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Enumerative combinatorics, representations and quasisymmetric functions

Dissertation
by
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It is dedicated to my parents,
Anastasia and Petros.

Abstract

The present thesis consists of two parts whose main protagonists are colored quasisymmetric functions. In 1984, Gessel introduced quasisymmetric functions, a generalization of symmetric functions. In 1993, together with Reutenauer they studied specializations of families of quasisymmetric functions associated to subsets of the symmetric group, which have many desirable properties, such as symmetry and Schur-positivity. In 1998, Poirier introduced colored quasisymmetric functions, a colored analogue of quasisymmetric functions. In the first part, we develop a general theory of specializations of colored quasisymmetric functions in the spirit of Gessel and Reutenauer's work. This allows us to systematically prove refined Euler–Mahonian identities on colored permutation groups and subsets of these, such as derangements and involutions. In 2017, Elizalde and Roichman proved that the quasisymmetric function of the product of a collection of permutations whose quasisymmetric generating function equals the Frobenius characteristic of some character χ of the symmetric group and an inverse descent class equals the Frobenius characteristic of the character of the tensor product of χ and the corresponding descent representation of the symmetric group. The second part deals with proving a colored analogue of Elizalde and Roichman's result. More precisely, we introduce a notion of colored ribbons and prove that the (colored) Frobenius characteristic of the descent representation of colored permutation groups equals the colored quasisymmetric generating function of colored ribbon shaped tableaux. This provides a colored analogue of Gessel's zig-zag shape approach to descent representations of the symmetric group. In addition, exploiting Hsiao–Petersen's theory of colored P-partitions and the method developed in the first part, we prove a colored analogue of Stanley's shuffling theorem.

Introduction

It is said that if the twentieth century was the century of symmetric functions, then perhaps the twenty-first century will be defined by the explosion of developments in the theory of quasisymmetric functions [26, Section 1]. The aim of this thesis is to present further evidence for this hypothesis. In particular, it consists of roughly two parts. The first part presents several generalizations of the author's published results [70, 71] on specializations of colored quasisymmetric functions and the second part presents material on descent representations of colored permutation groups and colored quasisymmetric functions.

A major theme in enumerative combinatorics is the study of distributions of permutation statistics. It can be traced all the way back to MacMahon's 1900 work on plane partitions (see [53, Section 3]). Stanley in his 1971 Ph.D. thesis [86] developed the theory of P-partitions and used it to study, what we now call, Euler–Mahonian distributions. Euler–Mahonian distributions appear often at the intersection of algebra, combinatorics and geometry (see, for example, [74]).

The most prominent example of an Eulerian distribution is the number of descents, encoded via Eulerian polynomials. The n-th Eulerian polynomial is the numerator on the right-hand side of the identity

$$\sum_{m\geq 0}(m+1)^n\, x^m \;=\; \frac{\sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)}}{(1-x)^{n+1}},$$

often used as its definition (for all missing definitions we refer to Chapter 1). Several interesting q-analogues of Eulerian polynomials exist in the literature (see, for example, the notes of [90, Chapter 1]). The one involving the major index will be of particular interest in this thesis. The generating polynomial for the distribution of the Euler-Mahonian pair $(\mathrm{des}, \mathrm{maj})$ satisfies

$$\sum_{m\geq 0}(1+q+\cdots+q^m)^n x^m \;=\; \frac{\sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)}q^{\mathrm{maj}(w)}}{(1-x)(1-xq)\cdots(1-xq^n)}.$$

Identities of this type are called Euler–Mahonian identities.

Since the symmetric group is a Coxeter group, a common theme in algebraic combinatorics is to generalize a combinatorial statement involving the symmetric group to other Coxeter groups or complex reflection groups, starting from the hyperoctahedral group, namely the group of symmetries of the higher dimensional cube. In the past two decades, several combinatorialists such as Adin, Brenti, Foata, Gessel, Mansour, Roichman and Zeng, to name a few, have studied such generalizations of Euler–Mahonian identities. A systematic approach to the case of colored permutation groups was established recently in 2013 by Beck and Braun [20] using techniques from polyhedral geometry.

Another possible approach is via symmetric/quasisymmetric functions. Quasisymmetric functions appeared in Stanley's work as generating functions of P-partitions (see [26, Section 2]). Gessel in his 1984 paper [52] formalized this new tool and studied the algebra of quasisymmetric functions. Gessel and Reutenauer in a 1993 seminal paper [55] initiated a study of specializations of families of quasisymmetric functions associated to subsets of the symmetric group, which have many desirable properties, such as symmetry and Schur-positivity.

Poirier in his 1998 Ph.D. thesis [76] introduced a colored generalization of quasisymmetric functions. It is our purpose in the first part of this thesis to develop a general theory of specializations of colored quasisymmetric functions in the spirit of Gessel and Reutenauer's work. This will allow us to systematically prove refined Euler–Mahonian identities on colored permutation groups. The main advantage of this theory is that it allows us to prove new Euler–Mahonian identities on several important classes of colored permutations, such as involutions and colored permutations without fixed points.

A major theme in algebraic combinatorics and combinatorial representation theory is the study of character formulas which express the values of characters of the symmetric group as weighted enumerations of combinatorial objects. An example of such character formula is the Murnaghan–Nakayama rule [89, Section 7.17], where the enumerated objects are border strip tableaux. Particularly interesting is the discovery of such formulas expressing the values of characters in terms of the distribution of the descent set over certain classes of permutations. The archetypal example is Roichman's rule [78], a formula for the irreducible characters of the symmetric group where the enumerated objects are standard Young tableaux. It turns out that the existence of such formulas is closely related to Schur-positivity.

Representation theory of the symmetric group is connected to the theory of symmetric functions via the Frobenius characteristic map. In particular, it maps the irreducible characters of the symmetric group to Schur functions, which in turn, by a result of Stanley, are the quasisymmetric generating functions of standard Young tableaux. Adin and Roichman in 2015 [8] developed an abstract framework to capture this phenomenon. More recently, in 2017, Adin, Athanasiadis, Elizalde and Roichman [1] introduced and studied a signed analogue, in which colored quasisymmetric functions, in the special case of two colors, play a central role.

The set of elements in a Coxeter group having a given descent set carries a natural representation of the group, called the descent representation. The study of descent representations has its origin's in Solomon's work [82] on Weyl groups, where they appear as alternating sums of certain permutation representations. This concept was first extended to the hyperoctahedral group by Adin, Brenti and Roichman in a 2005 paper [3]. Shortly after, in 2007, Bagno and Biagioli [15] further extended it to complex reflection groups. This construction involves the coinvariant algebra as the representation space.

Descent representations are closely related to inverse descent classes of permutations. It is well known that the Frobenius characteristic of the descent representation of the symmetric group can be expressed as the quasisymmetric generating function of the corresponding inverse descent classes. Elizalde and Roichman [39] proved a strengthening of this fact, where the descent representations are replaced by their tensor product with some representation of the symmetric group. The second part of this thesis aims to provide a colored analogue of this result, via studying descent representations of colored permutation groups. In particular, a colored version of Gessel's approach to descent representations, which uses ziz-zag (or ribbon) shapes, is developed by introducing a notion of colored ribbons. This could potentially yield many instances of Schur-positive classes of colored permutations.

One of the main ingredients of the proof of the aforementioned result is of particular interest. Hsiao and Petersen [62] developed a colored analogue of Stanley's theory of P-partitions in order to exploit several connections between Hopf algebras arising from colored quasisymmetric functions. Building upon their theory, we aim to provide a colored analogue of Stanley's shuffling theorem. Stanley's shuffling theorem asserts that the distribution of the descent set over all shuffles of two disjoint permutations u and v depends only on the descent sets of u and v and their lengths. Recently, Gessel and Zhuang [56] formalized this remarkable property of the descent set, called shuffle-compatibility and studied in depth several shuffle-compatible permutation statistics such as the descent number, major index, peak set, peak number, left peak set, left peak number and so on. In particular, we aim to present a colored analogue of a small portion of their work and thus initiate the study of shuffle-compatible colored permutation statistics.

This thesis is structured as follows. Chapter 1 reviews background material needed for chapters that follow. In particular, it surveys Euler–Mahonian distributions on colored permutation statistics and reviews the combinatorics of colored permutation groups, symmetric/quasisymmetric functions and the connection between Schur-positivity, quasisymmetric functions and the representation theory of the symmetric group. Chapters 2 and 3 and Chapters 4 and 5 comprise the first and second part of this thesis, respectively.

Chapter 2 reviews the notion of colored quasisymmetric functions and develops a method for specializing them to derive general refined formulas for the distribution of a Mahonian statistic and the pair of an Eulerian and a Mahonian statistic. In addition, it introduces the (k, ℓ)-flag major index of signed permutations, a notion which generalizes both the major index and the flag major index and derives general refined formulas for the distribution of this statistic and its joint distribution with some Eulerian partner.

Chapter 3 applies the method developed in the previous chapter to prove refined Euler–Mahonian identities. In particular, Section 3.1 refines known Euler–Mahonian identities on colored permutations. Section 3.2 refines known Euler–Mahonian identities on colored derangements and proves several new ones. Section 3.3 studies refinements of fix–Euler–Mahonian distributions on colored involutions. Lastly, Section 3.4 studies refinements of multivariate distributions, involving Eulerian and Mahonian statistics on colored permutations.

Chapter 4 reviews Hsiao and Petersen's theory of colored P-partitions and uses it to prove colored analogues of Stanley's shuffling theorem. Afterwards, it briefly reviews Gessel and Zhuang's theory of shuffle-compatible permutation statistics and proves that the colored descent set and the colored peak composition are shuffle-compatible.

Chapter 5 reviews elements of the combinatorial theory of colored permutation groups, especially Poirier's version of the characteristic map and proves useful identities for the sequel. In addition, it reviews descent representations and their connection to quasisymmetric functions and reviews Schur-positivity in the colored context. A notion of colored ribbons is introduced, similar to that of regular ribbons (or zig-zag shapes), which is used to describe colored descent representations for colored permutation groups. The rest of the chapter is devoted to the statement and proof of the colored analogue of Elizalde and Roichman's result mentioned above.

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Contents

A stagecoach passed by on the road and went on;
And the road didn't become more beautiful or even more ugly.
That's human action on the outside world.
We take nothing away and we put nothing back, we pass by and we forget;
And the sun is always punctual every day.

Fernando Pessoa

Chapter 1

Preliminaries

This chapter fixes notation and reviews background material on colored permutation statistics, colored permutation groups, symmetric/quasisymmetric functions and connections with the representation theory of the symmetric group.

Throughout this thesis we assume familiarity with basic combinatorics of the symmetric group, including combinatorics of permutations and tableaux, representations and symmetric functions, as presented in [90, Chapter 1], [7], [89, Chapter 7] and [80], as well as with basic theory of partially ordered sets (posets hereafter) as presented in [90, Section 3].

1.1 Permutation statistics and the Euler-Mahonian identity

For a positive integer n, let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. We will think of permutations $w \in \mathfrak{S}_n$ as n-element words $w = w_1 w_2 \cdots w_n$. For $w \in \mathfrak{S}_n$, an index $i \in [n-1]$ is called a *descent* of w, if $w_i > w_{i+1}$. The set of all descents of w, written $\mathrm{Des}(w)$, is called the *descent set* of w. The cardinality and the sum of all elements of $\mathrm{Des}(w)$ are written as $\mathrm{des}(w)$ and $\mathrm{maj}(w)$, respectively, and called the *descent number* and *major index* of w.

A statistic on $\mathfrak{S}_n$ which is equidistributed with des (resp. maj) is called *Eulerian* (resp. *Mahonian*). Let

$$A_n(x,q) \; := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} \tag{1.1}$$

be the generating polynomial for the joint distribution $(\mathrm{des}, \mathrm{maj})$ on $\mathfrak{S}_n$, sometimes called the n-th *q-Eulerian polynomial*[1]. The polynomial $A_n(x) := A_n(x, 1)$ is called the n-th *Eulerian polynomial* and constitutes one of the most important polynomials

[1] Several q-analogues of Eulerian polynomials appear in the literature (see, for example, the references in [90, Section 1]). In this thesis we focus on those involving the pair $(\mathrm{des}, \mathrm{maj})$ and variants of those.

in combinatorics. They have been in the spotlight of research in combinatorics in recent years and throughout the second half of the 20th century. For a detailed exposition of their importance in combinatorics, algebra and geometry we refer to Petersen's excellent book [74].

MacMahon [65, Vol.2, Section IX] proved a formula which specializes to

$$\sum_{m\geq 0} [m+1]_q^n \, x^m \;=\; \frac{A_n(x,q)}{(1-x)(1-xq)\cdots(1-xq^n)}, \tag{1.2}$$

where $[n]_q := 1+q+\cdots+q^{n-1}$ is the q-analogue of n. This formula is usually attributed to Carlitz [30] and hence called the *Carlitz identity*. We will call it the *Euler–Mahonian identity*. Several proofs of the Euler–Mahonian identity are known in the literature. For example, one can prove it using a "balls into boxes" argument similar to that of [74, Exercise 1.14]. In Section 2.2, we provide a proof of Equation (1.2) using (quasi)symmetric functions, which serves as the motivation for the method developed in that chapter.

Remark 1.1.1. For $q=1$, Equation (1.2) reduces to the following identity [90, Proposition 1.4.4]

$$\sum_{m\geq 0} (m+1)^n x^m \;=\; \frac{A_n(x)}{(1-x)^{n+1}}, \tag{1.3}$$

which is sometimes used as the definition of Eulerian polynomials.

1.2 Colored permutation statistics

Fix a positive integer r and let $\mathbb{Z}_r$ be the additive cyclic group of order r. The elements of $\mathbb{Z}_r$ will be represented by those of $[0, r-1]$ and will be thought of as colors. We think of the set $[n]\times\mathbb{Z}_r$ as the set of r*-colored integers*

$$\Omega_{n,r} := \{1^0, 2^0, \dots, n^0, 1^1, 2^1, \dots, n^1, \dots, 1^{r-1}, 2^{r-1}, \dots, n^{r-1}\}.$$

We may often identify colored integers i^0 with i.

The r*-colored permutation group*, denoted by $\mathfrak{S}_{n,r}$, consists of all permutations of $\Omega_{n,r}$, i.e. bijective maps $\sigma : \Omega_{n,r} \to \Omega_{n,r}$, such that

$$\sigma(a^0) = b^j \;\Rightarrow\; \sigma(a^i) = b^{i+j}, \tag{1.4}$$

where $i+j$ is computed modulo r and the product of $\mathfrak{S}_{n,r}$ is composition of permutations. The r-colored permutation group can be realized as the *wreath product* $\mathbb{Z}_r \wr \mathfrak{S}_n$, that is the semidirect product $\mathbb{Z}_r^n \rtimes \mathfrak{S}_n$ for the usual permutation action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{Z}_r^n$. It can be also realized as a complex reflection group, consisting of all $n\times n$ matrices whose non-zero entries are rth roots of unity such that there is exactly one non-zero entry in every row and every column.

Elements of $\mathfrak{S}_{n,r}$ are called r*-colored permutations*. We will think of colored permutations $w^\epsilon \in \mathfrak{S}_{n,r}$ as n-element words $w^\epsilon = w_1^{\epsilon_1} w_2^{\epsilon_2} \cdots w_n^{\epsilon_n}$ on $\Omega_{n,r}$. We will call $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ the *underlying permutation* and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_r^n$

the *color vector* of w^{ϵ}[2]. The product of two colored permutations in $\mathfrak{S}_{n,r}$ is given by

$$u^{\epsilon} \cdot v^{\delta} \ = \ (uv)^{v(\epsilon)+\delta}$$

where $uv = u \circ v$ is evaluated from right to left, $v(\epsilon) := (\epsilon_{v_1}, \epsilon_{v_2}, \ldots, \epsilon_{v_n})$ and the addition is coordinatewise modulo r. The inverse of $w^{\epsilon} \in \mathfrak{S}_{n,r}$ is the element $\left(w^{-1}\right)^{\delta}$, where w^{-1} is the inverse of w in $\mathfrak{S}_n$ and $\delta = -w^{-1}(\epsilon)$, where

$$-\epsilon := (-\epsilon_1, -\epsilon_2, \ldots, -\epsilon_n)$$

where the entries are computed modulo r. Also, define $\overline{w^{\epsilon}} := w^{-\epsilon}$[3]. The following observation will be useful in Chapter 5.

Observation 1.2.1. For all $w^{\epsilon} \in \mathfrak{S}_{n,r}$, we have $\left(\overline{w^{\epsilon}}\right)^{-1} = \left(w^{-1}\right)^{w^{-1}(\epsilon)}$. In addition, if $u = \left(\overline{w^{\epsilon}}\right)^{-1}$, then

(1) $\overline{u} = (w^{\epsilon})^{-1}$,

(2) $\overline{u}^{-1} = w^{\epsilon}$, and

(3) $u^{-1} = \overline{w^{\epsilon}}$.

The case $r = 2$ is of special interest: $\mathfrak{S}_{n,2}$ is the hyperoctahedral group, written $\mathfrak{B}_n$, the group of *signed permutations* of length n. The hyperoctahedral group is a Coxeter group of type B_n which can be realized as the group of symmetries of the n-dimensional cube (see, for example, [74, Part III]). In this case, following [1], we use the bar notation to indicate 1-colored integers, i.e.

$$\Omega_n \ := \ \Omega_{n,2} \ = \ \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\},$$

and the words color and colored are replaced by bar and barred, respectively. In this case also, we think of signed permutations as n-element words $w = w_1 w_2 \cdots w_n$ on Ω_n.

Following the success of the study of permutation statistics in the second half of the twentieth century (although its origin traces back to the work of MacMahon [65]) many authors studied combinatorial properties of the distributions of signed permutation statistics. In particular, Brenti [29] studied a notion of descent for signed permutations that arises when we view $\mathfrak{B}_n$ as a Coxeter group. Around the same time, Reiner [77] studied a notion of descent based on the usual geometric description of $\mathfrak{B}_n$ using root systems. Later, Adin, Brenti and Roichman [2] addressed Foata's problem of extending the Euler–Mahonian distribution of $(\mathrm{des}, \mathrm{maj})$ to the hyperoctahedral group $\mathfrak{B}_n$.

We discuss several developments on Foata's problem for colored permutation groups. For this purpose, we need to specify what colored permutation statistics qualify as Eulerian, Mahonian and Euler–Mahonian.

[2]When there is no case of confusion we will represent both the colored permutation and its underlying permutation by the same letter.

[3]This is essentially the complex conjugate of w^{ϵ}, when viewed as a complex reflection group.

1.2.1 Eulerian statistics

Steingrímsson in his Ph.D. thesis [92] initiated the study of colored permutation statistics, by introducing a notion of descent for colored permutations. For $w^\epsilon \in \mathfrak{S}_{n,r}$, an index $i \in [n]$ is called a *descent* of w^ϵ, if

- $1 \le i \le n-1$ and either $\epsilon_i > \epsilon_{i+1}$, or $\epsilon_i = \epsilon_{i+1}$ and $w_i > w_{i+1}$
- $i = n$ and $\epsilon_n \neq 0$.

Consider the right lexicographic order on $[n] \times \mathbb{Z}_r$, or in other words, the following total order

$$1^0 <_{\mathrm{St}} \cdots <_{\mathrm{St}} n^0 <_{\mathrm{St}} 1^1 <_{\mathrm{St}} \cdots <_{\mathrm{St}} n^1 <_{\mathrm{St}} \cdots <_{\mathrm{St}} 1^{r-1} <_{\mathrm{St}} \cdots <_{\mathrm{St}} n^{r-1}$$

on $\Omega_{n,r}$. We may equivalently[4] define a descent of w^ϵ as an index $i \in [n]$ such that

- $1 \le i \le n-1$ and $w_i^{\epsilon_i} >_{\mathrm{St}} w_{i+1}^{\epsilon_{i+1}}$
- $i = n$ and $\epsilon_n \neq 0$.

Let $\mathrm{Des}_{<_{\mathrm{St}}}(w^\epsilon)$ be the set of all descents of w^ϵ and $\mathrm{des}_{<_{\mathrm{St}}}(w^\epsilon)$ be its cardinality. Steingrímsson [92, Theorem 17] proved that

$$\sum_{m \ge 0} (rm+1)^n \, x^m \;=\; \frac{\sum_{w \in \mathfrak{S}_{n,r}} x^{\mathrm{des}_{<_{\mathrm{St}}}(w)}}{(1-x)^{n+1}}. \tag{1.5}$$

This formula reduces to Equation (1.3) for $r = 1$ and generalizes a formula of Brenti [29, Theorem 3.4 (ii)] for $r = 2$.

Biagioli and Caselli [22] studied a notion of descents by considering the right lexicographic order on $[n] \times \mathbb{Z}_r$, when the elements of $\mathbb{Z}_r$ are ordered as $r-1 <' \cdots <' 1 <' 0$. This is often called the *color order* and in terms of colored integers is the following total order

$$1^{r-1} <_c \cdots <_c n^{r-1} <_c \cdots <_c 1^1 <_c \cdots <_c n^1 <_c 1^0 <_c \cdots <_c n^0$$

on $\Omega_{n,r}$. For $w^\epsilon \in \mathfrak{S}_{n,r}$, we define $\mathrm{Des}_{<_c}(w^\epsilon)$ to be the set of all indices $i \in [n-1]$ such that $w_i^{\epsilon_i} >_c w_{i+1}^{\epsilon_{i+1}}$ together with 0, whenever $\epsilon_1 \neq 0$ and write $\mathrm{des}_{<_c}(w^\epsilon)$ for its cardinality. It follows from their work [22, Corollary 5.3 for $p = s = q = 1$] that Equation (1.5) holds if we replace $<_{\mathrm{St}}$ with $<_c$. This fact can also be proved bijectively (see, for example, the proof of [11, Proposition 2.2]).

Another notion of descent set was studied by Biagioli and Zeng [24]. In particular, let $<_\ell$ be the following total order on $\Omega_{n,r}$

$$n^{r-1} <_\ell \cdots <_\ell n^1 <_\ell \cdots <_\ell 1^{r-1} <_\ell \cdots <_\ell 1^1 <_\ell 1^0 <_\ell \cdots <_\ell n^0,$$

called the *length order*. For $w^\epsilon \in \mathfrak{S}_{n,r}$, we define $\mathrm{Des}_{<_\ell}(w^\epsilon)$ to be the set of all indices $1 \le i \le n-1$ such that $w_i^{\epsilon_i} >_\ell w_{i+1}^{\epsilon_{i+1}}$ together with 0, whenever $\epsilon_1 \neq 0$

[4] For the purposes of this thesis it is more convinient to consider descents on colored permutations according to some fixed total order on $\Omega_{n,r}$.

and write $\mathrm{des}_{<_\ell}(w^\epsilon)$ for its cardinality. They proved [24, Proposition 8.1 for $q=1$] that Equation (1.5) still holds if we replace $<_{\mathrm{St}}$ with $<_\ell$. It also follows from [22, Proposition 7.1]. So, the above three mentioned distributions are all equidistributed on $\mathfrak{S}_{n,r}$ and define the *Eulerian distribution* on colored permutations.

1.2.2 Mahonian statistics

As a group, $\mathfrak{S}_{n,r}$ is generated by the set $S := \{s_0, s_1, \ldots, s_{n-1}\}$, where $s_0 := (1^0\ 1^1)$ and $s_i := (i^0\ (i+1)^0)$ in cycle notation, for all $1 \le i \le n-1$ (see, for example, [6, Section 2] and [14, Section 2.2]). The length function, written ℓ_S, with respect to S satisfies [14, Theorem 4.4]

$$\sum_{w \in \mathfrak{S}_{n,r}} q^{\ell_S(w)} \;=\; \prod_{i=1}^{n} [i]_q (1 + q^i [r-1]_q). \tag{1.6}$$

For $r=1$, Equation (1.6) reduces to MacMahon's celebrated formula

$$A_n(1,q) \;=\; [1]_q [2]_q \cdots [n]_q \tag{1.7}$$

for the distribution of the major index over $\mathfrak{S}_n$ (see, for example, [90, Chapter 1, Notes]).

From this point of view, a Mahonian statistic on $\mathfrak{S}_{n,r}$, is expected to be equidistributed with the length function. Bagno [14, Theorem 5.2] introduced such a statistic by using the length order. We recall its definition. For $w^\epsilon \in \mathfrak{S}_{n,r}$, let

$$\mathrm{lmaj}(w^\epsilon) \;:=\; \mathrm{maj}_{<_\ell}(w^\epsilon) + \sum_{c_i \neq 0} (w_i - 1) + \mathrm{csum}(w^\epsilon),$$

where $\mathrm{maj}_{<_\ell}(w^\epsilon)$ is the sum of all elements of $\mathrm{Des}^*_{<_\ell}(w^\epsilon) := \mathrm{Des}_{<_\ell}(w^\ell) \setminus \{0\}$[5] and

$$\mathrm{csum}(w^\epsilon) \;:=\; \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$$

is the *color sum statistic*.

It is worth noticing, as the authors in [22, Section 7] point out, that the length order seems to be the suitable order for proving a combinatorial interpretation of the length function of $\mathfrak{S}_{n,r}$, whereas the color order is often used in the study of some algebraic aspects, such as the invariant theory of $\mathfrak{S}_{n,r}$.

Another Mahonian candidate, the flag major index, was introduced by Adin and Roichman in their seminal paper [6]. We use the following combinatorial interpretation [6, Theorem 3.1] as our definition. The *flag major index* of $w \in \mathfrak{S}_{n,r}$ is defined by

$$\mathrm{fmaj}_{<_c}(w) \;:=\; r\,\mathrm{maj}_{<_c}(w) + \mathrm{csum}(w),$$

where $\mathrm{maj}_{<_c}(w)$ is the sum of all elements of $\mathrm{Des}^*_{<_c}(w) := \mathrm{Des}_{<_c}(w) \setminus \{0\}$. The authors remark, after the proof of [6, Theorem 2.2], that the flag major index is not

[5] The $*$-descent set is often called *type A descent set* because it does not take into account the descents in positions 0 or n (see, for example, [20, Definition 5.4]).

equidistributed with the length function on $\mathfrak{S}_{n,r}$ for $r \geq 3$. Haglund, Loehr and Remmel [61, Equation (34)] were the first to explicitly compute a formula for the distribution of the flag major index

$$\sum_{w \in \mathfrak{S}_{n,r}} q^{\mathrm{fmaj}_{<_c}(w)} \;=\; [r]_q[2r]_q \cdots [nr]_q. \tag{1.8}$$

Because the right-hand side does not coincide with that of Equation (1.6) for $r \geq 3$, considering the flag major index as a Mahonian statistic on $\mathfrak{S}_{n,r}$ is an abuse of terminology, which is motivated by the following reasons. On the one hand, for $r = 1$ Equation (1.8) reduces to MacMahon's formula (1.7). On the other hand, it is known that $r, 2r, \ldots, nr$ are the degrees of $\mathfrak{S}_{n,r}$, when viewed as a complex reflection group, so the right-hand side of Equation (1.8) is the Hilbert series for the coinvariant algebra of $\mathfrak{S}_{n,r}$ (see [18, Equation (1.4)]).

Chow and Mansour [36, Theorem 5] prove a different interpretation for the flag major index using Steingrímsson's total order on $\Omega_{n,r}$, namely

$$\mathrm{fmaj}_{<_c}(w) \;=\; r\,\mathrm{maj}_{<_{\mathrm{St}}}(w) - \mathrm{csum}(w),$$

where $\mathrm{maj}_{<_{\mathrm{St}}}(w)$ is the sum of all elements of $\mathrm{Des}_{<_{\mathrm{St}}}(w)$, for all $w \in \mathfrak{S}_{n,r}$. We denote by $\mathrm{fmaj}_{<_{\mathrm{St}}}$ the right-hand side of the above equation. It is also true that Equation (1.8) holds if we replace $<_c$ by $<_\ell$ as the authors remark in [22, Propostion 7.1] and [24, Remark 2.2]. Thus, $\mathrm{fmaj}_<$ for all $< \in \{<_c, <_{\mathrm{St}}, <_\ell\}$ on $\Omega_{n,r}$ can be called Mahonian statistics on colored permutations[6].

1.2.3 Euler–Mahonian statistics

Now that we have explained what it means for a colored permutation statistic to be Eulerian or Mahonian, we discuss Euler–Mahonian pairs of statistics. In particular, we will be interested in (what we call) *colored Euler–Mahonian identities*, meaning colored generalizations of Equation (1.2) involving generating polynomials of distributions of triples (csum, eul, mah), where eul is an Eulerian statistic and mah is a Mahonian statistic on colored permutations. For ease of notation, we will write

$$A_{n,r}^{(\mathrm{eul},\mathrm{mah})}(x,q,p) \;:=\; \sum_{w \in \mathfrak{S}_{n,r}} x^{\mathrm{eul}(w)} q^{\mathrm{mah}(w)} p^{\mathrm{csum}(w)},$$

for an Eulerian (resp. Mahonian) statistic eul (resp. mah) on $\mathfrak{S}_{n,r}$.

Biagioli and Caselli [22, Theorem 5.2 for $s = p = 1$] prove, in the more general setting of projective reflection groups, the following colored Euler–Mahonian identity

$$\sum_{m \geq 0} \left([m+1]_{q^r} + pq[m]_{q^r}[r-1]_{pq}\right)^n x^m \;=\; \frac{A_{n,r}^{(\mathrm{des}_{<_c},\mathrm{fmaj}_{<_c})}(x,q,p)}{(1-x)(1-xq^r)\cdots(1-xq^{nr})}. \tag{1.9}$$

Equation (1.9) reduces to Equations (1.2) and (1.5) for $r = 1$ and $q = 1$, respectively, and $p = 1$ and generalizes a formula of Chow and Gessel [34, Theorem 3.7] for $r = 2$

[6] For the applications, in Chapter 3, we will deal with the color order.

and $p = 1$. Chow and Mansour [36, Theorem 9 (iv)] prove that the above identity holds if we replace $<_c$ with $<_{\mathrm{St}}$. It is worth mentioning that, in the case $r = 2$, Chow–Mansour's identity was first noticed by Biagioli and Zeng [23, Section 3].

In a subsequent paper, these authors showed [24, Proposition 8.1] that it also holds if we replace $<_c$ by $<_\ell$. Furthermore, they prove [24, Equation (8.1)] that

$$\sum_{m\geq 0}\left([m+1]_q + p[r-1]_p[m]_q\right)^n x^m \;=\; \frac{A_{n,r}^{(\mathrm{des}_{<_\ell},\mathrm{maj}_{<_\ell})}(x,q,p)}{(1-x)(1-xq)\cdots(1-xq^n)}, \tag{1.10}$$

which reduces to Equations (1.2) and (1.5) for $r = 1$ and $q = 1$, respectively and generalizes a formula of Chow and Gessel [34, Equation (26)] for $r = 2$. We will demonstrate how one can prove Equation (1.10) for $<_{\mathrm{St}}$ using the classical method of "balls into boxes" [90, Section 1.9], [73].

Proposition 1.2.2. *For every positive integer n,*

$$\sum_{m\geq 0}\left([m+1]_q + p[r-1]_p[m]_q\right)^n x^m \;=\; \frac{A_{n,r}^{(\mathrm{des}_{<_{\mathrm{St}}},\mathrm{maj}_{<_{\mathrm{St}}})}(x,q,p)}{(1-x)(1-xq)\cdots(1-xq^n)}. \tag{1.11}$$

Proof. Given a colored permutation $w = w_1^{\epsilon_1}w_2^{\epsilon_2}\cdots w_n^{\epsilon_n} \in \mathfrak{S}_{n,r}$ there are $n+1$ positions at the beginning, between letters and at the end of w. The right-hand side of Equation (1.11) is the generating function of the number of bars, the sum of all colors and the positions of bars (counted by x, p and q, respectively) over all colored permutation $w^\epsilon \in \mathfrak{S}_{n,r}$ with any numbers of bars inserted in the $n+1$ available positions such that there must be a bar between $w_i^{\epsilon_i}$ and $w_{i+1}^{\epsilon_{i+1}}$ for every $i \in \mathrm{Des}_{<_{\mathrm{St}}}(w^\epsilon)$. In particular, if $\epsilon_n > 0$, then there must be a bar after $w_n^{\epsilon_n}$.

We will show that this is also equal to the left-hand side of Equation (1.11). Suppose we have a colored permutation $w \in \mathfrak{S}_{n,r}$ and m bars. These bars create $m+1$ boxes for increasing colored integers according to Steingrímsson's total order to be placed. For each $1 \leq i \leq n$, we have to make two choices:

- choose a color for i and
- choose in which box to put it.

The letters in each box are then placed in increasing order. Notice that the last box cannot contain a positive-colored integer; otherwise n would be a descent, but there would not be a bar succeeding the last letter. Therefore, if i is a letter of color $1 \leq j \leq r-1$, then it contributes

$$p^j(1+q+\cdots+q^{m-1})$$

to the sum and can be put in any of the first m boxes. But, if i is a letter of color 0, then it contributes

$$1+q+\cdots+q^m$$

in the sum and can be put in any of the $m+1$ boxes. Therefore, we have a total contribution of

$$\left(\sum_{j=1}^{r-1}(p^j[m]_q)+[m+1]_q\right)^n \;=\; ([m+1]_q+p[r-1]_p[m]_q)^n.$$

The proof follows by summing over all $m \geq 0$. □

Remark 1.2.3. The preceding proof can be refined, so as to keep track of each color that appears in a colored permutation individually and not only the sum of those colors. In particular, if $\mathbf{p}=(p_0,p_1,\ldots,p_{r-1})$ is a sequence of indeterminates, then for every positive integer n

$$\sum_{m\geq 0}(p_0[m+1]_q+(p_1+p_2+\cdots+p_{r-1})[m]_q)^n\, x^m = \frac{A_{n,r}^{\mathrm{des}_{<_{\mathrm{St}}},\mathrm{maj}_{<_{\mathrm{St}}}}(x,q,\mathbf{p})}{(1-x)(1-xq)\cdots(1-xq^n)}, \tag{1.12}$$

where[7]

$$A_{n,r}^{\mathrm{des}_{<_{\mathrm{St}}},\mathrm{maj}_{<_{\mathrm{St}}}}(x,q,\mathbf{p}) := \sum_{w^\epsilon\in\mathfrak{S}_{n,r}} x^{\mathrm{des}_{<_{\mathrm{St}}}(w^\epsilon)}q^{\mathrm{maj}_{<_{\mathrm{St}}}(w^\epsilon)}p_0^{\mathrm{n}_0(w^\epsilon)}p_1^{\mathrm{n}_1(w^\epsilon)}\cdots p_{r-1}^{\mathrm{n}_{r-1}(w^\epsilon)},$$

and $\mathrm{n}_j(w^\epsilon) := |\{i\in[n] : \epsilon_i = j\}$ for each $0\leq j\leq r-1$. In Chapter 3, we will prove several formulas like Equation (1.12), which refine known colored Euler–Mahonian identities in the literature.

Bagno and Biagioli [15] defined the *flag descent number* of a colored permutation $w^\epsilon\in\mathfrak{S}_{n,r}$

$$\mathrm{fdes}_{<_c}(w^\epsilon) \;:=\; r\,\mathrm{des}^*_{<_c}(w^\epsilon)+\epsilon_1,$$

where $\mathrm{des}^*_{<_c}(w^\epsilon)$ is the cardinality of $\mathrm{Des}^*_{<_c}(w^\epsilon)$, generalizing a notion first introduced by Adin, Brenti and Roichman [2, Section 4] for the hyperoctahedral group $\mathfrak{B}_n$ (see also [11, Section 2.2]). They proved [15, Theorem A. 1] that

$$\sum_{m\geq 0}[m+1]_q^n\, x^m \;=\; \frac{A_{n,r}^{(\mathrm{fdes}_{<_c},\mathrm{fmaj}_{<_c})}(x,q,1)}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}, \tag{1.13}$$

which generalizes Adin, Brenti and Roichman's formula [2, Theorem 4.2] for $r=2$.

Biagioli and Caselli [22, Theorem 5.4] further generalized Equation (1.13) in order to include the color sum statistic. For a nonnegative integer m, we write $m = r\mathrm{Q}(m)+\mathrm{R}(m)$ for some nonnegative integer $\mathrm{Q}(m)$ and $0\leq\mathrm{R}(m)<r$. Then,

$$\sum_{m\geq 0}([\mathrm{Q}(m)+1]_{q^r}+pq[r-1]_{pq}[\mathrm{Q}(m)]_{q^r}+pq^{r\mathrm{Q}(m)+1}[\mathrm{R}(m)]_{pq})^n\, x^m$$
$$= \frac{A_{n,r}^{(\mathrm{fdes}_{<_c},\mathrm{fmaj}_{<_c})}(x,q,p)}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}. \tag{1.14}$$

[7] In general throughout this thesis we will use boldface letters to represent r-partite concepts.

For the sake of completeness, we remark that Bagno [14] introduced an Eulerian partner to his lmaj statistic, defined by

$$\mathrm{ldes}(w) \ := \ \mathrm{des}^{*}_{<_\ell}(w) + \mathrm{csum}(w),$$

for every $w \in \mathfrak{S}_{n,r}$ and proved the following Euler–Mahonian identity

$$\sum_{m\geq 0} [m+1]_q^n \, x^m \ = \ \frac{A_{n,r}^{(\mathrm{ldes},\mathrm{maj})}(x,q,1)}{(x;q)_{n+1}(-x[r-1]_{qx};q)_{n+1}}, \tag{1.15}$$

where $(x;q)_n := (1-x)(1-xq)\cdots(1-xq^n)$. In Corollary 3.1.4, we prove refined colored Euler–Mahonian identities for the pairs $(\mathrm{ldes},\mathrm{maj})$ and $(\mathrm{ldes},\mathrm{fmaj})$, where both statistics are computed using the color order.

Furthermore, Bagno introduced an Euler–Mahonian pair $(\mathrm{ndes},\mathrm{nmaj})$ on colored permutations, which together with csum satisfy Equation (3.11). This serves as a generalization of the "negative" statistics first considered by Adin, Brenti and Roichman [2, Section 3] for the hyperoctahedral group $\mathfrak{B}_n$.

1.3 Compositions, partitions and Young tableaux

1.3.1 Compositions and sets

A *composition* of a positive integer n, written $\alpha \vDash n$, is a sequence $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_k)$ of positive integers, called *parts*, summing to n. Compositions of n are in one to one correspondence with subsets of $[n-1]$. In particular, let $\mathrm{S}(\alpha) = \{r_1,r_2,\ldots,r_{k-1}\}$ be the set of partial sums $r_i := \alpha_1+\alpha_2+\cdots+\alpha_i$, for each $1 \leq i \leq k$. Also, if $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$, then let $\mathrm{co}(S) = (s_1, s_2-s_1,\ldots,s_k-s_{k-1},n-s_k)$. The maps $\alpha \mapsto \mathrm{S}(\alpha)$ and $S \mapsto \mathrm{co}(S)$ are bijections and mutual inverses.

It is often convenient to work with subsets of $[n-1]$ which contain n. For this purpose, we define $\widehat{S} := S \cup \{n\}$ for each $S \subseteq [n-1]$. We remark that the maps[8] $\alpha \to \widehat{\mathrm{S}}(\alpha)$ and $\widehat{S} \mapsto \mathrm{co}(S)$ remain bijections and are mutual inverses. In other words, compositions of n are in one-to-one correspondence with subsets of $[n]$ containing n. Let $\mathrm{Comp}(n)$ be the set of all compositions of n.

We consider the partial order of *reverse refinement* on $\mathrm{Comp}(n)$, whose covering relations are of the form

$$(\alpha_1,\ldots,\alpha_i+\alpha_{i+1},\ldots,\alpha_k) \prec (\alpha_1,\ldots,\alpha_i,\alpha_{i+1},\ldots,\alpha_k).$$

The corresponding partial order on the set $2^{[n-1]}$ of all subsets of $[n-1]$ is just inclusion of subsets.

[8] Notice that $\widehat{\mathrm{S}}(\alpha) = \{r_1,r_2,\ldots,r_k\}$.

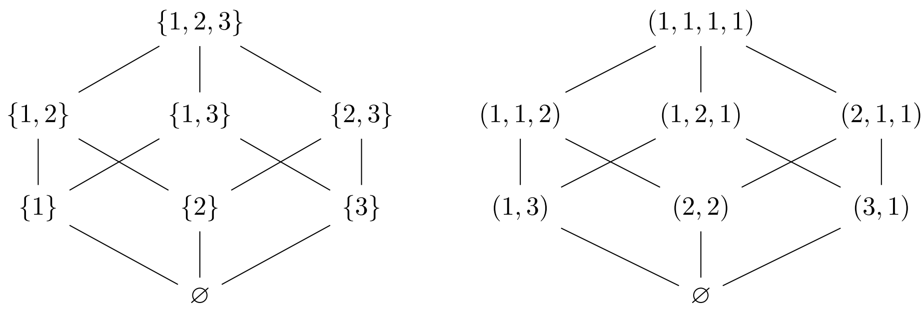


Figure 1.1: The posets $2^{[3]}$ and Comp(4).

1.3.2 Partitions and Young tableaux

A *partition* of n, written $\lambda \vdash n$, is a composition λ of n whose parts appear in weakly decreasing order. The number of parts of λ, written $\ell(\lambda)$, is called the *length* of λ. It is well known that partitions of n are in one-to-one correspondence with conjugacy classes of $\mathfrak{S}_n$.

The *Young diagram* of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ is an array of n boxes into left-justified rows such that the ith row contains λ_i boxes. A *standard Young tableau* (resp. *semistandard Young tableau*) of shape λ is a bijective filling (resp. filling) of the boxes of the Young diagram of λ with the integers $1, 2, \ldots, n$ (resp. positive integers) such that

- rows increase (resp. weakly increase) from left to right and
- columns increase from top to bottom.

We denote by $\mathrm{SYT}(\lambda)$ and $\mathrm{SSYT}(\lambda)$ the set of standard and semistandard Young tableaux, respectively. We also write SYT_n for the set of all standard Young tableaux of size n.

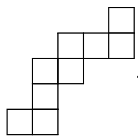
There exists a notion of descent for standard Young tableaux, which we now recall. A *descent* of a standard Young tableau Q is an entry $1 \le i \le n-1$ such that $i+1$ appears in a lower row than i. We denote by $\mathrm{Des}(Q)$ the descent set of Q and write $\mathrm{des}(Q)$ for its cardinality. Also, we write $\mathrm{maj}(Q)$ for the sum of all elements of $\mathrm{Des}(Q)$.

The *Robinson–Schensted correspondence* [89, Section 7.11] is a bijection from the symmetric group $\mathfrak{S}_n$ to the set of pairs of standard Young tableaux of the same shape and size n with the properties that $\mathrm{Des}(w) = \mathrm{Des}(Q(w))$ and $\mathrm{Des}(w^{-1}) = \mathrm{Des}(P(w))$, where $(P(w), Q(w))$ is the pair of tableaux associated to $w \in \mathfrak{S}_n$. The *Knuth class* [89, Appendix 1], written K_T, corresponding to a standard Young tableau T of size n is the set of all permutations $w \in \mathfrak{S}_n$, such that $P(w) = T$, where $P(w)$ is defined as before. If T has shape λ, then we say that K_T is a *Knuth class of shape* λ and by abuse of notation we may write K_λ.

1.3.3 Ribbons

A *zig-zag diagram*, also called *ribbon*[9], *skew hook* and *border strip* is a connected skew shape that does not contain a 2×2 square. Ribbons with n squares are in one-to-one correspondence with compositions of n (and therefore subsets of $[n-1]$).

In particular, for $\alpha \in \mathrm{Comp}(n)$ let Z_α be the ribbon with n cells whose row lengths, when read from bottom to top, are the parts of α. Now, given $S \subseteq [n-1]$ we construct a skew shape with n cells, labelled $1, 2, \ldots, n$ as follows: Start with a single cell labelled 1. For every $1 \leq i \leq n-1$, place a cell labelled $i+1$ directly north (resp. east) of the cell labelled i whenever $i \in S$ (resp. $i \notin S$). Let $\mathrm{Z}_{n,S}$ be the underlying ribbon. The maps $\alpha \mapsto \mathrm{Z}_\alpha$ and $S \mapsto \mathrm{Z}_{n,S}$ are bijections between $\mathrm{Comp}(n)$ (resp. $2^{[n-1]}$) and the set of ribbons with n cells. For example, for $n = 9$ and $S = \{2, 3, 5, 8, 9\}$ we have

$\mathrm{Z}_{n,S} =$  .

This ribbon is also equal to $\mathrm{Z}_{\mathrm{co}(S)}$, where $\mathrm{co}(S) = (2, 1, 2, 3, 1)$.

For $S \subseteq [n-1]$, let

$$\begin{aligned} \mathrm{D}_{n,S} &:= \{w \in \mathfrak{S}_n : \mathrm{Des}(w) = S\} \\ \mathrm{D}^{-1}_{n,S} &:= \{w \in \mathfrak{S}_n : \mathrm{Des}(w^{-1}) = S\} \end{aligned}$$

be the *descent class* and the *inverse descent class*, respectively, corresponding to S. Also, define

$$\begin{aligned} \mathrm{R}_{n,S} &:= \{w \in \mathfrak{S}_n : \mathrm{Des}(w) \subseteq S\} \\ \mathrm{R}^{-1}_{n,S} &:= \{w \in \mathfrak{S}_n : \mathrm{Des}(w^{-1}) \subseteq S\}. \end{aligned}$$

It is well known that permutations of $\mathfrak{S}_n$ correspond bijectively to standard Young tableaux of ribbon shape with n cells. The following refinement of this fact explains the connection between (inverse) descent classes and tableaux of ribbon shape (see, for example, [7, Propositions 3.5 and 10.12]).

Lemma 1.3.1. *For every $S \subseteq [n-1]$, there exists a bijection from the set $\mathrm{SYT}(\mathrm{Z}_{n,S})$ to the descent class $\mathrm{D}_{n,S}$ such that $\mathrm{Des}(Q) = \mathrm{Des}(w^{-1})$, where w is the permutation associated to $Q \in \mathrm{SYT}(\mathrm{Z}_{n,S})$. In particular, the distribution of the descent set is the same over $\mathrm{D}^{-1}_{n,S}$ and $\mathrm{SYT}(\mathrm{Z}_{n,S})$.*

[9] We will use the term ribbon.

1.4 Colored compositions, r-partite partitions and Young tableaux

1.4.1 Colored compositions and colored sets

An r-*colored composition* of a positive integer n [67, Definition 6.3][10] is a pair (γ, ϵ), where γ is a composition of n and $\epsilon \in \mathbb{Z}_r^{\ell(\gamma)}$ is a sequence of colors. We will also represent a colored composition by γ^ϵ (or simply γ). Equivalently, it can be viewed as a sequence of colored integers which when forgetting all colors sum to n. For example,

$$\gamma \;=\; (2^0,\ 2^1,\ 1^1,\ 1^3,\ 3^1,\ 1^2)$$

is a 4-colored composition of 10. Let $\mathrm{Comp}(n, r)$ be the set of all r-colored compositions of n.

Colored compositions correspond bijectively to colored subsets of $[n]$. An r-*colored subset* of $[n]$ is a pair $\sigma = (\widehat{S}, \epsilon)$, where $S \subseteq [n-1]$ and $\epsilon : \widehat{S} \to \mathbb{Z}_r$ is the color map which assigns to each element of $\widehat{S}$ a color from $\mathbb{Z}_r$. The correspondence between r-colored compositions of n and r-colored subsets of $[n]$ is given by $(\gamma, \epsilon) \mapsto (\widehat{\mathrm{S}}(\gamma), \epsilon)$ with inverse $(\widehat{S}, \epsilon) \mapsto (\mathrm{co}(S), \epsilon)$. We write $\Sigma(n, r)$ for the set of all r-colored subsets of $[n]$ which can also be viewed as subsets of $\Omega_{n,r}$ where each i^j appears at most once for $1 \le i \le n-1$ and exactly once for $i = n$[11]. For example, the colored subset of $[10]$ corresponding to $(2^0,\ 2^1,\ 1^1,\ 1^3,\ 3^1,\ 1^2)$ is

$$\sigma \;=\; \{2^0, 4^1, 5^1, 6^3, 9^1, 10^2\}.$$

For an r-colored composition $\gamma = (\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \ldots, \gamma_k^{\epsilon_k})$ of n, we may extend the sequence $\epsilon \in \mathbb{Z}_r^k$ to $\tilde{\epsilon} = (\tilde{\epsilon}_1, \tilde{\epsilon}_2, \ldots, \tilde{\epsilon}_n) \in \mathbb{Z}_r^n$, called the *color vector* of γ, by defining

$$\tilde{\epsilon} \;:=\; (\underbrace{\epsilon_1, \epsilon_1, \ldots, \epsilon_1}_{\gamma_1 \text{ times}}, \underbrace{\epsilon_2, \epsilon_2, \ldots, \epsilon_2}_{\gamma_2 \text{ times}}, \ldots, \underbrace{\epsilon_k, \epsilon_k, \ldots, \epsilon_k}_{\gamma_k \text{ times}}).$$

Equivalently, for an r-colored subset $\sigma = (\widehat{S}, \epsilon)$ of $[n]$, with $\widehat{S} = \{s_1 < s_2 < \cdots < s_{k-1} < s_k := n\}$ we extend ϵ to a map $\tilde{\epsilon} : [n] \to \mathbb{Z}_r$ by setting $\tilde{\epsilon}(j) := \epsilon(s_i)$ for every $s_{i-1} < j \le s_i$ for each $i \in [k]$ where $s_0 := 0$. We refer to $\tilde{\epsilon}$ as the *color vector* of σ and write $(\tilde{\epsilon}_1, \tilde{\epsilon}_2, \ldots, \tilde{\epsilon}_n)$ instead. For example, the color vector of $\sigma = \{2^0, 4^1, 5^1, 6^3, 9^1, 10^2\}$ (and $(2^0,\ 2^1,\ 1^1,\ 1^3,\ 3^1,\ 1^2)$) is $(0, 0, 1, 1, 1, 3, 1, 1, 1, 2)$.

