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The error term of quadrature formulae for analytic functions

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Dedicated to my family,
Adam, Helen, Theodoros and Valeria.

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Abstract

In certain spaces of analytic functions, the error term of a quadrature formula is a bounded linear functional. The purpose of this thesis is to provide the methods used in order to compute explicitly the norm of the error functional, which subsequently can be used in order to derive estimates for the error term. In the first chapter, an introduction is made to orthogonal polynomials, presenting some of their most important properties and making a special reference to Chebyshev polynomials. The second chapter deals with quadrature formulae, focusing, mainly, on Gauss quadrature formulae, along with some crucial properties, which indicate their superiority compared to other quadrature formulae. This chapter concludes with the computation of the nodes and weights of the Gauss-Chebyshev quadrature formula of any of the four kinds. The third chapter is dedicated to estimating the error in Gauss quadrature formulae for analytic functions, which is done by Hilbert space methods or contour integration techniques. Finally, in the fourth chapter, some numerical experiments are carried out, which demonstrate the effectiveness of the bounds obtained in the previous chapter.

Chapter 1

Orthogonal Polynomials

The main aim of this chapter is to present, briefly, a review of orthogonal polynomials, along with their properties. For a comprehensive study, one can look at Gautschi (2004) (cf. [3]) and Szegö (1975) (cf. [9]).

1.1 Definition and basic theory

Let $\lambda(t)$ be an non-decreasing function on the real line $\mathbb{R}$ having finite limits as $t \to \pm\infty$, and assume that the induced positive measure $d\lambda$ has finite moments of all orders,

$$\mu_r = \mu_r(d\lambda) := \int_{\mathbb{R}} t^r d\lambda(t), \;\; r = 0, 1, 2, ..., \tag{1.1}$$

with $\mu_0 > 0$.

Let $\mathbb{P}$ be the space of real polynomials and $\mathbb{P}_d \subset \mathbb{P}$ the space of polynomials of degree $\leq d$. For any u, v in $\mathbb{P}$, the inner product can be defined as

$$(u, v) = \int_{\mathbb{R}} u(t)v(t)d\lambda(t). \tag{1.2}$$

If $(u, v) = 0$, then u is said to be orthogonal to v. If $u = v$, then

$$\|u\| = \sqrt{(u, u)} = \left(\int_{\mathbb{R}} u^2(t)d\lambda(t) \right)^{1/2} \tag{1.3}$$

is called the norm of u. We write $(u, v)_{d\lambda}$ and $\|u\|_{d\lambda}$, if we want to point out the measure $d\lambda$. Obviously, $\|u\| \geq 0$ for all $u \in \mathbb{P}$, and $\|u\| = 0$ only for $u = 0$, since $d\lambda$ is a positive measure. Schwarz's inequality states that

$$|(u, v)| \leq \|u\|\|v\|. \tag{1.4}$$

Definition 1.1. The monic real polynomials $\pi_k(t) = t^k + \dots,\ k = 0, 1, 2, \dots,$ are called monic orthogonal polynomials, with respect to the measure $d\lambda$, and will be denoted by $\pi_k(\cdot) = \pi_k(\cdot; d\lambda)$, if

$$(\pi_k, \pi_m)_{d\lambda} = 0,\ \ k \neq m,\ \ k, m = 0, 1, 2, \dots, \tag{1.5}$$

and

$$\|\pi_k\| > 0,\ \ k = 0, 1, 2, \dots. \tag{1.6}$$

Remark 1.2. If the index set $k = 0, 1, 2, \dots$ is unbounded then there are infinitely many orthogonal polynomials. In the same sense, there are finitely many orthogonal polynomials, if the index set is bounded.

Definition 1.3. The normalization $\tilde{\pi}_k = \frac{\pi_k}{\|\pi_k\|},\ \ k = 0, 1, 2 \dots,$ yields the orthonormal polynomials. They satisfy

$$(\tilde{\pi}_k, \tilde{\pi}_m)_{d\lambda} = \delta_{km} := \begin{cases} 0, & k \neq m, \\ 1, & k = m, \end{cases} \tag{1.7}$$

and will be denoted by $\tilde{\pi}_k(\cdot) = \tilde{\pi}_k(\cdot; d\lambda)$.

1.2 Properties of orthogonal polynomials

1.2.1 General properties

Lemma 1.4. ([3], Lemma 1.4) Let $\pi_k,\ \ k = 0, 1, 2, \dots, n,$ be monic orthogonal polynomials. If $q \in \mathbb{P}_n$ satisfies $(q, \pi_k) = 0$ for $k = 0, 1, 2, \dots, n$, then $q \equiv 0$.

Proof. The polynomial q can be written as $q(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$. Then, one has

$$\begin{aligned} 0 = (q, \pi_n) =& (a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0, \pi_n) \\ =& a_n (t^n, \pi_n) + a_{n-1}(t^{n-1}, \pi_n) + \cdots + a_1 (t, \pi_n) + a_0 (1, \pi_n) \\ =& a_n (t^n, \pi_n) = a_n (\pi_n, \pi_n), \end{aligned}$$

by orthogonality of π_n to polynomials of lower degree. Since $(\pi_n, \pi_n) > 0$, this yields that $a_n = 0$. Similarly, one can show that $a_{n-1} = a_{n-2} = \cdots = a_0 = 0$. Hence, $q \equiv 0$. □

Lemma 1.5. ([3] , Lemma 1.5) A set $\pi_0, \pi_1, \dots, \pi_n$ of monic orthogonal polynomials is linearly independent. Furthermore, any polynomial $q \in \mathbb{P}_n$ can be uniquely represented in the form

$$q = \sum_{k=0}^{n} c_k \pi_k, \tag{1.8}$$

for $c_k \in \mathbb{R}$. The set $\pi_0, \pi_1, \ldots, \pi_n$ forms a basis in $\mathbb{P}_n$.

Proof. Assuming that

$$\sum_{k=0}^{n} c_k \pi_k = 0,$$

and taking the inner product of both sides with π_i, $i = 0, 1, 2, \ldots, n$, one has that

$$\left(\sum_{k=0}^{n} c_k \pi_k, \pi_i \right) = 0 \iff \sum_{k=0}^{n} c_k \big(\pi_k, \pi_i\big) = 0,$$

from which there follows, by orthogonality,

$$c_i \big(\pi_i, \pi_i\big) = 0,$$

that is,

$$c_i = 0, \quad i = 0, 1, 2, \ldots, n.$$

This proves linear independence. Now, writing q in the form (1.8) and taking the inner product of both sides with π_i, gives, by orthogonality,

$$\big(q, \pi_i\big) = \left(\sum_{k=0}^{n} c_k \pi_k, \pi_i \right) \iff \big(q, \pi_i\big) = c_i \big(\pi_i, \pi_i\big),$$

which yields

$$c_i = \frac{(q, \pi_i)}{(\pi_i, \pi_i)}, \quad i = 0, 1, 2, \ldots, n.$$

With the coefficients c_k so defined, $q - \sum_{k=0}^{n} c_k \pi_k$ is orthogonal to $\pi_0, \pi_1, \ldots, \pi_n$, since

$$\begin{aligned} \left(q - \sum_{k=0}^{n} c_k \pi_k, \pi_i \right) =& (q, \pi_i) - \left(\sum_{k=0}^{n} c_k \pi_k, \pi_i \right) \\ =& (q, \pi_i) - c_i (\pi_i, \pi_i) = 0, \quad i = 0, 1, 2, \ldots, n, \end{aligned}$$

hence, by Lemma 1.4, identically zero. □

1.2.2 Symmetry

Definition 1.6. An absolutely continuous measure $d\lambda(t) = w(t)dt$ is symmetric if its support interval is $[-a, a]$, $0 < a \leq \infty$, and $w(-t) = w(t)$ for all $t \in \mathbb{R}$.

Theorem 1.7. ([3], Theorem 1.17) If $d\lambda$ is symmetric, then

$$\pi_k(-t; d\lambda) = (-1)^k \pi_k(t; d\lambda). \tag{1.9}$$

Hence, π_k is an even or odd polynomial, depending on the parity of k.

Proof. Define $\hat{\pi}_k(t) = (-1)^k \pi_k(-t; d\lambda)$. All $\hat{\pi}_k$ are monic, and

$$\begin{aligned}(\hat{\pi}_k, \hat{\pi}_m)_{d\lambda} &= \int_{-a}^{a} (-1)^k \pi_k(-t)(-1)^m \pi_m(-t) d\lambda(t) \\ &= (-1)^{k+m} \int_{-a}^{a} \pi_k(-t)\pi_m(-t) w(t) d(t).\end{aligned}$$

Set $-t = u$. Then, $w(t) = w(-u) = w(u)$, since $d\lambda$ is symmetric, and

$$\begin{aligned}(\hat{\pi}_k, \hat{\pi}_m)_{d\lambda} &= (-1)^{k+m} \int_{a}^{-a} \pi_k(u)\pi_m(u) w(u)(-du) \\ &= (-1)^{k+m} \int_{-a}^{a} \pi_k(u)\pi_m(u) w(u) du \\ &= (-1)^{k+m} (\pi_k, \pi_m)_{d\lambda} = 0, \quad k \neq m.\end{aligned}$$

By the uniqueness of monic orthogonal polynomials, $\hat{\pi}_k(t) \equiv \pi_k(t; d\lambda)$, which proves (1.9). □

1.2.3 Three-term recurrence relation

The three-term recurrence relation satisfied by orthogonal polynomials is, by far, the most important property for the constructive and computational use of orthogonal polynomials. The reason why there exists a three-term recurrence relation is the shift property

$$(tu, v)_{d\lambda} = (u, tv)_{d\lambda}, \tag{1.10}$$

for all $u, v \in \mathbb{P}$, obviously enjoyed by the inner product (1.2). The shift property (1.10) shall be used in the proof of the following theorem.

Theorem 1.8. ([3], Theorem 1.27) Let $\pi_k(\cdot) = \pi_k(\cdot; d\lambda)$, $k = 0, 1, 2, \ldots$, be the monic orthogonal polynomials with respect to the measure $d\lambda$. Then,

$$\begin{aligned}\pi_{k+1}(t) &= (t - a_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \quad k = 0, 1, 2, \ldots, \\ \pi_{-1}(t) &= 0, \quad \pi_0(t) = 1,\end{aligned} \tag{1.11}$$

where

$$a_k = \frac{(t\pi_k, \pi_k)_{d\lambda}}{(\pi_k, \pi_k)_{d\lambda}}, \quad k = 0, 1, 2, \ldots, \tag{1.12}$$

$$\beta_k = \frac{(\pi_k, \pi_k)_{d\lambda}}{(\pi_{k-1}, \pi_{k-1})_{d\lambda}}, \quad k = 1, 2, \ldots. \tag{1.13}$$

Proof. Since π_k are monic, $\pi_{k+1} - t\pi_k$ is a polynomial of degree $\leq k$. By Lemma 1.5, it can be written as

$$\pi_{k+1}(t) - t\pi_k(t) = -a_k\pi_k(t) - \beta_k\pi_{k-1}(t) + \sum_{i=0}^{k-2} c_{ki}\pi_i(t), \tag{1.14}$$

where a_k, β_k, c_{ki} are certain constants, $\pi_{-1}(t) = 0$ and $\pi_0(t) = 1$. Taking the inner product of both sides with π_k, we have

$$\begin{aligned} (\pi_{k+1} - t\pi_k, \pi_k) &= \left(-a_k\pi_k - \beta_k\pi_{k-1} + \sum_{i=0}^{k-2} c_{ki}\pi_i, \pi_k\right) \\ \iff \big(\pi_{k+1}, \pi_k\big) - \big(t\pi_k, \pi_k\big) &= -\, a_k\big(\pi_k, \pi_k\big) - \beta_k\big(\pi_{k-1}, \pi_k\big) + \sum_{i=0}^{k-2} c_{ki}\big(\pi_i, \pi_k\big), \end{aligned}$$

that is, by orthogonality,

$$-\big(t\pi_k, \pi_k\big) = -a_k\big(\pi_k, \pi_k\big) \iff a_k = \frac{(t\pi_k, \pi_k)}{(\pi_k, \pi_k)}, \quad k = 0, 1, 2 \ldots,$$

which proves (1.12). Similarly, taking the inner product of both sides in (1.14) with π_{k-1}, we have

$$-\,\big(t\pi_k, \pi_{k-1}\big) = -\beta_k\big(\pi_{k-1}, \pi_{k-1}\big). \tag{1.15}$$

Now, by (1.10),

$$\big(t\pi_k, \pi_{k-1}\big) = \big(\pi_k, t\pi_{k-1}\big), \tag{1.16}$$

and writing $t\pi_{k-1} = \pi_k + p_{k-1}$, where p_{k-1} is a polynomial of degree $\leq k-1$, we get, by orthogonality,

$$\big(\pi_k, t\pi_{k-1}\big) = \big(\pi_k, \pi_k + p_{k-1}\big) = \big(\pi_k, \pi_k\big) + \big(\pi_k, p_{k-1}\big) = \big(\pi_k, \pi_k\big).$$

This, together with (1.15) and (1.16), yields (1.13). Finally, taking the inner product of both sides in (1.14) with π_i, $i = 0, 1, 2, \ldots, k-2$, gives

$$-\big(t\pi_k, \pi_i\big) = c_{ki}\big(\pi_i, \pi_i\big).$$

Again, we use that

$$\big(t\pi_k, \pi_i\big) = \big(\pi_k, t\pi_i\big) = 0,$$

as $t\pi_i$ is a polynomial of degree $\leq k-1$. Hence, $c_{ki} = 0$ for $i = 0, 1, 2, \ldots, k-2$. □

Remark 1.9. Relations (1.12) and (1.13) are well defined as the inner product $(\cdot, \cdot)_{d\lambda}$ is positive definite.

The coefficients in the three-term recurrence relation (1.11) will be denoted by $a_k(d\lambda)$ and $\beta_k(d\lambda)$. Although β_0 can be arbitrary, since it multiplies $\pi_{-1} = 0$, it is convenient to define

$$\beta_0 = \big(\pi_0, \pi_0\big) = \int_{\mathbb{R}} d\lambda(t). \tag{1.17}$$

From (1.13), all β_k are positive, and the combination of (1.13) and (1.17) gives

$$\begin{aligned}\big(\pi_n, \pi_n\big) =& \frac{\big(\pi_n, \pi_n\big)}{\big(\pi_{n-1}, \pi_{n-1}\big)} \frac{\big(\pi_{n-1}, \pi_{n-1}\big)}{\big(\pi_{n-2}, \pi_{n-2}\big)} \cdots \frac{\big(\pi_1, \pi_1\big)}{\big(\pi_0, \pi_0\big)} \big(\pi_0, \pi_0\big)\\ =& \beta_n \beta_{n-1} \ldots \beta_1 \beta_0, \;\; n = 0, 1, 2 \ldots .\end{aligned}$$

Hence,

$$\|\pi_n\|^2 = \beta_n \beta_{n-1} \ldots \beta_1 \beta_0, \;\; n = 0, 1, 2 \ldots . \tag{1.18}$$

Theorem 1.10. ([3], Theorem 1.29) Let $\tilde{\pi}_k(\cdot) = \tilde{\pi}_k(\cdot; d\lambda)$, $k = 0, 1, 2, \ldots$, be the orthonormal polynomials with respect to the measure $d\lambda$. Then

$$\begin{aligned}\sqrt{\beta_{k+1}} \tilde{\pi}_{k+1}(t) =& (t - a_k) \tilde{\pi}_k(t) - \sqrt{\beta_k} \tilde{\pi}_{k-1}(t), \;\; k = 0, 1, 2, \ldots ,\\ \tilde{\pi}_{-1}(t) = 0,& \;\; \tilde{\pi}_0(t) = \frac{1}{\sqrt{\beta_0}},\end{aligned} \tag{1.19}$$

where a_k and β_k are given by (1.12), (1.13) and (1.17).

Proof. By Definition 1.3, we have

$$\pi_k(t) = \|\pi_k\| \tilde{\pi}_k(t). \tag{1.20}$$

Obviously, $\tilde{\pi}_{-1}(t) = 0$ and, from (1.18),

$$\tilde{\pi}_0(t) = \frac{\pi_0(t)}{\|\pi_0\|} = \frac{1}{\sqrt{\beta_0}}.$$

Inserting (1.20) into the three-term recurrence relation (1.11), we get

$$\|\pi_{k+1}\| \tilde{\pi}_{k+1}(t) = (t - a_k) \|\pi_k\| \tilde{\pi}_k(t) - \beta_k \|\pi_{k-1}\| \tilde{\pi}_{k-1}(t),$$

and, dividing by $\|\pi_{k+1}\|$, we obtain

$$\begin{aligned}\tilde{\pi}_{k+1}(t) =& (t - a_k) \frac{\|\pi_k\| \tilde{\pi}_k(t)}{\|\pi_{k+1}\|} - \beta_k \frac{\|\pi_{k-1}\| \tilde{\pi}_{k-1}(t)}{\|\pi_{k+1}\|}\\ =& (t - a_k) \frac{\|\pi_k\|}{\|\pi_{k+1}\|} \tilde{\pi}_k(t) - \beta_k \frac{\|\pi_{k-1}\|}{\|\pi_k\|} \frac{\|\pi_k\|}{\|\pi_{k+1}\|} \tilde{\pi}_{k-1}(t),\end{aligned}$$

which, by (1.13), gives

$$\tilde{\pi}_{k+1}(t) = (t - a_k) \frac{\tilde{\pi}_k(t)}{\sqrt{\beta_{k+1}}} - \frac{\beta_k}{\sqrt{\beta_k \beta_{k+1}}} \tilde{\pi}_{k-1}(t).$$

Finally, multiplying both sides by $\sqrt{\beta_{k+1}}$, yields (1.19). □

Definition 1.11. The Jacobi matrix associated with the measure $d\lambda$ is the $n \times n$ symmetric, tridiagonal matrix

$$J_n = J_n(d\lambda) = \begin{bmatrix} a_0 & \sqrt{\beta_1} & 0 & \dots & 0 \\ \sqrt{\beta_1} & a_1 & \sqrt{\beta_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sqrt{\beta_{n-1}} & a_{n-1} \end{bmatrix}. \tag{1.21}$$

Let

$$\tilde{\pi}(t) = \left[\tilde{\pi}_0(t), \tilde{\pi}_1(t), \dots, \tilde{\pi}_{n-1}(t)\right]^T. \tag{1.22}$$

Theorem 1.12. ([3], Theorem 1.31) The zeros $\tau_\nu^{(n)}$ of $\pi_n(\cdot; d\lambda)$ (or $\tilde{\pi}_n(\cdot; d\lambda)$) are the eigenvalues of the $n \times n$ Jacobi matrix J_n, and $\tilde{\pi}\left(\tau_\nu^{(n)}\right)$ are the corresponding eigenvectors.

Proof. The three-term recurrence relation (1.19) can, equivalently, be written as

$$t\tilde{\pi}_k(t) = \sqrt{\beta_k}\tilde{\pi}_{k-1}(t) + a_k\tilde{\pi}_k(t) + \sqrt{\beta_{k+1}}\tilde{\pi}_{k+1}(t), \;\; k = 0, 1, 2, \dots, n-1. \tag{1.23}$$

Then, in view of (1.22), (1.23) can be expressed in matrix form as

$$t\tilde{\pi}(t) = J_n\tilde{\pi}(t) + \sqrt{\beta_n}\tilde{\pi}_n(t)e_n,$$

where $e_n = [0, 0, \dots, 1]^T$. Now, put $t = \tau_\nu^{(n)}$ and note that $\tilde{\pi}\left(\tau_\nu^{(n)}\right) \neq 0$, as $\tilde{\pi}_0\left(\tau_\nu^{(n)}\right) = \frac{1}{\sqrt{\beta_0}}$. Thus, we get

$$J_n\tilde{\pi}\left(\tau_\nu^{(n)}\right) = \tau_\nu^{(n)}\tilde{\pi}\left(\tau_\nu^{(n)}\right),$$

which proves our assertion. □

An important consequence of the three-term recurrence relation (1.19) is the Christoffel-Darboux formula.

