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to my supervisor Prof. Hatziafratis Telemachos

to my husband,

and to my son

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1 Introduction

In this work we present several results concerning mostly applications of Baire's Category theorem in Complex Analysis in one and in several complex variables. An important problem in complex analysis is whether there exists a holomorphic function f, in a given open set Ω in $\mathbb{C}^n$, which is singular at every boundary point of Ω in the sense that whenever U and V are open subsets of $\mathbb{C}^n$, with U being connected and $\varnothing \neq V \subseteq U \cap \Omega \neq U$, then there is no holomorphic function F in U which extends $f\big|_V$, i.e., $F(z) = f(z)$ for $z \in V$. See for example [2], [7], [10], [11], [14], [16] and [20]. Also the problem of constructing singular functions with specific properties – for example satisfying certain growth conditions near the boundary or having certain smoothness upto the boundary – has been studied in various directions. See for example [7], [8], [10], and [11]. In this work we will show – under certain restrictions on the open set Ω – that the set of the $\mathcal{O}L^p$ (holomorphic and L^p with respect to Lebesgue measure) and $H^p(\mathbb{B})$ (holomorphic and H^p with respect to the Euclidean surface area measure on the sphere $\partial\mathbb{B}$), $1 \leq p < \infty$, functions in Ω and $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$, which are totally unbounded, is dense and $\mathcal{G}_\delta$ in the space $\mathcal{O}L^p(\Omega)$ and $H^p(\mathbb{B})$ respectively. In fact we work mostly with the spaces $\bigcap\limits_{p<q} \mathcal{O}L^p(\Omega)$ -see chapter 3- and $\bigcap\limits_{1 \leq p<q} H^p(\Omega)$ -see chapter 4-, $q \in (1, +\infty)$ endowed with its natural topology.

The second chapter is an introduction for the chapters 3 and 4. Here we present the definitions of concepts which appear in this thesis, propositions and theorems which we apply in order to prove the theorems presented in the chapters 3 and 4.

In the third chapter we consider Bergman spaces $\mathcal{O}L^p(\Omega)$ and variations of them on domains Ω in one or several complex variables. For certain domains Ω we show that the generic function in these spaces is totaly unbounded in Ω and hence non-extendable. We also show that generically these functions do not belong – not even locally – in Bergman spaces of higher order. Finally, in certain domains Ω, we give examples of bounded non-extendable holomorphic functions – a generic result in the spaces $A^s(\Omega)$ of holomorphic functions in Ω whose derivatives of order $\leq s$ extend continuously to $\overline{\Omega}$ ($0 \leq s \leq \infty$).

In the fourth chapter we study some Hardy type spaces $\bigcap\limits_{1 \leq p<q} H^p(\Omega)$ and we prove that the set of the holomorphic functions which are totally unbounded in certain domains is dense and $\mathcal{G}_\delta$ in these spaces. These totally unbounded functions are non-extendable, despite the fact that they have non-tangential limits at the boundary of the domain. Similarly we show that the set of the holomorphic functions in these spaces which are non-extendable is dense and $\mathcal{G}_\delta$ in these spaces. Following a suggestion of Nestoridis, we also consider local Hardy spaces $H^p(\mathbb{B}, G)$, for open subsets G of the sphere $\partial\mathbb{B}$

(the precise definition is given in the subsection 4.3.) as another way of measuring how singular a holomorphic function is near a boundary point. In this chapter we show that the set of the functions in the space $\bigcap_{1\le p<q} H^p(\mathbb{B})$ which do not belong to any local H^q-space is dense and $\mathcal{G}_\delta$ (Theorem 4.3.2.). We first work in the case of the unit ball of $\mathbb{C}^n$ where the calculations are easier and the results are somehow better, and then we extend them to the case of strictly pseudoconvex domains. In sections 4.4.and 4.6., we will extend these results from the ball to the case of strictly pseudoconvex domains. In this more general case we have to modify the definition of local Hardy spaces which we give in the case of the ball. Thus if Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, we consider the space $H^p(\Omega,U)$, where U is an open subset of $\mathbb{C}^n$ so that $U\cap(\partial\Omega)\neq\varnothing$ (For the precise definition, see section 4.5.). In the last section of this chapter we extend the results for strictly pseudoconvex domains in $\mathbb{C}^n$ to the case of harmonic functions in domains of $\mathbb{R}^n$.

Last it remains an open question if the results presented in chapters 3 and 4 could be extended in other spaces as Nevanlinna or in convex sets respectively.

2 Preliminaries

2.1. Basic Theorems and Definitions

We will use the following theorems to prove the main results of this work.

Definition 2.1.1. A function $f:\Omega\to\mathbb{C}$, defined on an open subset $\Omega\subseteq\mathbb{C}^n, n\geq 1$, is said to be *holomorphic* in Ω if $f\in C(\Omega)$ and is holomorphic in each variable separately. The classes of all holomorphic functions in Ω will be denoted by $\mathcal{O}(\Omega)$.

Definition 2.1.2. The *hermitian inner product* is defined by

$$\langle z,w\rangle=\sum_{j=1}^{n} z_j\overline{w}_j,\quad z,w\in\mathbb{C}^n$$

and the associated norm is: $|z|=\langle z,z\rangle^{1/2},\ z\in\mathbb{C}^n$.

Definition 2.1.3. Let X be a topological space. A $\mathcal{G}_\delta$ set in X is a countable intersection of open sets in X. Furthermore, a subset E of X is called *dense* if intersects every nonempty open subset of X.

Theorem 2.1.4. (Baire's Theorem, [23, Theorem 5.6]) Any countable family of open and dense sets in a complete metric space has a non-empty and in fact dense intersection.

Theorem 2.1.5. (Taylor's Theorem, [7, Problem 25]) Let $\Omega\subseteq\mathbb{C}^n$ be an open set, and ρ a real valued C^m– function in Ω. Then, for all $\zeta\in\Omega$ and for $z\to\zeta$,

$$\begin{aligned}\rho(z)=\rho(\zeta)&+\sum_{j=1}^{2n}\frac{\partial\rho(\zeta)}{\partial x_j}\big(x_j(z)-x_j(\zeta)\big)+\\ &+\frac{1}{2!}\sum_{j,k=1}^{2n}\frac{\partial^2\rho(\zeta)}{\partial x_j\partial x_k}\big(x_j(z)-x_j(\zeta)\big)\big(x_k(z)-x_k(\zeta)\big)+\ldots+\\ &+\frac{1}{m!}\sum_{j_1,\ldots,j_m=1}^{2n}\frac{\partial^m\rho(\zeta)}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_m}}\big(x_{j_1}(z)-x_{j_1}(\zeta)\big)\big(x_{j_2}(z)-x_{j_2}(\zeta)\big)\ldots\big(x_{j_m}(z)-x_{j_m}(\zeta)\big)+\circ\left(|\zeta-z|^m\right),\end{aligned}$$

where $x_j=x_j(\zeta)$ are the real coordinates of $\zeta\in\mathbb{C}^n$ such that $\zeta_j=x_j(\zeta)+ix_{j+n}(\zeta)$, $j=1,\ldots,n$.

Theorem 2.1.6. (Hölder's inequality, [28, Proposition 3.3.2]) Let $(X,\mathcal{A},\mu)$ be a measure space and let $p,q\in[1,+\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f\in L^p(X,\mathcal{A},\mu)$ and $g\in L^q(X,\mathcal{A},\mu)$ then fg belongs to $L^1(X,\mathcal{A},\mu)$ and satisfies

$$\int |fg| \mathrm{d}\mu \le \|f\|_p \|g\|_q .$$

Theorem 2.1.7. (Fatou's Lemma, [6, Theorem 1.17]) Let $f_k : X \to [0,\infty]$ be $\mu-$ measurable functions for $k = 1,\ldots$. Then

$$\int \lim_{k\to\infty} f_k \, d\mu \le \liminf_{k\to\infty} \int f_k \, d\mu.$$

Theorem 2.1.8. (Monotone convergence Theorem, [6, Theorem 1.18]) Let $f_k : X \to [0,\infty]$ be μ–measurable functions ($k = 1,\ldots$), with

$$f_1 \le \ldots \le f_k \le f_{k+1} \le \ldots .$$

Then

$$\lim_{k\to\infty} \int f_k \, d\mu = \int \lim_{k\to\infty} f_k \, d\mu .$$

Theorem 2.1.9. (Fubini's theorem, [6, Theorem 1.22]) Let μ be a measure on X and v be a measure on Y. If f is $(\mu \times v)$- integrable and f is σ- finite with respect to $\mu \times v$ (in particular, if f is $(\mu \times v)$- summable), then the mapping $y \mapsto \int\limits_X f(x,y) \, d\mu(x)$ is v- integrable, the mapping $x \mapsto \int\limits_Y f(x,y) \, dv(y)$ is μ- integrable, and

$$\int\limits_{X\times Y} f \, d(\mu \times v) = \int\limits_Y \left[\int\limits_X f(x,y) \, d\mu(x) \right] dv(y) = \int\limits_X \left[\int\limits_Y f(x,y) \, dv(y) \right] d\mu(x).$$

Theorem 2.1.10. (Inverse Function Theorem, [18, Theorem 8.3], [10, Theorem 1.1.18]) Let $F : \Omega \to \mathbb{R}^n$, where Ω is an open set in $\mathbb{R}^n$, be of class C^1 and p a point in Ω such that $JF(p)$ is invertible. Then, there exist an open set X containing p, an open set Y containing $F(p)$, and a function $G : Y \to X$ of class C^1 that satisfies $FG(y) = y$, for all y in Y, and $GF(x) = x$, for x in X. Moreover, $JG(y) = JF(G(\mathrm{y}))^{-1}$, for all y in Y.

Theorem 2.1.11. (Montel's Theorem, [18, Theorem 5.2]) Let $\mathcal{F}$ be a bounded family of holomorphic functions on an open set $\Omega \subset \mathbb{C}^n$. Then, each sequence of functions in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets.

Theorem 2.1.12. ([25, Proposition 5.2]) Let $\mathcal{V}$ be a topological vector space over $\mathbb{C}$, X a non-empty set, and let $\mathbb{C}^X$ denote the vector space of all complex-valued functions on X. Suppose $T : \mathcal{V} \to \mathbb{C}^X$ is a linear (or sublinear) operator with the property that, for every $x \in X$, the functional $T_x : \mathcal{V} \to \mathbb{C}$, defined by $T_x(f) = T(f)(x)$, for $f \in \mathcal{V}$, is continuous. Let

$$\mathcal{E} = \{ f \in \mathcal{V} : T(f) \ \textit{is unbounded on} \ X \}.$$

Then either $\mathcal{E} = \varnothing$ or $\mathcal{E}$ is dense and $\mathcal{G}_\delta$ set in the space $\mathcal{V}$.

Proof. That $\mathcal{E}$ is a $\mathcal{G}_\delta$ set follows from the fact that

$$\mathcal{E} = \bigcap_{m=1}^{\infty} \bigcup_{x \in X} \{ f \in \mathcal{V} : |T(f)(x)| > m \}$$

and the continuity of $f \to T(f)(x)$.

Next we show that $\mathcal{E}$ is dense in the space $\mathcal{V}$, if it is not empty. Let $g \in \mathcal{E}$, i.e., $g \in \mathcal{V}$ and $T(g)$ is unbounded on X, and let $f \in \mathcal{V} - \mathcal{E}$. Then $T(f)$ is bounded on X, let us say by κ_2. Also for fixed $n \geq 1$, the function $T\left(f + \frac{1}{n} g\right)$ is unbounded on X. Indeed, suppose that it is bounded on X by a positive number κ_1. Then, if $x \in X$, by the linearity of T, we would have

$$\begin{aligned} |T(g)(x)| = n \left|T(\tfrac{1}{n} g)(x)\right| &= n \left|T\left(f + \tfrac{1}{n} g\right)(x) - T(f)(x)\right| \\ &\leq n\left(\left|T\left(f + \tfrac{1}{n} g\right)(x)\right| + |T(f)(x)|\right) \leq n\,\kappa_1 + n\,\kappa_2, \end{aligned}$$

which contradicts the fact that $T(g)$ is unbounded on X.

In the more general case in which T is sublinear (not necessarily linear), i.e.,

$$|T(f+g)| \leq |T(f)| + |T(g)| \ \text{and} \ |T(\lambda f)| = |\lambda||T(f)|, \ \text{for} \ f, g \in \mathcal{V} \ \text{and} \ \lambda \in \mathbb{C},$$

we would have

$$\left|\tfrac{1}{n} T(g)(x)\right| = \left|T(\tfrac{1}{n} g)(x)\right| \leq \left|T\left(f + \tfrac{1}{n} g\right)(x)\right| + |T(-f)(x)| = \left|T\left(f + \tfrac{1}{n} g\right)(x)\right| + |T(f)(x)|,$$

and this would give again the contradiction that $T(g)$ is bounded on X by $n(\kappa_1 + \kappa_2)$.

Therefore $T\left(f + \frac{1}{n} g\right)$ is unbounded on X, i.e., $f + \frac{1}{n} g \in \mathcal{E}$ for every $n \geq 1$, and $f + \frac{1}{n} g$ converges to f, in $\mathcal{V}$, as $n \to \infty$. Since f was an arbitrary function in $\mathcal{V} - \mathcal{E}$, it follows that $\mathcal{E}$ is indeed dense in $\mathcal{V}$. □

Remark 2.1.13. One can prove more general versions of the above theorem. For example the operator T may be assumed to satisfy the weaker condition:

$$|T(\lambda(f+g))| \leq |\lambda|^{\alpha} |T(f)|^{\alpha} + |\lambda|^{\beta} |T(g)|^{\beta}, \ \text{for some} \ \alpha, \beta > 0.$$

The following Theorem was proved by Nestoridis. (See [19, Theorem 3.3].)

Theorem 2.1.14. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $\mathcal{V}$ be a vector subspace of $\mathcal{O}(\Omega)$. Suppose that in $\mathcal{V}$ there is defined a complete metric whose topology makes $\mathcal{V}$ a topological vector space and such that convergence in $\mathcal{V}$ implies pointwise convergence in $\mathcal{O}(\Omega)$. If for every pair of balls (B, b) with $b \subset\subset B \cap \Omega \neq B$, there exists $f_{(B,b)} \in \mathcal{V}$ such that the restriction $f_{(B,b)}|_b$ (of the function $f_{(B,b)}$ to b) does not have any bounded holomorphic extention to B, then the set of the functions $g \in \mathcal{V}$ which are non-extendable is dense and $\mathcal{G}_\delta$ in $\mathcal{V}$.

Proof. Let $\mathcal{A}=\{f\in\mathcal{V}: f \textit{ is non-extendable}\}$. In order to prove that $\mathcal{A}$ is dense and $\mathcal{G}_\delta$ in $\mathcal{V}$, it suffices to show that its complement $\mathcal{V}-\mathcal{A}$ is a countable union of closed subsets of $\mathcal{V}$ with empty interior.

For this purpose we consider the set $\mathcal{Y}$ of the couples (B,b) of open Euclidean balls so that $b\subset\subset B\cap\Omega\neq B$ with the centers of B and b belonging to $(\mathbb{Q}+i\mathbb{Q})^n$ and the radii of B, b belonging to $(0,+\infty)\cap\mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. It is clear that this set $\mathcal{Y}$ is countable. Also it is easy to see that

$$\mathcal{V}-\mathcal{A}=\bigcup_{(B,b)\in\mathcal{Y}}\ \bigcup_{M\in\mathbb{N}}T(B,b,M)$$

where we have set

$$T(B,b,M)=\{f\in\mathcal{V}:\exists F\in\mathcal{O}(B),\, \textit{bounded by } M,\, \textit{so that } F|_b=f|_b\}.$$

Since the set $\{T(B,b,M):(B,b)\in\mathcal{Y}\ \textit{and}\ M\in\mathbb{N}\}$ is countable, it remains to show that, for fixed $(B,b)\in\mathcal{Y}$ and $M\in\mathbb{N}$,

$$T(B,b,M) \text{ is closed (in } \mathcal{V}\text{) and } \operatorname{int}[T(B,b,M)]=\varnothing.$$

Let us consider a sequence $f_n\in T(B,b,\mathrm{M})$ such that $f_n\to f$ in the topology of $\mathcal{V}$ (with $f\in\mathcal{V}$). For each $n=1,2,3,\ldots$, there exists a holomorphic function F_n on B, bounded by M, such that $F_n|b=f_n|b$. By Montel's theorem (see Theorem 2.1.11.), there exists a subsequence F_{k_n} of F_n which converges uniformly on compact subsets of B towards a function F which is holomorphic on B and bounded by M.

Since the convergence $f_n\to f$ in the topology of $\mathcal{V}$ implies pointwise convergence in Ω by our assumption, it follows that $f|b=\lim_n f_n|b=\lim_n f_{k_n}\big|b=\lim_n F_{k_n}\big|b=F\big|b$. Since $f\in\mathcal{V}$ and F is holomorphic on B and bounded by M, $f\in T(B,b,\mathrm{M})$. This proves that $T(B,b,\mathrm{M})$ is closed in $\mathcal{V}$.

Finally, to prove that the interior of $T(B,b,\mathrm{M})$ in $\mathcal{V}$ is empty, let us assume, in order to reach a contradiction, that there exists an $f\in\operatorname{int}[T(B,b,M)]$. By our assumption there exists a function $f_{(B,b)}\in\mathcal{V}$ such that its restriction to b does not have any bounded holomorphic extention to B. Since $f+\frac{1}{n}f_{(B,\mathrm{b})}\to f$ in the topology of $\mathcal{V}$ and f is in the interior of $T(B,b,\mathrm{M})$, it follows that for some $n\in\{1,2,3,\ldots\}$ the function $f+\frac{1}{n}f_{(B,\mathrm{b})}$ belongs to $T(B,b,\mathrm{M})$. The same holds also for the function f. Thus, both functions $f+\frac{1}{n}f_{(B,\mathrm{b})}$ and f, restricted to b, admit holomorphic extensions to B which are bounded by M. Thus, their difference $\frac{1}{n}f_{(B,\mathrm{b})}$, restricted to b, admits a holomorphic extension on B bounded by $2M$. It follows that the function $f_{(B,\mathrm{b})}$ restricted to b admits a holomorphic extension on B bounded by

$2nM$. This contradicts the fact that $f_{(B,b)}|_b$ does not admit any bounded holomorphic extension on B. Thus $\operatorname{int}[T(B,b,M)] = \varnothing$ and the proof is complete. □

2.2. Totally unbounded Holomorphic functions

Let $\Omega \subset \mathbb{C}^n$ be an open set. We will say that a holomorphic function $f : \Omega \to \mathbb{C}$ is *totally unbounded* in Ω, if for every $w \in \partial\Omega$, every $\delta > 0$, and every connected component E of the set

$$B(w,\delta) \cap \Omega = \{z \in \Omega : |z - w| < \delta\},$$

the function $f|_E$ is unbounded, i.e., $\sup_{z \in E} |f(z)| = \infty$. Notice that such a function is *singular* at every point of $\partial\Omega$.

More precisely the following proposition holds.

Proposition 2.2.1. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $f : \Omega \to \mathbb{C}$ be a totally unbounded holomorphic function. Then for every open sets $U, V \subset \mathbb{C}^n$, with U being connected and $\varnothing \neq V \subseteq U \cap \Omega \neq U$, there does not exist a holomorphic function F on U which extends $f|_V$, i.e., $F|_V = f|_V$.

Proof. Suppose – to reach a contradiction – that for some pair of sets U and V, there exists a function F, which extends f in the way described above. Let E_1 be the connected component of $U \cap \Omega$ which contains V. Then $F|_{E_1} = f|_{E_1}$ and $\overline{E}_1 \cap \partial\Omega \neq \varnothing$, so that we may take a point $w \in \overline{E}_1 \cap \partial\Omega$, and a ball $B(w,\delta)$ with $\overline{B(w,\delta)} \subset U$. Then $B(w,\delta) \cap E_1 \neq \varnothing$, and if $c \in B(w,\delta) \cap E_1$ then for the connected component E of the set $B(w,\delta) \cap \Omega$, which contains the point c, we have $\sup_{z \in E} |f(z)| = \infty$ (since f is assumed to be totally unbounded). But this contradicts the equation $F|_E = f|_E$, which follows from the principle of unique analytic continuation, applied to the connected open set E and the fact that open set $E \cap E_1 \neq \varnothing$. This completes the proof.□

Remark 2.2.2. In the above proof we used the fact that $\overline{E}_1 \cap \partial\Omega \neq \varnothing$. To justify this elementary topological fact, let us observe that, since $U \cap \Omega \neq \varnothing$, $U \cap (\mathbb{C} - \Omega) \neq \varnothing$ and U is connected, it follows that $U \cap \partial\Omega \neq \varnothing$. Let $a \in V$ and $b \in U \cap \partial\Omega$, and let Γ be a curve which lies in U and connects the points a and b. If C is the connected component of $U \cap \Omega$ which contains a, then C is open, $a \in C \cap \Gamma$ and $b \notin C \cap \Gamma$. Since the set Γ is connected, we must have $\Gamma \cap \partial C \neq \varnothing$. Then for a point $\tau \in \Gamma \cap \partial C$, we will have $\tau \in \partial\Omega$

and $\tau \in \overline{C}$, and therefore $\overline{C} \cap \partial\Omega \neq \varnothing$. Finally, since $E_1 \supseteq C$, we obtain that, indeed, $\overline{E}_1 \cap \partial\Omega \neq \varnothing$.

2.3. Integrals over level sets

Lemma 2.3.1. (Integration in polar coordinates, [28, Lemma 1.8]) Let dv denote the volume measure on $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$, normalized so that $v(\mathbb{B}) = 1$ and $d\sigma$ is the Euclidean surface area measure on the sphere $\partial\mathbb{B} = \mathbb{S}$, $\sigma(\mathbb{S}) = 1$. Then the measures v and σ are related by the formula

$$\int_{\mathbb{B}} f(z)dv(z) = 2n\int_0^1 r^{2n-1}dr \int_{\mathbb{S}} f(r\zeta)d\sigma(\zeta).$$

Proposition 2.3.2. ([22, Proposition 1.4.10]) For $z \in \mathbb{B}$, c real, t > 1 define

$$I_c(z) = \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{\left|1 - \langle z, \zeta\rangle\right|^{n+c}}$$

and

$$J_{c,t}(z) = \int_{\mathbb{B}} \frac{\left(1 - |w|^2\right)dv(w)}{\left|1 - \langle z, w\rangle\right|^{n+1+t+c}}$$

When $c < 0$, then I_c and $J_{c,t}$ are bounded in $\mathbb{B}$.

When $c > 0$, then

$$I_c(z) \approx \left(1 - |z|^2\right)^{-c} \approx J_{c,t}(z).$$

Finally,

$$I_0(z) \approx \log\frac{1}{1 - |z|^2} \approx J_{0,t}(z).$$

Theorem 2.3.3. (Integration over level sets, [6, Theorem 3.13]) Assume $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous.

(i) Then

$$\int_{\mathbb{R}^n} |Df|\, dx = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}\left(\{f = t\}\right)dt,$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbb{R}^n$ – $\mathcal{H}^n = L^n$ on $\mathbb{R}^n$ –.

(ii) Assume also $ess\,inf\, |Df| > 0$, and suppose $g : \mathbb{R}^n \to \mathbb{R}$ is L^n–summable i.e., $|f|$ has a finite integral. Then

$$\int_{\{f>t\}} g\,dx = \int_t^{\infty} \left(\int_{\{f=s\}} \frac{g}{|Df|} \, d\mathcal{H}^{n-1} \right) ds\,,$$

where $ess\,inf\ f = sup\left\{b \in \mathbb{R} : \mu\left(\{x : f < b\}\right) = 0\right\}$.

(iii) In particular,

$$\frac{d}{dt}\left(\int_{\{f>t\}} g\,dx \right) = -\int_{\{f=t\}} \frac{g}{|Df|} \, d\mathcal{H}^{n-1}, \quad \text{for } L^1 \text{ a.e. } t,$$

where the expression a.e., means almost everywhere with respect the space L^1 .

2.4. Convex sets

Definition 2.4.1. A set $U \subset \mathbb{R}^n$ is *convex* if the line segment between any two points in U lies in U , i.e. if for any $x, y \in U$ and any $t \in \mathbb{R}$ with $0 \le t \le 1$, we have $tx + (1-t)\,y \in U$.

Definition 2.4.2. A set $C \subset \mathbb{R}^n$ is *affine* if the line through any distinct points in C lies in C , i.e., if for any $x, y \in C$ and $t \in \mathbb{R}$, we have $tx + (1-t)\,y \in C$.

In other words, C contains the linear combination of any two points in C , provided the coefficients in the linear combination sum to one.

Definition 2.4.3. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ *is affine* if it is a sum of linear function and a constant, i.e., if it has the form $f(x) = Ax + b$, where $A \in \mathbb{R}^{n+m}$ and $b \in \mathbb{R}^m$.

Remark 2.4.4. Suppose $S \subset \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine function. Then the image of S under f , $f(S) = \{ f(x) : x \in S \}$ is convex.

Definition 2.4.5. A *hyperplane* H is a set of the form $H(a,b) = \left\{ x \in \mathbb{R}^n : \langle x,a \rangle = b \right\}$, where $a \in \mathbb{R}^n, a \ne 0$ and $b \in \mathbb{R}$.

Remarks 2.4.6.

(i) Geometrically, the hyperplane can be interpreted as the set of points with a constant inner product to a given vector a .

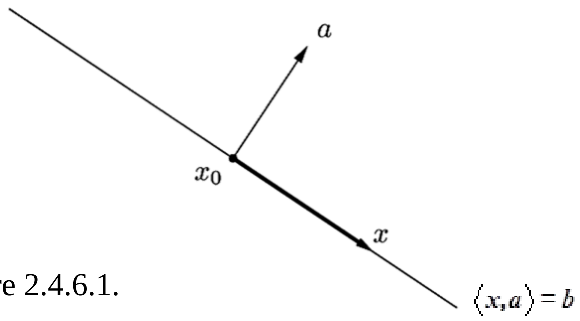

figure 2.4.6.1.

The figure 2.4.6.1. illustrates the hyperplane in $\mathbb{R}^2$ with normal vector *a* and a point x_0 in the hyperplane. For any point *x* in the hyperplane, $x - x_0$ (shown as the darker arrow) is orthogonal to *a*.

(ii) A hyperplane divides $\mathbb{R}^n$ into two half spaces. (see figure 2.4.6.2.) A (closed) halfspace is a set of the form $\{x \in \mathbb{R}^n : \langle x,a \rangle \le b\}$, where $a \neq 0$.

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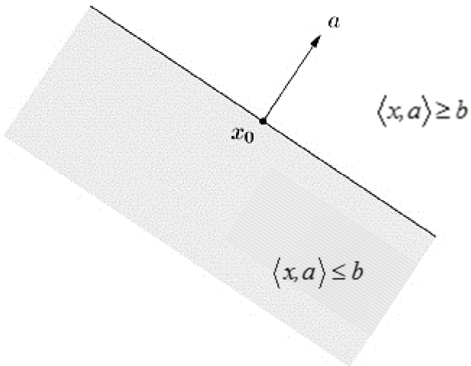

figure 2.4.6.2.

(iii) Halfspaces are convex but not affine.

Theorem 2.4.7. (Separating hyperplane Theorem, [3, Theorem 2.5.1]) Let C and D are two convex sets that do not intersect, i.e., $C \cap D \neq \varnothing$. Then there exist $a \neq 0$ and b such that $\langle x,a \rangle \le b$ for all $x \in C$ and $\langle x,a \rangle \ge b$ for all $x \in D$. In other words, the affine function $\langle x,a \rangle - b$ is nonpositive on C and nonnegative on *D.* This is illustrated in figure 2.4.7.1..

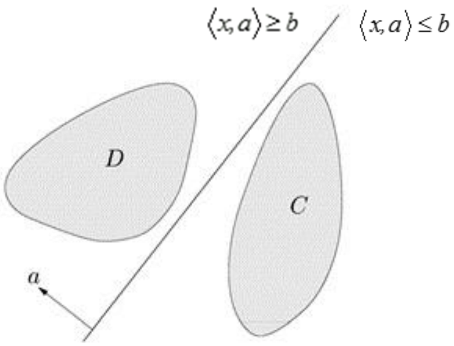

figure 2.4.7.1.

