Master's Thesis
in Pure Mathematics

Boundary Actions and C*-Algebraic Properties of Discrete Groups

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Ευχαριστίες

Κατά μία έννοια, η παρούσα εργασία είναι το απόσταγμα της μέχρι τώρα ακαδημαϊκής μου πορείας. Ως εκ τούτου, θα μπορούσα να γράψω μια δεύτερη εργασία αφιερωμένη στην πλήρη απόδοση των απαραίτητων ευχαριστιών. Ελλείψει χώρου και χρόνου, οφείλω να αρκεστώ στα εξής. Ευχαριστώ:

1. Τους κ.κ. Δοδό, Ανούση και Κατάβολο, για τη συμμετοχή τους στην τριμελή επιτροπή.

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      1.1.1. Τον κ. Κατάβολο, για τον καθοριστικό ρόλο που έπαιξε στη διαμόρφωση των μαθηματικών μου ενδιαφερόντων.

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Πάνω από όλα όμως, θέλω να ξεχωρίσω τρεις πραγματικά σπάνιους ανθρώπους, και ακόμη σπανιότερους φίλους. Νέστωρ, Όλγα, Στέφανε, σας ευχαριστώ θερμά. Για όλα.

Σπύρος Πετράκος
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Περίληψη

Ανάμεσα στα πολλά σπουδαία μαθηματικά επιτεύγματα του H. Furstenberg, ένα από τα πλέον αναγνωρισμένα είναι η θεωρία συνόρων του. Η θεωρία αυτή μπορεί να μελετηθεί σε δύο επίπεδα: το μετρήσιμο και το τοπολογικό. Για τουλάχιστον τέσσερις δεκαετίες, το ενδιαφέρον σχεδόν μονοπωλούνταν από το πρώτο. Ωστόσο, τα έργα των Kalantar και Kennedy, και Breuillard, Kalantar, Kennedy και Ozawa το 2014 έφεραν τα τοπολογικά σύνορα στο προσκήνιο τόσο του κόσμου των τοπολογικών δυναμικών συστημάτων, όσο και αυτού των αλγεβρών τελεστών.

Ακρογωνιαίος λίθος αυτής της αναζωπύρωσης ενδιαφέροντος είναι η ταύτιση του καθολικού τοπολογικού συνόρου μιας διακριτής ομάδας Γ, γνωστό ως (τοπολογικό) σύνορο του Furstenberg, με το σύνορο του Hamana, έναν τοπολογικό χώρο που προκύπτει στη θεωρία των Γ-εμφυτευτικών καλυμμάτων συστημάτων τελεστών που ανέπτυξε ο M. Hamana. Η διττή φύση (δυναμική και C^*-αλγεβρική) του συνόρου αυτού, που θα το συμβολίζουμε με $\partial_{\mathrm{FH}}\Gamma$, επιτρέπει τον χαρακτηρισμό ιδιοτήτων της ανηγμένης C^*-άλγεβρας $\mathrm{C}^*_r(\Gamma)$ μέσω του τρόπου που η Γ δρα στο $\partial_{\mathrm{FH}}\Gamma$.

Συγκεκριμένα, το πρώτο μεγάλο επίτευγμα της θεωρίας αυτής είναι ο χαρακτηρισμός των ακόλουθων τριών ιδιοτήτων της Γ

1. C^*-απλότητα (απλότητα της $\mathrm{C}^*_r(\Gamma)$).

2. Ιδιότητα μοναδικού ίχνους (η $\mathrm{C}^*_r(\Gamma)$ δεν έχει άλλο ίχνος πέραν του κανονικού).

3. Τετριμμένο amenable ριζικό (η μοναδική amenable κανονική υποομάδα της Γ είναι η τετριμμένη).

και η αποσαφήνιση της μεταξύ τους σχέσης, πρόβλημα που παρέμενε ανοικτό από το 1975 και τη δουλειά του Powers. Επιπλέον, δόθηκε ένας νέος χαρακτηρισμός της ακρίβειας της Γ, δηλαδή της ακρίβειας της $\mathrm{C}^*_r(\Gamma)$ ως C^*-άλγεβρας, σε γραμμές παρόμοιες με αυτές των σχετικών έργων των Anantharaman-Delaroche και Renault, Ozawa, και Anantharaman-Delaroche μεταξύ 1998 και 2002.

Στην παρούσα εργασία μελετώνται οι χαρακτηρισμοί αυτοί, καθώς και μερικοί ακόμη που ακολούθησαν και βασίστηκαν σε αυτούς.

Abstract

Among the many important mathematical contributions of H. Furstenberg, one of the most recognised is his boundary theory. This theory can be studied on two levels: the measurable and the topological. For at least four decades, interest was almost monopolised by the former. However, the works of Kalantar and Kennedy, and Breuillard, Kalantar, Kennedy and Ozawa in 2014 brought topological boundaries on the spotlight of the world of topological dynamical systems, as well as that of operator algebras.

The cornerstone of this resurgence is the identification between the universal topological boundary of a discrete group Γ, known as the (topological) Furstenberg boundary, and the Hamana boundary, a topological space introduced in M. Hamana's theory of Γ-injective envelopes of operator systems. The dual nature (dynamical and C^*-algebraic) of this boundary, which will be denoted by $\partial_{FH}\Gamma$, allows the characterisation of properties of the reduced C^*-algebra $C^*_r(\Gamma)$ via the action of Γ on $\partial_{FH}\Gamma$.

In particular, the first great achievement of this theory is the characterisation of the following three properties of Γ

1. C^*-simplicity (simplicity of $C^*_r(\Gamma)$).
2. The unique trace property ($C^*_r(\Gamma)$ admits no traces other than the canonical one).
3. Triviality of the amenable radical (the only amenable normal subgroup of Γ is the trivial one).

and the disambiguation of their relationship, a problem that stood since 1975 and the work of Powers. Furthermore, exactness of Γ, i.e. the exactness of $C^*_r(\Gamma)$ as a C^*-algebra, was also given a new characterisation, in the spirit of the work done on amenable actions by Anantharaman-Delaroche and Renault, Ozawa, and Anantharaman-Delaroche between 1998 and 2002.

In this work we will study those characterisations, as well as some later results that relied on them.

Contents

Chapter 1

Preliminaries

In this chapter we develop the basic tools that will be used throughout this work. If the reader is unfamiliar with the fundamentals of C^*-algebras, a concise summary is given in the Appendix.

1.1 Notational Conventions

Throughout this chapter, and throughout this whole work, Γ will denote a discrete group. Its identity element will be denoted by e. No assumption on the cardinality of Γ is made.

1.2 Unitary Representations

The theory of unitary representations is a rich one. It serves both as an extension and an analytic analogue of classical representation theory, and it is similarly both an invaluable tool and an independent field of research. In this section we will only present the (relatively few) elements required later on. The interested reader can refer to Part II of [BHV08] for more (from the general viewpoint of topological groups).

Definition 1.2.1. A *unitary representation* $(\pi, \mathcal{H})$ of Γ in a Hilbert space $\mathcal{H}$ is a group homomorphism $\pi : \Gamma \to U(\mathcal{H})$, where $U(\mathcal{H}) \subseteq B(\mathcal{H})$ is the group of unitaries.

Example 1.2.2. We define the *unit* or *trivial* representation $1_\Gamma : \Gamma \to \mathbb{C}$ of Γ by mapping all elements to 1. Similarly, for any Hilbert space $\mathcal{H}$ we can define the trivial representation $1_{\mathcal{H}} : \Gamma \to B(\mathcal{H})$ by mapping all elements to the identity operator.

Example 1.2.3. We define the *left regular* representation $\lambda_\Gamma : \Gamma \to B(\ell^2(\Gamma))$ by

$$\lambda_\Gamma(s)\delta_t = \delta_{st}$$

for all $s, t \in \Gamma$ (it can easily be verified that this extends to a unitary, since it is obviously isometric and $\lambda_\Gamma(s)^* = \lambda_\Gamma(s^{-1})$). We oftentimes write $\lambda(s)$ or λ_s, instead of $\lambda_\Gamma(s)$. Similarly, we define the *right regular* representation $\rho_\Gamma : \Gamma \to B(\ell^2(\Gamma))$ by

$$\rho_\Gamma(s)\delta_t = \delta_{ts^{-1}}$$

for all $s, t \in \Gamma$.

Example 1.2.4. Let $\Lambda \leqslant \Gamma$ be a subgroup. We define the *(left) quasi-regular* representation $\lambda_{\Gamma/\Lambda} : \Gamma \to B(\ell^2(\Gamma/\Lambda))$ of Γ associated to Λ by

$$\lambda_{\Gamma/\Lambda}(s)\delta_{t\Lambda} = \delta_{st\Lambda}$$

for all $s, t \in \Gamma$ (again, the reader can check that the above truly defines a unitary).

Definition 1.2.5. Two unitary representations $(\pi, \mathcal{H}), (\sigma, \mathcal{K})$ of Γ are called *equivalent* (denoted by $\pi \simeq \sigma$) iff there exists an isometric linear isomorphism $T : \mathcal{H} \to \mathcal{K}$ that *intertwines* them, i.e.

$$T\pi(s) = \sigma(s)T$$

for all $s \in \Gamma$.

Definition 1.2.6. Let $(\pi_i, \mathcal{H}_i)$ be a family of unitary representations of Γ and $\mathcal{H} = \bigoplus \mathcal{H}_i$ be the Hilbert direct sum of the $\mathcal{H}_i$'s. We define the *direct sum* of the representations π_i to be the unitary representation $(\pi, \mathcal{H})$ defined by

$$\pi(s)(\oplus\xi_i) = \oplus\pi_i(s)\xi_i$$

for all $s \in \Gamma$ and $\oplus\xi_i \in \mathcal{H}$. We denote the direct sum $\oplus_I \pi$ of copies of a representation π by $|I|\pi$.

Proposition 1.2.7. *Let $(\pi, \mathcal{H})$ be a unitary representation of Γ and $\mathcal{K} \subseteq \mathcal{H}$ be a Γ-invariant subspace. Then $\mathcal{K}^\perp$ is also Γ-invariant.*

Proof. We have

$$\langle \pi(s)\xi, \eta \rangle = \langle \xi, \pi(s)^*\eta \rangle = \left\langle \xi, \pi(s^{-1})\eta \right\rangle = 0$$

for all $\xi \in \mathcal{K}^\perp, \eta \in \mathcal{K}$ and $s \in \Gamma$. ∎

Definition 1.2.8. For a unitary representation $(\pi, \mathcal{H})$ of Γ and a closed Γ-invariant subspace $\mathcal{K} \subseteq \mathcal{H}$, we will denote by $\pi^{\mathcal{K}}$ the representation $s \mapsto \pi(s)|_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$. We will call $\pi^{\mathcal{K}}$ a *subrepresentation* of π.

Corollary 1.2.9. *With the above notation, we have $\pi = \pi^{\mathcal{K}} \oplus \pi^{\mathcal{K}^\perp}$.*

Proposition 1.2.10. *Let* $(\pi, \mathcal{H}_1)$ *and* $(\sigma, \mathcal{H}_2)$ *be unitary representations of* Γ *and* $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ *be an intertwining operator between them. Then* $\mathcal{K}_1 = (\ker T)^\perp$ *and* $\mathcal{K}_2 = \overline{\operatorname{im} T}$ *are* Γ*-invariant and* $\pi^{\mathcal{K}_1} \simeq \sigma^{\mathcal{K}_2}$*.*

Proof. Γ-invariance is immediate from the intertwining relation and Proposition 1.2.7. Since T intertwines π and σ, T^* intertwines σ and π. Thus, because $|T|$ is a limit of polynomials in T^*T, we have that $|T|$ intertwines π with itself. Let $T = U|T|$ be the polar decomposition of T. We know that U restricts to an isometric isomorphism between $\mathcal{K}_1$ and $\mathcal{K}_2$, so it only remains to show U intertwines $\pi^{\mathcal{K}_1}$ and $\sigma^{\mathcal{K}_2}$. Indeed, we have

$$\sigma(s)U|T|\xi = \sigma(s)T\xi = T\pi(s)\xi = U|T|\pi(s)\xi = U\pi(s)|T|\xi$$

for all $s \in \Gamma$ and $\xi \in \mathcal{H}_1$, i.e. $\sigma(s)U = U\pi(s)$ for all $\eta \in \operatorname{im}|T|$. Since $\overline{\operatorname{im}|T|} = \mathcal{K}_1$, we are done. ■

Definition 1.2.11. A unitary representation $(\pi, \mathcal{H})$ of Γ is called *cyclic* iff there exists $\xi \in \mathcal{H}$ such that $\overline{\operatorname{span}\{\pi(\Gamma)\xi\}} = \mathcal{H}$. Then, we say that ξ is a *cyclic vector* for π.

Proposition 1.2.12. *Every unitary representation* $(\pi, \mathcal{H})$ *of* Γ *can be decomposed as a direct sum of cyclic ones.*

Proof. Let $\mathcal{X}$ be the set of all families of mutually orthogonal closed Γ-invariant subspaces of $\mathcal{H}$, partially ordered by inclusion. By Zorn's lemma, $\mathcal{X}$ contains a maximal family $(\mathcal{H}_i)$. If $\bigoplus \mathcal{H}_i \subsetneq \mathcal{H}$, then there exists a non-zero $\xi \in \mathcal{H}$ which is orthogonal to all $\mathcal{H}_i$'s. But then, the family $(\mathcal{H}_i) \cup \{\mathcal{K}\}$, where $\mathcal{K} = \overline{\operatorname{span}\{\pi(\Gamma)\xi\}}$, is contained in $\mathcal{X}$, contradicting maximality. Therefore, $\mathcal{H} = \bigoplus \mathcal{H}_i$ and $\pi = \oplus \pi^{\mathcal{H}_i}$. ■

Forming the direct sum of all (up to unitary equivalence) cyclic representations of Γ gives us the *universal representation* π_u (the observant reader might realise that it is not completely trivial that this sum is well-defined, but it is not that hard to convince themselves, since the cardinality of any Hilbert space on which Γ is cyclically represented is bounded).

Definition 1.2.13. Let $(\pi, \mathcal{H})$, $(\sigma, \mathcal{K})$ be unitary representations of Γ. The *tensor product* $\pi \otimes \sigma$ of π and σ is the unitary representation of Γ on $\mathcal{H} \otimes \mathcal{K}$ defined by

$$(\pi \otimes \sigma)(s)(\xi \otimes \eta) = \pi(s)\xi \otimes \sigma(s)\eta$$

for all $s \in \Gamma$, $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ (as usual, we extend linearly to $\mathcal{H} \odot \mathcal{K}$ and then by density to $\mathcal{H} \otimes \mathcal{K}$).

