The AdS/CFT Correspondence

Master's Thesis
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Abstract

The AdS/CFT correspondence is the conjecture of an equivalence relation between two seemingly different theories, quantum Gravity (IIB supergravity) in an Anti-de-Sitter (AdS) spacetime and a non gravitational Conformal Field theory (CFT), $\mathcal{N} = 4$ $\mathcal{SYM}$, on the boundary of the latter spacetime. In the correspondence, that was initially formulated by J.Maldacena (1998), the couplings of the two theories have an inverse relation, i.e. a CFT with a strong coupling is dual to a quantum Gravity theory with a weak coupling. Due to this *strong/weak duality*, the correspondence has been a center of attention since it provides great computational tools for both theories in the duality. One of them is the computation of the Entanglement Entropy in CFT, which through the correspondence translates to a geometrical problem, according to the Ryu-Takayanagi conjecture. This thesis is restricted to the part of the correspondence that includes a strongly coupled CFT, which is dual to a weakly coupled quantum Gravity in AdS, and is constituted by two parts. In the first part we present the basic aspects of the correspondence and study its application to the computation of correlation functions in the CFT through Gravity, while in the second part, we apply the Ryu-Takayanagi conjecture at the computation of the Entanglement Entropy in the special case of a CFT on a hyperspherical region of the AdS boundary.

Keywords: Gauge/Gravity duality, AdS/CFT correspondence, Holography, Conformal Field Theory, Anti-de-Sitter, Entanglement Entropy, Holographic Entropy, Large N limit, Yang-Mills Theory

Περίληψη

Η αντιστοιχία AdS/CFT είναι μια εικασία η οποία προβλέπει την αντιστοιχία μεταξύ μιας θεωρίας κβαντικής Βαρύτητας (IIB υπερσυμμετρική Βαρυτητα) σε έναν χώρο Anti-de-Sitter και μίας Σύμμορφης θεωρίας πεδίου $\mathcal{N} = 4$ $\mathcal{SYM}$, χωρίς Βαρύτητα, στο σύνορο του Anti-de-Sitter χωρόχρονου. Η αντιστοιχία αυτή, η οποία προτάθηκε το 1998 από τον J.Maldacena, συνδέει τις δυο θεωρίες με συζεύξεις οι οποίες εχουν αντίστροφη σχέση μεταξύ τους, δηλαδή μία Θεωρία Πεδίου ισχυρής σύζευξης αντιστοιχεί σε μία θεωρία κβαντικής Βαρύτητας ασθενούς σύζευξης. Εξαιτίας αυτής της ισχυρής/ασθενούς διυκότητας, η αντοιστοιχία έχει ανοίξει δρόμους σε νέα υπολογιστικά εργαλεία και για τις δύο θεωρίες που περιλαμβάνοται στην αντιστοιχία. Ένα από αυτά είναι και ο υπολογισμός της Εντροπίας Διεμπλοκής στην Σύμμορφη Θεωρία πεδίου, η οποία μέσω της αντιστοιχίας μεταφράζεται από ένα δύσκολο υπολογιστικά πρόβλημα σε ένα γεωμετρικό πρόβλημα, σύμφωνα με την εικασία Ryu-Takayanagi (2008). Στα πλαίσια της παρούσας εργασίας περιοριζόμαστε στο κομμάτι της αντιστοιχίας που περιλαμβάνει ασθενή υπερσυμμετρική Βαρύτητα η οποία αντιστοιχεί σε μία Σύμμορφη θεωρία πεδίου ισχυρής σύζευξης. Η εργασία αποτελείται από δύο μέρη. Στο πρώτο, παρουσιάζουμε τις βασικές έννοιες της αντιστοιχίας και μελετάμε την εφαρμογή της στον υπολογισμό των συναρτήσεων συσχέτησης της Σύμμορφης θεωρίας, ενώ, στο δεύτερο μέρος, εξετάζουμε την εικασία Ryu-Takayanagi στην ειδική περίπτωση που αναζητούμε την Εντροπία Διεμπλοκής της Σύμμορφης Θεωρίας σε ένα υπερσφαιρικό τμήμα του συνοριακού χωρόχρονου.

Λέξεις-Κλειδιά: Δυϊκότητα Βαρύτητας/Βαθμίδας, AdS/CFT correspondence, Αντιστοιχία AdS/CFT, Ολογραφία, Σύμμορφη θεωρία πεδίου, Anti-de-Sitter, Εντροπία, Ολογραφική Εντροπία, Large N limit, Yang-Mills Θεωρία.

Summary

The goal of this thesis is the presentation and study of the AdS/CFT correspondence that connects Conformal Field theories with a Gravitational theory in a higher dimensional space, as well as the application of the correspondence in the calculation of Entanglement Entropy. The motivation for the formulation of the correspondence was the theoretical understanding of the quark confinement in QCD.

Quark confinement in QCD is the phenomenon according to which quarks are confined within hadrons, unable to escape from them. This phenomenon is explained by the fact that the running coupling constant, $a_s = \frac{g^2}{2\pi}$, of the theory is increasing as energy decreases. Thus, in low energies, the running coupling is strong and it is impossible to perform perturbation analysis. An approach to this problem was proposed, in 1973, by G.t'Hooft, who suggested a perturbative scheme for the SU(N) Yang-Mills theory, where N is the number of colour charges in the theory. The idea is to consider very large N in a way that the product $\lambda = g_{YM}^2 N$, known as the *t'Hooft coupling*, remains constant. This limit is known as the *large N limit.* This way, there are certain simplifications to the theory. The first and very important simplification is based on the fact that, in that limit, it is possible to perform a perturbative expansion in powers of $\frac{\lambda}{N}$ or equivalently $\frac{1}{N}$, the so-called $\frac{1}{N}$-expansion. The first term of the expansion, that is independent of N, is the sum of all planar Feynman diagrams, while the rest terms ($\frac{1}{N}$, $\frac{1}{N^2}$,...) represent the sums of diagrams of higher genus. Thus, the first simplification is that only planar diagrams have important contribution in the large N limit and their sum defines the initial term in the $\frac{1}{N}$-expansion, which we should be able to define in order to understand QCD confinement. But, that is a particularly difficult problem. The second simplification is derived from the factorisation of the correlation functions, that suggests that quantum correlations do not contribute at this limit. Without this contribution, the action integral is eliminated and the action of the theory is defined by a single field configuration, called the *Master Field.* Now, an equally difficult problem emerges and that is the definition of the *Master Field.*

The initial approach to the problem was given by J.Maldacena in 1998, who formulated his conjecture of the correspondence between an $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory, in a $\mathbb{R}^{3+1}$ Minkowski spacetime with 4 supersymmetries, and a quantum supergravity IIB theory (in String theory) in an Anti-de-Sitter,

AdS_5, spacetime in $4+1$ dimensions, whose boundary coincides with the $\mathbb{R}^{3+1}$ Minkowski spacetime, where we have the Yang-Mills theory. According to the correspondence, every observable measure on $\mathbb{R}^{3+1}$, the boundary of AdS_5, can be computed, according to the correspondence, in terms of the gravity theory in the AdS_5 spacetime. The practical significance of the correspondence is that the two theories concerned have inverse couplings, which means that a strongly coupled $\mathcal{N}=4$ SU(N) Yang-Mills is dual to a weakly coupled Gravity theory and reversely. This way, it is possible to to sum all planar diagrams, or, equivalently, to compute the *Master field*, through the Gravity theory that is weakly coupled and permits a perturbative analysis. So, Maldacena's suggestion was that the *Master field* is expressed in terms of Gravity! Additionally to its theoretical value, the correspondence can also be a powerful computational tool.

In terms of this thesis, we are solely concerned with the case of strong coupling on the boundary, that corresponds to a weakly coupled Gravity theory in AdS_5 spacetime. In the large N limit, we are concerned with a $\mathcal{N}=4$ SU(N) Yang-Mills with conformal symmetries, so we generalise our discussion to an arbitrary strongly coupled Conformal Field Theory (CFT) on the boundary. The relation that captures Maldacena's proposal in the most descriptive way was proposed by Gubser, Klebanov, Polyakov and Witten in 1998:

$$\mathcal{Z}_{\text{bulk (Q.Gravity)}}|_{boundary} = \mathcal{Z}_{\text{CFT}} \tag{0.1}$$

and it suggests the identification of the partition functions of the two theories when the gravity theory is studied on the limit that it approaches its boundary. The first part of the thesis includes the verification of this identification through the identification of the correlation functions produced by each theory for the scalar fields alone.

The AdS/CFT correspondence revealed novel computational tools. One of them is the computation of Entanglement Entropy in CFT through the correspondence, as proposed by Ryu and Takayanagi in 2008, that is discussed in the second part of the thesis. Their idea was that, since CFT is dual to Gravity, it should be possible to compute Entanglement Entropy through Gravity. Inspired by black hole entropy and the respective Bekenstein-Hawking formula, $S_{BH} = \frac{c^3 A_H}{4\hbar G_N} = \frac{A_H}{4l_P^2}$, where the Entropy is proportional to the surface area of the black hole, they proposed that the Entanglement Entropy of a CFT subsystem should be proportional to the area of a surface extending in the interior of AdS. This surface, that we shall call A_m, has 2 characteristics. The first is that on the AdS boundary it should coincide with the boundary of the subsystem, while the second is that it should be of minimum area. The proposed formula is:

$$S(A) = \frac{\text{Area}\,(A_m)}{4G_N^{(D+1)}} \tag{0.2}$$

where $G_N^{(D+1)}$ is the Newton constant in the dimensions of AdS_{D+1} spacetime. Thus, the second part of the thesis is concerned with the application of the Ryu-Takayanagi proposal in the special case that the boundary subsystem is a hypersphere in D dimensions, and the comparison of this result with the expected outcomes from direct computation through CFT.

Σύνοψη

Σκοπός της παρούσας εργασίας είναι η παρουσίαση και μελέτη της αντιστοιχίας AdS/CFT που συνδέει Σύμμορφες Θεωρίες πεδίου με τη Βαρυτική θεωρία σε ένα χώρο με μία παραπάνω διάσταση, καθώς και την εφαρμογή της αντιστοιχίας στον υπολογισμό της Εντροπίας συμπλοκής. Η αφορμή για την ανάπτυξη της αντιστοιχίας δώθηκε κατά την αναζήτηση θεωρητικής θεμελίωσης του εγκλωβισμού των κουαρκ στην κβαντικη Χρωμοδυναμική.

Ο εκγλωβισμός των κουαρκ στην κβαντική Χρωμοδυναμική είναι το φαινόμενο κατά το οποίο τα κουαρκ είναι περιορισμένα μέσα στα αδρόνια, χωρίς να μπορούν να διαφύγουν από αυτά. Το φαινόμενο αυτό εξηγείται με το γεγονός ότι η τρέχουσα σταθερά σύζευξης, $a_s = \frac{g^2}{2\pi}$, της θεωρίας αυξάνεται με την ελλάτωση της ενέργειας και σε χαμηλές ενέργειες είναι μεγάλη και δεν επιτρέπει την μελέτη της θεωρίας με διαταρραχές. Μία προσέγγιση στο πρόβλημα δώθηκε το 1973 από τον G.t'Hooft ο οποίος πρότεινε έναν τρόπο διαταρρακτικής ανάπτυξης για θεωρίες SU(N) Yang-Mills με N χρωματικούς βαθμούς ελευθερίας. Η ιδέα είναι να θεωρήσουμε το N πολύ μεγάλο με τέτοιο ομως τρόπο ώστε το γινόμενο $\lambda = g_{YM}^2 N$, γνωστό και ως *σύζευξη t'Hooft* να παραμένει σταθερό. Το όριο αυτό ονομάζεται *όριο των μεγάλων N* ή *large N limit* . Με αυτό τον τρόπο μπορούμε να κάνουμε διαταρρακτική ανάπτυξη ως προς τη σταθερά σύζευξης $g_{YM} = \frac{\lambda}{N}$ ή ισοδύναμα ως προς το κλάσμα $\frac{1}{N}$. Σε αυτό το όριο προκύπτουν στη θεωρία ορισμένενες απλοποιήσεις. Η πρώτη και πολύ βασική απλοποίηση βασίζεται στο ότι στο διαταρρακτικό ανάντυγμα ως προς $\frac{1}{N}$, ο πρώτος όρος, που δεν περιλαμβανει το N, αποτελεί το άθροισμα των επίπεδων διαγραμμάτων Φεψνμανν, ενώ οι υπόλοιποι ($\frac{1}{N}$, $\frac{1}{N^2}$, ...) αποτελούν το άθροισμα διαγραμμάτων μεγαλύτερου γένους. Έτσι, η πρώτη απλοποίηση είναι ότι στο όριο των μεγάλων N τα διαγράμματα Feynman με σημαντική συνεισφορά στη θεωρία είναι μόνο τα επίπεδα διαγράμματα, δηλαδή αυτά που μπορούν να εγγραφούν πάνω σε μία σφαίρα. Για να μπορεσουμε λοιπόν σε αυτή την προσέγγιση να μελετήσουμε τον εγκλωβισμό των κουάρκ θα πρέπει να αθροίσουμε όλα τα επίπεδα διαγράμματα, το οποίο όμως είναι ενα δύσκολο πρόβλημα. Μια άλλη απλοποίηση που προκύπτει είναι η παραγοντοποίηση των συναρτήσεων συσχέτισης, το οποίο υποδεικνύει τη μη συνεισφορά των κβαντικών συσχετίσεων σε αυτό το όριο της θεωρίας. Χωρίς αυτή τη συνεισφορά η δράση που χρησιμοποιείται για των υπολογισμό των συσχετήσεων προσδιορίζεται αποκλειστικά από μία διαμόρφωση πεδίου, το οποίο ονομάζουμε *Master field* ή *Κύριο Πεδίο.* Εξίσου δύσκολο πρόβλημα λοιπόν είναι ο προσδιορισμός αυτού του κύριου πεδίου στην Yang-Mills θεωρία.

Η πρώτη προσέγγιση στο πρόβλημα δώθηκε το 1998 από τον J.Maldacena, ο οποίος διατύπωσε την εικασία της αντιστοιχίας μιας SU(N) θεωρίας Yang-Mills σε ένα 3+1 χώρο Minkowski ($\mathbb{R}^{3+1}$) με 4 υπερσυμμετρίες $\mathcal{N}=4$, με μία κβαντική θεωρία υπερβαρύτητας τύπου IIB (στη θεωρία χορδών) σε ένα Anti-de-Sitter χώρο στις 4+1 διαστάσεις (AdS_5) , του οποίου το σύνορο στο άπειρο είναι ο Minkowski χώρος $\mathbb{R}^{3+1}$ όπου έχουμε τη Yang-Mills θεωρία. Σύμφωνα με την αντιστοιχία κάθε υπολογίσμο μέγεθος της θεωρίας στο σύνορο μπορέι να υπολογιστεί σύμφωνα με την αντιστοιχία στη θεωρία της βαρύτητας στον AdS_5 χώρο. Η πρακτική σημαντικότητα της αντιστοιχίας είναι ότι η σύνδεση μεταξύ των σταθερών σύζευξης των θεωριών είναι αντίστροφη, υπό την έννοια ότι μία ισχυρά συζευγμένη θεωρία $\mathcal{N}=4$ SU(N) Yang-Mills είναι δυϊκή μίας ασθενούς θεωρίας βαρύτητας και αντίστροφα. Με τον τρόπο αυτό δίνεται η δυνατότητα να αθροίσουμε τα επίπεδα διαγράμματα ή ισοδύναμα να υπολογίσουμε το Κύριο Πεδίο μέσω της ασθενώς συζευγμένης θεωρίας βαρύτητος, όπου μπορεί να γίνει διαταρρακτική ανάλυση. Το πόρισμα λοιπόν του Μαλδαςενα ήταν ότι το Κύριο Πεδίο εκφράζεται μέσω της βαρύτητας. Εκτός, λοιπόν, από τη θεωρητική διορατικότητα που μπορεί να προσφέρει η αντιστοιχία, είναι και ένα ισχυρό υπολογιστικό εργαλείο.

Στα πλαίσια της εργασίας μας απασχολεί αποκλειστικά η περίπτωση που έχουμε ισχυρή σύζευξη στο σύνορο που αντιστοιχεί σε μία θεωρία βαρύτητας στο κλασσικό της όριο στον AdS_5 χώρο. Στο όριο των μεγάλων N, η $\mathcal{N}=4$ SU(N) Yang-Mills θεωρία είνα μία θεωρία με σύμμορφη συμμετρία, και για αυτό το λόγο γενικεύουμε τη συζήτησή μας και μιλάμε πλέον για τη γενική περίπτωση Σύμμορφων Θεωριών Πεδίου στο σύνορο με ισχυρή σύζευξη. Η βασική σχέση η οποία αποτυπώνει την αντιστοιχία του Maldacena, διατυπώθηκε από τους Gubser, Klebanov, Polyakov και Witten:

$$\mathcal{Z}_{\text{bulk (Q.Gravity)}}|_{boundary} = \mathcal{Z}_{\text{CFT}} \tag{0.3}$$

η οποία εξισώνει τις συναρτήσεις επιμερισμού της Σύμμορφης θεωρίας και της Βαρύτητας όταν η τελευταία μελετάται στο όριο του συνόρου. Το πρώτο μέρος της εργασίας περιλαμβάνει την επιβεβαίωση της ταυτοποίησης των συναρτήσεων επιμερισμού μέσω των ταυτοποιήσεων των συναρτήσεων συσχέτησης που προκύπτουν από κάθε θεωρία, στην περίπτωση μόνο των βαθμωτών πεδίων.

Η αντιστοιχία AdS/CFT άνοιξε νέους δρόμους σε νέα υπολογιστικά εργαλεία. Ένα από αυτά αφορά τον υπολογισμό της Εντροπίας Διεμπλοκής στη Σύμμορφη θεωρία πεδίου, θέμα το οποίο αναπτύσσεται στο τελευταίο μέρος της διπλωματικής. Συγκεκριμένα, το 2008 οι Ryu-Takayanagi πρότειναν μια σχέση για τον υπολογισμό της Εντροπίας Διεμπλοκής στη Σύμμορφη Θεωρία Πεδίου μέσω της αντιστοιχίας. Η ιδέα των τελευταίων ήταν ότι εφόσον η Σύμμορφη Θεωρία αντιστοιχεί σε μία θεωρία Βαρύτητας, και ο υπολογισμός της Εντροπίας θα πρέπει να μπορεί να γίνει και αυτός μέσω της Βαρύτητας. Εμπνεόμενοι, λοιπόν, από την εντροπία στις μαύρες τρύπες, η οποία σύμφωνα με τη σχέση των Bekenstein-Hawking, $S_{BH} = \frac{c^3 A_H}{4\hbar G_N} = \frac{A_H}{4 l_P^2}$, είναι ανάλογη της επιφάνειας της μαύρης τρύπας A_H, πρότειναν ότι η εντροπία ενός υποσυστήματος της Σύμμορφης Θεωρίας πεδίου στο σύνορο του AdS χώρου μπορεί να υπολογιστεί από μία επφάνεια στο εσωτερικό του AdS χώρου. Η επιφάνεια αυτή, την οποία λέμε A_m έχει 2 χαρακτηριστικά. Το πρώτο είναι ότι πρέπει στο σύνορο να έχει κοινά όρια με το χωρίο που ορίζει το εκάστοτε υποσύστημα και το δεύτερο είναι ότι θα πρέπει η επιφάνεια αυτή να έχει το ελάχιστο δυνατό εμβαδόν. Η προτεινόμενη σχέση τότε έχει τη μορφή:

$$S(A) = \frac{\mathrm{Area}\,(A_m)}{4G_N^{(D+1)}} \tag{0.4}$$

όπου $G_N^{(D+1)}$ είναι η σταθερά του Νεύτωνα στις διαστάσεις του AdS χώρου. Το δεύτερο, λοιπόν, μέρος της εργασίας περιλαμβάνει την εφαρμογή της σχέσης Ryu-Takayanagi στην ειδική περίπτωση που έχουμε ένα υπερσφαιρικό -στις D διαστάσεις- χωρίο στο σύνορο και τη σύγκριση του αποτελέσματος αυτού με τα αναμενόμενα αποτελέσματα που παράγει ανεξάρτητα η Θεωρία Πεδίου.

Contents

Chapter 1

Introduction

1.1 The holographic nature of Gravity

In the last decades, the Standard Model appeared to be a great success in unifying the fundamental forces, from quantum to macroscopic level. Electromagnetism, the Weak and the Strong Nuclear Forces are all included in the Standard Model, accept for one, Gravity! But, why is Gravity so different after all?

To begin with, in contrast to all other forces, gravity is extremely weak. Compared to the strong nuclear force it is 10^{-39} time weaker and compared to the most empirically familiar electromagnetism it is 10^{-37} times more weak. To put in perspective how small this number is, the ratio of the radius of a Hydrogen atom to the size of the observable universe is almost $3 \cdot 10^{-37}$! Additionally, it is an exclusively attractive force, due to the positivity of gravitational mass, which is assumed solely because a negative mass has just neither been observed or realised.

The attempt to quantise the oldest known fundamental force, gravity, has been a puzzle yet to be solved. A quantisation similar to conventional Quantum Field Theory techniques is actually rather unsuccessful. The underlying reason is that the dimensionless gravitational coupling grows quadratically with energy, so a perturbation theory would be more and more divergent at each order of perturbation, thus non-renormalisable. The failure of QFT to properly quantise gravity demonstrates the need for new Physics in the UV limit of general relativity. Up to date, the most dominant way to overcome these divergences is String Theory. Particles are considered to be different excitation modes of an 1-dimensional string, with gravity carried by a particle called *graviton*, which is represented by a closed string of zero mass and spin 2. The long wavelength interactions of the graviton reproduce general relativity [1].

Lastly, another important impediment in the quantisation of gravity is that it must include quantum fluctuations of the spacetime. Gravity is the only fundamental force that is implicated with the geometry of spacetime itself. In fact, gravity is the geometry of the spacetime. On the contrary, all other forces propagate along a stationary spacetime background without ever interacting with it.

So, in the quest do derive a proper description of gravity in the quantum scale some crucial questions arise. What is the nature of spacetime, is it even a continuum? Associated to the latter is another question: is Gravity fundamental or is it emergent?

If gravity is emergent, then its carrier, the graviton should not be fundamental but rather a bound state of other particles. Maybe two gauge bosons of spin 1? On the other hand, Weinberg and Witten [2], in 1980, proved two theorems that state that this is impossible. The two theorems concerned a Lagrangian field theory of massless particles:

- *Theorem 1*: : A theory that allows the construction of a Lorentz covariant and conserved current J_μ cannot contain massless particles with non-zero symmetry charge $\left(Q = \int d^3\vec{x}J^0\right)$ of spin greater that $\frac{1}{2}$
- *Theorem 2*: A theory that allows the construction of a Lorentz covariant and conserved energy-momentum tensor $T_{\mu\nu}$ cannot contain massless particles of spin greater than 1.

Consequently, the second theorem does not allow the graviton to be a composite particle in that theory and the graviton has to be fundamental. And exactly here lies the detail that can still allow the graviton to be expressed as a bound state. That is the assumption that the graviton propagates in the same spacetime as its "ingredients". What if it does not? What if it moves in an additional dimension? [3]

An insight to that question is given by the *Holographic Principle*. The Holographic Principle was proposed by G. t'Hoooft in 1993 and states that any theory of quantum gravity in a volume of space can be encoded in a lower-dimensional boundary, such as a light-like boundary in a gravitational horizon [4]. Thus, gravity lives in a spacetime of higher dimension, but all of its information can be projected to the boundary of that spacetime, or, to put it in other words, all the degrees of freedom can be encoded on a surface!

The very first concrete example of the holographic principle was presented by J. Maldacena, in 1998. In a string theory framework, he conjectured that a String/M theory in an Anti-de-Sitter (AdS) spacetime corresponds to a non-gravitational Conformal Field Theory - that is a quantum field theory with conformal symmetries- living in the boundary of AdS. The example he proposed states that **type IIB String Theory on** $AdS_5 \times S_5$ is dual to 4 dimensional **N = 4 super–Yang–Mills (SYM)** with gauge group $SU(N)$ and coupling constant g_{SYM}, which is a Conformal Field Theory (CFT).

Quantum Gravity in D+1 dimensions	$\longleftrightarrow$	Conformal Field Theory in D dimensions

So, to answer our very first question, what the *Holographic Principle* suggests and what Maldacena indicated is that maybe Gravity cannot be different because it is not comparable to Quantum Field Theory, but emergent from it. Thus, it's quantisation may correspond to the already existing quantisation of QFTs.

1.2 History and Development

Maldacena's conjecture may not have been an absolute surprise, but it has clearly been a game changer! In his lectures J. Polchinski [3] would even state that the importance of the conjecture is of the same rank as Maxwell's equations or Euler's equation. So, let's take a look to the series of fortunate events that led to the conjecture.

The similarity between gravity of General Relativity and hydrodynamics, that is an effective theory derived from microscopic molecular interactions, could be an indicator of an emergent nature in gravity. In fact, in 1967, Sakharov suggested exactly this [6]. In particular, he considered quantum matter fields in a curved background without any gravity dynamics and expanded the effective action in a Taylor series with respect to curvature. He showed that at one loop order the effective action contains terms proportional to the cosmological constant, the Einstein–Hilbert action, plus higher-order terms. So, he suggested that considering a dependence of quantum fluctuations on the space curvature, it is possible to get equations of gravity [5].

But, the first strong indication originates from black holes. Bekenstein introduced, in 1973 [6], the interpretation of black hole entropy as the inaccessibility of information to an external observer, in analogy to the way thermodynamic entropy expresses the lack of information about microscopic configurations of a system that a macroscopic observer experiences. His approach using information theory, along with the observation that the black hole horizon area is, similarly to the entropy, non-decreasing -as proved by Hawking [7]- led him to the conclusion *that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity.* Later, in 1975 [8], Hawking proved the well-known emission of *Hawking Radiation* corresponding to a certain temperature, *Hawking Temperature*, and additionally, confirmed Bekenstein's conjecture specifying the dimensionless constant that Bekenstein missed in his result to be $\frac{1}{4}$. Hence, the famous expression of the maximal black hole entropy, the *Hawking - Bekenstein formula*:

$$S_{BH} = \frac{c^3 A_H}{4\hbar G_N} = \frac{A_H}{4l_P^2} \tag{1.1}$$

where A_H is the surface area of the black hole horizon, G_N is the Newton constant and $l_P = \left(\frac{\hbar G_N}{c^3}\right)^{\frac{1}{D-2}}$ is the Planck length in D dimensions.

It was not until twenty years after the Bekenstein-Hawking formula, that the next radical interpretation of the area-law would be presented by Gerald t'Hooft, in 1993 [4]. As mentioned above, what he proposed was the *Holographic Principle*, that, once again, suggests that *any* quantum gravity theory must be holographic, meaning that the quantum states of a volume must be encoded in a lower-dimensional boundary. The generality of the argument lies above his method. He used a description of the degrees of freedom as Boolean variables and cellular automata -hence strict mathematical logic arguments- to prove the necessity of imposing certain constrains to the quantisation of Gravity, based only on unitarity, entropy and counting arguments. As a result, he proved that at Planckian length scales where quantum gravity takes over, the degrees of freedom of quantum gravity in 3+1 dimensions actually live in a 2+1 dimensional surface governed by a gauge theory.

Nevertheless, t'Hooft was not the first one to connect quantum gravity with gauge field theory or present a holographic nature of gravity. But, truths and objects are invisible until they are observed! Since 1978, C. Thorn [9] suggested that string theory admits a lower dimensional description in which gravity emerges. He described strings as composite systems of more fundamental pointlike objects rather than fundamental objects themselves. Apparently, he was not the only one to see the connection. Klebanov and Susskind demonstrated, in 1988, that one of the phases of a lightcone, 2+1 dimensional, lattice gauge theory with infinite number of colours $N \to \infty$, known as the t'Hooft limit, exactly describes free fundamental strings.

A year later, Susskind [10] inspired by the similarities of their prior observation with Klebanov and the Holographic Principle, refined the idea of C. Thorn and gave a precise string theory interpretation of the Holographic Principle.

Finally, in 1998, J.Maldacena publishes his work on AdS/CFT correspondence, giving life to the Holographic Principle. His argument was in terms of String/M theory and describes the correspondence between IIB string theory in AdS spacetime and $N = 4$ Super-Yang-Mills field theory, a CFT, in the Minkowski boundary of AdS. The original argument will be demonstrated later in the thesis.