We consider the partial order of reverse refinement on consecutive parts of the same color on $\mathrm{Comp}(n, r)$, whose covering relations are of the form

$$(\gamma_1^{\epsilon_1}, \ldots, (\gamma_i + \gamma_{i+1})^{\epsilon_i}, \ldots, \gamma_k^{\epsilon_k}) \prec (\gamma_1^{\epsilon_1}, \ldots, \gamma_i^{\epsilon_i}, \gamma_{i+1}^{\epsilon_i}, \ldots, \gamma_k^{\epsilon_k}).$$

The corresponding partial order on the set $\Sigma(n, r)$ is inclusion of subsets of the same color vector. Notice that the posets $\mathrm{Comp}(n, r)$ and $\Sigma(n, r)$ are not connected, in contrast to their uncolored counterparts.

[10] See also [62, Section 3.2], [19, Section 5.2], [21, Section 2.1] and for the case $r = 2$ see [1, Section 2.1].

[11] For a similar notion of colored subset see [88, Section 6] for $r = 2$ and [11, Section 3.2] for $r \ge 3$.

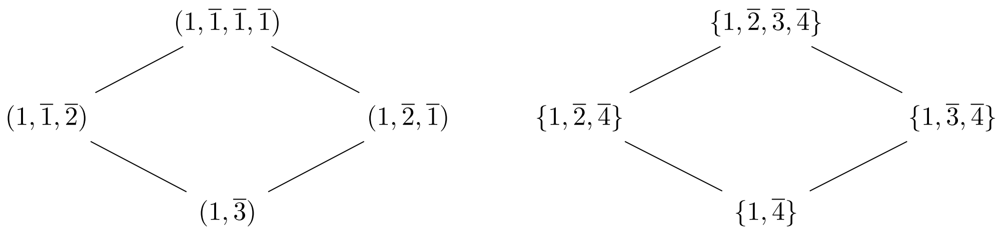


Figure 1.2: A connected component of the poset $\Sigma(4,2)$ and its corresponding component in $\mathrm{Comp}(4,2)$.

In enumerative combinatorics it is common to associate a composition (resp. set) to each permutation in order to record its increasing runs (resp. descents), called the descent composition (resp. descent set). Mantaci and Reutenauer [67, Page 53] introduced a similar notion, called *descent shape* to record the lengths and colors of increasing runs of constant color of colored permutations. Following [1] and [62], we define the colored descent set and colored descent composition.

Definition 1.4.1. Let $w^\epsilon \in \mathfrak{S}_{n,r}$ be a colored permutation.

- The *colored descent set* of w^ϵ, written $\mathrm{sDes}(w^\epsilon)$, is the colored subset $(\widehat{S},\epsilon)$ of $[n]$, where S consists of all $1 \le i \le n-1$ such that $\epsilon_i \neq \epsilon_{i+1}$, or $\epsilon_i = \epsilon_{i+1}$ and $w_i > w_{i+1}$ and $\epsilon : \widehat{S} \to \mathbb{Z}_r$ is the map given by $\epsilon(s) = \epsilon_s$ for each $s \in \widehat{S}$.
- The *colored descent composition* of w^ϵ, written $\mathrm{co}(w^\epsilon)$, is the colored composition corresponding to $\mathrm{sDes}(w^\epsilon)$.

Two remarks on Definition 1.4.1 are in order. We denote both the color vector of w^ϵ and the color map of $\mathrm{sDes}(w^\epsilon)$ by the same letter. This notation is unambiguous since the latter is completely determined by the former. The colored descent set is called signed descent set in [1] and therefore, by analogy, it would be reasonable to write cDes for the colored descent set. We choose to keep the notation sDes to avoid any confusion created by the fact that cDes is widely used in the literature to denote the cyclic descent set of a permutation, introduced by Cellini [31].

Example 1.4.2. To illustrate Definition 1.4.1, the colored descent set and the colored descent composition of $w^\epsilon = 5^4 1^3 4^1 6^1 2^0 7^0 3^0 \in \mathfrak{S}_{7,6}$ are

$$\begin{aligned} \mathrm{sDes}(w^\epsilon) &= (\{1,2,4,6,7\},(4,3,1,0,0)) \\ \mathrm{co}(w^\epsilon) &= (1^4,1^3,2^1,2^0,1^0). \end{aligned}$$

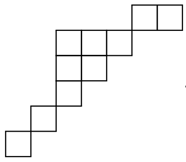
1.4.2 r-partite partitions and standard Young r-partite tableaux

An *r-partite partition* of n is a r-tuple $\boldsymbol{\lambda} = (\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(r-1)})$ of (possibly empty) integer partitions of total sum n. In this case we write $\boldsymbol{\lambda} \vdash n$. For example,

$$\boldsymbol{\lambda} \;=\; (2,321,1,1)$$

is a 4-partite partition of $n = 10$.

The *direct sum* of two partitions λ and μ, written $\lambda \oplus \mu$, is the skew shape whose diagram is obtained by placing the diagrams of λ and μ in such a way that the lower-left vertex of the diagram of μ coincides with the upper-right vertex of that of λ. One can also think of an r-partite partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ as the direct sum of r partitions $\lambda^{(r-1)} \oplus \cdots \oplus \lambda^{(1)} \oplus \lambda^{(0)}$. For example, the 4-partite partition $\boldsymbol{\lambda} = (2, 321, 1, 1)$ of 10 can be thought of as



A *standard Young r-partite tableau* of shape $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)}) \vdash n$ is an r-tuple $\boldsymbol{Q} = (Q^{(0)}, \ldots, Q^{(r-1)})$ of tableaux which are strictly increasing along rows and columns such that $Q^{(i)}$ has shape $\lambda^{(i)}$, for all $0 \le i \le r-1$ and every element of $[n]$ appears exactly once as an entry of $Q^{(i)}$ for some $0 \le i \le r-1$. These tableaux are called *parts* of $\boldsymbol{Q}$. Let $\mathrm{SYT}(\boldsymbol{\lambda})$ (resp. $\mathrm{SYT}_{n,r}$) be the set of all standard Young r-partite tableaux of shape $\boldsymbol{\lambda} \vdash n$ (resp. of any shape and size n).

For $\boldsymbol{Q} = (Q^{(0)}, \ldots, Q^{(r-1)}) \in \mathrm{SYT}_{n,r}$, an integer $i \in [0, n-1]$ is called a *descent* of $\boldsymbol{Q}$, if

- i and $i+1$ belong in the same part of $\boldsymbol{Q}$ and $i+1$ appears in a lower row than i does, or
- $i \in Q^{(j)}$ and $i+1 \in Q^{(k)}$, for some $0 \le j < k \le r-1$, or
- $i = 0$ and 1 appears in $Q^{(j)}$ for some $j \neq 0$.

The set of all descents of $\boldsymbol{Q}$, written $\mathrm{Des}(\boldsymbol{Q})$, is called the *descent set* of $\boldsymbol{Q}$. The cardinality of $\mathrm{Des}(\boldsymbol{Q})$, denoted by $\mathrm{des}(\boldsymbol{Q})$, is called the *descent number* of $\boldsymbol{Q}$. Also, let $\mathrm{Des}^*(\boldsymbol{Q})$ be the set obtained from $\mathrm{Des}(\boldsymbol{Q})$ by removing the zero, if present and write $\mathrm{maj}(\boldsymbol{Q})$ for the sum of all elements of $\mathrm{Des}^*(\boldsymbol{Q})$.

For example, an element of $\mathrm{SYT}(2, 321, 1, 1)$ is

$$\boldsymbol{Q} = \left(\begin{array}{|c|c|}\hline 1 & 9 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 3 & 5 & 6 \\ \hline 4 & 10 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 8 \\ \hline\end{array} \right).$$

We have $\mathrm{Des}(\boldsymbol{Q}) = \{1, 3, 6, 7, 9\}$ and $\mathrm{des}(\boldsymbol{Q}) = 5$. Using the representation of an r-partite partition as the direct sum of r partitions mentioned above, one sees that the *-descent set of $\boldsymbol{Q}$ is essentially the descent set of the standard Young tableau of this skew shape. In our running example, we have

$$\begin{array}{ccccccc} & & & & & 1 & 9 \\ & & 3 & 5 & 6 & & \\ & & 4 & 10 & & & \\ & & 7 & & & & \\ & 2 & & & & & \\ 8 & & & & & & \end{array} \quad \xmapsto{\mathrm{Des}} \quad \{1, 3, 6, 7, 9\}.$$

The following definition of the colored descent set of an r-partite tableau first appeared in [1, Definition 2.3] for the case of $r = 2$ and analogously to the case of colored permutations it records the lengths and colors of increasing runs in each part of the r-partite tableau.

Definition 1.4.3. The *colored descent set* of $\boldsymbol{Q} \in \mathrm{SYT}_{n,r}$, denoted $\mathrm{sDes}(\boldsymbol{Q})$, is the colored set $(\widehat{S}, \epsilon) \in \Sigma(n,r)$, where

- the set S consists of all entries $1 \le i \le n-1$ of $\boldsymbol{Q}$ for which either i and $i+1$ appear in different parts or they appear in the same part of $\boldsymbol{Q}$ and $i+1$ appears in a lower row than i,
- the map $\epsilon : \widehat{S} \to \mathbb{Z}_r$ is defined by $\epsilon(i) = j$, where j is the color of the part of $\boldsymbol{Q}$ in which $i \in \widehat{S}$ belongs.

Also, we define the *color vector* $\tilde{\epsilon} : [n] \to \mathbb{Z}_r$ of $\boldsymbol{Q}$ by letting $\tilde{\epsilon}(i) = j$, where j is the color of the part in which i belongs. We will use sequence notation for the color vector of $\boldsymbol{Q}$. The 4-partite tableau of our running example has color vector $(0,2,1,1,1,1,1,3,0,1)$ and colored descent set

$$\mathrm{sDes}(\boldsymbol{Q}) \;=\; (\{1,2,3,6,7,8,9,10\},(0,2,1,1,1,3,0,1)).$$

The Robinson–Schensted correspondence has a natural colored analogue, first considered by White [98, Corollary 9 and Remark 11] and further studied by Stanton and White [91]. It is a bijection from the r-colored permutation group $\mathfrak{S}_{n,r}$ to the set of pairs $(\boldsymbol{P},\boldsymbol{Q})$ of standard Young r-partite tableaux of the same shape and size n. It has the following property (cf. [18, Proposition 6.2], [5, Lemma 5.2]). If $w \mapsto (\boldsymbol{P}(w),\boldsymbol{Q}(w))$ via this map, then

$$\mathrm{sDes}(w) \;=\; \mathrm{sDes}(\boldsymbol{Q}(w))$$

and

$$\mathrm{sDes}(\overline{w}^{-1}) \;=\; \mathrm{sDes}(\boldsymbol{P}(w)).$$

The *Knuth class*, written $\mathrm{K}_{\boldsymbol{T}}$, corresponding to a standard Young r-partite tableau $\boldsymbol{T}$ of size n is the set of all r-colored permutations $w \in \mathfrak{S}_{n,r}$, such that $\boldsymbol{P}(w) = \boldsymbol{T}$, where $P(w)$ is defined as before. If $\boldsymbol{T}$ has shape $\boldsymbol{\lambda}$, then we say that $\mathrm{K}_{\boldsymbol{T}}$ is a *Knuth class of shape* $\boldsymbol{\lambda}$ and by abuse of notation we may write $\mathrm{K}_{\boldsymbol{\lambda}}$.

1.5 Symmetric and quasisymmetric functions and their specializations

For a sequence $\mathbf{x} = (x_1, x_2, \dots)$ of commuting indeterminates, let $\mathbb{C}[[\mathbf{x}]]$ be the $\mathbb{C}$-algebra of formal power series in $\mathbf{x}$ over the complex numbers $\mathbb{C}$. The multiplication in $\mathbb{C}[[\mathbf{x}]]$ is the usual multiplication of formal power series. We denote by $\mathrm{Sym}(\mathbf{x})$ (resp. $\mathrm{QSym}(\mathbf{x})$) the $\mathbb{C}$-algebra of *symmetric* (resp. *quasisymmetric*) *functions* in $\mathbf{x}$ with complex coefficients.

Quasisymmetric functions are certain power series in infinitely many variables that generalize the notion of symmetric functions. In particular, a quasisymmetric function $f(\mathbf{x})$ is an element of $\mathbb{C}[[\mathbf{x}]]$ of bounded degree such that for every

$\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{Z}_{>0}$, we have

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}] f(\mathbf{x}) \;=\; [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}] f(\mathbf{x}),$$

for all $i_1 > i_2 > \cdots > i_k$ and $j_1 > j_2 > \cdots > j_k$, where $[x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}] f(\mathbf{x})$ is the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ in $f(x)$. There is an inclusion of $\mathbb{C}$-algebras $\mathrm{Sym}(\mathbf{x}) \to \mathrm{QSym}(\mathbf{x})$, but not every quasisymmetric function is necessarily a symmetric function. For example,

$$\sum_{i>j} x_i^2 x_j$$

is quasisymmetric, but it is not symmetric because $x_2^2 x_1$ appears as term whereas $x_1^2 x_2$ does not. Notice that adding

$$\sum_{i>j} x_i x_j^2$$

makes it symmetric.

Quasisymmetric functions first appeared, not with this name yet, in Stanley's thesis [87] as generating functions of P-partitions (for a detailed description of Stanley's contribution to symmetric and quasisymmetric functions see [26]) and were later defined and studied systematically by Gessel [52] (see also [89, Section 7.19] and [53, Section 8.5]). In Section 4.1 we review the connection between the theory of P-partitions and quasisymmetric functions.

Both $\mathrm{Sym}(\mathbf{x})$ and $\mathrm{QSym}(\mathbf{x})$ are graded $\mathbb{C}$-algebras. We write $\mathrm{Sym}_n(\mathbf{x})$ (resp. $\mathrm{QSym}_n(\mathbf{x})$) for their n-th homogeneous components. The dimension of $\mathrm{Sym}_n(\mathbf{x})$ (resp. $\mathrm{QSym}_n(\mathbf{x})$) is the number $p(n)$ of partitions of n (resp. 2^{n-1}). It follows immediately from the definition of $\mathrm{QSym}_n(\mathbf{x})$ that the set of

$$M_\alpha(\mathbf{x}) \;:=\; \sum_{i_1 > i_2 > \cdots > i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

for all composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of n forms a basis for this vector space. These elements are called *monomial quasisymmetric functions*. For $S \subseteq [n-1]$, we write $M_{n,S(\alpha)}(\mathbf{x}) := M_\alpha(\mathbf{x})$.

Apart from the monomial basis, $\mathrm{QSym}_n(\mathbf{x})$ has another very interesting basis, called the *fundamental basis*. The *fundamental quasisymmetric function* associated to $S \subseteq [n-1]$ is defined by

$$F_{n,S}(\mathbf{x}) \;:=\; \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}, \tag{1.16}$$

and $F_{0,\varnothing}(\mathbf{x}) := 1$. By groupping together the sequences $i_1 \geq i_2 \geq \cdots \geq i_n$ according to whether $i_j > i_{j+1}$ or $i_j = i_{j+1}$ we get

$$F_{n,S}(\mathbf{x}) \;=\; \sum_{S \subseteq T \subseteq [n-1]} M_{n,T}(\mathbf{x}).$$

For example, we have

$$\begin{aligned}
F_{4,\{1\}}(\mathbf{x}) &= \sum_{i_1>i_2\geq i_3\geq i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4}\\
&= \sum_{i_1>i_2=i_3=i_4} x_{i_1}x_{i_2}^3 + \sum_{i_1>i_2>i_3=i_4} x_{i_1}x_{i_2}x_{i_3}^2 + \sum_{i_1>i_2>i_3>i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4}\\
&= M_{4,\{1\}}(\mathbf{x}) + M_{4,\{1,2\}}(\mathbf{x}) + M_{4,[3]}(\mathbf{x}).
\end{aligned}$$

Remark 1.5.1. The original definition of quasisymmetric functions (see [52]and [89, Section 7.19]) requires that the inequalities $i_1 > i_2 > \cdots > i_k$ and $j_1 > j_2 > \cdots > j_k$ are reversed. These two approaches are equivalent. In particular, consider the automorphism * : QSym $\to$ QSym defined by $F^*_{n,S}(\mathbf{x}) = F_{n,n-S}(\mathbf{x})$ where $n - S := \{n - s : s \in S\}$ for any $S \subseteq [n-1]$ and extending linearly (see, for example, [63, Section 3.6]). It is not hard to see that

$$F^*_{n,S}(\mathbf{x}) = \sum_{\substack{i_1\leq i_2\leq\cdots\leq i_n\\ j\in S\Rightarrow i_j<i_{j+1}}} x_{i_1}x_{i_2}\cdots x_{i_n}.$$

The main theme of Chapters 2 and 3 concerns specializations of symmetric and quasisymmetric functions. Specializations of symmetric functions date back to Stanley's work on the enumeration of plane partitions [86]. Formally, a specialization of $\mathrm{Sym}(\mathbf{x})$ (resp. $\mathrm{QSym}(\mathbf{x})$) is an algebra homomorphism $\mathrm{Sym}(\mathbf{x}) \to A$ (resp. $\mathrm{QSym}(\mathbf{x}) \to A$) where A is a (commutative) $\mathbb{C}$-algebra. In this thesis, we will be interested in specializations that arise from substituting elements of A for the variables x_i, when this substitution makes sense.

A well studied pair of specializations of this kind is that of principal specializations [89, Section 7.8]. In particular,

- the *stable principal specialization* is the homomorphism $\mathrm{ps}_q : R \to \mathbb{C}[[q]]$ defined by the substitutions

$$x_1 = 1,\ x_2 = q,\ x_3 = q^2, \ldots$$

- and the *principal specialization of order m* is the homomorphism $\mathrm{ps}_{q,m} : R \to \mathbb{C}[q]$ defined by the substitutions

$$x_1 = 1,\ x_2 = q,\ \ldots,\ x_m = q^{m-1},\ x_{m+1} = x_{m+2} = \cdots = 0,$$

where $R \in \{\mathrm{Sym}(\mathbf{x}), \mathrm{QSym}(x)\}$.

We conclude this section by recalling some notable bases of $\mathrm{Sym}_n(\mathbf{x})$. The *Schur basis* is perhaps the most important basis. We say that a semistandard Young tableau Q has *type* (or *content*) $\alpha = (\alpha_1, \alpha_2, \ldots)$ if it has α_i entries equal to i. The *Schur function* associated to λ is

$$s_\lambda(\mathbf{x}) = \sum_{Q\in\mathrm{SSYT}(\lambda)} \mathbf{x}^Q,$$

where $\mathbf{x}^Q := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For more information on Schur functions and their importance in enumerative and algebraic combinatorics we refer to Stanley's excellent exposition [89, Chapter 7]. Here, we just recall the following well known expansion[12] [89, Theorem 7.19.7]

$$s_\lambda(\mathbf{x}) \;=\; \sum_{Q \in \mathrm{SYT}(\lambda)} F_{n,\mathrm{Des}(w)}(\mathbf{x}) \tag{1.17}$$

which will be used in the sequel. Other notable bases, which will be used throughout this thesis are the *complete homogeneous symmetric functions*, written $h_\lambda(\mathbf{x})$, the *elementary symmetric functions*, written $e_\lambda(\mathbf{x})$, the *power sum symmetric functions*, written $p_\lambda(\mathbf{x})$ and the *monomial symmetric functions*, written $m_\lambda(\mathbf{x})$. Notice that $F_{n,\varnothing}(\mathbf{x}) = h_n(\mathbf{x})$ and $F_{n,[n-1]}(\mathbf{x}) = e_n(\mathbf{x})$.

1.6 Schur-positivity and quasisymmetric functions

It is well known that (complex, finite-dimensional) irreducible $\mathfrak{S}_n$-characters are indexed by integer partitions. Let $\mathcal{R}(\mathfrak{S}_n)$ be the $\mathbb{Z}$-module generated by irreducible $\mathfrak{S}_n$-characters[13] and let

$$\mathcal{R}(\mathfrak{S}) \;:=\; \mathbb{Z} \oplus \mathcal{R}(\mathfrak{S}_1) \oplus \mathcal{R}(\mathfrak{S}_2) \oplus \cdots .$$

The $\mathbb{Z}$-module $\mathcal{R}(\mathfrak{S})$ has a ring structure, induced by the induction product. The *induction product*, written $f \circ g$, of an $\mathfrak{S}_i$-character f and an $\mathfrak{S}_j$-character g is defined by

$$f \circ g \;:=\; (f \otimes g) \uparrow_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}},$$

where $\uparrow$ denotes *induction* and $\mathfrak{S}_n \times \mathfrak{S}_m$ is viewed as a subgroup of $\mathfrak{S}_{n+m}$ in the obvious way, that is $\mathfrak{S}_n$ permutes the elements of $[n]$ and $\mathfrak{S}_m$ permutes the elements of $[n+1, n+m]$. The ring $\mathcal{R}(\mathfrak{S})$ is closely related to Sym via the Frobenius characteristic map.

The *Frobenius characteristic map* $\mathrm{ch} : \mathcal{R}(\mathfrak{S}) \to \mathrm{Sym}(\mathbf{x})$ is a ring isomorphism[14], with the property that

$$\mathrm{ch}(\chi^\lambda)(\mathbf{x}) \;=\; s_\lambda(\mathbf{x}),$$

where χ^λ is the irreducible $\mathfrak{S}_n$-character corresponding to $\lambda \vdash n$. In particular, we have

$$\mathrm{ch}(1 \uparrow_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n})(\mathbf{x}) \;=\; h_\alpha(\mathbf{x})$$

where $\mathfrak{S}_\alpha$ is the *Young subgroup* corresponding to α and 1_n denotes the trivial $\mathfrak{S}_n$-character.

One of the primary objectives of *combinatorial representation theory* according to [17] is the study of character formulas of the type

$$\chi(\alpha) \;=\; \sum_T \mathrm{weight}_\alpha(T)$$

[12] This equation holds for every skew shape λ/μ (see [89, Proposition 7.19.7]).

[13] Elements of $\mathcal{R}(\mathfrak{S}_n)$ are often called *virtual* $\mathfrak{S}_n$*-characters*.

[14] Here $\mathrm{Sym}(\mathbf{x})$ is viewed as a $\mathbb{Z}$-algebra, for which the complete symmetric functions h_n are algebraically independent generators (see [89, Corollary 7.6.2]). In addition, Schur functions constitute a basis for the $\mathbb{Z}$-module $\mathrm{Sym}(\mathbf{x})$ (see the discussion after [89, Corollary 7.10.6]).

where the sum runs over all T in a set of nice combinatorial objects and $\mathrm{weight}_\alpha(T)$ is a weight on those objects which depends on α. A well celebrated character formula of this sort the *Murnaghan–Nakayama rule* [89, Section 7.17] for the irreducible $\mathfrak{S}_n$-character, where the enumerated objects are border strip tableaux.

Many instances of these formulas occur in the literature (see, for example, [1, 8] and references therein). Adin and Roichman [8] proposed an abstract setting which captures these phenomena for the symmetric group and has close connection with Schur-positivity. A symmetric function is called *Schur-positive* if its expansion in the Schur basis has nonnegative coefficients. Examples of Schur-positive symmetric functions often reveal hidden structures in the symmetric group and thus the problem of Schur-positivity is of interest.

A permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ is called *unimodal* if there exists an index $1 \le k \le n$ such that

$$w_1 > \cdots > w_k < \cdots < w_n.$$

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n and each $1 \le i \le k$, let

$$\mathcal{B}_i(\alpha) := [r_{i-1} + 1, r_i],$$

be the ith block of α of cardinality α_i where r_i are the partial sums of α and $r_0 := 0$. A permutation $w \in \mathfrak{S}_n$ is called *α-unimodal* if the restriction of π to each block of α is unimodal. A subset $S \subseteq [n-1]$ is called *α-unimodal* if it is the descent set of an α-unimodal permutation of $\mathfrak{S}_n$. For example, for $\alpha = (n)$ the set $[n-1]$ is α-unimodal because it is the descent set of $n(n-1)\cdots 21$. Let U_α be the set of all α-unimodal subsets of $[n-1]$. For more information on α-unimodality we refer to [8, 9].

We will say that a collection $\mathcal{A}$ of permutations of $\mathfrak{S}_n$ is *Schur-positive* if the *quasisymmetric generating function*

$$F(\mathcal{A}; \mathbf{x}) \;:=\; \sum_{w \in \mathcal{A}} F_{n,\mathrm{Des}(w)}(\mathbf{x})$$

associated to $\mathcal{A}$ is symmetric and Schur-positive. Adin and Roichman [8, Theorem 1.5] proved that $\mathcal{A}$ is Schur-positive if and only if the function $\chi^{\mathcal{A}} : \mathrm{Comp}(n) \to \mathbb{Z}$ defined by

$$\chi^{\mathcal{A}}(\mu) \;=\; \sum_{w \in \mathcal{A} \cap U_\mu} (-1)^{|\,\mathrm{Des}(w) \setminus S(\mu)|}$$

is a nonvirtual character of the symmetric group $\mathfrak{S}_n$. Adin and Roichman called such $\mathcal{A}$ a *fine set*. Later, Adin et al. [1, Theorem 3.2] proved that $\chi^{\mathcal{A}}$ is a virtual $\mathfrak{S}_n$-character if and only if

$$\mathrm{ch}(\chi^{\mathcal{A}})(\mathbf{x}) \;=\; F(\mathcal{A}; \mathbf{x})$$

and therefore in this case the distribution of the descent set over $\mathcal{A}$ is uniquely determined by $\chi^{\mathcal{A}}$. We will say that $\mathcal{A} \subseteq \mathfrak{S}_n$ is *Schur-positive for the $\mathfrak{S}_n$-character χ*, if $\mathcal{A}$ is Schur-positive and $\mathrm{ch}(\chi)(\mathbf{x}) = F(\mathcal{A}; \mathbf{x})$.

To illustrate the phenomenon with a specific example, consider $\mathcal{A} = \mathfrak{S}_n$ to be the whole symmetric group. On the one hand, the Robinson–Schensted correspondence implies (see [89, Corollary 7.12.5])

$$F(\mathfrak{S}_n; \mathbf{x}) \;=\; h_1(\mathbf{x})^n. \tag{1.18}$$

The right-hand side of Equation (1.18) is known to equal[15] the Frobenius characteristic of $1_n \uparrow_{\mathfrak{S}_1 \times \cdots \times \mathfrak{S}_1}^{\mathfrak{S}_n}$. But, this is exactly the character of the *regular representation* of $\mathfrak{S}_n$, denoted by χ^{reg}. On the other hand, it follows by a result of Roichman (see [8, Corollary 3.8]) that $\chi^{\mathrm{reg}} = \chi^{\mathfrak{S}_n}$, which completes the picture.

The concept of a Schur-positive set, as well as the results of this section, can be generalized to any set of combinatorial objects equipped with a descent map. They can also be generalized to multisets instead of just sets. In particular, for a collection (with repetitions) $\mathcal{A}$ of combinatorial objects equipped with a descent map $\mathrm{Des} : \mathcal{A} \to 2^{[n-1]}$, we write

$$F(\mathcal{A}; \mathbf{x}) \;:=\; \sum_{a \in \mathcal{A}} m_{\mathcal{A}}(a) F_{n, \mathrm{Des}(a)}(\mathbf{x}),$$

where $m_{\mathcal{A}}(a)$ denotes the multiplicity of a in $\mathcal{A}$, for the quasisymmetric generating function of $\mathcal{A}$.

For every partition λ, Equation (1.17) rewritten as

$$s_\lambda(\mathbf{x}) \;= F(\mathrm{SYT}(\lambda); \mathbf{x})$$

is an instance of a Schur-positive set of combinatorial objects endowed with a descent map. The Robinson–Schensted correspondence implies that the distribution of Des over $\mathrm{SYT}(\lambda)$ is equal to its distribution over all permutations in the Knuth class K_P for some $P \in \mathrm{SYT}(\lambda)$ and therefore

$$s_\lambda(\mathbf{x}) \;=\; F(K_P; \mathbf{x}).$$

Lemma 1.6.1. (cf. [39, Remark 3.4]) *A (multi)set $\mathcal{A}$ of combinatorial objects endowed with a descent map* $\mathrm{Des} : \mathcal{A} \to 2^{[n-1]}$ *is Schur-positive if and only if there exists a (multi)set partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_m$ and* Des*-preserving bijections*

$$\mathcal{A}_i \;\longrightarrow K_{P^i}$$

for $P^i \in \mathrm{SYT}(\lambda^i)$ and a partition $\lambda^i \vdash n$, for all $1 \le i \le m$.

Another consequence of the Robinson–Schensted correspondence is that inverse descent classes are disjoint unions of Knuth classes and therefore Schur-positive. In particular, for $S \subseteq [n-1]$ we have

$$\mathrm{D}_{n,S}^{-1} \;=\; \bigsqcup_{\substack{P \in \mathrm{SYT}_n \\ \mathrm{Des}(P) = S}} \mathrm{K}_P\,.$$

[15] See the discussion in [89, Example 7.18.8(c)].

Inverse descent classes are connected to ribbon Schur functions.

Each ribbon defines a skew Schur function, sometimes called *ribbon Schur function*. For a subset $S \subseteq [n-1]$ (resp. composition α of n) we define $r_{n,S}(\mathbf{x}) := s_{Z_{n,S}}(\mathbf{x})$ (resp. $r_\alpha(\mathbf{x}) := s_{Z_\alpha}(\mathbf{x})$). Their study goes back to MacMahon [65] and in recent years they appeared in seminal works of Gessel [52] and Gessel–Reutenauer [55] and in Schur-positivity problems [25, 54]. Ribbon Schur functions are the images via the Frobenius characteristic map of the $\mathfrak{S}_n$-characters of Specht modules of ribbon shape, often called the *Foulkes characters*. The latter were initially considered by Foulkes [47] and later studied by Diaconis and Fulman [38] and others. Their connection with descent representations of the symmetric group will be reviewed in Section 5.2

Combining Lemma 1.3.1 and Equation (1.17) yields the following result of Gessel [52, Theorem 7] (see also [89, Corollary 7.23.4]).

Lemma 1.6.2. *For every $S \subseteq [n-1]$, we have*

$$r_{n,S}(\mathbf{x}) \;=\; F(\mathrm{D}_{n,S}^{-1}; \mathbf{x}) \;=\; F(\mathrm{SYT}(\mathrm{Z}_{n,S}); \mathbf{x}).$$

In particular, inverse descent classes are Schur-positive for the Foulkes characters of $\mathfrak{S}_n$ and

$$r_{n,S}(\mathbf{x}) \;=\; \sum_{\lambda \vdash n} c_\lambda(S) s_\lambda(\mathbf{x}), \tag{1.19}$$

where $c_\lambda(S)$ counts the number of standard Young tableaux $Q \in \mathrm{SYT}(\lambda)$ such that $\mathrm{Des}(Q) = S$*.*

Foulkes characters appear in disguised form in Stanley's work on group actions on finite posets [88] as well as in Louis Solomon's work on group algebras of Coxeter groups [83] . In particular, we have (cf. [88, Theorem 4.3] and [83, Section 6])

$$\phi_\alpha \;=\; \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \; 1_\beta \uparrow_{\mathfrak{S}_\beta}^{\mathfrak{S}_n} \tag{1.20}$$

for the Foulkes character ϕ_α associated to the composition α of n, where 1_β is the trivial $\mathfrak{S}_\beta$-character. The symmetric function version of Equation (1.20)

$$r_\alpha(\mathbf{x}) \;=\; \sum_{\beta \preceq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \; h_\beta(\mathbf{x}), \tag{1.21}$$

which also follows from the work of MacMahon, was used by Gessel [52] as the definition of ribbon Schur functions. We write $\phi_{n,S}$ for the Foulkes character associated to the subset S of $[n-1]$. Both Equations (1.20) and (1.21) can be stated in terms of subsets of $[n-1]$.

Chapter 2

Specializations of colored quasisymmetric functions; General formulas

This chapter reviews the notion of colored quasisymmetric functions and then develops a method for specializing them to derive general formulas for the joint distribution of

- a Mahonian statistic and the statistics which count the number of entries of a colored permutation of a certain color
- an Eulerian statistic, a Mahonian statistic and the statistics which count the number of entries of a colored permutation of a certain color

on colored permutation groups. In addition, it introduces the notion of (k, ℓ)-flag major index on signed permutations which generalizes both the major and the flag major index and derives general formulas involving the distribution of this statistic.

Let $q, p, p_0, \ldots, p_{r-1}$ be indeterminates and $\mathbf{p} = (p_0, p_1, \ldots, p_{r-1})$. For a function $f : \mathbb{Z}_r \to \mathbb{N}$, we define

$$\mathbf{p}^f \ := \ p_0^{f(0)} p_1^{f(1)} \cdots p_{r-1}^{f(r-1)}.$$

Similarly, for $\mathbf{a}$ and other such bold variables. For a nonnegative integer n, let

$$(x; q)_n := \begin{cases} 1, & \text{if } n = 0 \\ (1-x)(1-xq)\cdots(1-xq^{n-1}), & \text{if } n \geq 1 \end{cases}$$

and set $(q)_n := (q, q)_n$. Also, for a statement P, let $\chi(P) = 1$, if P is true and $\chi(P) = 0$, otherwise.

2.1 Colored symmetric and quasisymmetric functions

A signed analogue of the algebra of quasisymmetric functions was introduced in Chow's Ph.D. thesis [32, Chapter 2]. Chow's homogeneous type B quasisymmetric functions of degree n are indexed by *pseudo-compositions* of n, which are compositions of n whose first part is a nonnegative integer. Although, this notion may be useful when viewing $\mathfrak{B}_n$ as a Coxeter group (see, for example, [16, 68]) this does not seem to be the case when $\mathfrak{B}_n$ is viewed as a colored permutation group.

In this direction, Poirier [75, 76] introduced a colored analogue of the algebra of quasisymmetric functions. For a comparison between Chow and Poirier's notion of signed quasisymmetric functions we refer to Petersen's comprehensive note [72]. Throughout this thesis we deal with Poirier's colored quasisymmetric functions. This choice is motivated by their role in Adin, Athanasiadis, Elizalde and Roichman's recent work [1] on developing a signed analogue of the concept of fine sets and fine characters of Adin and Roichman [8], which will be reviewed in Section 5.3.

Since their introduction, colored quasisymmetric functions were studied by several authors, including Baumann and Hohlweg [19] and Bergeron and Hohlweg [21] from a Hopf algebra point of view. Later, Hsiao and Petersen [62] developed a colored P-partition theory in which Poirier's fundamental colored quasisymmetric functions play the role of Gessel's fundamental quasisymmetric functions to Stanley's P-partition theory. We will review the relation between quasisymmetric functions and P-partitions, as well as Hsiao–Petersen's theory of colored P-partitions in Chapter 4.

We mostly follow the exposition of [1] and [62], although some adjustments have to be made. Let $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots)$ be sequences of commuting indeterminates, for each $0 \le j \le r-1$ and $\mathbb{C}[[\mathbf{X}^{(r)}]]$ be the $\mathbb{C}$-algebra of formal power series in $\mathbf{X}^{(r)} := (x_i^{(0)}, x_i^{(1)}, \dots, x_i^{(r-1)})_{i\ge 1}$. The multiplication in $\mathbb{C}[[\mathbf{X}^{(r)}]]$ is again the usual multiplication of formal power series. For ease of notation we will not mention the variables $\mathbf{x}^{(i)}$, except when it is needed. Let $\mathrm{Sym}^{(r)}$ be the subalgebra of all elements of $\mathbb{C}[[\mathbf{X}^{(r)}]]$, which are symmetric in each $\mathbf{x}^{(i)}$ separately. We call this the *algebra of colored symmetric functions*.

The algebra $\mathrm{Sym}^{(r)}$ can be viewed in two more ways. Firstly, as discussed in [64, Chapter 1, Appendix B] it is the graded $\mathbb{C}$-algebra generated by the independent indeterminates $p_n^{(0)}, p_n^{(1)}, \dots, p_n^{(r-1)}$, for $n \ge 1$, where each of $p_n^{(j)}$ is of degree n. With this in mind, every $p_n^{(j)}$ may be regarded as the n-th power sum symmetric function in $\mathbf{x}^{(j)}$. Secondly, we can think of $\mathrm{Sym}^{(r)}$ as the tensor product $\mathrm{Sym} \otimes \mathrm{Sym} \otimes \cdots \otimes \mathrm{Sym}$ (r times). Since, the algebra of symmetric functions is generated by the power sum symmetric functions, $p_n^{(j)}$ may be regarded as $1 \otimes \cdots \otimes p_n \otimes \cdots \otimes 1$, where p_n lies in the jth component.

Consider the left lexicographic order on $[n] \times \mathbb{Z}_r$ or equivalently, the following total order

$$1^0 <_{\mathrm{llex}} 1^1 <_{\mathrm{llex}} \cdots <_{\mathrm{llex}} 1^{r-1} <_{\mathrm{llex}} \cdots <_{\mathrm{llex}} n^0 <_{\mathrm{llex}} n^1 <_{\mathrm{llex}} \cdots <_{\mathrm{llex}} n^{r-1}$$

on $\Omega_{n,r}$. The following observation follows immediately from the definition of $<_{\mathrm{llex}}$.

Observation 2.1.1. Let $i, j \in [n]$ and $\alpha, \beta \in \mathbb{Z}_r$.

- If $\alpha \leq \beta$, then $i^\alpha >_{\text{llex}} j^\beta$ if and only if $i > j$.
- If $\alpha > \beta$, then $i^\alpha >_{\text{llex}} j^\beta$ if and only if $i \geq j$.

Definition 2.1.2. (cf. [62, Page 269]) A *colored quasisymmetric function* f is an element of $\mathbb{C}[[\mathbf{X}^{(r)}]]$ of bounded degree such that for all $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{Z}_{>0}$ and all $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \mathbb{Z}_r$ we have[1]

$$[{x_{i_1}^{(\epsilon_1)}}^{\alpha_1} {x_{i_2}^{(\epsilon_2)}}^{\alpha_2} \cdots {x_{i_k}^{(\epsilon_k)}}^{\alpha_k}]f \;=\; [{x_{j_1}^{(\epsilon_1)}}^{\alpha_1} {x_{j_2}^{(\epsilon_2)}}^{\alpha_2} \cdots {x_{j_k}^{(\epsilon_k)}}^{\alpha_k}]f$$

for all $i_1^{\epsilon_1} >_{\text{llex}} i_2^{\epsilon_2} >_{\text{llex}} \cdots >_{\text{llex}} i_k^{\epsilon_k}$ and $j_1^{\epsilon_1} >_{\text{llex}} j_2^{\epsilon_2} >_{\text{llex}} \cdots >_{\text{llex}} j_k^{\epsilon_k}$.

Let $\mathrm{QSym}^{(r)}$ be the $\mathbb{C}$-algebra of colored quasisymmetric functions. There is a natural inclusion of algebra $\mathrm{Sym}^{(r)} \to \mathrm{QSym}^{(r)}$, but not every colored quasisymmetric function belongs to $\mathrm{Sym}^{(r)}$. For example, the element

$$\sum_{i^0 >_{\text{llex}} j^1} {x_i^{(0)}}^2 x_j^{(1)} \;=\; \sum_{i>j} {x_i^{(0)}}^2 x_j^{(1)} \;\in\; \mathrm{QSym}^{(2)}$$

does not belong to $\mathrm{Sym}^{(2)}$ because ${x_2^{(0)}}^2 x_1^{(1)}$ appears as term, whereas ${x_1^{(0)}}^2 x_1^{(1)}$ does not. Notice that adding

$$\sum_{i^1 >_{\text{llex}} j^0} x_i^{(1)} {x_j^{(0)}}^2 \;=\; \sum_{i>j} x_i^{(1)} {x_j^{(0)}}^2 + \sum_{i=j} x_i^{(1)} {x_i^{(0)}}^2 \;\in\; \mathrm{QSym}^{(2)}$$

makes it an element of $\mathrm{Sym}^{(2)}$.

It is apparent from the descriptions above that $\mathrm{Sym}^{(r)}$ is a graded algebra. Let $\mathrm{Sym}_n^{(r)}$ be its homogeneous n-th component, whose dimension as a vector space equals the number of r-partite partitions of n. A natural basis of $\mathrm{Sym}_n^{(r)}$ is spanned by elements

$$m_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}) \;:=\; m_{\lambda^{(0)}}(x^{(0)}) m_{\lambda^{(1)}}(x^{(1)}) \cdots m_{\lambda^{(r-1)}}(x^{(r-1)})$$

for every r-partite partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ of n.

The algebra of colored quasisymmetric functions $\mathrm{QSym}^{(r)}$ is also a graded algebra. Let $\mathrm{QSym}_n^{(r)}$ be its homogeneous n-th component, whose dimension as a vector space equals $r(r+1)^{n-1}$, the number of r-colored compositions of n. It follows immediately from Definition 2.1.2 that the set of

$$M_{\gamma^\epsilon}(\mathbf{X}^{(r)}) \;:=\; \sum_{i_1^{\epsilon_1} >_{\text{llex}} i_2^{\epsilon_2} >_{\text{llex}} \cdots >_{\text{llex}} i_k^{\epsilon_k}} {x_{i_1}^{(\epsilon_1)}}^{\gamma_1} {x_{i_2}^{(\epsilon_2)}}^{\gamma_2} \cdots {x_{i_k}^{(\epsilon_k)}}^{\gamma_k}$$

for all r-colored composition $\gamma^\epsilon = (\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \ldots, \gamma_k^{\epsilon_k})$ of n forms a basis for this vector space. These are called *monomial colored quasisymmetic functions*. The following lemma explains the connection between the monomial bases of $\mathrm{Sym}^{(r)}$ and $\mathrm{QSym}^{(r)}$.

[1] The notation is somewhat complicated but we will not need to exponentiate any colored variable in what follows, expect for the current section.

Lemma 2.1.3. (cf. [76, Lemma 12]) *For an r-partite partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ of n, we have*

$$m_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}) \;=\; \sum_{\gamma} M_\gamma(\mathbf{X}^{(r)}) \tag{2.1}$$

where the sum runs through all r-colored compositions γ of n whose ith colored component[2]*, when arranged in decreasing order yields the underlying composition of $\lambda^{(i)}$.*

Proof. The proof follows by expanding each $m_{\lambda^{(i)}}(\mathbf{x}^{(i)})$ on the left-hand side of Equation (2.1) via the rule [89, Proof of Proposition 7.19.9]

$$m_{\lambda^{(i)}}(\mathbf{x}^{(i)}) \;=\; \sum M_{\gamma^{(i)}}(\mathbf{x}^{(i)})$$

where the sum runs through every composition $\gamma^{(i)} \vDash |\lambda^{(i)}|$ such that when rearranging its parts in decreasing order yields the underlying composition $\lambda^{(i)}$. $\square$

Example 2.1.4. To illustrate Lemma 2.1.3, for the bipartition $((2),(1))$ of 3 we explicitly compute

$$\begin{aligned}
m_{((2),(1))}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}) &= m_{(2)}(\mathbf{x}^{(0)}) m_{(1)}(\mathbf{x}^{(1)}) \\
&= M_{(2)}(\mathbf{x}^{(0)}) M_{(1)}(\mathbf{x}^{(1)}) \\
&= ({x_1^{(0)}}^2 + {x_2^{(0)}}^2 + \cdots)(x_1^{(1)} + x_2^{(1)} + \cdots) \\
&= {x_1^{(0)}}^2 x_1^{(1)} + {x_1^{(0)}}^2 x_2^{(1)} + \cdots + {x_2^{(0)}}^2 x_1^{(1)} + {x_2^{(0)}}^2 x_1^{(1)} + \cdots \\
&= \underbrace{\sum_{i>j} {x_i^{(0)}}^2 x_j^{(1)}} + \underbrace{\sum_{i>j} x_i^{(1)} {x_j^{(0)}}^2 + \sum_{i=j} x_i^{(1)} {x_i^{(0)}}^2} \\
&= \sum_{i^0 >_{\mathrm{llex}} j^1} {x_i^{(0)}}^2 x_j^{(1)} + \sum_{i^1 >_{\mathrm{llex}} j^0} x_i^{(1)} {x_j^{(0)}}^2 \\
&= M_{(2^0,1^1)} + M_{(1^1,2^0)}.
\end{aligned}$$

It is quite interesting how the (left) lexicographic order comes into play in the second to last equality.

We will now introduce the fundamental colored quasisymmetric functions. The *fundamental colored quasisymmetric function* associated to $\gamma \in \mathrm{Comp}(n,r)$ is defined by

$$F_\gamma(\mathbf{X}^{(r)}) \;:=\; \sum_{\gamma \preceq \beta} M_\beta(\mathbf{X}^{(r)}).$$

Grouping together inequalities in the left lexicographic order as was done in Section 1.5 yields an alternative formula for $F_\gamma(\mathbf{X}^{(r)})$ similar to Equation (1.16). For

[2] By component of a colored composition we simply mean the composition obtained by merging together all parts of certain color in order of appearance and forgetting the color.

an r-colored composition $\gamma^\epsilon = (\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \ldots, \gamma_k^{\epsilon_k})$ of n, we have

$$\begin{aligned} F_{\gamma^\epsilon}(\mathbf{X}^{(r)}) &= \sum_{\substack{i_1^{\tilde{\epsilon}_1} \geq_{\text{llex}} i_2^{\tilde{\epsilon}_2} \geq_{\text{llex}} \cdots \geq_{\text{llex}} i_n^{\tilde{\epsilon}_n} \\ i_{r_j}^{\tilde{\epsilon}_j} >_{\text{llex}} i_{r_{j+1}}^{\tilde{\epsilon}_{j+1}}, \text{ for all } 1 \leq j \leq k-1}} x_{i_1}^{\tilde{\epsilon}_1} x_{i_2}^{\tilde{\epsilon}_2} \cdots x_{i_n}^{\tilde{\epsilon}_n} \\ &= \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \\ \epsilon_j \leq \epsilon_{j+1} \Rightarrow i_{r_j} > i_{r_{j+1}}, \text{ for all } 1 \leq j \leq k-1}} x_{i_1}^{\tilde{\epsilon}_1} x_{i_2}^{\tilde{\epsilon}_2} \cdots x_{i_n}^{\tilde{\epsilon}_n}, \end{aligned} \tag{2.2}$$

where the second equality follows from Observation 2.1.1. Therefore, if we define

$$\operatorname{Des}(\gamma^\epsilon) := \{r_i : \epsilon_i \leq \epsilon_{i+1}, \text{ for each } 1 \leq i \leq k-1\} \subseteq [n-1],$$

Equation (2.2) becomes

$$F_{\gamma^\epsilon}(\mathbf{X}^{(r)}) = \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \\ j \in \operatorname{Des}(\gamma^\epsilon) \Rightarrow i_j > i_{j+1}}} x_{i_1}^{\tilde{\epsilon}_1} x_{i_2}^{\tilde{\epsilon}_2} \cdots x_{i_n}^{\tilde{\epsilon}_n}. \tag{2.3}$$

Example 2.1.5. To illustrate this argument, consider the 2-colored composition $\gamma = (1^0, 1^1, 2^1)$ of 4 with color vector $(0, 1, 1, 1)$. We have

$$\begin{aligned} F_{(1^0,1^1,2^1)} &= M_{(1^0,1^1,2^1)} + M_{(1^0,1^1,1^1,1^1)} \\ &= \sum_{i_1^0 >_{\text{llex}} i_2^1 >_{\text{llex}} i_3^1 = i_4^1} x_{i_1}^{(0)} x_{i_2}^{(1)} {x_{i_3}^{(1)}}^2 \\ &\quad + \sum_{i_1^0 >_{\text{llex}} i_2^1 >_{\text{llex}} i_3^1 >_{\text{llex}} i_4^1} x_{i_1}^{(0)} x_{i_2}^{(1)} x_{i_3}^{(1)} x_{i_4}^{(1)} \\ &= \sum_{i_1^0 >_{\text{llex}} i_2^1 >_{\text{llex}} i^3 \geq_{\text{llex}} i_4^1} x_{i_1}^{(0)} x_{i_2}^{(1)} x_{i_3}^{(1)} x_{i_4}^{(1)} \\ &= \sum_{i_1 > i_2 > i_3 \geq i_4} x_{i_1}^{(0)} x_{i_2}^{(1)} x_{i_3}^{(1)} x_{i_4}^{(1)} \end{aligned}$$

and $\operatorname{Des}(\gamma^\epsilon) = \{1, 2\}$.