Theorem 1.2.14 (Fell's absorption principle). *Let* $(\pi, \mathcal{H})$ *be a unitary representation of* Γ*. Then* $\lambda_\Gamma \otimes \pi$ *is unitarily equivalent to* $\lambda_\Gamma \otimes 1_{\mathcal{H}}$*.*

Proof. Consider the unitary $U \in B(\ell^2(\Gamma) \otimes \mathcal{H})$ defined by

$$U(\delta_t \otimes \xi) = \delta_t \otimes \pi(t)\xi$$

for all $t \in \Gamma$ and $\xi \in \mathcal{H}$. Then, we have

$$\begin{aligned} U^*(\lambda_\Gamma \otimes \pi)(s)U(\delta_t \otimes \xi) &= U^*(\lambda_\Gamma \otimes \pi)(s)(\delta_t \otimes \pi(t)\xi) \\ &= U^*(\delta_{st} \otimes \pi(s)\pi(t)\xi) \\ &= \delta_{st} \otimes \xi \\ &= (\lambda_\Gamma \otimes 1_{\mathcal{H}})(s)(\delta_t \otimes \xi) \end{aligned}$$

for all $s, t \in \Gamma$ and $\xi \in \mathcal{H}$. ∎

Consider now a subgroup $\Lambda \leqslant \Gamma$ and a unitary representation $(\pi, \mathcal{H})$ of Λ. Let

$$\mathcal{H}_\Gamma = \{\xi : \Gamma \to \mathcal{H} : \sum_{s\Lambda \in \Gamma/\Lambda} \|\xi(s)\|^2 < \infty \text{ and } \xi(st) = \pi(t^{-1})\xi(s)\ \forall s \in \Gamma, t \in \Lambda\},$$

where the second condition guarantees that the sum in the first one is well-defined. We can equip $\mathcal{H}_\Gamma$ with an inner product defined by

$$\langle \xi, \eta \rangle = \sum_{s\Lambda \in \Gamma/\Lambda} \langle \xi(s), \eta(s) \rangle$$

for all $\xi, \eta \in \mathcal{H}_\Gamma$, which turns it into a Hilbert space (it is just $l^2(\Gamma/\Lambda, \mathcal{H})$ in disguise, as every element in $\mathcal{H}_\Gamma$ is uniquely determined by its values on a fixed set of representatives of Γ/Λ).

Definition 1.2.15. With the above notation, we define the representation $\operatorname{ind}_\Lambda^\Gamma \pi : \Gamma \to B(\mathcal{H}_\Gamma)$ by

$$(\operatorname{ind}_\Lambda^\Gamma \pi(s)\xi)(t) = \xi(s^{-1}t)$$

for all $s, t \in \Gamma$ and $\xi \in \mathcal{H}_\Gamma$. $\operatorname{ind}_\Lambda^\Gamma \pi$ is called the representation of Γ *induced* by π.

Example 1.2.16. $\operatorname{ind}_\Lambda^\Gamma \lambda_\Lambda = \lambda_\Gamma$.

Example 1.2.17. $\operatorname{ind}_\Lambda^\Gamma 1_\Lambda = \lambda_{\Gamma/\Lambda}$.

Definition 1.2.18. For a unitary representation $(\pi, \mathcal{H})$ of Γ, we call the functions $\langle \pi(.)\xi, \eta \rangle$, for $\xi, \eta \in \mathcal{H}$, the *matrix coefficients* of π. We call the diagonal matrix coefficients (i.e. those of the form $\langle \pi(.)\xi, \xi \rangle$) the *functions of positive type* associated with π.

Definition 1.2.19. Let $(\pi, \mathcal{H}), (\sigma, \mathcal{K})$ be unitary representations of Γ. We say that π is *weakly contained* in σ (and write $\pi \prec \sigma$) iff every function of positive type associated with π can be approximated uniformly on finite subsets of Γ by finite sums of functions of positive type associated with σ. We say that π and σ are *weakly equivalent* (and write $\pi \sim \sigma$) iff $\pi \prec \sigma$ and $\sigma \prec \pi$.

Remark. The reader can easily check that weak containment is transitive and depends only on the equivalence class of the representations involved. Furthermore, it does not take multiplicities into account, i.e. for unitary representations π, σ and cardinal numbers α, β we have $\alpha\pi \prec \beta\sigma \iff \pi \prec \sigma$.

It is immediate thet we can restrict the functions of positive type that need to be checked to those of the form $\langle \pi(.)\xi, \xi \rangle$ where ξ is a unit vector (we call those *normalised*). Moreover, such a function is approximated as in the definition iff it is approximated by convex combinations of normalised ones (this is not immediate, but it is reasonably easy). However, a more useful and less trivial restriction can be achieved using the following lemma.

Lemma 1.2.20. *Let $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$ be unitary representations of Γ and $\mathcal{V} \subseteq \mathcal{H}$ be such that $\pi(\Gamma)\mathcal{V}$ is total in $\mathcal{H}$. Then $\pi \prec \sigma$ iff every function of positive type of the form $\langle \pi(.)\xi, \xi \rangle$, $\xi \in \mathcal{V}$ can be approximated uniformly on finite subsets of Γ by finite sums of functions of positive type associated with σ.*

Proof. Let $\mathcal{X}$ be the set of all vectors $\xi \in \mathcal{H}$ such that the corresponding functions of positive type $\langle \pi(.)\xi, \xi \rangle$ can be approximated as described above. We have to show that $\mathcal{X} = \mathcal{H}$.

First of all, $\mathcal{X}$ is closed. Indeed, let $\xi \in \overline{\mathcal{X}}$ and notice that

$$|\langle \pi(s)\xi, \xi \rangle - \langle \pi(s)\xi', \xi' \rangle| \leqslant (\|\xi\| + \|\xi'\|)\, \|\xi - \xi'\|$$

for all $s \in \Gamma$ and $\xi' \in \mathcal{X}$. The right-hand side can be made ε-small for any ε, so any ε-approximation for $\langle \pi(.)\xi', \xi' \rangle$ is a 2ε-approximation for $\langle \pi(.)\xi, \xi \rangle$ and thus $\xi \in \mathcal{X}$.

Next, for $s_1, s_2 \in \Gamma$, $z_1, z_2 \in \mathbb{C}$ and $\xi \in \mathcal{X}$, let $\xi' = z_1\pi(s_1)\xi + z_2\pi(s_2)\xi$ and $\varphi(.) = \langle \pi(.)\xi, \xi \rangle$. We have

$$\langle \pi(s)\xi', \xi' \rangle = |z_1|^2\varphi(s_1^{-1}ss_1) + |z_2|^2\varphi(s_2^{-1}ss_2) + z_1\bar{z}_2\varphi(s_2^{-1}ss_1) + z_2\bar{z}_1\varphi(s_1^{-1}ss_2)$$

for all $s \in \Gamma$. Therefore, for any finite set F, uniformly approximating φ on $(s_1^{-1}Fs_1) \cup (s_2^{-1}Fs_2) \cup (s_2^{-1}Fs_1) \cup (s_1^{-1}Fs_2)$ (which is still finite) allows us to uniformly approximate $\langle \pi(.)\xi', \xi' \rangle$. Thus, if $\xi \in \mathcal{X}$, then $\mathcal{X}$ also contains the closed Γ-invariant subspace generated by ξ in $\mathcal{H}$.

Finally, we can show that $\mathcal{X}$ is closed under addition (and since it contains $\mathcal{V}$, we will be done). To that end, consider $\xi_1, \xi_2 \in \mathcal{X}$ and let $\mathcal{H}_1, \mathcal{H}_2$ be the respective Γ-invariant subspaces they generate in $\mathcal{H}$. Denote by $\mathcal{K}$ the closure of $\mathcal{H}_1 + \mathcal{H}_2$ in $\mathcal{H}$, which is also Γ-invariant. Let P be the orthogonal projection from $\mathcal{H}_2$ to the orthogonal complement $\mathcal{H}_1^\perp$ of $\mathcal{H}_1$ in $\mathcal{K}$. Notice now that $P(\mathcal{H}_2)$ is dense in $\mathcal{H}_1^\perp$, and P intertwines $\pi^{\mathcal{H}_2}$ and $\pi^{\mathcal{H}_1^\perp}$. Hence, by Proposition 1.2.10, $\pi^{(\ker P)^\perp} \simeq \pi^{\mathcal{H}_1^\perp}$. In particular, since $(\ker P)^\perp) \subseteq \mathcal{H}_2 \subseteq \mathcal{X}$, we also have $\mathcal{H}_1^\perp \subseteq \mathcal{X}$. Writing now $\xi = \xi_1 + \xi_2$ as $\xi_1' + \xi_2'$ with $\xi_1' = P(\xi) \in \mathcal{H}_1^\perp$

and $\xi_2' = \xi - P(\xi) \in \mathcal{H}_1$, we have

$$\langle \pi(s)\xi, \xi\rangle = \langle \pi(s)\xi_1', \xi_1'\rangle + \langle \pi(s)\xi_2', \xi_2'\rangle$$

for all $\sigma \in \Gamma$. Therefore $\xi \in \mathcal{X}$. ■

Theorem 1.2.21 (continuity of induction). *Let* $(\pi, \mathcal{H})$ *and* $(\sigma, \mathcal{K})$ *be unitary representations of a subgroup* $\Lambda \leqslant \Gamma$. *Then,* $\pi \prec \sigma$ *implies* $\operatorname{ind}_\Lambda^\Gamma \pi \prec \operatorname{ind}_\Lambda^\Gamma \sigma$.

Proof. For simplicity, we will denote $\operatorname{ind}_\Lambda^\Gamma \pi$ and $\operatorname{ind}_\Lambda^\Gamma \sigma$ by $(\pi', \mathcal{H}')$ and $(\sigma', \mathcal{K}')$, respectively. Let Σ be a transversal of the left coset space Γ/Λ containing e. Notice that the set $\pi(\Gamma)\mathcal{V}$, where

$$\mathcal{V} = \{\xi \in \mathcal{H}' : \operatorname{supp}(\xi) \subseteq \Lambda\},$$

is total in $\mathcal{H}'$. Therefore, from the previous lemma, we can restrict our attention to functions of positive type of the form $\langle \pi'(.)\xi, \xi\rangle$ for $\xi \in \mathcal{V}$. But for such ξ we have

$$\begin{aligned}
\langle \pi'(s)\xi', \xi'\rangle &= \sum_{t\in\Sigma} \left\langle \xi(s^{-1}t), \xi(t)\right\rangle \\
&= \left\langle \xi(s^{-1}), \xi(e)\right\rangle \\
&= \begin{cases} \langle \xi(s^{-1}), \xi(e)\rangle, & s \in \Lambda \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \langle \pi(s)\xi(e), \xi(e)\rangle, & s \in \Lambda \\ 0, & \text{otherwise} \end{cases} ,
\end{aligned}$$

where we have used the definition of $\mathcal{H}'$ and $\mathcal{V}$. Similar calculations show that for any $\eta \in \mathcal{K}$ and $s \in \Lambda$,

$$\langle \sigma(s)\eta, \eta\rangle = \langle \sigma'(s)\eta', \eta'\rangle ,$$

where $\eta' \in \mathcal{K}'$ is the function that maps e to η and all other elements of Σ to 0. Thus, since we can approximate $\langle \pi(.)\xi(e), \xi(e)\rangle$ by sums of $\langle \sigma(.)\eta, \eta\rangle$'s, and 0 by 0's, we can also approximate $\langle \pi'(.)\xi, \xi\rangle$ by sums of $\langle \sigma'(.)\eta', \eta'\rangle$'s. ■

1.3 Group C*-Algebras

In this section we will introduce the C*-algebras associated with unitary representations of Γ. Naturally, we will focus on the properties of the reduced C*-algebra, $C_r^*(\Gamma)$, and the full C*-algebra, $C^*(\Gamma)$. The former is especially important in this work, as all the main results have to do with its properties.

Let as begin by considering the group algebra $\mathbb{C}[\Gamma]$, which is essentially $C_c(\Gamma, \mathbb{C})$ equipped with the convolution product. We can endow $\mathbb{C}[\Gamma]$ with an involution by declaring $s^* = s^{-1}$ and extending antilinearly.

It is apparent that unitary representations of Γ correspond exactly to *-representations of $\mathbb{C}[\Gamma]$. For that reason, whenever we have a unitary representation $(\pi, \mathcal{H})$ of Γ, the corresponding *-representation will also be denoted by π. For any such representation, we will denote by $C^*_\pi(\Gamma)$ the $\|.\|$-closure of $\pi(\mathbb{C}[\Gamma])$ inside $B(\mathcal{H})$.

Definition 1.3.1. We call $C^*_{\lambda_\Gamma}(\Gamma)$ the *reduced* C^* *algebra* of Γ, and denote it by $C^*_r(\Gamma)$.

Definition 1.3.2. We call $C^*_{\pi_u}$ the *full* or *universal* C^**-algebra* of Γ, and denote it by $C^*(\Gamma)$.

Using Proposition 1.2.12, it is clear that $C^*(\Gamma)$ has the following universal property.

Proposition 1.3.3. *For every* **-representation* π *of* $\mathbb{C}[\Gamma]$*, there exists a surjective* **-homomorphism* $C^*(\Gamma) \to C^*_\pi(\Gamma)$ *such that* $\pi_u(s) \mapsto \pi(s)$ *for all* $s \in \Gamma$*. We denote the kernel of this* **-homomorphism by* $C^* \ker \pi$*.*

We give now a useful characterisation of weak containment.

Proposition 1.3.4. *Let* $(\pi, \mathcal{H})$ *and* $(\sigma, \mathcal{K})$ *be unitary representations of* Γ*. Then the following are equivalent:*

1. $\pi \prec \sigma$.
2. $C^* \ker \pi \supseteq C^* \ker \sigma$.

Proof. First of all, note that condition (2) is equivalent to $\|\pi(a)\| \leqslant \|\sigma(a)\|$ for all $a \in \mathbb{C}[\Gamma]_+$, as both simply state that the *-homomorphism

$$\begin{gathered} C^*_\sigma(\Gamma) = C^*(\Gamma)/C^* \ker \sigma \to C^*(\Gamma)/C^* \ker \pi = C^*_\pi(\Gamma) \\ \sigma(a) \mapsto \pi(a) \end{gathered}$$

is well-defined.

