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On the growth problem for Hadamard matrices

Master Thesis

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Abstract

In the present thesis, we study one of the most famous problems in Numerical Analysis: the growth problem for Hadamard matrices. It concerns the determination of the growth factor of Hadamard matrices, a quantity associated with the stability of the Gaussian elimination algorithm, and it is a very challenging problem; it has only been solved for matrices of small orders and pertinent research is ongoing.

We begin with a brief presentation of the definition and basic properties of determinants of matrices in Chapter 1. Determinants play a major role in solving linear systems of equations and they are also of great importance in our study for the evaluation of the growth factor.

In Chapter 2, we present Gaussian elimination, the most useful numerical method for evaluating determinants and solving linear systems. It consists of a sequence of elementary row and column operations that transform a given matrix to an equivalent upper triangular one. From a numerical point of view, Gaussian elimination is an efficient algorithm, however, if implemented in its original form, it can be unstable. To overcome this issue, we use a technique called pivoting. There are two types of pivoting: partial and complete. In the last section of the chapter , we examine in more detail the stability of Gaussian elimination, with and without pivoting, in terms of backward error analysis and we present the notion of growth factor, a quantity with which the stability is closely associated.

In Chapter 3, we present a special category of matrices called Hadamard matrices. These are characterized by unique properties, one of which being that they seem to be the only matrices that attain growth factor equal to their size. Their special structure allows us to find formulae and values for their minors, a generally very difficult task. We conclude with a brief presentation of some special cases and generalizations of Hadamard matrices and an overview of their applications in a variety of fields.

In Chapter 4, we restrict our attention to the growth factor of Hadamard matrices, associated with Gaussian elimination with complete pivoting. Its determination is one of the most famous and challenging open problems in Numerical Analysis; it has been achieved only for orders 1 to 16 and a lot of investigation concerning it is ongoing. A key element in the study of the growth factor of a matrix is the evaluation of its pivots and its minors, thus we focus on deriving useful formulae for their computation and extensively examining the possible values that they can take. In the last section of the chapter, we proceed to the computation of the growth factor of Hadamard matrices of orders 1 to 12. We also introduce a new lower bound for pivots that emerged in our study and employ it to rediscover the computation for case 12. We give a brief overview

of the, significantly more complicated, proof for order 16 and finally, we display some more results obtained from further research on the growth problem.

Περίληψη

Στην παρούσα εργασία, μελετάμε ένα από τα πιο διάσημα προβλήματα της Αριθμητικής Ανάλυσης: το πρόβλημα του συντελεστή μεγέθυνσης για τους πίνακες Hadamard, μιας ποσότητας που σχετίζεται με την ευστάθεια του αλγορίθμου της απαλοιφής του Gauss. Το πρόβλημα αυτό είναι πολύ απαιτητικό· έχει λυθεί μόνο για μικρά μεγέθη πινάκων και η έρευνα που γίνεται στο σχετικό πεδίο είναι μεγάλη.

Ξεκινάμε με μία σύντομη παρουσίαση του ορισμού και των βασικών ιδιοτήτων οριζουσών πινάκων στο Κεφάλαιο 1. Οι ορίζουσες παίζουν μεγάλο ρόλο στην επίλυση γραμμικών συστημάτων εξισώσεων και είναι επίσης πολύ σημαντικές για την μελέτη του υπολογισμού του συντελεστή μεγέθυνσης.

Στο Κεφάλαιο 2, παρουσιάζουμε την απαλοιφή Gauss, την πιο διαδεδομένη αριθμητική μέθοδο υπολογισμού οριζουσών και επίλυσης γραμμικών συστημάτων. Αποτελείται από μία σειρά στοιχειωδών πράξεων γραμμών και στηλών, οι οποίες μετατρέπουν ένα δοσμένο πίνακα σε έναν ισοδύναμο άνω τριγωνικό. Από αριθμητικής άποψης, η απαλοιφή Gauss είναι ένας αποτελεσματικός αλγόριθμος, όμως αν εφαρμοστεί στην κλασσική του εκδοχή, μπορεί να είναι ασταθής. Για να αντιμετωπίσουμε αυτό το ζήτημα, χρησιμοποιούμε μία τεχνική που λέγεται οδήγηση. Υπάρχουν δύο είδη οδήγησης: η μερική και η ολική. Στην τελευταία παράγραφο του κεφαλαίου, εξετάζουμε πιο λεπτομερώς την ευστάθεια της απαλοιφής Gauss, με και χωρίς οδήγηση, με όρους της backwards ανάλυσης σφάλματος και παρουσιάζουμε την έννοια του συντελεστή μεγέθυνσης, μιας ποσότητας με την οποία η ευστάθεια συνδέεται στενά.

Στο Κεφάλαιο 3, παρουσιάζουμε μία ειδική κατηγορία πινάκων που λέγονται πίνακες Hadamard. Αυτοί χαρακτηρίζονται από μοναδικές ιδιότητες, μία από τις οποίες είναι ότι φαίνεται να αποτελούν τους μόνους πίνακες των οποίων ο συντελεστής μεγέθυνσης είναι ίσος με το μέγεθός τους. Η ειδική δομή τους μας επιτρέπει να βρούμε μαθηματικούς τύπους και τιμές για τις υποορίζουσές τους, κάτι το οποίο είναι γενικά πολύ δύσκολο να γίνει. Το κεφάλαιο ολοκληρώνεται με μία σύντομη παρουσίαση κάποιων ειδικών περιπτώσεων και γενικεύσεων των πινάκων Hadamard και μια σκιαγράφηση των εφαρμογών τους σε ένα πλήθος επιστημονικών πεδίων.

Στο Κεφάλαιο 4, εστιάζουμε την προσοχή μας στον συντελεστή μεγέθυνσης πινάκων Hadamard που αφορά την απαλοιφή Gauss με ολική οδήγηση. Ο καθορισμός του είναι ένα από τα πιο διάσημα και απαιτητικά ανοικτά προβλήματα στην Αριθμητική Ανάλυση· έχει επιτευχθεί μόνο για τις τάξεις 1 έως 16 και η σχετική έρευνα που γίνεται είναι μεγάλη. Ένα στοιχείο – κλειδί στην μελέτη του συντελεστή μεγέθυνσης ενός πίνακα είναι ο υπολογισμός των οδηγών στοιχείων και των υποοριζουσών του, συνεπώς δίνουμε έμφαση στην εύρεση χρήσιμων μαθηματικών τύπων για τον καθορισμό τους και στην λεπτομερή εξέταση των πιθανών τιμών που μπορούν να πάρουν. Στην τελευταία παράγραφο του κεφαλαίου, προχωράμε στον καθορισμό του συντελεστή μεγέθυνσης πινάκων Hadamard τάξεων 1 έως 12. Επίσης παρουσιάζουμε ένα νέο κάτω φράγμα για οδηγά στοιχεία το οποίο προέκυψε στην

έρευνά μας και το χρησιμοποιούμε για να δώσουμε έναν νέο τρόπο υπολογισμού του συντελεστή μεγέθυνσης για την περίπτωση 12. Κάνουμε μία σύντομη περιγραφή της, αρκετά πιο πολύπλοκης, απόδειξης για την τάξη 16 και τέλος παρουσιάζουμε κάποια επιπλέον αποτελέσματα που έχουν προκύψει από την περαιτέρω έρευνα πάνω σε αυτό το πεδίο.

Acknowledgments

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Contents

Chapter 1

Determinants

In the first chapter, we make a brief presentation of the basic properties of determinants. Determinants play an important part in the determination of the solutions of linear systems. We will later see that they are of crucial importance in our study, since they give insightful information about the growth factor of a matrix. The chapter is largely based on paragraphs 3.4 and 6.3 of [39].

1.1 Definition and properties

In the following, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.1.1. *A function $D : \mathbb{F}^{n\times n} \to \mathbb{F}$, from the set of $n \times n$ matrices over $\mathbb{F}$,$n \in \mathbb{N}$, to $\mathbb{F}$, is called a* determinant function *if it satisfies the following properties for every $A = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \in \mathbb{F}^{n\times n}$, $r_i \in \mathbb{F}^{1\times n}$:*

D_1: If $r_i = r + r'$ for some $i \in \{1, \dots, n\}$, $r, r' \in \mathbb{F}^{1\times n}$, then

$$D\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} = D\begin{pmatrix} r_1 \\ \vdots \\ r + r' \\ \vdots \\ r_n \end{pmatrix} = D\begin{pmatrix} r_1 \\ \vdots \\ r \\ \vdots \\ r_n \end{pmatrix} + D\begin{pmatrix} r_1 \\ \vdots \\ r' \\ \vdots \\ r_n \end{pmatrix}$$

D_2: If $r_i = \lambda r, \lambda \in \mathbb{F}$, $r \in \mathbb{F}^{1\times n}$, then

$$D\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} = \lambda D\begin{pmatrix} r_1 \\ \vdots \\ r \\ \vdots \\ r_n \end{pmatrix}$$

D_3: If A has two identical rows, i.e. there exist $i, j \in \{1, \dots, n\}$ so that $r_i = r_j$, then $D(A) = 0$
D_4: If $A = I_n$, then $D(A) = 1$

Remark From properties D_1 and D_2 follows by induction that

$$D\begin{pmatrix} r_1 \\ \vdots \\ \sum_{s=1}^{m} \lambda_s \rho_s \\ \vdots \\ r_n \end{pmatrix} = \sum_{s=1}^{m} \lambda_s D\begin{pmatrix} r_1 \\ \vdots \\ \rho_s \\ \vdots \\ r_n \end{pmatrix}.$$

Also from property D_2 follows that if $r_i = 0$ for some $i \in \{1, \ldots, n\}$, then

$$D\begin{pmatrix} r_1 \\ \vdots \\ 0 \\ \vdots \\ r_n \end{pmatrix} = D\begin{pmatrix} r_1 \\ \vdots \\ 0 \cdot r_i \\ \vdots \\ r_n \end{pmatrix} = 0D\begin{pmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{pmatrix} = 0.$$

Elementary row operations and determinants

There are three types of elementary row operations:

- row multiplication (multiplication of all elements of a row by a non-zero scalar)
- row addition (replacement of a row by the sum of that row and a multiple of another row)
- row switching (interchange of two rows)

Property D_2 describes how a determinant function behaves under row multiplication. The following proposition describes the behavior of determinant functions under row addition and row switching.

Proposition 1.1.2. *Assume that D is a determinant function and $A \in \mathbb{F}^{n\times n}$ is an $n \times n$ matrix.*
(i) If B is the matrix that results when we apply row addition to A, then $D(B) = D(A)$.
(ii) If C is the matrix that results when we apply row switching on A, then $D(C) = -D(A)$.

Corollary 1.1.3. *Assume that D is a determinant function and $A, B \in \mathbb{F}^{n\times n}$ are two row-equivalent $n \times n$ matrices (i.e. one can be obtained from the other by applying a sequence of elementary row operations). Then $D(A) = 0$ if and only if $D(B) = 0$.*

Definition 1.1.4. *A square matrix $A \in \mathbb{F}^{n\times n}$ is* invertible *or* non singular *if there exists a square matrix $B \in \mathbb{F}^{n\times n}$ of the same order that satisfies $AB = BA = I$. In that case, B is called the* inverse *of A and is denoted $B = A^{-1}$.*

Proposition 1.1.5. *Assume that $A \in \mathbb{F}^{n\times n}$ is an $n \times n$ matrix. Then A is invertible if and only if $det(A) \neq 0$.*

Examples 1.1.6. *(i) For $n = 1$,*

$$D : \mathbb{F}^{1\times 1} \to \mathbb{F}$$

$$D([a_{11}]) = a_{11}$$

is a determinant function.
(ii) For $n = 2$,

$$D : \mathbb{F}^{2\times 2} \to \mathbb{F}$$

$$D\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

is a determinant function.
(iii) Assume that $D : \mathbb{F}^{(n-1)\times(n-1)} \to \mathbb{F}$ is a determinant function from the set of all $(n-1)\times(n-1)$ matrices and A is an $n \times n$ matrix. Then for every $j \in \{1, \dots, n\}$, the function

$$f_j : \mathbb{F}^{n\times n} \to \mathbb{F}$$

$$f_j(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} D(A_{ij}),$$

where A_{ij} is the $(n-1)\times(n-1)$ matrix that results by removing the i-th row and the j-th column from A, is a determinant function froom the set of all $n \times n$ matrices.