Theorem 2.4.8. (Strong Separating Hyperplane Theorem ([3]) Let C and D are two disjoint nonempty convex subsets of $\mathbb{R}^n$. Suppose C is compact and D is closed. Then there exist a nonzero $a \in \mathbb{R}^n$ that strongly separates C and D, i.e., there exist b such that $\langle x,a \rangle < b$ for all $x \in C$ and $\langle x,a \rangle > b$ for all $x \in D.$

Theorem 2.4.9. (Supporting Hyperplane Theorem, [3, Theorem 2.5.2]) Let $C \subset \mathbb{R}^n$ be a nonempty convex set and x_0 is a point in its boundary. Then there exist $a \in \mathbb{R}^n, a \neq 0$ satisfies $\langle x,a \rangle \leq \langle x_0,a \rangle$ for all $x \in C.$- see figure 2.4.9.1.-

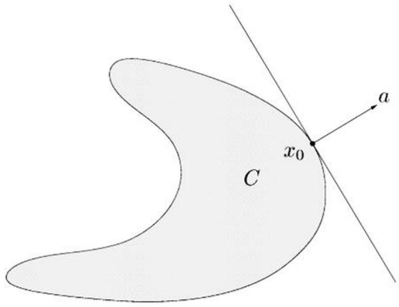

figure 2.4.9.1.

Definition 2.4.10. (Boundary of C^k class) Let $k \in (0,\infty]$. An open set $\Omega \subseteq \mathbb{R}^n$ has a C^k*–boundary,* if for every $y \in \partial\Omega$ there exists a neighborhood U of y and a function $\rho \in C^k(U)$ such that $d\rho(x) \neq 0$ for every $x \in U$ and such that $\Omega \cap U = \{x \in U : \rho(x) < 0\}$.

Remark 2.4.11. A function ρ defined as in the above definition is called C^k–local defining function for Ω.

Theorem 2.4.12. Let $\Omega \subseteq \mathbb{R}^n$ be an open, convex and bounded set with C^k–boundary, and ρ C^1–defining function for Ω - see definition 2.7.1.-. Then for $x \in \partial\Omega$ and $y \in \Omega$ we have that:

$$\sum_{j=1}^{n} \frac{\partial\rho(x)}{\partial x_j}(x_j - y_j) > 0.$$

Theorem 2.4.13. Let $\Omega \subseteq \mathbb{R}^n$ be an open, convex and bounded set with C^2–boundary, and ρ C^2– defining function for Ω. Then for $x \in \partial\Omega$ we have that:

$$\sum_{1\leq j,k\leq n} \frac{\partial^2\rho(x)}{\partial x_j \partial x_k} t_j t_k \geq 0 \text{ for every } t \in \mathbb{R}^n \text{ with } \sum_{j=1}^{n} \frac{\partial^2\rho(x)}{\partial x_j} t_j = 0.$$

Definition 2.4.14. A bounded domain $\Omega \subseteq \mathbb{R}^n$ is called *strictly convex* if there exist a C^2–defining function for Ω, ρ, such that for all $x \in \partial\Omega$ we have that:

$$\sum_{1\le j,k\le n} \frac{\partial^2 \rho(x)}{\partial x_j \partial x_k} t_j t_k > 0 \text{ for every } t \in \mathbb{R}^n,\ t \neq 0 \text{ with } \sum_{j=1}^{n} \frac{\partial^2 \rho(x)}{\partial x_j} t_j = 0.$$

2.5. Harmonic, Subharmonic and Plurisubharmonic functions

Let recall that the Laplace operator Δ in $\mathbb{C}$ is defined by $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \overline{z}}$,

where $z = x + iy$. Then we have the following definition in one complex variable.

Definition 2.5.1. A C^2- function u on a region $\Omega \subset \mathbb{C}$ is called *harmonic* if $\Delta u = 0$ on Ω .

We state some of well- known elementary properties of harmonic functions.

(2.5.1.1.) A real valued function u is harmonic if and only if u is locally the real part of a holomorphic function. In particular, harmonic functions are C^∞ .

(2.5.1.2.) **The mean value property.** If u is harmonic on $\Omega \subset \mathbb{C}$, then $u(a) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(a + re^{i\theta})\,\mathrm{d}\theta$, for every disk $\overline{D(a,r)} = \{z : |z-a| \le r\} \subset \Omega.$

(2.5.1.3.) **The maximum principle.** If u is a real valued and harmonic on $\Omega \subset \mathbb{C}$, then we have the following:

(i) **(Strong version)** If u has a local maximum at the point $a \in \Omega$, then u is constant in a neighborhood of a (and hence on the connected component of Ω which contains a).

(ii) **(Weak version)** If $\Omega \subset\subset \mathbb{C}$ and u extends continuously to $\overline{\Omega}$, then $u(z) \le \max_{z\in\partial\Omega} u(z)$ for $z \in \Omega.$

(2.5.1.4.) **The Dirichlet Problem.** If $D(a,r) = \{z : |z-a| < r\}$ and $g \in C(\partial D(a,r))$, then there is a unique continuous function u on $\overline{D(a,r)}$ which is harmonic in $D(a,r)$, such that $u(z) = g(z)$ for $z \in \partial D(a,r)$. This harmonic extention u is given explicitly by the Poisson integral of g, i.e., $u(a+\zeta) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \frac{r^2 - |\zeta|^2}{|re^{i\theta} - \zeta|^2} g(a + re^{i\theta})d\theta$, for $|\zeta| < r$.

Definition 2.5.2. Let Ω be an open set of $\mathbb{C}$ a function $u:\Omega\to\mathbb{R}\cup\{-\infty\}$ is called *subharmonic* if u is upper semicontinuous and if for every compact set $K\subset\Omega$ and for every function $h\in C(K)$ which is harmonic on the interior of K and satisfies $u\le h$ on ∂K it follows that $u\le h$ on K.

Recall that u is upper semicontinuous on Ω if $\limsup\limits_{z\to a} u(z)\le u(a)$ for $a\in\Omega$, or equivalently, $\{z\in\Omega : u(z)<c\}$ is open for every $c\in\mathbb{R}$.

Remark 2.5.3. From the weak version of maximum principle (2.5.1.3. (ii)) one can see that harmonic functions are subharmonic.

Next we will mention some properties of the subharmonic functions.

Lemma 2.5.4. Let $\Omega\subset\mathbb{C}$ be open.
(i) If u is subharmonic on Ω, so is cu for $c>0$.
(ii) If $\{u_a : a\in A\}$ is a locally upper bounded family of subharmonic functions on Ω such that $u=\sup u_a$ is upper semicontinuous, then u is subharmonic.
(iii) If $\{u_j : \mathrm{j}=1,2,...\}$ is a decreasing sequence of subharmonic functions on Ω, then $u=\lim\limits_{j\to\infty} u_j$ is subharmonic.

The following corollary is an application of the previous Lemma

Corollary 2.5.5. For every open set Ω in $\mathbb{C}$ the function $u(z)=-\log dist(z,\partial\Omega)$ is subharmonic on Ω.

Proof. If $\Omega=\mathbb{C}$, then $u(z)\equiv-\infty$, and there is nothing to prove. If $\Omega\ne\mathbb{C}$, then $u(z)$ is continuous. Indeed, fix $z\in\Omega$ and $\varepsilon>0$ Suppose $w\in\Omega$ and $|\mathrm{z}-\mathrm{w}|<\varepsilon$. We have

$$\log dist(z,\partial\Omega)=\inf_{z\in\partial\Omega}|z-\zeta|\le\inf_{z\in\partial\Omega}\left(|w-\zeta|+|z-\zeta|-|w-\zeta|\right)\le$$
$$\inf_{\zeta\in\partial\Omega}\left(|w-\zeta|+|z-w|\right)\le\inf_{\zeta\in\partial\Omega}\left(|w-\zeta|+\varepsilon\right)\le\inf_{\zeta\in\partial\Omega}|w-\zeta|+\varepsilon=dist(w,\partial\Omega)+\varepsilon$$

The same argument shows that $dist(\mathrm{w},\partial\Omega)\le dist(z,\partial\Omega)+\varepsilon$. We have shown that $|\mathrm{z}-\mathrm{w}|<\varepsilon$ implies that $|dist(\mathrm{w},\partial\Omega)-dist(z,\partial\Omega)|\le\varepsilon$, so $dist(.,\partial\Omega)$ is continuous on Ω. Since for $z\in\Omega$ one has $u(z)=-\log dist(z,\partial\Omega)=\sup\{-\log|z-\zeta| : \zeta\in\partial\Omega\}$. The function $-\log|z-\zeta|$ isharmonic since is the real part of the holomorphic function $-\log|z-\zeta|$, and hence subharmonic (see Remark 2.5.3.). By lemma 2.5.4. (ii) the proof is complete.□

Next we discuss some other characterizations of subharmonic functions which show that the subharmonicity is a local property.

Theorem 2.5.6. (Submean value property) Let Ω be open in $\mathbb{C}$. A continuous function $u:\Omega\to\mathbb{R}$ is subharmonic if and only if for every disc $\overline{D(a,r)}\subset\Omega$,

$$u(a)\le\frac{1}{2\pi}\int_0^{2\pi}u(a+re^{i\theta})\,\mathrm{d}\theta.$$

Proof. Suppose u is subharmonic and $\overline{D(a,r)}\subset\Omega$. Since $P_D^u=\frac{1}{2\pi}\int_0^{2\pi}u(a+re^{i\theta})\,\mathrm{d}\theta$ is continuous on $\overline{D(a,r)}$ and harmonic on $D(a,r)$ and since $P_D^u=u$ on $\partial D(a,r)$, it follows form the definition of subharmonicity $u\big|_D\le P_D^u$.

Conversely, let K be a compact set, $K\subset\overline{D(a,r)}\subset\Omega$ and suppose $u\big|_D\le P_D^u$ then for every function $h\in C\left(\overline{D(a,r)}\right)$ which is harmonic on $D(a,r)$ and satisfies $u\le h$ on $\partial D(a,r)$ it suffices to show that $u\le h$ on $\overline{D(a,r)}$. Since $P_D^u=u$ on $\partial D(a,r)$ then $P_D^u=u\le h$ on $\partial D(a,r)$. By the maximum principle for harmonic functions, $P_D^u\le h$ on $\overline{D(a,r)}$ Hence, $u\big|_{\overline{D}}\le P_D^u\le h\big|_{\overline{D}}$ and so $u\le h$ on $\overline{D(a,r)}$.□

2.5.1. Examples of subharmonic functions

Example 2.5.7. If f is holomorphic on an open set Ω of $\mathbb{C}$, then $u=|f|^a, a>0$ is subharmonic.

Example 2.5.8. Every convex function u is subharmonic.
Indeed, let u be a convex function at a neighborhood of the closed unite disk $\overline{D(0,1)}\subset\mathbb{C}$, then $u(0)\le\frac{1}{2}\left[u(e^{i\theta})+u(e^{i(\theta+\pi)})\right]$, for $\theta\in[0,2\pi]$ and therefore we have $u(0)\le\frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}\left[u(e^{i\theta})+u(e^{i(\theta+\pi)})\right]d\theta=\frac{1}{2\pi}\int_0^{2\pi}u(e^{i\theta})\,d\theta$. By theorem 2.5.6. u is subharmonic.

The following proposition gives a simple computational test for subharmonicity.

Proposition 2.5.9. Let Ω be open in $\mathbb{C}$. A real valued function $u\in C^2(\Omega)$ is subharmonic on Ω if and only if $\Delta u=4\frac{\partial^2}{\partial z\partial\overline{z}}\ge 0$ on Ω.

The local equivalence between harmonic functions and real parts of holomorphic functions does not hold in more than one complex variable - see example 2.5.10. - .

Moreover, the class of subharmonic functions in $2n$ real variables on an open subset of $\mathbb{C}^n$ is not invariant under biholomorphic maps except for $n=1$. A generalization of harmonic and subharmonic in several complex variables is pluriharmonic and plusrisubharmonic respectively and are those functions whose restrictions to complex lines are harmonic or subharmonic – see definitions 2.5.11. and 2.5.13.-.

Example 2.5.10. If $z_j = x_j + iy_j,\ j=1,2$ the function $u(x_1, y_1, x_2, y_2) = x_1^2 + x_2^2$ is harmonic butisnot the real part of any holomorphic function- not even locally -.
Indeed, suppose there were locally a holomorphic function $f(z_1, z_2)$ such that $f = u + iv$. Then, for fixed z_2, the function $f(z_1) = f(z_1, z_2)$ would be holomorphic and hence the real part $u_1(x_1, y_1) = x_1^2 - x_2^2$ would be harmonic in (x_1, y_1) which is not.

A complex line in $\mathbb{C}^n$ is a set of the form $\ell = \{z : z = a + \lambda b, \lambda \in \mathbb{C}\}$, where a and b are fixed points in $\mathbb{C}^n$, with $b \neq 0$. Let us say that ℓ is the complex line through a in the "direction" b. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{C}^n$. Thus, the coordinates of e_j are given by the Kronecker delta δ_k^j. The complex line through a in the direction of e_j is called the complex line through a in the direction of the j-th coordinate.

Definition 2.5.11. A real- valued function u defined in an open set Ω of $\mathbb{C}^n$ is said to be *pluriharmonic*in Ω if $u \in C^2(\Omega)$ and the restriction of u to $\ell \cap \Omega$ is harmonic for each complex line ℓ.

Remark 2.5.12. Unlike the holomorphic situation, this is not equivalent to being harmonic in each coordinate direction.

Definition 2.5.13. Let Ω be an open set in $\mathbb{C}^n$. A function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be *plurisubharmonic* on Ω if u is upper semicontinuous, and if for every $z \in \Omega$ and $w \in \mathbb{C}^n$ the function $\lambda \mapsto u(z + \lambda w)$ is subharmonic on the region $\{\lambda \in \mathbb{C} : z + \lambda w \in \Omega\}$. The class of plurisubharmonic functions on Ω is denoted by $psh(\Omega)$.

The following proposition gives a characterization for plurisubharmonic functions of $C^2(\Omega)$ class.

Proposition 2.5.14. Let Ω be an open set in $\mathbb{C}^n$ and $u \in C^2(\Omega)$ is a real valued.
Then $u \in psh(\Omega)$ if and only if the complex Hessian of u,

$L_u(z,t)=\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j\partial \overline{z}_k}(z)t_j\overline{t}_k$ is positive semi-definite on $\mathbb{C}^n$ at every point $z\in\Omega$ and $t\in\mathbb{C}^n$.

In the case that $L_u(z,t)$ is positive definite, i.e., $L_u(z,t)=\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j\partial \overline{z}_k}(z)t_j\overline{t}_k>0$, for every $z\in\Omega$ and $t\in\mathbb{C}^n-\{0\}$ u is strictly plurisubharmonic. The class of strictly plurisubharmonic functions on Ω is denoted by $s.psh(\Omega)$.

Remarks 2.5.15.

(i) The complex Hessian of u, $L_u(z,t)=\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j\partial \overline{z}_k}(z)t_j\overline{t}_k$ is called *Levi form* of u at z.

(ii) For u strictly plurisubharmonic at p we have that $\gamma=\min\{L_u(z,t):|t|=1\}$ is positive, and hence $L_u(z,t)\geq\gamma|t|^2$ for $t\in\mathbb{C}^n$. and all $z\in U$, where U is some neighborhood of p, i.e., p is strictly plurisubharmonic at all points near p as well.

2.5.2. Examples of plurisubharmonic functions

Example 2.5.16. Every convex function is plurisubharmonic.

Example 2.5.17. If f is holomorphic on an open set Ω of $\mathbb{C}^n$, then $|f|^a, a>1$ is plurisubharmonic, since s^a is a convex function.

Example 2.5.18. If f is holomorphic on an open set Ω of $\mathbb{C}^n$, then $\log|f|$ is plurisubharmonic.

Example 2.5.19. In $\Omega=\mathbb{C}^2\setminus\{0\}$ the function $u(z)=\log|z|$ is not plurisubharmonic. To see this, we show that the restriction of u to complex line $\ell=\{(1,z_2):z_2\in\mathbb{C}\}$ is not subharmonic, because it does not satisfy the mean value inequality at the point $a=(1,0)$. Consider the disc $D=\{z=(1,z_2)\in\ell:|z_2|=1\}$.

For $z\in\partial D$, $|z|^2=1^2+|z_2|^2=2$. Thus, $|z|>|a|$, since $|a|=1$ and $u(a)=-\log|a|>\log|z|=u(z)$.

Therefore, $u(a)$ is than its average on the boundary of the disc D and so does not satisfy the mean value inequality at a. Thus, $u|\ell$ is not subharmonic and consequently u is not plurisubharmonic in Ω.

2.5.3. Properties of plurisubharmonic functions

The plurisubharmonicity is a local property of the function.

Suppose $\Omega \subseteq \mathbb{C}^n$ and $D \subseteq \mathbb{C}^n$ are open. Then the following properties hold.
(i) If $u, v \in psh(\Omega)$ then the sum $u+v$ is also a plurisubharmonic function on Ω, and so is the $\max\{u, v\}$.
(ii) If $u \in psh(\Omega)$ and $\lambda > 0$ then $\lambda u \in psh(\Omega)$.
(iii) Let $\mathcal{U}$ be a locally upper bounded family of plurisubharmonic functions on Ω, then the function $u^* = \sup\{u : u \in \mathcal{U}\}$ is also plurisubharmonic on Ω.
(iv)) If $\{u_j\}_{j \in J}$ is a family of plurisubharmonic functions on Ω and $\sup\limits_{j \in J} u_j$ is continuous in Ω, then $\sup\limits_{j \in J} u_j \in psh(\Omega)$.
(v) If u_j is a sequence of plurisubharmonic functions on Ω and $u_j \to u$ converges uniformly to u on the compact subsets of Ω, then $u \in psh(\Omega)$.
(vi) Let $F : D \to \Omega$ a holomorphic function then the composition $u \circ F \in psh(D)$, for every $u \in psh(\Omega)$.

Theorem 2.5.20. (Submean value property ([21 Lemma 4.11.]) Let Ω be open in $\mathbb{C}^n$ and $u : \Omega \to \mathbb{R}$ is plurisubharmonic then for every polydisc $\overline{P(a,r)} \subset \Omega$,

$$u(a) \leq \frac{1}{vol(P(a,r))} \int\limits_{P(a,r)} u(z) dv(z).$$

Proof. By applying the submean value property-see Theorem 2.5.6.- in each coordinate separately, one obtains

$$u(a) \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \ldots \int_0^{2\pi} u(a + \rho e^{i\theta}) d\theta_1 \ldots d\theta_n,$$

for all $\rho = (\rho_1, \ldots, \rho_n)$ with $0 \leq \rho \leq r$, where $\rho e^{i\theta} = \left(\rho_1 e^{i\theta_1}, \ldots \rho_n e^{i\theta_n}\right)$. After multiplying by $\rho_1 \ldots \rho_n \, d\rho_1 \ldots d\rho_n$ and integrating in ρ_j from 0 to r_j, $1 \leq j \leq n$, it follows that

$$u(a) \leq \frac{1}{vol(P(a,r))} \int\limits_{P(a,r)} u(z) dv(z).$$

2.6. Domains of Holomorphy and Pseudoconvexity

Definition 2.6.1. An open set $\Omega \subset \mathbb{C}^n$ is called a *domain of holomorphy* if there exists a holomorphic function $f : \Omega \to \mathbb{C}$ satisfying the following condition: For every two open sets $U, V \subset \mathbb{C}^n$ such that $\emptyset \neq V \subset U \cap \Omega \neq U$ and U being connected, it is not possible to find a holomorphic function $\tilde{f} : U \to \mathbb{C}$ *with* $\tilde{f} = f$ in *V*.
In figure 2.6.1.1, we illustrate the sets in the definition.

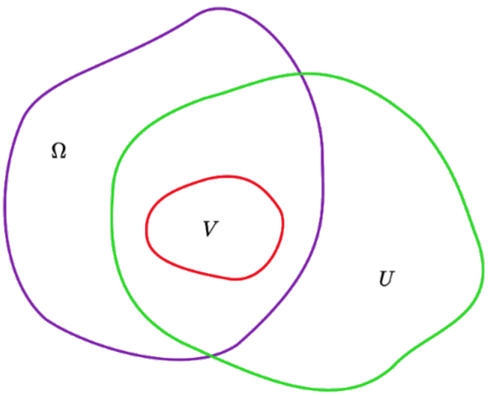

Figure 2.6.1.1

Definition 2.6.2. Let $\Omega \subset \mathbb{C}^n$ be open set and $f : \Omega \to \mathbb{C}$ a holomorphic function. Then f is called *extendable* if there exist two open sets $U, V \subset \mathbb{C}^n$ such that $\emptyset \neq V \subset U \cap \Omega \neq U$, and U connected and a holomorphic function $\tilde{f} : U \to \mathbb{C}$ with $\tilde{f} = f$ in *V*. Otherwise f is called *non-extendable*.

Remark 2.6.3. It easy to see that an open set $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if there exist $f : \Omega \to \mathbb{C}$ which is non-extendable.

Definition 2.6.4. For a compact subset K of an open set $\Omega \subset \mathbb{C}^n$, its holomorphically convex hull $\hat{K}_{\mathcal{O}(\Omega)}$ in Ω is defined by

$$\hat{K}_{\mathcal{O}(\Omega)} = \left\{ z \in \Omega : |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)| \text{ for all } f \in \mathcal{O}(\Omega) \right\}.$$

$\hat{K}_{\mathcal{O}(\Omega)}$ is also called the $\mathcal{O}(\Omega)$*- hull of* K *and* $K \subset \Omega$ is called $\mathcal{O}(\Omega)$*- convex* if $\hat{K}_{\mathcal{O}(\Omega)} = K$.

The following theorem of Cartan-Thullen gives equivalent definitions of domains of holomorphy.

Theorem 2.6.5. ([10, Theorem 1.3.7) For an open set $\Omega \subset \mathbb{C}^n$, the following conditions are equivalent:

(i) Ω is a domain of holomorphy.

(ii) For every couple of open sets $U,V \subset \mathbb{C}^n$ such that $\varnothing \neq V \subset U \cap \Omega \neq U$ and U connected, there exist a holomorphic function $f:\Omega \to \mathbb{C}$ so that it is not possible to find a holomorphic function $\tilde{f}: U \to \mathbb{C}$ with $\tilde{f} = f$ in V.

(iii) For every compact subset $K \subset\subset \Omega$, $\operatorname{dist}(K,\partial\Omega) = \operatorname{dist}\left(\hat{K}_{\mathcal{O}(\Omega)},\partial\Omega\right)$,

where $\operatorname{dist}(K,\partial\Omega) = \inf\left\{|w-z| : w \in K, z \in \partial\Omega\right\}$.

(iv) For every compact subset $K \subset\subset \Omega$, $\hat{K}_{\mathcal{O}(\Omega)} \subset\subset \Omega$.

(v) For every infinite set $X \subset \Omega$, which is discrete in Ω, there exists a holomorphic function which is unbounded on X.

Remark 2.6.6. An open set $\Omega \subset \mathbb{C}^n$ is called a weak domain of holomorphy if it satisfies the condition (ii) of the above theorem.

2.6.1. Examples of domains of Holomorphy

Example 2.6.7. In the case $n = 1$, every open set is a domain of holomorphy. To see this, let $U, V \subset \mathbb{C}$ be two open sets such that $\varnothing \neq V \subset U \cap \Omega \neq U$ and U connected. Let $\zeta \in \partial\Omega \cap U$ and define $f_\zeta : \Omega \to \mathbb{C}$, $f_\zeta(z) = 1/(z-\zeta)$, $z \in \Omega$. Then we see that f_ζ is holomorphic on Ω and cannot be extended to a holomorphic function on U.

For $n \geq 2$, this is no longer true, as it follows from Hartogs's Theorem.

Theorem 2.6.8. (Hartogs's extension phenomenon, [16, Theorem 1.2.6]) Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be an open set and K a compact subset of Ω such that $\Omega - K$ is connected. Then each holomorphic function $f : \Omega - K \to \mathbb{C}$ can be extended to a holomorphic function F: $\Omega \to \mathbb{C}$.

Example 2.6.9. The unit ball $\Omega = \{z \in \mathbb{C}^n : |z| < 1\}$ is a domain of holomorphy. Indeed for $\zeta \in \partial\Omega$, we consider the holomorphic function

$$f_\zeta(z) = \frac{1}{1-\langle z,\zeta\rangle} = \frac{1}{1-\sum_{j=1}^n \bar{\zeta}_j z_j}, \quad z \in \Omega.$$

It is easy to see that f_ζ is singular at ζ so the assertion follows from condition (ii) of Theorem 2.6.5.

Example 2.6.10. Every convex set $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy.

Indeed, let $U,V \subset \mathbb{C}^n$ be two open sets such that $\varnothing \neq V \subset U \cap \Omega \neq U$ and U connected. For $\zeta \in \partial\Omega \cap U$ by the separation theorem of convex sets and points there exist $a_j, b_j, \lambda \in \mathbb{R}$, $j = 1,2,\ldots,n$, so that $\sum[a_j x_j(z) + b_j y_j(z)] < \lambda$ for every

$z \in \Omega$ while $\sum[a_j x_j(\zeta) + b_j y_j(\zeta)] = \lambda$. (Here we are using the notation $x_j(z) = \operatorname{Re} z_j$ and $y_j(z) = \operatorname{Im} z_j$.) But

$$\sum[a_j x_j(z) + b_j y_j(z)] = \operatorname{Re}(\sum c_j z_j) \text{ where } c_j = a_j - ib_j.$$

Therefore the function

$$f_\zeta(z) = \frac{1}{\sum c_j z_j - \lambda} = \frac{1}{\sum c_j (z_j - \zeta_j)}$$

is holomorphic for $z \in \Omega$ and cannot be extended as a holomorphic function to any neighborhood of the point ζ.

Example 2.6.11. Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy and $h \in \mathcal{O}(\Omega)$. Then the set $G = \Omega - Z_h$, where Z_h is the zero set of h in Ω, is also a domain of holomorphy.
Indeed, this can be proved by Theorem 2.6.5 (ii). It is easy to see that if $\zeta \in \partial G - \partial\Omega$ then $h(\zeta) = 0$ and the function $1/h \in \mathcal{O}(\mathrm{G})$ is singular at ζ.

Example 2.6.12. Let $\Omega \subset \mathbb{C}^n$ be an open set, $F : \Omega \to \mathbb{C}^m$ a holomorphic mapping and $G \subset \mathbb{C}^m$ a domain of holomorphy. Then the set $F^{-1}(\mathrm{G})$ is a domain of holomorphy if at least one of the following conditions holds:
(1) Ω is a domain of holomorphy.
(2) $F^{-1}(G) \subset\subset \Omega$.
Indeed, let ζ be a point of the boundary of $F^{-1}(G)$. If $F^{-1}(G) \subset\subset \Omega$ then $\zeta \in \Omega$ and therefore there is defined the point $F(\zeta) \in \partial G$. Since G is domain of holomorphy, there is a function $h \in \mathcal{O}(G)$ which is singular at the point $F(\zeta)$. But then the function $h \circ F \in \mathcal{O}(F^{-1}(G))$ and is singular at ζ.
Now in the case (2) does not hold, a point ζ of the boundary $F^{-1}(G)$ may not belong to Ω in which case $\zeta \in \partial\Omega$. But since Ω is a domain of holomorphy, there exists a function $f \in \mathcal{O}(\Omega)$ which is singular at ζ, and clearly $f \in \mathcal{O}(F^{-1}(G))$.

Example 2.6.13. Each analytic polyhedron is a domain of holomorphy. Firstly, a bounded open set $A \subset \mathbb{C}^n$ is called an analytic polyhedron if there is an open neighborhood U of $\overline{A}$ and functions $f_1, \dots, f_N \in \mathcal{O}(U)$ such that

$$A = \{z \in U : |f_1(z)| < 1, \dots, |f_N(z)| < 1\}.$$

That an analytic polyhedron is domain of holomorphy follows from the previous example since $A = F^{-1}(G)$ where

$$F = (f_1, \dots, f_N) \text{ and } G = \{w \in \mathbb{C}^m : |w_1| < 1, \dots, |w_N| < 1\}.$$

(It is clear of course that $A \subset\subset U$.)

