστηκαν μαζί μου, καθώς επίσης και την Στέλλα Μπουλά για τις συμβουλές της σχετικά με την παρουσίαση της παρούσας εργασίας. Ευχαριστώ επίσης τους φίλους μου Ευγενία, Ανδρέα και Σταμάτη καθώς και τους Θοδωρή, Μανώλη και Φίλιππο για την υποστήριξή τους κατά την διάρκεια αυτής της χρονιάς. Ευχαριστώ πολύ και την κα. Σοφία Ζαρμπούτη για την υποστήριξή της. Τέλος, θα ήθελα να ευχαριστήσω τους γονείς μου χάρη στην στήριξη και τις θυσίες των οποίων είχα την δυνατότητα να ολοκληρώσω τις σπουδές μου.

Contents

Chapter 1

Introduction

Magnetohydrodynamic flows have been a topic of study for well over half a century, ever since the publication of Hannes Alfvén's paper [1] on the existence of electromagnetic-hydrodynamic waves, which propagate in a conducting fluid when it is placed in a magnetic field. In the latter half of the 20th century, the theory of ideal (perfectly conductive) MHD, both in the classical and relativistic regimes, has been applied in the study of flows found in astrophysical environments and often constitute the main driving mechanism behind observed astrophysical phenomena. Examples of such flows are stellar winds, most notably studied by Parker (1958) [2] and Weber & Davis (1967) [3], and astrophysical outflows, namely disk winds and jets, emanating from the accretion disks of compact objects.

While astrophysical MHD flows of infinite conductivity have been studied extensively and to great success, both analytically and numerically, the properties of flows of finite conductivity are not as well understood. The motivation for a semianalytic exploration of the dynamics of resistive MHD flows is quite significant, due to both the crucial role of these flows in certain astrophysical processes as well as the absence of an analytic model describing them in detail.

One magnetohydrodynamic process in which resistivity is an important factor is magnetic reconnection. Magnetic reconnection is the process during which oppositely directed and topologically distinct magnetic field lines connect. This diffusion of magnetic fields in regions where reconnection takes place is oftentimes aided by the resistivity of the current sheet separating them. During this reconfiguration of the magnetic field's topology, magnetic energy is converted into heat, kinetic energy which accelerates the flow, as well as particle acceleration. [4]

While traditionally applied in the study of solar flares, the Earth's magnetosphere, and fusion plasmas, the relativistic generalization of the aforementioned theory also finds application in the field of high energy astrophysics, in which strongly magnetized and relativistic flows are of great significance. Magnetic reconnection occuring in such extreme environments may have an important role in both the dynamics of these flows (AGN jets, magnetized disk winds), as well as in the high energy emission from these regions, as an efficient mechanism for particle acceleration. [5]

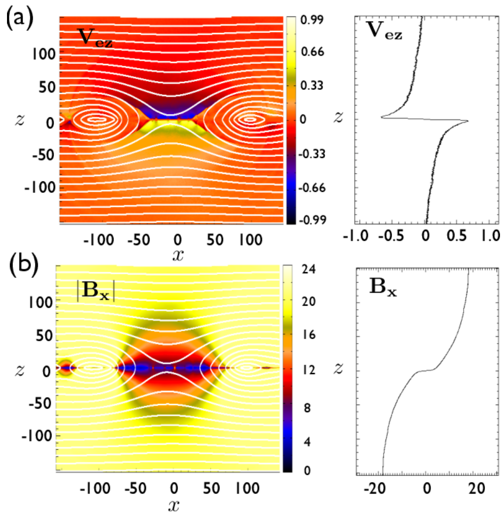


Figure 1.1: The morphology of a relativistic magnetic reconnection site. Graph (a) depicts the electron z velocity, while graph (b) the x magnetic field component. The white contours are the magnetic flux lines. [6]

The theory of resistive, relativistic MHD may also be of use in the magnetohydrodynamical description of pulsar magnetospheres. While traditionally described by vacuum or force-free solutions, these two limits can not provide the full picture regarding high energy emission from pulsars, as they can not simultaneously determine the behaviour of both the accelerating electric fields, and the currents flowing in their magnetospheres. Numerical simulations have shown that solutions assuming a plasma of finite resistivity have been shown to be able to bridge the vacuum and force-free limits, by modifying the plasma's resistivity from an infinite value to zero respectively. [7]

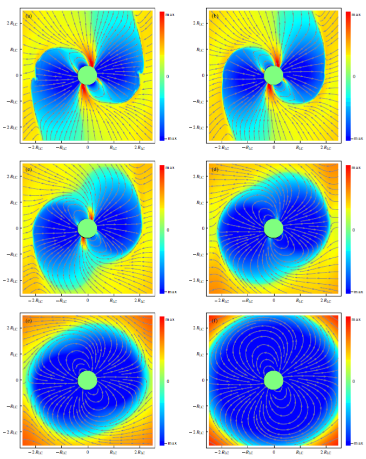


Figure 1.2: Magnetic field lines for (a) force-free, (b)-(e) resistive, and (f) vacuum magnetospheres. [7]

One more application of resistive MHD is the study of accretion disks around neutron stars and black holes, as well as the formation of jets and their acceleration. The energy dissipated through ohmic dissipation, due to the accretion disk's plasma resistivity, may account for the disk's thermal radiation. Moreover, numerical simulations of jets of a uniform resistivity have shown that the jet's main driving mechanism past the slow magnetosonic point is, as in the case of ideal jets, the magnetocentrifugal force, while the plasma is ejected from the disk due to its thermal pressure. [8] [9]

The effects of resistivity on the accretion rate in disks around black holes has also been investigated recently. Numerical simulations show that the accretion rate decreases as the value of the plasma's resistivity increases. [10]

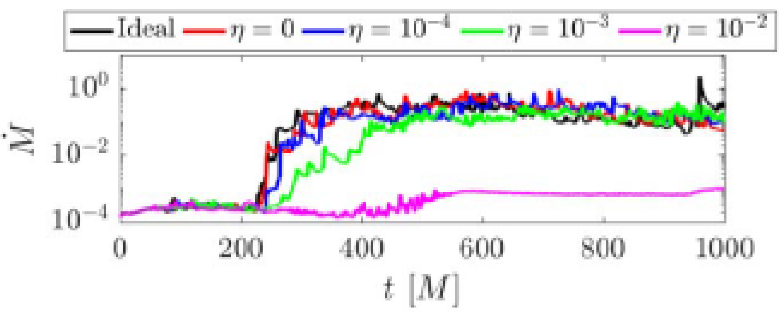


Figure 1.3: Accretion rates for various resistivity values η. [10]

Due to the numerous possible applications of resistive MHD in many extreme astrophysical phenomena, the motivation for a semianalytic exploration of the dynamics and properties of resistive and relativistic magnetized flows is strong, as it can shed light on the geometrical configuration of the electromagnetic and velocity fields, which is significantly modified from that of ideal MHD flows, as well as on the resistivity's impact on the flow's energetics. Namely, the plasma can heat up significantly due to ohmic dissipation, which as will be shown by the solutions presented in this thesis, is the mechanism through which energy is extracted from the flow's electromagnetic field and deposited in the fluid.

Chapter 2

The Steady-State Equations of Resistive, Relativistic Magnetohydrodynamics

A flat space-time is assumed, as well as the convention $\eta^{\mu\nu} = diag(-1, +1, +1, +1)$ in regard to the metric. When used as indices, greek characters take the values $(0, 1, 2, 3)$, while latin characters the values $(1, 2, 3)$, denoting only the spatial components of a tensor.

2.1 Fundamental Tensors

The fundamental tensors of Relativistic Magnetohydrodynamics, through which the governing equations are derived, are the particle number density four-current, or the mass density four-current, the stress-energy tensor, and the electromagnetic tensor.

2.1.1 The Mass Density Four-Current:

One conserved quantity in a fluid is the particle number density, which in an arbitrary space-time point in its comoving frame of reference shall be represented by n_0. Assuming the fluid is ideal and consisting of particles of mean mass $\overline{m}$, its mass density in the comoving frame is defined as: $\rho_0 = n_0\overline{m}$.

The mass density four-current may now be defined as a four-vector, whose temporal and spatial components in the fluid's comoving frame are respectively:

$$\tilde{D}^0 = \rho_0 c \text{ and } \tilde{D}^i = 0,$$

thus creating the four vector:

$$\tilde{\boldsymbol{D}} = \begin{pmatrix} \rho_0 c \\ 0 \end{pmatrix} \tag{2.1}$$

In the lab frame, or observer's frame, in which the fluid's macroscopic velocity is $\vec{u}$, the mass density four-current $\boldsymbol{D}$ is calculated through the Lorentz transformation of $\tilde{\boldsymbol{D}}$ as follows:

$$D^{\mu}_{\nu} = \Lambda(-\vec{u})^{\mu}_{\kappa}\tilde{D}^{\kappa}_{\nu} \tag{2.2}$$

where

$$\Lambda(-\vec{u}) = \begin{pmatrix} \gamma & \gamma\dfrac{\vec{u}^T}{c} \\ \gamma\dfrac{\vec{u}}{c} & I + \dfrac{\gamma - 1}{u^2}\vec{u}\vec{u}^T \end{pmatrix}$$

is the 4×4 matrix for a generalized Lorentz boost to an inertial frame in which the fluid, and the comoving frame, move with velocity $\vec{u}$. [11]

The temporal and spatial components of $\boldsymbol{D}$ given by (2.2) for $\mu = 0, i$ respectively, are:

$$D^0 = \gamma\rho_0 c \tag{2.3}$$

$$D^i = \gamma\rho_0 u^i \tag{2.4}$$

Equations (2.3) and (2.4) suggest that D can be written as:

$$D^\mu = \rho_0 U^\mu \tag{2.5}$$

with $\boldsymbol{U} = \begin{pmatrix} \gamma c \\ \gamma\vec{u} \end{pmatrix}$ the fluid's four-velocity in the lab frame. [11]

2.1.2 The Stress-Energy Tensor

The stress-energy tensor is a symmetric, four-dimensional, second order tensor, the components of which contain all information regarding the energy density, energy flux and momentum density flux through the surface eclosing an arbitrary volume inside the fluid of interest. In the context of relativistic MHD, it is comprised of a matter component, concerning the fluid, an electromagnetic component, concerning the electromagnetic field, and a component describing the flow's radiative losses. In the comoving frame, the stress-energy tensor's matter component is defined as [11]:

$$\tilde{T}_M^{00} = \rho_0 c^2 + \rho_0 e$$

$$\tilde{T}_M^{0i} = 0$$

$$\tilde{T}_M^{ij} = P\delta_{ij}$$

where P is the fluid's pressure, and e its internal energy per unit mass.

The above definition suggests that it can be written in the form:

$$\tilde{T}_M^{\mu\nu} = (\rho_0 c^2 + \rho_0 e + P)\frac{\tilde{U}^\mu \tilde{U}^n}{c^2} + P\eta^{\mu\nu} \tag{2.6}$$

In the above expression, $\tilde{\boldsymbol{U}} = \begin{pmatrix} c \\ \\ \vec{0} \end{pmatrix}$ is the fluid's four-velocity in its comoving frame.

The Lorentz transformation of (2.6) will give $T_M^{\mu\nu}$ in the lab frame:

$$T_M^{\mu\nu} = \Lambda(-\vec{u})^\mu_\kappa \Lambda(-\vec{u})^\nu_\lambda \tilde{T}^{\kappa\lambda} \tag{2.7}$$

By use of the property:

$$\Lambda(\vec{u})^\mu_\kappa \Lambda(\vec{u})^\nu_\lambda \eta^{\kappa\lambda} = \eta^{\mu\nu}$$

and:

$$U^\mu = \Lambda(-\vec{u})^\mu_\kappa \tilde{U}^\kappa$$

equation (2.7) gives:

$$T_M^{\mu\nu} = (\rho_0 c^2 + \rho_0 e + P)\frac{U^\mu U^\nu}{c^2} + P\eta^{\mu\nu}$$

Assuming the ideal fluid is characterised by an adiabatic index Γ, its specific enthalpy ξ may be defined as:

$$\xi \rho_0 c^2 = \rho_0 c^2 + \rho_0 e + P = \rho_0 c^2 + \frac{\Gamma}{\Gamma - 1} P \Rightarrow$$

$$\xi = 1 + \frac{\Gamma}{\Gamma - 1} \frac{P}{\rho_0 c^2}$$

with Γ the fluid's adiabatic index. An equation of state of the form:

$$P = K \rho_0^{\Gamma}$$

is also assumed. [13]

With the specific enthalpy ξ defined, the matter component of the stress-energy tensor can finally be written as:

$$T_M^{\mu\nu} = \xi \rho_0 U^\mu U^\nu + P \eta^{\mu\nu} \tag{2.8}$$

The electromagnetic component of the stress-energy tensor, in any inertial frame of reference, is defined as [12]:

$$T_{EM}^{\mu\nu} = \frac{1}{4\pi}(F^{\mu\kappa} F^{\nu}_{\kappa} - \frac{1}{4} \eta^{\mu\nu} F^{\kappa\lambda} F_{\kappa\lambda}) \tag{2.9}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, which will be discussed in the next subsection. The components of $T_{EM}^{\mu\nu}$ in the lab frame are [13]:

$$T_{EM}^{00} = \frac{E^2 + B^2}{8\pi}$$

$$T_{EM}^{0i} = T_{EM}^{i0} = \frac{(\vec{E} \times \vec{B})_i}{4\pi} = \frac{S_i}{c}$$

$$T_{EM}^{ij} = T_{EM}^{ji} = \frac{E^2 + B^2}{8\pi} \delta_{ij} - \frac{E_i E_j + B_i B_j}{4\pi} = -\sigma_{ij}$$

where S_i is the $i-$th component of the Poynting vector, and σ_{ij} are components of the Maxwell stress tensor. [15]

The final component of the stress-energy tensor, concerning the radiative losses, is defined as follows:

In the comoving frame, the four-force acting on the flow due to its radiative losses can be defined as:

$$\tilde{G}^\mu = \begin{pmatrix} -\frac{\Lambda}{c} \\ \\ \vec{0} \end{pmatrix}$$

The four-force due to radiative losses in the lab frame then is simply:

$$G^{\mu} = \Lambda^{\mu}_{\nu}(-\vec{u})\tilde{G}^{\nu} = \begin{pmatrix} -\gamma\dfrac{\Lambda}{c} \\ \\ -\gamma\dfrac{\Lambda\vec{u}}{c^2} \end{pmatrix}$$

The radiative losses component $T^{\mu\nu}_{RL}$ may now be defined as:

$$T^{\mu\nu}_{RL;\nu} = -G^{\mu} = -\begin{pmatrix} -\gamma\dfrac{\Lambda}{c} \\ \\ -\gamma\dfrac{\Lambda\vec{u}}{c^2} \end{pmatrix} \tag{2.10}$$

In conclusion, the stress-energy tensor fully describing the fluid and the electromagnetic field is:

$$T^{\mu\nu} = T^{\mu\nu}_{M} + T^{\mu\nu}_{EM} + T^{\mu\nu}_{RL} \tag{2.11}$$

with $T^{\mu\nu}_{M}$, $T^{\mu\nu}_{EM}$, $T^{\mu\nu}_{RL}$, as given by equations (2.8), (2.9), and (2.10) respectively.

2.1.3 The Electromagnetic Tensor

In order to properly define the Lorentz transformation of the electromagnetic field between two inertial frames of reference, the electromagnetic tensor $F^{\mu\nu}$ is introduced. [11]

The electromagnetic tensor is an antisymmetric, four-dimensional, second order tensor. In its contravariant form, $F^{\mu\nu}$ is [12]:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{2.12}$$

or more concisely:

$$\begin{aligned} F^{00} &= 0 \\ F^{0i} &= E_i \\ F^{ij} &= \epsilon^{ijk} B_k \end{aligned} \tag{2.13}$$

The electromagnetic tensor is the fundamental tool in the covariant formulation of classical electrodynamics. By use of $F^{\mu\nu}$, Maxwell's equations may be written in covariant form as:

$$F^{\mu\nu}_{;\mu} = -\frac{4\pi}{c} J^{\nu} \tag{2.14}$$

$$F_{\mu\nu;\kappa} + F_{\kappa\mu;\nu} + F_{\nu\kappa;\mu} = 0 \tag{2.15}$$

where J^{μ} is the current density four-vector:

$$J^{\mu} = \begin{pmatrix} J^0 \\ \vec{J} \end{pmatrix}$$

Its temporal component J^0 is simply the charge density multiplied by c, while its spatial component is the current density $\vec{J}$.

Equation (2.14) for $\nu = 0$ gives Gauss' Law, while for $\nu = i$ Ampère's Law. Faraday's Law of Induction and the conservation of magnetic flux in their can be retrieved from equation (2.15).

The electromagnetic tensor also provides a way to write the equation of motion for a charged particle of charge q in an electromagnetic field in covariant form:

$$\frac{dp^{\mu}}{d\tau} = \frac{q}{c} F^{\mu\nu} U_{\nu} \tag{2.16}$$

where p^μ is the particle's four-momentum, U^μ its four-velocity, and $qF^\mu_\nu U^\nu$ the four-force acting on it due to the electromagnetic field. [11]

It is also noted that the electromagnetic tensor may be defined as the exterior derivative of the one form A_μ:

$$F = dA$$

or

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

A_μ is the four-potential, defined as:

$$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}$$

where Φ and $\vec{A}$ are the familiar scalar and vector potentials respectively. [12]

2.2 The Generalization of Ohm's Law in Special Relativity

In the case of a non-moving conductor, Ohm's Law takes the familiar form:

$$\vec{J} = \sigma \vec{E} \tag{2.17}$$

with σ the conductor's conductivity. [15]

In the flow's comoving frame, the plasma may be treated as a non-moving conductor of conductivity σ. Equation (2.17) then is the appropriate expression of Ohm's Law for the system in the flow's comoving frame, which will be rewritten as:

$$\tilde{\vec{J}} = \sigma \tilde{\vec{E}} \tag{2.18}$$

Since the equations that fully describe the system of the fluid and the electromagnetic field will be derived in the lab frame, Ohm's Law must be defined in that inertial frame of reference. This can be achieved by writing equation (2.18) in a covariant form, and then transforming the resulting expression via two consecutive Lorentz transformations. It should also be noted that the resulting covariant expression of Ohm's Law should be Lorentz covariant, as it describes a physical law.

The right hand side (RHS) of equation (2.18) may be written in covariant form as:

$$\frac{\sigma}{c} \tilde{F}^{\mu\nu} \tilde{U}_\nu$$

where $\tilde{U}_\nu = \begin{pmatrix} -c \\ 0 \end{pmatrix}$, is the covariant four-velocity of the conductor, in this case the plasma, in its comoving frame. Indeed, it is simple to prove that the result of the product $\tilde{F}^{\mu\nu}\tilde{U}_\nu$ is:

$$\tilde{F}^{\mu\nu}\tilde{U}_\nu = \begin{pmatrix} 0 \\ c\tilde{\vec{E}} \end{pmatrix}$$

[11]

Because the current density $\tilde{\vec{J}}$ appears in the LHS of equation (2.18), it is clear that the covariant form of Ohm's Law must include the current density four-vector $\tilde{J}^\mu$ in its LHS. In order to derive a proper expression for the LHS, two things must be taken into account. The first is that the RHS is a first order contravariant tensor. The second is that for $\mu = 0$: $\tilde{F}^{0\nu}\tilde{U}_\nu = 0$. The LHS of the covariant expression for Ohm's Law must also be equal to the current density $\tilde{\vec{J}}$, for $\mu = i$, in the comoving frame.

The expression

$$\tilde{J}^\mu + \tilde{J}_\nu \frac{\tilde{U}^\nu \tilde{U}^\mu}{c^2}$$

satisfies all three of these requirements. First of all, it is a first order contravariant tensor, as can easily be seen. Secondly, for $\mu = 0$:

$$\tilde{J}^0 + \tilde{J}_\nu \frac{\tilde{U}^\nu \tilde{U}^0}{c^2} = \tilde{J}^0 - \frac{c^2}{c^2}\tilde{J}^0 = 0$$

since $\tilde{J}_\mu = \begin{pmatrix} -\tilde{J}^0 \\ \tilde{\vec{J}} \end{pmatrix}$. Also, for $\mu = i$ it gives:

$$\tilde{J}^i + \tilde{J}_\nu \frac{\tilde{U}^\nu \tilde{U}^i}{c^2} = \tilde{J}^i - \tilde{J}^0 \frac{\tilde{U}^i}{c} = \tilde{J}^i$$

The covariant expression for Ohm's Law in the flow's comoving frame is, finally:

$$\tilde{J}^\mu + \tilde{J}_\nu \frac{\tilde{U}^\nu \tilde{U}^\mu}{c^2} = \frac{\sigma}{c}\tilde{F}^{\mu\nu}\tilde{U}_\nu$$

or more compactly

$$\tilde{J}_\nu(\eta^{\mu\nu} + \frac{\tilde{U}^\mu \tilde{U}^\nu}{c^2}) = \frac{\sigma}{c}\tilde{F}^{\mu\nu}\tilde{U}_\nu \tag{2.19}$$

Ohm's Law in any other inertial frame moving with velocity $\vec{u}$ with respect to the comoving frame is given by the Lorentz transformation of the above expression:

$$\Lambda(\vec{u})^\lambda_\nu \tilde{J}_\lambda(\Lambda(\vec{u})^\mu_\kappa \Lambda(\vec{u})^\nu_\lambda \eta^{\kappa\lambda} + \Lambda(\vec{u})^\mu_\kappa \Lambda(\vec{u})^\nu_\lambda \frac{\tilde{U}^\kappa \tilde{U}^\lambda}{c^2}) = \frac{\sigma}{c}\Lambda(\vec{u})^\mu_\kappa \Lambda(\vec{u})^\nu_\lambda \Lambda(\vec{u})^\lambda_\nu \tilde{F}^{\kappa\lambda}\tilde{U}_\lambda$$

which results in:

$$J_\nu(\eta^{\mu\nu} + \frac{U^\mu U^\nu}{c^2}) = \frac{\sigma}{c} F^{\mu\nu} U_\nu \tag{2.20}$$

Equation (2.20) is the covariant generalization of Ohm's Law for flat spacetimes, an expression which is indeed Lorentz covariant, as is evident from the comparison of equations (2.19) and (2.20).

In the lab frame, in which the flow moves with velocity $\vec{u}$ and its four-velocity is $U^\mu = \begin{pmatrix} \gamma c \\ \gamma \vec{u} \end{pmatrix}$, Ohm's Law is given by equation (2.20). Equation (2.20) for $\mu = 0$ gives:

$$J_\nu \eta^{0\nu} + \gamma J_\nu \frac{U^\nu}{c} = \frac{\sigma}{c} F^{0i} U_i \Rightarrow$$

$$J^0 - \gamma^2 J^0 + \gamma^2 \frac{\vec{u} \cdot \vec{J}}{c} = \sigma\gamma \frac{\vec{u} \cdot \vec{E}}{c} \Rightarrow$$

$$(\gamma^2 - 1) J^0 = \gamma^2 \frac{\vec{u} \cdot \vec{J}}{c} - \sigma\gamma \frac{\vec{u} \cdot \vec{E}}{c} \tag{2.21}$$

For $\mu = i$:

$$J^i + J_\nu U^\nu \frac{U^i}{c^2} = \frac{\sigma}{c} F^{i\nu} U_\nu \Rightarrow$$

$$J_i + \gamma^2 \frac{u_i}{c}(-J^0 + \frac{\vec{u} \cdot \vec{J}}{c}) = \sigma\gamma(E_i + \epsilon_{ijk} \frac{u^j B^k}{c}) \Rightarrow$$

$$\vec{J} + \gamma^2 \frac{\vec{u}}{c}(\frac{\vec{u} \cdot \vec{J}}{c} - J^0) = \sigma\gamma(\vec{E} + \frac{\vec{u} \times \vec{B}}{c}) \tag{2.22}$$

It is clear that equation (2.21) is simply the projection of equation (2.22) along the velocity $\vec{u}$. It may then be disregarded, as equation (2.22) is sufficient as a mathematical expression of Ohm's Law in the lab frame.

By multiplying both sides of (2.21) with $\frac{\vec{u}}{c}$, and then adding the resulting expression to (2.22), the following expression for the current density $\vec{J}$ is derived:

$$\vec{J} = J^0 \frac{\vec{u}}{c} + \sigma\gamma(\vec{E} + \frac{\vec{u} \times \vec{B}}{c} - \frac{\vec{u} \cdot \vec{E}}{c} \frac{\vec{u}}{c}) \tag{2.23}$$

The two terms appearing on the RHS of equation (2.23) are the convection current:

$$\vec{J}_{conv} = J^0 \frac{\vec{u}}{c}$$

and the conduction current:

$$\vec{J}_{cond} = \sigma\gamma(\vec{E} + \frac{\vec{u} \times \vec{B}}{c} - \frac{\vec{u} \cdot \vec{E}}{c} \frac{\vec{u}}{c})$$

[14]

The electric field $\vec{E}$, as given by (2.22), generally has three terms:

$$\vec{E} = -\frac{\vec{u} \times \vec{B}}{c} + \frac{\vec{J}}{\gamma\sigma} + \frac{\gamma}{\sigma}\frac{\vec{u}}{c}(\frac{\vec{u} \cdot \vec{J}}{c} - J^0) \tag{2.24}$$

The first term is the induction or dynamo term, which appears due to the motion of the fluid, while the second term is the familiar ohmic term, now divided by the conductor's (fluid) Lorentz factor γ in the lab frame. The last term is a relativistic correction due to the the relativistic motion of the fluid in the lab frame. This term becomes practically zero for a non-relativistic motions ($\frac{u}{c} \simeq 0$, $\gamma \simeq 1$). [?]

In the classical limit, equaton (2.23) reduces to:

$$\vec{E} = -\frac{\vec{u} \times \vec{B}}{c} + \frac{\vec{J}}{\sigma} \tag{2.25}$$

In the limit of extremely high conductivity ($\sigma \to \infty$), equations (2.23) and (2.24) both reduce to the "frozen-in" condition of Ideal (Relativistic) MHD:

$$\vec{E} + \frac{\vec{u} \times \vec{B}}{c} = 0 \tag{2.26}$$

[1]

When the plasma's conductivity is practically infinite and this condition is satisfied, the magnetic field lines are frozen into the plasma, following its motion, a phenomenon known as "flux-freezing". Additionally, the electric field $\vec{E}$ is always vertical to both the magnetic field $\vec{B}$ and the velocity $\vec{u}$. [?]

If, on the other hand, the plasma's conductivity is finite, this condition breaks down. The plasma flow is no longer forced to follow the topology of the magnetic field lines, while the electric field gains components parallel to the magnetic field lines and the velocity vector.

It is noted that in the previous analysis and in the entirety of this thesis, the conductivity, and in consequence the resistivity, are assumed to be scalar quantities, and not tensors.

2.3 Governing Equations

The Governing Equations of Resistive Relativistic Magnetohydrodynamics (RRMHD) are the laws of conservation of mass, momentum and energy, describing the fluid, Maxwell's equations describing the electromagnetic field, and the relativistic generalization of Ohm's Law, which closes the system of equations, along with the fluid's Equation of State (EOS).

2.3.1 Maxwell's Equations

As described in the previous section, Maxwell's equations in their familiar differential form can be derived from the electromagnetic tensor though the covariant expressions:

$$F^{\mu\nu}_{;\mu} = -\frac{4\pi}{c} J^{\nu} \tag{2.14}$$

$$F_{\mu\nu;\kappa} + F_{\kappa\mu;\nu} + F_{\nu\kappa;\mu} = 0 \tag{2.15}$$

Equation (2.14) for $\nu = 0$ gives:

$$F^{i0}_{;i} = -\frac{4\pi}{c} J^0 \Rightarrow$$

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c} J^0 \tag{2.27}$$

since $F^{00} = 0$ and $F^{i0} = -E_i$.

For $\nu = i$:

$$F^{0i}_{;0} + F^{ji}_{;j} = -\frac{4\pi}{c} J^i \Rightarrow$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J} \tag{2.28}$$

since $F^{0i} = E_i$ and $F^{ji} = \epsilon^{jik} B_k = -\epsilon^{ijk} B_k$.

Equation (2.15) provides the remaining two of Maxwell's equations:

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{2.29}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.30}$$

[11]

In summary, the four Maxwell Equations describing the electromagnetic field are:

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{c} J^0 \tag{2.26}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J} \tag{2.27}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{2.28}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.29}$$

2.3.2 Conservation Laws

The three conservation laws describing the motion of the fluid are the conservation of mass (continuity equation), the conservation of momentum, and the conservation of energy (conservation of four-momentum).

The continuity equation is derived by setting the covariant divergence of the mass density four-current equal to zero, assuming no sources or sinks of mass:

$$D^{\mu}_{;\mu} = 0$$

The above expression gives:

$$\frac{1}{c}\frac{\partial(\gamma\rho_0 c)}{\partial t} + \vec{\nabla}\cdot(\gamma\rho_0\vec{u}) = 0 \Rightarrow$$

$$\frac{\partial(\gamma\rho_0)}{\partial t} + \vec{\nabla}\cdot(\gamma\rho_0\vec{u}) = 0 \tag{2.31}$$

,the relativistic generalization of the continuity equation. [11]

The conservation laws for the fluid's energy and the momentum are given by:

$$T^{\mu\nu}_{;\nu} = 0 \tag{2.32}$$

The temporal component of (2.32) ($\mu = 0$) provides the equation of energy conservation

$$T^{0\nu}_{M;\nu} + T^{0\nu}_{EM;\nu} + T^{0\nu}_{RL;\nu} = 0 \Rightarrow$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c^2 - P) + \vec{\nabla}\cdot(\gamma^2\xi\rho_0 c\vec{u}) + \frac{1}{c}\frac{\partial}{\partial t}(\frac{E^2+B^2}{8\pi}) + \vec{\nabla}\cdot(\frac{\vec{E}\times\vec{B}}{4\pi}) - \gamma\frac{\Lambda}{c} = 0 \Rightarrow$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c^2 - P + \frac{E^2+B^2}{8\pi}) + \vec{\nabla}\cdot(\gamma^2\xi\rho_0 c\vec{u} + \frac{\vec{E}\times\vec{B}}{4\pi}) - \gamma\frac{\Lambda}{c} = 0 \tag{2.33}$$

The spatial component ($\mu = i$) of (2.32) will give the conservation of momentum, in the absence of any external forces and in flat spacetime:

$$T^{i\nu}_{M;\nu} + T^{i\nu}_{EM;\nu} + T^{i\nu}_{RL;\nu} = 0 \Rightarrow$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c\vec{u}) + \vec{\nabla}\cdot(\gamma^2\xi\rho_0(\vec{u}\otimes\vec{u})) + \vec{\nabla}P - \frac{J^0\vec{E}+\vec{J}\times\vec{B}}{c} + \frac{\gamma\Lambda\vec{u}}{c} = 0 \Rightarrow$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c\vec{u}) + \vec{\nabla}\cdot(\gamma^2\xi\rho_0(\vec{u}\otimes\vec{u})) = -\vec{\nabla}P + \frac{J^0\vec{E}+\vec{J}\times\vec{B}}{c} - \frac{\gamma\Lambda\vec{u}}{c} \tag{2.34}$$

where for the Lorentz force, the relation:

$$T^{i\nu}_{EM;\nu} = -\frac{1}{c}F^{i}_{\nu}J^{\nu}$$

was used. [11]

Instead of equation (2.33), it is preferable to use the conservation of entropy (First Law of Thermodynamics) as an energy equation [13]:

$$U_{\mu}T^{\mu\nu}_{;\nu} = 0 \tag{2.35}$$

The matter component of equation (2.35) is:

$$U_\mu T^{\mu\nu}_{M;\nu} = U_\mu \nabla_\nu (\xi\rho_0 U^\mu U^\nu + P\eta^{\mu\nu}) \Rightarrow$$

$$U_\mu T^{\mu\nu}_{M;\nu} = \xi U_\mu U^\mu \nabla_\nu (U^\nu \rho_0) + \rho_0 U_\mu U^\nu \nabla_\nu (U^\mu \xi) + U_\mu \nabla^\mu P \Rightarrow$$

$$U_\mu T^{\mu\nu}_{M;\nu} = -c^2\xi[\frac{\partial}{\partial t}(\gamma\rho_0) + \vec{\nabla}\cdot(\gamma\rho_0\vec{u})] + \gamma[\frac{\partial P}{\partial t} + (\vec{u}\cdot\vec{\nabla})P]$$

$$-\gamma^2 c^2\rho_0[\frac{\partial}{\partial t}(\gamma\xi) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi)] + \gamma^2\rho_0\vec{u}\cdot[\frac{\partial}{\partial t}(\gamma\xi\vec{u}) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})]$$

The first term in the above expression is the continuity equation (2.31), and so it is set equal to zero. The matter component of the First Law of Thermodynamics then becomes:

$$U_\mu T^{\mu\nu}_{M;\nu} = +\gamma[\frac{\partial P}{\partial t} + (\vec{u}\cdot\vec{\nabla})P] - \gamma^2 c^2\rho_0[\frac{\partial}{\partial t}(\gamma\xi) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi)] + \gamma^2\rho_0\vec{u}\cdot[\frac{\partial}{\partial t}(\gamma\xi\vec{u}) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})]$$

The second term may now be broken down to:

$$-\gamma^2 c^2\rho_0\gamma[\frac{\partial}{\partial t}(\xi) + \vec{u}\cdot\vec{\nabla}(\xi)] - \gamma^2 c^2\rho_0\xi[\frac{\partial}{\partial t}(\gamma) + \vec{u}\cdot\vec{\nabla}(\gamma)] \qquad (2.36)$$

while the third term may be written as:

$$\gamma^2\rho_0\vec{u}\cdot[\frac{\partial}{\partial t}(\gamma\xi\vec{u}) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})] = \gamma\rho_0 c^2\gamma^2\frac{u^2}{c^2}[\frac{\partial}{\partial t}(\xi) + \vec{u}\cdot\vec{\nabla}(\xi)] + \xi\rho_0 c^2\gamma^2\frac{u^2}{c^2}[\frac{\partial}{\partial t}(\gamma) + \vec{u}\cdot\vec{\nabla}(\gamma)] +$$

$$\gamma\xi\rho_0\gamma^2\vec{u}\cdot[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}] =$$

$$\gamma\rho_0 c^2\gamma^2\frac{u^2}{c^2}[\frac{\partial}{\partial t}(\xi) + \vec{u}\cdot\vec{\nabla}(\xi)] + \xi\rho_0 c^2\gamma^2\frac{u^2}{c^2}[\frac{\partial}{\partial t}(\gamma) + \vec{u}\cdot\vec{\nabla}(\gamma)] + \frac{1}{2}\gamma\xi\rho_0\gamma^2[\frac{\partial u^2}{\partial t} + (\vec{u}\cdot\vec{\nabla})u^2]$$

With use of the relation:

$$\frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

the third term finally becomes:

$$\gamma^2\rho_0\vec{u}\cdot[\frac{\partial}{\partial t}(\gamma\xi\vec{u}) + (\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})] = (\gamma^2-1)\gamma\rho_0 c^2[\frac{\partial}{\partial t}(\xi) + \vec{u}\cdot\vec{\nabla}(\xi)] + (\gamma^2-1)\xi\rho_0 c^2[\frac{\partial}{\partial t}(\gamma) + \vec{u}\cdot\vec{\nabla}(\gamma)] -$$

$$-\frac{1}{2}\gamma^3\rho_0 c^2[\frac{\partial}{\partial t}(\frac{1}{\gamma^2}) + (\vec{u}\cdot\vec{\nabla})(\frac{1}{\gamma^2})] =$$

$$(\gamma^2-1)\gamma\rho_0 c^2[\frac{\partial}{\partial t}(\xi) + \vec{u}\cdot\vec{\nabla}(\xi)] + \gamma^2\xi\rho_0 c^2[\frac{\partial}{\partial t}(\gamma) + \vec{u}\cdot\vec{\nabla}(\gamma)] \qquad (2.37)$$

Due to (2.36) and (2.37), $U_\mu T^{\mu\nu}_{M;\nu}$ simplifies to:

$$U_\mu T^{\mu\nu}_{M;\nu} = \gamma[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})P - \rho_0 c^2(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})\xi]$$

Using the definition of the specific enthalpy:

$$\xi = 1 + \frac{\Gamma}{\Gamma - 1}\frac{P}{\rho_0 c^2}$$

the above expression can be written as:

$$U_\mu T^{\mu\nu}_{M;\nu} = \gamma[-\frac{1}{\Gamma-1}(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})P - \frac{\Gamma}{\Gamma-1}\rho_0(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{1}{\rho_0})] =$$

$$-\frac{\rho_0^\Gamma}{\Gamma-1}\gamma[\frac{1}{\rho_0^\Gamma}(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})P + \frac{\Gamma}{\rho_0^{\Gamma-1}}(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{1}{\rho_0})] =$$

$$-\gamma\frac{\rho_0^\Gamma}{\Gamma-1}[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma})]$$

The electromagnetic component of the entropy conservation equation is:

$$U_\mu T^{\mu\nu}_{EM;\nu} = -\frac{1}{c}U_\mu F^\mu_\nu J^\nu = -\frac{U_0}{c}F^0_j J^j - \frac{U_i}{c}F^i_0 J^0 - \frac{U_i}{c}F^i_j J^j =$$

$$\gamma\vec{J}\cdot(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) - \gamma J^0\frac{\vec{u}\cdot\vec{E}}{c}$$

Lastly, the component due to the radiative losses:

$$U_\mu T^{\mu\nu}_{RL;\nu} = -U_0 G^0 - U_i G^i = -\gamma^2\Lambda + \gamma^2\frac{u^2}{c^2}\Lambda = -\Lambda$$

The First Law of Thermodynamics can finally be expressed as:

$$\gamma\frac{\rho_0^\Gamma}{\Gamma-1}[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma})] = \gamma\vec{J}\cdot(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) - \gamma J^0\frac{\vec{u}\cdot\vec{E}}{c} - \Lambda \quad (2.38)$$

Summarizing, the system of equations which fully describes a resistive, relativistic, magnetohydrodynamic flow in flat spacetime is:

Ohm's Law:

$$\vec{J} + \gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot\vec{J}}{c} - J^0) = \sigma\gamma(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) \quad (2.22)$$

Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c}J^0 \quad (2.27)$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \tag{2.28}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{2.29}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.30}$$

Conservation of Mass, Momentum, and Energy:

$$\frac{\partial(\gamma\rho_0)}{\partial t} + \vec{\nabla} \cdot (\gamma\rho_0\vec{u}) = 0 \tag{2.31}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c\vec{u}) + \vec{\nabla} \cdot (\gamma^2\xi\rho_0(\vec{u}\otimes\vec{u})) = -\vec{\nabla}P + \frac{J^0\vec{E} + \vec{J}\times\vec{B}}{c} - \frac{\gamma\Lambda\vec{u}}{c} \tag{2.34}$$

$$\gamma\frac{\rho_0^\Gamma}{\Gamma - 1}[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma})] = \gamma\vec{J}\cdot(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) - \gamma J^0\frac{\vec{u}\cdot\vec{E}}{c} - \Lambda \tag{2.38}$$

2.3.3 The Limit $\sigma \to 0$

In the limit of zero conductivity, or infinite resistivity, the current density $\vec{J}$, as given by (2.23), becomes equal to the convection current $\vec{J}_{conv}$:

$$\vec{J} = J^0\frac{\vec{u}}{c} \tag{2.39}$$

since the conduction current $\vec{J}_{cond}$ vanishes.

After substituting $\vec{J}$ as given by (2.39) in the energy equation (2.38), the resulting equation is:

$$\gamma\frac{\rho_0^\Gamma}{\Gamma - 1}[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma})] = -\Lambda$$

or for a non-radiative flow ($\Lambda = 0$):

$$\gamma\frac{\rho_0^\Gamma}{\Gamma - 1}[(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma})] = 0$$

Ohmic dissipation then vanishes, in the limit of zero conductivity, or practically, in flows characterized by large resistivities.