From the examples above, one can prove by induction the following proposition

Proposition 1.1.7. *For every $n \geq 1$, there exists at least one determinant function from the set of all $n \times n$ matrices over $\mathbb{F}$ that satisfies the properties of Definition 1.1.1.*

Proposition 1.1.8. *If $D, D' : \mathbb{F}^{n\times n} \to \mathbb{F}$ are determinant functions from the set of all $n \times n$ matrices, then $D = D'$.*

The previous propositions lead us to the following theorem

Theorem 1.1.9. *For every $n \geq 1$, there exists a unique determinant function from the set of all $n \times n$ matrices over $\mathbb{F}$.*

Definition 1.1.10. *If $A \in \mathbb{F}^{n\times n}$ is an $n\times n$ matrix over $\mathbb{F}$ and D is the uniquely defined determinant function from the set of all $n \times n$ matrices, then we define the* determinant *of A, we denote* det(A)*, to be the image of A under D, i.e. $det(A) = D(A)$.*

Remark Let $A \in \mathbb{F}^{n\times n}$ be an $n \times n$ matrix over $\mathbb{F}$ and $f_j : \mathbb{F}^{n\times n} \to \mathbb{F}, j = 1, \dots, n$ be the determinant functions defined in Example 1.1.6 (iii). From Theorem 1.1.9 and Definition 1.1.10 follows that $f_j(A) = det(A), \forall j$. Expression $f_j(A)$ is called the *Laplace expansion along the j-th column*. Similarly, we can define the *Laplace expansion along the i-th row* and prove that it satisfies all the properties of a determinant function. The uniqueness of the determinant function guarantees that for every i, j, all the above expressions are equivalent, hence we can calculate the determinant of a matrix by expanding along any row or column.

Proposition 1.1.11. *The determinant of an $n \times n$ triangular matrix is the product of its diagonal elements.*

Proof. By induction. For upper triangular matrices: For $n = 1$ the statement is true. For the induction step, we assume that the statement is true for $n-1, n \geq 2$. Let $A = (a_{ij})$ be an upper triangular $n \times n$ matrix. Using the Laplace expansion along the first column, we get $det(A) = \sum_{i=1}^{n}(-1)^{i+1}a_{i1}det(A_{i1})$ and since $a_{i1} = 0, i > 1$, $det(A) = a_{11}det(A_{11})$. A_{11} is an $(n-1) \times (n-1)$ upper triangular matrix with diagonal elements $a_{22}, \ldots, a_{nn}$, therefore $det(A_{11}) = a_{22} \ldots a_{nn}$. In conclusion $det(A) = a_{11}a_{22} \ldots a_{nn}$.
For lower triangular matrices: we work similarly, using the Laplace expansion along the first row. □

Theorem 1.1.12. *For every $A, B \in \mathbb{F}^{n \times n}$, $det(AB) = det(A)det(A)$.*

Corollary 1.1.13. *If A is invertible, $det(A^{-1}) = \dfrac{1}{det(A)}$.*

Theorem 1.1.14. *For every $A \in \mathbb{F}^{n \times n}$ $det(A) = det(A^T)$.*

Geometric interpretation of determinants

For real matrices, the determinant represents the volume of the parallelepiped defined by the rows (or columns) of the matrix. More specifically, if $A = [a_1 \ldots a_n]$, $a_i \in \mathbb{R}^{n \times 1}$, is an $n \times n$ real matrix, then the n-parallelepiped defined by the n-dimensional vectors $a_1, \ldots, a_n$ (the columns of A) is

$$P = \{c_1a_1 + \ldots + c_na_n | 0 \leq c_i \leq 1, \forall i\}.$$

The absolute value of the determinant of A can be proven to be the n-dimensional volume of P, i.e. $|det(A)| = vol(P)$. The parallelepiped defined by the rows of A is a different parallelepiped in general, but it has the same volume. That follows from the fact that this parallelepiped will be the parallelepiped defined by the columns of A^T, hence its volume will be $|det(A^T)|$ and from Theorem 1.1.14, that is equal to $|det(A)|$.

Determinants and linear systems of equations

Determinants can be used to express explicitly the solutions of linear systems. A linear system of n equations in n variables $x_1, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be expressed in the matrix form

$$A \cdot x = b,$$

where $A = (a_{ij})_{i,j=1,\ldots,n}$, $x = [x_1 \ldots x_n]^T$ and $b = [b_1 \ldots b_n]^T$. The system has a unique solution if and only if A is invertible, i.e. $det(A) \neq 0$, and in that case it is given by

$$x_i = \frac{1}{det(A)} \sum_{j=1}^{n} (-1)^{i+j} b_j det(A_{ji}), \quad i = 1, \ldots, n.$$

This formula is known as *Cramer's rule*.

Computational Complexity

Computational complexity concerns the amount of operations in floating-point arithmetic that are required for the evaluation of a quantity by a computer. The study of the computational complexity is of great importance in Numerical Analysis and algorithm design, since it provides information about the performance of an algorithm. A floating-point operation (flop) is a calculation of the general form $d \leftarrow a + b \cdot c$, where a, b, c and d are machine numbers and $\leftarrow$ denotes the assignment operation.

We are now going to evaluate the number of operations required for the computation of a determinant using the Laplace expansion. For an $n \times n$ matrix A,

$$det(A) = \sum_{k=1}^{n} a_{1k} (-1)^{1+k} det(A_{1k}).$$

This expression requires n multiplications and each one of them requires the evaluation of the determinant of an $(n-1) \times (n-1)$ matrix. Using again the Laplace expansion, we will see that each determinant evaluation includes $(n-1)$ multiplications that require the evaluation of an $(n-2) \times (n-2)$ determinant. Following this procedure, we can calculate the total number of operations to be

$$n(n-1) \ldots 1 = n!$$

Chapter 2

Gaussian Elimination

In this chapter, we will present the method of Gaussian Elimination. It is the most useful numerical method for solving linear systems and computing determinants of matrices. The chapter is mostly based on paragraph 3.2 of [1] and Chapter 3 of [25].

2.1 Gaussian elimination without pivoting

As we have seen in Chapter 1, the evaluation of the determinant of an $n \times n$ matrix requires $n!$ operations. When n is large, the number of operations increases drastically, making the computation impossible. Naturally, we are looking for a more efficient method of evaluating determinants.

From Proposition 1.1.11, we see that when a matrix is triangular, the evaluation of its determinant is significantly simpler, since it is the product of the diagonal entries. This observation leads us to the search for a method that triangularizes a given matrix while maintaining its determinant. Such a method is Gaussian elimination (GE), which consists of a sequence of elementary row operations that transform a matrix into an upper triangular one that is equivalent to the original.

We will present the method for the case of square invertible matrices. However, the algorithm can be extended to include non invertible as much as non square matrices.

The method

Consider $A = (a_{ij})_{i,j}$ to be an $n \times n$ matrix with $det(A) \neq 0$.
1st step: We assume that a_{11} is non-zero (otherwise, by interchanging rows, we can get a non-zero element in the upper left corner of the matrix, since $det(A) \neq 0$). We define the multipliers $m_{i1}, i = 2, \ldots, n$ as follows:
$m_{i1} := \dfrac{a_{i1}}{a_{11}}, i = 2, \ldots, n.$
We then multiply, for every $i = 2, \ldots, n$, the first row of A by $-m_{i1}$ and add it to the i-th row.
This procedure results in the equivalent matrix:

$$A^{(1)} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix}$$

where $a_{ij}^{(1)} = a_{ij} - m_{i1}a_{1j}, i, j = 2, \dots, n$. We also denote $a_{ij} = a_{ij}^{(0)}$.
2nd step: We continue by following the same procedure for the $(n-1)\times(n-1)$ lower right submatrix of $A^{(1)}$. If the upper left element of the submatrix is 0, we can find a non-zero element in the first column of the submatrix, since its determinant is non-zero (which is implied by the invertibility of A), and interchange rows. We define the multipliers $m_{i2} := \dfrac{a_{i2}^{(1)}}{a_{22}^{(1)}}, i = 3, \dots, n$ and for each $i = 3, \dots, n$ we multiply the first row of the submatrix (the second row of $A^{(1)}$) by $-m_{i2}$ and add it to the (i-1)-th row of the submatrix (the i-th row of $A^{(1)}$), resulting in the matrix

$$A^{(2)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix}$$

r-th step: In this step we begin with matrix

$$A^{(r-1)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1r}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2r}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3r}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{rr}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nr}^{(r-1)} & \dots & a_{nn}^{(r-1)} \end{bmatrix}$$

Again, like before, we can have a non-zero element in the upper left corner of the submatrix $(a_{ij}^{(r-1)})_{i,j=r,\dots,n}$. We define the multipliers $m_{ir} := \dfrac{a_{ir}^{(r-1)}}{a_{rr}^{(r-1)}}, i = r+1, \dots, n$ and for each $i = r+1, \dots, n$ we multiply the r-th row of $A^{(r-1)}$ by

$-m_{ir}$ and we add it to the i-th row. Hence we get matrix

$$A^{(r)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1r}^{(0)} & a_{1r+1}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2r}^{(1)} & a_{2r+1}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3r}^{(2)} & a_{3r+1}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{rr}^{(r-1)} & a_{rr+1}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ 0 & 0 & 0 & \dots & 0 & a_{r+1r+1}^{(r)} & \dots & a_{r+1n}^{(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{nr}^{(r)} & \dots & a_{nn}^{(r)} \end{bmatrix}$$

(n-1)-th step: in the last step, after following the above procedure, we end up with the upper triangular matrix

$$A^{(n-1)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1r}^{(0)} & a_{1r+1}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2r}^{(1)} & a_{2r+1}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3r}^{(2)} & a_{3r+1}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{rr}^{(r-1)} & a_{rr+1}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ 0 & 0 & 0 & \dots & 0 & a_{r+1r+1}^{(r)} & \dots & a_{r+1n}^{(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & a_{nn}^{(n-1)} \end{bmatrix}$$

Elements $a_{ii}^{(i-1)}$ of the diagonal of $A^{(n-1)}$ are called *pivots*. As we observe in the implementation of the method, these elements are the denominators of the multipliers defined in every step.

Computational Complexity

We will first calculate the number of operations required in each step. In step r the number of operations needed for the evaluation of the multipliers is $n-r$. For the evaluation of the lower right $(n-r)\times(n-r)$ submatrix we need $(n-r)^2$ operations (one for each element of the submatrix). Hence, in the r-th step we need $(n-r)^2+(n-r)$ operations.

The total number of operations that are required for all $(n-1)$ steps of Gaussian Elimination is

$$\sum_{r=1}^{n-1}[(n-r)^2+(n-r)] = \frac{n^3-n}{3}.$$

The row operations that take place during Gaussian elimination do not affect the determinant of the matrix, except for row switching which changes the sign of the determinant. Hence, if k is the number of the row permutations that took place, $det(A) = (-1)^k a_{11}^{(0)} a_{22}^{(1)} \dots a_{nn}^{(n-1)}$. The number of operations required to evaluate this expression is $n-1$, therefore computing the deteminant of a matrix using Gaussian elimination requires

$$\frac{n^3 - n}{3} + (n - 1)$$

operations. We observe that the amount of computations required is significantly less, compared to the $n!$ computations needed when we use the Lagrange expansion formula. This is why, in almost all cases, GE is the most efficient method for the evaluation of determinants.

Gaussian elimination and LU factorization

LU factorization (or LU decomposition) factors an $n \times n$ matrix as a product of a lower triangular matrix L and an upper triangular matrix U.