Theorem 1.13. ([3], Theorem 1.32) Let $\tilde{\pi}_k(\cdot) = \tilde{\pi}_k(\cdot; d\lambda)$ be the orthonormal polynomials with respect to the measure $d\lambda$. Then,

$$\sum_{k=0}^{n} \tilde{\pi}_k(x)\tilde{\pi}_k(t) = \sqrt{\beta_{n+1}}\frac{\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)}{x - t} \tag{1.24}$$

and

$$\sum_{k=0}^{n} \left[\tilde{\pi}_k(t)\right]^2 = \sqrt{\beta_{n+1}}\left[\tilde{\pi}'_{n+1}(t)\tilde{\pi}_n(t) - \tilde{\pi}'_n(t)\tilde{\pi}_{n+1}(t)\right]. \tag{1.25}$$

Proof. Writing the first relation of (1.19) as

$$(t - a_k)\tilde{\pi}_k(t) = \sqrt{\beta_{k+1}}\tilde{\pi}_{k+1}(t) + \sqrt{\beta_k}\tilde{\pi}_{k-1}(t),$$

and multiplying both sides with $\tilde{\pi}_k(x)$ gives

$$(t - a_k)\tilde{\pi}_k(t)\tilde{\pi}_k(x) = \sqrt{\beta_{k+1}}\tilde{\pi}_{k+1}(t)\tilde{\pi}_k(x) + \sqrt{\beta_k}\tilde{\pi}_{k-1}(t)\tilde{\pi}_k(x). \tag{1.26}$$

Interchange x and t in (1.26) in order to get

$$(x - a_k)\tilde{\pi}_k(x)\tilde{\pi}_k(t) = \sqrt{\beta_{k+1}}\tilde{\pi}_{k+1}(x)\tilde{\pi}_k(t) + \sqrt{\beta_k}\tilde{\pi}_{k-1}(x)\tilde{\pi}_k(t). \tag{1.27}$$

Subtracting (1.26) from (1.27), we have

$$(x-t)\tilde{\pi}_k(x)\tilde{\pi}_k(t) = \sqrt{\beta_{k+1}}\big[\tilde{\pi}_{k+1}(x)\tilde{\pi}_k(t) - \tilde{\pi}_k(x)\tilde{\pi}_{k+1}(t)\big] - \sqrt{\beta_k}\big[\tilde{\pi}_k(x)\tilde{\pi}_{k-1}(t) - \tilde{\pi}_{k-1}(x)\tilde{\pi}_k(t)\big].$$

Dividing both sides by $x - t$, summing from $k = 0$ to $k = n$, and observing that $\tilde{\pi}_{-1}(t) = 0$, we get

$$\begin{aligned}
\sum_{k=0}^{n} \tilde{\pi}_k(x)\tilde{\pi}_k(t) =& \frac{1}{x-t}\sum_{k=0}^{n}\Big\{\sqrt{\beta_{k+1}}\big[\tilde{\pi}_{k+1}(x)\tilde{\pi}_k(t) - \tilde{\pi}_k(x)\tilde{\pi}_{k+1}(t)\big] - \sqrt{\beta_k}\big[\tilde{\pi}_k(x)\tilde{\pi}_{k-1}(t) - \tilde{\pi}_{k-1}(x)\tilde{\pi}_k(t)\big]\Big\}\\
=& \frac{1}{x-t}\Big\{\sqrt{\beta_1}\big[\tilde{\pi}_1(x)\tilde{\pi}_0(t) - \tilde{\pi}_0(x)\tilde{\pi}_1(t)\big] - \sqrt{\beta_0}\big[\tilde{\pi}_0(x)\tilde{\pi}_{-1}(t) - \tilde{\pi}_{-1}(x)\tilde{\pi}_0(t)\big]\\
&+ \sqrt{\beta_2}\big[\tilde{\pi}_2(x)\tilde{\pi}_1(t) - \tilde{\pi}_1(x)\tilde{\pi}_2(t)\big] - \sqrt{\beta_1}\big[\tilde{\pi}_1(x)\tilde{\pi}_0(t) - \tilde{\pi}_0(x)\tilde{\pi}_1(t)\big] + \cdots +\\
&+ \sqrt{\beta_n}\big[\tilde{\pi}_n(x)\tilde{\pi}_{n-1}(t) - \tilde{\pi}_{n-1}(x)\tilde{\pi}_n(t)\big] - \sqrt{\beta_{n-1}}\big[\tilde{\pi}_{n-1}(x)\tilde{\pi}_{n-2}(t) - \tilde{\pi}_{n-2}(x)\tilde{\pi}_{n-1}(t)\big]\\
&+ \sqrt{\beta_{n+1}}\big[\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)\big] - \sqrt{\beta_n}\big[\tilde{\pi}_n(x)\tilde{\pi}_{n-1}(t) - \tilde{\pi}_{n-1}(x)\tilde{\pi}_n(t)\big]\Big\}\\
=& \sqrt{\beta_{n+1}}\frac{\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)}{x-t},
\end{aligned}$$

which proves (1.24). Now, writing

$$\begin{aligned}
\sum_{k=0}^{n} \tilde{\pi}_k(x)\tilde{\pi}_k(t) =& \sqrt{\beta_{n+1}}\frac{\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)}{x-t}\\
=& \sqrt{\beta_{n+1}}\frac{\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_{n+1}(t)\tilde{\pi}_n(t) + \tilde{\pi}_{n+1}(t)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)}{x-t}\\
=& \sqrt{\beta_{n+1}}\left[\frac{\tilde{\pi}_{n+1}(x)\tilde{\pi}_n(t) - \tilde{\pi}_{n+1}(t)\tilde{\pi}_n(t)}{x-t} + \frac{\tilde{\pi}_{n+1}(t)\tilde{\pi}_n(t) - \tilde{\pi}_n(x)\tilde{\pi}_{n+1}(t)}{x-t}\right]\\
=& \sqrt{\beta_{n+1}}\left[\frac{\tilde{\pi}_{n+1}(x) - \tilde{\pi}_{n+1}(t)}{x-t}\tilde{\pi}_n(t) - \frac{\tilde{\pi}_n(x) - \tilde{\pi}_n(t)}{x-t}\tilde{\pi}_{n+1}(t)\right],
\end{aligned}$$

and taking the limit as $x \to t$, yields (1.25). □

1.2.4 Zeros

Theorem 1.14. ([3], Theorem 1.19) All zeros of $\pi_n(\cdot) = \pi_n(\cdot; d\lambda)$, $n \geq 1$, are real, simple and located in the interior of the support interval $[a, b]$ of $d\lambda$.

Proof. Let $\tau_1, \tau_2, \ldots, \tau_n$ be the zeros of π_n. At least one of them must exist in the interior of $[a, b]$. Assume that this is not the case. Then, π_n keeps constant sign in (a, b), so,

$$\int_{\mathbb{R}} \pi_n(t) d\lambda(t) = \int_a^b \pi_n(t)\pi_0(t) d\lambda(t) \neq 0,$$

which contradicts the orthogonality of π_n to polynomials of lower degree.

If we assume that there is a double zero τ_d and define

$$p_d = \frac{\pi_n}{(t - \tau_d)^2} \in \mathbb{P}_{n-2},$$

then

$$(\pi_n, p_d) = \int_{\mathbb{R}} \pi_n(t) p_d(t) d\lambda(t) = \int_a^b \frac{\pi_n^2(t)}{(t - t_d)^2} d\lambda(t) > 0,$$

which contradicts the orthogonality of π_n to polynomials of lower degree.

Now, assume that there are k zeros of π_n in (a, b), $k < n$, and define

$$p_k = (t - \tau_1)(t - \tau_2) \ldots (t - \tau_k) \in \mathbb{P}_k.$$

Hence,

$$(\pi_n, p_k) = \int_{\mathbb{R}} \pi_n(t) p_k(t) d\lambda(t) = \int_a^b (t - \tau_1)^2 (t - \tau_2)^2 \ldots (t - \tau_k)^2 (t - \tau_{k+1}) \ldots (t - \tau_n) d\lambda(t) \neq 0,$$

since $\pi_n p_k$ has constant sign in $[a, b]$. Again, it contradicts the orthogonality of π_n to polynomials of lower degree. Thus, it must be $k = n$. □

Theorem 1.15. ([3], Theorem 1.20) The zeros of π_{n+1} alternate with those of π_n, that is,

$$\tau_{n+1}^{(n+1)} < \tau_n^{(n)} < \tau_n^{(n+1)} < \tau_{n-1}^{(n)} < \cdots < \tau_1^{(n)} < \tau_1^{(n+1)}, \tag{1.28}$$

where $\tau_i^{(n+1)}$ and $\tau_j^{(n)}$ are the zeros, in descending order, of π_{n+1} and π_n, respectively.

Proof. From (1.25), we have

$$\tilde{\pi}'_{n+1}(t)\tilde{\pi}_n(t) - \tilde{\pi}'_n(t)\tilde{\pi}_{n+1}(t) > 0. \tag{1.29}$$

Let $\tau_i^{(n+1)}$ and $\tau_{i+1}^{(n+1)}$ be two consecutive zeros of $\tilde{\pi}_{n+1}$. Since $\tilde{\pi}_{n+1} = \pi_{n+1}/\|\pi_{n+1}\|$, $\tau_i^{(n+1)}$ and $\tau_{i+1}^{(n+1)}$ are two consecutive zeros of π_{n+1} as well. From Theorem 1.14, $\tau_i^{(n+1)}$ and $\tau_{i+1}^{(n+1)}$ are real and simple, so $\tilde{\pi}'_{n+1}\left(\tau_i^{(n+1)}\right)$ and $\tilde{\pi}'_{n+1}\left(\tau_{i+1}^{(n+1)}\right)$ have opposite signs. Equivalently,

$$\tilde{\pi}'_{n+1}\left(\tau_i^{(n+1)}\right)\tilde{\pi}'_{n+1}\left(\tau_{i+1}^{(n+1)}\right) < 0. \tag{1.30}$$

Setting $t = \tau_i^{(n+1)}$ and $t = \tau_{i+1}^{(n+1)}$ in (1.29), we get

$$\tilde{\pi}'_{n+1}\left(\tau_i^{(n+1)}\right)\tilde{\pi}_n\left(\tau_i^{(n+1)}\right) > 0$$

and

$$\tilde{\pi}'_{n+1}\left(\tau_{i+1}^{(n+1)}\right)\tilde{\pi}_n\left(\tau_{i+1}^{(n+1)}\right) > 0,$$

respectively. Multiplying these two relations,

$$\tilde{\pi}'_{n+1}\left(\tau_i^{(n+1)}\right)\tilde{\pi}_n\left(\tau_i^{(n+1)}\right)\tilde{\pi}'_{n+1}\left(\tau_{i+1}^{(n+1)}\right)\tilde{\pi}_n\left(\tau_{i+1}^{(n+1)}\right) > 0,$$

and noting (1.30), we obtain

$$\tilde{\pi}_n\left(\tau_i^{(n+1)}\right)\tilde{\pi}_n\left(\tau_{i+1}^{(n+1)}\right) < 0.$$

Hence, between $\tau_i^{(n+1)}$ and $\tau_{i+1}^{(n+1)}$ there is one zero of $\tilde{\pi}_n$. Since $\tilde{\pi}_{n+1}$ has n pairs of consecutive zeros, the result follows. □

1.3 Classical orthogonal polynomials

Orthogonal polynomials' contribution in Computational Analysis and Approximation Theory is indisputable. In this section, we shall introduce the Chebyshev and Legendre polynomials, providing their three-term recurrence relation, the corresponding weight function and some important formulas and properties. Also, in Table A.1, one can find the recurrence coefficients, weight functions and support intervals of some of the most widely used orthogonal polynomials. To begin with, classical orthogonal polynomials, are neither monic nor orthonormal. We identify the leading coefficients k_n and squared norms h_n, that is,

$$p_n(t) = k_n t^n + \dots, \tag{1.31}$$

$$h_n = \|p_n\|^2, \tag{1.32}$$

where p_n is the respective orthogonal polynomial of degree n.

1.3.1 Chebyshev polynomials

There is almost no area of numerical analysis where Chebyshev polynomials do not drop in like surprise visitors, and indeed there are now a number of subjects in which these polynomials take a significant position in modern developments, including orthogonal polynomials and numerical integration. Here, we make a brief introduction to Chebyshev polynomials, along with some of their properties. For the relevant theory, Mason and Handscomb (2002) (cf. [7]) is the best source.

Definition 1.16. The Chebyshev polynomials $T_n(t)$ of the first kind is a polynomial in t of degree n, defined by the relation

$$T_n(t) = \cos n\theta \quad \text{when } t = \cos\theta, \tag{1.33}$$

so that $T_n(1) = 1$.

Obviously, if the range of the variable t is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$.

Note that

$$\cos 0\theta = 1,\ \ \cos 1\theta = \cos\theta,\ \ \cos 2\theta = 2\cos^2\theta - 1,\ \ \cos 3\theta = 4\cos^3\theta - 3\cos\theta, \ldots,$$

hence, we get

$$T_0(t) = 1,\ \ T_1(t) = t,\ \ T_2(t) = 2t^2 - 1,\ \ T_3(t) = 4t^3 - 3t, \ldots.$$

So, one has that $k_0 = 1$, $k_n = 2^{n-1}$, $n \geq 1$, while $h_0 = \pi$, $h_n = \frac{1}{2}\pi$, $n \geq 1$. Indeed, assuming that $n \geq 1$ and writing $d\lambda(t) = \left(1 - t^2\right)^{-1/2} dt$, the Chebyshev measure of the first kind (Table 1.1), we get

$$h_n = \|T_n\|^2 = \int_{-1}^{1} T_n^2(t) \left(1 - t^2\right)^{-1/2} dt.$$

Setting $t = \cos\theta$, gives

$$\begin{aligned} h_n &= \int_0^{\pi} \cos^2 n\theta \left(1 - \cos^2\theta\right)^{-1/2} \sin\theta d\theta \\ &= \int_0^{\pi} \cos^2 n\theta d\theta = \frac{1}{2}\int_0^{\pi} (\cos 2n\theta + 1)\, d\theta \\ &= \frac{1}{2}\left[\frac{\sin 2n\theta}{2n} + \theta\right]_0^{\pi} = \frac{1}{2}\pi. \end{aligned}$$

If $n = 0$, working similarly as in the case of $n \geq 1$, yields

$$\begin{aligned} h_0 = \|T_0\|^2 = & \int_{-1}^{1} T_0^2(t) \left(1 - t^2\right)^{-1/2} dt \\ = & \int_{-1}^{1} \left(1 - t^2\right)^{-1/2} dt = \int_0^{\pi} 1 d\theta \\ = & \pi. \end{aligned}$$

The zeros of $T_n(t)$, for t in $[-1, 1]$, must correspond to the zeros of $\cos n\theta$, for θ in $[0, \pi]$, that is,

$$n\theta = \frac{2k-1}{2}\pi, \ \ k = 1, 2, \ldots, n.$$

Hence, the zeros of $T_n(t)$ are

$$\tau_k^{(1)} = \cos \frac{2k-1}{2n}\pi, \ \ k = 1, 2, \ldots, n. \tag{1.34}$$

The importance of the Chebyshev polynomials T_n in Approximation Theory stems from the extremal property in the uniform norm

$$\|u\|_\infty = \max_{-1 \leq t \leq 1} |u(t)|,$$

satisfied by the monic Chebyshev polynomial $T_n^\circ = 2^{1-n} T_n$, $n \geq 1$, $T_0^\circ = T_0$,

$$\|p\|_\infty \geq \|T_n^\circ\|_\infty = 2^{1-n} \text{ for all } p \in \mathbb{P}_n^\circ, \tag{1.35}$$

where $\mathbb{P}_n^\circ$ is the class of monic polynomials of degree n. Equality in (1.35) holds if and only if $p = T_n^\circ$.

Definition 1.17. The Chebyshev polynomials $U_n(t)$ of the second kind is a polynomial in t of degree n, defined by the relation

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \text{ when } t = \cos\theta, \tag{1.36}$$

so that $U_n(1) = n + 1$.

The ranges of t and θ are the same as those in the case of $T_n(t)$.

Note that

$$\sin 1\theta = \sin\theta, \ \ \sin 2\theta = 2\sin\theta\cos\theta, \ \ \sin 3\theta = \sin\theta(4\cos^2\theta - 1), \ldots,$$

hence, we get

$$U_0(t) = 1, \ \ U_1(t) = 2t, \ \ U_2(t) = 4t^2 - 1, \ldots.$$

So, one has that $k_n = 2^n$, while $h_n = \frac{1}{2}\pi$. Indeed, writing $d\lambda(t) = \left(1-t^2\right)^{1/2} dt$, the Chebyshev measure of the second kind (Table 1.1), we get

$$h_n = \|U_n\|^2 = \int_{-1}^{1} U_n^2(t) \left(1-t^2\right)^{1/2} dt.$$

Setting $t = \cos\theta$, gives

$$\begin{aligned} h_n &= \int_0^\pi \frac{\sin^2(n+1)\theta}{\sin^2\theta} \left(1-\cos^2\theta\right)^{1/2} \sin\theta d\theta \\ &= \int_0^\pi \sin^2(n+1)\theta d\theta = \frac{1}{2}\int_0^\pi \left[1-\cos 2(n+1)\theta\right] d\theta \\ &= \frac{1}{2}\left[\theta - \frac{\sin 2(n+1)\theta}{2(n+1)}\right]_0^\pi = \frac{1}{2}\pi. \end{aligned}$$

The zeros of $U_n(t)$, for t in $[-1,1]$, must correspond to the zeros of $\sin(n+1)\theta$, for θ in $[0,\pi]$, that is,

$$(n+1)\theta = k\pi, \quad k = 1, 2, \ldots, n.$$

Hence, the zeros of $U_n(t)$ are

$$\tau_k^{(2)} = \cos\frac{k}{n+1}\pi, \quad k = 1, 2, \ldots, n. \tag{1.37}$$

The monic polynomial $U_n^\circ = 2^{-n}U_n$, $n \geq 0$, also satisfies an extremal property, but this time in the L_1-norm

$$\|u\|_1 = \int_{-1}^{1} |u(t)| dt,$$

which is,

$$\|p\|_1 \geq \|U_n^\circ\|_1 \text{ for all } p \in \mathbb{P}_n^\circ. \tag{1.38}$$

Equality in (1.38) holds if and only if $p = U_n^\circ$.

Definition 1.18. The Chebyshev polynomials $V_n(t)$ and $W_n(t)$ of the third and fourth kind, respectively, are polynomials in t of degree n, defined by

$$V_n(t) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{1}{2}\theta} \tag{1.39}$$

and

$$W_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \tag{1.40}$$

where $t = \cos\theta$, so that $V_n(1) = 1$ and $W_n(1) = 2n+1$.

The ranges of t and θ are the same as those in the case of $T_n(t)$.

Similarly, as in the cases of $T_n(t)$ and $U_n(t)$, we can show that

$$V_0(t) = 1,\ \ V_1(t) = 2t - 1,\ \ V_2(t) = 4t^2 - 2t - 1, \ldots$$

and

$$W_0(t) = 1,\ \ W_1(t) = 2t + 1,\ \ W_2(t) = 4t^2 + 2t + 1, \ldots.$$

So, one has that $k_n = 2^n$, while $h_n = \pi$ for both V_n and W_n.

Furthermore, the zeros of $V_n(t)$ and $W_n(t)$ are given by

$$\tau_k^{(3)} = \cos \frac{2k-1}{2n+1}\pi,\ \ k = 1, 2, \ldots, n, \tag{1.41}$$

and

$$\tau_k^{(4)} = \cos \frac{2k}{2n+1}\pi,\ \ k = 1, 2, \ldots, n, \tag{1.42}$$

respectively.

From the evenness/oddness of $U_n(t)$ for n even/odd, we may immediately deduce that

$$\begin{aligned} W_n(t) =& V_n(-t) \qquad \text{(n even)}; \\ W_n(t) =& -V_n(-t) \ \text{(n odd)}, \end{aligned}$$

that is,

$$W_n(t) = (-1)^n V_n(-t). \tag{1.43}$$

This means that the third and fourth kind polynomials essentially transform into each other if the range $[-1, 1]$ of t is reversed, and it is therefore sufficient for us to study only one of these kinds of polynomials.