1967	Sakharov	Suggests that General Relativity is emergent from condensed matter systems
1971	Hawking / Christodoulou	The surface area of a black hole is non-decreasing
1973	Bekenstein	The maximal entropy of a black hole is proportional to its area
1973	t'Hooft	In Yang-Mills large N limit only planar diagrams survive
1975	Hawking	Confirms Bekenstein's area law, presents a precise result and proposes Hawking Radiation
1978	Thorn	Strings are not fundamental, but composite systems of pointlike objects
1980	Weinberg / Witten	Present two theorems that restrict the renormalisable QFTs
1988	Klebanov / Susskind	String theory emerges from 2+1 gauge theory on a lightcone
1993	t'Hooft	The holographic principle
1994	Susskind	Precise string interpretation of Thorn ideas and the holographic principle
1998	Maldacena	AdS/CFT Correspondence

Chapter 2

Anti-de-Sitter Spacetime

In order to comprehend the AdS/CFT duality, firstly, we must understand the spacetime it discusses, the *Anti-de-Sitter spacetime* [11]. Very briefly, the *Anti-de-Sitter spacetime* is a *maximally symmetric spacetime with a negative cosmological constant*.

But, what does maximally symmetric mean? A spacetime of D dimensions is called maximally symmetric if it has $\frac{D(D+1)}{2}$ symmetries. The highest symmetrical spacetime we can imagine is the flat Euclidean spacetime $\mathbb{R}^D$. What symmetries does $\mathbb{R}^D$ have? It has a translation symmetry with D possible translations and a rotational symmetry with $\binom{D}{2} = \frac{D(D-1)}{2}$ possible rotations. Totally, $\mathbb{R}^D$ has $\frac{D(D+1)}{2}$ independent symmetries. Then, all spacetimes with the same number of independent symmetries, are called *maximally symmetric spacetimes* [1] Now, if we want to find maximally symmetric curved spacetimes, the symmetries dictate that the spacetime we are seeking has a constant curvature R, otherwise at least one of the symmetries is violated. This leaves us with three choices:

- Positive constant curvature $R > 0$.
  In spacetimes of positive curvature, we have elliptic geometry, at which parallel lines, finally converge to the same point. Such a spacetime is the *de-Sitter spacetime* dS_D, which is an analog to the $D - sphere$. The symmetry group of de Sitter spacetime is $SO(D, 1)$, that is the group of rotations with 1 temporal dimension and D spacial dimensions[2].

- Negative constant curvature $R < 0$.
  Spacetimes of negative curvature have a hyperbolic geometry, where parallel lines, diverge and their distance increases exponentially. Such a spacetime is the *Anti-de-Sitter spacetime* AdS_D, that has a symmetry group $SO(D-1, 2)$. So, AdS_D is symmetric under rotations with 2 temporal

[1] In terms of general relativity, this translates to $\frac{D(D+1)}{2}$ *Killing vectors*.
[2] Notice that there is one extra spacial dimension.

dimensions and $D-1$ spacial dimensions [3].

- Zero constant curvature $R = 0$
  Zero curvature corresponds to a flat geometry. The Euclidean and the Minkowski spacetime are the most common examples of maximally symmetric flat spacetimes. The corresponding symmetry groups is the Poincaré group. The Poincaré group corresponds to rotational as well as translational symmetry in all directions. Rotations involving time correspond to boosts.

In this thesis, we consider an AdS_{D+1} spacetime with 1 temporal and D spacial coordinates, with a constant negative curvature $R < 0$. According to the above, AdS_{D+1} has $\frac{(D+1)(D+2)}{2}$ symmetries, thus, the group of symmetries that acts on AdS_{D+1} is $SO(D,2)$.

2.1 Definition

The Anti-de-Sitter spacetime AdS_{D+1} can embedded to $\mathbb{R}^{(D,2)}$,with a Minkowski metric $\eta_{\mu\nu} = (-1, 1..., 1, -1)$, providing a convenient definition of AdS_{D+1}:

$$-X_0^2 - X_{D+1}^2 + \sum_{i=1}^{D} X_i^2 = -b^2 \tag{2.1}$$

where, b is the *AdS radius of curvature*, the embedding coordinates X_0 and X_{D+1} are temporal and all others X_i are spacial. Thus, AdS_{D+1} spacetime can be embedded in a Minkowski spacetime with two temporal dimensions. The resulting space is called *Embedding Spacetime* of AdS_{D+1}, while, the metric of the embedding spacetime is:

$$ds^2 = -dX_0^2 - dX_{D+1}^2 + \sum_{i=1}^{D} dX_i^2 \tag{2.2}$$

The scalar curvature of AdS_{D+1} can be found to be $R = -\frac{D(D+1)}{b^2} < 0$, that of course is negative and the, also, negative cosmological constant is $\Lambda = -\frac{D(D-1)}{2b^2}$.

[3]Notice that there is one extra temporal dimension.

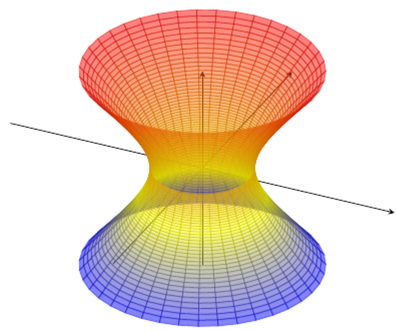

Figure 2.1: The AdS spacetime in embedded coordinates.

The shape of AdS in the embedding space is a two sided hyperboloid. At the center of the spacial dimensions $X_i = 0$ the equation (2.1) yields: $X_0^2 + X_{D+1}^2 = b^2$ which is a circle, in time. For any other point $X_i = x_i$ the time coordinates form a circle, again, but with a greater radius $X_0^2 + X_{D+1}^2 = b^2 + \sum_{i=1}^{D} x_i^2 > b^2$. So, at the origin of space, there is a temporal circle with the smallest possible radius. This region is called the *'neck'* of the hyperboloid. Also, these circles are the timelike geodesics of AdS.

2.2 Coordinate systems

There many coordinate systems that describe AdS_{D+1}, but we present only the ones needed for the purposes of the thesis. Further, coordinate systems and representations of the AdS spacetime can be found in [12] and [13] [14].

2.2.1 Embedding Coordinates

The embedding coordinates have already been presented in the definition of the spacetime. Through them we can define all other coordinate systems.

2.2.2 Global Coordinates

Even though we will not directly use global coordinates, we present them here because they provide a deeper understanding of the geometry. Global coordinates cover the whole hyperboloid, thus their name is justified.

In embedding coordinates, there are two temporal coordinates. Even so, AdS_{D+1} is a surface, i.e. a lower dimensional space, in $\mathbb{R}^{(D,2)}$, so the actual time is one-dimensional. For different stationary points in space, the time evolves on a circle! For different points the circles are concentric, due to rotational symmetry in space. This circular formation of the temporal dimension becomes evident in the formulation of Global coordinates.

The global coordinates are $(\tau, \tilde{\rho}, \Omega_i)$ and are defined by:

$$X_0 = bcosh\tilde{\rho}sin\tau \quad X_i = bsinh\tilde{\rho}\Omega_i \quad X_{D+1} = bcosh\tilde{\rho}cos\tau \tag{2.3}$$

$$\tau\epsilon[0, 2\pi) \quad \tilde{\rho}\epsilon\mathbf{R} \quad \sum_{i=1}^{D}\Omega_i = 1 \tag{2.4}$$

where, τ is a temporal coordinate, Ω_i parametrise the unit sphere S^{D-1} and $\tilde{\rho}$ is a radius-like variable, with a slightly different notion of this ih Euclidean space, since AdS is hyperbolic. The metric in global coordinates is:

$$ds^2 = b^2(-cosh^2\tilde{\rho}d\tau^2 + d\tilde{\rho}^2 + sinh^2\tilde{\rho}d\Omega^2_{D-1}) \tag{2.5}$$

The two temporal coordinates X_0 and X_1 have a harmonic decomposition with respect to the time τ in the global coordinates, that reveals exactly that time evolves in circles, or 1-dimensional spheres S^1, for objects not moving in space. Translations in τ correspond to rotations in the $X_0 - X_1$ plane in embedding coordinates. So, time-like geodesics in AdS are circles. This periodicity in time means that any event happening in a spacetime coordinate will eventually return back to the same time. This creates a problem with causality. How can we eliminate this repetition? By unwrapping S^1, which means that now τ and $\tau + 2\pi n, n \in \mathbb{Z}$ are no longer equivalent. This unwrapping is as if the hyperboloid's 'neck' rolls on a straight line, unfolding the time. The procedure results in values of time τ in all of $\mathbb{R}$, and the resulting spacetime is called the *universal cover of AdS* or $CAdS_{D+1}$. A visualisation of the process can be seen in figure (2.2). The resulting spacetime is called the *universal covering* of Anti-de-Sitter and is denoted as $\tilde{AdS}$.

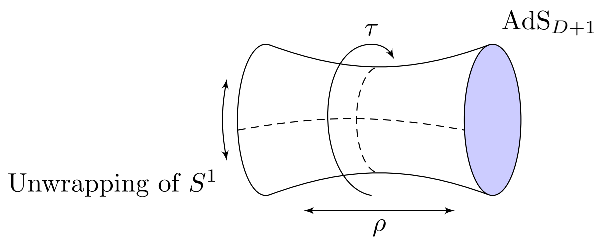

Figure 2.2: The unwrapping or AdS_{D+1}. The $\tau \in \mathbb{R}$ is rolled around the hyperboloid in circles S^1. Points on the dashed circle at the neck of the hyperboloid are at the same spacial coordinate and all points on the vertical dashed line, exist at the exact same time.

To investigate the conformal boundary of AdS space in global coordinates, it is more convenient to perform the change of coordinates $\sinh(\tilde{\rho}) = tan(\rho)$, with $\rho \in \left[0, \frac{\pi}{2}\right)$. Then, it is straightforward to obtain the metric:

$$ds^2 = \frac{b^2}{\cos^2 \rho} \left(-d\tau^2 + d\rho^2 + \sin^2 \rho d\Omega_{D-1}^2\right), \quad \rho \in \left[0, \frac{\pi}{2}\right), \quad \tau \in [0, 2\pi) \tag{2.6}$$

This form of the metric is conformally flat up to a factor $\frac{b^2}{\cos^2 \rho}$. Omitting the conformal factor, we get a cylinder known as the *Penrose diagram*, shown in 2.3.

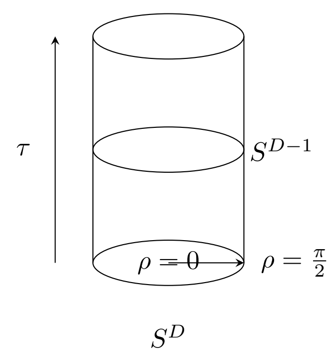

Figure 2.3: The Penrose diagram of AdS spacetime.

The boundary is at $\rho = \frac{\pi}{2}$, which reveals the topology of the boundary $S^1 \times S^{D-1}$ that at the universal cover becomes $\mathbb{R} \times S^{D-1}$.

2.2.3 Poincaré Coordinates

These are the coordinates we use in the majority of the thesis. Let us begin with their definition. The Poincaré coordinates (t, x_i, u) are defined by the embedding space as:

$$\begin{aligned} X_0 &= \frac{z}{2}\left(1 + \frac{1}{z^2}\left(b^2 + \vec{x}^2 - t^2\right)\right), \qquad X_i = \frac{1}{z} x_i \\ X_D &= \frac{z}{2}\left(1 - \frac{1}{z^2}\left(b^2 - \vec{x}^2 + t^2\right)\right), \quad X_{D+1} = \frac{1}{z} t \end{aligned} \tag{2.7}$$

$$t, x_i \in \mathbb{R}, \quad z \in \mathbb{R}^+ \tag{2.8}$$

The metric is:

$$ds^2 = \frac{b^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right) \tag{2.9}$$

where $x^\mu = (t, \vec{x})$.

The resulting spacetime is called the *Poincaré patch.*A major difference between Poincaré and Global coordinates is that the Poincaré coordinates only cover a portion of AdS space. That is where the term 'patch' comes from. In order to cover it, we need another set of the same coordinates for $z < 0$. The Poincaré patch covers the portion of the Penrose diagram shown in figure (2.4).

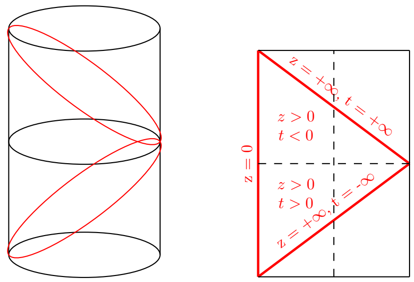

Figure 2.4: The Poincaré patch in the Penrose diagram. From the diagram we can see that the boundary $z \to 0$ corresponds to $\rho = \frac{\pi}{2}$ in global coordinates, so it correspond to the boundary. Also $z \to \infty$ corresponds to $\rho = \frac{\pi}{2}$ but only for one point on the cylinder, where $\tau = 0$ and $\Omega_i = 0$. Thus, the boundary in the Poincaré patch includes also a unique point at infinity $z \to \infty$.

If we want the whole AdS space covered, we can use the Euclidean metric for the flat component by 'rotating time' $t \to it$. That gives the Euclidean signature on the boundary, but we work on the Lorentz signature. This form of parametrisation of AdS is very convenient, because of the flat component $dx^\mu dx_\mu = \eta_{\mu\nu} dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the Minkowskian metric. The coordinate of time t is included in this part of the metric. On the other hand, z is a spacial coordinate. For a constant z dimension the spacetime acts like an ordinary Minkowski flat spacetime. The existence of the factor $\frac{b^2}{z^2}$ has an effect when we try to move in the z-direction. What happens is that as we move to smaller z the distances between point become greater, while, when we move towards greater z values the distances contract and for $z \to \infty$ all points converge to the same singular point.

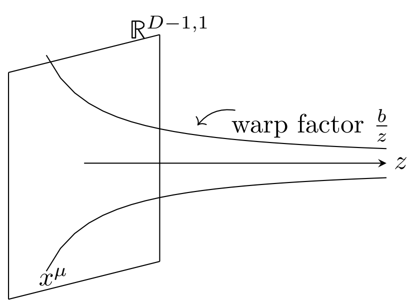

Figure 2.5: Poincaré coordinates

A very important property of the Poincaré patch is that its metric is unaffected by dilatations, so dilatations are an isometry of AdS, thus it preserves distances, which means that the metric doesn't change. Indeed, for $z \to \lambda z$ and

$x^\mu \to \lambda x^\mu$ the metric becomes:

$$d\tilde{s}^2 = b^2 \frac{\lambda^2 z^2}{(\lambda^2 dz^2 + \lambda^2 dx_\mu dx^\mu)} = ds^2 \tag{2.10}$$

2.3 The Boundary of AdS

In the AdS/CFT correspondence, the conformal field theory (CFT) exists on the boundary of AdS, so we are interested in the behaviour of AdS at its boundary [14] [11] [12]. The boundary of AdS is at the asymptotic infinity of the space. In terms of global coordinates, the boundary is then at $\rho \to \infty$. That is the two spheres S^{D-1} at the two antipodal boundaries of the hyperboloid, as well as the hyperbolic side surface. At the figure (2.1), the boundary is the coloured surface and the two asymptotic spheres (or disks) in the upper and lower limit, when the size of the hyperboloid is taken to ∞. At the figure (2.6), the boundary is the coloured surface and the two asymptotic spheres (or disks) in the upper and lower limit, when the size of the hyperboloid is taken to ∞. For instance, in the AdS_2 case, the boundary is a temporal circle of infinite radius as shown in 2.6. The boundary metric in global coordinates is conformally equivalent to a cylinder.

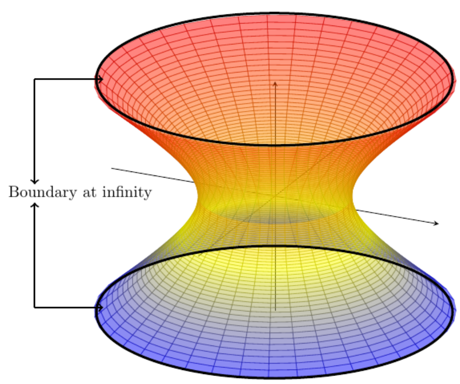

Figure 2.6: For AdS_2 the boundary at spacial infinity is two temporal circles, one at each infinity.

The boundary at the Penrose diagram is, once again, at $\rho = \frac{\pi}{2}$. The topology of the boundary at the universal cover $\tilde{AdS}$ is $\mathbb{R} \times S^{D-1}$

In terms of the Poincaré coordinates, the boundary is a copy of $\mathbb{R}^{\mathbb{D}}$ situated at $z = 0$ and also includes a single point located at infinity $z = \infty$, as explained in the figure (2.4). At the boundary limit $z \to 0$ the space is conformally equivalent to a flat spacetime , as the metric takes the form:

$$ds^2 \to \frac{b^2}{z^2} dx^\mu dx_\mu \tag{2.11}$$

where $dx^\mu dx_\mu$ corresponds to the flat Minkowski metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 - d\vec{x}^2$. So, the AdS boundary is conformally equivalent to $R^{1,D-1}$.

Moreover, from the metric (2.6), the boundary is at $\rho = \frac{\pi}{2}$ and the emergent topology of the boundary is $\mathbb{R} \times S^{D-1}$. At any point of the boundary there exists a $(D-1)$-sphere.

In any representation of the metric the boundary is conformally flat and infinitely away from the interior. Massive particles can never reach the boundary, while, on the contrary, light rays can reach it in finite time.

2.4 Anti-de-Sitter Geodesics

The motion of free particles in AdS spacetime differs drastically between massive and massless particles. A detailed study on the properties and propagation in AdScan be found in [11]. For massive particles the equations of motion in AdS are given by the geodesic equations:

$$\frac{\partial^2 x^\rho}{\partial \tau^2} + \Gamma^\rho_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} \tag{2.12}$$

where τ is the proper time and $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{2.13}$$

From the Global metric (2.6) we only need the following components:

$$\begin{aligned} g_{00} &= -\frac{b^2}{\cos^2\rho} \\ g_{11} &= \frac{b^2}{\cos^2\rho} \\ g_{ij} &= b^2\tan^2\rho \prod_{m=1}^{i-2}\sin^2\theta_m \delta_{ij}, \quad i,j = 2,\dots,d \end{aligned} \tag{2.14}$$

Solving (2.13) for x with initial condition $\rho_0 = \rho(\tau_0)$ we get the trajectory of a free falling particle:

$$\rho(\tau) = |\arcsin\left[\sin\rho_0\cos\left(\tau - \tau_0\right)\right]| \tag{2.15}$$

The emergent oscillatory motion shows that a massive particle can never practically escape the interior of AdS space, consequently it can never reach the boundary and will return back in a finite amount of time $\Delta\tau = \frac{\pi}{2}$.

On the contrary, light does reach the boundary in finite time. A lightlike trajectory in Anti-de-Sitter space is given by the geodesic equation:

$$\frac{d^2 x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{2.16}$$

that is the same as for the massive case with the difference that the parameter of differentiation is not τ but an affine parameter λ [4], since proper time is not defined for light velocity. The trajectory is linear with a slope of 45 deg, as can be deduced from the solution of the equation (2.16):

$$\rho = \pm\tau \tag{2.17}$$

The time needed to a massless particle to reach the boundary is $\Delta\tau = \pi$

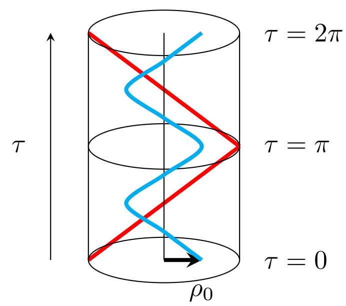

Figure 2.7: Geodesics in the Penrose diagram. A null (light) geodesic is a straight line of slope 45 deg reaching the boundary at proper time $\tau = \pi$. The trajectory of a moving particle is oscillating and the particle does never escape the interior. In the proper time the light needs to reach the boundary, the massive particle returns to its original distance ρ_0 twice.

[4] Affine parameter is one which makes the acceleration perpendicular to the velocity, $g^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$

Chapter 3

Conformal Field Theory in D dimensions

3.1 The Conformal Group

Conformal transformations are a group of more general than the more familiar transformations, such as translations, rotations and Lorentz transformations. The latter preserves lengths as well as the relativistic space-time interval, while Conformal transformations preserve angles, a more general property. Conformal symmetry is a property that is not preserved in Quantum Electrodynamics, because it produces causality problems when coupled with matter. Mass introduces a length braking scale invariance. Thus,in order to characterise a QFT as a CFT, a necessary condition is that the coupling constants are invariant under Energy scaling, which means that at RG fixed points the β-functions should vanish and if this leads to zero trace for the energy momentum tensor the fixed point defines a CFT. So, conformal invariant fields include free massless scalar and Dirac fields and, also, pure gauge fields.

In this chapter, we will present the most important concepts of Conformal Field theories in $D > 2$ dimensions in the bosonic sector that are necessary for the understanding and formulating of the AdS/CFT correspondence, in the next chapter. A more detailed presentation of Conformal Field Theories can be found in [15], [16], [17], [18] and [19]. Briefly, the conformal transformations dictate the preservation of angles but permit dilatations of the spacetime and inversions, that are encoded in the special conformal transformations. In this 'stretching' of the spacetime, there are certain fields that transform in a very particular way, called *primary fields*, that for scalar fields this way is only dependent on the dilatation of the spacetime at their position, $\phi'(x') = \Omega^{-\Delta}\phi(x)$. Each field in a CFT corresponds to an operator, so primary fields correspond to the *primary operators*. Primary fields and operators are characterised by their *conformal scaling* or *conformal dimension*, Δ. This behaviour of conformal fields introduces major simplifications in the calculations of measurables of CFTs, i.e.

correlation functions, and will prove to be very important in the calculation of n-point functions in terms of the AdS/CFT correspondence.

What is more, the conformal transformations permit the mapping of flat spacetime to a cylindrical one. This mapping reveals a very interesting way of 'slicing' the spacetime, called the *radial quantization*, that, whilst simple, visualises the correspondence of operators and quantum states in CFT, and, also, provides a delicate derivation of upper bounds for the conformal dimensions of fields. Further details on the above can be found in [15], [17] and [20]. Among all fields, our main concern will be scalar fields.

3.1.1 Infinitesimal Transformations

The conformal transformations can be classified into 4 categories: translations, Lorentz transformations, i.e. rotations in Euclidean spacetime, dilatations and special conformal transformations. Let $x^\mu \to x'^\mu(x)$ be a coordinate transformation, then, it is called a Conformal transformation if the line elements are invariant up to a local scale factor.

$$dx' = \Omega(x)^2 dx \qquad g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x^\nu} g_{\mu\nu}(x) \equiv \Omega(x)^2 g_{\mu\nu}(x) \tag{3.1}$$

This condition dictates the preservation of relative vector's angle $\frac{\alpha^\mu \beta_\mu}{\sqrt{\alpha^2\beta^2}}$, the primary characteristic of a Conformal transformation. The factor $\Omega(x)$ is called a *Scale factor* and it has to be positive, in order to preserve the metric signature and the causality. For the rest of the chapter, we shall work on Conformal transformations on flat spacetime with a Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, or $g_{\mu\nu} = \delta_{\mu\nu}$ for a Euclidean metric. Let the coordinate x^μ be infinitesimally transformed as:

$$x'^\mu = x^\mu + v^\mu \qquad \Omega(x) = 1 + \sigma(x) \tag{3.2}$$

where v^μ is an infinitesimal vector and $\sigma(x)$ the infinitesimal scaling of the metric at a point x. Then, it can be proved that v^μ is of the form:

$$v_\mu = \overbrace{\alpha_\mu}^{\text{Translations}} - \overbrace{\omega_{\mu\nu}x^\nu}^{\text{Rotations/Lorentz}} + \overbrace{\lambda x^\mu}^{\text{Dilatations}} + \overbrace{\beta_\mu x^2 - 2x^\mu \beta_\nu x^\nu}^{\text{Special Conformal Tranformations}} \tag{3.3}$$

with $\omega_{\mu\nu}$ antisymmetric, $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Obviously, Poincaré transformations - Lorentz transformations and Translations are a subset of conformal transformations. Thus, a conformal transformation includes combinations of these four types of transformations. In order to specify v_μ, thus the exact transformation, one needs $\frac{1}{2}(D+1)(D+2)$ parameters, as occurs from adding:

- Translation $\to D$ parameters
- Rotation $\to \frac{1}{2}D(D-1)$ parameters
- Dilatation $\to 1$ parameter

- Special Conformal Transformation $\to D$ parameters

Next, we want to see how distances are altered under an infinitesimal conformal transformation. The infinitesimal variation of the distance between two points x and y results in:

$$\delta_v(x-y)^2 = (\sigma(x)+\sigma(y))(x-y)^2 \tag{3.4}$$

so, the infinitesimal variations of distances depends on σ, i.e. the dilatation of the metric, as occurs from (3.1) and (3.2). So, after a conformal transformation, distances change according to the dilatation of the metric (3.1) under the transformation. The finite form of the distance transformation is, of course, dependent on the finite scaling of the metric Ω.

$$(x'-y')^2 = \Omega(x)\Omega(y)(x-y)^2 \tag{3.5}$$

The infinitesimal and finite form of the transformations of a CFT are presented in Table (3.1)

Transformation	Infinitesimal Form	Finite Form
Translation	$x^\mu \to x^\mu + \alpha_\mu$	$x^\mu \to x^\mu + \alpha_\mu$
Rotation	$x^\mu \to x^\mu + \omega_{\mu\nu} x^\nu$	$x^\mu \to \Lambda^\mu_\nu x^\nu$
Dilatation	$x^\mu \to x^\mu + \lambda x^\mu$	$x^\mu \to \lambda x^\mu$
Special Conformal	$x^\mu \to x^\mu + \beta^\mu x^2 - 2\beta_\nu x^\nu x^\mu$	$x^\mu \to \frac{x^\mu - x^2\beta^\mu}{1-2\beta_\nu x^\nu + \beta^2 x^2}$

Table 3.1: Conformal Coordinate Transformations

Conformal Invariants

An important property of a Conformal Field theory is that it is possible to construct ratios that are invariant under conformal transformations, $x \to x'$. These ratios are called *anharmonic ratios* or *cross ratios* [16], with a general formulation:

$$u_{ijkl} = \frac{(x_i - x_j)^2 (x_k - x_l)^2}{(x_i - x_k)^2 (x_j - x_l)^2} \tag{3.6}$$

For n points with $n \leq D+1$ there are $\frac{1}{2}n(n-3)$ such ratios. Because of these invariants, the quantities that can be computed by the conformal symmetry in the field theory are restricted.

Conformal Transformations of Fields

An important class of fields in CFT is the *primary fields.* A field ϕ_I with spin I that will transform under a conformal transformation $x \to x'$ as:

$$\phi'_I(x') = \Omega(x)^{-\Delta} R_I^J(x)\phi_J(x) \tag{3.7}$$

is called a *Primary Field*, where Δ is called the *scaling dimension* of the field ϕ_I. Also, R_I^J is the rotation matrix in the representation determined by ϕ_I, similarly to R_μ^ν. We see that conformal fields are affected by dilatations according to the metric scaling Ω and their scaling dimension Δ. For the rest of the thesis we will only refer to scalar fields, for whom the transformation yields:

$$\phi'\left(x'\right) = \Omega(x)^{-\Delta}\phi(x) \tag{3.8}$$

For dilatations, $x^\mu \to \lambda x^\mu$ the primary fields transform as:

$$\phi(x) \to \phi(\lambda x) = \lambda^{-\Delta}\phi(x) \tag{3.9}$$

3.1.2 Representations of the Conformal Group

To determine the conformal group, it is necessary to determine its generators. A more elaborate derivation can be found in [15], [21] and [16]. In short, we can express the coordinate transformation as a Taylor expansion with respect to the infinitesimal parameter v^α, where $v^\mu \to v^\alpha \frac{\partial x^\mu}{\partial v^\alpha}$. Then, $x'^\mu = x^\mu + v^\alpha \frac{\partial x^\mu}{\partial v^\alpha}$. Three relations are necessary for the derivation of the generators. Firstly, the transformation of a field can be written as:

$$\phi(x) \to \phi'\left(x'\right) = \phi(x) + v^\alpha \frac{\partial}{\partial v^\alpha} F_\alpha[\phi(x)] \tag{3.10}$$

Secondly, by the definition of generators we have that:

$$\phi'(x) \equiv e^{-iv^\alpha G_\alpha}\phi(x) \overset{\text{infinitesimal}}{=} \left(1 - iv^\alpha G_\alpha\right)\phi(x) \tag{3.11}$$

Lastly, we can perform a Taylor expansion on $\phi'(x')$:

$$\phi'\left(x'\right) = \phi'(x) + \frac{\partial \phi'}{\partial x'^\mu}\left(v^\alpha \frac{\partial x^\mu}{\partial v^\alpha}\right) \tag{3.12}$$

Combining the three relations we can derive the expression that results into the conformal generators that are presented in Table (3.2). The generators, then, can be derived from:

$$G_\alpha \phi(x) = i\left(\frac{\partial x^\mu}{\partial v^\alpha}\partial_\mu \phi - \frac{\partial}{\partial v^\alpha}F_\alpha[\phi(x)]\right) \tag{3.13}$$

Transformation	Infinitesimal Form	Generator
Translation	$x^\mu \to x^\mu + \alpha_\mu$	$P_\mu = -i\partial_\mu$
Rotation	$x^\mu \to x^\mu + \omega_{\mu\nu}x^\nu$	$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$
Dilatation	$x^\mu \to x^\mu + \lambda x^\mu$	$D = -ix^\mu\partial_\mu$
Special Conformal	$x^\mu \to x^\mu + \beta^\mu x^2 - 2\beta_\nu x^\nu x^\mu$	$K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)$

Table 3.2: Conformal Generators

3.2 Primary Operators

Fields that transform according to (3.7) and (3.9) are called *Primary Fields*. Their transformation under dilatation (3.9) is more appropriate for the definition of the scaling dimension.

$$\phi(x) \to \phi(\lambda x) = \lambda^{-\Delta}\phi(x) \tag{3.14}$$

where Δ is the scaling dimension of the field. Such fields are also called quasi-primary fields.