With that in mind, assume $\pi \prec \sigma$ and let $a = \sum_{s \in \Gamma} a_s s \in \mathbb{C}[\Gamma]_+$, $F = \operatorname{supp}(a)$, and $\xi \in \mathcal{H}$ be a unit vector. Consider now unit vectors $\eta_1, \ldots, \eta_n \in \mathcal{K}$ and real numbers numbers $c_1, \ldots, c_n \geqslant 0$ such that $c_1 + \cdots + c_n = 1$. We have

$$\left| \langle \pi(a)\xi, \xi \rangle - \sum_{j=1}^n c_j \langle \sigma(a)\eta_j, \eta_j \rangle \right| \leqslant \sum_{s \in F} |a_s| \left| \langle \pi(s)\xi, \xi \rangle - \sum_{j=1}^n c_j \langle \sigma(s)\eta_j, \eta_j \rangle \right|.$$

Since we can approximate $\|\pi(a)\|$ from below by $\langle \pi(a)\xi, \xi \rangle$'s due to positivity, and $\langle \pi(.)\xi, \xi \rangle$'s by $(\sum_{j=1}^n c_j \langle \sigma(.)\eta_j, \eta_j \rangle)$'s uniformly on F due to weak containment, we immediately get $\|\pi(a)\| \leqslant \|\sigma(a)\|$.

For the converse, let $\xi \in \mathcal{H}$ be a unit vector. We can extend $\langle \pi(.)\xi, \xi \rangle$ to a state on $C^*_\pi(\Gamma)$. Composing it with the *-homomorphism $C^*_\sigma(\Gamma) \to C^*_\pi(\Gamma)$ provided by condition (2), we get a state φ on $C^*_\sigma(\Gamma)$, which restricted to Γ is still $\langle \pi(.)\xi, \xi \rangle$. It suffices to prove that φ is contained in the weak-*-closed convex hull K of states on $C^*_\sigma(\Gamma)$ coming from normalised functions of positive type associated with σ.

Assume φ is not contained in K. By the Hahn-Banach separation theorem, there exists an element $a \in C^*_\sigma(\Gamma)_{sa}$ and $c \in \mathbb{R}$ such that

$$\varphi(a) < c \leqslant \psi(a)$$

for all $\psi \in K$. In particular, $\langle a\eta, \eta \rangle \geqslant c$ for all unit vectors $\eta \in \mathcal{K}$. Replacing a with $a - c1_{C^*_\sigma(\Gamma)}$, we can assume $c = 0$, and thus a is positive. But then $\varphi(a) \geqslant c = 0$, a contradiction. ■

Notice that, by its universal property, $C^*(\Gamma)$ always has a character (coming from 1_Γ), and thus a trace. However, much more important is the existence of a trace in the reduced case.

Proposition 1.3.5. *The vector state*

$$\tau_0 : C^*_r(\Gamma) \to \mathbb{C} : a \mapsto \langle a\delta_e, \delta_e \rangle$$

is tracial and faithful. This map is called the canonical trace*.*

Proof. Since

$$\langle \lambda_s \lambda_t \delta_e, \delta_e \rangle = \langle \delta_{st}, \delta_e \rangle = \langle \delta_{ts}, \delta_e \rangle = \langle \lambda_t \lambda_s \delta_e, \delta_e \rangle$$

for all $s, t \in \Gamma$, we immediately have that τ_0 is a trace.

Now, notice that λ_s and ρ_t commute for all $s, t \in \Gamma$, and thus ρ_t commutes with all elements in $C^*_r(\Gamma)$. Therefore, if $a \in C^*_r(\Gamma)$ is such that $a\delta_e = 0$, then

$$a\delta_s = a\rho_{s^{-1}}\delta_e = \rho_{s^{-1}} a\delta_e = 0$$

for all $s \in \Gamma$, and hence $a = 0$. With that in mind, for $a \in C^*_r(\Gamma)$ we have

$$\tau_0(a^* a) = \|a\delta_e\|^2 ,$$

which is 0 iff $a = 0$ iff $a^* a = 0$, proving faithfulness. ■

We close this section by taking a look at how $C^*_r(\Gamma)$ relates to $C^*_r(\Lambda)$ for a subgroup $\Lambda \leqslant \Gamma$.

Proposition 1.3.6. *If $\Lambda \leqslant \Gamma$, then $C^*_r(\Lambda) \subseteq C^*_r(\Gamma)$ canonically.*

Proof. It suffices to note that

$$\ell^2(\Gamma) = \bigoplus_{s \in \Sigma} \ell^2(\Lambda s)$$

where Σ is a transversal of the right coset space $\Gamma \setminus \Lambda$. This implies that the restriction of λ_Γ to Λ is a multiple of λ_Λ, and thus the mapping $\lambda_\Lambda(s) \mapsto \lambda_\Gamma(s)$ extends to the desired isometric *-homomorphism. ■

Proposition 1.3.7. *If $\Lambda \leqslant \Gamma$, then there exists a canonical conditional expectation $E_\Lambda : C^*_r(\Gamma) \to C^*_r(\Lambda)$.*

Proof. It suffices to show that the *-homomorphism $E_\Lambda : \mathbb{C}[\Gamma] \subseteq C^*_r(\Gamma) \to C^*_r(\Lambda)$ defined by $\lambda_s \mapsto \mathbb{1}_\Lambda(s)\lambda_s$ is contractive, and then use Tomiyama's theorem to conclude that its extension on $C^*_r(\Gamma)$ (denoted also by E_Λ) is a conditional expectation (since it is obviously a projection).

To that end, let $\xi \in \ell^2(\Lambda) \subseteq \ell^2(\Gamma)$ and $a = \sum_{s \in \Gamma} a_s \lambda_s$. We have

$$\begin{aligned}
\|a\xi\|^2 &= \left\| \sum_{s \in \Lambda} a_s \lambda_s \xi + \sum_{s \in \Lambda^c} a_s \lambda_s \xi \right\|^2 \\
&= \left\| \sum_{s \in \Lambda} a_s \lambda_s \xi \right\|^2 + \left\| \sum_{s \in \Lambda^c} a_s \lambda_s \xi \right\|^2 \\
&\geqslant \left\| \sum_{s \in \Lambda} a_s \lambda_s \xi \right\|^2 \\
&= \|E_\Lambda(a)\xi\|^2
\end{aligned}$$

and thus

$$\begin{aligned}
\|E_\Lambda(a)\| &= \sup\{\|E_\Lambda(a)\xi\| : \xi \in \ell^2(\Lambda),\ \|\xi\| = 1\} \\
&\leqslant \sup\{\|a\xi\| : \xi \in \ell^2(\Lambda),\ \|\xi\| = 1\} \\
&\leqslant \sup\{\|a\xi\| : \xi \in \ell^2(\Gamma),\ \|\xi\| = 1\} \\
&= \|a\|,
\end{aligned}$$

as required. ■

1.4 Amenability

Amenability as a notion has its roots in measure theory and the Banach-Tarski paradox, but its significance has spread to many areas of mathematics, including geometric group theory, dynamics, ergodic theory, and (of course!) operator algebras. Introduced by von Neumann in the 1920's, amenable groups have since been characterised in a wide variety of wildly different ways. So wide in fact, that it has become somewhat of a running joke among the initiated. We, however, will take a very modest approach, presenting only the few equivalent definitions that will be useful in this work. We will also explore some of the properties of the class of amenable groups.

Definition 1.4.1. Γ is called *amenable* iff there exists a state ω on $\ell^\infty(\Gamma)$ which is invariant under the left translation action of Γ, i.e.

$$\omega(sf) = \omega(f)$$

for all $f \in \ell^\infty$ and $s \in \Gamma$, where $(sf)(t) = f(s^{-1}t)$. Such a state ω is called an *invariant mean.*

We will denote by $\mathcal{P}(\Gamma)$ the space of probability measures on Γ, i.e. the positive part of the unit sphere of $\ell^1(\Gamma)$. Note that $\mathcal{P}(\Gamma)$ is invariant under the left translation action defined above on $\ell^\infty(\Gamma)$.

Definition 1.4.2. We say that Γ has an *approximate invariant mean* iff for any finite subset $F \subseteq \Gamma$ and $\varepsilon > 0$, there exists $\mu \in \mathcal{P}(\Gamma)$ such that

$$\sup_{s\in F} \|s\mu - \mu\|_1 < \varepsilon.$$

Definition 1.4.3. We say that Γ satisfies the *Følner condition* iff for any finite subset $E \subseteq \Gamma$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq \Gamma$ such that

$$\sup_{s\in E} \frac{|sF \triangle F|}{|F|} < \varepsilon.$$

A net of finite subsets $F_i \subseteq \Gamma$ such that

$$\frac{|sF_i \triangle F_i|}{|F_i|} \to 0$$

for all $s \in \Gamma$ is called a *Følner net* (obviously Γ satisfies the Følner condition iff it has a Følner net).

Theorem 1.4.4. *The following are equivalent:*

1. Γ *is amenable.*
2. Γ *has an approximate invariant mean.*
3. Γ *satisfies the Følner condition.*
4. *There exist unit vectors* (ξ_i) *in* $\ell^2(\Gamma)$ *such that* $\|\lambda_s\xi_i - \xi_i\| \to 0$ *for all* $s \in \Gamma$.
5. $1_\Gamma \prec \lambda_\Gamma$.
6. $C^*_r(\Gamma)$ *has a character.*

Proof. ($1 \implies 2$) Let $\omega \in \ell^\infty(\Gamma)^*$ be an invariant mean. Since $\ell^1(\Gamma)$ is the predual of $\ell^\infty(\Gamma)$, it is weak-*-dense in $\ell^\infty(\Gamma)^*$. Therefore, we can find a net (μ_i) in $\mathcal{P}(\Gamma)$ such that $\mu_i \xrightarrow{w^*} \omega$, which implies that $s\mu_i - \mu_i \xrightarrow{w} 0$ for all $s \in \Gamma$. Hence, for any finite subset $F \subseteq \Gamma$, the weak closure of $\bigoplus_{s\in F}\{s\mu - \mu : \mu \in \mathcal{P}(\Gamma)\}$ contains 0. But this set is convex in $\bigoplus_{s\in F} \ell^1(\Gamma)$, so norm and weak closures coincide.

($2 \implies 3$) Let $E \subseteq \Gamma$ be finite and $\varepsilon > 0$. Choose $\mu \in \mathcal{P}(\Gamma)$ such that

$$\sum_{s\in E} \|s\mu - \mu\|_1 < \varepsilon.$$

For each $r > 0$ and $f \in \ell^1(\Gamma)_+$, define $F(f, r) = \{s \in \Gamma : f(s) > r\}$. We have

$$\begin{aligned}
\|s\mu - \mu\|_1 &= \sum_{t\in\Gamma} |s\mu(t) - \mu(t)| \\
&= \sum_{t\in\Gamma} \int_0^1 |\mathbb{1}_{F(s\mu,r)}(t) - \mathbb{1}_{F(\mu,r)}(t)|\, dr \\
&= \int_0^1 |F(s\mu, r) \triangle F(\mu, r)|\, dr \\
&= \int_0^1 |sF(\mu, r) \triangle F(\mu, r)|\, dr,
\end{aligned}$$

and therefore

$$\int_0^1 \sum_{s\in E} |sF(\mu, r) \triangle F(\mu, r)|\, dr < \varepsilon = \varepsilon \int_0^1 |F(\mu, r)|\, dr.$$

Hence, for some r we must have

$$\sum_{s\in E} |sF(\mu, r) \triangle F(\mu, r)| < \varepsilon |F(\mu, r)|$$

and thus the Følner condition is satisfied (since $F(\mu, r)$ is finite for any $r > 0$).

($3 \implies 4$) Let (F_i) be a Følner net. Then the ξ_i's defined by $\xi_i = |F_i|^{-1/2}\mathbb{1}_{F_i}$ do the job.

($4 \implies 5$) The only normalised function of positive type associated with 1_Γ is the constant function 1. For (ξ_i) in $\ell^2(\Gamma)$ satisfying condition (4), we have

$$|\langle \lambda_s \xi_i, \xi_i \rangle - 1| = |\langle \lambda_s \xi_i - \xi_i, \xi_i \rangle| \leqslant \|\lambda_s \xi_i - \xi_i\| \to 0$$

for all $s \in \Gamma$. Thus $1_\Gamma \prec \lambda_\Gamma$.

($5 \implies 6$) Proposition 1.3.4 implies the existence of a *-homomorphism

$$C^*_r(\Gamma) = C^*_{\lambda_\Gamma}(\Gamma) \to C^*_{1_\Gamma}(\Gamma) = \mathbb{C},$$

i.e. a character.

(6 $\implies$ 1) Consider a character $C^*_r(\Gamma) \to \mathbb{C}$ and extend it to a state ω on $B(\ell^2(\Gamma))$, which restricts to a state on $\ell^\infty(\Gamma)$. Now, for $s \in \Gamma$ and $f \in \ell^\infty(\Gamma)$ we have

$$\omega(sf) = \omega(\lambda_s f \lambda_s^*) = \omega(\lambda_s)\omega(f)\omega(\lambda_s^*) = \omega(f),$$

where we have used that the Γ-action on $\ell^\infty(\Gamma)$ is spatially implemented (as a very simple calculation shows), and that $C^*_r(\Gamma)$ is contained in the multiplicative domain of ω (which is completely positive because $\mathbb{C}$ is abelian). Thus, Γ is amenable. ■

Condition (5) and continuity of induction imply the following, which will be used later on.

Corollary 1.4.5. *If a subgroup* $\Lambda \leqslant \Gamma$ *is amenable, then* $\lambda_{\Gamma/\Lambda} \prec \lambda_\Gamma$.

Theorem 1.4.6 (Day's fixed point). *The following are equivalent:*

1. Γ *is amenable.*
2. *Every* Γ*-action on a compact convex subset* K *of a locally convex space* X *has a fixed point.*

Proof. (2 $\implies$ 1) Immediate, since the state space of $\ell^\infty(\Gamma)$ is a weak-*-compact convex subset of $\ell^\infty(\Gamma)^*$.

(1 $\implies$ 2) Fix $x_0 \in K$ and consider an invariant mean $\omega \in \ell^\infty(\Gamma)^*$. Let $A(K)$ be the set of continuous affine maps $K \to \mathbb{C}$. For each $\varphi \in A(K)$, define $f_\varphi : \Gamma \to \mathbb{C} : s \mapsto \varphi(sx_0)$ and notice that $f_\varphi \in \ell^\infty(\Gamma)$. We want to show that there exists $x_\Gamma \in K$ such that $\omega(f_\varphi) = \varphi(x_\Gamma)$ for all $\varphi \in A(K)$, which will turn out to be the desired fixed point.