Definition 2.1.1. *Let A be an $m \times n$ matrix. The determinant of any $k \times k$ submatrix of A is called a $k \times k$* minor *of A.*

We have the following theorem

Theorem 2.1.2 (LU factorization)**.** *Let A be an $n \times n$ matrix whose leading principal minors are non zero. Then A admits a unique LU factorization*

$$A = L \cdot U,$$

where L is a lower triangular matrix with diagonal entries 1 and U is an upper triangular matrix.

One can prove that

$$U = A^{(n-1)} \text{ and } L = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ m_{21} & 1 & \dots & 0 & 0 \\ m_{31} & m_{32} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn-1} & 1 \end{bmatrix},$$

i.e. U is the upper triangular matrix of the final step of Gaussian Elimination, while L is the lower triangular matrix whose diagonal elements are 1 and whose entries below the diagonal are the multipliers defined in the GE method.

The hypothesis concerning the leading principal minors guarantees that in every step r of GE, pivot element $a_{rr}^{(r-1)}$ will be non zero, hence no row permutations will be required. In the more general case of an invertible matrix A whose leading principal minors are not necessarily non zero, row permutations will be needed. These can be applied by left-multiplying A with a permutation matrix P. In that case, matrix PA represents matrix A when all row switches have been done in advance and will have non zero leading principal minors. Thus, from Theorem 2.1.2, $PA = LU$.

LU factorization can also be attained for non invertible matrices. In that case, some of the diagonal elements of U will be zero.

2.2 Stability

As we mentioned before, pivot elements $a_{ii}^{(i-1)}$ of the diagonal are the denominators of the multipliers defined in every step of GE. In case a pivot is too small (i.e. its magnitude is small), the magnitudes of the corresponding multipliers will be too large. In the evaluation of the lower right submatrix, we will probably have to add large numbers to relatively small numbers. When working with machine numbers, this often leads to large rounding errors, since there are limitations on the amount of digits used to represent numbers.

In order to avoid such errors, we have to replace these elements with others whose magnitudes are bigger. This process is called *pivoting*. There are two pivoting techniques:

- Partial Pivoting
- Complete Pivoting

2.2.1 Gaussian elimination with pivoting

Gaussian elimination with Partial Pivoting (GEPP)

In each step r of Gaussian elimination with partial pivoting, we chose the pivot to be the element with the maximum absolute value from the first column of the lower right $(n-r+1)\times(n-r+1)$ submatrix:

r-th step: we begin with matrix

$$A^{(r-1)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1r}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2r}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3r}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{rr}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nr}^{(r-1)} & \dots & a_{nn}^{(r-1)} \end{bmatrix}$$

From all elements $\{a_{rr}^{(r-1)}, \dots, a_{nr}^{(r-1)}\}$, we chose the one that has the largest magnitude and we apply row switching so that it is placed in the upper left corner of the lower $(n-r+1)\times(n-r+1)$ submatrix. We then continue by defining the multipliers and applying the appropriate row operations as described in the previous section.

Gaussian Elimination with Complete Pivoting (GECP)

In each step r of Gaussian elimination with complete pivoting, we chose the pivot to be the element with the maximum absolute value from all entries of the lower right $(n-r+1)\times(n-r+1)$ submatrix:

r**-th step**: we begin with matrix

$$A^{(r-1)} = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1r}^{(0)} & \dots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2r}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3r}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{rr}^{(r-1)} & \dots & a_{rn}^{(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nr}^{(r-1)} & \dots & a_{nn}^{(r-1)} \end{bmatrix}$$

From all elements of the lower right $(n-r+1)\times(n-r+1)$ submatrix, we chose the one with the largest magnitude and we apply row and column switching so that it is placed in the upper left corner of the submatrix. We then continue by defining the multipliers and applying the appropriate row operations as described in the previous section.

2.2.2 Backward Error Analysis - Growth factor

In this segment, we will examine the stability of the GE algorithm in terms of backward error analysis. Backward error analysis is used to estimate the numerical stability of an algorithm and is based on the concept that the calculated result, generally incorrect due to rounding errors, will be the exact solution to a nearby problem with slightly perturbed data.

As we previously mentioned, GE provides the LU factorization of a square $n \times n$ matrix A. When implementing the GE algorithm to compute matrices L and U, rounding errors may appear, hence we will get matrices $\bar{L}$ and $\bar{U}$, probably different from L, U. The computed matrices will satisfy

$$\bar{L}\bar{U} = A + E,$$

where E is a matrix with small entries (in magnitude), i.e. $\bar{L}$ and $\bar{U}$ will be the exact LU factorization of a slightly perturbed matrix. Backward error analysis obtains an upper bound for E:

$$||E||_\infty \leq n^2 \frac{\max\limits_{i,j,r} |a_{ij}^{(r)}|}{\max\limits_{i,j} |a_{ij}^{(0)}|} u||A||_\infty, \qquad (*)$$

where

- $||\cdot||_\infty$ is the infinity norm defined as $||A||_\infty = \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{n} |a_{ij}|$ (the maximum absolute row sum) and
- u is the unit round off (machine epsilon) that represents the precision of the machine.

We observe that upper bound $(*)$ depends on quantity

$$g(n, A) = \frac{\max\limits_{i,j,r} |a_{ij}^{(r)}|}{\max\limits_{i,j} |a_{ij}^{(0)}|},$$

which is called the *growth factor* of matrix A. If we want to examine the stability of GE, we must investigate the possible values that the growth factor can take.

Growth factor in GE with complete pivoting

In 1961, James H. Wilkinson proved that when applying GE with complete pivoting, the growth factor is bounded as follows

$$g(n, A) \leq \{n 2^1 3^{1/2} 4^{1/3} \dots n^{1/(n-1)}\}^{1/2}$$

([36]). The right hand of the inequality is a function that slowly grows. Furthermore, in practice, $g(n, A)$ never attains that value. However, it is very difficult to obtain a better bound.

The growth factor associated with GECP is almost always less than n and thus it was conjectured by Wilkinson in 1965 that it is bounded by the size of the matrix. In 1991, Gould constructed a 13×13 matrix whose growth factor when applying GE with complete pivoting was 13.0205 ([9]), thus proving the conjecture false. Gould also constructed matrices of orders 14, 15 and 16, whose growth factor exceeded their size. We note, though, that such matrices are extremely rare (Gould used sophisticated optimization methods in order to construct them) and, in practice, they never appear in applications. In fact, in almost all cases, the growth factor is significantly smaller than the matrix's size. Indicatively, for the purposes of this thesis we have evaluated the growth factors of some random matrices of various sizes with elements in the interval $(0, 1)$. We display the results in the following table

n	max $g(n, A)$ that appeared	average $g(n, A)$ (approx.)
10	1.1083...	1.034445
100	3.2195...	2.788235
1000	12.085...	10.224195

We observe that in all cases $g(n, A)$ is much smaller than n.

Even though the aforementioned bound does not guarantee stability, GE with complete pivoting is considered a stable algorithm.

Remark From the definition of the growth factor, for the case of GECP we have

$$g(n, A) = \frac{\max\{|a_{11}^{(0)}|, |a_{22}^{(1)}|, \dots, |a_{nn}^{(n-1)}|\}}{|a_{11}^{(0)}|}$$

i.e. the growth factor is the ratio of the largest pivot to the first pivot (in magnitude).

Growth factor in GE with partial pivoting

Wilkinson showed that when applying GE with partial pivoting, the growth factor is bounded by

$$g(n, A) \leq 2^{n-1}.$$

Matrices that attain growth factor equal to 2^{n-1} when GE with partial pivoting is applied, can be easily constructed ([37], p.212). However, GE with partial pivoting is considered stable in practice, but without a theoretical confirmation.

Growth factor in GE without pivoting

When applying GE without pivoting, the growth factor can take large values. As we mentioned before, in GE without pivoting, large rounding errors often appear, thus the algorithm is considered unstable.

Remark We note that, while a small growth factor leads to a decrease of the upper bound $(*)$ for $||E||_\infty$ and, thus, improves the stability of GE, a large growth factor does not necessarily imply that the algorithm is unstable. In GE with pivoting, experience shows that when $||E||_\infty$ is large, it is not due to the large growth factor of A but due to the large *condition number* of A, which is defined as $cond(A) = ||A||_\infty ||A^{-1}||_\infty$.

Chapter 3

Hadamard Matrices

In this chapter, we present a special category of square matrices, called Hadamard matrices. These matrices appear in a lot of applications and they are characterized by beautiful mathematical properties. Especially in the field of numerical analysis, they are of great interest, since they seem to be the only matrices whose growth factor is equal to their size.

3.1 Definition and basic properties

Definition 3.1.1. *A* Hadamard matrix of order n *is an $n \times n$ matrix H whose entries are ± 1 and satisfies $HH^T = H^T H = nI_n$.*
Equivalently, a Hadamard matrix of order n is a matrix whose entries are ± 1 and whose rows (and columns) are mutually orthogonal.

Remark $HH^T = nI_n$ implies that H is non singular and its inverse is $n^{-1}H^T$.Hence we get $H^T H = nI_n$

Hadamard's maximal determinant problem

In 1893, Jacques Hadamard presented the following bound concerning the determinants of complex matrices

Theorem 3.1.2 (Hadamard's inequality, [10])**.** *If $M = (m_{ij})$ is an $n \times n$ complex matrix, then*

$$|det(M)| \leq \left[\prod_{i=1}^{n} \left(\sum_{j=1}^{n} |m_{ij}|^2 \right) \right]^{1/2}$$

In the special case where M is a real matrix whose elements satisfy $|m_{ij}| \leq 1$, we obtain the following result

Theorem 3.1.3 (Hadamard's Inequality)**.** *If $M = (m_{ij})$ is an $n \times n$ real matrix and $|m_{ij}| \leq 1$, $\forall\ i, j$, then*

$$|det(M)| \leq \left[\prod_{i=1}^{n} \left(\sum_{j=1}^{n} m_{ij}^2 \right) \right]^{1/2} \leq n^{n/2}.$$

and equality holds if and only if M is a Hadamard matrix.

Proof. One can give a geometrical proof of Hadamard's inequality, since $det(M)$ is the n-volume of the parallelepiped in $\mathbb{R}^n$, spanned by the row vectors of M. Such a volume is maximized when the vectors are mutually orthogonal and every entry is ± 1, i.e when M is a Hadamard matrix [1]. □

Existence and Construction of Hadamard matrices

Proposition 3.1.4 ([30],[31]). *If H is a Hadamard matrix of order n, then $n = 1, 2$ or $4t, t \in \mathbb{N}$.*

Proof. For $n = 1, 2$, Hadamard matrices can be easily constructed (Examples 3.1.6). For $n > 2$, since every two rows of a Hadamard matrix have ± 1 entries and they are orthogonal (their inner product is zero), n must be an even number. Knowing that these two rows must also be orthogonal to a third row, n must be a multiple of 4. □

While it is necessary that the order of a Hadamard matrix be 1,2 or a multiple of 4, it is still an open problem whether Hadamard matrices exist for every order $4t$, $t \in \mathbb{N}$. The smallest order for which the problem is still unsolved is 668.

Definition 3.1.5 (H-equivalence). *We call two Hadamard matrices* Hadamard eqivalent *or* H-equivalent *if one can be obtained from the other by a sequence of the operations:*
1. interchange any pair of rows and/or columns
2. multiply any row and/or column through by -1.

Examples 3.1.6. *The unique (up to H-equivalence) Hadamard matrices of orders 1, 2 and 4 are:*

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Such matrices were first constructed and studied by James Joseph Sylvester in 1867 ([31]). Sylvester observed that if H is a Hadamard matrix then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is also a Hadamard matrix. Beginning with $H_1 = \begin{bmatrix} 1 \end{bmatrix}$ and using Sylvester's construction we get the following sequence of Hadamard matrices:

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

[1] Hadamard matrices were named after this property: they are the ± 1 matrices that make Hadamard's inequality sharp

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\vdots$$

$$H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}$$

that are also called Sylvester-Hadamard matrices.

Hence, we have the following lemma

Lemma 3.1.7 (Sylvester, 1867)**.** *There exists a Hadamard matrix of order* 2^t *for all* $t \in \mathbb{N}$.