All four Chebyshev polynomials satisfy the same three-term recurrence relation

$$y_{k+1} = 2ty_k - y_{k-1},\ \ k = 1, 2, \ldots, \tag{1.44}$$

where

$$y_0 = 1 \ \text{ and } \ y_1 = \begin{cases} t & \text{for } T_n(t), \\ 2t & \text{for } U_n(t), \\ 2t-1 & \text{for } V_n(t), \\ 2t+1 & \text{for } W_n(t). \end{cases} \tag{1.45}$$

Multiplying both sides of (1.44) with $2^{-(k+1)}$ for U_{k+1}, V_{k+1} and W_{k+1}, or with 2^{-k} for T_{k+1}, one has that U_k°, V_k° and W_k° satisfy the same three-term recurrence relation

$$y_{k+1}^\circ = ty_k^\circ - \frac{1}{4}y_{k-1}^\circ, \quad k = 1, 2, \ldots, \tag{1.46}$$

while T_{k+1}° satisfy (1.46) for $k \geq 2$ and (1.46) with 1/2 in the place of 1/4 for $k = 1$. Furthermore

$$y_0^\circ = 1 \quad \text{and} \quad y_1^\circ = \begin{cases} t & \text{for } T_n^\circ(t), \\ t & \text{for } U_n^\circ(t), \\ t - \frac{1}{2} & \text{for } V_n^\circ(t), \\ t + \frac{1}{2} & \text{for } W_n^\circ(t). \end{cases} \tag{1.47}$$

Due to their significance, one can collect all of the above information, along with the corresponding weight functions, in the following table.

Kind	y_0	y_1	k_n	h_n	$w(t)$
T_n	1	t	2^{n-1}	π $(n = 0)$ $\frac{1}{2}\pi$ $(n \geq 1)$	$\left(1 - t^2\right)^{-1/2}$
U_n	1	$2t$	2^n	$\frac{1}{2}\pi$	$\left(1 - t^2\right)^{1/2}$
V_n	1	$2t - 1$	2^n	π	$(1 - t)^{-1/2}(1 + t)^{1/2}$
W_n	1	$2t + 1$	2^n	π	$(1 - t)^{1/2}(1 + t)^{-1/2}$

Table 1.1: Chebyshev polynomials

1.3.2 Legendre polynomials

Legendre polynomials arise from the orthogonalisation (Gram-Schmidt) process for polynomials with the weight function $w(t) = 1$ on $[-1, 1]$. The usual notation for the nth-degree Legendre polynomial is P_n and corresponds to the normalization $P_n(1) = 1$.

Remark 1.19. Consider the Jacobi polynomials, which are denoted by $P_n^{(a,\beta)}$ and normalized by $P_n^{(a,\beta)}(1) = \binom{n+a}{n}$, and the Gegenbauer polynomials which are defined in terms of the Jacobi polynomials $P_n^{(a,\beta)}$ by

$$P_n^{(\lambda)}(t) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma\left(n + \lambda + \frac{1}{2}\right)} P_n^{\left(\lambda - \frac{1}{2}, \lambda - \frac{1}{2}\right)}(t), \quad \lambda \neq 0, \tag{1.48}$$

where Γ is the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The Legendre polynomials are special case of the Jacobi polynomials with $a = \beta = 0$. They are, also, special case of the Gegenbauer polynomials with $\lambda = 1/2$, that is,

$$P_n(t) = P_n^{(0,0)}(t) = P_n^{(1/2)}(t). \tag{1.49}$$

The first few Legendre polynomials are

$$\begin{aligned}
P_0(t) &= 1, \\
P_1(t) &= t, \\
P_2(t) &= \frac{1}{2}(3t^2 - 1), \\
P_3(t) &= \frac{1}{2}(5t^3 - 3t), \\
P_4(t) &= \frac{1}{8}(35t^4 - 30t^2 + 3), \\
&\vdots
\end{aligned}$$

One has that

$$k_n = \frac{(2n)!}{2^n (n!)^2} \tag{1.50}$$

and

$$h_n = \frac{1}{n + \frac{1}{2}}. \tag{1.51}$$

The Legendre polynomials P_n satisfy the three-term recurrence relation

$$\begin{gathered}
(k+1)P_{k+1}(t) = (2k+1)tP_k(t) - kP_{k-1}(t), \;\; k = 1, 2, \ldots, \\
P_0(t) = 1, \;\; P_1(t) = t.
\end{gathered} \tag{1.52}$$

Remark 1.20. Each P_n is bounded by 1 on $[-1, 1]$.

Chapter 2

Quadrature Formulae

2.1 Introduction

Let $d\lambda$ be a measure with bounded or unbounded support, which shall be considered to be positive. An n-point quadrature formula for the measure $d\lambda$ is a formula of the type

$$\int_{\mathbb{R}} f(t)d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R_n(f), \tag{2.1}$$

where the sum on the right provides an approximation for the integral on the left and R_n is the respective error. The τ_ν, assumed distinct, are called the nodes, and λ_ν the weights of the quadrature formula.

Definition 2.1. The quadrature formula (2.1) is said to have degree of exactness d if

$$R_n(p) = 0 \text{ for } p \in \mathbb{P}_d. \tag{2.2}$$

It is said to have precise degree of exactness d, if it has degree of exactness d but not $d+1$, that is, if (2.2) holds, but $R_n(p) \neq 0$ for some $p \in \mathbb{P}_{d+1}$.

2.2 Interpolatory quadrature formulae

Consider the Lagrange interpolation formula

$$f(t) = \sum_{\nu=1}^{n} f(\tau_\nu)\ell_\nu(t) + r_{n-1}(f;t), \tag{2.3}$$

where ℓ_ν are the Lagrange polynomials, defined by

$$\ell_\nu(t) = \prod_{\substack{\mu=1\\ \mu\neq\nu}}^{n} \frac{t-\tau_\mu}{\tau_\nu-\tau_\mu}. \tag{2.4}$$

Note that

$$\ell_\nu(\tau_\mu) = \delta_{\nu\mu} = \begin{cases} 1, & \mu=\nu, \\ 0, & \mu\neq\nu. \end{cases} \tag{2.5}$$

Multiplying both sides of (2.3) with the measure $d\lambda(t)$ and integrating gives

$$\begin{aligned} \int_{\mathbb{R}} f(t)d\lambda(t) &= \int_{\mathbb{R}} \left[\sum_{\nu=1}^{n} f(\tau_\nu)\ell_\nu(t) + r_{n-1}(f;t)\right] d\lambda(t) \\ &= \int_{\mathbb{R}} \sum_{\nu=1}^{n} f(\tau_\nu)\ell_\nu(t)d\lambda(t) + \int_{\mathbb{R}} r_{n-1}(f;t)d\lambda(t) \\ &= \sum_{\nu=1}^{n} \left[\int_{\mathbb{R}} \ell_\nu(t)d\lambda(t)\right] f(\tau_\nu) + \int_{\mathbb{R}} r_{n-1}(f;t)d\lambda(t). \end{aligned}$$

Hence, setting

$$\lambda_\nu = \int_{\mathbb{R}} \ell_\nu(t)d\lambda(t) = \int_{\mathbb{R}} \prod_{\substack{\mu=1\\ \mu\neq\nu}}^{n} \frac{t-\tau_\mu}{\tau_\nu-\tau_\mu} d\lambda(t), \ \ \nu=1,2,\dots,n, \tag{2.6}$$

and

$$R_n(f) = \int_{\mathbb{R}} r_{n-1}(f;t)d\lambda(t), \tag{2.7}$$

we obtain (2.1), which in this case is called interpolatory as it was obtained by interpolation. Such a formula has degree of exactness (at least) $d = n-1$, as if $p \in \mathbb{P}_{n-1}$, then $r_{n-1}(p;t) = 0$, which gives $R_n(p) = 0$.

Given a set of n distinct points τ_ν, one can construct an interpolatory formula having as nodes the τ_ν. The following theorem provides conditions on the τ_ν, such that an interpolatory formula has degree of exactness greater than $n-1$. First, we introduce the node polynomial

$$\omega_n(t) = \prod_{\nu=1}^{n} (t-\tau_\nu). \tag{2.8}$$

Theorem 2.2. ([4], Theorem 3.2.1) Given an integer k with $0 \le k \le n$, the quadrature formula (2.1) has degree of exactness $d = n-1+k$ if and only if both of the following conditions are satisfied:

(a) Formula (2.1) is interpolatory.

(b) The node polynomial (2.8) satisfies

$$\int_{\mathbb{R}} \omega_n(t)p(t)d\lambda(t) = 0 \text{ for all } p \in \mathbb{P}_{k-1}.$$

Proof. Assume that the formula has degree o exactness $n-1+k$. As $n-1+k \geq n-1$, it has degree of exactness at least $n-1$, so formula (2.1) is interpolatory, which proves (a).

Now, let $p \in \mathbb{P}_{k-1}$. Then $\omega_n p \in \mathbb{P}_{n-1+k}$, as $\omega_n \in \mathbb{P}_n$. Therefore,

$$\int_{\mathbb{R}} \omega_n(t)p(t)d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu \omega_n(\tau_\nu)p(\tau_\nu) = 0,$$

since the formula has degree of exactness $d = n-1+k$ and $\omega_n(\tau_\nu) = 0$. This proves the necessity of (b).

Conversely, if (a) and (b) hold, one must show that under these conditions formula (2.1) has degree of exactness $n-1+k$, that is, $R_n(p) = 0$ for any $p \in \mathbb{P}_{n-1+k}$. Let $p \in \mathbb{P}_{n-1+k}$ and dividing p by ω_n, we have

$$p = q\omega_n + r, \quad q \in \mathbb{P}_{k-1}, \quad r \in \mathbb{P}_{n-1},$$

so,

$$\begin{aligned} \int_{\mathbb{R}} p(t)d\lambda(t) &= \int_{\mathbb{R}} \big[q(t)\omega_n(t) + r(t)\big] d\lambda(t) \\ &= \int_{\mathbb{R}} q(t)\omega_n(t)d\lambda(t) + \int_{\mathbb{R}} r(t)d\lambda(t). \end{aligned}$$

Since, $q \in \mathbb{P}_{k-1}$ and $\omega_n \in \mathbb{P}_n$, because of (b), the first integral on the right equals to zero. For the second integral, we have, as $r \in \mathbb{P}_{n-1}$, by virtue of (a),

$$\int_{\mathbb{R}} r(t)d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu r(\tau_\nu).$$

Note that

$$r(\tau_\nu) = p(\tau_\nu) - q(\tau_\nu)\omega_n(\tau_\nu) = p(\tau_\nu),$$

as $\omega_n(\tau_\nu) = 0$. Finally,

$$\int_{\mathbb{R}} p(t)d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu p(\tau_\nu),$$

that is $R_n(p) = 0$. $\square$

Condition (b) requires ω_n to be orthogonal to polynomials of degree $k-1$ relative to the measure $d\lambda$. Since $d\lambda$ is a positive measure, choosing $k=n$ in Theorem 2.2 is optimal; indeed, the choice $k=n+1$ would require ω_n to be orthogonal to all polynomials of degree at most n, in particular, orthogonal to itself, which is obviously impossible.

The optimal quadrature formula (2.1) with $k=n$, having degree of exactness $2n-1$, is called the Gauss quadrature formula, with respect to the measure $d\lambda$. Condition (b) in Theorem 2.2 shows that ω_n is the nth-degree orthogonal polynomial relative to the measure $d\lambda$, that is, $\omega_n(t)=\pi_n(t;d\lambda)$. Hence, the nodes τ_ν of the Gauss quadrature formula are the zeros of π_n, while the weights λ_ν can be computed by (2.6).

2.3 Gauss quadrature formulae

The n-point Gauss quadrature formula will be written in the form

$$\int_{\mathbb{R}} f(t)d\lambda(t)=\sum_{\nu=1}^{n}\lambda_\nu^G f\left(\tau_\nu^G\right)+R_n^G(f), \tag{2.9}$$

with

$$R_n^G\left(\mathbb{P}_{2n-1}\right)=0. \tag{2.10}$$

2.3.1 Properties of the Gauss quadrature formulae

The Gauss formula (2.9), apart from being optimal as to the degree of exactness, has a number of other remarkable properties, which shall be presented in the following theorems (cf. [1] and [4]).

Theorem 2.3. All nodes $\tau_\nu=\tau_\nu^G$ are distinct and contained in the interior of the support interval $[a,b]$ of $d\lambda$.

Proof. This is a property of orthogonal polynomials and follows from Theorem 1.14, since τ_ν are the zeros of $\pi_n(\cdot;d\lambda)$.

□

Theorem 2.4. All weights $\lambda_\nu=\lambda_\nu^G$ are positive.

Proof. Set $f=\ell_\mu^2$ in (2.9), with ℓ_μ given by (2.4), and note that

$$\int_{\mathbb{R}} \ell_\mu^2(t)d\lambda(t)>0. \tag{2.11}$$

As $\ell_\mu^2 \in \mathbb{P}_{2n-2} \subset \mathbb{P}_{2n-1}$, formula (2.9) is exact, that is, $R_n\left(\ell_\mu^2\right) = 0$, $\mu = 1, 2, \ldots, n$. Hence,

$$\int_{\mathbb{R}} \ell_\mu^2(t) d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu \ell_\mu^2(\tau_\nu) = \sum_{\nu=1}^{n} \lambda_\nu \delta_{\mu\nu}^2 = \lambda_\mu, \;\; \mu = 1, 2, \ldots, n. \tag{2.12}$$

Combining (2.11) and (2.12), proves our assertion. □

Theorem 2.5. If $[a, b]$ is a finite interval, then formula (2.9) converges for any continuous function f in $[a, b]$, that is, $R_n(f) \to 0$ as $n \to \infty$.

Proof. Let f be a continuous function in $[a, b]$ and $\hat{p}_{2n-1}(f; \cdot)$ be the polynomial of degree $2n - 1$ that approximates f best on $[a, b]$ from $\mathbb{P}_{2n-1}$. By the Weierstrass Approximation Theorem, there holds

$$\lim_{n \to \infty} \|f(\cdot) - \hat{p}_{2n-1}\left(f; \cdot\right)\|_\infty = 0. \tag{2.13}$$

Since $\hat{p}_{2n-1} \in \mathbb{P}_{2n-1}$, formula (2.9) is exact, so $R_n\left(\hat{p}_{2n-1}\right) = 0$. As the measure $d\lambda$ is positive and $\lambda_\nu > 0$, $\nu = 1, 2, \ldots, n$, one has that

$$\begin{aligned}
|R_n(f)| &= |R_n(f) - R_n(\hat{p}_{2n-1})| = |R_n\left(f - \hat{p}_{2n-1}\right)| \\
&= \left| \int_a^b \left[f(t) - \hat{p}_{2n-1}(f; t)\right] d\lambda(t) - \sum_{\nu=1}^{n} \lambda_\nu \left[f(\tau_\nu) - \hat{p}_{2n-1}(f; \tau_\nu)\right] \right| \\
&\le \int_a^b |f(t) - \hat{p}_{2n-1}(f; t)| \, d\lambda(t) + \sum_{\nu=1}^{n} \lambda_\nu \left|f(\tau_\nu) - \hat{p}_{2n-1}(f; \tau_\nu)\right| \\
&\le \|f(\cdot) - \hat{p}_{2n-1}(f; \cdot)\|_\infty \left[\int_a^b d\lambda(t) + \sum_{\nu=1}^{n} \lambda_\nu\right].
\end{aligned}$$

Note that $1 = t^0 \in \mathbb{P}_0$, so formula (2.9) is again exact, that is,

$$\int_a^b d\lambda(t) = \int_a^b 1 \cdot d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu \cdot 1 = \sum_{\nu=1}^{n} \lambda_\nu,$$

and, by virtue of (1.1),

$$\int_a^b d\lambda(t) = \mu_0.$$

Hence, from (2.13),

$$|R_n(f)| \le 2\mu_0 \|f(\cdot) - \hat{p}_{2n-1}(f; \cdot)\|_\infty \to 0 \text{ as } n \to \infty.$$

□

Theorem 2.6. The nodes of the $(n+1)$-point formula (2.9) alternate with those of the n-point formula (2.9), that is,

$$\tau_{n+1}^{(n+1)} < \tau_n^{(n)} < \tau_n^{(n+1)} < \tau_{n-1}^{(n)} < \cdots < \tau_1^{(n)} < \tau_1^{(n+1)}, \tag{2.14}$$

where $\tau_i^{(n+1)}$ and $\tau_j^{(n)}$ are the nodes in descending order of the $(n+1)$-point and the n-point formula (2.9), respectively.

Proof. The result follows from Theorem 1.15, since $\tau_i^{(n+1)}$ and $\tau_j^{(n)}$ are the zeros in descending order of π_{n+1} and π_n, respectively. □

In Chapter 1, we proved that if $\pi_k(\cdot) = \pi_k(\cdot; d\lambda)$, $k = 0, 1, 2, \ldots$, is the monic orthogonal polynomial with respect to the measure $d\lambda$, then (1.11) holds,

$$\begin{aligned} \pi_{k+1}(t) =& (t - a_k)\pi_k(t) - \beta_k \pi_{k-1}(t), \;\; k = 0, 1, 2, \ldots, \\ & \pi_{-1}(t) = 0, \;\; \pi_0(t) = 1, \end{aligned}$$

where a_k and β_k are given by (1.12) and (1.13), and β_0 was defined, conventionally, in (1.17) as

$$\beta_0 = \int_{\mathbb{R}} d\lambda(t).$$

Also, we defined the $n \times n$ Jacobi matrix (1.21), which is given by

$$J_n = J_n(d\lambda) = \begin{bmatrix} a_0 & \sqrt{\beta_1} & 0 & \ldots & 0 \\ \sqrt{\beta_1} & a_1 & \sqrt{\beta_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \sqrt{\beta_{n-1}} & a_{n-1} \end{bmatrix}.$$

In Theorem 1.12, we proved that the zeros $\tau_k^{(n)}$ of $\pi_n(\cdot; d\lambda)$ (or $\tilde{\pi}_n(\cdot; d\lambda)$), i.e., the nodes of the Gauss formula (2.9), are the eigenvalues of the Jacobi matrix J_n, and $\tilde{\pi}\left(\tau_k^{(n)}\right)$ are the corresponding eigenvectors, where

$$\tilde{\pi}(t) = \left[\tilde{\pi}_0(t), \tilde{\pi}_1(t), \ldots, \tilde{\pi}_{n-1}(t)\right]^T.$$

Dividing $\tilde{\pi}\left(\tau_k^{(n)}\right)$ by $\left\|\tilde{\pi}\left(\tau_k^{(n)}\right)\right\|_2$, yields the normalized eigenvector v_k, that is,

$$v_k = \frac{\tilde{\pi}\left(\tau_k^{(n)}\right)}{\left\|\tilde{\pi}\left(\tau_k^{(n)}\right)\right\|_2} = \frac{\tilde{\pi}\left(\tau_k^{(n)}\right)}{\left(\sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2\right)^{1/2}}, \tag{2.15}$$

where $\|\cdot\|_2$ is the euclidean norm on $\mathbb{R}^n$.