To that end, let ω_i be a net of finitely supported positive elements of norm 1 in $\ell^1(\Gamma) \subseteq \ell^\infty(\Gamma)^*$ such that $\omega_i \xrightarrow{w^*} \omega$. Notice that for such an element $\omega_i = \sum_{k=1}^n c_{i,k}\delta_{s_k}$, $c_{i,k} > 0$, $\sum_{k=1}^n c_{c_i,k} = 1$ and for all $\varphi \in A(K)$ we have

$$\omega_i(f_\varphi) = \sum_{k=1}^n c_{i,k}\varphi(s_k x_0) = \varphi(\sum_{k=1}^n c_{i,k} s_k x_0),$$

i.e. for each i there exists a point $x_i \in K$ such that $\omega_i(f_\varphi) = \varphi(x_i)$. By compactness, we can assume (x_i) converges to some $x_\Gamma \in K$. But then

$$\omega(f_\varphi) = \lim_i \omega_i(f_\varphi) = \lim_i \varphi(x_i) = \varphi(x_\Gamma)$$

for all $\varphi \in A(K)$.

Now, because the Γ-action is affine, we have that the map $\varphi_{s,\psi} : K \to \mathbb{C} : x \mapsto \psi(sx)$ belongs to $A(K)$ for all $s \in \Gamma$ and $\psi \in X^*$. Furthermore, we have

$$f_{\varphi_{s,\psi}}(t) = \varphi_{s,\psi}(tx_0) = \psi(stx_0) = (s^{-1}f_{\varphi_{e,\psi}})(t),$$

and thus

$$\begin{aligned}\psi(s x_\Gamma) &= \varphi_{s,\psi}(x_\Gamma) = \omega(f_{\varphi_{s,\psi}}) = \omega(s^{-1} f_{\varphi_{e,\psi}}) \\ &= \omega(f_{\varphi_{e,\psi}}) = \varphi_{e,\psi}(x_\Gamma) = \psi(x_\Gamma)\end{aligned}$$

for all $s \in \Gamma$ and $\psi \in X^*$. Since X^* separates points in X, x_Γ is indeed fixed by Γ. ■

Let us now take a look at how amenability behaves under standard group theoretic operations.

Proposition 1.4.7. *Amenability is closed under taking subgroups.*

Proof. If $\Lambda \leqslant \Gamma$, then $C_r^*(\Lambda) \subseteq C_r^*(\Gamma)$ canonically. Thus, any character on $C_r^*(\Gamma)$ restricts to a character on $C^*(\Lambda)$. ■

Proposition 1.4.8. *Amenability is closed under taking quotients.*

Proof. Let Γ be amenable, $\Lambda \trianglelefteq \Gamma$, and $\omega \in \ell^\infty(\Gamma)^*$ be an invariant mean. If $\pi : \Gamma \to \Gamma/\Lambda$ is the canonical projection, we can define a state $\widetilde{\omega}$ on $\ell^\infty(\Gamma/\Lambda)$ by

$$f \mapsto \omega(f \circ \pi).$$

We have

$$((s\Lambda f) \circ \pi)(t) = f(s^{-1} t\Lambda) = (f \circ \pi)(s^{-1}t) = (s(f \circ \pi))(t)$$

and thus

$$\widetilde{\omega}(s\Lambda f) = \omega((s\Lambda f) \circ \pi) = \omega(s(f \circ \pi)) = \omega(f \circ \pi) = \widetilde{\omega}(f)$$

for all $f \in \ell^\infty(\Gamma/\Lambda)$ and $s\Lambda \in \Gamma/\Lambda$. ■

Proposition 1.4.9. *Amenability is closed under extensions.*

Proof. Let $\Lambda \trianglelefteq \Gamma$ and Γ/Λ be amenable, and $\omega_\Lambda \in \ell^\infty(\Lambda)^*$, $\omega_{\Gamma/\Lambda} \in \ell^\infty(\Gamma/\Lambda)^*$ be invariant means. For $f \in \ell^\infty(\Gamma)$ define $\tilde{f} \in \ell^\infty(\Gamma/\Lambda)$ by $\tilde{f}(s\Lambda) = \omega_\Lambda((s^{-1}f)|_\Lambda)$, which is well-defined due to the Λ-invariance of ω_Λ. Define now $\omega_\Gamma \in \ell^\infty(\Gamma)^*$ by

$$f \mapsto \omega_{\Gamma/\Lambda}(\tilde{f}),$$

which is obviously a state. Now we have

$$\widetilde{sf}(t\Lambda) = \omega_\Lambda((t^{-1}sf)|_\Lambda) = \tilde{f}(s^{-1}t\Lambda) = (s\Lambda\tilde{f})(t)$$

and thus

$$\omega_\Gamma(sf) = \omega_{\Gamma/\Lambda}(\widetilde{sf}) = \omega_{\Gamma/\Lambda}(s\Lambda\tilde{f}) = \omega_{\Gamma/\Lambda}(\tilde{f}) = \omega_\Gamma(f)$$

for all $f \in \ell^\infty(\Gamma)$ and $s \in \Gamma$. ■

Proposition 1.4.10. *Amenability is closed under direct unions.*

Proof. Let (Γ_i) be a direct system of amenable groups, $\varepsilon > 0$ and $E \subseteq \Gamma := \cup_i \Gamma_i$ be a finite set. Then there exists i such that $E \subseteq \Gamma_i$. Thus, there exists a finite $F \subseteq \Gamma_i \leqslant \Gamma$ such that

$$\sup_{s \in F} \frac{|sF \triangle F|}{|F|} < \varepsilon,$$

meaning that Γ satisfies the Følner condition. ∎

We will now use the above to establish the following fact, which will come into play later on.

Proposition 1.4.11. *There exists a normal amenable subgroup $\Lambda_0 \leqslant \Gamma$ which contains all other normal amenable subgroups of Γ. We call this the* amenable radical *of Γ and denote it by $R_a(\Gamma)$.*

Proof. Let $\{\Lambda_i\}$ be the family of normal amenable subgroups of Γ. Since this family is closed under direct unions, we can invoke Zorn's lemma to obtain a maximal element Λ_0 of $\{\Lambda_i\}$. Assume that $\Lambda_i \nsubseteq \Lambda_0$ for some i. Then Λ_0 is normal in $\Lambda_0\Lambda_i$ and $\Lambda_0\Lambda_i/\Lambda_0 \simeq \Lambda_i/(\Lambda_i \cap \Lambda_0)$. Thus, since amenability is closed under extensions, $\Lambda_0\Lambda_i$ is amenable and normal (since Λ_0, Λ_i are), contradicting maximality of Λ_0. ∎

1.5 Crossed Products

Crossed products sit in the heart of the interplay between dynamics and operator theory, so their usefulness in this work should be quite unsurprising. The context in which they arise is that of C^*-dynamical systems, and it is an effective way of encoding them. This section serves as an introduction to the topic, presenting the constructions as well as some key properties.

Definition 1.5.1. A C^**-dynamical system* is a triplet $(\mathcal{A}, \alpha, \Gamma)$, where $\mathcal{A}$ is a (unital in this work) C^*-algebra and α is a Γ-action on $\mathcal{A}$ by *-automorphisms. We will call such an $\mathcal{A}$ a Γ-C^**-algebra*.

For the rest of the section, unless otherwise specified, $(\mathcal{A}, \alpha, \Gamma)$ will denote a C^*-dynamical system, the notation covering the individual parts of the triplet, too (e.g. $\mathcal{A}$ will denote a Γ-C^*-algebra, even on its own).

We want to construct a single C^*-algebra which minimally contains $\mathcal{A}$ and Γ (i.e. it is generated by them) in a way that makes α inner (a property reminiscent, uncoincidentally, of the semidirect product of groups), so that we will be able to recover from it information about $(\mathcal{A}, \alpha, \Gamma)$, and vice versa.

To that end, we start with the α-twisted group algebra $\mathcal{A}[\Gamma; \alpha]$ $(= C_c(\Gamma, \mathcal{A})$ as linear spaces), i.e. the usual group algebra, equipped instead with the

α-twisted convolution product defined by

$$(\sum_{s\in\Gamma} a_s s) *_\alpha (\sum_{t\in\Gamma} b_t t) = \sum_{s,t\in\Gamma} a_s \alpha_s(b_t) st.$$

We also define an involution on $\mathcal{A}[\Gamma;\alpha]$ by

$$(\sum_{s\in\Gamma} a_s s)^* = \sum_{s\in\Gamma} \alpha_{s^{-1}}(a_s^*) s^{-1},$$

which turns it into a *-algebra that seems to be doing exactly what we want. It remains to find a suitable completion.

Definition 1.5.2. A *covariant representation* $(\pi, u, \mathcal{H})$ of $\mathcal{A}$ consists of a unitary representation $u : \Gamma \to B(\mathcal{H})$ and a *-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ such that the Γ-action on $\mathcal{A}$ is spatially implemented inside $B(\mathcal{H})$, i.e. $\pi(\alpha_s(a)) = u_s \pi(a) u_s^*$ for all $s \in \Gamma$ and $a \in \mathcal{A}$.

It is clear that covariant representations of $\mathcal{A}$ correspond exactly to *-representations of $\mathcal{A}[\Gamma;\alpha]$. For a covariant representation $(\pi, u, \mathcal{H})$, we will denote the corresponding *-representation by $\pi \times u$.

We can construct an abundance of covariant representations in the following way. Consider any *-representation $(\pi, \mathcal{H})$ of $\mathcal{A}$. This induces a *-representation $(\tilde{\pi}, \mathcal{H} \otimes \ell^2(\Gamma))$ of $\mathcal{A}$ defined by

$$\tilde{\pi}(a)(\xi \otimes \delta_s) = (\pi(\alpha_{s^{-1}}(a))\xi) \otimes \delta_s$$

for all $a \in \mathcal{A}$, $s \in \Gamma$ and $\xi \in \mathcal{H}$. Then, $(\tilde{\pi}, 1_{\mathcal{H}} \otimes \lambda_\Gamma, \mathcal{H} \otimes \ell^2(\Gamma))$ is a covariant representation. Notice that $\tilde{\pi} \times (1_{\mathcal{H}} \otimes \lambda_\Gamma)$ is faithful whenever π is. In particular, the universal norm defined below is, indeed, a norm.

Definition 1.5.3. The *full* or *universal crossed product* of $(\mathcal{A}, \alpha, \Gamma)$, denoted by $\mathcal{A} \rtimes_\alpha \Gamma$ (or simply $\mathcal{A} \rtimes \Gamma$), is the completion of $\mathcal{A}[\Gamma;\alpha]$ with respect to the norm defined by

$$\|x\|_u = \sup\{\|\pi(x)\| : \pi \text{ is a } {}^*\text{-representation of } \mathcal{A}[\Gamma;\alpha]\}$$

for all $x \in \mathcal{A}[\Gamma;\alpha]$.

Notice that if $\mathcal{A} = \mathbb{C}$ and α is the trivial action, then $\mathcal{A}[\Gamma; a] = \mathbb{C}[\Gamma]$ and $\mathbb{C} \rtimes_\alpha \Gamma = C^*(\Gamma)$. Similarly to $C^*(\Gamma)$, the full crossed product satisfies (and is characterised by) the following universal property.

Proposition 1.5.4. *For every covariant representation* $(\pi, u, \mathcal{H})$ *of* $\mathcal{A}$*, there exists a* **-representation* $(\pi \rtimes u, \mathcal{H})$ *of* $\mathcal{A} \rtimes_\alpha \Gamma$ *such that*

$$\pi \rtimes u(\sum_{s\in\Gamma} a_s s) = \sum_{s\in\Gamma} \pi(a_s) u_s$$

for all $\sum_{s\in\Gamma} a_s s \in \mathcal{A}[\Gamma;\alpha] \subseteq \mathcal{A} \rtimes_\alpha \Gamma$ *(in other words,* **-representations of* $\mathcal{A}[\Gamma;\alpha]$ *extend to* **-representations of* $\mathcal{A} \rtimes_\alpha \Gamma$*).*

Definition 1.5.5. The *reduced crossed product* of $(\mathcal{A}, \alpha, \Gamma)$, denoted by $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ (or simply $\mathcal{A} \rtimes_r \Gamma$), is the $\|.\|$-closure of $(\tilde{\pi} \times (1_{\mathcal{H}} \otimes \lambda_\Gamma))(\mathcal{A}[\Gamma; \alpha])$ inside $B(\mathcal{H} \otimes \ell^2(\Gamma))$ for some faithful representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ (using the notation introduced right before defining the full crossed product).

For now, the above definition seems a bit shaky, but the next proposition should remedy that.

Proposition 1.5.6. *The reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ does not depend on the choice of faithful representation $\pi : \mathcal{A} \to B(\mathcal{H})$.*

Proof. Consider a finite $F \subseteq \Gamma$ and let $P \in B(\ell^2(\Gamma))$ be the orthogonal projection onto $\operatorname{span}\{\delta_s : s \in F\}$. We would like to show that the norm of any $x \in \mathcal{A}[\Gamma; \alpha] \subseteq B(H \otimes \ell^2(\Gamma))$ is independent of π. It suffices to show that this holds for the compression of x by $1_{B(\mathcal{H})} \otimes P$, and take the limit over F.

To that end, let $\{e_{s,t}\}_{s,t\in\Gamma}$ denote the matrix units inside $B(\ell^2(\Gamma))$, $a \in \mathcal{A}$, and $s \in \Gamma$. Notice that

$$\tilde{\pi}(a) = \sum_{t\in\Gamma} \pi(\alpha_t^{-1}(a)) \otimes e_{t,t},$$

where the convergence is in the strong operator topology (i.e. the topology of pointwise convergence). Therefore, $\tilde{\pi}(a)$ commutes with $1_{B(\mathcal{H})} \otimes P$.