We define the monomial and fundamental colored quasisymmetric functions associated to a colored subset σ of $[n]$ as the monomial and fundamental colored quasisymmetric functions associated to its corresponding colored composition. We do the same with $\operatorname{Des}(\sigma)$. The case where σ is the colored descent set of an r-colored permutation w^ϵ or a standard Young r-partite tableau $\boldsymbol{Q} \in \mathrm{SYT}_{n,r}$ is of particular interest. In particular,

$$\begin{aligned} \operatorname{Des}(\operatorname{sDes}(w^\epsilon)) &= \operatorname{Des}^*_{<_c}(w^\epsilon) \\ \operatorname{Des}(\operatorname{sDes}(\boldsymbol{Q})) &= \operatorname{Des}^*(\boldsymbol{Q}) \end{aligned}$$

and therefore Equation (2.3) becomes

$$F_{\operatorname{sDes}(w^\epsilon)}(\mathbf{X}^{(r)}) = \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \\ j \in \operatorname{Des}^*_{<_c}(w^\epsilon) \Rightarrow i_j > i_{j+1}}} x_{i_1}^{(\epsilon_1)} x_{i_2}^{(\epsilon_2)} \cdots x_{i_n}^{(\epsilon_n)} \tag{2.4}$$

and similarly for $F_{\mathrm{sDes}(\boldsymbol{Q})}$. For ease of notation we will write $F_{w^\epsilon} := F_{\mathrm{sDes}(w)}$ and $F_{\boldsymbol{Q}} = F_{\mathrm{sDes}(\boldsymbol{Q})}$.

The original definition of colored quasisymmetric functions (see [19, Page 1533], [21, Section 5.2] and [62, Page 269]) required that the inequalities in Definition 2.1.2 are reversed. This approach leads to a different definition of $\mathrm{Des}(\gamma^\epsilon)$ than ours. The advantage of our approach will be unveiled in the next section. Nevertheless, we will explain why these two approaches are equivalent (see, also, Remark 1.5.1).

Consider the following operation on $\mathrm{Comp}(n,r)$

$$(\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \ldots, \gamma_k^{\epsilon_k}) \overset{*}{\longmapsto} (\gamma_k^{\epsilon_k}, \gamma_{k-1}^{\epsilon_{k-1}}, \ldots, \gamma_1^{\epsilon_1}).$$

In terms of colored subsets of $[n]$, it reads

$$\{s_1^{\epsilon(s_1)}, s_2^{\epsilon(s_2)}, \ldots, s_k^{\epsilon(s_k)}, n^{\epsilon(n)}\} \overset{*}{\longmapsto} \{n^{\epsilon(1)}, (n-s_1)^{\epsilon(s_2)}, \ldots, (n-s_k)^{\epsilon(n)}\}.$$

To emphasize the interplay between passing from a colored composition to a colored subset and backwards and applying *, we notice the following commutative diagram

$$\begin{tikzcd}
\mathrm{Comp}(n,r) \arrow[r] \arrow[d, "*"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma(n,r) \arrow[d, "*"] \\
\mathrm{Comp}(n,r) & \Sigma(n,r). \arrow[l]
\end{tikzcd}$$

For example, for $n = 11$ and $r = 3$ we have

$$(2^2, 2^2, 2^0, 1^0, 2^0, 1^1, 1^0) \overset{*}{\mapsto} (1^0, 1^1, 2^0, 1^0, 2^0, 2^2, 2^2) \mapsto \{1^0, 2^1, 4^0, 5^0, 7^0, 9^2, 11^2\}$$

and

$$(2^2, 2^2, 2^0, 1^0, 2^0, 1^1, 1^0) \mapsto \{2^2, 4^2, 6^0, 7^0, 9^0, 10^1, 11^0\} \overset{*}{\mapsto} \{11^2, 9^2, 7^0, 5^0, 4^0, 2^1, 1^0\}.$$

Lemma 2.1.6. *The map $^* : \mathrm{QSym}^{(r)} \to \mathrm{QSym}^{(r)}$ defined by $F^*_{\gamma^\epsilon} := F_{\gamma^{\epsilon *}}$, for all $\gamma^\epsilon \in \mathrm{Comp}(n,r)$ and extending linearly is an algebra automorphism and*[3]

$$\begin{aligned}
F^*_{\gamma^\epsilon}(\mathbf{X}^{(r)}) &= \sum_{\substack{i_1^{\tilde{\epsilon}_1} \le_{\mathrm{llex}} i_2^{\tilde{\epsilon}_2} \le_{\mathrm{llex}} \cdots \le_{\mathrm{llex}} i_n^{\tilde{\epsilon}_n} \\ i_{r_j}^{\tilde{\epsilon}_j} <_{\mathrm{llex}} i_{r_{j+1}}^{\tilde{\epsilon}_{j+1}}, \text{ for all } 1 \le j \le k-1}} x_{i_1}^{\tilde{\epsilon}_1} x_{i_2}^{\tilde{\epsilon}_2} \cdots x_{i_n}^{\tilde{\epsilon}_n} \\
&= \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ \epsilon_j \ge \epsilon_{j+1} \Rightarrow i_{r_j} < i_{r_{j+1}}, \text{ for all } 1 \le j \le k-1}} x_{i_1}^{\tilde{\epsilon}_1} x_{i_2}^{\tilde{\epsilon}_2} \cdots x_{i_n}^{\tilde{\epsilon}_n}.
\end{aligned}$$

Closing this section, we recall the relation between the Schur basis of $\mathrm{QSym}^{(r)}$, spanned by the elements

$$s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}) := s_{\lambda^{(0)}}(\mathbf{x}^{(0)}) s_{\lambda^{(1)}}(\mathbf{x}^{(1)}) \cdots s_{\lambda^{(r-1)}}(\mathbf{x}^{(r-1)}),$$

[3] This formula coincides with the definition of the fundamental colored quasisymmetric function used in [19, 21, 62].

for all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ of n and the fundamental basis of $\mathrm{QSym}^{(r)}$. Recently, Adin et al. [1, Proposition 4.2] provided a signed analogue of Equation (1.17). The following colored analogue

$$s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}) = \sum_{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda})} F_{\boldsymbol{Q}}(\mathbf{X}^{(r)}), \tag{2.5}$$

follows from a trivial generalization of the proof of [1, Proposition 4.2] and will be used in Section 3.1.

2.2 Specializations of symmetric/quasisymmetric functions; Motivation

Gessel and Reutenauer, in their seminal paper [55], used the stable principal specialization and the principal specialization of order m of fundamental quasisymmetric functions, together with the fact that the quasisymmetric generating function of the set of permutations of fixed cycle type is symmetric, to derive formulas for the joint distribution of the descent statistic and major index on cycles, involutions and derangements.

The principal specialization of order m and the stable principal specialization of $F_{n,S}(\mathbf{x})$ satisfy the following formulas [55, Lemma 5.2] (see also the first half of [52, Section 4])

$$\sum_{m \geq 1} \mathrm{ps}_{q,m}(F_{n,S}(\mathbf{x}))\, x^{m-1} = \frac{x^{|S|}\, q^{\mathrm{sum}(S)}}{(x;q)_n} \tag{2.6}$$

$$\mathrm{ps}_q(F_{n,S}(\mathbf{x})) = \frac{q^{\mathrm{sum}(S)}}{(q)_n}, \tag{2.7}$$

where $\mathrm{sum}(S)$ stands for the sum of all elements of S. The standard way to connect quasisymmetric functions with permutation statistics is by letting $S = \mathrm{Des}(w)$ in Equation (1.16), as done in [55, Section 5]. Equations (2.6) and (2.7) allow us to study the Euler–Mahonian distribution on $\mathfrak{S}_n$ by specializing the quasisymmetric generating function associated to a subset $\mathcal{A} \subseteq \mathfrak{S}_n$. In particular, one has [55, Theorem 5.3]

$$\sum_{m \geq 1} \mathrm{ps}_{q,m}(F(\mathcal{A};\mathbf{x}))\, x^{m-1} = \frac{\sum_{w \in \mathcal{A}} x^{\mathrm{des}(w)}\, q^{\mathrm{maj}(w)}}{(x;q)_n} \tag{2.8}$$

$$\mathrm{ps}_q(F(\mathcal{A};\mathbf{x})) = \frac{\sum_{w \in \mathcal{A}} q^{\mathrm{maj}(w)}}{(q)_n}. \tag{2.9}$$

Gessel and Reutenauer [55] studied subsets of the symmetric group whose quasisymmetric generating function is, in fact, symmetric and exploited a connection with the representation theory of the symmetric group. Let us now illustrate how one can specialize fundamental quasisymmetric functions in order to prove Equations (1.2) and (1.7). Although the relation between these formulas is not obvious, the above mentioned machinery allows us to easily prove both in a uniform way. This proof will serve as a prototype for all proofs of our applications in Chapter 3. For a similar approach, see Gessel and Zhuang's recent work [57].

Proof of Equations (1.2) *and* (1.7). As we saw in Equation (1.18), the quasisymmetric generating function of $\mathfrak{S}_n$ has the following nice form

$$F(\mathfrak{S}_n;\mathbf{x}) = (x_1 + x_2 + \cdots)^n. \tag{2.10}$$

Taking the principal specialization of order m and the stable principal specialization of Equation (2.10) yields

$$\begin{aligned} \mathrm{ps}_{q,m}(F(\mathfrak{S}_n;\mathbf{x})) &= [m]_q^n \\ \mathrm{ps}_q(F(\mathfrak{S}_n;\mathbf{x})) &= \frac{1}{(1-q)^n}, \end{aligned}$$

respectively. Then, Equations (1.2) and (1.7) follow by substituting these computations in Equations (2.8) and (2.9), for $\mathcal{A} = \mathfrak{S}_n$, respectively. □

This proof suggests that whenever $F(\mathcal{A};\mathbf{x})$ has a nice form, then one can use Equations (2.8) and (2.9) to prove Euler–Mahonian identities on $\mathcal{A}$. Particularly interesting examples include (among others) the following collections of permutations

- *derangements* (permutations without fixed points)
- *involutions* (permutations which are equal to their own inverses)
- *conjugacy classes* (permutations of a given cycle type)
- *inverse descent classes* (sets of permutations whose inverse has a fixed descent set)

In the following section we will provide various colored generalizations of Equations (2.8) and (2.9)

2.3 Main formulas

Fix a total order $<$ on $\Omega_{n,r}$. For an r-colored permutation $w^\epsilon \in \mathfrak{S}_{n,r}$, we define the following statistics

$$\begin{aligned} \mathrm{Des}_<(w^\epsilon) &:= \{i \in [n-1] : w_i^{\epsilon_i} > w_{i+1}^{\epsilon_{i+1}}\} \cup \{0 : \epsilon_1 \neq 0\} \\ \mathrm{Des}^*_<(w^\epsilon) &:= \mathrm{Des}_<(w^\epsilon) \setminus \{0\} \\ \mathrm{des}_<(w^\epsilon) &:= |\,\mathrm{Des}_<(w^\epsilon)| \\ \mathrm{des}^*_<(w^\epsilon) &:= |\,\mathrm{Des}^*_<(w^\epsilon)| \\ \mathrm{fdes}_<(w^\epsilon) &:= r\,\mathrm{des}^*_<(w^\epsilon) + \epsilon_1 \\ \mathrm{maj}_<(w^\epsilon) &:= \mathrm{sum}(\mathrm{Des}^*_<(w^\epsilon)) \\ \mathrm{fmaj}_<(w^\epsilon) &:= r\,\mathrm{maj}_<(w^\epsilon) + \mathrm{csum}(w) \\ \mathrm{n}_j(w^\epsilon) &:= |\{i \in [n] : \epsilon_i = j\}|, \text{ for all } 0 \le j \le r-1 \\ \mathrm{n}(w^\epsilon) &:= (\mathrm{n}_0(w^\epsilon), \mathrm{n}_1(w^\epsilon), \ldots, \mathrm{n}_{r-1}(w^\epsilon)). \end{aligned}$$

For the rest of this section we omit the total order subscript in colored statistics for ease of notation. Also, when there is no need to specify the color vector of a colored permutation, we will simply write w instead of w^ϵ.

We begin by considering the specialization $\mathrm{ps}^{(r)}_{q,\mathbf{p}}$ defined by the substitutions $x^{(j)}_i = q^{i-1}p_j$ for every $i \geq 1$ and $0 \leq j \leq r-1$, the specialization $\mathrm{ps}^{(r)}_{q,\mathbf{p},m}$ defined by the substitutions

$$\begin{cases} x^{(0)}_i = q^{i-1}p_0, & 1 \leq i \leq m \\ x^{(j)}_i = q^{i-1}p_j, & 1 \leq j \leq r-1 \text{ and } 1 \leq i \leq m-1 \\ x^{(j)}_i = 0, & \text{otherwise} \end{cases}$$

and the specialization $\widetilde{\mathrm{ps}}^{(r)}_{q,\mathbf{p},m}$, defined by the substitutions $x^{(j)}_i = q^{i-1}p_j$ for every $1 \leq i \leq m$ and $0 \leq j \leq r-1$ and $x^{(j)}_i = 0$ otherwise.

Theorem 2.3.1. *For a positive integer n and every $w \in \mathfrak{S}_{n,r}$, we have*

$$\mathrm{ps}^{(r)}_{q,\mathbf{p}}(F_w(\mathbf{X}^{(r)})) = \frac{q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(q)_n} \tag{2.11}$$

$$\sum_{m\geq 1} \mathrm{ps}^{(r)}_{q,\mathbf{p},m}(F_w(\mathbf{X}^{(r)}))\, x^{m-1} = \frac{x^{\mathrm{des}(w)}q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}} \tag{2.12}$$

$$\sum_{m\geq 1} \widetilde{\mathrm{ps}}^{(r)}_{q,\mathbf{p},m}(F_w(\mathbf{X}^{(r)}))\, x^{m-1} = \frac{x^{\mathrm{des}^*(w)}q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}}. \tag{2.13}$$

Proof. We prove Equations (2.11) and (2.12) in parallel. Equation (2.13) follows in a similar way. For a colored permutation $w^\epsilon \in \mathfrak{S}_{n,r}$, we have

$$\mathrm{ps}^{(r)}_{q,\mathbf{p}}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n\geq 1\\ j\in\mathrm{Des}^*(w^\epsilon)\Rightarrow i_j>i_{j+1}}} q^{i_1+i_2+\cdots+i_n-n}\,\mathbf{p}^{\mathrm{n}(w^\epsilon)} \tag{2.14}$$

$$\mathrm{ps}^{(r)}_{q,\mathbf{p},m}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{\substack{m:=i_0\geq i_1\geq i_2\geq\cdots\geq i_n\geq 1\\ j\in\mathrm{Des}(w^\epsilon)\Rightarrow i_j>i_{j+1}}} q^{i_1+i_2+\cdots+i_n-n}\,\mathbf{p}^{\mathrm{n}(w^\epsilon)}. \tag{2.15}$$

Under the specialization $\mathrm{ps}^{(r)}_{q,\mathbf{p},m}$, substitutions $x^{(1)}_m, x^{(2)}_m, \ldots, x^{(r-1)}_m$ occur only if $\epsilon_1 \neq 0$, which in turn is exactly when 0 is considered a descent of w^ϵ, explaining the first inequality under the sum on the right-hand side of Equation (2.15). Define

$$\begin{aligned} i'_j &= i_j - \chi_j - \cdots - \chi_{n-1} \\ i'_n &= i_n, \end{aligned}$$

where $\chi_j := \chi(j \in \mathrm{Des}(w^\epsilon))$, for every $0 \leq j \leq n-1$. Then, Equations (2.14) and (2.15) become

$$\mathrm{ps}^{(r)}_{q,\mathbf{p}}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{i'_1\geq i'_2\geq\cdots\geq i'_n\geq 1} q^{i'_1+i'_2+\cdots+i'_n-n+\mathrm{maj}(w^\epsilon)}\mathbf{p}^{\mathrm{n}(w^\epsilon)} \tag{2.16}$$

$$\mathrm{ps}^{(r)}_{q,\mathbf{p},m}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{m-\mathrm{des}(w^\epsilon)\geq i'_1\geq i'_2\geq\cdots\geq i'_n\geq 1} q^{i'_1+i'_2+\cdots+i'_n-n+\mathrm{maj}(w^\epsilon)}\mathbf{p}^{\mathrm{n}(w^\epsilon)}, \tag{2.17}$$

because

$$\begin{aligned}\mathrm{des}(w^\epsilon) &= \sum_{j=0}^{n-1} \chi_j \\ \mathrm{maj}(w^\epsilon) &= \sum_{j=1}^{n-1} j\chi_j.\end{aligned}$$

Now, let

$$\begin{aligned}a_0 &= m - \mathrm{des}(w^\epsilon) - i'_1 \\ a_j &= i'_j - i'_{j+1} \\ a_n &= i'_n - 1,\end{aligned}$$

for every $1 \leq j \leq n-1$. On the one hand, Equation (2.16) becomes

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{a_1,a_2,\ldots,a_n\in\mathbb{N}} q^{a_1+2a_2+\cdots+na_n+\mathrm{maj}(w^\epsilon)}\,\mathbf{p}^{\mathrm{n}(w^\epsilon)} = \frac{q^{\mathrm{maj}(w^\epsilon)}\mathbf{p}^{\mathrm{n}(w^\epsilon)}}{(q)_n}.$$

On the other hand, Equation (2.17) becomes

$$\mathrm{ps}_{q,\mathbf{p},m}^{(r)}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum q^{a_1+2a_2+\cdots+na_n+\mathrm{maj}(w^\epsilon)}\,\mathbf{p}^{\mathrm{n}(w^\epsilon)}$$

where the sum runs through all $\mathbb{N}$-solutions of $a_0 + a_1 + \cdots + a_n = m - \mathrm{des}(w^\epsilon) - 1$. This is exactly the coefficient of x^{m-1} in the expansion of the right-hand side of Equation (2.12) and the proof follows. □

Notice that in the proof of [55, Lemma 5.2] (see also [89, Lemma 7.19.10]) the authors deal with the comajor index, instead of the major index. Our choice of the direction of inequalities in the definition of the fundamental colored quasisymmetric functions (see Equation (2.3)) allows us to deal with the major index directly. This observation explains the motivation behind our choice. The following formulas are immediate consequences of Theorem 2.3.1.

Corollary 2.3.2. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we have*

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)})) = \frac{\sum_{w\in\mathcal{A}} q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(q)_n} \tag{2.18}$$

$$\sum_{m\geq 1} \mathrm{ps}_{q,\mathbf{p},m}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)}))\,x^{m-1} = \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}(w)}q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}} \tag{2.19}$$

$$\sum_{m\geq 1} \widetilde{\mathrm{ps}}_{q,\mathbf{p},m}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)}))\,x^{m-1} = \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}^*(w)}q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}}. \tag{2.20}$$

Remark 2.3.3. Setting $p_j = p^j$, we have that $\mathbf{p}^{\mathrm{n}(w^\epsilon)} = p^{\mathrm{csum}(w^\epsilon)}$ because

$$\sum_{j=0}^{r-1} j\,\mathrm{n}_j(w^\epsilon) = \sum_{j=0}^{r-1} j|\{i\in[n] : \epsilon_i = j\}| = \sum_{i=0}^{n} \epsilon_i = \mathrm{csum}(w^\epsilon)$$

for every colored permutation $w^\epsilon \in \mathfrak{S}_{n,r}$. Therefore, in this case, formulas presented in Theorem 2.3.1 and Corollary 2.3.2 would involve the color sum statistic.

Next, we consider the specialization $\psi_{q,\mathbf{p}}^{(r)}$ defined by the substitutions $x_i^{(j)} = q^{r(i-1)+j}p_j$ for every $i \geq 1$ and $0 \leq j \leq r-1$, the specialization $\psi_{q,\mathbf{p},m}^{(r)}$ defined by the substitutions

$$\begin{cases} x_i^{(0)} = q^{r(i-1)}p_0, & 1 \leq i \leq m \\ x_i^{(j)} = q^{r(i-1)+j}p_j, & 1 \leq j \leq r-1 \text{ and } 1 \leq i \leq m-1 \\ x_i^{(j)} = 0, & \text{otherwise} \end{cases}$$

and the specialization $\widetilde{\psi}_{q,\mathbf{p},m}^{(r)}$ defined by the substitutions $x_i^{(j)} = q^{r(i-1)+j}p_j$ for every $1 \leq i \leq m$ and $0 \leq j \leq r-1$ and $x_i^{(j)} = 0$ otherwise.

Theorem 2.3.4. *For a positive integer n and every $w \in \mathfrak{S}_{n,r}$, we have*

$$\psi_{q,\mathbf{p}}^{(r)}(F_w(\mathbf{X}^{(r)})) = \frac{q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(q)_n} \tag{2.21}$$

$$\sum_{m\geq 1} \psi_{q,\mathbf{p},m}^{(r)}(F_w(\mathbf{X}^{(r)}))\, x^{m-1} = \frac{x^{\mathrm{des}(w)}q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}} \tag{2.22}$$

$$\sum_{m\geq 1} \widetilde{\psi}_{q,\mathbf{p},m}^{(r)}(F_w(\mathbf{X}^{(r)}))\, x^{m-1} = \frac{x^{\mathrm{des}^*(w)}q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}}. \tag{2.23}$$

Proof. The proof follows from Theorem 2.3.1 by setting $q \to q^r$ and $p_j \to q^j p_j$, for all $0 \leq j \leq r-1$. □

The following formulas are immediate consequences of Theorem 2.3.4.

Corollary 2.3.5. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we have*

$$\psi_{q,\mathbf{p}}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)})) = \frac{\sum_{w\in\mathcal{A}} q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(q)_n} \tag{2.24}$$

$$\sum_{m\geq 1} \psi_{q,\mathbf{p},m}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)}))\, x^{m-1} = \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}(w)}q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}} \tag{2.25}$$

$$\sum_{m\geq 1} \widetilde{\psi}_{q,\mathbf{p},m}^{(r)}(F(\mathcal{A};\mathbf{X}^{(r)}))\, x^{m-1} = \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}^*(w)}q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)}}{(x;q)_{n+1}}. \tag{2.26}$$

Lastly, we consider a more complicated specialization $\phi_{q,\mathbf{p},m}^{(r)}$ defined as follows: If $m = rs + t$, for some $1 \leq t \leq r$ and $s \geq 0$, then let $x_i^{(j)} = 0$ if the pair (i,j) is lexicographically greater than the pair $(rs+1, t-1)$ and otherwise

$$x_i^{(j)} = \begin{cases} q^{i+j-1}p_j, & \text{if } i \equiv 1 \pmod{r} \\ 0, & \text{if } i \not\equiv 1 \pmod{r}. \end{cases}$$

We illustrate the definition of $\phi^{(r)}_{q,p,m}$ by considering a specific example for $r=3$ and $m \in \{7,8,9\}$. The substitutions become

$$(x_i^{(j)})_{\substack{0\le j\le 2\\ i\ge 1}} = \begin{cases} \begin{pmatrix} p_0 & 0 & 0 & q^3p_0 & 0 & 0 & q^6p_0 & 0 & \cdots \\ qp_1 & 0 & 0 & q^4p_1 & 0 & 0 & 0 & 0 & \cdots \\ q^2p_2 & 0 & 0 & q^5p_2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m=7 \\ \begin{pmatrix} p_0 & 0 & 0 & q^3p_0 & 0 & 0 & q^6p_0 & 0 & \cdots \\ qp_1 & 0 & 0 & q^4p_1 & 0 & 0 & q^7p_1 & 0 & \cdots \\ q^2p_2 & 0 & 0 & q^5p_2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m=8 \\ \begin{pmatrix} p_0 & 0 & 0 & q^3p_0 & 0 & 0 & q^6p_0 & 0 & \cdots \\ qp_1 & 0 & 0 & q^4p_1 & 0 & 0 & q^7p_1 & 0 & \cdots \\ q^2p_2 & 0 & 0 & q^5p_2 & 0 & 0 & q^8p_2 & 0 & \cdots \end{pmatrix}, & \text{if } m=9. \end{cases}$$

Theorem 2.3.6. *For a positive integer n and every $w \in \mathfrak{S}_{n,r}$, we have*

$$\sum_{m\ge 1} \phi^{(r)}_{q,\mathbf{p},m}(F_w(\mathbf{X}^{(r)}))\, x^{m-1} = \frac{x^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)} \mathbf{p}^{\mathrm{n}(w)}}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}. \tag{2.27}$$

Furthermore, for every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$ we have

$$\sum_{m\ge 1} \phi^{(r)}_{q,\mathbf{p},m}(F(\mathcal{A};\mathbf{X}^{(r)}))\, x^{m-1} = \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{fdes}(w)} q^{\mathrm{fmaj}(w)} \mathbf{p}^{\mathrm{n}(w)}}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}. \tag{2.28}$$

Proof. Taking the specialization $\phi^{(r)}_{q,\mathbf{p},m}$ of the fundamental colored quasisymmetric function associated to $w^\epsilon \in \mathfrak{S}_{n,r}$ yields

$$\phi^{(r)}_{q,\mathbf{p},m}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{\substack{m:=i_0\ge i_1\ge i_2\ge\cdots\ge i_n\ge 1\\ j\in\mathrm{Des}(w^\epsilon)\Rightarrow i_j>i_{j+1}\\ i_1,\dots,i_n \equiv 1 \pmod r}} q^{i_1+i_2+\cdots+i_n-n+\mathrm{csum}(w^\epsilon)}\, \mathbf{p}^{\mathrm{n}(w^\epsilon)}, \tag{2.29}$$

because as in Equation (2.15), $x_m^{(1)}, x_m^{(2)}, \dots, x_m^{(r-1)}$ occur only if $\epsilon_1 \neq 0$, which is equivalent to 0 being a descent of w^ϵ. Define

$$\begin{aligned} i'_0 &= i_0 - \epsilon_1 - r\chi_1 - \cdots - r\chi_{n-1} \\ i'_j &= i_j - r\chi_j - \cdots - r\chi_{n-1} \\ i'_n &= i_n, \end{aligned}$$

where $\chi_j := \chi(j \in \mathrm{Des}(w^\epsilon))$, for every $1 \le j \le n-1$. Then, Equation (2.29) becomes

$$\phi^{(r)}_{q,\mathbf{p},m}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum_{\substack{m-\mathrm{fdes}(w^\epsilon)\ge i'_1\ge i'_2\ge\cdots\ge i'_n\ge 1\\ i'_1,i'_2\dots,i'_n \equiv 1 \pmod r}} q^{i'_1+i'_2+\cdots+i'_n-n+\mathrm{fmaj}(w^\epsilon)}\, \mathbf{p}^{\mathrm{n}(w^\epsilon)}, \tag{2.30}$$

because

$$\begin{aligned}\mathrm{fdes}(w^\epsilon) &= c_1 + r\sum_{j=1}^{n-1}\chi_j\\ \mathrm{fmaj}(w^\epsilon) &= r\sum_{j=1}^{n-1} j\chi_j + \mathrm{csum}(w^\epsilon).\end{aligned}$$

The first inequality under the summation on the right-hand side of Equation (2.30) is justified by the fact that the last nonzero substitution is $x_{m-t+1}^{(t-1)} = q^{m-1}$, where $m \equiv t \pmod r$, for every $1 \le t \le r$. Now, making the substitution

$$\begin{aligned}a_0 &= m - \mathrm{fdes}(w^\epsilon) - i_1'\\ a_j &= i_j' - i_{j+1}'\\ a_n &= i_n' - 1,\end{aligned}$$

for every $1 \le j \le n-1$, Equation (2.30) becomes

$$\phi^{(r)}_{q,\mathbf{p},m}(F_{w^\epsilon}(\mathbf{X}^{(r)})) = \sum q^{a_1+2a_2+\cdots+na_n+\mathrm{fmaj}(w^\epsilon)}\mathbf{p}^{\mathrm{n}(w^\epsilon)}, \tag{2.31}$$

where the sum runs through all $\mathbb{N}$-solutions of $a_0+a_1+a_2+\cdots+a_n = m-\mathrm{fdes}(w^\epsilon)-1$ with the requirement that $a_1, a_2, \ldots, a_n \equiv 0 \pmod r$, because they are differences of two positive integers congruent to $1 \pmod r$. The right-hand side of Equation (2.31) is precisely the coefficient of x^{m-1} in the expansion of Equation (2.27) and the proof follows. □

Remark 2.3.7. All formulas presented in this section can be stated for general r-colored subsets of $[n]$. For example, for $\sigma = (\widehat{S}, \epsilon) \in \Sigma(n,r)$ we have

$$\mathrm{ps}^{(r)}_{q,\mathbf{p}}(F_\sigma(\mathbf{X}^{(r)}) = \frac{q^{\mathrm{maj}(\sigma)}\mathbf{p}^{\mathrm{n}(\sigma)}}{(q)_n}, \tag{2.32}$$

where

$$\begin{aligned}\mathrm{maj}(\sigma) &:= \sum_{i\in\mathrm{Des}(\sigma)} i\\ \mathrm{n}(\sigma) &:= \mathrm{n}_0(\sigma)+\mathrm{n}_1(\sigma)+\cdots+\mathrm{n}_{r-1}(\sigma),\\ \mathrm{n}_j(\sigma) &:= |\{i\in[n] : \widetilde{\epsilon}_i = j\}|.\end{aligned}$$

Thus, letting σ be the colored descent set of some colored permutation yields Equation (2.11).

2.4 The (k,ℓ)-flag major index

For this section we work in the case $r = 2$. We will write $\mathbf{x}$ and $\mathbf{y}$ instead of $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$, respectively. We fix a total order $<$ on Ω_n and assume the notation introduced at the beginning of the previous section with the following modifications to match the barred notation

$$\begin{aligned}
\mathrm{n}_-(w) &:= |\{i \in [n] : w_i \text{ is barred}\}| \\
\mathrm{n}_+(w) &:= |\{i \in [n] : w_i \text{ is not barred}\}|,
\end{aligned}$$

for every signed permutation $w \in \mathfrak{B}_n$. We will also write $\mathrm{neg}(w)$ for $\mathrm{n}_-(w)$ which is the notation widely used in the literature.

Let k and ℓ be a positive and a nonnegative, respectively, integer. For $w \in \mathfrak{B}_n$, we define

$$\mathrm{fmaj}_{k,\ell}(w) \ := \ k\,\mathrm{maj}(w) + \ell\mathrm{neg}(w)$$

to be the (k,ℓ)-*flag major index* of w. The $(1,0)$-flag major index coincides with the major index and the $(2,1)$-flag major index is just the flag major index on signed permutations. We are going to derive general formulas for the joint distribution of $\mathrm{des}, \mathrm{fmaj}_{k,\ell}$ and neg, by considering a (k,ℓ)-variation of the specializations of Theorem 2.3.4 for $r=2$.

Consider the specialization $\vartheta^{k,\ell}_{q,a,b}$ defined by substitutions $x_i = q^{k(i-1)}a$ and $y_i = q^{k(i-1)+\ell}b$, for every $i \geq 1$, the specialization $\vartheta^{k,\ell}_{q,a,b,m}$ defined by the substitutions

$$\begin{cases} x_i = q^{k(i-1)}a, & \text{for every } 1 \leq i \leq m \\ y_i = q^{k(i-1)+\ell}b, & \text{for every } 1 \leq i \leq m-1 \\ x_i = y_i = 0, & \text{otherwise} \end{cases}$$

and the specialization $\widetilde{\vartheta}^{k,\ell}_{q,a,b,m}$ defined as $\vartheta^{k,\ell}_{q,a,b,m}$, but including the substitution $x_m = q^{k(m-1)+\ell}b$. For $(k,\ell)=(1,0)$ and $(k,\ell)=(2,1)$, these specializations coincide with $\mathrm{ps}^{(2)}_{q,\mathbf{p}}, \mathrm{ps}^{(2)}_{q,\mathbf{p},m}, \widetilde{\mathrm{ps}}^{(2)}_{q,\mathbf{p},m}$ and $\psi^{(2)}_{q,\mathbf{p}}, \psi^{(2)}_{q,\mathbf{p},m}, \widetilde{\psi}^{(2)}_{q,\mathbf{p},m}$ for $\mathbf{p}=(a,b)$, respectively.

Theorem 2.4.1. *For a positive integer n and every $w \in \mathfrak{B}_n$, we have*

$$\vartheta^{k,\ell}_{q,a,b}(F_w(\mathbf{x},\mathbf{y})) \ = \ \frac{q^{\mathrm{fmaj}_{k,\ell}(w)}a^{\mathrm{n}_+(w)}b^{\mathrm{n}_-(w)}}{(q^k)_n} \tag{2.33}$$

$$\sum_{m\geq 1} \vartheta^{k,\ell}_{q,a,b,m}(F_w(\mathbf{x},\mathbf{y}))\,x^{m-1} \ = \ \frac{x^{\mathrm{des}(w)}q^{\mathrm{fmaj}_{k,\ell}(w)}a^{\mathrm{n}_+(w)}b^{\mathrm{n}_-(w)}}{(x;q^k)_{n+1}} \tag{2.34}$$

$$\sum_{m\geq 1} \widetilde{\vartheta}^{k,\ell}_{q,a,b,m}(F_w(\mathbf{x},\mathbf{y}))\,x^{m-1} \ = \ \frac{x^{\mathrm{des}^*(w)}q^{\mathrm{fmaj}_{k,\ell}(w)}a^{\mathrm{n}_+(w)}b^{\mathrm{n}_-(w)}}{(x;q^k)_{n+1}}. \tag{2.35}$$

The proof of Theorem 2.4.1 is a variant of the proof of Theorem 2.3.1 and is therefore omitted. The following formulas are immediate consequences of Theorem 2.4.1.

Corollary 2.4.2. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{B}_n$, we have*

$$\vartheta^{k,\ell}_{q,a,b}(F(\mathcal{A};\mathbf{x},\mathbf{y})) \;=\; \frac{\sum_{w\in\mathcal{A}} q^{\mathrm{fmaj}_{k,\ell}(w)} a^{\mathrm{n}_+(w)} b^{\mathrm{n}_-(w)}}{(q^k)_n} \tag{2.36}$$

$$\sum_{m\geq 1} \vartheta^{k,\ell}_{q,a,b,m}(F(\mathcal{A};\mathbf{x}^{(0)},\mathbf{x}^{(1)}))\, x^{m-1} \;=\; \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}(w)} q^{\mathrm{fmaj}_{k,\ell}(w)} a^{\mathrm{n}_+(w)} b^{\mathrm{n}_-(w)}}{(x;q^k)_{n+1}} \tag{2.37}$$

$$\sum_{m\geq 1} \widetilde{\vartheta}^{k,\ell}_{q,a,b,m}(F(\mathcal{A};\mathbf{x},\mathbf{y}))\, x^{m-1} \;=\; \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}^*(w)} q^{\mathrm{fmaj}_{k,\ell}(w)} a^{\mathrm{n}_+(w)} b^{\mathrm{n}_-(w)}}{(x;q^k)_{n+1}}. \tag{2.38}$$

Chapter 3

Specializations of colored quasisymmetric functions; Applications

This chapter applies the corollaries of Section 2.3 to prove color sum Euler–Mahonian identities on colored permutation groups, colored derangements and absolute involutions. In particular, Section 3.1 proves most of the Euler–Mahonian identities mentioned in Section 1.2 and introduces new examples. Section 3.2 studies color sum Mahonian and color sum Euler–Mahonian distributions on derangements, providing a colored analogue of a result of Wachs [96, Theorem 4] (see [55, page 209]). Section 3.3 studies Eulerian and fix-Euler-Mahonian distributions on involutions and their colored analogues and generalizes a formula of Désarménien and Foata [37, Equation (1.8)] and Gessel and Reutenauer [55, Equation (7.3)] (see also [57, Section 5]) and a formula of Athanasiadis [12, Equation (40)]. Lastly, Section 3.4 studies color sum bimahonian and multivariate distributions, involving Eulerian and Mahonian statistics on colored permutations. In what follows, we use the color order for colored permutation statistics.

3.1 Colored permutations

The following observation, which appears in [75, Proposition 1.13] in the more general setting of wreath products of the symmetric group and a finite abelian group, is a generalization of Equation (2.10) and computes the colored quasisymmetric generating function associated to the r-colored permutation group $\mathfrak{S}_{n,r}$. It is the key that allows us to pass from general formulas to Euler–Mahonian identities by suitable specialization. We will give two independent proofs of this fact, one in Section 4.4 using a theory of colored P-partitions and one in Section 5.1 using the Frobenius formula for $\mathfrak{S}_{n,r}$.

Lemma 3.1.1. *For a nonnegative integer n, we have*

$$F(\mathfrak{S}_{n,r}; \mathbf{X}^{(r)}) \; = \; \left(h_1(\mathbf{x}^{(0)}) + h_1(\mathbf{x}^{(1)}) + \cdots + h_1(\mathbf{x}^{(r-1)})\right)^n, \qquad (3.1)$$

where $h_1(\mathbf{x}^{(j)}) := \sum_{i\geq 1} x_i^{(j)}$, *for every* $0 \leq j \leq r-1$.

For a positive integer n, let

$$\begin{aligned} A_{n,r}^{\text{eul,mah}}(x,q,\mathbf{p}) &:= \sum_{w\in\mathfrak{S}_{n,r}} x^{\text{eul}(w)} q^{\text{mah}(w)} \mathbf{p}^{\text{n}(w)} \\ A_{n,r}^{\text{mah}}(q,\mathbf{p}) &:= A_{n,r}^{\text{eul,mah}}(1,q,\mathbf{p}), \end{aligned}$$

where eul and mah is an Eulerian and a Mahonian statistic on $\mathfrak{S}_{n,r}$, respectively. We will specialize Equation (3.1) and apply the corollaries of Section 2.3 to prove colored Euler–Mahonian identities.

Corollary 3.1.2. *For a positive integer n, we have*

$$A_{n,r}^{\text{maj}}(q,\mathbf{p}) = (p_0+p_1+\cdots+p_{r-1})^n [n]_q! \tag{3.2}$$

and

$$\sum_{m\geq 0} \left(p_0[m+1]_q + (p_1+\cdots+p_{r-1})[m]_q\right)^n x^m = \frac{A_{n,r}^{\text{des,maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}} \tag{3.3}$$

$$\sum_{m\geq 0} \left((p_0+p_1+\cdots+p_{r-1})[m+1]_q\right)^n x^m = \frac{A_{n,r}^{\text{des}^*,\text{maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}}. \tag{3.4}$$

Proof. Specializing Equation (3.1) as in Theorem 2.3.1 yields

$$\begin{aligned} \mathrm{ps}_{q,\mathbf{p}}^{(r)}(F(\mathfrak{S}_{n,r};\mathbf{X}^{(r)})) &= \left(\frac{p_0+p_1+\cdots+p_{r-1}}{(q)_1}\right)^n \\ \mathrm{ps}_{q,\mathbf{p},m}^{(r)}(F(\mathfrak{S}_{n,r};\mathbf{X}^{(r)})) &= \left(p_0[m]_q + (p_1+\cdots+p_{r-1})[m-1]_q\right)^n \\ \widetilde{\mathrm{ps}}_{q,\mathbf{p},m}^{(r)}(F(\mathfrak{S}_{n,r};\mathbf{X}^{(r)})) &= \left((p_0+p_1+\cdots+p_{r-1})[m]_q\right)^n. \end{aligned}$$

The proof follows by substituting in Corollary 2.3.2 for $\mathcal{A} = \mathfrak{S}_{n,r}$. □

Notice that Equation (3.3) appeared as Equation (1.12) in Section 1.2.3, but in that section the descent number and the major index were computed using the Steingrímsson's order. Equations (3.2) and (3.3) for $p_j = p^j$ are due to Assaf [10, Equation (13)] and Biagioli and Zeng [24, Equation (8.1)], respectively.

Corollary 3.1.3. *For a positive integer n, we have*

$$A_{n,r}^{\text{fmaj}}(q,\mathbf{p}) = (p_0+p_1q+\cdots+p_{r-1}q^{r-1})^n [n]_{q^r}! \tag{3.5}$$

and

$$\sum_{m\geq 0} \left(p_0[m+1]_q + (p_1q+\cdots+p_{r-1}q^{r-1})[m]_{q^r}\right)^n x^m = \frac{A_{n,r}^{\text{des,fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}} \tag{3.6}$$

$$\sum_{m\geq 0} \left((p_0+p_1q+\cdots+p_{r-1}q^{r-1})[m+1]_q\right)^n x^m = \frac{A_{n,r}^{\text{des}^*,\text{fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}}. \tag{3.7}$$

Proof. The proof follows from Corollary 3.1.2 by setting $q \to q^r$ and $p_j \to q^j p_j$ for each $0 \leq j \leq r-1$. $\square$

Equation (3.5) refines a computation due to Haglund, Loehr and Remmel [61, Equation (34)] for the distribution of the flag major index over colored permutations and Equation (3.6) for $p_j = p^j$ appears in the work of Biagioli and Caselli [22, Theorem 5.2].

Corollary 3.1.4. *For a positive integer n, we have*

$$\sum_{m\geq 0} [m+1]_q^n \, x^m = \frac{A_{n,r}^{\mathrm{ldes,maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}(p_0 + xp_1 + \cdots + x^{r-1}p_{r-1})^n} \tag{3.8}$$

$$\sum_{m\geq 0} [m+1]_{q^r}^n \, x^m = \frac{A_{n,r}^{\mathrm{ldes,fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}(p_0 + xqp_1 + \cdots + (xq)^{r-1}p_{r-1})^n}. \tag{3.9}$$

Proof. The proof follows by setting $p_j \to x^j p_j$ for each $0 \leq j \leq r-1$ in Equations (3.4) and (3.7), respectively. $\square$

Remark 3.1.5. Equations (3.3) and (3.6) for $q = p_0 = \cdots = p_{r-1} = 1$ become

$$\sum_{m\geq 0} (m[r]_p + 1)^n x^m = \frac{\sum_{w\in\mathfrak{S}_{n,r}} x^{\mathrm{des}(w)} p^{\mathrm{csum}(w)}}{(1-x)^{n+1}}, \tag{3.10}$$

which reduces to an identity of Brenti [29, Equation (12)] for $r = 2$. In particular, the polynomial $\sum_{w\in\mathfrak{S}_{n,r}} x^{\mathrm{des}(w)} p^{\mathrm{csum}(w)} := \sum_{i=0}^{n} a_{n,r,i}(p) x^i$ satisfies the formula

$$(m[r]_p + 1)^n = \sum_{i=0}^{n} a_{n,r,i}(p) \binom{m+n-i}{n}$$

and therefore has only real roots[1] for every positive integer n and every $p \geq 1$ (cf. [29, Corollary 3.7]). Although this result may not be new, it served as a motivation to introduce the parameters p_j which keep track of the number of entries of a colored permutation of each color.

The following corollary for $p_j \to p^j$ appears in the work of Biagioli and Caselli [22, Theorem 5.4].

Corollary 3.1.6. *For a nonnegative integer m, we write $m = r\mathrm{Q}(\mathrm{m}) + \mathrm{R}(\mathrm{m})$ for some nonnegative integer $\mathrm{Q}(\mathrm{m})$ and $0 \leq \mathrm{R}(\mathrm{m}) < \mathrm{r}$. Then*

$$\sum_{m\geq 0} \left(\sum_{j=0}^{\mathrm{R}(m)} p_j q^j [\mathrm{Q}(m)+1]_{q^r} + \sum_{j=\mathrm{R}(m)+1}^{r-1} p_j q^j [\mathrm{Q}(m)]_{q^r} \right)^n x^m = \frac{[r]_x \, A_{n,r}^{\mathrm{fdes,fmaj}}(x,q,\mathbf{p})}{(x^r;q^r)_{n+1}}. \tag{3.11}$$

[1] Real-rooted polynomials appear often in combinatorics, algebra and geometry (see, for example, [28])

Proof. Suppose $m = rs + t$, for some $1 \le t \le r$. In order to compute $\phi^{(r)}_{q,p,m} F(\mathfrak{S}_{n,r})$, imagine the defining substitutions as entries of the following $r \times (s-t+1)$-matrix and then take the sum of all its elements

$$\begin{pmatrix} x_1^{(0)} & \cdots & x_{r+1}^{(0)} & \cdots & x_{r(s-1)+1}^{(0)} & \cdots & x_{rs+1}^{(0)} \\ x_1^{(1)} & \cdots & x_{r+1}^{(1)} & \cdots & x_{r(s-1)+1}^{(1)} & \cdots & x_{rs+1}^{(1)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_1^{(t-1)} & \cdots & x_{r+1}^{(t-1)} & \cdots & x_{r(s-1)+1}^{(t-1)} & \cdots & x_{rs+1}^{(t-1)} \\ x_1^{(t)} & \cdots & x_{r+1}^{(t)} & \cdots & x_{r(s-1)+1}^{(t)} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ x_1^{(r-1)} & \cdots & x_{r+1}^{(r-1)} & \cdots & x_{r(s-1)+1}^{(r-1)} & \cdots & 0 \end{pmatrix}.$$

Notice that, by definition, the last nonzero substitution occurs in the

$$(t-1, s-t+1) \;=\; (t-1, rs+t-t+1) \;=\; (t-1, rs+1)$$

position. Therefore, we have

$$\begin{aligned} \psi^{(r)}_{q,\mathbf{p},m} F(\mathfrak{S}_{n,r}) &= (p_0(1 + q^r + \cdots + q^{r(s-1)} + q^{rs}) + \\ &\quad p_1(q + q^{r+1} + \cdots + q^{r(s-1)+1} + q^{rs+1}) + \cdots + \\ &\quad p_{t-1}(q^{t-1} + q^{r+(t-1)} + \cdots + q^{r(s-1)+(t-1)} + q^{rs+(t-1)}) + \\ &\quad p_t(q^t + q^{r+t} + \cdots + q^{r(s-1)+t}) + \cdots + \\ &\quad p_{r-1}(q^{r-1} + q^{r+(r-1)} + \cdots + q^{r(s-1)+(r-1)}))^n \\ &= \left(\sum_{j=0}^{t-1} p_j q^j [s+1]_{q^r} + \sum_{j=t}^{r-1} p_j q^j [s]_{q^r} \right)^n . \end{aligned}$$

The proof follows by substituting in Equation (2.27) for $\mathcal{A} = \mathfrak{S}_{n,r}$ and noticing that going from m to $m+1$ leaves s intact and changes t to $t+1$. □

For the remainder of this section we assume that $r = 2$. The following corollary gives a formula for the joint distribution of the (k,ℓ)-flag major index and the pair $(\mathrm{n}_-, \mathrm{n}_+)$ on signed permutations of $\mathfrak{B}_n$. Equation (3.12) below refines Adin and Roichman's formula [6, Theorem 2] for the distribution of the flag major index. Furthermore, it computes signed Euler–Mahonian identities on $\mathfrak{B}_n$ for the Mahonian statistic $\mathrm{fmaj}_{k,\ell}$. We remark that Equation (3.13) reduces to Chow and Gessel's formulas [34, Equation (26)] for $(k,\ell) = (1,0)$ and $a = 1$ and refines [34, Theorem 3.7] for $(k,\ell) = (2,1)$ (see also [23, Remark 5.2]).

Corollary 3.1.7. *For a positive integer n, we have*

$$A^{\mathrm{fmaj}_{k,\ell}}_{n,2}(q,a,b) \;=\; (a + bq^\ell)^n [n]_{q^k}!, \tag{3.12}$$

and

$$\sum_{m\geq 0}\left(a[m+1]_{q^k}+bq^{\ell}[m]_{q^k}\right)^n x^m \;=\; \frac{A_{n,r}^{\mathrm{des},\mathrm{fmaj}_{k,\ell}}(x,q,a,b)}{(x;q^k)_{n+1}} \tag{3.13}$$

$$\sum_{m\geq 0}(a+bq^{\ell})^n[m+1]_{q^k}^n\, x^m \;=\; \frac{A_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}_{k,\ell}}(x,q,a,b)}{(x;q^k)_{n+1}}. \tag{3.14}$$

Proof. The proof follows by specializing Equation (3.1) for $r=2$ as in Theorem 2.4.1 and substituting in Corollary 2.4.2 for $\mathcal{A}=\mathfrak{B}_n$. □

In view of Theorem 2.4.1 and Corollary 2.4.2 we pose the following question.

Question 3.1.8. Does the (k,ℓ)-flag major index have some algebraic meaning for $k\geq 1$ and $\ell\geq 2$, similar to that of the flag major and major indices?