The momentum equation (2.34) becomes:

$$\frac{1}{c}\frac{\partial}{\partial t}(\gamma^2\xi\rho_0 c\vec{u}) + \vec{\nabla}\cdot(\gamma^2\xi\rho_0(\vec{u}\otimes\vec{u})) = -\vec{\nabla}P + \frac{J^0}{c}(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) - \frac{\gamma\Lambda\vec{u}}{c}$$

2.4 The Equations of Steady-State Axisymmetric RRMHD

Upon the hypothesis that the properties of the system flow-electromagnetic field change on timescales much larger than the timescales of interest, its behaviour can be assumed to be time-independent ($\partial_t = 0$), and the aforementioned equations can reduce to:

Ohm's Law

$$\vec{J} + \gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot\vec{J}}{c} - J^0) = \sigma\gamma(\vec{E} + \frac{\vec{u}\times\vec{B}}{c}) \tag{2.22}$$

Maxwell's Equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c}J^0 \tag{2.27}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \tag{2.40}$$

$$\vec{\nabla}\times\vec{E} = 0 \tag{2.41}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{2.30}$$

Conservation of Mass, Momentum, and Energy:

$$\vec{\nabla}\cdot(\gamma\rho_0\vec{u}) = 0 \tag{2.42}$$

$$\gamma\rho_0(\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u}) = -\vec{\nabla}P + \frac{J^0\vec{E} + \vec{J}\times\vec{B}}{c} - \frac{\gamma\Lambda\vec{u}}{c} \tag{2.43}$$

$$\gamma\frac{\rho_0^{\Gamma}}{\Gamma-1}(\vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^{\Gamma}})=\gamma\vec{J}\cdot(\vec{E}+\frac{\vec{u}\times\vec{B}}{c})-\gamma J^0\frac{\vec{u}\cdot\vec{E}}{c}-\Lambda \tag{2.44}$$

The term:

$$\vec{\nabla}\cdot(\gamma^2\xi\rho_0(\vec{u}\otimes\vec{u}))$$

in the momentum conservation equation, may be written as:

$$\vec{\nabla}\cdot((\gamma\rho_0\vec{u})\otimes(\gamma\xi\vec{u}))=(\gamma\xi\vec{u})\vec{\nabla}\cdot(\gamma\rho_0\vec{u})+\gamma\rho_0(\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})=\gamma\rho_0(\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})$$

by use of the identity:

$$\vec{\nabla}(\vec{B}\otimes\vec{A})=\vec{A}(\vec{\nabla}\cdot\vec{B})+(\vec{B}\cdot\vec{\nabla})\vec{A}$$

and the continuity equation (2.42). After substituting equations (2.27) and (2.40) in equations (2.22), (2.43), and (2.44), the system of equations reduces even further to:

$$\vec{\nabla}\cdot(\gamma\rho_0\vec{u})=0 \tag{2.42}$$

$$\vec{\nabla}\cdot\vec{B}=0 \tag{2.31}$$

$$\vec{\nabla}\times\vec{E}=0 \tag{2.41}$$

$$(\vec{\nabla}\times\vec{B})+\gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot(\vec{\nabla}\times\vec{B})}{c}-\vec{\nabla}\cdot\vec{E})=\frac{4\pi\sigma}{c}\gamma(\vec{E}+\frac{\vec{u}\times\vec{B}}{c}) \tag{2.45}$$

$$\gamma\rho_0(\vec{u}\cdot\vec{\nabla})(\gamma\xi\vec{u})=-\vec{\nabla}P+\frac{(\vec{\nabla}\cdot\vec{E})\vec{E}+(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}-\frac{\gamma\Lambda\vec{u}}{c^2} \tag{2.46}$$

$$\gamma\frac{\rho_0^{\Gamma}}{\Gamma-1}(\vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^{\Gamma}})=\frac{\gamma c}{4\pi}(\vec{\nabla}\times\vec{B})\cdot(\vec{E}+\frac{\vec{u}\times\vec{B}}{c})-\gamma(\vec{\nabla}\cdot\vec{E})\frac{\vec{u}\cdot\vec{E}}{4\pi}-\Lambda \tag{2.47}$$

Equations (2.42), (2.41), and (2.30), along with the assumption of axisymmetry, allow for the velocity field, the magnetic field, and the electric field to be written, in cylindrical coordinates, as:

$$\vec{u} = \frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0\varpi} + u_\phi\hat{\phi} = \vec{u}_p + u_\phi\hat{\phi} \tag{2.48}$$

$$\vec{B} = \frac{\vec{\nabla}A \times \hat{\phi}}{\varpi} + B_\phi\hat{\phi} = \vec{B}_p + B_\phi\hat{\phi} \tag{2.49}$$

$$\vec{E} = -\vec{\nabla}\Phi = -\frac{\partial\Phi}{\partial\varpi}\hat{\varpi} - \frac{\partial\Phi}{\partial z}\hat{z} \tag{2.50}$$

where $\vec{u}_p$, $\vec{B}_p$ are the poloidal components (ϖ, z) of the velocity and the magnetic field. The ϕ component of $\vec{E}$, E_ϕ, is zero due to the azimuthal symmetry of the scalar potential Φ.

The function Ψ is the mass flux function, while A is the magnetic flux function. These functions can be defined due to the conservation of the respective fluxes:

$$\vec{\nabla} \cdot (4\pi\gamma\rho_0\vec{u}) = \vec{\nabla} \cdot (4\pi\gamma\rho_0\vec{u}_p) = 0$$
$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \vec{B}_p = 0$$

Since these vector fields have a zero divergence, and are both axisymmetric, they may be generated through the curl of the vector potentials $\vec{\Psi} = |\vec{\Psi}|\hat{\phi}$ and $\vec{A} = |\vec{A}|\hat{\phi}$ respectively.

According to Stokes' Theorem:

$$\int\int 4\pi\gamma\rho_0\vec{u}_p \cdot d\vec{S} = 2\pi\varpi \int \vec{\Psi} \cdot \hat{\phi}d\phi = 2\pi\varpi|\vec{\Psi}|$$
$$\int\int \vec{B}_p \cdot d\vec{S} = 2\pi\varpi \int \vec{A} \cdot \hat{\phi}d\phi = 2\pi\varpi|\vec{A}|$$

Ψ and A may then be defined as:

$$\Psi = \varpi|\vec{\Psi}| = \frac{1}{2\pi}\int\int 4\pi\gamma\rho_0\vec{u}_p \cdot d\vec{S}$$
$$A = \varpi|\vec{A}| = \frac{1}{2\pi}\int\int \vec{B}_p \cdot d\vec{S}$$

The poloidal components of $\vec{u}$ and $\vec{B}$ then are:

$$\vec{u}_p = \frac{\vec{\nabla} \times \vec{\Psi}}{4\pi\gamma\rho_0} = \frac{\vec{\nabla} \times (\frac{\Psi}{\varpi}\hat{\phi})}{4\pi\gamma\rho_0} = \frac{(\vec{\nabla}\Psi) \times \hat{\phi}}{4\pi\gamma\rho_0} \tag{2.51}$$
$$\vec{B}_p = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\frac{A}{\varpi}\hat{\phi}) = \frac{(\vec{\nabla}A) \times \hat{\phi}}{\varpi}$$

The electric field may be generated through the gradient of a scalar potential Φ, thanks to its independence on time t (equation (2.41)).

It must be noted that in the case of finite conductivity, the relation between the poloidal components of the magnetic field and the velocity:

$$\vec{u}_p = \frac{\Psi_A}{4\pi\gamma\rho_0}\vec{B}_p$$

where $\Psi_A = \dfrac{d\Psi}{dA} = const.$, is no longer valid. Due to the assumption of a finite conductivity, Ψ is not a function of A, as is in the case of an ideal steady-state flow with one ignorable coordinate. [?], [17]

The components of the current density four-vector, in accordance with Gauss' and Ampère's Laws (eqs (2.27) and (2.40)), are written as:

$$J^0 = \frac{c}{4\pi}\vec{\nabla}\cdot\vec{E} = -\frac{c}{4\pi}(\vec{\nabla}\cdot\vec{\nabla})\Phi = -\frac{c}{4\pi}\nabla^2\Phi \tag{2.52}$$

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = \frac{c}{4\pi}\vec{\nabla}\times[\vec{\nabla}\times(\frac{A}{\varpi}\hat{\phi})] + \frac{c}{4\pi}\vec{\nabla}\times\vec{B}_\varphi =$$
$$-\frac{c}{4\pi}\nabla^2(\frac{A}{\varpi}\hat{\phi}) + \frac{c}{4\pi}\vec{\nabla}\times\vec{B}_\phi$$

by use of the identity:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$$

and the magnetic field's vector potential axisymmetry:

$$\vec{\nabla}\cdot(\frac{A}{\varpi}\hat{\phi}) = 0$$

By use of the vector Laplacian definition in cylyndrical coordinates, the current density $\vec{J}$ becomes:

$$\vec{J} = \frac{c}{4\pi}[\frac{A}{\varpi^3} - \nabla^2(\frac{A}{\varpi})]\hat{\phi} + \frac{c}{4\pi}\vec{\nabla}\times\vec{B}_\phi$$

The term $\nabla^2(\dfrac{A}{\varpi})$ can be expressed as:

$$\nabla^2(\frac{A}{\varpi}) = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi\frac{\partial}{\partial\varpi}(\frac{A}{\varpi})) + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} = \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(-\frac{A}{\varpi} + \frac{\partial A}{\partial\varpi}) + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} =$$
$$\frac{A}{\varpi^3} - \frac{2}{\varpi^2}\frac{\partial A}{\partial\varpi} + \frac{1}{\varpi^2}\frac{\partial A}{\partial\varpi} + \frac{1}{\varpi}\frac{\partial^2 A}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} = \frac{A}{\varpi^3} - \frac{2}{\varpi^2}\frac{\partial A}{\partial\varpi} + \frac{1}{\varpi}\nabla^2 A$$

using the definition for the Laplacian of a scalar in cylindrical coordinates. [15]

Finally, the expression for the current density is:

$$\vec{J} = \frac{c}{4\pi}(\frac{2}{\varpi}\frac{\partial A}{\partial\varpi} - \nabla^2 A)\frac{\hat{\phi}}{\varpi} + \frac{c}{4\pi}\frac{\vec{\nabla}(\varpi B_\phi)\times\hat{\phi}}{\varpi} \tag{2.53}$$

After substituting (2.48)-(2.52) into Ohm's Law (eq. (2.22)), the resulting expression is:

$$(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)\frac{\hat{\phi}}{\varpi} + \frac{\vec{\nabla}(\varpi B_\phi) \times \hat{\phi}}{\varpi} +$$

$$\frac{\gamma^2}{c}(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0\varpi} + u_\phi\hat{\phi})[\frac{1}{c}(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0\varpi} + u_\phi\hat{\phi}) \cdot ((\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)\frac{\hat{\phi}}{\varpi} + \frac{\vec{\nabla}(\varpi B_\phi) \times \hat{\phi}}{\varpi}) + \nabla^2 \Phi] =$$

$$\frac{4\pi}{c}\sigma\gamma[-\vec{\nabla}\Phi + \frac{1}{c}(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0\varpi} + u_\phi\hat{\phi}) \times (\frac{\vec{\nabla}A \times \hat{\phi}}{\varpi} + B_\phi\hat{\phi})]$$

By defining the flow's resistivity η as:

$$\eta = \frac{1}{4\pi\sigma}$$

the factor 4π on the RHS cancels out, and Ohm's Law is expressed as:

$$(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)\frac{\hat{\phi}}{\varpi} + \frac{\vec{\nabla}(\varpi B_\phi) \times \hat{\phi}}{\varpi} +$$

$$\gamma^2(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0 c\varpi} + \frac{u_\phi}{c}\hat{\phi})[\frac{\vec{\nabla}\Psi \cdot \vec{\nabla}(\varpi B_\phi)}{4\pi\gamma\rho_0 c\varpi^2} + \frac{u_\phi}{c\varpi}(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A) + \nabla^2\Phi] =$$

$$\frac{\gamma}{\eta c}(-\vec{\nabla}\Phi + \frac{\vec{\nabla}\Psi \times \vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^2} + \frac{u_\phi}{c\varpi}\vec{\nabla}A - \frac{B_\phi}{c\varpi}\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0}) \tag{2.54}$$

by use of the identities:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D})$$

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = |\vec{A}\vec{B}\vec{D}|\vec{C} - |\vec{A}\vec{B}\vec{C}|\vec{D}$$

and the fact that the gradient of any of the scalar quantities is perpendicular to $\hat{\phi}$, due to their independence on the azimuthal angle ϕ.

$|\vec{A}\vec{B}\vec{C}|$ denotes the scalar triple product:

$$|\vec{A}\vec{B}\vec{C}| = \vec{A} \cdot (\vec{B} \times \vec{C})$$

Before the equations of energy and momentum conservation are derived, the steady-state material derivative $\vec{u} \cdot \vec{\nabla}$ must be expressed in terms of Ψ and $\vec{\nabla}$:

$$\vec{u} \cdot \vec{\nabla} = \frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0\varpi} \cdot \vec{\nabla} \tag{2.55}$$

or

$$\vec{u} \cdot \vec{\nabla} = -\frac{\vec{\nabla}\Psi \times \vec{\nabla}}{4\pi\gamma\rho_0\varpi} \cdot \hat{\phi} \tag{2.56}$$

Also, the term $\vec{E}+\frac{\vec{u}\times\vec{B}}{c}$ is written as:

$$\vec{E}+\frac{\vec{u}\times\vec{B}}{c}=-\vec{\nabla}\Phi+\frac{\vec{\nabla}\Psi\times\vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^2}-B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi}+u_\phi\frac{\vec{\nabla}A}{c\varpi} \tag{2.57}$$

The equation of energy conservation (2.44), using (2.52) and (2.55), takes the following form:

$$-\frac{\rho_0^\Gamma}{\Gamma-1}\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}(\frac{P}{\rho_0^\Gamma})}{4\pi\rho_0\varpi}\right)\cdot\hat{\phi}=\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}\Phi}{(4\pi)^2\rho_0\varpi}\cdot\hat{\phi}\right)\nabla^2\Phi-\Lambda+$$

$$\frac{c}{4\pi}\left[\left(\frac{2}{\varpi}\frac{\partial A}{\partial\varpi}-\nabla^2 A\right)\frac{\hat{\phi}}{\varpi}+\frac{c}{4\pi}\frac{\vec{\nabla}(\varpi B_\phi)\times\hat{\phi}}{\varpi}\right]\cdot\left[-\vec{\nabla}\Phi+\frac{\vec{\nabla}\Psi\times\vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^2}-B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi}+u_\phi\frac{\vec{\nabla}A}{c\varpi}\right]\Rightarrow$$

$$\frac{\rho_0^\Gamma}{\Gamma-1}\left(\frac{\vec{\nabla}(\frac{P}{\rho_0^\Gamma})\times\vec{\nabla}\Psi}{4\pi\rho_0\varpi}\right)\cdot\hat{\phi}=\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}\Phi}{(4\pi)^2\rho_0\varpi}\cdot\hat{\phi}\right)\nabla^2\Phi+\frac{\gamma c}{4\pi}\left(\frac{2}{\varpi}\frac{\partial A}{\partial\varpi}-\nabla^2 A\right)\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^3}\right)\cdot\hat{\phi}+$$

$$\frac{\gamma c}{4\pi}\left[\left(-\vec{\nabla}\Phi-B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi}+u_\phi\frac{\vec{\nabla}A}{c\varpi}\right)\times\frac{\vec{\nabla}(\varpi B_\phi)}{\varpi}\right]\cdot\hat{\phi}-\Lambda \tag{2.58}$$

The energy conservation equation may also be written in terms of the specific enthalpy ξ:

$$\left(\frac{\rho_0 c^2\vec{\nabla}\xi\times\vec{\nabla}\Psi-\vec{\nabla}P\times\vec{\nabla}\Psi}{4\pi\rho_0\varpi}\right)\cdot\hat{\phi}=\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}\Phi}{(4\pi)^2\rho_0\varpi}\cdot\hat{\phi}\right)\nabla^2\Phi+\frac{\gamma c}{4\pi}\left(\frac{2}{\varpi}\frac{\partial A}{\partial\varpi}-\nabla^2 A\right)\left(\frac{\vec{\nabla}\Psi\times\vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^3}\right)\cdot\hat{\phi}+$$

$$\frac{\gamma c}{4\pi}\left[\left(-\vec{\nabla}\Phi-B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi}+u_\phi\frac{\vec{\nabla}A}{c\varpi}\right)\times\frac{\vec{\nabla}(\varpi B_\phi)}{\varpi}\right]\cdot\hat{\phi}-\Lambda \tag{2.59}$$

Lastly, the momentum conservation equation (2.43), with the help of (2.51) and the aforementioned vector identities, can be written as:

$$\left(\frac{\vec{\nabla}\Psi\times\hat{\phi}}{4\pi\varpi}\cdot\vec{\nabla}\right)\left(\xi\frac{\vec{\nabla}\Psi\times\hat{\phi}}{4\pi\rho_0\varpi}+\gamma\xi u_\phi\hat{\phi}\right)=-\vec{\nabla}P+$$

$$\frac{1}{4\pi}\left[(\nabla^2\Phi)\vec{\nabla}\Phi+\left(\frac{2}{\varpi}\frac{\partial A}{\partial\varpi}-\nabla^2 A\right)\frac{\vec{\nabla}A}{\varpi^2}-\frac{\vec{\nabla}A\times\vec{\nabla}(\varpi B_\phi)}{\varpi^2}-\frac{B_\phi}{\varpi}\vec{\nabla}(\varpi B_\phi)\right]-$$

$$\frac{\Lambda}{c^2}\left(\frac{\vec{\nabla}\Psi\times\hat{\phi}}{4\pi\rho_0\varpi}+\gamma u_\phi\hat{\phi}\right) \tag{2.60}$$

The unit vector defined as:

$$\hat{n}_A = \frac{\vec{\nabla} A}{|\vec{\nabla} A|}$$

is perpendicular to the poloidal magnetic field lines and defines the transfield direction, with regard to the magnetic field. Projecting the momentum equation (2.55) along the transfield direction, gives the transfield equation:

$$(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\varpi} \cdot \vec{\nabla})(\xi(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\rho_0\varpi}) \cdot \hat{n}_A) = -(\vec{\nabla}P) \cdot \hat{n}_A +$$

$$\frac{1}{4\pi}[(\nabla^2\Phi)\vec{\nabla}\Phi + (\frac{2}{\varpi}\frac{\partial A}{\partial\varpi} - \nabla^2 A)\frac{\vec{\nabla}A}{\varpi^2} - \frac{\vec{\nabla}A \times \vec{\nabla}(\varpi B_\phi)}{\varpi^2} - \frac{B_\phi}{\varpi}\vec{\nabla}(\varpi B_\phi)] \cdot \hat{n}_A -$$

$$\frac{\Lambda}{c^2}(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\rho_0\varpi}) \cdot \hat{n}_A \tag{2.61}$$

In the context of Ideal MHD, in which the poloidal magnetic field lines and the stream lines coincide, equation (2.55) can provide all necessary information regarding the flow's collimation and shape [**?**].

When studying a resistive flow on the other hand, in which case the plasma's conductivity is finite and the "frozen-in" condition does not apply, the stream lines will in general diverge from the magnetic field lines. In that case, a second "transfield" equation is required, one which expresses the equilibrium of forces perpendicular to the mass flux lines. This equation, in complete analogy with equation (2.60), is derived by projecting the momentum equation (2.54) on the direction perpendicular to the stream lines, as defined by the unit vector:

$$\hat{n}_\Psi = \frac{\vec{\nabla}\Psi}{|\vec{\nabla}\Psi|}$$

Similarly, the momentum equation projected on the unit vector:

$$\hat{n}_\Phi = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}$$

will give the equilibrium of forces perpendicular to the electric equipotential lines.

Of these three transfield equations, only the one expressing the equilibrium of forces perpendicular to the mass flux lines can provide information regarding the flow's collimation.

In conclusion, the equations that govern a steady-state, axisymmetric RRMHD flow are:

Ohm's Law:

$$(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)\frac{\hat{\phi}}{\varpi} + \frac{\vec{\nabla}(\varpi B_\phi) \times \hat{\phi}}{\varpi} +$$
$$\gamma^2(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\gamma\rho_0 c\varpi} + \frac{u_\phi}{c}\hat{\phi})[\frac{\vec{\nabla}\Psi \cdot \vec{\nabla}(\varpi B_\phi)}{4\pi\gamma\rho_0 c\varpi^2} + \frac{u_\phi}{c\varpi}(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A) + \nabla^2 \Phi] =$$
$$\frac{\gamma}{\eta c}(-\vec{\nabla}\Phi + \frac{\vec{\nabla}\Psi \times \vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^2} + \frac{u_\phi}{c\varpi}\vec{\nabla}A - \frac{B_\phi}{c\varpi}\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0}) \tag{2.49}$$

Energy conservation:

$$\frac{\rho_0^\Gamma}{\Gamma - 1}(\frac{\vec{\nabla}(\frac{P}{\rho_0^\Gamma}) \times \vec{\nabla}\Psi}{4\pi\rho_0\varpi})\cdot\hat{\phi} = (\frac{\vec{\nabla}\Psi \times \vec{\nabla}\Phi}{(4\pi)^2\rho_0\varpi}\cdot\hat{\phi})\nabla^2\Phi + \frac{\gamma c}{4\pi}(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)(\frac{\vec{\nabla}\Psi \times \vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^3})\cdot\hat{\phi} +$$
$$\frac{\gamma c}{4\pi}[(-\vec{\nabla}\Phi - B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi} + u_\phi\frac{\vec{\nabla}A}{c\varpi}) \times \frac{\vec{\nabla}(\varpi B_\phi)}{\varpi}] \cdot \hat{\phi} - \Lambda \tag{2.54}$$

or:

$$(\frac{\rho_0 c^2\vec{\nabla}\xi \times \vec{\nabla}\Psi - \vec{\nabla}P \times \vec{\nabla}\Psi}{4\pi\rho_0\varpi})\cdot\hat{\phi} = (\frac{\vec{\nabla}\Psi \times \vec{\nabla}\Phi}{(4\pi)^2\rho_0\varpi}\cdot\hat{\phi})\nabla^2\Phi + \frac{\gamma c}{4\pi}(\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)(\frac{\vec{\nabla}\Psi \times \vec{\nabla}A}{4\pi\gamma\rho_0 c\varpi^3})\cdot\hat{\phi} +$$
$$\frac{\gamma c}{4\pi}[(-\vec{\nabla}\Phi - B_\phi\frac{\vec{\nabla}\Psi}{4\pi\gamma\rho_0 c\varpi} + u_\phi\frac{\vec{\nabla}A}{c\varpi}) \times \frac{\vec{\nabla}(\varpi B_\phi)}{\varpi}] \cdot \hat{\phi} - \Lambda \tag{2.55}$$

Momentum Conservation:

$$(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\varpi} \cdot \vec{\nabla})(\xi\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\rho_0\varpi} + \gamma\xi u_\phi\hat{\phi}) = -\vec{\nabla}P +$$
$$\frac{1}{4\pi}[(\nabla^2\Phi)\vec{\nabla}\Phi + (\frac{2}{\varpi}\frac{\partial A}{\partial \varpi} - \nabla^2 A)\frac{\vec{\nabla}A}{\varpi^2} - \frac{\vec{\nabla}A \times \vec{\nabla}(\varpi B_\phi)}{\varpi^2} - \frac{B_\phi}{\varpi}\vec{\nabla}(\varpi B_\phi)] -$$
$$\frac{\Lambda}{c^2}(\frac{\vec{\nabla}\Psi \times \hat{\phi}}{4\pi\rho_0\varpi} + \gamma u_\phi\hat{\phi}) \tag{2.56}$$

Chapter 3

Radially Self-Similar RRMHD Flows

Radial self-similarity is a frequently encountered concept in both classical and relativistic Magnetohydrodynamics, especially in the search of disk-wind and jet solutions. This assumption was first applied by Bardeen & Berger (1978) [18], in their construction of a purely hydrodynamic model of a hot wind emanating from the disk of a spiral galaxy. It was subsequently popularized by Blandford & Payne (1982) [19], who via the addition of a rotating magnetic field, showed that an ideal, non-relativistic cold flow may be accelerated magnetocentrifugally from the plane of an accretion disk, by following magnetic field lines extending from the disk to large distances, thus introducing the Blandford-Payne mechanism of energy and angular momentum extraction. Vlahakis & Tsinganos (1998) [20] then presented a systematic way of constructing radially self-similar disk wind models. Exact semianalytic solutions for ideal, cold, relativistic, electromagnetically driven, radially self-similar jets and winds were derived independently by Li et al. (1992) [21] and Contopoulos & Lovelace (1994) [22]. This work was later generalized to a hot, relativistic, radially self-similar flow by Vlahakis & Königl (2003) [13].

The assumption of radial self-similarity, henceforth r self-similarity, is a powerful tool in the process of finding semianalytic solutions, as it offers a way, through the separations of variables, of simplifying the Partial Differential Equations (PDEs) of (R)MHD into Ordinary Differential Equations (ODEs) of one variable, in this case the polar angle θ. [20]

3.1 Separation of Variables

As a concept, r self-similarity can be simply understood as follows. Firstly, due to the axial symmetry of the flow and the electromagnetic field, any scalar quantity G describing the flow is a function of only the radial distance r and the polar angle θ. If the flow is assumed to be r self-similar, the quantity G will have the following property:
Suppose a conical surface $\theta = \theta_0$, intersecting with the contours of G correspoding to the values G_1 and G_2, at distance r_1 and r_2 respectively. The values G_1 and G_2 must be related through:

$$\frac{G_1}{G_2} = (\frac{r_1}{r_2})^p$$

Additionally, while moving along $\hat{\theta}$ on any spherical surface of radius r, the quantity G must have the exact same normalized profile, irregardles of the radial distance r.

The quantity G then can be expressed as a function of (r, θ), by separation of these variables as follows:

$$G(r, \theta) = r^p G_0(\theta) \tag{3.1}$$

where r^p is its radial dependence, while $G_0(\theta)$ is a function of only the polar angle θ. [23]

The starting point in expressing all of the scalar quantities of the system flow-electromagnetic field in the form (3.1) is the magnetic flux function $A(r,\theta)$. The magnetic flux function may be written as:

$$A(r,\theta) = B_0 \varpi_0^2 (\frac{r}{\varpi_0})^F A_0(\theta) \tag{3.2}$$

where B_0 and ϖ_0 are normalization constants with magnetic field and distance units respectively. Their introduction allows for the function $A_0(\theta)$ to be unitless, which will simplify the process of deriving the r self-similar equations. The index F is the magnetic flux function's radial dependence and is considered a free parameter.

The gradient of A is:

$$\vec{\nabla} A = \frac{\partial A}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial A}{\partial \theta}\hat{\theta} = \frac{1}{\varpi_0}[\frac{\partial A}{\partial(\frac{r}{\varpi_0})}\hat{r} + \frac{1}{(\frac{r}{\varpi_0})}\frac{\partial A}{\partial \theta}\hat{\theta}] =$$

$$B_0 \varpi_0 (\frac{r}{\varpi_0})^{F-1}[F A_0(\theta)\hat{r} + A_0'(\theta)\hat{\theta}]$$

The poloidal magnetic field, as given by eq. (2.49) with $\varpi = \varpi_0(\frac{r}{\varpi_0})\sin\theta$ is:

$$\vec{B}_p = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{F A_0(\theta)}{\sin\theta}(\hat{r}\times\hat{\phi}) + \frac{A_0'(\theta)}{\sin\theta}(\hat{\theta}\times\hat{\phi})] \Rightarrow$$

$$\vec{B}_p = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{A_0'(\theta)}{\sin\theta}\hat{r} - \frac{F A_0(\theta)}{\sin\theta}\hat{\theta}]$$

The magnetic field's ϕ component should follow the same scaling with the radial distance. It may the be written as:

$$\vec{B}_\phi = B_0(\frac{r}{\varpi_0})^{F-2} B_{\phi 0}(\theta)\hat{\phi}$$

Finally, the magnetic field is:

$$\vec{B} = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{A_0'(\theta)}{\sin\theta}\hat{r} - \frac{F A_0(\theta)}{\sin\theta}\hat{\theta} + B_{\phi 0}(\theta)\hat{\phi}] \tag{3.3}$$

All components of the electromagnetic field should follow the same scaling with r, a statement equivalent to the one that both the temporal and spatial components of the four-potential A^μ should scale the same with the radial distance, as well as have the same units.

The vector potential $\vec{A}$ is:

$$\vec{A} = \frac{A(r,\theta)}{\varpi}\hat{\phi} = \frac{A(r,\theta)}{r\sin\theta}\hat{\phi} = B_0\varpi_0(\frac{r}{\varpi_0})^{F-1}\frac{A_0(\theta)}{\sin\theta}\hat{\phi}$$

Consequently, the scalar potential Φ should be expressed as:

$$\Phi(r,\theta) = B_0\varpi_0(\frac{r}{\varpi_0})^{F-1}\Phi_0(\theta) \tag{3.4}$$

The resulting electric field is:

$$\vec{E} = -\vec{\nabla}\Phi(r,\theta) = -B_0(\frac{r}{\varpi_0})^{F-2}[(F-1)\Phi_0(\theta)\hat{r} + \Phi_0'(\theta)\hat{\theta}] \tag{3.5}$$

The separation of variables for all remaining quantities will be performed based on the demand that all terms of a given equation must have the same radial dependence, so that said equation can be reduced to an ODE of the polar angle θ. Ohm's Law (eq. (2.22)) suggests that:

$$[\vec{E}] = [\frac{\vec{u}\times\vec{B}}{c}]$$

with [ ] denoting the radial dependence of the enclosed quantity. Due to (3.3), (3.5), it follows that:

$$[u] = (\frac{r}{\varpi_0})^0$$

The flow's velocity then is independent of the radial distance r.

Assuming the stream function Ψ has a radial dependence n:

$$\Psi(r,\theta) = V_0\varpi_0^2(\frac{r}{\varpi_0})^n\Psi_0(\theta)$$

the velocity field may be written, according to (2.48), as:

$$\vec{u} = \frac{V_0}{4\pi\gamma\rho_0}(\frac{r}{\varpi_0})^{n-2}[\frac{\Psi_0'(\theta)}{\sin\theta}\hat{r} - \frac{n\Psi_0}{\sin\theta}\hat{\theta}] + cu_{\phi 0}(\theta)\hat{\phi} \tag{3.6}$$

where V_0 a normalization constant with appropriate units, and $\vec{u}_\phi = cu_\phi\hat{\phi}$. The Lorentz factor $\gamma = \dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$ must also be a function of only the polar angle θ.

As is evident from equation (3.6), in order to determine the stream function's radial dependence, the density $\rho_0(r,\theta)$ must first be defined. Assuming the density is:

$$\rho_0(r,\theta) = \rho(\frac{r}{\varpi_0})^m\rho_{00}(\theta)$$

where ρ a normalization constant with density units, its radial dependence m can be calculated from the momentum equation (2.43), by taking the specific enthalpy ξ to be a function of only θ.

Then:

$$[\gamma\rho_0\vec{u}\cdot\vec{\nabla}(\gamma\xi\vec{u})] = [\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}] \Rightarrow$$

$$\frac{c^2}{\varpi_0}\rho(\frac{r}{\varpi_0})^{m-1} = \frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}$$

where the magnetic field's curl's dependence is $F-3$. The constant ϖ_0 appears in both sides due to $\vec{\nabla}$, while the speed of light appears in order to preserve the velocity units, which appear squared in the former expression. The density then is:

$$\rho_0(r,\theta) = \frac{B_0^2}{4\pi c^2}(\frac{r}{\varpi_0})^{2F-4}\rho_{00}(\theta) \tag{3.7}$$

After substituting eq. (3.7) in eq. (3.6), the velocity becomes:

$$\vec{u} = \frac{V_0 c^2}{B_0^2}(\frac{r}{\varpi_0})^{n-2F+2}[\frac{\Psi_0'(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{r} - \frac{n\Psi_0(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{\theta}] + cu_{\phi 0}(\theta)\hat{\phi}$$

The stream function's dependence n must then be equal to $2F-2$, in order for the velocity to be independent of r. So:

$$\Psi(r,\theta) = \frac{B_0^2\varpi_0^2}{c}(\frac{r}{\varpi_0})^{2F-2}\Psi_0(\theta) \tag{3.8}$$

The constant V_0 was chosen to be: $V_0 = \dfrac{B_0^2\varpi^2}{c}$, so that the velocity may be expressed in units of c as:

$$\vec{u} = c[\frac{\Psi_0'(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{r} - \frac{(2F-2)\Psi_0(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{\theta} + u_{\phi 0}(\theta)\hat{\phi}] \tag{3.9}$$

The thermal pressure $P(r,\theta)$ can be defined in accordance with the definition of the specifc enthalpy as:

$$P(r,\theta) = \frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma}\rho_0(r,\theta)c^2 \Rightarrow$$

$$P(r,\theta) = \frac{B_0^2}{4\pi}(\frac{r}{\varpi_0})^{2F-4}\frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma}\rho_{00}(\theta) \tag{3.10}$$

or

$$P(r,\theta) = \frac{B_0^2}{4\pi}(\frac{r}{\varpi_0})^{2F-4}P_0(\theta) \tag{3.11}$$

with

$$P_0(\theta) = \frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma}\rho_{00}(\theta)$$

The radiative losses term $\Lambda(r,\theta)$ will be determined with the help of the energy equation (2.38). As can be seen from eq. (2.38)

$$[\Lambda(r,\theta)] = [\frac{c}{4\pi}(\vec{\nabla}\times\vec{B})\cdot\vec{E}] = \frac{cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}$$

In consequence, $\Lambda(r,\theta)$ finally is:

$$\Lambda(r,\theta) = \frac{cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\Lambda_0(\theta) \tag{3.12}$$

The resistivity $\eta(r,\theta)$ remains to be defined. From Ohm's Law (eq. (2.22)):

$$[\frac{\gamma\vec{E}}{4\pi\eta}] = [\frac{c}{4\pi}(\vec{\nabla}\times\vec{B})] \Rightarrow$$

$$[\eta] = \frac{\varpi_0}{c}(\frac{r}{\varpi_0})^{F-2}(\frac{r}{\varpi_0})^{3-F} = \frac{r}{c}$$

The resistivity is consequently:

$$\eta(r,\theta) = \frac{r}{c}\eta_0(\theta) \tag{3.13}$$

Summarizing, after performing the separation of variables, based on the assumption of r self-similarity, on the scalar quantities describing the system flow-electromagnetic field with the goal of expressing them in the form given by eq. (3.1), they were found to be:

$$A(r,\theta) = B_0\varpi_0^2(\frac{r}{\varpi_0})^F A_0(\theta) \tag{3.2}$$

$$\Psi(r,\theta) = \frac{B_0^2\varpi_0^2}{c}(\frac{r}{\varpi_0})^{2F-2}\Psi_0(\theta) \tag{3.8}$$

$$\Phi(r,\theta) = B_0\varpi_0(\frac{r}{\varpi_0})^{F-1}\Phi_0(\theta) \tag{3.4}$$

$$\rho_0(r,\theta) = \frac{B_0^2}{4\pi c^2}(\frac{r}{\varpi_0})^{2F-4}\rho_{00}(\theta) \tag{3.7}$$

$$P(r,\theta) = \frac{B_0^2}{4\pi}(\frac{r}{\varpi_0})^{2F-4}\frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma}\rho_{00}(\theta) \tag{3.10}$$

$$\Lambda(r,\theta) = \frac{cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\Lambda_0(\theta) \tag{3.12}$$

$$\eta(r,\theta) = \frac{r}{c}\eta_0(\theta) \tag{3.13}$$

The velocity and electromagnetic fields:

$$\vec{u} = c[\frac{\Psi_0'(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{r} - \frac{2(F-1)\Psi_0(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}\hat{\theta} + u_{\phi 0}(\theta)\hat{\phi}] \tag{3.9}$$

$$\vec{B} = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{A_0'(\theta)}{\sin\theta}\hat{r} - \frac{FA_0(\theta)}{\sin\theta}\hat{\theta} + B_{\phi 0}(\theta)\hat{\phi}] \tag{3.3}$$

$$\vec{E} = -\vec{\nabla}\Phi(r,\theta) = -B_0(\frac{r}{\varpi_0})^{F-2}[(F-1)\Phi_0(\theta)\hat{r} + \Phi_0'(\theta)\hat{\theta}] \tag{3.5}$$

The electric field's divergence is:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varpi_0}[\frac{1}{(\frac{r}{\varpi_0})^2}\frac{\partial}{\partial(\frac{r}{\varpi_0})}((\frac{r}{\varpi_0})^2 E_r) + \frac{1}{(\frac{r}{\varpi_0})\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta)] \Rightarrow$$

$$\vec{\nabla}\cdot\vec{E} = -\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}[F(F-1)\Phi_0(\theta) + \frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))] \tag{3.14}$$

The magnetic field's curl:

$$(\vec{\nabla}\times\vec{B})_r = \frac{1}{\varpi_0}(\frac{1}{(\frac{r}{\varpi_0})\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\phi)) \Rightarrow$$

$$(\vec{\nabla}\times\vec{B})_\theta = -\frac{1}{(\frac{r}{\varpi_0})}\frac{\partial}{\partial(\frac{r}{\varpi_0})}((\frac{r}{\varpi_0})B_\phi) \Rightarrow$$

$$(\vec{\nabla}\times\vec{B})_\phi = \frac{1}{(\frac{r}{\varpi_0})}(\frac{\partial}{\partial(\frac{r}{\varpi_0})}(\frac{r}{\varpi_0}B_\theta) - \frac{\partial B_r}{\partial\theta}) \Rightarrow$$

$$(\vec{\nabla}\times\vec{B})_r = \frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta)) \tag{3.15}$$

$$(\vec{\nabla}\times\vec{B})_\theta = -\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}(F-1)B_{\phi 0}(\theta) \tag{3.16}$$

$$(\vec{\nabla}\times\vec{B})_\phi = -\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}[\frac{F(F-1)A_0(\theta)}{\sin\theta} + \frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta})] \tag{3.17}$$

The thermal pressure gradient:

$$(\vec{\nabla} P)_r = \frac{1}{\varpi_0}\frac{\partial P}{\partial(\frac{r}{\varpi_0})} \Rightarrow$$

$$(\vec{\nabla} P)_\theta = \frac{1}{\varpi_0}\frac{1}{(\frac{r}{\varpi_0})}\frac{\partial P}{\partial \theta} \Rightarrow$$

$$(\vec{\nabla} P)_r = \frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\rho_{00}(\theta) \tag{3.18}$$

$$(\vec{\nabla} P)_\theta = \frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}[\xi'(\theta)\rho_{00}(\theta)+(\xi(\theta)-1)\rho'_{00}(\theta)] \tag{3.19}$$

The components of the Lorentz force acting on the fluid:

$$(\frac{(\vec{\nabla}\cdot\vec{E})\vec{E}}{4\pi})_r = \frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}(F-1)\Phi_0(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi'_0(\theta))] \tag{3.20}$$

$$(\frac{(\vec{\nabla}\cdot\vec{E})\vec{E}}{4\pi})_\theta = \frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\Phi'_0(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi'_0(\theta))] \tag{3.21}$$

$$(\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi})_r = -\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}[(F-1)B_{\phi 0}^2(\theta)+ \frac{FA_0(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A'_0(\theta)}{\sin\theta}))] \tag{3.22}$$

$$(\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi})_\theta = -\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}[\frac{B_{\phi 0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+ \frac{A'_0(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A'_0(\theta)}{\sin\theta}))] \tag{3.23}$$

$$(\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi})_\phi = -\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}[\frac{FA_0(\theta)}{\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))- \frac{(F-1)A'_0(\theta)B_{\phi 0}(\theta)}{\sin\theta}] \tag{3.24}$$