Thus, we have

$$\begin{aligned}
&(1_{B(\mathcal{H})} \otimes P)\tilde{\pi}(a)(1_{B(\mathcal{H})} \otimes \lambda_s)(1_{B(\mathcal{H})} \otimes P) \\
&= (\sum_{t\in\Gamma} \pi(\alpha_t^{-1}(a)) \otimes e_{t,t})(1_{B(\mathcal{H})} \otimes P\lambda_s P) \\
&= (\sum_{t\in\Gamma} \pi(\alpha_t^{-1}(a)) \otimes e_{t,t})(\sum_{t\in F\cap sF} 1_{B(\mathcal{H})} \otimes e_{t,s^{-1}t}) \\
&= \sum_{t\in F\cap sF} \pi(\alpha_t^{-1}(a)) \otimes e_{t,s^{-1}t}
\end{aligned}$$

which lives inside $\mathbb{M}_F(\mathcal{A}) \hookrightarrow B(\mathcal{H} \otimes \ell^2(\Gamma))$. Hence,

$$(1_{B(\mathcal{H})} \otimes P)(\tilde{\pi} \rtimes (1_{\mathcal{H}} \times \lambda_\Gamma))(\mathcal{A}[\Gamma; \alpha])(1_{B(\mathcal{H})} \otimes P) \hookrightarrow \mathbb{M}_F(\mathcal{A}),$$

which admits a unique C^*-norm. ∎

Notationally, we will henceforth completely forget about π and the fact that we amplified λ_Γ, and denote the image of a typical element $x = \sum_{s\in\Gamma} a_s s \in \mathcal{A}[\Gamma; \alpha]$ inside $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ by $\sum_{s\in\Gamma} a_s \lambda_s$.

Unsurprisingly, if $\mathcal{A} = \mathbb{C}$ and α is the trivial action, then the associated reduced crossed product is just $C_r^*(\Gamma)$.

Proposition 1.5.7 (Fell's absorption principle - C^*-dynamical version). *Let $(\pi, u, \mathcal{H})$ be a covariant representation. Then $\tilde{\pi} \times (1_{\mathcal{H}} \otimes \lambda_\Gamma)$ is unitarily* *equivalent to* $(\pi \otimes 1_{B(\ell^2(\Gamma))}) \otimes (u \otimes \lambda_\Gamma)$ *(where we have abused notation for the* $*$*-representation* $a \mapsto \pi(a) \otimes 1_{B(\ell^2(\Gamma))}$ *of* $\mathcal{A}$*). In particular,*

$$\mathcal{A} \rtimes_{\alpha,r} \Gamma \simeq C^*(\pi(\mathcal{A}) \otimes 1_{B(\ell^2(\Gamma))}, (u \otimes \lambda_\Gamma)(\Gamma)) \subseteq B(\mathcal{H}) \otimes C^*_r(\Gamma)$$

whenever π *is faithful.*

Proof. Consider the unitary U defined in the proof of Fell's absorption property (Theorem 1.2.14), which, as we already know, intertwines $1_{\mathcal{H}} \otimes \lambda_\Gamma$ and $u \otimes \lambda_\Gamma$. Using covarience, we also have

$$\begin{aligned} U^*(\pi \otimes 1_{B(\ell^2(\Gamma))})U(\xi \otimes \delta_t) &= (u_t^* \pi(a) u_t \xi) \otimes \delta_t \\ &= (\pi(\alpha_{t^{-1}}(a))\xi) \otimes \delta_t \\ &= \tilde{\pi}(a)(\xi \otimes \delta_t) \end{aligned}$$

for all $\xi \in \mathcal{H}, t \in \Gamma$, and $a \in \mathcal{A}$. The result is now immediate. ∎

We will close this section (and this chapter) with the introduction of a very useful tool. But first, the following lemma is required.

Lemma 1.5.8. *Let* $\mathcal{A}, \mathcal{B}$ *be* C^**-algebras and* φ *be a faithful state on* $\mathcal{B}$*. Then* $\mathrm{id}_{\mathcal{A}} \otimes \varphi ; \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ *is faithful.*

Proof. Assume $\mathcal{A} \subseteq B(\mathcal{H})$ and $\mathcal{B} \subseteq B(\mathcal{K})$, and thus $\mathcal{A} \otimes \mathcal{B} \hookrightarrow B(\mathcal{H} \otimes \mathcal{K})$. Since vector states corresponding to elementary tensors separate operators in $B(\mathcal{H} \otimes \mathcal{K})$, we have that elementary tensors of states in $\mathcal{A}^* \odot \mathcal{B}^*$ separate the points of $\mathcal{A} \otimes \mathcal{B}$ (because the aforementioned vector states are elementary tensors of vector states in $B(\mathcal{H})^* \odot B(\mathcal{K})^*$).

Therefore, for $x \in (\mathcal{A} \otimes \mathcal{B})_+$, there exists a state ψ on $\mathcal{A}$ such that $(\psi \otimes \mathrm{id}_{\mathcal{B}})(x) > 0$. Since φ is faithful, we have that

$$\psi((\mathrm{id}_{\mathcal{A}} \otimes \varphi)(x)) = \varphi((\psi \otimes \mathrm{id}_{\mathcal{B}})(x)) > 0,$$

which implies $(\mathrm{id}_{\mathcal{A}} \otimes \varphi)(x) \neq 0$. ∎

Proposition 1.5.9. *There exists a faithful conditional expectation* $E : \mathcal{A} \rtimes_{\alpha,r} \Gamma \to \mathcal{A}$ *such that* $E(\lambda_s) = 1_{\mathcal{A}} \delta_e(s)$ *for all* $s \in \Gamma$*.*

Proof. By Fell's absorption property, $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ can be seen as a C^*-subalgebra of $B(\mathcal{H}) \otimes C^*_r(\Gamma)$. Restricting $\mathrm{id}_{B(\mathcal{H})} \otimes \tau_0$ to $\mathcal{A} \rtimes_{\alpha,r} \Gamma$ gives us E, which is faithful by the previous lemma. ∎

Chapter 2

The Furstenberg-Hamana Boundary

In the second half of the 1970's and the first half of the 1980's, Masamichi Hamana published a series of papers introducing the notion of an injective envelope for several categories of operator algebraic interest [Ham78; Ham79a; Ham79b; Ham85]. The concept was by no means new or groundbreaking for analysts, as, for example, Cohen had already done the same for Banach spaces [Coh64]; and even before that, the idea existed in the algebraic realm of modules under the cloak of injective hulls [ES53]. Hamana's work, however, has seen a lot of use in the last decade, as an observation he had already made was rediscovered and subsequently exploited to make a connection between the dynamical and the C^*-algebraic properties of discrete groups.

In this chapter we briefly go over Hamana's theory of Γ-injective envelopes, in order to introduce the Hamana boundary of Γ and study its properties. As we shall see – and this is the cornerstone of this whole work – this boundary, albeit C^*-algebraic in nature, can be identified with the Furstenberg boundary of the group, which was defined in dynamical terms by Hillel (Harry) Furstenberg.

2.1 Injective Envelopes of Γ-Operator Systems

We begin this section by introducing the main categorical concept behind it.

Definition 2.1.1. An object I in a category $\mathfrak{C}$ is called *injective* iff every morphism $X \to I$ factors through every monomorphism $X \to Y$.

Well-known examples of injective objects in analysis include $\mathbb{C}$ in the category of Banach spaces (Hahn-Banach theorem), c_0 in the category of separable Banach spaces (Sobczyk's theorem) and $B(\mathcal{H})$ in the category $\mathfrak{G}$ defined below (Arveson's extension theorem) and in the category of operator spaces with completely contractive maps as morphisms (Wittstock's extension theorem).

Let us now focus on the categories we are interested in. In the rest of the chapter, $\mathcal{S}$ will denote an operator system, unless otherwise specified.

Definition 2.1.2. We call $\mathcal{S}$ a Γ-*operator system* (or Γ-*module*) iff Γ acts on it by unital complete order isomorphisms. A unital completely positive (u.c.p.) Γ-equivariant map between Γ-operator systems will be simply called a Γ-*map* or a Γ-*homomorphism*.

With that in mind, we define $\Gamma\mathfrak{O}_1$ to be the category consisting of Γ-operator systems as objects and Γ-maps as morphisms. We also define the categories $\mathfrak{O}$ and $\mathfrak{O}_1$ of operator systems with completely positive (c.p.) and u.c.p. maps as morphisms, respectively.

Definition 2.1.3. $\mathcal{S}$ is called Γ-*injective* iff it is injective in $\Gamma\mathfrak{O}_1$.

Example 2.1.4. $\ell^\infty(\Gamma)$ equipped with the action

$$(sf)(t) = f(s^{-1}t),\ \ s \in \Gamma, f \in \ell^\infty(\Gamma)$$

is Γ-injective.

More generally, we have the following.

Lemma 2.1.5. *If $\mathcal{S}$ is injective in $\mathfrak{O}$, then $\ell^\infty(\Gamma, \mathcal{S})$ equipped with the action*

$$(sf)(t) = f(s^{-1}t),\ \ s \in \Gamma, f \in \ell^\infty(\Gamma, \mathcal{S})$$

is Γ-injective.

Proof. Let $\varphi : \mathcal{T} \to \ell^\infty(\Gamma, \mathcal{S})$ be a Γ-map and $\iota : \mathcal{T} \hookrightarrow \mathcal{U}$ a Γ-monomorphism. Consider the u.c.p. map $\psi = ev_e \circ \varphi : \mathcal{T} \to \mathcal{S}$, where ev_e is the evaluation at the identity element e of Γ. Since $\mathcal{S}$ is injective, there exists a u.c.p. map $\widehat{\psi} : \mathcal{U} \to \mathcal{S}$ which extends ψ, i.e. $\widehat{\psi} \circ \iota = \psi$. Then the map $\widehat{\varphi} : \mathcal{U} \to \ell^\infty(\Gamma, \mathcal{S}) : x \mapsto (\widehat{\psi}(s^{-1}x))_{s\in\Gamma}$ is a Γ-map extending φ.

$$\begin{tikzcd}
\mathcal{U} \arrow[dr, dashed, "\widehat{\varphi}"] \arrow[drr, dashed, bend left, "\widehat{\psi}"] & & \\
\mathcal{T} \arrow[u, hook, "\iota"] \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi"'] & \ell^\infty(\Gamma, \mathcal{S}) \arrow[r, "ev_e"] & \mathcal{S}
\end{tikzcd}$$

∎

We want now to shift our attention to a more specific kind of Γ-injectivity. We require a few more definitions.

Definition 2.1.6. A Γ-*extension* of $\mathcal{S}$ is a pair $(\mathcal{T}, \iota)$, where $\mathcal{T}$ is a Γ-operator system and $\iota : \mathcal{S} \to \mathcal{T}$ is a completely isometric Γ-equivariant map.

Definition 2.1.7. A Γ-extension $(\mathcal{T}, \iota)$ is called:

- Γ-*injective* iff $\mathcal{T}$ is Γ-injective.
- Γ-*essential* iff for every Γ-map $\varphi : \mathcal{T} \to \mathcal{U}$, φ is completely isometric whenever $\varphi \circ \iota$ is.
- Γ-*rigid* iff for every Γ-map $\varphi : \mathcal{T} \to \mathcal{T}$, φ is the identity on $\mathcal{T}$ whenever $\varphi \circ \iota = \iota$.

Definition 2.1.8. A Γ-extension of $\mathcal{S}$ that is both Γ-injective and Γ-essential is called a Γ-*injective envelope* of $\mathcal{S}$.

A couple of remarks are now in order. Firstly, every Γ-operator system $\mathcal{S} \subseteq B(\mathcal{H})$ has a Γ-injective extension. To see this, we simply notice that the map $j : \mathcal{S} \to \ell^\infty(\Gamma, B(\mathcal{H})) : x \mapsto (s^{-1}x)_{s\in\Gamma}$ is a Γ-monomorphism. Since $\ell^\infty(\Gamma, B(\mathcal{H}))$ is Γ-injective (by Lemma 2.1.5), it is a Γ-injective extension .

Secondly, if $\mathcal{S}$ is Γ-injective, then $\mathrm{id}_{\mathcal{S}}$ factors through j, producing a Γ-map $\varphi : \ell^\infty(\Gamma, \mathcal{S}) \to \mathcal{S}$ such that $\varphi \circ j = \mathrm{id}_{\mathcal{S}}$. Furthermore, if $\psi : \mathcal{T} \to \mathcal{S}$ is a c.p. map and $i : \mathcal{T} \to \mathcal{U}$ is a complete isometry, then ψ factors through i as seen in the commutative diagram below. Thus, $\mathcal{S}$ is also injective in $\mathfrak{G}$ (as is $\ell^\infty(\Gamma, B(\mathcal{H}))$; an easy consequence of Arveson's theorem).

$$\begin{tikzcd}
\mathcal{U} \arrow[drr, dashed, "\widehat{\psi\circ j}"] \arrow[drrr, dashed, bend left, "\hat{\psi}"] & & & \\
\mathcal{T} \arrow[u, "i"] \arrow[r, "\psi"] \arrow[rrr, bend right, "\psi"'] & \mathcal{S} \arrow[r, "j"] & \ell^\infty(\Gamma, B(\mathcal{H})) \arrow[r, "\varphi"] & \mathcal{S}
\end{tikzcd}$$

Conversely, if $\mathcal{S}$ is injective in $\mathfrak{G}$ and there exists a Γ-map φ extending $\mathrm{id}_{\mathcal{S}}$ as above, then $\mathcal{S}$ is Γ-injective (using practically the same diagram).

From now on we will freely assume $\mathcal{S} \subseteq \ell^\infty(G, \mathcal{S})$, forgetting the Γ-monomorphism involved (thus the map φ defined above will be regarded as a projection).

As we shall see next, every Γ-operator system also has a Γ-injective envelope - and a unique one at that.

Let us fix operator systems $\mathcal{S} \subseteq \mathcal{T} \subseteq B(\mathcal{H})$ such that $\mathcal{T}$ is Γ-injective and $\mathcal{S}$ is a Γ-operator subsystem of $\mathcal{T}$.

Definition 2.1.9. An $\mathcal{S}$-*map* on $\mathcal{T}$ is a Γ-map $\varphi : \mathcal{T} \to \mathcal{T}$ that fixes $\mathcal{S}$ pointwise. An $\mathcal{S}$-*seminorm* on $\mathcal{T}$ is a seminorm p on $\mathcal{T}$ such that $p(.) = \|\varphi(.)\|$ for some $\mathcal{S}$-map φ on $\mathcal{T}$. An $\mathcal{S}$-map is called an $\mathcal{S}$-*projection* iff it is idempotent.

Lemma 2.1.10. *There exists a minimal $\mathcal{S}$-seminorm on $\mathcal{T}$.*

Proof. We first note that the unit ball, call it B_1, of $B(\mathcal{T}, \ell^\infty(\Gamma, B(\mathcal{H})))$, the space of bounded linear maps from T into $\ell^\infty(\Gamma, B(\mathcal{H}))$, is compact in the point-σ-topology, i.e. the topology of pointwise convergence where $\ell^\infty(\Gamma, B(\mathcal{H}))$ is endowed with the σ-weak topology.