Example 2.6.14. Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy and $F = (f_1, f_2, \dots, f_m) : \Omega \to \mathbb{C}^m$ be a holomorphic mapping. Then the set

$$D = \{z \in \Omega : |f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_m(z)|^2 < 1\}$$

is a domain of holomorphy. This follows from the example 2.6.12, case (1). Indeed, it suffices to notice that $D = F^{-1}(G)$ where G is the open unit ball of $\mathbb{C}^m$.

Example 2.6.15. Let $n \geq 2$. In this example we consider a region Ω in $\mathbb{C}^n \simeq \mathbb{C} \times \mathbb{C}^{n-1}$ defined as follows. Let $\mathcal{H}' \subseteq \mathcal{H}$ be open sets in $\mathbb{C}^{n-1}$ with $\mathcal{H}$ connected and let $r \in \mathbb{R}_{\geq 0}$ and $R \in \mathbb{R}_{>0}$ satisfy $r < R.$ Define

$$\Omega = \left\{ (z_0, z_1) \in \mathbb{C} \times \mathbb{C}^{n-1} : z_0 \in D^1(0, \mathrm{R}) \setminus \overline{\mathrm{D}^1}(0, \mathrm{r}), z_1 \in \mathcal{H} \right\}$$
$$\cup \left\{ (z_0, z_1) \in \mathbb{C} \times \mathbb{C}^{n-1} : z_0 \in D^1(0, \mathrm{R}), z_1 \in \mathcal{H}' \right\}$$

A set defined in this manner is called a *Hartogs figure*. -see figure 2.6.15.1.-

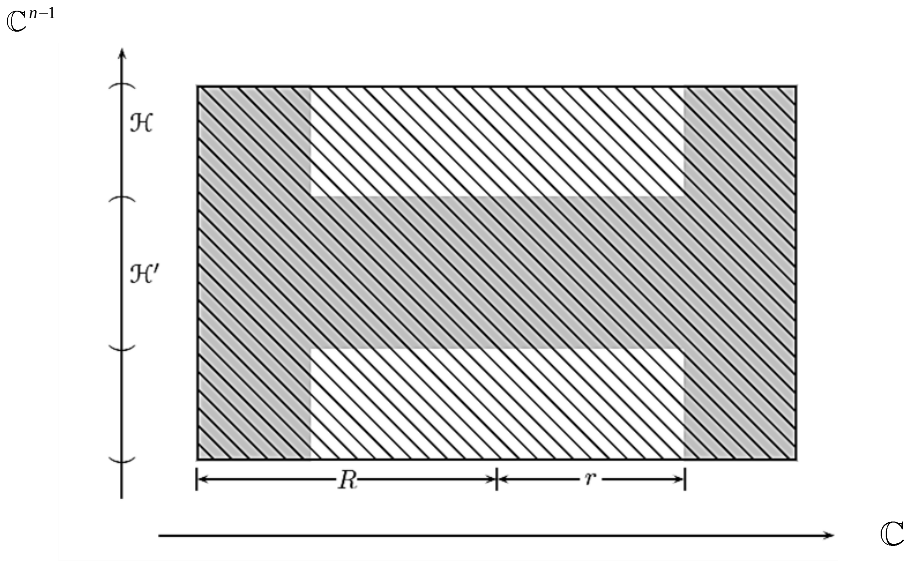

Figure 2.6.15.1. A depiction of a set that is not a domain of holomorphy

The shaded region is how one can think of Ω. We will now show that Ω is not a domain of holomorphy. To do this, we consider.
In figure 2.6.15.1 the hatched region depicts $\mathcal{V}$. Let

$$\mathcal{W} = \left\{ (z_0, z_1) \in \mathbb{C} \times \mathbb{C}^{n-1} : z_0 \in D^1(0, \mathrm{R}), z_1 \in \mathcal{H}' \right\},$$

$f \in \mathcal{O}(\Omega)$, $(z_0, z_1) \in \mathcal{V}$ and $\rho \in \mathbb{R}_{>0}$ be such that $\max\{|z_0|, r\} < \rho < R.$

We consider the holomorphic function $\hat{f} : \mathcal{V} \to \mathbb{C}$ defined as follows

$$\hat{f}(z_0, z_1) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{f(\zeta, z_1)}{\zeta - z_0} d\zeta .$$

By Cauchy Integral Formula, $\hat{f} \mid \mathcal{W} = f \mid \mathcal{W}$. Therefore, since Ω is connected, $\hat{f} \mid \Omega = f$. Thus $\hat{f}$ is an extension of f to $\mathcal{V}$ and this prohibits Ω from being a domain of holomorphy.

Theorem 2.6.16. Let $\Omega \subset \mathbb{C}^n$ be a domain of holomorphy. Then the set

$$\mathcal{A} = \left\{ f \in \mathcal{O}(\Omega) : f \textit{ is non-extendable} \right\}$$

is a dense and $\mathcal{G}_\delta$ subset of the space $\mathcal{O}(\Omega)$.

Proof. We will apply theorem 2.1.14 with $\mathcal{V} = \mathcal{O}(\Omega)$. For this purpose, let us consider a pair of balls (B,b) with $b \subset\subset B \cap \Omega \neq B$. Then $B \cap \Omega \neq \varnothing$, $B \cap (\mathbb{C}^n - \Omega) \neq \varnothing$, and, since B is connected, $B \cap \partial\Omega \neq \varnothing$. Let us consider a point $\sigma \in B \cap \partial\Omega \neq \varnothing$ and a sequence z_k in $B \cap \Omega$ which converges to σ. Since Ω is a domain of holomorphy, there exists a function f, holomorphic in Ω, such that $\sup\limits_k \left| f(z_k) \right| = \infty$. Then $f \in \mathcal{V} = \mathcal{O}(\Omega)$ and the restriction $f|_b$, of f to b, has no bounded holomorphic extension to B. Therefore, from Theorem 2.1.14, the set $\mathcal{A}$ is dense and $\mathcal{G}_\delta$ in the space $\mathcal{V} = \mathcal{O}(\Omega)$.□

2.6.2. Pseudoconvex sets

If Ω is an open set of $\mathbb{C}^n$ then it is clear that the function $z \to dist(z, \partial\Omega)$ is continuous and positive in Ω. Consequently the function $-\log dist\left(z, \partial\Omega\right)$ is a continuous real-valued function in Ω. Sometimes the function $-\log dist\left(z, \partial\Omega\right)$ is not plurisubharmonic, even though in one dimension it always is, as we saw in Corollary 2.5.5. Let us consider an example of this. We let $\Omega = \mathbb{C}^n - \{0\}$, with $n \geq 2$, and let us take $z = e_1$ and $w = e_2$, with $\left(e_1, e_2, ..., e_n\right)$ denoting the standard basis. We then have

$$-\log dist\left(z + \zeta w, \partial\Omega\right) = -\log dist\left(e_1 + \zeta e_2, \partial\Omega\right) = -\log \sqrt{1 + |\zeta|^2}\,.$$

Note that the function $\zeta \mapsto -\log \sqrt{1 + |\zeta|^2}$ has a strict maximum at $\zeta = 0$ and therefore it is not subharmonic.

In order to give the definition of pseudoconvex sets we need to introduce the notion of exhaustion functions.

Definition 2.6.17. Let $\Omega \subset \mathbb{C}^n$ be an open set. A function $u : \Omega \to \mathbb{R}$ is called an *exhaustion function* of Ω if $\{z \in \Omega : u(z) < a\}$ is relatively compact in Ω for all $a \in \mathbb{R}$.

Definition 2.6.18. An open set Ω of $\mathbb{C}^n$ is said to be *pseudoconvex* if and only if Ω has a continuous plurisubharmonic exhaustion function.

Example 2.6.19. $\mathbb{C}^n$ is pseudoconvex set. Indeed, the function u defined by $u(z) = \sum_{j=1}^{n} z_j \overline{z_j}$ is a plurisubharmonic function, since

$$L_u(z,w)=\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\, w_j \overline{w_k} = \sum_{j=1}^{n} w_j \overline{w_j} \geq 0 .$$

Also it is clear that u is an exhaustion function for $\mathbb{C}^n$.

Example 2.6.20. Let $\Omega \subseteq \mathbb{C}^n$ bounded and define the boundary distance function as in the single variable case $\delta_\Omega : \Omega \to \mathbb{R}_{\geq 0}$, $\delta_\Omega(z) = dist\left(z, \mathbb{C}^n \setminus \Omega\right)$. We have that δ_Ω is continuous –see Corollary 2.5.5. – . Since Ω is bounded, then $-\log \delta_\Omega$ is an exhaustion function since $\left(-\log \delta_\Omega\right)^{-1}\left((-\infty, a)\right)$ has bounded closure.

Remark 2.6.21. For general Ω, $-\log \delta_\Omega$ may not be an exhaustion function. For example, if $\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane, then, for $a \in \mathbb{R}_{>0}$,

$\left(-\log \delta_\Omega\right)^{-1}(-\infty, a) = \{z \in \Omega : \mathrm{Im}(z) < e^{-a}\}$ which is not relatively compact.

As we showed, $-\log \delta_\Omega$ is not an exhaustion function when Ω is unbounded, it is easy to modify it so as to produce an exhaustion function. Indeed, the function $u : \mathrm{U} \to \mathbb{R}$ defined by is easy to verify that $u(z) = \max\left\{\|z\|^2, -\log \delta_\Omega\right\}$, where

$\|z\|^2 = \sum_{j=1}^{n} z_j \overline{z_j}$ is a continuous exhaustion function.

In order to give a characterization of pseudoconvex sets we need the following definitions.

Definition 2.6.22. Let $\Omega \subset \mathbb{C}^n$ be an open set and K a compact subset K of Ω. Then the *plurisubharmonic convex hull* $\hat{K}^p_{\mathcal{O}(\Omega)}$ *in* Ω is defined to be the set

$$\hat{K}_{\mathcal{O}(\Omega)} = \left\{ z \in \Omega : |u(z)| \leq \sup_{\zeta \in K} |u(\zeta)| \text{ for all } u \in psh(\Omega) \right\}.$$

$\hat{K}^p_{\mathcal{O}(\Omega)}$ is also called the $\mathcal{P}(\Omega)$ - *hull of K. If* $\hat{K}^p_{\mathcal{O}(\Omega)} = K$ then $K \subset \Omega$ is called $\mathcal{P}(\Omega)$ - *convex.*

It is clear that that the $\mathcal{P}(\Omega)$ - hull of *K* is contained in the $\mathcal{O}(\Omega)$ - hull of *K*.

Definition 2.6.23. An *analytic disc* Δ in Ω is a continuous function $\varphi : \{|\lambda| \leq 1\} \to \Omega$ which is holomorphic in $\{|\lambda| \leq 1\}$. Then we may write $\Delta = \varphi\left(\{|\lambda| \leq 1\}\right)$ and $\partial\Delta = \varphi\left(\{|\lambda| = 1\}\right)$.

Theorem 2.6.24. (Characterizations of pseudoconvex sets ([18, Proposition 14.1], [24]) If $\Omega \subset \mathbb{C}^n$ is an open set then the following conditions are equivalent:

(i) Ω is pseudoconvex.

(ii) $-\log dist(z,\partial\Omega)$ is plurisubharmonic in Ω.

(iii) $\hat{K}^{p}_{\mathcal{O}(\Omega)} \subset\subset \Omega$ if $K \subset\subset \Omega$.

(iv) For every analytic disc $\Delta \subset \Omega,\ dist(\Delta,\partial\Omega) = dist(\partial\Delta,\partial\Omega)$.

(v) For every family of analytic disc $\{\Delta_j\}_{j\in J}$ in Ω, $\bigcup_{j\in J}\partial\Delta_j \subset\subset \Omega$ we have $\bigcup_{j\in J}\Delta_j \subset\subset \Omega$.

Proposition 2.6.25. (Basic properties of pseudoconvex sets ([11, Theorem 2.6.9])
The following statements hold:

(i) If $D\subset\mathbb{C}^n$ and $G\subset\mathbb{C}^m$ are pseudoconvex open sets, then $D\times G\subset\mathbb{C}^{n+m}$ is pseudoconvex.

(ii) If $\{\Omega_j\}_{j\in J}$ is a family of pseudoconvex open sets in $\mathbb{C}^n$, then the interior of $\bigcap_{j\in J}\Omega_j$ is pseudoconvex.

(iii) If Ω_j is a sequence of pseudoconvex open sets in $\mathbb{C}^n$ for which $\Omega_j\subset\Omega_{j+1},\ j\in\mathbb{N}$ then $\bigcup_{j\in J}\Omega_j$ is also pseudoconvex.

Proof. (i) Let $u\in psh(D)\cap C(D),\ v\in psh(G)\cap C(G)$ be exhaustion functions. Let $\hat{u}: D\times G\to\mathbb{R}$ and $\hat{v}: D\times G\to\mathbb{R}$ be defined by

$$\hat{u}(z,w)=u(z),\ \hat{v}(z,w)=v(w).$$

Both $\hat{u}$, $\hat{v}$ are plurisubharmonic -see 2.5.3. (vi)-, since $\hat{u}=u\circ\pi_z$, and $\hat{v}=v\circ\pi_w$, where $\pi_z: D\times G\to D$, $\pi_w: D\times G\to G$ are the projection maps, which are holomorphic.
We define

$$\sigma(z,w)=\max\{\hat{u}(z,w),\hat{v}(z,w)\}.$$

The function σ is obviously continuous and plurisubharmonic -see 2.5.3. (iv)-.
Since $\sigma^{-1}((-\infty,a))\subset u^{-1}((-\infty,a))\times v^{-1}((-\infty,a))$ it follows that σ is also an exhaustion function, and that completes the proof.

(ii) Let Δ be an analytic disc in $\Omega=\operatorname{int}\left(\bigcap_{j\in J}\Omega_j\right)$ then $dist(\Delta,\partial\Omega_j)=dist(\partial\Delta,\partial\Omega_j)$.

Hence $dist(\partial\Delta,\partial\Omega)=\inf_j dist(\partial\Delta,\partial\Omega_j)=\inf_j dist(\Delta,\partial\Omega_j)=dist(\Delta,\partial\Omega)$.

(iii) This follows from the theorem 2.6.24.(v). □

Remark 2.6.26. If $\Omega\subset\mathbb{C}^n$ is a domain of holomorphy, then it is also pseudoconvex. Indeed, it is easy to see that $\hat{K}^{p}_{\mathcal{O}(\Omega)}\subset\hat{K}_{\mathcal{O}(\Omega)}$, since if $f\in\mathcal{O}(\Omega)$ then $u=|f|\in psh(\Omega)$.

The following theorem shows that pseudoconvexity is a local property of the boundary. The condition in this theorem is only a restriction on the boundary.

Theorem 2.6.27. ([11, Theorem 2.6.10]) Let $\Omega \subset \mathbb{C}^n$ be an open set, then Ω is pseudoconvex if and only if for every point $\zeta \in \partial\Omega$ there exists an open neighborhood U_ζ of ζ such that $\Omega \cap U_\zeta$ is pseudoconvex.

Proof. One direction is trivial, since if Ω is pseudoconvex and consider U_ζ a convex set then $\Omega \cap U_\zeta$ is pseudoconvex.-see Proposition 2.6.25.(ii) and Remark 2.6.26.-

For the converse we will first prove it for bounded sets and after for unbounded. Let Ω be bounded pseudoconvex set. Since plurisubharmonicity is a local property then if $-\log dist\left(z, \partial\left(\Omega \cap U_\zeta\right)\right)$ are plurisubharmonic in each $\Omega \cap U_\zeta$, $\zeta \in \partial\Omega$, the function $-\log dist\left(z, \partial\Omega\right)$ will be plurisubharmonic in a set of the form $\Omega \cap W$, where W is a neighborhood of the boundary of Ω.
Since $\Omega - W \subset\subset \Omega$ -Ω is bounded- then we have that

$$A = \sup\left\{-\log dist\left(z, \partial\Omega\right) : z \in \Omega - W\right\} < \infty.$$

Now we consider the function $u(z) = \max\left\{-\log dist\left(z, \partial\Omega\right), |z|^2 + A + 1\right\}$ which is a continuous plurisubharmonic exhaustion function for Ω, and hence Ω is pseudoconvex.
For the case where Ω is undounded we have that if the boundary of Ω is locally pseudoconvex then the same applies for the sets $\Omega_j = \Omega \cap B(0, j)$, $j \in \mathbb{N}$. Since the sets Ω_j are bounded, by the previous case, they are pseudoconvex. Hence the set $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$ is pseudoconvex -see Proposition 2.6.25.(iii)-. □

2.7. Stein Lemma

Definition 2.7.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary. Let ρ be a real valued function defined in a neighborhood of $\overline{\Omega}$ so that $\rho : \mathbb{C}^n \to \mathbb{R}$ is a C^2 function, $\Omega = \{\rho < 0\}$, $\partial\Omega = \{\rho = 0\}$, $\mathbb{C}^n - \overline{\Omega} = \{\rho > 0\}$ and $\nabla\rho \neq 0$ at the points of $\partial\Omega$. A function ρ of the above type will be called *defining function* for Ω.

Remark 2.7.2. There are infinitely many such characterizing functions. Each characterizing function determines a family of approximating subdomains $\mathfrak{D}_\varepsilon$ as follows: $\mathfrak{D}_\varepsilon = \left\{\rho < -\varepsilon\right\}$. Their boundaries $\partial\mathfrak{D}_\varepsilon$ are then the level surfaces $\{\rho = -\varepsilon\}$, and for ε sufficiently small and positive ρ + ε is a defining function for $\mathfrak{D}_\varepsilon$.

Proposition 2.7.3. Let Ω be an open subset of $\mathbb{R}^n$ with smooth boundary. Let W be a neighbourhood of $\partial\Omega$ and $\rho: W \to \mathbb{R}$ a C^∞ function so that $\Omega \cap W = \{\rho < 0\}$, $\partial\Omega \cap W = \{\rho = 0\}$, $\left(\mathbb{R}^n - \overline{\Omega}\right) \cap W = \{\rho > 0\}$ and $\nabla\rho \neq 0$ at the points of $\partial\Omega$. If ρ_1, ρ_2 two defining functions for Ω then there exist h a C^∞ function in a neighbourhood of $\partial\Omega$, $h(x) > 0$ $(\forall x)$ so that. $\rho_2 = h\rho_1$

Proof. Let $\mathcal{E} = \{x \in W : \rho_1(x) < 0\}, \nabla\rho_1(x) \neq 0$ $x \in \mathcal{E}$ then $\rho_2(x) = 0$ for $x \in \mathcal{E}$ ($\nabla\rho_2(x) \neq 0, \quad \forall x \in \partial\Omega$) and therefore, there exist a function h, $h: W \to \mathbb{R}, \quad \mathrm{h} > 0$ so that $\rho_2 = h\rho_1$. Also, it is obvious that $d\rho_2(x) = h(x)d\rho_1(x)$, $x \in \partial\Omega$, thus the proof is complete. □

Lemma Stein 2.7.4. ([26, Lemma 3]) Let $\mathfrak{D}$ be a bounded smooth domain in $\mathbb{R}^n$**.** Let ρ_1, ρ_2 two defining functions for $\mathfrak{D}$, and $\partial\mathfrak{D}_\varepsilon^i = \{\rho_i = -\varepsilon\}$, $i = 1,2$. Then for each p, $p \geq 1$ and each harmonic function u in $\mathfrak{D}$ the two conditions

$$\sup_{\varepsilon > 0} \int_{\partial\mathfrak{D}_\varepsilon^i} |u(x)|^p \mathrm{d}\sigma_\varepsilon^i(\mathrm{x}) < \infty, \quad i = 1,2, \qquad (2.7.4.1)$$

are equivalent.

Proof. It suffices to show that the condition (2.7.4.1.) for $i = 1$ implies the same condition for $i = 2$.

Now there exist positive constants κ, κ_1 and κ_2 (independent of ε) so that if $x \in \{\rho_2(x) = -\varepsilon\}$ (i.e. $\rho_2(x) = -\varepsilon$) then

$$B(x, \kappa\varepsilon) \subset \Lambda_\varepsilon := \{x \in \mathbb{R}^\mathrm{n} : -\kappa_1\varepsilon < \rho_1(x) < -\kappa_2\varepsilon\}.$$

(The positive parameter ε is assumed to be sufficiently small so that the various assertions in this proof hold true.) By the mean value property,

$$|u(x)|^\mathrm{p} \leq \frac{\kappa_3}{\varepsilon^n} \int_{w \in \mathbb{R}^n} \chi_\varepsilon(x, y)|u(y)|^\mathrm{p}\, dy \text{ for } x \in \{\rho_2(x) = -\varepsilon\},$$

where $\chi_\varepsilon(x, y) = 1$ for $y \in B(x, \kappa\varepsilon)$ and $\chi_\varepsilon(x, y) = 0$ for $y \in \mathbb{R}^n - B(x, \kappa\varepsilon)$.

In what follows, κ_j, $j = 3,4,5,6$, are appropriate constants independent of ε. Then

$$\int_{\partial\mathfrak{D}_\varepsilon^2} |u(x)|^p\, d\sigma_\varepsilon^2(x) \leq \frac{\kappa_3}{\varepsilon^n} \int_{y \in \mathbb{R}^n} \left(\int_{\partial\mathfrak{D}_\varepsilon^2} \chi_\varepsilon(x, y) d\sigma_\varepsilon^2(x) \right) |u(y)|^\mathrm{p}\, dy,$$

where we used Fubini's theorem (see Theorem 2.1.9.) and the measurability of the function $\chi_\varepsilon(x, y)$ for $(x, y) \in \partial\mathfrak{D}_\varepsilon^2 \times \mathbb{R}^n$ with respect to the product measure $d\sigma_\varepsilon^2(x) \times dy$
.

Since $B(x, \kappa\varepsilon) \subset \Lambda_\varepsilon$ for $x \in \partial\mathfrak{D}_\varepsilon^2$.

Then

$\int_{\partial\mathfrak{D}_\varepsilon^2} \chi_\varepsilon(x, y) d\sigma_\varepsilon^2(x) = 0$ if $y \in \mathbb{R}^n - \Lambda_\varepsilon$ and

$$\int\limits_{\partial\mathfrak{D}_\varepsilon^{2}} \chi_\varepsilon(x,y)d\sigma_\varepsilon^2(x) \le \kappa_4\, \varepsilon^{2n-1} \text{ for } y\in\Lambda_\varepsilon .$$

It follows that

$$\int\limits_{\partial\mathfrak{D}_\varepsilon^{2}} |u(x)|^p\, d\sigma_\varepsilon^2(x) \le \frac{\kappa_5}{\varepsilon} \int\limits_{\Lambda_\varepsilon} |u(y)|^p\, d\sigma_\varepsilon^2(y) \le \frac{\kappa_6}{\varepsilon} \int\limits_{\kappa_2\varepsilon}^{\kappa_1\varepsilon} \left(\int\limits_{\partial\mathfrak{D}_\eta^{1}} |u(y)|^p\, d\sigma_\eta^1(y) \right) d\eta .$$

(The existence of the constant κ_6 follows from the coarea formula (see Theorem 2.3.3.) Thus

$$\sup_{\varepsilon>0} \int\limits_{\partial\mathfrak{D}_\varepsilon^{2}} |u(x)|^p\, d\sigma_\varepsilon^2(x) \le \kappa_6(\kappa_1-\kappa_2) \sup_{\eta>0} \int\limits_{\partial\mathfrak{D}_\varepsilon^{1}} |u(x)|^p\, d\sigma_\eta^1(x),$$

and this implies $\sup\limits_{\varepsilon>0} \int\limits_{\partial\mathfrak{D}_\varepsilon^{2}} |u(x)|^p\, d\sigma_\varepsilon^2(x) < \infty$, since the condition (2.7.4.1.) holds for $i=1$. □

Definition 2.7.5. Let $B=\{x\in\mathbb{R}^n : |x|<1\}$ be the unit ball of $\mathbb{R}^n$. The Poisson kernel for the unit ball has the following form

$$P(x,y)=\frac{1}{\omega_{n-1}}\frac{1-|x|^2}{|x-y|^n}\ ,\ \left(|x|<1, |y|=1\right),$$

ω_{n-1} is the surface area of the unit $(n\text{-}1)$-sphere.

Definition 2.7.6. Let $\mathbb{B}=\{z\in\mathbb{C}^n : |z|<1\}$ be the unit ball of $\mathbb{C}^n, n>1$. The (invariant) Poisson kernel for the ball has the following form

$$P(z,\zeta)=\frac{\left(1-|z|^2\right)^n}{|1-\langle z,\zeta\rangle|^{2n}}\ \left(z\in\mathbb{B}, \zeta\in\mathbb{S}\right).$$

For $n=1$ the Poisson kernel for the unit ball of the complex plane $\mathbb{C}\approx\mathbb{R}^2$ has the following form

$$P(z,\zeta)=\frac{1}{\omega_{2n-1}}\frac{1-|z|^2}{|z-\zeta|^{2n}}\ \left(z,\zeta\in\mathbb{C}, |z|<1, |\zeta|=1\right),$$

ω_{2n-1} is the surface area of the unit $(2n\text{-}1)$-sphere.

Definition 2.7.7. (The Green's Function) Let $\Omega\subseteq\mathbb{R}^n$ be a bounded domain with C^2 boundary. A function $G:(\Omega\times\Omega)\setminus\{\text{diagonal}\}\to\mathbb{R}$ is the Green's function on Ω if:

1. G is C^2 on $(\Omega\times\Omega)\setminus\{\text{diagonal}\}$ and, for any small $\varepsilon>0$, is $C^{2-\varepsilon}$ up to $(\Omega\times\overline{\Omega})\setminus\{\text{diagonal}\}$;
2. $\Delta_y G(x,y)=0$ for $x\neq y, y\in\Omega$;

3. For each fixed $x \in \Omega$ the function $G(x,y)+\Gamma_n(y-x)$,

$$\Gamma_n(x)=\Gamma(x)=\begin{cases}(2\pi)^{-1}\log|x| & \text{if } n=2,\\ (2-n)^{-1}\omega_{n-1}^{-1}|x|^{-n+2} & \text{if } n>2.\end{cases}$$ is harmonic as a function of

$y \in \Omega$ (even at the point x), ω_{n-1} denotes the σ measure of the $(n-1)$ dimensional unit sphere in $\mathbb{R}^n$;

4. $G(x,y)\big|_{y\in\partial\Omega}=0$ for each fixed $x \in \Omega$.

Proposition 2.7.8. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Then Ω has a Green's function.

Theorem 2.7.9. (Poisson Integral Formula ([16, Theorem 1.3.12]) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Let ν represent the unit outward normal vector field on $\partial\Omega$. Let the Poisson kernel on Ω be the function

$$P(x,y)=-\nu_y G(x,y).$$

If $u \in C(\overline{\Omega})$ is harmonic on Ω, then

$$u(x)=\int_{\partial\Omega} P(x,y)u(y)\,\mathrm{d}\sigma(y) \text{ for all } x \in \Omega.$$

Corollary 2.7.10. For each fixed $y \in \partial\Omega$, $P(x,y)$ is harmonic in x.

Proposition 2.7.11. The Poisson Kernel for $B=\{x \in \mathbb{R}^n : |x|<1\} \subseteq \mathbb{R}^n$ has the following properties:

1. $P(x,y) \geq 0$.
2. $\int_{\partial B} P(x,y)\,\mathrm{d}\sigma(y)=1$, all $x \in B$.
3. For any $\delta>0$, any fixed $\zeta_0 \in \partial B$,

$$\lim_{\substack{x\to\zeta_0\\ x\in B}} \int_{|\zeta_0-y|>\delta} P(x,y)\,\mathrm{d}\sigma(y)=0.$$

Remark 2.7.12. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. It follows from the maximum principle that $G(x,y) \geq 0$. Hence by the Hopf lemma- see Lemma 2.7.14. -, we conclude that $P(x,y)>0$. Therefore, for each $x \in \Omega$ the argument in the previous proposition shows that $\|P(x,y)\|_{L^1(\partial\Omega,\mathrm{d}\sigma)}=1$. Thus for $\phi \in C(\partial\Omega)$, the functional

$$\phi \mapsto \int_{\partial\Omega} P(x,y)\phi(y)\,\mathrm{d}\sigma(y)$$

is bounded.