Corollary 2.7. The weights λ_k of the Gauss formula (2.9) can be expressed in terms of the first component $v_{k,1}$ of the corresponding normalized eigenvectors v_k, by

$$\lambda_k = \beta_0 v_{k,1}^2, \quad k = 1, 2, \ldots, n. \tag{2.16}$$

Proof. Writing (2.15) in vector form

$$\begin{bmatrix} v_{k,1} \\ v_{k,2} \\ \vdots \\ v_{k,n-1} \end{bmatrix} = \frac{1}{\left(\sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2\right)^{1/2}} \begin{bmatrix} \tilde{\pi}_0\left(\tau_k^{(n)}\right) \\ \tilde{\pi}_1\left(\tau_k^{(n)}\right) \\ \vdots \\ \tilde{\pi}_{n-1}\left(\tau_k^{(n)}\right) \end{bmatrix},$$

and comparing the first components of both sides, gives

$$v_{k,1} = \frac{\tilde{\pi}_0\left(\tau_k^{(n)}\right)}{\left(\sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2\right)^{1/2}}. \tag{2.17}$$

Since $\tilde{\pi}_0(t) = \frac{1}{\sqrt{\beta_0}}$, squaring both sides of (2.17), we get

$$\beta_0 v_{k,1}^2 = \frac{1}{\sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2}. \tag{2.18}$$

Noting that $\tilde{\pi}_i\tilde{\pi}_j \in \mathbb{P}_{2n-2} \subset \mathbb{P}_{2n-1}$, $i, j = 0, 1, \ldots, n-1$, the Gauss formula (2.9) must be exact, that is,

$$\int_{\mathbb{R}} \tilde{\pi}_i(t)\tilde{\pi}_j(t)d\lambda(t) = \sum_{k=1}^{n} \lambda_k \tilde{\pi}_i\left(\tau_k^{(n)}\right)\tilde{\pi}_j\left(\tau_k^{(n)}\right), \quad i, j = 0, 1, \ldots, n-1. \tag{2.19}$$

Since $\tilde{\pi}_i$ and $\tilde{\pi}_j$ are orthonormal with respect to the measure $d\lambda$, there holds, by (1.7),

$$\int_{\mathbb{R}} \tilde{\pi}_i(t)\tilde{\pi}_j(t)d\lambda(t) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n-1. \tag{2.20}$$

Combining (2.19) and (2.20), we obtain

$$\sum_{k=1}^{n} \lambda_k \tilde{\pi}_i\left(\tau_k^{(n)}\right)\tilde{\pi}_j\left(\tau_k^{(n)}\right) = \delta_{ij}, \quad i, j = 0, 1, \ldots, n-1, \tag{2.21}$$

which can be written in matrix form as

$$\Pi^T \Lambda \Pi = I, \tag{2.22}$$

where

$$\Pi = \begin{bmatrix} \tilde{\pi}_0\left(\tau_1^{(n)}\right) & \tilde{\pi}_1\left(\tau_1^{(n)}\right) & \dots & \tilde{\pi}_{n-1}\left(\tau_1^{(n)}\right) \\ \tilde{\pi}_0\left(\tau_2^{(n)}\right) & \tilde{\pi}_1\left(\tau_2^{(n)}\right) & \dots & \tilde{\pi}_{n-1}\left(\tau_2^{(n)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\pi}_0\left(\tau_n^{(n)}\right) & \tilde{\pi}_1\left(\tau_n^{(n)}\right) & \dots & \tilde{\pi}_{n-1}\left(\tau_n^{(n)}\right) \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

Theorem 2.4 states that all weights λ_k are positive, so Λ is invertible. By virtue of (2.22), we have

$$\det\left(\Pi^T \Lambda \Pi\right) = \det\left(I\right) = 1,$$

which is equivalent to

$$\det\left(\Pi^T\right)\det\left(\Lambda\right)\det\left(\Pi\right) = 1.$$

Hence, Π and Π^T are also invertible, so by (2.22),

$$\Lambda\Pi = \left(\Pi^T\right)^{-1},$$

that is,

$$\Lambda = \left(\Pi^T\right)^{-1}\Pi^{-1} = \left(\Pi\Pi^T\right)^{-1}.$$

Therefore,

$$\Lambda^{-1} = \Pi\Pi^T,$$

where

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda_n} \end{bmatrix},$$

from which there follows that

$$\frac{1}{\lambda_k} = \sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2,$$

that is,

$$\lambda_k = \frac{1}{\sum_{i=0}^{n-1}\left[\tilde{\pi}_i\left(\tau_k^{(n)}\right)\right]^2}. \tag{2.23}$$

Now, (2.23), together with (2.18), proves our assertion. □

Theorem 2.8. ([3], Theorem 1.48) Given the Hermite interpolation polynomial $p_{2n-1}(f;\cdot)$ of degree $2n-1$, which satisfies

$$p_{2n-1}\left(f;\tau_\nu^G\right) = f\left(\tau_\nu^G\right),\ \ p'_{2n-1}\left(f;\tau_\nu^G\right) = f'\left(\tau_\nu^G\right),\ \ \nu = 1,2,\ldots,n, \tag{2.24}$$

there holds

$$\int_{\mathbb{R}} p_{2n-1}(f;t)d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^G f\left(\tau_\nu^G\right). \tag{2.25}$$

Proof. From properties of the Hermite interpolation polynomials, writing $\tau_\nu = \tau_\nu^G$, one has

$$p_{2n-1}(f;t) = \sum_{\nu=1}^{n} \left[a_\nu(t)f(\tau_\nu) + \beta_\nu(t)f'(\tau_\nu)\right], \tag{2.26}$$

where

$$\begin{aligned} a_\nu(t) &= \left(1 - 2(t-\tau_\nu)\ell'_\nu(\tau_\nu)\right)\ell_\nu^2(t), \\ \beta_\nu(t) &= (t-\tau_\nu)\ell_\nu^2(t), \end{aligned} \tag{2.27}$$

and ℓ_ν are the Lagrange polynomials satisfying (2.4) and (2.5). Multiplying both sides of (2.26) with the measure $d\lambda(t)$ and then integrating, gives

$$\begin{aligned} \int_{\mathbb{R}} p_{2n-1}(f;t)d\lambda(t) &= \int_{\mathbb{R}} \left\{\sum_{\nu=1}^{n} \left[a_\nu(t)f(\tau_\nu) + \beta_\nu(t)f'(\tau_\nu)\right]\right\} d\lambda(t) \\ &= \sum_{\nu=1}^{n} \left\{\left[\int_{\mathbb{R}} a_\nu(t)d\lambda(t)\right] f(\tau_\nu) + \left[\int_{\mathbb{R}} \beta_\nu(t)d\lambda(t)\right] f'(\tau_\nu)\right\}. \end{aligned}$$

Formula (2.9) shall be exact, since $a_\nu, \beta_\nu \in \mathbb{P}_{2n-1}$, so,

$$\begin{aligned} \int_{\mathbb{R}} a_\nu(t)d\lambda(t) &= \sum_{\mu=1}^{n} \lambda_\mu a_\nu(\tau_\mu) = \sum_{\mu=1}^{n} \left[\lambda_\mu \left(1 - 2(\tau_\mu - \tau_\nu)\ell'_\nu(\tau_\nu)\right)\ell_\nu^2(\tau_\mu)\right] \\ &= \sum_{\mu=1}^{n} \left[\lambda_\mu \left(1 - 2(\tau_\mu - \tau_\nu)\ell'_\nu(\tau_\nu)\right)\delta_{\nu\mu}^2\right] = \lambda_\nu \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \beta_\nu(t)d\lambda(t) &= \sum_{\mu=1}^{n} \lambda_\mu \beta_\nu(\tau_\mu) = \sum_{\mu=1}^{n} \left[(\tau_\mu - \tau_\nu)\ell_\nu^2(\tau_\mu)\right] = \\ &= \sum_{\mu=1}^{n} \left[(\tau_\mu - \tau_\nu)\delta_{\nu\mu}^2\right] = 0, \end{aligned}$$

hence, our assertion follows. □

Corollary 2.9. If $f \in C^{2n}[a,b]$, then the remainder $R_n^G(f)$ in formula (2.9) can be expressed as

$$R_n^G(f) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_{\mathbb{R}} [\pi_n(t; d\lambda)]^2 \, d\lambda(t), \;\; \xi \in (a,b). \tag{2.28}$$

Proof. For functions $f \in C^{2n}[a,b]$, it is well known from the Hermite interpolation theory that

$$f(t) = p_{2n-1}(f;t) + r_{2n-1}(f;t), \tag{2.29}$$

where

$$r_{2n-1}(f;t) = \frac{f^{(2n)}(\zeta(t))}{(2n)!} \prod_{\nu=1}^{n} (t-\tau_\nu)^2, \;\; \zeta(t) \in (a,b). \tag{2.30}$$

Multiplying both sides of (2.29) with the measure $d\lambda(t)$ and then integrating, gives

$$\begin{aligned}\int_{\mathbb{R}} f(t) d\lambda(t) &= \int_{\mathbb{R}} [p_{2n-1}(f;t) + r_{2n-1}(f;t)] \, d\lambda(t) \\ &= \int_{\mathbb{R}} p_{2n-1}(f;t) d\lambda(t) + \int_{\mathbb{R}} r_{2n-1}(f;t) d\lambda(t),\end{aligned}$$

hence, from (2.25),

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{\nu=1}^{n} \lambda_\nu^G f\left(\tau_\nu^G\right) + \int_{\mathbb{R}} r_{2n-1}(f;t) d\lambda(t). \tag{2.31}$$

Now, comparing (2.31) with the Gauss formula (2.9), yields that

$$R_n^G(f) = \int_{\mathbb{R}} r_{2n-1}(f;t) d\lambda(t) = \int_{\mathbb{R}} \frac{f^{(2n)}(\zeta(t))}{(2n)!} \prod_{\nu=1}^{n} (t-\tau_\nu)^2 d\lambda(t).$$

By the Mean Value Theorem of integration, there exists a $\xi \in (a,b)$ such that

$$\int_{\mathbb{R}} \frac{f^{(2n)}(\zeta(t))}{(2n)!} \prod_{\nu=1}^{n} (t-\tau_\nu)^2 d\lambda(t) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_{\mathbb{R}} \prod_{\nu=1}^{n} (t-\tau_\nu)^2 d\lambda(t).$$

Noting that $\tau_\nu = \tau_\nu^G$, so,

$$\prod_{\nu=1}^{n} (t-\tau_\nu) = \prod_{\nu=1}^{n} \left(t - \tau_\nu^G\right) = \pi_n(t; d\lambda),$$

we finally obtain (2.28). □

2.3.2 Gauss-Chebyshev quadrature formulae

Consider the Gauss formula (2.9) with $d\lambda(t) = \left(1-t^2\right)^{-1/2} dt$, the Chebyshev measure of the first kind (Table A.1). The nodes $\tau_\nu = \tau_\nu^{(1)}$ are the zeros of the Chebyshev polynomial T_n, given by

$$\tau_\nu = \cos\theta_\nu, \;\; \theta_\nu = \frac{2\nu-1}{2n}\pi, \;\; \nu = 1, 2, \ldots, n. \tag{2.32}$$

The weights $\lambda_\nu = \lambda_\nu^{(1)}$ can be computed by (2.6),

$$\lambda_\nu = \int_{-1}^{1} \ell_\nu(t)\big(1-t^2\big)^{-1/2} dt, \ \ \nu = 1, 2, \ldots, n,$$

where $\ell_\nu(t)$ can be written as

$$\ell_\nu(t) = \prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{n} \frac{t-\tau_\mu}{\tau_\nu-\tau_\mu}.$$

Noting that

$$\prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{n} (t-\tau_\mu) = \prod_{\mu=1}^{n} (t-\tau_\mu) \cdot \frac{1}{t-\tau_\nu} = \frac{T_n(t)}{t-\tau_\nu}, \tag{2.33}$$

and letting $t \to \tau_\nu$, as $T_n(\tau_\nu) = 0$, $\nu = 1, 2, \ldots, n$, gives

$$\prod_{\substack{\mu=1 \\ \mu\neq\nu}}^{n} (\tau_\nu-\tau_\mu) = T_n'(\tau_\nu). \tag{2.34}$$

Now, (2.33), together with (2.34), implies that

$$\ell_\nu(t) = \frac{T_n(t)}{(t-\tau_\nu)T_n'(\tau_\nu)},$$

hence,

$$\lambda_\nu = \frac{1}{T_n'(\tau_\nu)} \int_{-1}^{1} \frac{T_n(t)}{t-\tau_\nu} \big(1-t^2\big)^{-1/2} dt, \ \ \nu = 1, 2, \ldots, n. \tag{2.35}$$

In order to compute the $T_n'(\tau_\nu)$, we use the trigonometric representation for T_n and the formula for the zeros. Indeed, as $T_n(\cos\theta) = \cos n\theta$, there holds

$$T_n'(\cos\theta) = n\frac{\sin n\theta}{\sin\theta},$$

which leads to

$$T_n'(\tau_\nu) = n\frac{\sin n\theta_\nu}{\sin\theta_\nu} = n\frac{(-1)^{\nu+1}}{\sin\theta\nu}, \ \ \nu = 1, 2, \ldots, n. \tag{2.36}$$

For the computation of the integral in (2.35), we use the Christoffel-Darboux formula (1.24). Noting that $\tilde{T_n} = T_n/\|T_n\|$, where $\|T_n\|^2 = h_n$ (Table 1.1) and β_{n+1} is the recurrence coefficient which can be computed by (1.13), yields

$$\frac{T_{n+1}(t)T_n(\tau_\nu) - T_n(t)T_{n+1}(\tau_\nu)}{\pi(t-\tau_\nu)} = \sum_{k=0}^{n} \frac{T_k(t)T_k(\tau_\nu)}{h_k},$$

which, since $T_n(\tau_\nu) = 0$, $\nu = 1, 2, \dots, n$, takes the form

$$-\frac{T_n(t)T_{n+1}(\tau_\nu)}{\pi(t-\tau_\nu)} = \sum_{k=0}^{n} \frac{T_k(t)T_k(\tau_\nu)}{h_k}.$$

So,

$$\begin{aligned}\frac{T_n(t)}{t-\tau_\nu} &= -\frac{\pi}{T_{n+1}(\tau_\nu)}\left[\frac{T_0(t)T_0(\tau_\nu)}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n} T_k(t)T_k(\tau_\nu)\right] \\ &= -\frac{\pi}{T_{n+1}(\tau_\nu)}\left[\frac{1}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n} T_k(t)T_k(\tau_\nu)\right],\end{aligned}$$

and integrating both sides with respect to the measure $\left(1-t^2\right)^{-1/2}dt$, gives

$$\begin{aligned}&\int_{-1}^{1} \frac{T_n(t)}{t-\tau_\nu}\left(1-t^2\right)^{-1/2} dt \\ &= -\frac{\pi}{T_{n+1}(\tau_\nu)}\left[\frac{1}{\pi}\int_{-1}^{1}\left(1-t^2\right)^{-1/2} dt + \frac{2}{\pi}\sum_{k=1}^{n} T_k(\tau_\nu)\int_{-1}^{1} T_k(t)\left(1-t^2\right)^{-1/2} dt\right].\end{aligned}$$

Note that the first integral on the right-hand side equals to π, while the second integral vanishes, due to the orthogonality of the Chebyshev polynomial T_k, hence,

$$\int_{-1}^{1} \frac{T_n(t)}{t-\tau_\nu}\left(1-t^2\right)^{-1/2} dt = -\frac{\pi}{T_{n+1}(\tau_\nu)}. \tag{2.37}$$

Now, one can compute that

$$\begin{aligned}T_{n+1}(\tau_\nu) &= \cos(n+1)\frac{2\nu-1}{2n}\pi \\ &= \cos n\frac{2\nu-1}{2n}\pi\cos\theta_\nu - \sin n\frac{2\nu-1}{2n}\pi\sin\theta_\nu \\ &= -(-1)^{\nu+1}\sin\theta_\nu.\end{aligned}$$

The latter, combined with (2.35), (2.36) and (2.37), yields

$$\lambda_\nu = \frac{\pi}{n}, \quad \nu = 1, 2, \dots, n, \tag{2.38}$$

thus, the Gauss-Chebyshev quadrature formula with the Chebyshev measure of the first kind, or the so-called Gauss-Chebyshev quadrature formula of the first kind, is

$$\int_{-1}^{1} f(t)\left(1-t^2\right)^{-1/2} dt = \frac{\pi}{n}\sum_{\nu=1}^{n} f\left(\cos\frac{2\nu-1}{2n}\pi\right) + R_n^G(f). \tag{2.39}$$

One can, also, get (2.38) and subsequently (2.39), by noting that (2.9) must be exact for $f(t) = T_k(t)$, $k = 0, 1, \ldots, n-1$, since (2.9) is interpolatory and has degree of exactness $d = n-1$. As $\tau_\nu = \cos\theta_\nu$, one has

$$\int_{-1}^{1} T_k(t)\left(1-t^2\right)^{-1/2} dt = \sum_{\nu=1}^{n} \lambda_\nu T_k(\tau_\nu), \quad k = 0, 1, \ldots, n-1,$$

that is,

$$\pi\delta_{k0} = \sum_{\nu=1}^{n} \lambda_\nu \cos k\theta_\nu, \quad k = 0, 1, \ldots, n-1, \tag{2.40}$$

due to the orthogonality of the Chebyshev polynomial T_n.

Lemma 2.10. With θ_ν and θ_μ given by (2.32), there holds

$$\sum_{k=0}^{n-1}{}' \cos k\theta_\nu \cos k\theta_\mu = \frac{1}{2} n\delta_{\nu\mu}, \quad \nu, \mu = 1, 2, \ldots, n, \tag{2.41}$$

where the prime means that the first term (for $k = 0$) is to be halved.

Proof. In view of (2.32), writing

$$\begin{aligned}\cos k\theta_\nu \cos k\theta_\mu =& \frac{1}{2}\left(\cos k(\theta_\nu - \theta_\mu) + \cos k(\theta_\nu + \theta_\mu)\right)\\ =& \frac{1}{2}\left(\cos k\frac{\nu-\mu}{n}\pi + \cos k\frac{\nu+\mu-1}{n}\pi\right), \quad \nu, \mu = 1, 2, \ldots, n,\end{aligned}$$

and using Euler's formula for the cosine, we get

$$\cos k\frac{\nu-\mu}{n}\pi = \frac{1}{2}\left(e^{ik\frac{\nu-\mu}{n}\pi} + e^{-ik\frac{\nu-\mu}{n}\pi}\right)$$

and

$$\cos k\frac{\nu+\mu-1}{n}\pi = \frac{1}{2}\left(e^{ik\frac{\nu+\mu-1}{n}\pi} + e^{-ik\frac{\nu+\mu-1}{n}\pi}\right).$$

So,

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \cos k\theta_\nu \, \cos k\theta_\mu =& \frac{1}{4}\sum_{k=0}^{n-1}{}' \left[e^{ik\frac{\nu-\mu}{n}\pi} + e^{-ik\frac{\nu-\mu}{n}\pi} + e^{ik\frac{\nu+\mu-1}{n}\pi} + e^{-ik\frac{\nu+\mu-1}{n}\pi}\right]\\ =& \frac{1}{4}\sum_{k=0}^{n-1}{}' \left[\left(e^{i\frac{\nu-\mu}{n}\pi}\right)^k + \left(e^{-i\frac{\nu-\mu}{n}\pi}\right)^k + \left(e^{i\frac{\nu+\mu-1}{n}\pi}\right)^k + \left(e^{-i\frac{\nu+\mu-1}{n}\pi}\right)^k\right]\\ =& \frac{1}{4}\left[\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu-\mu}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{\mu-\nu}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu+\mu-1}{n}\pi}\right)^k\right.\\ &\left. + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{1-\nu-\mu}{n}\pi}\right)^k\right].\end{aligned}$$

Note that

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \left(e^{i\frac{x}{n}\pi}\right)^k &= \sum_{k=0}^{n-1} \left(e^{i\frac{x}{n}\pi}\right)^k - \frac{1}{2} \\ &= \frac{1-\left(e^{i\frac{x}{n}\pi}\right)^n}{1-e^{i\frac{x}{n}\pi}} - \frac{1}{2} \\ &= \frac{1-e^{ix\pi}}{1-e^{i\frac{x}{n}\pi}} - \frac{1}{2},\end{aligned} \tag{2.42}$$

where $x = \nu-\mu$, $\mu-\nu$, $\nu+\mu-1$ or $1-\nu-\mu$, $\nu,\mu = 1,2,\ldots,n$. Now, let $\nu=\mu$, so, $\nu-\mu=0$, $\mu-\nu=0$, $\nu+\mu-1 = 2\mu-1$ and $1-\nu-\mu = 1-2\mu$. By the first equation of (2.42),