At the quantum level, functions of the fields are treated as operators $f[\phi(x)] \rightleftarrows O(x)$. Of course, we can refer also to plain fields as operators. A more frequent annotation for an operator would be $\hat{O}(x)$, but for simplicity, we will use plain $O(x)$. The transformation of primary operators under dilatation is then:

$$O(x) \to O(\lambda x) = \lambda^{-\Delta}O(x) \tag{3.15}$$

There is a special group of primary operators called *quasi-primary operators*. These are all operators that are annihilated by the special conformal generator K_μ. More explicitly:

$$K_\mu O(0) = 0 \tag{3.16}$$

which is, also, valid for any coordinates x^μ, that can be proven via a translation.

$$K_\mu O(x) = 0 \tag{3.17}$$

Operators that do not have this property, but do have a definite scaling dimension $O_\Delta(\lambda x) = \lambda^{-\Delta}O_\Delta(x)\big)$ are called *descendant operators*. Actually, their name is not random, they are descendants of some quasi-primary operator. The generators K_μ of the special conformal transformations and P_μ of the translations, act as ladder operators for the eigenvalues of the dilatation generator, increasing and decreasing the conformal scaling dimension by step 1. To be more precise, K_μ is the 'annihilation' or 'descending' operator, while P_μ is the 'creation' or 'ascending' operator. This is revealed by the commutation relations:

$$\begin{aligned} [D, K_\mu] &= -iK_\mu \\ [D, P_\mu] &= +iP_\mu \end{aligned} \tag{3.18}$$

Consequently, a quasi-primary operator, that does not 'survive' the action of K_μ has a scaling dimension Δ_0, can generate a countable set of other descendant operators with scaling dimensions $\Delta_0 + n$, where n is the times P_μ acted. For $n = 0, 1, 3, \ldots$ we get a set called a *conformal family.* Reversely, the annihilation condition for a field, leads to the definition of a primary field (3.8), equivalently:

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{D}} \phi(x) = \Omega^{-\Delta}\phi(x) \tag{3.19}$$

3.3 Operator Product Expansion

Poincaré invariance suggests that correlation functions should only depend on the distance of the operators. A tool to formulate this relation is the *Operator Product Expansion*, or OPE, which is a tool to approximate products of operators at nearby points. The idea is that these products can be approximated by a series of different operators at one of these points. To put this in a mathematical formulation for two points:

$$O_i(x_i)\,O_j(x_j) = \sum_k c_{ij}^k(|x_i - xj|)\,O_k(x_i) \tag{3.20}$$

The coefficients c_{ij}^k rely only on the distance of the two points, due to Poincaré invariance, as it includes translations. Then, the expansion can be applied to expectation values:

$$\langle O_i(x_i)\,O_j(x_j)\ldots\rangle = \sum_k c_{ij}^k(|x_i - xj|)\,\langle O_k(x_i)\ldots\rangle \tag{3.21}$$

The singular behaviour of the OPE at the limit $x_i - x_j \to 0$ omits the *central charge*.

3.4 Bounds of the Conformal dimension

The radial quantization and the state operator map in CFT,[15], [21] and [16], provide restrictions on the minimum values the conformal scaling can have. The state operator map describes a way to match quantum states of the CFT to operators in a conformally equivalent cylindrical geometry. From the positivity of the norm of the quantum states:

$$\langle \chi \mid \chi \rangle \geq 0 \tag{3.22}$$

we can derive the lower bounds of the conformal scalings of various fields. These values are indicative of the scalings of quasi-primary operators, that have the lower possible conformal scaling Δ_0. For scalar fields, for whom the spin is zero, the bound is:

$$\Delta_{s=0} \geq \frac{D}{2} - 1 \tag{3.23}$$

Therefore, the minimum conformal scaling is subject to the dimensions of the spacetime. For scalar fields must exceed the value $\frac{D}{2} - 1$. [15] includes a rigorous proof of the above bound.

3.5 CFT Correlators

Generally, a n-point correlation function is determined by the integral

$$\langle \phi_1(x_1)\ldots\phi_n(x_n)\rangle = \frac{1}{Z}\int d\phi\phi_1(x_1)\ldots\phi_n(x_n)\,e^{-S[\phi]} \tag{3.24}$$

where ϕ are the fields of the theory, S the respective action and Z the partition function. Respectively, in CFT, the correlation functions of quasi-primary operators are calculated by:

$$\langle O_1(x_1)\dots O_n(x_n)\rangle = \frac{1}{Z}\int d\phi O_1(x_1)\dots O_n(x_n)\, e^{-S[\phi]} \tag{3.25}$$

where $O_i(x_i)$ are primary operators acting on a poin x_i and ϕ are the fundamental fields of the CFT. Then conformal invariance of the action and the path integral yields (3.19):

$$\langle O_1(x_1)\dots O_n(x_n)\rangle = \Omega^{\Delta_1}(x_1)\dots\Omega^{\Delta_n}(x_n)\langle O_1(x_1')\dots O_n(x_n')\rangle \tag{3.26}$$

In order to prove this argument, we consider a conformal transformation $x \to x'$. Then the n -point function transforms becomes:

$$\begin{aligned}
\langle O_1(x_1')\dots O_n(x_n')\rangle &= \frac{1}{Z}\int \mathcal{D}\phi O_1(x_1')\dots O_n(x_n')\, e^{-S[\phi]} \\
&= \frac{1}{Z'}\int \mathcal{D}\phi' O_1'(x_1)\dots O_n'(x_n)\, e^{-S[\phi']} \\
&= \frac{1}{Z}\int \mathcal{D}\phi O_1'(x_1)\dots O_n'(x_n)\, e^{-S[\phi]} \\
&= \langle O_1'(x_1)\dots O_n'(x_n)\rangle \\
&= \left|\frac{\partial x}{\partial x'}\right|_{x'=x_1'}^{-\frac{\Delta_1}{D}}\cdots\left|\frac{\partial x}{\partial x'}\right|_{x'=x_n'}^{\frac{-\Delta_n}{D}}\langle O_1(x_1)\dots O_n(x_n)\rangle
\end{aligned}$$

With the inverse transformation we finally get:

$$\begin{aligned}
\langle O_1(x_1)\dots O_n(x_n)\rangle &= \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{D}}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\frac{\Delta_n}{D}}\langle \phi_1(x_1')\dots\phi_n(x_n')\rangle \\
&= \Omega^{\Delta_1}(x_1)\dots\Omega^{\Delta_n}(x_n)\langle O_1(x_1')\dots O_n(x_n')\rangle
\end{aligned}$$

It is noteworthy that correlation functions can be fully determined by conformal invariance for $n < 4$, thus, until 3-point functions. For higher orders, the existence of conformally invariant cross-ratios does not allow a full definition of the correlation function. Nevertheless, it permits the reduction of the number of variables. The manifestation of this argument will become evident in the next chapter, where a full calculation is carried through in terms of the AdS/CFT correspondence.

Chapter 4

The original argument for the AdS/CFT correspondence

4.1 Large N Yang-Mills Theory

Before presenting the formulation of AdS/CFT correspondence, it is important to understand a crucial feature of the theory on the one side of the correspondence, CFT. In Maldacena's conjecture on the one side of the correspondence there is a $\mathcal{N}=4$ super-Yang-Mills theory, that obeys the supersymmetries and is also conformal.

A super-Yang-Mills theory is a 'toy' theory based in Yang-Mills theory, having assumed supersymmetry, and introduces useful simplifications for understanding essential aspects of complex problems. A Yang-Mills theory is a gauge theory based on a special unitary group $SU(N)$ and it is the core of the unification of Electromagnetism, the Weak force and QCD. The number N represents the number of different kinds of fields or, equivalently, the number of gauge charges/colours. Generally, N is the number of degrees of freedom of the theory. The theory that is concerned in the correspondence includes a large number of colours N, known as the *Large N limit*, which introduces a number of simplifications [22] [23]. The first is that only planar diagrams are important as this limit. The Lagrangian density of the theory is:

$$\mathcal{L}_{\mathcal{YM}} = -\frac{1}{2g_{YM}^2}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig_{YM}\left[A^\mu A^\nu\right] \tag{4.1}$$

where A_μ are the gauge fields. Let T^α be the $SU(N)$ generators and $F^{\mu\nu} = F_\alpha^{\mu\nu}T^\alpha$, then the Lagrangian density can, also be expressed as:

$$\mathcal{L}_{YM} = -\frac{1}{4g_{YM}^2}F_\alpha^{\mu\nu}F_{\mu\nu}^\alpha \tag{4.2}$$

Under the special unitary group the transformations of the gauge fields are:

$$A^{\mu} \rightarrow U A^{\mu} U^{\dagger} - \frac{i}{g_{YM}} \partial^{\mu} U U^{\dagger}$$
$$F^{\mu\nu} \rightarrow U F^{\mu\nu} U^{\dagger}$$

where the transformation matrices satisfy the unitarity conditions $UU^{\dagger} = U^{\dagger}U = \mathbb{I}$, $detU = 1$. Moreover, the propagator of the theory in phase space takes the form:

$$D^{\mu\nu}_{\alpha\beta}(k) \equiv \left\langle A^{\mu}_{\alpha} A^{\nu}_{\beta} \right\rangle = -\frac{i}{k^2} \delta_{\alpha\beta} \left(\eta_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{k^2} \right) \tag{4.3}$$

where α and β are the indices of the *Lie algebra*. Up to this point, the number of colours N has nowhere appeared. The reason is that it is hidden within the $SU(N)$ indices. So, if we consider as fundamental fields the matrix components $A^{\mu}_{ij} = A^{\mu}_{\alpha} T^{\alpha}_{ij}$, where T^{α}_{ij} are the generators in the adjoint representation, then, the propagator becomes:

$$D^{\mu\nu}_{\alpha\beta}(k) \equiv \left\langle A^{\mu}_{\alpha} A^{\nu}_{\beta} \right\rangle = -\frac{i}{k^2} \delta_{\alpha\beta} \left(\eta_{\mu\nu} + \frac{k_{\mu} k_{\nu}}{k^2} \right) \tag{4.4}$$

because the relation $T^{\alpha}_{ij} T^{\beta}_{kl} = \left(\delta_{il}\delta_{kj} - \frac{1}{N} \delta_{ij}\delta_{kl} \right)$ holds for the adjoint generators of $SU(N)$. From the form (4.4) it is obvious that for large N the second term in the parenthesis vanishes, simplifying the propagator and the remaining δ-functions demonstrate the replacement of the propagator with a *double line notation*, as shown in figure (4.1).

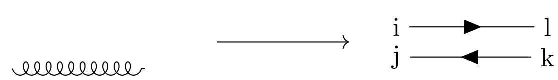

Figure 4.1: Double line notation in the large N limit of $\mathcal{N} = 4$ SYM

But, why is the *large-N limit* so important? First of all, taking the limit is indicative of the degree of planarity of the theory that occurs. By the end of the paragraph, it will be evident why in the large N limit, only planar diagrams are important in $\mathcal{N} = 4$ SYM theory. What is more, by allowing the number of colours to become infinitely large, the theory gets simplified and can capture non-perturbative dynamics with regard to expansion in g_{YM}, that allows a classical description of the theory through a *master field*, as it will be described in the next paragraph.

Nevertheless, the large N limit introduces a new problem. The perturbative expansion is non-normalisable, as the β-functions diverge $\beta(g) = \mu \frac{\partial g}{\partial \mu} \rightarrow \infty$. The solution, proposed by t'Hooft (1974), is to get the limit of $g_{YM} \rightarrow 0$ in a way that the product of the coupling with the number of colours is kept constant.

$$\lambda = g^2_{YM} N = const. \tag{4.5}$$

This number is called the *t'Hooft coupling* and, as a constant, it must be of unity order $\lambda \sim 1$, which is known as *the t'Hooft limit*.

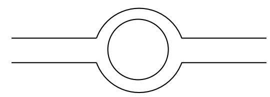

Figure 4.2: Loop diagram with double line notation at large N

To put the t'Hooft coupling into perspective, let us consider a diagram like the one in figure (4.2). On the one hand, the Yang-Mills theory has coupling g_{YM}^2 and, on the other hand, the effective coupling coming from the diagram is $g_{YM}^2 N$, because the loop can have N possible colours. Consequently, the effective coupling must be finite, in order to obtain a normalisable loop expansion. Thus, $g_{YM}^2 N = \lambda$ is considered finite and, for $\lambda \ll 1$, we have the case of weak coupling.

A very interesting simplification that occurs in the t'Hooft limit is the suppression of all non-planar diagrams, which we will show below. Solving for $g_{YM}^2 = \frac{\lambda}{N}$, the contributions to the Feynman diagrams take the form:

- Propagator $\rightarrow g_{YM}^2 = \frac{\lambda}{N}$
- 3-vertex $\rightarrow \frac{1}{g_{YM}^2} = \frac{N}{\lambda}$
- 4-vertex $\rightarrow \frac{1}{g_{YM}^2} = \frac{N}{\lambda}$
- Loops $\rightarrow N$

We are interested in vacuum diagrams, thus, diagrams with no external lines. The geometrical topology of a vacuum Feynman diagram is dictated by its *Euler characteristic*, χ which is a pure number that determines the genus of the diagram, g. The genus of a diagram is also a pure number that counts the number of holes or handles of the surface the diagram defines. For a diagram with E edges/propagators, V vertices and F faces/index loops, the Euler characteristic is:

$$\chi = 2 - 2g = F - E + V \tag{4.6}$$

For instance:

- $g = 0 \rightarrow$ planar diagram
- $g = 1 \rightarrow$ toroidal diagram
- $g = 2 \rightarrow$ double-toroidal diagram

Calculating the total contribution to a vacuum diagram with E edges/propagators, V vertices and F faces/index loops, we get a function of the t'Hooft coupling and the number of colours N.

$$\left(\frac{\lambda}{N}\right)^E \left(\frac{N}{\lambda}\right)^V N^F$$
$$= N^{-E+V+F} \lambda^{E-V}$$
$$= N^{2-2g} \lambda^{E-V}$$

We notice that the effect of N to the vacuum contribution depends only on the genus of the surface. The effective action of the corresponding theory is:

$$logZ = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \tag{4.7}$$

with $f_g(\lambda)$ the sum of diagrams that can be drawn on the surface of genus g. The effective action, now, reveals that in the large N limit, only diagrams of genus $g = 0$, i.e. planar diagrams, have important contributions. So, in the large N limit, only planar diagrams survive [24] [25].

4.2 Motivation of the correspondence

The initial problem that motivated Maldacena's conjecture [25] was the problem of confinement in strong interactions. Confinement in QCD is the phenomenon in Yang-Mills theory of strong attraction between quarks at low energies, that must form colour-neutral baryonic bound states, the mesons and hadrons (protons, neutrons) that form ordinary matter. Even though this is an expected phenomenon, as well as proved in experiment and simulations, it lacks a solid theoretical demonstration. The goal is then, to prove theoretically that the potential energy of these colourless bound-states grows to infinity when the distances between the quarks increase, thus quarks should be confined.

This is difficult as confinement is not a perturbative effect. In QCD the effective coupling constant, i.e. the coupling after renormalisation according to a scale μ, is increasing for low values of μ. As a result, at low energies the effective coupling is large and does not permit a perturbative analysis. A solution was proposed by t'Hooft [22], who formulated a perturbative expansion in the case that QCD has N colours instead of 3. The appropriate expansion parameter is then $\frac{1}{N}$, where N is the number of colours of the theory, and the method is known as the $\frac{1}{N}$*-expansion* or the *large N limit*. As demonstrated in the next paragraph, for the $N \to \infty$ limit only planar Feynman diagrams have important contributions, resulting to *tree-level diagrams*.

Knowing how to provide a perturbative expansion in this special case, the next problem, is how one can calculate the leading term in the $\frac{1}{N}$ expansion. This entails the summation of all planar diagrams, a particularly difficult task. Nevertheless, at the large N-limit, the problem translates to a different one, providing another perspective, for which two observations are needed.

The first observation is that large N Yang-Mills is asymptotically free. The reason is that the correlation functions factorise. To see how this occurs let us consider the Yang-Mills action (4.1):

$$S_{\mathrm{YM}} = -\frac{1}{2g_{YM}^2}\int d^4x \,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} \tag{4.8}$$

The t'Hooft coupling (4.5), $\lambda = g_{YM}^2 N$, suggests that the factor in front of the integral in the action must be proportional to N:

$$S_{\mathrm{YM}} = -\frac{N}{2\lambda}\int d^4x \,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} \tag{4.9}$$

In order for the N factor to be maintained, we can rescale fields and operators $G \to G' = \frac{G}{N}$. Then, the connected Feynman diagrams are computed by the partition function as follows:

$$\langle \mathcal{G}_1 \ldots \mathcal{G}_p \rangle_c = \frac{1}{N^p}\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_p}\log Z[J] \tag{4.10}$$

, thus all the connected diagrams are of the order : $\langle \mathcal{G}_1 \ldots \mathcal{G}_p \rangle_c \sim N^{2-p}$, where c denotes that the diagram is connected,. So, the two point connected diagram $\langle G'G' \rangle_c$ is of the order $\sim N^0$. On the other hand, the product $\langle G' \rangle \langle G' \rangle$ is of the order $\sim N^1$, which entails that for $N \to \infty$:

$$\langle G'G' \rangle = \langle G'G' \rangle_c + \langle G' \rangle \langle G' \rangle \overset{N\to\infty}{\longrightarrow} \langle G' \rangle \langle G' \rangle \tag{4.11}$$

This means that the theory is free of quantum fluctuations at this limit. As such, emerges the concept of the *master field*, the new perspective. Since the theory is asymptotically free, there should be a configuration of the gauge fields $\bar{A}_\mu$ that would evaluate the correlation functions. Of course, as it is the classical limit the integration of the action in the correlation computation no longer exists and all correlations would be calculable.

$$\langle G_1'\left(A_\mu\left(x_1\right)\right)\ldots G_n'\left(A_\mu\left(x_n\right)\right)\rangle = G_1'\left(\overline{A}_\mu\left(x_1\right)\right)\ldots G_n'\left(\overline{A}_\mu\left(x_n\right)\right) \tag{4.12}$$

Consequently, the problem transforms to that of finding the *master field*. That would solve the problem of summation of all planar diagrams, but creates a new one. How can we calculate that master field? That is equivalently difficult since the $\bar{A}_\mu$ are $\infty \times \infty$ matrices!

Maldacena managed to propose a calculation of the master field in terms of gravity! Certainly, that is a link that does not go unnoticed. The conjecture states that the $SU(N)$ gauge theory of $\mathcal{N} = 4$ super-Yang-Mills is equivalent to a quantum gravity theory in an extra dimensional space. Through the correspondence he proposed a method of calculation of the master field, where the latter is actually a gravitational field. For a strongly coupled $\mathcal{N} = 4$ super-Yang-Mills, the master field is the classical gravitational field in a hyperbolic spacetime.

Of course, his conjecture is not analytically proved, because a proof would require the knowledge of the master field by another computation, that, evidently, does not exist. However, it has been tested in several simplified cases.

4.3 Maldacena's Conjecture

In the following section we present the outline of J.Maldacena's first formulation of his conjecture on the AdS/CFT correspondence [25]. The conjecture was derived in terms of string theory and M-theory, for which we present the fundamental notions before discussing the derivation [26] [24].

In string theory there can be two types of strings, open and closed strings. The string coupling is denoted as g_s and is indicative of the energy. The gravitational force is carried by closed massless strings. The open string's endpoints end on hypersurfaces called a *D-branes*, where 'D' denotes the Dirichlet boundary conditions of the string on the brane. The endpoints of open strings are particles propagating on the D-brane and the energy of a D-brane scales as $\sim \frac{1}{g_s}$. D-branes are often denoted as Dp-branes, where p refers to the number of dimensions in which the D-branes expand. In the 10-dimensional supergravity, for instance, a $D3$-brane is a 3-dimensional surface expanding in 4 embedding coordinates out of the total 10 dimensions.

In his paper [25] Maldacena conjectured the AdS/CFT correspondence working in terms of type IIB superstring theory, that is out of the purposes of the thesis, so we will only present the general idea of the derivation. To begin with, Maldacena considered a stack of N parallel $D3$-branes embedded in a $(9+1)$-Minkowski spacetime and studied their behaviour in two different perspectives, the open string perspective and the closed string perspective. The choice of perspective depends on the value of the effective coupling constant of D-branes, $g_s N$.

Before presenting the two perspectives, let us present a coordinate set up in order to facilitate the discussion. In a $(9+1)$-Minkowski spacetime $\mathbb{R}^{9+1}$ described by the embedding coordinates $X_0, X_1, ...X_9$, the $D3$-branes extend in X_0, X_1, X_2, X_3 and are set at the origin of the rest coordinates $X_4...X_9 = 0$. We will denote the distance from the branes $r^2 = \sum_{i=4}^{9} X_i^2$.

- **Open string perspective**
  In the case that the effective coupling constant $g_s N$ is small $g_s N \ll 1$, the $D3$-branes are flat and only massless strings are taken into account. The theory is described by open and closed strings. The open strings are massless, they begin and end at each one of the N parallel branes and can be viewed as excitations of the $D3$-branes, i.e. $(3+1)$-dimensional hyperplane. On the other hand the closed strings are the excitations of $(9+1)$-dimensional flat spacetime. At the $g_s N \ll 1$ limit the effective action is composed by three parts:

$$S = S_{open} + S_{closed} + S_{int} \tag{4.13}$$

  where S_{open} refers to open strings, S_{closed} refers to closed strings and S_{int} to their interactions. But for $g_s N \ll 1$, the interaction part S_{int} is negligible, so the closed stings decouple from the branes. What is more, S_{open} that describes the dynamics of open strings, turns out to describe an $SU(N)$ gauge theory, $\mathcal{N} = 4$ super-Yang-Mills theory on the $\mathbb{R}^{3+1}$

hyperplane of the branes parametrised by X_0, X_1, X_2, X_3, while S_{closed} describes supergavity in $\mathbb{R}^{9+1}$ flat spacetime. The relation between the string coupling and the Yang-Mills coupling is:

$$
\begin{aligned}
2\pi g_s &= g_{YM}^2 \\
2\pi g_s N &= g_{YM}^2 N
\end{aligned}
\tag{4.14}
$$

Notice that $g_s N$ is the effective coupling of the branes, while $g_{YM}^2 N$ is the effective coupling of the Yang-Mills theory. This means that $g_s N \ll 1 \Rightarrow g_{YM}^2 N \ll 1$.

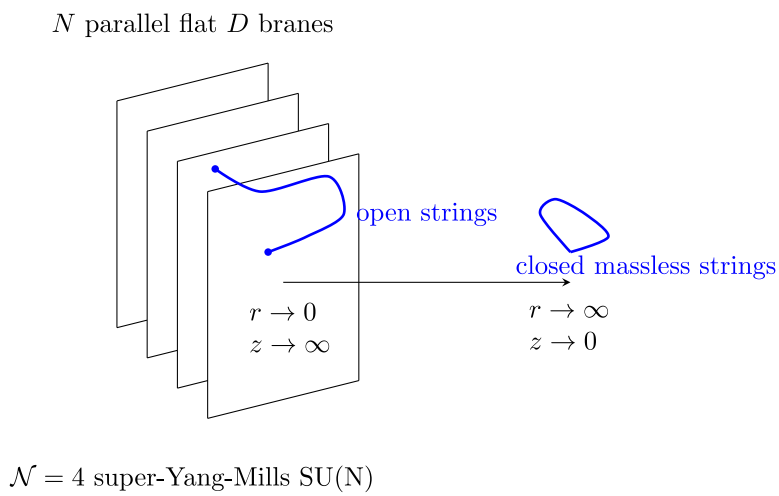

Figure 4.3: Open string perspective

- **Closed string perspective**
  In the case that the effective coupling constant $g_s N$ is $g_s N \gg 1$, the $D3$-branes are strongly coupled and curve spacetime and, also, only closed strings are considered. It is important to note the even though the effective coupling is large, we are still in the low energy limit where $g_s \ll 1$.

  The metric at this limit diverges near the branes and we say that $r \to 0$ is near the horizon region. Near the horizon the spacetime takes the form of $AdS_5 \times S_5$ and away from the spacetime is a $\mathbb{R}^{9+1}$ flat spacetime. It turns out that there can be massless closed strings propagating away from the horizon in the $\mathbb{R}^{9+1}$ flat spacetime, as well as closed strings, that can be massive, propagating near the horizon in the curved $AdS_5 \times S_5$ and the two types of closed strings are decoupled from one another.

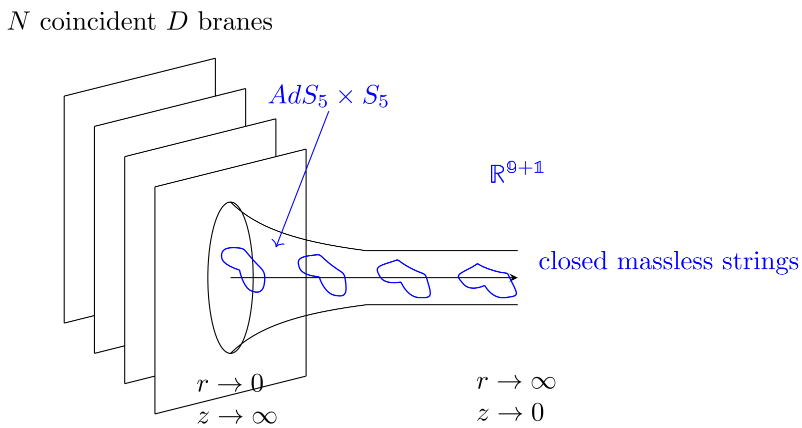

Figure 4.4: Closed string perspective

We see that in both perspectives, we have two types of strings decoupled and, also IIB supergravity propagating in a flat Minkowski spacetime $\mathbb{R}^{9+1}$ away from the brane. But, the two perspectives describe the same physical reality. Additionally, $\mathcal{N} = 4$ super-Yang-Mills theory is well defined for strong coupling $g_{YM}^2 N = \lambda \sim O(1)$. These observations led Maldacena [25] to conjecture that, assuming that we can relax the low energy limit such that $\mathcal{N} = 4$ super-Yang-Mills becomes strongly coupled, then the strongly coupled $SU(N)$ gauge theory on the flat Minkowski $3+1$-spacetime $\mathbb{R}^{3+1}$ should be equivalent to IIB string theory on $AdS_5 \times S_5$.

So, the precise statement is that

$$\boxed{D = 4, \mathcal{N} = 4, SU(N) \text{ Yang-Mills } \rightleftarrows \text{ IIB string theory on } AdS_5 \times S^5\ .} \tag{4.15}$$

This observation opened the way to calculating the summation of all planar, i.e. large N limit, diagrams of $\mathcal{N} = 4$ super-Yang-Mills, a simplified model of QCD, or equivalently computing the master field. Nevertheless, Maldacena's statement remains a conjecture and has not been rigorously proved, but it has passed many crucial tests, some of which we will see in the next chapter.

Chapter 5

Tests of Correspondence

5.1 Symmetries

As we saw in Chapter 2 the AdS_{D+1} spacetime has a symmetry group $SO(D,2)$. A theory on the boundary of AdS_{D+1} generated by the marginal behaviour of bulk fields should have the same number of symmetries. That is because the Hilbert spaces of the two theories should match and, thus, the quantum fields of the theories should respect all transformation symmetries of the space. But $SO(D,2)$ is exactly the conformal group on a Minkowski spacetime $R^{D-1,1}$, as we saw in chapter 3. Consequently, the appropriate dual theory of a gravity theory in AdS_{D+1}, that transforms under the same symmetry group, is a Conformal Field Theory in D dimensions CFT_D. Then, if we want a theory on the boundary that is not a CFT then the corresponding gravity theory cannot be a pure AdS spacetime. Actually, there are variations of the correspondence, where the corresponding gravity dual of a field theory on the boundary is asymptotically an AdS spacetime.