Some more calculations which will simplify the derivation of the r self-similar ODEs follow:
In Ohm's Law (eq. (2.45)):

$$\frac{\gamma\vec{u}}{c}\cdot(\vec{\nabla}\times\vec{B}) = \frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)}-$$
$$\gamma(\theta)u_{\phi 0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta})) \quad (3.25)$$

The terms appearing on the RHS of Ohm's Law (2.45), and of the energy equation (2.47):

$$\gamma\frac{\vec{u}\cdot\vec{E}}{c} = B_0(\frac{r}{\varpi_0})^{F-2}(F-1)\frac{2\Psi_0(\theta)\Phi_0'(\theta)-\Psi_0'(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)\sin\theta} \quad (3.26)$$

$$(\frac{\vec{u}\times\vec{B}}{c})_r = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{FA_0(\theta)u_{\phi 0}(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}] \quad (3.27)$$

$$(\frac{\vec{u}\times\vec{B}}{c})_\theta = B_0(\frac{r}{\varpi_0})^{F-2}[\frac{u_{\phi 0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi 0}(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}] \quad (3.28)$$

$$(\frac{\vec{u}\times\vec{B}}{c})_\phi = B_0(\frac{r}{\varpi_0})^{F-2}\frac{2(F-1)A_0'(\theta)\Psi_0(\theta)-FA_0(\theta)\Psi_0'(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin^2\theta} \quad (3.29)$$

$$\gamma(\vec{\nabla}\times\vec{B})\cdot(\vec{E}+\frac{\vec{u}\times\vec{B}}{c}) =$$
$$\frac{B_0^2}{\varpi_0}(\frac{r}{\varpi_0})^{2F-5}[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))(-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta}-$$
$$\frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta})-(F-1)B_{\phi 0}(\theta)(-\gamma(\theta)\Phi_0'(\theta)+\frac{\gamma(\theta)u_{\phi 0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta})-$$
$$\frac{2(F-1)\Psi_0(\theta)A_0'(\theta)-F\Psi_0'(\theta)A_0(\theta)}{\sin^2\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))] \quad (3.30)$$

The LHS of the energy equation may be written as:

$$\gamma\frac{\rho_0^\Gamma}{\Gamma-1}(\vec{u}\cdot\vec{\nabla})(\frac{P}{\rho_0^\Gamma}) = \gamma\rho_0c^2(\vec{u}\cdot\vec{\nabla})\xi-\gamma(\vec{u}\cdot\vec{\nabla})P$$

as shown in subsection (2.3.2) of the second chapter. It is preferable to choose this form of the energy equation, as the terms on the RHS of the last expression are easier to calculate. These terms are:

$$\gamma(\vec{u}\cdot\vec{\nabla})P = \frac{\gamma u_r}{\varpi_0}\frac{\partial P}{\partial(\frac{r}{\varpi_0})}+\frac{\gamma}{\varpi_0}\frac{u_\theta}{(\frac{r}{\varpi_0})}\frac{\partial P}{\partial\theta} \Rightarrow$$

$$\gamma(\vec{u}\cdot\vec{\nabla})P =$$

$$\frac{2cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}(\frac{(F-2)\Psi_0'(\theta)(\xi(\theta)-1)}{\sin\theta}+\frac{\xi'(\theta)}{\sin\theta}-\frac{(F-1)\Psi_0(\theta)\rho_{00}'(\theta)(\xi(\theta)-1)}{\rho_{00}(\theta)\sin\theta}) \tag{3.31}$$

$$\gamma\rho_0 c^2(\vec{u}\cdot\vec{\nabla})\xi = -\frac{2cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{(F-1)\Psi_0(\theta)}{\sin\theta}\xi'(\theta) \tag{3.32}$$

The calculation of the momentum equation's (2.46) inertia terms (LHS) can be simplified by mutiplying said equation by $\xi(\theta)$ and rewritting the LHS as:

$$\xi\gamma\rho_0(\vec{u}\cdot\vec{\nabla})(\xi\gamma\vec{u}) = \rho_0\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2}) + \rho_0[\vec{\nabla}\times(\xi\gamma\vec{u})]\times(\xi\gamma\vec{u})$$

by use of the identity:

$$(\vec{A}\cdot\vec{\nabla})\vec{A} = \vec{\nabla}(\frac{A^2}{2}) - \vec{A}\times(\vec{\nabla}\times\vec{A})$$

So:

$$\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2}) = \frac{1}{\varpi_0}\frac{1}{(\frac{r}{\varpi_0})}\frac{d}{d\theta}(\frac{\xi^2\gamma^2u^2}{2})\hat{\theta} \Rightarrow$$

$$\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2}) = \frac{c^2}{\varpi_0}(\frac{r}{\varpi_0})^{-1}\frac{d}{d\theta}[\frac{\xi^2(\theta)}{2}((\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta})^2+(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2+\gamma^2(\theta)u_{\phi0}^2(\theta))]\hat{\theta} \tag{3.33}$$

$$(\vec{\nabla}\times(\xi\gamma\vec{u}))_r = \frac{c}{\varpi_0}(\frac{r}{\varpi_0})^{-1}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi0}(\theta)) \tag{3.34}$$

$$(\vec{\nabla}\times(\xi\gamma\vec{u}))_\theta = -\frac{c}{\varpi_0}(\frac{r}{\varpi_0})^{-1}\xi(\theta)\gamma(\theta)u_{\phi0}(\theta) \tag{3.35}$$

$$(\vec{\nabla}\times(\xi\gamma\vec{u}))_\phi = -\frac{c}{\varpi_0}(\frac{r}{\varpi_0})^{-1}[\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}+\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta})] \tag{3.36}$$

$$[(\vec{\nabla}\times(\xi\gamma\vec{u}))\times(\xi\gamma\vec{u})]_r =$$

$$-\frac{c^2}{\varpi_0}(\frac{r}{\varpi_0})^{-1}[\xi^2(\theta)\gamma^2(\theta)u_{\phi0}^2(\theta)+\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}(\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}+\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}))] \tag{3.37}$$

$$[(\vec{\nabla}\times(\xi\gamma\vec{u}))\times(\xi\gamma\vec{u})]_\theta =$$

$$-\frac{c^2}{\varpi_0}(\frac{r}{\varpi_0})^{-1}\xi(\theta)[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}+\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}))+$$

$$\frac{\gamma(\theta)u_{\phi0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi0}(\theta))] \tag{3.38}$$

$[(\vec{\nabla}\times(\xi\gamma\vec{u}))\times(\xi\gamma\vec{u})]_\phi =$

$$\frac{c^2}{\varpi_0}(\frac{r}{\varpi_0})^{-1}[\frac{\xi^2(\theta)\gamma(\theta)u_{\phi0}(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}-\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi0}(\theta))] \tag{3.39}$$

3.2 The r Self-Similar ODEs

By substituting the relations calculated in the last section in the steady-state equations (2.45), (2.46), (2.47), a 7×7 system of second order linear differential equations will be derived. Three of these equations correspond to the three components of Ohm's Law, with three more corresponding to the three components of the momentum equation. The last equation is derived from the energy equation (2.47).

The $\hat{r}$ component of Ohm's Law (2.45), using the definitions of the velocity and the electric field (3.9) and (3.5), as well as the relations (3.14), (3.15), (3.25), and (3.27):

$$[(\vec{\nabla}\times\vec{B})+\gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot(\vec{\nabla}\times\vec{B})}{c}-\vec{\nabla}\cdot\vec{E})]_r=\frac{\gamma}{\eta c}[\vec{E}+\frac{\vec{u}\times\vec{B}}{c}]_r\Rightarrow$$

$$\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+$$
$$\frac{2(F-1)^2\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta}-\gamma(\theta)u_{\phi0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+$$
$$F(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]=$$
$$\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{\gamma(\theta)}{\eta_0(\theta)}[-(F-1)\Phi_0(\theta)+\frac{FA_0(\theta)u_{\phi0}(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}]\Rightarrow$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+\frac{2(F-1)^2\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta}-$$
$$\gamma(\theta)u_{\phi0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+F(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-$$
$$\frac{1}{\eta_0(\theta)}[-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{FA_0(\theta)\gamma(\theta)u_{\phi0}(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta}]=0 \tag{3.40}$$

The $\hat{\theta}$ component, with the help of (3.5), (3.9), (3.14), (3.16), (3.25), and (3.28):

$$[(\vec{\nabla}\times\vec{B})+\gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot(\vec{\nabla}\times\vec{B})}{c}-\vec{\nabla}\cdot\vec{E})]_\theta=\frac{\gamma}{\eta c}[\vec{E}+\frac{\vec{u}\times\vec{B}}{c}]_\theta\Rightarrow$$

$$-\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}(F-1)B_{\phi 0}(\theta)-\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+$$
$$\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-\gamma(\theta)u_{\phi 0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+$$
$$F(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]=$$
$$\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{\gamma(\theta)}{\eta_0(\theta)}[-\Phi_0'(\theta)+\frac{u_{\phi 0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi 0}(\theta)}{\gamma(\theta)\rho_{00}(\theta)\sin\theta}]\Rightarrow$$

$$-(F-1)B_{\phi 0}(\theta)-\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-$$
$$\gamma(\theta)u_{\phi 0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+F(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-$$
$$\frac{1}{\eta_0(\theta)}[-\gamma(\theta)\Phi_0'(\theta)+\frac{\gamma(\theta)u_{\phi 0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}]=0 \tag{3.41}$$

The $\hat{\phi}$ component, by use of (3.5), (3.9), (3.14), (3.17), (3.25), and (3.29):

$$[(\vec{\nabla}\times\vec{B})+\gamma^2\frac{\vec{u}}{c}(\frac{\vec{u}\cdot(\vec{\nabla}\times\vec{B})}{c}-\vec{\nabla}\cdot\vec{E})]_\phi=\frac{\gamma}{\eta c}[\vec{E}+\frac{\vec{u}\times\vec{B}}{c}]_\phi\Rightarrow$$
$$-\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))$$
$$\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\gamma(\theta)u_{\phi 0}(\theta)[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-$$
$$\gamma(\theta)u_{\phi 0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+F(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]=$$
$$\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}\frac{\gamma(\theta)}{\eta_0(\theta)}\frac{2(F-1)\Psi_0(\theta)A_0'(\theta)-F\Psi_0'(\theta)A_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}\Rightarrow$$
$$-\frac{F(F-1)A_0(\theta)}{\sin\theta}-\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta})+\gamma(\theta)u_{\phi 0}(\theta)[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+$$
$$\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-\gamma(\theta)u_{\phi 0}(\theta)(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+F(F-1)\gamma(\theta)\Phi_0(\theta)+$$
$$\frac{\gamma(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-\frac{1}{\eta_0(\theta)}\frac{2(F-1)\Psi_0(\theta)A_0'(\theta)-F\Psi_0'(\theta)A_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}=0 \tag{3.42}$$

The energy equation (2.47), with the help of (3.12), (3.26), (3.30), (3.31), and (3.32):

$$\gamma\rho_0 c^2(\vec{u}\cdot\vec{\nabla})\xi-\gamma(\vec{u}\cdot\vec{\nabla})P=\frac{\gamma c}{4\pi}(\vec{\nabla}\times\vec{B})\cdot(\vec{E}+\frac{\vec{u}\times\vec{B}}{c})-\gamma(\vec{\nabla}\cdot\vec{E})\frac{\vec{u}\cdot\vec{E}}{4\pi}-\Lambda\Rightarrow$$

$$-\frac{2cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{(F-1)\Psi_0(\theta)}{\sin\theta}\xi'(\theta)-$$
$$\frac{2cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}\frac{(F-2)\Psi_0'(\theta)(\xi(\theta)-1)-(F-1)\Psi_0(\theta)(\rho_{00}'(\theta)(\xi(\theta)-1)+\rho_{00}(\theta)\xi'(\theta))}{\rho_{00}(\theta)\sin\theta}=$$
$$\frac{cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}(\frac{r}{\varpi_0})^{2F-5}[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))(-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi0}(\theta)A_0(\theta)}{\sin\theta}-$$
$$\frac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta})-(F-1)B_{\phi0}(\theta)(-\gamma(\theta)\Phi_0'(\theta)+\frac{\gamma(\theta)u_{\phi0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta})-$$
$$\frac{2(F-1)\Psi_0(\theta)A_0'(\theta)-F\Psi_0'(\theta)A_0(\theta)}{\sin^2\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))]+$$
$$B_0(\frac{r}{\varpi_0})^{2F-5}(F-1)\frac{2\Psi_0(\theta)\Phi_0'(\theta)-\Psi_0'(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{B_0}{\varpi_0}(\frac{r}{\varpi_0})^{F-3}[F(F-1)\Phi_0(\theta)+$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-\frac{cB_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\Lambda_0(\theta)\Rightarrow$$

$$\frac{\Gamma-1}{\Gamma}\frac{2(F-1)\Psi_0(\theta)(\xi(\theta)-1)}{\sin\theta}\frac{\rho_{00}'(\theta)}{\rho_{00}(\theta)}-\frac{\Gamma-1}{\Gamma}\frac{(2F-4)\Psi_0'(\theta)(\xi(\theta)-1)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)}{\Gamma\sin\theta}\xi'(\theta)-$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))(-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi0}(\theta)A_0(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta})+$$
$$(F-1)B_{\phi0}(\theta)(-\gamma(\theta)\Phi_0'(\theta)+\frac{\gamma(\theta)u_{\phi0}(\theta)A_0'(\theta)}{\sin\theta}-\frac{\Psi_0'(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta})+$$
$$\frac{2(F-1)\Psi_0(\theta)A_0'(\theta)-F\Psi_0'(\theta)A_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))-$$
$$\frac{2\Psi_0(\theta)\Phi_0'(\theta)-\Psi_0'(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)\sin\theta}(F-1)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]+\Lambda_0(\theta)=0 \tag{3.43}$$

The $\hat{r}$ component of the momentum equation (2.47), using (3.18), (3.20), (3.22), (3.37), and the definitions (3.9), (3.12):

$$[\rho_0\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2})+\rho_0[\vec{\nabla}\times(\xi\gamma\vec{u})]\times(\xi\gamma\vec{u})]_r = -[\xi(\vec{\nabla}P)+\xi\frac{(\vec{\nabla}\cdot\vec{E})+(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}-\xi\frac{\gamma\Lambda\vec{u}}{c^2}]_r \Rightarrow$$

$$-\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}[\xi^2(\theta)\rho_{00}(\theta)\gamma^2(\theta)u_{\phi0}^2(\theta)+\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\sin\theta}(\frac{2(F-1)\xi(\theta)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}+$$
$$\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}))] = -\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}(2F-4)\xi(\theta)\rho_{00}(\theta)(\xi(\theta)-1)+$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}(F-1)\xi(\theta)\Phi_0(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\xi(\theta)[(F-1)B_{\phi0}^2(\theta)+\frac{FA_0(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))]-$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\xi(\theta)\Psi_0'(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$$

$$-\xi(\theta)\rho_{00}(\theta)\gamma^2(\theta)u_{\phi0}^2(\theta)-\frac{2(F-1)\Psi_0(\theta)}{\sin\theta}(\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}+\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}))+$$
$$\frac{\Gamma-1}{\Gamma}(2F-4)\rho_{00}(\theta)(\xi(\theta)-1)-(F-1)\Phi_0(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]+$$
$$(F-1)B_{\phi0}^2(\theta)+\frac{FA_0(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))+\frac{\Psi_0'(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta}=0 \quad (3.44)$$

The $\hat{\theta}$ component, with the help of (3.19), (3.21), (3.23), (3.33), (3.38), and (3.9), (3.12):

$$[\rho_0\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2})+\rho_0[\vec{\nabla}\times(\xi\gamma\vec{u})]\times(\xi\gamma\vec{u})]_r = -[\xi(\vec{\nabla}P)+\xi\frac{(\vec{\nabla}\cdot\vec{E})+(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}-\xi\frac{\gamma\Lambda\vec{u}}{c^2}]_\theta \Rightarrow$$

$$\frac{B_0^2}{8\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\rho_{00}(\theta)\frac{d}{d\theta}[\xi^2(\theta)((\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta})^2+(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2+\gamma^2(\theta)u_{\phi0}^2(\theta))]-$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\rho_{00}(\theta)\xi(\theta)[\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}(\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}+\frac{d}{d\theta}(\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}))+$$
$$\frac{\gamma(\theta)u_{\phi0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi0}(\theta))] = -\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{\Gamma-1}{\Gamma}\xi(\theta)[\xi'(\theta)\rho_{00}(\theta)+$$
$$(\xi(\theta)-1)\rho_{00}'(\theta)]+\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\xi(\theta)\Phi_0'(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]-$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\xi(\theta)[\frac{B_{\phi0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+\frac{A_0'(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))]+$$
$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\frac{2(F-1)\xi(\theta)\Psi_0(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$$

$$-\frac{\rho_{00}(\theta)\gamma(\theta)u_{\phi0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi0}(\theta)) - \frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{\Psi_0'(\theta)}{\sin\theta}+$$
$$\frac{\rho_{00}(\theta)}{2\xi(\theta)}\frac{d}{d\theta}[(\frac{2(F-1)\xi(\theta)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2+\xi^2(\theta)\gamma^2(\theta)u_{\phi0}^2(\theta)]+$$
$$\frac{\Gamma-1}{\Gamma}[\xi'(\theta)\rho_{00}(\theta)+(\xi(\theta)-1)\rho_{00}'(\theta)]-\Phi_0'(\theta)[F(F-1)\Phi_0(\theta)+\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Phi_0'(\theta))]+$$
$$\frac{B_{\phi0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi0}(\theta))+\frac{A_0'(\theta)}{\sin\theta}(\frac{F(F-1)A_0(\theta)}{\sin\theta}+\frac{d}{d\theta}(\frac{A_0'(\theta)}{\sin\theta}))-\frac{2(F-1)\Psi_0(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta} = 0 \quad (3.45)$$

The $\hat{\phi}$ component, by use of (3.24), (3.39), (3.9), (3.12):

$$[\rho_0\vec{\nabla}(\frac{\xi^2\gamma^2u^2}{2})+\rho_0[\vec{\nabla}\times(\xi\gamma\vec{u})]\times(\xi\gamma\vec{u})]_r = -[\xi(\vec{\nabla}P)+\xi\frac{(\vec{\nabla}\cdot\vec{E})+(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}-\xi\frac{\gamma\Lambda\vec{u}}{c^2}]_\phi \Rightarrow$$

$$\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\rho_{00}\xi(\theta)(\theta)[\gamma(\theta)u_{\phi 0}(\theta)\frac{\xi(\theta)\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta}-\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi 0}(\theta))] =$$

$$-\frac{B_0^2}{4\pi\varpi_0}(\frac{r}{\varpi_0})^{2F-5}\xi(\theta)[\frac{FA_0(\theta)}{\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))-\frac{(F-1)A_0'(\theta)B_{\phi 0}(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)\Lambda_0(\theta)] \Rightarrow$$

$$\gamma(\theta)u_{\phi 0}(\theta)\frac{\xi(\theta)\Psi_0'(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)}{\sin^2\theta}\frac{d}{d\theta}(\sin\theta\xi(\theta)\gamma(\theta)u_{\phi 0}(\theta))+$$

$$\frac{FA_0(\theta)}{\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))-\frac{(F-1)A_0'(\theta)B_{\phi 0}(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)\Lambda_0(\theta)=0 \tag{3.46}$$

Equations (3.40)-(3.46) comprise the 7×7 system of ODEs that describe the behaviour of a steady-state, axisymmetric, radially self-similar, magnetized flow with a finite conductivity. One additional equation for the flow's Lorentz factor $\gamma(\theta)$ is required. This equation will be derived through the Lorentz invariant: $U_\mu U^\mu$.

This is:

$$U_\mu U^\mu = -c^2 \Rightarrow -\gamma^2c^2+\gamma^2u^2=-c^2 \Rightarrow$$

$$\gamma^2 = 1+\gamma^2\frac{u^2}{c^2} \Rightarrow \gamma = \sqrt{1+(\frac{\gamma u_r}{c})^2+(\frac{\gamma u_\theta}{c})^2+(\frac{\gamma u_\phi}{c})^2}$$

By use of (3.9), the last relation can be written as:

$$\gamma(\theta) = \sqrt{1+(\frac{\Psi_0'(\theta)}{\rho_{00}(\theta)\sin\theta})^2+(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2+(\gamma(\theta)u_{\phi 0}(\theta))^2} \tag{3.47}$$

In conclusion, the full set of equations to be solved is the 7×7 system (3.40)- (3.46), together with equation (3.47) which determines the flow's Lorentz factor as a function of θ, by relating it to the flow's four-velocity's spatial components.

3.3 Specification of the Flow's Critical Surfaces

The study of this system of ordinary differential equations with respect to its singular points is a necessary step to be taken before attepting to determine any solutions, especially if one's goal is to determine solutions describing outflows from disks and central objects. The reason for this necessity is that the number of singular points of this system of ODEs, as well as their positions θ_c^i, coincide with the number and positions of the conical critical surfaces of the flow. As will be shown in greater detail, the polar angles θ_{cr}^i at which the singular points are located are those angles at which the flow's $\hat{\theta}$ four-velocity becomes equal to the propagation velocities of the different types of perturbations that may appear and propagate in a magnetized flow. In order for a solution describing a radially self-similar, relativistic outflow to have physical meaning, its $\hat{\theta}$ four-velocity should in general become equal to these characteristic velocities, modified properly as a result of r self-similarity, at the respective conical critical surfaces, and take larger values downstream. In the case of an ideal flow these perturbations are the Alfvén waves and the slow and fast magnetosonic waves. So, solutions describing an ideal outflow in general cross these three critical points.

The crossing of the modified fast magnetosonic critical surface is most important, as it is the limiting surface for the causal connection of the flow's source to the flow at infinity (large distances). Any perturbation which starts beyond this surface, for example at infinite distance form the source, and propagates downstream along $\hat{\theta}$, will never be able to reach and affect the base of the flow, as due to the correct crossing of this critical surface, the flow has a velocity larger than the largest possible propagation velocity of a perturbation in an ideal plasma, which is the modified fast magnetosonic velocity. The term modified arises from the fact that the regular expression for the propagation velocity of fast magnetosonic waves is modified, as a result of the geometry of a radially self-similar flow. [13], [16], [24]

In the case of a radially self-similar, relativistic, magnetized flow of finite conductivity, the number and type of critical points changes. From a mathematical standpoint, this is due to the fact that the system of equations describing such a flow is fundamentally different to the one describing a flow of infinite conductivity.

In order to specify the singular points of the system of equations derived in the previous section, it should first be simplified into a system of linear first order ODEs:

$$Y' = F(\theta, Y)$$

with Y the column vector of the system's unknowns, and F a function linear with respect to Y. The system may then be further simplified into the form:

$$Y' = DY + P \tag{3.48}$$

with D the matrix of coefficients, and P a column vector.

Due to the complex forms of the equations comprising this system, it is much simpler for it to be written in the following form:

$$GY' = H \tag{3.49}$$

in contrast to the usual way of writing linear systems of first order ODEs in matrix form (3.48).

The system's singular points are the points at which the derivatives of the unknown functions become indefinite. If the elements of Y' are written in the form $\frac{N}{D}$, then at a singular point:

$$\frac{dY_i}{d\theta} = \frac{0}{0}$$

The singular points can be determined by examining the determinant of the square matrix G. Specifically, the angles θ^i_{cr} at which the determinant vanishes are the locations of the systems's singular points.

Before attempting to write the system of equations in the form (3.49), the equations must be transformed into first order ODEs by properly defining some of the terms appearing in them as new unknown functions. After that is done, the elements of Y, G, and H may be determined. By examining equations (3.40)-(3.46), it can be made clear that these new functions must be defined as:

$$A_1(\theta) = \frac{A_0'(\theta)}{\sin\theta} \tag{3.50}$$

$$\Psi_1(\theta) = \frac{\Psi_0'(\theta)}{\sin\theta} \tag{3.51}$$

$$\Phi_1(\theta) = \sin\theta \Phi_0'(\theta) \tag{3.52}$$

Equations (3.40)-(3.46) then take the following forms:

$\hat{r}$ Ohm's Law component (eq. (3.40)):

$$\left(\frac{1}{\sin\theta}+\frac{\Psi_1^2(\theta)}{\rho_{00}^2(\theta)\sin\theta}\right)\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta)) - \frac{\gamma(\theta)u_{\phi 0}(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)}\frac{dA_1(\theta)}{d\theta} + \frac{\gamma(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{d\Phi_1(\theta)}{d\theta} =$$
$$-\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}\left[\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta} - \frac{F(F-1)\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta} + F(F-1)\gamma(\theta)\Phi_0(\theta)\right] +$$
$$\frac{1}{\eta_0(\theta)}\left[-(F-1)\gamma(\theta)\Phi_0(\theta) + \frac{FA_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta} - \frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}\right] \tag{3.53}$$

$\hat{\theta}$ Ohm's Law component (eq. (3.41)):

$$-\frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)}{\rho_{00}^2(\theta)\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+\frac{2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{dA_1(\theta)}{d\theta}-$$

$$\frac{2(F-1)\gamma(\theta)\Psi_0(\theta)}{\rho_{00}(\theta)\sin^2\theta}\frac{d\Phi_1(\theta)}{d\theta}=(F-1)B_{\phi 0}(\theta)+\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}[\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-$$

$$\frac{F(F-1)\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta}+F(F-1)\gamma(\theta)\Phi_0(\theta)]+$$

$$\frac{1}{\eta_0(\theta)}[-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)A_1(\theta)-\frac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)}] \quad (3.54)$$

$\hat{\phi}$ Ohm's Law component (eq. (3.42)):

$$\frac{\gamma(\theta)u_{\phi 0}(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+(-1-\gamma^2(\theta)u_{\phi 0}^2(\theta))\frac{dA_1}{d\theta}+\frac{\gamma^2(\theta)u_{\phi 0}(\theta)}{\sin\theta}\frac{d\Phi_1(\theta)}{d\theta}=$$

$$\frac{F(F-1)A_0(\theta)}{\sin\theta}-\gamma(\theta)u_{\phi 0}(\theta)[\frac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}-\frac{F(F-1)\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta}+$$

$$F(F-1)\gamma(\theta)\Phi_0(\theta)]+\frac{1}{\eta_0(\theta)}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} \quad (3.55)$$

The energy equation (3.43):

$$-\frac{1}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}]\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))+$$

$$\frac{\Gamma-1}{\Gamma}\frac{2(F-1)(\xi(\theta)-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{d\rho_{00}(\theta)}{d\theta}-\frac{2(F-1)\Psi_0(\theta)}{\Gamma\sin\theta}\frac{d\xi(\theta)}{d\theta}+$$

$$\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{dA_1(\theta)}{d\theta}-\frac{F-1}{\sin\theta}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta}-\frac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)})\frac{d\Phi_1(\theta)}{d\theta}=$$

$$\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)+\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta}-\Psi_1(\theta)\Phi_0(\theta))-$$

$$(F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)A_1(\theta)-\frac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)})-$$

$$\frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}-\Lambda_0(\theta) \quad (3.56)$$

$\hat{r}$ momentum equation component (eq. (3.44)):

$$\frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}^2(\theta)\sin\theta}\frac{d\rho_{00}(\theta)}{d\theta} - \frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{d\xi(\theta)}{d\theta} + \frac{FA_0(\theta)}{\sin\theta}\frac{dA_1(\theta)}{d\theta} -$$

$$\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{d\Psi_1(\theta)}{d\theta} - \frac{(F-1)\Phi_0(\theta)}{\sin\theta}\frac{d\Phi_1(\theta)}{d\theta} = \xi(\theta)\rho_{00}(\theta)\gamma^2(\theta)u_{\phi 0}^2(\theta) +$$

$$\frac{4(F-1)^2\Psi_0^2(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\rho_{00}(\theta) -$$

$$\frac{F^2(F-1)A_0^2(\theta)}{\sin^2\theta} - (F-1)B_{\phi 0}^2(\theta) + F(F-1)^2\Phi_0^2(\theta) - \frac{\Psi_1(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)} \quad (3.57)$$

$\hat{\theta}$ momentum equation component (eq. (3.45)):

$$\frac{B_{\phi 0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta)) + (\xi(\theta)\gamma(\theta)u_{\phi 0}(\theta) - \xi(\theta)\gamma(\theta)u_{\phi 0}(\theta))\frac{d(\gamma(\theta)u_{\phi 0}(\theta))}{d\theta} +$$

$$[\frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma} - \xi(\theta)(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2]\frac{d\rho_{00}(\theta)}{d\theta} + [\frac{\Gamma-1}{\Gamma} + (\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2]\rho_{00}(\theta)\frac{d\xi(\theta)}{d\theta} +$$

$$A_1(\theta)\frac{dA_1(\theta)}{d\theta} - \frac{\Phi_1(\theta)}{\sin^2\theta}\frac{d\Phi_1(\theta)}{d\theta} = \rho_{00}(\theta)\xi(\theta)\gamma^2(\theta)u_{\phi 0}^2\theta\cot\theta + \frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta} -$$

$$\frac{4(F-1)^2\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}(\Psi_1(\theta) - \Psi_0(\theta)\frac{\cot\theta}{\sin\theta}) + \frac{F(F-1)\Phi_0(\theta)\Phi_1(\theta)}{\sin\theta} -$$

$$\frac{F(F-1)A_0(\theta)A_1(\theta)}{\sin\theta} + \frac{2(F-1)\Psi_0(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$$

$$\frac{B_{\phi 0}(\theta)}{\sin\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta)) + [\frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma} - \xi(\theta)(\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2]\frac{d\rho_{00}(\theta)}{d\theta}$$

$$+[\frac{\Gamma-1}{\Gamma} + (\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2]\rho_{00}(\theta)\frac{d\xi(\theta)}{d\theta} + A_1(\theta)\frac{dA_1(\theta)}{d\theta} - \frac{\Phi_1(\theta)}{\sin^2\theta}\frac{d\Phi_1(\theta)}{d\theta} =$$

$$\rho_{00}(\theta)\xi(\theta)\gamma^2(\theta)u_{\phi 0}^2\theta\cot\theta + \frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta} - \frac{4(F-1)^2\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}(\Psi_1(\theta) - \Psi_0(\theta)\frac{\cot\theta}{\sin\theta})$$

$$+\frac{F(F-1)\Phi_0(\theta)\Phi_1(\theta)}{\sin\theta} - \frac{F(F-1)A_0(\theta)A_1(\theta)}{\sin\theta} + \frac{2(F-1)\Psi_0(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)\sin\theta} \quad (3.58)$$

$\hat{\phi}$ momentum equation component (eq. (3.46)):

$$\frac{FA_0(\theta)}{\sin^2\theta}\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta)) - \frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\sin\theta}\frac{d(\gamma(\theta)u_{\phi 0}(\theta))}{d\theta} -$$
$$\frac{2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta}\frac{d\xi(\theta)}{d\theta} = 2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta)\frac{\cot\theta}{\sin\theta} -$$
$$\Psi_1(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta) + (F-1)A_1(\theta)B_{\phi 0}(\theta) - \gamma(\theta)u_{\phi 0}(\theta)\Lambda_0(\theta) \quad (3.59)$$

Equation (3.47) will then be rewritten as:

$$\gamma(\theta) = \sqrt{1 + (\frac{\Psi_1(\theta)}{\rho_{00}(\theta)})^2 + (\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2 + (\gamma(\theta)u_{\phi 0}(\theta))^2} \quad (3.60)$$

3.3.1 The Modified Sonic Critical Surface

In order to calculate the determinant of the matrix G, its elements must first be defined, by writing the system of equations (3.50-3.59) in the form (3.49). Firstly, the matrix Y is defined as the column vector:

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \\ Y_{10} \end{pmatrix} = \begin{pmatrix} A_0(\theta) \\ \Psi_0(\theta) \\ \Phi_0(\theta) \\ \sin\theta B_{\phi 0}(\theta) \\ \gamma(\theta)u_{\phi 0}(\theta) \\ \rho_{00}(\theta) \\ \xi(\theta) \\ A_1(\theta) \\ \Psi_1(\theta) \\ \Phi_1(\theta) \end{pmatrix} \quad (3.61)$$

The unknown functions of the system are the elements Y_1-Y_{10} of the column matrix Y, as those are defined in (3.61).

Having defined the matrix Y, and in consequence the matrix of the derivatives Y', the elements of G and H may then be defined by use of (3.49), and of the fact that each line of (3.49) corresponds to one equation of the system (3.50)-(3.59), in the order that these equations appear in the previous pages. Specifically, the first three lines of eq. (3.49), correspond to the definitions (3.50)-(3.52), while the fourth, fifth, and sixth line to the $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ components of Ohms' Law respectively (eqs. (3.53), (3.54), (3.55)).

The seventh line is the energy equation (3.56), while the eighth, ninth, and tenth lines the $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ components of the momentum equation (eqs. (3.57), (3.58), (3.59)). Equations (3.50)-(3.59) can then be written in the more compact form:

$$\sum_j G_{ij} Y'_j = H_i \tag{3.62}$$

The first three lines of G, as well as the first three elements of H, according to (3.50)-(3.52), are:

- $G_{11} = 1$, $G_{1j} = 0$, for $j \neq 1$, and $H_1 = \sin\theta A_1(\theta)$
- $G_{22} = 1$, $G_{2j} = 0$, for $j \neq 2$, and $H_2 = \sin\theta \Psi_1(\theta)$
- $G_{33} = 1$, $G_{3l} = 0$, for $j \neq 3$, and $H_3 = \dfrac{\Phi_1(\theta)}{\sin\theta}$

Lines 4, 5, and 6, according to (3.53)-(3.54):

- $G_{44} = \dfrac{1}{\sin\theta}(1 + \dfrac{\Psi_1^2(\theta)}{\rho_{00}^2(\theta)}) = \dfrac{1}{\sin\theta}(1 + (\dfrac{\gamma u_r}{c})^2)$, since $\dfrac{\Psi_1(\theta)}{\rho_{00}(\theta)} = \dfrac{\gamma u_r}{c}$, where $\dfrac{\gamma u_r}{c} = \dfrac{\Psi_1(\theta)}{\rho_{00}(\theta)}$, by use of (3.9) and (3.51)
- $G_{48} = -\dfrac{\gamma(\theta) u_{\phi 0}(\theta) \Psi_1(\theta)}{\rho_{00}(\theta)} = -\dfrac{\gamma u_\phi}{c}\dfrac{\gamma u_r}{c}$, with $\dfrac{\gamma u_\phi}{c} = \gamma(\theta) u_{\phi 0}(\theta)$, according to (3.9)
- $G_{410} = \dfrac{\gamma(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = \dfrac{\gamma}{\sin\theta}\dfrac{\gamma u_r}{c}$
- $G_{4j} = 0$, for $j \neq 4, 8, 10$
- $H_4 = -\dfrac{\Psi_1(\theta)}{\rho_{00}(\theta)}[\dfrac{2(F-1)^2\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta} - \dfrac{F(F-1)\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta} + F(F-1)\gamma(\theta)\Phi_0(\theta)] + \dfrac{1}{\eta_0(\theta)}[-(F-1)\gamma(\theta)\Phi_0(\theta) + \dfrac{FA_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta} - \dfrac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}] \Rightarrow$
  $H_4 = -\dfrac{\gamma u_r}{c}[(F-1)\dfrac{\gamma u_\phi}{c}B_{\theta 0} - (F-1)\dfrac{\gamma u_\theta}{c}B_{\phi 0} - F\gamma E_{r0}] + \dfrac{1}{\eta_0(\theta)}[\gamma E_{r0} - \dfrac{\gamma u_\phi}{c}B_{\theta 0} + \dfrac{\gamma u_\theta}{c}B_{\phi 0}]$
  where $E_{r0} = -(F-1)\Phi_0(\theta)$, $B_{\theta 0} = -\dfrac{FA_0(\theta)}{\sin\theta}$, $\dfrac{\gamma u_\theta}{c} = -\dfrac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}$, from (3.3), (3.5), (3.9), and (3.51)

- $G_{54} = -\dfrac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)}{\rho_{00}^2(\theta)\sin^2\theta} = \dfrac{\gamma u_r}{c}\dfrac{\gamma u_\theta}{c}$

- $G_{58} = \dfrac{2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta} = -\dfrac{\gamma u_\theta}{c}\dfrac{\gamma u_\phi}{c}$

- $G_{510} = -\dfrac{2(F-1)\gamma(\theta)\Psi_0(\theta)}{\rho_{00}(\theta)\sin^2\theta} = \dfrac{\gamma}{\sin\theta}\dfrac{\gamma u_\theta}{c}$

- $G_{5j} = 0$, for $j \neq 4,\ 8,\ 10$

- $H_5 = (F-1)B_{\phi0}(\theta) + \dfrac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}[\dfrac{2(F-1)^2\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta} - \dfrac{F(F-1)A_0(\theta)\gamma(\theta)u_{\phi0}(\theta)}{\sin\theta} + F(F-1)\gamma(\theta)\Phi_0(\theta)] + \dfrac{1}{\eta_0(\theta)}[-\dfrac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta} + \gamma(\theta)u_{\phi0}(\theta)A_1(\theta) - \dfrac{\Psi_1(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)}] \Rightarrow$
  $H_5 = (F-1)B_{\phi0} - \dfrac{\gamma u_\theta}{c}[(F-1)\dfrac{\gamma u_\phi}{c}B_{\theta0} - (F-1)\dfrac{\gamma u_\theta}{c}B_{\phi0} - F\gamma E_{r0}] + \dfrac{1}{\eta_0}[\gamma E_{\theta0} + \dfrac{\gamma u_\phi}{c}B_{r0} - \dfrac{\gamma u_r}{c}B_{\phi0}]$,
  where $E_{\theta0} = -\dfrac{\Phi_1(\theta)}{\sin\theta}$ and $Br_0 = A_1(\theta)$, according to (3.3), (3.5), (3.50), and (3.52)

- $G_{64} = \dfrac{\gamma(\theta)u_{\phi0}(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = \dfrac{1}{\sin\theta}\dfrac{\gamma u_r}{c}\dfrac{\gamma u_\phi}{c}$

- $G_{68} = -1 - \gamma^2(\theta)u_{\phi0}^2(\theta) = -1 - (\dfrac{\gamma u_\phi}{c})^2$

- $G_{610} = \dfrac{\gamma^2(\theta)u_{\phi0}(\theta)}{\sin\theta} = \dfrac{\gamma}{\sin\theta}\dfrac{\gamma u_\phi}{c}$

- $G_{6j} = 0$, for $j \neq 4,\ 8,\ 10$

- $H_6 = \dfrac{F(F-1)A_0(\theta)}{\sin\theta} - \gamma(\theta)u_{\phi0}(\theta)[\dfrac{2(F-1)^2\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta} - \dfrac{F(F-1)\gamma(\theta)u_{\phi0}(\theta)A_0(\theta)}{\sin\theta} + F(F-1)\gamma(\theta)\Phi_0(\theta)] + \dfrac{1}{\eta_0(\theta)}\dfrac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$
  $H_6 = -FB_{\theta0} - \dfrac{\gamma u_\phi}{c}[(F-1)\dfrac{\gamma u_\phi}{c}B_{\theta0} - (F-1)\dfrac{\gamma u_\theta}{c}B_{\phi0} - F\gamma E_{r0}] + \dfrac{1}{\eta_0}[\dfrac{\gamma u_r}{c}B_{\theta0} - \dfrac{\gamma u_\theta}{c}B_{r0}]$

The seventh line corresponding to the energy equation (3.56):

- $G_{74} = -\dfrac{1}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta) + \dfrac{F\gamma(\theta)u_{\phi0}(\theta)A_0(\theta)}{\sin\theta} - \dfrac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta}] = -\dfrac{1}{\sin\theta}[\gamma E_{r0} - \dfrac{\gamma u_\phi}{c}B_{\theta0} + \dfrac{\gamma u_\theta}{c}B_{\phi0}]$