3.2 Colored derangements

Permutations in $\mathfrak{S}_n$ without fixed points are called *derangements*. Let $\mathcal{D}_n$ be the set of all derangements in $\mathfrak{S}_n$. For a positive integer n, let

$$\mathcal{D}_n(x,q) \;:=\; \sum_{w\in\mathcal{D}_n} x^{\mathrm{des}(w)}q^{\mathrm{maj}(w)}$$

be the n-th (x,q)-*derangement polynomial* and $d_n(q):=\mathcal{D}_n(1,q)$ the q-*derangement numbers*. The q-derangement numbers satisfy the following formula

$$d_n(q) \;=\; [n]_q!\sum_{k=0}^{n}(-1)^k\,\frac{q^{\binom{k}{2}}}{[k]_q!}. \tag{3.15}$$

proved bijectively by Wachs [96, Theorem 4] and later by Gessel and Reutenauer [55, page 209]. Eulerian and Mahonian distributions on derangements have been studied by many authors (see, for example, [12, Section 2.1.4] and references therein). The following theorem provides an Euler–Mahonian identity on derangements in $\mathfrak{S}_n$, which refines Wachs' formula (3.15). For $0\leq k\leq n$, the q-*binomial coefficient*, written $\binom{n}{k}_q$ is defined by

$$\binom{n}{k}_q \;:=\; \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[n]_q! := [1]_q[2]_q\cdots[n]_q$ for all $n\geq 1$ and $[0]_q! := 1$.

Theorem 3.2.1. *For a positive integer n, we have*

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\binom{m+1}{k}_q[m+1]_q^{n-k}\,x^m \;=\; \frac{\mathcal{D}_n(x,q)}{(x;q)_{n+1}}. \tag{3.16}$$

Proof. Setting $\mathcal{A} = \mathcal{D}_n$, Equation (2.8) becomes

$$\sum_{m\geq 1} \mathrm{ps}_{q,m}(F(\mathcal{D}_n;\mathbf{x}))\, x^{m-1} \;=\; \frac{\mathcal{D}_n(x,q)}{(x;q)_{n+1}}. \tag{3.17}$$

Gessel and Reutenauer [55, Theorem 8.1] computed the quasisymmetric generating function of $\mathcal{D}_n$

$$F(\mathcal{D}_n;\mathbf{x}) \;=\; \sum_{k=0}^{n} (-1)^k e_k(\mathbf{x}) h_1(\mathbf{x})^{n-k}. \tag{3.18}$$

The principal specialization of order m of elementary symmetric functions is given by [89, Proposition 7.8.3]

$$\mathrm{ps}_{q,m}(e_k(\mathbf{x})) \;=\; q^{\binom{k}{2}} \binom{m}{k}_q. \tag{3.19}$$

Taking the principal specialization of order m of Equation (3.18) and computing, using Equation (3.19), yields

$$\mathrm{ps}_{q,m}(F(\mathcal{D}_n;\mathbf{x})) \;=\; \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{m}{k}_q [m]_q^{n-k}. \tag{3.20}$$

The proof follows by substituting Equation (3.20) in Equation (3.17). □

Another proof of Equation (3.15) can be obtained by considering the stable principal specialization of Equation (3.18) instead and following the steps of the previous proof, as done by Gessel and Reutenauer [55, Theorem 8.4]. Next we consider a colored analogue of Theorem 3.2.1.

An element of $\mathfrak{S}_{n,r}$ without fixed points of zero color is called a *colored derangement*. Let $\mathcal{D}_{n,r}$ be the set of all colored derangements in $\mathfrak{S}_{n,r}$. Faliharimalala and Zeng [40, Equation (2.7)] (see also [10, Theorem 2.1], where colored derangements are called cyclic derangements) proved the following formula

$$|\mathcal{D}_{n,r}| \;=\; r^n n! \sum_{k=0}^{n} \frac{(-1)^k}{r^k k!}, \tag{3.21}$$

which generalizes the well known formula [90, Equation (2.11)] for the number of derangements in the symmetric group $\mathfrak{S}_n$. In a subsequent paper [41, Equation (2.5)], where the authors use the color order, they provide a formula for the colored q-derangement numbers

$$\sum_{w\in\mathcal{D}_{n,r}} q^{\mathrm{fmaj}(w)} \;=\; [r]_q[2r]_q\cdots[nr]_q \sum_{k=0}^{n} (-1)^k \frac{q^{r\binom{k}{2}}}{[r]_q[2r]_q\cdots[kr]_q}, \tag{3.22}$$

which reduces to Wachs' formula (3.15) for $r = 1$ and generalizes a formula of Chow [33, Theorem 5] for $r = 2$.

For a positive integer n, let

$$\begin{aligned}\mathcal{D}_{n,r}^{\mathrm{eul,mah}}(x,q,\mathbf{p}) &:= \sum_{w\in\mathcal{D}_{n,r}} x^{\mathrm{eul}(w)}q^{\mathrm{mah}(w)}\mathbf{p}^{\mathrm{n}(w)}\\ \mathcal{D}_{n,r}^{\mathrm{mah}}(q,\mathbf{p}) &:= \mathcal{D}_{n,r}^{\mathrm{eul,mah}}(1,q,\mathbf{p}),\end{aligned}$$

where eul and mah is an Eulerian and a Mahonian statistic on $\mathfrak{S}_{n,r}$, respectively. The corollaries which follow provide a refinement of Equation (3.22) as well as several colored Euler–Mahonian identities on colored derangements. Our starting point will the following colored analogue of Gessel and Reutenauer's formula (3.18)

$$F(\mathcal{D}_{n,r};\mathbf{X}^{(r)}) = \sum_{k=0}^{n}(-1)^k e_k(\mathbf{x}^{(0)})(h_1(\mathbf{x}^{(0)})+\cdots+h_1(\mathbf{x}^{(r-1)}))^{n-k}. \tag{3.23}$$

It was recently proven for $r=2$ by Adin et al. [1, Theorem 7.3]. Equation (3.23) can be proved by trivially generalizing Adin et al.'s argument in the proof of [1, Theorem 7.3] for general r and using [76, Theorem 16].

Corollary 3.2.2. *For a positive integer n, we have*

$$\mathcal{D}_{n,r}^{\mathrm{maj}}(q,\mathbf{p}) = (p_0+p_1+\cdots+p_{r-1})^n[n]_q!\sum_{k=0}^{n}(-1)^k\frac{q^{\binom{k}{2}}}{(p_0+p_1+\cdots+p_{r-1})^k[k]_q!} \tag{3.24}$$

and

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\binom{m+1}{k}_q(p_0[m+1]_q+(p_1+\cdots+p_{r-1})[m]_q)^{n-k}x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{des,maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}} \tag{3.25}$$

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\binom{m+1}{k}_q((p_0+p_1+\cdots+p_{r-1})[m+1]_q)^{n-k}x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{des^*,maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}}. \tag{3.26}$$

Proof. The proof follows by specializing Equation (3.23) as in Theorem 2.3.1 and substituting in Corollary 2.3.2 for $\mathcal{A}=\mathcal{D}_{n,r}$. □

Equation (3.24) for $p_j = p^j$ can be found in Assaf's work [10, Theorem 3.2]. The following corollary can be proven by setting $q\to q^r$ and $p_j \to q^j p_j$ for each $0\leq j\leq r-1$ in Corollary 3.2.2. Equation (3.27) below refines Equation (3.22) and Equation (3.28) reduces to Equation (3.16) for $r=1$.

Corollary 3.2.3. *For a positive integer n, we have*

$$\mathcal{D}_{n,r}^{\mathrm{fmaj}}(q,\mathbf{p}) = (p_0+p_1q+\cdots+p_{r-1}q^{r-1})^n[n]_{q^r}!\sum_{k=0}^{n}(-1)^k\frac{q^{r\binom{k}{2}}}{(p_0+p_1q+\cdots+p_{r-1}q^{r-1})^k[k]_{q^r}!} \tag{3.27}$$

and

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k q^{r\binom{k}{2}}\binom{m+1}{k}_{q^r}(p_0[m+1]_{q^r}+(p_1q+\cdots+p_{r-1}q^{r-1})[m]_{q^r})^{n-k}\,x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{des},\mathrm{fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}} \tag{3.28}$$

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k q^{r\binom{k}{2}}\binom{m+1}{k}_{q^r}((p_0+p_1q+\cdots+p_{r-1}q^{r-1})[m+1]_{q^r})^{n-k}\,x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}}. \tag{3.29}$$

Setting $p_j \to x^j p_j$ for each $0 \leq j \leq r-1$ in Equations (3.26) and (3.29) yields the following colored Euler–Mahonian identities for the pairs $(\mathrm{ldes}, \mathrm{maj})$ and $(\mathrm{ldes}, \mathrm{fmaj})$ on colored derangements.

Corollary 3.2.4. *For every positive integer n, we have*

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k\frac{q^{\binom{k}{2}}\binom{m+1}{k}_q[m+1]_q^{n-k}}{(p_0+p_1x+\cdots+p_{r-1}x^{r-1})^k}\,x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{ldes},\mathrm{maj}}(x,q,\mathbf{p})}{(x;q)_{n+1}(p_0+p_1x+\cdots+p_{r-1}x^{r-1})^n} \tag{3.30}$$

$$\sum_{m\geq 0}\sum_{k=0}^{n}(-1)^k\frac{q^{r\binom{k}{2}}\binom{m+1}{k}_{q^r}[m+1]_{q^r}^{n-k}}{(p_0+p_1qx+\cdots+p_{r-1}(qx)^{r-1})^k}\,x^m = \frac{\mathcal{D}_{n,r}^{\mathrm{ldes},\mathrm{fmaj}}(x,q,\mathbf{p})}{(x;q^r)_{n+1}(p_0+p_1qx+\cdots+p_{r-1}(qx)^{r-1})^n}. \tag{3.31}$$

Concluding this section, we turn our attention to the signed case $r = 2$. The following corollary computes signed Euler–Mahonian identities on $\mathcal{D}_{n,2}$ involving the (k,ℓ)-flag major index, the first of which refines a formula due to Chow [33, Theorem 5]. Furthermore, it computes a formula for the joint distribution of the (k,ℓ)-flag major index and the statistics which keep track of the number of barred and unbarred entries of a signed permutation over signed deragnements, which refines another formula of Chow [33, Theorem 5].

Corollary 3.2.5. *For a positive integer n, we have*

$$\mathcal{D}_{n,2}^{\mathrm{fmaj}_{k,\ell}}(q,p_0,p_1) = (p_0+p_1q^{\ell})^n[n]_{q^k}!\sum_{i=0}^{n}(-1)^i\frac{q^{k\binom{i}{2}}}{(p_0+p_1q^{\ell})^i[i]_{q^k}!} \tag{3.32}$$

and

$$\sum_{m\geq 0}\sum_{i=0}^{n}(-1)^i q^{k\binom{i}{2}}\binom{m+1}{i}_{q^k}(p_0[m+1]_{q^k}+p_1q^{\ell}[m]_{q^k})^{n-i}\,x^m = \frac{\mathcal{D}_{n,2}^{\mathrm{des},\mathrm{fmaj}_{k,\ell}}(x,q,p_0,p_1)}{(x;q^k)_{n+1}} \tag{3.33}$$

$$\sum_{m\geq 0}\sum_{i=0}^{n}(-1)^i q^{k\binom{i}{2}}\binom{m+1}{i}_{q^k}(p_0+p_1q^{\ell})^{n-i}[m+1]_{q^k}^{n-i}\,x^m = \frac{\mathcal{D}_{n,2}^{\mathrm{des}^*,\mathrm{fmaj}_{k,\ell}}(x,q,p_0,p_1)}{(x;q^k)_{n+1}}. \tag{3.34}$$

Proof. The proof follows by specializing Equation (3.23) for $r = 2$ as in Theorem 2.4.1 and substituting in Corollary 2.4.2 for $\mathcal{A} = \mathcal{D}_{n,2}$. $\square$

3.3 Colored involutions and absolute involutions

Permutations in $\mathfrak{S}_n$ which are equal to their own inverse are called *involutions*. Let $\mathcal{I}_n$ be the set of all involutions in $\mathfrak{S}_n$. For a positive integer n, let

$$\mathcal{I}_n(x,q,a) \;:=\; \sum_{w\in\mathcal{I}_n} x^{\mathrm{des}(w)}q^{\mathrm{maj}(w)}a^{\mathrm{fix}(w)},$$

where $\mathrm{fix}(w)$ is the number of fixed points of w and a is an indeterminate. Also, set $\mathcal{I}_n(x,q) := \mathcal{I}_n(x,q,1)$ and $\mathcal{I}_n(q) := I_n(1,q,1)$. This polynomial was considered by Désarménien and Foata [37, Section 6] and later by Gessel and Reutenauer [55, Section 7], where they computed a generating function for $\mathcal{I}_n(x,q,p)$. In particular, Désarménien and Foata proved [37, Equation (6.2)] (where $(q;q)_n$ is to be replaced by $(t;q)_{n+1}$)

$$\sum_{n\geq 0}\frac{\mathcal{I}_n(x,q,a)}{(x;q)_{n+1}}z^n \;=\; \sum_{m\geq 0}(az;q)_{m+1}^{-1}\prod_{0\leq i<j\leq m}(1-z^2q^{i+j})^{-1}\,x^m, \tag{3.35}$$

where $\mathcal{I}_0(x,q,a) := 1$.

One can prove Equation (3.35) by taking the principal specialization of order m of the quasisymmetric generating function for involutions according to fixed points [55, Equation (7.1)]

$$\sum_{n\geq 0}\sum_{w\in\mathcal{I}_n}F_{n,\mathrm{Des}(w)}(\mathbf{x})\,a^{\mathrm{fix}(w)}z^n \;=\; \prod_{i\geq 1}(1-azx_i)^{-1}\prod_{1\leq i<j}(1-z^2x_ix_j)^{-1} \tag{3.36}$$

This is essentially the approach of Gessel and Reutenauer in the proof of [55, Equation (7.2)]. The connecting link between the Désarménien–Foata and Gessel–Reutenauer approaches is Equation (1.17).

In particular, one has [89, Corollary 7.13.8 and Exercise 7.28]

$$\sum_{\lambda} s_\lambda(\mathbf{x})\, a^{\mathrm{col}^{\mathrm{o}}(\lambda)} z^{|\lambda|} \;=\; \prod_{i\geq 1}(1-azx_i)^{-1} \prod_{1\leq i<j}(1-z^2x_ix_j)^{-1}, \tag{3.37}$$

where the sum runs through all partitions λ and

- $\mathrm{col}^{\mathrm{o}}(\lambda)$ is the number of columns of λ of odd length and
- $|\lambda|$ is the size of λ, that is the sum of all of its parts.

Therefore Equation (3.36) follows from Equation (3.37), when you combine Equation (1.17) and the fact [89, Corollary 7.13.9] that the Robinson–Schensted correspondence restricts to a des-preserving bijection between the set of involutions of $\mathfrak{S}_n$ and the set of all standard Young tableaux of size n.

An Euler–Mahonian identity on involutions involving the "hook-content formula" for Schur functions can be derived in the following way. Recall from [64, Example 1 of Section 3] the following notation[2]

$$\binom{n}{\lambda}_q := \prod_{u\in\lambda} \frac{1-q^{n-\mathrm{c}(u)}}{1-q^{\mathrm{h}(u)}},$$

slightly altered to match our notation, where for a cell $u \in \lambda$, $\mathrm{c}(u)$ and $\mathrm{h}(u)$ are the *content* and the *hook length* of u, respectively[3]. Then, Macdonald's interpretation of Stanley's well celebrated "hook-content formula" [89, Theorem 7.21.2] becomes

$$\mathrm{ps}_{q,m}(s_\lambda(\mathbf{x})) \;=\; q^{\mathrm{b}(\lambda)}\binom{m}{\lambda'}_q, \tag{3.38}$$

where $\mathrm{b}(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$, for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and λ' denotes the conjugate partition to λ.

The quasisymmetric generating function associated to $\mathcal{I}_n$ is known to satisfy

$$F(\mathcal{I}_n;\mathbf{x}) \;=\; \sum_{\lambda\vdash n} s_\lambda(\mathbf{x}). \tag{3.39}$$

Applying the principal specialization of order m to this formula, substituting in Equation (2.8) for $\mathcal{A} = \mathcal{I}_n$ and using Equation (3.38) yields the following Euler–Mahonian identity on $\mathcal{I}_n$.

Proposition 3.3.1. *For a positive integer n, we have*

$$\sum_{m\geq 0}\sum_{\lambda\vdash n} q^{\mathrm{b}(\lambda)}\binom{m+1}{\lambda'}_q x^m \;=\; \frac{\mathcal{I}_n(x,q)}{(x;q)_{n+1}}. \tag{3.40}$$

[2]This notation is justified, as the author in [64, Example 1 of Section 3] mentions, because letting $\lambda = (k)$, the partition with one part of length k, yields $\binom{n}{(k)}_q = \binom{n}{k}_q$.

[3]We refer to [89, Section 7.21] for more details on these concepts.

Now, recall from [89, Corollary 7.21.3] the the stable principal specialization of the Schur function

$$\mathrm{ps}_q(s_\lambda(\mathbf{x})) \;=\; q^{\mathrm{b}(\lambda)}\prod_{u\in\lambda}(1-q^{\mathrm{h}(u)})^{-1}. \tag{3.41}$$

Applying the stable principal specialization to Equation (3.39), substituting in Equation (2.9) for $\mathcal{A}=\mathcal{I}_n$ and using Equation (3.41) yields

$$I_n(q) \;=\; \sum_{\lambda\vdash n} q^{\mathrm{b}(\lambda)}\frac{[n]_q!}{\prod_{u\in\lambda}[\mathrm{h}(u)]_q}, \tag{3.42}$$

a formula which bears connection with the well known Stanley's q-hook length formula [89, Corollary 7.21.5].

The purpose of this section is to discuss colored analogues of Equations (3.35), (3.36), (3.40) and (3.42). We deal with two types of involutions in colored permutation groups, the colored involutions and the absolute involutions. A *colored (resp. absolute) involution* is an element $w\in\mathfrak{S}_{n,r}$, such that $w^{-1}=w$ (resp. $\overline{w}^{-1}=w$)[4]. Let $\mathcal{I}_{n,r}$ (resp. $\mathcal{I}^{\mathrm{abs}}_{n,r}$) be the set of all colored (resp. absolute) involutions in $\mathfrak{S}_{n,r}$. Absolute involutions do not coincide with colored involutions for $r\geq 3$. For example, the colored permutation $3^1 2^0 1^3 4^2 6^3 5^1\in\mathfrak{S}_{6,4}$ is an involution, but not an absolute involution and on the other hand the colored permutation $3^1 2^0 1^1 4^2 6^3 5^3\in\mathfrak{S}_{6,4}$ is an absolute involution, but not an involution.

Chow and Mansour [35, Section 4] studied colored involutions. In a similar fashion, for $w^\epsilon\in\mathcal{I}^{\mathrm{abs}}_{n,r}$ we see that

- $w\in\mathcal{I}_n$ and
- if $w(i)=j$, then $\epsilon_{w(i)}=\epsilon_j$, computed modulo r

for some $i,j\in[n]$. Modifying the arguments of Chow and Mansour [35, Proposition 7] yields the following formulas

$$|\mathcal{I}^{\mathrm{abs}}_{n,r}| \;=\; r^n n!\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(1/2r)^k}{k!(n-2k)!}$$

$$\sum_{n\geq 0}|\mathcal{I}^{\mathrm{abs}}_{n,r}|\frac{x^n}{n!} \;=\; e^{r(x^2/2+x)},$$

where $|\mathcal{I}^{\mathrm{abs}}_{0,r}|:=1$ and the following recurrence formula for the number of absolute involutions in $\mathfrak{S}_{n,r}$,

$$|\mathcal{I}^{\mathrm{abs}}_{n+1,r}| \;=\; r(|\mathcal{I}^{\mathrm{abs}}_{n,r}|+n|\mathcal{I}^{\mathrm{abs}}_{n-1,r}|),$$

for every positive integer $n\geq 1$, with initial condition $|\mathcal{I}^{\mathrm{abs}}_{1,r}|=r$.

A polynomial $f(x)$ with real coefficients is called γ-positive if

$$f(x) \;=\; \sum_{i=0}^{\lfloor n/2\rfloor}\gamma_i x^i(1+x)^{n-2i},$$

[4] Recall that $\overline{w^\epsilon}$ is defined as the colored permutation $w^{-\epsilon}$.

for some $n \in \mathbb{N}$ and nonnegative reals $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$. Chow and Mansour implicitly proved [35, Proposition 8] that the generating polynomial of the excedance statistic on colored involutions is γ-positive for every even color r. The *number of excedances* of $w^\epsilon \in \mathfrak{S}_{n,r}$, written $\mathrm{exc}(w)$, is defined to be the number of all indices $i \in [n]$, such that $w_i > i$, or $w_i = i$ and $\epsilon_i > 0$. Recall from [92, Theorem 15] that the excedance statistic is Eulerian on colored permutations.

For every even color r, [35, Proposition 8] states that

$$\sum_{w \in \mathcal{I}_{n,r}} x^{\mathrm{exc}(w)} \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i}\, x^i (1+x)^{n-2i}, \tag{3.43}$$

where $\gamma_{n,i}$ is the number of $w \in \mathcal{I}_n$ having i two-cycles multiplied by r^i. Gamma-positivity is an elementary property that implies symmetry and unimodality and appears often in combinatorics. For more information we refer the reader to Athanasiadis' comprehensive survey [12].

In fact, one can further argue as in [35, Proposition 8] and prove the following

$$\sum_{w \in \mathcal{I}^{\mathrm{abs}}_{n,r}} x^{\mathrm{exc}(w)} \;=\; \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_{n,i}\, x^i (1+(r-1)x)^{n-2i}, \tag{3.44}$$

where $\gamma_{n,i}$ is the same as in Equation (3.43). Equation (3.44) coincides with the corresponding formulas of Chow and Mansour [35] for $r \le 2$.

Absolute involutions appeared in Adin, Postnikov and Roichman's study [5] of Gelfand models for colored permutation groups $\mathfrak{S}_{n,r}$. They are suitable for providing a colored analogue of Désarménien and Foata's Formula (3.35). In particular, the colored Robinson–Schensted correspondence restricts to a des-preserving bijection between $\mathcal{I}^{\mathrm{abs}}_{n,r}$ and $\mathrm{SYT}_{n,r}$, the set of all standard Young r-partite tableaux of size n. In addition, from its description (see, for example, [5, Section 5]), the number of fixed points of color j of an absolute involution in $\mathfrak{S}_{n,r}$ equals the number of odd columns of the jth part of the P-tableau, which corresponds to w via the colored Robinson–Schensted correspondence.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{r-1})$ be a sequence of indeterminates. For a positive integer n, we consider

$$F(\mathcal{I}^{\mathrm{abs}}_{n,r}; \mathbf{X}^{(r)}, \mathbf{a}) \;:=\; \sum_{w \in \mathcal{I}^{\mathrm{abs}}_{n,r}} F_w(\mathbf{X}^{(r)})\, \mathbf{a}^{\mathrm{fix}(w)},$$

the quasisymmetric generating function for absolute involutions according to fixed points of various colors, where $\mathrm{fix}(w) = (\mathrm{fix}^0(w), \mathrm{fix}^1(w), \ldots, \mathrm{fix}^{r-1}(w))$ and $\mathrm{fix}^j(w)$ is the number of fixed points of $w \in \mathfrak{S}_{n,r}$ of color j. The following theorem provides a colored anagolue of Equation (3.36) by computing the generating function of $F(\mathcal{I}^{\mathrm{abs}}_{n,r}; \mathbf{X}^{(r)}, \mathbf{a})$.

Theorem 3.3.2. *We have*

$$\sum_{n \ge 0} F(\mathcal{I}^{\mathrm{abs}}_{n,r}; \mathbf{X}^{(r)}, \mathbf{a})\, z^n \;=\; \prod_{t=0}^{r-1} \prod_{i \ge 1} (1 - z a_t x_i^{(t)})^{-1} \prod_{1 \le i < j} (1 - z^2 x_i^{(t)} x_j^{(t)})^{-1}. \tag{3.45}$$

In particular, for a positive integer n we have

$$F(\mathcal{I}^{\mathrm{abs}}_{n,r};\mathbf{X}^{(r)},\mathbf{a}) \;=\; \sum_{\boldsymbol{\lambda}\vdash n} s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)})\,\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}, \tag{3.46}$$

where $\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda}) = (\mathrm{col}^{\mathrm{o}}(\lambda^{(0)}), \mathrm{col}^{\mathrm{o}}(\lambda^{(1)}), \ldots, \mathrm{col}^{\mathrm{o}}(\lambda^{(r-1)}))$ *for a r-partite partition* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$.

Proof. The discussion before the statement of the theorem implies that

$$\sum_{n\geq 0} F(\mathcal{I}^{\mathrm{abs}}_{n,r};\mathbf{X}^{(r)},\mathbf{a})\,z^n \;=\; \sum_{n\geq 0}\sum_{\boldsymbol{\lambda}\vdash n}\sum_{\boldsymbol{Q}\in\mathrm{SYT}(\boldsymbol{\lambda})} F_{\boldsymbol{Q}}(\mathbf{X}^{(r)})\,\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})} z^n.$$

Thus, by Equation (1.17) we have

$$\sum_{n\geq 0} F(\mathcal{I}^{\mathrm{abs}}_{n,r};\mathbf{X}^{(r)},\mathbf{a})\,z^n \;=\; \sum_{\boldsymbol{\lambda}}\prod_{j=0}^{r-1} s_{\lambda^{(j)}}(\mathbf{x}^{(j)})\,a_j^{\mathrm{col}^{\mathrm{o}}(\lambda^{(j)})} z^{|\,\boldsymbol{\lambda}\,|}, \tag{3.47}$$

where the sum runs through all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ and $|\,\boldsymbol{\lambda}\,| := |\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}|$. Now, Equation (3.46) follows by extracting the coefficient of z^n in Equation (3.47) and Equation (3.45) follows by expanding the right-hand side of Equation (3.47) according to Equation (3.37) for every color. $\square$

We will now specialize Equations (3.45) and (3.46) and via Theorems 2.3.1, 2.3.4, 2.3.6 and 2.4.1 we will derive colored analogues of Désarménien and Foata's Formula (3.35) and Equations (3.40) and (3.42) respectively. For a positive integer n, let

$$\begin{aligned}\mathcal{I}^{\mathrm{eul,mah}}_{n,r}(x,q,\mathbf{p},\mathbf{a}) \;&:=\; \sum_{\pi\in\mathcal{I}^{\mathrm{abs}}_{n,r}} x^{\mathrm{eul}(\pi)} q^{\mathrm{mah}(\pi)} \mathbf{p}^{\mathrm{n}(w)} \mathbf{a}^{\mathrm{fix}(w)}\\ \mathcal{I}^{\mathrm{mah}}_{n,r}(q,\mathbf{p},\mathbf{a}) \;&:=\; \mathcal{I}^{\mathrm{eul,mah}}_{n,r}(1,q,\mathbf{p},\mathbf{a})\end{aligned}$$

where eul and mah is an Eulerian and a Mahonian statistic on colored permutations, respectively. Also, set $\mathcal{I}^{\mathrm{eul,mah}}_{0,r}(x,q,p,\mathbf{p},\mathbf{a}) := 1$. We start by specializing Equation (3.46).

Corollary 3.3.3. *For a positive integer n, we have*

$$\mathcal{I}^{\mathrm{maj}}_{n,r}(q,\mathbf{p},\mathbf{a}) \;=\; \sum_{\boldsymbol{\lambda}\vdash n} q^{\mathrm{b}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\frac{[n]_q!}{\prod_{j=0}^{r-1}\prod_{u\in\lambda^{(j)}}[\mathrm{h}(u)]_q} \tag{3.48}$$

and

$$\sum_{m\geq 0}\sum_{\boldsymbol{\lambda}} q^{\mathrm{b}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\binom{m+1}{\lambda^{(0)\prime}}_q\prod_{j=0}^{r-1}\binom{m}{\lambda^{(j)\prime}}_q x^m \;=\; \frac{\mathcal{I}^{\mathrm{des,maj}}_{n,r}(x,q,\mathbf{p},\mathbf{a})}{(x;q)_{n+1}} \tag{3.49}$$

$$\sum_{m\geq 0}\sum_{\boldsymbol{\lambda}} q^{\mathrm{b}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\prod_{j=0}^{r-1}\binom{m+1}{\lambda^{(j)\prime}}_q x^m \;=\; \frac{\mathcal{I}^{\mathrm{des}^*,\mathrm{maj}}_{n,r}(x,q,\mathbf{p},\mathbf{a})}{(x;q)_{n+1}}, \tag{3.50}$$

where the sums run through all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ *of n,* $\mathrm{b}(\boldsymbol{\lambda}) := \mathrm{b}(\lambda^{(0)}) + \cdots + \mathrm{b}(\lambda^{(r-1)})$ *and* $\mathrm{size}(\boldsymbol{\lambda}) = (|\lambda^{(0)}|, |\lambda^{(1)}|, \ldots, |\lambda^{(r-1)}|)$.

Proof. Specializing Equation (3.46), as in Theorem 2.3.1 yields

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(F(\mathcal{I}_{n,r}^{\mathrm{abs}};\mathbf{X}^{(r)},\mathbf{a})) \;=\; \sum_{\boldsymbol{\lambda}}\prod_{j=0}^{r-1} s_{\lambda^{(j)}}(1,p_jq,p_jq^2,\dots)\,a_j^{\mathrm{col}(\lambda^{(j)})}.$$

But, because of the homogeneousness of Schur functions we have

$$s_{\lambda^{(j)}}(1,p_jq,p_jq^2,\dots)=p_j^{|\lambda^{(j)}|}\,\mathrm{ps}_q(s_{\lambda^{(j)}}(\mathbf{x}))=p_j^{|\lambda^{(j)}|}q^{\mathrm{b}(\lambda^{(j)})}\prod_{u\in\lambda^{(j)}}(1-q^{\mathrm{h}(u)})^{-1},$$

where the last equality follows from Equation (3.41). Combining these equations yields

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(F(\mathcal{I}_{n,r}^{\mathrm{abs}};\mathbf{X}^{(r)},\mathbf{a})) \;=\; \sum_{\boldsymbol{\lambda}} q^{\mathrm{b}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\prod_{j=0}^{r-1} a_j^{\mathrm{col}(\lambda^{(j)})}\prod_{u\in\lambda^{(j)}}(1-q^{\mathrm{h}(u)})^{-1}.$$

The proof of Equation (3.48) follows by substituting in Equation (2.18) for $\mathcal{I}_{n,r}^{\mathrm{abs}}$. The proofs of the remaining two equations follow in a similar manner, but using Equation (3.38) instead and therefore are omitted. □

For $r=1$, Equation (3.48),(3.72) (and (3.73)) become Equations (3.40) and (3.42), respectively. Also, setting $q\to q^r$ and $p_j\to q^jp_j$ in the formulas of Corollary 3.3.3 yields analogous formulas for the polynomials $\mathcal{I}_{n,r}^{\mathrm{fmaj}}(q,\mathbf{p},\mathbf{a}),\mathcal{I}_{n,r}^{\mathrm{des,fmaj}}(x,q,\mathbf{p},\mathbf{a})$ and $\mathcal{I}_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}}(x,q,\mathbf{p},\mathbf{a})$.

Corollary 3.3.4. *For a positive integer n, we have*

$$\mathcal{I}_{n,r}^{\mathrm{fmaj}}(q,\mathbf{p},\mathbf{a}) \;=\; \sum_{\boldsymbol{\lambda}\vdash n} q^{r\,\mathrm{b}(\boldsymbol{\lambda})+\mathrm{r}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\frac{[n]_{q^r}!}{\prod_{j=0}^{r-1}\prod_{u\in\lambda^{(j)}}[\mathrm{h}(u)]_{q^r}} \tag{3.51}$$

and

$$\sum_{m\geq 0}\sum_{\boldsymbol{\lambda}} q^{r\,\mathrm{b}(\boldsymbol{\lambda})+\mathrm{r}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\binom{m+1}{\lambda^{(0)\prime}}_{q^r}\prod_{j=0}^{r-1}\binom{m}{\lambda^{(j)\prime}}_{q^r}x^m = \frac{\mathcal{I}_{n,r}^{\mathrm{des,fmaj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}} \tag{3.52}$$

$$\sum_{m\geq 0}\sum_{\boldsymbol{\lambda}} q^{r\,\mathrm{b}(\boldsymbol{\lambda})+\mathrm{r}(\boldsymbol{\lambda})}\mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})}\mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\boldsymbol{\lambda})}\prod_{j=0}^{r-1}\binom{m+1}{\lambda^{(j)\prime}}_{q^r}x^m = \frac{\mathcal{I}_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}}, \tag{3.53}$$

where the sums run through all r-partite partitions $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(r-1)})$ of n and $\mathrm{r}(\boldsymbol{\lambda}):=\sum_{j=0}^{r-1}j|\lambda^{(j)}|$.

Remark 3.3.5. For a $\boldsymbol{Q}\in\mathrm{SYT}_{n,r}$, letting $\sigma=\mathrm{sDes}(\boldsymbol{Q})$ in Equation (2.32) and summing over all r-partite standard Young tableau of shape $\boldsymbol{\lambda}$ and size n yields

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(F(\mathrm{SYT}(\boldsymbol{\lambda});\mathbf{X}^{(r)})) \;=\; \frac{\sum_{\boldsymbol{Q}\in\mathrm{SYT}(\boldsymbol{\lambda})}q^{\mathrm{maj}(\boldsymbol{Q})}\mathbf{p}^{\mathrm{n}(\boldsymbol{Q})}}{(q)_n}, \tag{3.54}$$

where $F(\mathrm{SYT}(\boldsymbol{\lambda}); \mathbf{X}^{(r)}), \mathrm{maj}(\boldsymbol{Q})$ and $\mathrm{n}(\boldsymbol{Q})$ are defined analogously to colored permutations. Now, substituting Equation (2.5) in Equation (3.54) yields the following colored analogue of Stanley's q-hook length formula [89, Corollary 7.21.5]

$$\sum_{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda})} q^{\mathrm{maj}(\boldsymbol{Q})} \mathbf{p}^{\mathrm{n}(\boldsymbol{Q})} = q^{\mathrm{b}(\boldsymbol{\lambda})} \mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})} \frac{[n]_q!}{\prod_{j=0}^{r-1} \prod_{u \in \lambda^{(j)}} [\mathrm{h}(u)]_q}. \tag{3.55}$$

This formula refines one of Stembridge [93, Equation 5.6] (see also [7, Corollary 10.28]. Following this reasoning, one can prove Euler–Mahonian identities on (r-partite) standard Young tableaux of a fixed shape.

Setting $p_j \to x^j p_j$ for each $0 \leq j \leq r-1$ in Equations (3.53) and (3.73) yields colored Euler–Mahonian identities for the pairs $(\mathrm{ldes}, \mathrm{maj})$ and $(\mathrm{ldes}, \mathrm{fmaj})$ on absolute involutions.

Corollary 3.3.6. *For a positive integer n, we have*

$$\sum_{m \geq 0} \sum_{\boldsymbol{\lambda}} q^{\mathrm{b}(\boldsymbol{\lambda})} \mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})} \mathbf{a}^{\mathrm{col^o}(\boldsymbol{\lambda})} \prod_{j=0}^{r-1} \binom{m+1}{\lambda^{(j)\prime}}_q x^{m+\mathrm{r}(\boldsymbol{\lambda})} = \frac{\mathcal{I}_{n,r}^{\mathrm{ldes},\mathrm{maj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q)_{n+1}}, \tag{3.56}$$

$$\sum_{m \geq 0} \sum_{\boldsymbol{\lambda}} q^{r\,\mathrm{b}(\boldsymbol{\lambda})+\mathrm{r}(\boldsymbol{\lambda})} \mathbf{p}^{\mathrm{size}(\boldsymbol{\lambda})} \mathbf{a}^{\mathrm{col^o}(\boldsymbol{\lambda})} \prod_{j=0}^{r-1} \binom{m+1}{\lambda^{(j)\prime}}_{q^r} x^{m+\mathrm{r}(\boldsymbol{\lambda})} = \frac{\mathcal{I}_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}}, \tag{3.57}$$

where the sums run through all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ of n.

One could specialize Equation (3.46) as in Theorem 2.3.6, but the resulting formula would be too complicated to write it in a nice form and we therefore omit it. We continue by specializing Equation (3.45). For the next few corollaries, we need to introduce one more piece of notation

$$(x;q)_\infty := \prod_{i \geq 0} (1 - xq^i).$$

Corollary 3.3.7. *We have*

$$\sum_{n \geq 0} \frac{\mathcal{I}_{n,r}^{\mathrm{maj}}(q,\mathbf{p},\mathbf{a})}{(q)_n} z^n = \prod_{t=0}^{r-1} (za_t p_t; q)_\infty^{-1} \prod_{0 \leq i < j} (1 - z^2 p_t^2 q^{i+j})^{-1} \tag{3.58}$$

and

$$\begin{aligned}\sum_{n \geq 0} \frac{\mathcal{I}_{n,r}^{\mathrm{des},\mathrm{maj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q)_{n+1}} z^n = &\sum_{m \geq 0} (za_0p_0;q)_{m+1}^{-1} \prod_{0 \leq i < j \leq m} (1 - z^2 p_0^2 q^{i+j})^{-1} \times \\ &\prod_{t=1}^{r-1} (za_t p_t; q)_m^{-1} \prod_{0 \leq i < j \leq m-1} (1 - z^2 p_t^2 q^{i+j})^{-1} x^m\end{aligned} \tag{3.59}$$

$$\sum_{n \geq 0} \frac{\mathcal{I}_{n,r}^{\mathrm{des}^*,\mathrm{maj}}(x,q,\mathbf{p},\mathbf{a})}{(x;q)_{n+1}} z^n = \sum_{m \geq 0} \prod_{t=0}^{r-1} (za_t p_t; q)_{m+1}^{-1} \prod_{0 \leq i < j \leq m} (1 - z^2 p_t^2 q^{i+j})^{-1} x^m. \tag{3.60}$$

Proof. The proof follows by specializing Equation (3.45) as in Theorem 2.3.1 and substituting in Corollary 2.3.2 for $\mathcal{A} = \mathcal{I}^{\mathrm{abs}}_{n,r}$. □

Corollary 3.3.8. *We have*

$$\sum_{n\geq 0} \frac{\mathcal{I}^{\mathrm{fmaj}}_{n,r}(q,\mathbf{p},\mathbf{a})}{(q^r)_n}\, z^n \;=\; \prod_{t=0}^{r-1} (za_tp_tq^t;q^r)^{-1}_{\infty} \prod_{0\leq i<j} (1-z^2p_t^2q^{r(i+j)+2t})^{-1} \tag{3.61}$$

and

$$\begin{aligned}\sum_{n\geq 0} \frac{\mathcal{I}^{\mathrm{des,fmaj}}_{n,r}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}}\, z^n \;=\;& \\ &\sum_{m\geq 0} (za_0p_0;q^r)^{-1}_{m+1} \prod_{0\leq i<j\leq m} (1-z^2p_0^2q^{r(i+j)})^{-1} \times \\ &\prod_{t=1}^{r-1} (za_tp_tq^t;q^r)^{-1}_{m} \prod_{0\leq i<j\leq m-1} (1-z^2p_t^2q^{r(i+j)+2t})^{-1}\, x^m \end{aligned} \tag{3.62}$$

$$\begin{aligned}\sum_{n\geq 0} \frac{\mathcal{I}^{\mathrm{des}^*,\mathrm{fmaj}}_{n,r}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}}\, z^n \;=\;& \\ &\sum_{m\geq 0} \prod_{t=0}^{r-1} (za_tp_tq^t;q^r)^{-1}_{m+1} \prod_{0\leq i<j\leq m} (1-z^2p_t^2q^{r(i+j)+2t})^{-1}\, x^m. \end{aligned} \tag{3.63}$$

Proof. The proof follows from Corollary 3.3.7 by setting $q \to q^r$ and $p_j \to q^j p_j$ for each $0 \leq j \leq r-1$. □

Corollary 3.3.9. *We have*

$$\begin{aligned}\sum_{n\geq 0} \frac{[x]_r\, \mathcal{I}^{\mathrm{fdes,fmaj}}_{n,r}(x,q,\mathbf{p},\mathbf{a})}{(x;q^r)_{n+1}}\, z^n \;=\;& \\ &\sum_{m\geq 0} (za_0p_0;q^r)^{-1}_{\lfloor m/r\rfloor} \prod_{0\leq i<j\leq \lfloor m/r\rfloor} (1-z^2p_0^2q^{r(i+j)})^{-1} \times \\ &\prod_{t=1}^{r-1} (za_tp_tq^t;q^r)^{-1}_{\lfloor \frac{m-1}{r}\rfloor} \prod_{0\leq i<j\leq \lfloor \frac{m-1}{r}\rfloor} (1-z^2p_t^2q^{r(i+j)+2t})^{-1}\, x^m. \end{aligned} \tag{3.64}$$

Proof. The proof follows by specializing Equation (3.45) as in Theorem 2.3.6 and substituting in Equation (2.28) for $\mathcal{A} = \mathcal{I}^{\mathrm{abs}}_{n,r}$. □

Although formulas presented in Corollaries 3.3.7 to 3.3.9 look complicated and not so easy to handle, one may consider appropriate specializations of $\mathcal{I}^{\mathrm{eul,mah}}_{n,r}(x,q,\mathbf{p},\mathbf{a})$ to study enumerative aspects concerning Euler–Mahonian statistics on absolute involutions. For example, letting $q = p, a_0 = a_1 = \cdots = a_{r-1} = 1$ and $p_j = p^j$ for each $0 \leq j \leq r-1$, Equation (3.59) becomes

$$\sum_{n\geq 0} \frac{\mathcal{I}^{\mathrm{des}}_{n,r}(x,p)}{(1-x)^{n+1}}\, z^n \;=\; \sum_{m\geq 0} \frac{x^m}{(1-z)(1-z^2)^m (z;p)_r^m (z^2;p^2)_r^{\binom{m}{2}}},$$

where

$$\mathcal{I}^{\mathrm{des}}_{n,r}(x,p) \;:=\; \sum_{w\in\mathcal{I}^{\mathrm{abs}}_{n,r}} x^{\mathrm{des}(w)} p^{\mathrm{csum}(w)}.$$

This equation for $r = 2$ and $p = 1$ was recently considered in [69], where the author used it to prove a linear recurrence for the coefficients of $\mathcal{I}^{\mathrm{des}}_{n,2}(x)$, which eventually leads to its unimodality via an inductive argument (see [69, Sections 3 and 4]).

We continue by providing a colored version of a formula due to Athanasiadis [12, Proposition 2.2.] (see also [57, Corollary 5.7 (b)]), which expresses the generating polynomials of the distribution of the number of descents on $\mathcal{I}_n$ and $\mathcal{I}_{n,2}$ in terms of Eulerian polynomials $A_n(x)$ and $A_{n,2}(x)$, respectively.

The following result assumes familiarity with the cycle type of a colored permutation, a colored version of the Frobenius characteristic map introduced by Poirier [76] and the colored power sum basis of $\mathrm{Sym}^{(r)}$. Although all of these notions are defined in Section 5.1, we find it more suitable to include the following result here. For $w \in \mathfrak{S}_{n,r}$, let $\mathrm{c}^j(w)$ be the number of colored cycles of w of color j, for every $0 \leq j \leq r-1$. The following corollary reduces to [12, Proposition 2.22] for $r \leq 2$ and $p = 1$.

Corollary 3.3.10. *For a positive integer n, we have*

$$\mathcal{I}^{\mathrm{des}}_{n,r}(x,p) \;=\; \frac{1}{r^n n!} \sum_{w\in\mathfrak{S}_{n,r}} (1-x)^{n-\mathrm{c}^0(w\overline{w})}\, A^{\mathrm{des}}_{\mathrm{c}^0(w\overline{w}),r}(x,p), \tag{3.65}$$

where $A^{\mathrm{des}}_{n,r}(x,p) := A^{\mathrm{des,mah}}_{n,r}(x,1,p)$ for any mahonian statistic mah *on colored permutations*[5]*.*

Proof. Substituting $\mathcal{A} = \mathcal{I}^{\mathrm{abs}}_{n,r}, q = 1$ and $p_j = p^j$ for each $0 \leq j \leq r-1$ in Equation (2.19) yields

$$\frac{\mathcal{I}^{\mathrm{des}}_{n,r}(x,p)}{(1-x)^{n+1}} \;=\; \sum_{m\geq 1} F(\mathcal{I}^{\mathrm{abs}}_{n,r}; 1^m, p^{m-1}, \ldots, (p^{r-1})^{m-1})\, x^{m-1}, \tag{3.66}$$

where $(p^j)^s$ in the above notation means that $x^{(j)}_1 = x^{(j)}_2 = \cdots = x^{(j)}_s = p^j$ and $x^{(j)}_{s+1} = x^{(j)}_{s+2} = \cdots = 0$ etc.. Applying Equation (3.46) for $a_0 = a_1 = \cdots = a_{r-1} = 1$, Equation (3.66) becomes

$$\frac{\mathcal{I}^{\mathrm{des}}_{n,r}(x,p)}{(1-x)^{n+1}} \;=\; \sum_{m\geq 1} \sum s_{\lambda^{(0)}}(1^m) s_{\lambda^{(1)}}(p^{m-1}) \cdots s_{\lambda^{(r-1)}}((p^{r-1})^{m-1})\, x^{m-1}, \tag{3.67}$$

[5] Recall this notation from Section 1.2.3

where the second sum runs through all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ of n. From the properties of the characteristic map, we know that

$$s_{\lambda^{(0)}}(1^m) s_{\lambda^{(1)}}(p^{m-1}) \cdots s_{\lambda^{(r-1)}}((p^{r-1})^{m-1}) = \mathrm{ch}_r(\chi^{\boldsymbol{\lambda}})(1^m, p^{m-1}, \ldots, (p^{r-1})^{m-1})$$

and we can use Equation (5.1) to expand it in the colored power sum basis, as follows

$$\begin{aligned}\mathrm{ch}_r(\chi^{\boldsymbol{\lambda}})(1^m, p^{m-1}, \ldots, (p^{r-1})^{m-1}) &= \\ &\frac{1}{r^n n!} \sum_{w \in \mathfrak{S}_{n,r}} \chi^{\boldsymbol{\lambda}}(w) p_{\mathrm{ct}(w^{-1})}(1^m, p^{m-1}, \ldots, (p^{r-1})^{m-1}),\end{aligned}$$

where $\chi^{\boldsymbol{\lambda}}$ is the irreducible $\mathfrak{S}_{n,r}$-character associated with the r-partite partition $\boldsymbol{\lambda} \vdash n$. But,

$$\begin{aligned}p_{\mathrm{ct}(w^{-1})}&(1^m, p^{m-1}, \ldots, (p^{r-1})^{m-1}) \\ &= \prod_{j=0}^{r-1} (m + (p\zeta^j + p^2\zeta^{2j} + \cdots + p^{r-1}\zeta^{(r-1)j})(m-1))^{\mathrm{c}^j(w^{-1})} \\ &= ([r]_p(m-1)+1)^{\mathrm{c}^0(w^{-1})} \prod_{j=1}^{r-1} (m-(m-1))^{\mathrm{c}^j(w^{-1})} \\ &= ([r]_p(m-1)+1)^{\mathrm{c}^0(w)},\end{aligned}$$

because

$$p\zeta^j + p^2\zeta^{2j} + \cdots + p^{r-1}\zeta^{(r-1)j} = \frac{1-p^r\zeta^{jr}}{1-p\zeta^j} - 1 = -1,$$

for every $1 \le j \le r-1$ and $\mathrm{c}^0(w^{-1}) = \mathrm{c}^0(w)$. Combining these calculations, substituting in Equation (3.67) and changing the order of summation yields

$$\frac{\mathcal{I}^{\mathrm{des}}_{n,r}(x,p)}{(1-x)^{n+1}} = \frac{1}{r^n n!} \sum_{w \in \mathfrak{S}_{n,r}} \left(\sum_{\boldsymbol{\lambda} \vdash n} \chi^{\boldsymbol{\lambda}}(w) \right) \left(\sum_{m \ge 0} ([r]_p m + 1)^{\mathrm{c}^0(w)} x^m \right). \tag{3.68}$$

A special case of a well known result due to Frobenius and Schur (see, for example, [89, Exercise 7.69 (c)] and references therein) is that the sum of all irreducible $\mathfrak{S}_n$-characters computed in the conjugacy class corresponding to the cycle type of $w \in \mathfrak{S}_n$ is equal to the number of square roots of w in $\mathfrak{S}_n$. Adin, Postnikov and Roichman [5, Theorem 3.4] extended this result to colored permutation groups by proving the following

$$\sum_{\boldsymbol{\lambda} \vdash n} \chi^{\boldsymbol{\lambda}}(w) = |\{u \in \mathfrak{S}_{n,r} : u\overline{u} = w\}|, \tag{3.69}$$

for every $w \in \mathfrak{S}_{n,r}$. Therefore, the proof follows by substituting Equation (3.69) in Equation (3.68) and using Equation (3.10). $\square$

Remark 3.3.11. Continuing the arguments of Remark 3.3.5 one can prove a formula for the polynomials

$$\sum_{\boldsymbol{Q}\in\mathrm{SYT}(\boldsymbol{\lambda})} x^{\mathrm{des}(\boldsymbol{Q})} q^{\mathrm{csum}(\boldsymbol{Q})}$$

similar to that of Equation (3.65). In particular, we have

$$\sum_{\boldsymbol{Q}\in\mathrm{SYT}(\boldsymbol{\lambda})} x^{\mathrm{des}(\boldsymbol{Q})} q^{\mathrm{csum}(\boldsymbol{Q})} = \frac{1}{r^n n!} \sum_{w\in\mathfrak{S}_{n,r}} \chi^{\boldsymbol{\lambda}}(w) A_{\mathrm{c}^0(w),r}(x,p)(1-x)^{n-\mathrm{c}^0(w)} \tag{3.70}$$

for every r-partite partition $\boldsymbol{\lambda}$ of n.