Definition 3.1.8 (Kronecker product)**.** *If* A *is an* $m \times n$ *matrix and* B *is a* $p \times q$ *matrix, the* Kronecker product *of* A *and* B*, denoted* $A \otimes B$*, is the* $mp \times nq$ *matrix:*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

The Kronecker product of k *copies of* A*,* $A \otimes \ldots \otimes A$*, which is well defined since the Kronecker product is associative, will be denoted* $\otimes^k A$.

For Sylvester-Hadamard matrices, $H_{2^k} = H_2 \otimes H_{2^{k-1}} = \ldots = \otimes^k H_2$. The following lemma generalizes the construction presented above:

Lemma 3.1.9 (Sylvester)**.** *Let* H_{h_1} *and* H_{h_2} *be Hadamard matrices of orders* h_1 *and* h_2*. The Kronecker product* $H_{h_1} \otimes H_{h_2}$ *is a Hadamard matrix of order* $h_1 h_2$.

There exist several other techniques for constructing Hadamard matrices ([8],[11],[29]). Some of them are of great importance since they lead to large families of H-inequivalent Hadamard matrices.

Construction techniques along with further investigation on the subject have proven the existence of Hadamard matrices of special orders. The following proposition summarizes these results.

Proposition 3.1.10 ([2])**.** *Hadamard matrices exist at the following orders*
(i) 2^t, $t \in \mathbb{N}$.
(ii) $p^a + 1$*, where* p *is prime and* $p^a \equiv 3 mod 4$.
(iii) $2(p^a + 1)$*, where* p *is prime and* $p^a \equiv 1 mod 4$.
(iv) $p(p+2) + 1$*, where* p *and* $p+2$ *are twin primes.*
(v) $4p^{4t}$*, where* p *is prime and* $t \geq 1$.
(vi) $4t$*, for all values of* $t \leq 250$ *except for* $t \in \{167, 179, 223\}$.
(vii) $n = ab/2$ *or* $n = abcd/16$*, where* a, b, c, d *are orders of Hadamard matrices.*
(viii) If t *is an odd integer, then there exist constants* a *and* b *such that there exists a Hadamard matrix of order* $2^{\lceil a + b \log_2(t) \rceil} t$.

3.2 Minors of Hadamard matrices

In general, the evaluation of the minors of a matrix is a very difficult task. Hadamard matrices, however, have a special structure that allows us to establish values for their minors as well as formulae that reveal the connections between their submatrices.

Notation For a matrix A, $A(k)$ denotes the upper left $k \times k$ minor of A, while $A[k]$ denotes the lower right $k \times k$ minor of A.

Theorem 3.2.1 ([5],[24],[32]). *Let* $H_n = \begin{bmatrix} M_k & B \\ C & M_{n-k} \end{bmatrix}$ *be a Hadamard matrix of order* n*, where* M_k *is the* $k \times k$ *leading principal submatrix of* H_n*. Then*

$$det(M_{n-k}) = \pm n^{\frac{n}{2}-k} det(M_k)$$

or, using the above notation,

$$H[n-k] = \pm n^{\frac{n}{2}-k} H(k).$$

Proof. from the definition of a Hadamard matrix follows that

$$H_n(n^{-1}H_n^T) = I_n \iff$$

$$\iff \begin{bmatrix} M_k & B \\ C & M_{n-k} \end{bmatrix} \cdot \begin{bmatrix} n^{-1}M_k^T & n^{-1}C^T \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix}.$$

Thus, $M_k C^T + B M_{n-k}^T = 0$
Consider that

$$\begin{bmatrix} M_k & B \\ 0 & I_{n-k} \end{bmatrix} \cdot \begin{bmatrix} n^{-1}M_k^T & n^{-1}C^T \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ n^{-1}B^T & n^{-1}M_{n-k}^T \end{bmatrix}.$$

By taking determinants, we have

$$det(M_k)det(I_{n-k})det(n^{-1}H_n^T) = det(I_k)det(n^{-1}M_{n-k}^T) \Rightarrow$$

$$det(M_k)n^{-n}det(H_n) = n^{-(n-k)}det(M_{n-k})$$

and since $det(H_n) = \pm n^{\frac{n}{2}}$

$$det(M_{n-k}) = \pm n^{\frac{n}{2}-k} det(M_k).$$

□

Values of minors

Proposition 3.2.2 ([4]). *Let* A *be a an* $n \times n$ *matrix whose elements are* ± 1*. Then*
(i) $det(A)$ *is an integer and* 2^{n-1} *divides* $det(A)$*.*
(ii) When $n \leq 6$*, the only possible values for* $|det(A)|$ *are these and they do all occur:*

n	1	2	3	4	5	6
$\|det(A)\|$	1	$0, 2$	$0, 4$	$0, 8, 16$	$0, 16, 32, 48$	$0, 32, 64, 96, 128, 160$

Proof. (i) By doing one step of Gaussian elimination on A we obtain matrix $A^{(1)}$ that has ± 1 entries in the first row and 0 or ± 2 in the rows 2 through n

$$A^{(1)} = \begin{bmatrix} \pm 1 & a_{1\times(n-1)} \\ 0_{(n-1)\times 1} & B \end{bmatrix}.$$

$(n-1)\times(n-1)$ matrix B has entries 0 and ± 2 and thus can be written as $2^{n-1}C$, where C is a matrix with entries 0 and ± 1. Therefore $det(A) = \pm 1 det(B) = (\pm 1)2^{n-1}det(C)$ and since $det(C)$ is an integer, we conclude that 2^{n-1} divides $det(A)$.
(ii) The values listed above, are the multiples of 2^{n-1} less than or equal to $n^{n/2}$ (the upper bound by Hadamard's inequality). For $n = 6$, value 192 is excluded, since it doesn't occur. □

The authors in [21] extended the above results for the 7×7 case

Proposition 3.2.3. *If A is a 7×7 matrix with elements ± 1, then the possible values of $|det(A)|$ are the following and they do all occur:*

$$0, 64, 128, 192, 256, 320, 384, 448, 512, 576$$

Theorem 3.2.4 ([15],[30])**.** *For a Hadamard matrix of order n:*
(i) All the $(n-1)\times(n-1)$ minors are $\pm n^{\frac{n}{2}-1}$.
(ii) All the $(n-2)\times(n-2)$ minors are 0 or $\pm 2n^{\frac{n}{2}-2}$.
(iii) All the $(n-3)\times(n-3)$ minors are 0 or $\pm 4n^{\frac{n}{2}-3}$.
(iv) All the $(n-4)\times(n-4)$ minors are 0,$\pm 8n^{\frac{n}{2}-4}$ or $\pm 16n^{\frac{n}{2}-4}$.

Proof. Without loss of generality we assume that the $(n-i)\times(n-i)$ minors occur in the lower right corner of the matrix. If not, by using row and column interchanges, we can place the minor in the lower right corner. From Theorem 3.2.1, using the values listed in Proposition 3.2.2, we obtain the listed possible values:
(i) $H[n-1] = n^{\frac{n}{2}-1}H(1) = \pm n^{\frac{n}{2}-1}$, since $H(1) = \pm 1$.
(ii) $H[n-2] = n^{\frac{n}{2}-2}H(2) = 0$ or $\pm 2n^{\frac{n}{2}-2}$, since $H(2) = 0$ or ± 2.
(iii) $H[n-3] = n^{\frac{n}{2}-3}H(3) = 0$ or $\pm 4n^{\frac{n}{2}-3}$, since $H(3) = 0$ or ± 4.
(iv)$H[n-4] = n^{\frac{n}{2}-4}H(4) = 0, \pm 8n^{\frac{n}{2}-4}$ or $\pm 16n^{\frac{n}{2}-4}$, since $H(4) = 0, \pm 8$ or ± 16.

□

Similarly, we can prove the following theorem

Theorem 3.2.5 ([14])**.** *For a Hadamard matrix of order n:*
(i) All the $(n-5)\times(n-5)$ minors are 0, $\pm 16n^{\frac{n}{2}-5}$, $\pm 32n^{\frac{n}{2}-5}$ or $\pm 48n^{\frac{n}{2}-5}$.
(ii) All the $(n-6)\times(n-6)$ minors are 0, $\pm 32n^{\frac{n}{2}-6}$, $\pm 64n^{\frac{n}{2}-6}$, $\pm 96n^{\frac{n}{2}-6}$, $\pm 128n^{\frac{n}{2}-6}$ or $\pm 160n^{\frac{n}{2}-6}$.
(iii)All the $(n-7)\times(n-7)$ minors are $0, \pm 64n^{\frac{n}{2}-7}, \pm 128n^{\frac{n}{2}-7}, \pm 192n^{\frac{n}{2}-7}$, $\pm\, 256n^{\frac{n}{2}-7}, \pm 320n^{\frac{n}{2}-7}, \pm 384n^{\frac{n}{2}-7}, \pm 448n^{\frac{n}{2}-7}, \pm 512n^{\frac{n}{2}-7}$ or $\pm 576n^{\frac{n}{2}-7}$

We note that all the possible values displayed in Theorems 3.2.4 and 3.2.5 occur.

Definition 3.2.6 (D-optimal design)**.** *A $n \times n$, ± 1 matrix with maximum possible determinant (in magnitude) is called a* D-optimal design of order n *and is denoted D_n. We also denote $d_n = |det(D_n)|$.*

Remark A Hadamard matrix of order n is a D-optimal design.

Lemma 3.2.7 ([5]). *For $n = 2, \ldots, 7$, if an $n \times n$ matrix is D_n, then a D_{n-1} must be embedded in it.*

Remark This is not true for $n = 8$, since all 7×7 minors of a D_8 have magnitude 512.

Theorem 3.2.8 ([24]). *For a Hadamard matrix H_n of order n, if $\frac{d_k}{d_{n-k}} > n^{k-\frac{n}{2}}$, then D_k is not embedded in H_n.*

Proof. If a D_k is embedded in H_n, then by applying row and column permutations appropriately, we can place D_k in the upper left corner of H_n. From Theorem 3.2.1 and the definition of a D-optimal design, we have

$$d_k = |det(D_k)| = n^{k-\frac{n}{2}}|H[n-k]| \leq n^{k-\frac{n}{2}} d_{n-k} \iff \frac{d_k}{d_{n-k}} \leq n^{k-\frac{n}{2}}$$

□

Lemma 3.2.7 and Theorem 3.2.8 lead us to the following corollary

Corollary 3.2.9. *D_5, D_6 and D_7 are not embedded in H_8.*

Proof. Since $\frac{d_5}{d_3} = \frac{48}{4} = 12 > 8^{5-4}$, D_5 is not embedded in H_8. If D_6 was embedded in H_8, then from Lemma 3.2.7, D_5 would also be embedded in H_8 and that leads to a contradiction. Hence, D_6 is not embedded in H_8 . Using the same argument we conclude that D_7 is not embedded in H_8 either. □

More about embedded D-optimal designs in Hadamard matrices can be found in [26].

3.3 Special cases and generalizations of Hadamard matrices

In this section, we briefly present some special cases and generalizations of Hadamard matrices. Many of them, appear very often in applications.

3.3.1 Special cases

Skew Hadamard matrices

Definition 3.3.1. *A* skew Hadamard matrix *is a Hadamard matrix H that can be written as $H = I + S^T$, where $S^T = -S$.*

Walsh matrices

Definition 3.3.2. *A* Walsh matrix *is a ± 1 square matrix of order $2^t, t \in \mathbb{N}$, whose rows and columns are orthogonal and whose each row corresponds to a Walsh function.*

Sylvester-Hadamard matrices are Walsh matrices.

Regular Hadamard matrices

Definition 3.3.3. *A* regular Hadamard matrix *is a Hadamard matrix whose row and column sums are all equal.*

The order of a regular Hadamard matrix must be a perfect square ([45]).