- $G_{76} = \dfrac{\Gamma - 1}{\Gamma} \dfrac{2(F-1)\Psi_0(\theta)(\xi(\theta)-1)}{\rho_{00}(\theta)\sin\theta} = -\dfrac{(\Gamma-1)(\xi-1)}{\Gamma}\dfrac{\gamma u_\theta}{c}$
- $G_{77} = -\dfrac{2(F-1)\Psi_0(\theta)}{\Gamma \sin\theta} = \dfrac{\rho_{00}}{\Gamma}\dfrac{\gamma u_\theta}{c}$
- $G_{78} = \dfrac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = \dfrac{\gamma u_r}{c}B_{\theta 0} - \dfrac{\gamma u_\theta}{c}B_{r0}$
- $G_{710} = -\dfrac{F-1}{\sin\theta}(\dfrac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \dfrac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)}) =$
  $-\dfrac{1}{\sin\theta}(\dfrac{\gamma u_r}{c}E_{r0} + \dfrac{\gamma u_\theta}{c}E_{\theta 0})$
- $G_{7j} = 0$, for $j \neq 4, 6, 7, 8, 10$
- $H_7 = \dfrac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta) + \dfrac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\dfrac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta} -$
  $\Psi_1(\theta)\Phi_0(\theta)) - (F-1)B_{\phi 0}(\theta)(-\dfrac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta} + \gamma(\theta)u_{\phi 0}(\theta)A_1(\theta) - \dfrac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)}) -$
  $\dfrac{F(F-1)A_0(\theta)}{\sin\theta}\dfrac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} - \Lambda_0(\theta) \Rightarrow$
  $H_7 = (2F-4)\dfrac{(\Gamma-1)(\xi-1)\rho_{00}}{\Gamma}\dfrac{\gamma u_r}{c} - FE_{r0}(\dfrac{\gamma u_r}{c}E_{r0} - \dfrac{\gamma u_\theta}{c}E_{\theta 0}) -$
  $(F-1)B_{\phi 0}(\gamma E_{\theta 0} + \dfrac{\gamma u_\phi}{c}B_{r0} - \dfrac{\gamma u_r}{c}B_{\phi 0}) + (F-1)B_{\theta 0}(\dfrac{\gamma u_r}{c}B_{\theta 0} - \dfrac{\gamma u_\theta}{c}B_{r0}) -$
  Λ_0

The last three lines, which are the three components of the momentum equation:

- $G_{86} = \dfrac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}^2(\theta)\sin\theta} = -\dfrac{\gamma u_r}{c}\dfrac{\gamma u_\theta}{c}\xi$
- $G_{87} = -\dfrac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = \dfrac{\gamma u_r}{c}\dfrac{\gamma u_\theta}{c}\rho_{00}$
- $G_{88} = \dfrac{FA_0(\theta)}{\sin\theta} = -B_{\theta 0}$
- $G_{89} = -\dfrac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta} = \dfrac{\gamma u_\theta}{c}\xi$
- $G_{810} = -\dfrac{(F-1)\Phi_0(\theta)}{\sin\theta} = \dfrac{E_{r0}}{\sin\theta}$
- $G_{8j} = 0$, for $j \neq 6, 7, 8, 9, 10$
- $H_8 = \xi(\theta)\rho_{00}(\theta)\gamma^2(\theta)u_{\phi 0}^2(\theta) + \dfrac{4(F-1)^2\Psi_0^2(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \dfrac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-$
  $1)\rho_{00}(\theta) - \dfrac{F^2(F-1)A_0^2(\theta)}{\sin^2\theta} - (F-1)B_{\phi 0}^2 + F(F-1)^2\Phi_0^2(\theta) - \dfrac{\Psi_1(\theta)\Lambda_0(\theta)}{\rho_{00}(\theta)} \Rightarrow$

$H_8 = \xi\rho_{00}((\frac{\gamma u_\phi}{c})^2 + (\frac{\gamma u_\theta}{c})^2) - (2F-4)\frac{(\mathit{\Gamma}-1)(\xi-1)}{\mathit{\Gamma}}\rho_{00} - (F-1)(B_{\theta 0}^2 + B_{\phi 0}^2) + FE_{r0}^2 - \frac{\gamma u_r}{c}\mathit{\Lambda}_0$

- $G_{94} = \frac{B_{\phi 0}(\theta)}{\sin\theta}$
- $G_{96} = \frac{(\mathit{\Gamma}-1)(\xi(\theta)-1)}{\mathit{\Gamma}} - (\frac{2(F-1)\mathit{\Psi}_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2\xi(\theta) = \frac{(\mathit{\Gamma}-1)(\xi-1)}{\mathit{\Gamma}} - (\frac{\gamma u_\theta}{c})^2\xi$
- $G_{97} = \rho_{00}(\theta)[\frac{\mathit{\Gamma}-1}{\mathit{\Gamma}} + (\frac{2(F-1)\mathit{\Psi}_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2] = \rho_{00}[\frac{\mathit{\Gamma}-1}{\mathit{\Gamma}} + (\frac{\gamma u_\theta}{c})^2]$
- $G_{98} = A_1(\theta) = B_{r0}$
- $G_{910} = -\frac{\mathit{\Phi}_1(\theta)}{\sin^2\theta} = \frac{E_{\theta 0}}{\sin\theta}$
- $G_{9j} = 0$, for $j \neq 4, 6, 7, 8, 10$
- $H_9 = \rho_{00}(\theta)\xi(\theta)\gamma^2(\theta)u_{\phi 0}^2(\theta)\cot\theta + \frac{2(F-1)\mathit{\Psi}_0(\theta)\mathit{\Psi}_1(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta} - \frac{4(F-1)^2\mathit{\Psi}_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}(\mathit{\Psi}_1(\theta) - \mathit{\Psi}_0(\theta)\frac{\cot\theta}{\sin\theta}) + \frac{F(F-1)\mathit{\Phi}_0(\theta)\mathit{\Phi}_1(\theta)}{\sin\theta} - \frac{F(F-1)A_0(\theta)A_1(\theta)}{\sin\theta} + \frac{2(F-1)\mathit{\Psi}_0(\theta)\mathit{\Lambda}_0(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$
  $H_9 = \rho_{00}\xi(\frac{\gamma u_\phi}{c})^2\cot\theta + \rho_{00}\xi\frac{\gamma u_\theta}{c}[2(F-3)\frac{\gamma u_r}{c} + \frac{\gamma u_\theta}{c}\cot\theta] - F\frac{E_{r0}}{\sin\theta}\frac{\gamma u_r}{c}\rho_{00}(\theta) + (F-1)B_{r0}B_{\theta 0} - \frac{\gamma u_\theta \mathit{\Lambda}_0}{c}$
- $G_{104} = \frac{FA_0(\theta)}{\sin^2\theta} = -\frac{B_{\theta 0}}{\sin\theta}$
- $G_{105} = -\frac{2(F-1)\mathit{\Psi}_0(\theta)\xi(\theta)}{\sin\theta} = \frac{\gamma u_\theta}{c}\rho_{00}\xi$
- $G_{107} = -\frac{2(F-1)\mathit{\Psi}_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta} = \frac{\gamma u_\theta}{c}\frac{\gamma u_\phi}{c}\rho_{00}$
- $H_{10} = 2(F-1)\mathit{\Psi}_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta)\frac{\cot\theta}{\sin\theta} - \mathit{\Psi}_1(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta) + (F-1)A_1(\theta)B_{\phi 0}(\theta) - \gamma(\theta)u_{\phi 0}(\theta)\mathit{\Lambda}_0(\theta) \Rightarrow$
  $H_{10} = -\rho_{00}\xi\frac{\gamma u_\phi}{c}(\frac{\gamma u_r}{c} + \frac{\gamma u_\theta}{c}\cot\theta) + (F-1)B_{r0}B_{\phi 0} - \frac{\gamma u_\phi \mathit{\Lambda}_0}{c}$

The determinant of G may now be calculated:

$$Det(G) = \frac{[(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1]\gamma\rho_{00}^2\xi^2}{\mathit{\Gamma}\sin\theta}(\frac{\gamma u_\theta}{c})^4[1 - (\frac{\gamma u_r}{c})^2(\sin\theta - 1)][\frac{(\mathit{\Gamma}-1)(\xi-1)}{(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1} - (\frac{\gamma u_\theta}{c})^2]$$

The angles at which $Det(G) = 0$ are the locations of the flow's critical surfaces. These angles will be determined by setting each factor of the determinant of G equal to zero.

The first factor:

$$\frac{[(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1]\gamma\rho_{00}^2\xi^2}{\mathit{\Gamma}\sin\theta}(\frac{\gamma u_\theta}{c})^4 = 0 \Rightarrow \frac{\gamma u_\theta}{c} = 0$$

So, the first critical surface is an arbitrary one found at the angle θ_{cr}^1 at which the flow's $\hat{\theta}$ four-velocity becomes equal to zero. When attempting to retrieve disk-wind and jet solutions, $\frac{\gamma u_\theta}{c} = 0$ can be used as boundary condition for the flow's $\hat{\theta}$ four-velocity on the midplane ($\theta = \frac{\pi}{2}$ of the disk-source of the flow. So, the first critical point is located on the midplane at $\theta_{cr}^1 = \frac{\pi}{2}$.

The second factor:

$$[1 - (\frac{\gamma u_r}{c})^2(\sin\theta - 1)]$$

is positive for all possible values of θ.

The third factor:

$$[\frac{(\mathit{\Gamma}-1)(\xi-1)}{(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1} - (\frac{\gamma u_\theta}{c})^2] = 0$$

gives the location of the second critical point at the angle θ_{cr}^2 at which:

$$\frac{\gamma u_\theta}{c} = \sqrt{\frac{(\mathit{\Gamma}-1)(\xi-1)}{(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1}} \tag{3.63}$$

The expression on the RHS of (3.63) is the modified sonic speed:

$$\frac{u_s}{c} = \sqrt{\frac{(\mathit{\Gamma}-1)(\xi-1)}{(2-\mathit{\Gamma})\xi + \mathit{\Gamma} - 1}} = \frac{\frac{c_s}{c}}{\sqrt{1 - \frac{c_s^2}{c^2}}}$$

where $c_s = \sqrt{\frac{\mathit{\Gamma} P}{\xi\rho_0 c^2}}$ the regular speed of sound for a hot gas. [13]

The conical surface $\theta = \theta_{cr}^2$ then, where (3.63) is satisfied, is called the modified sonic critical surface.

3.4 Specific Cases

Two specific cases of resistive, relativistic, r sef-similar flows, with respect to the radiative losses term $\Lambda_0(\theta)$, will be studied in the context of this thesis. The first is the case of a flow with no radiative losses, for which $\Lambda_0(\theta) = 0$. The second case to be investigated is a flow which radiates a percentage π of the total energy dissipated as heat due to the plasma's resistivity (Joule heating), with a Λ given by:

$$\Lambda = \pi[\gamma \vec{J} \cdot (\vec{E} + \frac{\vec{u} \times \vec{B}}{c}) - \gamma J^0 \frac{\vec{u} \cdot \vec{E}}{c}] \Rightarrow$$

$$\Lambda = \pi(\frac{\gamma c}{4\pi}(\vec{\nabla} \times \vec{B}) \cdot (\vec{E} + \frac{\vec{u} \times \vec{B}}{c}) - \gamma(\vec{\nabla} \cdot \vec{E})\frac{\vec{u} \cdot \vec{E}}{4\pi}) \quad (3.64)$$

As it will be shown in detail, the system of ordinary differential equations describing a radiative flow, displays significant differences to the one describing a flow with an undefined $\Lambda_0(\theta)$ (eqs. (3.50-3.59)), presented in section 3.3.

3.4.1 Non-Radiative r Self-Similar Flow

The system of ordinary differential equations describing a flow with zero radiative losses can be derived by setting the function $\Lambda_0(\theta)$ appearing in equations (3.56)-(3.59) equal to zero. This flow then is described by the set of equations (3.50)-(3.59), with $\Lambda_0(\theta) = 0$.

The system of equations describing a non-radiative flow may be expressed in a form analogous to (3.62):

$$\sum_j G^0_{ij} Y'_j = H^0_i \quad (3.65)$$

The elements G^0_{ij} of G^0 are exactly the same as the elements of G defined in the previous section. The elements H^0_7, H^0_8, H^0_9, H^0_{10}, are the only elements of H^0 which differ from their respective elements of H. They can be calculated by setting $\Lambda_0(\theta) = 0$, in H_7, H_8, H_9, H_{10}.

3.4.2 Radiative r Self-Similar Flow

The radiative losses term profile $\Lambda_0(\theta)$ of a flow radiating a fraction π of the total energy dissipated due through ohmic dissipation, can be defined by applying the separation of variables presented in section 3.1 on equation (3.64). The radiative losses profile $\Lambda_0(\theta)$ is:

$$\Lambda_0(\theta) =$$

$$\pi[\frac{1}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}]\frac{d}{d\theta}(\sin\theta B_{\phi 0}(\theta))-$$
$$\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{dA_1(\theta)}{d\theta}+\frac{F-1}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta}-\Psi_1(\theta)\Phi_0(\theta))\frac{d\Phi_1(\theta)}{d\theta}+$$
$$\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta}-\Psi_1(\theta)\Phi_0(\theta))-(F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)A_1(\theta)-$$
$$\frac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)})-\frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}] \quad (3.66)$$

The system of equations describing such a flow is the one that emerges after substituting $\Lambda_0(\theta)$ in equations (3.56)-(3.59). The substitution can be made easier by expressing $\Lambda_0(\theta)$ in terms of the elements of G, H, and Y' appearing in (3.65). To that end:

$$\Lambda_0(\theta) = \pi[-G_{74}\frac{dY_4}{d\theta}-G_{78}\frac{dY_8(\theta)}{d\theta}-G_{710}\frac{dY_{10}(\theta)}{d\theta}+H_7-\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)] \quad (3.67)$$

Additionally, the ordinary differential equations describing the flow will be put in the form:

$$\sum_j \tilde{G}_{ij}Y'_j = \tilde{H}_i \quad (3.68)$$

The matrices $\tilde{G}$, $\tilde{H}$ are in essence the matrices G^0 and H^0, now modified, due to the existence of $\Lambda_0(\theta)$, as defined by (3.67). The elements of the new matrices $\tilde{G}$ and $\tilde{H}$ will be defined by substituting (3.67) in equations (3.56)-(3.59), expressed in the form (3.68).

The energy equation (3.56) then is modified as:

$$(1-\pi)[G_{74}\frac{dY_4}{d\theta}+G_{78}\frac{dY_8(\theta)}{d\theta}+G_{710}\frac{dY_{10}(\theta)}{d\theta}]+G_{76}\frac{dY_6(\theta)}{d\theta}+G_{77}\frac{dY_7(\theta)}{d\theta} =$$
$$\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)+$$
$$(1-\pi)[\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta}-\Psi_1(\theta)\Phi_0(\theta))-$$
$$(F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi 0}(\theta)A_1(\theta)-\frac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)})-$$
$$\frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}] \quad (3.69)$$

As can be seen from the previous equation, the elements G_{74}, G_{78}, and G_{710} of the seventh line of G have been modified:

- $\tilde{G}_{74} = (1-\pi)G_{74} = -\frac{1-\pi}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta) + \frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta} - \frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}] = -\frac{1-\pi}{\sin\theta}[\gamma E_{r0} - \frac{\gamma u_\phi}{c}B_{\theta 0} + \frac{\gamma u_\theta}{c}B_{\phi 0}]$

- $\tilde{G}_{78} = (1-\pi)G_{78} = (1-\pi)\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = (1-\pi)(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})$

- $\tilde{G}_{710} = (1-\pi)G_{710} = -(1-\pi)\frac{F-1}{\sin\theta}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \frac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}}) = -\frac{(1-\pi)}{\sin\theta}(\frac{\gamma u_r}{c}E_{r0} + \frac{\gamma u_\theta}{c}E_{\theta 0}),$

while the elements G_{76}, G_{77} have remained unchanged:

- $\tilde{G}_{76} = G_{76} = \frac{\Gamma-1}{\Gamma}\frac{2(F-1)\Psi_0(\theta)(\xi(\theta)-1)}{\rho_{00}(\theta)\sin\theta} = -\frac{(\Gamma-1)(\xi-1)}{\Gamma}\frac{\gamma u_\theta}{c}$

- $\tilde{G}_{77} = G_{77} = \frac{2(F-1)\Psi_0(\theta)}{\Gamma\sin\theta} = \frac{\rho_{00}}{\Gamma}\frac{\gamma u_\theta}{c}$

The RHS H_7 is also modified :

$$\tilde{H}_7 = \frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta) + (1-\pi)[\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta} - \Psi_1(\theta)\Phi_0(\theta)) - (F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta} + \gamma(\theta)u_{\phi 0}(\theta)A_1(\theta) - \frac{\Psi_1(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)}) - \frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}] \Rightarrow$$

$$\tilde{H}_7 = (1-\pi)H_7 + \pi\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)$$

The system of equations describing a radiative flow is consequently characterized by the new matrices $\tilde{G}$ and $\tilde{H}$, which differ significantly from the ones describing a non-radiative flow, as will be shown next.

After substituting (3.67) in the $\hat{r}$ component of the momentum equation (eq. (3.57)), the resulting equation in the form (3.62) is:

$$G_{86}\frac{dY_6}{d\theta} + G_{87}\frac{dY_7}{d\theta} + G_{88}\frac{dY_8}{d\theta} + G_{89}\frac{dY_9}{d\theta} + G_{810}\frac{dY_{10}}{d\theta} =$$

$$H_8 - \pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}[-G_{74}\frac{dY_4}{d\theta} - G_{78}\frac{dY_8(\theta)}{d\theta} - G_{710}\frac{dY_{10}(\theta)}{d\theta} + H_7 - \frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)] \Rightarrow$$

$$-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{74}\frac{dY_4}{d\theta}+G_{86}\frac{dY_6}{d\theta}+G_{87}\frac{dY_7}{d\theta}+(G_{88}-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{78})\frac{dY_8}{d\theta}+G_{89}\frac{dY_9}{d\theta}+$$

$$(G_{810}-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{710})\frac{dY_{10}}{d\theta}=H_8-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}(H_7-\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)) \tag{3.70}$$

The non-zero elements of the eighth line of $\tilde{G}$ appearing in equation (3.69) are:

- $\tilde{G}_{84}=-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{74}=\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta)+\frac{F\gamma(\theta)u_{\phi0}(\theta)A_0(\theta)}{\sin\theta}-\frac{2(F-1)\Psi_0(\theta)B_{\phi0}(\theta)}{\rho_{00}(\theta)\sin\theta}]=\pi\frac{\gamma u_r}{c\sin\theta}[\gamma E_{r0}-\frac{\gamma u_\phi}{c}B_{\theta0}+\frac{\gamma u_\theta}{c}B_{\phi0}]$
- $\tilde{G}_{86}=G_{86}=\frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}^2(\theta)\sin\theta}=-\frac{\gamma u_r}{c}\frac{\gamma u_\theta}{c}\xi$
- $\tilde{G}_{87}=G_{87}=-\frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}=\frac{\gamma u_r}{c}\frac{\gamma u_\theta}{c}\rho_{00}$
- $\tilde{G}_{88}=G_{88}-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{78}=\frac{FA_0(\theta)}{\sin\theta}-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}=-B_{\theta0}-\pi\frac{\gamma u_r}{c}(\frac{\gamma u_r}{c}B_{\theta0}-\frac{\gamma u_\theta}{c}B_{r0})$
- $\tilde{G}_{89}=G_{89}=-\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}=\frac{\gamma u_\theta}{c}\xi$
- $\tilde{G}_{810}=G_{810}-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}G_{710}=-\frac{(F-1)\Phi_0(\theta)}{\sin\theta}+\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}\frac{F-1}{\sin\theta}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta}-\frac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}})=\frac{E_{r0}}{\sin\theta}+\pi\frac{\gamma u_r}{c\sin\theta}(\frac{\gamma u_r}{c}E_{r0}+\frac{\gamma u_\theta}{c}E_{\theta0})$

The element $\tilde{H}_8$ of $\tilde{H}$:

- $\tilde{H}_8=H_8-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}(H_7-\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta))\Rightarrow$
  $\tilde{H}_8=\xi(\theta)\rho_{00}(\theta)\gamma^2(\theta)u_{\phi0}^2(\theta)+\frac{4(F-1)^2\Psi_0^2(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin^2\theta}-\frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\rho_{00}(\theta)-\frac{F^2(F-1)A_0^2(\theta)}{\sin^2\theta}-(F-1)B_{\phi0}^2+F(F-1)^2\Phi_0^2(\theta)-\pi\frac{\Psi_1(\theta)}{\rho_{00}(\theta)}[\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta}-\Psi_1(\theta)\Phi_0(\theta))-(F-1)B_{\phi0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta}+\gamma(\theta)u_{\phi0}(\theta)A_1(\theta)-\frac{\Psi_1(\theta)B_{\phi0}}{\rho_{00}(\theta)})-\frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta)-FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}]\Rightarrow$

$$\tilde{H}_8 = \xi\rho_{00}((\frac{\gamma u_\phi}{c})^2 + (\frac{\gamma u_\theta}{c})^2) - (2F-4)\frac{(\Gamma-1)(\xi-1)}{\Gamma}\rho_{00} - (F-1)(B_{\theta 0}^2 + B_{\phi 0}^2) + FE_{r0}^2 - \pi\frac{\gamma u_r}{c}[-FE_{r0}(\frac{\gamma u_r}{c}E_{r0} - \frac{\gamma u_\theta}{c}E_{\theta 0}) - (F-1)B_{\phi 0}(\gamma E_{\theta 0} + \frac{\gamma u_\phi}{c}B_{r0} - \frac{\gamma u_r}{c}B_{\phi 0}) + (F-1)B_{\theta 0}(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})]$$

The $\hat{\theta}$ component of the momentum equation, after substituting (3.67) in (3.58), in the form (3.62):

$$(G_{94} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{74})\frac{dY_4}{d\theta} + G_{96}\frac{dY_6}{d\theta} + G_{97}\frac{dY_7}{d\theta} + (G_{98} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{78})\frac{dY_8}{d\theta} +$$

$$(G_{910} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{710})\frac{dY_{10}}{d\theta} = H_9 + \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}(H_7 - \frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta))$$

The non-zero elements of the ninth line of $\tilde{G}$:

- $\tilde{G}_{94} = G_{94} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{74} = \frac{B_{\phi 0}(\theta)}{\sin\theta} + \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{1}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta) + \frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta} - \frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}] = \frac{B_{\phi 0}}{\sin\theta} - \pi\frac{\gamma u_\theta}{c}\frac{1}{\sin\theta}[\gamma E_{r0} - \frac{\gamma u_\phi}{c}B_{\theta 0} + \frac{\gamma u_\theta}{c}B_{\phi 0}]$
- $\tilde{G}_{96} = G_{96} = \frac{(\Gamma-1)(\xi(\theta)-1)}{\Gamma} - (\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2\xi(\theta) = \frac{(\Gamma-1)(\xi-1)}{\Gamma} - (\frac{\gamma u_\theta}{c})^2\xi$
- $\tilde{G}_{97} = G_{97} = \rho_{00}(\theta)[\frac{\Gamma-1}{\Gamma} + (\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta})^2] = \rho_{00}[\frac{\Gamma-1}{\Gamma} + (\frac{\gamma u_\theta}{c})^2]$
- $\tilde{G}_{98} = G_{98} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{78} = A_1(\theta) - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = B_{r0} + \pi\frac{\gamma u_\theta}{c}(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})$
- $\tilde{G}_{910} = G_{910} - \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}G_{710} = -\frac{\Phi_1(\theta)}{\sin^2\theta} + \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}\frac{F-1}{\sin\theta}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \frac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}}) = \frac{E_{\theta 0}}{\sin\theta} - \pi\frac{\gamma u_\theta}{c\sin\theta}(\frac{\gamma u_r}{c}E_{r0} + \frac{\gamma u_\theta}{c}E_{\theta 0})$

The element $\tilde{H}_9$:

- $\tilde{H}_9 = H_9 + \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}(H_7 - \frac{\Gamma-1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)) \Rightarrow$

$$\tilde{H}_9 = \rho_{00}(\theta)\xi(\theta)\gamma^2(\theta)u_{\phi 0}^2(\theta)\cot\theta + \frac{2(F-1)\Psi_0(\theta)\Psi_1(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta} - \frac{4(F-1)^2\Psi_0(\theta)\xi(\theta)}{\rho_{00}(\theta)\sin\theta}(\Psi_1(\theta) - \Psi_0(\theta)\frac{\cot\theta}{\sin\theta}) + \frac{F(F-1)\Phi_0(\theta)\Psi_1(\theta)}{\sin\theta} - \frac{F(F-1)A_0(\theta)A_1(\theta)}{\sin\theta} + \pi\frac{2(F-1)\Psi_0(\theta)}{\rho_{00}(\theta)\sin\theta}(\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta} - \Psi_1(\theta)\Phi_0(\theta)) - (F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta} + \gamma(\theta)u_{\phi 0}(\theta)A_1(\theta) - \frac{\Psi_1(\theta)B_{\phi 0}}{\rho_{00}(\theta)}) - \frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} \Rightarrow$$

$$\tilde{H}_9 = \rho_{00}\xi(\frac{\gamma u_\phi}{c})^2\cot\theta + \rho_{00}\xi\frac{\gamma u_\theta}{c}[2(F-3)\frac{\gamma u_r}{c} + \frac{\gamma u_\theta}{c}\cot\theta] - F\frac{E_{r0}}{\sin\theta}\frac{\gamma u_r}{c}\rho_{00}(\theta) + (F-1)B_{r0}B_{\theta 0} - \pi\frac{\gamma u_\theta}{c}[-FE_{r0}(\frac{\gamma u_r}{c}E_{r0} - \frac{\gamma u_\theta}{c}E_{\theta 0}) - (F-1)B_{\phi 0}(\gamma E_{\theta 0} + \frac{\gamma u_\phi}{c}B_{r0} - \frac{\gamma u_r}{c}B_{\phi 0}) + (F-1)B_{\theta 0}(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})]$$

Lastly, the $\hat{\phi}$ component of the momentum equation:

$$(G_{104} - \pi\gamma(\theta)u_{\phi 0}(\theta)G_{74})\frac{dY_4}{d\theta} + G_{105}\frac{dY_5}{d\theta} + G_{107}\frac{dY_7}{d\theta} - \pi\gamma(\theta)u_{\phi 0}(\theta)G_{78}\frac{dY_8}{d\theta} -$$

$$\pi\gamma(\theta)u_{\phi 0}(\theta)G_{710}\frac{dY_{10}}{d\theta} = H_{10} - \pi\gamma(\theta)u_{\phi 0}(\theta)(H_7 - \frac{\Gamma - 1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta)$$

So, the non-zero elements of the tenth line of $\tilde{G}$:

- $\tilde{G}_{104} = G_{104} - \pi\gamma(\theta)u_{\phi 0}(\theta)G_{74} = \frac{FA_0(\theta)}{\sin^2\theta} + \pi\frac{\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta}[-(F-1)\gamma(\theta)\Phi_0(\theta) + \frac{F\gamma(\theta)u_{\phi 0}(\theta)A_0(\theta)}{\sin\theta} - \frac{2(F-1)\Psi_0(\theta)B_{\phi 0}(\theta)}{\rho_{00}(\theta)\sin\theta}] = -\frac{B_{\theta 0}}{\sin\theta} + \pi\frac{\gamma u_\phi}{c\sin\theta}[\gamma E_{r0} - \frac{\gamma u_\phi}{c}B_{\theta 0} + \frac{\gamma u_\theta}{c}B_{\phi 0}]$
- $\tilde{G}_{105} = G_{105} = -\frac{2(F-1)\Psi_0(\theta)\xi(\theta)}{\sin\theta} = \frac{\gamma u_\theta}{c}\rho_{00}\xi$
- $\tilde{G}_{107} = G_{107} = -\frac{2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)}{\sin\theta} = \frac{\gamma u_\theta}{c}\frac{\gamma u_\phi}{c}\rho_{00}$
- $\tilde{G}_{108} = -\pi\gamma(\theta)u_{\phi 0}(\theta)G_{78} = -\pi\gamma(\theta)u_{\phi 0}(\theta)\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta} = -\pi\frac{\gamma u_\phi}{c}(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})$
- $\tilde{G}_{1010} = -\pi\gamma(\theta)u_{\phi 0}(\theta)G_{710} = \pi\gamma(\theta)u_{\phi 0}(\theta)\frac{F-1}{\sin\theta}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\rho_{00}(\theta)\sin^2\theta} - \frac{\Psi_1(\theta)\Phi_0(\theta)}{\rho_{00}(\theta)}) = \frac{\gamma u_\phi}{c\sin\theta}(\frac{\gamma u_r}{c}E_{r0} + \frac{\gamma u_\theta}{c}E_{\theta 0})$

The element $\tilde{H}_{10}$:

- $\tilde{H}_{10} = H_{10} - \pi\gamma(\theta)u_{\phi 0}(\theta)(H_7 - \frac{\Gamma - 1}{\Gamma}(2F-4)(\xi(\theta)-1)\Psi_1(\theta) \Rightarrow$
  $\tilde{H}_{10} = 2(F-1)\Psi_0(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta)\frac{\cot\theta}{\sin\theta} - \Psi_1(\theta)\gamma(\theta)u_{\phi 0}(\theta)\xi(\theta) +$
  $(F-1)A_1(\theta)B_{\phi 0}(\theta) - \pi\gamma(\theta)u_{\phi 0}(\theta)[\frac{\Gamma-1}{\Gamma}\frac{F(F-1)^2\Phi_0(\theta)}{\rho_{00}(\theta)}(\frac{2\Psi_0(\theta)\Phi_1(\theta)}{\sin^2\theta} -$
  $\Psi_1(\theta)\Phi_0(\theta)) - (F-1)B_{\phi 0}(\theta)(-\frac{\gamma(\theta)\Phi_1(\theta)}{\sin\theta} + \gamma(\theta)u_{\phi 0}(\theta)A_1(\theta) - \frac{\Psi_1(\theta)B_{\phi 0}}{\rho_{00}(\theta)}) -$
  $\frac{F(F-1)A_0(\theta)}{\sin\theta}\frac{2(F-1)\Psi_0(\theta)A_1(\theta) - FA_0(\theta)\Psi_1(\theta)}{\rho_{00}(\theta)\sin\theta}] \Rightarrow$
  $\tilde{H}_{10} = -\rho_{00}\xi\frac{\gamma u_\phi}{c}(\frac{\gamma u_r}{c} + \frac{\gamma u_\theta}{c}\cot\theta) + (F-1)B_{r0}B_{\phi 0} -$
  $\pi\gamma(\theta)u_{\phi 0}(\theta)[-FE_{r0}(\frac{\gamma u_r}{c}E_{r0} - \frac{\gamma u_\theta}{c}E_{\theta 0}) -$
  $(F-1)B_{\phi 0}(\gamma E_{\theta 0} + \frac{\gamma u_\phi}{c}B_{r0} - \frac{\gamma u_r}{c}B_{\phi 0}) + (F-1)B_{\theta 0}(\frac{\gamma u_r}{c}B_{\theta 0} - \frac{\gamma u_\theta}{c}B_{r0})]$

3.5 Field Line Geometry-Energy Fluxes

The flow's finite resistivity affects its field line geometry, which differs significantly from that of an ideal magnetohydrodynamic flow. In the case of a perfectly conductive flow, Ohm's Law:

$$\vec{E} + \frac{\vec{u} \times \vec{B}}{c} = 0 \tag{2.26}$$

and Faraday's Law:

$$\vec{\nabla} \times \vec{E} = 0 \tag{2.41}$$

dictate that mass flux lines (poloidal velocity field lines), magnetic flux lines (poloidal magnetic field lines), and equipotential lines of the electric potential Φ (Poynting flux lines) coincide. [17]

It is simple to show that the electric equipotential lines are the Poynting flux lines. The Poynting vector is defined as:

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

By use of equations (2.49) and (2.50), it may be written as:

$$\vec{S} = \frac{c}{4\pi} (-\vec{\nabla}\Phi \times (B_\phi \hat{\phi}) - \vec{\nabla}\Phi \times (\frac{\vec{\nabla}A \times \hat{\phi}}{\Phi})) \Rightarrow$$

$$\vec{S} = \frac{c}{4\pi} (\varpi B_\phi \frac{(-\vec{\nabla}\Phi) \times \hat{\phi}}{\varpi} + \frac{\vec{\nabla}\Phi \cdot \vec{\nabla}A}{\varpi} \hat{\phi})$$

Its poloidal component finally is:

$$\vec{S}_p = -\frac{c}{4\pi} \varpi B_\phi \vec{\nabla} \times (\frac{\Phi}{\varpi} \hat{\phi})$$

So, Poynting flux lines are the curves $\Phi = const.$ on the poloidal plane, in the same fashion that mass flux lines and magnetic flux lines are the curves corresponding to $\Psi = const.$ and $A = const.$ respectively.

In the case of a flow of finite conductivity, Ohm's Law is expressed by equation (2.22). The aforementioned three families of lines on the poloidal plane no longer coincide, but each take a unique shape.

Flux line shapes are determined with the help of (3.8), (3.2) and (3.4). For example, the mass flux lines' shape is determined as follows:
Suppose that the flow described by the solution of the system of ODEs has a conical boundary at the polar angle θ_i. Then, from $\Psi = const.$ and (3.8):

$$(\frac{r(\theta_i)}{\varpi_0})^{2F-2} \Psi_0(\theta_i) = (\frac{r(\theta)}{\varpi_0})^{2F-2} \Psi_0(\theta) \tag{3.71}$$

With $r(\theta_i) = r_i$ and $\Psi_0(\theta_i) = \Psi_{0i}$ the shape of the mass flux lines on the poloidal planes finally is:

$$\frac{r_\Psi(\theta)}{r_i} = (\frac{\Psi_{0i}}{\Psi_0(\theta)})^{\frac{1}{2F-2}} \tag{3.72}$$

Similarly, the shape of the magnetic fux lines is given by:

$$\frac{r_A(\theta)}{r_i} = (\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}} \tag{3.73}$$

, while for the Poynting flux lines:

$$\frac{r_S(\theta)}{r_i} = (\frac{\Phi_{0i}}{\Phi_0(\theta)})^{\frac{1}{F-1}} \tag{3.74}$$

For the three-dimensional plots of the field lines, the azimuthal angle ϕ as a function of θ must be determined, by solving the following equation:

$$\frac{V_\phi}{\sin\theta d\phi} = \frac{V_\theta}{d\theta} \Rightarrow \frac{d\phi}{d\theta} = \frac{1}{\sin\theta}\frac{V_\theta}{V_\phi}$$

with $V = u$ for the velocity field, $V = B$ for the magnetic field, and $V = E$ for the electric field.

An important quantity that must also be defined is the ratio of the total energy flux to the fluid's rest energy flux along the mass flux lines, a quantity which in the context of ideal MHD is preserved. [16] The matter component of the total energy flux, which is effectively the flux of energy contained in the fluid's volume, is the $0p$ components of the matter stress energy tensor, where p denotes the poloidal direction.

$$T^{0p} = \gamma^2\xi\rho_0 c^2 u_p = \gamma\rho_0 u_p c^2 \mu_{HD}$$

The electromagnetic component is the Poynting flux projected on the flow's poloidal velocity.

$$\vec{S}\cdot\frac{\vec{u}_p}{u_p} = \frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot\frac{\vec{u}_p}{u_p} = \gamma\rho_0 u_p c^2 \mu_{EM}$$

The ratio then of total energ flux to the mass flux is:

$$\mu = \mu_{HD} + \mu_{EM} = \frac{\gamma^2\xi\rho_0 c^2 u_p + \frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot\frac{\vec{u}_p}{u_p}}{\gamma\rho_0 u_p c^2}$$

Substituting all quantities from the respective expressions derived in section 3.1, the resulting expression for μ is:

$$\mu(\theta) = \xi(\theta)\gamma(\theta) - \frac{B_{\phi 0}(\theta)\sin\theta(\Phi_1(\theta)\Psi_1(\theta) + 2(F-1)^2\Phi_0(\theta)\Psi_0(\theta))}{\Psi_1^2(\theta)\sin^2\theta + 4(F-1)^2\Psi_0^2(\theta)} \tag{3.75}$$

which makes it evident that μ is independent of the radial distance r.

Chapter 4

Effects of Resistivity in Field Line Geometry and Flow Dynamics

In this chapter the effects of different resistivity profiles and values on the flow's dynamical properties, the energy exchange between the plasma and the electromagnetic field, as well as the fields' spatial configuration are explored, assuming a non-radiative flow ($\Lambda = 0$). The two resistivity profiles examined are:

- an exponential resistivity profile: $\eta_0(\theta) = \dfrac{e^{\chi\theta^2}}{e^{(\chi+\chi_0)\theta_i^2}}$, with χ as a free parameter controlling the resistivity's rate of increase as a function of θ, and θ_i the boundary value of θ at which the integration of the equations starts.
- a Gaussian resistivity profile: $\eta_0(\theta) = 3e^{-\chi(\theta-\theta_0)^2}$, with θ_0 the middle of the solution domain (θ_i, θ_f).

4.1 Boundary Conditions

The system of equations governing the system flow-electromagnetic field is solved using the Residual Function Method for systems of Differential-Algebraic Equations. In all solutions presented in the folowing sections, $\theta_i = 0.9$, $\theta_f = \dfrac{\pi}{2}$, $\chi_0 = 8$.

At $\theta_i = 0.9$, the flow is assumed to be in an ideal MHD state. In consequence, the boundary values for the components of the velocity, magnetic, and electric field satisfy the condition: $\vec{E} + \dfrac{\vec{u} \times \vec{B}}{c} = 0$. The components of the magnetic, electric, and velocity field at θ_i must then satisfy the following relations:

- $E_r = \dfrac{u_\phi B_\theta - u_\theta B_\phi}{c}$
- $E_\theta = \dfrac{u_r B_\phi - u_\phi B_r}{c}$
- $u_r B_\theta = u_\theta B_r$

After substituting the field components from (3.3), (3.5), and (3.9), the above equations give the following boundary conditions:

- $\Phi_0(\theta_i) = \dfrac{F A_0(\theta_i) u_{\phi 0}(\theta_i)}{(F-1)\sin\theta_i} - \dfrac{2\Psi_0(\theta_i) B_{\phi 0}(\theta_i)}{\gamma(\theta_i)\rho_{00}(\theta_i)\sin\theta_i}$ (4.1)
- $\Phi_1(\theta_i) = A_1(\theta_i) u_{\phi 0}(\theta_i)\sin\theta_i - \dfrac{\Psi_1(\theta_i) B_{\phi 0}(\theta_i)\sin\theta_i}{\gamma(\theta_i)\rho_{00}(\theta_i)}$ (4.2)
- $A_1(\theta_i) = \dfrac{F}{2(F-1)} \dfrac{\Psi_1(\theta_i)}{\Psi_0(\theta_i)} A_0(\theta_i)$ (4.3)

Equations (4.1), (4.2), (4.3) are used to determine the boundary values of $\Phi_0(\theta)$, $\Phi_1(\theta)$, and $A_1(\theta)$ at θ_i, based on the freely chosen boundary values of $\Psi_0(\theta_i)$, $A_0(\theta_i)$, $u_{\phi 0}(\theta_i)$, and $\rho_{00}(\theta_i)$.

In all solutions presented in the following sections, the boundary values of the mass density profile $\rho_{00}(\theta)$ and of the specific enthalpy $\xi(\theta)$ are: $\rho_{00}(\theta_i) = 1$, $\xi(\theta_i) = 1.0000001$. The values of the radial dependence F and of the polytropic index Γ are chosen to be: $F = 1.1$, $\Gamma = \dfrac{4}{3}$.