For the remainder of this section, we limit our discussion to the signed case $r=2$. The next few corollaries compute formulas similar to those of Corollaries 3.3.3, 3.3.7 and 3.3.8 for signed involutions, involving the (k,ℓ)-flag major index. The proofs of the following corollaries are entirely similar to that of Corollaries 3.3.3 and 3.3.7 and are therefore omitted.

Corollary 3.3.12. *For a positive integer n, we have*

$$\mathcal{I}_{n,2}^{\mathrm{fmaj}_{k,\ell}}(q,\mathbf{p},\mathbf{a}) = \sum_{(\lambda,\mu)\vdash n} q^{k\,\mathrm{b}(\lambda,\mu)+\ell|\mu|} \mathbf{p}^{\mathrm{size}(\lambda,\mu)} \mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\lambda,\mu)} \frac{[n]_{q^k}!}{\prod_{\substack{u\in\lambda\\ v\in\mu}} [h(u)]_{q^k}[h(v)]_{q^k}} \tag{3.71}$$

and

$$\sum_{m\geq 0}\sum_{(\lambda,\mu)\vdash n} q^{k\,\mathrm{b}(\lambda,\mu)+\ell|\mu|} \mathbf{p}^{\mathrm{size}(\lambda,\mu)} \mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\lambda,\mu)} \binom{m+1}{\lambda'}_{q^k} \binom{m}{\mu'}_{q^k} x^m = \frac{\mathcal{I}_{n,2}^{\mathrm{des},\mathrm{fmaj}_{k,\ell}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^k)_{n+1}} \tag{3.72}$$

$$\sum_{m\geq 0}\sum_{(\lambda,\mu)\vdash n} q^{k\,\mathrm{b}(\lambda,\mu)+\ell|\mu|} \mathbf{p}^{\mathrm{size}(\lambda,\mu)} \mathbf{a}^{\mathrm{col}^{\mathrm{o}}(\lambda,\mu)} \binom{m+1}{\lambda'}_{q^k} \binom{m+1}{\mu'}_{q^k} x^m = \frac{\mathcal{I}_{n,2}^{\mathrm{des}^*,\mathrm{fmaj}_{k,\ell}}(x,q,p,p_0,p_1)}{(x;q^k)_{n+1}}. \tag{3.73}$$

Corollary 3.3.13. *We have*

$$\sum_{n\geq 0} \frac{\mathcal{I}_{n,r}^{\mathrm{fmaj}_{k,\ell}}(q,\mathbf{p},\mathbf{a})}{(q^k)_n} z^n = (za_0p_0;q^k)_\infty^{-1}\,(zp_1a_1q^\ell p;q^k)_\infty^{-1} \times \prod_{0\leq i<j} (1-z^2p_0^2q^{k(i+j)})^{-1}(1-z^2p_1^2q^{k(i+j)+2\ell})^{-1} \tag{3.74}$$

and

$$\sum_{n\geq 0}\frac{\mathcal{I}_{n,r}^{\mathrm{des},\mathrm{fmaj}_{k\ell}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^k)_{n+1}}\,z^n \;=\; \sum_{m\geq 0}(za_0p_0;q^k)_{m+1}^{-1}(za_1p_1q^\ell;q^k)_m^{-1}\prod_{0\leq i<j\leq m}(1-z^2p_0^2q^{k(i+j)})^{-1}\times \prod_{0\leq i<j\leq m-1}(1-z^2p_1^2q^{k(i+j)+2\ell})^{-1}\,x^m \tag{3.75}$$

$$\sum_{n\geq 0}\frac{\mathcal{I}_{n,r}^{\mathrm{des}^*,\mathrm{fmaj}_{k\ell}}(x,q,\mathbf{p},\mathbf{a})}{(x;q^k)_{n+1}}\,z^n \;=\; \sum_{m\geq 0}(za_0p_0;q^k)_{m+1}^{-1}(za_1p_1q^\ell;q^k)_{m+1}^{-1}\times \prod_{0\leq i<j\leq m}(1-z^2p_0^2q^{k(i+j)})^{-1}(1-z^2p_1^2)^2q^{k(i+j)+2\ell})^{-1}\,x^m. \tag{3.76}$$

3.4 Multivariate colored permutation statistics

In what follows, the generating polynomials of permutation statistics are set to equal 1 for $n=0$. For a colored permutation statistic stat and a colored permutation w, we write $\mathrm{istat}(w) := \mathrm{stat}(w^{-1})$ and $\overline{\mathrm{istat}}(w) := \mathrm{stat}(\overline{w}^{-1})$. A pair of statistics is called *bimahonian* if it is equidistributed with $(\mathrm{maj},\mathrm{imaj})$. A celebrated result, due to Foata and Schützenberger [46, Theorem 1], states that the pair $(\mathrm{maj},\mathrm{inv})$ is bimahonian on $\mathfrak{S}_n$. Gessel [51, Theorem 8.5] computed the following generating function for the bimahonian statistic $(\mathrm{inv},\mathrm{maj})$

$$\sum_{n\geq 0}\frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)}p^{\mathrm{maj}(w)}}{(q)_n(p)_n}\,z^n \;=\; \frac{1}{(z;q,p)_{\infty,\infty}}, \tag{3.77}$$

where

$$(z;q,p)_{\infty,\infty} \;:=\prod_{i\geq 1}\prod_{j\geq 1}(1-zq^{i-1}p^{j-1}).$$

Equation (3.77) also holds for the bimahonian pair $(\mathrm{maj},\mathrm{imaj})$. This appeared implicitly in Gordon's work [58] and made explicit by Roselle [79]. Because of that, Equation (3.77) for the bimahonian pair $(\mathrm{maj},\mathrm{imaj})$ is often called *Roselle identity.* Garsia and Gessel [49] studied bieulerian-bimahonian distributions, meaning the four-variate distribution $(\mathrm{des},\mathrm{ides},\mathrm{maj},\mathrm{imaj})$ and proved the following generating function

$$\sum_{n\geq 0}\frac{\sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)}y^{\mathrm{ides}(w)}q^{\mathrm{maj}(w)}p^{\mathrm{imaj}(w)}}{(x;q)_{n+1}(y;p)_{n+1}}\,z^n \;=\; \sum_{m_1\geq 0}\sum_{m_2\geq 0}\frac{x^{m_1}y^{m_2}}{(z;q,p)_{m_1+1,m_2+1}}, \tag{3.78}$$

where

$$(z;q,p)_{k,l} := \prod_{i=1}^{k}\prod_{j=1}^{l}(1-zq^{i-1}p^{j-1}),$$

for every all integers k,l.

Equations (3.77) and (3.78) can be proved by taking the stable principal specialization and the principal specialization of order m of the following identity [89, Equation (7.114) and Equation (7.44)]

$$\sum_{n\geq 0}\sum_{w\in\mathfrak{S}_n} F_{n,\mathrm{Des}(w)}(\mathbf{x})F_{n,\mathrm{Des}(w^{-1})}(\mathbf{y})\,z^n = \prod_{i\geq 1}\prod_{j\geq 1}(1-zx_iy_j)^{-1}, \tag{3.79}$$

and using Equations (2.8) and (2.9) for $\mathcal{A}=\mathfrak{S}_n$, respectively. This is essentially the approach of [89, Corollary 7.23.9] (see also [37]). This section develops a colored analogue of this approach and provides colored analogues of Equations (3.77) and (3.78) for bimahonian and bieulerian-bimahonian distributions on colored permutations.

For every $0\leq j\leq r-1$, let $\mathbf{y}^{(j)}=(y_1^{(j)},y_2^{(j)},\dots)$ be another sequence of commuting indeterminates and let $\mathbf{Y}^{(r)} := (y_i^{(0)},y_i^{(1)},\dots,y_i^{(r-1)})_{i\geq 1}$. The following lemma, essentially due to Poirier [76, Lemma 4], is a colored analogue of Equation (3.79).

Lemma 3.4.1. *We have*

$$\sum_{n\geq 0}\sum_{w\in\mathfrak{S}_{n,r}} F_w(\mathbf{X}^{(r)})F_{\overline{w}^{-1}}(\mathbf{Y}^{(r)})\,z^n = \prod_{t=0}^{r-1}\prod_{i\geq 1}\prod_{j\geq 1}(1-zx_i^{(t)}y_j^{(t)})^{-1}. \tag{3.80}$$

Proof. Recall from [76, Lemma 4] that

$$\sum_{n\geq 0}\sum_{\boldsymbol{\lambda}\vdash n} s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)})s_{\boldsymbol{\lambda}}(\mathbf{Y}^{(r)})\,z^n = \prod_{t=0}^{r-1}\prod_{i\geq 1}\prod_{j\geq 1}(1-zx_i^{(t)}y_j^{(t)})^{-1}, \tag{3.81}$$

The proof follows by expanding the product $s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)})s_{\boldsymbol{\lambda}}(\mathbf{Y}^{(r)})$ in the left-hand side of Equation (3.81) according to Equation (2.5) and then applying the colored Robinson–Schensted correspondence. $\square$

For a positive integer n, let

$$A_{n,r}^{\mathrm{eul},\overline{\mathrm{ieul}},\mathrm{mah},\overline{\mathrm{imah}}}(x,y,q,p,a,b) := \sum_{w\in\mathfrak{S}_{n,r}} x^{\mathrm{eul}(w)}y^{\overline{\mathrm{ieul}}(w)}q^{\mathrm{mah}(w)}p^{\overline{\mathrm{imah}}(w)}a^{\mathrm{csum}(w)}b^{\overline{\mathrm{icsum}}(w)}$$

$$A_{n,r}^{\mathrm{mah},\overline{\mathrm{imah}}}(q,p,a,b) := A_{n,r}^{\mathrm{eul},\overline{\mathrm{ieul}},\mathrm{mah},\overline{\mathrm{imah}}}(1,1,q,p,a,b),$$

where $\mathrm{eul},\mathrm{mah}$ are an Eulerian and a Mahonian statistic on colored permutations, respectively and a,b are indeterminates. For ease of notational complexity we will deal with the color sum statistic instead of the statistic which counts the number of entries of certain color of a permutation. Therefore in specializations that follow we are referring to those versions involving csum instead of $(\mathrm{n}_1,\mathrm{n}_2,\dots,\mathrm{n}_{r-1})$. The following corollary specializes Equation (3.80) as in Theorem 2.3.1 and obtains formulas for the generating functions for the distributions of the following tuples

- $(\mathrm{maj}, \overline{\mathrm{imaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$
- $(\mathrm{des}, \overline{\mathrm{ides}}, \mathrm{maj}, \overline{\mathrm{imaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$
- $(\mathrm{des}^*, \overline{\mathrm{ides}^*}, \mathrm{maj}, \overline{\mathrm{imaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$.

Biagioli and Zeng [24, Theorem 7.1] computed the generating function for the second tuple. In this paper the authors use the length order and by Reiner [77, Corollary 7.3], where he considers $i \in [n]$ to be a descent of $w \in \mathfrak{S}_{n,r}$, if $\ell_S(ws_i^{-1}) = \ell_S(w) - 1$, where $s_1, s_2, \dots, s_{n-1}$ and $s_n := s_0$, as defined in Section 1.2. For $r = 2$, Equation (3.83) below coincides with Biagioli and Zeng's formula.

Corollary 3.4.2. *We have*

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{maj},\overline{\mathrm{imaj}}}(q,p,a,b)}{(q)_n(p)_n} z^n = \prod_{t=0}^{r-1} \frac{1}{(z(ab)^t;q,p)_{\infty,\infty}}, \tag{3.82}$$

and

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{des},\overline{\mathrm{ides}},\mathrm{maj},\overline{\mathrm{imaj}}}(x,y,q,p,a,b)}{(x;q)_{n+1}(y;p)_{n+1}} z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{(z;q,p)_{m_1+1,m_2+1}\prod_{t=1}^{r-1}(z(ab)^t;q,p)_{m_1,m_2}} \tag{3.83}$$

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{des}^*,\overline{\mathrm{ides}^*},\mathrm{maj},\overline{\mathrm{imaj}}}(x,y,q,p,a,b)}{(x;q)_{n+1}(y;p)_{n+1}} z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{\prod_{t=0}^{r-1}(z(ab)^t;q,p)_{m_1+1,m_2+1}}. \tag{3.84}$$

Proof. The proof follows by combining Equation (3.80) and Equations (2.11) to (2.13). □

Setting $q \to q^r$, $p \to p^r$ and $a \to aq^r$, $b \to bp^r$ in Corollary 3.4.2 yields formulas for the generating functions for the distributions of the following tuples

- $(\mathrm{fmaj}, \overline{\mathrm{ifmaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$
- $(\mathrm{des}, \overline{\mathrm{ides}}, \mathrm{fmaj}, \overline{\mathrm{ifmaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$
- $(\mathrm{des}^*, \overline{\mathrm{ides}^*}, \mathrm{fmaj}, \overline{\mathrm{ifmaj}}, \mathrm{csum}, \overline{\mathrm{icsum}})$.

Generating functions for the distribution of first tuple have been proved by Foata and Han [43, Equation (4.3)] for $r = 2$, where the authors use the integer order, by Biagioli and Zeng [24, Proposition 8.5], where the authors use the length order and by Biagioli and Caselli [22, Equation (21) for $p = s = 1$], where the authors use the color order. The latter also proved [22, Proposition 6.2] a generating function for the distribution of the second tuple. For $r = 2$, Equation (3.85) below coincides with Biagioli–Zeng's formula.

Corollary 3.4.3. *We have*

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{fmaj},\overline{\mathrm{ifmaj}}}(q,p,a,b)}{(q^r)_n(p^r)_n} z^n = \prod_{t=0}^{r-1} \frac{1}{(z(qpab)^t;q^r,p^r)_{\infty,\infty}}, \tag{3.85}$$

and

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{des},\overline{\mathrm{ides}},\mathrm{fmaj},\overline{\mathrm{ifmaj}}}(x,y,q,p,a,b)}{(x;q^r)_{n+1}(y;p^r)_{n+1}} z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{(z;q^r,p^r)_{m_1+1,m_2+1}\prod_{t=1}^{r-1}(z(qpab)^t;q^r,p^r)_{m_1,m_2}} \tag{3.86}$$

$$\sum_{n\geq 0} \frac{A_{n,r}^{\mathrm{des}^*,\overline{\mathrm{ides}^*},\mathrm{fmaj},\overline{\mathrm{ifmaj}}}(x,y,q,p,a,b)}{(x;q^r)_{n+1}(y;p^r)_{n+1}} z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{\prod_{t=0}^{r-1}(z(qpab)^t;q^r,p^r)_{m_1+1,m_2+1}}. \tag{3.87}$$

The following corollary computes the generating function for the distribution of $(\mathrm{fdes},\overline{\mathrm{ifdes}},\mathrm{fmaj},\overline{\mathrm{ifmaj}},\mathrm{csum},\overline{\mathrm{icsum}})$. Its proof is analogous to the proof of Corollary 3.4.2 and is therefore omitted. For a similar formula for $r=2$ see [43, Theorem 1.1], where the authors use the integer order.

Corollary 3.4.4. *We have*

$$\sum_{n\geq 0} \frac{[r]_x[r]_y\, A_{n,r}^{\mathrm{fdes},\overline{\mathrm{ifdes}},\mathrm{fmaj},\overline{\mathrm{ifmaj}}}(x,y,q,p,a,b)}{(x^r;q^r)_{n+1}(y^r;p^r)_{n+1}} z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{(z;q^r,p^r)_{\lfloor\frac{m_1}{r}\rfloor+1,\lfloor\frac{m_2}{r}\rfloor+1}\prod_{t=1}^{r-1}(z(qpab)^t;q^r,p^r)_{\lfloor\frac{m_1-1}{r}\rfloor+1,\lfloor\frac{m_2-1}{r}\rfloor+1}}. \tag{3.88}$$

For the next corollary we limit our discussion to the signed case $r=2$. It computes the generating functions for the distributions of the following tuples

- $(\mathrm{fmaj}_{k,\ell},\mathrm{ifmaj}_{k,\ell},\mathrm{neg},\mathrm{ineg})$
- $(\mathrm{des},\overline{\mathrm{ides}},\mathrm{fmaj}_{k,\ell},\mathrm{ifmaj}_{k,\ell},\mathrm{neg},\mathrm{ineg})$
- $(\mathrm{des}^*,\overline{\mathrm{ides}^*},\mathrm{fmaj}_{k,\ell},\mathrm{fmaj}_{k,\ell},\mathrm{neg},\mathrm{ineg})$.

for positive integers k,k' and nonnegative integers ℓ,ℓ'.

Corollary 3.4.5. *We have*

$$\sum_{n\geq 0} \frac{A_{n,2}^{\mathrm{fmaj}_{k,\ell},\mathrm{ifmaj}_{k',\ell'}}(q,p,a,b)}{(q^k)_n(p^{k'})_n} z^n = \frac{1}{(z;q^k,p^{k'})_{\infty,\infty}(zq^\ell p^{\ell'};q^k,p^{k'})_{\infty,\infty}} \tag{3.89}$$

and

$$\sum_{n\geq 0} \frac{A_{n,2}^{\mathrm{des},\mathrm{ides},\mathrm{fmaj}_{k,\ell},\mathrm{ifmaj}_{k',\ell'}}(x,y,q,p,a,b)}{(x;q^k)_{n+1}(y;p^{k'})_{n+1}}\, z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{(z;q^k,p^{k'})_{m_1+1,m_2+1}(zabq^{\ell}p^{\ell'};q^k,p^{k'})_{m_1,m_2}} \tag{3.90}$$

$$\sum_{n\geq 0} \frac{A_{n,2}^{\mathrm{des}^*,\mathrm{ides}^*,\mathrm{fmaj}_{k,\ell},\mathrm{ifmaj}_{k',\ell'}}(x,y,q,p,a,b)}{(x;q^k)_{n+1}(y;p^{k'})_{n+1}}\, z^n = \sum_{m_1,m_2\geq 0} \frac{x^{m_1}y^{m_2}}{(z;q^k,p^{k'})_{m_1+1,m_2+1}(zabq^{\ell}p^{\ell'};q^k,p^{k'})_{m_1+1,m_2+1}} \tag{3.91}$$

Proof. The proof follows by combining Equation (3.80) and Equations (2.33) to (2.35). ∎

Remark 3.4.6. Assigning different values to k, k', ℓ, ℓ' in Corollary 3.4.5 yield formulas for the distributions of the following tuples

- $(\mathrm{maj}, \mathrm{ifmaj}, \mathrm{neg}, \mathrm{ineg})$
- $(\mathrm{fmaj}, \mathrm{imaj}, \mathrm{neg}, \mathrm{ineg})$
- $(\mathrm{des}, \mathrm{ides}, \mathrm{maj}, \mathrm{ifmaj}, \mathrm{neg}, \mathrm{ineg})$
- $(\mathrm{des}, \mathrm{ides}, \mathrm{fmaj}, \mathrm{imaj}, \mathrm{neg}, \mathrm{ineg})$
- $(\mathrm{des}^*, \mathrm{ides}^*, \mathrm{maj}, \mathrm{ifmaj}, \mathrm{neg}, \mathrm{ineg})$
- $(\mathrm{des}^*, \mathrm{ides}^*, \mathrm{fmaj}, \mathrm{imaj}, \mathrm{neg}, \mathrm{ineg})$

among others, which may be of interest.

Chapter 4

A colored shuffling theorem and shuffle-compatibility

This chapter reviews Hsiao–Petersen's theory of colored P-partitions and proves a colored analogue of Stanley's shuffling theorem. Furthermore, it proves that the colored descent set is shuffle-compatible and provides further examples of shuffle-compatible colored permutation statistics.

4.1 P-partitions, quasisymmetric functions and shuffle-compatibility

As we mentioned in the introduction, quasisymmetric functions first appeared as generating functions of P-partitions in Stanley's work [87]. We review basic concepts of this theory related to fundamental quasisymmetric functions. We also recall a connection with shuffle-compatibility and Stanley's shuffling theorem. For a thorough treatment on P-partitions we refer to [90, Section 3.15], [89, Section 7.19] and [74, Chapter 3].

Let $(P, <_P)$ be a naturally labeled poset with n elements. A *P-partition* is a function $f : P \to \mathbb{Z}_{>0}$ such that

(I) $i <_P j$ implies $f(i) \geq f(j)$

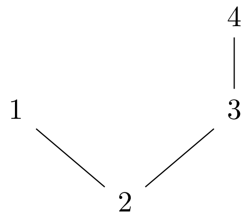
(II) $i <_P j$ and $i >_{\mathbb{Z}} j$ implies $f(i) > f(j)$,

for all $i, j \in P$. Let $\mathcal{A}(P)$ be the set of all P-partitions and consider the generating function

$$\Gamma(P; \mathbf{x}) \ := \ \sum_{f \in \mathcal{A}(P)} \prod_{i \in P} x_{f(i)}.$$

This is a (homogeneous) quasisymmetric function of degree n.

Example 4.1.1. If P is the poset



then

$$\Gamma(P;\mathbf{x}) \;=\; \sum_{f(1)<f(2)\geq f(3)\geq f(4)} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)} \;\in\; \mathrm{QSym}_4(\mathbf{x}).$$

If P is a permutation $w \in \mathfrak{S}_n$, viewed as the chain $w_1 <_P w_2 <_P \cdots <_P w_n$, then

$$\begin{aligned}\mathcal{A}(P) \;=\; \{f : P \to \mathbb{Z}_{>0} : &\, f(w_1) \geq f(w_2) \geq \cdots \geq f(w_n)\\ &\text{and } i \in \mathrm{Des}(w) \;\Rightarrow\; f(w_i) > f(w_{i+1})\}\end{aligned}$$

and therefore $\Gamma(P;\mathbf{x}) = F_{n,\mathrm{Des}(w)}(\mathbf{x})$. Also, notice that if P is an antichain with n elements, then $\Gamma(P;\mathbf{x}) = h_1(\mathbf{x})^n$. The *fundamental lemma of P-partitions* [89, Theorem 7.19.4] states that

$$\Gamma(P;\mathbf{x}) \;=\; \sum_{w\in\mathcal{L}(P)} F_{n,\mathrm{Des}(w)}(\mathbf{x}),$$

where $\mathcal{L}(P)$[1] is the set of all linear extensions of P. In our working example, we have $\mathcal{L}(P) = \{2134, 2314, 2341\}$ and therefore

$$\Gamma(P;\mathbf{x}) \;=\; F_{4,\{1\}}(\mathbf{x}) + F_{4,\{2\}}(\mathbf{x}) + F_{4,\{4\}}(\mathbf{x}).$$

Given two disjoint permutations $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_m$, a *shuffle* of u and v is a permutation of length $n+m$, in which both u and v appear as subsequences. We write $u \sqcup\!\sqcup\, v$ for the set of all shuffles of u and v. The multiplication in $\mathrm{QSym}(\mathbf{x})$ in terms of the fundamental basis amounts to shuffling permutations. Specifically, we have

$$F_{n,\mathrm{Des}(u)}(\mathbf{x})F_{m,\mathrm{Des}(v)}(\mathbf{x}) \;=\; \sum_{w\,\in\, u\sqcup\!\sqcup v} F_{n+m,\mathrm{Des}(w)}(\mathbf{x}).$$

This is a consequence of the fact that the set of linear extensions of the disjoint union of two chains is the set of shuffles of those chains. In general, the set $\mathcal{A}(P+Q)$ of $(P+Q)$-partitions of the disjoint union of two posets P and Q with n and m elements is in one-to-one correspondence with the cartesian product $\mathcal{A}(P)\times\mathcal{A}(Q)$ and therefore

$$\Gamma(P+Q;\mathbf{x}) \;=\; \Gamma(P;\mathbf{x})\Gamma(Q;\mathbf{x}).$$

The discussion of the previous paragraph leads to the following remarkable fact, observed by Stanley [90, Exercise 161], about the descent set statistic and shuffles:

[1] It is often called the *Jordan–Hölder set* of P (see [90, Section 3.15]).

for two disjoint permutations u and v, the multiset $\{\mathrm{Des}(w) : w \in u \sqcup v\}$ depends only on $\mathrm{Des}(u), \mathrm{Des}(v)$ and the lengths of u and v. Recently, Gessel and Zhuang [56] initiated a systematic study of shuffle-compatible permutation statistics. A permutation statistic stat is called *shuffle-compatible* if for any disjoint permutations u and v, the multiset $\{\mathrm{stat}(w) : w \in u \sqcup v\}$ depends only on $\mathrm{stat}(u), \mathrm{stat}(v)$ and the lengths of u and v.

To each shuffle-compatible statistic stat, we can associate a $\mathbb{C}$-algebra in the following way. We say that permutations u and v are stat-*equivalent* if they have the same length and $\mathrm{stat}(u) = \mathrm{stat}(v)$. Let $[u]_{\mathrm{stat}}$ be the stat-equivalence class of u. Let $\mathcal{A}_{\mathrm{stat}}$ be the complex vector space whose basis is the set of stat-equivalence classes of permutations. This vector space becomes a $\mathbb{C}$-algebra with multiplication given by

$$[u]_{\mathrm{stat}}[v]_{\mathrm{stat}} = \sum_{w \in u \sqcup v} [w]_{\mathrm{stat}},$$

which is well defined because stat is shuffle-compatible. This is called the *shuffle algebra of* stat. As noticed by Gessel and Zhuang [56, Corollary 4.2], the shuffle-compatibility of the descent set implies that the shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ of Des is isomorphic to QSym as a graded complex algebra via

$$[w]_{\mathrm{stat}} \mapsto F_{|w|,\mathrm{Des}(w)}(\mathbf{x})$$

where $|w|$ is the length of w.

More examples of shuffle compatible statistics include the major index, the descent number, the peak set and the peak number, the left peak number and the pair $(\mathrm{des}, \mathrm{maj})$. These are all *descent statistics*, in the sense that they depend only on the descent composition (see [56, Section 2.2]). Given a permutation $w \in \mathfrak{S}_n$, we recall their definitions:

- The *comajor index*, written $\mathrm{comaj}(w)$, of w is defined by

$$\mathrm{comaj}(w) := \sum_{i \in \mathrm{Des}(w)} (n-i).$$

- The *peak set*, written $\mathrm{Pk}(w)$, of w is defined by

$$\mathrm{Pk}(w) := \{i \in [2, n-1] : w_{i-1} < w_i > w_{i+1}\}$$

  and the *peak number* of w is defined as $\mathrm{pk}(w) := |\,\mathrm{Pk}(w)|$.

- The *left peak set*, written $\mathrm{LPk}(w)$, of w is defined by[2]

$$\mathrm{LPk}(w) := \{i \in [n-1] : w_{i-1} < w_i > w_{i+1}\},$$

  where $w_0 := 0$ and the *left peak number* is defined as $\mathrm{lpk}(w) := |\,\mathrm{LPk}(w)|$.

We urge the reader to recall [56, Theorems 4.5-4.10], where the authors describe the shuffle algebras associated to $\mathrm{des}, (\mathrm{des}, \mathrm{comaj}), \mathrm{Pk}, \mathrm{pk}, \mathrm{LPk}$ and lpk.

[2] In other words it consists of all peaks of w plus 1 whenever it is a descent of w.

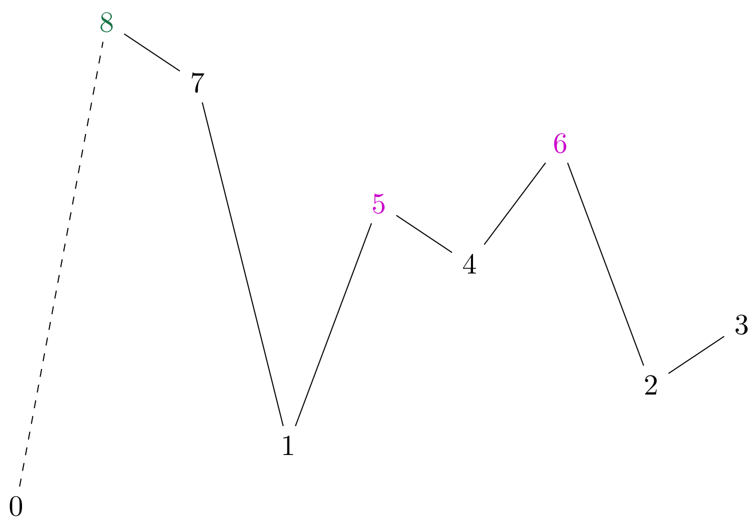


Figure 4.1: The permutation $w = 87154623 \in \mathfrak{S}_8$ has $\mathrm{Pk}(w) = \{4, 6\}$ and $\mathrm{LPk}(w) = \{1, 4, 6\}$, both determined by the descent composition $\mathrm{co}(w) = (1, 1, 2, 2, 2)$.

Lastly, we recall the celebrated *Stanley's shuffling theorem*. It computes the distribution of the major index over shuffles of disjoint permutations with a given number of descents in terms of q-binomial coefficients.

Theorem 4.1.2. (Stanley [87, Proposition 12.6 and Equation (24)]) *For any disjoint permutations u and v of length n and m, respectively, and any integer $0 \le k \le n$,*

$$\sum_{\substack{w \in u \sqcup v \\ \mathrm{des}(w)=k}} q^{\mathrm{maj}(w)} = q^{\mathrm{maj}(u)+\mathrm{maj}(v)+(k-\mathrm{des}(u))(k-\mathrm{des}(v))} \times \binom{n-\mathrm{des}(u)+\mathrm{des}(v)}{k-\mathrm{des}(u)}_q \binom{m-\mathrm{des}(v)+\mathrm{des}(u)}{k-\mathrm{des}(v)}_q . \tag{4.1}$$

In particular,

$$\sum_{w \in u \sqcup v} q^{\mathrm{maj}(w)} = q^{\mathrm{maj}(u)+\mathrm{maj}(v)} \binom{n+m}{n}_q . \tag{4.2}$$

4.2 Colored P-partitions and colored quasisymmetric functions

Hsiao and Petersen [62] developed a theory of colored P-partitions in which colored quasisymmetric functions appear naturally as generating functions of those. We recall their construction.

Let $(P, <_p)$ be a finite poset of cardinality n. A *colored labeling* of P is an injection $\omega : P \to \Omega_{n,r}$ such that exactly one element of $\{i^0, i^1, \ldots, i^{r-1}\}$ appears in the image of ω for every $i \in [n]$. A poset equipped with a colored labeling will be called a *colored poset*. For the rest of this chapter we assume that posets are equipped with a colored labeling, unless otherwise stated. For example, a colored labeling of an n-element chain is just a colored permutation of length n in window notation[3].

Let $\mathbb{P}^{(r)} := \mathbb{Z}_{>0} \times \mathbb{Z}_r$ be the set of colored integers. We regard the elements of $\mathbb{P}^{(r)}$ as colored integers i^α, for $i \in \mathbb{Z}_{>0}$ and $\alpha \in \mathbb{Z}_r$. We assume that $\mathbb{P}^{(r)}$ is equipped with the left lexicographic order, written $<_{\text{llex}}$.

Definition 4.2.1. (cf. [62, Definition 3.3]) A *colored P-partition* is a function $f : P \to \mathbb{P}^{(r)}$ such that

(I) $i <_P j$ implies $f(i) \geq_{\text{llex}} f(j)$

(II) $i <_P j$ and $i >_c j$ implies $f(i) >_{\text{llex}} f(j)$

(III) the color of i, determined by the colored labeling of P, and the color of $f(i)$ is the same and we denote it by $\epsilon(i)$,

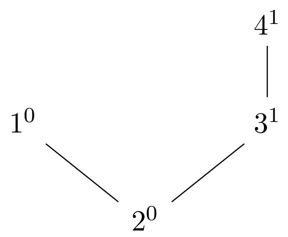
for all i, j of P.

Let $\mathcal{A}^{(r)}(P)$ be the set of all colored P-partitions and consider the colored quasisymmetric generating function

$$\Gamma(P; \mathbf{X}^{(r)}) \ := \sum_{f \in \mathcal{A}^{(r)}(P)} \prod_{i \in P} x_{f(i)}^{\epsilon(i)}.$$

This is a homogeneous element of $\mathrm{QSym}_n^{(r)}$.

Example 4.2.2. For $n = 4$ and $r = 2$, if P is the poset



then $2^0 <_P 1^0$ and $2^0 >_c 1^0$, $2^0 <_P 3^1$ and $2^0 >_c 3^1$, and $3^1 <_P 4^1$ and $3^1 <_c 4^1$ and therefore

$$\Gamma(P; \mathbf{x}^{(0)}, \mathbf{x}^{(1)}) \ = \sum_{f(1^0) <_{\text{llex}} f(2^0) >_{\text{llex}} f(3^1) \geq_{\text{llex}} f(4^1)} x_{f(1^0)}^{(0)} x_{f(2^0)}^{(0)} x_{f(3^1)}^{(1)} x_{f(4^1)}^{(1)}.$$

Any colored permutation $w = w^\epsilon \in \mathfrak{S}_{n,r}$ can be viewed as the n-element chain

$$w_1^{\epsilon_1} <_w w_2^{\epsilon_2} <_w \cdots <_w w_n^{\epsilon_n}$$

[3] One reconstructs the whole permutation of $\Omega_{n,r}$ via the rule (1.4).

with the obvious colored labelling and the set of colored w-partitions can be shown to equal

$$\begin{aligned}\mathcal{A}^{(r)}(w) \;=\; \{f : w \to \mathbb{P}^{(r)} : f(w_1^{\epsilon_1}) \geq f(w_2^{\epsilon_2}) \geq \cdots \geq f(w_n^{\epsilon_n}) \\ \text{and } i \in \mathrm{Des}^*_{<_c}(w^\epsilon) \;\Rightarrow\; f(w_i^{\epsilon_i}) > f(w_{i+1}^{\epsilon_{i+1}})\}.\end{aligned}$$

Therefore, in view of Equation (2.3), we have

$$\Gamma(w; \mathbf{X}^{(r)}) \;=\; F_{w^\epsilon}(\mathbf{X}^{(r)}). \tag{4.3}$$

The *fundamental lemma of colored P-partitions* implies that [62, Corollary 3.6]

$$\Gamma(P; \mathbf{X}^{(r)}) \;=\; F(\mathcal{L}^{(r)}(P); \mathbf{X}^{(r)}) \;=\; \sum_{w \in \mathcal{L}^{(r)}(P)} \Gamma(w; \mathbf{X}^{(r)})$$

where $\mathcal{L}^{(r)}(P)$ is the set of linear extensions of P, viewed as a subset of $\mathfrak{S}_{n,r}$.

Example 4.2.3. The set of linear extensions of the poset P in Example 4.2.2 is

$$\mathcal{L}^{(2)}(P) \;=\; \{2^0 1^0 3^1 4^1,\, 2^0 3^1 1^0 4^1,\, 2^0 3^1 4^1 1^0\}$$

and therefore

$$\Gamma(P) \;=\; F_{(1^0,1^0,2^1)} + F_{(1^0,1^1,1^0,1^1)} + F_{(1^0,2^1,1^0)}.$$

Now, consider two finite, colored posets P, Q of cardinality n and m, respectively. Hsiao and Petersen [62, Section 3.1] in order to introduce the Hopf algebra of colored posets defined a poset $P \sqcup_r Q$ reminiscent of the disjoint sum of usual (uncolored) posets as follows. If as subsets of colored integers P and Q have any elements of the same underlying integer, then replace Q by another colored poset with m elements that is *label-equivalent*[4] to Q and whose elements have different underlying integers from those of P. The poset $P \sqcup_r Q$ is just the disjoint sum $P + Q$. In the case that P and Q have distinct colored labelings, then $P \sqcup_r Q$ is just their disjoint sum and we will simply write $P + Q$. We can define $P_1 \sqcup_r P_2 \sqcup_r \cdots \sqcup_r P_k$ for a finite number of colored posets $P_1, P_2, \ldots, P_k$ in a similar fashion.

Hsiao and Petersen [62, Lemma 3.7] prove that the set of r-colored $(P \sqcup_r Q)$-partitions is in one-to-one correspondence with the cartesian product of $\mathcal{A}^{(r)}(P)$ and $\mathcal{A}^{(r)}(Q)$. This observation implies the following useful formula[5]

$$\Gamma(P \sqcup_r Q; \mathbf{X}^{(r)}) \;=\; \Gamma(P; \mathbf{X}^{(r)})\Gamma(Q; \mathbf{X}^{(r)}). \tag{4.4}$$

If P and Q are two disjoint[6] chains u, v of length n and m, respectively, then the set of colored $(u + v)$-partitions depends only on $\mathrm{sDes}(u)$ and $\mathrm{sDes}(v)$ and $\mathcal{L}^{(r)}(u + v) = u \sqcup\!\sqcup v$. Here, $u \sqcup\!\sqcup v$ denotes the set of all r-colored permutations of length

[4] Two posets are label-equivalent if there exists a color-preserving poset isomorphism which respects the left lexicographic order (for the precise definition we refer to [62, Section 3.1]).

[5] This formula holds for any finite number of disjoint posets.

[6] In the sense that they have distinct labelings.

$n+m$ in which both u and v appear as subsequences. For example, for $u = 1^2 2^0$ and $v = 3^1$, we have

$$\mathcal{L}^{(3)}\left(\begin{array}{cc} 2^0 & \\ | & 3^1 \\ 1^2 & \end{array}\right) = \left\{\begin{array}{ccc} 3^1 & 2^0 & 2^0 \\ | & | & | \\ 2^0, & 3^1, & 1^2 \\ | & | & | \\ 1^2 & 1^2 & 3^1 \end{array}\right\} = \{1^2 2^0 3^1,\ 1^2 3^1 2^0,\ 3^1 1^2 2^0\}.$$

Applying Equation (4.4) in this case leads to a formula for computing the product in $\mathrm{QSym}^{(r)}$ in terms of the fundamental basis which involves shuffles of colored permutations.

Theorem 4.2.4. (Hsiao–Petersen [62, Equation (3.4)]) *Let $c^{\tau}_{\sigma,\varrho}$ be the number of colored permutations $w \in u \sqcup\!\sqcup v$ such that $\mathrm{sDes}(w) = \tau$, where u and v are disjoint colored permutations with $\mathrm{sDes}(u) = \sigma$ and $\mathrm{sDes}(v) = \varrho$. Then*

$$F_\sigma(\mathbf{X}^{(r)})F_\varrho(\mathbf{X}^{(r)}) = \sum_{\tau\in\Sigma(n,r)} c^{\tau}_{\sigma,\varrho} F_\tau(\mathbf{X}^{(r)}). \tag{4.5}$$

4.3 Colored shuffling theorem

For a finite, colored poset P with n elements, we define

$$\begin{aligned} A_P^{\mathrm{eul,mah}}(x,q,\mathbf{p}) &:= \sum_{w\in\mathcal{L}^{(r)}(P)} x^{\mathrm{eul}(w)} q^{\mathrm{mah}(w)} \mathbf{p}^{\mathrm{n}(w)} \\ A_P^{\mathrm{mah}}(q,\mathbf{p}) &:= A_P^{\mathrm{eul,mah}}(1,q,\mathbf{p}) \end{aligned}$$

where eul and mah are an Eulerian and a Mahonian statistic on colored permutations and $\mathrm{n}(w)$ is defined as in Section 2.3.

Suppose P is an antichain with n elements. Each color vector $\epsilon \in \mathbb{Z}_r^n$ determines a colored labeling on P. We write P_ϵ for the one corresponding to ϵ. The discussion at the end of the previous section implies that

$$\mathcal{L}^{(r)}(P_\epsilon) = 1^{\epsilon_1} \sqcup\!\sqcup 2^{\epsilon_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup n^{\epsilon_n},$$

and therefore by Equation (4.4) we have

$$\Gamma(P_\epsilon; \mathbf{X}^{(r)}) = h_1(\mathbf{x}^{(\epsilon_1)}) h_2(\mathbf{x}^{(\epsilon_2)}) \cdots h_n(\mathbf{x}^{(\epsilon_n)}). \tag{4.6}$$

On the other hand, summing over all $\epsilon \in \mathbb{Z}_r^n$ the left-hand side of Equation (4.6) becomes $F(\mathfrak{S}_{n,r};\mathbf{X}^{(r)})$ and therefore

$$\begin{aligned}
F(\mathfrak{S}_{n,r};\mathbf{X}^{(r)}) &= \sum_{\epsilon\in\mathbb{Z}_r^n} F(\mathcal{L}^{(r)}(P_\epsilon);\mathbf{X}^{(r)}) \\
&= \sum_{\substack{1\le i\le n\\ 0\le\epsilon_i\le r-1}} h_1(\mathbf{x}^{(\epsilon_1)})h_2(\mathbf{x}^{(\epsilon_2)})\cdots h_n(\mathbf{x}^{(\epsilon_n)}) \\
&= \prod_{i=1}^{n}\left(h_1(\mathbf{x}^{(0)})+h_1(\mathbf{x}^{(1)})+\cdots+h_1(\mathbf{x}^{(r-1)})\right) \\
&= \left(h_1(\mathbf{x}^{(0)})+h_1(\mathbf{x}^{(1)})+\cdots+h_1(\mathbf{x}^{(r-1)})\right)^n,
\end{aligned}$$

in agreement with Equation (3.1). We write $\mathrm{n}_{=j}(\epsilon),\mathrm{n}_{\le j}(\epsilon),\mathrm{n}_{>j}(\epsilon)$ and $\mathrm{n}_{\neq j}(\epsilon)$ for the number of $1\le i\le n$ for which $\epsilon_i=j,\epsilon_i\le j,\epsilon_i>j$ and $\epsilon_i\neq j$, respectively, for all $0\le j\le r-1$.

Theorem 4.3.1. *For every $\epsilon=(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)\in\mathbb{Z}_r$, we have*

$$A^{\mathrm{maj}}_{P_\epsilon}(x,q,\mathbf{p}) \;=\; p_{\epsilon_1}\cdots p_{\epsilon_n}[n]_q!$$

and

$$\sum_{m\ge0}[m+1]_q^{\mathrm{n}_{=0}(\epsilon)}[m]_q^{\mathrm{n}_{\neq0}(\epsilon)}\prod_{i=1}^{n}p_{\epsilon_i}\,x^m \;=\; \frac{A^{\mathrm{des,maj}}_{P_\epsilon}(x,q,\mathbf{p})}{(x;q)_{n+1}}$$

In addition, assuming the notation of Corollary 3.1.6 we have

$$\sum_{m\ge0}[\mathrm{Q}(m)+1]_{q^r}^{\mathrm{n}_{\le\mathrm{R}(m)}(\epsilon)}[\mathrm{Q}(m)]_{q^r}^{\mathrm{n}_{>\mathrm{R}(m)}(\epsilon)}\prod_{i=1}^{n}p_{\epsilon_i}q^{\epsilon_i}\,x^m \;=\; \frac{[r]_xA^{\mathrm{fdes,fmaj}}_{P_\epsilon}(x,q,\mathbf{p})}{(x^r;q^r)_{n+1}}.$$

Proof. The proof follows by specializing Equation (4.6) as in Theorems 2.3.1 and 2.3.6 and substituting in Equations (2.18), (2.19) and (2.28), respectively, for $\mathcal{A}=\mathcal{L}^{(r)}(P_\epsilon)$. For example, we compute

$$\begin{aligned}
\sum_{m\ge1}\mathrm{ps}^{(r)}_{q,\mathbf{p},m}(\Gamma(P_\epsilon))x^{m-1} &= \sum_{m\ge1}\prod_{i=1}^{n}p_{\epsilon_i}(1+q+\cdots+q^{m-2}+q^{m-1}\chi(\epsilon_i=0))\,x^{m-1} \\
&= \sum_{m\ge1}[m]_q^{\mathrm{n}_{=0}(\epsilon)}[m-1]_q^{\mathrm{n}_{\neq0}(\epsilon)}\prod_{i=1}^{n}p_{\epsilon_i}\,x^{m-1}.
\end{aligned}$$

Substituting this to Equation (2.19) yields the second formula. □

Remark 4.3.2. Notice that

$$\begin{aligned}
A^{\mathrm{eul,mah}}_{n,r}(x,q,\mathbf{p}) &= \sum_{\epsilon\in\mathbb{Z}_r^n} A^{\mathrm{eul,mah}}_{P_\epsilon}(x,q,\mathbf{p}) \\
A^{\mathrm{mah}}_{n,r}(q,\mathbf{p}) &= \sum_{\epsilon\in\mathbb{Z}_r^n} A^{\mathrm{mah}}_{P_\epsilon}(q,\mathbf{p})
\end{aligned}$$

where eul and mah is an Eulerian and a Mahonian statistic on colored permutations. Therefore summing over all $\epsilon\in\mathbb{Z}_r^n$ in both sides of the equations that appear in

Theorem 4.3.1 yields alternative proofs of Equations (3.2), (3.3) and (3.11). For example, we compute

$$\begin{aligned}\sum_{\substack{0\le\epsilon_i\le r-1\\1\le i\le n}}&[m+1]_q^{\mathrm{n}_{=0}(\epsilon)}[m]_q^{\mathrm{n}_{\neq 0}(\epsilon)}\prod_{i=1}^n p_{\epsilon_i}\\&=\sum_{k=0}^n\binom{n}{k}\left(p_0[m+1]_q\right)^k\left((p_1+\cdots+p_{r-1})[m]_q\right)^{n-k}\\&=\left(p_0[m+1]_q+(p_1+\cdots+p_{r-1})[m]_q\right)^n,\end{aligned}$$

which is in agreement with the left-hand side of Equation (3.3).

The following theorem generalizes the first formula of Theorem 4.3.1 to any poset. It also serves as a colored analogue of the second part of Stanely's shuffling theorem (cf. [87, Equation (24)]).

Theorem 4.3.3. *If P and Q are colored posets of cardinality n and m, respectively, then*

$$A^{\mathrm{maj}}_{P\sqcup_r Q}(q,\mathbf{p}) = \binom{n+m}{n}_q A^{\mathrm{maj}}_P(q,\mathbf{p})A^{\mathrm{maj}}_Q(q,\mathbf{p}) \tag{4.7}$$

$$A^{\mathrm{fmaj}}_{P\sqcup_r Q}(q,\mathbf{p}) = \binom{n+m}{n}_{q^r} A^{\mathrm{fmaj}}_P(q,\mathbf{p})A^{\mathrm{fmaj}}_Q(q,\mathbf{p}) \tag{4.8}$$

In particular, for any disjoint r-colored permutations u and v of length n and m, respectively, we have

$$\sum_{w\,\in\, u\sqcup v} q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)} = \binom{n+m}{n}_q q^{\mathrm{maj}(u)+\mathrm{maj}(v)}\,\mathbf{p}^{\mathrm{n}(u)+\mathrm{n}(v)} \tag{4.9}$$

$$\sum_{w\,\in\, u\sqcup v} q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)} = \binom{n+m}{n}_{q^r} q^{\mathrm{fmaj}(u)+\mathrm{fmaj}(v)}\,\mathbf{p}^{\mathrm{n}(u)+\mathrm{n}(v)}, \tag{4.10}$$

where

$$\mathbf{p}^{\mathrm{n}(u)+\mathrm{n}(v)} := \prod_{i=1}^n p_i^{\mathrm{n}_i(u)+\mathrm{n}_i(v)}.$$

Proof. Specializing Equation (4.4) as in Theorems 2.3.1 and 2.3.4 and substituting in Equations (2.18) and (2.24) yields Equations (4.7) and (4.8), respectively. Equations (4.9) and (4.10) are immediate consequences of Equations (4.7) and (4.8), respectively. □

For a mahonian statistic mah on colored permutations, we define

$$A^{\mathrm{des,mah}}_P(x,q,\mathbf{p}) := \sum_{k=0}^n A^{\mathrm{mah}}_{P,k}(q,\mathbf{p})x^k,$$

where

$$A^{\mathrm{mah}}_{P,k}(q,\mathbf{p}) := \sum_{\substack{w\in\mathcal{L}^{(r)}(P)\\ \mathrm{des}(w)=k}} q^{\mathrm{mah}(w)}\mathbf{p}^{\mathrm{n}(w)}.$$

The next theorem provides a colored analogue of the first part of Stanely's shuffling theorem (cf. [87, Proposition 12.6]).