Circulant Hadamard matrices

Definition 3.3.4. *A* circulant matrix *is a square matrix in which all row vectors are composed of the same elements and each row vector is rotated one element to the right compared to the preceding row vector.*

Definition 3.3.5. *A* circulant Hadamard matrix *is a Hadamard matrix that is also circulant.*

A circulant matrix is regular, hence a circulant Hadamard matrix is a regular Hadamard matrix and its order must be a perfect square. Moreover, if there exists a circulant Hadamard matrix of order $n > 1$, then $n = 4u^2$, where u is an odd number ([34]). It is conjectured that, apart from the known 1×1 and 4×4 examples, no such matrices exist.

Definition 3.3.6. *A* generalized circulant Hadamard matrix with diagonal d *is a circulant matrix whose off-diagonal entries are* ± 1*, the diagonal entries are* $d \in \mathbb{R}$ *and whose rows are mutually orthogonal.*

3.3.2 Generalizations

Weighing matrices

Definition 3.3.7. *An* $n \times n$ $(0, 1, -1)$ *matrix* $W = W(n, k)$ *that satisfies* $WW^T = kI_n$*, is called a* weighing matrix of order n and weight k.

A $W(n, n)$, $n \equiv 0 mod 4$, is a Hadamard matrix of order n.

Complex Hadamard matrices

Definition 3.3.8. *A* complex Hadamard matrix H *is a square* $n \times n$ *matrix with unimodular entries, i.e.* $|h_{ij}| = 1 \; \forall \; i, j$*, that satisfies* $HH^* = nI_n$*, where* H^* *is the conjugate transpose.*

As opposed to the real case, complex Hadamard matrices exist for every order $n \in \mathbb{N}$.

3.4 Applications

Hadamard matrices and their generalizations appear in a wide variety of applications in many fields. Apart from their evident connection to maximal determinant problems ([11],[28]), they can be transformed to other mathematical objects and be used in Design theory, Statistics, Telecommunications, Information technology, Signal processing, Harmonic Analysis, Operator theory and Combinatorics. Here we have made a brief list of examples. A more detailed presentation of these and other applications can be found in the cited references.

Design Theory and Statistics
- Balanced Incomplete Block designs, Group divisible designs and Youden designs ([11])
- Optimal fractional factorial designs ([11])
- Optimal weighing designs ([6],[11],[29])
- Orthogonal arrays ([11])
- Orthogonal F-square designs ([6],[11])
- Balanced repeated replication (technique used to estimate the variance of a statistical estimator) ([40])
- Robust parameter design (for investigating noise factor impacts on responses) ([46])
- Plackett-Burman design (for investigating the dependence of some measured quantity on a number of independent variables)([44])

Telecommunications, Information technology and Signal processing
- Error control codes (error correcting capabilities of codes derived from Hadamard matrices)([11],[29])
- Walsh functions (defined from Sylvester-Hadamard matrices)([6],[29])
- Direct sequence spread spectrum CDMA systems ([6],[29])
- Hadamard transform spectrometry ([11])
- Boolean functions ([6])
- Barker sequences ([11])
- Quantum information technology (Quantum Hadamard gate and Hadamard transform) ([43],[47])

Operator theory, Harmonic Analysis and Combinatorics
- Spectral sets ([12],[33])
- Constructions of bases of unitaries ([35])
- Construction of mutually unbiased bases of Hilbert spaces ([38])

Chapter 4

CP Hadamard matrices and the growth problem

In the final chapter of this thesis, we restrict our attention to the properties of Hadamard matrices related to the growth factor associated with GECP.

The growth factor of a matrix when GECP is applied is

$$g(n, A) = \frac{\max\{|a_{11}^{(0)}|, |a_{22}^{(1)}|, \ldots, |a_{nn}^{(n-1)}|\}}{|a_{11}^{(0)}|} = \frac{max\{|p_1|, \ldots, |p_n|\}}{|p_1|},$$

where p_i denotes the i-th pivot of A. We therefore observe that in order to evaluate the growth factor of a matrix, we must compute the values of its pivots.

4.1 Definition and properties

Definition 4.1.1. *A matrix A is called* Completely Pivoted (CP) *if during Gaussian elimination with complete pivoting no row or column interchanges are required.*

A CP matrix can be viewed as a matrix on which all row and column permutations have been applied in advance, so that when GE with complete pivoting is applied, no row or column interchanges will be needed. Every matrix can be transformed to a CP one, such that when we apply GE on it and GECP on the original matrix, the resulting pivots are the same. This is why, in our theoretical approach, we will assume without loss of generality that all matrices are CP.

Notation If A is an $m \times n$ matrix, $A(i_1 \ldots i_p | j_1 \ldots j_p)$ denotes the determinant of the $p \times p$ submatrix of A obtained from the intersection of rows $i_1, \ldots, i_p$ with columns $j_1, \ldots, j_p$. When the two set of indices are the same, the abbreviation $A(i_1 \ldots i_p)$ will be used.

Proposition 4.1.2 ([4],[7])**.** *Let A be a non singular CP matrix on which GECP is applied. Then, after r steps, $1 \leq r < n$, the (i, j) entry of A for $i, j > r$ is*

$$a_{ij}^{(r)} = \frac{A(1 \ldots ri | 1 \ldots rj)}{A(1 \ldots r)}$$

Proof. Knowing that matrix A is non singular and CP, i.e. no row or column interchanges will be done during GECP, and since the determinant is invariant under row addition, we have that the leading principal minors of A and $A^{(r)}$ are the same and they are non zero. Hence, $A(1\dots r) = a_{11}^{(0)}a_{22}^{(1)}\dots a_{rr}^{(r-1)}$.

Now we adjoin $r+1$ entries of row i and column j, to get the submatrix of A whose determinant is $A(1\dots ri|1\dots rj)$. Since A is CP and $i,j>r$, the submatrix will also be CP and if we apply GECP, we will get that $A(1\dots ri|1\dots rj) = a_{11}^{(0)}a_{22}^{(1)}\dots a_{rr}^{(r-1)}a_{ij}^{(r)}$. Evaluating the quotient, we have

$$\frac{A(1\dots ri|1\dots rj)}{A(1\dots r)} = \frac{a_{11}^{(0)}\dots a_{rr}^{(r-1)}a_{ij}^{(r)}}{a_{11}^{(0)}\dots a_{rr}^{(r-1)}} = a_{ij}^{(r)}.$$

□

Lemma 4.1.3 ([3],[4]). *Let A be a non singular CP matrix, and let $A(i)$ denote its $i\times i$ leading principal minor.*
(i) The value of the i-th pivot p_i, appearing after application of GE on A, is given by

$$p_i = \frac{A(i)}{A(i-1)}, i=1,\dots,n, \quad A(0)=1.$$

(ii) The maximum (in magnitude) $i\times i$ leading principal minor of A when the first $i-1$ rows and columns are fixed, is $A(i)$.

Proof. (i) Follows from Proposition 4.1.2:

$$p_i = a_{ii}^{(i-1)} = \frac{A(1\dots i)}{A(1\dots(i-1))} = \frac{A(i)}{A(i-1)}.$$

(ii) Since the first $i-1$ rows and columns are fixed, if there was an $i\times i$ submatrix of A with magnitude greater than $A(i)$, from (i) we would get that after applying $i-1$ steps of GE on A there is an element in the lower right $(n-(i-1))\times(n-(i-1))$ submatrix with magnitude greater than p_i and that leads to a contradiction, since A is CP. □

Lemma 4.1.4 ([5]). *If H is a CP Hadamard matrix of order n, then the k-th pivot from the end is $p_{n+1-k} = n\dfrac{H[k-1]}{H[k]}$.*

Proof. It follows from Theorem 3.2.1 and Lemma 4.1.3. □

Corollary 4.1.5 ([5]). *If H is a CP Hadamard matrix of order n and $k<n$, then, for all $(k-1)\times(k-1)$ minors M_{k-1} of the $k\times k$ lower right submatrix of H, we have $|H[k-1]| \geq |M_{k-1}|$.*

Proof. This follows from Lemma 4.1.4 and the CP property of H, for otherwise we could permute rows and columns of the lower right $k\times k$ submatrix of H to obtain a larger magnitude for p_{n+1-k}. □

For the next proposition, we will need the following lemma

Lemma 4.1.6 ([4]). *If $g(n) := sup\{g(n,A)|A$ is an $n\times n$ matrix $\}$, then*

$$g(3) = \frac{9}{4}.$$

Proposition 4.1.7 ([4]). *Let $H_n = (h_{ij})$ be a CP Hadamard matrix of order n. The magnitudes of the first four pivots after applying GE on H_n are $1, 2, 2$ and 4.*

Proof. $p_1 = h_{11} = \pm 1$. After 1 step of GE, every entry of the $(n-1) \times (n-1)$ lower right submatrix of $H_n^{(1)}$ is 0 or ± 2. Thus $p_2 = h_{22}^{(1)} = \pm 2$, since H_n is CP. After 2 steps of GE, every entry of the lower right $(n-2) \times (n-2)$ submatrix of $H_n^{(2)}$ must be 0, ± 2 or ± 4. Since $g(3) < 4$ from Lemma 4.1.6, there cannot be any ± 4's, so $p_3 = h_{33}^{(2)} = \pm 2$. After 3 steps of GE, every entry of the lower right $(n-3) \times (n-3)$ submatrix of $H_n^{(3)}$ must be 0,± 2 or ± 4. We will show that there is an entry ± 4, thus, from the CP property, we will have that $h_{44}^{(3)} = \pm 4$. We know that the size of H_n is $n = 4t$. Without destroying the CP property of H_n, we can multiply rows and columns by -1, so that all entries of the first row and the first column are 1. Then each column of H_n has one of the following four sign patterns in its first three entries:

I	**II**	**III**	**IV**
+	+	+	+
+	-	+	-
+	+	-	-

The mutual orthogonality of the first three rows implies that there are exactly t columns of each type[1]. Since $H_n(3) \neq 0$, the first three columns must be of three different types.
We choose any column j of the type not represented among columns 1,2 and 3 (there is at least one). For the purpose of finding a row i so that $A(123i|123j) = \pm 16$, it does not matter if columns 2,3 and j are rearranged, thus we may assume that H_n has the pattern presented above in its first three rows and the four columns 1,2,3,j. Like columns, rows can be divided into the same four types of groups and each one of them has t rows. Thus, there are t rows having pattern $+ - -$ in the first three entries and all these lie below the first three rows. At least one of these rows must have $h_{ij} = 1$, for otherwise column j has more than t negative entries, which contradicts its being orthogonal to column 1. Thus $A(123i|123j)$ is the determinant of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

and that is 16. Then from Proposition 4.1.2, $h_{ij}^{(3)} = \dfrac{A(123i|123j)}{A(123)} = \pm\dfrac{16}{4} = \pm 4$. □

For ease of reference, since we are focusing on magnitudes of pivots, from now on whenever a minor or a pivot is mentioned, we will mean its absolute value.