4.2 Exponential Resistivity Profile

4.2.1 Effects of the Value of χ on the Flow

The resistivity's boundary value at θ_i is: $e^{-\chi_0 \theta_i^2} \simeq 1.5 \times 10^{-3}$, corresponding to an initial value for the magnetic Reynolds number: $R_m \sim 10^4$. The resistivity profile $\eta_0(\theta)$, as well as the resistivity, in units of $\dfrac{\varpi_0}{c}$ as a function of r along a mass flux line are presented below, along with the magnetic Reynolds number profile $R_{m0}(\theta)$ and the magnetic Reynolds number R along a mass flux line.

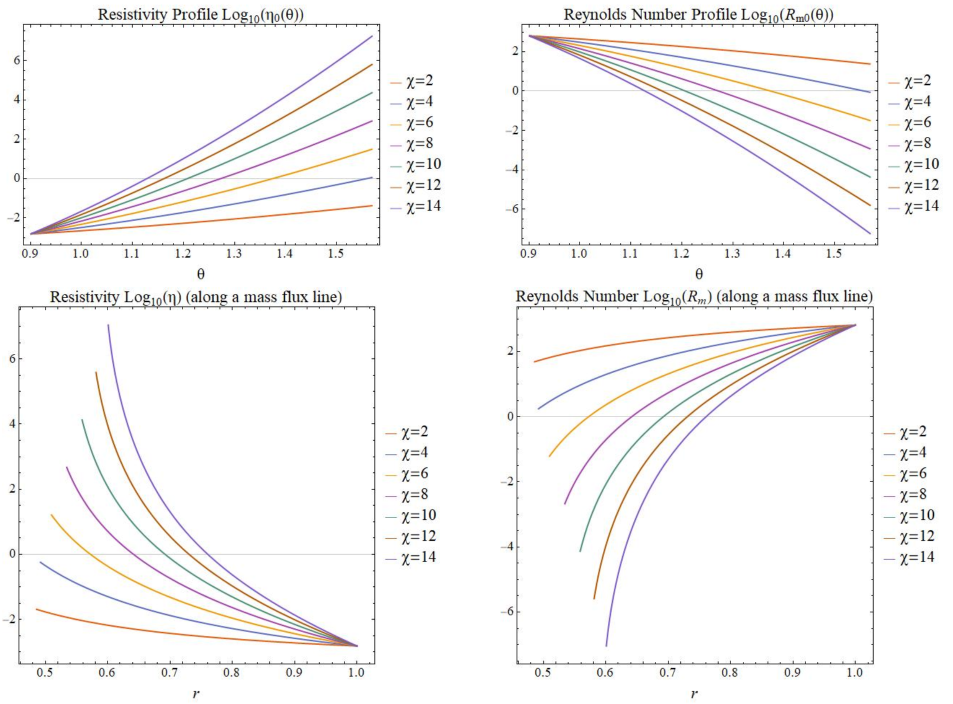


Figure 4.1: Top: The resistivity profile $\eta_0(\theta)$, and magnetic Reynolds number profile $R_{m0}(\theta)$ as functions of the polar angle θ. Bottom: Resistivity η and magnetic Reynolds number R_m along a mass flux line.

The magnetic Reynolds number R_m is calculated through the flow's resistivity as follows. The magnetic Reynolds number's definition is:

$$R_m = \frac{UL}{m} \tag{4.1}$$

where U is a characteristic velocity scale of the flow, L a characteristic length scale, and $m = c^2\eta$ the magnetic fields diffusivity. By choosing $U = c$ and $L = \varpi_0$, R_m becomes:

$$R_m = \frac{\varpi_0}{c\eta} = (\frac{r}{\varpi_0})^{-1}\frac{1}{\eta_0(\theta)} \tag{4.2}$$

The magnetic Reynolds number profile $R_{m0}(\theta)$ then is: $R_{m0}(\theta) = \dfrac{1}{\eta_0(\theta)}$.

The flow's Lorentz factor:

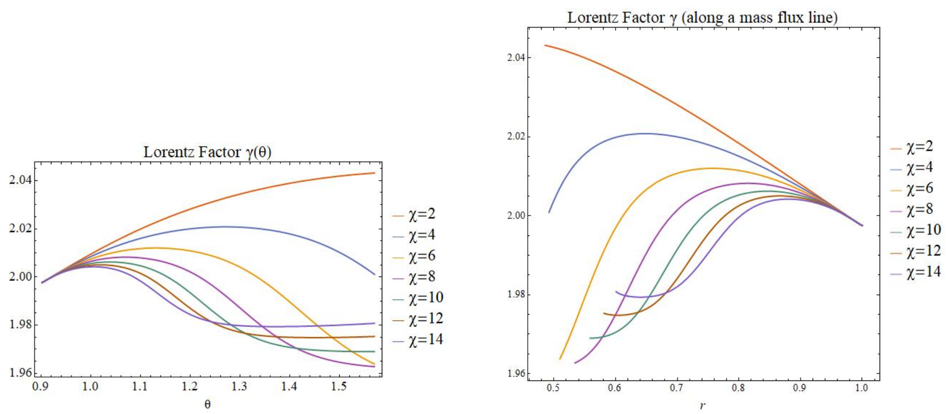


Figure 4.2: Left: The flow's Lorentz factor as a function of the polar angle θ. Right: The flow's Lorentz factor along a mass flux line.

For $\chi = 2$, that is for a resistivity profile whose values remain under 10^{-1}, the flow is accelerated over the whole domain of the solution. For all other values of χ, the flow is accelerated up to a maximum Lorentz factor, located at a smaller values of θ as the value of χ increases.

By allowing the solver to determine a solution in the larger domain $(0.9, 2)$ for θ, it can be seen that the flow characterized by $\chi = 2$ will also reach a maximum Lorentz factor and then begin to decelerate.

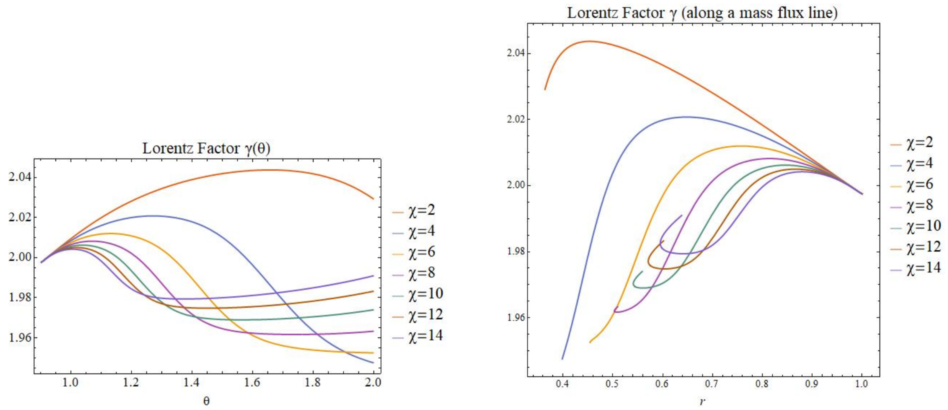


Figure 4.3: Left: The flow's Lorentz factor as a function of the polar angle θ. Right: The flow's Lorentz factor along a mass flux line.

As evidenced by the following graphs, acceleration stops at angles θ at which the resistivity profile takes values of the order of 10^{-2}, or the magnetic Reynolds number R_m becomes: $R_m \sim 10^3$.

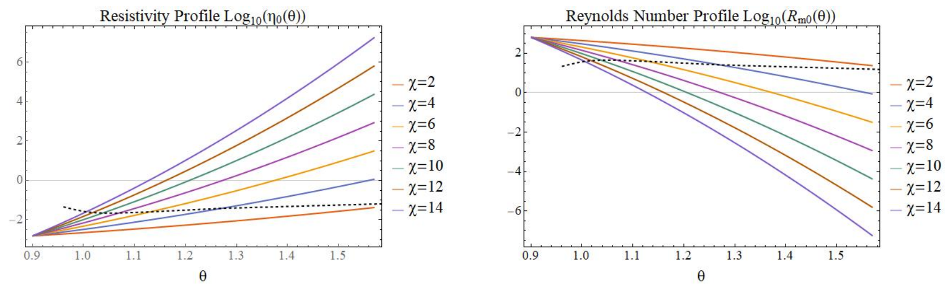


Figure 4.4: For resistivity profile values below, or magnetic Reynolds number values above the dashed lines in the respective graphs, flow decelaration occurs.

Plotting the Lorentz factor for values of χ larger than $\chi = 10$ reveals that the flow begins to accelerate again after a certain value of θ for each value of χ.

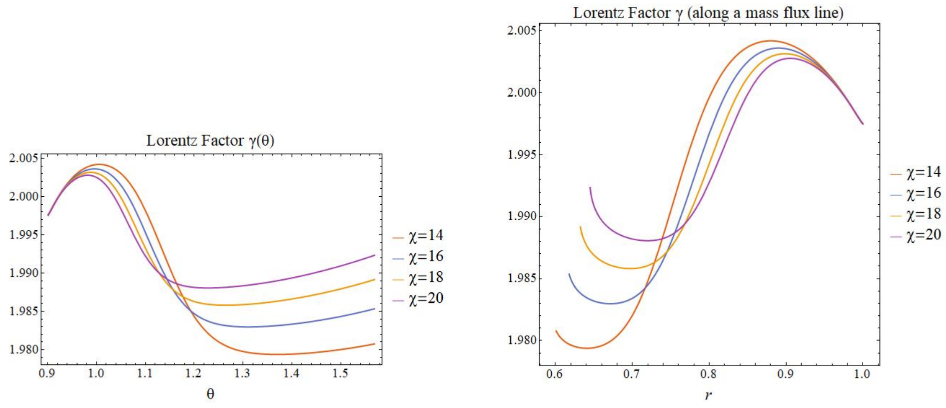


Figure 4.5: Lorentz factor for larger values of χ.

This acceleration takes place when the resistivity profile takes values close to 10^5, as shown in the following figures.

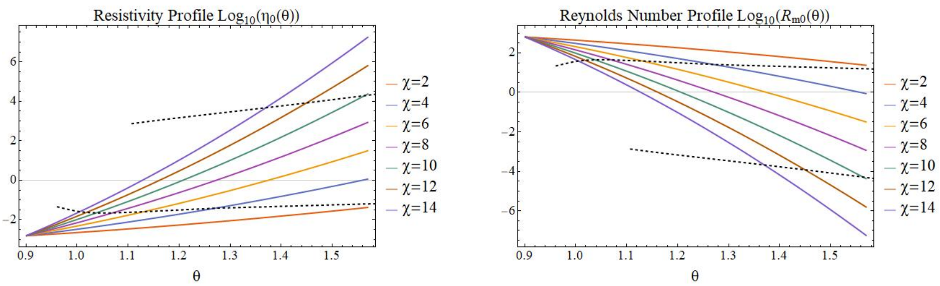


Figure 4.6: For resistivity profile values or magnetic Reynolds number values between the dashed lines in their respectve graphs, decelaration occurs.

The different values of χ have interesting effects on the heating of the plasma due to ohmic dissipation. The flow's sepcific enthalpy ξ and temperature, assuming a proton-electron plasma, both as functions of θ and along a mass flux line are presented in the following figures.

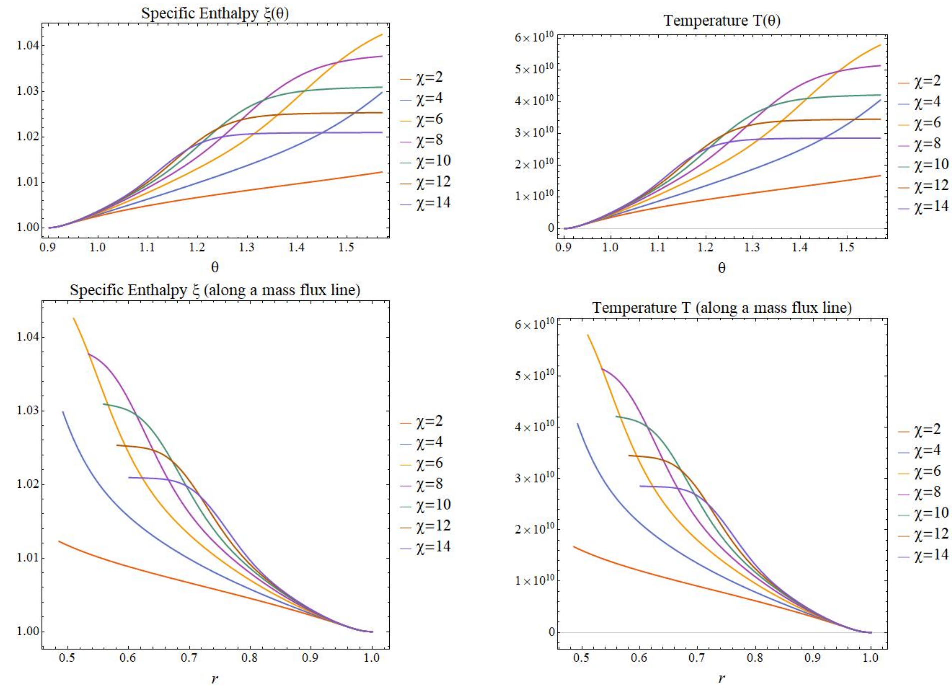


Figure 4.7: Top: The specific enthalpy and temperature as functions of the polar angle θ. Bottom: The specific enthalpy and temperature along a mass flux line.

The flow's specific enthalpy ξ and temperature increase over the entirety of the solution's domain for $\chi = 2, 4, 6, 8$, with the greatest increase in temperature, achieved for $\chi = 6$. For $\chi = 12, 14$ the plasma's temperature reaches a maximum value and then plateaus. This is due to the power dissipated through ohmic dissipation becoming negligible for large values of the resistivity profile, as can be seen in the next figure.

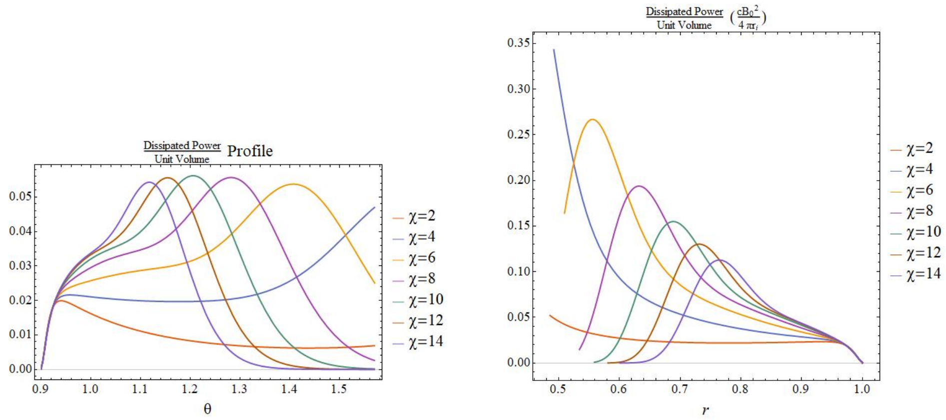


Figure 4.8: Left: The dissipated power per unit volume profile. Right: Power per unit volume dissipated along a mass flux line.

Since Joule heating is the only source of heating for the plasma, when no power is dissipated through ohmic dissipation, the plasma's temperature can no longer increase. As it has been shown, just as in the case of an ideal, perfectly conductive plasma, ohmic dissipation is negligible for extremely high values of resistivity.

The behaviour of the flow's mass density is presented next.

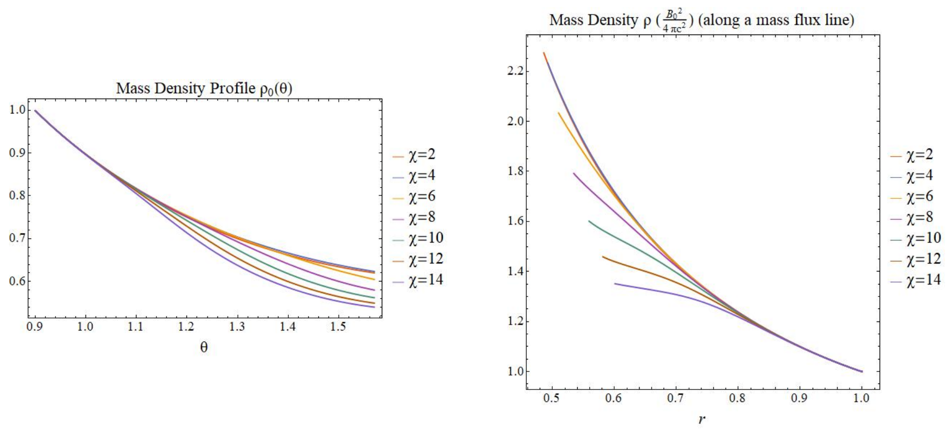


Figure 4.9: Left: Mass density profile $\rho_0(\theta)$. Right: Mass density along a mass flux line.

While the mass density's profile $\rho_0(\theta)$ s a decreasing function of the polar angle θ, the right figure renders it clear that the flow's mass density increases along a mass flux line.

Graphs of the magnetic pressure, the thermal pressure, and of the plasma

β follow.

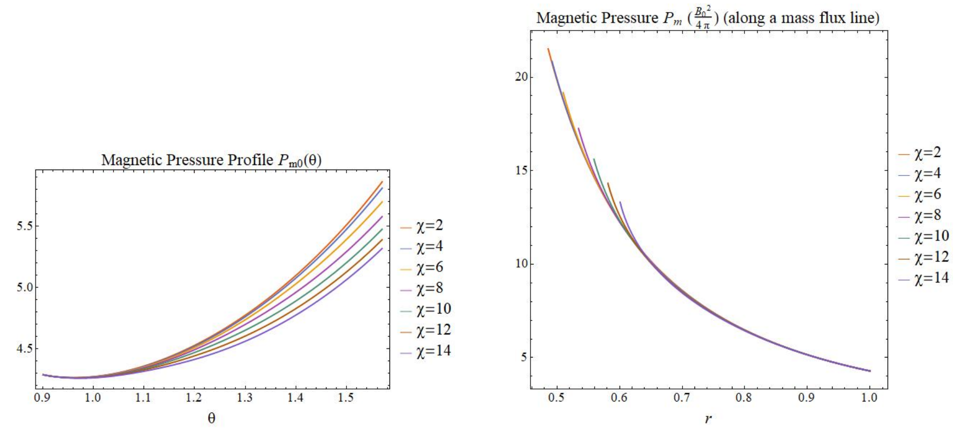


Figure 4.10: Left: Magnetic pressure profile. Bottom: Magnetic pressure along a mass flux line.

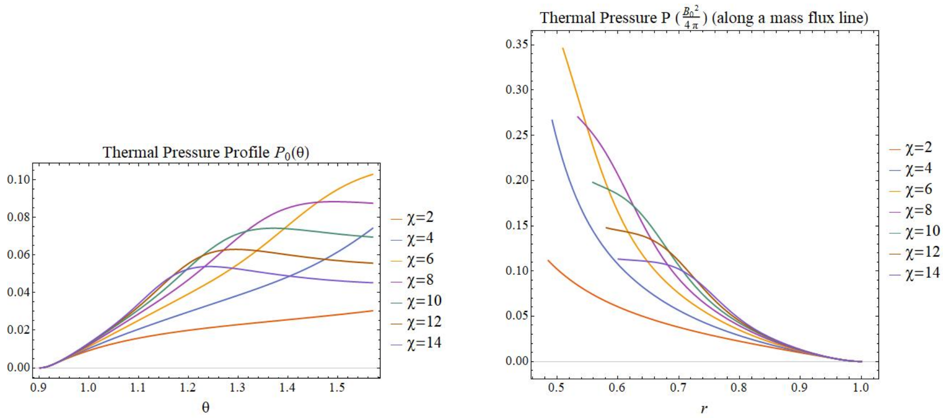


Figure 4.11: Left: Thermal pressure profile. Right: Thermal pressure along a mass flux line.

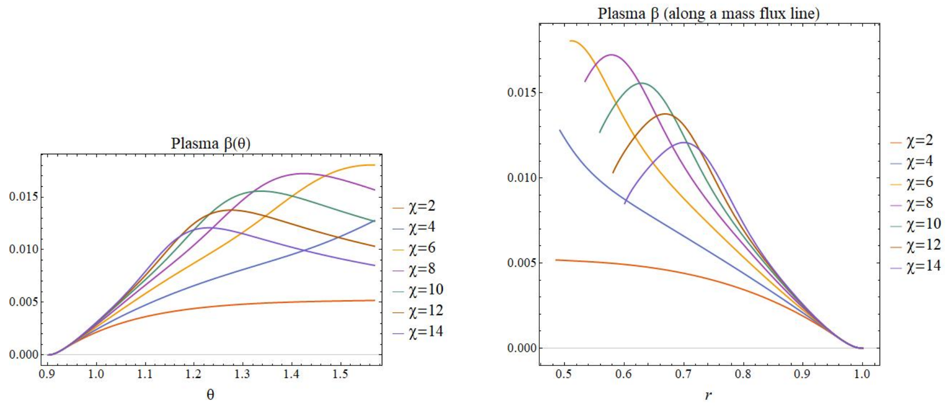


Figure 4.12: Left: Plasma β as a function of θ.
Right: Plasma β along a mass flux line.

The total energy flux per unit volume, over the rest energy flux per unit volume, both projected on the flow's mean velocity, μ, is presented in the next figure. μ has two components, a hydrodynamic one, μ_{HD}, expressing the total energy carried by the fluid, and the electromagnetic μ_{EM}, which is the Poynting flux projected on the flow's velocity, i.e. along the mass flux lines, over the rest energy flux, also along the mass flux lines.

The graphs of μ and its components clearly show that μ presents a minimum value for $\chi \geq 6$ in the solution's domain. This mimimum value appears at values of the polar angle θ where the Poynting flux over the mass flux μ_{EM} becomes minimum. These values of θ appear to coincide with the locations of the dissipated power's maximums. Additionally, for the same values of χ, the hydrodynamic term μ_{HD} appears to plateau. This is an indication that ohmic dissipation is the mechanism through which energy is transferred from the electromagnetic field to the fluid, causing the latter's heating.

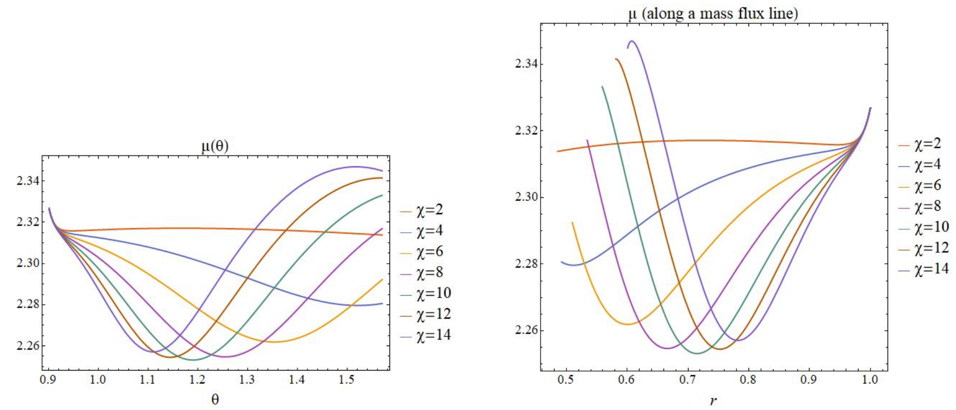


Figure 4.13: Right: The total energy flux per unit volume projected on the flow's velocity over the mass flux as a function of θ. Left: Along a mass flux line.

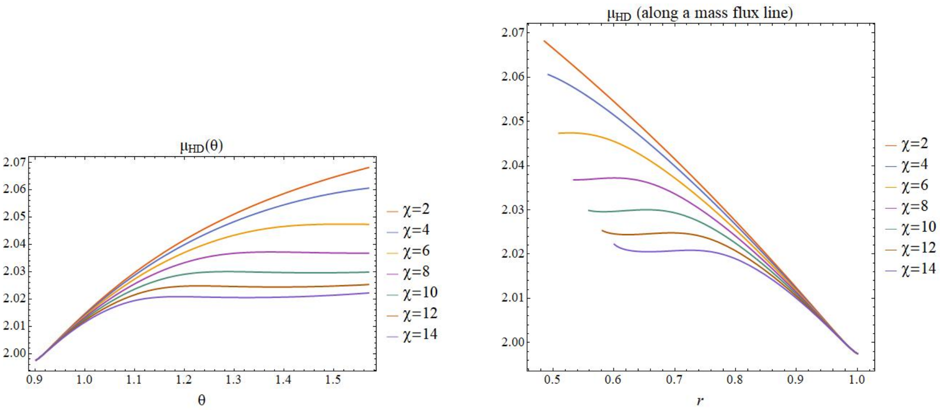


Figure 4.14: The matter component of μ.

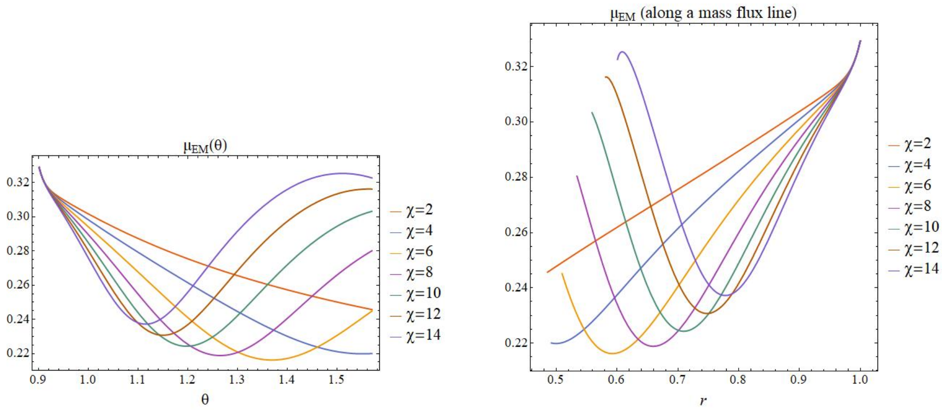


Figure 4.15: The electromagnetic component of μ (Poynting flux projected on the flow velocity $\vec{u}$, over the mass flux).

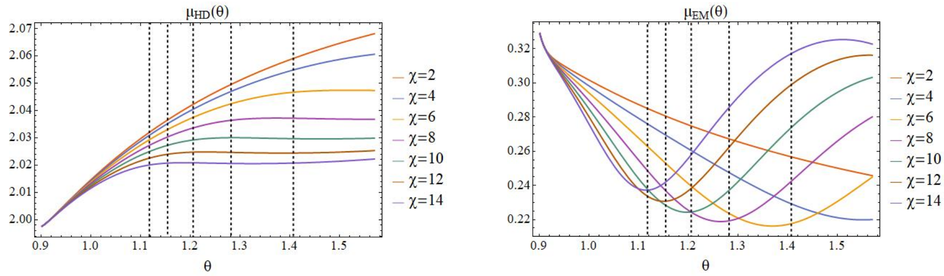


Figure 4.16: The vertical dashed lines indicate the locations of the dissipated power's maximums for each $\chi \geq 6$.

The increase of μ_{EM} after its minimum is attributed to two factors. The first is the increase of its nominator, mostly due to the angle between the mass flux and Poynting flux lines becoming smaller, The second factor is the decrease in the denominator which is the mass flux line.

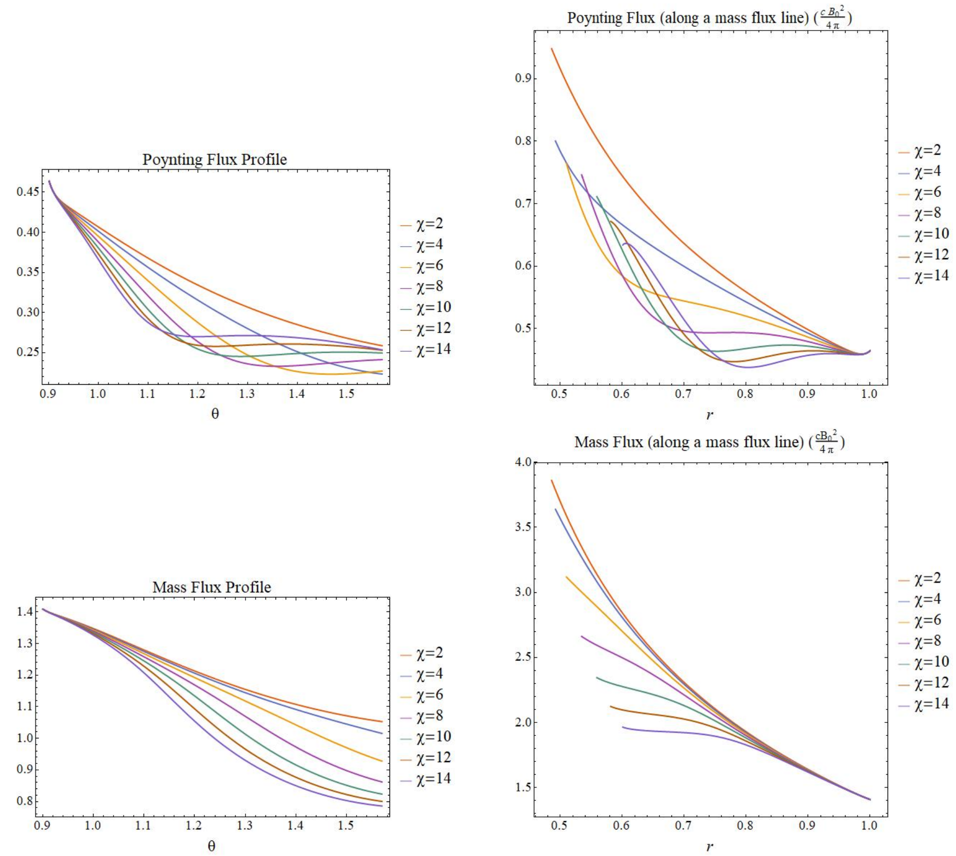


Figure 4.17: Top: The Poynting flux projection on the poloidal velocity. Bottom: The mass flux.

The value of χ has significant impact on the geometrical configuration of the velocity, magnetic, and electric fields, as can be seen by the following graphs of the angle between the electric field and the velocity and magnetic fields.

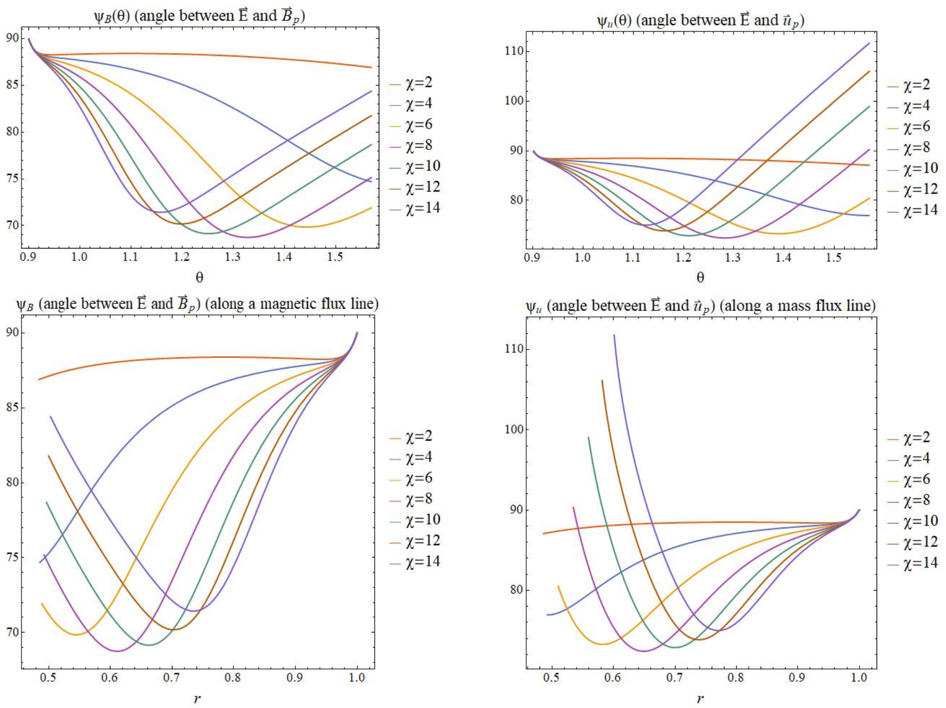


Figure 4.18: Top: The angles between the electic field and the magnetic and velocity fields as functions of θ. Bottom: The angles between the electic field and the magnetic and velocity fields along a magnetic and a mass flux line respectively.

The boundary conditions at $\theta_i = 0.9$ were given assuming the flow is in and ideal MHD state, with the electric field perpendicular to both the poloidal flow velocity and the poloidal magnetic field. This choice of boundary conditions can be seen in the above graphs, where the angles ψ_B and ψ_u are equal to $90°$. Both angles deviate from this boundary value over the domain of the solution, even for small resistivity values ($\chi = 2$), indicating that there exists an electric field component parallel to both the poloidal magnetic field and the poloidal velocity field, leading to the conclusion that particle acceleration is possible along magnetic flux or mass flux lines.

The mass flux, magnetic flux, and electric equipotential lines are presented next, for different values of χ.

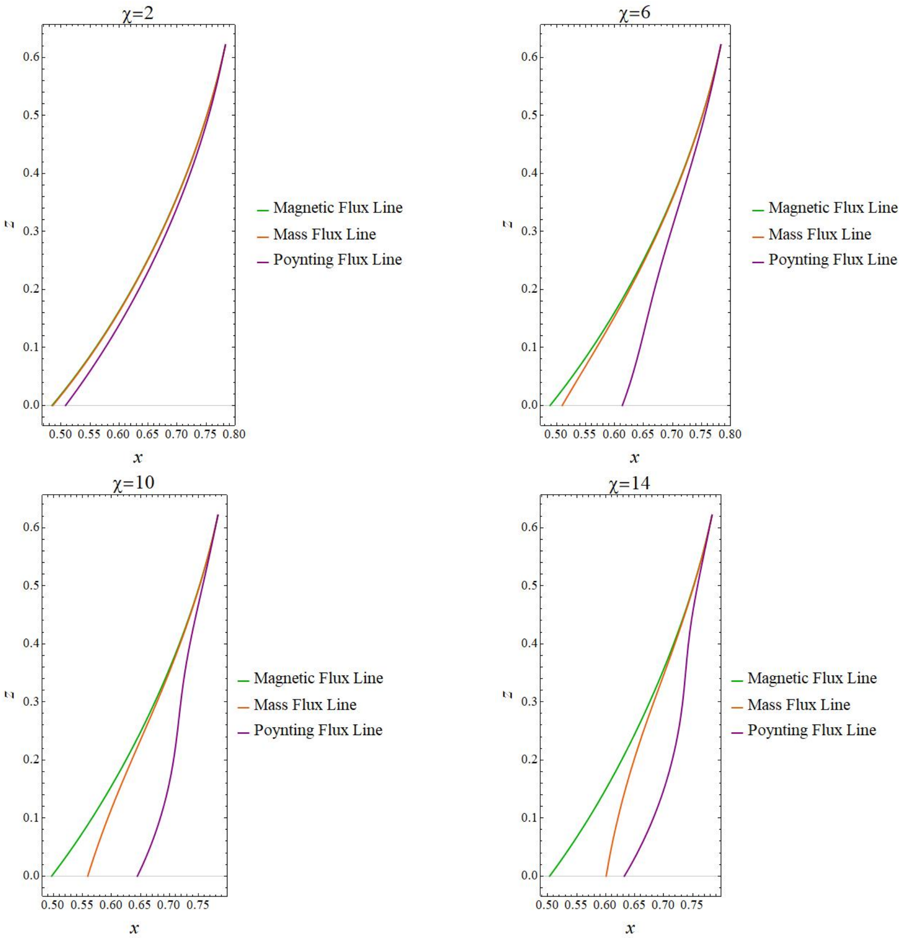


Figure 4.19: Mass flux, magnetic flux, and electric equipotential lines for $\chi = 2, 6, 10, 14$.

The larger the values of χ are, i.e. the higher the resistivity becomes over the solution's domain, the lines on the poloidal field deviate more from their initial ideal state, where all three kinds of lines coincide.

4.2.2 Effects of ϕ_{Bi} on the Flow

The effects of the flow's resistivity on its dynamics and field line configuration are modified when the angle ϕ_B between the otal magnetic field $\vec{B}$ and its azimuthal component $\vec{B}_\phi$ changes. The previous results concern a flow with an initial value of ϕ_B satisfying: $\phi_{Bi} \simeq 49°$.

In order to better understand the way the initial value of ϕ_B affects the flow's dynamics and the field line geometry, solutions for $\chi = 2, 6, 10, 14$ will be presented next, with ϕ_{Bi} as a free parameter. The flow's initial velocity in all solutions presented hereafter is the same as the one in the solutions presented above.

It is noted that the black dashed line in all graphs for $\chi = 2$, represents the respective quantity in the case that the magnetic and electric field are configured in such away that the Poynting flux in the direction of the flow velocity is near zero.

The flow's Lorentz factor along a mass flux line:

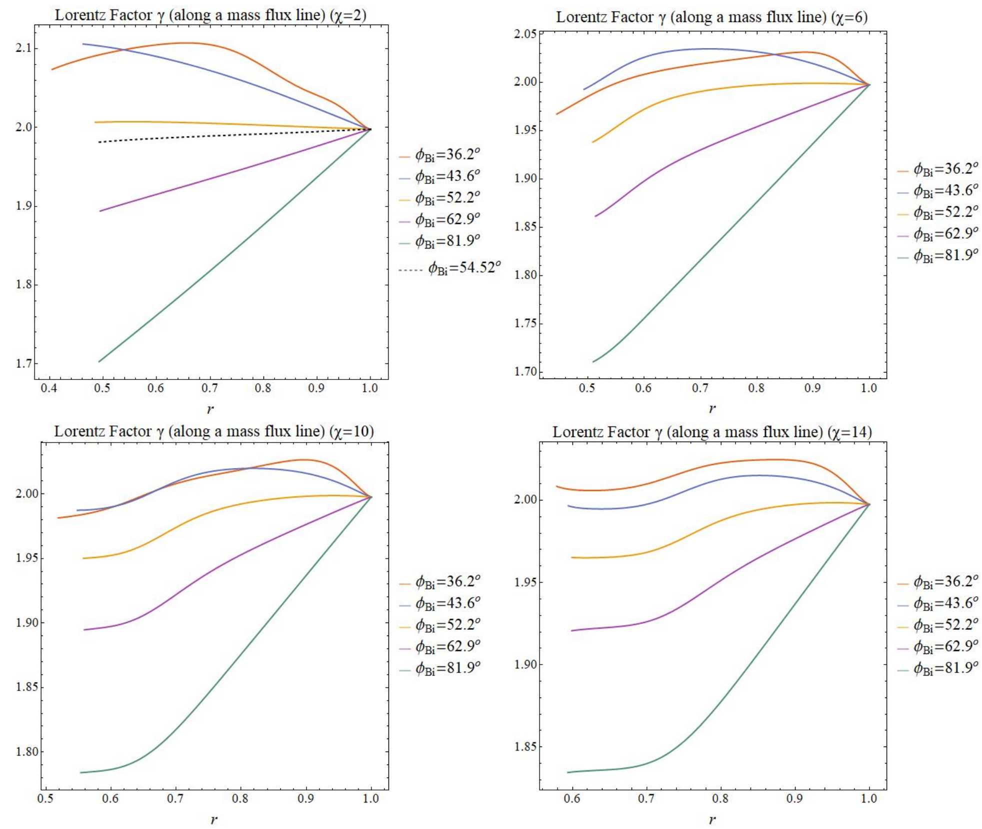


Figure 4.20: Top: Lorentz factor for $\chi = 2, 6$. Bottom: Lorentz factor for $\chi = 10, 14$.