Theorem 4.3.4. *If P and Q are colored posets of cardinality n and m, respectively, then*

$$A^{\mathrm{maj}}_{P\sqcup_r Q}(q,\mathbf{p}) = \sum_{i=0}^{n}\sum_{j=0}^{m} q^{(k-i)(k-j)} \binom{n+j-i}{n-i}_q \binom{n+i-j}{n-j}_q A^{\mathrm{maj}}_{P,i}(q,\mathbf{p}) A^{\mathrm{maj}}_{Q,j}(q,\mathbf{p}) \quad (4.11)$$

$$A^{\mathrm{fmaj}}_{P\sqcup_r Q}(q,\mathbf{p}) = \sum_{i=0}^{n}\sum_{j=0}^{m} q^{r(k-i)(k-j)} \binom{n+j-i}{n-i}_{q^r} \binom{n+i-j}{n-j}_{q^r} A^{\mathrm{fmaj}}_{P,i}(q,\mathbf{p}) A^{\mathrm{fmaj}}_{Q,j}(q,\mathbf{p}) \quad (4.12)$$

In particular, for any disjoint r-colored permutations u and v of length n and m, respectively, we have

$$\sum_{\substack{w\,\in\, u\sqcup v\\ \mathrm{des}(w)=k}} q^{\mathrm{maj}(w)}\mathbf{p}^{\mathrm{n}(w)} = q^{\mathrm{maj}(u)+\mathrm{maj}(v)+(k-\mathrm{des}(u))(k-\mathrm{des}(v))}\mathbf{p}^{\mathrm{n}(u)+\mathrm{n}(v)} \times \binom{n-\mathrm{des}(u)+\mathrm{des}(v)}{k-\mathrm{des}(u)}_q \binom{m-\mathrm{des}(v)+\mathrm{des}(u)}{k-\mathrm{des}(v)}_q \quad (4.13)$$

$$\sum_{\substack{w\,\in\, u\sqcup v\\ \mathrm{des}(w)=k}} q^{\mathrm{fmaj}(w)}\mathbf{p}^{\mathrm{n}(w)} = q^{\mathrm{fmaj}(u)+\mathrm{fmaj}(v)+r(k-\mathrm{des}(u))(k-\mathrm{des}(v))}\mathbf{p}^{\mathrm{n}(u)+\mathrm{n}(v)} \times \binom{n-\mathrm{des}(u)+\mathrm{des}(v)}{k-\mathrm{des}(u)}_{q^r} \binom{m-\mathrm{des}(v)+\mathrm{des}(u)}{k-\mathrm{des}(v)}_{q^r}. \quad (4.14)$$

Proof. Equations (4.13) and (4.14) are immediate consequences of Equations (4.11) and (4.12), respectively and it suffices to prove Equation (4.11). The proof follows the exact same steps as Stanley's proof of [87, Proposition 12.6]. We reconstruct the argument.

We introduce the following notation

$$U^r_s(P;q,\mathbf{p}) \;:=\; \mathrm{ps}^{(r)}_{q,\mathbf{p},s+1}(\Gamma(P;\mathbf{X}^{(r)})).$$

Equation (2.19) for $\mathcal{A}=\mathcal{L}^{(r)}(P)$ becomes

$$\sum_{s\geq 0} U^r_s(P;q,\mathbf{p})x^s \;=\; \frac{A^{\mathrm{des,maj}}_P(x,q,\mathbf{p})}{(x;q)_{n+1}}. \quad (4.15)$$

Recall from [90, Equation (1.87)] the following idendities[7]

$$(x;q)_{n+1} = \sum_{k=0}^{n+1} (-1)^k q^{\binom{k}{2}} \binom{n+1}{k}_q x^m \tag{4.16}$$

$$\frac{1}{(x;q)_{n+1}} = \sum_{s\geq 0} \binom{n+m}{n}_q x^s. \tag{4.17}$$

On the one hand, rewriting Equation (4.15) as

$$A_P^{\mathrm{des,maj}}(x,q,\mathbf{p}) = (x;q)_{n+1} \left(\sum_{s\geq 0} U_s^r(P;q,\mathbf{p}) x^s \right),$$

substituting Equation (4.16) and extracting the coefficient of x^k yields

$$A_{P,k}^{\mathrm{maj}}(q,\mathbf{p}) = \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \binom{n+1}{i}_q U_{k-i}^r(P;q,\mathbf{p}). \tag{4.18}$$

On the other hand, substituting Equation (4.17) in Equation (4.15) and extracting the coefficient of x^m yields

$$U_s^r(P;q,\mathbf{p}) = \sum_{k=0}^{s} \binom{n+k}{n}_q A_{P,s-k}^{\mathrm{maj}}(P). \tag{4.19}$$

Next, applying Equation (4.18) for the poset $P \sqcup_r Q$ of cardinality $n+m$ and substituting Equation (4.19) yields

$$\begin{aligned} A_{P\sqcup_r Q,k}^{\mathrm{maj}}(q,\mathbf{p}) &= \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \binom{n+m+1}{i}_q U_{k-i}^r(P\sqcup_r Q;q,\mathbf{p}) \\ &= \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \binom{n+m+1}{i}_q U_{k-i}^r(P;q,\mathbf{p}) U_{k-i}^r(Q;q,\mathbf{p}) \\ &= \sum_{i=0}^{k} (-1)^i q^{\binom{i}{2}} \binom{n+m+1}{i}_q \times \\ &\quad \left(\sum_{j=0}^{k-i} \binom{n+j}{n}_q A_{P,k-i-j}^{\mathrm{maj}}(q,\mathbf{p}) \right) \left(\sum_{j=0}^{k-i} \binom{m+j}{m}_q A_{Q,k-i-j}^{\mathrm{maj}}(q,\mathbf{p}) \right), \end{aligned} \tag{4.20}$$

where the second equality follows from the fact that the specialization $\mathrm{ps}_{q,\mathbf{p},m}^{(r)}$ is an algebra homomorphism. Now, as Stanley points out the coefficient of

$$A_{P,t}^{\mathrm{maj}}(q,\mathbf{p}) A_{Q,s}^{\mathrm{maj}}(q,\mathbf{p})$$

[7] Equation (4.16) for $x \to -x$ is often called the *q-binomial theorem*.

in the far right-hand side of Equation (4.20) is equal to

$$q^{(k-t)(k-s)}\binom{n+s-t}{n-t}_q\binom{n+t-s}{n-s}_q$$

and the proof follows. □

In view of the bijective proof of Stanley's shuffling theorem that exist in the literature [59, 85] we propose the following problem.

Problem 4.3.5. *Find bijective proofs of Theorems 4.3.3 and 4.3.4.*

4.4 Shuffle-compatible colored permutation statistics

We say that a colored permutation statistic stat is a *colored descent statistic* if it depends only on the colored descent composition of sDes. Examples of colored descent statistics include[8]

$$\mathrm{sDes},\ \mathrm{Des},\ \mathrm{des},\ \mathrm{maj},\ \mathrm{fmaj},\ (\mathrm{des},\mathrm{maj}),\ (\mathrm{fdes},\mathrm{fmaj}),\ \mathrm{csum}$$

as well as the colored peak composition, introduced by Bergeron and Hohlweg [21], which will be defined later in this section. We will now describe the shuffle algebras associated to some of these statistics following the arguments of Gessel and Zhuang [56].

Theorem 4.2.4 implies that $\mathrm{QSym}^{(r)}$ is isomorphic to the shuffle algebra for the colored descent set with the fundamental basis corresponding to the basis of sDes-equivalence classes.

Theorem 4.4.1. *The colored descent set* sDes *is shuffle compatible and the linear map on* $\mathcal{A}_{\mathrm{sDes}}$ *defined by*

$$[w]_{\mathrm{sDes}} \mapsto F_w$$

is a $\mathbb{C}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{sDes}}$ *to* $\mathrm{QSym}^{(r)}$.

Theorem 4.4.2. (a) *The shuffle algebra of* maj *is isomorphic to the shuffle algebra of the uncolored major index as described in [56, Theorem 3.1].*

(b) *The linear map on* $\mathcal{A}_{\mathrm{fmaj}}$ *defined by*

$$[w]_{\mathrm{fmaj}} \mapsto \frac{q^{\mathrm{fmaj}(w)}}{[|w|]_{q^r}!}\,x^{|w|}$$

is a $\mathbb{C}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{fmaj}}$ *to the span of*

$$\left\{\frac{q^j}{[n]_{q^r}!}\,x^n : n \geq 0,\ 0 \leq j \leq r\binom{n}{2} + n(r-1)\right\},$$

a subalgebra of $\mathbb{C}[[q]][x]$. *The* n*-th homogeneous component of* $\mathcal{A}_{\mathrm{fmaj}}$ *has dimension* $r\binom{n}{2} + n(r-1) + 1$.

[8] We assume that they are computed using the color order, although it does not really depend on the total order of $\Omega_{n,r}$.

Proof. The proof follows the exact steps of the proof of [56, Theorem 3.1] but one needs to apply Equation (4.10) for $p_0 = p_1 = \cdots = p_{r-1} = 1$ instead of Stanley's shuffling theorem. The possible values for the flag major index for r-colored permutations of length n range from 0 to $r\binom{n}{2} + n(r-1)$, the latter being attained for $n^{r-1} \cdots 2^{r-1} 1^{r-1}$. □

Following [56], we denote by $\mathbb{C}[[x*, q]][t]$ the algebra of polynomials in t whose coefficients are formal power series in x and q, where multiplication is ordinary multiplication in the variables t and q, but is the Hadamard product in x. More precisely, the *Hadamard product* $*$ on formal power series in x is given by

$$\left(\sum_{n\geq 0} a_n x^n\right) * \left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} a_n b_n x^n.$$

In addition, we write $\mathbb{C}[q,t]^{\mathbb{N}}$ for the algebra of functions $\mathbb{N} \to \mathbb{C}[q,t]$.

Theorem 4.4.3. (a) *The linear map on $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ defined by*

$$[w]_{(\mathrm{des},\mathrm{maj})} \mapsto q^{\mathrm{maj}(w)} \binom{m - \mathrm{des}(w) + |w| - 1}{|w|}_q t^{|w|}$$

is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ to the span of

$$\{1\} \bigcup \left\{ q^k \binom{m-j+n-1}{n}_q t^n : n \geq 1,\ 0 \leq j \leq n, \binom{j}{2} \leq k \leq nj - \binom{j+1}{2} \right\},$$

a subalgebra of $\mathbb{C}[q,t]^{\mathbb{N}}$[9].

(b) *The linear map on $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ defined by*

$$[w]_{(\mathrm{des},\mathrm{maj})} \mapsto \begin{cases} \dfrac{x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(x;q)_{|w|+1}} t^{|w|}, & \text{if } |w| \geq 1 \\ \dfrac{1}{1-x}, & \text{if } |w| = 0 \end{cases}$$

is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ to the span of

$$\left\{\frac{1}{1-x}\right\} \bigcup \left\{ \frac{x^{j+1} q^k}{(x;q)_{n+1}} t^n : n \geq 1,\ 0 \leq j \leq n \ \binom{j}{2} \leq k \leq nj - \binom{j+1}{2} \right\},$$

a subalgebra of $\mathbb{C}[[x*, q]][t]$.

[9] Functions $\mathbb{N} \to \mathbb{C}[q,x]$ are understood in the variable m.

(c) *The n-th homogeneous component of $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ has dimension $\binom{n+1}{3}+n+1$.*

The proof is essentially a trivial generalization to the colored case of Gessel and Zhuang's proof of [56, Theorem 4.3]. The nontrivial part is to replace the principal specialization of order m with the specialization $\mathrm{ps}^{(r)}_{q,\mathbf{p},m}$ for $p_0 = p_1 = \cdots = p_{r-1} = 1$. We record a criterion for the shuffle-compatibility of a colored descent statistic, which is a colored version of [56, Theorem 4.3].

Lemma 4.4.4. *A colored descent statistic* stat *is shuffle-compatible if and only if there exists a $\mathbb{C}$-algebra homomorphism $\varphi_{\mathrm{stat}} : \mathrm{QSym}^{(r)} \to A$, where A is a $\mathbb{C}$-algebra with basis $\{u_\alpha\}$ indexed by* stat*-equivalence classes α of r-colored compositions*[10]*, such that $\varphi_{\mathrm{stat}}(F_\gamma) = u_\alpha$ whenever γ belongs in the class α. In this case, the linear map on $\mathcal{A}_{\mathrm{stat}}$ defined by*

$$[w]_{\mathrm{stat}} \mapsto u_\alpha$$

where $\mathrm{co}(w)$ belongs in the class α is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{\mathrm{stat}}$ to A.

We are ready to prove Theorem 4.4.3.

Proof of Theorem 4.4.3. We first prove (a). Our goal is to apply Lemma 4.4.4 for A being the subalgebra of $\mathbb{C}[q,t]^{\mathbb{N}}$ spanned by

$$\{1\} \bigcup \left\{ q^k \binom{m-j+n-1}{n}_q t^n : n \geq 1,\ 0 \leq j \leq n \right.$$
$$\left. \binom{j}{2} \leq k \leq nj - \binom{j+1}{2} \right\}.$$

For a positive integer m and an r-colored quasisymmetric function f we define

$$\Phi^{(m)}_{(\mathrm{des},\mathrm{maj})}(f) \;=\; \mathrm{ps}^{(r)}_{m,q}(f)\, t^{\deg(f)},$$

where $\mathrm{ps}^{(r)}_{m,q}$ is just $\mathrm{ps}^{(r)}_{m,q,\mathbf{p}}$ when $p_0 = p_1 = \cdots = p_{r-1} = 1$ and $\deg(f)$ denotes the degree of f. Also, let $\Phi^{(0)}_{(\mathrm{des},\mathrm{maj})}(f)$ be the constant term in f. Since $\Phi^{(m)}_{(\mathrm{des},\mathrm{maj})} : \mathrm{QSym}^{(r)} \to \mathbb{C}[q,t]$ is a homomorphism, the map $\varphi_{(\mathrm{des},\mathrm{maj})} : \mathrm{QSym}^{(r)} \to \mathbb{C}[q,t]^{\mathbb{N}}$ that takes a colored quasisymmetric function f to the function

$$m \;\mapsto\; \Phi^{(m)}_{(\mathrm{des},\mathrm{maj})}(f),$$

is an algebra homomorphism.

The proof of Equation (2.12) and Remark 2.3.7 imply that

$$\varphi_{(\mathrm{des},\mathrm{maj})}(F_\gamma(\mathbf{X}^{(r)}))(m) \;=\; q^{\mathrm{sum}(\mathrm{Des}(\gamma))} \binom{m - |\mathrm{Des}(\gamma)| + n - 1}{n}_q t^n$$

[10] A colored descent statistic stat induces an equivalence relation on colored compositions, since colored permutations with the same colored descent composition are necessarily stat-equivalent.

for every r-colored composition γ of n and all $m \geq 1$ and $\varphi_{(\mathrm{des},\mathrm{maj})}(F_{\varnothing}(\mathbf{X}^{(r)}))(0) = 1$. Furthermore, similarly to the classical case, we have

$$\sum_{m\geq 0} \binom{m - |\operatorname{Des}(\gamma)| + n - 1}{n}_q x^m = \sum_{m \geq 0} \binom{m+n}{n}_q x^{m+|\operatorname{Des}(\gamma)|+1} = \frac{x^{|\operatorname{Des}(\gamma)|+1}}{(x;q)_{n+1}} \tag{4.21}$$

for every r-colored composition γ of n and therefore the functions

$$m \mapsto q^k \binom{m-j+n-1}{n}_q t^m$$

are linearly independent. The proof follows from Lemma 4.4.4 and the fact that (cf. [56, Proposition 2.4])

- for any $w \in \mathfrak{S}_{n,r}$ with $\operatorname{des}(w) = j$, we have[11]
$$\binom{j}{2} \leq \operatorname{maj}(w) \leq nj - \binom{j+1}{2},$$
- and if $j \in [0, n-1]$ and $\binom{j}{2} \leq k \leq nj - \binom{j+1}{2}$, then there exists $w \in \mathfrak{S}_{n,r}$ such that $\operatorname{des}(w) = j$ and $\operatorname{maj}(w) = k$.

To prove (b), we notice that the map $\mathbb{C}[q,t]^{\mathbb{N}} \to \mathbb{C}[q,t][[x*]]$ defined by

$$f \mapsto \sum_{m \geq 0} f(m)\, x^m$$

is an isomorphism and by Equation (4.21), the images of the basis elements in (a) are those given in (b), which belong to $\mathbb{C}[[x*, q]][t]$.

The dimension of the n-th homogeneous component of $\mathcal{A}_{(\mathrm{des},\mathrm{maj})}$ is

$$\sum_{j=0}^{n} \left(\left(nj - \binom{j+1}{2} \right) - \binom{j}{2} + 1 \right) = \sum_{j=0}^{n} (nj - j^2 + 1) = \binom{n+1}{3} + n + 1,$$

which settles (c). □

Theorem 4.4.5. (a) *The linear map on $\mathcal{A}_{\mathrm{des}}$ defined by*

$$[w]_{\mathrm{des}} \mapsto \binom{m - \operatorname{des}(w) + |w| - 1}{|w|} t^{|w|}$$

is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{\mathrm{des}}$ to the span of

$$\{1\} \bigcup \left\{ \binom{m-j+n-1}{n} t^n : n \geq 1,\ 0 \leq j \leq n \right\},$$

a subalgebra of $\mathbb{C}[m,t]$.[12].

[11] Among all r-colored permutations with j descents, the smallest possible value of the major index is attained when the descent set is $[0, j-1]$.

[12] Polynomial functions in characteristic zero may be indentified with polynomials.

(b) $\mathcal{A}_{\mathrm{des}}$ *is isomorphic to the span of*

$$\{1\} \cup \{(r(m-1)+1)^j\, t^n : n \geq 1,\ 1 \leq j \leq n+1\},$$

a subalgebra of $\mathbb{C}[m,t]$.

(c) *The linear map on* $\mathcal{A}_{\mathrm{des}}$ *defined by*

$$[w]_{\mathrm{des}} \ \mapsto \ \begin{cases} \dfrac{x^{\mathrm{des}(w)+1}}{(1-x)^{|w|+1}}\, t^{|w|}, & \textit{if } |w| \geq 1 \\ \dfrac{1}{1-x}, & \textit{if } |w| = 0 \end{cases}$$

is a $\mathbb{C}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{des}}$ *to the span of*

$$\left\{\frac{1}{1-x}\right\} \bigcup \left\{\frac{x^{j+1}}{(1-x)^{n+1}}\, t^n : n \geq 1\ 0 \leq j \leq n\right\},$$

a subalgebra of $\mathbb{C}[[x*]][t]$.

(d) *The* n*-th homogeneous component of* $\mathcal{A}_{\mathrm{des}}$ *has dimension* $n+1$.

As with the proof of Theorem 4.4.3, the proof of Theorem 4.4.5 is essentially a trivial generalization to the colored case of Gessel and Zhuang's proof of [56, Theorem 4.6]. The nontrivial part is the specialization. We omit the proof, but notice that it uses a colored version of [56, Theorem 3.3] and Theorem 4.4.5. One can prove a similar description for the shuffle algebra of the flag descent number.

Stembridge [95] studied a subalgebra of QSym using a variant of P-partitions, called *enriched* P*-partitions*. A composition of n such that all but its last part must greater than 1 is called *peak composition*. These compositions appear as the compositions associated to the peak set of some permutation. For example, for $w = 87154623 \in \mathfrak{S}_8$ we have $\mathrm{Pk}(w) = \{4,6\}$ and the corresponding peak composition is $\mathrm{co}(\mathrm{Pk}(w)) = (4,2,2)$. The number of peak compositions of n equals the n-th Fibonacci number (see, for example, [62, Section 2.5]). Starting from a composition α of n, one can obtain a peak composition $\widehat{\alpha}$ of n by replacing consecutive 1s with their sum added to the next part to the right:

$$(\dots, \underbrace{\alpha_i}_{>1}, \underbrace{1,1,\dots,1}_{m}, \underbrace{\alpha_{i+m+1}}_{>1}, \dots) \ \mapsto \ (\dots, \alpha_i, \alpha_{i+m+1}+m, \dots).$$

In our running example, we have $\mathrm{co}(w) = (1,1,2,2,2)$ and therefore $\widehat{\mathrm{co}}(w) = (4,2,2)$, in agreement with our previous computation.

For $\alpha \in \mathrm{Comp}(n)$, let

$$K_\alpha(\mathbf{x}) \ = \sum_{\alpha \preceq \beta^*} 2^{\ell(\beta)} M_\beta(\mathbf{x}),$$

where β^* is the refinement of β obtained by replacing every part $\beta_i \geq 2$ with two parts $(1, \beta_i - 1)$ for all $i > 1$. Let Π_n be the subalgebra of QSym_n spanned by the set of all $K_\alpha(\mathbf{x})$ for a peak composition α of n. Stembridge [95] proved that the map QSym $\to \Pi$ defined by

$$F_\alpha(\mathbf{x}) \;\mapsto\; K_{\hat{\alpha}}(\mathbf{x})$$

is a surjective algebra homomorphism, where $\Pi := \mathbb{C} \oplus \Pi_1 \oplus \Pi_2 \oplus \cdots$. This is called the *algebra of peak functions* and each $K_\alpha(\mathbf{x})$ indexed by a peak composition α is called a *peak function*.

Peak functions arise as generating functions of enriched P-partitions. The theory of P-partitions provides a rule for multiplying peak functions which involves shuffles of permutations. In particular, for two disjoint permutations u and v of length n and m, respectively, we have

$$K_{n,\mathrm{Pk}(u)}(\mathbf{x}) K_{m,\mathrm{Pk}(v)}(\mathbf{x}) \;=\; \sum_{w \in u \sqcup v} K_{n+m,\mathrm{Pk}(w)}(\mathbf{x}), \tag{4.22}$$

where $K_{n,\mathrm{Pk}(u)}(\mathbf{x}) := K_{\mathrm{co}(\mathrm{Pk}(u))}(\mathbf{x})$. Gessel and Zhuang [56, Theorem 4.7] noticed that Equation (4.22) implies that the peak set Pk is shuffle-compatible and that the linear map on $\mathcal{A}_{\mathrm{Pk}}$ defined by

$$[w]_{\mathrm{Pk}} \;\mapsto\; K_{|w|,\mathrm{Pk}(w)}(\mathbf{x})$$

is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{\mathrm{Pk}}$ to Π.

Begeron and Hohlweg [21] introduced a generalization of Stembridge's peak algebra, introducing the notion of colored peak composition[13]. Our goal for the remainder of this section is to derive a colored analogue of Gessel and Zhuang's result for the shuffle compatibility of the colored peak composition.

Following [21, Section 2] and [62, Section 3.6], the *rainbow decomposition* of an r-colored composition γ of n is the unique expression

$$\gamma \;=\; \gamma_{(1)}^{\epsilon(1)} \gamma_{(2)}^{\epsilon(2)} \cdots \gamma_{(k)}^{\epsilon(k)},$$

of γ as the concatenation of compositions $\gamma_{(i)}$, such that all the parts of $\gamma_{(i)}$ have the same color $\epsilon_{(i)}$ and no two consecutive $\gamma_{(i)}$ have the same color. For example, for $n = 12$ and $r = 6$ we have

$$(3^4, 1^3, 1^1, 3^1, 2^0, 2^0) \;=\; (3^4)(1^3)(1^1, 3^1)(2^0, 2^0).$$

An *r-colored peak composition* is an r-colored composition such that each part of its rainbow decomposition, when forgetting the color, is a peak composition. For example, the 6-colored composition we considered in the previous example is not a colored peak composition, because $(1,3)$ is not a peak composition. Colored peak compositions arise as peak compositions of colored permutations in the following way. For $w^\epsilon \in \mathfrak{S}_{n,r}$, we write[14]

$$w \;=\; w_{(1)}^{\epsilon(1)} w_{(2)}^{\epsilon(2)} \cdots w_{(k)}^{\epsilon(k)},$$

[13] Petersen studied a subalgebra of Chow's type B quasisymmetric functions, which is isomorphic to the shuffle algebra of the left peak set. For more information we refer to [56, Section 4.4] and references therein.

[14] This is essentially the rainbow decomposition of a colored permutation.

where each $w_{(i)}^{\epsilon(i)}$ is an r-colored permutation in which each letter has the same color $\epsilon_{(i)}$ and no two consecutive words have the same color, that is $\epsilon_{(i)} \neq \epsilon_{(i+1)}$. We define the *peak composition* of w^ϵ, written $\widehat{\mathrm{co}}(w^\epsilon)$, as the concatenation of the peak compositions of each $w_{(i)}$ with color $\epsilon_{(i)}$. For example, the 6-colored permutation

$$w \;=\; \underbrace{5^4 8^4 9^4}_{w_{(1)}^4}\;\underbrace{1^3}_{w_{(2)}^3}\;\underbrace{6^1 4^1 10^1 12^1}_{w_{(3)}^1}\;\underbrace{2^0 7^0 3^0 11^0}_{w_{(4)}^0}$$

has peak composition

$$\widehat{\mathrm{co}}(w) \;=\; (3^4)(1^3)(4^1)(2^0,2^0).$$

Starting from a colored composition one can obtain a colored peak composition by applying the operation $\widehat{\ }$ to every part of its rainbow decomposition. For example,

$$(3^4,1^3,1^1,3^1,2^0,2^0) \;\overset{\widehat{\ }}{\longmapsto}\; (3^4)(1^3)(4^1)(2^0,2^0).$$

For $\gamma = \gamma_{(1)}^{\epsilon(1)}\gamma_{(2)}^{\epsilon(2)}\cdots\gamma_{(k)}^{\epsilon(k)} \in \mathrm{Comp}(n,r)$, let

$$K_\gamma(\mathbf{X}^{(r)}) \;=\; \sum_{\gamma_{(i)}^{\epsilon(i)} \preceq \beta_{(i)}^{*\,\epsilon(i)}} 2^{\ell(\beta_{(1)})+\ell(\beta_{(2)})+\cdots+\ell(\beta_{(k)})} M_{\beta_{(1)}^{\epsilon(1)}\beta_{(2)}^{\epsilon(2)}\cdots\beta_{(k)}^{\epsilon(k)}}(\mathbf{X}^{(r)}).$$

Let $\Pi_n^{(r)}$ be the subalgebra of $\mathrm{QSym}_n^{(r)}$ spanned by the set of all $K_\gamma(\mathbf{X}^{(r)})$ for an r-colored peak composition γ of n. Bergeron and Hohlweg [21, Theorem 5.3] proved that the map $\mathrm{QSym}^{(r)} \to \Pi^{(r)}$ defined by

$$F_\gamma(\mathbf{X}^{(r)}) \;\mapsto\; K_{\hat{\gamma}}(\mathbf{X}^{(r)})$$

is a surjective algebra homomorphism, where $\Pi^{(r)} := \mathbb{C} \oplus \Pi_1^{(r)} \oplus \Pi_2^{(r)} \oplus \cdots$. This is called the *algebra of colored peak functions* and each $K_\gamma(\mathbf{X}^{(r)})$, indexed by a colored peak composition γ, is called a *colored peak function*.

Hsiao and Petersen [62, Section 3.7] developed a variant of their colored P-partitions, called *colored enriched P-partitions*, in which colored peak functions arise as generating functions in a similar fashion to the uncolored case. The theory of colored enriched P-partitions provides a rule [62, Equation 3.13] for multiplying colored peak functions which involves shuffles of colored permutations. In particular, for two disjoint r-colored permutations u and v of length n and m, we have

$$K_{\mathrm{sPk}(u)}(\mathbf{X}^{(r)})K_{\mathrm{sPk}(v)}(\mathbf{X}^{(r)}) \;=\; \sum_{w\in u\sqcup v} K_{\mathrm{sPk}(w)}(\mathbf{X}^{(r)}) \;\in\; \Pi_{n+m}^{(r)}. \tag{4.23}$$

where $K_{n,\mathrm{sPk}(u)}(\mathbf{X}^{(r)}) := K_{\mathrm{co}(\mathrm{sPk}(u))}(\mathbf{X}^{(r)})$ and $\mathrm{sPk}(u)$ is the r-colored subset of $[n]$ corresponding to the peak composition of u, called the *colored peak set* of u. Equation (4.23) implies the following theorem.

Theorem 4.4.6. *The colored peak set* sPk *is shuffle-compatible. In addition, the linear map on $\mathcal{A}_{\mathrm{sPk}}$ defined by*

$$[w]_{\mathrm{sPk}} \;\mapsto\; K_{\mathrm{sPk}}(\mathbf{X}^{(r)})$$

is a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}_{\mathrm{sPk}}$ to $\Pi^{(r)}$.

Chapter 5

Descent representations and quasisymmetric functions

5.1 Combinatorial representation theory of colored permutation groups

The (complex) character theory of colored permutation groups was developed by Wilhelm Specht [84]. For a complete account of it we refer to [19, Section 4.2], [60, Section 4.5] and [93, Section 4]. Irreducible complex $\mathfrak{S}_{n,r}$-characters are indexed by r-partite partitions of n. Let $\mathcal{R}(\mathfrak{S}_{n,r})$ be the $\mathbb{Z}$-module generated by irreducible $\mathfrak{S}_{n,r}$-characters and let

$$\mathcal{R}(\mathfrak{S}^{(r)}) = \mathbb{Z} \oplus \mathcal{R}(\mathfrak{S}_{1,r}) \oplus \mathcal{R}(\mathfrak{S}_{2,r}) \oplus \cdots .$$

As in the uncolored case, the $\mathbb{Z}$-module $\mathcal{R}(\mathfrak{S}^{(r)})$ has a ring structure induced by the induction product. The induction product $f \circ g$ of an $\mathfrak{S}_{i,r}$-character and an $\mathfrak{S}_{j,r}$-character g is defined by

$$f \circ g \ := \ (f \otimes g) \uparrow_{\mathfrak{S}_{i,r} \times \mathfrak{S}_{j,r}}^{\mathfrak{S}_{i+j,r}},$$

where $\mathfrak{S}_{i,r} \times \mathfrak{S}_{j,r} \cong \mathbb{Z}_r \wr \mathfrak{S}_{i+j}$ is viewed as a subgroup of $\mathfrak{S}_{i+j,r}$. The ring $\mathcal{R}(\mathfrak{S}^{(r)})$ is closely connected to $\mathrm{Sym}^{(r)}$ via a colored version of the Frobenius characteristic map. We will describe Poirier's version of the colored characteristic map [76, Section 2], which is a variant of Macdonald's [64, Appendix B][1] (see, for example, [60, Chapter 4] and [19, Section 4.2]). For this we need to introduce some notation.

As we discussed in Section 1.4.2, conjugacy classes in $\mathfrak{S}_{n,r}$ are in one-to-one correspondence with r-partite partitions of n. We now describe the cycle type of a colored permutation. Starting with a colored permutation, first form the cycle decomposition of the underlying permutation and then provide the entries with their

[1]Stembridge [94, Section 5] considered yet another variation of Macdonald's map in the case $r = 2$.

original color, forming *colored cycles*. Then, define the color of a colored cycle to be the sum of colors of all its entries computed modulo r. Now, the cycle type of $w \in \mathfrak{S}_{n,r}$, written $\mathbf{ct}(w)$, is the r-partite partition of n, whose j-th part is the integer partition formed by the lengths of the colored cycles of w having color j, for every $0 \leq j \leq r-1$. For example,

$$w \;=\; 6^0 2^5 4^4 3^1 1^6 5^3 \;=\; \underbrace{(1^6\ 6^0\ 5^3)}_{\text{1-colored cycle}}\ \underbrace{(3^1\ 4^4)}_{\text{5-colored cycle}}\ \underbrace{(2^5)}_{\text{5-colored cycle}} \;\in \mathfrak{S}_{6,8}$$

has cycle type $(\varnothing, (3), \varnothing, \varnothing, (2,1), \varnothing, \varnothing)$. It is well known that colored permutations are conjugate if and only if they have the same cycle type (see, for example, [76, Proposition 1]).

Fix ζ a primitive r-th root of unity. For a nonnegative integer k and any $0 \leq j \leq r-1$, let

$$p_k^{(j)}(\mathbf{X}^{(r)}) \;:=\; p_k(\mathbf{x}^{(0)}) + \zeta^j p_k(\mathbf{x}^{(1)}) + \cdots + \zeta^{j(r-1)} p_k(\mathbf{x}^{(r-1)}) \;\in\; \mathrm{Sym}^{(r)}.$$

For example, for $r = 2$, we have $\zeta = -1$ and therefore[2]

$$\begin{aligned} p_k^{(0)}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}) &= p_k(\mathbf{x}^{(0)}) + p_k(\mathbf{x}^{(1)}) \\ p_k^{(1)}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}) &= p_k(\mathbf{x}^{(0)}) - p_k(\mathbf{x}^{(1)}). \end{aligned}$$

Also, for an integer partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ and any $0 \leq j \leq r-1$, let

$$p_\lambda^{(j)}(\mathbf{X}^{(r)}) \;:=\; p_{\lambda_1}^{(j)}(\mathbf{X}^{(r)}) p_{\lambda_2}^{(j)}(\mathbf{X}^{(r)}) \cdots \;\in\; \mathrm{Sym}_n^{(r)}.$$

To an r-partite partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r-1)})$ of n we associate the following element of $\mathrm{Sym}_n^{(r)}$

$$p_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}) \;:=\; p_{\lambda^{(0)}}^{(0)}(\mathbf{X}^{(r)}) p_{\lambda^{(1)}}^{(1)}(\mathbf{X}^{(r)}) \cdots p_{\lambda^{(r-1)}}^{(r-1)}(\mathbf{X}^{(r)})$$

which we call the *colored power sum symmetric function*. The set $\{p_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \vdash n\}$ forms yet another basis for $\mathrm{Sym}_n^{(r)}$.

The *colored Frobenius characteristic map* $\mathrm{ch}_r : \mathcal{R}(\mathfrak{S}^{(r)}) \to \mathrm{Sym}^{(r)}$ defined by

$$\mathrm{ch}_r(\chi)(\mathbf{X}^{(r)}) \;:=\; \frac{1}{r^n n!} \sum_{w \in \mathfrak{S}_{n,r}} \chi(w) p_{\mathbf{ct}(w^{-1})}(\mathbf{X}^{(r)}) \tag{5.1}$$

is a ring isomorphism[3] with the property that [76, Section 2]

$$\mathrm{ch}_r(\chi^{\boldsymbol{\lambda}})(\mathbf{X}^{(r)}) \;=\; s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}),$$

where $\chi^{\boldsymbol{\lambda}}$ is the irreducible $\mathfrak{S}_{n,r}$-character associated to the r-partite partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r-1)})$ of n.

[2] They are denoted by p_k^+ and p_k^- respectively in [1, Section 2.3].

[3] As in the uncolored case, we view $\mathrm{Sym}^{(r)}$ as a $\mathbb{Z}$-algebra for which $p_n^{(j)}$, for all $n \geq 0$, $0 \leq j \leq r-1$ are algebraically independent generators (see Section 2.1). In addition, the elements $s_{\boldsymbol{\lambda}}$ form a basis of the $\mathbb{Z}$-module $\mathrm{Sym}^{(r)}$ (see the discussion in [19, page 1501]).

We are in position now to prove Lemma 3.1.1, as we promised at the beginning of Chapter 3.

Proof of Lemma 3.1.1. The Frobenius formula [15, Theorem 6.1] for $\mathfrak{S}_{n,r}$ states

$$p_{\boldsymbol{\mu}}(\mathbf{X}^{(r)}) \;=\; \sum_{\boldsymbol{\lambda}\vdash n} \chi^{\boldsymbol{\lambda}}(\boldsymbol{\mu}) s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}), \tag{5.2}$$

where $\chi^{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is the value of $\chi^{\boldsymbol{\lambda}}$ at the elements of $\mathfrak{S}_{n,r}$ of cycle type $\boldsymbol{\mu} \vdash n$ (see [89, Equation (7.76)] for the $r = 1$ case and [1, Equation (2.6)] for the $r = 2$ case). Notice that the conjugacy class of the identity element in $\mathfrak{S}_{n,r}$ corresponds to the r-partite partition of n, whose part of color 0 is (1^n) and all the other parts are the empty partitions, written $(1^n, \varnothing^{r-1})$. Therefore, we have

$$p_{(1^n,\varnothing^{r-1})}(\mathbf{X}^{(r)}) \;=\; p^{(0)}_{(1^n)}(\mathbf{X}^{(r)}) \;=\; \left(p^{(0)}_1(\mathbf{X}^{(r)})\right)^n \;=\; \left(p_1(\mathbf{x}^{(0)}) + \cdots + p_1(\mathbf{x}^{(r-1)})\right)^n,$$

which is exactly the right-hand side of Equation (3.1). Furthermore, $\chi^{\boldsymbol{\lambda}}(1^n, \varnothing^{r-1})$ equals the dimension of $\chi^{\boldsymbol{\lambda}}$, which is known to equal the number of r-partite standard Young tableaux of shape $\boldsymbol{\lambda}$ (see, for example, [5, Section 2]). Therefore, Equation (5.2) for $\boldsymbol{\mu} = (1^n, \varnothing^{r-1})$ becomes

$$p_{(1^n,\varnothing^{r-1})}(\mathbf{X}^{(r)}) = \sum_{\boldsymbol{\lambda}\vdash n} \sum_{\boldsymbol{P},\boldsymbol{Q}\in \mathrm{SYT}(\boldsymbol{\lambda})} F_{\boldsymbol{Q}}(\mathbf{X}^{(r)}),$$

using the expansion (2.5). This in turn is exactly the left-hand side of Equation (3.1) by the colored Robinson–Schensted correspondence and its properties. The proof follows by combining the two calculations. $\square$

There are $2r$ one-dimensional $\mathfrak{S}_{n,r}$-characters (see, for example, [22, Section 4]), which are of the form

$$\chi_{\pm,j}(w^{\epsilon}) \;=\; (\pm 1)^{\mathrm{inv}(w)} \zeta^{j\,\mathrm{csum}(w^{\epsilon})}$$

where $\mathrm{inv}(w) := |\{1 \le i < j \le n : w(i) > w(j)\}|$ is the *inversion number* of w for every $0 \le j \le r-1$ and $w^{\epsilon} \in \mathfrak{S}_{n,r}$. The character[4] $\mathbb{1}_{n,j} := \chi_{+,j}$ corresponds to the irreducible $\mathfrak{S}_{n,r}$-character corresponding to $\chi^{\varnothing,\ldots,(n),\ldots,\varnothing}$, where the nonempty part occurs in the jth color.

For $w \in \mathfrak{S}_{n,r}$, recall that $\mathrm{c}^j(w)$ denotes the number of j-colored cycles, for all $0 \le j \le r-1$. The color sum of w is also given by

$$\mathrm{csum}(w) \;=\; \sum_{j=0}^{r-1} j\, \mathrm{c}^j(w).$$

Therefore, $\mathbb{1}_{n,j}$ acts on the conjugacy class $\mathrm{K}(\boldsymbol{\lambda})$ of $\mathfrak{S}_{n,r}$ corresponding to $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ as follows

$$\mathbb{1}_{n,j}(\boldsymbol{\lambda}) \;=\; \zeta^{j\left(\ell(\lambda^{(1)})+2\ell(\lambda^{(2)})+\cdots+(r-1)\ell(\lambda^{(r-1)})\right)}. \tag{5.3}$$

[4] For $r = 2$, the $\mathfrak{B}_n$-character $\chi_{+,1}$ corresponding to the bipartition $(\varnothing, (n))$ is sometimes called the *parity character* (see, for example, [50, Section II] and [94, Section 1], where it is denoted by δ).

We will now examine what is the effect of tensoring some $\mathfrak{S}_{n,r}$-character with $\mathbb{1}_{n,j}$ on the level of elements of $\mathrm{Sym}^{(r)}$ via the colored characteristic map. But first, let us recall some facts about tensor products (see [48] and [60, Section 4.1]).

Representation Theory Digression. Let G be a finite group. Given two finite, complex G-representations V and U, G acts on their tensor product $V \otimes U$ via

$$g \cdot v \otimes u \;:=\; (g \cdot v, g \cdot u).$$

This is called the *diagonal action* of G on $V \otimes U$. The resulting G-representation, denoted also by $V \otimes U$, is sometimes called the *inner tensor product* (or *Kronecker product*) of V and U. The character $\chi^{V\otimes U}$ of $V \otimes U$ is given by the product $\chi^V \chi^U$ of the characters χ^V and χ^U of V and U, respectively. Sometimes, to emphasize this fact it is convinient to write $\chi^{V\otimes U} \;:=\; \chi^V \otimes \chi^U$.

Now, let G and G' be finite groups and let V (resp. V') be a finite, complex G (resp. G')-representation. The direct product $G \times G'$ acts on the tensor product $V \otimes V'$ via

$$(g, g') \cdot (v, v') \;:=\; (g \cdot v, g' \cdot v').$$

The resulting $G \times G'$-representation, denoted by $V \boxtimes U$, is sometimes called the *outer tensor product* of V and U. The (pointwise) product $\phi \times \psi \in \mathrm{CF}(G \times G')$ of two characters ϕ of G and ψ of G', given by

$$(\phi \times \psi)(g, g') \;=\; \phi(g)\psi(g'),$$

where $\mathrm{CF}(G \times G')$ is the space of (complex) class functions of $G \times G'$ is also a character corresponding to some outer tensor product just described. In general, if V and V' are irreducible G and G'-characters respectively, then $V \boxtimes V'$ is an irreducible $G \times G'$-character and every irreducible $G \times G'$ arises in this way (see, for example, [48, Exercise 2.36]).

For a color $j \in \mathbb{Z}_r$, we define an operator shift_j on colored partitions and elements of $\mathbb{C}[[\mathbf{X}^{(r)}]]$ as follows

$$\begin{aligned}
\mathrm{shift}_j\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)}\right) \;&:=\; (\lambda^{(j)}, \lambda^{(j+1)}, \ldots, \lambda^{(j-1)}) \\
\mathrm{shift}_j\left(f(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)})\right) \;&:=\; f(\mathbf{x}^{(j)}, \mathbf{x}^{(j+1)}, \ldots, \mathbf{x}^{(j-1)}),
\end{aligned}$$

for each r-partite partition $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ and $f \in \mathbb{C}[[\mathbf{X}^{(r)}]]$, where the exponents are computed modulo r.

Lemma 5.1.1. *For every partition λ, we have $\mathrm{shift}_j(p_\lambda^{(0)}) = p_\lambda^{(0)}$ and*

$$\mathrm{shift}_j(p_\lambda^{(i)}) \;=\; \zeta^{j(r-i)\ell(\lambda)} p_\lambda^{(i)}$$

for every $0 \le j \le r-1$ and $1 \le i \le r-1$.

Proof. The first equality is obvious by the definition of $p_\lambda^{(0)}$. For every $1 \le i \le r-1$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ we have

$$\begin{aligned}
\zeta^{j(r-i)\ell(\lambda)} p_\lambda^{(i)} &= \zeta^{j(r-i)\ell(\lambda)} \prod_{k=1}^{\ell(\lambda)} \left(p_{\lambda_k}(\mathbf{x}^{(0)}) + \zeta^i p_{\lambda_k}(\mathbf{x}^{(1)}) + \cdots + \zeta^{(r-1)i} p_{\lambda_k}(\mathbf{x}^{(r-1)}) \right) \\
&= \prod_{k=1}^{\ell(\lambda)} \Big(\zeta^{j(r-i)} p_{\lambda_k}(\mathbf{x}^{(0)}) + \zeta^{j(r-i)+i} p_{\lambda_k}(\mathbf{x}^{(1)}) + \cdots \\
&\qquad + \zeta^{j(r-i)+(r-1)i} p_{\lambda_k}(\mathbf{x}^{(r-1)}) \Big).
\end{aligned}$$

Now, since $j(r-i) + ji = jr \equiv 0 \pmod r$, the far right-hand side above becomes

$$p_\lambda^{(i)}(\mathbf{x}^{(j)}, \mathbf{x}^{(j+1)}, \dots, \mathbf{x}^{(j-1)})$$

and the proof follows. □

Lemma 5.1.2. (cf. [50, Proposition II.1] and [76, Lemma 21]) *For every finite, complex $\mathfrak{S}_{n,r}$-character χ we have*

$$\mathrm{ch}_r(\chi \otimes \mathbb{1}_{n,j}) \;=\; \mathrm{shift}_j(\mathrm{ch}_r(\chi)).$$

In particular,

$$\mathrm{ch}_r(\mathbb{1}_{n,j})(\mathbf{X}^{(r)}) \;=\; h_n(\mathbf{x}^{(j)}),$$

for every $0 \le j \le r-1$.

Proof. If a colored permutation has cycle type $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r-1)})$, then its inverse has cycle type $(\lambda^{(0)}, \lambda^{(r-1)}, \dots, \lambda^{(1)})$ (see [75, Proposition 7.8]). Therefore, by Equation (5.1) we have

$$\begin{aligned}
\mathrm{ch}_r(\chi \otimes \mathbb{1}_{n,j}) &= \frac{1}{r^n n!} \sum_{\boldsymbol{\lambda} \vdash n} |K(\boldsymbol{\lambda})| \chi(\boldsymbol{\lambda}) \mathbb{1}_{n,j}(\boldsymbol{\lambda}) p^{(0)}_{\lambda^{(0)}} p^{(1)}_{\lambda^{(r-1)}} \cdots p^{(r-1)}_{\lambda^{(1)}} \\
&= \frac{1}{r^n n!} \sum_{\boldsymbol{\lambda} \vdash n} |K(\boldsymbol{\lambda})| \chi(\boldsymbol{\lambda}) p^{(0)}_{\lambda^{(0)}} \left(\zeta^{j(r-1)\ell(\lambda^{(r-1)})} p^{(1)}_{\lambda^{(r-1)}} \right) \cdots \left(\zeta^{j\ell(\lambda^{(1)})} p^{(r-1)}_{\lambda^{(1)}} \right) \\
&= \frac{1}{r^n n!} \sum_{\boldsymbol{\lambda} \vdash n} |K(\boldsymbol{\lambda})| \chi(\boldsymbol{\lambda}) \,\mathrm{shift}_j(p^{(0)}_{\lambda^{(0)}}) \,\mathrm{shift}_j(p^{(1)}_{\lambda^{(r-1)}}) \cdots \mathrm{shift}_j(p^{(r-1)}_{\lambda^{(1)}}) \\
&= \frac{1}{r^n n!} \sum_{\boldsymbol{\lambda} \vdash n} |K(\boldsymbol{\lambda})| \chi(\boldsymbol{\lambda}) \,\mathrm{shift}_j \left(p^{(0)}_{\lambda^{(0)}} p^{(1)}_{\lambda^{(r-1)}} \cdots p^{(r-1)}_{\lambda^{(1)}} \right) \\
&= \mathrm{shift}_j(\mathrm{ch}_r(\chi)),
\end{aligned}$$

where the third equality follows from Lemma 5.1.1. The second implication follows from the first for $\chi = \mathbb{1}_{n,0} = 1_n$ the trivial $\mathfrak{S}_{n,r}$-character and the fact that

$$\mathrm{ch}_r(1_n)(\mathbf{X}^{(r)}) \;=\; h_n(\mathbf{x}^{(0)}).$$

□

5.2 Descent representations and products of Schur-positive sets

The set of elements of a Coxeter group having a fixed descent set carries a natural representation of the group, called the *descent representation*. Descent representations first appeared in Solomon's work [81] on Weyl groups as alternating sums of permutation representations. This concept was extended by Bagno and Biagioli [15] to complex reflection groups. Their construction of descent representations involves the coinvariant algebra as the representation space and builds upon work by Adin, Brenti and Roichman [3] who treated the case of Weyl groups of type A and B.

There is another description for descent representations by Gessel [52] using ribbons. Lemma 1.6.2 asserts that the quasisymmetric generating function of an inverse descent class of the symmetric group equals the corresponding ribbon Schur function. Thus, the $\mathfrak{S}_n$-characters of descent representations for the symmetric group correspond to Foulkes characters. Recall the discussion of Section 1.6 and in particular Lemma 1.6.2.