Corollary 4.1.8. *If H is a CP Hadamard matrix, then*
-$H(1) = 1$

[1] This result is also known as the distribution lemma ([15])

-$H(2) = 2$
-$H(3) = 4$
-$H(4) = 16$

Proposition 4.1.9 ([22]). *For every pivot p_i, $i = 2, \dots n$ of a CP Hadamard matrix H_n of order n, it holds $p_i > 1$. Furthermore, the leading principal minors form an increasing sequence, namely $H(i) > H(i-1)$, $i = 1, \dots, n$.*

Proof. Suppose $p_{k+1} \leq 1$ for some $1 \leq k < n$. Then the matrix after the k-th GE step will be

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix},$$

where the diagonal entries of A are $p_1, \dots, p_k$ and B is the square submatrix of order $n-k$, which has p_{k+1} as its upper left entry. Since the matrix is CP, p_{k+1} -lying on the pivoting position- is the maximum, in magnitude, entry of B. From the hypothesis $p_{k+1} \leq 1$, we get that B has entries with magnitudes ≤ 1 and we can state Hadamard's inequality for B:

$$|det(B)| \leq (n-k)^{\frac{n-k}{2}}. \tag{1}$$

It holds

$$|det(A)| = p_1 p_2 \dots p_k = \frac{H(1)}{H(0)} \frac{H(2)}{H(1)} \cdots \frac{H(k)}{H(k-1)} = H(k)$$

because of Lemma 4.1.3. But since $|det(A)|$ equals the $k \times k$ leading principal minor of H_n with entries ± 1, we can apply Hadamard's inequality for $H(k)$ and get:

$$|det(A)| = H(k) \leq k^{\frac{k}{2}}. \tag{2}$$

Since the determinant of a matrix is invariant under GE, we have

$$|det(A)det(B)| = |det(H_n)| = n^{\frac{n}{2}}.$$

Multiplying (1) and (2), we get $n^{\frac{n}{2}} \leq (n-k)^{\frac{n-k}{2}} k^{\frac{k}{2}}$, or, after squaring it,

$$n^n \leq (n-k)^{n-k} k^k.$$

We observe that both $n-k$ and k are less than n. Hence the product on the right hand side of the inequality is less than $n^{n-k} n^k = n^n$ and we are led to a contradiction. Therefore the pivots $p_2, \dots p_n$ of a CP Hadamard matrix have magnitudes greater than 1. The second part of the proposition follows straightforwardly considering Lemma 4.1.3. □

The fact that the principal minors of a CP Hadamard matrix appear in ascending order, can be utilized for demonstrating explicitly some possible values of their their theoretically admissible minors (resp. pivots):

Corollary 4.1.10 ([22]). *Let H_n be a Hadamard matrix of order n. The possible values of the leading principal minors of orders 5,6,7 and 8 are:*
-$H(5) = 32, 48$
-$H(6) = 64, 96, 128, 160$
-$H(7) = 128, 192, 256, 320, 384, 448, 512, 576$
-$H(8) = 256, 384, 512, 640, 768, 896, 1024, 1152, 1280, 1408, 1536, 1664, 1792, 1920, 2048, 2176, 2304, 2560, 3072, 4096$

Proof. Since $H(4) = 16$ for a CP Hadamard matrix, from Propositions 3.2.2 and 4.1.9 follows that $H(5) = 32$ or 48. The same reasoning leads to the evaluation of all possible values of $H(6)$ and $H(7)$. For $H(8)$, we first consider all possible values that a determinant of a 8×8, ± 1 matrix can take. From Proposition 3.2.2, we get that $H(8) = k \cdot 2^7 = k \cdot 128$, $k \in \mathbb{N}$ and from Theorem 3.1.3 we have $H(8) \leq 8^4 = 4096$, thus $H(8) \in \{k \cdot 128, k = 1, \ldots, 32\}$. Value 128 is excluded due to Proposition 4.1.9, since $H(7) \geq 128$, while it has been proved that some of the rest of the possible values do not occur. □

From the values listed in Corollary 4.1.10, we obtain all possible values of the 5th, 6th and 7th pivot

Proposition 4.1.11 ([4],[22]). *If H is a CP Hadamard matrix, then $p_5 = 2$ or 3.*

Proof. $p_5 = \dfrac{H(5)}{H(4)}$ and from Corollary 4.1.10, we conclude that $p_5 = \dfrac{32}{16} = 2$ or $p_5 = \dfrac{48}{16} = 3$. □

Proposition 4.1.12 ([22]). *If H is Hadamard matrix, then the 6th and 7th pivot are bounded as follows:*

$$\frac{4}{3} \leq p_6 \leq 4$$
$$\frac{6}{5} \leq p_7 \leq 8$$

Proof. Combining the result of Proposition 4.1.9, concerning the ascending order in which the minors appear, with Corollary 4.1.10, we evaluate all possible values for p_6 and p_7:

$$p_6 = \frac{4}{3}, 2, \frac{8}{3}, 3, \frac{10}{3}, 4, 5$$
$$p_7 = \frac{6}{5}, \frac{4}{3}, \frac{8}{5}, 2, \frac{12}{5}, \frac{5}{2}, \frac{8}{3}, \frac{14}{5}, 3, \frac{16}{5}, \frac{10}{3}, \frac{7}{2}, \frac{18}{5}, 4, \frac{9}{2}, \frac{14}{3}, 5, \frac{16}{3}, 6, 7, 8, 9$$

It has been proved that p_6 and p_7 cannot take the values 5 and 9 respectively ([22]), hence we obtain the bounds presented above. □

Proposition 4.1.13 ([4]). *If H is a CP Hadamard matrix, then the four last pivots are $\frac{n}{2}$ or $\frac{n}{4}$, $\frac{n}{2}$, $\frac{n}{2}$ and n.*

Proof. Combining Lemma 4.1.3 with Theorem 3.2.4 and the fact that since H is CP, all its leading principal submatrices are non singular, we obtain:

$$p_n = \frac{H(n)}{H(n-1)} = \frac{n^{\frac{n}{2}}}{n^{\frac{n}{2}-1}} = n,$$
$$p_{n-1} = \frac{H(n-1)}{H(n-2)} = \frac{n^{\frac{n}{2}-1}}{2n^{\frac{n}{2}-2}} = \frac{n}{2},$$
$$p_{n-2} = \frac{H(n-2)}{H(n-3)} = \frac{2n^{\frac{n}{2}-2}}{4n^{\frac{n}{2}-3}} = \frac{n}{2} \text{ and}$$
$$p_{n-3} = \frac{H(n-3)}{H(n-4)} = \frac{4n^{\frac{n}{2}-3}}{8n^{\frac{n}{2}-4}} \text{ or } \frac{4n^{\frac{n}{2}} - 3}{16n^{\frac{n}{2}-4}}, \text{ i.e. } p_{n-3} = \frac{n}{2} \text{ or } \frac{n}{4}$$

□

4.2 The growth conjecture

Conjecture (Cryer's growth conjecture, 1968, [3]) $g(n, A) \leq n$ *with equality if and only if A is a CP Hadamard matrix.*

Cryer's growth conjecture is one of the most famous open problems in Numerical Analysis. As we mentioned in Chapter 2, the inequality part of the conjecture was proven false in 1991 by Gould, who constructed a 13×13 matrix that had growth factor 13.0205 ([9]). The part concerning Hadamard matrices is still unsolved and it is commonly referred as the *growth problem for Hadamard matrices*. Investigation on the properties of CP Hadamard matrices has lead to the following, more refined, conjecture ([13])

Conjecture (The growth conjecture for Hadamard matrices) *Let H_n be an $n \times n$ Hadamard matrix. Reduce H_n by GE. Then*
(i) $g(n, H_n) = n$
(ii) The four last pivots are $\frac{n}{2}$ or $\frac{n}{4}$, $\frac{n}{2}$, $\frac{n}{2}$, n.
(iii) The fifth last pivot can take the values $\frac{n}{3}$ or $\frac{n}{2}$
(iv) The sixth last pivot can take the values $\frac{n}{4}$, $\frac{n}{10/3}$ or $\frac{n}{8/3}$.
(v) Every pivot before the last has magnitude at most $\frac{n}{2}$.
(vi) The first six pivots are $1, 2, 2, 4, 2$ *or* 3, $\frac{10}{3}$ *or* $\frac{8}{3}$ *or* 4.

Statement (ii) of the conjecture has be proven (Proposition 4.1.13) as well as the part of statement (vi) concerning the first five pivots (Propositions 4.1.7 and 4.1.11). Even though it appears to be easy, the determination of the growth factor of a matrix is an extremely difficult task, thus making the proof of the conjecture very challenging. The growth problem has only been solved for small orders of Hadamard matrices and a lot of research is ongoing, concerning bigger orders.

4.2.1 Proof for orders 1 to 16

H_1,H_2,H_4 and H_8

The results concerning pivots presented in the previous section, prove easily the conjecture for the cases of Hadamard matrices of orders 1,2,4 and 8.

Proof for cases 1,2, and 4, follows directly from Proposition 4.1.7 and the unique pivot patterns are

$$1$$

$$1, 2$$

$$1, 2, 2, 4$$

Proof for case 8 is a combination of Propositions 4.1.7 and 4.1.13 and the unique pivot pattern is

$$1, 2, 2, 4, 2, 4, 4, 8$$

H_{12}

The first proof for the case of Hadamard matrices of order 12 was published in 1995 by W. Edelman and W. Mascarenhas ([5]). In their paper, they proved the following lemma

Lemma 4.2.1. *If H is a 12×12 CP Hadamard matrix then $H(5) = 48$.*

and employed it to evaluate the unique pivot pattern that occurs

Theorem 4.2.2. *The unique pivot pattern of a CP 12×12 Hadamard matrix is*

$$1, 2, 2, 4, 3, \frac{10}{3}, \frac{18}{5}, 4, 3, 6, 6, 12$$

Proof. We know that the first four pivots are 1,2,2,4. From Lemma 4.2.1, $H(5) = 48$ and thus $p_5 = 3$. From Theorem 3.2.1, we get that $H[7] = 12 \cdot H(5) = 576$, which is the maximum value attained by a 7×7, ± 1 matrix. Lemma 3.2.7 tells us that a 6×6 matrix with maximal determinant is embedded in the 7×7 lower right corner and as a consequence of Corollary 4.1.5, $H[6] = 160$. The same argument leads us to the conclusion that $H[5] = 48, H[4] = 16, H[3] = 4, H[2] = 2$ and $H[1] = 1$. The last seven pivots follow from Lemma 4.1.4. □

We observe that the pivot pattern satisfies all the conditions of the conjecture for Hadamard matrices and the growth factor is $g(n, H_{12}) = \max\{p_1, \dots, p_{12}\} = 12$.

The authors in [22] presented another proof of Cryer's growth conjecture for Hadamard matrices of order 12. Using the bounds and values for pivots presented in the previous section, they showed that all pivots are less than or equal to 12 and since the last pivot is 12, $g(n, H_{12}) = 12$.

A new result on pivot values and a rediscovery of the proof

In the following proposition, we introduce a new result concerning values of pivots, that emerged in our study

Proposition 4.2.3. *If H is a CP Hadamard matrix of order n, then the following lower bound holds for every pivot p_i*

$$p_i \geq \frac{n}{n-i+1}.$$

Proof. We use the following notation: for every $k = 1, \dots, n-1$, we can write

$$H = \begin{bmatrix} H_k & B_{k\times(n-k)} \\ C_{(n-k)\times k} & M_{n-k} \end{bmatrix},$$

where $|det(M_{n-k})| = H[n-k]$.
For $i > 1$, from Lemma 4.1.4, we have

$$p_i = p_{n+1-(n-i+1)} = n\frac{H[n-i]}{H[n-i+1]} \iff$$

$$\frac{1}{p_i} = \frac{1}{n}\frac{H[n-i+1]}{H[n-i]}.$$

Evaluating $H[n-i+1] = |det(M_{n-i+1})|$ using the Laplace expansion along the first row $[m_{11} \dots m_{1n-i+1}]$ of M_{n-i+1}, we get

$$H[n-i+1] = \left|\sum_{s=1}^{n-i+1} m_{1s}(-1)^{1+s}(det(M_{n-i+1})_{1s})\right| \leq$$

$$\sum_{s=1}^{n-i+1} |m_{1s}||(-1)^{1+s}||det(M_{n-i+1})_{1s}| = \sum_{s=1}^{n-i+1} |det(M_{n-i+1})_{1s}|,$$

since $|m_{1s}| = 1, \forall s$. Each $(M_{n-i+1})_{1s}$ is an $(n-i)\times(n-i)$ submatrix of M_{n-i+1}, (which is the $(n-i+1)\times(n-i+1)$ lower right submatrix of H). Since $i > 1$,we have $n-i+1 < n$ and from Corollary 4.1.5, we get $|det(M_{n-i+1})_{1s}| \leq H[n-i]$ for every $s = 1, \dots, n-i+1$, thus

$$\sum_{s=1}^{n-i+1} |det(M_{n-i+1})_{1s}| \leq (n-i+1)H[n-i].$$

Finally, for $i > 1$

$$\frac{1}{p_i} = \frac{1}{n}\frac{H[n-i+1]}{H[n-i]} \leq \frac{(n-i+1)H[n-i]}{nH[n-i]} = \frac{n-i+1}{n} \iff p_i \geq \frac{n}{n-i+1}.$$