Furthermore, in the example of $AdS_5 \times S^5$ that was used in the original conjecture, AdS_5 has an isometry group of $SO(4,2)$ and S^5 of $SO(6)$. The same symmetry groups are found in the other side of the correspondence, $\mathcal{N} = 4$ SYM. The $SO(4,2)$ is the conformal group, and $SU(4) \simeq SO(6)$ is the R-symmetry group associated with the theory. R-symmetry is the symmetry that transforms different supercharges in the theory into each other, these transformations act only on spinors. In $\mathcal{N} = 4$ SYM there is 1 vector field, 4 spinors and 6 scalar fields, which are related by 4 supersymmetries.Thus, the R-symmetry is $SU(4) = SO(6)$. Lastly, both theories have 32 supersymmetries [25].

5.2 UV/IR Correspondence

One of the most crucial characteristics of the correspondence is *UV/IR duality*. The duality suggests that objects of small size in the bulk of AdS correspond to objects of big size in the boundary, and inversely. The same argument holds for the energy, high energies in the interior correspond to low energies in the boundary and low energies in the interior correspond to high energies in the boundary. It was observed and introduced by L.Susskind and E.Witten in 1998 [27].

To begin with, let us see how this duality manifests itself in terms of the behaviour of the metric in Poincaré coordinates, $ds^2 = \frac{b^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right)$. The boundary metric is that of a Minkowski spacetime, $ds^2 = \eta_{\mu\nu} dx^\nu dx^\mu$, in the sense that the divergence of $\frac{b^2}{z^2}$ is not taken into account for the description of a theory on the boundary. Next, let us consider an object of a certain size at $z = z_0$ in the bulk. Using the scaling isometry of AdS, moving the object closer to the boundary is equivalent to performing the dilatation $(z, x^\mu) \to (\lambda z, \lambda x^\mu)$ for $\lambda \to 0$. Then, since the dilatation is an isometry the size of the object remains invariant in the bulk, but, on the flat boundary we have $x^\mu \to \lambda x^\mu$ so the corresponding boundary object shrinks to a point-like object. Inversely, the behaviour in the interior can be obtained by a scaling with $\lambda \to \infty$. Then, the corresponding object on the boundary will be infinitely large. This is known as the *scale/radius relation*, that suggests that objects of large spacial extend in the CFT of the boundary, correspond to a dual object in the gravitational bulk away from the boundary and, inversely, objects of small spacial extend on the boundary are dual to bulk objects close to the boundary. Of course, by 'close' we mean at a small radial coordinate z and not actually at a small distance from the boundary, since it is still infinitely away.

The same argument can also be seen in terms of the dictionary. According to the AdS/CFT dictionary, each field in the AdS bulk has a conformal dual on the boundary. For instance let us consider a scalar field $\phi(z, x^\mu)$, that is extensively studied in 5.3, and its dual $\phi_0(x^\mu)$ on the boundary, that is obtained by the near boundary behaviour of $\phi(z, x^\mu)$, neglecting the z- divergence or decay. The two fields form a single parameter family of dual fields $\phi(\lambda z, \lambda x^\mu)$ and $\phi_0(\lambda x^\mu)$. For small λ the boundary field ϕ_0 extends along a restricted region on the Minkowski space, while its dual ϕ in the bulk is at a small z-position. On the other hand, for large λ, the boundary field covers a big area in the boundary, while the bulk dual extends deeper in the bulk.

A significant difference between the two processes is that going close to the boundary entails covering a large distance (IR), while going deeper in the bulk requires a smaller distance (UV), as the metric is contracting. So, short distances in the CFT of the boundary correspond to long distances in the gravity description in the bulk. The dual behaviour of the bulk and the boundary theories in the UV and IR limits is actually a characteristic appearing in a wide range of contexts of the correspondence.

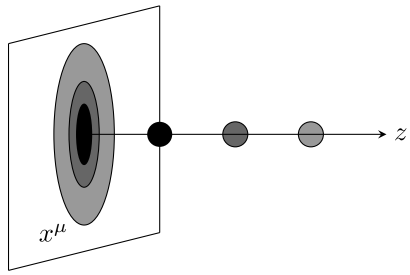

Figure 5.1: Different positions in the bulk translate to different sizes at the boundary.

As for the energy of the field theory of the CFT, under the scaling $(z, x^\mu) \to (\lambda z, \lambda x^\mu)$ for $\lambda \to 0$, the CFT energy scales as $E_{CFT} \to \frac{1}{\lambda} E_{CFT}$. Thus, the energy of the field theory on the boundary scales inversely of the z direction, so $E \sim \frac{1}{z}$. The radial direction z is then the inverse energy scale of the CFT. So, an excitation closer to the boundary, creates a localised, high energy excitation on the boundary, whereas, an excitation in the interior induces a spread excitation on the boundary. For the field in the bulk on the other hand, its energy going near the boundary vanishes (IR), due to the metric divergence that translates to dilatation of time and vanishing frequencies, while moving towards the interior means contraction of distances and time, high frequencies and energies (UV).

The fact that short distances and high energies in the AdS bulk correspond to long distances and low energies on the boundary, is called the *UV/IR duality* or *correspondence*

5.2.1 Counting the degrees of freedom

For the correspondence to make physical sense, the number of degrees of freedom of the gravity theory in the the bulk must be equal to the number of degrees of freedom of the gauge (CFT) theory on the boundary. In fact in their paper [27], E.Witten and L.Susskind demonstrated, via a counting procedure, that their observations about the UV/IR relation and the parameters matching between the two theories lead to the equation of the degrees of freedom.

The matching of the degrees of freedom can actually show how the physics of the two regions are actually related in terms of energy. We assume that we discretize the D dimensional flat boundary to a lattice with spacing ϵ, i.e. the UV cutoff of the gauge theory, and also impose a IR cutoff at L. Then each cell in the lattice has a volume of ϵ^3 and the total number of cells is $\left(\frac{L}{\epsilon}\right)^{D-1}$. We assume that at each cell there can exist only one degree of freedom and each degree can store one piece of information, such as a quantum field, that is seen as a quantum oscillator. Since the Yang-Mills gauge theory on the boundary is an $SU(N)$ theory, the number of field degrees of freedom is N^2 in each cell, i.e. the possible oscillators. Consequently, the overall degrees of freedom on the boundary conformal SYM theory is:

$$N_{dof}^{CFT} \sim \left(\frac{L}{\epsilon}\right)^{D-1} N^2 \tag{5.1}$$

For the gravity side, introducing a UV cutoff on the boundary, suggests the same UV cutoff in the AdS coordinates. Then the bulk boundary shifts at $z = \epsilon$, that is infinitely away from $z = 0$. According to the holographic principle [4] the maximum entropy in the bulk is $\sim \frac{A}{4G_N^{(D+1)}}$ (see [27]), with A the area of the boundary region. Interpreting the entropy as a counter for the degrees of freedom, we can say that the number of the degrees of freedom in the gravity side is:

$$N_{dof}^{AdS} \sim \frac{A}{4G_N^{(D+1)}} \tag{5.2}$$

where $G_N^{(D+1)}$ is the $(D+1)$-dimensional Newton's constant, and the area A of the boundary can be computed for $z = \epsilon$:

$$A = \int_{\mathbb{R}^{D-1}} d^{D-1}\vec{x}\sqrt{-g} = \left(\frac{b}{\epsilon}\right)^{D-1} \int_{\mathbb{R}^{D-1}} d^{D-1}\vec{x} = \left(\frac{bL}{\epsilon}\right)^{D-1} \tag{5.3}$$

so, the number of degrees of freedom is:

$$N_{dof}^{AdS} \sim \frac{b^{D-1}}{4G_N^{(D+1)}} \left(\frac{L}{\epsilon}\right)^{D-1} \tag{5.4}$$

Consequently, the identification between the degrees of freedom between the two regions yields:

$$\begin{aligned} N_{dof}^{CFT} &= N_{dof}^{AdS} \Rightarrow \\ N^2 &\sim \frac{b^{D-1}}{4G_N^{(D+1)}} \end{aligned} \tag{5.5}$$

Moreover, the CFT quantity that corresponds to the degrees of freedom in a general CFT theory is the central charge c_{CFT}. Performing the same steps for the field degrees of freedom described by c_{CFT} rather that N^2, yields: $c_{CFT} \sim \frac{b^{D-1}}{4G_N^{(D+1)}}$

What the latter result suggests is that a large the number of field degrees of freedom N, or colours in terms of QCD, corresponds to a large AdS radius. Using the equation $G_{N+1} = l_P^{D-1}$, where l_P is the Planck length, we get that the AdS radius is large compared to the Planck length, $N^2 \gg 1$ gives $L \gg G_N^{(D+1)} = l_P^{D-1}$. Thus the gravity theory is described by its classical limit. That is the limit that we discuss in the thesis. Since, large N corresponds to a strongly coupled theory, there is a duality between the boundary and the bulk *weak/strong* coupling. A strongly coupled CFT corresponds to a weakly coupled (classical) gravity theory.

It is noteworthy that the result(5.5) is independent of the cutoffs and is valid at the continuum limit as well.

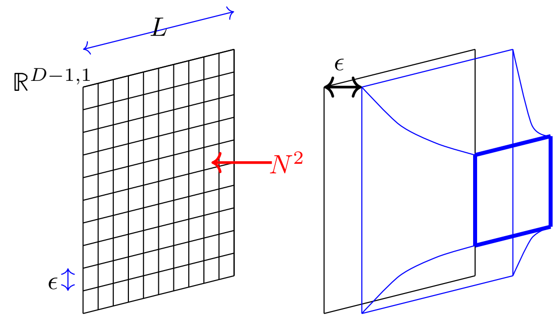

Figure 5.2: Given a UV cutoff ϵ, the boundary can be discretized to a lattice. There are $\left(\frac{L}{\epsilon}\right)^{D-1}$ lattice cells each can store a single bit of information, or a single degree of freedom. There can exist N^2 different field degrees of freedom in each cell for a SU(N) theory, such as SYM.

5.3 Correlation functions

To begin with, the goal of this section is to provide a verification of the partition function correspondence (5.98) as presented in the AdS/CFT dictionary in paragraph 5.4, in the bosonic sector, i.e. only scalar fields are concerned. The relation (5.98) suggests the correspondence of the partition functions of the two theories when gravity is studied near the boundary.

$$\mathcal{Z}_{\text{bulk (Q.Gravity)}}|_{\partial M} = \mathcal{Z}_{\text{CFT}} \tag{5.6}$$

The correspondence between the partition functions entails the correspondence of the correlation functions produced by the two theories. In short, for the duality to hold the boundary correlations computed by the gravity in the bulk must reproduce the correct result, as it would be computed directly by the CFT.

Initially, we must investigate the behaviour of the scalar fields in the interior of AdS_{D+1} spacetime and as they approach the boundary. It occurs that the fields exhibit a conformal scaling behaviour near the boundary, that is actually the key to correspondence of the correlators. Their behaviour induces sources in the boundary theory that correspond to conformal operators acting on the CFT. That is the field-operator correspondence of the theory.

The second step, is apparently to calculate the AdS_{D+1} propagators. The plural should not be of surprise, because the behaviour near the boundary is the objective, so we study separately the way the propagator behaves when its source is on the boundary. An overview of the result is that each operator on the conformal boundary is expected to act as a source of a dual field theory propagating in the bulk and, inversely, the asymptotic value of scalar fields in the Anti-de-Sitter geometry induces a primary conformal operator on the boundary. The propagation between the boundary and the bulk is done via the Boundary-to-Bulk propagator, that describes the extension of the boundary fields in the interior of AdS. Thus, there is a holographic relation between the two theories.

5.3.1 Field-Operator Correspondence

We begin with the behaviour of scalar fields in the interior ofAdS_{D+1} spacetime and as they approach the boundary. The fields considered are scalar fields of positive squared mass $m^2 \geq 0$. We will work in Poincaré coordinates with the metric $ds^2 = \frac{b^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right)$. As mentioned in chapter 2, the boundary in this system is located at $z = 0$ and, also, includes a single point at infinity $z = \infty$. Before going to a detailed analysis, let us discuss how we expect the fields to behave. As already mentioned above in 2.2.3 curvature is included explicitly in the z direction, while for constant z the spacetime is conformally a flat Minkowski spacetime described by the x^μ coordinates. Consequently, we should expect fields to propagate in the x^μ plane as flat waves, just like what we see in non-gravitational theories. The perception is altered when it comes to the behaviour along the z direction. The result, eventually, includes modified Bessel functions, that coincides with the fact that the boundary is conformally a cylinder.

For future reference, we present the metric components that we need for future calculations. In Poincaré coordinates (z, x_μ), $\mu = 0, 1, ..., D-1$, the metric of the manifold $\mathcal{M} = PAdS_{D+1}$ is:

$$ds^2 = \frac{b^2}{z^2}(dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu) \tag{5.7}$$

and the metric components are:

$$g_{zz} = \frac{b^2}{z^2} \qquad g_{\mu\nu} = \frac{b^2}{z^2}\eta_{\mu\nu} \qquad \sqrt{-g} = \sqrt{-(\frac{b^2}{z^2})^{D+1}} \Rightarrow \sqrt{-g} = \frac{b^{D+1}}{z^{D+1}} \tag{5.8}$$

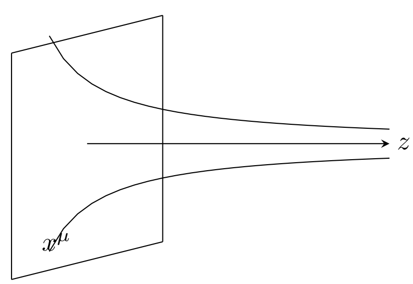

Figure 5.3: Poincaré coordinates. The warp factor $\frac{b^2}{z^2}$ dictates that at the boundary distances become infinitely large, while at $z \to \infty$ the distances become infinitely small.

Firstly, the dynamics of scalar fields in the $\mathcal{M} = AdS_{D+1}$ manifold are described by the action:

$$S = \frac{1}{2}\int_M d^{D+1}x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) \tag{5.9}$$

that can be written as:

$$\begin{aligned}
S &= \frac{1}{2}\int_M d^{D+1}x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2)\\
S &= \frac{1}{2}\int_M d^{D+1}x[-\phi\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + \partial_\mu(\phi\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + \sqrt{-g}m^2\phi^2] \Rightarrow\\
S &= \frac{1}{2}\int_M d^{D+1}x\sqrt{-g}\phi(-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + m^2)\phi]\\
&+ \frac{1}{2}\int_M d^{D+1}x\partial_\mu(\phi\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \qquad (5.10)\\
&= \frac{1}{2}\int_M d^{D+1}x\sqrt{-g}\phi(-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + m^2)\phi]\\
&+ \frac{1}{2}\int_{\partial M} d^{D}x\phi\sqrt{-\gamma}n^\mu\partial_\mu\phi \qquad (5.11)
\end{aligned}$$

where n^μ is the normal vector of the boundary and $\gamma^{\mu\nu}$ the induced metric on the boundary ∂M. We see that the action is composed of two terms, whose variation yields the equation of motion (5.14). The second term is the action on the boundary, that, after substitution from (5.8), can be expressed as:

$$\begin{aligned}
S_{\partial M} &= \frac{1}{2}\int_{\partial M} d^{D}x\phi\sqrt{-\gamma}n^\mu\partial_\mu\phi\\
&= \frac{1}{2}\left(\frac{b}{z}\right)^D\int_{\partial M} d^{D}x\phi(z,x)\partial_z\phi(z,x)\Big|_{z\to 0}
\end{aligned} \tag{5.12}$$

Since, at the $z \to 0$ limit, the boundary action may diverge, we introduce a UV cutoff at $z = \epsilon$, with ϵ arbitrarily small. Actually, that is not a distance UV cut-off but a coordinate UV cut-off, since the distance from the boundary is infinite. The boundary action, then becomes:

$$S^\epsilon_{\partial M} = \frac{1}{2}\left(\frac{b}{\epsilon}\right)^D\int_{\partial M} d^{D}x\phi(\epsilon,x)\partial_\epsilon\phi(\epsilon,x) \tag{5.13}$$

Obviously, the form of the boundary action depends on the boundary conditions. It is apparent that for Dirichlet or Neumann boundary conditions the variation of the boundary action is zero and thus the equation of motion (5.14) is derived from the first term of the action (5.10). Whereas, for mixed boundary conditions the boundary action is not invariant and one has to modify the bulk action [28]. In this thesis, we consider solely Dirichlet boundary conditions.

Free Scalar Field

A free-scalar field propagating in the bulk must obey the equation of motion that occurs from the variation of the action (5.10), that results in the Klein-Gordon equation :

$$(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - m^2)\phi = 0 \tag{5.14}$$

For simplicity, the Laplacian operator may be referred to as:

$$\mathcal{L}_X = D^A D_A = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) \tag{5.15}$$

Substituting (5.8) of Poincaré coordinates to the equation of motion yields :

$$\begin{aligned}
(\frac{z}{b})^{D+1}(\partial_z(\frac{b}{z})^{D+1}g^{zz}\partial_z + \partial_\mu(\frac{b}{z})^{D+1}\frac{z^2}{b^2}\eta_{\mu\nu}\partial_\nu)\phi - m^2\phi = 0 \Rightarrow \\
(\frac{z}{b})^{D+1}(\frac{b^{D-1}}{z^D}(-D+1)\partial_z + \frac{b^{D-1}}{z^{D-1}}\partial_{zz} + (\frac{b}{z})^{D-1}\partial_\mu\eta_{\mu\nu}\partial_\nu)\phi - m^2\phi = 0 \Rightarrow \\
[\frac{z^2}{b^2}\partial_{zz} - \frac{z}{b^2}(D-1)\partial_z - \frac{z^2}{b^2}\Box_{(D)} - m^2]\phi = 0
\end{aligned}$$

where $\Box_{(D)}$ is the D'Alambertian in the boundary Minkowski space ∂M :

$$\Box = -\partial^\mu\partial_\mu = -\eta^{\mu\nu}\partial_\nu\partial_\mu = +\partial t^2 + \partial\vec{x}$$

Then, the form of the equation suggests a separation of variables x^μ and z : $\phi(x,z) = \phi(x)f(z)$ that results in two solvable differential equations:

$$\begin{aligned}
[\frac{z^2}{b^2}f^{''}(z)\phi(x) - \frac{z}{b^2}(D-1)f^{'}(z)\phi(x) - \frac{z^2}{b^2}f(z)\Box_{(D)}\phi(x) - m^2\phi(x)f(z)] = 0 \Rightarrow \\
[-\frac{z^2}{b^2}f^{''}(z) + \frac{z}{b^2}(D-1)f^{'}(z) + m^2 f(z)]\phi(x) = -\frac{z^2}{b^2}f(z)\Box_{(D)}\phi(x) \Rightarrow \\
\frac{1}{f(z)}[-f^{''}(z) + \frac{(D-1)}{z}f^{'}(z) + \frac{m^2b^2}{z^2}f(z)] = -\frac{-\Box_{(D)}\phi(x)}{\phi(x)} = -k^2
\end{aligned}$$

with the constant k^2 being the norm of a vector k^μ. The resulting equations are:

$$-\Box_{(D)}\phi(x) + k^2\phi(x) = 0 \tag{5.16}$$

$$f^{''}(z) - \frac{(D-1)}{z}f^{'}(z) - (\frac{m^2b^2}{z^2} + k^2)f(z) = 0 \tag{5.17}$$

Equation (5.16) simply describes plane waves propagating along the x-plane, that coincides with the non-variation of the curvature along all x^μ directions.

$$\phi_k(x) = \frac{e^{ikx}}{(2\pi)^D} \tag{5.18}$$

Then, the general solution of (5.14) takes the form:

$$\phi(x,z) = \int dx^D \phi_k(x) f_k(z) \tag{5.19}$$

That is the Fourier transformation of the yet unknown function $f_k(z)$. Consequently, the latter is the solution of the equation of motion in the phase-space, corresponding to the x-plane. As for the second equation, along the z-axis, with the appropriate substitution, (5.17) can be transformed into a modified Bessel equation. The full proof of the solution of the equation (5.17) is presented in [29] and [30]. Then:

$$f_k(z) = c_k (kz)^{\frac{D}{2}} K_\nu(kz)$$

where c_k is the coefficient of the k-mode and K_ν is the modified Bessel function of the second kind.

It can be shown that the (5.17) equation gives stable solutions only if $\nu^2 > 0$, where $\nu^2 \equiv (\frac{D^2}{4} + m^2 b^2)$. This restriction impose the so-called *Breitenlohner-Freedman bound* [31]that restricts the mass squared possible values:

$$\nu^2 > 0 \Rightarrow m^2 b^2 > -\frac{D^2}{4} \tag{5.20}$$

we observe that there are no restrictions imposing a non-negative condition on mass. On the contrary, as long as the mass is not too negative there are no instabilities in the theory. In all following calculations, mass is assumed to be real, thus, $m^2 > 0$. For positive mass squared, we get the restriction on ν :

$$\nu = \pm \frac{D}{2} \sqrt{1 + (\frac{2mb}{D})^2} \Rightarrow \nu^2 \geq \frac{D^2}{4} \tag{5.21}$$

So, ν is required to be at least greater than half the dimensions of the Minkowski spacetime.

Next, we are interested in the near the boundary behaviour of the solution that can be described by setting $z = \epsilon$:

$$K_\nu(kz) \approx \frac{\Gamma(\nu)}{2} (\frac{2}{kz})^\nu (1 + O((kz)^2)) \tag{5.22}$$

where based on the property of the modified Bessel of the second kind $K_{-\nu} = K_\nu$ one can write:

$$\begin{aligned} f_k(z) &= c_k (kz)^{\frac{D}{2}} (\frac{\Gamma(\nu)}{2} (\frac{2}{kz})^\nu + \frac{\Gamma(-\nu)}{2} (\frac{2}{kz})^{-\nu}) \Rightarrow \\ &= \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+} \end{aligned}$$

with:

$$\begin{aligned} \phi_0(k) &= c_k 2^{\nu-1} (k)^{\frac{D}{2}-\nu} \Gamma(-\nu) \\ \phi_1(k) &= c_k 2^{-\nu-1} (k)^{\frac{D}{2}+\nu} \Gamma(\nu) \end{aligned}$$

and

$$\Delta_+ = \frac{D}{2} + \nu = +\frac{D}{2}\left(1 + \sqrt{1 + (\frac{2mb}{D})^2}\right)$$
$$\Delta_- = \frac{D}{2} - \nu = -\frac{D}{2}\left(1 - \sqrt{1 + (\frac{2mb}{D})^2}\right)$$

Since, we work on the case $m^2 > 0$, the quantities Δ_+, Δ_- are of opposite sign, $\Delta_+ > 0$ and $\Delta_- < 0$. This will be very important to the interpretation of the behaviour of the scalar fields as they approach the boundary and to the form of the propagators. The derivation of Δ_+, Δ_-, will soon prove to be conformal scalings of the boundary fields.

Finally, the field satisfying the Klein-Gordon equation (5.14) is given by the Fourier transformation:

$$\phi(x,z) = \int \frac{d^D k}{(2\pi)^D} a_k K_\nu(kz) e^{ikx} \tag{5.23}$$

Now, let us examine the behaviour of these solutions as they approach the boundary of AdS, at $z \to 0$ or equivalently at $z = \epsilon$.

$$\phi(x,z) \approx z^{\Delta_-}\phi_0(x) + z^{\Delta_+}\phi_1(x) \tag{5.24}$$

where

$$\begin{aligned} \phi_0(x) &= \int \frac{d^D k}{(2\pi)^D} \phi_0(k) e^{ikx} \\ \phi_1(x) &= \int \frac{d^D k}{(2\pi)^D} \phi_1(k) e^{ikx} \end{aligned} \tag{5.25}$$

The result shows that the field is composed by a vanishing component $z^{\Delta_+}\phi_1(x)$ and a divergent and non-renormalisable term $z^{\Delta_-}\phi_0(x)$, since $\Delta_+ > 0$ and $\Delta_- < 0$. In fact, for the case that $m^2 < 0$, both Δ_+ and Δ_- are positive, and it can be shown that, as long as the mass is in the range of the allowed negative values (5.20), the the two boundary fields ϕ_1 and ϕ_0 are related by a canonical transformation [29] [32].

The latter produces a $\phi_0(x)$ field on the boundary that has a scaling dimension Δ_-, as shown below. So, the boundary behaviour is described by the field $\phi_0(x)$. The boundary field ϕ_0 has a conformal scaling dimension Δ_- near the boundary.

$$\begin{aligned}
\phi(z,\epsilon) &\approx \epsilon^{\Delta_-}\phi_0(x) \Rightarrow \\
\phi_0(\lambda x) &\approx \epsilon^{-\Delta_-}\phi(\lambda x,\epsilon) \\
&= \lambda^{-\Delta_-}\epsilon^{-\Delta_-}\lambda^{\Delta_-}\phi(\lambda x,\epsilon) \\
&= \lambda^{-\Delta_-}\lim_{z\to 0^+}\left(\frac{z}{\lambda}\right)^{-\Delta_-}\phi(\lambda x,z) \\
\text{change of variables:}\quad z^{'} &= \lambda^{-1}z \\
&= \lambda^{-\Delta_-}\lim_{z^{'}\to 0^+}(z^{'})^{-\Delta_-}\phi(\lambda x,\lambda z^{'}) \\
&= \lambda^{-\Delta_-}\phi_0(x)
\end{aligned}$$

where, the scaling invariance of the $\phi(x,z)$ field was used. Thus,

$$\phi_0(\lambda x) = \lambda^{-\Delta_-}\phi_0(x) \tag{5.26}$$

To conclude, as long as the mass respects the Breitenlohner-Freedman bound (5.20), the field $\phi(x,z)$ inside the bulk induces a boundary field $\phi_0(x)$. This field is a source for a primary conformal operator of scaling dimension Δ_- on the boundary, which will be shown in detail in paragraph 5.3.3. But, the fact that we have a source on the boundary indicates that it can propagate back in the bulk! So, we need to see how this back propagation comes to life with a Green function from the boundary back to the AdS bulk. Later, this relation will become evident as we will actually find ϕ_0 to be inside the expression of the *Boundary-to-Bulk propagator*.

Below the Breitenlohner-Freedman bound If the mass violates the *Breitenlohner-Freedman bound* [31], then it leads to instabilities. More explicitly, if the bound is violated, then the conformal scalings become complex. In order to see the instability take place, one must rewrite the Klein-Gordon equation (5.14) as a Schröedinger equation. This will give rise to negative energy and modes that will grow exponentially in time as $\phi \sim \epsilon^{|\omega|t}$.

5.3.2 AdS Propagators

As discussed above, the boundary sources ϕ_0 induce fields in the bulk, of whom the calculation requires a propagator. Additionally, the propagators are obviously necessary for the calculation of correlation functions. Since, we have two regions of interest, the bulk and the boundary, we need to calculate three kinds of propagators:

- Bulk-to-Bulk propagator
- Boundary-to-Bulk propagator
- Boundary-to-Boundary propagator

Bulk-to-Bulk Propagator

Initially, we want t study propagation in the interior of AdS_{D+1}. In order to determine the scalar field propagator in the bulk we will take advantage of the already known modes of scalar field, as computed above, and express the propagator as Fourier integral.

The Bulk-to-Bulk propagator $-iG_{\Delta}$ is defined by the Green differential equation below, where $X = (x, z)$:

$$\left(-\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}) + m^2\right) G_{\Delta}(X, X') = \frac{1}{\sqrt{-g}}\delta^{D+1}(X, X') \tag{5.27}$$

Then, the solution of Klein-Gordon equation with a source $J(X)$:

$$(-\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}) + m^2)\phi = J \tag{5.28}$$

can be expressed as the convolution of the source with the Green function:

$$\phi(X) = \int d^{D+1}X'G(X, X')J(X') \tag{5.29}$$

Any scalar field satisfying the homogeneous Klein-Gordon equation, can be expressed by an initial bulk configuration $\phi(X')$ and the Green function, as follows:

$$\begin{aligned}\phi(X) = \int d^{D+1}X'\sqrt{-g}\,&\big(\phi(X')\left(-\mathcal{L}_{X'} + m^2\right) G(X, X') \\ &- \left(-\mathcal{L}_{X'} + m^2\right)\phi(X')G(X, X')\big)\end{aligned} \tag{5.30}$$

where $\mathcal{L}_{X'}$ the Laplacian operator given from (5.15). Integration by parts yields:

$$\begin{aligned}\phi(X) &= -\int_M d^{D+1}X'\sqrt{-g}D^A\left(\phi(X')D_AG(X, X') - G(X, X')D_A\phi(X')\right) \Rightarrow \\ \phi(X) &= -\int_{\partial M} d^Dy'\sqrt{-\gamma}\left(\phi(y')n^AD_AG(X; y') - G(X; y')n^AD_A\phi(y')\right)\end{aligned} \tag{5.31}$$

Here γ is the induced metric on the boundary. The relation above shows that:

- if G vanishes on the boundary ∂M then $\phi(X)$ is determined by the Dirichlet boundary conditions of $\phi(X')$.
- if $n^AD_AG(X, X')$ vanishes on the boundary then $\phi(X)$ is determined by the von Neumann boundary conditions of $\phi(X')$.
- if neither vanishes then the boundary conditions are mixed.