From this, Theorem (Poisson Integral Formula), and the maximum principle, we have the next result.

Proposition 2.7.13. The Poisson kernel for a C^2 domain Ω is uniquely determined by the property that it is positive and solves the Dirichlet problem.

Lemma 2.7.14. (Hopf ([16, Exc. 1.6.22) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2 boundary. Let $f : \overline{\Omega} \to \mathbb{R}$ be harmonic and nonconstant on Ω, C^1 on $\overline{\Omega}$. Suppose that f assumes a (not necessarily strict) maximum at $P \in \partial\Omega$. If $v = v_p$ is the unit outward normal to $\partial\Omega$ at P, then $\left({\partial f}/{\partial v}\right)(P) > 0.$

Proposition 2.7.15. ([16], Proposition 8.2.1. and [25]) Let $\Omega \subseteq \mathbb{R}^n$ be a domain with C^2 boundary. Let $P = P_\Omega : \Omega \times \partial\Omega \to \mathbb{R}$ be its Poisson Kernel. Then for each $x \in \Omega$, there is a positive constant C_x such that

$$0 < C_x \le P(x,\mathrm{y}) \le \frac{C}{|x - \mathrm{y}|^{\mathrm{n}}} \le \frac{C}{dist(x,\partial\Omega)^{\mathrm{n}}}$$

2.8. Strictly Pseudoconvex sets and the Levi polynomial

Definition 2.8.1. Let Ω be a bounded open set. Ω is called *strictly pseudoconvex* if there exists a strictly plurisubharmonic C^2 – function ρ in some neighborhood U of the boundary of Ω such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$. If moreover ρ is smooth of class C^k $(k = 2,3,...)$, then Ω is said to be a C^k strictly pseudoconvex open set.

Remark 2.8.2. The boundary of a strictly pseudoconvex open set $\Omega \subset\subset \mathbb{C}^n$ need not be smooth. For example, $\Omega = \{z = x + iy \in \mathbb{C} : 2x^2 - y^2 + y^4 < 0\}$ - see figure 2.8.2.1.- is a strictly pseudoconvex open set with no smooth boundary.

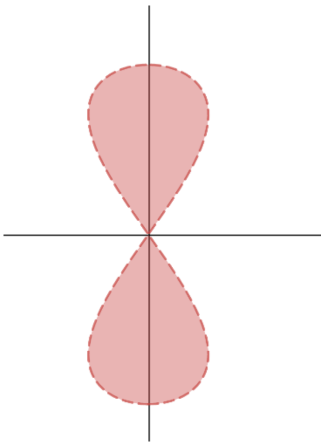

Figure 2.8.2.1.

Indeed, if the boundary of Ω is C^2, then $d\rho \neq 0$ at the boundary of Ω, where $\rho(z) = 2x^2 - y^2 + y^4,\ z = x + iy,\ x, y \in \mathbb{R}$ is the strictly plurisubharmonic C^2-function for Ω, and hence is a defining function for Ω. Since the boundary of Ω is C^2 then there exists λ a C^2-function in a neighborhood $V \subset U$ of the boundary of Ω such that $\Omega \cap V = \{\lambda < 0\}$, $\partial\Omega = \{\lambda = 0\}$ and $d\lambda \neq 0$ at the boundary of Ω. We have that $\phi = \dfrac{\rho}{\lambda}$ is C^1 in V and $d\rho(\zeta) = \phi(\zeta) d\lambda(\zeta)$, for $\zeta \in \partial\Omega$. So, we need to show that $\phi \neq 0$ at every boundary point of Ω. Suppose $\phi(\zeta_1) = 0$ for some $\zeta_1 \in \partial\Omega$ then since $\phi \geq 0$ we have that $d\phi = 0$. Hence all the second degree derivatives at the point ζ_1 will equal to zero, which contradicts with the fact that ρ is a strictly plurisubharmonic.

Moreover, the strictly pseudoconvex sets may consist of infinitely many components.

Remark 2.8.3. Every strictly pseudoconvex set is pseudoconvex. Indeed, since Ω is strictly pseudoconvex there exists a strictly plurisubharmonic C^2-function ρ in some neighborhood U of the boundary of Ω such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$. Let $\zeta \in \partial\Omega$ and $B_\zeta \subset\subset U$ a small open ball centered at ζ -see figure 2.8.3.1.- .
We consider the function $u = -1/\rho$ which is plurisubharmonic for all $w \in \Omega \cap B_\zeta$, since for $z \in \partial\Omega \cap B_\zeta$, $u(z) \to \infty$. For $z \in \overline{\Omega} \cap \partial B_\zeta$ we have $1/\left(r^2 - |z|^2\right) \to \infty$, hence $\Omega \cap B_\zeta$ is pseudoconvex and therefore by Theorem 2.6.27 Ω is pseudoconvex.

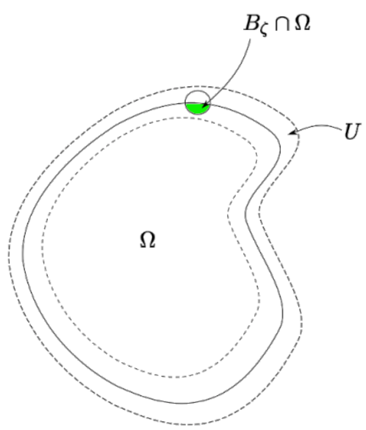

Figure 2.8.3.1

Examples 2.8.4.
(i) The set $\Omega = \{z = x + iy \in \mathbb{C} : x^2 + y^4 < 1\}$ is strictly pseudoconvex.
(ii) Let $\Omega = \left\{z = (z_1, ..., z_n) \in \mathbb{C}^n : \operatorname{Im} z_n > |z_1|^2 + ... + |z_{n-1}|^2\right\}$. Every boundary point is a is a strictly speudoconvex point.

(iii) Let $\Omega=\left\{z\in\mathbb{C}^n : |f_1(z)|^2+|f_2(z)|^2 \dots+|f_n|^2<1\right\}$ where $f_j\in\mathcal{O}(\overline{\Omega})$. Then Ω is pseudoconvex. If $p\in\partial\Omega$ and $\det\left[\frac{\partial f_j}{\partial z_k}(p)\right]\neq 0$ then p is a point of strict pseudoconvexity.

Theorem 2.8.5. (Solution of $\bar{\partial}$- equation ([12, Theorem 6.16], [10, Lemma 2.4.1]) Let $\Omega\subset\subset\mathbb{C}^n$ be a strictly pseudoconvex open set with C^2 boundary and a smooth $(p,q)-$form f with bounded coefficients in Ω with $\bar{\partial} f=0$. Then there is a bounded C^∞ $(p,q-1)-$form u in Ω such that $\bar{\partial} u = f$.

The Levi polynomial plays an important role for strictly plurisubharmonic functions.

2.8.1. The Levi polynomial.

Definition 2.8.6. Let $\rho : U\to\mathbb{R}$ be a strictly plurisubharmonic C^2-function in some neighborhood U of the boundary of Ω such that

$$\Omega\cap U=\left\{z\in U:\rho(z)<0\right\}.$$

The *Levi polynomial of the function* ρ is the following second degree polynomial of z :

$$F(z,\zeta)=-\left[2\sum_{j=1}^{n}\frac{\partial\rho(\zeta)}{\partial\zeta_j}(z_j-\zeta_j)+\sum_{1\le j,k\le n}\frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\zeta_k}(z_j-\zeta_j)(z_k-\zeta_k)\right],\ z,\zeta\in\Omega.$$

Remark 2.8.7. The Levi polynomial is only continuous in ζ. At the Henkin's construction-see Section 2.9.- the continuous derivatives $\frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\zeta_k}$ are replaced by sufficiently close C^1-functions. The obtained modification of $F(z,\zeta)$ is denoted by $Q(z,\zeta)$, and it is called the modified Levi polynomial.

The following lemma describes the connection between the Levi polynomial and the Levi form-see Remark 2.5.15.(i)-.

Lemma 2.8.8. Let $\Omega\subseteq\mathbb{C}^n$ be an open set, and ρ be a real valued C^2-function in Ω. Then, for all $\zeta\in\Omega$ and $z\to\zeta$,

$$\rho(z)=\rho(\zeta)-\operatorname{Re}F(z,\zeta)+\sum_{j,k=1}^{n}\frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\overline{\zeta_k}}(z_j-\zeta_j)(\overline{z_k}-\overline{\zeta_\kappa})+\circ\left(|\zeta-z|^2\right),\qquad(2.8.8.1)$$

where $\operatorname{Re} F(z,\zeta)$ is the real part of $F(z,\zeta)$.

Proof. Let $x_j = x_j(\zeta)$ be the real coordinates of $\zeta \in \mathbb{C}^n$ such that $\zeta_j = x_j(\zeta) + i x_{j+n}(\zeta)$, $j = 1,\dots,n$ .Then a computation gives

$$\sum_{j=1}^{2n} \frac{\partial \rho(\zeta)}{\partial x_j}\left(x_j(z) - x_j(\zeta)\right) = 2\operatorname{Re}\left[\sum_{j,k=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j}\left(z_j - \zeta_j\right)\right].$$

and

$$\frac{1}{2}\sum_{j,k=1}^{2n} \frac{\partial^2 \rho(\zeta)}{\partial x_j \partial x_k}\left(x_j(z) - x_j(\zeta)\right)\left(x_k(z) - x_k(\zeta)\right) =$$

$$= \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \overline{\zeta_k}}\left(z_j - \zeta_j\right)\left(\overline{z_k} - \overline{\zeta_k}\right) + \operatorname{Re}\left[\sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k}\left(z_j - \zeta_j\right)\left(z_k - \zeta_k\right)\right].$$

Consequently, (2.8.8.1.) is the Taylor expansion of ρ at ζ. -see Taylor Theorem 2.1.5.- □

Proposition 2.8.9. ([21, Proposition 2.16]) Let $\Omega \subset \mathbb{C}^n$ be an open set, and ρ be a strictly plurisubharmonic real valued C^2–function in Ω. Given $U \subset\subset \Omega$, there are constants $c > 0$ and $\varepsilon > 0$, such that the function $F(z,\zeta)$ defined on $\Omega \times \mathbb{C}^n$ by

$$F(z,\zeta) = \sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j}(z_j - \zeta_j) - \frac{1}{2}\sum_{1 \le j,k \le n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k}(z_j - \zeta_j)(z_k - \zeta_k)$$

satisfies the estimate

$$2\operatorname{Re} F(z,\zeta) \ge \rho(\zeta) - \rho(z) + c\left|\zeta - z\right|^2 \text{ for } \zeta \in U \text{ and} \left|\zeta - z\right| < \varepsilon. \quad (2.8.9.1.)$$

Proof. From Proposition 2.5.14 and Lemma 2.8.8, we see that the Taylor expansion of $\rho(z)$ at ζ is given by

$$\rho(z) = \rho(\zeta) - 2\operatorname{Re} F(z,\zeta) + L_\rho(\zeta, z - \zeta) + \mathrm{o}\left(\left|\zeta - z\right|^2\right) \qquad (2.8.9.2.)$$

If $\overline{U} \subset \Omega$ is compact, then by Remark 2.5.15.(ii) there is $c > 0$, such that $L_\rho(\zeta, z - \zeta) \ge 2c\left|\zeta - z\right|^2$ for $\zeta \in U$ and $z \in \mathbb{C}^n$. Taylor Theorem and the uniform continuity on U of the derivatives of ρ up to order 2 imply that the error term in (2.8.9.2.) is uniform in $\zeta \in U$, that is $\varepsilon > 0$, so that $\left|\circ\left(\left|\zeta - z\right|^2\right)\right| \le c\left|\zeta - z\right|^2$ for $\zeta \in U$ and $\left|\zeta - z\right| < \varepsilon$. Equation (2.8.9.1.) now follows by using these estimates in (2.8.9.2.) and rearranging.□

2.9. Henkin's Construction

Locally the Levi polynomial – see definition 2.8.6 – can be used as the support function $\Phi(z,\zeta)$. To obtain $\Phi(z,\zeta)$ globally, we have to solve some $\bar{\partial}$– equation which depends continuously differentiable on a parameter. This can be done by using the following lemma 2.9.1. and certain arguments which follow from Banach's open mapping theorem. First we give some notations.

Notation. If $Y \subset \mathbb{C}^n$ is a measurable set and $f \in C(Y)$, $0 < a < 1$ wedenote by $\|f\|_{a,Y}$ the a– Hölder norm, $\|f\|_{a,Y} = \sup_{z\in Y}\|f\| + \sup_{z,\zeta\in Y} \frac{|f(z)-f(\zeta)|}{|\zeta - z|}$.

Set $H^a(Y) := \left\{ f \in C(Y) : \|f\|_{a,Y} < \infty \right\}$, $H^a(Y)$ endowed with the norm $\|.\|_{a,Y}$ forms a Banach space, which is called the space of a– Hölder continuous functions(Hölder space).

The notations $H^a_{(p,q)}(Y)$, $C_{(p,q)}(Y)$, $C^\infty_{(p,q)}(Y)$ will be used for the spaces of differential forms of bidegree (p,q) and with coefficients in $H^a_{(p,q)}(Y)$, $C_{(p,q)}(Y)$, $C^\infty_{(p,q)}(Y)$, respectively.

If $\Omega \subset \mathbb{C}^n$ be an open set, then we denote by $C^\infty(\Omega)$ the Fréchet space of all complex- valued C^∞– functions in Ω endowed with the topology of uniform convergence on compact sets together with all derivatives. By $Z^\infty_{(0,1)}(\Omega)$ will be denoted the Fréchet space (endowed with the same topology) of all $C^\infty_{(0,1)}$– forms f in Ω such that $\bar{\partial} f = 0$ in Ω.

Lemma 2.9.1. ([10, Lemma 2.3.4.]) Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set, and let f be a continuous $(0,q)$– form in some neighbourhood $U_{\bar{\Omega}}$ of $\bar{\Omega}$ such that $\bar{\partial} f = 0$ in $U_{\bar{\Omega}}$, $1 \le q \le n$. Then there exists a $u \in H^{1/2}_{(0,q-1)}(\bar{\Omega})$ such that $\bar{\partial} u = f$ in Ω.

Lemma 2.9.2. ([10, Lemma 2.4.1.]) Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set and let $U_{\bar{\Omega}}$ be a neighbourhood of $\bar{\Omega}$. Then there exists a continuous linear operator $T : Z^\infty_{(0,1)}(U_{\bar{\Omega}}) \to C^\infty(\Omega)$ C^∞– function such that

$$\bar{\partial} Tf = f \text{ in } \Omega \text{ for all } f \in Z^\infty_{(0,1)}(U_{\bar{\Omega}}).$$

Lemma 2.9.3. (Henkin's contruction ([10, Lemma 2.4.2.]) Let us consider an open set $\Theta \subset\subset \mathbb{C}^n$ and a C^2 strictly plurisubharmonic function ρ in a neighbourhood of $\overline{\Theta}$. If we set

$$\beta = \frac{1}{3}\min\left\{ \sum_{1\le j,k\le n} \frac{\partial^2 \rho(\zeta)}{\partial\zeta_j \partial\bar{\zeta}_k}\xi_j\bar{\xi}_k : \zeta \in \overline{\Theta},\ \xi \in \mathbb{C}^n \ with\ |\xi| = 1 \right\}$$

then $\beta > 0$ and there exist C^1 functions a_{jk} in a neighbourhood of $\overline{\Theta}$ such that

$$\max\left\{ \left| a_{jk}(\zeta) - \frac{\partial^2 \rho(\zeta)}{\partial\zeta_j \partial\zeta_k} \right| : \zeta \in \overline{\Theta} \right\} < \frac{\beta}{n^2}.$$

Let $\varepsilon > 0$ be sufficiently small so that

$$\max\left\{ \left| \frac{\partial^2 \rho(\zeta)}{\partial x_j \partial x_k} - \frac{\partial^2 \rho(z)}{\partial x_j \partial x_k} \right| : \zeta, z \in \overline{\Theta}\ with\ |\zeta - z| \le \varepsilon \right\} < \frac{\beta}{2n^2} \text{ for } j,k = 1,2,\dots,2n,$$

where $x_j = x_j(\xi)$ are the real coordinates of $\xi \in \mathbb{C}^n$ such that $\xi_j = x_j(\xi) + ix_{j+n}(\xi)$.
For $z,\zeta \in \overline{\Theta}$ we consider the modified Levi polynomial

$$Q(z,\zeta) = -\left[2\sum_{j=1}^{n} \frac{\partial\rho(\zeta)}{\partial\zeta_j}(z_j - \zeta_j) + \sum_{1\le j,k\le n} a_{jk}(z_j - \zeta_j)(z_k - \zeta_k) \right].$$

Then we have the estimate

$\operatorname{Re} Q(z,\zeta) \ge \rho(\zeta) - \rho(z) + \beta|\zeta - z|^2$ for $z,\zeta \in \overline{\Theta}$ with $|\zeta - z| \le \varepsilon$. (2.9.3.1)

Proof. The proof follows from Lemma 2.8.8. and Taylor Theorem 2.1.5. □

Theorem 2.9.4. (Henkin's construction ([10, Theorem 2.4.3.]) Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set, let Θ be an open neighbourhood of $\partial\Omega$, and let ρ be a C^2 strictly plurisubharmonic function in a neighbourhood of $\overline{\Theta}$ such that

$$\Omega \cap \Theta = \{z \in \Theta : \rho(z) < 0\}.$$

Let us choose ε, β, and $Q(z,\zeta)$, as above, and let us make the positive number ε smaller so that

$\{z \in \mathbb{C}^n : |\zeta - z| \le 2\varepsilon\} \subseteq \Theta$ for every $\zeta \in \partial\Omega$. (2.9.4.1.)

Then there exists a function $\Phi(z,\zeta)$ defined for ζ in some open neighbourhood $U_{\partial\Omega} \subseteq \Theta$ of $\partial\Omega$ and $z \in U_{\overline{\Omega}} = \Omega \cup U_{\partial\Omega}$, which is C^1 in $(z,\zeta) \in U_{\overline{\Omega}} \times U_{\partial\Omega}$, holomorphic in $z \in U_{\overline{\Omega}}$, and such that $\Phi(z,\zeta) \ne 0$ for $(z,\zeta) \in U_{\overline{\Omega}} \times U_{\partial\Omega}$ with $|\zeta - z| \ge \varepsilon$, and

$$\Phi(z,\zeta) = Q(z,\zeta)C(z,\zeta) \text{ for } (z,\zeta) \in U_{\overline{\Omega}} \times U_{\partial\Omega} \text{ with } |\zeta - z| \le \varepsilon,$$

for some C^1-function $C(z,\zeta)$ defined for $(z,\zeta) \in U_{\overline{\Omega}} \times U_{\partial\Omega}$ and $\ne 0$ when $|\zeta - z| \le \varepsilon$

Proof. It follows from (2.9.3.1) that

$$\operatorname{Re} Q(z,\zeta) \geq \rho(\zeta) - \rho(z) + \beta\varepsilon^2 \text{ for } (z,\zeta) \in \Theta \text{ with } \varepsilon < |\zeta - z| < 2\varepsilon. \qquad (2.9.4.2.)$$

Since $\rho = 0$ on $\partial\Omega$ and by (2.9.4.1.), we can choose a neighbourhood $V_{\partial\Omega} \subseteq \Theta$ of $\partial\Omega$ so small that $|\rho| \leq \dfrac{\beta\varepsilon^2}{3}$ on $V_{\partial\Omega}$ and, for every $\zeta \in V_{\partial\Omega}$, the ball $|\zeta - z| \leq 2\varepsilon$ is contained in Θ. Set $V_{\overline{\Theta}} :\subset \Omega \cup V_{\partial\Omega}$. Then, for every $(z,\zeta) \in V_{\overline{\Omega}} \times V_{\partial\Omega}$, both ζ and z belong to Θ and it follows from (2.9.4.2.) that $\operatorname{Re} Q(z,\zeta) \geq \dfrac{\beta\varepsilon^2}{3}$ for all $z \in V_{\overline{\Omega}}$ and $\zeta \in V_{\partial\Omega}$ with $\varepsilon < |\zeta - z| < 2\varepsilon$. Therefore, we can define $\ln Q(z,\zeta)$ for $z \in V_{\overline{\Omega}}$ and $\zeta \in V_{\partial\Omega}$ with $\varepsilon < |\zeta - z| < 2\varepsilon$. Choose a C^∞– function $\chi : \mathbb{C}^n \to \mathbb{R}$ such that $\chi(\xi) = 1$ for $|\xi| \leq \varepsilon + \varepsilon/4$ and $\chi(\xi) = 0$ for $|\xi| \geq 2\varepsilon - \varepsilon/4$.

For $z \in V_{\overline{\Omega}}$ and $\zeta \in V_{\partial\Omega}$ we define

$$f(z,\zeta) := \begin{cases} \partial_z \chi(\zeta - z) \ln Q(z,\zeta) & \text{if } \varepsilon < |\zeta - z| \leq 2\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Then the map $V_{\partial\Omega} \ni \zeta \to f(.,\zeta)$ is continuously differentiable with values in the Fréchet space $Z^\infty_{(0,1)}\left(V_{\overline{\Omega}}\right)$. Now we choose a neighbourhood $U_{\partial\Omega} \subseteq V_{\partial\Omega}$ such that $U_{\overline{\Omega}} := \Omega \cup U_{\partial\Omega}$ is strictly pseudoconvex.

Then by Lemma 2.9.2., there is a continuous linear operator $T : Z^\infty_{(0,1)}\left(V_{\overline{\Omega}}\right) \to C^\infty\left(U_{\overline{\Omega}}\right)$ such that $\bar{\partial} T\varphi = \varphi$ on $U_{\overline{\Omega}}$ for all $\varphi \in Z^\infty_{(0,1)}\left(V_{\overline{\Omega}}\right)$.

For $z \in U_{\overline{\Omega}}$ and $\zeta \in U_{\partial\Omega}$ we define

$u(z,\zeta) := \left(Tf(.,\zeta)\right)(z)$, $\mathrm{C}(z,\zeta) := \exp\left(-u(z,\zeta)\right)$ and

$$\Phi(z,\zeta) := \begin{cases} Q(z,\zeta)\mathrm{C}(z,\zeta) & \text{if } |\zeta - z| \geq \varepsilon, \\ \exp \chi(\zeta - z) \ln Q(z,\zeta) - u(z,\zeta) & \text{if } |\zeta - z| \leq \varepsilon. \end{cases}$$

This completes the proof. □

3 Bergman type spaces

3.1. The Bergman spaces $\mathcal{O}L^p(\Omega)$

Definition 3.1.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. We recall that for $p \geq 1$, the Bergman space $\mathcal{O}L^p(\Omega)$ is defined to be the set of holomorphic functions $f:\Omega \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_\Omega |f(z)|^p dv(z) \right)^{1/p} < +\infty,$$

where dv is the Lebesgue measure in $\mathbb{C}^n$. Then the quantity $\|\cdot\|_p$ is a norm, and with this norm, $\mathcal{O}L^p(\Omega)$ is a Banach space.

Theorem 3.1.2. ([11], Theorem 1.2.4) Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. For every compact set $K \subset \Omega$ there are constants $c(\alpha, K)$ such that

$$\sup_K \left| \frac{\partial^\alpha f}{\partial z^\alpha} \right| \leq c(\alpha, K) \|f\|_1, \text{ for } f \in \mathcal{O}L^1(\Omega),$$

where $c(\alpha, K)$ is a constant depending on K and the multi-index α.

Remarks 3.1.3. We also recall that if a sequence $f_k \in \mathcal{O}L^1(\Omega)$ converges to f, in the $L^1(\Omega)-$norm, then the convergence is uniform on compact subsets of Ω. Indeed, this follows from the inequality of Theorem 3.1.2.
In particular, $\mathcal{O}L^1(\Omega)$ is closed subspace of $L^1(\Omega)$, and, more generally, $\mathcal{O}L^p(\Omega)$ is closed subspace of $L^p(\Omega)$, for $p \geq 1$. Since we assume Ω to be bounded, $\mathcal{O}L^q(\Omega) \subset \mathcal{O}L^p(\Omega)$ when $q > p$. Similarly we define the space $\mathcal{O}L^\infty(\Omega)$, of bounded holomorphic functions $f:\Omega \to \mathbb{C}$, which becomes a Banach space with the norm $\|f\|_\infty = \sup_{z\in\Omega} |f(z)|$.

For a fixed $q > 1$, we will also consider the spaces

$$\bigcap_{p<q} \mathcal{O}L^p(\Omega)$$

endowed with the metric

$$d(f,g) := \sum_{j=1}^\infty \frac{1}{2^j} \frac{\|f-g\|_{p_j}}{1+\|f-g\|_{p_j}}, \quad f,g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega),$$

where p_j is a sequence with $1 < p_1 < p_2 < \cdots < p_j < \cdots < q$ and $p_j \to q$ (as $j \to \infty$). Then $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ becomes a complete metric space, its topology being independent of the choice of the sequence p_j. In fact, a sequence f_k converges to f, in the space

$\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, if and only if $\|f_k - f\|_p \to 0$ for every $p<q$. Thus, Baire's theorem holds in $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$: A countable intersection of open and dense subsets of $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ is dense and $\mathcal{G}_\delta$ in this space. Moreover, we point out that the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, with the above topology, is also a topological vector space. In particular, if $f_k, f \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$ with $d(f_k, f) \to 0$ ($k \to \infty$), and $\lambda_k, \lambda \in \mathbb{C}$ with $\lambda_k \to \lambda$, then $d(\lambda_k f_k, \lambda f) \to 0$.
Finally, we observe that all the above hold in the case $q = \infty$ too, defining the space $\bigcap_{p<\infty} \mathcal{O}L^p(\Omega)$, and that this space contains the space of bounded holomorphic functions in Ω:

$$\bigcap_{p<\infty} \mathcal{O}L^p(\Omega) \supset \mathcal{O}L^\infty(\Omega).$$

3.2. The case of totally unbounded functions in $\mathcal{O}L^p(\Omega)$

We will show that under certain assumptions on Ω, the set of the functions in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, which are totally unbounded in Ω, is dense and $\mathcal{G}_\delta$ (in this space). We will also give examples of specific domains in which this $\mathcal{G}_\delta$ – density conclusion holds.

Theorem 3.2.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and $q \in \mathbb{R}$, $q > 1$. Suppose that for every point $\zeta \in \partial\Omega$, there exists a function f_ζ such that

$$f_\zeta \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) \quad \text{and} \quad \lim_{\substack{z \to \zeta \\ z \in \Omega}} f_\zeta(z) = \infty.$$

Then the set of the functions g in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, which are totally unbounded in Ω, is dense and $\mathcal{G}_\delta$ in this space. In particular, the set of the functions h in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, which are singular at every boundary point of Ω is dense and $\mathcal{G}_\delta$ in this space.