For $i = 1$, we observe that $p_1 \geq \dfrac{n}{n-1+1} = 1$ holds. □

We now employ this new lower bound for pivots, to rediscover the evaluation of the growth factor of H_{12}

Proposition 4.2.4. *The growth factor of a Hadamard matrix of order 12 is 12.*

Proof. Proof: We know that the first four pivots are $p_1 = 1, p_2 = 2, p_3 = 2$ and $p_4 = 4$.
The fifth pivot can take the values 2 or 3, hence $p_5 \geq 2$.
The four last pivots are $p_9 = 3$ or 6, i.e. $p_9 \geq 3$, $p_{10} = 6, p_{11} = 6$ and $p_{12} = 12$.
The eighth pivot is $p_8 = \dfrac{H(8)}{H(7)} = \dfrac{H(12-4)}{H(12-5)}$ and from the values displayed in Theorems 3.2.4 and 3.2.5, we get that

$$p_8 = \frac{8\cdot 12^{6-4}}{16\cdot 12^{6-5}}, \frac{8\cdot 12^{6-4}}{32\cdot 12^{6-5}}, \frac{8\cdot 12^{6-4}}{48\cdot 12^{6-5}}, \frac{16\cdot 12^{6-4}}{16\cdot 12^{6-5}}, \frac{16\cdot 12^{6-4}}{32\cdot 12^{6-5}} \text{ or } \frac{16\cdot 12^{6-4}}{48\cdot 12^{6-5}},$$

thus $p_8 = 2, 3, 4, 6$ or 12. From Proposition 4.2.3, $p_8 \geq \dfrac{12}{5} > 2$, hence $p_8 = 3, 4, 6$ or 12.
If $p_8 = 3 = \dfrac{8\cdot 12^{6-4}}{32\cdot 12^{6-5}} = \dfrac{H(12-4)}{H(12-5)}$, then p_9 must be $\dfrac{H(12-3)}{H(12-4)} = \dfrac{4\cdot 12^{6-3}}{8\cdot 12^{6-4}} = 6$. Thus

$$1\cdot 2\cdot 2\cdot 4\cdot 2\cdot p_6\cdot p_7\cdot 3\cdot 6\cdot 6\cdot 6\cdot 12 \leq p_1 \dots p_{12} = det(H_{12}) = 12^6$$

$$\Rightarrow p_6 \cdot p_7 \leq \frac{12^6}{2 \cdot 2 \cdot 4 \cdot 2 \cdot 3 \cdot 6 \cdot 6 \cdot 6 \cdot 12} = 12$$

and since $p_6, p_7 > 1$ (Proposition 4.1.9), we have that $p_6 < 12$ and $p_7 < 12$. Otherwise, $p_8 \geq 4$ and we get

$$1 \cdot 2 \cdot 2 \cdot 4 \cdot 2 \cdot p_6 \cdot p_7 \cdot 4 \cdot 3 \cdot 6 \cdot 6 \cdot 12 \leq p_1 \ldots p_{12} = det(H_{12}) = 12^6$$

$$\Rightarrow p_6 \cdot p_7 \leq \frac{12^6}{2 \cdot 2 \cdot 4 \cdot 2 \cdot 4 \cdot 3 \cdot 6 \cdot 6 \cdot 12} = 18.$$

From Proposition 4.2.3, $p_7 \geq \frac{12}{6} = 2$, hence

$$p_6 \cdot 2 \leq p_6 \cdot p_7 \leq 18 \Rightarrow p_6 \leq 9.$$

Similarly, $p_6 \geq \frac{12}{7}$, hence

$$\frac{12}{7} \cdot p_7 \leq p_6 \cdot p_7 \leq 18 \Rightarrow p_7 \leq \frac{21}{2}.$$

We notice that in every case, all pivots have magnitudes less than or equal to 12 and since $p_{12} = 12$, the growth factor is $g(12, H_{12}) = 12$. □

H_{16}

The growth conjecture for Hadamard matrices of order 16 was proved in 2009 by C. Kravvaritis and M. Mitrouli ([20], [23]). The proof for that case is significantly more difficult and complicated, compared to the cases of smaller orders.

The authors used sophisticated numerical techniques and they developed a strategy to show that an arbitrary CP Hadamard matrix of order 16 will have one of 34 possible pivot patterns. Here, we give a brief overview of their method.

Two algorithms have a significant role in proving the conjecture: algorithm *Exist*, that specifies the existence of given submatrices in Hadamard matrices, and algorithm *Minors*, that, for a given j, computes all possible $(n-j) \times (n-j)$ minors of Hadamard matrices.

The proof is divided in two parts:
In the first part, all possible values of the first eight pivots are evaluated. Beginning with matrix

$$A = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix},$$

all probable $(j+1) \times (j+1)$ extensions of the $j \times j$, $j = 4, 5, 6, 7$, matrices that always exist in the upper left corner of a CP Hadamard matrix of order 16 are evaluated. From them only the CP ones are kept and then algorithm *Exist* keeps only the ones that can appear. From all possible existing submatrices that result, the first eight pivots are evaluated using Lemma 4.1.3.
In the second part, all possible values of the eight last pivots are evaluated.

Using algorithm *Minors*, the $(16-j)\times(16-j)$, $j = 4, \ldots, 7$ minors of Hadamard matrices of order 16 are computed. From Lemma 4.1.3, all possible values of p_{10}, p_{11} and p_{12} are evaluated. With the implementation of algorithm *Exist*, the values that do not appear are excluded. The possible values of the ninth pivot are derived from the values of the rest of the pivots.

The list of all 34 pivot patterns can be found in Appendix B.

4.2.2 Open cases and further research

The growth conjecture for Hadamard matrices of order n has been proved only for $n = 1, 2, 4, 8, 12$ and 16. The next unsolved case is $n = 20$, for which they have been observed at least 1128 different pivot patterns. For the purposes of this thesis, some pivot patterns were evaluated, using MATLAB functions. They are presented in Appendix B, along with some comments.

As one can observe, the proof of the conjecture is an extremely difficult task, even for small orders. A lot of investigation is ongoing, concerning possible extensions of the methods used in the existing proofs to bigger orders.

The authors in [23] suggest that the methods introduced in their paper for the proof of case 16, can be used as a basis for calculating pivot patterns of Hadamard matrices of higher orders. Due to the high complexity of such problems, they point out the need for developing algorithms that implement effectively their or other more elaborate ideas.

A more theoretical approach of the problem involves the techniques used in the proofs for case $n = 12$ presented in the previous section and in [5] and [22]. Possible extensions of these results, that could also be combined with new techniques, are under consideration.

We notice that the examination of pivot patterns is a key element in our study. Thus, a large part of the ongoing research is focused on them.

A theoretical classification of the pivot patterns with respect to equivalence classes would be of great importance, since it might give insightful information on the possible connection between the pivot values and the structure of the matrices. While, in general, a pivot pattern may occur in matrices from different classes, some patterns seem to appear only in matrices from a specific class. For instance, it has been observed that value $\frac{n}{2}$ as the fourth pivot from the end only appears in matrices that are H-equivalent to Sylvester-Hadamard matrices. It has also been noticed that some patterns appear more frequently than others. Such detailed observations on pivot patterns can be found in [13].

An other significant remark concerns the possible values of Hadamard matrices' minors (resp. pivots). It has been noticed that from all possible values of $H(7)$ and $H(8)$ displayed in Proposition 4.1.10, only a few occur. More specifically, in case 16, $H(7)$ takes only values 256, 384, 512 and 576 while $H(8)$ takes the only values 1024, 1536, 2048, 2304, 2560, 3072 and 4096. In the still unsolved case 20, the only values that have been observed for $H(7)$ are 512 and 576. A lot of questions arise on why some values of minors never occur. If answered, they might provide significant information for developing strategies that will exclude values of minors and, respectively, pivots.

An other important remark on pivot patterns is the following proposition

Proposition 4.2.5 ([4])**.** *Suppose A is an $n \times n$ CP matrix. Then $A \otimes H_2$ is CP and its pivots (in magnitude) are the Kronecker product of the pivots of A*

with those of H_2:

$$\begin{bmatrix} |a_{11}| & \dots & |a_{nn}^{(n-1)}| \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \end{bmatrix}$$

That leads to the following corollary that proves the growth conjecture for Sylvester-Hadamard matrices

Corollary 4.2.6 ([4])**.** *The pivot pattern (in magnitude) of $H_{2^k} = \otimes^k H_2$ is*

$$\otimes^k \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 & 2 & 4 & 4 & 8 & \dots & \end{bmatrix}.$$

The growth conjecture can also be proved for Hadamard matrices that have the *good pivots property.* This notion is introduced and extensively examined in [22]. A *good pivot pattern* is of the form

$$\begin{bmatrix} p_1 & p_2 & \dots & p_{\frac{n}{2}} & \frac{n}{p_{\frac{n}{2}}} & \dots & \frac{n}{p_2} & \frac{n}{p_1} \end{bmatrix},$$

i.e. for every $i = 1, \dots, n,\ p_i p_{n-i+1} = n$. Hadamard matrices are the only matrices known that lead to good pivot patterns. We have the following proposition

Proposition 4.2.7 ([22])**.** *If a CP Hadamard matrix H has good pivots, then $g(n, H) = n$.*

Proof. It is sufficient to prove that $p_i \le n$, $i = 2 \dots, n-1$, since $p_1 = 1, p_n = n$ and $g(n, H) = \max\{p_1 \dots p_n\}$.
If $p_i \ge n$ for some i, then $p_{n-i+1} n \le p_{n-i+1} p_i = n \Rightarrow p_{n-i+1} \le 1$, which cannot hold because of Proposition 4.1.9. We are led to a contradiction, thus $p_i \le n$. □

The next result follows from Proposition 4.2.5 and proves the existence of an infinite family of Hadamard matrices with good pivots

Proposition 4.2.8 ([22])**.** *Suppose A is an $n \times n$ CP matrix with good pivots. Then the pivots of $A \otimes H_2$ are also good.*

Deriving similar results concerning other special classes of Hadamard matrices, e.g. symmetric Hadamard matrices, is under consideration.

The examination of the growth problem for Hadamard matrices has also led to the study of the growth factor of other classes of matrices, related to Hadamard matrices. Some examples are weighting matrices ([16]), $(1, -1)$ incidence matrices of SBIBDs ([17],[18]) and circulant Hadamard matrices ([27]).

Conclusions

The need for the determination of the growth factor of a matrix followed from backward error analysis, when it was shown that the stability of the Gaussian elimination algorithm is connected to it. However, experience shows that Gaussian elimination with pivoting is stable in practice and hence the problem of evaluating the growth factor has not been an issue in Numerical Analysis for years. It is still, though, a very interesting mathematical problem.

The attention is restricted to Hadamard Matrices, for whom it is conjectured that they are the only matrices whose growth factor equals their size. Hadamard matrices are characterized by unique properties and their special structure allows us to obtain useful formulae that can be employed for the evaluation of their growth factor.

Despite the fact that the growth conjecture for Hadamard matrices is seemingly an easy problem, the determination of their growth factor is an incredibly difficult task and it has been achieved only for the orders 1 to 16. A lot of investigation on this open problem is ongoing and main research is focused on the extension of existing methods (numerical and theoretical) for the evaluation of the growth factor and the extensive study of their pivot patterns (their classification, their possible connection to the matrices' structure, the exclusion of possible values etc).