We are interested in the case of Dirichlet boundary conditions, as the bulk field ϕ corresponds to a boundary field ϕ_0 (5.25). Substituting the metric components defined at (5.8), equation (5.27) becomes:

$$\left(-\frac{z^2}{b^2}\partial_{zz} + \frac{z}{b^2}(D-1)\partial_z + \frac{z^2}{b^2}\Box_{(D)} + m^2\right) G_\Delta(X, X') = \delta^{D+1}(X, X') \quad (5.32)$$

In order to solve this equation, one can recall the notion of Retarded and Advanced Green functions and set:

$$\begin{aligned} &\text{(Retarded - receding from the boundary)} \\ &G^R_\Delta(X, X') = \Theta(z, z')F(z, z')K_0(x, x') \quad (5.33) \\ &\text{(Advanced - approaching the boundary)} \\ &G^A_\Delta(X, X') = \Theta(z', z)F(z', z)K_0(x, x') \quad (5.34) \end{aligned}$$

The idea analogous to the one used for solving the wave Green equation, with the difference that the ordering is space-like and not time-like. From the Retarded function:

$$\left(-\frac{z^2}{b^2}\partial_{zz} + \frac{z}{b^2}(D-1)\partial_z + \frac{z^2}{b^2}\Box_{(D)} + m^2\right) F(z, z')K_0(x, x') = 0 \Rightarrow$$

$$\left(-\frac{z^2}{b^2}\partial_{zz} + \frac{z}{b^2}(D-1)\partial_z + m^2\right) F(z, z')K_0(x, x') + \frac{z^2}{b^2}\Box_{(D)}F(z, z')K_0(x, x') = 0$$

An identical procedure used for solving the Klein-Gordon equation (5.14) leads to the results:

$$F(z, z') = z^{\frac{D}{2}}\left(A_k(z')K_\nu(kz) + B_k(z')I_\nu(kz)\right) \quad (5.35)$$

$$K_0(x, x') = \int \frac{dk^D}{(2\pi)^D}\frac{e^{-ik\cdot(x-x')}}{(2\pi)^D} \quad (5.36)$$

To prevent divergence of the solution in the interior of the bulk ($z \to \infty$), $B_k(z')$ must be set to zero, because $I_\nu(kz)$ diverges at this limit.

Solving for the Advanced function, knowing $F(z, z') = (z')^{\frac{D}{2}}\left(A_k(z)K_{\nu(kz')}\right)$ gives a similar result:

$$A_k(z) = z^{\frac{D}{2}}\left(C_kK_\nu(kz) + D_kI_\nu(kz)\right)$$

Here, z is closer to the boundary than z', so it cannot go to infinity, but the solution should not diverge when $z \to 0$. Thus, we set $C_k = 0$. Then:

$$F(z, z') = (zz')^{\frac{D}{2}}\left(D_kI_\nu(kz')K_{\nu(kz)}\right)$$

Substituting the above in (5.33), the Bulk propagator that obeys the Green equation (5.27) takes the form [33] :

$$\begin{aligned} G_\Delta(X, X') = \int d^{D+1}\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(x-x')}(zz')^{\frac{D}{2}}\,&[\Theta(z, z')K_\nu(kz)I_\nu(kz') \\ &+\Theta(z', z)K_\nu(kz')I_\nu(kz)] \end{aligned} \quad (5.37)$$

The integration can be performed and the solution is written in terms of a hypergeometric function, given by [34] [33]:

$$G_\Delta(X,X') = \frac{2C_\Delta}{\nu}\left(\frac{\xi}{2}\right)^\Delta F\left(\frac{\Delta}{2},\frac{\Delta}{2}+\frac{1}{2};\nu+1;\xi^2\right) \tag{5.38}$$
$$\xi = \frac{2zz'}{z^2+z'^2+(x-x')^2}$$

,where $\Delta = \Delta_+$ or Δ_-, obeying the equation $\Delta(D-\Delta)+m^2=0$ and C_Δ is a constant given by:

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{D}{2}}\Gamma(\nu)}$$

The normalisation constant is selected in a way that normalises the Boundary-to-Bulk propagator. If there is a cutoff at $z=\epsilon$, then the Green function should vanish at that point near the boundary. The relative Green function that vanishes at $z=\epsilon$ can be found [33] [29]:

$$G_\epsilon(X,X') = G_\Delta(X,X') + \int d^{D+1}\frac{d^Dk}{(2\pi)^D}e^{-ik\cdot(x-x')}\left(zz'\right)^{\frac{D}{2}}K_\nu(kz')K_\nu(kz)\frac{I_\nu(k\epsilon)}{K_\nu(k\epsilon)} \tag{5.39}$$

Boundary-to-Bulk propagator

In the case the source point X' in on the boundary $z=0$, the Boundary-to-Bulk propagator obeys the equation:

$$\left(-\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_n u)+m^2\right)K_\Delta(x,z;x') = \delta^{D+1}(x,z;,x',0) \tag{5.40}$$

The latter is exactly the Green equation (5.27) with a source on the boundary $z=0$. Assuming a source on the boundary $\phi_0(X')$, the solution in the bulk should, again, be the convolution with the propagator, as already seen in (5.29):

$$\phi(X) = \int d^Dx' K_\Delta(x,z;x')\phi_0(x') \tag{5.41}$$

and from Fourier transformation in x-plane, the phase space equivalent is:

$$f_k(z) = K_k(z)\phi_0(k)$$

Witten's Method

The Boundary-to-Bulk propagator can be obtained directly from the Bulk-to-Bulk propagator taking the limit where the source is at the boundary $z=0$. This derivation is explained in [29].

A method richer in physical intuition for the calculation of the Boundary-to-Bulk propagator is Witten's method [35]. It involves two tricks: The first is to remember that the boundary includes a singular point at infinity and solve the equation near the center $z \to \infty$, at the singular boundary point (see figure (2.4)). Then, the second is to obtain the wanted result, using the isometries of AdS spacetime.

At the infinity limit, the space shrinks to a single point and the metric is simplified, due to the warp factor $w(z) = \frac{1}{z}$.

$$\frac{b^2}{z^2}\eta_{\mu\nu} \to 0 \tag{5.42}$$

and the space can be thought to have degenerated to a single point. The equation (5.40) then simplifies to (5.17), with $k = 0$.

$$K_{\Delta}^{''}(z) - \frac{(D-1)}{z}K_{\Delta}^{'}(z) - \frac{m^2b^2}{z^2}K_{\Delta}(z) = 0 \tag{5.43}$$

From f_k solutions we have for $z \to \infty$:

$$\begin{aligned} f_k(z) &= \overbrace{\phi_0(k)z^{\Delta_-}}^{0} + \phi_1(k)z^{\Delta_+} \\ \text{where: } \phi_1(0) &= C_{\Delta} \end{aligned}$$

In the latter, the k dependence has been absorbed in the constant c_0. So the solution near the center of AdS $z = \infty$ is the simple expression [1]:

$$K_{\Delta}(z) = C_{\Delta}z^{\Delta_+} \tag{5.44}$$

All that remains is to use AdS isometries [35] to map the $z \to \infty$ region back to the whole bulk and ensure that the result actually acts as a delta function at the boundary. The required isometries are an inversion I to map the solution to finite x, and a translation T in the x-plane.

$$I : \begin{cases} x^{\mu} \to \frac{x^{\mu}}{z^2+x^2} \\ z \to \frac{z}{z^2+x^2} \end{cases}$$

$$T : x^{\mu} \to (x^{\mu} - x'_{\mu})$$

The Boundary-to-Bulk propagator becomes:

$$K_{\Delta_+}(z, x; x') = C_{\Delta_+}\left(\frac{z}{z^2 + (x - x')^2}\right)^{\Delta_+} \tag{5.45}$$

The last quantity that must be evaluated is the normalisation constant C_{Δ_+}. The propagator must have a δ behaviour when the source goes to the boundary.

[1] An alternative way to obtain the same result, could be to solve the equation (5.43) anew. Similarly to solving for f_k, substituting $f_k = z^{\frac{D}{2}}g(z)$ yields an Euler differential equation: $z^2g^{''}(z) + zg^{'}(z) - \frac{(\frac{D^2}{4}+m^2b^2)}{(z)^2}g(z) = 0$ with an indicial equation: $\alpha(\alpha-1) + \alpha - \nu^2 = 0$ for $\nu^2 \equiv (\frac{D^2}{4} + m^2b^2)$. The obvious solution is $g(z)\ z^{\pm\nu}$. The final result, following the same reasoning is (5.44).

From (5.24) the boundary field is $\phi_0(x)$ and not $z^{\Delta_-}\phi_0(x)$. Similarly, K_{Δ_+} is expected to diverge $\propto z^{\Delta_-}$, so the correct normalisation should cancel out this divergence, as follows:

$$\lim_{z\to 0^+} z^{-\Delta_-} K_{\Delta_+}(z,x;x') = \delta^{(D)}(x,x') \tag{5.46}$$

From the normalisation, the normalisation constant can be found (see [29]):

$$C_{\Delta_+} = \frac{\Gamma(\Delta_+)}{\pi^{\frac{D}{2}}\Gamma(\nu)} \tag{5.47}$$

From (5.46) emerges the near the boundary, $z = \epsilon$, behaviour of the Boundary-to-Bulk propagator:

$$K_{\Delta_+}(x,\epsilon;x') = \epsilon^{\Delta_-}\delta^{(D)} + C_{\Delta_+}\left(\frac{\epsilon^{\Delta_+}}{(x-x')^{2\Delta_+}}\right) \tag{5.48}$$

The importance of the Boundary-to-Bulk propagator lays on the computation of correlation functions of boundary operators, the observables of the theory, which are calculated via Witten diagrams in the interior. These diagrams consist of combinations of Boundary-to-Bulk propagators.

Boundary-to-Boundary propagator

Similarly to the Boundary-to-Bulk propagator, the Boundary-to-Boundary propagator is given by the double limit [29]:

$$\beta_\Delta(x,x') = \lim_{z,z'\to 0^+} \frac{2\nu^2}{(zz')^{\Delta_+}} G_\Delta(X,X') \tag{5.49}$$

which yields:

$$\beta_\Delta(x,x') = C_{\Delta_+}\left(\frac{1}{(x-x')}\right)^{2\Delta_+} \tag{5.50}$$

That is exactly the 2-point function of a Conformal Field Theory [16], which should be anticipated due to the correspondence.

Back-propagation of Boundary Fields

As it is obvious from (5.48), the Boundary-to-Bulk propagator has a behaviour near the boundary that decays in the bulk , $\epsilon^{\Delta_-}\delta(x-x')$. Also, a field in the Bulk can be expressed as (5.41) the convolution of a boundary source with the Boundary-to-Bulk propagator. In this paragraph, we will show that the behaviour of a field near the boundary can be derived both from the convolution with (5.48) and, also, if we consider the boundary behaviour of a free field in the Bulk $\phi_{free}(x,z) \approx z^{\Delta_-}\phi_0(x)$ to be a source. The first way of computation is obvious. As for the second, the source ϕ_0 will produce a field:

$$\phi(x,z) = \int d^D x' K_\Delta(x,z;x')\phi_0(x') \Rightarrow$$
$$\phi'(x,z) = \int d^D x' C_{\Delta_+}\left(\frac{z}{z^2+(x-x')^2}\right)^{\Delta_+}\phi_0(x') \Rightarrow$$
$$\phi'(x,z) = z^{\Delta_+}\int d^D x' C_{\Delta_+}\left(\frac{1}{z^2+(x-x')^2}\right)^{\Delta_+}\phi_0(x')$$

so, near the boundary:

$$\phi'(x,z) \to z^{\Delta_+}\int d^D x' C_{\Delta_+}\frac{1}{(x-x')^{2\Delta_+}}\phi_0(x') \equiv z^{\Delta_+}\phi_1(x) \tag{5.51}$$

This results to a full solution of the Klein-Gordon equation with a source $\phi_0(x)$:

$$\phi(x,z) = \phi_{free}(x,z) + \phi'(x,z) \to z^{\Delta_-}\phi_0(x) + z^{\Delta_+}\phi_1(x)$$

This behaviour is identical to the one we found analytically in (5.24). Thus, we conclude that considering ϕ_0 as a source in the boundary reproduces the marginal behaviour of the bulk scalar fields. Consequently, the solution of the Klein-Gordon equation near the boundary is can be expressed as:

$$\phi(x,z) \to \int d^D x' \left(z^{\Delta_-}\delta(x,x') + z^{\Delta_+}C_{\Delta_+}\frac{1}{(x-x')^{2\Delta_+}}\right)\phi_0(x') \tag{5.52}$$

where the the expression in the integral is exactly the Boundary-to-Bulk propagator when there is a cut-off imposed at $z = \epsilon$.

5.3.3 Evaluation of the action

Free Action

Due to the homogeneous Klein-Gordon equation, from the free action (5.10) only the second term remains and it can be expressed solely in terms of boundary conditions:

$$S[\phi] = \frac{1}{2}\int_{\partial M} d^D x \phi\sqrt{-\gamma}n^\mu\partial_\mu\phi \tag{5.53}$$

It can be proven [29] that the action can be written as:

$$S[\phi_0] = \nu b^{D-1}C_{\Delta_+}\int d^D x_1 d^D x_2 \frac{\phi_0(x_1)\phi_0(x_2)}{(x_1-x_2)^{2\Delta_+}} \tag{5.54}$$

According to the correspondence formula (5.98), it is now easy to see that the free action results in the correct form of the CFT two point function, which is calculated below in (5.70).

$$\langle O(x)O(y)\rangle = -\frac{\delta}{\delta\phi_0(x)}\frac{\delta}{\delta\phi_0(y)}S[\phi_0]|_{\phi_0=0} = -\nu b^{D-1}C_{\Delta_+}|x-y|^{-2\Delta_+} \tag{5.55}$$

The confirmation of this match emerges for the second time, as the CFT two point function already appeared in the Boundary-to-Boundary propagator calculation (5.50).

In the special case that ν is an integer [30], the computation needs special treatment as it includes a logarithmic term. Its contribution is only local so, the relative treatment is similar, but will not be discussed in the thesis.

With the present form of the action, it is possible to prove that the "source" fields ϕ_1 and ϕ_0 are conjugate [29]

$$\begin{aligned}\frac{\delta S[\phi_0]}{\delta\phi_0} &= \nu b^{D-1} C_{\Delta_+} \int d^D x' \frac{\phi_0(x')}{(x-x')^{2\Delta_+}} \\ &= \nu b^{D-1}\phi_1(x)\end{aligned} \tag{5.56}$$

with ϕ_1 the boundary field of conformal scale Δ_+ defined in (5.51).

General Expression of the Action

For the computation of many point correlation functions in the AdS side it is necessary to introduce interaction terms in the action, so we write:

$$S_t[\phi] = S[\phi] + S_{int}[\phi] \tag{5.57}$$

where $S[\phi]$ is the free action (5.9). Thus,

$$S_t[\phi] = \frac{1}{2}\int_M d^{D+1}x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2) + S_{int}[\phi] \tag{5.58}$$

The equation of motion then includes a source (5.28)

$$(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - m^2)\phi(X) = J(X) \tag{5.59}$$

with $J(X)$ occurring from the functional variation of $S_{int}[\phi]$.

$$J(X) = \frac{\delta S_{int}}{\delta\phi(X)} \tag{5.60}$$

The solution of (5.28) can be written as the sum of a homogeneous solution with partial solution:

$$\phi(x,z) = \phi_{(0)}(x,z) + \phi_{(p)}(x,z) \tag{5.61}$$

with $\phi_{(0)}$ obtained from the first half of (5.31), determined by Dirichlet boundary conditions, and $\phi_{(p)}$ from the second, being the partial solution.

$$\phi_{(0)}(x,z) = -\int_{\partial M} d^D x' \sqrt{-\gamma}\phi(x') n^A D_A G(x,z;x')|_{x'\in\partial M} \tag{5.62a}$$

$$\overset{(5.41)}{=} \int d^D x' K_\Delta(x,z;x')\phi_0(x') \tag{5.62b}$$

$$\phi_{(p)}(x,z) = \int_M d^{D+1}X G_\Delta(X,X')J(X') \tag{5.63}$$

The homogeneous solution is, of course the solution already found in 5.3.1. Substituting (5.61) back to the action, utilising the known homogeneous solution, results in an action [30] of the form:

$$S_t[\phi] = \nu b^{D-1} C_{\Delta_+} \int d^D x_1 d^D x_2 \frac{\phi_0(x_1)\phi_0(x_2)}{(x_1-x_2)^{2\Delta_+}}$$
$$+\frac{1}{2}\int_M d^{D+1}X_1 d^{D+1}X_2 J(X_1)G(X_1,X_2)J(X_2) + S_{int} \tag{5.64}$$

For the computation of correlations, the last two terms need to be expanded perturbatively.

5.3.4 N-point Functions Correspondence

The purpose of this paragraph is to verify the correspondence between the correlation functions computed by each theory. The calculations go up to the 4-point correlation functions, but similar procedure holds for arbitrary n-point functions, see [30]. The aim of those calculations is to verify the identification of the partition functions of the two theories (5.98), the classical gravity in AdS and CFT on the flat boundary. The correlators are calculated in each theory separately and in the end their match is concluded. CFT correlators are computed solely via their symmetries, but only up to 3-point functions. For higher order correlators, there are invariant quantities that do not permit a full computation. Then, the details of the CFT theory are necessary to determine the exact form of the correlators, but the correspondence with their gravitational dual is still evident from their general formulation. AdS correlators emerge from the standard definition of correlation functions in a QFT in curved space with the *Boundary-to-Bulk propagators* and *Boundary-to-Boundary propagators* the appropriate Green functions.

1-point functions

CFT side Let $\phi(x)$ be a *quasi-primary operator* of scaling dimension Δ. The 1-point correlation function, then, transforms under a dilatation transformation as:

$$\langle\phi(x)\rangle = \lambda^\Delta \langle\phi(\lambda x)\rangle$$

Setting $\phi(x) = f(x)$ and expand $f(x)$ in Taylor series, the 1-point function reduces to a δ function, that demands that the scaling dimension is zero.

$$\begin{aligned} f(x) &= \lambda^{\Delta} f(\lambda x) \Rightarrow \\ \sum_{n=1}^{\infty} c_n x^n &= \sum_{n=1}^{\infty} c_n \lambda^{\Delta+n} x^n \Rightarrow \end{aligned}$$

$$\Delta = -n \quad \text{and} \quad f(x) = c_{-\Delta} x^{-\Delta}$$

Using the translation invariance $x^{\mu} \to x^{\mu} + a^{\mu}$:

$$\begin{aligned} x^{-\Delta} &= (x+\alpha)^{-\Delta} \Rightarrow \\ \Delta &= 0 \Rightarrow \\ f(x) &= c_0 \delta_{\Delta,0} \end{aligned}$$

Finally the 1-point function takes the form:

$$< \phi(x) >= c_0 \delta_{\Delta,0} \tag{5.65}$$

Performing rescaling the field becomes:

$$< \phi(x) >= \delta_{\Delta,0} \tag{5.66}$$

This means that there is no dilation invariant field other than the unit operator $\phi(x) = \mathbb{1}$, thus the vacuum.

AdS side The 1-point function in AdS $< \phi(x) >$ vanishes and after renormalisation matches the CFT side.

2-point functions

CFT side As seen in (3.15) the conformal symmetry dictates that, under dilatation, primary operators must satisfy:

$$\langle O_1(x_1) O_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle O_1(\lambda x_1) O_2(\lambda x_2) \rangle \tag{5.67}$$

Due to Poincaré invariance the correlation function can only depend on the distance between two points, thus:

$$\begin{aligned} \langle O_1(x_1) O_2(x_2) \rangle &= f(|x_1 - x_2|) \Rightarrow \\ f(|x_1 - x_2|) &= f(\lambda |x_1 - x_2|) \end{aligned}$$

If the latter is then Taylor expanded, using the translation invariance one can find:

$$\begin{aligned}
\sum_{n=0}^{\infty}|x_1-x_2|^n &= \sum_{n=0}^{\infty} c_n \lambda^{\Delta_1+\Delta_2+n}|x_1-x_2|^n \Rightarrow \\
\Delta_1+\Delta_2+n &= 0 \Rightarrow \\
n &= -\Delta_1-\Delta_2 \Rightarrow \\
f(|x_1-x_2|) &= c_{-\Delta_1-\Delta_2}|x_1-x_2|^{-\Delta_1-\Delta_2} \Rightarrow \\
\langle\phi(x_1)\phi(x_2)\rangle &= \frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}
\end{aligned}$$

where, $d_{12} = c_{-\Delta_1-\Delta_2}$

In order to determine d_{12} we use invariance under special conformal transformations,

$$\begin{aligned}
x_i' &= \frac{x_i^\mu - x_i b^\mu}{\gamma_i} \\
\gamma_i &= \frac{1}{1-2b\cdot x_i + b^2 x_i^2}
\end{aligned}$$

that according to [16]:

$$|x_1'-x_2'| = \frac{|x_1-x_2|}{\sqrt{\gamma_1\gamma_2}} \tag{5.68}$$

and

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{d_{12}}{|x_1'-x_2'|^{\Delta_1+\Delta_2}} \tag{5.69}$$

Thus, the two point function becomes:

$$\begin{aligned}
(5.69) \Rightarrow \frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} &= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{d_{12}}{|x_1'-x_2'|^{\Delta_1+\Delta_2}} \Rightarrow \\
\frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} &= \frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}\frac{(\gamma_1\gamma_2)^{\frac{\Delta_2+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \Rightarrow \\
\frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} &= \frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}\left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{\Delta_1-\Delta_2}{2}} \Rightarrow \\
\Delta_1 &= \Delta_2
\end{aligned}$$

This means that the 2-point function vanishes unless the two fields have the same scaling dimension. Finally, the 2 point correlation function takes the form:

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{d_{12}}{|x_1-x_2|^{2\Delta_1}}\delta_{\Delta_1\Delta_2} \tag{5.70}$$

After normalisation it becomes:

$$\langle\phi(x_1)\phi(x_2)\rangle = \frac{d_{12}}{\tilde{d}_1\tilde{d}_2}\frac{1}{|x_1-x_2|^{2\Delta_1}}\delta_{\Delta_1\Delta_2} \tag{5.71}$$

AdS side The two point function is simply the Boundary-to-Boundary propagator (5.50):

$$\beta_\Delta(x,x') = C_{\Delta_+}\frac{1}{|x-x'|^{2\Delta_+}} \tag{5.72}$$

and can be computed according to (5.55), as well.

Obviously, both ways of deriving the 2-point function match perfectly for $\Delta_+ = \Delta_1$ and $C_{\Delta_+} = d_{12}$.

3-point functions

CFT side The 2 and 3-point functions of conformal primary fields are the only ones that can be determined uniquely, up to an overall coefficient, for reasons explained in the next paragraph. This is a result of the fact that conformal transformations map any three points on $\mathbb{R}^{D-1,1}$ to any other three points [16].

For primary fields, dilatation symmetry translates to:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \lambda^{\Delta_1+\Delta_2+\Delta_3}\langle O_1(\lambda x_1)O_2(\lambda x_2)O_3(\lambda x_3)\rangle \tag{5.73}$$

In an identical fashion, the 3-point function can be determined by invariance under translations and special conformal transformations. Poincaré symmetry dictates:

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = f(|x_1-x_2|,|x_2-x_3|,|x_3-x_1|) \Rightarrow\\ f(|x_1-x_2|,|x_2-x_3|,|x_3-x_1|) = f(\lambda|x_1-x_2|,\lambda|x_2-x_3|,\lambda|x_3-x_1|)\end{aligned}$$

and from Taylor expansion:

$$\begin{aligned}&\sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}|x_1-x_2|^{n_1}|x_2-x_3|^{n_2}|x_3-x_1|^{n_3}\\ &= \sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\lambda^{\Delta_1+\Delta_2+\Delta_3+n_1+n_2+n_3}|x_1-x_2|^{n_1}|x_2-x_3|^{n_2}|x_3-x_1|^{n_3} \Rightarrow\\ &\Delta_1+\Delta_2+\Delta_3+n_1+n_2+n_3 = 0 \Rightarrow\\ &n_1+n_2+n_3 = -(\Delta_1+\Delta_2+\Delta_3)\end{aligned}$$

On the other hand, special conformal transformation, [16], gives:

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle &= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}\langle O_1(x_1')O_2(x_2')O_3(x_3')\rangle \Rightarrow\\ f(|x_1-x_2|,|x_2-x_3|,|x_3-x_1|) &= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} f(|x_1'-x_2'|,|x_2'-x_3'|,|x_3'-x_1'|) \Rightarrow\\ f(|x_1-x_2|,|x_2-x_3|,|x_3-x_1|) &= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} f(\frac{|x_1-x_2|}{\sqrt{\gamma_1\gamma_2}},\frac{|x_2-x_3|}{\sqrt{\gamma_2\gamma_3}},\frac{|x_3-x_1|}{\sqrt{\gamma_1\gamma_3}}) \Rightarrow\end{aligned}$$

above we used the relation (5.68).

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}|x_1-x_2|^{n_1}|x_2-x_3|^{n_2}|x_3-x_1|^{n_3}\\
&=\sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}\frac{|x_1-x_2|^{n_1}}{\sqrt{\gamma_1\gamma_2}^{\frac{n_1}{2}}}\frac{|x_2-x_3|^{n_2}}{\sqrt{\gamma_2\gamma_3}^{\frac{n_2}{2}}}\frac{|x_3-x_1|^{n_3}}{\sqrt{\gamma_1\gamma_3}^{\frac{n_3}{2}}}\\
&=\sum_{n_1,n_2,n_3=0}^{\infty} c_{n_1,n_2,n_3}\frac{1}{\gamma_1^{\Delta_1+\frac{n_1+n_3}{2}}\gamma_2^{\Delta_2+\frac{n_1+n_3}{2}}\gamma_3^{\Delta_3+\frac{n_1+n_2}{2}}}|x_1-x_2|^{n_1}|x_2-x_3|^{n_2}|x_3-x_1|^{n_3}\Rightarrow\\
&\qquad\qquad\left\{\begin{array}{l}\Delta_1+\frac{n_1+n_3}{2}=0\\ \Delta_2+\frac{n_1+n_3}{2}=0\\ \Delta_3+\frac{n_1+n_2}{2}=0\end{array}\right.\Rightarrow\left\{\begin{array}{l}n_1=\Delta_3-\Delta_1-\Delta_2\\ n_2=\Delta_1-\Delta_3-\Delta_2\\ n_3=\Delta_2-\Delta_1-\Delta_3\end{array}\right.\Rightarrow\\
&\qquad\qquad\left\{\begin{array}{l}n_1=-\Sigma+2\Delta_3\\ n_2=-\Sigma+\Delta_1\\ n_3=-\Sigma+2\Delta_2\end{array}\right.
\end{aligned}
$$

where, Σ is defined as $\Sigma=\sum_{n=1}^{3}\Delta_n$. Thus, the conformal scales of the primary operators $O_i(x_i)$ completely define the 3-point correlation function, due to conformal symmetries. The latter becomes:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle=\frac{d_{123}}{|x_1-x_2|^{\Sigma-2\Delta_3}|x_2-x_3|^{\Sigma-\Delta_1}|x_3-x_1|^{\Sigma-2\Delta_2}} \tag{5.74}$$

with the constant $d_{123}=c_{n_1,n_2,n_3}=c_{-\Sigma+2\Delta_3,-\Sigma+\Delta_1,-\Sigma+2\Delta_2}$

After normalisation the 3-point function may become:

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle=\frac{\tilde{d_{123}}}{|x_1-x_2|^{\Sigma-2\Delta_3}|x_2-x_3|^{\Sigma-\Delta_1}|x_3-x_1|^{\Sigma-2\Delta_2}} \tag{5.75}$$

for :

$$\tilde{d_{123}}=\frac{d_{123}}{d_1d_2d_3}$$

AdS side In order to get a 3-point interaction it is necessary to include a cubic interaction term in the action (5.9), which becomes (5.58):

$$S=\frac{1}{2}\int_M d^{D+1}x\sqrt{-g}\prod_{i=1}^{3}(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i+m^2\phi_i^2)+\frac{\lambda}{3!}\phi_1\phi_2\phi_3 \tag{5.76}$$

The corresponding equation of motion (5.59) reads:

$$(-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)+m^2)\phi_i=\frac{\lambda}{3!}\phi_j\phi_k \tag{5.77}$$

Then, ϕ_i can be approximated perturbatively, as described in (5.61), according to (5.62) and (5.63). The resulting field up to first order is:

$$\phi_i(x,z) = \int d^D x' K_{\Delta_i}(x,z;x')\phi_{0,i}(x') \tag{5.78}$$

$$+\frac{\lambda}{3!}\int d^{D+1}X' G(x,z;x',z')\times \tag{5.79}$$

$$\int d^D x_j d^D x_k K_{\Delta_j}(x',z';x_j)K_{\Delta_k}(x',z';x_k)\phi_{0,j}(x_j)\phi_{0,k}(x_k) \tag{5.80}$$

$$+O(\lambda^2) \tag{5.81}$$

Each term of the perturbative expression corresponds to a Witten diagram. A Witten diagram represents the propagation of fields in the AdS bulk, where the latter is represented as a slice of the Penrose cylinder (see figure 2.3). The boundary is at the surface of the cylinder.