Proof. Let us fix a pair (B,b), where B is a «small» open ball whose center lies on $\partial\Omega$ and b is a «smaller» open ball with $b \subset\subset B \cap \Omega$, and let $E(B,b)$ be the connected component of $B \cap \Omega$ which contains b, i.e., $E(B,b) \supset b$. We are going to apply Theorem 2.1.12. with $\mathcal{V} = \bigcap_{p<q} \mathcal{O}L^p(\Omega)$ and $X = E(B,b)$. For this purpose we consider the linear operator

$$T : \bigcap_{p<q} \mathcal{O}L^p(\Omega) \to \mathbb{C}^{E(B,b)},\ T(f)(z) := f(z) \text{ for } z \in E(B,b).$$

For each fixed $z \in E(B,b)$, the functional

$$T_z : \bigcap_{p<q} \mathcal{O}L^p(\Omega) \to \mathbb{C}, \text{ defined by } T_z(f) = T(f)(z) = f(z), \text{ for } f \in \bigcap_{p<q} \mathcal{O}L^p(\Omega),$$

is continuous. (This follows from the fact that convergence in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ implies pointwise convergence.) We also observe that, in this case, the set $S = \{f \in \mathcal{V} : T(f) \text{ } \textit{is unbounded on } X\}$ is equal to

$$S(B,b) = \left\{ f \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : \sup_{z \in E(B,b)} |f(z)| = +\infty \right\}.$$

We claim that $S(B,b) \neq \varnothing$. Indeed, since the set $\overline{E(B,b)}$ meets the boundary of Ω, there exists a point $\zeta \in \overline{E(B,b)} \cap \partial\Omega$. (See the Remark 2.2.2.) By the hypotheses, there is a function $f_\zeta \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$ such that $\lim_{\substack{z \to \zeta \\ z \in \Omega}} f_\zeta(z) = \infty$, and, therefore $f_\zeta \in S(B,b)$. It follows from Theorem 2.1.12 that $S(B,b)$ is dense and $\mathcal{G}_\delta$ in $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

To complete the proof of the theorem, we consider a countable dense subset $\{w_1, w_2, w_3, \ldots\}$ of $\partial\Omega$, and the set $\mathcal{B} = \{B(w_j, \tau) : \tau \in \mathbb{Q}^+,\ j = 1,2,3,\ldots\}$.

For each $B \in \mathcal{B}$, let Γ_B be the countable set of the balls b with centers in $(\mathbb{Q} + i\mathbb{Q})^n$ and rational radii, so that $b \subset\subset B \cap \Omega$. By Baire's theorem, the set

$$\bigcap_{B \in \mathcal{B}} \bigcap_{b \in \Gamma_B} S(B,b)$$

is dense and $\mathcal{G}_\delta$ in $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$. Notice that if f belongs to this set then f is totally unbounded in Ω.

Indeed suppose that $w \in \partial\Omega$, $\varepsilon > 0$, and E is a connected component of the set $B(w,\varepsilon) \cap \Omega$. Let b be a ball with «rational» center and rational radius, and $b \subset\subset E$. Then we may choose a ball $B \in \mathcal{B}$ so that $B \subset B(w,\varepsilon)$ and $b \subset\subset B$. Then the connected component $E(B,b)$ of $B \cap \Omega$ which contains b, is contained in E, i.e., $E(B,b) \subset E$. Since $\sup_{z \in E(B,b)} |f(z)| = +\infty$, it follows that $\sup_{z \in E} |f(z)| = +\infty$.

To prove the last assertion of the theorem, we will use Theorem 2.1.14. For this purpose let us consider a pair of balls (B,b) with $b \subset\subset B \cap \Omega \neq B$, and as before, let $E(B,b)$ be the connected component of $B \cap \Omega$ which contains b.

Then by the Remark 2.2.2, $E(B,b)\cap\partial\Omega\cap B\neq\varnothing$. If $\zeta\in E(B,b)\cap\partial\Omega\cap B$ then the function f_ζ (of the hypothesis of the theorem) belongs to $\mathcal{V}=\bigcap_{p<q}\mathcal{O}L^p(\Omega)$ and its restriction $f_\zeta\big|_b$ (to b) does not have any bounded holomorphic extension to B. Hence Theorem 2.1.14 gives the required conclusion. □

Remarks 3.2.2.

(i) By examining the above proof we see that if the sets $B\cap\Omega$ are connected (for those B's having sufficiently small radius – depending on the center of the each B) then the theorem holds under the weaker hypothesis of the existence of the functions f_ζ, not necessarily for all $\zeta\in\partial\Omega$, but only for ζ in a countable dense subset of $\partial\Omega$. This is the case – for example – in which the boundary of Ω is C^1.

(ii) Let us point out that the above theorem holds also in the case «$q=\infty$». The proof in this case is essentially the same. Although the case «$q=\infty$» is, in some sense, the most interesting one, it does not imply the case «$q<\infty$». Notices that changing the value of q in $\bigcap_{p<q}\mathcal{O}L^p(\Omega)$, changes not only the space but also the topology.

(iii) We can also prove an analogous theorem in the case of the spaces $\mathcal{O}L^p(\Omega)$ for each fixed p ($1\le p<\infty$). In this case we do not need to assume Ω to be bounded. Thus if $\Omega\subset\mathbb{C}^n$ is an open set and for every point $\zeta\in\partial\Omega$, there exists a function f_ζ such that $f_\zeta\in\mathcal{O}L^p(\Omega)$ and $\lim_{z\in\Omega,\, z\to\zeta} f_\zeta(z)=\infty$, then the set of functions g in the space $\mathcal{O}L^p(\Omega)$, which are totally unbounded in Ω, is dense and $\mathcal{G}_\delta$ in this space.

3.3. Functions in $\bigcap_{p<q}\mathcal{O}L^p(\Omega)$ which do not belong to $\mathcal{O}L^q(\Omega)$

In this section we will prove – under certain assumptions on the open set Ω – that generically the functions in $\bigcap_{p<q}\mathcal{O}L^p(\Omega)$ do not belong to the space $\mathcal{O}L^q(\Omega)$, not even 'locally'. More precisely we will prove the following theorem.

Theorem 3.3.1. Let $\Omega\subset\mathbb{C}^n$ be a bounded open set and $q\in\mathbb{R}$, $q>1$. Suppose that for every point $\eta\in\partial\Omega$ and $\varepsilon>0$, there exists a function $f_{\eta,\varepsilon}$ such that

$$f_{\eta,\varepsilon} \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) \text{ and } f_{\eta,\varepsilon} \notin \mathcal{O}L^q(B(\eta,\varepsilon) \cap \Omega) \text{ for every } \varepsilon > 0.$$

Then the set

$$\mathcal{S}(\Omega,q) = \left\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : g \notin \mathcal{O}L^q(B(\zeta,\varepsilon) \cap \Omega) \textit{ for every } \zeta \in \partial\Omega \textit{ and every } \varepsilon > 0 \right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Proof. Let us fix a point $w \in \partial\Omega$ and $\varepsilon > 0$. We are going to apply Theorem 2.1.12 with $\mathcal{V} = \bigcap_{p<q} \mathcal{O}L^p(\Omega)$ and X being the set of all compact subsets K of the intersection $B(w,\varepsilon) \cap \Omega$. For this purpose we consider the sublinear operator

$$T : \bigcap_{p<q} \mathcal{O}L^p(\Omega) \to \mathbb{C}^X, \ T(f)(K) := \left(\int_K |f|^q dv \right)^{1/q} \text{ for } K \in X.$$

For every $K \in X$, the functional

$$T_K : \bigcap_{p<q} \mathcal{O}L^p(\Omega) \to \mathbb{C}, \ T_K(f) = T(f)(K),$$

is continuous. Indeed, if f_k, $k = 1,2,3,\ldots$, is a sequence which converges to f, in $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, then f_k converges to f, uniformly on K, and therefore

$$\int_K |f_k|^q dv \to \int_K |f|^q dv, \text{ as } k \to \infty.$$

We also observe that, in this case, the set $S = \{ f \in \mathcal{V} : T(f) \textit{ is unbounded on } X \}$ is equal to

$$S(w,\varepsilon) = \left\{ f \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : \int_{B(w,\varepsilon)\cap\Omega} |f|^q dv = +\infty \right\}.$$

This follows from the fact that

$$\int_{B(w,\varepsilon)\cap\Omega} |f|^q dv = \sup_{K\in X} \int_K |f|^q dv.$$

Also $S(w,\varepsilon) \neq \varnothing$, since $f_w \in S(w,\varepsilon)$. Therefore, from Theorem 2.1.12, $S(w,\varepsilon)$ is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Next let us observe that if u_j is a sequence of points in $\partial\Omega$ which converges to a point $u \in \partial\Omega$, and $f \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$, then

$$\int_{B(u_j,\varepsilon)\cap\Omega} |f|^q dv = +\infty \ (\forall j) \ \Rightarrow \int_{B(u,2\varepsilon)\cap\Omega} |f|^q dv = +\infty.$$

This follows from the fact that if $|u_{j_0} - u| < \varepsilon$ (for some j_0) then $B(u,2\varepsilon) \supseteq B(u_{j_0},\varepsilon)$, which implies that

$$\int\limits_{B(u,2\varepsilon)\cap\Omega} |f|^q dv \geq \int\limits_{B(u_{j0},\varepsilon)\cap\Omega} |f|^q dv.$$

To complete the proof of the theorem we consider a countable dense subset $\{w_1, w_2, w_3, ...\}$ of $\partial\Omega$ and a decreasing sequence ε_s of positive numbers, with $\varepsilon_s \to 0.$ By the first part of the proof and Baire's theorem, the set

$$\bigcap_{j=1}^{\infty}\bigcap_{s=1}^{\infty} S(w_j, \varepsilon_s)$$

is dense and $\mathcal{G}_\delta$ in $\bigcap\limits_{p<q} \mathcal{O}L^p(\Omega)$. Notice that if f belongs to this set, and $\zeta \in \partial\Omega$, and w_{j_m} is a subsequence of w_j which converges to ζ, then $\int\limits_{B(w_{j_m},\varepsilon_s)\cap\Omega} |f|^q dv = +\infty$, and therefore $\int\limits_{B(\zeta,2\varepsilon_s)\cap\Omega} |f|^q dv = +\infty$. Since this holds for every $\zeta \in \partial\Omega$, and the sequence $\varepsilon_s \to 0$, this implies that $f \in \mathcal{S}(\Omega, q)$. This completes the proof of the theorem. □

Remarks 3.3.2.

(i) By examining the above proof, we see that this theorem holds under the weaker hypothesis of the existence of the functions $f_{\eta,\varepsilon}$, not necessarily for all $\eta \in \partial\Omega,$ but only for η in a countable dense subset of $\partial\Omega$.

(ii) The following version of the above theorem can be proved in a similar manner. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and $q, \tilde{q} \in \mathbb{R}$ with $\tilde{q} > q > 1$. Suppose that for every point $\zeta \in \partial\Omega$, there exists a function f_ζ such that

$$f_\zeta \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) \text{ and } f_\zeta \notin \mathcal{O}L^{\tilde{q}}(B(\zeta,\varepsilon)\cap\Omega) \text{ for every } \varepsilon > 0.$$

Then the set

$$\mathcal{S}(\Omega, q, \tilde{q}) = \left\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : g \notin \mathcal{O}L^{\tilde{q}}(B(\zeta,\varepsilon)\cap\Omega) \ \forall \zeta \in \partial\Omega \ \textit{and} \ \forall \varepsilon > 0 \right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap\limits_{p<q} \mathcal{O}L^p(\Omega)$.

(iii) If the boundary of Ω is C^1 and a function $g \notin \mathcal{O}L^q(B(\zeta,\varepsilon)\cap\Omega)$, for every $\zeta \in \partial\Omega$ and every $\varepsilon > 0$, then g is singular at every point of $\partial\Omega$.
Indeed, this follows from the fact that for sufficiently small $\varepsilon > 0$ (depending on each point $\zeta \in \partial\Omega$), the sets $B(\zeta,\varepsilon)\cap\Omega$ are connected.

(iv) In the above theorem if the sets $B(\zeta,\varepsilon)\cap\Omega$ are connected (for those $B's$ having sufficiently small radius – depending on the center of each B)

then the set of the functions h in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ which are singular at every boundary point of is dense and $\mathcal{G}_\delta$ in this space. This follows from the Theorem 2.1.14.

Theorem 3.3.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and $1 < q \le \infty$. Suppose that for every point $\zeta \in \partial\Omega$, there exists a function f_ζ such that

$$f_\zeta \in \bigcap_{p<q} \mathcal{O}L^p(\Omega), \quad f_\zeta \notin \mathcal{O}L^q(B(\zeta,\varepsilon) \cap \Omega) \text{ for every } \varepsilon > 0, \text{ and } \lim_{\substack{z\to\zeta \\ z\in\Omega}} f_\zeta(z) = \infty.$$

Then the set

$$\Big\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : g \textit{ is totally unbounded in } \Omega$$

$$\textit{and } g \notin \mathcal{O}L^q(B(\zeta,\varepsilon)\cap\Omega),\ \forall \zeta \in \partial\Omega \textit{ and } \forall \varepsilon > 0 \Big\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Proof. The conclusion follows from Theorems 3.2.1 and 3.3.1. Indeed, it suffices to notice that the set in this theorem is the intersection of the corresponding sets of the Theorems 3.2.1 and 3.3.1, and that the intersection of two dense and $\mathcal{G}_\delta$ sets in the complete metric space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ is again dense and $\mathcal{G}_\delta$, by Baire's theorem. □

3.4. Applications

3.4.1. Examples in the case $n=1$.

(i) Let $\Omega \subset \mathbb{C}$ be a bounded open set with C^1 boundary. For a fixed point $\zeta \in \partial\Omega$, let us consider the holomorphic function

$$f_\zeta : \Omega \to \mathbb{C}, \quad f_\zeta(z) = \frac{1}{z-\zeta}, \quad z \in \Omega.$$

Then $f_\zeta \in \bigcap_{p<2} \mathcal{O}L^p(\Omega)$ but $f_\zeta \notin \mathcal{O}L^2(\Omega)$. Indeed, for «small» $\delta > 0$,

$$\iint_{z\in B(\zeta,\delta)} \frac{dv(z)}{|z-\zeta|^p} < +\infty \text{ when } p<2, \text{ while } \iint_{z\in B(\zeta,\delta)\cap\Omega} \frac{dv(z)}{|z-\zeta|^2} = +\infty.$$

To prove the last equation, it suffices to notice that, since $\partial\Omega$ is assumed to be C^1, there is a small angular region A with vertex at ζ such that $\mathrm{A} \cap B(\zeta,\delta) \subset \Omega$, and, that the integral

$$\iint_{z\in \mathrm{A}\cap B(\zeta,\delta)} \frac{dv(z)}{|z-\zeta|^2} = +\infty,$$

as we can easily see if we integrate in polar coordinates with center at ζ.

Next, continuing to consider the point $\zeta \in \partial\Omega$ fixed, let $a \in \mathbb{C} - \overline{\Omega}$ be a point, sufficiently close to the point ζ, so that the line segment $[\zeta, a]$, which connects a and ζ, is contained in $\mathbb{C} - \Omega$. (Such a point exists since we assume that $\partial\Omega$ is C^1.) Then, in the set Ω, there exists a holomorphic branch of $\log\left(\frac{z-a}{z-\zeta}\right)$, i.e., there exists a holomorphic function $g_\zeta(z)$, $z \in \Omega$, such that $\exp(g_\zeta(z)) = \frac{z-a}{z-\zeta}$. Indeed, the Möbius transformation $(z-a)/(z-\zeta)$ maps the point a to 0, ζ to ∞, and the line segment $[a, \zeta]$ to a half line in the complex plane, starting at 0. We may also choose g_ζ so that $|\operatorname{Im} g_\zeta(z)| \le \pi$ for $z \in \Omega$. Then, for this function g_ζ, the integral

$$\iint_{z \in B(\zeta,\delta)\cap\Omega} |g_\zeta(z)|^p \, dv(z) < +\infty \text{ for every } p < \infty, \qquad (3.4.1.1)$$

while $\lim_{z\in\Omega,\, z\to\zeta} g_\zeta(z) = \infty$. To prove (3.4.1.1), it suffices to notice that

$$(\log x)^p \le (k!)^{p/k} x^{p/k} \quad \text{for every } x > 1,\ p \ge 1 \text{ and } k \in \mathbb{N},$$

and that if $\log w = \log|w| + i\theta$, then

$$|\log w|^p = [(\log|w|)^2 + \theta^2]^{p/2}, \text{ for } \theta \in \mathbb{R}.$$

Indeed, since $g_\zeta(z) = \log\left|\frac{z-a}{z-\zeta}\right| + i\theta$ (with $|\theta| \le \pi$), it follows that, for $z \in \Omega$ which are sufficiently close to the point ζ,

$$|g_\zeta(z)|^p = \left[\left(\log\left|\frac{z-a}{z-\zeta}\right|\right)^2 + \theta^2\right]^{p/2} \preceq (k!)^{p/k}\left|\frac{z-a}{z-\zeta}\right|^{p/k} \preceq (k!)^{p/k}\left|\frac{1}{z-\zeta}\right|^{p/k}.$$

Then (3.4.1.1) follows by an appropriate choice of $k \in \mathbb{N}$. Finally (3.4.1.1) implies that $g_\zeta \in \bigcap_{p<\infty} \mathcal{O}L^p(\Omega)$, while $g_\zeta \notin \mathcal{O}L^\infty(\Omega)$.

(ii) With notation as in the previous example, and for $1 < q < \infty$, let us consider the function

$$h_{q,\zeta}(z) = \exp\left[\frac{2}{q} g_\zeta(z)\right], \ z \in \Omega.$$

Then $h_{q,\zeta} \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$, while $h_{q,\zeta} \notin \mathcal{O}L^q(\Omega)$.

(iii) For $\alpha \in \mathbb{R}$, $\alpha \ge 1$, let us consider the domain

$$\Omega_\alpha = \{z = x + iy \in \mathbb{C} : 0 < x < 1 \ and \ 0 < y < x^\alpha\}.$$

Then

$$\frac{1}{z} \in \bigcap_{p<\alpha+1} \mathcal{O}L^p(\Omega_\alpha) \text{ and } \frac{1}{z} \notin \mathcal{O}L^{\alpha+1}(\Omega_\alpha),$$

$$\log z \in \bigcap_{p<\infty} \mathcal{O}L^p(\Omega_\alpha) \text{ and } \log z \notin \mathcal{O}L^\infty(\Omega_\alpha), \text{ and}$$

$$\frac{1}{z^{(\alpha+1)/q}} \in \bigcap_{p<q} \mathcal{O}L^p(\Omega_\alpha) \text{ and } \frac{1}{z^{(\alpha+1)/q}} \notin \mathcal{O}L^q(\Omega_\alpha) \text{ for } q \in \mathbb{R},\ q>0.$$

(iv) Let $\Omega = \{z = x+iy \in \mathbb{C} : 0<x<1 \ \mathit{and}\ 0<y<\exp(-1/x^2)\}$. Then

$$\frac{1}{z^N} \in \bigcap_{p<\infty} \mathcal{O}L^p(\Omega) \text{ and } \frac{1}{z^N} \notin \mathcal{O}L^\infty(\Omega), \text{ for every } N \in \mathbb{N}.$$

Theorem 3.4.1. (i) Let $\Omega \subset \mathbb{C}$ be an arbitrary bounded open set. Then the set of the functions $g \in \bigcap_{p<2} \mathcal{O}L^p(\Omega)$ which are totally unbounded in Ω is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<2} \mathcal{O}L^p(\Omega)$.

(ii) Suppose that $\Omega \subset \mathbb{C}$ is a bounded open set such that for every point $\zeta \in \partial\Omega$, the connected component C_ζ of $\mathbb{C} - \overline{\Omega}$ which contains ζ, contains at least one more point, i.e., $C_\zeta - \{\zeta\} \neq \varnothing$. Then, for each fixed q with $1<q<\infty$, the set of the functions $g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$ which are totally unbounded in Ω is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

(iii) Suppose that $\Omega \subset \mathbb{C}$ is a bounded open set with C^1 boundary and $1<q\leq\infty$. Then the set

$$\Big\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : g \text{ is totally unbounded in } \Omega$$
$$\text{and } g \notin \mathcal{O}L^q(B(\zeta,\varepsilon) \cap \Omega),\ \forall\, \zeta \in \partial\Omega \text{ and } \forall\, \varepsilon>0 \Big\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Proof. Having in mind the example (i) of Section 3.4.1, we easily obtain part (i), from Theorem 3.2.1., applied with the functions $\{f_\zeta : \zeta \in \partial\Omega\}$ where $f_\zeta : \Omega \to \mathbb{C}$, $f_\zeta(z) = 1/(z-\zeta)$, $z \in \Omega$. Part (iii) follows from Theorem 3.3.3, applied with the functions g_ζ of example (i) of Section 3.4.1 in the case $q=\infty$, and the functions $h_{q,\zeta}$ of example (ii) of Section 3.4.1 in the case $1<q<\infty$. It remains to prove part (ii). For this purpose let us take a point $a \in C_\zeta$, $a \neq \zeta$, and a compact curve K in C_ζ joining the points ζ and a. Then the Möbius transformation $(z-a)/(z-\zeta)$ maps the point a to 0, ζ to ∞, and the curve K to a connected set Γ joining the points 0 and ∞. Then in the open set $\mathbb{C} - \Gamma$, there is a holomorphic branch of the logarithm, and, therefore, there is a function $\varphi_\zeta(z)$, holomorphic in $z \in \Omega$, such that $\exp[\varphi_\zeta(z)] = (z-a)/(z-\zeta)$.

Also the function

$$\psi_\zeta(z) := \exp\left[\frac{2}{q}\varphi_\zeta(z)\right]$$

is holomorphic in Ω and

$$\left|\psi_\zeta(z)\right|^p = \exp\left[\frac{2p}{q}\operatorname{Re}\varphi_\zeta(z)\right] = \left|\frac{z-a}{z-\zeta}\right|^{2p/q}.$$

Therefore $\psi_\zeta \in \bigcap_{p<q} \mathcal{O}L^p(\Omega)$, and, since $\lim_{z\to\zeta,\, z\in\Omega} \psi_\zeta(z) = \infty$, part (ii) follows from Theorem 3.2.1. $\square$

3.4.2. The case of the unit ball of $\mathbb{C}^n$

Let us consider the unit ball $\Omega = \{z \in \mathbb{C}^n : |z| < 1\}$. For fixed $\zeta \in \partial\Omega$, we consider the function

$$f_\zeta(z) = \frac{1}{1-\langle z,\zeta\rangle} = \frac{1}{1-\sum_{j=1}^n \bar{\zeta}_j z_j},\ z \in \Omega.$$

Then

$$f_\zeta \in \bigcap_{p<n+1} \mathcal{O}L^p(\Omega) \text{ and } f_\zeta \notin \mathcal{O}L^{n+1}(\Omega).$$

Indeed, if $p < n+1$ then the integral

$$\int_\Omega \frac{dv(z)}{\left|1-\langle z,\eta\rangle\right|^p},$$

as a function of η, remains bounded for $\eta \in \Omega$ (see [22], Proposition 2.3.2), and, therefore, letting $\eta \to \zeta$,

$$\int_\Omega \frac{dv(z)}{\left|1-\langle z,\zeta\rangle\right|^p} = \int_\Omega \lim_{\eta\to\zeta} \frac{dv(z)}{\left|1-\langle z,\eta\rangle\right|^p} \le \liminf_{\eta\to\zeta} \int_\Omega \frac{dv(z)}{\left|1-\langle z,\eta\rangle\right|^p} < +\infty.$$

Next we show that

$$\int_\Omega \frac{dv(z)}{\left|1-\langle z,\zeta\rangle\right|^{n+1}} = +\infty. \qquad (3.4.2.1)$$

Indeed, for $r<1$ (sufficiently close to 1),

$$\int_\Omega \frac{dv(z)}{\left|1-\langle z,r\zeta\rangle\right|^{n+1}} \ge \lambda \log\frac{1}{1-r^2},$$

where λ is a positive constant independent of r (see [22], Proposition 2.3.2).

Since

$$\int_\Omega \frac{dv(z)}{\left|1-\langle z,r\zeta\rangle\right|^{n+1}} = \int_\Omega \frac{dv(z)}{\left|1-\langle rz,\zeta\rangle\right|^{n+1}} = \frac{1}{r^{2n}} \int_{r\Omega} \frac{dv(z)}{\left|1-\langle z,\zeta\rangle\right|^{n+1}}$$

(where $r\Omega = \{z \in \mathbb{C}^n : |z| < r\}$), it follows that

$$\int_{r\Omega} \frac{dv(z)}{|1-\langle z,\zeta\rangle|^{n+1}} \geq \lambda r^{2n} \log \frac{1}{1-r^2}.$$

Letting $r \to 1^-$, we obtain (3.4.2.1).

Observing that $\operatorname{Re}(1-\langle z,\zeta\rangle) > 0$, for $z \in \Omega$, we see that $\operatorname{Re} f_\zeta(z) > 0$, and therefore $\log f_\zeta(z)$ is defined and holomorphic for $z \in \Omega$, where $\log$ is the principal branch of the logarithm with $|\arg| \leq \pi$. Also $|\operatorname{Im}[\log f_\zeta(z)]| \leq \pi/2$. It follows, as in example (i) of Section 3.4.1, that

$$\log f_\zeta \in \bigcap_{p<\infty} \mathcal{O}L^p(\Omega), \text{ while } \log f_\zeta \notin \mathcal{O}L^\infty(\Omega).$$

Also the function $\left(f_\zeta\right)^{(n+1)/q} = \exp\left[\frac{n+1}{q} \log f_\zeta\right]$ satisfies

$$\left(f_\zeta\right)^{(n+1)/q} \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) \text{ and } \left(f_\zeta\right)^{(n+1)/q} \notin \mathcal{O}L^q(\Omega) \text{ for } q \in \mathbb{R},\ q > 0.$$

Theorem 3.4.2. Let $1 < q \leq \infty$. If Ω is the unit ball of $\mathbb{C}^n$, then the set

$$\Big\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : \textit{ g is totally unbounded in } \Omega$$
$$\textit{and } g \notin \mathcal{O}L^q(B(\zeta,\varepsilon) \cap \Omega),\ \forall \zeta \in \partial\Omega \textit{ and } \forall \varepsilon > 0 \Big\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Proof It suffices to apply Theorem 3.3.3, with appropriate choices from the set of the functions which were constructed in Section 3.4.2. □

3.4.3. The case of convex sets.

(i) Let $\Omega \subset \mathbb{C}^n$ be a bounded open and convex set with C^1 boundary and let us fix a point $\zeta \in \partial\Omega$. By the convexity of Ω, there exist real numbers $\alpha_j = \alpha_j(\zeta)$, $\beta_j = \beta_j(\zeta)$, $j = 1,2,\ldots,n$, such that $\sum\left[|\alpha_j|^2 + |\beta_j|^2\right] \neq 0$ and

$$\sum_{j=1}^{n} \left\{ \alpha_j [x_j(z) - x_j(\zeta)] + \beta_j [y_j(z) - y_j(\zeta)] \right\} > 0 \text{ for every } z \in \Omega,$$

where $x_j(z) = \operatorname{Re} z_j$, $y_j(z) = \operatorname{Im} z_j$, $x_j(\zeta) = \operatorname{Re} \zeta_j$, $y_j(\zeta) = \operatorname{Im} \zeta_j$. Setting $c_j := \alpha_j - i\beta_j$, we obtain

$$\mathrm{Re}\left[\sum_{j=1}^{n} c_j(z_j-\zeta_j)\right]>0 \text{ for every } z\in\Omega .$$

Then the conclusions of example (i) hold for the function f_ζ where

$$f_\zeta(z)=\frac{1}{\sum_{j=1}^{n} c_j(z_j-\zeta_j)},\ z\in\Omega . \qquad (3.4.3.1)$$

satisfies

$$f_\zeta \in \bigcap_{p<2}\mathcal{O}L^p(\Omega), \text{ while } f_\zeta \notin \mathcal{O}L^\infty(\Omega), \text{ and} \qquad (3.4.3.2)$$

$$\log f_\zeta \in \bigcap_{p<\infty}\mathcal{O}L^p(\Omega), \text{ while } \log f_\zeta \notin \mathcal{O}L^\infty(\Omega). \qquad (3.4.3.3)$$

To prove (3.4.3.2), we will show that for $p<2$,

$$\int_{B(\zeta,\delta)\cap\Omega} \left|f_\zeta(z)\right|^p dv(z)<+\infty, \text{ for «small» } \delta>0 . \qquad (3.4.3.4)$$

Assuming, without loss of generality, that $c_1\neq 0$, let us consider the $\mathbb{C}-$affine transformation

$$w_1(z)=\sum_{j=1}^{n} c_j(\zeta_j-z_j),\ w_2(z)=\zeta_2-z_2,\ \ldots,\ w_n(z)=\zeta_n-z_n .$$

Using this transformation we see that (3.4.3.2) follows from the fact that

$$\int_{|w|<\tilde{\delta}} \frac{dv(w)}{|w_1|^p}<+\infty \text{ (for } \tilde{\delta}>0\text{).}$$

To justify (3.4.3.3), let us recall that since

$$\mathrm{Re}\left[\sum_{j=1}^{n} c_j(z_j-\zeta_j)\right]>0 \text{ for every } z\in\Omega ,$$

the function $\log f_\zeta$ is well defined and holomorphic in Ω.