For large initial values of the angle ϕ_B, namely $\phi_{Bi} = 62.9°$ and $\phi_{Bi} = 81.9°$, the flow's Lorentz factor decreases over the entirety of the solution's domain, with the greatest increase for $\chi = 2, 6$. Another interesting observation is that for $\chi = 2$ and $\phi_{Bi} = 52.2°$, the Lorentz factor shows a very slight increase, remaining relatively steady around a value of 2.

The power per unit volume dissipated through ohmic dissipation is also modified, with significant impact on the heating of the fluid.

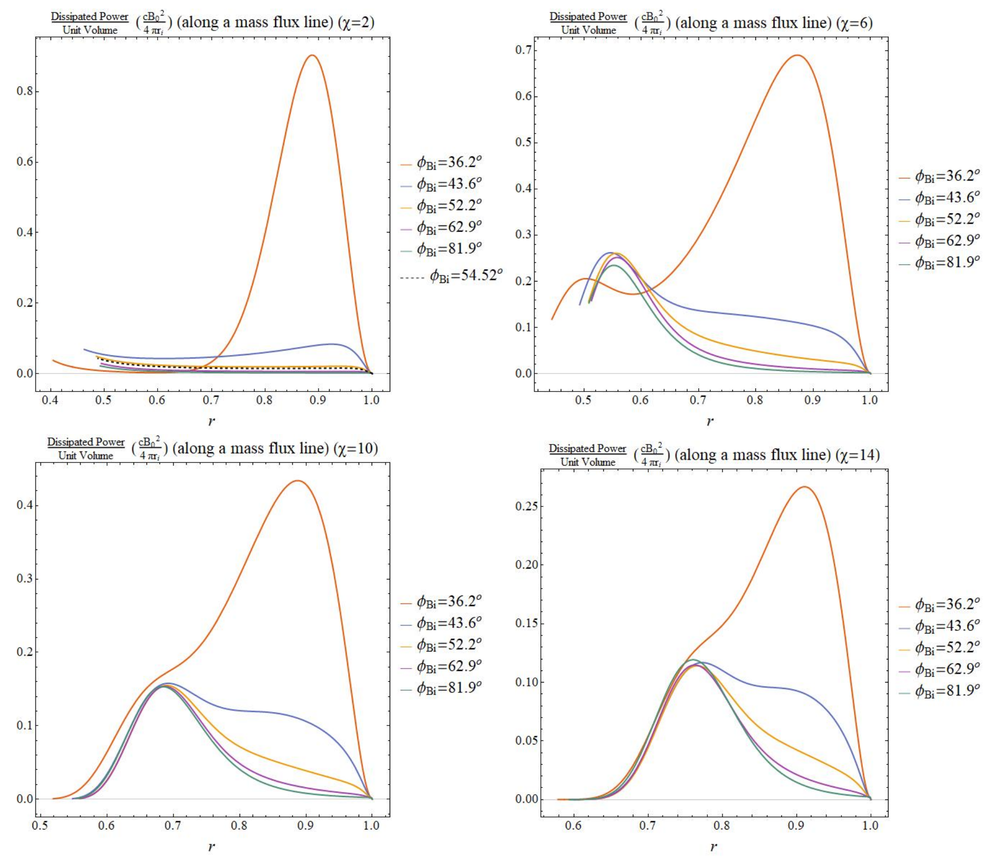


Figure 4.21: Top: Dissipated power per unit volume for $\chi = 2, 6$. Bottom: Dissipated power per unit volume for $\chi = 10, 14$.

For an initially strongly azimuthal magnetic field $\phi_{Bi} = 36.2°$, dissipation is strongest and displays a primary peak near $r(\theta_i) = \varpi_0$, for all four χ values, as well as a secondary one near $r(\theta_f)$, which is more clearly seen in the graphs corresponding to $\chi = 6, 10$. For $\chi = 6, 10, 14$, flows with an initially strongly poloidal field, display only the secondary peak, at the same $r(\theta)$ as the flow characterized by $\phi_{Bi} = 36.2°$.

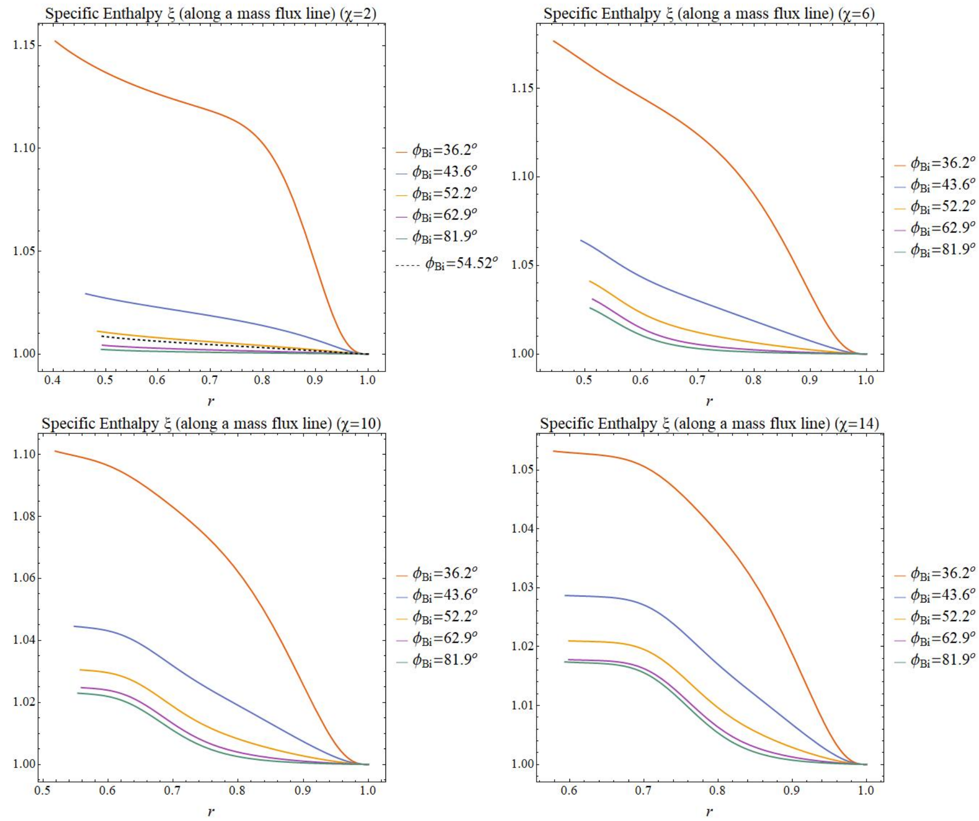


Figure 4.22: Top: Specific enthalpy for $\chi = 2, 6$. Bottom: Specific enthalpy for $\chi = 10, 14$.

As expected, flows with an initially mostly azimuthal magnetic field show the greatest increase in their specific enthalpy, since the display the most intense dissipation, of all the cases examined. Moreover, $\chi = 6$ leads to the most significant heating for all initial magnetic field confgurations.

The effect of ϕ_{Bi} on the plasma's mass density is presented in the next following figure.

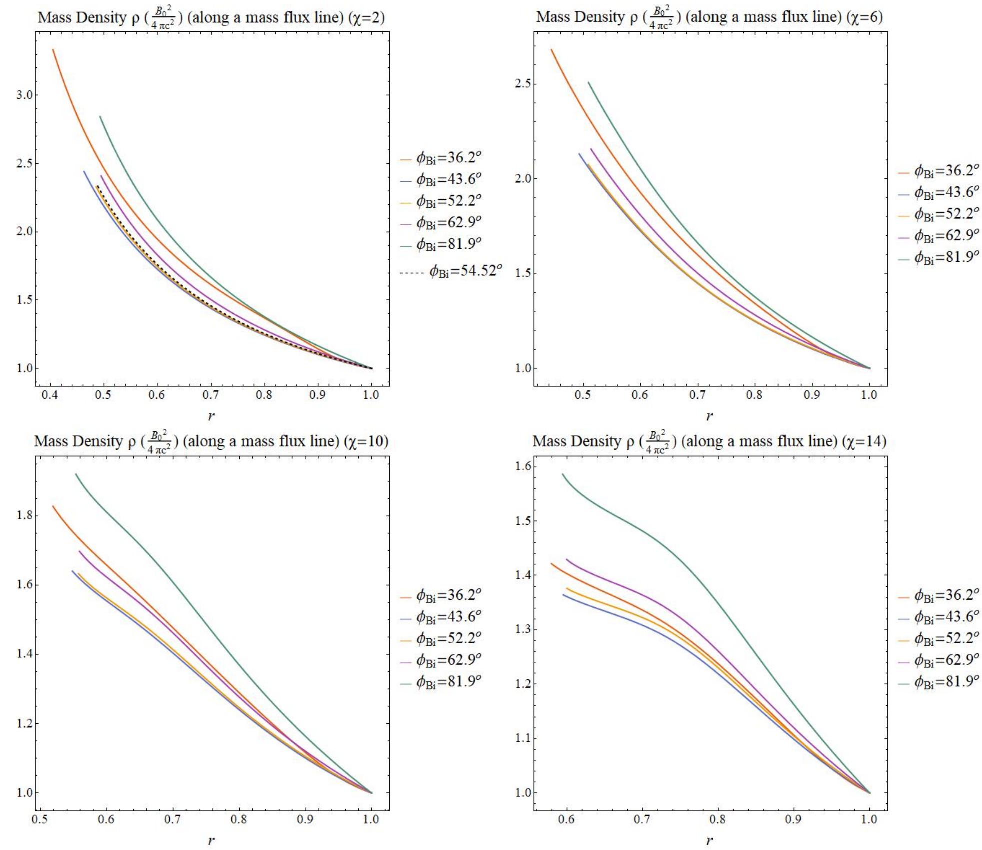


Figure 4.23: Top: Mass density for $\chi = 2, 6$. Bottom: Mass density for $\chi = 10, 14$.

The flow's mass density shows the same qualitative behaviour for all examined magnetic field configurations

Graphs of the magnetic pressure, the thermal pressure, and of the plasma beta follow.

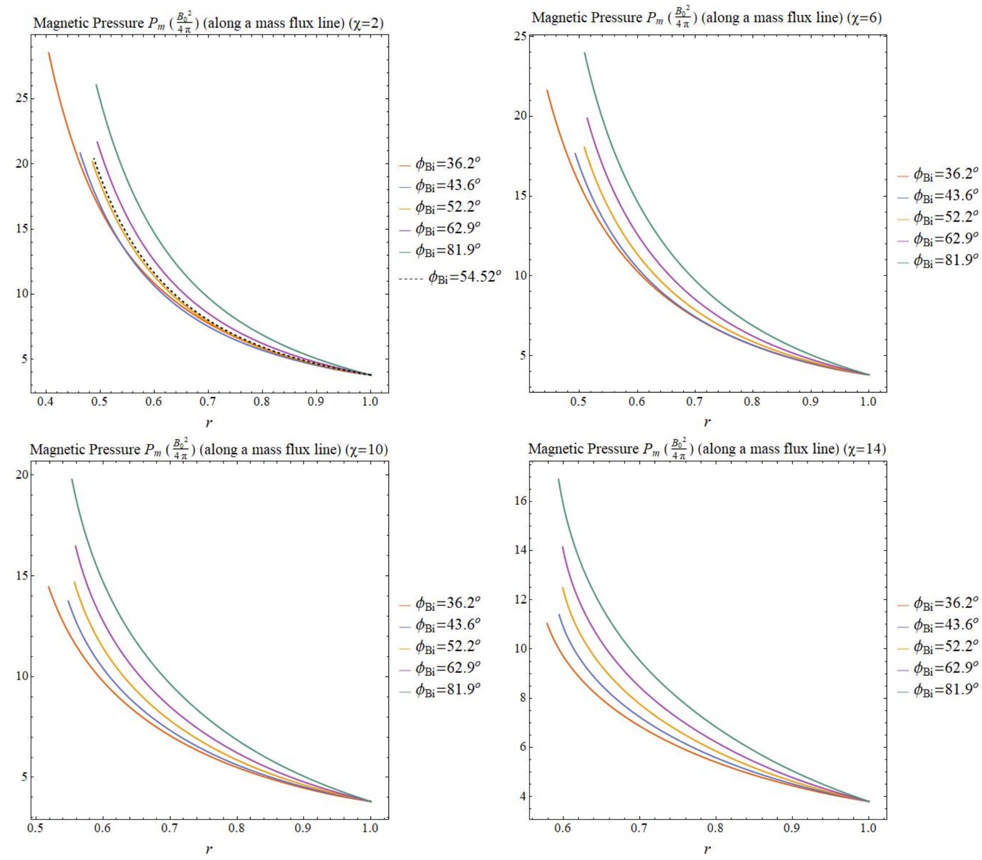


Figure 4.24: Top: Magnetic pressure for $\chi = 2, 6$. Bottom: Magnetic pressure for $\chi = 10, 14$.

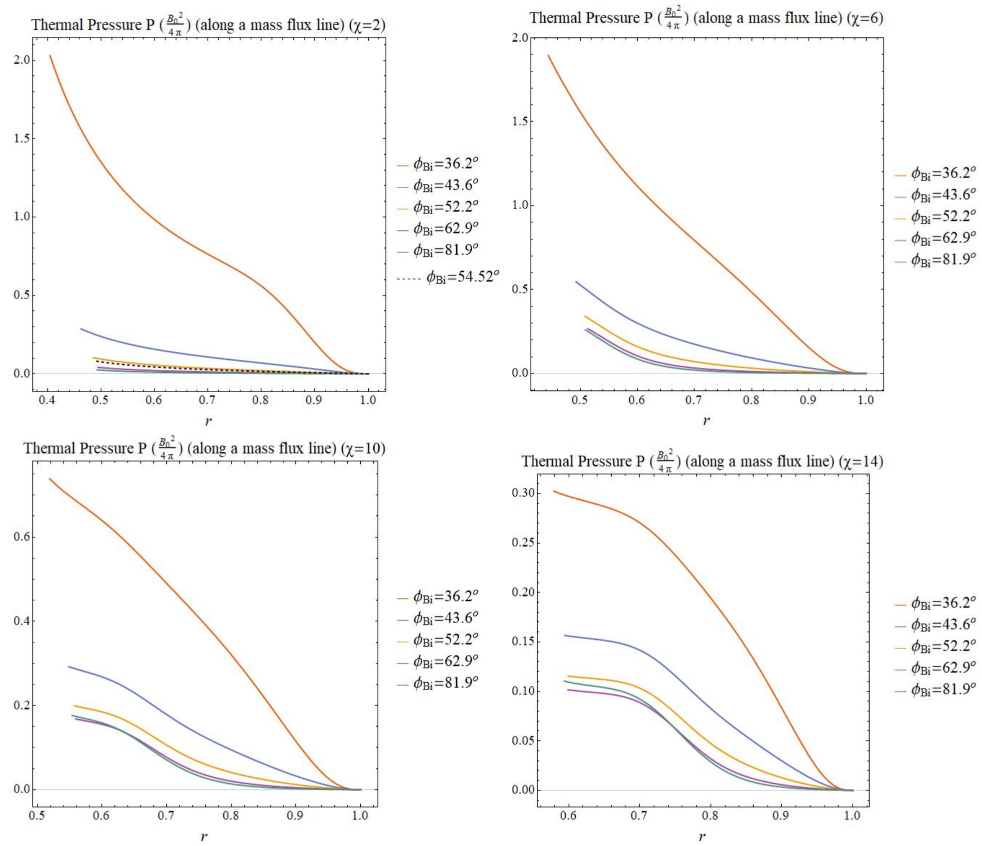


Figure 4.25: Top: Thermal pressure for $\chi = 2, 6$. Bottom: Thermal pressure for $\chi = 10, 14$.

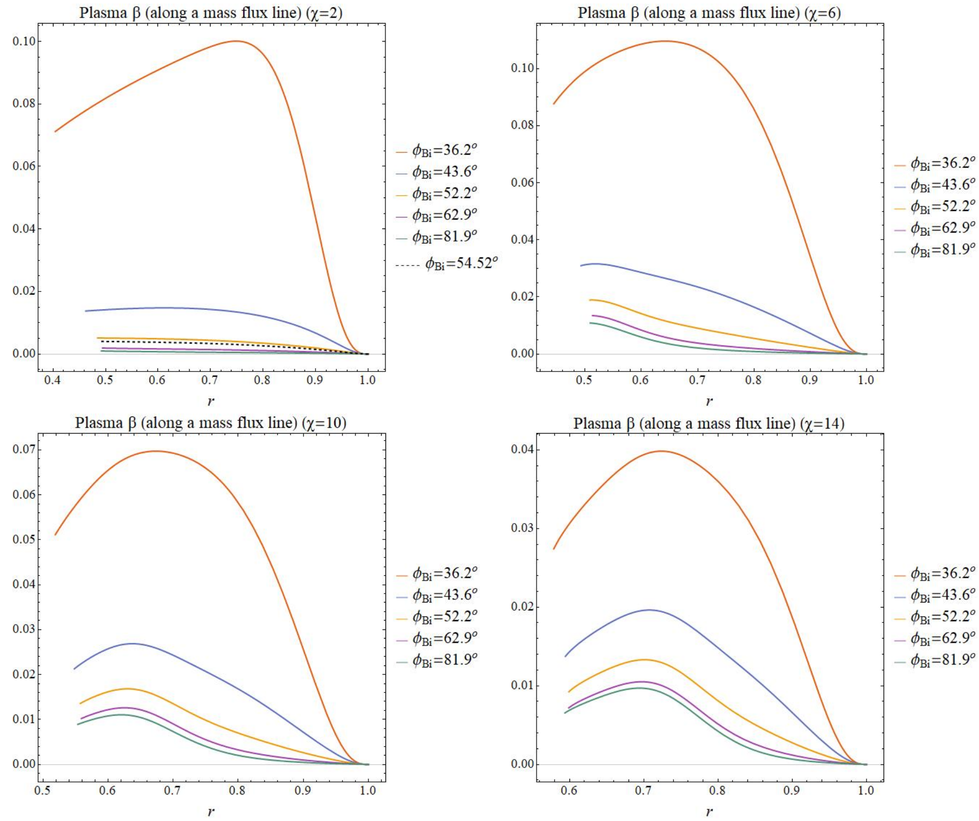


Figure 4.26: Top: Plasma β for $\chi = 2, 6$. Bottom: Plasma β for $\chi = 10, 14$.

The energy flux along the mass flux lines, especially its electromagnetic component (Poynting flux over mass flux), displays interesting behaviour for mostly poloidal initial magnetic fields (large values of ϕ_{Bi}).

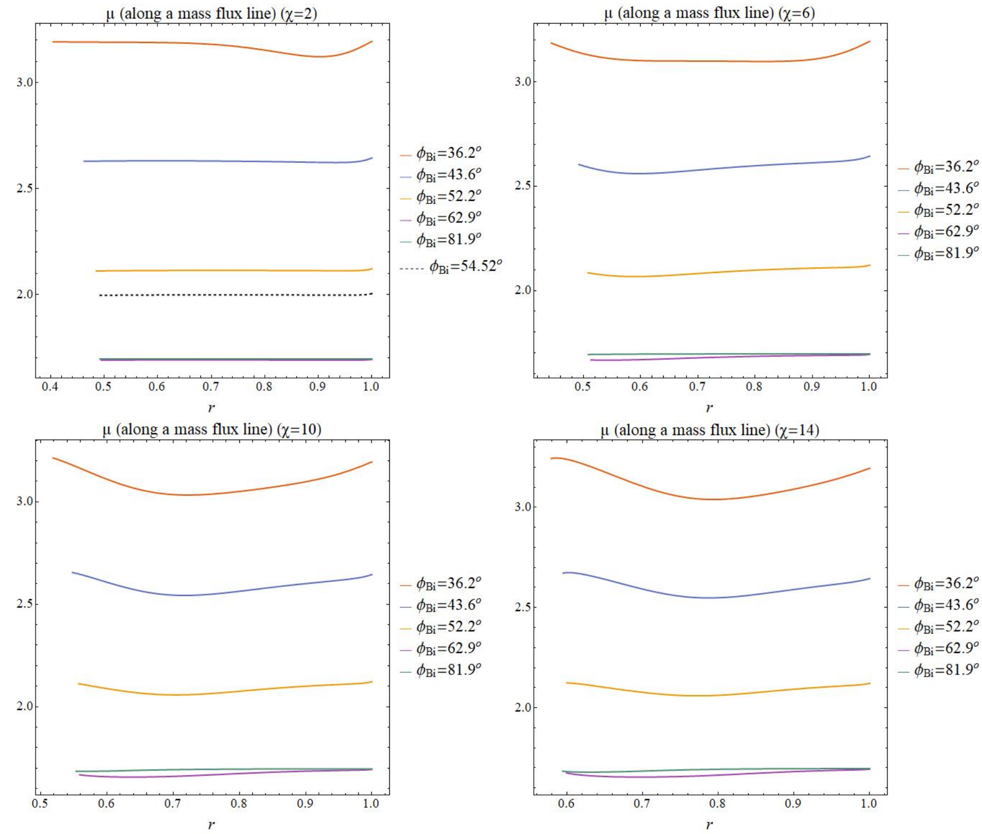


Figure 4.27: Top: Energy flux along mass flux lines μ for $\chi = 2, 6$. Bottom: Energy flux along mass flux lines μ for $\chi = 10, 14$.

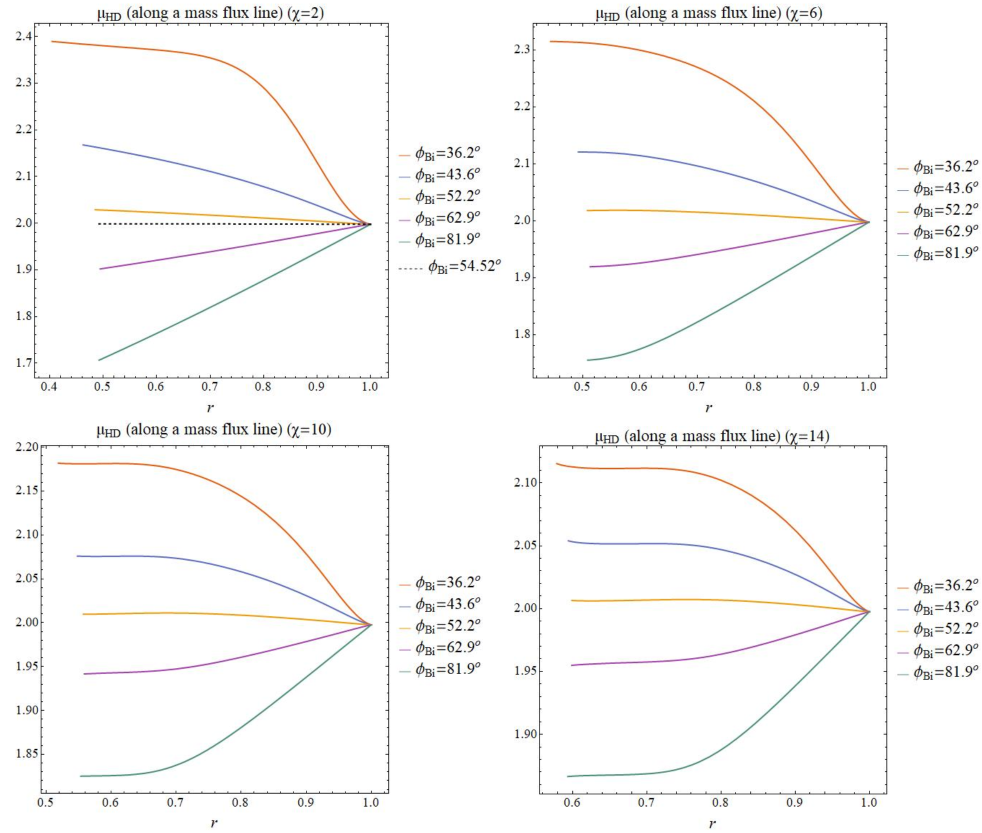


Figure 4.28: Top: The matter component of μ for $\chi = 2, 6$. Bottom: The matter component of μ for $\chi = 10, 14$.

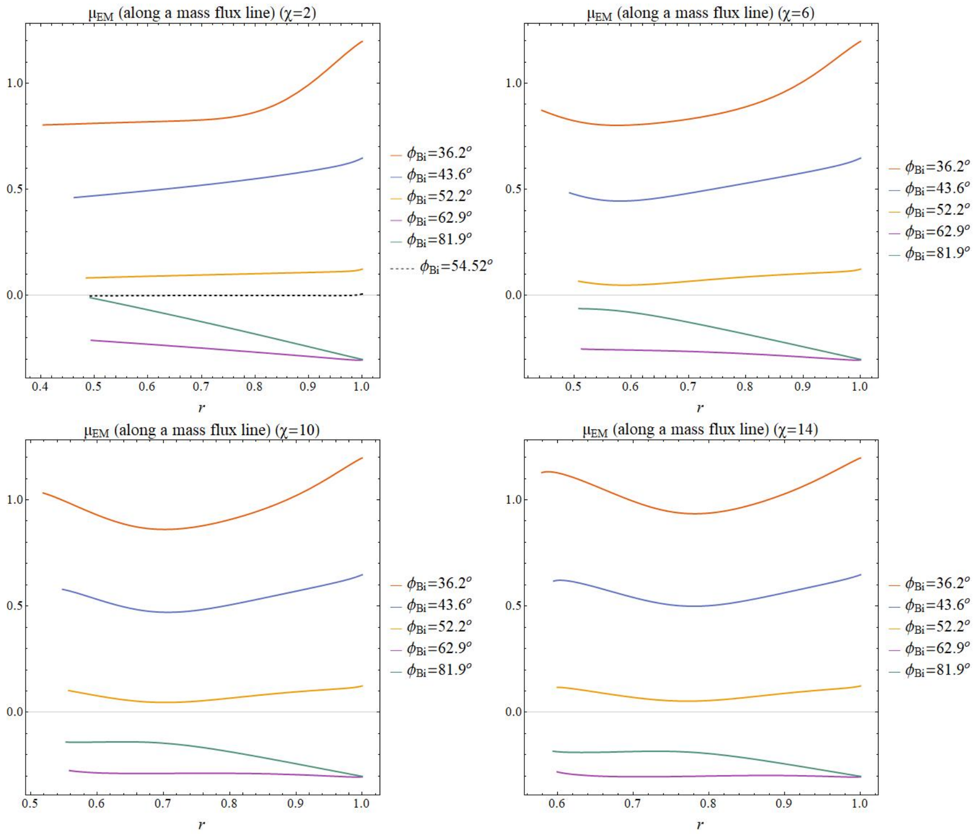


Figure 4.29: Top: The electromagnetic component of μ for $\chi = 2, 6$. Bottom: The electromagnetic component of μ for $\chi = 10, 14$.

For $\phi_{Bi} = 62.9°, 81.9°$, the matter component of μ decreases, while the electromagnetic component is negative and displays an increases, resulting in an increase of the total energy flux along the mass flux lines. For these same initial values of ϕ_B, the Poynting flux, projected on the flow velocity, over the mass flux μ_{EM} is negative, meaning that it flows from the equatorial plane $z = 0$ towards larger z values, in contrast with the mass flux, and the matter component of μ.

For $\chi = 2$ and $\phi_{Bi} = 54.52°$, the Poynting flux along mass flux lines is nearly zero. So, this initial configuration of the magnetic field leads to a Poynting vector almost vertical to the poloidal flow velocity, over the solution's domain.

This behaviour can be better understood by examining the geometric configuration of the poloidal fields and the Poynting flux lines.

Plots of the angle ψ between the magnetic field $\vec{B}$ and the flow velocity $\vec{u}$ follow.

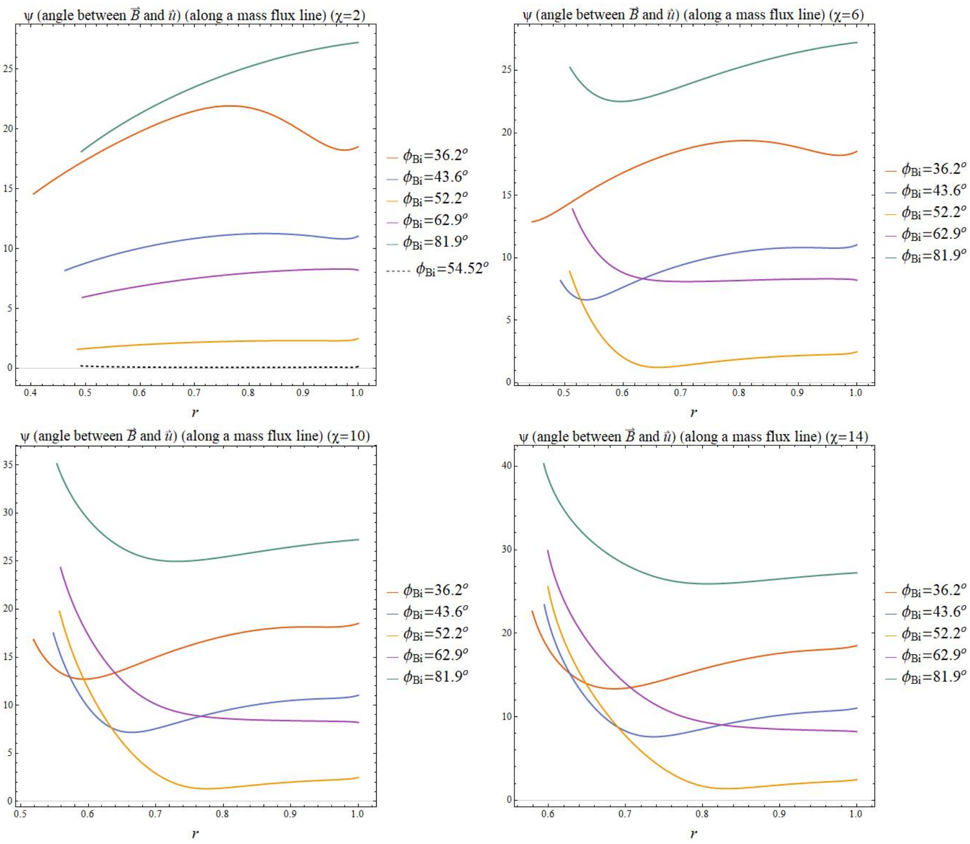


Figure 4.30: Top: The angle between $\vec{B}$ and $\vec{u}$ for $\chi = 2, 6$. Bottom: The angle between $\vec{B}$ and $\vec{u}$ for $\chi = 10, 14$.

For $\chi = 2$ and $\phi_{Bi} = 54.52°$, the magnetic field and flow velocity are approximately parallel, which is the reason why the Poynting flux is to a large degree perpendicular to the velocity field ($\mu_{EM} = 0$). Moreover, the electric field is strongly parallel to the magnetic field, renedering the Poynting flux much less significant compared to that of the othe magnetic field configurations, as shown in the following figure.

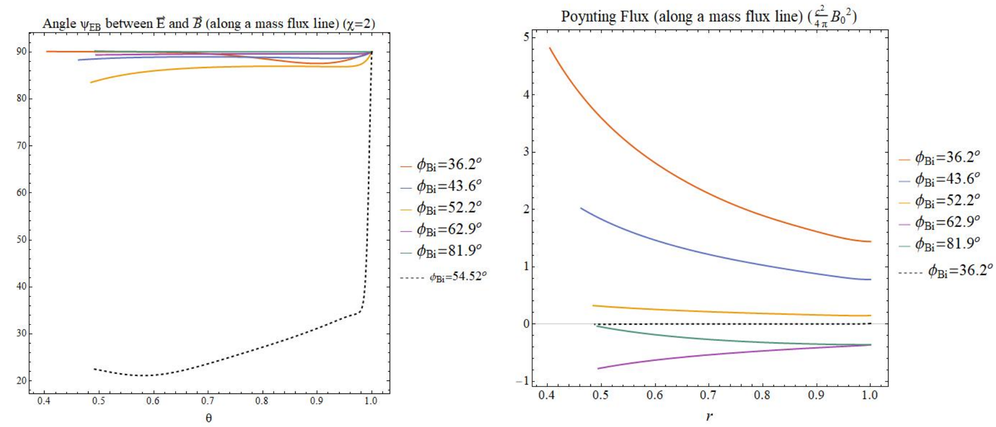


Figure 4.31: Left: The angle between $\vec{E}$ and $\vec{B}_p$ for $\chi = 2$, $\phi_{Bi} = 54.52°$. Right: The Poynting flux projected on the flow velocity for $\chi = 2$, $\phi_{Bi} = 54.52°$.

The mass flux, magnetic flux, and Poynting flux lines are presented in the next figure, for $\chi = 2$ and $\phi_{Bi} = 52.54°$ ($\mu_{EM} \simeq 0$).

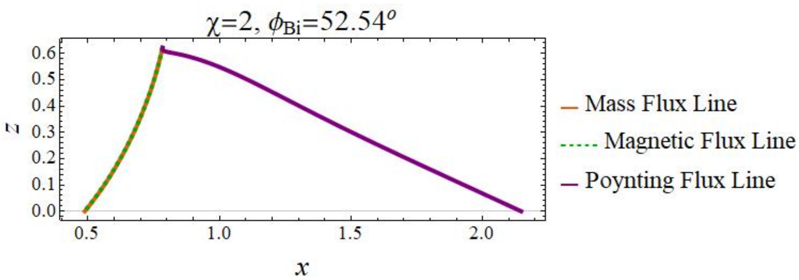


Figure 4.32: Poloidal mass, magnetic, and Poynting flux lines.

The poloidal field and Poynting flux lines are also presented for the case of $\chi = 2$, $\phi_{Bi} = 81.9°$.

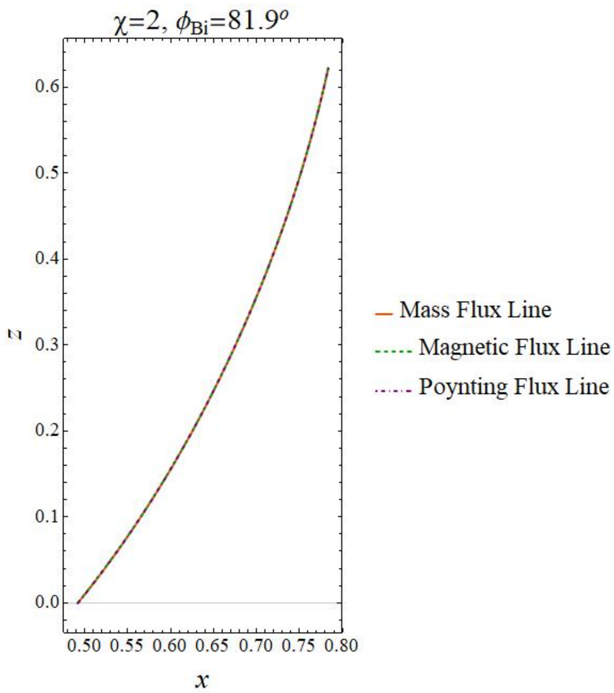


Figure 4.33: Poloidal mass, magnetic, and Poynting flux lines.

All three lines in figure 4.31 coincide, which is a characteristic of ideal, perfectly conductive magnetized flows. This is due to the fact that $\chi = 2$ does not lead to a strong increas of the resistivity's value.

4.3 Gaussian Resistivity Profile

Solutions for $\chi = 70, 80, 90$ are presented in this section. These values of χ were chosen so that the electric field of the flow, which is perpendicular to the velocity and magnetic fields at $\theta_i = 0$, ends up in the same configuration at θ_f.

The solutions presented below concern flows with the same boundary conditions at $\theta_i = 0.9$ as the ones in the first subsection of the previous section.

The three Gaussian resistivity profiles studied are shown in the next figure.

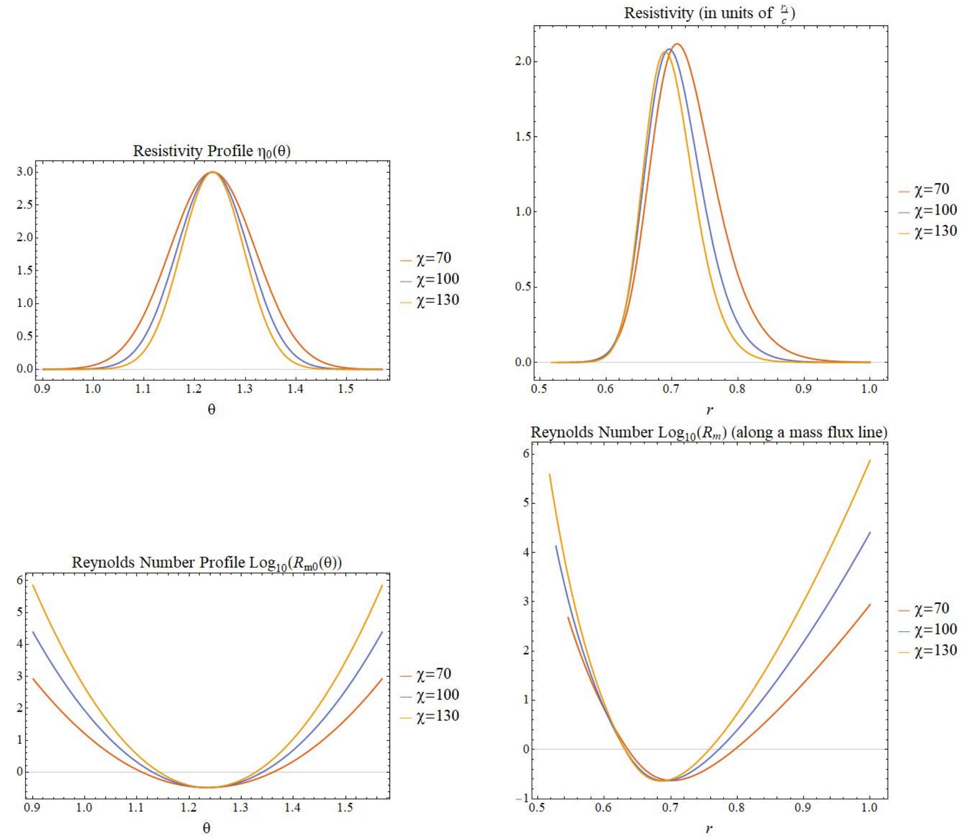


Figure 4.34: Top: The resistivity profile and resistivity along a mass flux line for $\chi = 70, 100, 130$. Bottom: The Reynolds number profile and Reynolds number along a mass flux line for $\chi = 70, 100, 130$.

The flow accelerates before decelarating slightly over a wide region of the solution's domain. Decelaration then becomes important as the resistivity's value begins to decrease, as can be seen in the following graph of the Lorentz factor.

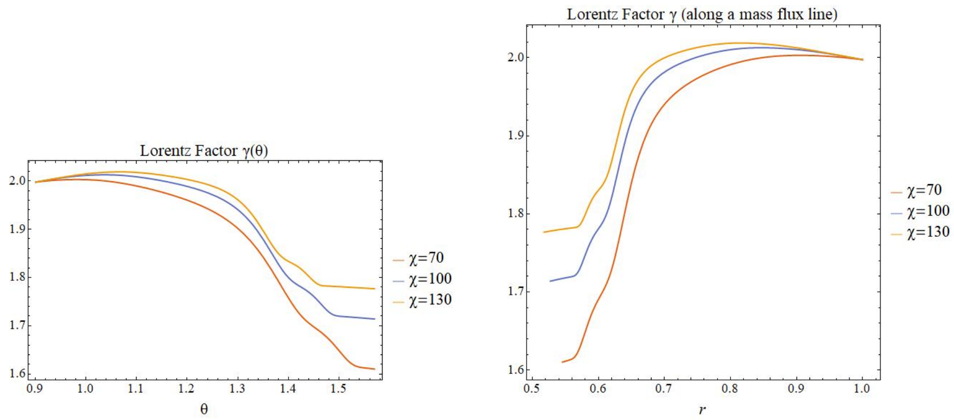


Figure 4.35: Left: The Lorentz factor for $\chi = 70, 100, 130$. Right: The Lorentz factor along a mass flux line.

The decrease of the Lorentz factor becomes less significant for greater values of χ.

The specific enthalpy and temperature exhibit the greatest increase for the smallest value of χ.