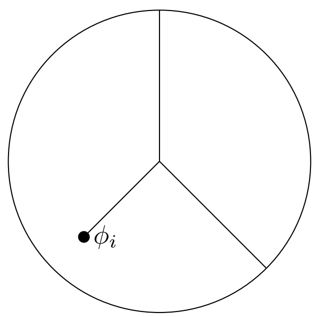

Figure 5.4: Witten diagram - Scalar field in the bulk from cubic interaction fig:WittenScalar3PF)

The part of the action omitting the three-point correlation function, is obtained by substituting the first term of (5.78) in the interaction term of the action (5.76) [29]. Thus,

$$S^{(3)}_{int} = \frac{\lambda}{3!}\int d^D x_1 d^D x_2 d^D x_3 \int d^{D+1}X\sqrt{-g}K_{\Delta_1}(x,z;x_1)K_{\Delta_2}(x,z;x_2)K_{\Delta_3}(x,z;x_3)$$
$$\phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)$$

The 3-point function is, finally, the product of three Boundary-to-Bulk propagators, from the boundary to the same point in the bulk. Of course we integrate over the possible positions in the Bulk.

$$\begin{aligned}&\langle\phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)\rangle = \\ -\lambda &\int d^{D+1}X\sqrt{-g}K_{\Delta_1}(x,z;x_1)K_{\Delta_2}(x,z;x_2)K_{\Delta_3}(x,z;x_3)\end{aligned} \tag{5.82}$$

for $X = (x, z)$. Performing the integration yields [30] [33]:

$$\langle \phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)\rangle \tag{5.83}$$

$$= \lambda C_3 \frac{1}{|x_1 - x_2|^{\Sigma-2\Delta_3}|x_2 - x_3|^{\Sigma-\Delta_1}|x_3 - x_1|^{\Sigma-2\Delta_2}} \tag{5.84}$$

$$\tag{5.85}$$

$$C_3 = \frac{\prod_{i=1}^{3}\Gamma(\frac{\Sigma}{2} - \Delta_i)}{2\pi^4\prod_{i=1}^{3}\Gamma(\Delta_i - \frac{D}{2})}\Gamma(\frac{\Sigma - D}{2}) \tag{5.86}$$

where, $\Sigma = \sum_{n=1}^{3}\Delta_n$ Consequently, the result is identical to the CFT 3-point function (5.75), given that the interaction strength λ is chosen properly.

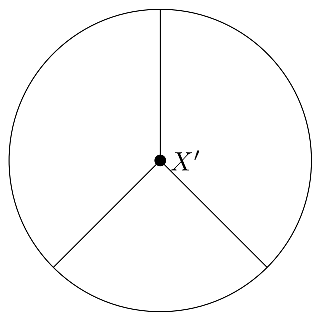

Figure 5.5: Witten diagram - 3-point function

4-point functions

CFT side With four points, it is possible to construct conformally invariant ratios, known as *cross ratios* or *anharmonic ratios*. Thus, it is impossible to determine exactly a n-point function for $n > 3$. In general,there are $\frac{n(n-3)}{2}$ independent *cross ratios* [16], they are given by the relation (3.6), and in the case $n = 4$:

$$u \equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^2 \text{ and } v \equiv \left(\frac{x_{12}x_{34}}{x_{14}x_{23}}\right)^2 \tag{5.87}$$

, where x_{ij} denotes the distance between two points $x_{ij} \equiv |x_i - x_j|$.

Nevertheless, the conformal symmetries can help us simplify the 4-point function. Using, the, familiar by now, translation and special conformal transformations, the 4-point function reduces to :

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = f(u,v)\prod_{i<j=1}^{4} x_{ij}^{\frac{\Sigma}{3}-\Delta_i-\Delta_j} \tag{5.88}$$

The complete evaluation of 4-point functions depend on the particular theory at hand. Consequently, when a duality needs to be confirmed it is crucial to confirm the 4-point function match.

AdS side The Witten diagrams corresponding to first order calculations of 4-point interactions are shown in figure (5.6) and the interaction term of the action is given by:

$$S_{int}[\phi] = \frac{\lambda}{4!}\int d^{D+1}X\sqrt{-g}\phi_1\phi_2\phi_3\phi_4 \tag{5.89}$$

The first order perturbation term of the action (5.64) is:

$$\begin{aligned} S_{int}^{(4)} = \frac{\lambda}{4!}\int & d^Dx_1d^Dx_2d^Dx_3d^Dx_4 \\ & \int d^{D+1}X\sqrt{-g}K(x,z;x_1)K(x,z;x_2)K(x,z;x_3)K(x,z;x_4) \\ & \phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)\phi_{0,4}(x_4) \end{aligned}$$

and the 4-point function is calculated through the integral:

$$\begin{aligned} & \langle\phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)\phi_{0,4}(x_4)\rangle \\ & = -\lambda\int d^{D+1}X\sqrt{-g}K_{\Delta_1}(x,z;x_1)K_{\Delta_2}(x,z;x_2)K_{\Delta_3}(x,z;x_3)K_{\Delta_4}(x,z;x_4) \end{aligned}$$

where K_{Δ_i} is the Boundary-to-Bulk propagator and $X = (x,z)$. Then, performing the, rather tricky, integration, with the simplification $\Delta_i = \Delta$ gives the result [30] [33]:

$$\langle\phi_{0,1}(x_1)\phi_{0,2}(x_2)\phi_{0,3}(x_3)\phi_{0,4}(x_4)\rangle = \lambda C_4\frac{1}{\prod_{i,j}|x_i - x_j|^{\frac{2\Delta}{3}}} \tag{5.90}$$

for

$$C_4 = \frac{2\pi^{\frac{D}{2}}\Gamma(2\Delta - \frac{D}{2})}{\Gamma(2\Delta)(uv)^{\frac{2\Delta}{3}}} \int_0^\infty dzF\left(\Delta, \Delta; 2\Delta; 1 - \frac{(u+v)^2}{(uv)^2} - \frac{4}{uv}sinh^2 z\right) \quad (5.91)$$

Once again the two theories lead to the same result.

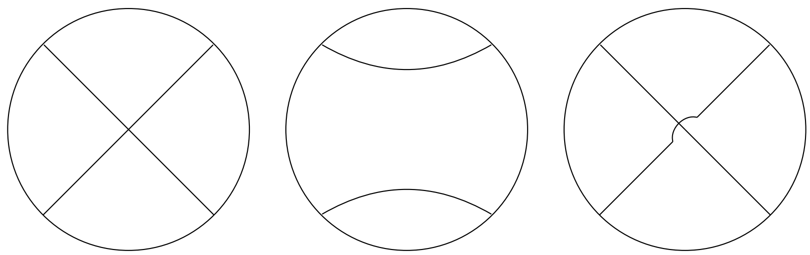

Figure 5.6: Witten diagrams of 4-point correlations

5.4 AdS/CFT Dictionary

The AdS/CFT correspondence was initially proposed by J.Maldacena, but a detailed dictionary of the correspondence was presented shortly after by Gubser, Klebanov, Polyakov [36] and Witten [35]. The most important of their results was the identification of the partition functions of the two theories $\mathcal{Z}_{\text{bulk (Q.Gravity)}} = \mathcal{Z}_{\text{CFT}}$. In this section, we present the most significant corresponding quantities of the two theories.

- **Geometry**
  Anti-de-Sitter AdS_{D+1} spacetime is a maximally symmetric hyperbolic spacetime with a conformally flat boundary at infinity. In the Poincaré patch coordinates the boundary is located at $z \to 0$ and the boundary metric is that of a Minkowski spacetime $\mathbb{R}^{D-1,1}$.

$$ds^2 = \frac{b^2}{z^2}\left(dz^2 + dx_\mu dx^\mu\right) \tag{5.92}$$

  where $x^\mu = (t, \vec{x})$. Isometries of AdS correspond to global symmetries in the CFT side.

- **Field-Operator relation**
  The boundary behaviour of scalar fields in the AdS_{D+1} bulk is (5.24):

$$\phi(x^\mu, z) \approx z^{\Delta_-}\phi_0(x^\mu) + z^{\Delta_+}\phi_1(x^\mu) \tag{5.93}$$

  where Δ_-, Δ_+ are the roots of the equation $\Delta^2 - D\Delta - m^2b^2 = 0$. The divergent part $z^{\Delta_-}\phi_0(x^\mu)$ exhibits a conformal scale Δ_- near the boundary $z = \epsilon$, while the emergent field $\phi_0(x^\mu)$ acts as a source at the boundary.

  According to the GKPW formula (5.98), the field $\phi_1(x^\mu)$ is the expectation value of the dual operator in the boundary, which has a conformal scale Δ_+.

$$\phi_1(x^\mu) = \langle O_{\Delta_+}(x^\mu)\rangle \tag{5.94}$$

  Consequently, each bulk field $\phi(x^\mu, z)$ corresponds to a conformal primary operator $O_\Delta(x^\mu)$ in the CFT on the boundary.

$$\boxed{\phi(x^\mu, z) \leftrightarrow O_\Delta(x^\mu)} \tag{5.95}$$

- **Mass-Conformal scale** The conformal scale Δ of operators in the CFT side correspond to the mass of the dual fields in the bulk.

$$\Delta \leftrightarrow m^2 = \frac{\Delta^2 - D\Delta}{b^2} \tag{5.96}$$

  Inversely, the mass of a bulk field corresponds to the conformal scale of the dual operator that is the positive root of $\Delta^2 - D\Delta - m^2b^2 = 0$.

- **Partition Functions**
  According to Gubser, Klebanov, Polyakov and Witten (GKPW) formula, the partition functions of the two theories should be identified, when the bulk fields approach the boundary.

$$\mathcal{Z}_{\text{bulk (Q.Gravity)}}\big|_{\phi \to z^{\Delta_-}\phi_0} = \mathcal{Z}_{\text{CFT}} \tag{5.97}$$

  The behaviour of the marginal fields ϕ_0 as sources on the boundary is encoded in the above relation as:

$$\boxed{\mathcal{Z}_{\text{bulk}}\ \left[\phi(x^\mu, z)\big|_{z=0} \sim \phi_0(x^\mu)\right] = \left\langle e^{\int d^d x \phi_0(x^\mu)\mathcal{O}(x^\mu)} \right\rangle_{\text{CFT}}} \tag{5.98}$$

  For weakly coupled gravity, i.e. strongly coupled CFT, we have classical supergravity and the partition function is:

$$\mathcal{Z}_{\text{bulk}}\left[\phi(x^\mu, z)\big|_{z=0} \sim \phi_0(x^\mu)\right] \simeq e^{-N^2 S_{\text{class}}[\phi] + O\left(\alpha'\right)} + O\left(g_s\right) \tag{5.99}$$

  Thus, the generating functional W_{bulk} is given by the classical gravity action.

- **Correlation Functions**
  The identification of the partition functions (5.98) of the two theories and the fact that ϕ_0 fields act as sources for the boundary theory suggest that CFT observables are computed by:

$$\boxed{\left\langle \mathcal{O}_1\left(x_1\right) \ldots \mathcal{O}_n\left(x_n\right)\right\rangle = i^{-n} \frac{\delta^n Z_{CFT}\left[\phi_0\right]}{\delta\phi_{1,0}\left(x_1\right) \ldots \delta\phi_{n,0}\left(x_n\right)}\bigg|_{\phi_0 = 0}} \tag{5.100}$$

  For strongly coupled CFT, i.e. classical gravity, the partition function is given by (5.99). So, we can compute all connected correlation functions by differentiations of the classical gravity action, or more generally by the bulk action:

$$\left\langle \mathcal{O}_1\left(x_1\right) \ldots \mathcal{O}_n\left(x_n\right)\right\rangle_c = (-1)^n \frac{\delta^n S_{\text{bulk}}\left[\phi_0\right]}{\delta\phi_{1,0}\left(x_1\right) \ldots \delta\phi_{n,0}\left(x_n\right)}\bigg|_{\phi_0 = 0} \tag{5.101}$$

  where $S_{\text{bulk}} = S_{\text{class}}$ is the classical supergravity action on the AdS space.

Even though they exceed the discussion of the thesis, we present some further dual quantities for completeness of the dictionary

- Spins and charges in the gravity theory correspond to the same spins and charges in the CFT.
- The bulk metric $g_{\mu\nu}$ corresponds to the energy momentum tensor $T^{\mu\nu}$ in the CFT.
- The vector fields A_μ in the bulk correspond to conserved currents J^μ on the conformal boundary. Therefore, gauge symmetries in the bulk translate to global symmetries on the boundary [35].

- Dirac fields ψ in the bulk, similarly to scalar fields, are dual to fermionic operators on the boundary. For spin $\frac{1}{2}$ or $\frac{3}{2}$ the relation between mass and the conformal scale is $|m|L = \Delta - d/2$ [26].

- In thermal CFT, the corresponding gravity dual includes a black hole in the interior. In that case, the temperature in the CFT translates to the Hawking temperature of the black hole. Also, instabilities of the black hole correspond to phase transitions in the CFT [35].

Chapter 6

The Ryu Takayanagi conjecture - Entanglement Entropy from Holography

A distinctive property of quantum mechanics is the phenomenon of quantum entanglement, according to which it is possible for two measurements to be correlated, independently of their spacial proximity.Quantum entanglement between two quantum systems that together form a composite system*Entanglement Entropy*. In higher dimensional QFTs, the computation of Entanglement Entropy is particularly demanding [37] [38]. Nevertheless, in the light of the AdS/CFT correspondence, the computation can be translated to a purely geometrical problem governed by Gravity! According to the Ryu-Takayanagi proposal (2008), the Entanglement Entropy of a subsystem of the CFT on the boundary is connected to the geometry of the bulk and can be expressed in terms of a surface area, just like the Bekenstein-Hawking formula for black holes. In this chapter we present the Ryu-Takayanagi formula and study its application on the special case of a hyperspherical CFT subsystem on the boundary. We are only concerned with the bosonic sector, thus only scalar fields will be taken into account.

6.1 Entropy and Entanglement

6.1.1 Density Matrix

A distinctive characteristic of quantum mechanics is the actual states of quantum systems and the observed states are not distinguishable. Nevertheless, it is possible to define an operator that describes simultaneously both notions. That operator is named *density matrix*, ρ, and is characterised by the properties [39]:

$$\rho^\dagger = \rho \qquad\qquad \rho \geq 0 \qquad\qquad Tr(\rho) = 1 \tag{6.1}$$

The density matrix is a matrix that describes the quantum states of physical systems and can be thought as a more general representation of wavefunctions or state vectors $|\psi\rangle$. Also, it allows the calculation of the expectation values of observables according to:

$$\langle O\rangle_\rho = Tr(O\rho) \tag{6.2}$$

A quantum state is called a pure state if the density matrix can be expressed in the form:

$$\rho = |\psi\rangle\langle\psi| \tag{6.3}$$

Otherwise, the state is called a mixed state. Mixed states represent our ignorance of the actual state of the system and allow the description of a quantum state as a statistical mixture of different state vectors. But, the probabilistic mixture of states should not be confused with the superposition of states. A superposition defines a new quantum state not a mixture of states. Let us clarify the difference with an example. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two state vectors. On the one hand, a superposition of the two states with equal probability amplitudes gives a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)$ with a corresponding density matrix describing in a pure state:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|\psi_1\rangle + |\psi_2\rangle)(\langle\psi_1| + \langle\psi_2|) \tag{6.4}$$

On the other hand, if the system's state is unknown, but there is 50% probability to be found either in the state $|\psi_1\rangle$ or $|\psi_2\rangle$, the density matrix is not in a pure state:

$$\rho = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) \tag{6.5}$$

In any case, the density matrix can be written in terms of its eigenvectors and eigenvalues:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \quad \sum_i p_i = 1, \quad p_i \geq 0 \tag{6.6}$$

If the eigenvectors are chosen be orthogonal, thus $\langle psi_i|\psi_j\rangle$, then, the density matrix is diagonal and, in that basis, the eigenvalues pi can be interpreted as the probabilities of the vector states' $|\psi_i\rangle$ occurrence. This is simply the process of diagonalization of lineal algebra. In that sense, a pure state has a single eigenvector $|\psi\rangle$ and an eigenvalue equal to 1.

6.1.2 Von Neumann Entropy

The von Neumann entropy is a measure of mixedness of a quantum state and is defined by the density matrix according to:

$$S(\rho) = -Tr(\rho ln(\rho)) = -\langle ln(\rho)\rangle_\rho \tag{6.7}$$

where (6.2) was used. Given a diagonalization of the density matrix, the von Neumann entropy is simply the sum:

$$S(\rho) = -\sum_i Tr(p_i ln(p_i)) \tag{6.8}$$

where p_i the probabilities corresponding to the quantum states, (6.6). From the form above, it is evident the the entropy of a pure state is zero since $S(\rho) = -Tr(1ln(1)) = 0$

$$\rho \quad \text{equal to } 1 \rightarrow S(\rho) = 0 \tag{6.9}$$

Moreover, the von Neumann entropy is invariant under unitary representations, so:

$$S(U\rho U^{-1}) = S(\rho) \tag{6.10}$$

6.1.3 Reduced Density Matrix

Let us consider a quantum system and its respective Hilbert space H. When we divide the system into two subsystems A and B with respective Hilbert spaces H_A and H_B, then, the total Hilbert space can be expressed as the tensor product $H = H_A \otimes H_B$. Assuming that an observer resides on subsystem A and cannot get any information from B, he does not perceive the state of the system to be described by the density matrix ρ, but by the *reduced density matrix* ρ_A. The reduced density matrix may also be called a *marginal state.* Since, information from B are inaccessible, ρ_A is computed by tracing out the states that reside on B, which is done by taking the partial trace of the total ρ on B.

$$\rho_A = Tr_B(\rho) \tag{6.11}$$

Then, any observable on A, $O_A \otimes \mathbb{I}_B$ is calculated according to (6.2) with use of ρ_A instead of ρ.

6.1.4 Definition of von Neumann and Entanglement Entropy

Of course, if a quantum state resides both in A and B, this conceptional division 'hides' information from the observer in subsystem A. If that happens, it is expected that the entropy measured by A, to be greater that the one measured in the total system. The corresponding von Neumann entropy is:

$$S(A) = -Tr(\rho_A ln(\rho_A)) \tag{6.12}$$

This quantity is a measure of entanglement of a given wavefunction $|\psi\rangle$. It is important to note here that this entropy is representative measure of the amount of entanglement between the two subsystems if the total system is in a pure state [39]. In that case, the von Neumann entropy of the relation (6.2) is called the *Entanglement Entropy.* It can be easily proved that, given a pure state for the total subsystem, the entanglement entropies of the two complementary subsystems are equal, thus $S(A) = S(B)$. What is more, it is important to note that the von Neumann or Entanglement entropy is time dependent, so every calculation must be specified in a fixed time-slice $t = t_0$.

6.1.5 Modular Hamiltonian

To begin with, let us define thermal entropy. For a system that is not in a pure state, the density matrix can be described by (6.6). If the system has many degrees of freedom, we are allowed to assume a canonical ensemble of temperature T, where the density matrix can be expressed as:

$$\rho_{thermal} = e^{-\beta \mathcal{H}} \tag{6.13}$$

where $\mathcal{H}$ is the Hamiltonian of the system and β the inverse temperature $\beta = \frac{1}{k_B T}$ or $\beta = \frac{1}{T}$, assuming units with the Boltzmann constant $k_B = 1$.

Now, let us get back to the general, not necessarily statistical case, where the density matrix is defined by (6.6) and obeys the constrains (6.1). The fact that the density matrix is hermitian and has positive eigenvalues dictates that we can express the density matrix as an exponent of a Hamiltonian-like operator, the *modular Hamiltonian* H_{mod}.

$$\rho = e^{-H_{mod}} \tag{6.14}$$

This form suggests that the density matrix is thermal in terms of the modular Hamiltonian H_{mod} with a respective temperature $T = \beta^{-1}$. Since, ρ has to be normalised, we can reformulate the H_{mod} to be $H_{mod} = \tilde{H} + lnZ$, where Z is the partition function of $\tilde{H}$, $Z = Tr(e^{-\tilde{H}})$. Then, the density matrix is written as:

$$\rho = \frac{1}{Z} e^{-\tilde{H}} \tag{6.15}$$

When we refer to the modular Hamiltonian usually we mean $\tilde{H}$.

This particular form of the density matrix is very useful for calculations of observables (6.2) using path integrals. Also, it has an interesting physical interpretation. Since the modular Hamiltonian is a Hamiltonian, it must produce translations of some kind of time. That is called a *modular time* and it can be proved that it is the time perceived by an observer in a point in spacetime that can be causally linked to the quantum system described by the Hamiltonian.

6.1.6 Properties of von Neumann Entropy

Mutual Information

An important measure of correlation between two subsystems A, B is the mutual information defined by:

$$I(A;B) = S(A) + S(B) - S(AB) \tag{6.16}$$

and it is a quantitative measure of both of correlation and entanglement. Mutual information has two important properties that produce the entanglement entropy's properties as well. The first is that it is always positive and this results in the subadditivity property of the entanglement entropy.

$$I(A;B) \geq 0 \tag{6.17}$$

While, the second is that it is non-decreasing under adjoining other systems to either A or B:

$$I(A;BC) \geq I(A;B) \tag{6.18}$$

this property results in the *Strong Subadditivity* property of entanglement entropy (6.21).

Properties

- **Entanglement Entropy of Complement** If the total subsystem $H = A \cup B$ is in a pure state, the entanglement entropies of A and B are equal. In other words, a subsystem and its complement have equal entanglement entropy given that the whole system is in a pure state, i.e. at finite temperature for large systems.

  $$S(A) = S(B) \tag{6.19}$$

  Let us note that this equation holds regardless of their size! This shows that the Entanglement entropy is not an extensive quantity.

- **Subadditivity**:
  For any system AB divided into two subsystems A and B, irrespective of their state, the total entropy is always less that the sum of the subsystems' entropies .

  $$S(AB) \leq S(A) + S(B) \tag{6.20}$$

  this inequality is a result of the positiveness of mutual information (6.17).

- **Strong Subadditivity**:
  For three disjoint subsystems A,B,C the following inequality holds:

  $$S(ABC) + S(B) \leq S(AB) + S(BC) \tag{6.21}$$

  This property is called *Strong Subadditivity* and is connected to the , mutual information property (6.18) and as a constrain it is stronger than subadditivity. Moreover, the substitution $AB \to A$, $BC \to B$, (6.21) yields:

  $$S(A \cup B) + S(A \cap B) \leq S(A) + S(B) \tag{6.22}$$

  For $A \cap B = \emptyset$ the property above is the subadditivity property (6.20).

- **Upper Bound**:
  From the definition of von Neumann entropy (6.7) it is evident that there is an upper limit:

  $$S(A) \leq ln(\dim(H_A)) \tag{6.23}$$

6.2 Entanglement Entropy in QFT

6.2.1 Computation of Entanglement Entropy in QFT

Evaluating the Entanglement Entropy in QFT [40] [37] can prove to be a rather complex procedure, especially in higher dimensions. A way to proceed when the spacetime is discretised is to directly determine the eigenvalues and eigenvectors of the density matrix, so that computations are carried through according to the relations (6.6) and (6.8). However, this process can be very demanding computationally.

An analytic way to compute entanglement entropy is using the *replica trick* [40] [41] [39]. The replica trick is more intuitively pictured in 2 dimensions in Euclidean spacetime, that is naturally generalised in D dimensions. Let us imagine as 2 dimensional Euclidean manifold $\mathcal{M} = \mathbb{R} \times \mathbb{R}$. A submanifold A would then be a line segment and, also, we call its complement B. We imagine that we cut the manifold at A, creating two almost coincident lines and a gap in between. At the one extremity of the cut, which is a line, let us call the fields ϕ_+ and on the other ϕ_-. The replica trick includes the idea of creating n copies of the same manifold, we can imagine them a sheets one above the other, all with the cut at A, and then perform a resewing of the manifolds such that for each copy, the one side of the cut, with ϕ_- is sewed to the opposite side of the successive manifold's cut, with ϕ_+. This procedure generates a Riemannian surface $\mathcal{R}_n$.

Each of the above steps corresponds to a computational step that we present below for arbitrary dimensions D. We assume a manifold $\mathcal{M} = \mathcal{N} \times \mathbb{R}$, a submanifold A and its complement B. The method is based on a differently defined entropy measure, the *Tsallis Entropy* that is defined for a subsystem A:

$$S_{n,\ \text{Tsallis}} = \frac{\operatorname{Tr} \rho_A^n - 1}{1-n} \tag{6.24}$$

Then, taking the limit of $n \to 1$, (6.24) yields the Von Neumann entropy, as follows:

$$\lim_{n\to 1} \frac{\operatorname{Tr} \rho_A^n - 1}{1-n} = - \frac{\partial}{\partial n} \operatorname{Tr} \rho_A^n \bigg|_{n=1} = - \operatorname{Tr} \rho_A \log \rho_A = S(A) \tag{6.25}$$

Consequently, the quantity we need to compute is $\operatorname{Tr} \rho_A^n$, that is relatively easier than computing $\operatorname{Tr}(\rho_A ln(\rho_A))$. We are interested in the case of a QFT at zero temperature, where the vacuum states $|\psi\rangle$ and $\langle\psi|$ can be computed by the integral:

$$|\psi(\phi_0)\rangle = \int_{\phi(-\infty,\vec{x})=0}^{\phi(0,\vec{x})=\phi_0(\vec{x})} [D\phi] e^{-S_E(\phi)}, \quad \langle\psi(\phi_0')| = \int_{\phi_0'(0,\vec{x})=\phi_0'(\vec{x})}^{\phi_0'(\infty,\vec{x})=0} [D\phi] e^{-S_E(\phi)} \tag{6.26}$$

where the integration is performed from Euclidean time $\tau_E = 0$ to $\tau_E = \infty$. Also, S_E is the Euclidean action, $\phi(\tau_E, \vec{x})$ are the fields that define the CFT,

ϕ_0 is the field for euclidean time $\tau_E = 0$. At zero temperature the system is in a pure state, thus we can write:

$$
\begin{aligned}
[\rho]_{\phi_-,\phi_+} &= |\psi(\phi_-)\rangle \langle \psi(\phi_+)| \\
&= \frac{1}{Z} \int_{\phi(-\infty,x)=0}^{\phi(\infty,x)=0} [D\phi] \prod_x \left[\delta\left(\phi(-\varepsilon,x) - \phi_-(x)\right) \delta\left(\phi(+\varepsilon,x) - \phi_+(x)\right)\right] e^{-S_E(\phi)}
\end{aligned}
\tag{6.27}
$$

with Z being the vacuum partition function, that is a normalisation factor such as $Tr(\rho) = 1$. Let us assume a subsystem A and its complement B. Then, the reduced density matrix ρ_A is computed by (6.27) by assuming $\phi_+ = \phi_-$ when $\vec{x} \in B$ and integrating $\phi_\pm$ over B. That is the part that we leave 'open' the part the submanifold A covers, as if we have cut through it. In order to evaluate the term $\mathrm{Tr}\, \rho_A^n$, we create n replicas of $[\rho_A]$ and write:

$$
[\rho_A]_{\phi_{1+}\phi_{1-}} [\rho_A]_{\phi_{2+}\phi_{2-}} \cdots [\rho_A]_{\phi_{n+}\phi_{n-}} \tag{6.28}
$$

We can imagine this as if we have stack n replicas of $\mathcal{M}$ one above the other. In order to obtain the trace, we then sew together the pairs of succeeding antidiametrical sides of the cuts, i.e. $\phi_{i-}(x) = \phi_{(i+1)+}(x)(i = 1, 2, \cdots, n)$. Finally, the trace is taken by sewing the last two remnant sides $\phi_{n-}(x) = \phi_{0+}(x)$:

$$
\mathrm{tr}_A\, \rho_A^n = (Z_1)^{-n} \int_{(t_E,x)\in\mathcal{R}_n} D\phi e^{-S(\phi)} \equiv \frac{Z_n}{(Z_1)^n} \tag{6.29}
$$

where Z_n is the partition function on the n-sheeted manifold $\mathcal{M}$.