Then, using (3.4.3.2) as in example (i) of Section 3.4.1, we see that, for «small» $\delta>0$,

$$\int_{B(\zeta,\delta)\cap\Omega} \left|\log f_\zeta(z)\right|^p dv(z)<+\infty, \text{ for every } p<\infty ,$$

and this implies (3.4.3.4).

We point out that in general the conclusion $f_\zeta \in \bigcap_{p<2}\mathcal{O}L^p(\Omega)$ cannot be improved in the sense that in some cases

$$\int_{B(\zeta,\delta)\cap\Omega} \left|f_\zeta(z)\right|^2 dv(z)=+\infty$$

(see the example (ii) below).

(ii) Let us consider the convex domain

$$D=\{z=(z_1,\dots,z_n)\in\mathbb{C}^n : |z|<1 \textit{ and } \operatorname{Re} z_1>0\}$$

and, as local defining function for D-see Remark 2.4.11- near its boundary point $\zeta=0$ $(0\in\partial D)$, $\rho(z)=-(z_1+\bar{z}_1)/2$. Then the function (3.4.3.1) becomes $f_\zeta(z)=1/z_1$. In this case

$$\int\limits_{B(\zeta,\delta)\cap D}|f_\zeta(z)|^2 dv(z)=\int\limits_{B(0,\delta)\cap D}\frac{1}{|z_1|^2}dv(z)=+\infty \text{, for every } \delta>0.$$

A similar computation can be done for every point ζ in the part of the boundary of ∂D where $\operatorname{Re}\zeta=0$ (and $|\zeta|<1$). Of course at the points $\zeta\in\partial D$ where $\operatorname{Re}\zeta>0$, the corresponding function f_ζ satisfies $f_\zeta\in\bigcap\limits_{p<n+1}\mathcal{O}L^p(D)$ and $f_\zeta\notin\mathcal{O}L^{n+1}(D)$, as we proved in Section 3.4.2.

(iii) Similarly to the previous example, if

$$R=\{z=(z_1,\dots,z_n)\in\mathbb{C}^n : 0<\operatorname{Re} z_j<1 \textit{ and } 0<\operatorname{Im} z_j<1,\ j=1,2,\dots,n\},$$

then for every point $\zeta\in\partial R$ (where ∂R is smooth), the function f_ζ satisfies

$$f_\zeta\in\bigcap_{p<2}\mathcal{O}L^p(R) \text{ and } f_\zeta\notin\mathcal{O}L^2(R).$$

Similar conclusions hold for «most» points in the boundary of the polydisk

$$P=\{z=(z_1,\dots,z_n)\in\mathbb{C}^n : |z_j|<1,\ j=1,2,\dots,n\}.$$

Theorem 3.4.3. Let $\Omega\subset\mathbb{C}^n$ be any bounded open and convex set and $1<q\le\infty$. Then the set of the functions g in $\bigcap\limits_{p<q}\mathcal{O}L^p(\Omega)$ such that g is totally unbounded in Ω, is dense and $\mathcal{G}_\delta$ in the space $\bigcap\limits_{p<q}\mathcal{O}L^p(\Omega)$.

Proof. It follows from Theorem 3.2.1 applied with the functions $\log f_\zeta$ of the above example (i). □

3.5. The case of strictly pseudoconvex domains

In this section we will show that some functions which are defined in terms of Henkin's support function belong to certain Bergman spaces. We describe the Henkin's support function $\Phi(z,\zeta)$ – as constructed in [10] – in Section 2.9.

First we will prove the following proposition. We use a set of coordinates – *the Levi coordinates* – which are appropriate when we are dealing with integrals involving the function $\Phi(z,\zeta)$ (for more details see [6], [10], [21]). As a matter of fact we will use a slight modification of the Levi coordinates.

Proposition 3.5.1. If, in addition, $\partial\Omega$ is C^1, then, for each fixed $\zeta \in \partial\Omega$ and for every $\delta > 0$,

$$\int_{z\in B(\zeta,\delta)\cap\Omega} \frac{dv(z)}{|\Phi(z,\zeta)|^p} < +\infty \text{ when } p < n+1, \text{ and } \int_{z\in B(\zeta,\delta)\cap\Omega} \frac{dv(z)}{|\Phi(z,\zeta)|^{2n}} = +\infty .$$

Therefore $\dfrac{1}{\Phi(\cdot,\zeta)} \in \bigcap\limits_{p<n+1} \mathcal{O}L^p(\Omega)$ and $\dfrac{1}{\Phi(\cdot,\zeta)} \notin \mathcal{O}L^{2n}(\Omega)$.

Furthermore, the functions $\dfrac{1}{\Phi(z,\zeta)}$ are C^1 in ζ.

Proof. Since we assume $\partial\Omega$ to be C^1, $\nabla\rho \neq 0$ at the points of $\partial\Omega$. Having fixed $\zeta \in \partial\Omega$, we consider a coordinate system

$$t = (t_1, t_2, t_3, \ldots, t_{2n}) = (t_1(z), t_2(z), t_3(z), \ldots, t_{2n}(z)),$$

of real C^1-functions, for points $z \in \mathbb{C}^n = \mathbb{R}^{2n}$, which are sufficiently close to the point ζ, as follows: We set

$$t_1(z) = -\rho(z) \text{ and } t_2(z) = \operatorname{Im} Q(z,\zeta).$$

Then $d_z Q(z,\zeta)\big|_{z=\zeta} = -2\sum\limits_{j=1}^{n} \dfrac{\partial\rho(\zeta)}{\partial\zeta_j} dz_j \Bigg|_{z=\zeta} = -2\partial\rho(\zeta)$ and, therefore,

$$d_z t_2(z)\big|_{z=\zeta} = d_z[\operatorname{Im} Q(z,\zeta)]\big|_{z=\zeta} = i[\partial\rho(\zeta) - \bar\partial\rho(\zeta)].$$

On the other hand,

$$d_z t_1(z)\big|_{z=\zeta} = d_z[-\rho(z)]\big|_{z=\zeta} = -[\partial\rho(\zeta) + \bar\partial\rho(\zeta)].$$

It follows that

$$\left(d_z t_1(z)\big|_{z=\zeta}\right) \wedge \left(d_z t_2(z)\big|_{z=\zeta}\right) = -2i\partial\rho(\zeta) \wedge \bar\partial\rho(\zeta) \neq 0.$$

Now the existence of C^1-functions $t_3(z), \ldots, t_{2n}(z)$ such that the mapping

$$z \to (t_1(z), t_2(z), t_3(z), \ldots, t_{2n}(z))$$

is a C^1-diffeomorphism, from an open neighbourhood of the point ζ to an open neighbourhood of $0 \in \mathbb{C}^n = \mathbb{R}^{2n}$ (with $t(\zeta) = 0$), follows from the inverse function theorem-see Theorem 2.1.10. Also let us point out that, for z sufficiently close to ζ, $z \in \Omega$ if and only if $t_1 = -\rho(z) > 0$.

We will show that, for every $\delta > 0$,

$$\int_{z\in B(\zeta,\delta)\cap\Omega} \frac{dv(z)}{|\Phi(z,\zeta)|^p} < +\infty \quad \text{for } p < n+1. \tag{3.5.1.1}$$

For points $z \in \Omega$ which are sufficiently close to ζ,

$$|\Phi(z,\zeta)| \approx |Q(z,\zeta)| \approx |\operatorname{Re} Q(z,\zeta)| + |\operatorname{Im} Q(z,\zeta)| \geq -\rho(z) + \beta|\zeta - z|^2 + |\operatorname{Im} Q(z,\zeta)|$$

and

$$|\zeta - z|^2 \approx t_1^2 + t_2^2 + t_3^2 + \cdots + t_{2n}^2 .$$

(When we write $\mathrm{A} \approx \mathrm{B}$, we mean that $\lambda\mathrm{B} \leq \mathrm{A} \leq \mu\mathrm{B}$, for some positive constants λ and μ which are independent of z.)

Therefore (for $z\in\Omega$ and sufficiently close to ζ)

$$|\Phi(z,\zeta)|\succeq t_1+t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2+|t_2|.$$

(When we write $\mathrm{A}\succeq\mathrm{B}$, we mean that $\mathrm{A}\geq\lambda\mathrm{B}$, for some positive constant λ which is independent of z.) Therefore (3.5.1.1) follows from

$$\int_{t_1>0}\frac{dt}{(t_1+|t_2|+t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2)^p}<+\infty$$

or equivalently from

$$\int_{t_1>0}\frac{dt}{(t_1+|t_2|+t_3^2+\cdots+t_{2n}^2)^p}<+\infty \quad (p<n+1).$$

(In the above integrals $dt=dt_1dt_2\cdots dt_{2n}$ and t is restricted in a «small» neighbourhood of $0\in\mathbb{C}^n=\mathbb{R}^{2n}$, i.e., $|t|$ is «small».)

We will also show that, for every $\delta>0$,

$$\int_{z\in B(\zeta,\delta)\cap\Omega}\frac{dv(z)}{|\Phi(z,\zeta)|^{2n}}=+\infty. \tag{3.5.1.2}$$

This time we will use the fact that, for points $z\in\Omega$ which are sufficiently close to ζ,

$$|\Phi(z,\zeta)|\approx|Q(z,\zeta)|\preceq|\zeta-z|\approx(t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2)^{1/2}.$$

Therefore (3.5.1.2) follows from

$$\int_{t_1>0}\frac{dv(z)}{(t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2)^n}=+\infty.$$

This completes the proof of the proposition. □

Theorem 3.5.2. Let $\Omega\subset\mathbb{C}^n$ be a strictly pseudoconvex open set with C^2–boundary, and $1<q<\infty$. Then the following hold:

(i) For every point $\zeta\in\partial\Omega$, there exists a function f_ζ such that

$$f_\zeta\in\bigcap_{p<\infty}\mathcal{O}L^p(\Omega) \text{ and } \lim_{\substack{z\to\zeta\\ z\in\Omega}}f_\zeta(z)=\infty.$$

(ii) For every point $\zeta\in\partial\Omega$, there exists a function h_ζ such that

$$h_\zeta\in\bigcap_{p<q}\mathcal{O}L^p(\Omega) \text{ and } h_\zeta\notin\mathcal{O}L^{2nq/(n+1)}(B(\zeta,\delta)\cap\Omega) \text{ for every } \delta>0\text{, and}$$

$$\lim_{\substack{z\to\zeta\\ z\in\Omega}}h_\zeta(z)=\infty.$$

(iii) The set

$$\left\{g\in\bigcap_{p<\infty}\mathcal{O}L^p(\Omega):\ g \textit{ is totally unbounded in } \Omega\right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<\infty}\mathcal{O}L^p(\Omega)$.

(iv) The set

$$\left\{ g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega) : \ g \ is \ totally \ unbounded \ in \ \Omega \right.$$

$$\left. and \ g \notin \mathcal{O}L^{2nq/(n+1)}(B(\zeta,\delta)\cap\Omega)\,, \forall\, \zeta \in \partial\Omega \ and \ \forall\, \delta > 0 \right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$.

Proof. Let ρ be a C^2 strictly plurisubharmonic defining function of Ω, defined in an open neighbourhood of $\overline{\Omega}$. Let us also fix a point $\zeta \in \partial\Omega$. Then, as it follows from Taylor's theorem and the strict plurisubharmonicity of ρ (see Proposition 2.8.9 and for more details [21] Proposition 2.16 page 60), the Levi polynomial of ρ

$$F(z,\zeta) = -\left[2\sum_{j=1}^{n} \frac{\partial\rho(\zeta)}{\partial\zeta_j}(z_j-\zeta_j) + \sum_{1\le j,k\le n} \frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\zeta_k}(z_j-\zeta_j)(z_k-\zeta_k) \right]$$

satisfies the inequality

$$\operatorname{Re} F(z,\zeta) \ge \rho(\zeta) - \rho(z) + \beta|\zeta - z|^2 \ \text{ for } z \in \mathbb{C}^n \text{ with } |\zeta - z| < \varepsilon,$$

for some «small» positive constants ε and β. In particular,

$$\operatorname{Re} F(z,\zeta) > 0 \ \text{ for } z \in B(\zeta,\varepsilon)\cap\overline{\Omega} - \{\zeta\}.$$

It follows that the function $\log[1/F(z,\zeta)]$ is defined and holomorphic for $z \in B(\zeta,\varepsilon)\cap\Omega$, and that $\lim_{z\in\Omega,\, z\to\zeta} \log[1/F(z,\zeta)] = \infty$. (Here $\log$ is the principal branch of the logarithm with $|\arg| \le \pi$.) Also we can prove, as in the proof of the Proposition 3.5.1., that if $q < n+1$,

$$\int_{z\in B(\zeta,\delta)\cap\Omega} \frac{dv(z)}{|F(z,\zeta)|^q} < +\infty \ \text{ for every } \delta > 0. \tag{3.5.2.1}$$

Then, using (3.5.2.1) (with $q = 1$, for example) as in example (i) of Section 3.4.1., we obtain

$$\int_{B(\zeta,2\varepsilon/3)\cap\Omega} \left| \log\left[\frac{1}{F(z,\zeta)} \right] \right|^p dv(z) < +\infty, \text{ for every } p < \infty. \tag{3.5.2.2}$$

Next we consider a C^∞ – function $\chi : \mathbb{C}^n \to \mathbb{R}$, $0 \le \chi(z) \le 1$, with compact support contained in $B(\zeta,2\varepsilon/3)$, and such that $\chi(z) = 1$ when $z \in B(\zeta,\varepsilon/3)$. Now the function

$$\chi(z)\log\left[\frac{1}{F(z,\zeta)} \right]$$

is extended to a C^∞ – function in Ω, by defining it to be 0 in $\Omega - B(\zeta,2\varepsilon/3)$. Then the $(0,1)$ – form

$$u(z) := \overline{\partial}\left\{ \chi(z)\log\left[\frac{1}{F(z,\zeta)} \right] \right\}$$

is defined and is C^∞ in a open neighbourhood $\overline{\Omega}$, it is zero for $z \in B(\zeta,\varepsilon/3)\cap\Omega$, and, in particular, it has bounded coefficients in Ω. In fact $u(z)$ extends to a C^∞

$(0,1)-$form for z in an open neighbourhood of $\overline{\Omega}$, since the function $\log\left[\dfrac{1}{F(z,\zeta)}\right]$ is holomorphic in an open neighbourhood of the compact set $\overline{[B(\zeta,2\varepsilon/3)-B(\zeta,\varepsilon/3)]\cap\Omega}$. It follows that there exists a bounded $C^{\infty}-$function $\psi:\Omega\to\mathbb{C}$ which solves the equation $\bar{\partial}\psi=u$ in Ω. (see Theorem 2.8.5. and for more details [12] Theorem 16.3.4). Then the function

$$f_{\zeta}(z):=\chi(z)\log\left[\frac{1}{F(z,\zeta)}\right]-\psi(z)$$

satisfies the requirements of (i)) (as it follows from (3.5.2.2).

A function h_{ζ} which satisfies the requirements of (ii) is

$$h_{\zeta}(z)=\exp\left[\frac{n+1}{q}f_{\zeta}(z)\right]=\exp\left\{\chi(z)\log\left[\frac{1}{F(z,\zeta)}\right]^{(n+1)/q}-\psi(z)\right\}.$$

Indeed, we have

$$\int\limits_{z\in B(\zeta,\delta)\cap\Omega}\frac{dv(z)}{|F(z,\zeta)|^{2n}}=+\infty \text{ (for every } \delta>0\text{)}$$

(this is proved in the same manner as the analogous result of Proposition 3.5.1) which implies that

$$\int\limits_{z\in B(\zeta,\delta)\cap\Omega}\frac{dv(z)}{|h_{\zeta}(z)|^{2nq/(n+1)}}=+\infty .$$

Notice that the behaviour of the above integral is not affected by the functions χ or ψ, since $\chi\equiv 1$ near ζ and ψ is bounded in Ω (so that $\exp(-\psi)$ is both bounded and bounded away from zero in Ω).

Finally assertions (iii) and (iv) follow from (i) and (ii), in combination with Theorems 3.2.1 and 3.3.1 (see also the Remark 3.3.2). □

Remark 3.5.3. It follows from the above theorem, in combination with Theorem 2.1.14. that the set of the functions h in the space $\bigcap\limits_{p<q}\mathcal{O}L^{p}(\Omega)$ which are singular at every boundary point of $\partial\Omega$ is dense and $\mathcal{G}_{\delta}$ in this space, for $1<q\leq\infty$. (Similar conclusions are reached also in the case of the convex domains, following Theorem 3.4.3. and certain – more general – domains in $\mathbb{C}$, following Theorem 3.4.1.)

3.6. Extensions of results in the case $0<p<1$

Let $\Omega\subset\mathbb{C}^{n}$ be a bounded open set. Recall that if $0<p<1$, we can define again the space $\mathcal{O}L^{p}(\Omega)$ as the set of holomorphic functions $f:\Omega\to\mathbb{C}$ such that $\int\limits_{\Omega}|f(z)|^{p}\,dv(z)<+\infty$, and that with the metric

$$d_p(f,g) := \int_{\Omega} |f(z)-g(z)|^p \, dv(z), \text{ for } f,g \in \mathcal{O}L^p(\Omega),$$

$\mathcal{O}L^p(\Omega)$ becomes a complete metric space. (This follows from the fact that convergence in the space $L^p(\Omega)$ implies uniform convergence on compact subsets of Ω, as we justify below.)

For a fixed q, with $0 < q \leq 1$, we may also define the spaces

$$\bigcap_{p<q} \mathcal{O}L^p(\Omega)$$

endowed with the metric

$$\tilde{d}(f,g) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_{p_j}(f,g)}{1+d_{p_j}(f,g)}, \; f,g \in \bigcap_{p<q} \mathcal{O}L^p(\Omega),$$

where p_j is a sequence with $0 < p_1 < p_2 < \cdots < p_j < \cdots < q$ and $p_j \to q$ (as $j \to \infty$). Then $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$ becomes a complete metric space, its topology being independent of the choice of the sequence p_j. In fact, a sequence f_k converges to f, in the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, if and only if $d_p(f_k, f) \to 0$ for every $p < q$. In particular Baire's theorem -see Theorem 2.1.4.- holds in $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$. Moreover we point out that the space $\bigcap_{p<q} \mathcal{O}L^p(\Omega)$, with the above topology, is also a topological vector space.

Let us recall also that if $P(a,r)$ is a polydisk, $P(a,r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j, \; j = 1,2,...,n\}$, and $f \in \mathcal{O}(\overline{P(a,r)})$, then – by the submean value property for the function $|f|^p$ (see Theorem 2.5.20 and for more details see [21]) we have

$$|f(a)|^p \leq \frac{1}{vol(P(a,r))} \int_{P(a,r)} |f(z)|^p \, dv(z) \;\; (p > 0).$$

Thus if $f \in \mathcal{O}L^p(\Omega)$ and K is a compact subset of Ω, then choosing $\delta > 0$, sufficiently small – depending on K, such that

$$P_a^{\delta} := \{z \in \mathbb{C}^n : |z_j - a_j| < \delta, \; j = 1,2,...,n\} \subset\subset \Omega, \text{ for every } a \in K,$$

we obtain

$$|f(a)|^p \leq \frac{1}{vol(P_a^{\delta})} \int_{P_a^{\delta}} |f(z)|^p \, dv(z) \leq \frac{1}{vol(P_a^{\delta})} \int_{\Omega} |f(z)|^p \, dv(z), \text{ for every } a \in K.$$

This gives the well-known inequality

$$\sup_{a \in K} |f(a)|^p \leq \frac{1}{vol(P_0^{\delta})} \int_{\Omega} |f(z)|^p \, dv(z).$$

In particular we see that convergence in the space $\mathcal{O}L^p(\Omega)$ implies uniform convergence on compact subsets of Ω.

The following conclusions can be reached for the case «$0 < p < 1$» in the same manner as in the case «$p \geq 1$».

Conclusions. Theorems 3.2.1., 3.3.1., 3.3.3., 3.4.1., 3.4.2., 3.4.3., 3.5.2., and Remark 3.5.3., hold also in the case $0 < q \leq 1$, and Remark 3.2.2. (iii) holds for the case $0 < p < 1$, too.

3.7. The spaces $\mathrm{A}^s(\Omega)$

As we pointed out in Section 2.2, a totally unbounded holomorphic function in an open set Ω, is singular at every point of $\partial\Omega$. On the other hand it is well-known that the converse of this is far from being correct. In fact, under some assumptions on the set Ω, there are holomorphic functions in Ω which are C^∞ up to the boundary of Ω and at the same time they are singular at every point of $\partial\Omega$. For deep results in this direction we refer to [14] and the bibliography given there. In this section we will use Theorem 2.1.14. in order to give a simple proof of the fact that in some pseudoconvex open sets there exist functions in $\mathrm{A}^s(\Omega)$, $s \in \{0,1,2,...\} \cup \{\infty\}$, which do not extend holomorphically beyond any boundary point of Ω. In fact we show, at the same time, that such functions form a dense and $\mathcal{G}_\delta$ set in the space $\mathrm{A}^s(\Omega)$ (in the natural topology of this space). To make this precise, we consider, for a bounded open set Ω in $\mathbb{C}^n$ and $s \in \{0,1,2,...\}$, the set $\mathrm{A}^s(\Omega)$ of all holomorphic functions f in Ω, whose derivatives

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$$

extend continuously to $\overline{\Omega}$, for every mult-index $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq s$. The topology in $\mathrm{A}^s(\Omega)$ is defined by the norm

$$|||f|||_s = \sup\left\{ \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| : \ z \in \overline{\Omega}, |\alpha| \leq s \right\}, \ f \in \mathrm{A}^s(\Omega),$$

and with this norm, $\mathrm{A}^s(\Omega)$ is complete.

Similaly $\mathrm{A}^\infty(\Omega)$ is the set of holomorphic functions f in Ω, whose derivatives $\partial^{|\alpha|} f / \partial z^\alpha$ extend continuously to $\overline{\Omega}$, for every multi-index $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{N}^n$. The topology in $\mathrm{A}^\infty(\Omega)$ is defined by the metric

$$\mu(f,g) = \sum_{N=0}^{\infty} \frac{1}{2^N} \frac{|||f-g|||_N}{1+|||f-g|||_N}, \ f,g \in \mathrm{A}^\infty(\Omega),$$

and, with this metric, $\mathrm{A}^{\infty}(\Omega)$ is complete. Furthermore, with the corresponding topology, $\mathrm{A}^{\infty}(\Omega)$ becomes a topological vector space. Thus, in particular, if $f_k, f \in \mathrm{A}^{\infty}(\Omega)$ with $\mu(f_k, f) \to 0$ ($k \to \infty$), and $\lambda_k, \lambda \in \mathbb{C}$ with $\lambda_k \to \lambda$, then $\mu(\lambda_k f_k, \lambda f) \to 0$.

The following theorem follows easily from Theorem 2.1.14. See also [2], [8] and [14] for related results.

Theorem 3.7.1. Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex open set such that its closure $\overline{\Omega}$ has a neighbourhood basis of pseudoconvex open sets, and $\operatorname{int}(\overline{\Omega}) = \Omega$.

If $s \in \{0,1,2,...\} \cup \{\infty\}$, then the set $\Xi^s(\Omega)$ of the functions in $\mathrm{A}^s(\Omega)$ which are not extendable, as holomorphic functions, beyond any point of the boundary $\partial\Omega$, is dense and $\mathcal{G}_\delta$ in the space $\mathrm{A}^s(\Omega)$.

In particular the conclusion holds if Ω is strictly pseudoconvex open set (not necessarily with smooth boundary) and $\operatorname{int}(\overline{\Omega}) = \Omega$.

Proof. We will apply Theorem 2.1.14. with $\mathcal{V} = \mathrm{A}^s(\Omega)$. For this purpose let us consider a pair (B,b) of open balls with $b \subset\subset B \cap \Omega \neq B$. We claim that $B \cap (\mathbb{C}^n - \overline{\Omega}) \neq \varnothing$. For if $B \cap (\mathbb{C}^n - \overline{\Omega}) = \varnothing$ then $B \subseteq \overline{\Omega}$ which would imply that $B \subseteq \operatorname{int}(\overline{\Omega})$, i.e., $B \subseteq \Omega$ (since we assume $\operatorname{int}(\overline{\Omega}) = \Omega$), and this contradicts the fact that $B \cap \Omega \neq B$. Let $\zeta \in B \cap (\mathbb{C}^n - \overline{\Omega})$. Since we assume that $\overline{\Omega}$ has a neighborhood basis of pseudoconvex open sets, there exists a pseudoconvex open set G such that $G \supset \overline{\Omega}$ and $\zeta \notin G$. Then $B \cap G \neq \varnothing$, $B \cap (\mathbb{C}^n - G) \neq \varnothing$, and B is connected, and therefore $B \cap \partial G \neq \varnothing$. Let us consider a point $\sigma \in B \cap \partial G$ and a sequence z_k in $B \cap G$ which converges to σ. Since G is pseudoconvex, there exists a function f, holomorphic in G, such that $\sup_k |f(z_k)| = \infty$ (see [15]). Then $f \in \mathcal{V} = \mathrm{A}^s(\Omega)$ and the restriction $f|_b$, of f to b, has no bounded holomorphic extension to B.

Therefore, from Theorem 2.1.14, the set $\Xi^s(\Omega)$ is dense and $\mathcal{G}_\delta$ in the space $\mathcal{V} = \mathrm{A}^s(\Omega)$.

The last conclusion of the theorem follows from the well-known fact that the closure of a strictly pseudoconvex open set has a neighbourhood basis of pseudoconvex open sets (for more details see [10]). □

4 Hardy type spaces

4.1. Hardy type spaces in the unit ball of $\mathbb{C}^n$

Definition 4.1.1. Let $\mathbb{B}=\{z\in\mathbb{C}^n : |z|<1\}$. We recall that the Hardy space $H^p(\mathbb{B})$, $1\le p<\infty$, is defined to be the set of holomorphic functions $f:\mathbb{B}\to\mathbb{C}$ such that

$$\|f\|_p=\sup_{r<1}\left(\int_{\zeta\in\partial\mathbb{B}}\left|f(r\zeta)\right|^p d\sigma(\zeta)\right)^{1/p}<+\infty,$$

where $d\sigma$ is the Euclidean surface area measure on the sphere $\partial\mathbb{B}$.
The space $H^p(\mathbb{B})$ endowed with the norm $\|\cdot\|_p$ is a Banach space.