Closely related to descent representations and quasisymmetric functions is another remarkable discovery by Solomon [82], the so-called *Solomon's descent algebra*. In particular, he proved (in the more general setting of Coxeter groups) that the set

$$\left\{ \sum_{w \in \mathrm{D}_{n,S}} w : S \subseteq [n-1] \right\}$$

spans a subalgebra, written Sol_n, of the group algebra $\mathbb{C}\mathfrak{S}_n$ of the symmetric group of dimension 2^{n-1}. Gessel [52, Section 4] showed that the comultiplication[5] Δ on QSym induced by the map

$$f(\mathbf{x}) \;\mapsto\; f(\mathbf{xy}),$$

where $\mathbf{xy}$ is the product of the sequences $\mathbf{x}$ and $\mathbf{y}$ of commutating indeterminates ordered lexicographically

$$x_{i_1}y_{j_1} <_{\mathrm{lex}} x_{i_2}y_{j_2} \;\Leftrightarrow\; \begin{cases} i_1 < i_2, & \text{or} \\ i_1 = i_2, & \text{and} \quad j_1 < j_2 \end{cases}$$

acts as follows on the fundamental basis

$$\Delta(F_\gamma) \;=\; \sum_{\alpha,\beta} c^{\gamma}_{\alpha,\beta}\, F_\alpha \otimes F_\beta$$

for every integer composition γ, where $c^{\gamma}_{\alpha,\beta}$ is the number of pairs (u,v) of permutations such that $\mathrm{co}(u) = \alpha$, $\mathrm{co}(v) = \beta$ and $uv = w$, where w is a permutation with $\mathrm{co}(w) = \gamma$. Therefore, Solomon's descent algebra $\mathrm{Sol} := \mathbb{C} \oplus \mathrm{Sol}_1 \oplus \mathrm{Sol}_2 \oplus \cdots$ is isomorphic to the graded dual QSym° of the algebra of quasisymmetric functions.

Remark 5.2.1. The interaction of all these structures; Sym, $\mathcal{R}(\mathfrak{S})$ (partitions), QSym, Sol (compositions), Π (peak compositions), and shuffle algebras is fascinating. For

[5] For the notion of comultiplication and graded dual, we refer to [60].

more information we refer to [60, 62, 72] and references therein. The following diagram illustrates the relations between them.

$$\begin{tikzcd}
\mathcal{A}_{\mathrm{Pk}} \arrow[r, two heads] \arrow[d, leftrightarrow, "\cong"'] & \mathcal{A}_{\mathrm{Des}} \arrow[d, leftrightarrow, "\cong"'] & & & \\
\Pi \arrow[r, two heads] & \mathrm{QSym} \arrow[r, dashed, no head, "\mathrm{dual}"] & \mathrm{QSym}^{\circ} \arrow[r, leftrightarrow, "\cong"] \arrow[d, two heads] & \mathrm{Sol} \arrow[d, two heads] & \Pi^{\circ} \arrow[l, hook'] \\
 & \mathrm{Sym} \arrow[u, hook] \arrow[r, dashed, no head, "\mathrm{dual}"] & \mathrm{Sym} \arrow[r, leftrightarrow, "\cong", "\text{as rings}"'] & \mathcal{R}(\mathfrak{S}) &
\end{tikzcd}$$

Elizalde and Roichman [39, Proposition 5.3] observed that the fact that Sol_n forms an algebra implies that the multiset and set products of inverse decent classes in $\mathfrak{S}_n$ are Schur-positive. The (*multiset-*)*product* of two subsets $A, B \subseteq \mathfrak{S}_n$, written AB, is defined to be the multiset of all permutations of the form uv for $u \in A$ and $v \in B$. They actually proved a significant strengthening about products of Schur-positive sets and inverse descent classes, in which Foulkes characters come into play.

Theorem 5.2.2. (Elizalde–Roichman [39, Theorem 5.12], Bloom [27, Theorem 3.3]) *Let $\mathcal{A} \subseteq \mathfrak{S}_n$ be a fine multiset for the $\mathfrak{S}_n$-character χ with corresponding $\mathfrak{S}_n$-representation ρ. For every $S \subseteq [n-1]$, the following hold.*

- *The product $\mathcal{A}\,\mathrm{R}_{n,S}^{-1}$ is a fine multiset for the $\mathfrak{S}_n$-character $(\chi\downarrow_{\mathfrak{S}_{\mathrm{co}(S)}})\uparrow^{\mathfrak{S}_n}$.*
- *The product $\mathcal{A}\,\mathrm{D}_{n,S}^{-1}$ is a fine multiset for the $\mathfrak{S}_n$-character $\chi \otimes \phi_{n,S}$ of the (internal) tensor product representation of ρ and the Foulkes representation corresponding to S.*
- *The distribution of the descent set over $\mathcal{A}\,\mathrm{D}_{n,S}^{-1}$ and over $\mathrm{D}_{n,S}^{-1}\,\mathcal{A}$ is the same. In particular,*
$$F(\mathcal{A}\,\mathrm{D}_{n,S}^{-1}; \mathbf{x}) \;=\; F(\mathrm{D}_{n,S}^{-1}\,\mathcal{A}; \mathbf{x}).$$

This result, as the authors in [39] illustrate, provides a general method for constructing Schur-positive sets and multisets. Adin, Athanasiadis, Elizalde and Roichman [1] extended the theory of fine sets and fine characters to the hyperoctahedral group. They managed to provide signed analogues for several interesting Schur-positive sets and their corresponding characters (see, for example, the discussion in [1, Section 1]), except for the case of inverse signed descent classes where the authors do not provide the corresponding characters.

Motivated by this, later in the chapter, we prove that inverse colored descent classes corresponding to colored sets are Schur-positive for the characters of descent representations of $\mathfrak{S}_{n,r}$, studied by Bagno and Biagioli [15]. Our proof involves a colored analogue of Gessel's approach to descent representations. More precisely, we introduce a notion of colored ribbons and associate the image of descent representations of $\mathfrak{S}_{n,r}$ via the characteristic map to Schur functions in $\mathrm{Sym}^{(r)}$ indexed by colored ribbons. We also provide a colored analogue of Theorem 5.2.2, which could potentially lead to many instances of Schur-positive subsets of colored permutations.

5.3 Schur-positivity of colored quasisymmetric functions

A (multi)set $\mathcal{A}$ endowed with a colored descent map $\mathrm{sDes} : \mathcal{A} \to \Sigma(n,r)$ is called *Schur-positive* if

$$F(\mathcal{A}; \mathbf{X}^{(r)}) \;:=\; \sum_{a\in\mathcal{A}} m_{\mathcal{A}}(a) F_{\mathrm{sDes}(a)}(\mathbf{X}^{(r)})$$

is a Schur-positive element of $\mathrm{Sym}^{(r)}$, meaning

$$F(\mathcal{A}; \mathbf{X}^{(r)}) \;=\; \sum_{\boldsymbol{\lambda}\vdash n} c_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)})$$

for some nonnegative integers $c_{\boldsymbol{\lambda}}$. By Equation (2.5), we have

$$s_{\boldsymbol{\lambda}} \;=\; F(\mathrm{SYT}(\boldsymbol{\lambda})) \;=\; \sum_{\sigma\in\Sigma(n,r)} |\{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda}) : \mathrm{sDes}(\boldsymbol{Q}) = \sigma\}|\, F_\sigma$$

for all r-partite partitions $\boldsymbol{\lambda}$ of n. Comparing with the definition of $F(\mathcal{A}; \mathbf{X}^{(r)})$ we get

$$m_{\mathcal{A}}(a)\, |\{a \in \mathcal{A} : \mathrm{sDes}(a) = \sigma\}| \;=\; \sum_{\boldsymbol{\lambda}\vdash n} c_{\boldsymbol{\lambda}}\, |\{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda}) : \mathrm{sDes}(\boldsymbol{Q}) = \sigma\}| \tag{5.4}$$

for all $\sigma \in \Sigma(n,r)$. Equation (5.4) is equivalent to the existence of a (multi)set partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_m$ and sDes-preserving bijections $\mathcal{A} \to \mathrm{SYT}(\boldsymbol{\lambda}^i)$, for some r-partite partitions $\boldsymbol{\lambda}^i$ of n, for all $1 \le i \le m$. Since the distribution of sDes over $\mathrm{SYT}(\boldsymbol{\lambda})$ and $K_{\boldsymbol{P}}$ is the same, for some $\boldsymbol{P} \in \mathrm{SYT}(\boldsymbol{\lambda})$ (recall the discussion towards the end of Section 1.4) we have the following criterion for Schur-positivity.

Theorem 5.3.1. *A (multi)set $\mathcal{A}$ endowed with a colored descent map* $\mathrm{sDes} : \mathcal{A} \to \Sigma(n,r)$ *is Schur-positive if and only if there exists a (multi)set partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_m$ and* sDes*-preserving bijections*

$$\mathcal{A}_i \;\longrightarrow\; K_{\boldsymbol{P}^i}$$

for $\boldsymbol{P}^i \in \mathrm{SYT}(\boldsymbol{\lambda}^i)$ and r-partite partitions $\boldsymbol{\lambda}^i$ of n, for all $1 \le i \le m$.

Adin et al. [1] developed an abstract framework to capture this phenomenon in the case $r = 2$, providing a signed analogue of Adin and Roichman's theory of fine sets [8]. We review (a small portion of) their work, stated for general r. Let $\sigma = (\widehat{S}, \epsilon) \in \Sigma(n,r)$ and $\gamma^\delta \in \mathrm{Comp}(n,r)$. The r-colored set σ is called γ^δ-*unimodal* if S is γ-unimodal, that is if S is unimodal with respect to the underlying composition of γ^δ.

Definition 5.3.2. (cf. [1, Definition 3.4]) The *weight* $\mathrm{weight}_{\gamma^\delta}(\sigma)$ of $\sigma = (\widehat{S}, \epsilon) \in \Sigma(n,r)$ with respect to $\gamma^\delta = (\gamma_1^{\delta_1}, \gamma_2^{\delta_2}, \ldots, \gamma_k^{\delta_k}) \in \mathrm{Comp}(n,r)$ is defined as follows:

- $\mathrm{weight}_{\gamma^\delta}(\sigma) := 0$, if either σ is not γ^δ-unimodal or for some index $1 \le i \le k$ the color vector $\tilde{\epsilon}$ of σ is not constant on $\mathcal{B}_i(\gamma)$[6].

[6] This is the ith block of γ of cardinality γ_i (recall the discussion in Section 1.6).

- Otherwise we set
$$\mathrm{weight}_{\gamma^\delta}(\sigma) \;:=\; \zeta^{\sum_{j=0}^{r-1} j\,\mathrm{n}_j(\gamma^\delta;\sigma)}(-1)^{|S\setminus S(\gamma)|}, \tag{5.5}$$
where $\mathrm{n}_j(\gamma^\delta;\sigma)$ denotes the number of indices $1 \le i \le k$ for which the elements of $\mathcal{B}_i(\gamma)$ are assigned the color j by both the color vectors $\tilde{\epsilon}$ and $\tilde{\delta}$ of σ and γ^δ, respectively.

Definition 5.3.3. (cf. [1, Definition 3.5]) Let χ be an $\mathfrak{S}_{n,r}$-character. A collection $\mathcal{A}$, endowed with a colored descent map $\mathrm{sDes} : \mathcal{A} \to \Sigma(n,r)$, is said to be *fine set* for χ if
$$\chi(\gamma) \;=\; \sum_{a\in\mathcal{A}} m_{\mathcal{A}}(a)\,\mathrm{weight}_\gamma(\mathrm{sDes}(a))$$
for every colored composition γ of n.

The following lemma states that the set of standard Young r-partite tableaux of shape $\boldsymbol{\lambda} \vdash n$ endowed with the colored descent map defined in Definition 1.4.3 is a fine set for the irreducible $\mathfrak{S}_{n,r}$character $\chi^{\boldsymbol{\lambda}}$. It is essentially [1, Theorem 4.1] for general r.

Lemma 5.3.4. *For all r-partite partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})$ and $\boldsymbol{\mu}$ of n and every r-colored composition γ of n whose ith colored component is a permutation of $\lambda^{(i)}$,*
$$\chi^{\boldsymbol{\lambda}}(\mu) \;=\; \sum_{\boldsymbol{Q}\in\mathrm{SYT}(\boldsymbol{\lambda})} \mathrm{weight}_\gamma(\mathrm{sDes}(\boldsymbol{Q})). \tag{5.6}$$

Proof. The proof is essentially a trivial generalization of the proof of [1, Theorem 4.1] for $r \ge 3$. We derive a formula for the left-hand side and a formula for the right-hand side of Equation (5.6) and show that they are equal. To compute $\chi^{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ for $\boldsymbol{\mu} = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(r-1)})$ we expand the left-hand side of Equation (5.2) as

$$\begin{aligned}
&p^{(0)}_{\mu^{(0)}}(\mathbf{X}^{(r)})p^{(1)}_{\mu^{(1)}}(\mathbf{X}^{(r)})\cdots p^{(r-1)}_{\mu^{(r-1)}}(\mathbf{X}^{(r)})\\
&\quad= \prod_{i=1}^{\ell(\mu^{(0)})}\Big(p_{\mu_i^{(0)}}(\mathbf{x}^{(0)}) + p_{\mu_i^{(0)}}(\mathbf{x}^{(1)}) + \cdots + p_{\mu_i^{(0)}}(\mathbf{x}^{(r-1)})\Big)\times\\
&\qquad\prod_{i=1}^{\ell(\mu^{(1)})}\Big(p_{\mu_i^{(1)}}(\mathbf{x}^{(0)}) + \zeta p_{\mu_i^{(1)}}(\mathbf{x}^{(1)}) + \cdots + \zeta^{r-1}p_{\mu_i^{(1)}}(\mathbf{x}^{(r-1)})\Big)\times\\
&\qquad\vdots\\
&\qquad\prod_{i=1}^{\ell(\mu^{(r-1)})}\Big(p_{\mu_i^{(r-1)}}(\mathbf{x}^{(0)}) + \zeta^{r-1}p_{\mu_i^{(r-1)}}(\mathbf{x}^{(1)}) + \cdots\\
&\qquad\qquad + (\zeta^{r-1})^{r-1}p_{\mu_i^{(r-1)}}(\mathbf{x}^{(r-1)})\Big)\\
&\quad= \sum_{\epsilon^{(j)}\in\mathbb{Z}_r^{\ell(\mu^{(j)})}} \zeta^{\sum_{j=0}^{r-1} j|\{1\le i\le \ell(\mu^{(j)}):\epsilon_i^{(j)}=j\}|}\times\\
&\qquad\qquad p_{\nu^{(0)}}(\mathbf{x}^{(0)})p_{\nu^{(1)}}(\mathbf{x}^{(1)})\cdots p_{\nu^{(r-1)}}(\mathbf{x}^{(r-1)}),
\end{aligned}$$

where $\nu^{(j)}$ is the composition consisting of the parts $\mu_k^{(0)}$ of $\mu^{(0)}$ with $\epsilon_k^{(0)} = j$, followed by those parts $\mu_k^{(1)}$ of $\mu^{(1)}$ with $\epsilon_k^{(1)} = j$, etc. until the parts $\mu_k^{(r-1)}$ of $\mu^{(r-1)}$ with $\epsilon_k^{(r-1)} = j$, for each $0 \le j \le r-1$. Now, expressing each $p_{\nu^{(j)}}(\mathbf{x}^{(j)})$ in the Schur-basis of $\mathrm{Sym}(\mathbf{x}^{(j)})$ and comparing to Equation (5.2) yields

$$\begin{aligned}\chi^{\boldsymbol{\lambda}}(\nu) \\ = \sum_{\epsilon^{(j)} \in \mathbb{Z}_r^{\ell(\mu^{(j)})}} \zeta^{\sum_{j=0}^{r-1} j |\{1 \le i \le \ell(\mu^{(j)}) : \epsilon_i^{(j)} = j\}|} \chi^{\lambda^{(0)}}(\nu^{(0)}) \chi^{\lambda^{(1)}}(\nu^{(1)}) \cdots \chi^{\lambda^{(r-1)}}(\nu^{(r-1)}).\end{aligned} \tag{5.7}$$

We now derive a formula for the right-hand side of Equation (5.6). Given color vectors $\epsilon^{(j)} = (\epsilon_i^{(j)}, \dots, \epsilon_{\ell(\mu^{(j)})}^{(j)}) \in \mathbb{Z}_r^{\ell(\mu^{(j)})}$ for all $0 \le j \le r-1$, we write $\gamma^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$ for the r-colored composition, whose underlying composition is that of γ and whose parts are colored according to the colors of the parts of $\mu^{(j)}$ assigned by $\epsilon^{(j)}$. Next, denote by $\gamma_j^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$ the composition obtained from $\gamma^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$ by removing all parts of colors other than i and forgetting the colors. The blocks of γ partition the set $[n]$, i.e.

$$[n] \;=\; \mathcal{B}_1(\gamma) \sqcup \mathcal{B}_2(\gamma) \sqcup \cdots \sqcup \mathcal{B}_{\ell(\gamma)}(\gamma).$$

We denote by $R_j^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$ the union of those segments which correspond to parts of $\gamma_j^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$, for all $0 \le j \le r-1$. Comparing Definitions 1.4.3 and 5.3.2 the right-hand side of Equation (5.6) equals

$$\begin{aligned}\sum_{\epsilon^{(j)} \in \mathbb{Z}_r^{\ell(\mu^{(j)})}} \zeta^{\sum_{j=0}^{r-1} j |\{1 \le i \le \ell(\mu^{(j)}) : \epsilon_i^{(j)} = j\}|} \left(\sum_{Q^{(0)}} (-1)^{|\operatorname{Des}(Q^{(0)}) \setminus S(\gamma_0^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}})|} \right) \times \cdots \\ \left(\sum_{Q^{(r-1)}} (-1)^{|\operatorname{Des}(Q^{(r-1)}) \setminus S(\gamma_{r-1}^{\epsilon^{(r-1)}, \dots, \epsilon^{(r-1)}})|} \right),\end{aligned}$$

where the jth sum ranges over all standard Young tableaux $Q^{(j)}$ of shape $\boldsymbol{\lambda}^{(j)}$ and content $R_j^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$ with $\gamma_j^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}$-unimodal descent set. By [1, Theorem 2.4] this becomes

$$\sum_{\epsilon^{(j)} \in \mathbb{Z}_r^{\ell(\mu^{(j)})}} \zeta^{\sum_{j=0}^{r-1} j |\{1 \le i \le \ell(\mu^{(j)}) : \epsilon_i^{(j)} = j\}|} \chi^{\lambda^{(0)}}(\gamma_0^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}}) \cdots \chi^{\lambda^{(r-1)}}(\gamma_{r-1}^{\epsilon^{(0)}, \dots, \epsilon^{(r-1)}})$$

and since the values $\chi(\alpha)$ of an irreducible $\mathfrak{S}_n$-character do not depend on the ordering of the parts of a composition α of n, comparing with Equation (5.7) yields Equation (5.6) and the proof follows □

The proof of the following theorem is essentially the same as the proof of [1, Theorem 3.6], but using Lemma 5.3.4 instead, and is therefore omitted.

Theorem 5.3.5. *Let χ be an $\mathfrak{S}_{n,r}$-character and $\mathcal{A}$ be a collection of combinatorial objects endowed with a colored descent map* $\mathrm{sDes} : \mathcal{A} \to \Sigma(n, r)$*. If*

$$\mathrm{ch}_r(\chi)(\mathbf{X}^{(r)}) \;=\; F(\mathcal{A}; \mathbf{X}^{(r)})$$

then $\mathcal{A}$ is a fine multiset for χ.

We will say that a (multi)set $\mathcal{A}$, endowed with a colored descent map $\mathrm{sDes} : \mathcal{A} \to \Sigma(n, r)$ is *Schur-positive for the $\mathfrak{S}_{n,r}$-character χ* if $\mathcal{A}$ is Schur-positive and

$$\mathrm{ch}_r(\chi)(\mathbf{X}^{(r)}) \;=\; F(\mathcal{A}; \mathbf{X}^{(r)}).$$

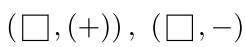
Examples of Schur-positive sets of colored permutations with their corresponding character include

- standard Young r-partite tableaux or Knuth classes of r-colored permutations and irreducible $\mathfrak{S}_{n,r}$-characters (see Lemma 5.3.4)
- absolute involutions of $\mathfrak{S}_{n,r}$ and the characters of Gelfand model of $\mathfrak{S}_{n,r}$ (see [1, Section 5] and [5]
- conjugacy classes on $\mathfrak{S}_{n,r}$ and the colored analogue of the Lie character (see [1, Section 7] and [76, Section 4])
- colored permutations of fixed flag inversion number[7] or flag major index and the characters of the $\mathfrak{S}_{n,r}$-action on the homogeneous components of the coinvariant algebra of $\mathfrak{S}_{n,r}$ (as a complex reflection group) (see [15], [1, Section 6]).

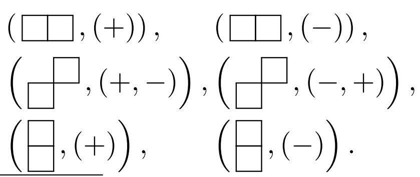
5.4 Colored ribbons, colored compositions and colored sets

Definition 5.4.1. An *r-colored ribbon* with n cells is a direct sum $Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$ of ribbons with a total number of n cells each summand of which has been assigned a color from $\mathbb{Z}_r$ with the property that consecutive summands cannot have the same color. Summands of Z are called parts. We will denote colored ribbons as pairs (Z, ϵ), where $\epsilon : [k] \to \mathbb{Z}_r$ is a color map.

For example, there exist two 2-colored ribbons with 1 cell



and six 2-colored ribbons with 2 cells



[7]The *flag inversion number* of $w^\epsilon \in \mathfrak{S}_{n,r}$ is defined as $\mathrm{finv}(w^\epsilon) := r\,\mathrm{inv}(w) + \mathrm{csum}(w^\epsilon)$ and was introduced by Foata–Han [44] for $r = 2$ and Fire [42] for general r. It is equidistributed with the flag major index over $\mathfrak{S}_{n,r}$.

Proposition 5.4.2. *The set of r-colored ribbons with n cells is in one-to-one correspondence with* $\mathrm{Comp}(n,r)$ *and therefore with* $\Sigma(n,r)$.

Proof. For an r-colored composition of n with rainbow decomposition

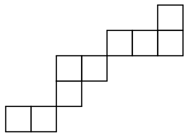
$$\gamma \;=\; \gamma_{(1)}^{\epsilon(1)}\gamma_{(2)}^{\epsilon(2)}\cdots\gamma_{(k)}^{\epsilon(k)}$$

we define

$$\mathrm{Z}_\gamma \;:=\; \mathrm{Z}_{\gamma_{(1)}} \oplus \mathrm{Z}_{\gamma_{(2)}} \oplus \cdots \oplus \mathrm{Z}_{\gamma_{(k)}},$$

where each summand $\mathrm{Z}_{\gamma_{(i)}}$, for all $1 \le i \le k$, has been assigned the color $\epsilon_{(i)}$. For example, for $n = 9, r = 2$ and $\gamma = (2^0)(1^1,2^1)(3^0,1^0)$ we have

$$\mathrm{Z}_\gamma \;=\; \mathrm{Z}_{(2)} \oplus \mathrm{Z}_{(1,2)} \oplus \mathrm{Z}_{(3,1)} \;=$$



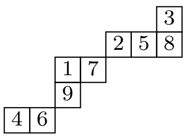
with colors 0, 1 and 0. The map $\gamma \mapsto \mathrm{Z}_\gamma$ is the desired bijection. $\square$

If (Z,ϵ) is a colored ribbon with k parts, then the colored descent set of a standard Young tableau $Q \in \mathrm{SYT}(Z)$ is defined to be the pair $\mathrm{sDes}(Q) = (\widehat{S},\delta) \in \Sigma(n,r)$ where

- δ is the restriction to $\widehat{S}$ of the map $\tilde{\delta} : [n] \to \mathbb{Z}_r$ defined as $\tilde{\delta}(i) = \tilde{\epsilon}(j)$, where $1 \le j \le k$ is the color of the part of Z in which i appears in Q, and
- $i \in S$, if $\delta_i \ne \delta_{i+1}$, or else if $\delta_i = \delta_{i+1}$ and $i \in \mathrm{Des}(Q)$.

Example 5.4.3. Let $Z = (\mathrm{Z}_{(2)} \oplus \mathrm{Z}_{(1,2)} \oplus \mathrm{Z}_{(3,1)}, (0,1,0))$ be the 2-colored ribbon with 9 cells and 3 parts considered in the proof of Proposition 5.4.2. If

$$Q \;=$$



then $\tilde{\delta} = (1,0,0,0,0,0,1,0,1)$ and

$$\mathrm{sDes}(Q) \;=\; \{1^1, 3^0, 5^0, 6^0, 7^1, 8^0, 9^1\}.$$

For $\sigma = (\widehat{S},\epsilon) \in \Sigma(n,r)$, let

$$\begin{aligned}
\mathrm{D}_\sigma &:= \{w \in \mathfrak{S}_{n,r} : \mathrm{sDes}(w) = \sigma\}\\
\mathrm{D}_\sigma^{-1} &:= \{w \in \mathfrak{S}_{n,r} : \mathrm{sDes}(w^{-1}) = \sigma\}\\
\overline{\mathrm{D}}_\sigma^{-1} &:= \{w \in \mathfrak{S}_{n,r} : \mathrm{sDes}(\overline{w}^{-1}) = \sigma\}
\end{aligned}$$

be the *colored descent class*, the *inverse colored descent class* and the *conjugate-inverse colored descent class* respectively, corresponding to σ. Also, define $\overline{\sigma}$ as the r-colored subset of $[n]$ with underlying set $\widehat{S}$ and color map $-\epsilon$.

Lemma 5.4.4. *For all $\sigma \in \Sigma(n,r)$,*

$$\mathrm{D}_{\sigma}^{-1} = \overline{\mathrm{D}}_{\overline{\sigma}}^{-1} \tag{5.8}$$
$$\overline{\mathrm{D}}_{\sigma}^{-1} = \mathrm{D}_{\overline{\sigma}}^{-1}. \tag{5.9}$$

Proof. We will prove Equation (5.9). Equation (5.8) can be proved in a similar way. Let $\sigma = (\widehat{S}, \epsilon)$. Notice that if $w^{\delta} \in \overline{\mathrm{D}}_{\sigma}^{-1}$, that is

$$\mathrm{sDes}(\overline{w^{\delta}}^{-1}) = \sigma,$$

then by Observation 1.2.1 we have $\epsilon = w^{-1}(\delta)$, since a colored permutation and its colored descent set have the same color vector. But, this is equivalent to

$$\mathrm{sDes}(w^{\epsilon^{-1}}) = \overline{\sigma}$$

and therefore $w^{\delta} \in \mathrm{D}_{\overline{\sigma}}^{-1}$. □

The following observation exploits the connection between tableaux of colored ribbon shape and (conjugate-inverse) colored descent classes and therefore constitutes a colored analogue to Lemma 1.3.1.

Proposition 5.4.5. *For every $\sigma \in \Sigma(n,r)$, there exists a bijection from the set* $\mathrm{SYT}(\mathrm{Z}_{\sigma})$ *to the colored descent class* D_{σ} *such that*

$$\mathrm{sDes}(Q) = \mathrm{sDes}(\overline{w}^{-1}),$$

where w is the r-colored permutation associated to the tableau $Q \in \mathrm{SYT}(\mathrm{Z}_{\sigma})$. In particular, the distribution of the sDes *is the same over $\overline{\mathrm{D}}_{\sigma}^{-1}$ and* $\mathrm{SYT}(\mathrm{Z}_{\sigma})$.

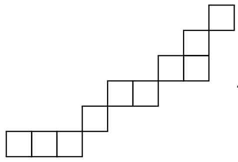
Proof. Given a tableau $Q \in \mathrm{SYT}(\mathrm{Z}_{\sigma})$, we define an r-colored permutation $w^{\epsilon} \in \mathfrak{S}_{n,r}$ as in the case $r = 1$ (see, for example, the proof of [7, Proposition 10.12]) by reading the cell entries of Q in the northeast direction, starting from the southwestern corner and assigning each cell entry a color from $\mathbb{Z}_r$ according to the color of each cell, which is determined by the color vector of σ. It follows directly from the definition that the colored descent set of w equals σ and therefore $w^{\epsilon} \in \mathrm{D}_{\sigma}$. This process can be reversed in a unique way and therefore resulting in a bijection.

For the second assertion, suppose $\mathrm{sDes}(Q) = (\widehat{T}, \delta)$. By the definition of $\mathrm{sDes}(Q)$, for all $i \in [n]$, $\tilde{\delta}(i)$ records the color of the part of Z in which i appears in Q. Since w^{ϵ} is the *reading word* of Q, this means $\tilde{\delta} = w^{-1}(\epsilon)$, which by Observation 1.2.1 is exactly the color vector of $\mathrm{sDes}(\overline{w^{\epsilon}}^{-1})$. To conclude the proof notice that if $\epsilon_{w_i^{-1}} = \epsilon_{w_{i+1}^{-1}}$ and $w_i^{-1} > w_{i+1}^{-1}$, then $i+1$ appears to the left of i in w and therefore $i+1$ appears in a lower cell than i does in Q which implies $i \in T$. □

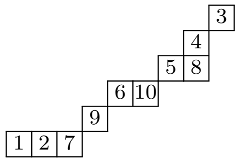
Example 5.4.6. To illustrate Proposition 5.4.5 let $n = 10, r = 3$ and consider

$$\sigma = \{3^2, 4^0, 6^1, 8^2, 9^2, 10^1\} \longmapsto \mathrm{co}(\sigma) = (3)^2\,(1)^0\,(2)^1\,(2,1)^2\,(1)^1$$

and therefore

Z_σ =  .

If, for example,

Q = 

$\in \mathrm{SYT}(\mathrm{Z}_\sigma)$

then the corresponding reading word is

$$w \;=\; 1^2\, 2^2\, 7^2\, 9^0\, 6^1\, 10^1\, 5^2\, 8^2\, 4^2\, 3^1 \;\; \in \mathrm{D}_\sigma\,.$$

Now, one computes

$$\begin{aligned} \overline{w}^{-1} &= 1^2\, 2^2\, 10^1\, 9^2\, 7^2\, 5^1\, 3^2\, 8^2\, 4^0\, 6^1 \\ \mathrm{sDes}(\overline{w}^{-1}) &= \{2^2, 3^1, 4^2, 5^2, 6^1, 8^2, 9^0, 10^1\} \end{aligned}$$

as well as

$$\mathrm{sDes}(Q) \;=\; \mathrm{sDes}(\overline{w}^{-1}).$$

5.5 Colored ribbons and descent representations for colored permutation groups

Recall that any product of (skew) Schur functions is a (skew) Schur function (see the discussion on page 339 of [89]). In particular, notice that for skew shapes $\lambda_1, \lambda_2, \ldots, \lambda_k$, we have

$$s_{\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_k} \;=\; s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_k}.$$

Each r-colored ribbon (Z, ϵ) with n cells defines an element of $\mathrm{Sym}_n^{(r)}$, which is the product of ribbon Schur functions in $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots,$ and $\mathbf{x}^{(r-1)}$. In particular, if $Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k$, let

$$s_{(Z,\epsilon)}(\mathbf{X}^{(r)}) \;:=\; s_{Z_1}(\mathbf{x}^{(\epsilon_1)}) s_{Z_2}(\mathbf{x}^{(\epsilon_2)}) \cdots s_{Z_k}(\mathbf{x}^{(\epsilon_k)}).$$

Definition 5.5.1. For an r-colored subset σ of $[n]$, the (unique) $\mathfrak{S}_{n,r}$-character χ_σ defined by

$$\mathrm{ch}_r(\chi_\sigma)(\mathbf{X}^{(r)}) \;=\; s_{\mathrm{Z}_{\mathrm{co}(\sigma)}}(\mathbf{X}^{(r)}) \tag{5.10}$$

is called the *descent character* of $\mathfrak{S}_{n,r}$ corresponding to σ. The corresponding $\mathfrak{S}_{n,r}$-representation, written ϕ_σ, is called the *descent representation* of $\mathfrak{S}_{n,r}$ corresponding to σ. We define the descent character (resp. representation) corresponding to an r-colored composition in a similar way and denote by $r_\sigma(\mathbf{X}^{(r)})$ the right-hand side of Equation (5.10).

Remark 5.5.2. Given $\sigma \in \Sigma(n,r)$, notice that for each tableau $Q \in \mathrm{SYT}(Z_\sigma)$, there exists a standard Young r-partite tableau $\boldsymbol{Q} \in \mathrm{SYT}(\mathrm{Z}_\sigma^{(0)}, \mathrm{Z}_\sigma^{(1)}, \ldots, \mathrm{Z}_\sigma^{(r-1)})$ (uniquely) obtained in the obvious way, where $\mathrm{Z}_\sigma^{(j)}$ is the direct sum of ribbons of color j, for all $0 \leq j \leq r-1$ appearing in Z_σ (in order of appearance), with the property that $\mathrm{sDes}(Q) = \mathrm{sDes}(\boldsymbol{Q})$. For example, the standard Young 3-partite tableau which corresponds to the tableau Q of Example 5.4.6 is

$$\boldsymbol{Q} \;=\; \left(\begin{ytableau} 9 \end{ytableau}\,, \begin{ytableau} \none & \none & 3 \\ 6 & 10 \end{ytableau}\,, \begin{ytableau} \none & \none & \none & \none & 4 \\ \none & \none & \none & 5 & 8 \\ 1 & 2 & 7 \end{ytableau} \right)$$

and

$$\mathrm{sDes}(\boldsymbol{Q}) \;=\; \{2^2, 3^1, 4^2, 5^2, 6^1, 8^2, 9^0, 10^1\},$$

which coincides with $\mathrm{sDes}(Q)$. Having in mind the construction of irreducible $\mathfrak{S}_{n,r}$-representations as presented, for example, in [93, Section 4] or [50, Section II], [94, Section 5] for $r=2$, one can see that descent representations of Definition 5.5.1 are actually extensions of Specht modules of skew shape from the symmetric group to the r-colored permutation group.

The following colored analogue of Lemma 1.6.2 proves that conjugate-inverse colored descent classes are Schur-positive sets for the descent characters of colored permutation groups and therefore contributes one more item to the list at the end of Section 5.3. The fact that inverse colored descent classes are Schur-positive (and in particular fine sets) was proved in [1, Poposition 5.5(i)]. The new result is that they correspond to descent characters of colored permutation groups, the lack of which was part of our motivation.

Theorem 5.5.3. *For all $\sigma \in \Sigma(n,r)$,*

$$F(\mathrm{D}_{\overline{\sigma}}^{-1}; \mathbf{X}^{(r)}) = F(\overline{\mathrm{D}}_\sigma^{-1}; \mathbf{X}^{(r)}) = F(\mathrm{SYT}(\mathrm{Z}_\sigma); \mathbf{X}^{(r)}) = r_\sigma(\mathbf{X}^{(r)}). \tag{5.11}$$

In particular, conjugate-inverse colored descent classes are Schur-positive for the descent characters of $\mathfrak{S}_{n,r}$ and

$$r_\sigma(\mathbf{X}^{(r)}) \;=\; \sum_{\boldsymbol{\lambda} \vdash n} c_{\boldsymbol{\lambda}}(\sigma) s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}), \tag{5.12}$$

where $c_{\boldsymbol{\lambda}}(\sigma)$ counts the number of standard Young r-partite tableaux $\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda})$ such that $\mathrm{sDes}(\boldsymbol{Q}) = \sigma$.

Proof. The first equality of Equation (5.11) follows from Lemma 5.4.4. The second equality follows directly from Proposition 5.4.5. Remark 5.5.2 together with Equation (2.5) yields the second equality of Equation (5.11). For the second assertion, notice that the colored Robinson–Schensted correspondence implies that

$$F(\overline{\mathrm{D}}_\sigma^{-1}; \mathbf{X}^{(r)}) \;=\; \sum_{\boldsymbol{\lambda} \vdash n} \sum_{\substack{\boldsymbol{P},\boldsymbol{Q} \,\in\, \mathrm{SYT}(\boldsymbol{\lambda}) \\ \mathrm{sDes}(\boldsymbol{P}) = \sigma}} F_{\mathrm{sDes}(\boldsymbol{Q})}(\mathbf{X}^{(r)})$$

which expands to the right-hand side of Equation (5.12) by Equation (2.5). $\square$

The expansion in Equation (5.12) coincides with [15, Theorem 10.5] for the case of colored permutation groups and emphasizes the connection with Bagno and Biagioli's descent representations for complex reflection groups. For a different notion of type B descent representations we refer to [76, Section 5].

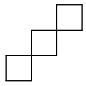
Example 5.5.4. We verify Theorem 5.5.3 in a specific example by computing[8] $F(\mathrm{D}_\sigma^{-1};\mathbf{x}^{(0)},\mathbf{x}^{(1)})$ and $r_\sigma(\mathbf{x}^{(0)},\mathbf{x}^{(1)})$ for $\sigma = \{1^0, 2^1, 3^0\} \in \Sigma(3,2)$. On the one hand, we have

$$\mathrm{D}_\sigma^{-1} \;=\; \{1^0 2^1 3^0,\ 1^0 3^0 2^1,\ 2^1 1^0 3^0,\ 2^1 3^0 1^0,\ 3^0 1^0 2^1,\ 3^0 2^1 1^0\}$$

and therefore

$$\begin{aligned} F(\mathrm{D}_\sigma^{-1};\mathbf{x}^{(0)},\mathbf{x}^{(1)}) &= F_{\{1,\overline{2},3\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) + F_{\{2,\overline{3}\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) + F_{\{\overline{1},3\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) \\ &\quad + F_{\{\overline{1},2,3\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) + F_{\{1,2,\overline{3}\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) + F_{\{1,\overline{2},3\}}(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) \\ &= s_{(2)}(\mathbf{x}^{(0)})s_{(1)}(\mathbf{x}^{(1)}) + s_{(1,1)}(\mathbf{x}^{(0)})s_{(1)}(\mathbf{x}^{(1)}). \end{aligned}$$

On the other hand, we have

$\mathrm{Z}_\sigma \;=$ 

with color vector $(0,1,0)$ and therefore

$$\begin{aligned} r_\sigma(\mathbf{x}^{(0)},\mathbf{x}^{(1)}) &= s_{(1)}(\mathbf{x}^{(0)})s_{(1)}(\mathbf{x}^{(1)})s_{(1)}(\mathbf{x}^{(0)}) \\ &= \left(s_{(2)}(\mathbf{x}^{(0)}) + s_{(1)}(\mathbf{x}^{(0)})\right) s_{(1)}(\mathbf{x}^{(1)}) \\ &= s_{(2)}(\mathbf{x}^{(0)})s_{(1)}(\mathbf{x}^{(1)}) + s_{(1,1)}(\mathbf{x}^{(0)})s_{(1)}(\mathbf{x}^{(1)}). \end{aligned}$$

The corresponding standard Young bitableaux of shape $((2),(1))$ and $((1,1),(1))$ with colored descent set σ are

$$\left(\boxed{1}\boxed{3}, \boxed{2}\right), \quad \left(\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \boxed{2}\right),$$

respectively.

Definition 5.5.5. For an r-colored composition $\gamma^\epsilon = (\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \dots, \gamma_k^{\epsilon_k})$ of n, let

$$\mathbb{1}_{\gamma^\epsilon} \;:=\; \mathbb{1}_{\gamma_1,\epsilon_1} \times \mathbb{1}_{\gamma_2,\epsilon_2} \times \cdots \times \mathbb{1}_{\gamma_k,\epsilon_k}$$

be the character of the outer tensor product representation

$$\mathbb{1}_{\gamma_1,\epsilon_1} \boxtimes \mathbb{1}_{\gamma_2,\epsilon_2} \boxtimes \cdots \boxtimes \mathbb{1}_{\gamma_k,\epsilon_k}$$

of $\mathfrak{S}_{\gamma,r} := \mathfrak{S}_{\gamma_1,r} \times \mathfrak{S}_{\gamma_2,r} \times \cdots \times \mathfrak{S}_{\gamma_k,r}$. For an r-colored subset σ of $[n]$ we write $\mathbb{1}_\sigma := \mathbb{1}_{\mathrm{co}(\sigma)}$.

Since the characteristic map ch_r is a ring homomorphism, it is easy to compute the image of $\mathbb{1}_{\gamma^\epsilon}$. In particular, we have

$$\begin{aligned} \mathrm{ch}_r(\mathbb{1}_{\gamma^\epsilon}\uparrow_{\mathfrak{S}_{\gamma,r}}^{\mathfrak{S}_{n,r}})(\mathbf{X}^{(r)}) &= \mathrm{ch}_r(\mathbb{1}_{\gamma_1,\epsilon_1})(\mathbf{X}^{(r)})\,\mathrm{ch}_r(\mathbb{1}_{\gamma_2,\epsilon_2})(\mathbf{X}^{(r)}) \cdots \mathrm{ch}_r(\mathbb{1}_{\gamma_k,\epsilon_k})(\mathbf{X}^{(r)}) \\ &= h_{\gamma_1}(\mathbf{x}^{(\epsilon_1)}) h_{\gamma_2}(\mathbf{x}^{(\epsilon_2)}) \cdots h_{\gamma_k}(\mathbf{x}^{(\epsilon_k)}), \end{aligned} \tag{5.13}$$

[8] In the case $r = 2$ notice that $\overline{w} = w$ for all signed permutations $w \in \mathfrak{B}_n$.

where the second equality follows from Lemma 5.1.1.

The following theorem provides a formula[9] for the descent characters as the alternating sums of inductions of the characters introduced in Definition 5.5.5, thus providing a colored analogue of Equation (1.20). It will play a central role later, in the proof of Theorem 5.6.5.

Theorem 5.5.6. *For all r-colored compositions γ of n*

$$\chi_\gamma \;=\; \sum_{\beta \preceq \gamma} (-1)^{\ell(\gamma)-\ell(\beta)}\, \mathbb{1}_\beta \uparrow_{\mathfrak{S}_{\beta,r}}^{\mathfrak{S}_{n,r}} . \tag{5.14}$$

Proof. Since the characteristic map is an isomorphism it suffices to prove

$$r_\gamma(\mathbf{X}^{(r)}) \;=\; \sum_{\beta \preceq \gamma} (-1)^{\ell(\gamma)-\ell(\beta)}\, \mathrm{ch}_r(\mathbb{1}_\beta \uparrow_{\mathfrak{S}_{\beta,r}}^{\mathfrak{S}_{n,r}})(\mathbf{X}^{(r)}), \tag{5.15}$$

for all r-colored compositions γ of n. Let γ^ϵ be an r-colored composition of n with rainbow decomposition

$$\gamma^\epsilon \;=\; \gamma_{(1)}^{\epsilon(1)} \gamma_{(2)}^{\epsilon(2)} \cdots \gamma_{(k)}^{\epsilon(k)} .$$

Expanding each term of the right-hand side of Equation (5.10) according to Equation (1.20) yields

$$\begin{aligned} r_\gamma(\mathbf{X}^{(r)}) &= \prod_{i=1}^{k} r_{\gamma_{(i)}}(\mathbf{x}^{(\epsilon_{(i)})}) \\ &= \prod_{i=1}^{k} \sum_{\beta_{(i)} \preceq \gamma_{(i)}} (-1)^{\ell(\gamma_{(i)})-\ell(\beta_{(i)})}\, h_{\beta_{(i)}}(\mathbf{x}^{(\epsilon_{(i)})}) \\ &= \sum_{\substack{1 \le i \le k \\ \beta_{(i)} \preceq \gamma_{(i)}}} (-1)^{\ell(\gamma)-\ell(\beta_{(1)})-\cdots-\ell(\beta_{(k)})} h_{\beta_{(1)}}(\mathbf{x}^{(\epsilon_{(1)})}) \cdots h_{\beta_{(k)}}(\mathbf{x}^{(\epsilon_{(k)})}), \end{aligned} \tag{5.16}$$

since the length of γ is the sum of all lengths of $\gamma_{(i)}$. Now, if we let $\beta := \beta_{(1)}\beta_{(2)} \cdots \beta_{(k)}$ and assign to each $\beta_{(i)}$ the color $\epsilon_{(i)}$, then

- $\ell(\beta) = \ell(\beta_{(1)}) + \ell(\beta_{(2)}) + \cdots + \ell(\beta_{(k)})$, and
- conditions $\beta_{(i)} \preceq \gamma_{(i)}$, for each $1 \le i \le k$ are precisely equivalent to $\beta \preceq \gamma$.

By this observation and Equation (5.13), Equation (5.16) becomes Equation (5.15) and the proof follows. □

5.6 Products of Schur-positive sets

Mantaci and Reutenauer [67] introduced a subalgebra of the group algebra $\mathbb{C}\mathfrak{S}_{n,r}$ of $\mathfrak{S}_{n,r}$ which arises naturally when working with colored descent sets and contains Solomon's descent algebra of type B_n for $r = 2$. We recall its statement, following the exposition of [72, Section 3].

[9] It was communicated to the author by Athanasiadis [13] for $r = 2$.

Theorem 5.6.1. (Mantaci–Reutenauer [67, Section 3]) *For an r-colored subset σ of $[n]$, let*

$$d_\sigma \;:=\; \sum_{w \in \mathrm{D}_\sigma} w.$$

For any $\sigma, \tau \in \Sigma(n,r)$, we have

$$d_\sigma\, d_\tau \;=\; \sum_{\varrho \,\in\, \Sigma(n,r)} \tilde{c}^{\varrho}_{\sigma,\tau}\; d_\varrho, \tag{5.17}$$

where the multiplication takes place in $\mathbb{C}\mathfrak{S}_{n,r}$ and $c^{\varrho}_{\sigma,\tau}$ counts pairs $(u,v) \in \mathfrak{S}_{n,r} \times \mathfrak{S}_{n,r}$ such that $\mathrm{sDes}(u) = \sigma, \mathrm{sDes}(v) = \tau$ and $\mathrm{sDes}(uv) = \varrho$. In particular, the set $\{d_\sigma : \sigma \in \Sigma(n,r)\}$ spans a subalgebra, written MR_n, of $\mathbb{C}\mathfrak{S}_{n,r}$ of dimension[10] *$r(r+1)^{n-1}$ called the* Mantaci–Reutenauer algebra*.*

It follows from the work of Baumann–Hohlweg [19], Bergeron–Hohlweg [21] and Petersen [72] that $\mathrm{QSym}^{(r)}_n$ can be given a structure of a coalgebra in a similar way to the uncolored case such that the Mantaci–Reutenauer algebra $\mathrm{MR} := \mathbb{C} \oplus \mathrm{MR}_1 \oplus \mathrm{MR}_2 \oplus \cdots$ is isomorphic to the graded dual $\mathrm{QSym}^{(r)^\circ}$ of the algebra of colored quasisymmetric functions. In addition, Bergeron and Hohlweg [21, Theorem 2.12 and Theorem 5.3] prove that the set

$$\left\{ \sum_{\substack{w \in \mathfrak{S}_{n,r} \\ \widehat{\mathrm{co}}(w) = \gamma}} w : \gamma \text{ is a colored peak composition of } n \right\}$$

spans a subalgebra of MR_n which is isomorphic to the graded dual $\Pi^{(r)^\circ}_n$ of the algebra of colored peak functions.

Remark 5.6.2. The following diagram illustrates the relations between the colored analogues of the structures which appear in 5.2.1: $\mathrm{Sym}^{(r)}, \mathcal{R}(\mathfrak{S}^{(r)})$ (r-partite partitions), $\mathrm{Sym}^{(r)}, \mathrm{MR}$ (r-colored compositions), $\Pi^{(r)}$ (colored peak compositions) and shuffle algebras of sDes and sPk.

$$\begin{tikzcd}
\mathcal{A}_{\mathrm{sPk}} \arrow[r, two heads] \arrow[d, "\cong"'] & \mathcal{A}_{\mathrm{sDes}} \arrow[d, leftrightarrow, "\cong"'] & & & \\
\Pi^{(r)} \arrow[r, two heads] & \mathrm{QSym}^{(r)} \arrow[r, dashed, no head, "\mathrm{dual}"] & \mathrm{QSym}^{(r)^\circ} \arrow[r, leftrightarrow, "\cong"] \arrow[d, two heads] & \mathrm{MR} \arrow[d, two heads] & \Pi^{(r)^\circ} \arrow[l, hook'] \\
 & \mathrm{Sym}^{(r)} \arrow[u, hook] \arrow[r, dashed, no head, "\mathrm{dual}"] & \mathrm{Sym}^{(r)} \arrow[r, leftrightarrow, "\cong", "\text{as rings}"'] & \mathcal{R}(\mathfrak{S}^{(r)}) &
\end{tikzcd}$$

The (*multiset-*)*product* of two subsets $A, B \subseteq \mathfrak{S}_{n,r}$, written AB, is defined to be the multiset of all r-colored permutations of the form uv for $u \in A$ and $v \in B$. Similarly to the uncolored case, the fact that MR_n forms an algebra implies that products of (conjugate-)inverse colored descent classes are Schur-positive.

[10] This is the number of r-colored subsets of $[n]$.