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu-\mu}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{\mu-\nu}{n}\pi}\right)^k &= \sum_{k=0}^{n-1} 1 - \frac{1}{2} + \sum_{k=0}^{n-1} 1 - \frac{1}{2} \\ &= n - \frac{1}{2} + n - \frac{1}{2} = 2n-1,\end{aligned} \tag{2.43}$$

and given that $e^{i(2\mu-1)\pi} = \cos(2\mu-1)\pi = -1$, we get

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu+\mu-1}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{1-\nu-\mu}{n}\pi}\right)^k &= \frac{1-e^{i(2\mu-1)\pi}}{1-e^{i\frac{2\mu-1}{n}\pi}} - \frac{1}{2} + \frac{1-e^{i(1-2\mu)\pi}}{1-e^{i\frac{1-2\mu}{n}\pi}} - \frac{1}{2} \\ &= \frac{2}{1-e^{i\frac{2\mu-1}{n}\pi}} + \frac{2}{1-e^{i\frac{1-2\mu}{n}\pi}} - 1 \\ &= \frac{4-2e^{i\frac{1-2\mu}{n}\pi} - 2e^{i\frac{2\mu-1}{n}\pi}}{2-e^{i\frac{1-2\mu}{n}\pi} - e^{i\frac{2\mu-1}{n}\pi}} - 1 = 1.\end{aligned} \tag{2.44}$$

The combination of (2.43) and (2.44), yields

$$\sum_{k=0}^{n-1}{}' \cos k\theta_\nu \cos k\theta_\mu = \frac{1}{4}\left(2n-1+1\right) = \frac{n}{2}, \quad \nu=\mu. \tag{2.45}$$

If $\nu \neq \mu$, suppose that $\nu-\mu$ is odd. Then, $\mu-\nu$ is also odd, while, $\nu+\mu-1$ and $1-\nu-\mu$ are both even. This implies that $e^{i(\nu-\mu)\pi} = e^{i(\mu-\nu)\pi} = -1$ and $e^{i(\nu+\mu-1)\pi} = e^{i(1-\nu-\mu)\pi} = 1$. So, in view of (2.42),

$$\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu-\mu}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{\mu-\nu}{n}\pi}\right)^k = 1,$$

similarly as in (2.44), and

$$\begin{aligned}\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu+\mu-1}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{1-\nu-\mu}{n}\pi}\right)^k &= \frac{1-e^{i(\nu+\mu-1)\pi}}{1-e^{i\frac{\nu+\mu-1}{n}\pi}} - \frac{1}{2} + \frac{1-e^{i(1-\nu-\mu)\pi}}{1-e^{i\frac{1-\nu-\mu}{n}\pi}} - \frac{1}{2} \\ &= -1.\end{aligned}$$

If $\nu - \mu$ is even, it is immediate that

$$\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu-\mu}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{\mu-\nu}{n}\pi}\right)^k = -1,$$

and

$$\sum_{k=0}^{n-1}{}' \left(e^{i\frac{\nu+\mu-1}{n}\pi}\right)^k + \sum_{k=0}^{n-1}{}' \left(e^{i\frac{1-\nu-\mu}{n}\pi}\right)^k = 1.$$

Hence,

$$\sum_{k=0}^{n-1}{}' \cos k\theta_\nu \cos k\theta_\mu = 0, \quad \nu \neq \mu,$$

which, together with (2.45), proves our assertion. □

Now, multiplying both sides of (2.40) by $\cos k\theta_\mu$ and then summing over k as in (2.41), in view of Lemma 2.10, gives

$$\begin{aligned}
\sum_{k=0}^{n-1}{}' [\pi\delta_{k0}\cos k\theta_\mu] &= \sum_{k=0}^{n-1}{}' \left[\sum_{\nu=1}^{n} \lambda_\nu \cos k\theta_\nu \cos k\theta_\mu\right] \\
\iff \quad \frac{\pi}{2} &= \sum_{\nu=1}^{n} \left[\lambda_\nu \sum_{k=0}^{n-1}{}' [\cos k\theta_\nu \cos k\theta_\mu]\right] \\
\iff \quad \frac{\pi}{2} &= \frac{1}{2}\sum_{\nu=1}^{n} \lambda_\nu n \delta_{\nu\mu} \\
\iff \quad \lambda_\nu &= \frac{\pi}{n}, \quad \nu = 1, 2, \ldots, n.
\end{aligned}$$

Remark 2.11. Note that for each $n = 1, 2, \ldots,$ the n-point Gauss-Chebyshev formula (2.39) has equal weights. Posse (1875) proved that this is the only Gauss formula that has this property.

Consider the Gauss formula (2.9) with $d\lambda(t) = \left(1 - t^2\right)^{1/2} dt$, the Chebyshev measure of the second kind (Table A.1). The nodes $\tau_\nu = \tau_\nu^{(2)}$ are the zeros of the Chebyshev polynomial U_n, given by

$$\tau_\nu = \cos\theta_\nu, \quad \theta_\nu = \frac{\nu\pi}{n+1}, \quad \nu = 1, 2, \ldots, n. \tag{2.46}$$

The weights $\lambda_\nu = \lambda_\nu^{(2)}$ can be computed by (2.6),

$$\lambda_\nu = \int_{-1}^{1} \ell_\nu(t)\left(1 - t^2\right)^{1/2} dt, \quad \nu = 1, 2, \ldots, n,$$

where $\ell_\nu(t)$ can be written similarly, as in the case of the Chebyshev measure of the fist kind, as

$$\ell_\nu(t) = \frac{U_n(t)}{(t - \tau_\nu)U_n'(\tau_\nu)},$$

hence,

$$\lambda_\nu = \frac{1}{U_n'(\tau_\nu)} \int_{-1}^{1} \frac{U_n(t)}{t-\tau_\nu}\left(1-t^2\right)^{1/2} dt, \;\; \nu = 1, 2, \ldots, n. \tag{2.47}$$

In order to compute the $U_n'(\tau_\nu)$, we use the trigonometric representation for U_n and the formula for the zeros. Indeed, as $U_n(\cos\theta) = \sin(n+1)\theta / \sin\theta$, there holds

$$U_n'(\cos\theta) = \frac{\cos\theta\sin(n+1)\theta - (n+1)\cos(n+1)\theta\sin\theta}{\sin^3\theta},$$

which leads to

$$U_n'(\tau_\nu) = -(n+1)\frac{\cos(n+1)\frac{\nu}{n+1}\pi}{\sin^2\theta_\nu} = -(n+1)\frac{(-1)^\nu}{1-\tau_\nu^2}. \tag{2.48}$$

Using the Christoffel-Darboux formula (1.24) and similar arguments, as in the case of the Chebyshev measure of the first kind, we obtain

$$\begin{aligned} \frac{U_n(t)}{t-\tau_\nu} &= -\frac{\pi}{U_{n+1}(\tau_\nu)}\left[\frac{2}{\pi}U_0(t)U_0(\tau_\nu) + \frac{2}{\pi}\sum_{k=1}^{n} U_k(t)U_k(\tau_\nu)\right] \\ &= -\frac{\pi}{U_{n+1}(\tau_\nu)}\left[\frac{2}{\pi} + \frac{2}{\pi}\sum_{k=1}^{n} U_k(t)U_k(\tau_\nu)\right], \end{aligned}$$

and integrating both parts with respect to the measure $\left(1-t^2\right)^{1/2} dt$, gives

$$\begin{aligned} &\int_{-1}^{1} \frac{U_n(t)}{t-\tau_\nu}\left(1-t^2\right)^{1/2} dt \\ &= -\frac{\pi}{U_{n+1}(\tau_\nu)}\left[\frac{2}{\pi}\int_{-1}^{1}\left(1-t^2\right)^{1/2} dt + \frac{2}{\pi}\sum_{k=1}^{n} U_k(\tau_\nu)\int_{-1}^{1} U_k(t)\left(1-t^2\right)^{1/2} dt\right]. \end{aligned}$$

Note that the first integral of the right-hand side equals to $\pi/2$, while the second integral vanishes, due to the orthogonality of the Chebyshev polynomial U_k, so,

$$\int_{-1}^{1} \frac{U_n(t)}{t-\tau_\nu}\left(1-t^2\right)^{1/2} dt = -\frac{\pi}{U_{n+1}(\tau_\nu)}. \tag{2.49}$$

Now, one can compute that

$$\begin{aligned} U_{n+1}(\tau_\nu) =& \frac{\sin(n+2)\theta_\nu}{\sin\theta_\nu} \\ =& \frac{\sin(n+1)\frac{\nu}{n+1}\pi\cos\theta_\nu + \cos(n+1)\frac{\nu}{n+1}\pi\sin\theta_\nu}{\sin\theta_\nu} \\ =& (-1)^\nu. \end{aligned}$$

The latter, combined with (2.47), (2.48) and (2.49), yields

$$\lambda_\nu = \frac{\pi}{n+1}\left(1-\tau_\nu^2\right), \;\; \nu = 1, 2, \ldots, n, \tag{2.50}$$

thus, the Gauss-Chebyshev quadrature formula with the Chebyshev measure of the second kind, or the so-called Gauss-Chebyshev quadrature formula of the second kind, is

$$\int_{-1}^{1} f(t)\big(1-t^2\big)^{1/2}dt = \frac{\pi}{n+1}\sum_{\nu=1}^{n}\big(1-\tau_\nu^2\big)f\Big(\cos\frac{\nu\pi}{n+1}\Big) + R_n^G(f). \tag{2.51}$$

Letting $d\lambda(t)$ be the Chebyshev measure of the third or fourth kind (Table A.1), one can similarly obtain the weights $\lambda_\nu^{(3)}$ and $\lambda_\nu^{(4)}$, respectively (see [7], Theorem 8.4). Indeed, if $d\lambda(t) = (1-t)^{-1/2}(1+t)^{1/2}\,dt$, then

$$\lambda_\nu^{(3)} = \frac{\pi}{n+\frac{1}{2}}\left(1+\tau_\nu^{(3)}\right),\ \ \nu = 1,2,\ldots,n, \tag{2.52}$$

while, if $d\lambda(t) = (1-t)^{1/2}(1+t)^{-1/2}dt$,

$$\lambda_\nu^{(4)} = \frac{\pi}{n+\frac{1}{2}}\left(1-\tau_\nu^{(4)}\right),\ \ \nu = 1,2,\ldots,n. \tag{2.53}$$

In the following table, we have collected the nodes and weights of the Gauss-Chebyshev quadrature formulae of any one of the four kinds.

Table 2.1: Gauss-Chebyshev quadrature formulae.

Kind	$d\lambda(t)$	τ_ν	λ_ν
#1 Chebyshev	$\big(1-t^2\big)^{-1/2}dt$	$\cos\frac{2\nu-1}{2n}\pi$	$\frac{\pi}{n}$
#2 Chebyshev	$\big(1-t^2\big)^{1/2}dt$	$\cos\frac{\nu}{n+1}\pi$	$\frac{\pi}{n+1}\big(1-\tau_\nu^2\big)$
#3 Chebyshev	$(1-t)^{-1/2}(1+t)^{1/2}dt$	$\cos\frac{2\nu-1}{2n+1}\pi$	$\frac{\pi}{n+\frac{1}{2}}(1+\tau_\nu)$
#4 Chebyshev	$(1-t)^{1/2}(1+t)^{-1/2}dt$	$\cos\frac{2\nu}{2n+1}\pi$	$\frac{\pi}{n+\frac{1}{2}}(1-\tau_\nu)$

Chapter 3

The Error Norm of Gauss Formulae for Analytic Functions

The most common method for estimating the error of a quadrature formula is by means of a high-order derivative of the function involved. Corollary 2.9 provides such an estimate. In this chapter, we shall present derivative-free error estimates that can be obtained by contour integration techniques or Hilbert space methods.

3.1 The norm of the error functional

Definition 3.1. A complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighborhood of the point, is called holomorphic.

Remark 3.2. The term analytic function is often used interchangeably with holomorphic function. The word "analytic" is defined in a broader sense to denote any function that can be written as a convergent power series in a neighborhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

Writing the measure $d\lambda$ in (2.1) as $d\lambda(t) = w(t)dt$, where w is a non-negative weight function, assumed to be integrable over $[-1, 1]$, yields the quadrature formula

$$\int_{-1}^{1} f(t)w(t)dt = \sum_{\nu=1}^{n} w_\nu f(\tau_\nu) + R_n(f), \tag{3.1}$$

where the τ_ν are certain distinct nodes in $[-1, 1]$, ordered decreasingly, and the w_ν are the corresponding weights.

Let f be a holomorphic function in $C_r = \{z \in \mathbb{C} : |z| < r\}$, $r > 1$. Then f can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in C_r. \tag{3.2}$$

Define

$$|f|_r = \sup\left\{|a_k| r^k \ : \ k \in \mathbb{N}_0 \quad \text{and} \quad R_n\big(t^k\big) \neq 0\right\}. \tag{3.3}$$

Then, $|\cdot|_r$ is a seminorm in the space

$$X_r = \{f \ : \ f \text{ holomorphic in } C_r \text{ and } |f|_r < \infty\}. \tag{3.4}$$

In view of (3.1), as $f \in C[-1,1]$, one has that

$$\begin{aligned}
|R_n(f)| = & \left|\int_{-1}^{1} f(t)w(t)dt - \sum_{\nu=1}^{n} w_\nu f(\tau_\nu)\right| \\
\leq & \left|\int_{-1}^{1} f(t)w(t)dt\right| + \left|\sum_{\nu=1}^{n} w_\nu f(\tau_\nu)\right| \\
\leq & \int_{-1}^{1} |f(t)|\, w(t)dt + \sum_{\nu=1}^{n} |w_\nu|\, |f(\tau_\nu)| \\
\leq & \|f\|_\infty \cdot \int_{-1}^{1} |w(t)|dt + \|f\|_\infty \cdot \sum_{\nu=1}^{n} |w_\nu|,
\end{aligned}$$

that is,

$$|R_n(f)| \leq \left(\|w\|_1 + \sum_{\nu=1}^{n} |w_\nu|\right) \|f\|_\infty, \tag{3.5}$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the L_1 and L_∞ norm of a function, respectively. Hence, R_n is a bounded and, equivalently, continuous linear functional on $(C[-1,1], \|\cdot\|_\infty)$. The continuity of R_n, combined with the uniform convergence of the series in (3.2) on $[-1,1]$, yields

$$R_n(f) = \sum_{k=0}^{\infty} a_k R_n\big(t^k\big). \tag{3.6}$$

Taking absolute values on both sides in (3.6), gives

$$|R_n(f)| = \left|\sum_{k=0}^{\infty} a_k R_n\big(t^k\big)\right| \leq \sum_{k=0}^{\infty} |a_k| \left|R_n\big(t^k\big)\right| = \sum_{k=0}^{\infty} |a_k| r^k \frac{\left|R_n\big(t^k\big)\right|}{r^k},$$

and by virtue of (3.3), we obtain

$$|R_n(f)| \leq \left[\sum_{k=0}^{\infty} \frac{\left|R_n\big(t^k\big)\right|}{r^k}\right] |f|_r. \tag{3.7}$$

Setting $f = t^k$ in (3.5), one has

$$|R_n(t^k)| \leq \left(\|w\|_1 + \sum_{\nu=1}^{n} |w_\nu| \right) \left\| t^k \right\|_\infty = \|w\|_1 + \sum_{\nu=1}^{n} |w_\nu|, \tag{3.8}$$

as $\left\| t^k \right\|_\infty = 1$ on $[-1, 1]$. Now, (3.7) together with (3.8), gives

$$|R_n(f)| \leq \left[\sum_{k=0}^{\infty} \frac{\|w\|_1 + \sum_{\nu=1}^{n} |w_\nu|}{r^k} \right] |f|_r,$$

from which we get that the series in (3.7) converges, as $\sum_{k=0}^{\infty} \frac{1}{r^k}$ converges. Therefore, R_n is a bounded linear functional on $(X_r, |\cdot|_r)$, with norm $\|R_n\|$, that is,

$$|R_n(f)| \leq \|R_n\| |f|_r, \tag{3.9}$$

where, from (3.7),

$$\|R_n\| \leq \sum_{k=0}^{\infty} \frac{\left| R_n\left(t^k\right) \right|}{r^k}. \tag{3.10}$$

In order to get the equality in (3.10), let

$$\phi(z) = \sum_{k=0}^{\infty} \operatorname{sign}\left(R_n\left(t^k\right) \right) \frac{z^k}{r^k},$$

and note that

$$\begin{aligned} |\phi|_r =& \sup \left\{ \left| \frac{\operatorname{sign}\left(R_n\left(t^k\right)\right)}{r^k} \right| r^k \ : \ k \in \mathbb{N}_0 \quad \text{and} \quad R_n\left(t^k\right) \neq 0 \right\} \\ =& \sup \left\{ \left| \operatorname{sign}\left(R_n\left(t^k\right) \right) \right| \ : \ k \in \mathbb{N}_0 \quad \text{and} \quad R_n\left(t^k\right) \neq 0 \right\} \\ =& 1. \end{aligned}$$

In view of (3.6), one can get

$$|R_n(\phi)| = \left| \sum_{k=0}^{\infty} \frac{\operatorname{sign}\left(R_n\left(t^k\right)\right) R_n\left(t^k\right)}{r^k} \right| = \sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^k} \cdot 1 = \left[\sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^k} \right] |\phi|_r.$$

Thus,

$$\|R_n\| = \sum_{k=0}^{\infty} \frac{\left| R_n\left(t^k\right) \right|}{r^k}. \tag{3.11}$$

The computation of the seminorm $|f|_r$ requires the knowledge of the coefficients a_k, $k \geq 0$, which are not always available, so $|f|_r$ often has to be estimated.

Definition 3.3. The space

$$H_2 = \left\{ f \ : \ f \text{ holomorphic in } C_r \text{ and } \int_{|z|=r} |f(z)|^2 \, |dz| < \infty \right\} \tag{3.12}$$

is called the Hardy space H_2 and the number

$$\|f\|_{2,r} = \left(\int_{|z|=r} |f(z)|^2 \, |dz| \right)^{1/2}, \tag{3.13}$$

is the corresponding H_2 norm.

Lemma 3.4. The polynomials

$$p_k(z) = \frac{z^k}{r^k\sqrt{2\pi r}}, \quad k = 0, 1, 2, \ldots, \tag{3.14}$$

form a complete orthonormal system in H_2.