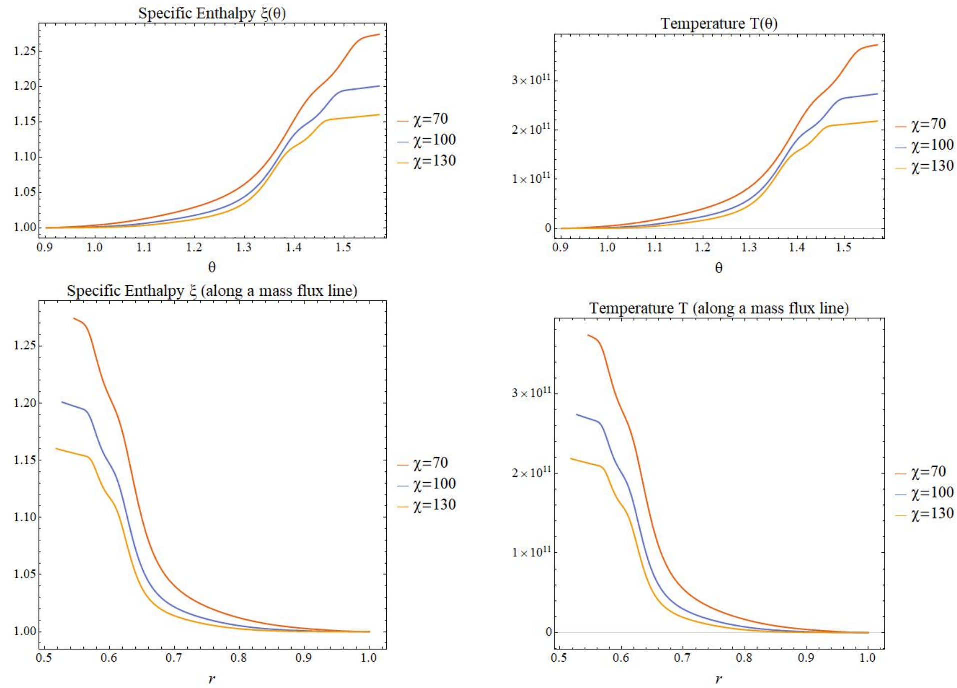


Figure 4.36: Top: The specific enthalpy and temperature as functions of the polar angle θ. Bottom: The specific enthalpy and temperature along a mass flux line.

The source of the fluid's heating, is once again the power dissipated through ohmic dissipation, due to the flow's finite resistivity.

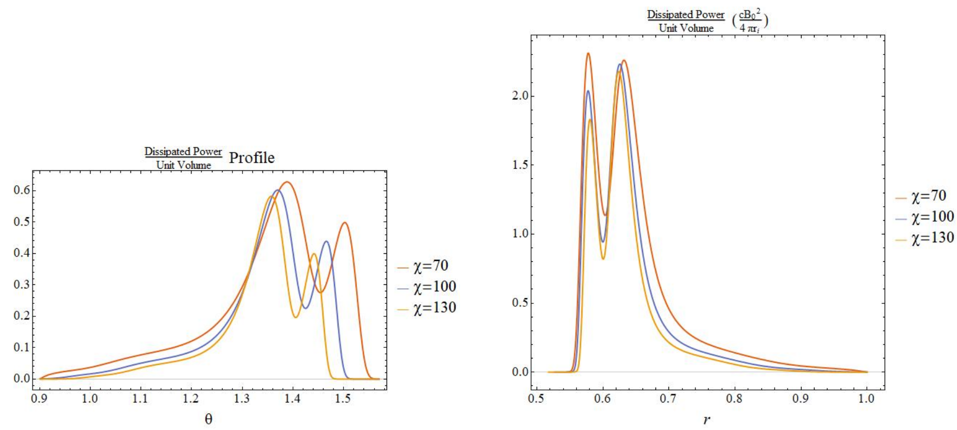


Figure 4.37: Left: The dissipated power per unit volume profile. Right: Power per unit volume dissipated along a mass flux line.

The flow's mass density, magnetic and thermal pressure, and plasma β

follow.

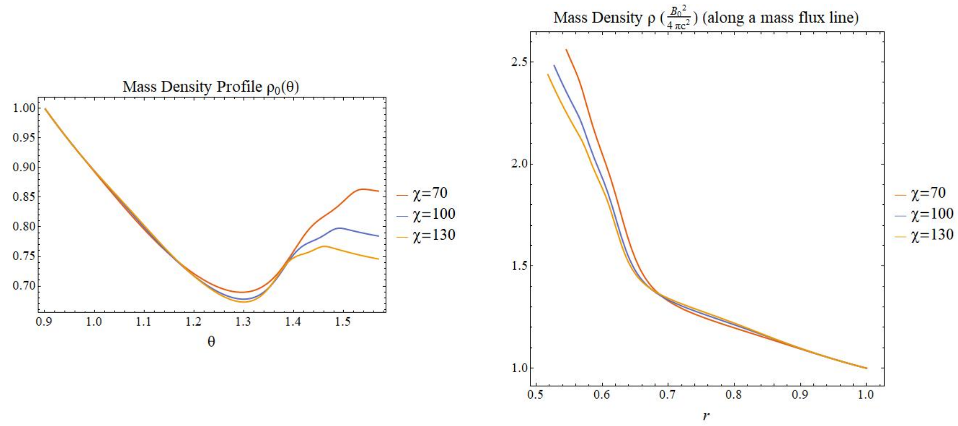


Figure 4.38: Left: Mass density profile $\rho_0(\theta)$. Right: Mass density along a mass flux line.

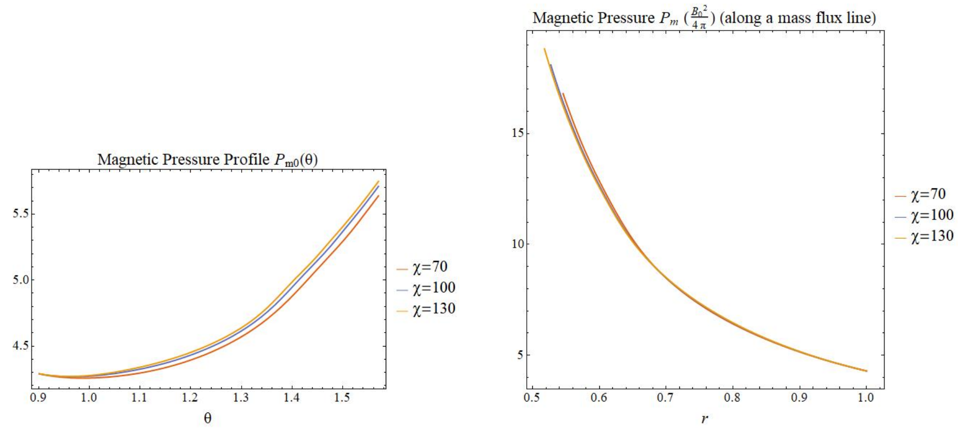


Figure 4.39: Left: Magnetic pressure profile. Bottom: Magnetic pressure along a mass flux line.

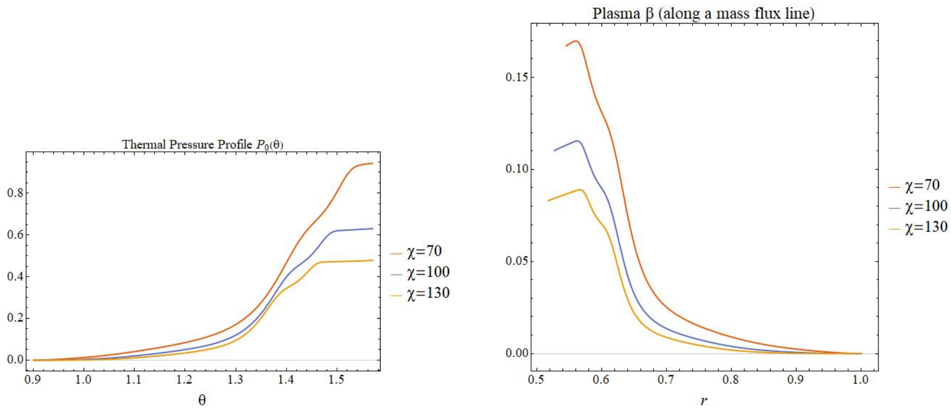


Figure 4.40: Left: Thermal pressure profile. Right: Thermal pressure along a mass flux line.

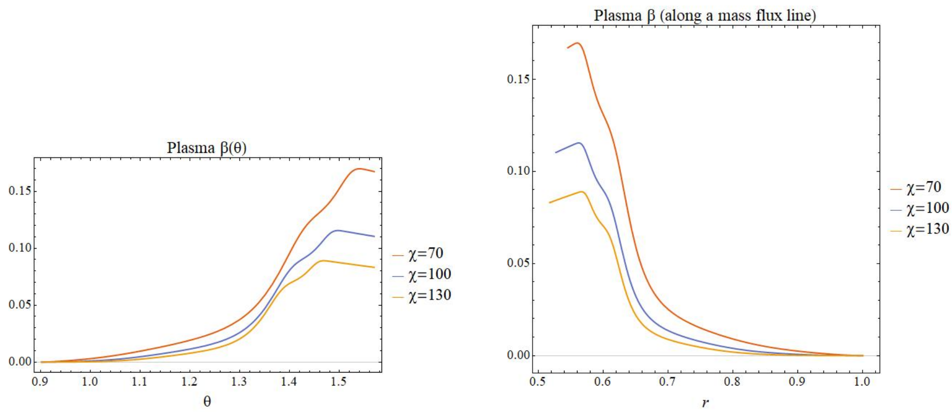


Figure 4.41: Left: Plasma β as a function of θ.
Right: Plasma β along a mass flux line.

The total energy flux projected on the poloidal flow velocity, over the mass flux, μ, its hydrodynamic component μ_{HD}, and its electromagnetic component (Poynting flux over mass flux) μ_{EM}, follow.

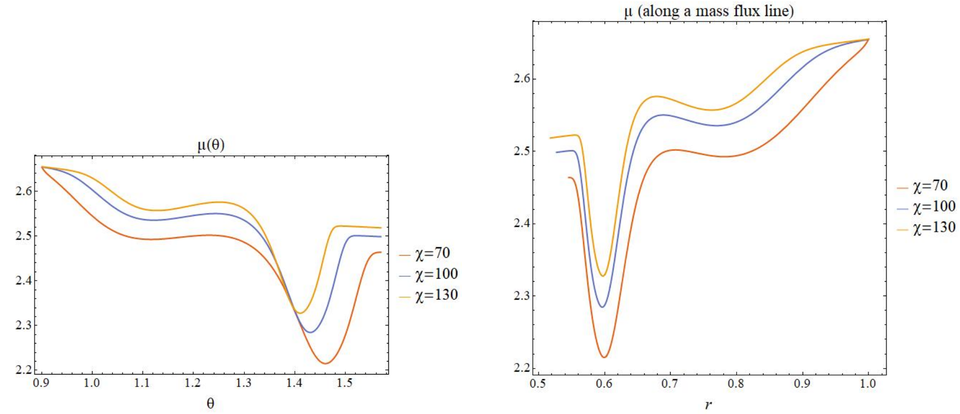


Figure 4.42: Right: The total energy flux per unit volume projected on the flow's velocity over the mass flux as a function of θ. Left: Along a mass flux line.

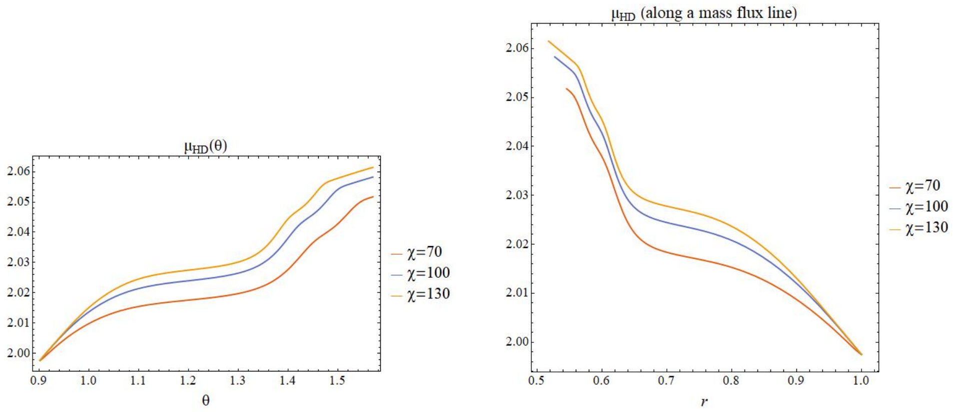


Figure 4.43: The matter component of μ.

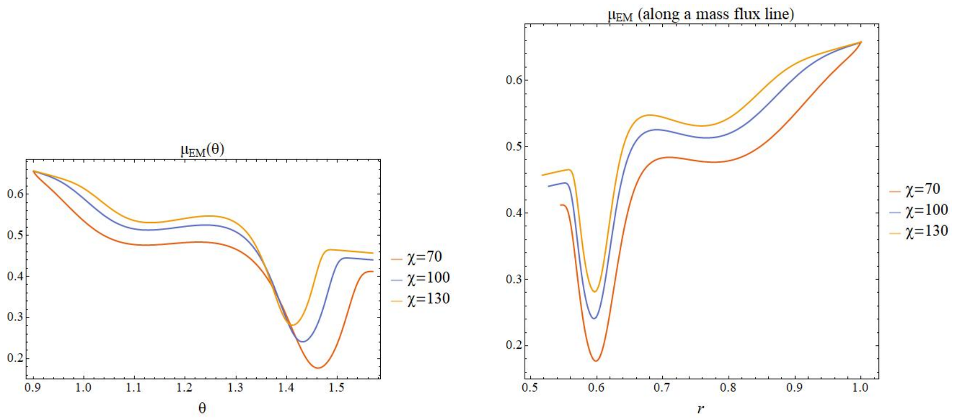


Figure 4.44: The electromagnetic component of μ (Poynting flux projected on the flow velocity $\vec{u}$, over the mass flux).

The strong decrease in the value of $\mu_{EM}(\theta)$ near $\theta = 1.4$, and its subsequent increase is attributed to the geometric configuration of the mass flux and Poynting flux lines. This can be seen clearly in figure 4.46 at the end of the section.

Field line geometry displays interesting behaviour as can be seen in the following plots of the angles between the electric field and the magnetic and velocity fields.

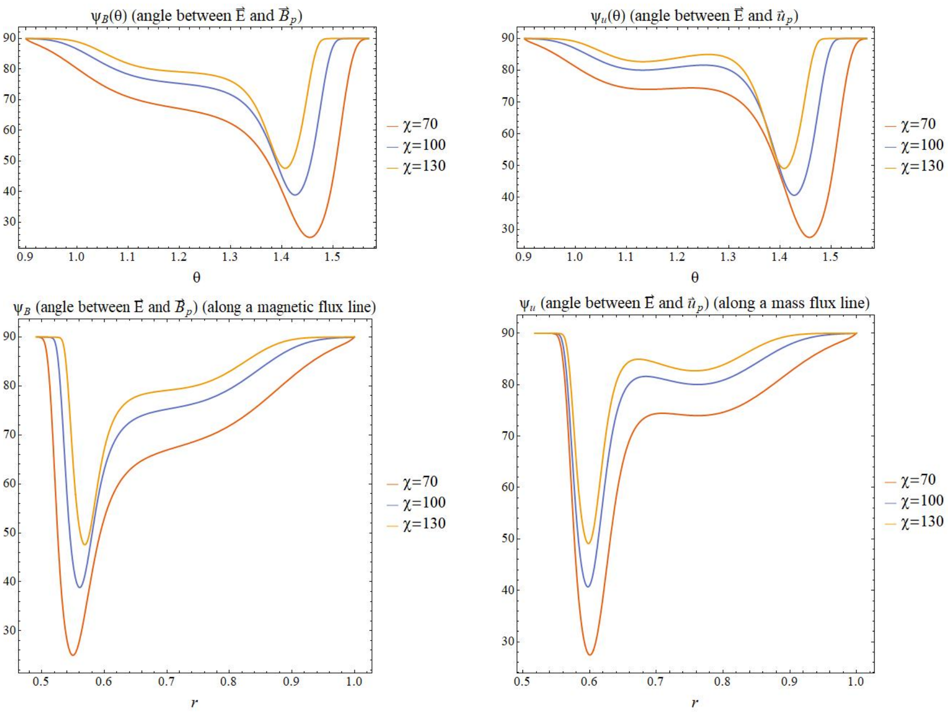


Figure 4.45: Top: The angles between the electic field and the magnetic and velocity fields as functions of θ. Bottom: The angles between the electic field and the magnetic and velocity fields along a magnetic and a mass flux line respectively.

The electric field at $\theta = 0.9$ is perpendicular to both the magnetic and velocity fields. The angles ψ_B and ψ_u take values smaller than $90°$, showing a significant decrease after the solution's domain's middle point, where the resistivity profile peaks. The electric field then has large parallel components to both the magnetic and velocity fields, before becoming perpendicular to both said fields near $\theta = \frac{\pi}{2}$.

Graphs of the mass flux, magnetic flux, and Poynting flux lines are presented next.

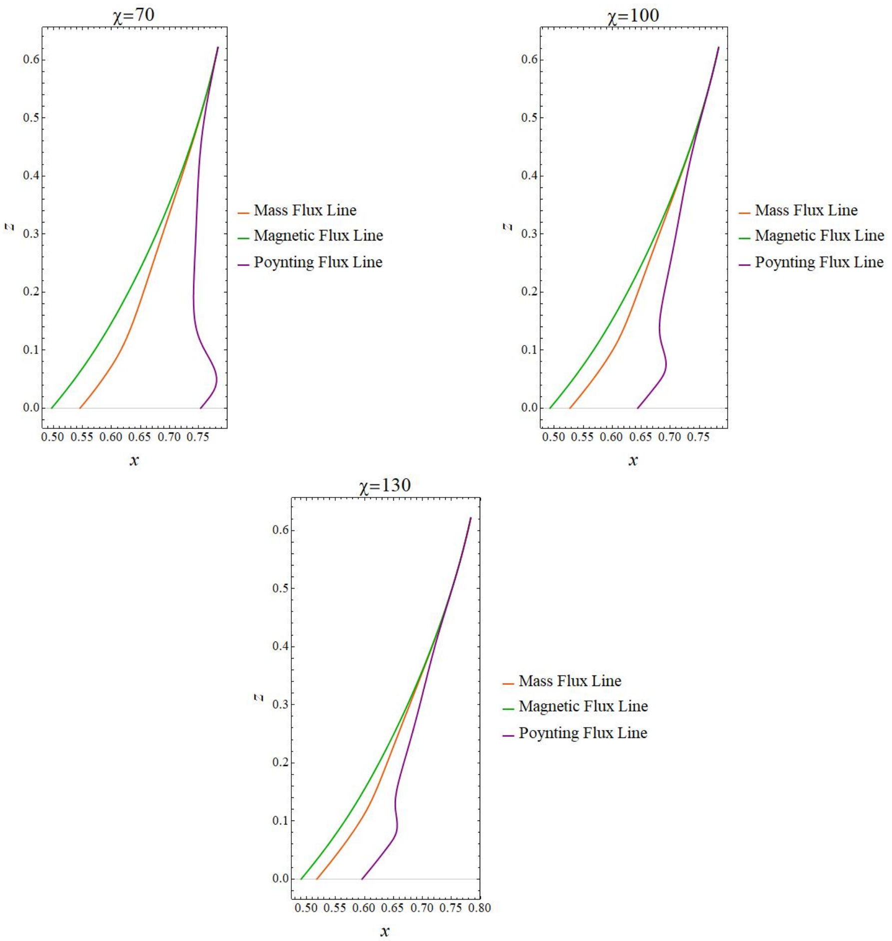


Figure 4.46: Mass flux, magnetic flux, and Poynting flux lines for $\chi = 70, 100, 130$.

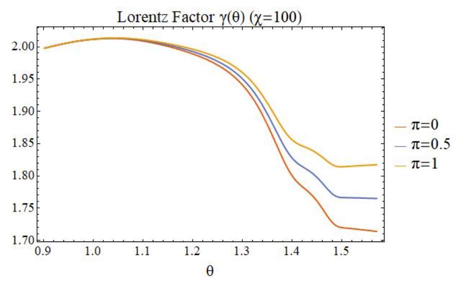
Chapter 5

Radiative Flows

The effects of two different resistivity profiles, with a wide range of values, on non-radiative flows was studied in detail in the previous chapter. In this chapter, the modification of the previously presented solutions due to radiative losses is examined. The flow's radiative losses are modeled as discussed in subsection 3.4.2 of the third chapter. The solutions presented describe radiative flows characterised by a Gaussian resistivity profile with $\chi = 100$.

5.1 Radiative Flow Solutions

The solutions presented in this section describe radiative flows characterised by $\pi = 0.5, 1$, with π the fraction of the power per unit volume dissipated through ohmic dissipation escaping the system in the form of radiative losses. The non-radiative solutions are included for comparison.



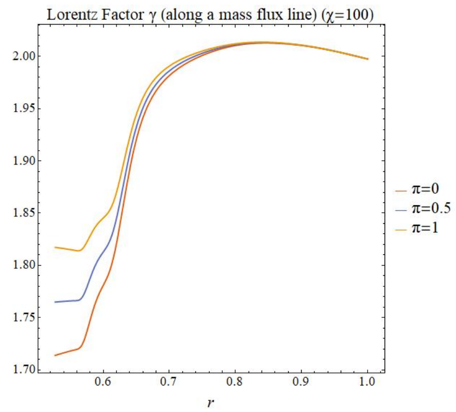


Figure 5.1: Left: The flow's Lorentz factor as a function of the polar angle θ. Right: The flow's Lorentz factor along a mass flux line.

The flow's Lorentz factor displays similar qualitative behaviour regardless of the value of π. Flow deceleration is less significant for $\pi = 1$, while near $\theta = \frac{\pi}{2}$, the Lorentz factor displays a slight increase.

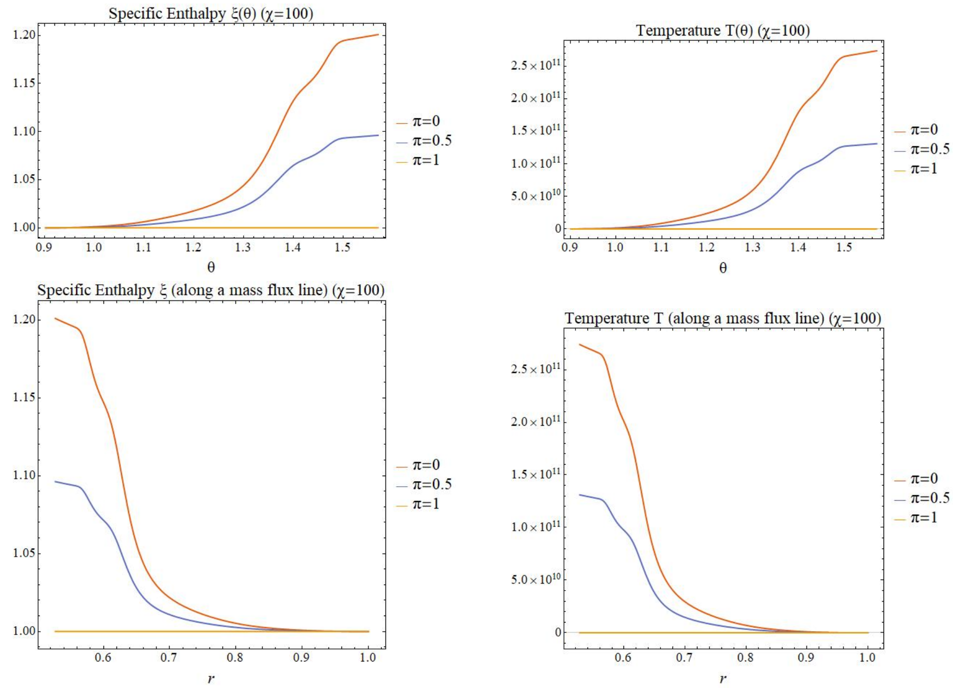


Figure 5.2: Top: The specific enthalpy and temperature as functions of the polar angle θ. Bottom: The specific enthalpy and temperature along a mass flux line.

As expected, for $\pi = 0.5$ the flow's temperature's value at $\theta = \dfrac{\pi}{2}$ is only half the respective value for the case of a non-radiative flow, since only half the dissipated energy is available for the flow's heating. For $\pi = 1$ the temperature stays practically constant, as all dissipated energy is radiated.

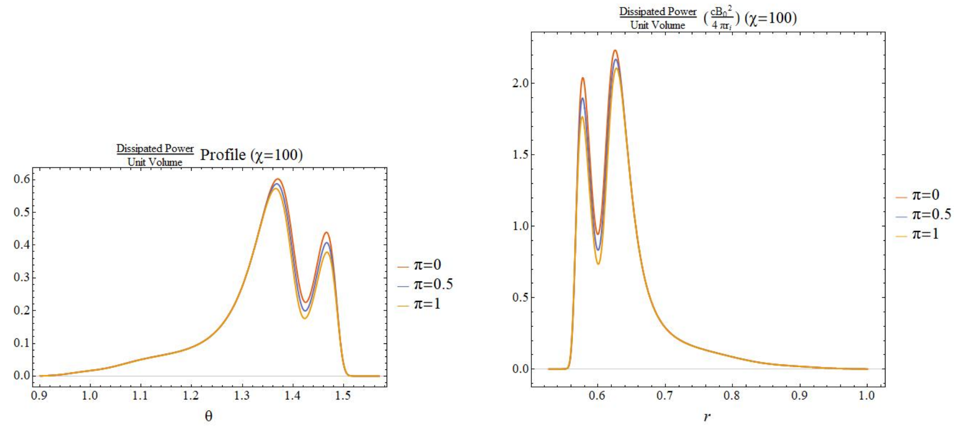


Figure 5.3: Left: The dissipated power per unit volume profile. Right: Power per unit volume dissipated along a mass flux line.

As evident in the previous graph, the power dissipated per unit volume due to ohmic dissipation is not modified significantly due to the flow's radiative losses.

The flow's mass density and magnetic pressure are similarly not significantly affected by the radiative losses.

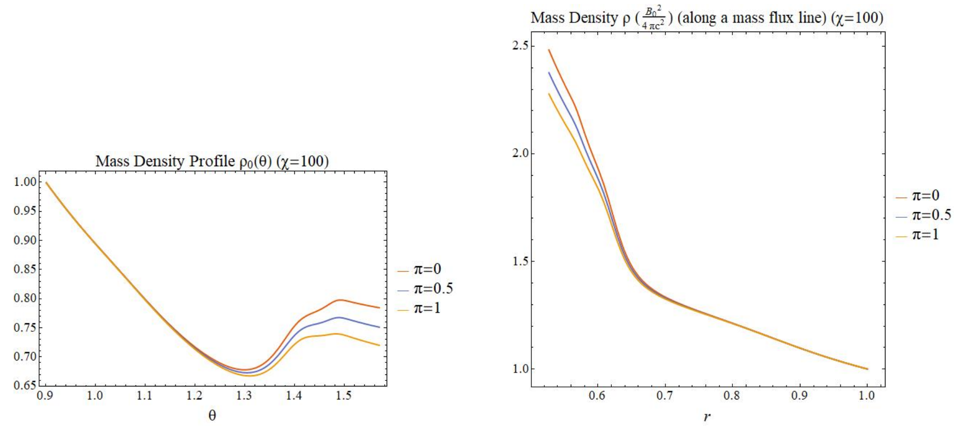
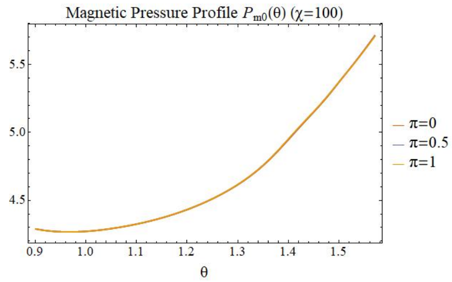


Figure 5.4: Left: Mass density profile $\rho_0(\theta)$. Right: Mass density along a mass flux line.



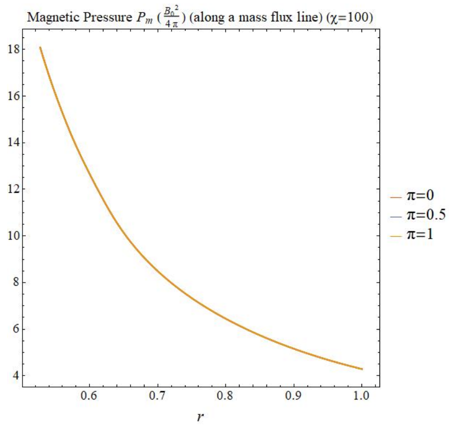
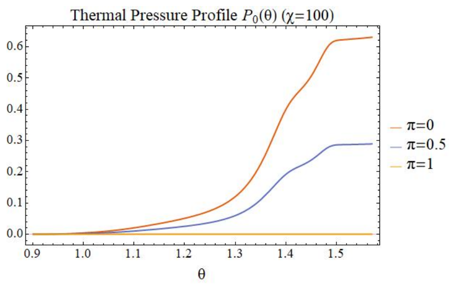


Figure 5.5: Left: Magnetic pressure profile. Bottom: Magnetic pressure along a mass flux line.

The magnetic pressure specifically is not affected at all by the radiative losses, as seen in the above figure. On the other hand, the thermal pressure, as expected, displays the same behaviour as the flow's temperature, being practically negligible for $\pi = 1$.



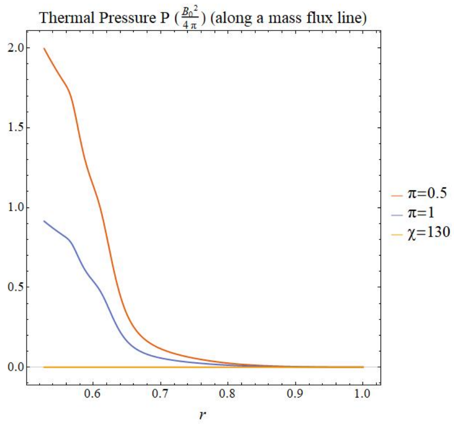


Figure 5.6: Left: Thermal pressure profile. Right: Thermal pressure along a mass flux line.

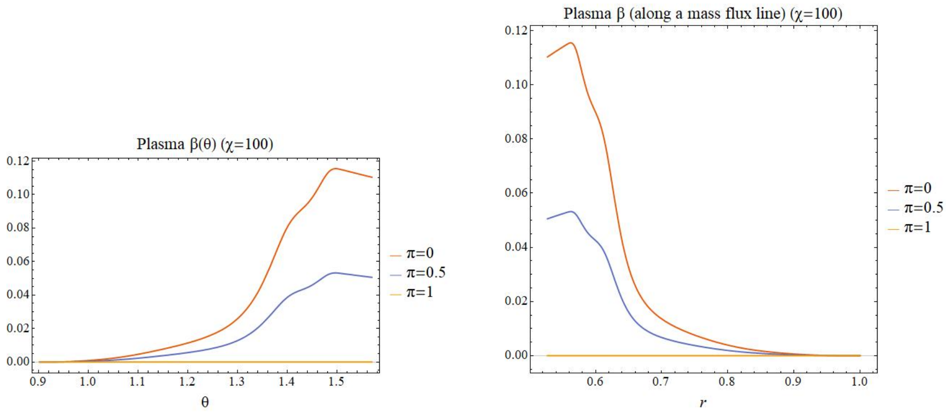


Figure 5.7: Left: Plasma β as a function of θ.
Right: Plasma β along a mass flux line.

The total energy flux to mass flux ratio along mass flux lines μ dislays the same behaviour for all values of π, but takes lower values as the radiative losses become more significant.

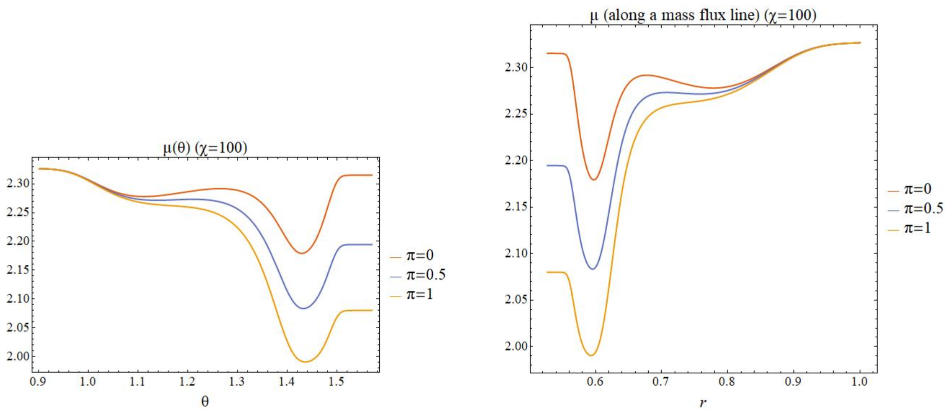


Figure 5.8: Right: The total energy flux per unit volume projected on the flow's velocity over the mass flux as a function of θ. Left: Along a mass flux line.

This is due to its matter component $\mu_{HD} = \gamma(\theta)\xi(\theta)$ being less significant for $\pi = 0.5$ and especially for $\pi = 1$. In those cases the energy available for the plasma's heating is half that of a non-radiative flow for $\pi = 0.5$, and practically zero for $\pi = 1$.

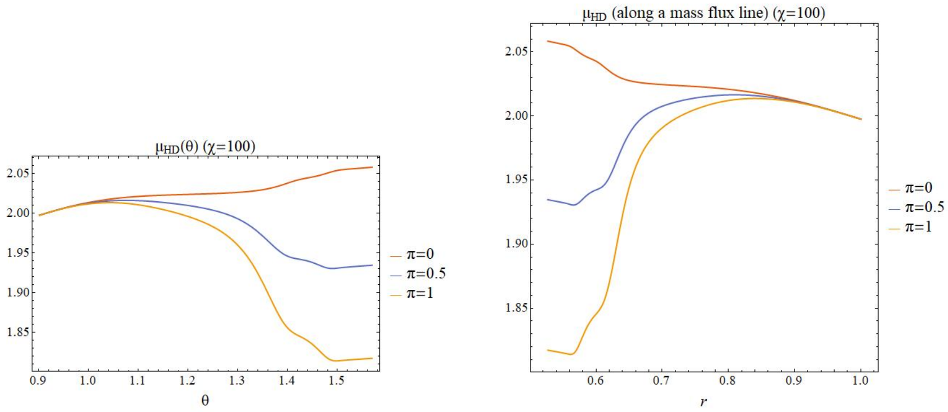


Figure 5.9: The matter component of μ.

The electromagnetic component μ_{EM} is not affected significantly by the radiative losses.

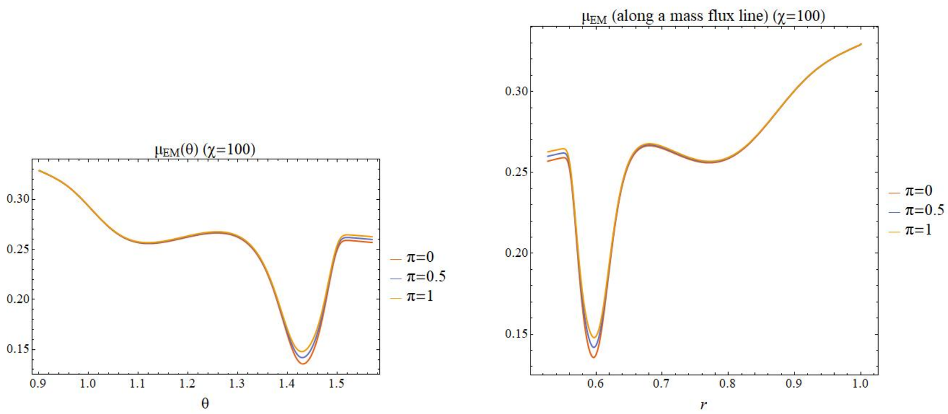


Figure 5.10: The electromagnetic component of μ (Poynting flux projected on the flow velocity $\vec{u}$, over the mass flux).

As evident in the following graphs of the angles ψ_B between the electric and magnetic fields and ψ_u between the electric and velocity fields, field line geometry is also not affected significantly by the existence of radiative losses.

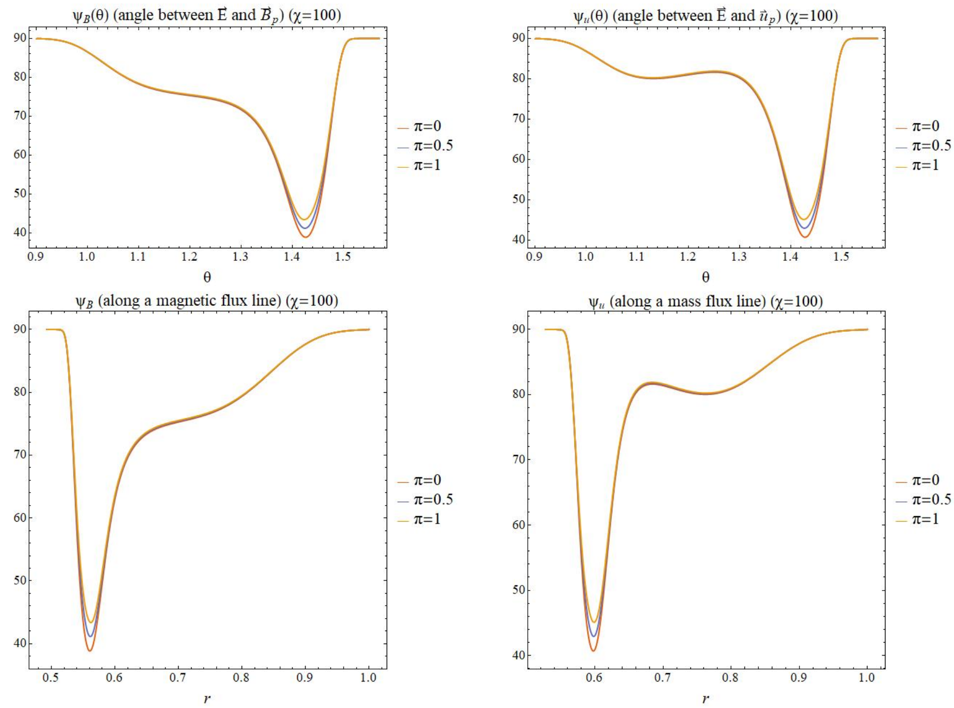


Figure 5.11: Top: The angles between the electic field and the magnetic and velocity fields as functions of θ. Bottom: The angles between the electic field and the magnetic and velocity fields along a mass flux line.

Chapter 6

Particle Accelaration

The solutions presented in the previous chapters render it clear that when the resistivity becomes high enough that dissipative effects become significant, the electric field acquires components parallel to both the magnetic and velocity fields. Especially in the Gaussian resistivity profile solutions, the electric field becomes mostly parallel before becoming again perpendicular to the aforementioned fields, as seen in Figure 4.45. This parallel to the magnetic field component of the electric field is able to accelerate charged particles injected into the flow to high energies. This accelaration of particles due to the parallel component of the electric field is studied in this chapter, by solving the equations of motion for a charged particle moving under the influence of the electromagnetic field determined by the solution of the equations describing the flow.

6.1 Equation of Motion for a Charged Particle

The equation describing the motion of a charged particle of charge q moving under the influence of the flow's electromagnetic field is:

$$\frac{dp^{\mu}}{d\tau} = \frac{q}{c} F^{\mu\nu} U_{\nu} \tag{2.16}$$

where p is the particle's four-momentum and U its four-velocity. Equation (2.16) will provide four equations, one for the particle's energy ($\mu = 0$) and three for the three components of its momentum.