Indeed, it is a closed subset of the topological space

$$\prod_{x \in \mathcal{T}} \{y \in \ell^\infty(\Gamma, B(\mathcal{H})) : \|y\| \leqslant \|x\|\},$$

which is compact by Tikhonov's theorem and the Banach-Alaoglu theorem.

Now we would like to invoke Zorn's lemma, so let (p_i) be a decreasing net of $\mathcal{S}$-seminorms on $\mathcal{T}$ and $\varphi_i : \mathcal{T} \to \mathcal{T}$ be corresponding $\mathcal{S}$-projections.

Regarding (φ_i) as a net in B_1, there exists a subnet (φ_j) and a $\varphi_0 \in B_1$ such that $\varphi_j(x) \to \varphi_0(x)$ σ-weakly for all $x \in \mathcal{T}$. It is immediate that φ_0 is completely positive (because the positive cone is closed in the σ-weak topology), $\varphi_0|_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$ (because $\mathcal{S}$ is fixed pointwise by every φ_i), and Γ-equivariant (because Γ acts by unital complete order isomorphisms, which are continuous). However, φ_0 need not be a Γ-projection, since its image is not necessarily contained in $\mathcal{T}$!

This is easily cured of course, because $\mathcal{T}$ is Γ injective and so there is a Γ-equivariant u.c.p. projection ψ from $\ell^\infty(\Gamma, B(\mathcal{H}))$ onto it, hence now $\psi \circ \varphi_0$ is a bona fide $\mathcal{S}$-projection on $\mathcal{T}$. Moreover, for all $x \in \mathcal{T}$ we have

$$p_0(x) := \|\psi \circ \varphi_0(x)\| \leqslant \|\varphi_0(x)\| \leqslant \limsup \|\varphi_j(x)\| = \lim p_i(x) \leqslant p_i(x) \ \forall i,$$

since p_i is decreasing.

Thus, Zorn's lemma can indeed be invoked and we are done. ∎

Lemma 2.1.11. *If $\varphi : \ell^\infty(\Gamma, B(\mathcal{H})) \to \ell^\infty(\Gamma, B(\mathcal{H}))$ is an $\mathcal{S}$-map corresponding to a minimal $\mathcal{S}$-seminorm, then φ is an $\mathcal{S}$-projection and $\mathcal{T} = \varphi(\ell^\infty(\Gamma, B(\mathcal{H})))$ is Γ-injective.*

Proof. To show that φ is an $\mathcal{S}$ projection, notice that $\varphi \circ \varphi$ is also an $\mathcal{S}$-map and $\|(\varphi \circ \varphi)(x)\| \leqslant \|\varphi(x)\|$ for all $x \in \ell^\infty(\Gamma, B(\mathcal{H}))$. By minimality, we get equality and inductively $\left\|\varphi^{(n)}(x)\right\| = \|\varphi(x)\|$ for all x and all n. By setting $\psi_n = (\varphi + \cdots + \varphi^{(n)})/n$ and using the same argument, we get $\|\psi_n(x)\| = \|\varphi(x)\|$ for all x and all n. Thus,

$$\begin{aligned}
\|\varphi(x) - (\varphi \circ \varphi)(x)\| &= \|\varphi(x - \varphi(x))\| = \|\psi_n(x - \varphi(x))\| \\
&= \left\| \frac{\varphi(x) + \cdots + \varphi^{(n)}(x)}{n} - \frac{\varphi^{(2)}(x) + \cdots + \varphi^{(n+1)}(x)}{n} \right\| \\
&\leqslant \frac{2\,\|\varphi(x)\|}{n} \to 0.
\end{aligned}$$

Hence, $\|\varphi(x) - (\varphi \circ \varphi)(x)\| = 0$ for all x, i.e. φ is an $\mathcal{S}$-projection.

Now Γ-injectivity is immediate, since $\ell^\infty(\Gamma, B(\mathcal{H}))$ is Γ-injective. ∎

Lemma 2.1.12. *Using the notation of the previous lemma, $(\mathcal{T}, \iota)$ is a Γ-rigid extension of $\mathcal{S}$ (ι is simply the inclusion map).*

Proof. Let $\psi : \mathcal{T} \to \mathcal{T}$ be a Γ-map such that $\psi|_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$. Since $\|(\psi \circ \varphi)(x)\| \leqslant \|\varphi(x)\|$ for all $x \in \ell^\infty(\Gamma, B(\mathcal{H}))$, we have $\|\psi(\varphi(x))\| = \|\varphi(x)\|$ by minimality. Thus ψ is an isometry. Furthermore, by the previous lemma, $\psi \circ \varphi$ is a projection, so $\psi \circ \varphi = \psi \circ \varphi \circ \psi \circ \varphi = \psi \circ \psi \circ \varphi$ (since φ is the identity on $\mathcal{T}$). Therefore, $\psi(\varphi - \psi \circ \varphi) = 0$. Now since ψ is an isometry, we get $\psi \circ \varphi = \varphi$, i.e. $\psi = \mathrm{id}_{\mathcal{T}}$. ∎

Lemma 2.1.13. *If $(\mathcal{T}, \iota)$ is a Γ-extension of $\mathcal{S}$ that is both Γ-injective and Γ-rigid, then it is also Γ-essential.*

Proof. Let $\varphi : \mathcal{T} \to \mathcal{U}$ be a Γ-map such that $\varphi \circ \iota$ is completely isometric. Consider the Γ-map $\iota \circ (\varphi \circ \iota)^{-1} : (\varphi \circ \iota)(\mathcal{S}) \to \mathcal{T}$. By Γ-injectivity, it extends to a Γ-map $\psi : \mathcal{U} \to \mathcal{T}$. But then the Γ-map $\omega = \psi \circ \varphi : \mathcal{T} \to \mathcal{T}$ satisfies $\omega \circ \iota = \iota$ and so $\omega = \mathrm{id}_{\mathcal{T}}$ by Γ-rigidity. Thus, φ is completely isometric and Γ-essentiality is proven. ∎

We now have all the ingredients needed for the main theorem of this section.

Theorem 2.1.14. *Every Γ-operator system $\mathcal{S} \subset B(\mathcal{H})$ has a Γ-injective envelope which is unique in the sense that if $(\mathcal{T}_1, \iota_1)$ and $(\mathcal{T}_2, \iota_2)$ are two Γ-injective envelopes of $\mathcal{S}$, then there exists a Γ-isomorphism $\omega : \mathcal{T}_1 \to \mathcal{T}_2$ such that $\omega \circ \iota_1 = \iota_2$.*

Proof. Let $\varphi : \ell^\infty(\Gamma, B(\mathcal{H})) \to \ell^\infty(\Gamma, B(\mathcal{H}))$ be an $\mathcal{S}$-map inducing a minimal $\mathcal{S}$-seminorm on $\ell^\infty(\Gamma, B(\mathcal{H}))$. By Lemma 2.1.11, $\mathcal{T} := \varphi(\ell^\infty(\Gamma, B(\mathcal{H}))$ is a Γ-injective extension of $\mathcal{S}$. By Lemmas 2.1.12 and 2.1.13, it is also Γ-rigid and thus Γ-essential. We have shown existence.

To prove uniqueness, let $(\mathcal{T}_1, \iota_1)$ be another Γ-injective envelope of $\mathcal{S}$. By Γ-injectivity of $\mathcal{T}_1$, there exists a Γ-map $\psi : \mathcal{T} \to \mathcal{T}_1$ extending ι_1. Similarly, there exists a Γ-map $\omega : \mathcal{T}_1 \to \mathcal{T}$ such that $\omega \circ \iota_1 = \mathrm{id}_{\mathcal{S}}$. Then $\omega \circ \psi$ fixes $\mathcal{S}$ pointwise, so by Γ-rigidity of $\mathcal{T}$, $\omega \circ \psi = \mathrm{id}_{\mathcal{T}}$. On the other hand, $\psi \circ \omega \circ \iota_1 = \iota_1$ and, by Γ-essentiality of $\mathcal{T}_1$, $\psi \circ \omega$ is completely isometric. Hence, ω is a Γ-isomorphism which, by definition, satisfies the condition we wanted.

$$\begin{tikzcd}
\mathcal{T}_1 \arrow[rr, dashed, shift left, "\omega"] & & \mathcal{T} \arrow[ll, dashed, shift left, "\psi"] \\
& \mathcal{S} \arrow[ul, "\iota_1"] \arrow[ur, hook] &
\end{tikzcd}$$

∎

Remark. A by-product of the above is that Γ-essentiality can be replaced by Γ-rigidity in the definition of the Γ-injective envelope. That, however, is not the only meaningful way the definition can be altered. The interested reader can refer to Paulsen's book [Pau03, Theorem 15.8] for more details.

We will denote the Γ-injective envelope of $\mathcal{S}$ by $I_\Gamma(\mathcal{S})$, omitting the Γ-monomorphism involved (since uniqueness renders it practically redundant in our line of study) and identifying $\mathcal{S}$ with its image inside $I_\Gamma(\mathcal{S})$.

Having established existence and uniqueness, we continue by proving that every Γ-injective envelope is, in a way, a C^*-algebra. More generally, Choi and Effros proved in [CE77] the following.

Theorem 2.1.15. *Let $\mathcal{S}$ be injective in $\mathfrak{G}$ and let $\varphi : \mathcal{A} \to \mathcal{S}$ be a u.c.p. projection from a C^*-algebra $\mathcal{A} \supseteq \mathcal{S}$ onto $\mathcal{S}$ (just extend $\mathrm{id}_{\mathcal{S}}$ by injectivity). Then the operation $x \circ y = \varphi(xy)$ defines a multiplication on $\mathcal{S}$ which, along with the original involution, turns it into a C^*-algebra.*

Proof. Clearly, $\circ$ is well-defined, distributive and has an identity (the same as the original). It remains to show associativity.

We will achieve this by showing that $\varphi(\varphi(a)x) = \varphi(ax)$ and $\varphi(x\varphi(a)) = \varphi(xa)$ for any $a \in \mathcal{A}$ and $x \in \mathcal{S}$.

To that end, consider the matrix $A = \begin{bmatrix} x^* & a \\ 0 & 0 \end{bmatrix}$. Using the Schwarz inequality for 2-positive maps on $\varphi_{(2)}$ and A, we obtain

$$\begin{bmatrix} \varphi(xx^*) & \varphi(xa) \\ \varphi(a^*x^*) & \varphi(a^*a) \end{bmatrix} - \begin{bmatrix} xx^* & x\varphi(a) \\ \varphi(a^*)x^* & \varphi(a)^*\varphi(a) \end{bmatrix} \geqslant 0.$$

and applying $\varphi_{(2)}$ yields

$$\begin{bmatrix} 0 & \varphi(xa) - \varphi(x\varphi(a)) \\ \varphi(a^*x^*) - \varphi(\varphi(a)x^*) & \varphi(a^*a) - \varphi(\varphi(a)^*\varphi(a)) \end{bmatrix} \geqslant 0.$$

Thus, $\varphi(xa) - \varphi(x\varphi(a)) = 0 = \varphi(a^*x^*) - \varphi(\varphi(a^*)x^*)$.

It remains to verify the C^*-condition. One direction is clear, since $\|x^* \circ x\| = \|\varphi(x^*x)\| \leqslant \|x^*x\| = \|x\|^2$. The other is not that hard either, since by the Schwarz inequality we have $\varphi(x^*x) \geqslant \varphi(x^*)\varphi(x) = x^*x$ and therefore $\|\varphi(x^*x)\| \geqslant \|x^*x\| = \|x\|^2$. ■

The multiplication defined in the above theorem will be henceforth referred to as the *Choi-Effros product*.

2.2 The Hamana Boundary

2.2.1 Definition and Universality

With the tools developed in the previous section at our disposal, we turn our attention to a particular case. Consider $\mathbb{C}$ equipped with the trivial

Γ-action. Then, as we saw, there is a u.c.p. projection $\ell^\infty(\Gamma) \to I_\Gamma(\mathbb{C})$. Thus, $I_\Gamma(\mathbb{C})$ equipped with the corresponding Choi-Effros product is turned into a commutative (since $\ell^\infty(\Gamma)$ is commutative) C^*-algebra.

Definition 2.2.1. We define the *Hamana boundary* of Γ, denoted by $\partial_H\Gamma$, to be the Gelfand spectrum of $I_\Gamma(\mathbb{C})$.

Remark. The action of Γ on $I_\Gamma(\mathbb{C})$ induces an action on $\mathcal{P}(\partial_H\Gamma)$, the space of probability measures on $\partial_H\Gamma$, by

$$\int f\, d(s\mu) = \int s^{-1}f\, d\mu,\;\; s \in \Gamma, f \in I_\Gamma(\mathbb{C}), \mu \in \mathcal{P}(\partial_H\Gamma).$$

The restriction of this action to the Dirac measures is, by the usual identification, a Γ-action on $\partial_H\Gamma$.

Definition 2.2.2. A locally compact Hausdorff space X is called a Γ-*space* iff Γ acts on it by homeomorphisms.

Hence the previous remark tells us that $\partial_H\Gamma$ is a compact Γ-space, as is $\mathcal{P}(\partial_H\Gamma)$ endowed with the weak-* topology. We will see that the former is of a very special kind.

Before proceeding, notice that $\partial_H\Gamma$ already has an immediate use; it detects amenability. Indeed, amenability is equivalent to the existence of a Γ-map $\ell^\infty(\Gamma) \to \mathbb{C}$ which in turn is equivalent to $I_\Gamma(\mathbb{C}) = \mathbb{C}$, i.e. $\partial_H\Gamma$ being a singleton.

We want now to introduce the notion of Γ-boundaries in the sense of Furstenberg [Fur73], in order to show that the Hamana boundary is properly titled in that context.

Definition 2.2.3. Let X be a compact Γ-space. The Γ-action on X is called:

- *minimal* iff $\overline{\Gamma x} = X$ for all $x \in X$.
- *proximal* iff for every pair of points $x, y \in X$ there exists a net (s_i) in Γ such that $\lim s_i x = \lim s_i y$.
- *strongly proximal* iff the induced Γ-action on $\mathcal{P}(X)$ is proximal iff for every $\mu \in \mathcal{P}(X)$ there exist a Dirac measure in $\overline{\Gamma\mu}^{w^*}$.