Proposition 5.6.3. *Products of (conjugate-)inverse colored descent classes are Schur-positive.*

Example 5.6.4. We verify Proposition 5.6.3 for $n = 3, r = 2$ and $\sigma = \{1^1, 2^0, 3^0\}, \tau = \{1^1, 3^0\}$. On the one hand, we have

$$\begin{aligned} \mathrm{D}_\sigma^{-1} &= \{1^1 3^0 2^0,\ 3^0 1^1 2^0,\ 3^0 2^0 1^1\} \\ \mathrm{D}_\tau^{-1} &= \{1^1 2^0 3^0,\ 2^0 1^1 3^0,\ 2^0 3^0 1^1\} \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{D}_\sigma^{-1}\,\mathrm{D}_\tau^{-1} &= \{1^0 3^0 2^0,\ 3^0 1^0 2^0,\ 3^0 2^0 1^0,\ 3^1 1^1 2^0,\ 1^1 3^1 2^0,\ 1^1 2^0 3^1,\ 3^1 2^0 1^1, \\ &\quad 2^0 3^1 1^1,\ 2^0 1^1 3^1\} \\ &= \mathrm{D}_{\{2^0,3^0\}}^{-1} \cup \mathrm{D}_{\{1^0,2^0,3^0\}}^{-1} \cup \mathrm{D}_{\{1^1,2^0,3^1\}}^{-1}\,. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{D}_\sigma &= \{1^1 3^0 2^0,\ 2^1 3^0 1^0,\ 3^1 2^0 1^0\} \\ \mathrm{D}_\tau &= \{1^1 2^0 3^0,\ 2^1 1^0 3^0,\ 3^1 1^0 2^0\} \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{D}_\tau\,\mathrm{D}_\sigma &= \{1^0 3^0 2^0,\ 2^1 3^0 1^1,\ 3^1 2^0 1^1,\ 2^0 3^0 1^0,\ 1^1 3^0 2^1,\ 3^1 1^1 2^1,\ 3^0 2^0 1^0, \\ &\quad 1^1 2^0 3^1,\ 2^1 1^0 3^1\} \\ &= \mathrm{D}_{\{2^0,3^0\}} \cup \mathrm{D}_{\{1^0,2^0,3^0\}} \cup \mathrm{D}_{\{1^1,2^0,3^1\}}\,. \end{aligned}$$

For all $\sigma \in \Sigma(n, r)$, let

$$\overline{\mathrm{R}}_\sigma^{-1} \;:=\; \{w \in \mathfrak{S}_{n,r} : \mathrm{sDes}(\overline{w}^{-1}) \preceq \sigma\} \;=\; \bigcup_{\tau \preceq \sigma} \overline{\mathrm{D}}_\tau^{-1}.$$

The following theorem is a strengthening of Proposition 5.6.3. It provides a method of constructing Schur-positive sets of colored permutations from known ones by taking the product with some (conjugate-)inverse colored descent class. It is essentially a colored analogue of Theorem 5.2.2.

Theorem 5.6.5. *Let $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$ be a Schur-positive multiset for the $\mathfrak{S}_{n,r}$-character χ. For all $\sigma \in \Sigma(n, r)$ the following hold.*

(1) *The product $\mathcal{A}\overline{\mathrm{R}}_\sigma^{-1}$ is Schur-positive for the character $(\chi \downarrow_{\mathfrak{S}_{\sigma,r}} \otimes \mathbb{1}_\sigma) \uparrow^{\mathfrak{S}_{n,r}}$.*

(2) *The product $\mathcal{A}\overline{\mathrm{D}}_\sigma^{-1}$ is Schur-positive for the character $\chi \otimes \chi_\sigma$ of the inner tensor product of the $\mathfrak{S}_{n,r}$-representation with character χ and the descent representation of $\mathfrak{S}_{n,r}$ corresponding to σ.*

Before proceeding to prove Theorem 5.6.5, we take a look at the special case $\sigma = \{n^j\}$, for all $0 \le j \le r - 1$. We will use the following consequence of Lemma 5.1.1. We remark that Lemma 5.6.6, below, holds also for Knuth classes.

Lemma 5.6.6. *For all r-partite partitions $\boldsymbol{\lambda}$ of n and all $0 \le j \le r-1$,*

$$F(\mathrm{SYT}(\mathrm{shift}_{-j}(\boldsymbol{\lambda})); \mathbf{X}^{(r)}) \;=\; \mathrm{ch}_r(\chi^{\boldsymbol{\lambda}} \otimes \mathbb{1}_{n,j})(\mathbf{X}^{(r)}) \tag{5.18}$$

and

$$F(\mathrm{SYT}(\mathrm{shift}_{-j}(\boldsymbol{\lambda})); \mathbf{X}^{(r)}) \;=\; \sum_{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda})} F_{\mathrm{shift}_j(\mathrm{sDes}(\boldsymbol{Q}))}(\mathbf{X}^{(r)}). \tag{5.19}$$

Proof. Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r-1)})$ be an r-partite partition of n. Since

$$\mathrm{ch}_r(\chi^{\boldsymbol{\lambda}})(\mathbf{X}^{(r)}) \;=\; s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)}),$$

Lemma 5.1.1 implies that

$$\begin{aligned}
\mathrm{ch}_r(\chi^{\boldsymbol{\lambda}} \otimes \mathbb{1}_{n,j})(\mathbf{X}^{(r)}) &= \mathrm{shift}_j\left(s_{\boldsymbol{\lambda}}(\mathbf{X}^{(r)})\right) \\
&= s_{\lambda^{(0)}}(\mathbf{x}^{(j)}) s_{\lambda^{(1)}}(\mathbf{x}^{(j+1)}) \cdots s_{\lambda^{(r-1)}}(\mathbf{x}^{(j-1)}) \\
&= s_{\lambda^{(-j)}}(\mathbf{x}^{(0)}) s_{\lambda^{(-j+1)}}(\mathbf{x}^{(1)}) \cdots s_{\lambda^{(-j+r-1)}}(\mathbf{x}^{(r-1)}) \\
&= s_{\mathrm{shift}_{-j}(\boldsymbol{\lambda})}(\mathbf{X}^{(r)}) \\
&= F(\mathrm{SYT}(\mathrm{shift}_{-j}(\boldsymbol{\lambda})); \mathbf{X}^{(r)}),
\end{aligned}$$

where the last equality follows from Equation (2.5) and the proof of Equation (5.18) follows. For the proof of Equation (5.19), notice that every element of $\mathrm{SYT}(\mathrm{shift}_{-j}(\boldsymbol{\lambda}))$ is of the form

$$\left(Q^{(-j)}, Q^{(-j+1)}, \dots, Q^{(-j+r-1)}\right)$$

for some $\boldsymbol{Q} = (Q^{(0)}, \dots, Q^{(r-1)}) \in \mathrm{SYT}(\boldsymbol{\lambda})$ and that

$$\mathrm{sDes}\left(Q^{(-j)}, Q^{(-j+1)}, \dots, Q^{(-j+r-1)}\right) \;=\; \mathrm{shift}_j(\mathrm{sDes}(\boldsymbol{Q})).$$

□

Theorem 5.6.7. *Let $0 \le j \le r-1$. If $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$ is a Schur-positive multiset for the $\mathfrak{S}_{n,r}$-character χ, then $\mathcal{A}\overline{\mathrm{D}}^{-1}_{\{n^j\}}$ is Schur-positive for the $\mathfrak{S}_{n,r}$-character $\chi \otimes \mathbb{1}_{n,j}$.*

Proof. Since $\mathcal{A}$ is Schur-positive for χ, by Theorem 5.3.1 there exists a multiset partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_m$ and sDes-preserving bijections $\mathcal{A}_i \to \mathrm{SYT}(\boldsymbol{\lambda}^i)$ for some r-partite partition $\boldsymbol{\lambda}^i$ of n. Now, $\overline{\mathrm{D}}^{-1}_{\{n^j\}}$ consists only of $1^j 2^j \cdots n^j$ and multiplying each $\mathcal{A}_i$ by $\overline{\mathrm{D}}^{-1}_{\{n^j\}}$ amounts to shifting all colors inside $\boldsymbol{\lambda}^i$ by $-j$. In particular, we have

$$\begin{aligned}
F(\mathcal{A}_i\overline{\mathrm{D}}^{-1}_{\{n^j\}}; \mathbf{X}^{(r)}) &= \sum_{\boldsymbol{Q} \in \mathrm{SYT}(\boldsymbol{\lambda}^i)} F_{\mathrm{shift}_j(\mathrm{sDes}(\boldsymbol{Q}))}(\mathbf{X}^{(r)}) \\
&= F(\mathrm{SYT}(\mathrm{shift}_{-j}(\boldsymbol{\lambda}^i)); \mathbf{X}^{(r)}) \\
&= \mathrm{ch}_r(\chi^{\boldsymbol{\lambda}^i} \otimes 1_{n,j})(\mathbf{X}^{(r)}),
\end{aligned} \tag{5.20}$$

where the second and third equalities follow from Equations (5.19) and (5.18), respectively. Taking the sum over all i in Equation (5.20) yields

$$F(\mathcal{A}\overline{\mathrm{D}}_{\{n^j\}}^{-1}) = \sum_i \mathrm{ch}_r(\chi^{\boldsymbol{\lambda}^i} \otimes 1_{n,j}) = \mathrm{ch}_r\left(\left(\sum_i \chi^{\boldsymbol{\lambda}^i}\right) \otimes \mathbb{1}_{n,j}\right) = \mathrm{ch}_r(\chi \otimes \mathbb{1}_{n,j})$$

and the proof follows. □

Most of the remaining of this section is devoted to the proof of Theorem 5.6.5. The strategy for the proof is essentially the same as Elizalde and Roichman's in [39, Section 5] for the unsigned case. We start by noticing that (1) implies (2) in Theorem 5.6.5.

Proof that (1) *implies* (2) *in Theorem 5.6.5.* Suppose that $\mathcal{A}\overline{\mathrm{R}}_\sigma^{-1}$ is a Schur-positive multiset for the $\mathfrak{S}_{n,r}$-character $(\chi \downarrow_{\mathfrak{S}_{\sigma,r}} \otimes \mathbb{1}_\sigma) \uparrow^{\mathfrak{S}_{n,r}}$. By definition, we have

$$F(\mathcal{A}\overline{\mathrm{R}}_\sigma^{-1}; \mathbf{X}^{(r)}) \;=\; \sum_{\tau \preceq \sigma} F(\mathcal{A}\overline{\mathrm{D}}_\tau^{-1}; \mathbf{X}^{(r)})$$

and therefore by the principle of inclusion–exclusion we get

$$F(\mathcal{A}\overline{\mathrm{D}}_\sigma^{-1}; \mathbf{X}^{(r)}) \;=\; \sum_{\tau \preceq \sigma} (-1)^{|\sigma|-|\tau|} F(\mathcal{A}\overline{\mathrm{R}}_\tau^{-1}; \mathbf{X}^{(r)}) \;\in\; \mathrm{Sym}^{(r)}, \tag{5.21}$$

where $|\sigma|$ denotes the cardinality of the underlying set of σ.

By the following observation in the representation theory of finite groups (see, for example, [97, Corollary 4.3.8]) we know that

$$(\chi \downarrow_{\mathfrak{S}_{\tau,r}} \otimes \mathbb{1}_\tau) \uparrow^{\mathfrak{S}_{n,r}} \;=\; \chi \otimes \mathbb{1}_\tau \uparrow^{\mathfrak{S}_{n,r}}$$

for all r-colored subsets τ of $[n]$. Taking the (alternating) sum over all $\tau \preceq \sigma$ on both sides yields

$$\sum_{\tau \preceq \sigma} (-1)^{|\sigma|-|\tau|} (\chi \downarrow_{\mathfrak{S}_{\tau,r}} \otimes \mathbb{1}_\tau) \uparrow^{\mathfrak{S}_{n,r}} = \chi \otimes \left(\sum_{\tau \preceq \sigma} (-1)^{|\sigma|-|\tau|} \mathbb{1}_\tau \uparrow^{\mathfrak{S}_{n,r}}\right) = \chi \otimes \chi_\sigma, \tag{5.22}$$

where the last equality follows from Equation (5.14). Finally, taking the characteristic map on the far left-hand side and the far right-hand side of Equation (5.22) and comparing with Equation (5.21), we conclude

$$F(\mathcal{A}\,\mathrm{D}_\sigma^{-1}; \mathbf{X}^{(r)}) \;=\; \mathrm{ch}_r(\chi \otimes \chi_\sigma)(\mathbf{X}^{(r)}),$$

as required. □

We will review the strategy of Elizalde–Roichman's proof in a specific example, before proceeding to the colored version. Consider the set of derangements in $\mathfrak{S}_3$

$$\mathcal{D}_3 \;=\; \{231,\; 312\}.$$

We have
$$F(\mathcal{D}_3;\mathbf{x}) \;=\; F_{(2,1)}(\mathbf{x})+F_{(1,2)}(\mathbf{x}) \;=\; \mathrm{ch}(\chi^{(2,1)}).$$
A key observation is that for all $S=\{s_1<s_2<\cdots<s_k\}\subseteq[n-1]$,
$$\mathrm{R}_{n,S}^{-1} \;=\; (1,2,\ldots,s_1)\sqcup(s_1+1,s_1+2,\ldots,s_2)\sqcup\cdots\sqcup(s_k+1,s_k+2,\ldots,n)$$
(see, for example, [4, Observation 2.1]). Thus, we have
$$\mathrm{R}_{3,S}^{-1} \;=\; \begin{cases} 123=\{123\}, & \text{if } S=\varnothing\\ 1\sqcup 23=\{123,\ 213,\ 231\}, & \text{if } S=\{1\}\\ 12\sqcup 3=\{123,\ 132,\ 312\}, & \text{if } S=\{2\}\\ 1\sqcup 2\sqcup 3=\mathfrak{S}_3, & \text{if } S=\{1,2\}. \end{cases}$$

Theorem 5.2.2 asserts that
$$\mathcal{D}_3\,\mathrm{R}_{3,S}^{-1} \;=\; \begin{cases} \mathcal{D}_3, & \text{if } S=\varnothing\\ (2\sqcup 31)\cup(3\sqcup 21), & \text{if } S=\{1\}\\ (23\sqcup 1)\cup(31\sqcup 2), & \text{if } S=\{2\}\\ (2\sqcup 3\sqcup 1)\cup(3\sqcup 1\sqcup 2), & \text{if } S=\{1,2\} \end{cases} \tag{5.23}$$
is fine multiset for the $\mathfrak{S}_n$-character
$$\chi^{(2,1)}\downarrow_{\mathfrak{S}_S}\uparrow^{\mathfrak{S}_3} \;=\; \begin{cases} \chi^{(2,1)}\downarrow_{\mathfrak{S}_\varnothing}\uparrow^{\mathfrak{S}_3}=\chi^{(2,1)}, & \text{if } S=\varnothing\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(1,2)}}\uparrow^{\mathfrak{S}_3}, & \text{if } S=\{1\}\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(2,1)}}\uparrow^{\mathfrak{S}_3}, & \text{if } S=\{2\}\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(1,1,1)}}\uparrow^{\mathfrak{S}_3}, & \text{if } S=\{1,2\}. \end{cases}$$

The proof proceeds as follows: The set $\mathcal{D}_3$ can be partitioned (see [39, Proposition 5.9]) in such a way that each block is in Des-preserving bijection with a Cartesian product of Knuth classes[11] determined by S
$$\mathcal{D}_3 \;\longrightarrow\; \begin{cases} K_{[3]}(2,1)\cup K_{[3]}(1,2), & \text{if } S=\varnothing\\ \big(K_{[1]}(1)\times K_{[2,3]}(1,1)\big)\cup\big(K_{[1]}(1)\times K_{[2,3]}(2)\big), & \text{if } S=\{1\}\\ \big(K_{[2]}(2)\times K_{\{3\}}(1)\big)\cup\big(K_{[2]}(1,1)\times K_{\{3\}}(1)\big), & \text{if } S=\{2\}\\ \big(K_{[1]}(1)\times K_{\{2\}}(1)\times K_{\{3\}}(1)\big)^2, & \text{if } S=\{1,2\}. \end{cases} \tag{5.24}$$
and each one of these is fine for the $\mathfrak{S}_S$-character
$$\chi^{(2,1)}\downarrow_{\mathfrak{S}_S} \;=\; \begin{cases} \chi^{(2,1)}\downarrow_{\mathfrak{S}_\varnothing}=\chi^{(2,1)}, & \text{if } S=\varnothing\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(1,2)}}, & \text{if } S=\{1\}\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(2,1)}}, & \text{if } S=\{2\}\\ \chi^{(2,1)}\downarrow_{\mathfrak{S}_{(1,1,1)}}, & \text{if } S=\{1,2\}. \end{cases}$$

[11] Here, abusing notation, we write $K_{[a,b]}(\lambda)$ for K_P, where $P\in\mathrm{SYT}(\lambda)$ with content $[a,b]$.

Because the descent set is shuffle compatible, (5.24) implies that the distribution of Des is the same over (5.23) and over

$$\begin{cases} K_{[3]}(2,1) \cup K_{[3]}(1,2), & \text{if } S = \varnothing \\ \left(K_{[1]}(1) \sqcup K_{[2,3]}(1,1)\right) \cup \left(K_{[1]}(1) \sqcup K_{[2,3]}(2)\right), & \text{if } S = \{1\} \\ \left(K_{[2]}(2) \sqcup K_{\{3\}}(1)\right) \cup \left(K_{[2]}(1,1) \sqcup K_{\{3\}}(1)\right), & \text{if } S = \{2\} \\ \left(K_{[1]}(1) \sqcup K_{\{2\}}(1) \sqcup K_{\{3\}}(1)\right)^2, & \text{if } S = \{1,2\}. \end{cases} \tag{5.25}$$

Now, because shuffles of Knuth classes correspond to induction of characters (see [39, Lemma 5.6]) the sets in (5.25) are fine for the $\mathfrak{S}_3$-characters

$$\begin{cases} \chi^{(2,1)}, & \text{if } S = \varnothing \\ \left(\chi^{(1)} \boxtimes \chi^{(1,1)}\right) \uparrow^{\mathfrak{S}_3} + \left(\chi^{(1)} \boxtimes \chi^{(2)}\right) \uparrow^{\mathfrak{S}_3}, & \text{if } S = \{1\} \\ \left(\chi^{(2)} \boxtimes \chi^{(1)}\right) \uparrow^{\mathfrak{S}_3} + \left(\chi^{(1,1)} \boxtimes \chi^{(1)}\right) \uparrow^{\mathfrak{S}_3}, & \text{if } S = \{2\} \\ 2\left(\chi^{(1)} \boxtimes \chi^{(1)} \boxtimes \chi^{(1)}\right) \uparrow^{\mathfrak{S}_3}, & \text{if } S = \{1,2\}. \end{cases}$$

Analysing $\chi^{(2,1)} \downarrow_{\mathfrak{S}_S}$ as a sum of irreducible $\mathfrak{S}_S$-characters via the decomposition (5.24) and keeping in mind that Knuth classes correspond to irreducible characters yields

$$\chi^{(2,1)} \downarrow_{\mathfrak{S}_S} = \begin{cases} \chi^{(2,1)}, & \text{if } S = \varnothing \\ \left(\chi^{(1)} \boxtimes \chi^{(1,1)}\right) + \left(\chi^{(1)} \boxtimes \chi^{(2)}\right), & \text{if } S = \{1\} \\ \left(\chi^{(2)} \boxtimes \chi^{(1)}\right) + \left(\chi^{(1,1)} \boxtimes \chi^{(1)}\right), & \text{if } S = \{2\} \\ 2\left(\chi^{(1)} \boxtimes \chi^{(1)} \boxtimes \chi^{(1)}\right), & \text{if } S = \{1,2\} \end{cases}$$

and the proof follows by basic properties of induction.

We turn our attention now to the colored case, by proving a couple of technical lemmas which will be used in the proof of Theorem 5.6.5. We begin with the following, a signed version of which can be found in [4, Lemma 4.1].

Lemma 5.6.8. *For all* $\sigma = (\widehat{S} = \{s_1 < s_2 < \cdots < s_k < s_{k+1} := n\}, \epsilon) \in \Sigma(n,r)$

$$\begin{aligned} \overline{\mathrm{R}}_\sigma^{-1} = \left(1^{\epsilon(s_1)}, \cdots, s_1^{\epsilon(s_1)}\right) \sqcup \left((s_1+1)^{\epsilon(s_2)}, \cdots, s_2^{\epsilon(s_2)}\right) \sqcup \\ \cdots \sqcup \left((s_k+1)^{\epsilon(s_{k+1})}, \cdots, s_{k+1}^{\epsilon(s_{k+1})}\right). \end{aligned} \tag{5.26}$$

In particular, if σ is a minimal element of $\Sigma(n,r)$ then the left-hand side of Equation (5.26) becomes $\overline{\mathrm{D}}_\sigma^{-1}$.

Proof. Given an r-colored permutation w^δ, we can describe $\mathrm{sDes}(\overline{w^\epsilon}^{-1})$ in the following way: We read numbers $1, 2, \ldots, n$ in the window notation of w^δ in their natural order until either

- their color, which is determined by δ, changes
- or we reach the end of w^δ

and then, we start at the beginning. To each $i \in [n]$ we assign the color $\delta_{w_i^{-1}}$. The set of all last entries of each run together with its assigned color forms the conjugate-inverse colored descent set of w^ϵ (see, also, the discussion in [90, Page 37] for the unsigned case). Using this observation one can determine Equation (5.26). $\square$

Example 5.6.9. For example, for $w = 3^1 4^1 5^0 6^0 1^2 2^2 \in \mathfrak{S}_{6,4}$ with color vector $(1,1,0,0,2,2)$ we have the following runs

$$1^2 2^2,\ 3^1 4^1,\ 5^0 6^0$$

and therefore $\mathrm{sDes}(\overline{w}^{-1}) = \{2^2, 4^1, 6^0\}$. In addition, $\overline{w}^{-1} = 5^2 6^2 1 6 1 2^1 3^0 4^0$ which coincides with our computation. This is an example of an element in

$$\overline{\mathrm{R}}^{-1}_{\{2^2,3^1,4^1,6^0\}} \ = \ 1^2 2^2 \sqcup 3^1 \sqcup 4^1 \sqcup 5^0 6^0.$$

Lemma 5.6.10. *Let $\gamma^\epsilon = (\gamma_1^{\epsilon_1}, \gamma_2^{\epsilon_2}, \ldots, \gamma_k^{\epsilon_k})$ be an r-colored composition of n and by abuse of notation consider the Knuth classes $K_{\mathcal{B}_i(\gamma)}(\boldsymbol{\lambda}^i)$ for some r-partite partitions $\boldsymbol{\lambda}^i$ of γ_i, for all $1 \le i \le k$. The set*

$$\mathrm{K}_{\mathcal{B}_1(\gamma)}(\boldsymbol{\lambda}^1) \sqcup \cdots \sqcup \mathrm{K}_{\mathcal{B}_k(\gamma)}(\boldsymbol{\lambda}^k)$$

is Schur-positive for the $\mathfrak{S}_{n,r}$-character

$$\left(\chi^{\boldsymbol{\lambda}^1} \boxtimes \cdots \boxtimes \chi^{\boldsymbol{\lambda}^k}\right) \uparrow_{\mathfrak{S}_{\gamma^\epsilon,r}}^{\mathfrak{S}_{n,r}} .$$

Proof. It suffices to prove it for the case $k = 2$. In this case, we have to prove that the set $\mathrm{K}_{[k]}(\boldsymbol{\lambda}^1) \sqcup \mathrm{K}_{[k+1,n]}(\boldsymbol{\lambda}^2)$, for all r-partite partitions $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ of k and $n-k$, respectively, is Schur-positive for the $\mathfrak{S}_{n,r}$-character $\left(\chi^{\boldsymbol{\lambda}^1} \boxtimes \chi^{\boldsymbol{\lambda}^2}\right) \uparrow_{\mathfrak{S}_{(k,n-k),r}}^{\mathfrak{S}_{n,r}}$. On the one hand, by the properties of the characteristic map , see that

$$\mathrm{ch}_r\left(\left(\chi^{\boldsymbol{\lambda}^1} \boxtimes \chi^{\boldsymbol{\lambda}^2}\right) \uparrow_{\mathfrak{S}_{(k,n-k),r}}^{\mathfrak{S}_{n,r}}\right) \ = \ s_{\boldsymbol{\lambda}^1} s_{\boldsymbol{\lambda}^2} \ = \ s_{\boldsymbol{\lambda}^1 \oplus \boldsymbol{\lambda}^2},$$

where the direct sum of two r-partite partitions $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ is defined to be the r-partite partition whose j-colored part is the direct sums of the j-colored parts of $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$. On the other hand, we also know that

$$F(K_{[k]}(\boldsymbol{\lambda}^1); \mathbf{X}^{(r)}) F(K_{[k]}(\boldsymbol{\lambda}^2); \mathbf{X}^{(r)}) \ = \ s_{\boldsymbol{\lambda}^1 \oplus \boldsymbol{\lambda}^2}(\mathbf{X}^{(r)})$$

and therefore by Equation (2.5) it suffices to prove

$$F(\mathrm{K}_{[k]}(\boldsymbol{\lambda}^1) \sqcup \mathrm{K}_{[k+1,n]}(\boldsymbol{\lambda}^2); \mathbf{X}^{(r)}) \ = \ F(\mathrm{SYT}(\boldsymbol{\lambda}^1 \oplus \boldsymbol{\lambda}^2); \mathbf{X}^{(r)}). \tag{5.27}$$

This essentially a colored version of [39, Theorem 2.5].

We describe an sDes-preserving bijection

$$\varphi : \mathrm{K}_{[k]}(\boldsymbol{\lambda}^1) \sqcup \mathrm{K}_{[k+1,n]}(\boldsymbol{\lambda}^2) \longrightarrow \mathrm{SYT}(\boldsymbol{\lambda}^1 \oplus \boldsymbol{\lambda}^2)$$

as follows. For a colored permutation $w^\delta \in \mathrm{K}_{[k]}(\boldsymbol{\lambda}^1) \sqcup \mathrm{K}_{[k+1,n]}(\boldsymbol{\lambda}^2)$ there exist unique $u \in \mathrm{K}_{[k]}(\boldsymbol{\lambda}^1)$ and $v \in \mathrm{K}_{[k+1,n]}(\boldsymbol{\lambda}^2)$ such that $w^\delta \in u \sqcup v$. Let

$$\begin{aligned} \{w_i^{-1} : 1 \le i \le k\} &= \{a_1 < a_2 < \cdots < a_k\} \\ \{w_i^{-1} : k+1 \le i \le n\} &= \{b_1 < b_2 < \cdots < b_{n-k}\} \end{aligned}$$

be the list of positions of the letters of u (resp. v) in w^δ. Now, consider the r-partite tableau $\boldsymbol{Q}(u) \oplus \boldsymbol{Q}(v)$ whose j-colored part is the direct sum of the j-colored parts of the recording tableau of u and v via the colored Robinson–Schensted correspondence. Let $\varphi(w^\delta)$ be the standard r-partite tableau of shape $\boldsymbol{\lambda}^1 \oplus \boldsymbol{\lambda}^2$ obtained from $\boldsymbol{Q}(u) \oplus \boldsymbol{Q}(v)$ by replacing each letter i by a_i if $i \le k$ and by b_{i-k} if $i > k$. Then one sees that $\mathrm{sDes}(w^\delta) = \mathrm{sDes}(\varphi(w^\epsilon))$ and the proof follows. $\square$

Example 5.6.11. We will illustrate the correspondence φ described in the proof of Lemma 5.6.10 with a specific example. Consider the Knuth classes corresponding to

$$\begin{aligned} \boldsymbol{P}^1 &= \left(\begin{ytableau} 1 & 3 & 4 \\ 5 \end{ytableau}, \begin{ytableau} \varnothing \end{ytableau}, \begin{ytableau} 2 \end{ytableau} \right) \in \mathrm{SYT}\left((3,1), \varnothing, (1)\right) \\ \boldsymbol{P}^2 &= \left(\begin{ytableau} 6 & 8 \end{ytableau}, \begin{ytableau} 7 \end{ytableau}, \begin{ytableau} 9 \end{ytableau} \right) \in \mathrm{SYT}\left((2), (1), (1)\right). \end{aligned}$$

Let $u = 1^0 3^0 5^0 2^2 4^0 \in \mathrm{K}_{\boldsymbol{P}^1}$, $v = 7^1 6^0 9^2 8^0 \in \mathrm{K}_{\boldsymbol{P}^2}$ and consider the shuffle

$$w = 7^1 6^0 1^0 3^0 5^0 2^2 9^2 8^0 4^0 \in \mathrm{K}_{\boldsymbol{P}^1} \sqcup \mathrm{K}_{\boldsymbol{P}^2}.$$

We compute

$$\begin{aligned} \{w_i^{-1} : 1 \le i \le k\} &= \{3 < 4 < 5 < 6 < 9\} \\ \{w_i^{-1} : k+1 \le i \le n\} &= \{1 < 2 < 7 < 8\} \end{aligned}$$

and

$$\boldsymbol{Q}(u) \oplus \boldsymbol{Q}(v) = \left(\begin{ytableau} \none & \none & \none & 7 & 9 \\ 1 & 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} \none & 6 \\ \varnothing \end{ytableau}, \begin{ytableau} \none & 8 \\ 4 \end{ytableau} \right) \in \mathrm{SYT}\left(((3,1), \varnothing, (1)) \oplus ((2), (1), (1))\right).$$

Therefore, we have

$$\varphi(w) = \left(\begin{ytableau} \none & \none & \none & 2 & 8 \\ 3 & 4 & 5 \\ 9 \end{ytableau}, \begin{ytableau} \none & 1 \\ \varnothing \end{ytableau}, \begin{ytableau} \none & 7 \\ 6 \end{ytableau} \right)$$

and thus

$$\mathrm{sDes}(w) = \mathrm{sDes}(\varphi(w)) = \{1^1, 2^0, 5^0, 7^2, 8^0, 9^0\}$$

as expected.

We are now in position to give the proof of Theorem 5.6.5.

Proof of Theorem 5.6.5 (1). Let

$$\sigma = \left(\widehat{S} = \{s_1 < s_2 < \cdots < s_k < s_{k+1} := n\}, \epsilon \right) \in \Sigma(n, r).$$

There exists a multiset partition $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_m$ and sDes-preserving bijections

$$\mathcal{A}_i \longrightarrow \mathrm{K}_{[s_1]}(\boldsymbol{\lambda}_1^i) \times \mathrm{K}_{[s_1+1,s_2]}(\boldsymbol{\lambda}_2^i) \times \cdots \times K_{[s_k+1,s_{k+1}]}(\boldsymbol{\lambda}_k^i)$$

for some r-partite partitions $\boldsymbol{\lambda}_j^i$ of $s_{j+1} - s_j$, for all $0 \le j \le k$. (cf. Theorem 5.3.1). Notice that in terms of $\mathfrak{S}_{n,r}$-character this implies that the decomposition of $\chi \downarrow_{\mathfrak{S}_{\sigma,r}}$ into irreducible $\mathfrak{S}_{\sigma,r}$-characters is

$$\chi \downarrow_{\mathfrak{S}_{\sigma,r}} = \sum_i \chi^{\boldsymbol{\lambda}_1^i} \boxtimes \cdots \boxtimes \chi^{\boldsymbol{\lambda}_k^i}. \tag{5.28}$$

Lemma 5.6.8 implies that $\mathcal{A}\overline{\mathrm{R}}_\sigma^{-1}$ consists of all elements of

$$\left(w_1^{\delta_1+\epsilon(s_1)}, \cdots, w_{s_1}^{\delta_{s_1}+\epsilon(s_1)}\right) \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left(w_{s_k+1}^{\delta_{s_k+1}+\epsilon(s_{k+1})}, \cdots, w_{s_{k+1}}^{\delta_{s_{k+1}}+\epsilon(s_{k+1})}\right)$$

for all $w^\delta \in \mathcal{A}$. Since

- multiplying each $\mathcal{A}_i$ by $\overline{\mathrm{R}}_\sigma^{-1}$ amounts to shifting all colors inside $\boldsymbol{\lambda}^i$ by $-\epsilon(s_i)$, and
- sDes is shuffle-compatible (see Theorem 4.4.1)

we get that the distribution of sDes is the same over $\mathcal{A}_i\overline{\mathrm{R}}_\sigma^{-1}$ and

$\mathrm{K}_{[s_1]}(\mathrm{shift}_{-\epsilon(s_1)}(\boldsymbol{\lambda}^1)) \sqcup\!\sqcup \mathrm{K}_{[s_1+1,s_2]}(\mathrm{shift}_{-\epsilon(s_2)}(\boldsymbol{\lambda}^2)) \sqcup\!\sqcup \cdots \sqcup\!\sqcup K_{[s_k+1,s_{k+1}]}(\mathrm{shift}_{-\epsilon(s_{k+1})}(\boldsymbol{\lambda}^k))$.

By Lemma 5.6.10 and Equation (5.18), the latter is Schur-positive for the $\mathfrak{S}_{n,r}$-character

$$\left(\left(\chi^{\boldsymbol{\lambda}_1^i} \otimes \mathbb{1}_{s_1,\epsilon(s_1)}\right) \boxtimes \cdots \boxtimes \left(\chi^{\boldsymbol{\lambda}_k^i} \otimes \mathbb{1}_{s_{k+1}-s_k,\epsilon(s_{k+1})}\right)\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}}.$$

Therefore $\mathcal{A}\overline{\mathrm{R}}_\sigma^{-1}$ is Schur-positive for the $\mathfrak{S}_{n,r}$-character

$$\begin{aligned}
&\sum_i \left(\left(\chi^{\boldsymbol{\lambda}_1^i} \otimes \mathbb{1}_{s_1,\epsilon(s_1)}\right) \boxtimes \cdots \boxtimes \left(\chi^{\boldsymbol{\lambda}_k^i} \otimes \mathbb{1}_{s_{k+1}-s_k,\epsilon(s_{k+1})}\right)\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}} \\
&\quad = \left(\sum_i \left(\chi^{\boldsymbol{\lambda}_1^i} \otimes \mathbb{1}_{s_1,\epsilon(s_1)}\right) \boxtimes \cdots \boxtimes \left(\chi^{\boldsymbol{\lambda}_k^i} \otimes \mathbb{1}_{s_{k+1}-s_k,\epsilon(s_{k+1})}\right)\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}} \\
&\quad = \left(\sum_i \left(\chi^{\boldsymbol{\lambda}_1^i} \boxtimes \cdots \boxtimes \chi^{\boldsymbol{\lambda}_k^i}\right) \otimes \left(\mathbb{1}_{s_1,\epsilon(s_1)} \boxtimes \cdots \boxtimes \mathbb{1}_{s_{k+1}-s_k,\epsilon(s_{k+1})}\right)\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}} \\
&\quad = \left(\left(\sum_i \chi^{\boldsymbol{\lambda}_1^i} \boxtimes \cdots \boxtimes \chi^{\boldsymbol{\lambda}_k^i}\right) \otimes \mathbb{1}_\sigma\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}} \\
&\quad = \left(\chi \downarrow_{\mathfrak{S}_{\sigma,r}} \otimes \mathbb{1}_\sigma\right) \uparrow_{\mathfrak{S}_{\sigma,r}}^{\mathfrak{S}_{n,r}},
\end{aligned}$$

where the last equality follows by Equation (5.28) and the proof is concluded. $\square$

Comparing Equation (5.21) and Equation (5.15) yields

$$F(\overline{\mathrm{R}}_\sigma^{-1}; \mathbf{X}^{(r)}) \;=\; \mathrm{ch}_r\left(\mathbb{1}_\sigma \uparrow^{\mathfrak{S}_{n,r}}\right)(\mathbf{X}^{(r)}), \tag{5.29}$$

for all $\sigma \in \Sigma(n,r)$, which can be easily computed via Equation (5.13). This enables us to compute a refined distribution of the (flag) major index over conjugate-inverse colored descent classes. Garsia and Gessel [49, Theorem 3.1] computed the distribution of the major index over inverse descent classes of $\mathfrak{S}_n$ and Adin, Brenti and Roichman [4, Theorem 3.3] computed the distribution of the flag major index over inverse descent classes of type B. We remark that the authors in [4] use a notion of descent of a signed permutation which arises when one views the hyperoctahedral group a Coxeter group of type B_n.

For nonnegative integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, we define the *$q$-multinomial coefficient* to be

$$\binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k}_q \;:=\; \frac{[n]_q!}{[\alpha_1]_q![\alpha_2]_q! \cdots [\alpha_k]_q!}.$$

For $k = 2$ we get the usual q-binomial coefficient, defined in Section 3.2. For $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$, let

$$\binom{n}{\Delta S}_q \;:=\; \binom{n}{s_1, s_2 - s_1, \ldots, n - s_k}_q.$$

Theorem 5.6.12. *For every r-colored subset $\sigma = (\widehat{S} = \{s_1 < s_2 < \cdots < s_k < s_{k+1} = n\}, \epsilon)$,*

$$\sum_{w \in \overline{\mathrm{R}}_\sigma^{-1}} q^{\mathrm{maj}(w)} \mathbf{p}^{\mathrm{n}(w)} \;=\; \binom{n}{\Delta S}_q \prod_{i=0}^{k} p_{\epsilon(s_i)}^{s_{i+1}-s_i}$$

$$\sum_{w \in \overline{\mathrm{R}}_\sigma^{-1}} q^{\mathrm{fmaj}(w)} \mathbf{p}^{\mathrm{n}(w)} \;=\; \binom{n}{\Delta S}_{q^r} q^{\sum_{i=0}^{k}(s_{i+1}-s_i)\epsilon(s_i)} \prod_{i=0}^{k} p_{\epsilon(s_i)}^{s_{i+1}-s_i},$$

where $s_0 := 0$.

Proof. In view of Equation (5.13), the proof follows by specializing Equation (5.29) as in Corollaries 2.3.2 and 2.3.5 for $\mathcal{A} = \overline{\mathrm{R}}_\sigma^{-1}$ and using the formulas

$$\mathrm{ps}_{q,\mathbf{p}}^{(r)}(h_n(\mathbf{x}^{(j)})) \;=\; \frac{p_j^n}{(q)_n}$$

$$\psi_{q,\mathbf{p}}^{(r)}(h_n(\mathbf{x}^{(j)})) \;=\; \frac{q^{nj} p_j^n}{(q^r)_n}.$$

□

Remark 5.6.13. An alternative proof of Theorem 5.6.12 can be given as follows. Consider the r-colored poset P which is the disjoint union of the chains

$$(s_i + 1)^{\epsilon(s_{i+1})} <_P (s_i + 2)^{\epsilon(s_{i+1})} <_P \cdots <_P (s_{i+1})^{\epsilon(s_{i+1})},$$

for all $0 \leq i \leq k$. Lemma 5.6.8 implies that

$$F(\overline{\mathrm{R}}_{\sigma}^{-1}; \mathbf{X}^{(r)}) = \Gamma(P; \mathbf{X}^{(r)}) = \prod_{i=0}^{k} h_{s_{i+1}-s_i}(\mathbf{x}^{(\epsilon(s_i))}),$$

where the second equality follows from Equation (4.4) and the proof follows in the same way as above.

We finish by this chapter by posing the following conjecture.

Conjecture 5.6.14. *For a Schur-positive multiset $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$ and every r-colored subset $\sigma \in \Sigma(n,r)$, the distribution of the colored descent set over $\mathcal{A}\,\mathrm{D}_{\sigma}^{-1}$ and over $\mathrm{D}_{\sigma}^{-1}\,\mathcal{A}$ is the same. In particular,*

$$F(\mathcal{A}\,\mathrm{D}_{\sigma}^{-1}; \mathbf{X}^{(r)}) \;=\; F(\mathrm{D}_{\sigma}^{-1}\,\mathcal{A}; \mathbf{X}^{(r)}).$$

Keep thy mind in hell and despair not.

Saint Silouan the Athonite

List of Symbols

Notation	**Description**
$[a,b]$	the integer interval $\{a, a+1, \dots, b\}$
$A_n(x,q)$	n-th q-Eulerian polynomial
$A_n(x)$	n-th Eulerian polynomial
$\mathcal{A}(P)$	set of P-partitions
$\mathcal{A}^{(r)}(P)$	set of r-colored P-partitions
$\mathcal{A}_{\mathrm{stat}}$	shuffle algebra of stat
$\mathfrak{B}_n$	hyperoctahedral group of order $2^n n!$
$\mathcal{B}_i(\alpha)$	ith block of a composition α
$\mathbb{C}$	complex numbers
$\mathbb{C}[[\mathbf{x}]]$	algebra of formal power series in $\mathbf{x}$ over $\mathbb{C}$
ch	Frobenius characteristic map
$\mathrm{c}^j(w)$	number of j-colored cycles of w
ct	cycle type of a colored permutation
$\mathrm{Comp}(n)$	set of all compositions of n
$\mathrm{Comp}(n,r)$	set of all r-colored compositions of n
co	(colored) composition corresponding to a (colored) set
$\mathrm{col}^{\mathrm{o}}(\lambda)$	number of columns of λ of odd length
comaj	comajor index
csum	color sum statistic
$\mathcal{D}_n$	set of all derangements of $\mathfrak{S}_n$
$\mathcal{D}_{n,r}$	set of all r-colored derangements of $\mathfrak{S}_{n,r}$
Des	descent set
Des*	descent set without 0 and n
des	descent number
$\mathrm{D}_{n,S}$	descent class corresponding to S
$\mathrm{D}_{n,S}^{-1}$	inverse descent class corresponding to S
D_{σ}	colored descent class corresponding to σ
D_{σ}^{-1}	inverse colored descent class corresponding to σ
$\overline{\mathrm{D}}_{\sigma}^{-1}$	conjugate-inverse colored descent class corresponding to σ

Notation	Description
e_λ	elementary symmetric function associated to λ
eul	Eulerian statistic
exc	number of excedances
fdes	flag descent number
$F_{n,S}$	fundamental quasisymmetric function corresponding to S
F_σ	fundamental colored quasisymmetric function corresponding to σ
fix	number of fixed points
fix^j	number of j-colored fixed points
F_{w^ϵ}	$F_{\mathrm{sDes}(w)}$
$F_{\boldsymbol{Q}}$	$F_{\mathrm{sDes}(\boldsymbol{Q})}$
$F(\mathcal{A};\mathbf{x})$	quasisymmetric generating function of $\mathcal{A}$
h_λ	complete homogeneous symmetric function associated to λ
$\mathcal{I}_n$	set of all involutions of $\mathfrak{S}_n$
$\mathcal{I}_{n,r}$	set of all r-colored involutions of $\mathfrak{S}_{n,r}$
$\mathcal{I}_{n,r}^{\mathrm{abs}}$	set of all absolute involutions of $\mathfrak{S}_{n,r}$
inv	number of inversions
K_P	Knuth class corresponding to P
K_α	peak quasisymmetric function associated to α
ldes	length descent number
lmaj	length flag major index
LPk	left peak set
lpk	left peak number
ℓ_S	length function with respect to S
$\ell(\lambda)$	length of λ
$\mathcal{L}(P)$	set of linear extensions of P
mah	Mahonian statistic
maj	major index
M_α	monomial quasisymmetric function corresponding to α
M_{γ^ϵ}	monomial colored quasisymmetric function corresponding to γ^ϵ
m_λ	monomial symmetric function associated to λ
MR	Mantaci-Reutenauer algebra
$\mathbb{N}$	nonnegative integers
$[n]_q$	$1+q+\cdots+q^{n-1}$
$[n]$	$\{1,2,\ldots,n\}$
$\mathrm{n}_j(w^\epsilon)$	number of $1 \le i \le n$ such that $\epsilon_i = j$
$\binom{n}{k}_q$	q-binomial coefficient
$\binom{n}{\alpha_1,\alpha_2,\ldots,\alpha_k}_q$	q-multinomial coefficient
$\mathbf{p}$	$(p_0,p_1,\ldots,p_{r-1})$; a sequence of indeterminates
p	indeterminate

Notation	Description
$\mathbb{P}^{(r)}$	$\mathbb{Z}_{>0} \times \mathbb{Z}_r$
Pk	peak set
pk	peak number
ps_q	stable principal specialization
$\mathrm{ps}_{q,m}$	principal specialization of order m
QSym	algebra of quasisymmetric functions
$\mathrm{QSym}^{(r)}$	algebra of colored quasisymmetric functions
q	indeterminate
$(q)_n$	$(1-q)(1-q^2)\cdots(1-q^n)$
$\mathcal{R}(\mathfrak{S}_n)$	$\mathbb{Z}$-module generated by irreducible $\mathfrak{S}_n$-characters
$\mathcal{R}(\mathfrak{S}_{n,r})$	$\mathbb{Z}$-module generated by irreducible $\mathfrak{S}_{n,r}$-characters
$\mathrm{R}_{n,S}$	union of descent classes corresponding to $T \subseteq S$
$\mathrm{R}_{n,S}^{-1}$	union of inverse descent classes corresponding to $T \subseteq S$
$\overline{\mathrm{R}}_{\sigma}^{-1}$	union of conjugate-inverse colored descent classes corresponding to $\tau \preceq \sigma$
r_α	ribbon Schur function associated to α
$r_{n,S}$	ribbon Schur function associated to S
$\lvert S\rvert$	cardinality of S
$\hat{S}$	$S \cup \{n\}$
$\mathrm{sum}(S)$	sum of all elements of S
s_λ	Schur function associated to λ
sDes	colored descent set
$\mathfrak{S}_\alpha$	Young subgroup corresponding to α
$\mathfrak{S}_n$	symmetric group of $[n]$
$\mathfrak{S}_{n,r}$	r-colored permutation group of order $r^n n!$
sPk	colored peak set
Sol	Solomon's descent algebra of type A
Sym	algebra of symmetric functions
$\mathrm{Sym}^{(r)}$	$\mathrm{Sym}^{\otimes r}$
$\mathrm{SYT}(\lambda)$	set of all standard Young tableaux of shape λ
$\mathrm{SYT}(\boldsymbol{\lambda})$	set of all standard Young r-partite tableaux of shape $\boldsymbol{\lambda}$
$\mathrm{SSYT}(\lambda)$	set of all semistandard Young tableaux of shape λ
SYT_n	set of all standard Young tableaux of size n
$\mathrm{SYT}_{n,r}$	set of all standard Young r-partite tableaux of size n
shift_j	shift operator; shifts all colors by j positions to the left
w^{-1}	inverse of a (colored) permutation w
w^ϵ	colored permutation with underlying permutation w and color vector ϵ
$\overline{w^\epsilon}$	colored permutation with underlying permutation w and color vector $-\epsilon$

Notation	Description
$\mathbf{x}$	sequence $(x_1, x_2, \dots)$ of commuting indeterminates
$\mathbf{x}^{(j)}$	sequence $(x_1^{(j)}, x_2^{(j)}, \dots)$ of commuting indeterminates
$\mathbf{X}^{(r)}$	$(x_i^{(0)}, x_i^{(1)}, \dots, x_i^{(r-1)})_{i\geq 1}$
$(x;q)_n$	$(1-x)(1-xq)\cdots(1-xq^n)$
$(x;q)_\infty$	$\prod_{i\geq 0}(1-xq^i)$
$\mathbb{Z}$	integers
$\mathbb{Z}_r$	additive cyclic group of order r
$\mathbb{Z}_r \wr \mathfrak{S}_n$	wreath product of $\mathfrak{S}_n$ and $\mathbb{Z}_r$
Z_α	(colored) ribbon corresponding to the (colored) composition α
$\widehat{\alpha}$	peak composition corresponding to α
$\tilde{\epsilon}$	color vector of ϵ
ζ	a primitive r-th root of unity
λ	integer partition
$\boldsymbol{\lambda}$	r-partite partition
$\lvert\lambda\rvert$	size of λ
Π	algebra of peak functions
$\Sigma(n,r)$	set of all r-colored subsets of $[n]$
χ^λ	irreducible $\mathfrak{S}_n$-character corresponding to λ
$\chi^{\boldsymbol{\lambda}}$	irreducible $\mathfrak{S}_{n,r}$-character corresponding to $\boldsymbol{\lambda}$
χ_σ	descent $\mathfrak{S}_{n,r}$-character corresponding to σ
Ω_n	set of all barred integers of $[n]$
$\Omega_{n,r}$	set of all r-colored integers of $[n]$
$\mathbb{1}_{n,j}$	one-dimensional $\mathfrak{S}_{n,r}$-character corresponding to the r-partite partition having (n) as j-colored part and all other parts are empty
$<_c$	color order on $\Omega_{n,r}$
$<_\ell$	length order on $\Omega_{n,r}$
$<_{\mathrm{St}}$	Steingrímsson's order on $\Omega_{n,r}$
$<_{\mathrm{llex}}$	left lexicographic order on $[n]\times\mathbb{Z}_r$
$\vdash$	integer partition or r-partite partition
$\downarrow$	restriction of representations
$\uparrow$	induction of representations
$\circ$	induction product
$\vDash$	integer composition
$\sqcup\!\sqcup$	shuffle
$\oplus$	direct sum
$\sqcup$	disjoint union
$\otimes$	inner (or Kronecker) tensor product
$\boxtimes$	outer tensor product
$:=$	equals by definition

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