In two dimensions, the replica trick yields the well-known formula for the Entanglement Entropy of CFT_2:

$$
S(A) = \frac{c_{CFT}}{3} \ln\left(\frac{L}{\epsilon}\right) \tag{6.30}
$$

, where c_{CFT} is the central charge of the theory and ϵ the UV-cutoff of the theory, i.e. the lattice spacing. In greater dimensions, the resulting Entanglement Entropy follows the *Area Law* described in the next paragraph:

$$
S(A) = \gamma \frac{\mathrm{Area}(\partial A)}{\epsilon^{D-2}} + O(\epsilon^{3-D}) \tag{6.31}
$$

where γ depends on the physics of the system.

6.2.2 The Area Law in QFT's Entanglement Entropy

Let us consider a D-dimensional manifold $\mathcal{M}$, with $D-1$ spacial and 1 temporal dimension and a QFT living on $\mathcal{M}$. In QFT, the quantum field's description is equivalent to a set of quantum oscillators in spacetime, whose state's, $\psi_i|$, ensemble is described by the density matrix $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$ of the whole system. Let us assume that the total system is in a pure state, which happens for instance when the system is at zero temperature. Considering a subsystem of the QFT in a submanifold $A \subset \mathcal{M}$, we want to find the entanglement entropy of the subsystem. So, if A' is the complement of A, we trace out of ρ the states of A', resulting in ρ_A Of course, this cannot be done as long as time is running, so we consider a time slice of $\mathcal{M}$ at $t = t_0$. For a given time, then, we can define the entanglement entropy of A, $S(A)$ according to (6.7). Nevertheless, the subsequent entropy is divergent and we are enforced to introduce a UV-cutoff ϵ. Then, the definition $S(A) = Tr(\rho_A ln(\rho_A))$ produces a sum of divergent and non-divergent terms that includes a very interesting term [42].

$$S(A) = \gamma \frac{\text{Area}(\partial A)}{\epsilon^{D-2}} + O(\epsilon^{3-D}) \tag{6.32}$$

where γ depends on the system.

The term of interest is the coefficient of the leading divergence, which is proportional to the area of the surface of the boundary ∂A. One would anticipate that the leading term should be proportional to the volume of A, rather that the area of the boundary. If so, entanglement entropy would be an extensive quantity, but, as argued in (6.19) it is not. In fact, for the bipartite quantum system $A \cup (\mathcal{M} - A)$ that is in a pure state, we have that $S(A) = S(\mathcal{M} - A)$. Consequently, entanglement entropy should depend on a quantity shared by both regions. This quantity is the area of the shared surface that separates them!

The latter observation was made by Srednicki [37], in 1993. Srednicki applied the results of entanglement entropy of quantum oscillators to a free Quantum Field theory and demonstrated numerically that entanglement entropy of a subsystem is proportional to the area of the subsystem's boundary. A similar result in the context of a QFT propagating in the interior of a black hole, was pointed out in [43], where the computed entanglement entropy was found proportional to the black hole surface area.

It is important to note that a 2-dimensional QFT $(1 + 1)$ is an exception to the rule, as, for instance, in CFT_2 the leading divergence is logarithmic $\propto \frac{c}{3} ln\left(\frac{l}{\epsilon}\right)$, where l is the length of the subsystem and ϵ the UV cutoff.

The so-called *Area Law* exhibits a profound similarity to the *Bekenstein-Hawking formula* for the black hole entropy:

$$S_{BH} = \frac{c^3 A_H}{4\hbar G_N} = \frac{A_H}{4 l_P^2} \tag{6.33}$$

where A_H is the area of the black hole horizon, G_N the Newton constant and l_P the Plank length. We can interpret this similarity to be rooted in the inaccessibility of information between the black hole and its exterior in the same way the entanglement entropy assumes that the subsystems A and B of a total system AB do not communicate. Thus, B $\rightarrow$ black hole , A $\rightarrow$ exterior. This observation justifies the Ryu-Takayanagi proposal for the calculation of entanglement entropy in the context of AdS/CFT correspondence.

6.3 The Ryu-Takayanagi Formula

Inspired by the *Area Law* and its similarity with the *Bekenstein-Hawking* formula, Shinsei Ryu and Tadashi Takayanagi (2008) [44], proposed a formula for the entanglement entropy of a conformal field theory in D dimensions, in the context of the AdS_{D+1}/CFT_D correspondence. According to the conjecture the entanglement entropy of the CFT_D theory can be computed as a geometrical quantity in the AdS_{D+1} space.

Before presenting the precise formula, let us argue on the physical interpretation of its conception. To begin with, all the following analysis is in terms of the Poincaré coordinates of AdS_{D+1} and a particular point in time $t = t_0$. Let us assume a region B on the boundary and an observer on its complement A. Considering the information of system B to be inaccessible to A, the observer will perceive an entanglement entropy $S(A)$. But, since the conformal boundary is the limit of the Anti-de-Sitter spacetime, the extension of the same setting in AdS should be a region one dimension higher, also inaccessible, that covers the boundary region B. Also, since in the concept of the duality the conformal boundary is just the marginal behaviour of the gravity theory in AdS, if the observer is allowed in the interior of AdS, but outside the extended B region, he should experience the same "fuzziness" of information, thus, the same entanglement entropy $S(A)$. We can think of the extension of B as an imaginary horizon A_m that covers B and their boundaries coincide, $\partial A_m = \partial B = \partial A$. Taking into account the already argued holographic nature of quantum gravity, the area law and its similarity to the Bekenstein-Hawking formula, the conjecture that the entanglement entropy should be proportional to the surface area of a minimal surface A_m in AdS covering the region B is more than justified. The exact proposal of a Bekenstein-Hawking like formula is:

$$S(A) = \frac{\text{Area}\,(A_m)}{4G_N^{(D+1)}} \tag{6.34}$$

where, A_m is $D-1$ the static minimal surface on AdS_{D+1} with a boundary ∂A_m on the CFT boundary. The respective area is Area(A_m). Moreover, $G_N^{(D+1)}$ is the Newton constant for $D+1$ dimensions. It should be noted that all quantities are static in time.

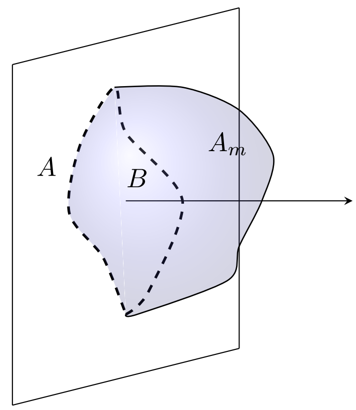

Figure 6.1: Minimal Surface

The proposal holds even if the temperature of the CFT_D is not zero, i.e. it is not in a pure state and $S(A) \neq S(B)$. In that case a temperature T of the CFT_D is equivalent to a black hole in AdS_{D+1} and the thermal entropy is dual to the black hole entropy in the gravity description. The existence of the black hole in the interior results to different entanglement entropies for A and B, since the surfaces induced differ due to the black hole horizon [44].

6.4 Holographic Entropy for Circular Disk

The aim of this section is provide an exact calculation of the Ryu-Takayanagi formula in the special case that the boundary region is a circular disk. The latter terminology is non actually correct, as in the general case of $D+1$ AdS dimensions the region is a sphere of $D-1$ spacial dimensions and not a 2D disk, but we shall call it a circular disk to distinguish its dimensionality from the one of the minimal surface.

According to (6.34) the problem of calculating the entanglement entropy of a subsystem $B = A_D$ on the boundary is altered to the problem of determination of the minimal surface A_m and its area $\text{Area}\,(A_m)$, whose boundary is ∂A_D. Given the induced metric on the surface $\hat{\gamma}_{ij}$ in $D-1$ dimensions, the area functional is given by the integral:

$$A_m = \int dX \sqrt{\hat{\gamma}} \tag{6.35}$$

where $\hat{\gamma}$ the determinant of the induced metric, that is defined by: $\hat{\gamma}_{ij} = g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$, where X^μ are the embedding coordinates of the surface in the general geometry of AdS. Reparametrisation invariance allows us to choose our coordinate system, so, we work with the Poincaré coordinates of AdS_{D+1}, whose metric is given by (2.9). The time is fixed at $t = t_0$ and the spacial dimensions are D for AdS and $D-1$ for the Minkowski space where CFT lives. What is

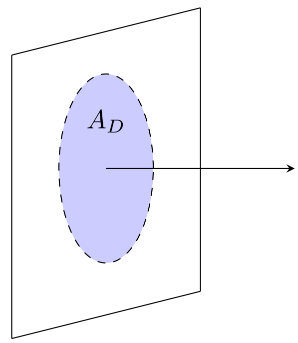

Figure 6.2: Spherical hypersurface on the boundary. The name 'circular disk' is only used to indicate lower dimensionality

more, let us assume that the boundary circular region has a radius l and, in the Minkowski space, is described by the equation:

$$x^\mu x_\mu = l^2 \tag{6.36}$$

Moreover, the embedding coordinates can be expressed as:

$$X^i = x^i \quad X^D = Z(\vec{x}) = z \quad i = 1, 2, \ldots, D-1 \tag{6.37}$$

Then the induced metric is:

$$\hat{\gamma}_{ij} \equiv g_{\mu\nu} \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j} = \frac{b^2}{z^2} \left(\delta_{ij} + \partial_i Z \partial_j Z\right) \quad i = 1, 2, \ldots, D-1 \tag{6.38}$$

We observe that $\hat{\gamma}_{ij}$ is the sum of a symmetric matrix with the identity matrix. It is known that, for symmetric matrices A the equation $det(\mathbb{I}+A) = 1+Tr(A)$, so the determinant $det(\hat{\gamma}_{ij}) = \hat{\gamma}$ can be written as:

$$\begin{aligned}
\det\left(\hat{\gamma}_{ij}\right) &= \left(\frac{b}{z}\right)^{2(D-1)} \left(1 + \mathrm{Tr}(\hat{\gamma})\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1 + \sum_{i=1}^{D} \partial_i Z \partial_i Z\right)
\end{aligned}$$

The symmetries of AdS introduce several simplifications. Poincaré invariance, allows us to study only the case where the center of the 'disk' is in the origin $(x^\mu, z) = (\vec{0}, 0)$. Also, Lorentz invariance in the x-plane suggests that the bulk coordinate $X^D = Z(\vec{x}) = z$ should only depend on the polar distance r from the symmetry axis z, or equivalently the line $x^\mu = 0$.

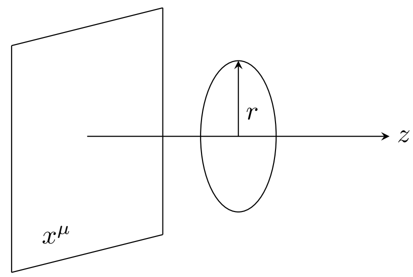

Figure 6.3: Poincaré coordinates, radial coordinate r

Considering these simplifications, the determinant $\hat{\gamma}$ can be written as:

$$\det\left(\hat{\gamma}_{ij}\right) = \left(\frac{b}{z}\right)^{2(D-1)} \left(1 + \left(\frac{\partial z}{\partial r}\right)^2\right)$$

and the volume element in the polar coordinates of the x-plane is:

$$d^D X\big|_A = d^{D-1}\vec{x} = dr d\Omega_{D-2} r^{D-2} \tag{6.39}$$

So, the area integral becomes:

$$\begin{aligned} A_m &= \int_0^l dr r^{D-2} \int d\Omega_{D-2} \left(\frac{b}{z}\right)^{D-1} \sqrt{1 + \left(\frac{\partial z}{\partial r}\right)^2} \\ &= b^{D-1} V\left(S^{D-2}\right) \int_0^l dr \underbrace{\frac{r^{D-2}\sqrt{1+\left(\frac{\partial z}{\partial r}\right)^2}}{z^{D-1}}}_{\mathcal{L}_{Area}} \end{aligned} \tag{6.40}$$

For simplicity the factor in front of the integral shall be called C:

$$C = b^{D-1} V\left(S^{D-2}\right) = \frac{2b^{D-1}\sqrt{\pi}^{D-1}}{\Gamma\left(\frac{D-1}{2}\right)} \tag{6.41}$$

and it is proportional to the surface of the S^{D-2} unitary sphere.

In order to specify the minimal surface, we demand the variation of (6.40) to vanish. This gives rise to the *Euler-Lagrange* equation:

$$\frac{\partial}{\partial r}\left(\frac{\partial \mathcal{L}_{\text{Area}}}{\partial\left(\partial_r z\right)}\right) - \frac{\partial \mathcal{L}_{\text{Area}}}{\partial z} = 0 \tag{6.42}$$

which transforms to the second order differential equation:

$$rzz'' + (D-2)z\left(z'\right)^3 + (D-2)zz' + (D-1)r\left(z'\right)^2 + (D-1)r = 0 \tag{6.43}$$

where $z' = \frac{\partial z}{\partial r}$ and $z'' = \frac{\partial^2 z}{\partial r^2}$.

In order to comprehend the resulting surface it is important to remember that AdS is a hyperbolic space and distances diverge as $z \to 0$. So, a minimal surface should not be very close to the boundary but extend inside the bulk. Also, we expect the surface to have a rotational symmetry around the z-axis. We can, then, guess the solution to be that of a semisphere $z = \sqrt{l^2 - r^2}$, $z > 0$, where the radius of the semisphere is $R = l$, and verify the solution by substitution. A detailed derivation is included in Appendix A.

$$z^2 + x^\mu x_\mu = l^2 \tag{6.44}$$

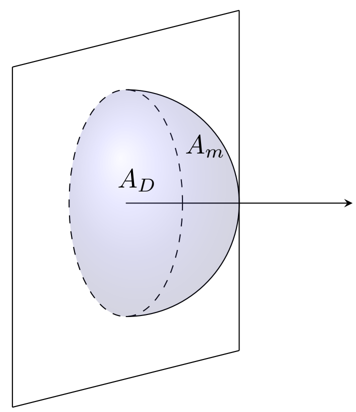

Figure 6.4: Minimal Area of Circular disk

So, the minimal surface induced in AdS is a semisphere of radius l, the same as the boundary disk's. From the equation of the semisphere we get: $rdr = -zdz$ and $\frac{\partial z}{\partial r} = -\frac{r}{z}$, from which we can calculate the integral (6.40). Nevertheless, the integral is divergent, consequently it is mandatory to introduce a UV cutoff at $z = \epsilon$, or equivalently at $r = \sqrt{l^2 - \epsilon^2}$.

$$\begin{aligned} A_m &= C \int_0^{\sqrt{l^2-\epsilon^2}} dr r^{D-2} \frac{\sqrt{1 + \left(-\frac{r}{z}\right)^2}}{z^{D-1}} \\ &= C \int_l^{\epsilon} (-zdz) \sqrt{l^2 - z^2}^{D-3} \frac{\sqrt{\left(\frac{r^2+z^2}{z^2}\right)}}{z^D} \\ &= C \int_\epsilon^l dz \frac{1}{l} \frac{\sqrt{1 - \frac{z^2}{l^2}}^{D-3}}{\left(\frac{z}{l}\right)^{D-1}} \end{aligned} \tag{6.45}$$

At this point, it is obvious that $D = 2$ requires special treatment. So, we examine each case separately.

D>2 Dimensions

With a change of variables $y = \frac{z}{l}$, $z = \epsilon \to y = \frac{\epsilon}{l}$ and $z = l \to y = 1$, the integral takes the form:

$$A_m = C \int_{\frac{\epsilon}{l}}^{1} dy \frac{\sqrt{1-y^2}^{D-3}}{(y)^{D-1}} \tag{6.46}$$

This is the hypergeometric function:

$$A_m = C \frac{1}{(2-D)} {}_2F_1 \left(\frac{3-D}{2}, \frac{2-D}{2}; \frac{2-D}{2} + 1; y^2 \right) y^{-D+2} \Big|_{\frac{\epsilon}{l}}^{1} \tag{6.47}$$

The hypergeometric function behaves differently depending upon the spacial dimension D of AdS_{D+1}. If the dimension D is even, i.e $D-1$ is odd, the expansion on $y = \frac{\epsilon}{l}$ includes, additionally, a logarithmic deviation. So, we examine the two cases separately, keeping also in mind that $D > 2$.

- $\underline{D = \text{odd}}$:
  For odd dimensions the third argument of the hypergeometric function is not a negative integer, so, it can be expressed as a well defined series:

  $${}_2F_1 \left(\frac{3-D}{2}, \frac{2-D}{2}; \frac{2-D}{2} + 1; y^2 \right) = \sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n \left(\frac{2-D}{2}\right)_n}{n! \left(\left(\frac{2-D}{2}+1\right)+n-1\right)_n} y^{2n} \tag{6.48}$$

  where $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$. The term for $z = 1$ can be found from Gauss's summation theorem that yields:

  $${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{6.49}$$

  Then, the area (6.47) has a constant term of the form:

  $$A_0 = C \frac{\Gamma\left(\frac{3-D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{2\sqrt{\pi}} \tag{6.50}$$

  Finally, expanding the hypergeometric function in the series (6.48) at $y = \frac{\epsilon}{i}$, in addition to to the Ryu-Takayanagi formula (6.34) and (6.50) we get the expression:

  $$\begin{aligned} S\left(A_m\right)|_{D=odd} = \frac{C}{4G_N^{(D+1)}} \Bigg[& \frac{1}{D-2} \left(\frac{l}{\epsilon}\right)^{(D-2)} - \frac{D-3}{2(D-4)} \left(\frac{l}{\epsilon}\right)^{(D-4)} + \ldots \\ & (-1)^{\left(\frac{D-3}{2}\right)} \left(\frac{D-3}{2}\right)! \left(\frac{l}{\epsilon}\right) + \frac{A_0}{C} \Bigg] \end{aligned} \tag{6.51}$$

  Note that the expression is divergent and, also, the series is not infinite at this case, because the arguments of ${}_2F_1$ are negative.

- $\underline{D = \text{even}}$:
  For an even number of dimensions, it can be proven that we get a similar expression, with two differences. The first is that we have an additional logarithmic deviation $\sim ln\left(\frac{l}{\epsilon}\right)$ and the second that the series is non-terminating. The final outcome is:

$$
\begin{aligned}
S\left(A_m\right)|_{D=even} = \frac{C}{4G_N^{(D+1)}}\Bigg[&\frac{1}{D-2}\left(\frac{l}{\epsilon}\right)^{(D-2)} - \frac{D-3}{2(D-4)}\left(\frac{l}{\epsilon}\right)^{(D-4)} + \dots \\
&(-1)^{\left(\frac{D-3}{2}\right)}\left(\frac{l}{\epsilon}\right)^{2}\left(\frac{D-3}{2}\right)! + a_{\left(\frac{D+2}{2}\right)} ln\left(\frac{l}{\epsilon}\right) + O\left(\frac{\epsilon}{l}\right) + \frac{A_0}{C}\Bigg]
\end{aligned}
\tag{6.52}
$$

  where $a_{\left(\frac{D+2}{2}\right)} = \frac{(D-3)!!}{(D+2)!!}$. Details on the derivation can be found in the appendix A. We observe that the terms of the order $O\left(\frac{\epsilon}{l}\right)$ vanish for $\epsilon \to 0$.

Comments

To begin with, in both cases there is a constant term $\sim O(1)$ independent of the UV cut-off ϵ, which means that it is completely unrelated to the details and depends solely on the dimensions of the theory as seen at (6.50). This quantity is related to the topological entropy [45], that is derived exclusively from the topology of the region or subsystem. Actually, the Γ functions in (6.50) reveal the topology of a sphere. The foresaid constant will be the same for continuous deformations of the surface, for which the area is not minimal, as the topology is unchanged.

Moreover, in the even dimensions D case, the coefficient of the logarithmic term $\ln\left(\frac{l}{\epsilon}\right)$ is related to the CFT central charge [42]. We will show this only for $D = 2$ below.

Lastly, but most importantly, the leading divergence of the entanglement entropy $\propto \epsilon^{2-D}$,in both cases, has a coefficient proportional to the area of the boundary disk $l^{D-2}V\left(S^{D-2}\right)$. This is the characteristic that verifies the validity of the conjecture in this case, as we have a reproduction of the *Area Law* [37] in field theories that we discussed above in paragraph 6.2.2. Consequently, in the leading divergence the holographic entanglement entropy is predicted by the *area law*:

$$
\begin{aligned}
S\left(A_m\right) &= \frac{C}{4G_N^{(D+1)}(D-2)}\left(\frac{l}{\epsilon}\right)^{(D-2)} \\
&\overset{(6.41)}{=} \frac{b^{D-1}V\left(S^{D-2}\right)}{4G_N^{(D+1)}(D-2)}\left(\frac{l}{\epsilon}\right)^{(D-2)}
\end{aligned}
\tag{6.53}
$$

that matches the expected behaviour:

$$
S(A) = \gamma\frac{\text{Area}(\partial A)}{\epsilon'^{D-2}} + O(\epsilon'^{3-D}) \tag{6.54}
$$

for equal UV-cutoff $\epsilon' = \epsilon$ and since $\text{Area}(\partial A) = V(S^{D-2})l^{D-2}$:

$$
\gamma = \frac{b^{D-1}}{4G_N^{(D+1)}(D-2)} \tag{6.55}
$$

D=2 Dimensions

In two dimensions the area integral (6.45) becomes:

$$\begin{aligned} A_2 &= 2b\int_0^{\sqrt{l^2-\epsilon^2}} dr\frac{\sqrt{1+\left(-\frac{r}{z}\right)^2}}{z} \\ &= b\int_{-\left(\sqrt{l^2-\epsilon^2}\right)}^{\sqrt{l^2-\epsilon^2}} dr\frac{l}{l^2-r^2} \\ &= \frac{b}{2}\ln\left(\frac{l+r}{l-r}\right)\Bigg|_{-\sqrt{l^2-\epsilon^2}}^{\sqrt{l^2-\epsilon^2}} \\ &= 2b\ln\left(\frac{2l}{\epsilon}\right) \end{aligned} \tag{6.56}$$

because the $D-2=0$ dimensional unitary sphere has $V(S_0)=2$, so the constant (6.41) is equal to $C=2b$. Also, in 2 dimensions the circular disk has reduced to a line with a 'diameter' $2\cdot l$. Therefore, the entanglement entropy according to the Ryu-Takayanagi formula is:

$$S\left(A_2\right) = \frac{b}{2G_N^{(3)}}\ln\left(\frac{l}{\epsilon}\right) \tag{6.57}$$

The similarity to the entanglement entropy of a CFT subsystem in 2 dimensions is evident, as the respective formula is:

$$S(A) = \frac{c_{CFT}}{3}\ln\left(\frac{L}{\epsilon}\right) \tag{6.58}$$

where, c_{CFT} the central charge of CFT_2 and L the length of the subsystem, that in our case is $2l$ in the Ryu-Takayanagi computation.

So, with the proper parameter matching the two expressions are identical. Actually, this matching defines the dual theory, which means that if we know the central charge c_{CFT} of the CFT_2 we can find the curvature b of the dual gravitational theory in AdS_3 and, reversely, if the curvature b is the originally known parameter, we can define a dual CFT_2 on the boundary with a central charge:

$$c_{CFT} = \frac{3b}{2G_N^{(3)}} \tag{6.59}$$

6.4.1 Properties Verification for the Holographic Entropy

In the light of the AdS/CFT correspondence, the Ryu-Takayanagi proposal introduces a transition from a statistical computation of the entanglement entropy in the CFT to the computation of a geometrical quantity, a surface area in a hyperbolic geometry. Of course since this area is a measure of entanglement entropy in the usual notion, it has to obey the same constrains and properties. So, for the example of a circular 'disk' on the boundary, we examine if the Ryu-Takayanagi formula satisfies the properties we mentioned in 6.1.6.

Subadditivity

The subadditivity property suggests that the entanglement entropy of a union of two disjoint subsystems is not greater than the sum of the entanglement entropies of the two subsystems. In other words, we miss less information if we can observe both subsystems simultaneously than we miss when we observe them separately. Translating the above argument in the holographic setup, the minimal surfaces A_m and B_m induced in the AdS spacetime by two subsystems A and B must have a total surface area greater than the minimal surface $(A \cup B)_m$ induced by $(A \cup B)$. So, the subadditivity translates to:

$$\text{Area}(A_m) + \text{Area}(B_m) = \text{Area}(A_m \cup B_m) \geq \text{Area}((A \cup B)_m) \tag{6.60}$$

This equality holds by the definition the surfaces, which are minimal according to (6.34). Thus the area of the surface $(A \cup B)$ has by default the minimum possible surface.

We can visualise the subadditivity property more easily in the case of $D = 2$, thus for AdS_3/CFT_2. In the case that the boundary of the joint subsystem $(A \cup B)$ coincides with the union of the boundaries of the two subsystems $\partial A \cup \partial B = \partial(A \cup B)$, thus, they simply touch, as shown in figure 6.5, the minimal surface $(A \cup B)_m$, which is actually a semicircle, can be exactly defined as the semicircle that covers the joint boundary. The respective entanglement entropy of the joint system is then:

$$S\left(AB\right) = \frac{b}{2G_N^{(3)}} \ln\left(\frac{l_A + l_B}{\epsilon}\right) \leq \frac{b}{2G_N^{(3)}} \ln\left(\frac{l_A l_B}{\epsilon^2}\right) \tag{6.61}$$

where l_A, l_B the lengths of the subsystems on the boundary

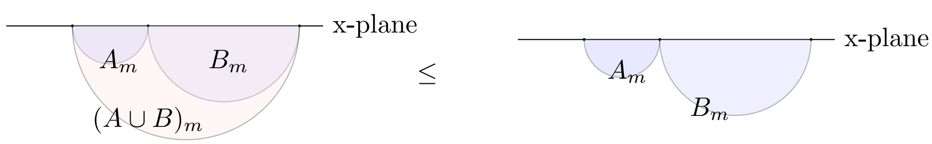

Figure 6.5: Subadditivity for Holographic Entanglement Entropy CFT_{1+1}

In the same context of CFT_2/AdS_3, if the two subsystems are disjoint, the problem of the minimal surface, or line more precisely, gets a little more complicated. In that case, there are two possible candidates for the minimal surface. The first one is two separate semicircles and the second is two semicircles connecting the two regions at their end points as shown in figure 6.6. But, how do we choose? The evident solution is to calculate both and take the minimum.

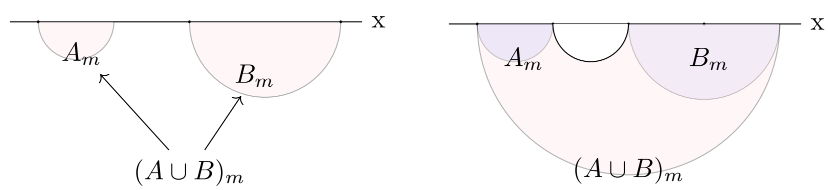

Figure 6.6: Holographic Entanglement Entropy for 2 disjoint regions in CFT_{1+1}

In any case, of course, the subadditivity property is satisfied. On the other hand, if we have more that two subsystems that are disjoint the calculation gets obviously much more complex. Nevertheless, subadditivity always provides the upper bound:

$$S(A_1 \cup A_2 \dots A_n) \leq \sum_{i=1}^{n} S(A_i) \tag{6.62}$$

For regions that are sufficiently far apart, the equal sign holds.

Strong Subadditivity

The strong subadditivity is a stronger property of the entanglement entropy and in holographic terms translates to:

$$\text{Area}((ABC)_m) + \text{Area}(B_m) \leq \text{Area}((AB)_m) + S((BC)_m) \tag{6.63}$$

In the simpler case of CFT_{1+1} the holographic interpretation of the formula is shown in figure 6.7.