Appendix A

MATLAB codes

```
function [H,p,q,r,c] = Hequiv(H)
% Randomly performs row and column operations on square matrix H to
% produce an H-equivalent one
% input:  square matrix H
% outputs:  matrix H after application of row/column operations,
% vectors p and q where row and column permutations are stored,
% vectors r and c where row and column multiplications by -1 are stored
n = size(H,1);
%random row switching
h = 1:n;
p = zeros(1,n);
for i = 1:n-1
  r = floor(1 + rand*(n+1-i));
  p(i) = h(r);
  h(r) = [];
end
p(n) = h;
H = H(p,:);
%random column switching
h = 1:n;
q = zeros(1,n);
for i = 1:n-1
  r = floor(1 + rand*(n+1-i));
  q(i) = h(r);
  h(r) = [];
end
q(n) = h;
H = H(:,q);
%random multiplication of rows by -1
r = ones(1,n);
for i = 1:n
  if rand>=0.5
    H(i,:)  = -1*H(i,:);
    (i) = -1;
  end
```

```
end
%random multiplication of columns by -1
c = ones(1,n);
for i = 1:n
  if rand>=0.5
    H(:,i) = -1*H(:,i);
    c(i) = -1;
  end
end
```

```
function piv = GECP(A);
%Applies GE with complete pivoting on square invertible matrix A and
%returns the pivot pattern piv
[n, n] = size(A);
for k=1:n-1
  %Search for the maximum element of the lower right submatrix
  [maxv, r] = max(abs(A(k:n, k:n)));
  %maxv contains the maximum element of every column of the lower right
  %submatrix
  %every entry of r is the index of the row (of the submatrix) the
  %maximum was found
  [maxv, c] = max(maxv);
  %now maxv is the total maximum element of the lower right submatrix
  %c is the index of the column (of the submatrix) in which that element
  %was found
  r = r(c); %r is the index of the row (of the submatrix) in which
  %the total maximum lies
  %Replacing pivot
  q = r+k-1; %q is the index of the row of A in which the maximum is
  %placed
  %row switching
  A([k q], :)  = A([q k], :);
  q = c+k-1; %q now represents the column index of A in which the maximum
  %lies
  %column switching
  A(:,[k q]) = A(:,[q k]);
  %Evaluation of the new matrix
  if A(k, k)  = 0
    %evaluation of the multipliers (they are stored below the
    %diagonal of A)
    A(k+1:n, k) = A(k+1:n, k)/A(k, k);
    %evaluation of the new elements of A (they replace the old ones)
    A(k+1:n, k+1:n) = A(k+1:n, k+1:n) - A(k+1:n, k)*A(k, k+1:n);
  end
end
%the diagonal entries of A are the pivots
piv = abs(diag(A));
```

Appendix B

Pivot patterns

The mentioned matrices were obtained from Neil Sloane's library of Hadamard matrices ([48]).

Pivots of H_{16}

Hadamard matices of order 16 can be classified into five classes of H - equivalence: I,II,III,IV and V (represented by matrices $H_{16}^{I}, H_{16}^{II}, H_{16}^{III}, H_{16}^{IV}$ and H_{16}^{V} respectively). Class I is the Sylvester-Hadamard class. Classes IV and V are one another's transpose and, therefore, identical for GE with complete pivoting (a matrix is CP if and only if its transpose is CP, in which case they give the same pivot pattern ([4])).

Here, we have evaluated 10 pivot patterns from each class, using the MATLAB functions presented in Appendix A. From every matrix H_{16}^{I} to H_{16}^{V}, we produced 10 H - equivalent matrices using MATLAB function `Hequiv` and then we applied GE with complete pivoting using MATLAB function `GECP`, to obtain the pivot patterns. The results are presented in tables B.1 through B.5.

We observe that H - equivalent matrices can lead to different pivot patterns. We also notice that the same pivot pattern can appear in matrices of different classes.

In Table B.6, the 34 different pivot patterns that occur when applying GE with complete pivoting in Hadamard matrices of order 16, are displayed. From these, 11 are good pivot patterns: the $1^{st}, 6^{th}, 9^{th}, 11^{th}, 15^{th}, 17^{th}, 21^{st}, 24^{th}$, $30^{th}, 33^{rd}$ and 34^{th}. The 12^{th} pivot pattern is the only one with $p_{13} = \frac{16}{2} = 8$ and was obtained from matrices from the Sylvester-Hadamard class.

Pivots of H_{20}

Hadamard matrices of order 20 can be classified into three classes of H-equivalence: I, II and III (represented by matrices H_{20}^{I}, H_{20}^{II} and H_{20}^{III} respectively). It is known that there exist at least 1128 different pivot patterns. For the purposes of this thesis, we evaluated 12 different pivot patterns from every class using the MATLAB functions presented in Appendix A. The results are displayed in Tables B.7 through B.9.

Again, we observe that matrices of the same class can lead to different pivot patterns. We also note that, even though the patterns presented here are all different, the same pivot pattern may appear in matrices from different classes.

There have been observed at least 47 good pivot patterns in Hadamard matrices of order 20. Here, 6 good pivot patterns appeared: 3 in class I (the $4^{th}, 10^{th}$ and 11^{th}), 2 in class II (the 5^{th} and 7^{th}) and 1 in class III (the 2^{nd}). Value $\frac{20}{2} = 10$ has never appeared as the fourth to last pivot.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	3	$\frac{8}{3}$	2	4	4	8	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16
1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16

Table B.1: class of H_{16}^{I}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	2	4	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{12}{5}$	4	$\frac{16}{3}$	5	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{12}{5}$	4	$\frac{16}{3}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{12}{5}$	4	$\frac{16}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{12}{5}$	4	$\frac{16}{3}$	5	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16

Table B.2: class of H_{16}^{II}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	4	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
1	2	2	4	2	$\frac{10}{3}$	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16

Table B.3: class of H_{16}^{III}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	2	4	4	5	$\frac{16}{5}$	5	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	4	4	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16

Table B.4: class of H_{16}^{IV}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	4	4	4	4	8	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	2	4	4	4	4	4	6	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{40}{9}$	$\frac{24}{5}$	$\frac{16}{3}$	4	8	8	16

Table B.5: class of H_{16}^{V}

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
1	1	2	2	4	2	4	4	4	4	4	4	8	4	8	8	16
2	1	2	2	4	2	4	4	4	4	4	6	$\frac{16}{3}$	4	8	8	16
3	1	2	2	4	2	4	4	4	4	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
4	1	2	2	4	2	4	4	4	$\frac{9}{2}$	$\frac{16}{18/5}$	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
5	1	2	2	4	2	4	4	$\frac{9}{2}$	4	$\frac{16}{18/5}$	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
6	1	2	2	4	2	4	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
7	1	2	2	4	2	4	4	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
8	1	2	2	4	2	4	4	5	$\frac{16}{5}$	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
9	1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
10	1	2	2	4	2	4	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
11	1	2	2	4	2	4	4	8	2	4	4	8	4	8	8	16
12	1	2	2	4	3	$\frac{8}{3}$	2	4	4	4	4	8	8	8	8	16
13	1	2	2	4	3	$\frac{8}{3}$	2	4	4	4	8	8	4	8	8	16
14	1	2	2	4	3	$\frac{8}{3}$	2	4	4	8	4	8	4	8	8	16
15	1	2	2	4	3	$\frac{8}{3}$	2	4	4	8	6	$\frac{16}{3}$	4	8	8	16
16	1	2	2	4	3	$\frac{8}{3}$	4	4	4	4	4	8	4	8	8	16
17	1	2	2	4	3	$\frac{8}{3}$	4	4	4	4	6	$\frac{16}{3}$	4	8	8	16
18	1	2	2	4	3	$\frac{8}{3}$	4	4	4	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
19	1	2	2	4	3	$\frac{8}{3}$	4	4	$\frac{9}{2}$	$\frac{16}{18/5}$	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
20	1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	4	4	8	4	8	8	16
21	1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
22	1	2	2	4	3	$\frac{8}{3}$	4	5	$\frac{16}{5}$	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
23	1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	4	8	4	8	8	16
24	1	2	2	4	3	$\frac{8}{3}$	4	6	$\frac{8}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
25	1	2	2	4	3	$\frac{10}{3}$	$\frac{8}{10/3}$	4	$\frac{16}{3}$	4	4	8	4	8	8	16
26	1	2	2	4	3	$\frac{10}{3}$	$\frac{8}{10/3}$	4	$\frac{16}{3}$	4	6	$\frac{16}{3}$	4	8	8	16
27	1	2	2	4	3	$\frac{10}{3}$	$\frac{8}{10/3}$	4	$\frac{16}{3}$	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
28	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	4	8	4	8	8	16
29	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	4	6	$\frac{16}{3}$	4	8	8	16
30	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	4	4	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
31	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	4	8	4	8	8	16
32	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	4	6	$\frac{16}{3}$	4	8	8	16
33	1	2	2	4	3	$\frac{10}{3}$	$\frac{16}{5}$	5	$\frac{16}{5}$	5	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16
34	1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{16}{18/5}$	$\frac{16}{10/3}$	$\frac{16}{3}$	4	8	8	16

Table B.6: All 34 pivot patterns for H_{16}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	$\frac{14}{3}$	$\frac{100}{21}$	4	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{46}{9}$	$\frac{120}{23}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	$\frac{14}{3}$	$\frac{100}{21}$	4	$\frac{25}{4}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	4	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{41}{9}$	$\frac{190}{41}$	$\frac{90}{19}$	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	$\frac{34}{9}$	$\frac{72}{17}$	$\frac{9}{2}$	$\frac{40}{9}$	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{9}{2}$	$\frac{380}{81}$	$\frac{100}{19}$	4	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{14}{3}$	5	$\frac{100}{21}$	4	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{40}{9}$	$\frac{23}{5}$	$\frac{100}{23}$	5	4	$\frac{25}{4}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{37}{9}$	$\frac{180}{37}$	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{40}{9}$	$\frac{22}{5}$	$\frac{50}{11}$	$\frac{9}{2}$	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{40}{9}$	$\frac{23}{5}$	$\frac{110}{23}$	$\frac{60}{11}$	$\frac{10}{3}$	5	5	10	5	10	10	20

Table B.7: Class of H^1_{20}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{37}{9}$	$\frac{204}{37}$	$\frac{70}{17}$	$\frac{100}{21}$	4	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{41}{9}$	$\frac{220}{41}$	$\frac{60}{11}$	$\frac{10}{3}$	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{38}{9}$	$\frac{82}{19}$	$\frac{220}{41}$	$\frac{60}{11}$	$\frac{10}{3}$	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{14}{3}$	$\frac{100}{21}$	6	$\frac{10}{3}$	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{9}{2}$	$\frac{40}{9}$	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	$\frac{13}{3}$	$\frac{60}{13}$	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	4	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{38}{9}$	$\frac{104}{19}$	$\frac{60}{13}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	$\frac{34}{9}$	$\frac{66}{17}$	$\frac{56}{11}$	5	$\frac{100}{21}$	4	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{44}{9}$	$\frac{60}{11}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{38}{9}$	$\frac{100}{19}$	4	8	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{38}{9}$	$\frac{90}{19}$	$\frac{40}{9}$	5	4	5	5	10	5	10	10	20

Table B.8: Class of H^2_{20}

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{3}$	4	$\frac{44}{9}$	$\frac{48}{11}$	5	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{38}{9}$	$\frac{90}{19}$	5	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	5	$\frac{14}{3}$	$\frac{100}{21}$	4	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{44}{9}$	$\frac{50}{11}$	5	4	$\frac{25}{4}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{44}{9}$	$\frac{46}{11}$	$\frac{120}{23}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	$\frac{32}{9}$	5	$\frac{26}{5}$	$\frac{60}{13}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{16}{3}$	$\frac{35}{8}$	$\frac{100}{21}$	4	$\frac{25}{4}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{41}{9}$	$\frac{200}{41}$	$\frac{9}{2}$	5	$\frac{50}{9}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	4	$\frac{14}{3}$	5	$\frac{100}{21}$	4	$\frac{25}{4}$	6	$\frac{20}{3}$	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{44}{9}$	$\frac{52}{11}$	$\frac{60}{13}$	$\frac{20}{3}$	$\frac{5}{2}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{5}$	4	$\frac{40}{9}$	5	$\frac{22}{5}$	$\frac{60}{11}$	$\frac{10}{3}$	5	5	10	5	10	10	20
1	2	2	4	3	$\frac{10}{3}$	$\frac{18}{3}$	4	$\frac{40}{9}$	$\frac{9}{2}$	$\frac{44}{9}$	$\frac{60}{11}$	$\frac{10}{3}$	5	$\frac{15}{2}$	$\frac{20}{3}$	5	10	10	20

Table B.9: Class of H^3_{20}

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