Theorem 4.1.2. ([22], Theorem 7.2.5) Suppose $n\ge 1, 0<p<\infty.$ If $f\in H^p(\mathbb{B})$ then

$$|f(z)|\le 2^{n/p}\|f\|_p(1-|z|)^{-n/p}$$

and

$$\lim_{|z|\to 1}(1-|z|)^{n/p}\left|f(z)\right|=0\ (z\in\mathbb{B}).$$

Remarks 4.1.3. We also recall that if a sequence $f_m\in H^p(\mathbb{B})$ converges to f, in the above norm, then f_m converges to f also uniformly on compact subsets of $\mathbb{B}$.
Indeed this follows from the inequality

$$\sup_{z\in K}|f(z)|\le C(p,K)\|f\|_p,$$

with K being a compact subset of $\mathbb{B}$ and $C(p,K)$ is a constant depending on p and K – see Theorem 4.1.2.
Also, $H^\infty(\mathbb{B})$ is the Banach space of bounded holomorphic functions $f:\mathbb{B}\to\mathbb{C}$, with the norm $\|f\|_\infty=\sup_{z\in\mathbb{B}}|f(z)|$.
For each $q>1$, we also consider the space $\bigcap_{1\le p<q}H^p(\mathbb{B})$, which becomes a complete metric space with the metric

$$d(f,g)=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{\|f-g\|_{p_j}}{1+\|f-g\|_{p_j}},$$

where $1<p_1<p_2<\cdots<p_j<\cdots<q$ and $p_j\to q$ ($j\to\infty$). Although this metric depends on the sequence p_j, the topology induced by this metric in the space $\bigcap_{1\le p<q}H^p(\mathbb{B})$ is independent of the choice of the sequence p_j. As a matter of fact, a sequence f_k converges to f in $\bigcap_{1\le p<q}H^p(\mathbb{B})$ if and only if $\|f_k-f\|_p\to 0$, for every

$p < q$. Indeed, if the sequence f_k converges to f, i.e., $d(f_k, f) \to 0$, then clearly $\| f_k - f \|_{p_j} \to 0$ for every $j = 1,2,3,\ldots$. But if $p < q$, we may choose a j_0 so that $p < p_{j_0} < q$. Then $\| f_k - f \|_{p_{j_0}} \to 0$ and therefore $\| f_k - f \|_p \to 0$. Conversely, we will show that if $\| f_k - f \|_p \to 0$, for every $p < q$, then $d(f_k, f) \to 0$. Let $\varepsilon > 0$. We choose $N = N(\varepsilon) \in \mathbb{N}$ so that $\sum_{j=N+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2}$. Since $\| f_k - f \|_{p_j} \to 0$ for $1 \le j \le N$, we may choose $k_0(\varepsilon) \in \mathbb{N}$ so that $\| f_k - f \|_{p_j} < \frac{\varepsilon}{2N}$ for $k \ge k_0(\varepsilon)$ and $1 \le j \le N$. Then it is easy to check that $d(f_k, f) < \varepsilon$ for $k \ge k_0(\varepsilon)$. This shows that $d(f_k, f) \to 0$.

Similarly, a sequence f_k in $\bigcap_{1 \le p < q} H^p(\mathbb{B})$ is Cauchy with respect to the metric d, i.e., $d(f_k, f_l) \to 0$ ($k, l \to \infty$) if and only if $\| f_k - f_l \|_p \to 0$ for every $p < q$.

Therefore the completeness of the metric space $\left(\bigcap_{1 \le p < q} H^p(\mathbb{B}), d \right)$ follows from the fact that each $H^p(\mathbb{B})$ is complete.

4.2. The case of the unit ball of $\mathbb{C}^n$

In this section we will first prove the following theorem.

Theorem 4.2.1. Let $q \in (1, +\infty]$. Then the set of the functions in the space $\bigcap_{1 \le p < q} H^p(\mathbb{B})$ which are totally unbounded in $\mathbb{B}$ is dense and $\mathcal{G}_\delta$ in this space.

The proof of this theorem will be based on the following lemma and theorem 2.1.12.

Lemma 4.2.2. For each point $\zeta \in \mathbb{S} = \partial \mathbb{B}$, we consider the functions

$$f_\zeta(z) = \frac{1}{1 - \langle z, \zeta \rangle} = \frac{1}{1 - \sum_{j=1}^n \bar{\zeta}_j z_j}, \quad h_\zeta(z) = \log f_\zeta(z) \text{ and } \varphi_{q,\zeta}(z) = \exp\left[\frac{n}{q} h_\zeta(z) \right]$$

defined for $z \in \mathbb{B}$. Then

(i) $f_\zeta \in \bigcap_{1 \le p < n} H^p(\mathbb{B})$ and $f_\zeta \notin H^n(\mathbb{B})$,

(ii) $h_\zeta \in \bigcap_{1 \le p < \infty} H^p(\mathbb{B})$ and $h_\zeta \notin H^\infty(\mathbb{B})$,

(iii) $\varphi_{q,\zeta} \in \bigcap_{1 \le p < q} H^p(\mathbb{B})$ and $\varphi_{q,\zeta} \notin H^q(\mathbb{B})$ for $1 < q < \infty$.

Proof. By Proposition 2.3.2, if $p < n$, the integral

$$\int_{w \in \mathbb{S}} \frac{d\sigma(w)}{|1-\langle z, w\rangle|^p},$$

as a function of z, remains bounded for $z \in \mathbb{B}$, and therefore since $r\zeta \in \mathbb{B}$ for $r < 1$,

$$\sup_{0<r<1} \int_{z\in\mathbb{S}} |f_\zeta(rz)|^p \, d\sigma(z) = \sup_{0<r<1} \int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle rz, \zeta\rangle|^p} = \sup_{0<r<1} \int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle r\zeta, z\rangle|^p} < \infty .$$

Thus $f_\zeta \in \bigcap_{1\le p<n} H^p(\mathbb{B})$.

Next we show that

$$\sup_{0<r<1} \int_{z\in\mathbb{S}} |f_\zeta(rz)|^n \, d\sigma(z) = \sup_{0<r<1} \int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle rz, \zeta\rangle|^n} = \infty .$$

Indeed, by Proposition 2.3.2, the integral

$$\int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle w, z\rangle|^n} \text{ behaves as } \log\frac{1}{1-|w|^2} \text{ for } w \in \mathbb{B},$$

and therefore

$$\sup_{0<r<1} \int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle rz, \zeta\rangle|^n} = \sup_{0<r<1} \int_{z\in\mathbb{S}} \frac{d\sigma(z)}{|1-\langle r\zeta, z\rangle|^n} = \sup_{0<r<1} \log\frac{1}{1-r^2} = \infty .$$

This proves (i). Next, observing that $\operatorname{Re}(1-\langle z,\zeta\rangle) > 0$, for $z \in \mathbb{B}$, we see that $\operatorname{Re} f_\zeta(z) > 0$ and therefore we may define $h_\zeta(z) = \log f_\zeta(z)$ using the principal branch of the logarithm with $-\pi < \arg \le \pi$. Then $|\operatorname{Im}[\log f_\zeta(z)]| < \pi/2$, i.e., $h_\zeta(z) = \log|f_\zeta(z)| + i\theta(z)$ with $|\theta(z)| < \pi/2$. It follows that if the point $rz \in \mathbb{B}$ and is sufficiently close to ζ,

$$|h_\zeta(rz)|^p = \left|\log\frac{1}{|1-\langle rz,\zeta\rangle|} + i\theta(rz)\right|^p = \left[\left(\log\frac{1}{|1-\langle rz,\zeta\rangle|}\right)^2 + \theta^2(rz)\right]^{p/2} \preceq (k!)^{p/k} \frac{1}{|1-\langle rz,\zeta\rangle|^{p/k}},$$

where we used the inequality $(\log x)^p \le (k!)^{p/k} x^{p/k}$ which holds for $x > 1$, $p \ge 1$ and $k \in \mathbb{N}$. (We also used the fact that, since $|1-\langle rz,\zeta\rangle| > 0$ for $rz \in \overline{\mathbb{B}}$ away from the point ζ, the quantity $|\log|1-\langle rz,\zeta\rangle||$ is bounded.) Fixing a $p < \infty$ and choosing $k > p/n$, we see (using also (i)) that $h_\zeta \in H^p(\mathbb{B})$ whence we obtain $h_\zeta \in \bigcap_{1\le p<\infty} H^p(\mathbb{B})$. Since obviously $\lim_{z\in\mathbb{B}, z\to\zeta} h_\zeta(z) = \infty$, (ii) follows. Finally observing that $|\varphi_{q,\zeta}| = |f_\zeta|^{n/q}$, we easily obtain (iii). □

Proof of Theorem 4.2.1. Let us consider a ball b, with sufficiently small radius, whose center lies on $\partial\mathbb{B}$, and let us set $X = b \cap \mathbb{B}$ and $\mathcal{V} = \bigcap_{1\le p<q} H^p(\mathbb{B})$. We define the linear operator

$$T:\mathcal{V}\to\mathbb{C}^X \text{ with } T(f)(z)=f(z) \text{ for } z\in X \text{ and } f\in\mathcal{V}.$$

For each fixed $z\in X$, the functional $T_z:\mathcal{V}\to\mathbb{C}$ defined by $T_z(f)=f(z)$, $f\in\mathcal{V}$, is continuous. It is easy to see that the set $\mathcal{E}=\{f\in\mathcal{V}: T(f) \textit{ is unbounded on } X\}$ in this case is equal to

$$\mathcal{E}(b)=\left\{f\in\bigcap_{1\le p<q}H^p(\mathbb{B}): \sup_{z\in b\cap\mathbb{B}}|f(z)|=\infty\right\}.$$

Also, by Lemma 4.1.5 (ii), $\mathcal{E}(b)\neq\varnothing$, since $h_\zeta\in\mathcal{E}(b)$ for $\zeta\in b\cap\partial\mathbb{B}$.

Therefore, by Theorem 2.1.12, $\mathcal{E}(b)$ is dense and $\mathcal{G}_\delta$ set in the space $\bigcap_{1\le p<q}H^p(\mathbb{B})$.

In order to complete the proof, we consider a countable dense subset $\{w_1,w_2,w_3,...\}$ of $\partial\mathbb{B}$, a decreasing sequence ε_s , $s=1,2,3,...$, of positive numbers with $\varepsilon_s\to 0$, and the balls $b(w_j,\varepsilon_s)$, centered at w_j and with radii ε_s . By the first part of the proof, each of the sets $\mathcal{E}(b(w_j,\varepsilon_s))$ is dense and $\mathcal{G}_\delta$ set in $\bigcap_{1\le p<q}H^p(\mathbb{B})$. It follows from Baire's theorem that the set

$$\mathcal{Y}=\bigcap_{j=1}^{\infty}\bigcap_{s=1}^{\infty}\mathcal{E}(b(w_j,\varepsilon_s)) \text{ is dense and } \mathcal{G}_\delta \text{ in the space } \bigcap_{1\le p<q}H^p(\mathbb{B}).$$

We claim that the set $\mathcal{Y}$ is exactly the set of the functions $f\in\bigcap_{1\le p<q}H^p(\mathbb{B})$ which are totally unbounded in $\mathbb{B}$. Indeed, if $f\in\mathcal{Y}$ and U is an open set with $U\cap\partial\mathbb{B}\neq\varnothing$, we may choose a point $w_{j_0}\in U\cap\partial\mathbb{B}$ and an ε_{s_0} so that $b(w_{j_0},\varepsilon_{s_0})\subset U$. Since $\sup\{|f(z)|: z\in b(w_{j_0},\varepsilon_{s_0})\cap\mathbb{B}\}=\infty$, it follows that $\sup\{|f(z)|: z\in U\cap\mathbb{B}\}=\infty$. Conversely, if $f\in\bigcap_{1\le p<q}H^p(\mathbb{B})$ and is totally unbounded then it is obvious that $f\in\mathcal{Y}$. This completes the proof. □

Next we define Hardy type spaces associated to open subsets of the sphere $\mathbb{S}=\partial\mathbb{B}$. These are local versions of the usual Hardy spaces and the main result is that, in general, the functions in $\bigcap_{1\le p<q}H^p(\mathbb{B})$ do not belong to Hardy spaces of higher order, not even locally.

4.3. Local Hardy spaces in the unit ball of $\mathbb{C}^n$

Definition 4.3.1. Let $G\subset\mathbb{S}$ be a non-empty open set (open in $\mathbb{S}$) and $1\le p<\infty$. A holomorphic function $f:\mathbb{B}\to\mathbb{C}$ is said to belong to the space $H^p(\mathbb{B},G)$ if

$$\sup_{r<1}\int_{z\in G}|f(rz)|^p\,d\sigma(z)<\infty .$$

Now we can state the following theorem.

Theorem 4.3.2. Let $q \in (1, +\infty)$. Then the set

$$\mathcal{A}_q = \left\{ g \in \bigcap_{1 \le p < q} H^p(\mathbb{B}) : \, g \notin H^q(\mathbb{B}, \mathbb{S} \cap b(\zeta, \varepsilon)) \ \textit{for any } \zeta \in \mathbb{S} \textit{ and any } \varepsilon > 0 \right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{1 \le p < q} H^p(\mathbb{B})$.

For the proof we will need the following lemma.

Lemma 4.3.3. If $\zeta \in G$ then for the functions f_ζ and $\varphi_{q,\zeta}$, defined in Lemma 4.2.2, we have:

(i) $f_\zeta \notin H^n(\mathbb{B}, G)$,

(ii) $\varphi_{q,\zeta} \notin H^q(\mathbb{B}, G)$ for $1 < q < \infty$.

Proof. Writing

$$\int_{z \in \mathbb{S}} \left| f_\zeta(rz) \right|^n d\sigma(z) = \int_{z \in G} \left| f_\zeta(rz) \right|^n d\sigma(z) + \int_{z \in \mathbb{S} - G} \left| f_\zeta(rz) \right|^n d\sigma(z) \ \text{ for } r < 1 ,$$

and taking into consideration the fact that

$$\sup_{r<1} \int_{z \in \mathbb{S}} \left| f_\zeta(rz) \right|^n d\sigma(z) = \infty ,$$

we see that it suffices to show that

$$\sup_{r<1} \int_{z \in \mathbb{S} - G} \left| f_\zeta(rz) \right|^n d\sigma(z) < \infty .$$

For this, let us notice that

$$\left| 1 - \langle rz, \zeta \rangle \right| \ge 1 - \mathrm{Re}(\langle rz, \zeta \rangle) = 1 - r(z \cdot \zeta) .$$

($z \cdot \zeta = \mathrm{Re}\langle z, \zeta \rangle$ is the inner product in $\mathbb{R}^{2n} = \mathbb{C}^n$.) Thus if $z \cdot \zeta < 0$ then $\left| 1 - \langle rz, \zeta \rangle \right| \ge 1$, and therefore

$$\int_{z \in (\mathbb{S} - G) \cap \{ z \cdot \zeta < 0 \}} \left| f_\zeta(rz) \right|^n d\sigma(z) \le \sigma(\mathbb{S}) .$$

On the other hand if $z \cdot \zeta \ge 0$ then $\left| 1 - \langle rz, \zeta \rangle \right| \ge 1 - z \cdot \zeta$. But

$$1 - z \cdot \zeta > 0 \ \text{ for } z \in (\mathbb{S} - G) \cap \{ z \cdot \zeta \ge 0 \},$$

since $\zeta \in G$ and $z \in (\mathbb{S} - G) \cap \{ z \cdot \zeta \ge 0 \}$ imply that z cannot be equal to $\lambda \zeta$ for any $\lambda > 0$, and therefore $z \cdot \zeta < |z||\zeta| = 1$. By the compactness of the set $(\mathbb{S} - G) \cap \{ z \cdot \zeta \ge 0 \}$

$$\alpha =: \inf \{ 1 - z \cdot \zeta : \, z \in (\mathbb{S} - G) \cap \{ z \cdot \zeta \ge 0 \} \} > 0 ,$$

whence

$$\int\limits_{z\in\mathbb{S}-G} \left|f_\zeta(rz)\right|^n d\sigma(z) \le \frac{\sigma(\mathbb{S})}{\alpha^n} \text{ for every } r<1 .$$

This proves (i). Now (ii) follows from (i), if we notice that $\left|\varphi_{q,\zeta}\right| = \left|f_\zeta\right|^{n/q}$. □

Proof of Theorem 4.3.2. Let us fix a point $w\in\mathbb{S}$ and $\delta>0$. With $X=\{r:\ 0<r<1\}$ and $\mathcal{V}=\bigcap\limits_{1\le p<q} H^p(\mathbb{B})$, we consider the sublinear operator $T:\mathcal{V}\to\mathbb{C}^X$ defined as follows:

$$T(f)(r)=\left(\int\limits_{z\in\mathbb{S}\cap B(w,\delta)} |f(rz)|^q d\sigma(z)\right)^{1/q} \text{ for } f\in\mathcal{V} \text{ and } r\in X .$$

Then, for each fixed $r\in X$, the functional $T_r:\mathcal{V}\to\mathbb{C}$, $T_r(f)=T(f)(r)$, $f\in\mathcal{V}$, is continuous. Indeed, if $f_m\in\mathcal{V}$ and $f_m\to f$ (in $\mathcal{V}$) then f_m converges to f uniformly on compact subsets of $\mathbb{B}$, as we pointed out in 4.1. Since $r<1$, it follows that

$$\int\limits_{z\in\mathbb{S}\cap B(w,\delta)} |f_m(rz)|^q d\sigma(z) \to \int\limits_{z\in\mathbb{S}\cap B(w,\delta)} |f(rz)|^q d\sigma(z) ,\ m\to\infty ,$$

i.e, $T_r(f_m)\to T_r(f)$.

On the other hand, by Lemma 4.3.3.(ii), the set

$$\mathcal{E}(w,\delta) := \left\{f\in\mathcal{V}: \sup\{T(f)(r):\ r\in X\}=\infty\right\}\neq\varnothing .$$

Therefore, by Theorem 2.1.12, the set $\mathcal{E}(w,\delta)$ is dense and $\mathcal{G}_\delta$ in the space $\bigcap\limits_{1\le p<q} H^p(\mathbb{B})$.

In order to complete the proof, we consider a countable dense subset $\{w_1,w_2,w_3,...\}$ of $\partial\mathbb{B}$ and a decreasing sequence δ_s, $s=1,2,3,...$, of positive numbers with $\delta_s\to 0$. By the first part of the proof and Baire's theorem, the set

$$\mathcal{Y}=\bigcap_{j=1}^{\infty}\bigcap_{s=1}^{\infty}\mathcal{E}(w_j,\delta_s) \text{ is dense and } \mathcal{G}_\delta \text{ in the space } \bigcap_{1\le p<q} H^p(\mathbb{B}) .$$

We claim that $\mathcal{Y}=\mathcal{A}_q$. Indeed if $f\in\mathcal{Y}$, $\zeta\in\partial\mathbb{B}$ and $\varepsilon>0$, we may choose $w_{j_0}\in B(\zeta,\varepsilon)$ and δ_{s_0} so that $B(w_{j_0},\delta_{s_0})\subset B(\zeta,\varepsilon)$, and since

$$\sup_{0<r<1}\int\limits_{z\in\mathbb{S}\cap B(w_{j_0},\delta_{s_0})} |f(rz)|^q d\sigma(z)=\infty ,$$

it follows that

$$\sup_{0<r<1}\int\limits_{z\in\mathbb{S}\cap B(\zeta,\varepsilon)} |f(rz)|^q d\sigma(z)=\infty ,$$

i.e., $f\notin H^q(\mathbb{B},\mathbb{S}\cap B(\zeta,\varepsilon))$. Thus $\mathcal{Y}\subset\mathcal{A}_q$, and since it is obvious that $\mathcal{A}_q\subset\mathcal{Y}$, the proof is complete. □

4.4. Hardy type spaces on bounded open sets with smooth boundary

First let us recall the definition of Hardy spaces in the case of bounded open sets with smooth boundary.

Definition 4.4.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary and let ρ be a defining function for this set, i.e., $\rho : \mathbb{C}^n \to \mathbb{R}$ is a C^2 function so that $\Omega = \{\rho < 0\}$, $\partial\Omega = \{\rho = 0\}$, $\mathbb{C}^n - \overline{\Omega} = \{\rho > 0\}$ and $\nabla\rho \neq 0$ at the points of $\partial\Omega$. For $p \geq 1$, the Hardy space $H^p(\Omega)$ is defined as follows:

$$H^p(\Omega) = \left\{ f : \Omega \to \mathbb{C},\ f \text{ holomorphic in } \Omega \text{ so that } \|f\|_{p,\rho} := \sup_{\varepsilon > 0} \left(\int_{\{\rho = -\varepsilon\}} |f(z)|^p d\sigma_\varepsilon^\rho(z) \right)^{1/p} < \infty \right\},$$

where $d\sigma_\varepsilon^\rho$ is the Euclidean surface area measure of the hypersurface $\{z \in \mathbb{C}^n :\ \rho(z) = -\varepsilon\}$ (with $\varepsilon > 0$ and sufficiently small).

Remarks 4.4.2.

(i) $H^p(\Omega)$ is independent of the defining function ρ.
In fact if λ is another defining function for Ω, the norms $\|f\|_{p,\rho}$ and $\|f\|_{p,\lambda}$ are equivalent. This follows from the proof of Stein Lemma 2.7.4.

(ii) Let us also observe that for compact subsets K of Ω,

$$\sup_{z \in K} |f(z)| \leq A(K,\rho) \|f\|_{p,\rho}, \quad f \in H^p(\Omega), \quad (4.4.2.1)$$

for some constant $A(K,\rho)$. To prove this inequality we may use the representation

$$f(z) = \int_{\partial\Omega_\varepsilon} f(\zeta) P_\varepsilon(\zeta, z) d\sigma_\varepsilon^\rho(\zeta), \ z \in K,$$

where $P_\varepsilon(\zeta, z)$ is the Poisson kernel of $\Omega_\varepsilon := \{\rho < -\varepsilon\}$ and $\varepsilon > 0$ and sufficiently small. (Once chosen, ε is fixed.) Since $P_\varepsilon(\zeta, z) \preceq [dist(z, \partial\Omega_\varepsilon)]^{-2n}$, Hölder's inequality (see Theorem 2.1.6) gives

$$|f(z)| \leq \left(\int_{\zeta \in \partial\Omega_\varepsilon} |f(\zeta)|^p d\sigma_\varepsilon(\zeta) \right)^{1/p} \left(\int_{\zeta \in \partial\Omega_\varepsilon} |P_\varepsilon(\zeta, z)|^{\tilde{p}} d\sigma_\varepsilon(\zeta) \right)^{1/\tilde{p}}$$

$$\preceq \frac{\|f\|_{p,\rho}}{[dist(z, \partial\Omega_\varepsilon)]^{2n}} \preceq \frac{\|f\|_{p,\rho}}{[dist(z, \partial\Omega)]^{2n}},$$

(with the point z restricted to the compact set K) and the inequality (4.4.2.1) follows, if $p > 1$. (For details concerning the Poisson kernel, see [16, 26] and Section 2.7) The case $p = 1$ is simpler.

(iii) $H^p(\Omega)$ becomes a Banach space with the norm $\|f\|_{p,\rho}$. This follows from (4.4.2.1) as in the case of the unit ball. (See also [28, Corollary 4.19].) From the same inequality also follows the fact that convergence in $H^p(\Omega)$ implies uniform convergence on compact subsets of Ω. As in the case of the unit ball, we define a metric in the space $\bigcap_{1\le p<q} H^p(\Omega)$, for a fixed $q>1$, as follows. We consider a sequence

$$1<p_1<p_2<\cdots<p_j<\cdots<q \text{ with } p_j \to q,$$

and we define the metric

$$d(f,g)=\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{\|f-g\|_{p_j,\rho}}{1+\|f-g\|_{p_j,\rho}}.$$

Then the topology of this metric induced on the space $\bigcap_{1\le p<q} H^p(\Omega)$ does not depend on the choice of the sequence p_j or on the choice of the defining function ρ. Indeed, a sequence f_k converges to f, in $\bigcap_{1\le p<q} H^p(\mathbb{B})$, if and only if $\|f_k - f\|_{p,\rho}\to 0$, for every $p<q$.

4.5. Local Hardy spaces

With Ω and ρ being as above, we consider an open set $U\subset\mathbb{C}^n$ with $U\cap\partial\Omega\neq\varnothing$ and we define the space $H^p_\rho(\Omega,U)$ to be the set of holomorphic functions $f:\Omega\to\mathbb{C}$ so that

$$\sup_{\varepsilon>0}\int_{\{\rho=-\varepsilon\}\cap U}|f(z)|^p\,d\sigma^\rho_\varepsilon(z)<\infty.$$

The space $H^p_\rho(\Omega,U)$ may depend on ρ. However we have the following lemma.

Lemma 4.5.1. Let ρ and λ be two defining functions for Ω. If U and V are two open subsets of $\mathbb{C}^n$ with $U\cap\partial\Omega\neq\varnothing$ and $V\cap\partial\Omega\neq\varnothing$, and if $V\subset\subset U$ then

$$H^p_\rho(\Omega,U)\subset H^p_\lambda(\Omega,V).$$

Proof. The following proof is essentially the proof of Stein Lemma 2.7.4 with some minor modifications. There exist positive constants κ, κ_1 and κ_2 (independent of ε) so that if $z\in\{\lambda=-\varepsilon\}$ (i.e. $\lambda(z)=-\varepsilon$) then

$$B(z,\kappa\varepsilon)\subset\Lambda_\varepsilon:=\{w\in\mathbb{C}^n: -\kappa_1\varepsilon<\rho(w)<-\kappa_2\varepsilon\}.$$

(The positive parameter ε is assumed to be sufficiently small so that the various assertions in this proof hold true.) By the submean value property, if $f \in H_\rho^p(\Omega,U)$,

$$|f(z)|^p \le \frac{\kappa_3}{\varepsilon^{2n}} \int_{w\in\mathbb{C}^n} \chi_\varepsilon(z,w)|f(w)|^p dw \text{ for } z\in\{\lambda=-\varepsilon\},$$

where $\chi_\varepsilon(z,w)=1$ for $w\in B(z,\kappa\varepsilon)$ and $\chi_\varepsilon(z,w)=0$ for $w\in\mathbb{C}^n - B(z,\kappa\varepsilon)$. In what follows, κ_j, $j=3,4,5,6$, are appropriate constants independent of ε. Then

$$\int_{\{\lambda=-\varepsilon\}\cap V} |f(z)|^p d\sigma_\varepsilon^\lambda(z) \le \frac{\kappa_3}{\varepsilon^{2n}} \int_{w\in\mathbb{C}^n} \left(\int_{\{\lambda=-\varepsilon\}\cap V} \chi_\varepsilon(z,w) d\sigma_\varepsilon^\lambda(z) \right) |f(w)|^p dw,$$

where we used Fubini's theorem (see Theorem 2.1.9) and the measurability of the function $\chi_\varepsilon(z,w)$ for $(z,w)\in\{\lambda=-\varepsilon\}\times\mathbb{C}^n$ with respect to the product measure $d\sigma_\varepsilon^\lambda(z)\times dw$. Since $V\subset\subset U$, making ε smaller – if necessary – we may assume that

$$B(z,\kappa\varepsilon)\subset\Lambda_\varepsilon\cap U \text{ for } z\in\{\lambda=-\varepsilon\}\cap V .$$

Then

$$\int_{\{\lambda=-\varepsilon\}\cap V} \chi_\varepsilon(z,w) d\sigma_\varepsilon^\lambda(z) = 0 \text{ if } w\in\mathbb{C}^n-(\Lambda_\varepsilon\cap U) \quad\text{and}\quad \int_{\{\lambda=-\varepsilon\}\cap V} \chi_\varepsilon(z,w) d\sigma_\varepsilon^\lambda(z) \le \kappa_4\varepsilon^{2n-1}$$

for $w\in\Lambda_\varepsilon\cap U$.

It follows that

$$\int_{\{\lambda=-\varepsilon\}\cap V} |f(z)|^p d\sigma_\varepsilon^\lambda(z) \le \frac{\kappa_5}{\varepsilon} \int_{w\in\Lambda_\varepsilon\cap U} |f(w)|^p dw \le \frac{\kappa_6}{\varepsilon} \int_{\kappa_2\varepsilon}^{\kappa_1\varepsilon} \left(\int_{\{\rho=-\eta\}\cap U} |f(w)|^p d\sigma_\eta^\rho(w) \right) d\eta .$$

(The existence of the constant κ_6 follows from the coarea formula – see Theorem 2.3.3.) Thus

$$\sup_{\varepsilon>0} \int_{\{\lambda=-\varepsilon\}\cap V} |f(z)|^p d\sigma_\varepsilon^\lambda(z) \le \kappa_6(\kappa_1-\kappa_2) \sup_{\eta>0} \int_{\{\rho=-\eta\}\cap U} |f(z)|^p d\sigma_\eta^\rho(z),$$

and this implies that $f\in H_\lambda^p(\Omega,V)$. □

4.6. The case of strictly pseudoconvex domains

In this section we will show that some functions which are defined in terms of Henkin's support function belong to certain Hardy spaces. We describe the Henkin's support function $\Phi(z,\zeta)$ – as constructed in [10] – in Section 2.9.

First we will prove the following lemma. We use a set of coordinates – *the Levi coordinates* – which are appropriate when we are dealing with integrals involving the Henkin's support function $\Phi(z,\zeta)$ (for more details see [10], [21]). As a matter of fact we will use a slight modification of the Levi coordinates.