Definition 2.2.4. A compact Γ-space X is called a Γ-*boundary* iff the Γ-action on X is minimal and strongly proximal.

Proposition 2.2.5. *The Γ-action on $\partial_H\Gamma$ is minimal.*

Proof. Let $x \in \partial_H\Gamma$ and consider the restriction map $r : C(\partial_H\Gamma) \to C(\overline{\Gamma x})$. Then r is a Γ-map which is completely isometric on $\mathbb{C}$. By the Γ-essentiality of $C(\partial_H\Gamma)$, r is completely isometric. That forces $\overline{\Gamma x} = \partial_H\Gamma$, because otherwise Urysohn's lemma guarantees $\ker r$ is non-empty. ∎

Remark. The above proof showcased a very nice property of $C(\partial_H\Gamma)$. Every Γ-map with $C(\partial_H\Gamma)$ as its domain is automatically completely isometric, since it is always completely isometric on $\mathbb{C}$.

Proposition 2.2.6. *The Γ-action on $\partial_H\Gamma$ is strongly proximal.*

Proof. Let $\mu \in \mathcal{P}(\partial_H\Gamma), x \in \partial_H\Gamma$ and $K = \overline{\operatorname{conv}\{\Gamma\mu\}}^{w^*}$. We will show that $\delta_x \in K$.

Indeed, if not, then by the Hahn-Banach separation theorem we can find a positive $f_0 \in C(\partial_H\Gamma)$ and $\varepsilon > 0$ such that

$$\int f_0 \, d(s\mu) \leqslant \int f_0 \, d\delta_x - \varepsilon \leqslant \|f_0\| - \varepsilon, \ \forall s \in \Gamma.$$

This implies that the so-called *Poisson map* $P_\mu : C(\partial_H\Gamma) \to \ell^\infty(\Gamma)$, defined by

$$P_\mu(f)(s) = \int f \, d(s\mu), \ f \in C(\partial_H\Gamma), s \in \Gamma,$$

satisfies $\|P_\mu(f_0)\| \leqslant \|f_0\| - \varepsilon$. But P_μ is a Γ-map and thus isometric by the above remark, a contradiction. Hence $\delta_x \in K$.

We just proved that K contains the extreme points of $\mathcal{P}(\partial_H\Gamma)$, so $K = \mathcal{P}(\partial_H\Gamma)$. Therefore, the Dirac measures are contained in $\overline{\Gamma\mu}^{w^*}$ by Milman's partial converse to the Krein-Milman theorem. ∎

Corollary 2.2.7. *The Hamana boundary $\partial_H\Gamma$ is a Γ-boundary.*

Furstenberg proved the existence and uniqueness of a universal object among Γ-boundaries, in the sense that every Γ-boundary is a continuous Γ-equivariant image of it, introducing what came to be known as the *Furstenberg boundary* $\partial_F\Gamma$ of Γ. In the remainder of this section we will prove, by completely different techniques, that $\partial_H\Gamma$ also satisfies this condition and thus the Hamana boundary and the Furstenberg boundary of a discrete group are one and the same. This identification was observed by Hamana himself in [Ham78, Remark 4] but was left unnoticed until Kalantar and Kennedy gave a formal proof in [KK17].

Lemma 2.2.8. *If M is a minimal compact Γ-space and X is a Γ-boundary, then any continuous Γ-equivariant map $M \to \mathcal{P}(X)$ has X as its range (i.e. the Dirac measures). Moreover, there exists at most one continuous Γ-equivariant map $M \to X$.*

Proof. Let $\alpha : M \to \mathcal{P}(X)$ be a continuous Γ-equivariant map. $\alpha(M) \subseteq \mathcal{P}(X)$ is compact, therefore closed, and Γ-invariant. Thus, since X is a boundary, $X \subseteq \alpha(M)$. In particular, we can choose $m \in M$ such that $\alpha(m) \in X$. By Γ-equivariance and minimality of M, $\alpha(M) = \alpha(\overline{\Gamma m}) \subseteq \overline{\Gamma\alpha(m)} = X$, so $\alpha(M) = X$.

Now if $\beta_1, \beta_2 : M \to X$ are continuous Γ-equivariant maps, then $\alpha : M \to \mathcal{P}(X) : m \mapsto \frac{1}{2}(\delta_{\beta_1(m)} + \delta_{\beta_2(m)})$ also is. Since $\alpha(M) = X$ and the Dirac measures are extreme points in $\mathcal{P}(X)$, we have $\beta_1 = \beta_2$. ∎

Corollary 2.2.9. *Let M and X be as in the previous lemma. There is at most one Γ-map $C(X) \to C(M)$. If such a map exists, then it is a unital *-homomorphism.*

Proof. Let $\varphi : C(X) \to C(M)$ be a Γ-map. Then the adjoint map φ^* restricts to a continuous Γ-equivariant map $\alpha : M \to \mathcal{P}(X)$, which, by the previous lemma, is unique and has X as its image. But α induces an injective *-homomorphism $C(X) \to C(M)$ by $f \mapsto f \circ \alpha$, which is actually the map φ we started with, since

$$(f \circ a)(m) := (\varphi^*(\delta_m))(f) = \delta_m(\varphi(f)) = \varphi(f(m)), \ f \in C(X), m \in M.$$

■

Theorem 2.2.10. *Let X be a Γ-boundary. Then there exists a continuous Γ-equivariant map $\partial_H\Gamma \twoheadrightarrow X$. Hence $\partial_H\Gamma$ and $\partial_F\Gamma$ can be identified.*

Proof. Fix any point $x \in X$ and consider the continuous Γ-equivariant map $\alpha_x : s \mapsto sx, s \in \Gamma$. By the universal property of the Stone-Čech compactification $\beta\Gamma$ of Γ, we can extend this map to a continuous Γ-equivariant map $\widehat{\alpha_x} : \beta\Gamma \to X$. Since $\widehat{\alpha_x}(\beta\Gamma)$ is compact and Γ-invariant, $\widehat{\alpha_x}$ is surjective.

Now $\widehat{\alpha_x}$ induces a unital isometric G-equivariant *-homomorphism $\iota : C(X) \to C(\beta\Gamma) = \ell^\infty(\Gamma)$ by $f \mapsto f \circ \widehat{\alpha_x}$. Composing ι with the idempotent u.c.p. Γ-equivariant projection $\ell^\infty(\Gamma) \twoheadrightarrow C(\partial_H\Gamma)$ produces a Γ-map $C(X) \to C(\partial_H\Gamma)$, which, as in the proof of the previous corollary, induces a continuous Γ-equivariant map $\partial_H\Gamma \twoheadrightarrow X$. ■

Henceforth, we will use the unifying $\partial_{FH}\Gamma$ to denote the Furstenberg-Hamana boundary of Γ.

2.2.2 Further Properties

Definition 2.2.11. A topological space is called *extremally disconnected* or *Stonean* iff the closure of every open set is open.

Proposition 2.2.12. *The Furstenberg-Hamana boundary of Γ is extremally disconnected.*

Proof. Let U be an open subset of $\partial_{FH}\Gamma$ and let $K = (\overline{U} \times \{0\}) \cup (U^c \times \{1\})$. Pick $x_0 \in \partial_{FH}\Gamma$ and define $\alpha : \Gamma \to K$ by

$$\alpha(s) = \begin{cases} (sx_0, 0), & \text{if } sx_0 \in U \\ (sx_0, 1), & \text{otherwise.} \end{cases}$$

K is compact in $\partial_{FH}\Gamma \times \{0,1\}$, so we can extend α to a continuous map $\widehat{\alpha} : \beta\Gamma \to K$.

We know there is a u.c.p. Γ equivariant projection $C(\beta\Gamma) = \ell^\infty(\Gamma) \to C(\partial_{FH}\Gamma)$, which induces a continuous Γ-equivariant $\beta : \partial_{FH}\Gamma \to \mathcal{P}(\beta\Gamma)$. Consider the composition $\gamma := \pi_* \circ \widehat{\alpha}_* \circ \beta : \partial_{FH}\Gamma \to \mathcal{P}(\partial_{FH}\Gamma)$ where $\pi : K \to \partial_{FH}\Gamma$ is the projection on the first coordinate (we are using the standard push-forward notation). Since $\pi \circ \alpha : s \mapsto sx_0$ is Γ-equivariant, so is $\pi \circ \widehat{\alpha}$ by continuity. Thus, γ is a continuous Γ-equivariant map $\partial_{FH}\Gamma \to \mathcal{P}(\partial_{FH}\Gamma)$. By Lemma 2.2.8, γ maps elements in $\partial_{FH}\Gamma$ to the corresponding Dirac measures and therefore $\mu_x := \widehat{\alpha}_* \circ b(x)$ is supported on $\{x\} \times \{0,1\}$ for any $x \in \partial_{FH}\Gamma$. This immediately gives us that $\mu_x(U^c \times \{1\}) = 0$ if $x \in U$, and $\mu_x(U^c \times \{1\}) = 1$ if $x \notin \overline{U}$. By the continuity of both the map $x \mapsto \mu_x$ and the indicator function $\mathbb{1}_{U^c \times \{1\}}$, we get that the map $x \mapsto \mu_x(U^c \times \{1\})$ is also continuous and hence forced to be the indicator function $\mathbb{1}_{U^c}$. This implies that U^c is clopen. ■

This property is, in a sense, quite discouraging, as extremally disconnected spaces are topologically not easy to grasp or pinpoint. It also eliminates any hope to generalize anything discussed in this work to the non-discrete case. The reason is that, although the Hamana boundary is always extremally disconnected, as the spectrum of an injective commutative C^*-algebra [Gle58, Theorem 5.1], the Furstenberg boundary can be much nicer (e.g. metrizable) when the group is not discrete (e.g. a semisimple Lie group) [Gla76]. Thus, we cannot, in general, make the crucial identification of the two.

Lemma 2.2.13. *For every $x \in \partial_{FH}\Gamma$, the point stabilizer $\Gamma_x = \{s \in \Gamma : sx = x\}$ is amenable.*

Proof. Let $x \in \partial_{FH}\Gamma$ and consider the u.c.p. Γ-equivariant projection $\varphi : \ell^\infty(\Gamma) \to C(\partial_{FH}\Gamma)$. The composition $ev_x \circ \varphi : \ell^\infty(\Gamma) \to \mathbb{C}$ is a Γ_x-invariant state, since

$$\begin{aligned}(s(ev_x \circ \varphi))(f) &= ev_x(s^{-1}\varphi(f)) = (s^{-1}\varphi(f))(x) \\ &= \varphi(f)(sx) = \varphi(f)(x) \\ &= (ev_x \circ \varphi)(f),\end{aligned}$$

for all $s \in \Gamma_x$ and $f \in \ell^\infty(\Gamma)$.

Now, let (s_i) be a transversal of the right coset space $\Gamma \backslash \Gamma_x$. Define a map $\psi : \ell^\infty(\Gamma_x) \to \ell^\infty(\Gamma)$ by $\psi(f)(s) = f(t)$, where $t \in \Gamma_x$ satisfies $s = ts_i$ for some i. Then ψ is a Γ_x-equivariant unital *-homomorphism and thus the composition $ev_x \circ \varphi \circ \psi$ is a Γ_x invariant state on $\ell^\infty(\Gamma_x)$, witnessing amenability. ■

Proposition 2.2.14. *The kernel of the action of Γ on $\partial_{FH}\Gamma$ coincides with the amenable radical $R_a(\Gamma)$ of Γ.*

Proof. Being amenable, $R_a(\Gamma)$ fixes a probability measure μ on $\partial_{FH}\Gamma$. Since it is also a normal subgroup, it fixes every measure in $\overline{\Gamma\mu}^{w^*}$. By strong proximality, $\overline{\Gamma\mu}$ contains a Dirac measure and thus, by minimality, all of them. Therefore, $R_a(\Gamma)$ fixes all of $\partial_{FH}\Gamma$, i.e. it is contained in the kernel of the action.

Conversely, the kernel of the action is the intersection of the point stabilizers, which is amenable by the previous lemma and thus contained in $R_a(\Gamma)$. ∎

Chapter 3

The Reduced Group C*-Algebra

In 1967, at a conference held in Baton Rouge, Dixmier posed the question of whether every simple C*-algebra is generated by its projections. At some point in 1968, Kadison, who already had conversations of a similar nature with Kaplansky back in 1949, suggested to Powers that the reduced C*-algebra of $\mathbb{F}_2$, the free group on two generators, should prove to be simple, yet projectionless. The latter managed to prove the first half of this statement, establishing C*-simplicity of $\mathbb{F}_2$, within a week. Powers, however, did not really care to publish this result, as he failed to prove the rest. In fact, it took him seven years to publish [Pow75], and only after a request by Akemann, who wanted to use it.

Nevertheless, Powers' work proved to be quite important. Powers' averaging property (the main ingredient in his proof) became, modulo variants and modifications of it, essentially the only tool to prove C*-simplicity, for decades. Spearheaded by de la Harpe's efforts, the list of C*-simple groups grew slowly but steadily using this kind of combinatorial methods and, along with it, new questions concerning the reduced C*-algebras of discrete groups quickly arose.

One main problem was to clarify the precise relationship between C*-simplicity and the unique trace property, as both implied triviality of the amenable radical, but no other direct connection between them had been found.

In this chapter, we will present dynamical characterisations of both properties, leading to a complete solution of the above problem, as well as a characterisation of exactness that fits nicely in our framework.

3.1 C*-Simplicity

3.1.1 The Main Theorem

The purpose of this subsection is to prove the following theorem, established in [KK17] and [BKKO17].

Theorem 3.1.1. *The following are equivalent:*

1. Γ *is* C^**-simple.*
2. *The reduced crossed product* $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$ *is simple.*
3. *The reduced crossed product* $C(X) \rtimes_r \Gamma$ *is simple for some* Γ*-boundary* X*.*
4. *The* Γ*-action on* $\partial_{FH}\Gamma$ *is (topologically) free.*
5. *The* Γ*-action on some* Γ*-boundary is (topologically) free.*

Before we proceed with this task, we have to give the necessary definitions.

Definition 3.1.2. Γ is called C^**-simple* iff the reduced C^*-algebra $C^*_r(\Gamma)$ is simple (i.e. it has no non-trivial closed ideals).

This definition, albeit completely natural, is not always the most useful. We will need the following characterisation, given as a definition for example in [Har07].