Proof. Taking the inner product of p_n, p_m, $n, m = 0, 1, 2, \ldots,$

$$(p_n, p_m)_{H_2} = \int_{|z|=r} p_n(z) \overline{p_m(z)} \, |dz| = \int_{|z|=r} \frac{z^n}{r^n\sqrt{2\pi r}} \frac{\overline{z}^m}{r^m\sqrt{2\pi r}} |dz|,$$

and setting $z = re^{it}$, gives

$$\begin{aligned}
(p_n, p_m)_{H_2} &= \int_0^{2\pi} \frac{e^{nit}}{\sqrt{2\pi r}} \frac{e^{-mit}}{\sqrt{2\pi r}} |rie^{it} dt| \\
&= \int_0^{2\pi} \frac{e^{(n-m)it}}{2\pi r} r dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{(n-m)it} \, dt = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}
\end{aligned}$$

This proves that the system in question is orthonormal. For completing the proof, it suffices to show that for an arbitrary $f \in H_2$ there holds

$$f = \sum_{k=0}^{\infty} (f, p_k) \, p_k.$$

Since $f \in H_2$, f is holomorphic and (3.2) holds, so,

$$\begin{aligned}
(f, p_k) &= \left(\sum_{j=0}^{\infty} a_j z^j, p_k \right) = \sum_{j=0}^{\infty} a_j \left(z^j, p_k \right) \\
&= \sum_{j=0}^{\infty} a_j r^j \sqrt{2\pi r} \, (p_j, p_k) \\
&= \sum_{j=0}^{\infty} a_j r^j \sqrt{2\pi r} \delta_{jk} = a_k r^k \sqrt{2\pi r}.
\end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned}\sum_{k=0}^{\infty}(f,p_k)\,p_k &= \sum_{k=0}^{\infty}\left(a_k r^k\sqrt{2\pi r}\right)p_k\\ &= \sum_{k=0}^{\infty}a_k r^k\sqrt{2\pi r}\frac{z^k}{r^k\sqrt{2\pi r}}\\ &= \sum_{k=0}^{\infty}a_k z^k = f,\end{aligned}$$

which proves our assertion. □

The fact that the polynomials p_k, $k = 0, 1, 2, \ldots$, form a complete orthonormal system in H_2, allows us to use Parseval's identity in order to compute the norm $\|f\|_{2,r}$, that is,

$$\|f\|_{2,r}^2 = \sum_{k=0}^{\infty}\left|(f,p_k)\right|^2.$$

Now, in view of (3.15), one has

$$\begin{aligned}\|f\|_{2,r}^2 =& \sum_{k=0}^{\infty}\left|a_k r^k\sqrt{2\pi r}\right|^2\\ =& 2\pi r\sum_{k=0}^{\infty}|a_k|^2\,r^{2k},\end{aligned}$$

thus,

$$\|f\|_{2,r} = \sqrt{2\pi r}\left(\sum_{k=0}^{\infty}|a_k|^2 r^{2k}\right)^{1/2}. \tag{3.16}$$

Note that

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k|^2 r^{2k} \geq& \sup\left\{|a_k|^2 r^{2k}\ :\ k\in\mathbb{N}_0\right\}\\ \geq& \sup\left\{|a_k|^2 r^{2k}\ :\ k\in\mathbb{N}_0 \text{ and } R_n(t^k)\neq 0\right\} = |f|_r^2,\end{aligned}$$

that is,

$$\sum_{k=0}^{\infty}|a_k|^2 r^{2k} \geq |f|_r^2. \tag{3.17}$$

Combining (3.16) and (3.17), we get

$$|f|_r \leq \frac{1}{\sqrt{2\pi r}}\|f\|_{2,r}. \tag{3.18}$$

Furthermore, from the definition of $\|f\|_{2,r}$, there holds

$$\begin{aligned}\|f\|_{2,r}^2 &= \int_{|z|=r} |f(z)|^2 |dz| \\ &\leq \left(\max_{|z|=r} |f(z)| \right)^2 \int_{|z|=r} |dz| \\ &= 2\pi r \left(\max_{|z|=r} |f(z)| \right)^2 .\end{aligned}$$

Taking the square root of both sides and inserting the latter into (3.18), yields

$$|f|_r \leq \max_{|z|=r} |f(z)|. \tag{3.19}$$

Although formula (3.11) is useful for obtaining an estimate for $\|R_n\|$, it cannot be used for computing $\|R_n\|$ explicitly. A practical representation for $\|R_n\|$ can be derived if we have some information on the sign of $R_n(t^k)$, $k \geq 0$. In addition, the representation becomes particularly useful if formula (3.1) is interpolatory.

Theorem 3.5. ([8], Theorem 2.1) Consider the quadrature formula (3.1). Let $\pi_n(t) = \prod_{\nu=1}^{n}(t-\tau_\nu)$ and $\epsilon \in \{-1, 1\}$.

(a) If $\epsilon R_n(t^k) \geq 0$, $k \geq 0$, then

$$\|R_n\| = r \left| R_n \left(\frac{1}{r-t} \right) \right| . \tag{3.20}$$

If, in addition, formula (3.1) is interpolatory, then

$$\|R_n\| = r \left| \frac{1}{\pi_n(r)} \int_{-1}^{1} \frac{\pi_n(t)}{r-t} w(t) dt \right| . \tag{3.21}$$

(b) If $\epsilon(-1)^k R_n(t^k) \geq 0$, $k \geq 0$, then

$$\|R_n\| = r \left| R_n \left(\frac{1}{r+t} \right) \right| . \tag{3.22}$$

If, in addition, formula (3.1) is interpolatory, then

$$\|R_n\| = r \left| \frac{1}{\pi_n(-r)} \int_{-1}^{1} \frac{\pi_n(t)}{r+t} w(t) dt \right| . \tag{3.23}$$

Proof. (a) Since R_n is continuous on $(C[-1,1], \|\cdot\|_\infty)$ and $\epsilon R_n(t^k) \geq 0$, $k \geq 0$, by (3.11), one has

$$\begin{aligned}\|R_n\| &= \sum_{k=0}^{\infty} \frac{|\epsilon|\left|R_n\left(t^k\right)\right|}{r^k} = \sum_{k=0}^{\infty} \frac{\left|\epsilon R_n\left(t^k\right)\right|}{r^k} \\ &= \sum_{k=0}^{\infty} \frac{\epsilon R_n\left(t^k\right)}{r^k} = \left|\sum_{k=0}^{\infty} \frac{\epsilon R_n\left(t^k\right)}{r^k}\right| \\ &= |\epsilon|\left|\sum_{k=0}^{\infty} \frac{R_n\left(t^k\right)}{r^k}\right| = \left|\sum_{k=0}^{\infty} R_n\left(\left(\frac{t}{r}\right)^k\right)\right| \\ &= \left|R_n\left(\sum_{k=0}^{\infty}\left(\frac{t}{r}\right)^k\right)\right| = \left|R_n\left(\frac{1}{1-t/r}\right)\right| = r\left|R_n\left(\frac{1}{r-t}\right)\right|.\end{aligned}$$

Assume that formula (3.1) is interpolatory and let p_{n-1} to be the polynomial of degree at most $n-1$, interpolating the function $\frac{1}{r-t}$ at the points $\tau_1, \tau_2, \ldots, \tau_n$. Then, there holds

$$\frac{1}{r-t} - p_{n-1}(t) = \frac{1-(r-t)p_{n-1}(t)}{r-t}. \tag{3.24}$$

Note that the left-hand side vanishes at the interpolating points, so the $\tau_1, \tau_2, \ldots, \tau_n$ must be zeros of the numerator on the right-hand side. As this is a polynomial of degree at most n, we have

$$1-(r-t)p_{n-1}(t) = c_n \pi_n(t). \tag{3.25}$$

Setting $t = r$ in (3.25), we get

$$c_n = \frac{1}{\pi_n(r)},$$

which, inserted into (3.25), along with (3.24), gives

$$\frac{1}{r-t} - p_{n-1}(t) = \frac{1}{\pi_n(r)} \frac{\pi_n(t)}{r-t}. \tag{3.26}$$

Finally, integrating (3.26) with respect to the weight function w and comparing this with (3.1), yields

$$R_n\left(\frac{1}{r-t}\right) = \frac{1}{\pi_n(r)} \int_{-1}^{1} \frac{\pi_n(t)}{r-t} w(t)dt,$$

which, inserted into (3.20), implies (3.21).

(b) The proof is similar to the proof of (a), but now, we use the fact that $\epsilon(-1)^k R_n(t^k) \geq 0$, $k \geq 0$.

So, by (3.11), one has

$$\begin{aligned}\|R_n\| &= \sum_{k=0}^{\infty} \frac{\left|\epsilon(-1)^k\right| \left|R_n\left(t^k\right)\right|}{r^k} = \sum_{k=0}^{\infty} \frac{\left|\epsilon(-1)^k R_n\left(t^k\right)\right|}{r^k} \\ &= \sum_{k=0}^{\infty} \frac{\epsilon(-1)^k R_n\left(t^k\right)}{r^k} = \left|\sum_{k=0}^{\infty} \frac{\epsilon(-1)^k R_n\left(t^k\right)}{r^k}\right| \\ &= |\epsilon| \left|\sum_{k=0}^{\infty} \frac{(-1)^k R_n\left(t^k\right)}{r^k}\right| = \left|\sum_{k=0}^{\infty} R_n\left((-1)^k \left(\frac{t}{r}\right)^k\right)\right| \\ &= \left|R_n\left(\sum_{k=0}^{\infty}\left(-\frac{t}{r}\right)^k\right)\right| = \left|R_n\left(\frac{1}{1+t/r}\right)\right| = r\left|R_n\left(\frac{1}{r+t}\right)\right|.\end{aligned}$$

Assume that formula (3.1) is interpolatory and let p_{n-1} to be the polynomial of degree at most $n-1$, interpolating the function $\frac{1}{r+t}$ at the points $\tau_1, \tau_2, \ldots, \tau_n$. Then, there holds

$$\frac{1}{r+t} - p_{n-1}(t) = \frac{1-(r+t)p_{n-1}(t)}{r+t}. \tag{3.27}$$

Note that the left-hand side vanishes at the interpolating points, so the $\tau_1, \tau_2, \ldots, \tau_n$ must be zeros of the numerator on the right-hand side. As this is a polynomial of degree at most n, we have

$$1-(r+t)p_{n-1}(t) = c_n \pi_n(t). \tag{3.28}$$

Setting $t=-r$ in (3.28), we get

$$c_n = \frac{1}{\pi_n(-r)},$$

which, inserted into (3.28), along with (3.27), gives

$$\frac{1}{r+t} - p_{n-1}(t) = \frac{1}{\pi_n(-r)} \frac{\pi_n(t)}{r+t}. \tag{3.29}$$

Finally, integrating (3.29) with respect to the weight function w and comparing this with (3.1), yields

$$R_n\left(\frac{1}{r+t}\right) = \frac{1}{\pi_n(-r)} \int_{-1}^{1} \frac{\pi_n(t)}{r+t} w(t)dt,$$

which, inserted into (3.22), implies (3.23). □

If f is a single-valued holomorphic function in a domain D containing $[-1,1]$ in its interior, and Γ is a contour in D surrounding $[-1,1]$, then applying the error term R_n, viewed as a bounded linear functional, on Cauchy's formula

$$f(t) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z-t} dz, \quad t \in [-1,1],$$

we get the representation

$$R_n(f) = \frac{1}{2\pi i}\oint_{\Gamma} K_n(z)f(z)dz, \tag{3.30}$$

where the function K_n, referred to as the kernel, is given by

$$K_n(z) = R_n\left(\frac{1}{z-\cdot}\right). \tag{3.31}$$

From (3.30), there follows

$$\begin{aligned}|R_n(f)| =&\left|\frac{1}{2\pi i}\oint_{\Gamma} K_n(z)f(z)dz\right| \\ \le&\frac{1}{2\pi}\max_{z\in\Gamma}|K_n(z)|\max_{z\in\Gamma}|f(z)|\oint_{\Gamma}|dz|,\end{aligned}$$

that is,

$$|R_n(f)| \le \frac{\ell\big(\Gamma\big)}{2\pi}\max_{z\in\Gamma}\big|K_n(z)\big|\max_{z\in\Gamma}|f(z)|, \tag{3.32}$$

where $\ell\big(\Gamma\big)$ denotes the length of Γ.

Theorem 3.6. ([2], Theorem 4.2) For each $r > 1$, there holds

$$\max_{|z|=r}|K_n(z)| = \begin{cases} |K_n(r)| \ \text{ if } \epsilon R_n\big(t^k\big) \ge 0, \ \ k \ge 0, \\ |K_n(-r)| \ \text{ if } \epsilon(-1)^k R_n\big(t^k\big) \ge 0, \ \ k \ge 0. \end{cases} \tag{3.33}$$

Proof. Expand

$$\frac{1}{z-t} = \frac{1}{z}\cdot\frac{1}{1-\frac{t}{z}}$$

in powers of $\frac{t}{z}$. Since R_n is continuous, we get

$$\begin{aligned}K_n(z) =&R_n\left(\frac{1}{z-t}\right) = \frac{1}{z}R_n\left(\frac{1}{1-\frac{t}{z}}\right) = \frac{1}{z}R_n\left(\sum_{k=0}^{\infty}\left(\frac{t}{z}\right)^k\right) \\ =&\frac{1}{z}\sum_{k=0}^{\infty}\frac{R_n\big(t^k\big)}{z^k} = \sum_{k=0}^{\infty}\frac{R_n\big(t^k\big)}{z^{k+1}},\end{aligned}$$

so,

$$\max_{|z|=r}|K_n(z)| = \max_{|z|=r}\left|\sum_{k=0}^{\infty}\frac{R_n\big(t^k\big)}{z^{k+1}}\right| \le \max_{|z|=r}\left(\sum_{k=0}^{\infty}\frac{\big|R_n\big(t^k\big)\big|}{|z|^{k+1}}\right) = \sum_{k=0}^{\infty}\frac{\big|R_n\big(t^k\big)\big|}{r^{k+1}}.$$

If $\epsilon R_n\big(t^k\big) \ge 0$, $k \ge 0$, writing

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\big|R_n\big(t^k\big)\big|}{r^{k+1}} =&\sum_{k=0}^{\infty}\frac{|\epsilon|\big|R_n\big(t^k\big)\big|}{r^{k+1}} = \sum_{k=0}^{\infty}\frac{\big|\epsilon R_n\big(t^k\big)\big|}{r^{k+1}} \\ =&\sum_{k=0}^{\infty}\frac{\epsilon R_n\big(t^k\big)}{r^{k+1}} = \epsilon\sum_{k=0}^{\infty}\frac{R_n\big(t^k\big)}{r^{k+1}} = \epsilon K_n(r),\end{aligned}$$

and noting that $\epsilon K_n(r) \geq 0$, yields

$$\sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^{k+1}} = |\epsilon K_n(r)| = |\epsilon|\,|K_n(r)| = |K_n(r)|\,,$$

that is,

$$\max_{|z|=r} |K_n(z)| \leq |K_n(r)|\,.$$

Obviously,

$$|K_n(r)| \leq \max_{|z|=r} |K_n(z)|\,,$$

hence,

$$\max_{|z|=r} |K_n(z)| = |K_n(r)|\,.$$

Similarly, if $\epsilon(-1)^k R_n\left(t^k\right)$, $k \geq 0$, writing

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^{k+1}} =& \sum_{k=0}^{\infty} \frac{\left|\epsilon(-1)^k\right|\left|R_n\left(t^k\right)\right|}{r^{k+1}} = \sum_{k=0}^{\infty} \frac{\left|\epsilon(-1)^k R_n\left(t^k\right)\right|}{r^{k+1}} = \sum_{k=0}^{\infty} \frac{\epsilon(-1)^k R_n\left(t^k\right)}{r^{k+1}} \\ =&\epsilon \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} R_n\left(t^k\right)}{(-1)^{k+1} r^{k+1}} = \epsilon \sum_{k=0}^{\infty} \frac{-R_n\left(t^k\right)}{(-r)^{k+1}} = -\epsilon K_n(-r),\end{aligned}$$

and noting that $-\epsilon K_n(-r) \geq 0$, yields

$$\sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^{k+1}} = |-\epsilon K_n(-r)| = |-\epsilon|\,|K_n(-r)| = |K_n(-r)|\,,$$

that is,

$$\max_{|z|=r} |K_n(z)| \leq |K_n(-r)|\,.$$

Obviously,

$$|K_n(-r)| \leq \max_{|z|=r} |K_n(z)|\,,$$

hence,

$$\max_{|z|=r} |K_n(z)| = |K_n(-r)|\,.$$

Thus, (3.33) follows. □

Taking $\Gamma = \partial C_r = \{z \in \mathbb{C} \,:\, |z| = r\}$, $r > 1$, by Theorem 3.6 and (3.11), there holds

$$\max_{|z|=r} \left|K_n(z)\right| = \sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^{k+1}} = \frac{\|R_n\|}{r}. \tag{3.34}$$

Combining (3.32), (3.34) and the fact that $\ell\left(\Gamma\right) = 2\pi r$, yields

$$|R_n(f)| \leq \|R_n\| \max_{|z|=r} |f(z)|. \tag{3.35}$$

3.2 The error norm of Gauss formulae

In this section, we mainly concentrate on Gauss quadrature formulae for which $\|R_n\|$ can be computed explicitly by means of (3.21) and (3.23).

If formula (3.1) is the Gauss formula for the weight function w on $[-1, 1]$ and τ_ν are the nodes, i.e., the zeros of the nth degree monic orthogonal polynomial $\pi_n(\cdot; w)$, then there exists the following important result of Gautschi.

Lemma 3.7. ([2], Lemma 4.1) Let w be a non-negative weight function, assumed to be integrable over $[-1, 1]$.
(a) If $w(t)/w(-t)$ is non-decreasing on $(-1, 1)$, then $R_n(t^k) \geq 0$, $k \geq 0$.
(b) If $w(t)/w(-t)$ is non-increasing on $(-1, 1)$, then $(-1)^k R_n(t^k) \geq 0$, $k \geq 0$.

The proof of Lemma 3.7 makes use of an interesting result of Hunter (cf. [6]).

Remark 3.8. In case that $w(t)/w(-t)$ is constant, then $w(t)/w(-t) = 1$, so w is an even function, and, by symmetry, $R_n\left(t^k\right) = 0$ for all k odd. Hence, both cases of Lemma 3.7 hold simultaneously.

Lemma 3.7 can be used in conjunction with Theorem 3.5 in order to compute $\|R_n\|$. First of all, note that for the Jacobi weight function $w(t) = (1-t)^a(1+t)^\beta$, $a, \beta > -1$, $-1 < t < 1$, there holds

$$\frac{w(t)}{w(-t)} = \frac{(1-t)^a(1+t)^\beta}{(1+t)^a(1-t)^\beta} = \left(\frac{1+t}{1-t}\right)^{\beta-a}.$$

Let $c(t) = w(t)/w(-t)$, then,

$$c'(t) = (\beta - a)\left(\frac{1+t}{1-t}\right)^{\beta-a-1}\frac{2}{(1-t)^2}.$$

So, it is immediate, that $w(t)/w(-t)$ is increasing on $(-1, 1)$ if $a < \beta$ and decreasing if $a > \beta$.

A special case of the Jacobi weight function are the Chebyshev weights of any of the four kinds

$$w^{(1)}(t) = \left(1-t^2\right)^{-1/2}, \quad -1 < t < 1, \tag{3.36}$$

$$w^{(2)}(t) = \left(1-t^2\right)^{1/2}, \quad -1 < t < 1, \tag{3.37}$$

$$w^{(3)}(t) = (1-t)^{-1/2}(1+t)^{1/2}, \quad -1 < t < 1, \tag{3.38}$$

$$w^{(4)}(t) = (1-t)^{1/2}(1+t)^{-1/2}, \quad -1 < t < 1. \tag{3.39}$$

The first two are even functions, so, according to what was stated before, $w^{(1)}, w^{(2)}$ and $w^{(3)}$ satisfy part (a) of Lemma 3.7, while $w^{(4)}$ satisfies part (b). Then $\|R_n\|$ can be computed by means of (3.21) and (3.23), respectively.

Theorem 3.9. ([8], Theorem 3.2) Consider the Gauss formula (3.1), and let $\tau = r - \sqrt{r^2-1}$.
(a) For $w = w^{(1)}$, we have

$$\|R_n^{(1)}\| = \frac{2\pi r\tau^{2n}}{(1+\tau^{2n})\sqrt{r^2-1}}, \quad n \geq 1. \tag{3.40}$$

(b) For $w = w^{(2)}$, we have

$$\|R_n^{(2)}\| = \frac{2\pi r\tau^{2n+2}\sqrt{r^2-1}}{1-\tau^{2n+2}}, \quad n \geq 1. \tag{3.41}$$

(c) For $w = w^{(3)}$ or $w = w^{(4)}$, we have

$$\|R_n^{(3)}\| = \frac{2\pi r\tau^{2n+1}}{1+\tau^{2n+1}}\sqrt{\frac{r+1}{r-1}}, \quad n \geq 1, \tag{3.42}$$

and $\|R_n^{(4)}\|$ is also given by (3.42).