By using the relation $\tau = \frac{t}{\gamma_p}$, with γ_p the particle's Lorentz factor, between the proper time and lab frame time, the four resulting equations, in spherical coordinates, are:

$$\frac{d(\gamma_p m u_r)}{dt} = q(E_r + \frac{u_\theta B_\phi - u_\phi B_\theta}{c})$$

$$\frac{d(\gamma_p m u_\theta)}{dt} = q(E_\theta + \frac{u_\phi B_r - u_r B_\phi}{c})$$

$$\frac{d(\gamma_p m u_\phi)}{dt} = q \frac{u_r B_\theta - u_\theta B_r}{c}$$

$$\frac{d(\gamma_p m c)}{dt} = q \frac{\vec{E} \cdot \vec{u}}{c}$$

With $\upsilon_i = \gamma_p u_i$ the spatial components of the particle's four momentum, the above equations are written as:

$$\frac{d\upsilon_r}{dt} = \frac{q}{m}(E_r + \frac{\upsilon_\theta B_\phi - \upsilon_\phi B_\theta}{\gamma_p c}) \tag{6.1}$$

$$\frac{d\upsilon_\theta}{dt} = \frac{q}{m}(E_\theta + \frac{\upsilon_\phi B_r - \upsilon_r B_\phi}{\gamma_p c}) \tag{6.2}$$

$$\frac{d\upsilon_\phi}{dt} = \frac{q}{m}\frac{\upsilon_r B_\theta - \upsilon_\theta B_r}{\gamma_p c} \tag{6.3}$$

$$\frac{d\gamma_p}{dt} = \frac{q}{mc}\frac{\vec{E}\cdot\vec{\upsilon}}{\gamma_p c} \tag{6.4}$$

Equations (6.1)-(6.4) will provide a system of four ordinary differential equations with the polar angle θ as the independent variable, the solution of which determines the particle's trajectory and its Lorentz factor as a function of θ. The first step towards deriving this system of ODEs is to write the components of the particle's four velocity in the following way:

$$\upsilon_i = c\upsilon_{0i}(\theta) \tag{6.5}$$

With the help of (6.5) and (3.3), (3.5), (3.9), equations (6.1)-(6.4) are written as:

$$\frac{d\upsilon_{0r}}{dt} = \frac{qB_0}{mc}(\frac{r}{\varpi_0})^{F-2}(-(F-1)\varPhi_0(\theta) + \frac{\upsilon_{0\theta}(\theta)B_{\phi 0}(\theta) + \upsilon_{0\phi}(\theta)\frac{FA_0(\theta)}{\sin\theta}}{\gamma_p(\theta)})$$

$$\frac{d\upsilon_{0\theta}}{dt} = \frac{qB_0}{mc}(\frac{r}{\varpi_0})^{F-2}(-\frac{\varPhi_1(\theta)}{\sin\theta} + \frac{\upsilon_{0\phi}(\theta)A_1(\theta) - \upsilon_{0r}(\theta)B_{\phi 0}(\theta)}{\gamma_p(\theta)})$$

$$\frac{d\upsilon_{0\phi}}{dt} = -\frac{qB_0}{mc}(\frac{r}{\varpi_0})^{F-2}\frac{\upsilon_{0r}(\theta)\frac{FA_0(\theta)}{\sin\theta} + \upsilon_{0\theta}(\theta)A_1(\theta)}{\gamma_p(\theta)}$$

$$\frac{d\gamma_p}{dt} = -\frac{qB_0}{mc}(\frac{r}{\varpi_0})^{F-2}\frac{(F-1)\varPhi_0(\theta)\upsilon_{0r}(\theta) + \frac{\varPhi_1(\theta)}{\sin\theta}\upsilon_{0\theta}(\theta)}{\gamma_p(\theta)}$$

The derivatives in the equations above must now be transformed into derivatives with respect to the polar angle θ. This can be achieved with the help of the chain rule:

$$\frac{d}{dt} = \frac{rd\theta}{dt}\frac{d}{rd\theta} = \upsilon_\theta(\frac{r}{\varpi_0})^{-1}\frac{d}{\varpi_0 d\theta}$$

The first three of the above differential equations are then written as:

$$\frac{d\upsilon_{0r}}{\varpi_0 d\theta} = \frac{qB_0}{mc^2}(\frac{r}{\varpi_0})^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}(-(F-1)\varPhi_0(\theta) + \frac{\upsilon_{0\theta}(\theta)B_{\phi 0}(\theta) + \upsilon_{0\phi}(\theta)\frac{FA_0(\theta)}{\sin\theta}}{\gamma_p(\theta)}) \tag{6.6}$$

$$\frac{d\upsilon_{0\theta}}{\varpi_0 d\theta} = \frac{qB_0}{mc^2}(\frac{r}{\varpi_0})^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}(-\frac{\varPhi_1(\theta)}{\sin\theta} + \frac{\upsilon_{0\phi}(\theta)A_1(\theta) - \upsilon_{0r}(\theta)B_{\phi 0}(\theta)}{\gamma_p(\theta)}) \tag{6.7}$$

$$\frac{d\upsilon_{0\phi}}{\varpi_0 d\theta} = -\frac{qB_0}{mc^2}(\frac{r}{\varpi_0})^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}\frac{\upsilon_{0r}(\theta)\frac{FA_0(\theta)}{\sin\theta} + \upsilon_{0\theta}(\theta)A_1(\theta)}{\gamma_p(\theta)} \tag{6.8}$$

It is also assumed that ϖ_0 is the particle's radial distance at $\theta = \theta_i$. The fourth ODE regarding the particle's energy, can be replaced by the following expression for its Lorentz factor:

$$\gamma_p = \sqrt{1 + \upsilon_{0r}^2 + \upsilon_{0\theta}^2 + \upsilon_{0\phi}^2} \tag{6.9}$$

, proven for the fluid's Lorentz factor in section 3.2 (equation 3.47). These four equations fully describe the kinematics of a charged particle in the flow's electromagnetic field.

The three-dimensional trajectory of the particle is determined by the solution of the two following differential equations:

$$\frac{dr}{\upsilon_r} = \frac{rd\theta}{\upsilon_\theta} \Rightarrow$$

$$\frac{dr}{d\theta} = r(\theta)\frac{\upsilon_{0r}(\theta)}{\upsilon_{0\theta}(\theta)} \tag{6.10}$$

$$\frac{\sin\theta d\phi}{\upsilon_\phi} = \frac{d\theta}{\upsilon_\theta} \Rightarrow$$

$$\frac{d\phi}{d\theta} = \frac{1}{\sin\theta}\frac{\upsilon_{0\phi}(\theta)}{\upsilon_{0\theta}(\theta)} \tag{6.11}$$

In conclusion, the system of ODEs which descibe a charged particle's motion and trajectory under the influence of the flow's electromagnetic field is:

$$\frac{d\upsilon_{0r}}{d\theta} = \frac{qB_0\varpi_0}{mc^2}(r_0(\theta))^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}(-(F-1)\Phi_0(\theta) + \frac{\upsilon_{0\theta}(\theta)B_{\phi 0}(\theta) + \upsilon_{0\phi}(\theta)\frac{FA_0(\theta)}{\sin\theta}}{\gamma_p(\theta)}) \tag{6.6}$$

$$\frac{d\upsilon_{0\theta}}{d\theta} = \frac{qB_0\varpi_0}{mc^2}(r_0(\theta))^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}(-\frac{\Phi_1(\theta)}{\sin\theta} + \frac{\upsilon_{0\phi}(\theta)A_1(\theta) - \upsilon_{0r}(\theta)B_{\phi 0}(\theta)}{\gamma_p(\theta)}) \tag{6.7}$$

$$\frac{d\upsilon_{0\phi}}{d\theta} = -\frac{qB_0\varpi_0}{mc^2}(r_0(\theta))^{F-1}\frac{\gamma_p(\theta)}{\upsilon_{0\theta}(\theta)}\frac{\upsilon_{0r}(\theta)\frac{FA_0(\theta)}{\sin\theta} + \upsilon_{0\theta}(\theta)A_1(\theta)}{\gamma_p(\theta)} \tag{6.8}$$

$$\gamma_p = \sqrt{1 + \upsilon_{0r}^2 + \upsilon_{0\theta}^2 + \upsilon_{0\phi}^2} \tag{6.9}$$

$$\frac{dr_0(\theta)}{d\theta} = r_0(\theta)\frac{\upsilon_{0r}(\theta)}{\upsilon_{0\theta}(\theta)} \tag{6.10}$$

$$\frac{d\phi}{d\theta} = \frac{1}{\sin\theta}\frac{\upsilon_{0\phi}(\theta)}{\upsilon_{0\theta}(\theta)} \tag{6.11}$$

, with $r_0(\theta) = \dfrac{r(\theta)}{\varpi_0}$.

The particle Lorentz factor resulting from the solution of the system of ODEs presented above, may be compared to the predicted Lorentz factor for a particle following the magnetic flux lines, meaning that its trajectory's projection on the poloidal plane coincides with the poloidal magnetic field lines. The Lorentz factor of a particle performing this particular motion can be calculated directly from the solution of the system of ODEs describing the flow, by solving the following ODE:

$$\frac{d\gamma_B}{dt} = \frac{q}{mc}\frac{\vec{E}\cdot\vec{u}}{c}$$

Since the particle's poloidal trajectory component is assumed to coincide with the poloidal magnetic field lines, its poloidal velocity may be written as:

$$\vec{u}_p = c\hat{p}_B$$

,with $|\vec{u}_p| \simeq c$ and $\hat{p}_B = \dfrac{\vec{B}_p}{|\vec{B}_p|}$. The dot product $\vec{E}\cdot\vec{u}$ may then be written as:

$$\vec{E}\cdot\vec{u} = c|\vec{E}|\cos\psi_B$$

, with ψ_B the angle between the electric and magnetic fields. The total derivative of γ_B with respect to lab frame time t can be transformed into:

$$\frac{d\gamma_B}{dt} = \frac{ds_B}{dt}\frac{d\gamma_B}{ds_B} \tag{6.12}$$

, where ds_B is the length traveled by the particle over the time interval dt along a magnetic flux line. So,

$$\frac{ds_B}{dt} = u_p \simeq c$$

and the differential equation for γ_B is written as:

$$\frac{d\gamma_B}{ds_B} = \frac{q}{mc^2}|\vec{E}|\cos\psi_B$$

The final step is the calculation of the arc length's s_B derivative with respect to the polar angle θ. Since a magnetic flux line is a curve on the poloidal plane, the differential of its arc length is:

$$ds_B = \sqrt{(dr)^2 + (rd\theta)^2}$$

The shape of a magnetic flux line is given by the function:

$$r(\theta) = r_i(\frac{A_{0i}}{A_0(\theta)})^{\dfrac{1}{F}} \tag{3.74}$$

Consequently:

$$ds_B = r(\theta)d\theta\sqrt{(\frac{1}{r(\theta)}\frac{dr}{d\theta})^2 + 1}$$

By use of (3.74):

$$\frac{dr}{d\theta} = -r_i\frac{1}{F}(\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}-1}\frac{A_{0i}}{A_0(\theta)}\frac{A_0'(\theta)}{A_0(\theta)} = -r_i\frac{1}{F}(\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}}\frac{A_1(\theta)\sin\theta}{A_0(\theta)}$$

The differential of s_B then becomes:

$$ds_B = r_i d\theta(\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}}\sqrt{(\frac{A_1(\theta)\sin\theta}{FA_0(\theta)})^2 + 1}$$

The derivative of γ_B with respect to the polar angle θ therefore is:

$$\frac{d\gamma_B}{ds_B} = \frac{r_i d\theta}{ds_B}\frac{d\gamma_B}{r_i d\theta} = \frac{1}{(\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}}\sqrt{(\frac{A_1(\theta)\sin\theta}{FA_0(\theta)})^2 + 1}}\frac{d\gamma_B}{r_i d\theta}$$

Finally, the differential equation for the Lorentz factor of a charged particle, the poloidal velocity of which is assumed to be $\vec{u}_p \simeq c\hat{p}_B$, moving under the influence of the flow's electromagnetic field is:

$$\frac{d\gamma_B}{d\theta} = \frac{qB_0 r_i}{mc^2}(\frac{A_{0i}}{A_0(\theta)})^{\frac{1}{F}}(\sqrt{(\frac{A_1(\theta)\sin\theta}{FA_0(\theta)})^2 + 1})E_0(\theta)\cos\psi_B(\theta) \qquad (6.13)$$

, where:

$$E_0(\theta) = \sqrt{(F-1)^2\Phi_0^2(\theta) + (\frac{\Phi_1(\theta)}{\sin\theta})^2}$$

$$\cos\psi_B(\theta) = \frac{\vec{E}\cdot\vec{B}}{|\vec{E}||\vec{B}|} = \frac{FA_0(\theta)\Phi_1(\theta) - (F-1)\Phi_0(\theta)A_1(\theta)\sin^2\theta}{\sqrt{(F-1)^2\Phi_0^2(\theta)\sin^2\theta + \Phi_1^2(\theta)}\sqrt{A_1^2(\theta)\sin^2\theta + F^2A_0^2(\theta)}}$$

The length constant r_i was also assumed to be $r_i = r(\theta_i) = \varpi_0$, since the goal of determining γ_B is its comparison to γ_p, therefore these two particles should start their respective motions at the same point.

6.2 Parameters and Boundary Values

The parameters appearing in the system of ODEs describing the motion of a charged particle in the flow's electromagnetic field are the particle's mass m and charge q, and the electromagnetic field's strength scaling constant B_0.

For positrons and electrons, the two types of particles the motion of which is studied in the following sections, the values of these parameters are:

- $m = m_e = 9.1093897 \times 10^{-28}\ gr$
- $q = \pm e = \pm 4.8040425 \times 10^{-10}\ esu$

The value of the electromagnetic field scaling constant B_0, and the scale of the accelerating region ϖ_0 were chosen to be:

- $B_0 = 1\ G$
- $\varpi_0 = 10^7\ cm$

The boundary values of the particles's four-velocity components at θ_{inj}, the angle at which it is injected into the flow are:

- $\upsilon_{0r}(\theta_{inj}) = \upsilon_{0i} c_{ri} = \sqrt{\gamma_{pi} - 1} c_{ri}$
- $\upsilon_{0\theta}(\theta_{inj}) = \upsilon_{0i} c_{\theta i} = \sqrt{\gamma_{pi} - 1} c_{\theta i}$
- $\upsilon_{0\phi}(\theta_{inj}) = \upsilon_{0i} c_{\phi i} = \sqrt{\gamma_{pi} - 1} c_{\phi i}$
- $\gamma_p(\theta_{inj}) = \gamma_{pi}$

In the expressions above, c_r, c_θ, c_ϕ are the particle's velocity's directional cosines, defined as:

- $c_r = \dfrac{u_r}{\sqrt{u_r^2 + u_\theta^2 + u_\phi^2}} = \dfrac{\upsilon_{0r}}{\sqrt{\upsilon_{0r}^2 + \upsilon_{0\theta}^2 + \upsilon_{0\phi}^2}}$
- $c_\theta = \dfrac{u_\theta}{\sqrt{u_r^2 + u_\theta^2 + u_\phi^2}} = \dfrac{\upsilon_{0\theta}}{\sqrt{\upsilon_{0r}^2 + \upsilon_{0\theta}^2 + \upsilon_{0\phi}^2}}$
- $c_\phi = \dfrac{u_\phi}{\sqrt{u_r^2 + u_\theta^2 + u_\phi^2}} = \dfrac{\upsilon_{0\phi}}{\sqrt{\upsilon_{0r}^2 + \upsilon_{0\theta}^2 + \upsilon_{0\phi}^2}}$

Also, the magnitude $\upsilon = c\upsilon_0$ of the spatial component of the particle's four-velocity is related to its Lorentz factor as follows:

$$-c^2 = -\gamma_p c^2 + c^2 \upsilon_0^2 \Rightarrow \upsilon_0 = \sqrt{\gamma_p^2 - 1}$$

So, the boundary value of υ_0 can be expressed in terms of the particle's Lorentz factor at θ_{inj} as:

$$\upsilon_{0i} = \sqrt{\gamma_{pi}^2 - 1}$$

This particular way of specifying the particle four-velocity's boundary values was chosen as it allows for all said boundary values to be expressed in terms of γ_{pi}. The system of ODEs may then be solved with the particle's Lorentz factor at θ_{inj}, γ_{pi}, as a free parameter.

The values of $c_{ri}, c_{\theta i}, c_{\phi i}$ are chosen before the process of solving the equations describing the particle's motion begins. The choice of their values has significant impact on the particle's trajectory, as they control the value of the pitch angle α at θ_{inj}. For example, for:

- $c_{ri} = \dfrac{B_{ri}}{|\vec{B}_i|} = \dfrac{A_1(\theta_{inj})}{\sqrt{A_1^2(\theta_{inj}) + (\frac{FA_0(\theta_{inj})}{\sin\theta_{inj}})^2 + B_{\phi 0}^2(\theta_{inj})}}$

- $c_{\theta i} = \dfrac{B_{\theta i}}{|\vec{B}_i|} = -\dfrac{\frac{FA_0(\theta_{inj})}{\sin\theta_{inj}}}{\sqrt{A_1^2(\theta_{inj}) + (\frac{FA_0(\theta_{inj})}{\sin\theta_{inj}})^2 + B_{\phi 0}^2(\theta_{inj})}}$

- $c_{\phi i} = \dfrac{B_{\phi i}}{|\vec{B}_i|} = \dfrac{B_{\phi 0}(\theta_{inj})}{\sqrt{A_1^2(\theta_{inj}) + (\frac{FA_0(\theta_{inj})}{\sin\theta_{inj}})^2 + B_{\phi 0}^2(\theta_{inj})}}$

the particle's pitch angle at θ_{inj} is $\alpha_i = 0°$ as its velocity is parallel to the magnetic field.

For the solution of (6.13), the boundary value of γ_B at the injection angle θ_{inj} is γ_{Bi} and also considered a free parameter.

It is also noted that the electromagnetic field under the influence of which particle motion is studied, is the one corresponding to the Gaussian resistivity profile solution for $\chi = 100$ and $\pi = 0$ presented in chapter 4.

6.3 Equation of Motion Solutions for e^+ and e^-

The solution to the system of equations (6.6)-(6.11) for positrons injected into the flow $\theta_{inj} = 0.9$ with a velocity perfectly parallel to its magnetic field is presented in this subsection. The electric field component parallel to the poloidal magnetic field, which is responsible for the particle acceleration is presented in the following graph.

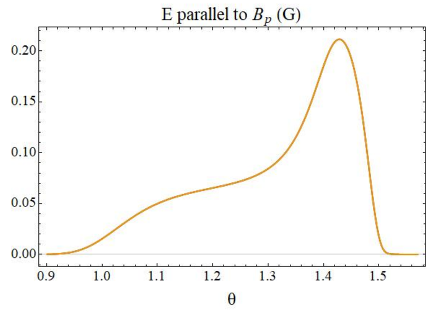


Figure 6.1: Electric field component parallel to the poloidal magnetic field.

The positrons' pitch angle as a function of the polar angle θ is demonstrated in the following figure, for positrons with $\gamma_{pi} = 1.01, 20, 40, 60$.

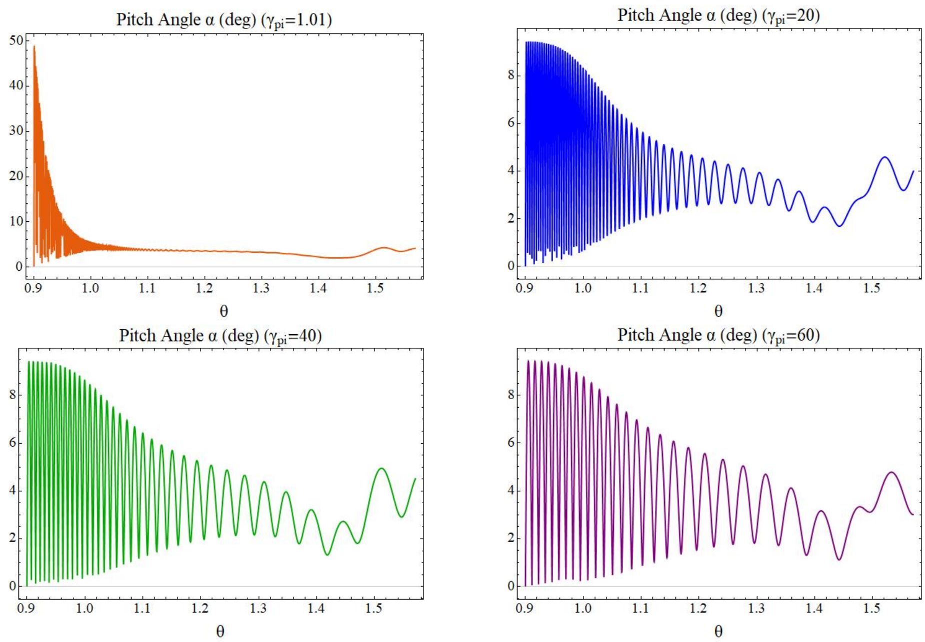


Figure 6.2: Positron pitch angle for $\gamma_{pi} = 1.01, 20, 40, 60$ as a function of θ.

The pitch angle of a positron with $\gamma_{pi} = 1.01$ varies extremely rapidly as a function of θ, as for such low positron kinetic energies the gyroradius is significantly smaller than the length travelled by the particle over the entirety of its motion. Furthermore, in the pitch angle plots for $\gamma_{pi} = 20, 40, 60$, the mounts and valleys of the pitch angle's graph are not symmetrical, which is an indication that the motion has a drift component due to the flow's electric field.

Graphs of the four-velocity components of positrons with $\gamma_{pi} = 1.01, 20, 40, 60$, are presented next, as well as a graph of the flow's four-velocity components.

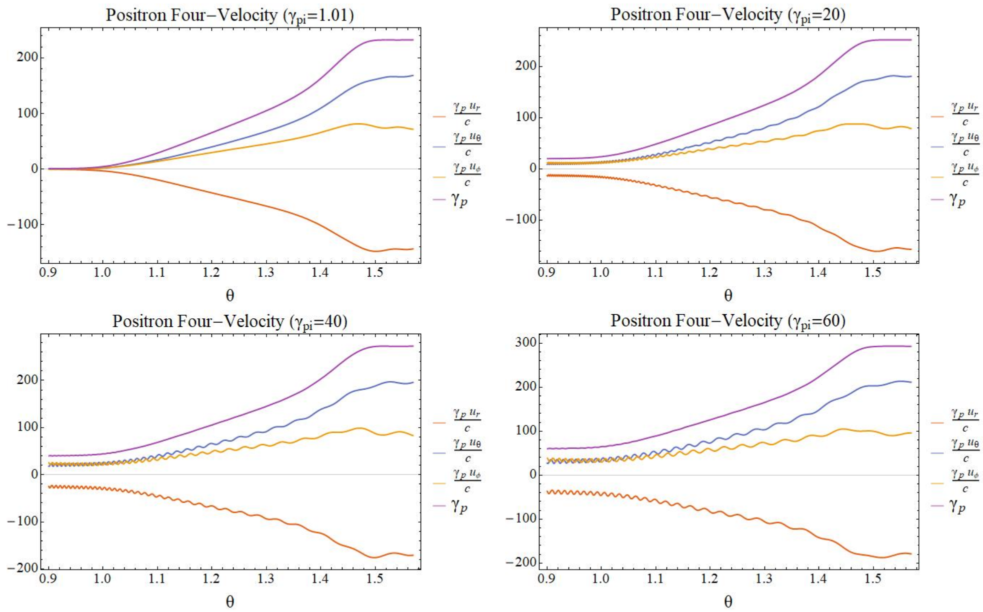


Figure 6.3: Positron four-velocity components for $\gamma_{pi} = 1.01, 20, 40, 60$.

The gyroradial component of the positrons' motion around the magnetic field lines is expressed through the oscillating graphs of the spatial components of their four-velocities.

The positron Lorentz factors, as determined through both the solution of (6.6)-(6.9) and (6.13) for $\gamma_{pi} = 1.01, 20, 40, 60$, follow.

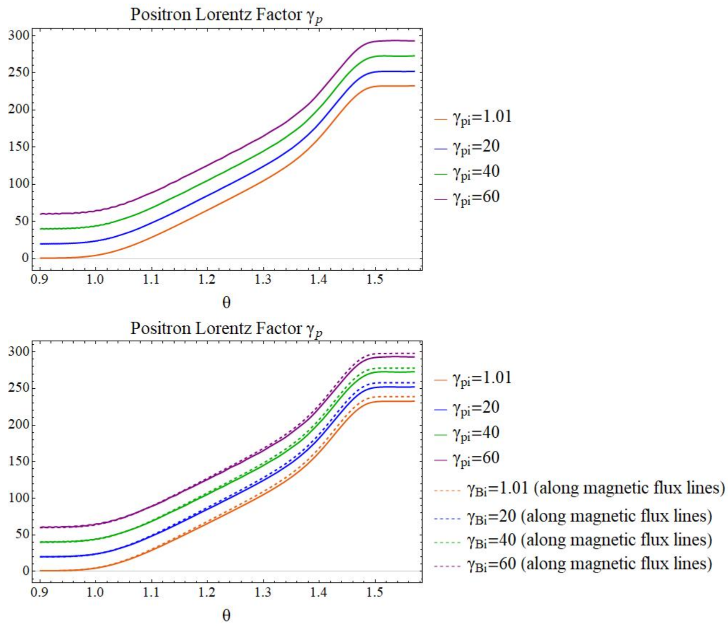


Figure 6.4: Top: Positron Lorentz factors as determined by the solution of the equations of motion (6.6)-(6.9).
Bottom: Comparison between the equations of motion solutions and the solutions of (6.13).

As can be seen in the bottom graph, the dashed line corresponding to the Lorentz factors given by the solution of equation (6.13) largely coincide with those determined by the solution of the equations of motion, especially for $\gamma_{pi} = 20, 40$. The dashed lines do not display the oscillatory shape of the equations of motion solutions, seen clearly in the solutions for $\gamma_{pi} = 40, 60$, which is a result of the characteristic motion of a charged particle around magnetic field lines, since equation (6.13) is an ODE describing only the energy gain of a particle which is assumed to move in parallel to a poloidal magnetic field line. The two solutions also seem to differ slightly near the equatorial plane. Another reason for any minor differences between the two families of solutions is the fact that the particles described by the solutions of (6.13) are assumed to move with a velocity of constant magnitude $\simeq c$.

Nevertheless, equation (6.13) can very accurately predict the energy gain of a particle injected into the flow electromagnetic field with a zero pitch angle.

The positrons' kinetic and electric potential energy variations are presented in the next figure, in units of $m_e c^2$.

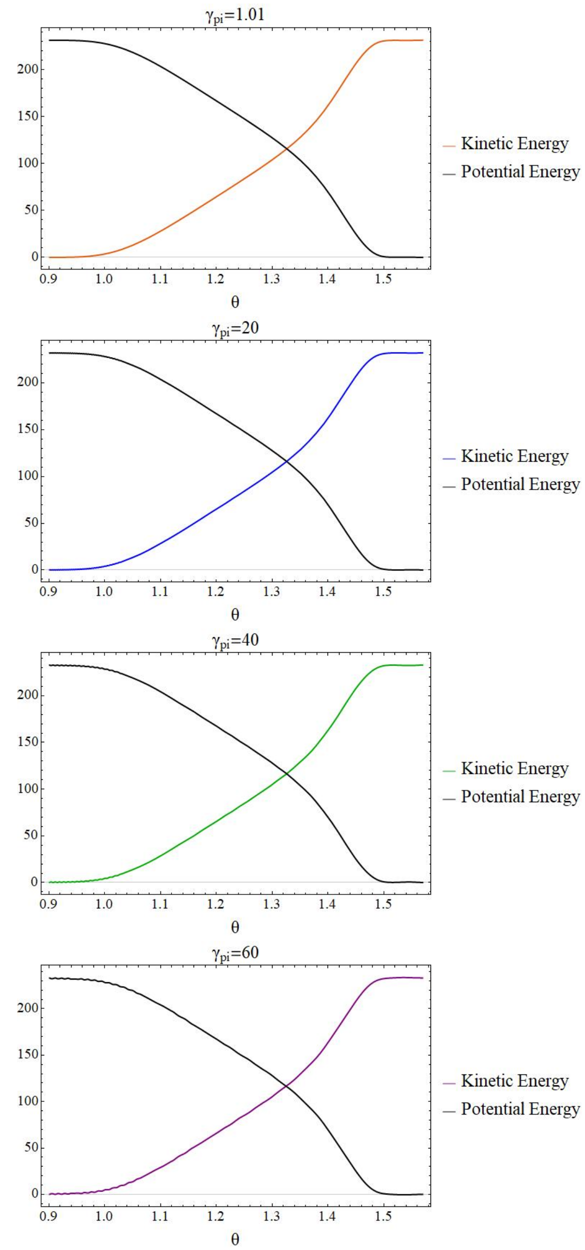


Figure 6.5: Positron kinetic energy and electric potential energy variation over its trajectory for $\gamma_{pi} = 1.01, 20, 40, 60$, in units of $m_e c^2$.

The graphs of the kinetic and potential energies perfectly coincide as expected, since the positrons are accelerated only due to the electric field component parallel to their trajectory on the poloidal plane.

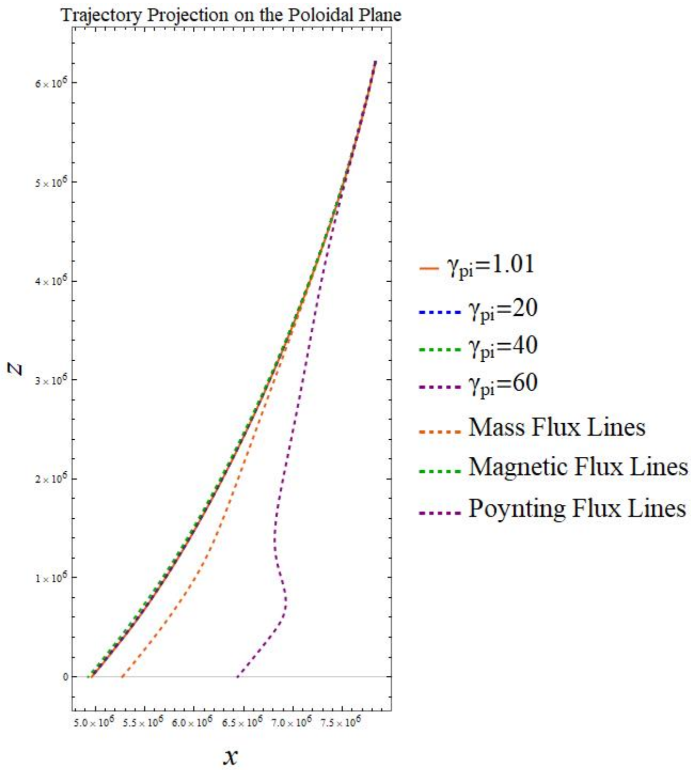


Figure 6.6: Positron trajectory projection on the poloidal plane for $\gamma_{pi} = 1.01, 20, 40, 60$.

The poloidal component of the positrons' trajectory coincides with the poloidal magnetic field lines for all values of γ_{pi}. The trajectories of positrons for $\gamma_{pi} = 1.01, 20, 40, 60$ and $\alpha_i = 0°$ are presented in the next figure, along with the magnetic field, flow velocity field, and Poynting flux lines.

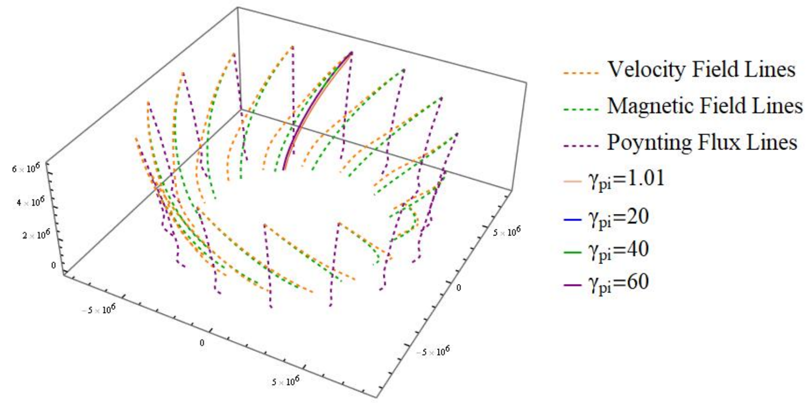


Figure 6.7: Positron trajectory for $\gamma_{pi} = 1.01, 20, 40, 60$.

Electrons are also accelerated by the electric field component parallel to the magnetic field. Due to their negative charge, they must be injected into this particular flow at $\theta_{inj} > 0.9$, since their acceleration requires an increase in their electric potential energy. The figure below depicts the Lorentz factors of electrons injected at $\theta = \dfrac{\pi}{2}$.

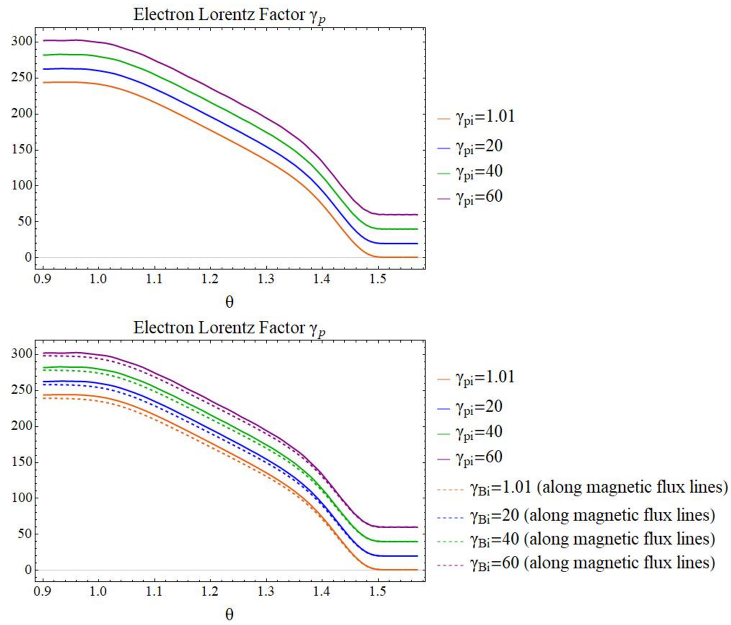


Figure 6.8: Top: Electron Lorentz factors as determined by the solution of the equations of motion (6.6)-(6.9).
Bottom: Comparison between the equations of motion solutions and the solutions of (6.13).

To summarize, the force acting on positrons and electrons injected into the flow by the electric field component parallel to the poloidal magnetic field constitutes an efficient acceleration mechanism, as it can cause an increase to their kinetic energy of about 120 MeV, for an electric field of the order of 0.1 G and an accelerating region scale $\varpi_0 = 10^7\ cm$. The particle trajectories on the poloidal plane deviate from the poloidal magnetic field lines by a negligible amount. This allows the calculation of the kinetic energy increase of positrons and electrons by simply determining the potential energy difference between the initial and final points of their motion in the flow's electromagnetic field.

Suppose that a particle (e^+ or e^-) is injected into the flow at the polar angle θ_{inj} and moves under the influence of the flow's electromagnetic field until it reaches θ_f. The increase to its kinetic energy $\varDelta K$ will then be equal to the difference in potential energy $\varDelta U_A$ between the initial and final point, $(r_A(\theta_{inj})\sin\theta_{inj}, r_A(\theta_{inj})\cos\theta_{inj})$ and $(r_A(\theta_f)\sin\theta_f, r_A(\theta_f)\cos\theta_f)$ of its trajectory on the poloidal plane, with $r_A(\theta)$ given by eq. (3.74). By use of eq. (3.4):

$$\varDelta U_A = B_0\varpi_0\left(\left(\frac{A_{0i}}{A_0(\theta_{inj})}\right)^{\frac{F-1}{F}}\varPhi_0(\theta_{inj}) - \left(\frac{A_{0i}}{A_0(\theta_f)}\right)^{\frac{F-1}{F}}\varPhi_0(\theta_f)\right) \quad (6.14)$$

The accuracy of this approximation can be seen in the following figure.

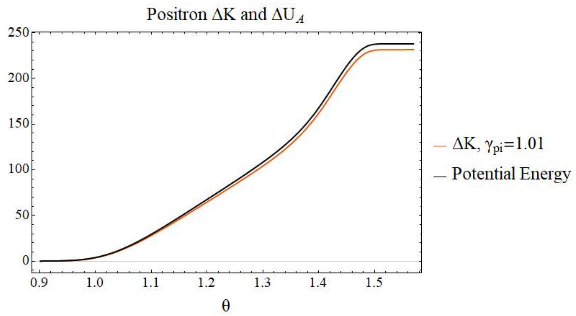


Figure 6.9: Kinetic energy increase ΔK for $\gamma_{pi} = 1.01$ and ΔU_A (black) in units of $m_e c^2$.

Chapter 7

Conclusions

In this thesis the properties of self-similar, resistive, relativistic MHD flows have been semianalyticaly explored. Starting from the conservation laws describing the fluid and Maxwell's equation for the electromagnetic field, with Ohm's Law expressed by equation (2.22) for a plasma of finite conductivity, the equations of steady-state, axisymmetric MHD were derived. In the next step, by assuming a separation of variables of the form: $G(r,\theta) = (\frac{r}{\varpi_0})^F G_0(\theta)$, the 10×10 differential-algebraic system of equations describing radially self-similar flows of this class was derived, the solution of which determined all quantities describing the flow as functions of the polar angle θ.

By examining the solutions for a flow characterized by an exponential resistivity profile, the range of resistivity or Reynolds number values between which the flow decelerates, was determined. This range of values coincides to a large degree with the range of resistivity values over which ohmic dissipation is significant, as it is the main mechanism behind the flow's deceleration and the conversion of electromagnetic energy into heat. This is supported by figures 4.3, 4.6, 4.8, 4.7 and 4.15. Dissipative effects were found to be most significant for intermediate resistivity values. For very small resistivity values, the flow's dynamics and geometric configuration resemble those expected of an ideal MHD flow to a great degree, while for extremely large resistivity values dissipation is again negligible, in accordance with the equations derived in subsection 2.3.3 regarding the infinite resistivity limit. The reason behind that is that in this limit, the conduction current $\vec{J}_{cond}$ becomes equal to zero, leaving the convection current $\vec{J}_{conv}$ as the only component of the total current density $\vec{J}$.

In the case of a gaussian resistivity profile, dissipative effects, as well as flux line separation was found to be more significant for more slowly varying, with respect to θ, resistivity profiles (figures 4.36, 4.37, 4.46). Figure 4.46 also clearly shows that flux lines become parallel again, when the resistivity drops back down to zero, but still no longer coincide, as in their initial configuration at $\theta_i = 0.9$.

Radiative losses, modelled as a fraction of the dissipated power per unit volume, were found to have significant impact on the flow's heating, as only the dissipated power's not radiated fraction can heat the flow (figure 5.2), as well as on the flow's Lorentz factor, since the radiation reaction force can significantly decelerate the flow. Radiative losses do not significantly affect dissipation (figure 5.3), or the fields' geometric configuration (figure 5.11).

Lastly, the electric field component parallel to the poloidal magnetic field, which appears due the flow's finite resistivity was examined, as an acceleration mechanism for electrons and positrons injected into the flow with a zero initial pitch angle. The increase to their kinetic energy due to the potential drop along their poloidal trajectory was determined both by solving their respective equations of motion, as well as calculating the potential drop along magnetic flux lines, between their injection point and their trajectory's final point. The two methods fow calculating the increase to positrons' and electrons' kinetic energy lead to very similar results, provided that particle energies remain low enough that they follow the magnetic field lines. Thus, calculating the potential drop between two points on a magnetic flux line was found to be an accurate way of estimating the kinetic energy increase for particles moving under the influence of the flow's electromagnetic field (figure 6.9). Moreover, the electric field component parallel to the flow's poloidal magnetic field is an efficient acceleration mechanism, as even a weak electric field (figure 6.1) acting over a small scale area ($\varpi_0 = 10^7\ cm$), can accelerate positrons or electrons to high energies $\Delta K \simeq 120\ MeV$.

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