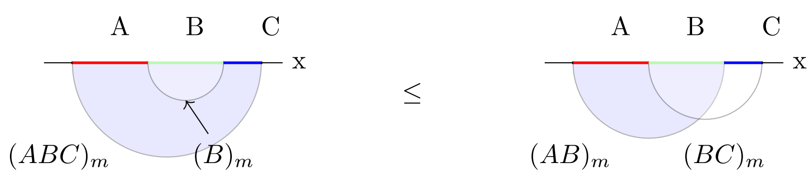

Figure 6.7: Subadditivity for Holographic Entanglement Entropy CFT_{1+1}

A general argument on why strong subadditivity holds for the Ryu-Takayanagi formula is the following: Let us consider 3 disjoint subsystems on the boundary A, B, C. Even though in figure (6.7) the boundary regions have coinciding edges, we do not make such an assumption for the proof. We define $r(A)$ as the region in AdS enclosed by the surface A_m and the same definition holds for all any surfaces. Inversely, the respective boundary of the region $r(A)$ is $\partial r(A) = A_m$. Then, we also define the regions $r(AB) \cup r(BC)$ and $r(AB) \cap r(BC)$. The boundary of the region $r(AB) \cap r(BC)$, that can be visualised as region below B in the first part of figure (6.7), is enclosing B but is not a minimal surface, so:

$$\text{Area}[\partial(r(AB) \cap r(BC))] \geq \text{Area}(B_m)$$

Similarly, the region $r(AB) \cup r(BC)$ is enclosing the boundary subsystem ABC but, again, its surface is not minimal, in contrast to $r(ABC)$. Consequently:

$$\text{Area}[\partial(r(AB) \cup r(BC))] \geq \text{Area}((ABC)_m)$$

Lastly, the fact that the union of the boundaries of the regions $r(AB) \cup r(BC)$ and $r(AB) \cap r(BC)$ is the sum of the areas of the minimal surfaces $(AB)_m$ and $(BC)_m$ yields the final result, by adding the two previous relations.

$$\begin{aligned}
&\text{Area}[\partial(r(AB) \cap r(BC))] + \text{Area}[\partial(r(AB) \cup r(BC))] \geq \\
&\text{Area}(B_m) + \text{Area}((ABC)_m) \Rightarrow
\end{aligned}$$

$$\text{Area}((AB)_m) + \text{Area}((BC)_m) \geq \text{Area}(B_m) + \text{Area}((ABC)_m) \tag{6.64}$$

which is exactly the strong subadditivity property in the holographic picture. Let us note that this proof was irrespective of the number of dimensions and of whether the regions have touching boundaries.

Conclusions

In terms of this thesis, we presented the AdS/CFT correspondence, according to which a strongly coupled CFT is dual to classical Gravity in a negatively curved Anti-de-Sitter spacetime. We only discussed the aspects of the correspondence concerning scalar fields and strong CFT. Nevertheless, the spectrum of the subject is very broad and extends to all kinds of fields, such as gauge fields and spinors, and also, to the limit where gravity is strong and CFT is weak, entering the Gravity's quantum limit. However, the correspondence already produces a very useful insight even in this limited discussion of scalar fields and very useful results such as the Holographic Entanglement Entropy.

The AdS/CFT correspondence is a surprising, yet very powerful tool for both the theoretical insight in the problem of QCD confinement and the previously difficult computations of CFT observables. It has shown new paths to research in topics that require strongly coupled Quantum Field theories, such us the quark-gluon plasma, condensed matter physics, black holes and quantum information. Moreover, the correspondence may unveil important information about the quantum nature of Gravity and the reasons behind its vast difference with the other fundamental forces.

On the other hand, it presents new challenges and questions on the topics it discusses. What other dualities there exist? What can we learn from them? Is it possible to rigorously prove the correspondence? And lastly, can we apply the correspondence to real physics problems?

Συμπεράσματα

Σε αυτή την εργασία παρουσιάσαμε την αντιστοιχία AdS/CFT, συμφωνα με την οποία μια Σύμμορφη Θεωρία Πεδίου ισχυρής σύζευξης είναι δυϊκή μίας θεωρίας κλασσικής Βαρύτητας σε έναν Anti-de-Sitter χωρόχρονο αρνητικής καμπυλότητας. Ασχολούμαστε μόνο με το κομμάτι της αντιστοιχίας που περιλαμβάνει τα βαθμωτά πεδία και ισχυρή σύζευξη στη Σύμμορφη Θεωρία Πεδίου. Παρόλα αυτά, η αντιστοιχία εκτείνεται σε όλα τα είδη των πεδίων, όπως σπινοριακά πεδία και πεδία βαθμίδας, καθώς και σε ασθενώς συζευγμένη Σύμμορφη θεωρία που είναι δυϊκή μίας ισχυρώς συζευγμένης κβαντικής θεωρίας Βαρύτητας. Ήδη όμως από τη μελέτη των βαθμωτών πεδίων, παράγονται πολύ χρήσιμα αποτελέσματα της αντιστοιχίας, όπως αυτό που μελετάμε στο δεύτερο κομμάτι της εργασίας, δηλαδή τον υπολογισμό της Εντροπίας Διεμπλοκής.

Η αντιστοιχία AdS/CFT είναι ένα αναπάντεχο και ταυτόχρονα πολύ χρήσιμο εργαλείο τόσο για τη θεωρητική κατανόηση του προβλήματος του περιορισμού των κουαρκ στην κβαντική Χρωμοδυναμική, όσο και για τον υπολογισμό μεγεθών σε ισχυρά συζευγμένες Σύμμορφες θεωρίες Πεδίου. Έχει ανοίξει το δρόμο σε νέα πεδία έρευνας που περιλμβάνουν ισχυρά συζευγμένες θεωρίες Πεδίου, όπως το πλάσμα κουαρκ-γκλουονίων, τη φυσική συμπυκνωμένης ύλης, τις μελανές οπές και τη κβαντική πληροφορία. Επιπροσθέτως, η αντιστοιχία μπορεί να δώσει απαντήσεις σχετικά με τη κβαντική φύση της Βαρύτητας και τη μεγάλη της διαφορά με τις υπόλοιπες θεμελειώδεις δυνάμεις της φύσης.

Τέλος, η αντιστοιχία γεννά νέες προκλήσεις καθώς και ερωτήσματα. Υπάρχουν άλλες αντιστοιχίες; Τι μπορούμε να μάθουμε από αυτές; Είναι δυνατή μια αυστηρή απόδειξη της αντιστοιχίας; Και τέλος, μπορούμε να εφαρμόσουμε την αντιστοιχία σε πραγματικά φυσικά προβλήματα;

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Appendix A

Detailed derivation of the Ryu-Takayanagi formula on a circular disk

According to the Ryu-Takayanagi proposal the entanglement entropy of a subsystem in CFT_D is given by the area of the minimal surface induced in AdS_{D+1} with the same boundary. The relation reads:

$$S(A) = \frac{\text{Area}\,(A_m)}{4G_N^{(D+1)}} \tag{A.1}$$

The area functional of a surface with an induced metric $\hat{\gamma}$ is given by:

$$A_m = \int dX \sqrt{\hat{\gamma}} \tag{A.2}$$

The surface for $z = 0$ must be a hypersphere of radius l. Due to Poincaré invariance we are allowed to center the disk at the origin $x^\mu = 0$:

$$x^\mu x_\mu \leq l^2 \tag{A.3}$$

that we may call a 'circular disk' because it is dimensionally lower. The area functional has a reparametrisation invariance, so we use the embedding coordinates:

$$X^i = x^i \quad X^D = Z(\vec{x}) = z \quad i = 1, 2, \ldots, D-1 \tag{A.4}$$

Then the induced metric is:

$$\hat{\gamma}_{ij} \equiv g_{\mu\nu} \frac{\partial X^\mu}{\partial x^i} \frac{\partial X^\nu}{\partial x^j} = \frac{b^2}{z^2} \left(\delta_{ij} + \partial_i Z \partial_j Z\right) \quad i = 1, 2, \ldots, D-1 \tag{A.5}$$

Lorentz invariance suggests that Z should only depend on the distance r, where $r = x^\mu x_\mu$ is the polar distance from the origin of the boundary. Also, for symmetric matrices $A: det(\mathbb{I}+A) = 1+Tr(A)$, so the determinant $det(\hat{\gamma}_{ij}) = \hat{\gamma}$ is:

$$
\begin{aligned}
\det\left(\hat{\gamma}_{ij}\right) &= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\operatorname{Tr}(\hat{\gamma})\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\sum_{i=1}^{D} \partial_i Z \partial_i Z\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\sum_{i=1}^{D} \frac{\partial Z}{\partial x^i}\frac{\partial Z}{\partial x^i}\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\sum_{i=1}^{D} \left(\frac{\partial Z}{\partial r}\frac{\partial r}{\partial x^i}\right)^2\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\left(\frac{\partial Z}{\partial r}\right)^2\right) \\
&= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\left(\frac{\partial z}{\partial r}\right)^2\right) \\
\Rightarrow \det\left(\hat{\gamma}_{ij}\right) &= \left(\frac{b}{z}\right)^{2(D-1)} \left(1+\left(\frac{\partial z}{\partial r}\right)^2\right)
\end{aligned}
\tag{A.6}
$$

where $V\left(S^{D-2}\right) = \frac{2\sqrt{\pi}^{D-1}}{\Gamma\left(\frac{D-1}{2}\right)}$ is the volume of the unitary $D-2$ -sphere. and the volume element in the polar coordinates of the x-plane is:

$$
d^D X\big|_A = d^{D-1}\vec{x} = dr d\Omega_{D-2} r^{D-2} \tag{A.7}
$$

So, the area integral becomes:

$$
\begin{aligned}
A_m &= \int_0^l dr r^{D-2} \int d\Omega_{D-2} \left(\frac{b}{z}\right)^{D-1} \sqrt{1+\left(\frac{\partial z}{\partial r}\right)^2} \\
&= b^{D-1} V\left(S^{D-2}\right) \int_0^l dr \underbrace{\frac{r^{D-2}\sqrt{1+\left(\frac{\partial z}{\partial r}\right)^2}}{z^{D-1}}}_{\mathcal{L}_{Area}}
\end{aligned}
\tag{A.8}
$$

For simplicity , let $C = b^{D-1} V\left(S^{D-2}\right) = \frac{2b^{D-1}\sqrt{\pi}^{D-1}}{\Gamma\left(\frac{D-1}{2}\right)}$. From the variation of the 'action' (A.8) we get the *Euler-Lagrange* equation:

$$
\frac{\partial}{\partial r}\left(\frac{\partial \mathcal{L}_{\text{Area}}}{\partial\left(\partial_r z\right)}\right) - \frac{\partial \mathcal{L}_{\text{Area}}}{\partial z} = 0 \tag{A.9}
$$

which transforms to the second order differential equation:

$$
rzz'' + (D-2)z\left(z'\right)^3 + (D-2)zz' + (D-1)r\left(z'\right)^2 + (D-1)r = 0 \tag{A.10}
$$

where $z' = \frac{\partial z}{\partial r}$ and $z'' = \frac{\partial^2 z}{\partial r^2}$.

Solution:
Let $\tilde{R} = R^2 = r^2 + z^2$, the distance from the coordinates' origin $(z, x^\mu) = (0, \vec{0})$, with $z = Z(r)$ then the derivatives are: $\frac{\tilde{R}'}{2} = r + zz'$ and $\frac{\tilde{R}''}{2} = 1 + zz'' + (z')^2$

$$\begin{aligned}(\mathrm{A}.9) \Rightarrow & r\left(zz'' + (z')^2 + 1\right) + (D-2)(z')^2(zz'+r) + (D-2)(zz'+r) = 0 \\ & r\frac{\tilde{R}''}{2} + (D-2)\left((z')^2 + 1\right)\tilde{R}' = 0 \\ & r\tilde{R}'' + (D-2)\left((z')^2 + 1\right)\tilde{R}' = 0\end{aligned} \tag{A.11}$$

Firstly, the equation of motion dictates a different approach for $D = 2$ and $D > 2$.

For $D = 2$:
The equation simplifies significantly since only the second derivative remains:

$$r\tilde{R}'' = 0 \Rightarrow z^2 + r^2 = c_1 r + c_2 \tag{A.12}$$

The r coordinate in this case is the absolute value of x, $r = |x|$. The boundary condition $z(l) = 0$ should reproduce the boundary shape (A.3). So, $c_1 = 0$, $c_2 = l^2$ and we get the line $z^2 + r^2 = l^2$, $r \in [-l, l]$.

For $D > 2$:
We argue that, if $\tilde{R}$ is a smooth function and if $z'(0), z''(0) < \infty$, then $\tilde{R}' = 0$ for all $l > r > 0$. Then, (A.11) yields for $r = 0$ that $\tilde{R}'(0) = 0$. Also, it suggests that $\tilde{R}'$ and $\tilde{R}''$ are either simultaneously zero or $\tilde{R}'(r)\tilde{R}''(r) < 0$ because $r, (D-2)\left((z')^2 + 1\right) > 0 \; \forall r$. We will prove the case that both $\tilde{R}'(r)$ and $\tilde{R}''(r)$ are negative is impossible.

Let us suppose that $\exists\, (r_1, r_2)$ where $\tilde{R}', \tilde{R}'' \neq 0$ and $\lim_{r \to r_1^-} \tilde{R}'(r) = 0$ and $\lim_{r \to r_2^-} \tilde{R}'(r) = 0$. Then there must be a $\xi \in (r_1, r_2)$ with $\tilde{R}''(\xi) = 0$, which is impossible. Thus there is not any interval with $\tilde{R}'(r) \to 0$ at both ends. So, at every interval $(0, r), \tilde{R}'(r) \neq 0$. As argued before, $\tilde{R}', \tilde{R}''$ must be of opposite sign inside this interval. But:
- if $\tilde{R}'' > 0 \Rightarrow \tilde{R}' \nearrow \Rightarrow \tilde{R}'(r) > \lim_{r \to 0^+} \tilde{R}(r) = 0$ which is impossible.
- if $\tilde{R}'' < 0 \Rightarrow \tilde{R}' \searrow \Rightarrow \tilde{R}'(r) < \lim_{r \to 0^+} \tilde{R}(r) = 0$ which is also impossible.
Consequently, $\tilde{R}', \tilde{R}'' \equiv 0 \; \forall r \in (0, l)$

So, the resulting minimal surface is the semisphere $r^2 + z^2 = l^2$ or in Poincaré coordinates:

$$z^2 + x^\mu x_\mu = l^2 \tag{A.13}$$

The corresponding area given by (A.2) can be calculated since $zdz = -rdr$ and $\frac{\partial z}{\partial r} = -\frac{r}{z}\cdot$ Also, we introduce a UV cutoff at $z = \epsilon \longleftrightarrow r = \sqrt{l^2 - \epsilon^2}$

$$\begin{aligned} A_m &= C \int_0^{\sqrt{l^2-\epsilon^2}} drr^{D-2} \frac{\sqrt{1 + \left(-\frac{r}{z}\right)^2}}{z^{D-1}} \\ &= C \int_l^{\epsilon} (-zdz) \sqrt{l^2 - z^2}^{D-3} \frac{\sqrt{\left(\frac{r^2+z^2}{z^2}\right)}}{z^D} \\ &= C \int_{\epsilon}^{l} dz \frac{1}{l} \frac{\sqrt{1 - \frac{z^2}{l^2}}^{D-3}}{\left(\frac{z}{l}\right)^{D-1}} \end{aligned} \tag{A.14}$$

At this point, it is obvious that $D = 2$ requires special treatment. So, we examine each case separately.

For $D > 2$:
With a change of variables $y = \frac{z}{l}$, $z = \epsilon \to y = \frac{\epsilon}{l}$ and $z = l \to y = 1$, the integral takes the form:

$$A_m = C \int_{\frac{\epsilon}{l}}^{1} dy \frac{\sqrt{1 - y^2}^{D-3}}{(y)^{D-1}} \tag{A.15}$$

Since, $\left|-y^2\right| < 1$ we can introduce the Taylor expansion, which is actually a binomial expansion: $\sqrt{1 - y^2}^{D-3} = \sum_{n=0}^{\infty} \binom{\frac{D-3}{2}}{n} \left(-y^2\right)^n$:

$$\begin{aligned} A_m &= C \int_f^1 dy \sum_{n=0}^{\infty} \frac{\left(\frac{D-3}{2}\right) \cdot \ldots \cdot \left(\frac{D-3}{2} - (n-1)\right) \left(-y^2\right)^n}{n!(y)^{D-1}} \\ &= C \int_{\hat{t}}^1 dy \sum_{n=0}^{\infty} \frac{\left(\frac{D-3}{2}\right) \left(\frac{D-3}{2} - 1\right) \cdot \ldots \cdot \left(\frac{D-3}{2} - (n-1)\right) (-1)^n y^{2n-D+1}}{n!} \\ &= C \int_{\frac{\epsilon}{\tau}}^1 dy \sum_{n=0}^{\infty} \frac{\left(\frac{3-D}{2}\right) \left(\frac{3-D}{2} + 1\right) \cdot \ldots \cdot \left(\frac{3-D}{2} + (n-1)\right) y^{2n-D+1}}{n!} \\ &= C \sum_{n=0}^{\infty} \frac{\left(\frac{3-D}{2}\right)_n}{n!} \int_{\frac{\epsilon}{T}}^1 dyy^{2n-D+1} \end{aligned}$$

where $(\alpha)_n = \alpha(\alpha + 1) \ldots (\alpha + n - 1)$

The result behaves differently depending upon the spacial dimension D of AdS_{D+1}. If the dimension is even then $D-1$ is odd, and the integration includes, additionally, a logarithmic deviation. So, we examine the two cases separately, keeping also in mind that $D > 2$:

- <u>For D odd</u>:
  At this case, the binomial expansion of $\sqrt{1 - y^2}^{D-3}$ terminates, because

$\frac{D-3}{2}$ is an integer and the area becomes:

$$\begin{aligned}
A_m^{\text{odd}} &= C\sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n}{n!}\frac{1}{2n-D+2}y^{2n-D+2}\Bigg|_i^1 \\
&= C\sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n}{n!}\frac{1}{2\left(\left(\frac{2-D}{2}+1\right)+n-1\right)}y^{2n-D+2}\Bigg|_{\frac{\pi}{t}}^1 \\
&= C\sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n}{n!}\frac{1}{2\left(\left(\frac{2-D}{2}+1\right)+n-1\right)}y^{2n-D+2}\Bigg|_i^1 \\
&= C\sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n}{n!}\frac{\left(\frac{2-D}{2}\right)^{-1}\left(\frac{2-D}{2}\right)_n}{2\left(\left(\frac{2-D}{2}+1\right)+n-1\right)_n}y^{2n-D+2}\Bigg|_{\bar{t}}^1 \\
&= C\sum_{n=0}^{\frac{D-3}{2}} \frac{1}{2\left(\frac{2-D}{2}\right)}\frac{\left(\frac{3-D}{2}\right)_n\left(\frac{2-D}{2}\right)_n}{n!\left(\left(\frac{2-D}{2}+1\right)+n-1\right)_n}y^{2n-D+2}\Bigg|_{\frac{\pi}{T}}^1 \\
&= C\frac{1}{2\left(\frac{1-D}{2}\right)}{}^2F_1\left(\frac{3-D}{2},\frac{2-D}{2};\frac{2-D}{2}+1;y^2\right)y^{-D+2}\Bigg|_{\bar{t}}^1
\end{aligned} \tag{A.16}$$

where we used the fact that:

$$\begin{aligned}
{}_2F_1\left(\frac{3-D}{2},\frac{2-D}{2};\frac{2-D}{2}+1;y^2\right) &= \sum_{n=0}^{\frac{D-3}{2}}\frac{\left(\frac{3-D}{2}\right)_n\left(\frac{2-D}{2}\right)_n}{n!\left(\left(\frac{2-D}{2}+1\right)+n-1\right)_n}y^{2n} \\
{}_2F_1\left(\frac{3-D}{2},\frac{2-D}{2};\frac{2-D}{2}+1;y^2\right) &= \sum_{n=0}^{\frac{D-3}{2}}\frac{\left(\frac{3-D}{2}\right)_n\left(\frac{2-D}{2}\right)_n}{n!\left(\left(\frac{2-D}{2}+1\right)+n-1\right)_n}y^{2n}
\end{aligned} \tag{A.17}$$

with ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$ the hypergeometric function. But, since the hypergeometric function has constrains on the values of the parameters a, b and c, it is necessary to be cautious. Firstly, we must verify that the function we wrote is actually defined. It is known that if $c \notin \mathbb{Z}^-$ then the hypergeometric function is defined for $|z| < 1$, which is satisfied as: $c = \frac{2-D}{2} + 1 \notin \mathbb{Z}^-$. Also, if either a of b are negative integers the series expansion terminates at $n = |a|$, which is also our case, since $a = \frac{D-3}{2} \in \mathbb{Z}^-$ and the termination takes place at $n = \frac{D-3}{2}$. The reason we introduced the hypergeometric function is to derive the constant term that comes from the limit $x \to 1$: From Gauss's summation theorem, the hypergeometric function yields

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{A.18}$$

Generally, the formula holds only if $\mathcal{R}e(c) > \mathcal{R}](a+b)$, which is satisfied in our case. So, at the calculation of the area, the constant term at $x \to 1$ is equal to:

$$\begin{aligned} A_0 &= C \frac{1}{2\left(\frac{2-D}{2}\right)} \frac{\Gamma\left(\frac{2-D}{2}+1\right)\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(1)} \\ &= C \frac{\Gamma\left(\frac{3-D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{2\sqrt{\pi}} \end{aligned} \tag{A.19}$$

Finally, the minimal surface area takes the form:

$$\begin{aligned} A_m|_{D=odd} &= A_0 - C \sum_{n=0}^{\frac{D-3}{2}} \frac{\left(\frac{3-D}{2}\right)_n}{n!} \frac{1}{2+2n-D} \left(\frac{\epsilon}{l}\right)^{2n-D+2} \\ &= C \left[-\frac{1}{2-D} \left(\frac{\epsilon}{l}\right)^{(2-D)} - \frac{3-D}{2(4-D)} \left(\frac{\epsilon}{l}\right)^{(4-D)} + \ldots + \frac{A_0}{C} \right] \\ &= C \left[\frac{1}{D-2} \left(\frac{l}{\epsilon}\right)^{(D-2)} - \frac{D-3}{2(D-4)} \left(\frac{l}{\epsilon}\right)^{(D-4)} + \ldots \right. \\ &\quad \left. (-1)^{\left(\frac{D-3}{2}\right)} \left(\frac{D-3}{2}\right)! \left(\frac{l}{\epsilon}\right) + \frac{A_0}{C} \right] \end{aligned} \tag{A.20}$$

According to the Ryu-Takayanagi formula (A.1) and the calculations above, the entanglement entropy takes the form:

$$\begin{aligned} S\left(A_m\right)|_{D=odd} &= \frac{C}{4G_N^{(D+1)}} \left[\frac{1}{D-2} \left(\frac{l}{c}\right)^{(D-2)} - \frac{D-3}{2(D-4)} \left(\frac{l}{c}\right)^{(D-4)} + \ldots \right. \\ &\quad \left. (-1)^{\left(\frac{D-3}{2}\right)} \left(\frac{D-3}{2}\right)! \left(\frac{l}{c}\right) + \frac{A_0}{C} \right] \end{aligned} \tag{A.21}$$

- <u>For D even</u>:
  For even spacial AdS_{D+1} coordinates, there is an odd number of boundary coordinates $D-1$. At that case, the binomial expansion of $\sqrt{1-y^2}^{D-3}$ does not terminate due to the non-integer power. What is more, the integral cannot be expressed in terms of a hypergeometric function if the dimension $D-1$ of the boundary is an odd number, for two reasons. To begin with, for $n = \frac{D+2}{2}$, the integration yields a logarithm. The

corresponding term is:

$$
\begin{aligned}
A_{\left(\frac{D+2}{2}\right)} &= C\int_{\frac{\hbar}{t}}^{1} dy \frac{\left(\frac{3-D}{2}\right)\left(\frac{3-D}{2}+1\right)\cdot\ldots\cdot\left(\frac{3-D}{2}+\left(\frac{D+2}{2}-1\right)\right)y}{\frac{D+2}{2}!} \\
&= C\int_{\frac{\epsilon}{l}}^{1} dy \frac{\left(\frac{D-3}{2}\right)!}{\frac{D+2}{2}!} y \\
&= C\frac{(D-3)!!}{(D+2)!!}\ln(y)\Big|_{\frac{\pi}{7}}^{1} \Rightarrow \\
A_{\left(\frac{D+2}{2}\right)} &= C\frac{(D-3)!!}{(D+2)!!}\ln\left(\frac{l}{\epsilon}\right)
\end{aligned}
\tag{A.22}
$$

The second reason is that the hypergeometric function ${}_2F_1(a,b;c;z)$ is not defined if $(1)c=-m, m\in\mathbb{N}$ and $(2)a$ or $b=-n, n\in\mathbb{N}$ and $(3)m<n$ which is exactly the case here.

So, we will only prove that the constant term that comes from the $x\to 1$ limit does not diverge. Indeed, the corresponding term is an alternating series with a vanishing coefficients, satisfying the Leibniz criterion.

$$
\begin{aligned}
A_0 &= C\sum_{n=0,n\neq\left(\frac{D+2}{2}\right)} \frac{\left(\frac{3-D}{2}\right)_n}{n!}\frac{1}{2n-D+2} \\
&= C\sum_{n=0,n\neq\left(\frac{D+2}{2}\right)} \frac{\left(\frac{D-3}{2}\right)\left(\frac{D-3}{2}-1\right)\cdot\ldots\cdot\left(\frac{D-3}{2}-(n-1)\right)(-1)^n}{n!} \Rightarrow \\
A_0 &\propto O(1)
\end{aligned}
\tag{A.23}
$$

Then , the minimal area surface for D even is:

$$
\begin{aligned}
A_m|_{D-even} = C\Bigg[&\frac{1}{D-2}\left(\frac{l}{\epsilon}\right)^{(D-2)} - \frac{D-3}{2(D-4)}\left(\frac{l}{\epsilon}\right)^{(D-4)} + \ldots \quad (7 \\
&(-1)^{\left(\frac{D-3}{2}\right)}\left(\frac{l}{\epsilon}\right)^{(2)}\left(\frac{D-3}{2}\right)! + \frac{A_{\left(\frac{D+2}{2}\right)}}{C} + O\left(\frac{\epsilon}{l}\right) + \frac{A_0}{C}\Bigg]
\end{aligned}
\tag{A.24}
$$

where $A_{\left(\frac{p+2}{2}\right)}$ is the logarithmic divergent term:

$$
A_{\left(\frac{D+2}{2}\right)} \propto \ln\left(\frac{l}{\epsilon}\right) \tag{A.25}
$$

Note that this series is not terminating, but the terms at some point become vanishing as they become of order at least $O\left(\frac{\epsilon}{l}\right)\to 0$.

For $D=2$:

For $D = 2$, the constant factor becomes $b^1 V(S0) = 2b$, then:

$$
\begin{aligned}
A_2 &= 2b \int_0^{\sqrt{l^2-\epsilon^2}} dr \frac{\sqrt{1+\left(-\frac{r}{z}\right)^2}}{z} \\
&= 2b \int_0^{\sqrt{l^2-\epsilon^2}} dr \frac{\sqrt{\left(\frac{l^2}{z^2}\right)}}{z} \\
&= 2b \int_0^{\sqrt{l^2-\epsilon^2}} dr \frac{l}{z^2} \\
&= 2b \int_0^{\sqrt{l^2-\epsilon^2}} dr \frac{l}{l^2-r^2} \\
&= 2\frac{b}{2} \int_{-\left(\sqrt{l^2-\epsilon^2}\right)}^{\sqrt{l^2-c^2}} dr \frac{l}{l^2-r^2} \\
&= 2\frac{b}{2}\frac{1}{2} \ln\left(\frac{l+r}{l-r}\right)\Bigg|_{-\sqrt{l^2-\epsilon^2}} \sqrt{l^2-c^2} \\
&= 2\frac{b}{2}\frac{1}{2} 2\ln\left(\frac{l+\sqrt{l^2-\epsilon^2}}{l-\sqrt{l^2-\epsilon^2}}\right)^2 \\
&= 2bln\left(\frac{l+\sqrt{l^2-\epsilon^2}}{l-\sqrt{l^2-\epsilon^2}}\right) \\
&= 2bln\left(\frac{2l}{\epsilon}\right)
\end{aligned}
\tag{A.26}
$$

The final result for the minimal surface area is:

$$
A_2 = 2b \cdot \ln\left(\frac{2l}{\epsilon}\right) \tag{A.27}
$$

where $2l$ is the 'diameter' of the circular disk that reduced to a 1 -dimensional line. The corresponding Entanglement entropy is:

$$
S\left(A_2\right) = \frac{b}{2G_N^{(3)}} \ln\left(\frac{l}{\epsilon}\right) \tag{A.28}
$$