Proof. (a) Applying (3.21) with $w = w^{(1)}$, from Theorem 3.5(a), in view of Lemma 3.7(a), we have

$$\begin{aligned}\|R_n^{(1)}\| =& r\left|\frac{1}{T_n(r)}\int_{-1}^{1}\frac{T_n(t)}{r-t}\left(1-t^2\right)^{-1/2}dt\right|\\ =&\frac{r}{T_n(r)}\int_{-1}^{1}\frac{T_n(t)}{r-t}\left(1-t^2\right)^{-1/2}dt,\end{aligned} \tag{3.43}$$

where T_n is the nth-degree Chebyshev polynomial of the first kind. Setting $t = \cos\theta$ in the integral on the right-hand side of (3.43), and using the trigonometric representation for T_n, we get

$$\begin{aligned}\int_{-1}^{1}\frac{T_n(t)}{r-t}\left(1-t^2\right)^{-1/2}dt =& \int_{\pi}^{0}\frac{\cos n\theta}{r-\cos\theta}\left(1-\cos^2\theta\right)^{-1/2}(-\sin\theta)\,d\theta\\ =&\int_{0}^{\pi}\frac{\cos n\theta}{r-\cos\theta}(\sin\theta)^{-1}\sin\theta d\theta\\ =&\int_{0}^{\pi}\frac{\cos n\theta}{r-\cos\theta}d\theta\\ =&\frac{\pi\tau^n}{\sqrt{r^2-1}}, \quad n = 0,1,2,\ldots\end{aligned} \tag{3.44}$$

(cf. [5], Equation 3.613.1 with $a = -1/r$). Also, using Euler's formula, we have

$$\begin{aligned}\cos n\theta =& \frac{1}{2}\left(e^{in\theta}+e^{-in\theta}\right)\\ =&\frac{1}{2}\left[(\cos\theta+i\sin\theta)^n+(\cos(-\theta)+i\sin(-\theta))^n\right]\\ =&\frac{1}{2}\left[\left(\cos\theta+i\sqrt{1-\cos^2\theta}\right)^n+\left(\cos\theta-i\sqrt{1-\cos^2\theta}\right)^n\right]\\ =&\frac{1}{2}\left[\left(\cos\theta-\sqrt{\cos^2\theta-1}\right)^n+\left(\cos\theta+\sqrt{\cos^2\theta-1}\right)^n\right],\end{aligned} \tag{3.45}$$

that is,

$$T_n(t) = \frac{1}{2}\left[\left(t+\sqrt{t^2-1}\right)^n + \left(t-\sqrt{t^2-1}\right)^n\right], \tag{3.46}$$

as $t = \cos\theta$. Replacing t with r in (3.46), since $\tau = r - \sqrt{r^2-1}$ and $(2r-\tau) = 1/\tau$, we get

$$\begin{aligned} T_n(r) =& \frac{1}{2}\left[\left(r+\sqrt{r^2-1}\right)^n + \left(r-\sqrt{r^2-1}\right)^n\right] \\ =& \frac{1}{2}\left[(2r-\tau)^n + \tau^n\right] \\ =& \frac{1}{2}\left(\frac{1}{\tau^n} + \tau^n\right) \\ =& \frac{1+\tau^{2n}}{2\tau^n}, \quad n = 0,1,2,\ldots. \end{aligned} \tag{3.47}$$

Hence, inserting (3.47), together with (3.44), into (3.43), yields

$$\|R_n^{(1)}\| = \frac{r}{\frac{1+\tau^{2n}}{2\tau^n}} \frac{\pi\tau^n}{\sqrt{r^2-1}} = \frac{2\pi r\tau^{2n}}{(1+\tau^{2n})\sqrt{r^2-1}},$$

which proves (3.40).

(b) Applying (3.21) with $w = w^{(2)}$, from Theorem 3.5(a), in view of Lemma 3.7(a), we have

$$\begin{aligned} \|R_n^{(2)}\| =& r\left|\frac{1}{U_n(r)}\int_{-1}^{1}\frac{U_n(t)}{r-t}\left(1-t^2\right)^{1/2} dt\right| \\ =& \frac{r}{U_n(r)}\int_{-1}^{1}\frac{U_n(t)}{r-t}\left(1-t^2\right)^{1/2} dt, \end{aligned} \tag{3.48}$$

where U_n is the nth-degree Chebyshev polynomial of the second kind. Setting $t = \cos\theta$ in the integral on the right-hand side of (3.48), and using the trigonometric representation for U_n, we get

$$\begin{aligned} \int_{-1}^{1}\frac{U_n(t)}{r-t}\left(1-t^2\right)^{1/2} dt =& \int_{\pi}^{0}\frac{\frac{\sin(n+1)\theta}{\sin\theta}}{r-\cos\theta}\left(1-\cos^2\theta\right)^{1/2}(-\sin\theta)\,d\theta \\ =& \int_0^{\pi}\frac{\sin(n+1)\theta}{r-\cos\theta}\frac{1}{\sin\theta}\sin^2\theta d\theta \\ =& \int_0^{\pi}\frac{\sin(n+1)\theta\sin\theta}{r-\cos\theta}d\theta \\ =& \frac{1}{2}\int_0^{\pi}\frac{\cos n\theta}{r-\cos\theta}d\theta - \frac{1}{2}\int_0^{\pi}\frac{\cos(n+2)\theta}{r-\cos\theta}d\theta \\ =& \frac{1}{2}\left(\frac{\pi\tau^n}{\sqrt{r^2-1}} - \frac{\pi\tau^{n+2}}{\sqrt{r^2-1}}\right) = \frac{1}{2}\frac{\pi\tau^n\left(1-\tau^2\right)}{\sqrt{r^2-1}}, \end{aligned}$$

by (3.44) with $n+2$ in the place of n in the second integral. Also, as $(2r-\tau)=1/\tau$, there holds

$$\begin{aligned}1-\tau^2 =&2\left(1-r^2+r\sqrt{r^2-1}\right)\\ =&2(1-r\tau)=2\tau(\frac{1}{\tau}-r)\\ =&2\tau(r-\tau)=2\tau\sqrt{r^2-1},\end{aligned} \tag{3.49}$$

which implies that

$$\int_{-1}^{1}\frac{U_n(t)}{r-t}(1-t^2)^{1/2}dt=\pi\tau^{n+1},\ \ n=0,1,2,\ldots. \tag{3.50}$$

Now, since $T_n(\cos\theta)=\cos n\theta$,

$$T_n'(\cos\theta)=n\frac{\sin n\theta}{\sin\theta},$$

which, as $t=\cos\theta$, leads to

$$T_n'(t)=nU_{n-1}(t),$$

hence,

$$U_n(t)=\frac{T_{n+1}'(t)}{n+1}.$$

Note that τ depends on r, so,

$$\tau'(r)=1-\frac{r}{\sqrt{r^2-1}}=\frac{\sqrt{r^2-1}-r}{\sqrt{r^2-1}}=\frac{-\tau}{\sqrt{r^2-1}},$$

from which, in view of (3.47), there follows that

$$\begin{aligned}T_n'(r)=&\frac{2n\tau^{2n-1}\tau^n(-\tau)-\left(1+\tau^{2n}\right)n\tau^{n-1}(-\tau)}{2\tau^{2n}\sqrt{r^2-1}}\\ =&\frac{n\left[\left(1+\tau^{2n}\right)\tau^n-2\tau^{3n}\right]}{2\tau^{2n}\sqrt{r^2-1}}\\ =&\frac{n\left(1+\tau^{2n}-2\tau^{2n}\right)}{2\tau^n\sqrt{r^2-1}}=\frac{n\left(1-\tau^{2n}\right)}{2\tau^n\sqrt{r^2-1}}.\end{aligned}$$

Thus,

$$\begin{aligned}U_n(r)=&\frac{T_{n+1}'(r)}{n+1}\\ =&\frac{1-\tau^{2n+2}}{2\tau^{n+1}\sqrt{r^2-1}},\ \ n=0,1,2,\ldots,\end{aligned} \tag{3.51}$$

and inserting (3.51), together with (3.50), into (3.48), yields

$$\|R_n^{(2)}\|=\frac{r}{\frac{1-\tau^{2n+2}}{2\tau^{n+1}\sqrt{r^2-1}}}\pi\tau^{n+1}=\frac{2\pi r\tau^{2n+2}\sqrt{r^2-1}}{1-\tau^{2n+2}},$$

which proves (3.41).

(c) Applying (3.21) with $w = w^{(3)}$, from Theorem 3.5(a), in view of Lemma 3.7(a), we have

$$\begin{aligned}\|R_n^{(3)}\| =& r\left|\frac{1}{V_n(r)}\int_{-1}^{1}\frac{V_n(t)}{r-t}(1-t)^{-1/2}(1+t)^{1/2}dt\right| \\ =& \frac{r}{V_n(r)}\int_{-1}^{1}\frac{V_n(t)}{r-t}(1-t)^{-1/2}(1+t)^{1/2}dt,\end{aligned} \tag{3.52}$$

where V_n is the nth-degree Chebyshev polynomial of the third kind. Setting $t = \cos\theta$ in the integral on the right-hand side of (3.52), and using the trigonometric representation for V_n, we get

$$\begin{aligned}\int_{-1}^{1}\frac{V_n(t)}{r-t}(1-t)^{-1/2}(1+t)^{1/2}dt &= \int_{\pi}^{0}\frac{\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}}{r-\cos\theta}(1-\cos\theta)^{-1/2}(1+\cos\theta)^{1/2}\,(-\sin\theta)\,d\theta \\ &= \int_{0}^{\pi}\frac{\cos(n+\frac{1}{2})\theta}{r-\cos\theta}\frac{1}{\cos\frac{1}{2}\theta}(1-\cos\theta)^{-1/2}(1+\cos\theta)^{1/2}\sin\theta d\theta.\end{aligned}$$

Since,

$$(1-\cos\theta)^{1/2} = \sqrt{2}\sin\frac{1}{2}\theta = \frac{\sqrt{2}}{2}\frac{\sin\theta}{\cos\frac{1}{2}\theta}$$

and

$$(1+\cos\theta)^{1/2} = \sqrt{2}\cos\frac{1}{2}\theta,$$

there follows

$$\begin{aligned}\int_{-1}^{1}\frac{V_n(t)}{r-t}(1-t)^{-1/2}(1+t)^{1/2}dt =& \int_{0}^{\pi}\frac{\cos(n+\frac{1}{2})\theta}{r-\cos\theta}\frac{1}{\cos\frac{1}{2}\theta}\frac{2}{\sqrt{2}}\frac{\cos\frac{1}{2}\theta}{\sin\theta}\sqrt{2}\cos\frac{1}{2}\theta\sin\theta d\theta \\ =& \int_{0}^{\pi}\frac{2\cos(n+\frac{1}{2})\theta\cos\frac{1}{2}\theta}{r-\cos\theta}d\theta \\ =& \int_{0}^{\pi}\frac{\cos n\theta}{r-\cos\theta}d\theta + \int_{0}^{\pi}\frac{\cos(n+1)\theta}{r-\cos\theta}d\theta \\ =& \frac{\pi\tau^n}{\sqrt{r^2-1}} + \frac{\pi\tau^{n+1}}{\sqrt{r^2-1}} \\ =& \frac{\pi\tau^n(1+\tau)}{\sqrt{r^2-1}}, \quad n = 0, 1, 2, \ldots,\end{aligned} \tag{3.53}$$

by (3.44) with $n+1$ in the place of n in the second integral. Also, there holds

$$V_n(t) = U_n(t) - U_{n-1}(t)$$

(cf. [7], Equation (1.17)). So, by (3.51), in view of (3.49), one has

$$
\begin{aligned}
V_n(r) =& U_n(r) - U_{n-1}(r) \\
=& \frac{1-\tau^{2n+2}}{2\tau^{n+1}\sqrt{r^2-1}} - \frac{1-\tau^{2n}}{2\tau^{n}\sqrt{r^2-1}} \\
=& \frac{(1-\tau)\left(1+\tau^{2n+1}\right)}{2\tau^{n+1}\sqrt{r^2-1}} \\
=& \frac{(1-\tau)\left(1+\tau^{2n+1}\right)}{\tau^{n}\left(1-\tau^2\right)} \\
=& \frac{1+\tau^{2n+1}}{\tau^n(1+\tau)}, \quad n=0,1,2,\ldots.
\end{aligned}
\tag{3.54}
$$

Hence, noting that

$$(\tau+1)^2 = 2r\tau + 2\tau = 2\tau(r+1),$$

and inserting (3.54), together with (3.53), into (3.52), yields

$$
\begin{aligned}
\|R_n^{(3)}\| =& \frac{r}{\frac{\tau^{2n+1}+1}{\tau^n(\tau+1)}} \frac{\pi\tau^n(1+\tau)}{\sqrt{r^2-1}} \\
=& \frac{\pi r\tau^{2n}(\tau+1)^2}{\left(\tau^{2n+1}+1\right)\sqrt{r^2-1}} \\
=& \frac{2\pi r\tau^{2n+1}(r+1)}{\left(\tau^{2n+1}+1\right)\sqrt{r^2-1}} \\
=& \frac{2\pi r\tau^{2n+1}}{1+\tau^{2n+1}}\sqrt{\frac{r+1}{r-1}},
\end{aligned}
$$

which proves (3.42). Finally, there holds

$$R_n^{(4)}\left(f(\cdot)\right) = R_n^{(3)}\left(f(-\cdot)\right),$$

thus,

$$R_n^{(4)}\left(t^k\right) = (-1)^k R_n^{(3)}\left(t^k\right), \quad k=0,1,2,\ldots.$$

The latter, combined with (3.11), implies that $\|R_n^{(3)}\| = \|R_n^{(4)}\|$. □

Chapter 4

Numerical Examples

In this chapter, we present some numerical experiments, in order to demonstrate the efficiency of our error estimates of the previous chapter. All computations were performed using our own Matlab routines.

Example 4.1. We estimate the error of the Gauss formula (3.1) for the integral

$$\int_{-1}^{1} e^{-t} w(t)dt,$$

where w is the Chebyshev weight of any of the four kinds (3.36)-(3.39). The integral was evaluated by the Gauss-Chebyshev quadrature formula of the first, second, third or fourth kind, respectively.

The function

$$f(z) = e^{-z} = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{k!}$$

is entire and

$$\begin{aligned} |f|_r &= \sup \left\{ \frac{r^k}{k!} \; : \; k \in \mathbb{N}_0 \quad \text{and} \quad R_n(t^k) \neq 0 \right\} \\ &= \begin{cases} \frac{r^{2n}}{(2n)!}, & 1 < r \leq 2n+1, \\ \frac{r^{2n+k}}{(2n+k)!}, & 2n+k < r \leq 2n+k+1, \;\; k = 1, 2, \ldots, \end{cases} \end{aligned}$$

so $f \in X_\infty$. Then, the bound $R_n(f)$ is estimated by means of (3.9), that is,

$$|R_n(f)| \leq \|R_n\| |f|_r, \tag{4.1}$$

where $\|R_n\|$ can be calculated by (3.40)-(3.42). Given that the right hand side of (4.1) depends on r, one can optimize (4.1), that is,

$$|R_n(f)| \leq \inf_{1<r<\infty} \left(\|R_n\| |f|_r \right). \tag{4.2}$$

Our results for various values of n are summarized in Tables 4.1-4.4. (Numbers in parentheses indicate decimal exponents.) The value r^* is the one for which we obtained the infinum in (4.2). When the number for the actual error is close to machine precision, the actual error could be larger than the error bound. In this case we enter "m.p." (for machine precision) in the last column.

n	r^*	Bound (4.2)	Error
2	5.000	1.739(−2)	1.720(−2)
5	11.000	1.733(−9)	1.729(−9)
10	21.000	2.494(−24)	m.p.
15	31.000	2.225(−41)	m.p.
20	41.000	7.048(−60)	m.p.

Table 4.1: Error bound (4.2) and actual error for Example 4.1 with $w = w^{(1)}$.

n	r^*	Bound (4.2)	Error
2	5.000	4.260(−3)	4.229(−3)
5	11.000	4.316(−10)	4.307(−10)
10	21.000	6.228(−25)	m.p.
15	31.000	5.558(−42)	m.p.
20	41.000	1.761(−60)	m.p.

Table 4.2: Error bound (4.2) and actual error for Example 4.1 with $w = w^{(2)}$.

n	r^*	Bound (4.2)	Error
2	5.000	1.054(−2)	6.964(−3)
5	11.000	9.475(−10)	7.878(−10)
10	21.000	1.307(−24)	m.p.
15	31.000	1.148(−41)	m.p.
20	41.000	3.610(−60)	m.p.

Table 4.3: Error bound (4.2) and actual error for Example 4.1 with $w = w^{(3)}$.

n	r^*	Bound (4.2)	Error
2	5.000	1.054(−2)	1.038(−2)
5	11.000	9.475(−10)	9.438(−10)
10	21.000	1.307(−24)	m.p.
15	31.000	1.148(−41)	m.p.
20	41.000	3.610(−60)	m.p.

Table 4.4: Error bound (4.2) and actual error for Example 4.1 with $w = w^{(4)}$.

As we can see, the bounds we have obtained are clearly sharp. However, it is interesting to see how bound (4.2) compares with other bounds. First, we choose a bound of the same type with the one in consideration. In view of (3.19), bound (4.1) can take the form

$$|R_n(f)| \leq \|R_n\| \max_{|z|=r} |f(z)| \,, \tag{4.3}$$

which, optimized, can be written as

$$|R_n(f)| \leq \inf_{1<r<\infty} \left(\|R_n\| \max_{|z|=r} |f(z)| \right),$$

where $\|R_n\|$ can be calculated, similarly, by (3.40)-(3.42). Since $|z| = r$, there holds

$$z = r(\cos\theta + i\sin\theta), \;\; 0 \leq \theta \leq \pi,$$

so,

$$\begin{aligned} |f(z)| = &\left| e^{-r(\cos\theta + i\sin\theta)} \right| = e^{-r\cos\theta} \left| e^{-ir\sin\theta} \right| \\ =&e^{-r\cos\theta} \leq e^r. \end{aligned}$$

Given that for $\theta = \pi$,

$$|f(z)| = e^r,$$

we obtain

$$\max_{|z|=r} |f(z)| = e^r.$$

Hence,

$$|R_n(f)| \leq \inf_{1<r<\infty} \left(\|R_n\| e^r \right). \tag{4.4}$$

The second bound we selected is a classical one. By Corollary 2.9, if $f \in C^{2n}[-1,1]$, there holds

$$R_n(f) = \frac{f^{(2n)}(\tau)}{(2n)!} \|\pi_n\|^2,$$

for some $\tau \in (-1,1)$, thus,

$$|R_n(f)| \leq \frac{\|\pi_n\|^2}{(2n)!} \max_{-1 \leq t \leq 1} \left| f^{(2n)}(t) \right|.$$

Given that π_n is the Chebyshev polynomial of any of the four kinds, $\|\pi_n\|^2 = h_n$, where h_n can be found in Table 1.1, and

$$\max_{-1 \leq t \leq 1} \left| f^{(2n)}(t) \right| = e,$$

we get

$$|R_n(f)| \leq \frac{h_n}{(2n)!} e. \tag{4.5}$$

Our results for both bounds (4.4) and (4.5) are summarized in Tables 4.5-4.7. The value r^* is the one for which we obtained the infinum in (4.4).

n	r^*	Bound (4.4)	Bound (4.5)
2	4.179	9.179(−2)	1.779(−1)
5	10.060	1.393(−8)	1.177(−6)
10	20.027	2.811(−23)	1.755(−18)
15	30.018	3.064(−40)	1.610(−32)
20	40.013	1.120(−58)	5.233(−48)

Table 4.5: Error bounds (4.4) and (4.5) for Example 4.1 with $w = w^{(1)}$.

n	r^*	Bound (4.4)	Bound (4.5)
2	4.122	2.227(−2)	1.779(−1)
5	10.150	3.464(−9)	1.177(−6)
10	20.025	7.018(−24)	1.755(−18)
15	30.017	7.657(−41)	1.610(−32)
20	40.012	2.799(−59)	5.233(−48)

Table 4.6: Error bounds (4.4) and (4.5) for Example 4.1 with $w = w^{(2)}$.

n	r^*	Bound (4.4)	Bound (4.5)
2	4.377	5.743(−2)	3.558(−1)
5	10.153	7.671(−9)	2.353(−6)
10	20.076	1.476(−23)	3.510(−18)
15	30.050	1.584(−40)	3.219(−32)
20	40.038	5.741(−59)	1.047(−47)

Table 4.7: Error bounds (4.4) and (4.5) for Example 4.1 with $w = w^{(3)}$ or $w = w^{(4)}$.

We notice that bound (4.4) provides a better estimate for the actual error than bound (4.5). Also, one can note that Table 4.7 refers to results for both $w^{(3)}$ and $w^{(4)}$. This was expected, since, by Theorem 3.9, $\|R_n^{(3)}\| = \|R_n^{(4)}\|$ and $h_n = \pi$ for both V_n and W_n.