Lemma 4.6.1. For each fixed point $\zeta \in \partial\Omega$,

$$\sup_{\varepsilon>0} \int_{\{\rho(z)=-\varepsilon\}} \frac{d\sigma_\varepsilon(z)}{|\Phi(z,\zeta)|^p} < \infty \text{ when } 1 < p < n, \tag{4.6.1.1}$$

and

$$\sup_{\varepsilon>0} \int_{\{\rho(z)=-\varepsilon\}} \frac{d\sigma_\varepsilon(z)}{|\Phi(z,\zeta)|^{2n-1}} = \infty. \tag{4.6.1.2}$$

Therefore $\dfrac{1}{\Phi(\cdot,\zeta)} \in \bigcap_{1\le p<n} H^p(\Omega)$ and $\dfrac{1}{\Phi(\cdot,\zeta)} \notin H^{2n-1}(\Omega)$.

Proof. We consider a coordinate system

$$t = (t_1, t_2, t_3, \ldots, t_{2n}) = (t_1(z), t_2(z), t_3(z), \ldots, t_{2n}(z)),$$

of real C^1 – functions, for points $z \in \mathbb{C}^n = \mathbb{R}^{2n}$, which are sufficiently close to the point ζ, as follows: We set

$$t_1(z) = -\rho(z) \text{ and } t_2(z) = \operatorname{Im} Q(z,\zeta).$$

Then $d_z Q(z,\zeta)\big|_{z=\zeta} = -2\sum_{j=1}^{n} \dfrac{\partial\rho(\zeta)}{\partial\zeta_j} dz_j\Big|_{z=\zeta} = -2\partial\rho(\zeta)$ and, therefore,

$$d_z t_2(z)\big|_{z=\zeta} = d_z[\operatorname{Im} Q(z,\zeta)]\big|_{z=\zeta} = i[\partial\rho(\zeta) - \bar\partial\rho(\zeta)].$$

On the other hand,

$$d_z t_1(z)\big|_{z=\zeta} = d_z[-\rho(z)]\big|_{z=\zeta} = -[\partial\rho(\zeta) + \bar\partial\rho(\zeta)].$$

It follows that

$$\left(d_z t_1(z)\big|_{z=\zeta}\right) \wedge \left(d_z t_2(z)\big|_{z=\zeta}\right) = 2i\partial\rho(\zeta) \wedge \bar\partial\rho(\zeta) \neq 0.$$

Now the existence of C^1 – functions $t_3(z), \ldots, t_{2n}(z)$ such that the mapping

$$z \to (t_1(z), t_2(z), t_3(z), \ldots, t_{2n}(z))$$

is a C^1 – diffeomorphism, from an open neighbourhood of the point ζ to an open neighbourhood of $0 \in \mathbb{C}^n = \mathbb{R}^{2n}$ (with $t(\zeta) = 0$), follows from the inverse function theorem (see Theorem 2.1.10). Also let us point out that, for z sufficiently close to ζ, $z \in \Omega$ if and only if $t_1 = -\rho(z) > 0$. For points $z \in \Omega$ which are sufficiently close to ζ,

$$|\Phi(z,\zeta)| \approx |Q(z,\zeta)| \approx |\operatorname{Re} Q(z,\zeta)| + |\operatorname{Im} Q(z,\zeta)| \geq -\rho(z) + \beta|\zeta - z|^2 + |\operatorname{Im} Q(z,\zeta)|$$

and

$$|\zeta - z|^2 \approx t_1^2 + t_2^2 + t_3^2 + \cdots + t_{2n}^2.$$

(When we write $\mathrm{A} \approx \mathrm{B}$, we mean that $\kappa\mathrm{B} \le \mathrm{A} \le \mu\mathrm{B}$, for some positive constants κ and μ which are independent of z.)

Therefore (for $z \in \Omega$ and sufficiently close to ζ)

$$|\Phi(z,\zeta)| \succeq t_1 + t_1^2 + t_2^2 + t_3^2 + \cdots + t_{2n}^2 + |t_2|.$$

Therefore (4.6.1.1) follows from

$$\sup_{\varepsilon>0} \int_{\{t_1=\varepsilon,t_2^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(t_1+|t_2|+t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2)^p} < \infty$$

or equivalently from

$$\sup_{\varepsilon>0} \int_{\{t_1=\varepsilon,t_2^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(t_1+|t_2|+t_3^2+\cdots+t_{2n}^2)^p} < \infty . \qquad (4.6.1.3)$$

But

$$\int_{\{t_1=\varepsilon,t_2^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(t_1+|t_2|+t_3^2+\cdots+t_{2n}^2)^p} \approx \int_{\{t_2^2+t_3^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(\varepsilon+|t_2|+t_3^2+\cdots+t_{2n}^2)^p}$$

Also, by Fubini's theorem - see Theorem 2.1.9,

$$\int_{\{t_2^2+t_3^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(\varepsilon+|t_2|+t_3^2+\cdots+t_{2n}^2)^p}$$

$$\approx \int_{\{t_3^2+\cdots+t_{2n}^2<1\}} \left(\int_{t_2=0}^{1} \frac{dt_2}{(\varepsilon+t_2+t_3^2+\cdots+t_{2n}^2)^p} \right) dt_3\cdots dt_{2n}$$

$$\approx \int_{\{t_3^2+\cdots+t_{2n}^2<1\}} \frac{dt_3\cdots dt_{2n}}{(\varepsilon+t_3^2+\cdots+t_{2n}^2)^{p-1}} .$$

Integrating in polar coordinates (see Lemma 2.3.1) we see that the last integral is equal to

$$\int_{r=0}^{1} \frac{r^{2n-3}dr}{(\varepsilon+r^2)^{p-1}} \le \int_{r=0}^{1} r^{2n-2p-1}dr < \infty .$$

This proves (4.6.1.3) and completes the proof of (4.6.1.1).
In order to prove (4.6.1.2), let us observe that for points $z \in \Omega$ which are sufficiently close to ζ ,

$$|\Phi(z,\zeta)| \approx |Q(z,\zeta)| \preceq |\zeta - z| \approx (t_1^2+t_2^2+t_3^2+\cdots+t_{2n}^2)^{1/2} ,$$

whence

$$\int_{\{\rho(z)=-\varepsilon\}} \frac{d\sigma_\varepsilon(z)}{|\Phi(z,\zeta)|^{2n-1}} \approx \int_{\{t_1=\varepsilon,t_2^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(t_1^2+t_2^2+\cdots+t_{2n}^2)^{(2n-1)/2}}$$

$$\succeq \int_{\{t_2^2+t_3^2+\cdots+t_{2n}^2<1\}} \frac{dt_2\cdots dt_{2n}}{(\varepsilon^2+t_2^2+\cdots+t_{2n}^2)^{(2n-1)/2}} .$$

By introducing polar coordinates in the last integral, we see that this integral behaves as

$$\int_{r=0}^{1} \frac{r^{2n-2}dr}{(\varepsilon^2+r^2)^{(2n-1)/2}} \approx \int_{r=0}^{1} \frac{r^{2n-2}dr}{(\varepsilon+r)^{2n-1}} .$$

But as ε decreases, the function $r^{2n-2}/(\varepsilon+r)^{2n-1}$ increases, and the monotone convergence theorem (see Theorem 2.1.8) gives that

$$\lim_{\varepsilon\to0^+} \int_{r=0}^{1} \frac{r^{2n-2}dr}{(\varepsilon+r)^{2n-1}} = \int_{r=0}^{1} \frac{dr}{r} = \infty ,$$

and proves (4.6.1.2). □

Lemma 4.6.2. Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set with C^2 boundary and let ρ be a C^2 strictly plurisubharmonic defining function of Ω defined in a neighbourhood of $\overline{\Omega}$. If $1 < q < \infty$ and $U \subset \mathbb{C}^n$ with $U \cap \partial\Omega \neq \varnothing$, then there exists a function $h_{q,U}$ so that

$$h_{q,U} \in \bigcap_{1 \le p < q} H^p(\Omega) \text{ and } h_{q,U} \notin H_\rho^{(2n-1)q/n}(\Omega, U).$$

Proof. Let us fix a point Let us fix a point $\zeta \in U \cap \partial\Omega$. Then, as it follows from Taylor's theorem and the strict plurisubharmonicity of ρ (see Proposition 2.8.9 and for more details [21] Proposition 2.16 page 60), the Levi polynomial of ρ

$$F(z,\zeta) = -\left[2\sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j}(z_j - \zeta_j) + \sum_{1 \le j,k \le n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k}(z_j - \zeta_j)(z_k - \zeta_k) \right]$$

satisfies the inequality

$$\operatorname{Re} F(z,\zeta) \ge \rho(\zeta) - \rho(z) + \gamma |\zeta - z|^2 \text{ for } z \in \mathbb{C}^n \text{ with } |\zeta - z| < \delta,$$

for some positive constants δ and γ. In particular,

$$\operatorname{Re} F(z,\zeta) > 0 \text{ for } z \in B(\zeta,\delta) \cap \overline{\Omega} - \{\zeta\}.$$

It follows that the function $\log[1/F(z,\zeta)]$ is defined and holomorphic for $z \in B(\zeta,\delta) \cap \Omega$, and that $\lim\limits_{z \in \Omega, z \to \zeta} \log[1/F(z,\zeta)] = \infty$. (Here $\log$ is the principal branch of the logarithm with $|\arg| \le \pi$.) Also we can prove, as in the proof of the Lemma 4.6.1, that if $q < n$,

$$\sup_{\varepsilon > 0} \int_{\{\rho(z) = -\varepsilon\} \cap B(\zeta,\delta)} \frac{d\sigma_\varepsilon(z)}{|F(z,\zeta)|^q} < \infty \tag{4.6.2.1}$$

and

$$\sup_{\varepsilon > 0} \int_{\{\rho(z) = -\varepsilon\} \cap B(\zeta,\delta)} \frac{d\sigma_\varepsilon(z)}{|F(z,\zeta)|^{2n-1}} = \infty. \tag{4.6.2.2}$$

Then, using (4.6.2.1), we obtain, as in Lemma 4.2.2,

$$\sup_{\varepsilon > 0} \int_{\{\rho(z) = -\varepsilon\} \cap B(\zeta,2\delta/3)} \left| \log\left[\frac{1}{F(z,\zeta)} \right] \right|^p d\sigma_\varepsilon(z) < \infty \quad \text{for every } p < \infty. \tag{4.6.2.3}$$

Next we consider a C^∞ – function $\chi : \mathbb{C}^n \to \mathbb{R}$, $0 \le \chi(z) \le 1$, with compact support contained in $B(\zeta, 2\delta/3)$, and such that $\chi(z) = 1$ when $z \in B(\zeta, \delta/3)$. Now the function

$$\chi(z) \log\left[\frac{1}{F(z,\zeta)} \right]$$

is extended to a C^∞ – function in Ω, by defining it to be 0 in $\Omega - B(\zeta, 2\delta/3)$. Then the $(0,1)$ – form

$$u(z) := \bar{\partial}\left\{\chi(z)\log\left[\frac{1}{F(z,\zeta)}\right]\right\}$$

is defined and is C^∞ in a open neighbourhood $\overline{\Omega}$, it is zero for $z \in B(\zeta, \delta/3) \cap \Omega$, and, in particular, it has bounded coefficients in Ω. In fact $u(z)$ extends to a C^∞ $(0,1)-$form for z in an open neighbourhood of $\overline{\Omega}$, since the function $\log\left[\frac{1}{F(z,\zeta)}\right]$ is holomorphic in an open neighbourhood of the compact set $\overline{[B(\zeta,2\delta/3) - B(\zeta,\delta/3)] \cap \Omega}$. It follows that there exists a bounded $C^\infty-$function $\psi : \Omega \to \mathbb{C}$ which solves the equation $\bar{\partial}\psi = u$ in Ω. (See Lemma 2.8.5 and [21], Theorem 16.3.4.) Then we may define the functions

$$f_\zeta(z) := \chi(z)\log\left[\frac{1}{F(z,\zeta)}\right] - \psi(z)$$

and

$$h_{q,\zeta}(z) := \exp\left[\frac{n}{q} f_\zeta(z)\right] = \exp\left\{\chi(z)\log\left[\frac{1}{F(z,\zeta)}\right]^{n/q} - \frac{n}{q}\psi(z)\right\}.$$

Then the functions $f_\zeta(z)$ and $h_{q,\zeta}(z)$ are holomorphic for $z \in \Omega$. We claim that

$$\sup_{\varepsilon>0} \int_{\{\rho(z)=-\varepsilon\}\cap B(\zeta,\delta)} \left|h_{q,\zeta}(z)\right|^p d\sigma_\varepsilon^\rho(z) < \infty \quad \text{for } p < q. \tag{4.6.2.4}$$

Notice that the behaviour of the above integral is not affected by the functions χ or ψ, since $\chi \equiv 1$ near ζ and ψ is bounded in Ω. Thus (4.6.2.4.) follows from (4.6.2.3). Also

$$\sup_{\varepsilon>0} \int_{\{\rho(z)=-\varepsilon\}\cap B(\zeta,\delta)} \left|h_{q,\zeta}(z)\right|^{(2n-1)q/n} d\sigma_\varepsilon^\rho(z) = \infty.$$

Indeed this follows from (4.6.2.2), since $\chi \equiv 1$ near ζ and $\exp(-\psi)$ is bounded away from zero in Ω.

Thus setting $h_{q,U} := h_{q,\zeta}$ we obtain the required function. □

Remark 4.6.3. The function $f_\zeta(z)$ which was constructed in the proof of the previous lemma has the following properties:

$$f_\zeta \in \bigcap_{1\le p<\infty} H^p(\Omega) \text{ and } \lim_{z\in\Omega,\, z\to\zeta} f_\zeta(z) = \infty.$$

(The first part follows from (4.6.2.1).)

Theorem 4.6.4. Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set with C^2 boundary and $q \in \mathbb{R} \cup \{\infty\}$, $q > 1$. Then the following hold:

(i) The set of the functions in the space $\bigcap_{1\le p<q} H^p(\Omega)$ which are totally unbounded in Ω is dense and $\mathcal{G}_\delta$ in this space.

(ii) The set of the functions in the space $\bigcap_{1\le p<q} H^p(\Omega)$ which are singular at every boundary point of Ω is dense and $\mathcal{G}_\delta$ in this space.

Proof. Let us consider a 'small' ball B whose center lies on $\partial\Omega$, and let us set $X = B\cap\Omega$ and $\mathcal{V} = \bigcap_{1\le p<q} H^p(\Omega)$. We define the linear operator

$$T:\mathcal{V}\to\mathbb{C}^X \text{ with } T(f)(z)=f(z) \text{ for } z\in X \text{ and } f\in\mathcal{V}.$$

For each fixed $z\in X$, the functional $T_z:\mathcal{V}\to\mathbb{C}$ defined by $T_z(f)=f(z)$, $f\in\mathcal{V}$, is continuous. It is easy to see that the set $\mathcal{E}=\{f\in\mathcal{V}: T(f) \textit{ is unbounded on } X\}$ in this case is equal to

$$\mathcal{E}(B)=\left\{ f\in \bigcap_{1\le p<q} H^p(\Omega): \sup_{z\in B\cap\Omega} |f(z)| = \infty \right\}.$$

Now we consider the function f_ζ which was constructed in the proof of Lemma 4.6.2. If the point $\zeta\in B\cap\partial\Omega$ then $f_\zeta\in\mathcal{E}(B)$, and therefore $\mathcal{E}(B)\neq\varnothing$. (see also Remark 4.6.3.) Therefore, by Theorem 2.1.12., $\mathcal{E}(B)$ is dense and $\mathcal{G}_\delta$ set in the space $\mathcal{V}$. In order to complete the proof, we consider a countable dense subset $\{w_1,w_2,w_3,...\}$ of $\partial\Omega$, a decreasing sequence ε_s, $s=1,2,3,...$, of positive numbers with $\varepsilon_s\to 0$, and the balls $B(w_j,\varepsilon_s)$, centered at w_j and with radious ε_s. By the first part of the proof, each of the sets $\mathcal{E}(B(w_j,\varepsilon_s))$ is dense and $\mathcal{G}_\delta$ set in $\bigcap_{1\le p<q} H^p(\Omega)$. It follows from Baire's theorem that the set

$$\mathcal{Y}=\bigcap_{j=1}^{\infty}\bigcap_{s=1}^{\infty}\mathcal{E}(B(w_j,\varepsilon_s)) \text{ is dense and } \mathcal{G}_\delta \text{ in the space } \bigcap_{1\le p<q} H^p(\Omega).$$

It is easy to see that $\mathcal{Y}$ is the set of the functions in the space $\bigcap_{1\le p<q} H^p(\Omega)$ which are totally unbounded in Ω, and this proves (i).
Finally, the assertion (ii) follows from (i) and Theorem 2.1.14, since a totally unbounded function in Ω is clearly singular (in Ω). □

Theorem 4.6.5. Let $\Omega\subset\subset\mathbb{C}^n$ be a strictly pseudoconvex open set with C^2 boundary and let ρ be a C^2 strictly plurisubharmonic defining function of Ω defined in a neighbourhood of $\overline{\Omega}$. If $q\in\mathbb{R}$, $q>1$, then the set

$$\mathcal{B}_q^\rho=\left\{ g\in \bigcap_{1\le p<q} H^p(\Omega): g\notin H_\rho^{(2n-1)q/n}(\Omega,U) \textit{ for any open set } U \textit{ with } U\cap\partial\Omega\neq\varnothing \right\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{1\le p<q} H^p(\Omega)$.

Proof. Let us fix a point $w \in \partial\Omega$ and a positive number δ. With $X = \{\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ (where ε_0 is a 'small' positive number) and $\mathcal{V} = \bigcap_{1 \le p < q} H^p(\Omega)$, we consider the sublinear operator $T : \mathcal{V} \to \mathbb{C}^X$ defined as follows:

$$T(f)(\varepsilon) = \left(\int_{\{\rho = -\varepsilon\} \cap B(w,\delta)} |f(z)|^{(2n-1)q/n} d\sigma_\varepsilon^\rho(z) \right)^{\frac{n}{(2n-1)q}} \quad \text{for } f \in \mathcal{V} \text{ and } \varepsilon \in X .$$

Then, for each fixed $\varepsilon \in X$, the functional $T_\varepsilon : \mathcal{V} \to \mathbb{C}$, $T_\varepsilon(f) = T(f)(\varepsilon)$, $f \in \mathcal{V}$, is continuous.
Also, by Lemma 4.6.2, the set $\mathcal{E}(w,\delta) := \{ f \in \mathcal{V} : \sup\{T(f)(\varepsilon) : \varepsilon \in X\} = \infty \} \neq \varnothing$. Therefore, by Lemma by Theorem 2.1.4, the set $\mathcal{E}(w,\delta)$ is dense and $\mathcal{G}_\delta$ in the space $\mathcal{V}$.
In order to complete the proof, we consider a countable dense subset $\{w_1, w_2, w_3, ...\}$ of $\partial\Omega$ and a decreasing sequence δ_s, $s = 1,2,3,...$, of positive numbers with $\delta_s \to 0$. By the first part of the proof and Baire's theorem, the set

$$\mathcal{Y} = \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \mathcal{E}(w_j, \delta_s) \text{ is dense and } \mathcal{G}_\delta \text{ in the space } \bigcap_{1 \le p < q} H^p(\Omega) .$$

Now it easy to see that $\mathcal{Y} = \mathcal{B}_q^\rho$, and this completes the proof. □

Combining Theorem 4.6.5 with Lemma 4.6.1, we will see that the set $\mathcal{B}_q^\rho$ is independent of ρ. Thus we have the following theorem.

Theorem 4.6.6. Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex open set with C^2 boundary. If $q \in \mathbb{R}$, $q > 1$, then the set

$$\mathcal{B}_q = \Big\{ g \in \bigcap_{1 \le p < q} H^p(\Omega) : g \notin H_\lambda^{(2n-1)q/n}(\Omega, U) \textit{ for any open set } U \textit{ with } U \cap \partial\Omega \neq \varnothing \textit{ and any defining function } \lambda \textit{ of } \Omega \Big\}$$

is dense and $\mathcal{G}_\delta$ in the space $\bigcap_{1 \le p < q} H^p(\Omega)$.

Proof. It is clear that $\mathcal{B}_q \subset \mathcal{B}_q^\rho$. Conversely, if $g \in \mathcal{B}_q^\rho$, U is any open set with $U \cap \partial\Omega \neq \varnothing$ and λ is any defining function of Ω, let us consider an open set V with $V \cap \partial\Omega \neq \varnothing$ and $V \subset\subset U$. By Lemma 4.6.1,

$$H_\lambda^{(2n-1)q/n}(\Omega, U) \subset H_\rho^{(2n-1)q/n}(\Omega, V) .$$

But $g \in \mathcal{B}_q^\rho$ implies that $g \notin H_\rho^{(2n-1)q/n}(\Omega, V)$, and therefore $g \notin H_\lambda^{(2n-1)q/n}(\Omega, U)$. It follows that $g \in \mathcal{B}_q$. Thus $\mathcal{B}_q = \mathcal{B}_q^\rho$. □

It is easy to see that one can obtain results analogous to the ones of Theorems 4.2.1, 4.3.2, 4.6.4, 4.6.6, with the spaces H^p in place of the intersections $\bigcap_{p<q} H^p$. Thus we have the following theorem.

Theorem 4.6.7. (i) For $1 \le p < \infty$, the set of the functions in the space $H^p(\mathbb{B})$ which are totally unbounded in $\mathbb{B}$ is dense and $\mathcal{G}_\delta$ in this space.
(ii) For $1 \le p < q < \infty$, the set

$$\{g \in H^p(\mathbb{B}) : g \notin H^q(\mathbb{B}, \mathbb{S} \cap B(\zeta, \varepsilon)) \;\; \textit{for any } \zeta \in \mathbb{S} \textit{ and any } \varepsilon > 0\}$$

is dense and $\mathcal{G}_\delta$ in the space $H^p(\mathbb{B})$.
(iii) If $\Omega \subset\subset \mathbb{C}^n$ is a strictly pseudoconvex open set with C^2 boundary and $1 \le p < \infty$, the set of the functions in the space $H^p(\Omega)$ which are totally unbounded in Ω is dense and $\mathcal{G}_\delta$ in this space.
(iv) If $\Omega \subset\subset \mathbb{C}^n$ is a strictly pseudoconvex open set with C^2 boundary, $1 \le p < \infty$ and $q > (2n-1)p/n$, then the set

$$\{g \in H^p(\Omega) : g \notin H^q_\lambda(\Omega, U) \;\; \textit{for any open set } U \textit{ with } U \cap \partial\Omega \ne \varnothing \;\; \textit{and any defining function } \lambda \textit{ of } \Omega\}$$

is dense and $\mathcal{G}_\delta$ in the space $H^p(\Omega)$.

4.7. Hardy Spaces of harmonic functions

The results of the previous sections can be extended to the case of harmonic functions in domains of $\mathbb{R}^n$. To describe this extension, let us consider a bounded open set $\Omega \subset \mathbb{R}^n$ with C^2 boundary. If ρ is a C^2 defining function of Ω then one can define the harmonic Hardy spaces $h^p(\Omega)$, $p \ge 1$, (see definition 4.7.1 below and for more details [1], [26] pages 3 and 117 respectively), the intersections $\bigcap_{p<q} h^p(\Omega)$, and the local Hardy spaces $h^p_\rho(\Omega, U)$, as before. ($U \subset \mathbb{R}^n$ is an open set with $U \cap \partial\Omega \ne \varnothing$.)

Definition 4.7.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set with C^2 boundary and let ρ be a defining function for this set. For $p \ge 1$, the harmonic Hardy space $h^p(\Omega)$ is defined as follows:

$$h^p(\Omega)=\left\{u:\Omega\to\mathbb{R},\ u \textit{ harmonic in } \Omega \textit{ so that } \|u\|_{p,\rho}:=\sup_{\varepsilon>0}\left(\int_{\{\rho=-\varepsilon\}}|u(z)|^p\, d\sigma_\varepsilon^\rho(z)\right)^{1/p}<\infty\right\},$$

where $d\sigma_\varepsilon^\rho$ is the Euclidean surface area measure of the hypersurface $\{z\in\mathbb{C}^n:\ \rho(z)=-\varepsilon\}$ (with $\varepsilon>0$ and sufficiently small).

Lemma 4.7.2. Let $n\geq 3$ and $y\in\partial\Omega$. Then the function $\varphi_y(x)=\dfrac{1}{|x-y|^{n-2}}$ ($x\neq y$) belongs to $h^p(\Omega)$ if and only if $p<\dfrac{n-1}{n-2}$. In particular $\varphi_y\notin h^{(n-1)/(n-2)}(\Omega)$.

Proof. We may assume that $y=0\in\partial\Omega$, in which case φ_y becomes the function

$$\varphi_0(x)=\frac{1}{|x|^{n-2}}=\frac{1}{(x_1^2+x_2^2+\cdots+x_n^2)^{(n-2)/2}}.$$

We must show that

$$\sup_{\varepsilon>0}\int_{\{\rho=-\varepsilon\}}\frac{d\sigma_\varepsilon^\rho(x)}{|x|^{(n-2)p}}<\infty \quad \text{if and only if } p<\frac{n-1}{n-2}. \tag{4.7.2.1}$$

Using a local diffeomorphism – near the point 0 of $\partial\Omega$ – we may assume that the hypersurface $\partial\Omega$, near 0, is defined by the equation $x_1=0$, and that $x_1>0$ for $x\in\Omega$ (close to 0). Then (4.7.2.1) is equivalent to

$$\sup_{\varepsilon>0}\int_{x_2^2+\cdots+x_n^2<1}\frac{dx_2\ldots dx_n}{(\varepsilon^2+x_2^2+\cdots+x_n^2)^{(n-2)p/2}}<\infty \quad \text{if and only if } p<\frac{n-1}{n-2}. \tag{4.7.2.2}$$

Integrating in polar coordinates (see Lemma 2.3.1) we see that the above integral behaves as

$$\int_{r=0}^{1}\frac{r^{n-2}dr}{(\varepsilon^2+r^2)^{(n-2)p/2}}.$$

By monotone convergence theorem - see Theorem 2.1.8 ,

$$\sup_{\varepsilon>0}\int_{r=0}^{1}\frac{r^{n-2}dr}{(\varepsilon^2+r^2)^{(n-2)p/2}}=\lim_{\varepsilon\to0^+}\int_{r=0}^{1}\frac{r^{n-2}dr}{(\varepsilon^2+r^2)^{(n-2)p/2}}\leq\int_{r=0}^{1}\frac{r^{n-2}dr}{r^{(n-2)p}},$$

and (4.7.2.2) follows. □

Lemma 4.7.3. Let $n\geq 3$ and $y\in\partial\Omega$. Then $\varphi_y\notin h_\rho^{(n-1)/(n-2)}(\Omega,U)$ for $y\in U$.

Proof. It follows easily from the previous lemma. □

With the above lemmas, we can prove the following theorems. Their proofs are similar to the proofs of Theorems 4.6.4 and 4.6.6.

Theorem 4.7.4. Let $1 < q \le \frac{n-1}{n-2}$. Then the set of the functions in the space $\bigcap_{1 \le p < q} h^p(\Omega)$ which are totally unbounded in Ω is dense and $\mathcal{G}_\delta$ in this space.

Theorem 4.7.5. *Let* $1 < q \le \frac{n-1}{n-2}$. *Then the set*

$$\mathcal{A}_q = \Big\{ g \in \bigcap_{1 \le p < q} h^p(\Omega) : \ g \notin h_\lambda^{(n-1)/n-2)}(\Omega, U) \ \ \textit{for any open set } U \textit{ with } U \cap \partial\Omega \neq \varnothing \ \textit{and any defining function } \lambda \textit{ of } \Omega \Big\}$$

is dense and $\mathcal{G}_\delta$ *in the space* $\bigcap_{1 \le p < q} h^p(\Omega)$.

Remark 4.7.6. According to Theorem 4.7.4, the functions in the space $\bigcap_{1 \le p < q} h^p(\Omega)$ are generically totally unbounded in Ω, despite the fact that all these functions have non-tangential limits almost everywhere at the points of the boundary of Ω (by Fatou's theorem – see Theorem 2.1.7. –). Similar remarks can be made for Theorems 4.1.4 and 4.6.4.

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