Example 4.2. We estimate the error of the Gauss formula for the integral

$$\int_{-1}^{1} \ln \frac{2}{2-t} w(t)dt,$$

where w is the Chebyshev weight of any of the four kinds (3.36)-(3.39). The true value of the integral was evaluated as in Example 4.1.

The function

$$f(z) = \ln \frac{2}{2-z} = \sum_{k=1}^{\infty} \frac{z^k}{2^k k},$$

is holomorphic in $C_2 = \{z \in \mathbb{C} : |z| < 2\}$ and

$$\begin{aligned} |f|_r =& \sup\left\{ \frac{r^k}{2^k k} \;:\; k \in \mathbb{N}_0 \quad \text{and} \quad R_n(t^k) \neq 0 \right\} \\ =& \frac{r^{2n}}{2^{2n} \cdot 2n} = \frac{r^{2n}}{2^{2n+1} n}, \end{aligned}$$

since $R_n\left(t^k\right) = 0$ for $k = 0, 1, \ldots, 2n-1$. So $f \in X_2$ and the bound for the error functional $R_n\left(f\right)$ is estimated by the analogous of bound (4.2), that is,

$$\begin{aligned} |R_n(f)| \leq& \inf_{1<r\leq 2} \left(\|R_n\| |f|_r\right) \\ =& \inf_{1<r\leq 2} \left(\|R_n\| \frac{r^{2n}}{2^{2n+1} n}\right), \end{aligned} \tag{4.6}$$

where $\|R_n\|$ can be calculated by (3.40)-(3.42). Also, we provide a second bound, which is the analogous of bound (4.4), that is,

$$|R_n(f)| \leq \inf_{1<r<2} \left(\|R_n\| \max_{|z|=r} |f(z)|\right). \tag{4.7}$$

Noting that

$$|f(z)| = \left|\sum_{k=1}^{\infty} \frac{z^k}{2^k k}\right| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{2^k k},$$

we get

$$\max_{|z|=r} |f(z)| \leq \sum_{k=1}^{\infty} \frac{r^k}{2^k k} = \ln \frac{2}{2-r}.$$

Given that $|z| = r$, there holds

$$z = r(\cos\theta + i\sin\theta), \quad 0 \leq \theta \leq \pi,$$

so, for $\theta = 0$,

$$|f(z)| = \left|\ln \frac{2}{2-r}\right| = \ln \frac{2}{2-r},$$

as $1 < r < 2$. Thus,

$$\max_{|z|=r} |f(z)| = \ln \frac{2}{2-r},$$

and bound (4.7) takes the form

$$|R_n(f)| \leq \inf_{1<r<2} \left(\|R_n\| \ln \frac{2}{2-r} \right). \tag{4.8}$$

Our results are shown in Tables 4.8-4.11. The value r^* is the one for which we obtained the infima in (4.6) and (4.8), respectively.

n	r^*	Bound (4.6)	r^*	Bound (4.8)	Error
2	2.000	9.302(−3)	1.864	1.424(−1)	8.076(−3)
5	2.000	1.384(−6)	1.958	6.888(−5)	1.999(−6)
10	2.000	1.320(−12)	1.982	1.538(−10)	1.132(−12)
15	2.000	1.679(−18)	1.989	3.179(−16)	m.p.
20	2.000	2.403(−24)	1.992	6.395(−22)	m.p.

Table 4.8: Error bounds (4.6),(4.8) and actual error for Example 4.2 with $w = w^{(1)}$.

The results for both bounds are quite satisfactory. One can note, that bound (4.8) overestimates the actual error by utmost two or three orders of magnitude, while bound (4.6) is very close to the actual error. Also, as expected, columns 2,3,4 and 5 of Tables 4.10 and 4.11 are exactly the same, as $\|R_n^{(3)}\| = \|R_n^{(4)}\|$.

n	r^*	Bound (4.6)	r^*	Bound (4.8)	Error
2	2.000	2.015(−3)	1.848	3.001(−2)	1.836(−3)
5	2.000	2.981(−7)	1.956	1.472(−5)	2.649(−7)
10	2.000	2.844(−13)	1.982	3.301(−11)	2.260(−13)
15	2.000	3.617(−19)	1.989	6.834(−17)	m.p.
20	2.000	5.175(−25)	1.992	1.375(−22)	m.p.

Table 4.9: Error bounds (4.6),(4.8) and actual error for Example 4.2 with $w = w^{(2)}$.

n	r^*	Bound (4.6)	r^*	Bound (4.8)	Error
2	2.000	7.503(−3)	1.882	1.189(−1)	5.974(−3)
5	2.000	1.113(−6)	1.960	5.594(−5)	9.272(−7)
10	2.000	1.061(−12)	1.983	1.242(−10)	6.501(−13)
15	2.000	1.350(−18)	1.989	2.563(−16)	m.p.
20	2.000	1.931(−24)	1.992	5.151(−22)	m.p.

Table 4.10: Error bounds (4.6),(4.8) and actual error for Example 4.2 with $w = w^{(3)}$.

n	r^*	Bound (4.6)	r^*	Bound (4.8)	Error
2	2.000	7.503(−3)	1.882	1.189(−1)	2.508(−3)
5	2.000	1.113(−6)	1.960	5.594(−5)	3.432(−7)
10	2.000	1.061(−12)	1.983	1.242(−10)	4.731(−13)
15	2.000	1.350(−18)	1.989	2.563(−16)	m.p.
20	2.000	1.931(−24)	1.992	5.151(−22)	m.p.

Table 4.11: Error bounds (4.6),(4.8) and actual error for Example 4.2 with $w = w^{(4)}$.

Example 4.3. We approximate the integral

$$\int_{-1}^{1} \frac{t^2}{\omega^2 - t^2}\sqrt{1-t^2}dt, \;\; \omega > 0,$$

by the Gauss quadrature formula (3.1) relative to the Chebyshev weight function of the second kind. The true value of the integral was evaluated by the Gauss-Chebyshev quadrature formula of the second kind.

Note that the bounds we provide can be used to estimate the error in approximating the integral only if $\omega > 1$. Furthermore, the function

$$\begin{aligned} f(z) =& \frac{z^2}{\omega^2 - z^2} = \frac{z^2}{\omega^2} \frac{1}{1 - \left(\frac{z}{\omega}\right)^2} \\ =& \sum_{k=0}^{\infty} \frac{z^{2k+2}}{\omega^{2k+2}}, \end{aligned}$$

is holomorphic in $C_\omega = \{z \in \mathbb{C} : |z| < \omega\}$, $\omega > 1$. Then,

$$\begin{aligned} |f|_r =& \sup \left\{ \left| \frac{1}{\omega^{2k}} \right| r^{2k} \; : \; k \in \mathbb{N}_0 \quad \text{and} \quad R_n(t^{2k}) \neq 0 \right\} \\ =& \frac{r^{2n}}{\omega^{2n}}, \end{aligned}$$

as $1 < r \le \omega$. So, $f \in X_\omega$ and the analogous of bounds (4.2) and (4.6) for this example is

$$\begin{aligned} |R_n(f)| \le& \inf_{1<r\le\omega} \left(\|R_n\| |f|_r\right) \\ =& \inf_{1<r\le\omega} \left(\|R_n\| \frac{r^{2n}}{\omega^{2n}} \right), \end{aligned} \tag{4.9}$$

where $\|R_n\| = \|R_n^{(2)}\|$ and can be calculated by (3.41). As a result of (3.19), one can estimate the seminorm $|f|_r$, $1 < r \le \omega$, by $\max_{|z|=r} |f(z)|$. Since $|z| = r$, there holds

$$z = r(\cos\theta + i \sin\theta), \;\; 0 \le \theta \le \pi,$$

so,

$$|f(z)| = \frac{|z|^2}{|\omega^2 - z^2|},$$

and noting that

$$\begin{aligned} \left|\omega^2 - z^2\right| =& \left|\omega^2 - r^2 \left(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta\right)\right| \\ =& \left|\omega^2 - r^2\cos 2\theta - ir^2 \sin 2\theta\right| \\ =& \left(\omega^4 - 2\omega^2 r^2 \cos 2\theta + r^4\right)^{1/2} \\ \ge& (\omega^4 - 2\omega^2 r^2 + r^4)^{1/2} = \omega^2 - r^2, \end{aligned}$$

we get

$$\max_{|z|=r} |f(z)| \le \frac{r^2}{\omega^2 - r^2}.$$

Given that for $\theta = 0$,

$$|f(z)| = \left| \frac{r^2}{\omega^2 - r^2} \right| = \frac{r^2}{\omega^2 - r^2},$$

as $1 < r < \omega$, we obtain

$$\max_{|z|=r} |f(z)| = \frac{r^2}{\omega^2 - r^2}.$$

Hence, the analogous of bounds (4.4) and (4.8) for this example is

$$|R_n(f)| \leq \inf_{1<r<\omega} \left(\|R_n\| \frac{r^2}{\omega^2 - r^2} \right). \tag{4.10}$$

Our results for various values of n and ω are shown in Table 4.12. The value r^* is the one for which we obtained the infima in (4.9) and (4.10), respectively.

	n	r^*	Bound (4.9)	r^*	Bound (4.10)	Error
$\omega = 2$	2	2.000	8.058(−3)	1.562	4.158(−2)	8.058(−3)
	5	2.000	2.981(−6)	1.824	4.335(−5)	2.981(−6)
	10	2.000	5.688(−12)	1.913	1.719(−10)	5.602(−12)
	15	2.000	1.085(−17)	1.942	4.983(−16)	m.p.
	20	2.000	2.070(−23)	1.957	1.275(−21)	m.p.
$\omega = 4$	2	4.000	4.087(−4)	2.912	1.740(−3)	4.087(−4)
	5	4.000	1.716(−9)	3.595	2.181(−8)	1.716(−9)
	10	4.000	1.876(−18)	3.802	5.018(−17)	m.p.
	15	4.000	2.050(−27)	3.869	8.364(−26)	m.p.
	20	4.000	2.241(−36)	3.902	1.229(−34)	m.p.
$\omega = 8$	2	8.000	2.435(−5)	5.700	9.892(−5)	2.435(−5)
	5	8.000	1.486(−12)	7.164	1.832(−11)	1.484(−12)
	10	8.000	1.406(−24)	7.593	3.660(−23)	m.p.
	15	8.000	1.330(−36)	7.731	5.284(−35)	m.p
	20	8.000	1.258(−48)	7.799	6.721(−47)	m.p
$\omega = 16$	2	16.000	1.504(−6)	11.336	6.039(−6)	1.504(−6)
	5	16.000	1.409(−15)	14.315	1.724(−14)	1.036(−15)
	10	16.000	1.264(−30)	15.181	3.269(−29)	m.p.
	15	16.000	1.133(−45)	15.459	4.476(−44)	m.p.
	20	16.000	1.017(−60)	15.596	5.399(−59)	m.p.

Table 4.12: Error bounds (4.9), (4.10) and actual error for Example 4.3 with $\omega = 2, 4, 8$ or 16.

Here, one can see that bound (4.9) is very accurate for smaller values of n and remains pretty close to the actual error as n increases. On the other hand, bound (4.10) overestimates the actual error by one or two orders of magnitude as n increases.

Example 4.4. We approximate the integral

$$\int_{-1}^{1} \frac{\cos t}{t^2+\omega^2}\sqrt{1-t^2}dt, \quad \omega > 0,$$

by the Gauss quadrature formula (3.1) relative to the Chebyshev weight function of the second kind. The true value of the integral was evaluated as in the previous example.

Both $\cos z$ and $1/(z^2+\omega^2)$ have Maclaurin series expansion for $z \in \mathbb{C}$ and $z \in C_\omega$, respectively. By the series multiplication theorem, as

$$\frac{1}{z^2+\omega^2} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{\omega^{2k+2}}$$

and

$$\cos z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!},$$

we have

$$\begin{aligned} f(z) =& \frac{\cos z}{z^2+\omega^2} = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^j \frac{z^{2j}}{(2j)!}(-1)^{k-j}\frac{z^{2(k-j)}}{\omega^{2(k-j)+2}}\right] \\ =& \sum_{k=0}^{\infty}(-1)^k \left[\sum_{j=0}^{k}\frac{\omega^{2j}}{(2j)!}\right]\frac{z^{2k}}{\omega^{2k+2}} \\ =& \sum_{k=0}^{\infty}(-1)^k\left[1+\frac{\omega^2}{2!}+\cdots+\frac{\omega^{2k}}{(2k)!}\right]\frac{z^{2k}}{\omega^{2k+2}}, \quad z \in C_\omega. \end{aligned}$$

The bounds under consideration can be used to estimate the error in approximating the integral only if $\omega > 1$. Also, the computation of the seminorm $|f|_r$, $1 < r < \omega$, is quite cumbersome, so, from (3.19), we estimate $|f|_r$ by $\max_{|z|=r}|f(z)|$. Writing $z = re^{i\theta}$, so $|z| = r$, and $\cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$, gives

$$\begin{aligned} |f(z)| =& \left|\frac{\cos z}{z^2+\omega^2}\right| = \frac{1}{2}\left|\frac{e^{iz}+e^{-iz}}{z^2+\omega^2}\right| \\ =& \frac{1}{2}\left|\frac{e^{ire^{i\theta}}+e^{-ire^{i\theta}}}{\left(re^{i\theta}\right)^2+\omega^2}\right| \\ =& \frac{1}{2}\left|\frac{e^{-r\sin\theta+ir\cos\theta}+e^{r\sin\theta-ir\cos\theta}}{r^2\cos^2\theta+2ir^2\cos\theta\sin\theta-r^2\sin^2\theta+\omega^2}\right| \\ =& \frac{1}{2}\left|\frac{e^{-r\sin\theta+ir\cos\theta}+e^{r\sin\theta-ir\cos\theta}}{r^2\cos 2\theta+ir^2\sin 2\theta+\omega^2}\right|. \end{aligned} \tag{4.11}$$

As,

$$\begin{aligned}\left|e^{-r\sin\theta+ir\cos\theta}+e^{r\sin\theta-ir\cos\theta}\right| \leq & e^{-r\sin\theta}\left|e^{ir\cos\theta}\right|+e^{r\sin\theta}\left|e^{-ir\cos\theta}\right| \\ = & e^{r\sin\theta}+e^{-r\sin\theta}\end{aligned}$$

and

$$\begin{aligned}\left|r^2\cos 2\theta+ir^2\sin 2\theta+\omega^2\right| &= \left(r^4\cos^2 2\theta+2r^2\omega^2\cos 2\theta+\omega^4+r^4\sin^2 2\theta\right)^{1/2} \\ &= \left(r^4+2r^2\omega^2\cos 2\theta+\omega^4\right)^{1/2} \\ &\geq \left(\omega^4-2r^2\omega^2+r^4\right)^{1/2}=\omega^2-r^2,\end{aligned}$$

(4.11) takes the form

$$|f(z)| \leq \frac{1}{2}\frac{e^{r\sin\theta}+e^{-r\sin\theta}}{\omega^2-r^2}=\frac{\cosh\left(r\sin\theta\right)}{\omega^2-r^2}.$$

Noting that the maximum of $\cosh\left(r\sin\theta\right)$ is $\cosh(r)$, we get

$$\max_{|z|=r}|f(z)| \leq \frac{\cosh(r)}{\omega^2-r^2},$$

and, given that for $\theta=\pi/2$,

$$|f(z)|=\frac{1}{2}\frac{e^{r}+e^{-r}}{\omega^2-r^2}=\frac{\cosh(r)}{\omega^2-r^2},$$

we finally obtain

$$\max_{|z|=r}|f(z)|=\frac{\cosh(r)}{\omega^2-r^2}.$$

Thus $f \in X_\omega$ and the analogous of bounds (4.4), (4.8) and (4.10) for this example is

$$|R_n\left(f\right)| \leq \inf_{1<r<\omega}\left(\|R_n\|\frac{\cosh(r)}{\omega^2-r^2}\right), \tag{4.12}$$

where $\|R_n\|=\|R_n^{(2)}\|$ and can be calculated by (3.41).

Our results for various values of n and ω are shown in Table 4.13. The value r^* is the one for which we obtained the infinum in (4.12).

We note that bound (4.12) worsens as ω decreases, as it overestimates the actual error up to three orders of magnitude. This is a consequence of the large value of $\max_{|z|=r}|f(z)|$ as $r^* \to \omega$. Also, for a fixed value of ω, the bound is worse for higher values of n.

	n	r^*	Bound (4.12)	Error
$\omega = 2$	2	1.602	4.211(−2)	4.429(−3)
	5	1.828	4.140(−5)	7.916(−7)
	10	1.913	1.626(−10)	4.277(−13)
	15	1.942	4.701(−16)	m.p.
	20	1.957	1.202(−21)	m.p.
$\omega = 3$	2	2.201	5.743(−3)	1.060(−3)
	5	2.687	4.665(−7)	2.223(−8)
	10	2.850	2.258(−14)	2.470(−15)
	15	2.902	7.717(−22)	m.p.
	20	2.927	2.309(−29)	m.p.
$\omega = 4$	2	2.927	1.829(−3)	4.354(−4)
	5	3.523	3.028(−8)	2.139(−9)
	10	3.783	7.745(−17)	*m.p.*
	15	3.861	1.336(−25)	m.p.
	20	3.897	1.996(−34)	m.p.

Table 4.13: Error bound (4.12) and actual error for Example 4.4 with $\omega = 2, 3$ or 4.

Appendix

Table of orthogonal polynomials

Table A.1: Recurrence coefficients and weight functions for classical monic orthogonal polynomials with respect to $d\lambda(t) = w(t)dt$.

Name	$w(t)$	Support	a_k	β_0	$\beta_k,\ k \geq 1$
#1 Chebyshev	$\left(1-t^2\right)^{-1/2}$	$[-1,1]$	0	π	$\frac{1}{2}\,(k=1),\ \frac{1}{4}\,(k>1)$
#2 Chebyshev	$\left(1-t^2\right)^{1/2}$	$[-1,1]$	0	$\frac{1}{2}\pi$	$\frac{1}{4}$
#3 Chebyshev	$(1-t)^{-1/2}(1+t)^{1/2}$	$[-1,1]$	$\frac{1}{2}\,(k=0)$ $0\,(k>0)$	π	$\frac{1}{4}$
#4 Chebyshev	$(1-t)^{1/2}(1+t)^{-1/2}$	$[-1,1]$	$-\frac{1}{2}\,(k=0)$ $0\,(k>0)$	π	$\frac{1}{4}$
Jacobi	$(1-t)^a(1+t)^\beta,\ a>-1, \beta>-1$	$[-1,1]$	a_k^J	β_0^J	β_k^J
Gegenbauer	$(1-t^2)^{\lambda-1/2},\ \lambda > -\frac{1}{2}$	$[-1,1]$	0	$\sqrt{\pi}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}$	$\frac{k(k+2\lambda-1)}{4(k+\lambda)(k+\lambda-1)}$
Legendre	1	$[-1,1]$	0	2	$\frac{1}{4-k^{-2}}$
Generalized Laguerre	$t^a e^{-t},\ a>-1$	$[0,\infty]$	$2k+a+1$	$\Gamma(1+a)$	$k(k+a)$
Laguerre	e^{-t}	$[0,\infty]$	$2k+1$	1	k^2
Hermite	e^{-t^2}	$[-\infty,\infty]$	0	$\sqrt{\pi}$	$\frac{1}{2}k$

$$a_k^J = \frac{\beta^2-a^2}{(2k+a+\beta)(2k+a+\beta+2)}^{*}$$

$$\beta_0^J = \frac{2^{a+\beta+1}\Gamma(a+1)\Gamma(\beta+1)}{\Gamma(a+\beta+1)},\quad \beta_k^J = \frac{4k(k+a)(k+\beta)(k+a+\beta)}{(2k+a+\beta)^2(2k+a+\beta+1)(2k+a+\beta-1)}^{**}.$$

*If $k=0$, the common factor $a+\beta$ in the numerator and denominator of a_0^J should be cancelled.

**If $k=1$, the last factors in the numerator and denominator of β_1^J should be cancelled.

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