Proposition 3.1.3. Γ *is* C^**-simple iff, for every unitary representation* π *of* Γ*, the conditions* $\pi \prec \lambda_\Gamma$ *and* $\pi \sim \lambda_\Gamma$ *are equivalent.*

Proof. The proof is a simple application of Proposition 1.3.4. Assume Γ is C^*-simple and let $\pi \prec \lambda_\Gamma$. Then, $C^* \ker \pi \supseteq C^* \ker \lambda_\Gamma$. Therefore, $C^* \ker \pi / C^* \ker \lambda_\Gamma$ is a closed ideal of $C^*(\Gamma)/C^* \ker \lambda_\Gamma \cong C^*_r(\Gamma)$ and thus $C^* \ker \pi = C^* \ker \lambda_\Gamma$. Hence, $\pi \sim \lambda_\Gamma$.

Conversely, assume Γ is not C^*-simple and let I be a non-trivial closed ideal of $C^*_r(\Gamma)$. Seeing $C^*_r(\Gamma)$ as the quotient $C^*(\Gamma)/C^* \ker \lambda_\Gamma$, I is of the form $J/C^* \ker \lambda_\Gamma$ for some closed ideal $J \supsetneq C^* \ker \lambda_\Gamma$ of $C^*(\Gamma)$. But J, being a closed ideal of $C^*(\Gamma)$, is of the form $C^* \ker \pi$ for some unitary representation π of Γ. Thus, we have $\pi \prec \lambda_\Gamma$ and $\pi \nsim \lambda_\Gamma$. ■

Definition 3.1.4. Let X be a Γ-space. The Γ-action on X is called *topologically free* iff $X^s := \{x \in X : sx = x\}$ has empty interior for all $s \in \Gamma \setminus \{e\}$.

We will split the proof of the theorem in several steps.

Lemma 3.1.5. *Let* X *be an extremally disconnected Hausdorff space and suppose* $f : X \to X$ *is a homeomorphism. Then the set* F *of fixed points of* f *is clopen.*

Proof. Call an open set $U \subseteq X$ f-*simple* iff $f(U) \cap U = \emptyset$. f-simplicity is closed under increasing unions and, thus, there exists a maximal f-simple set U_0. By extremal disconnectedness, $\overline{U_0}$ is also open and $f(\overline{U_0}) \cap U_0 \subseteq \overline{f(U_0)} \cap U_0 = \emptyset$. Hence $\overline{f(U_0)} \cap \overline{U_0} = \emptyset$, i.e. $\overline{U_0}$ is f-simple and so U_0 is clopen by maximality.

Clearly, $F \cap U_0 = \emptyset$ and therefore $F \cap f(U_0) = F \cap f^{-1}(U_0) = \emptyset$. Let $V = U_0 \cup f(U_0) \cup f^{-1}(U_0) \subseteq F^c$. We claim that equality holds, which would finish the proof. If not, let $x \in V^c$ be such that $f(x) \neq x$. Since X is Hausdorff and V is closed, there exists an open neighbourhood W of x such that $W \cap f(W) = W \cap V = \emptyset$. In particular, $W \cup U_0 \supsetneq U_0$ is f-simple, contradicting maximality of U_0. ∎

Corollary 3.1.6. *The Γ-action on $\partial_{FH}\Gamma$ is free iff it is topologically free.*

Proof. By Proposition 2.2.12, $\partial_{FH}\Gamma$ is extremally disconnected. By the previous lemma, the set $(\partial_{FH}\Gamma)^s$ is open for all $s \in \Gamma$, and thus it is empty iff it has empty interior. ∎

Lemma 3.1.7. *If the Γ-action on some Γ-boundary is topologically free, then so is the Γ-action on $\partial_{FH}\Gamma$.*

Proof. Let X be a Γ-boundary on which Γ acts topologically freely and assume that there exists an $s \in \Gamma$ such that $F := (\partial_{FH}\Gamma)^s$ has non-empty interior. Consider the Γ-map $\pi : \partial_{FH}\Gamma \twoheadrightarrow X$ provided by universality. Notice that $\pi(F)$ is compact (in particular closed) and has empty interior, as it is contained in X^s.

Let $U \subseteq F$ be open and non-empty. By minimality, we have $\Gamma U = \partial_{FH}\Gamma$. By compactness, there exist $s_1, s_2, \dots, s_n \in \Gamma$ such that $\cup_{i=1}^n s_i U = \partial_{FH}\Gamma$, and therefore $\cup_{i=1}^n s_i \pi(U) = X$. A fortiori, $\cup_{i=1}^n s_i \pi(F) = X$, a contradiction. ∎

We have proved (4) $\iff$ (5) of the theorem. Now we want to throw condition (1) into the mix.

Lemma 3.1.8. *Let X be a non-trivial Γ-boundary. Then X is infinite and contains no isolated points.*

Proof. If X were finite, then the uniform probability measure on it would be fixed by Γ, contradicting strong proximality. Now, if we assume $x \in X$ is an isolated point (i.e. $\{x\}$ is open), minimality implies that $\Gamma x = X$ and thus, by compactness, $Fx = X$ for some finite $F \subseteq \Gamma$. Therefore X is finite, a contradiction. ∎

Lemma 3.1.9. *Let X be a Γ-boundary. For every non-empty open set $U \subseteq X$ and every $\varepsilon > 0$, there exists a finite set $F \subseteq \Gamma \setminus \{e\}$ such that for every $\mu \in \mathcal{P}(\partial_{FH}\Gamma)$, there exists an $s \in F$ such that $\mu(sU) > 1 - \varepsilon$.*

Proof. If X is trivial (i.e. a singleton) there is nothing to prove, so we will assume otherwise. To that end, fix $x \in U$ and let $\mu \in \mathcal{P}(X)$.

If $\mu = \delta_x$, let $y \in U \setminus \{x\}$ (such y exists by the previous lemma) and let V be an open neighbourhood of y separating it from x. By minimality, there exists an $s_{\delta_x} \in \Gamma$ such that $s_{\delta_x}^{-1}x \in U \cap V$ and since $x \notin V$, s_{δ_x} is not the identity. Then of course $\delta_x(s_{\delta_x}U) = \delta_{s_{\delta_x}^{-1}x}(U) = 1 > 1 - \varepsilon$.

If $\mu \neq \delta_x$, by strong proximality and minimality there exists a net (s_i) in Γ such that $s_i\mu \to \delta_x$ and we can freely assume no s_i is the identity (by removing terms). By Urysohn's lemma, there exists $f \in C(X)$ such that $f(x) = 1$ and $0 \leqslant f \leqslant \mathbb{1}_U$. By definition, $s_i\mu \to \delta_x$ implies $(s_i\mu)(f) \to f(x) = 1$, thus we can pick an $s_\mu \in \Gamma$ such that $\mu(s_\mu U) = (s_\mu\mu)(U) = (s_\mu\mu)(\mathbb{1}_U) \geqslant (s_\mu\mu)(f) > 1 - \varepsilon$.

In any case, $\mu(s_\mu U) > 1 - \varepsilon$. Now, let $f \in C(X)$ be such that $0 \leqslant f \leqslant \mathbb{1}_{s_\mu U}$ and $\mu(f) > 1 - \varepsilon$. By continuity of the evaluation on f, there exists a weak-*-open neighbourhood V_μ of μ such that $\nu(s_\mu U) \geqslant \nu(f) \geqslant 1 - \varepsilon$ for all $\nu \in V_\mu$. By compactness, there are $V_{\mu_1}, V_{\mu_2}, \ldots, V_{\mu_n}$ that cover $\mathcal{P}(X)$, for some $n \in \mathbb{N}$, so we can pick $F = \{s_{\mu_1}, s_{\mu_2} \ldots, s_{\mu_n}\}$. ■

Proposition 3.1.10. *Let X be a Γ-boundary. If the Γ-action on X is not topologically free, then the left regular representation λ_Γ is not weakly contained in the quasi-regular representation $\lambda_{\Gamma/\Gamma_x}$ corresponding to Γ_x, for any $x \in X$.*

Proof. Let $s \in \Gamma \setminus \{e\}$ be such that X^s has non-empty interior U. Fix $\varepsilon = 1/3$ and let $F \subseteq \Gamma$ be as in the previous lemma.

Assuming $\lambda_\Gamma \prec \lambda_{\Gamma/\Gamma_x}$, there exist finitely many unit vectors $\xi_1, \xi_2, \ldots, \xi_n \in \ell^2(\Gamma/\Gamma_x)$ such that

$$\left| \left\langle \lambda_\Gamma(tst^{-1})\delta_e, \delta_e \right\rangle - \frac{1}{n}\sum_{i=1}^{n} \left\langle \lambda_{\Gamma/\Gamma_x}(tst^{-1})\xi_i, \xi_i \right\rangle \right| < \frac{1}{3} \tag{1}$$

for all $t \in F$.

Consider the probability measures on X defined by

$$\mu_i = \sum_{y \in \Gamma x} |\xi_i(y)|^2 \delta_y, \;\; \mu = \frac{1}{n}\sum_{i=1}^{n} \mu_i,$$

where we have used the natural identification between Γx and Γ/Γ_x. By definition, there exists $t_\mu \in F$ such that $\mu(t_\mu U^c) < \varepsilon$, i.e.

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{y \in \Gamma x} |\xi_i(y)|^2 \delta_y(t_\mu U^c) = \frac{1}{n}\sum_{i=1}^{n}\sum_{y \in U^c \cap \Gamma x} |\xi_i(t_\mu^{-1}y)|^2 < \frac{1}{3} \tag{2}$$

Moreover, denoting $\lambda_{\Gamma/\Gamma_x}(t_\mu^{-1})\xi_i$ by ν_i for each i, we obtain

$$\left\langle \lambda_{\Gamma/\Gamma_x}(s)\nu_i, \nu_i \right\rangle = \sum_{y \in \Gamma x} \nu_i(s^{-1}y)\overline{\nu_i(y)}$$

$$= \sum_{y \in U \cap \Gamma x} |\nu_i(y)|^2 + \sum_{y \in U^c \cap \Gamma x} \nu_i(s^{-1}y)\overline{\nu_i(y)}$$

and thus, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|1 - \left\langle \lambda_{\Gamma/\Gamma_x}(t_\mu s t_\mu^{-1})\xi_i, \xi_i \right\rangle\right| &= \left|1 - \left\langle \lambda_{\Gamma/\Gamma_x}(s)\nu_i, \nu_i \right\rangle\right| \\
&= \left|1 - \sum_{y \in U \cap \Gamma x} |\nu_i(y)|^2 - \sum_{y \in U^c \cap \Gamma x} \nu_i(s^{-1}y)\overline{\nu_i(y)}\right| \\
&= \left|\sum_{y \in U^c \cap \Gamma x} |\nu_i(y)|^2 - \sum_{y \in U^c \cap \Gamma x} \nu_i(s^{-1}y)\overline{\nu_i(y)}\right| \\
&\leqslant 2 \sum_{y \in U^c \cap \Gamma x} |\nu_i(y)|^2 \\
&= 2 \sum_{y \in U^c \cap \Gamma x} |\xi_i(t_\mu^{-1}y)|^2.
\end{aligned}$$

Averaging over i and using (2) we obtain

$$1 - \left|\frac{1}{n}\sum_{i=1}^{n} \left\langle \lambda_{\Gamma/\Gamma_x}(t_\mu s t_\mu^{-1})\xi_i, \xi_i \right\rangle\right| < \frac{2}{3},$$

which contradicts (1) (just notice that the term $\langle \lambda_\Gamma(tst^{-1})\delta_e, \delta_e \rangle$ always vanishes). ■

Corollary 3.1.11. *If Γ is C^*-simple, then the Γ-action on $\partial_{FH}\Gamma$ is topologically free.*

Proof. Assuming otherwise, by the previous proposition we get that $\lambda_\Gamma \not\prec \lambda_{\Gamma/\Gamma_x}$ for all $x \in \partial_{FH}\Gamma$. But, by Lemma 2.2.13, every Γ_x is amenable, and thus we have $\lambda_{\Gamma/\Gamma_x} \prec \lambda_\Gamma$. That contradicts Proposition 3.1.3. ■

Proposition 3.1.12. *If the Γ-action on $\partial_{FH}\Gamma$ is free, then Γ is C^*-simple.*

Proof. Let $\pi : C_r^*(\Gamma) \to B(\mathcal{H})$ be a non-trivial unital *-representation of $C_r^*(\Gamma)$. We need to show that π is injective. Since $C_r^*(\Gamma) = \mathbb{C} \rtimes_r \Gamma$ sits naturally inside $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$, we can extend π to a u.c.p. map $\varphi : C(\partial_{FH}\Gamma) \rtimes_r \Gamma \to B(\mathcal{H})$ by Arveson's extension theorem. We will show that φ is faithful, which is enough since $C_r^*(\Gamma)$ is contained in the multiplicative domain of φ.

First, notice that φ is a Γ-map (with respect to the natural Γ-actions by conjugation). Indeed, $\varphi(sa) = \varphi(\lambda_s a \lambda_s) = \pi(\lambda_s)\varphi(a)\pi(\lambda_s) = s\varphi(a)$. By the remark following Proposition 2.2.5, the restriction of φ to $C(\partial_{FH}\Gamma)$ is completely isometric. Thus, we can consider the inverse Γ-map $\varphi|_{C(\partial_{FH}\Gamma)}^{-1} : \varphi(C(\partial_{FH}\Gamma)) \to C(\partial_{FH}\Gamma)$ and extend it by Γ-injectivity to a Γ-map $\psi : \mathrm{im}(\varphi) \to C(\partial_{FH}\Gamma)$.

Now, the composition $\omega = \psi \circ \varphi : C(\partial_{FH}\Gamma) \rtimes_r \Gamma \to C(\partial_{FH}\Gamma)$ is a Γ-map which is the identity on $C(\partial_{FH}\Gamma)$ by Γ-rigidity. We will show that $\omega(\lambda_s) = 0$ for all $s \in \Gamma \setminus \{e\}$, implying ω is actually the canonical conditional expectation $E : C(\partial_{FH}\Gamma) \rtimes_r \Gamma \to C(\partial_{FH}\Gamma)$, which of course is faithful. To that end, let $s \in \Gamma\{e\}$ and $x \in \partial_{FH}\Gamma$. Since the Γ-action on $\partial_{FH}\Gamma$ is free, there exists $f \in C(\partial_{FH}\Gamma)$ such that $f(x) \neq 0 = f(s^{-1}x) = (sf)(x)$. Observe also that $C(\partial_{FH}\Gamma)$ is contained in the multiplicative domain of ω and thus

$$\omega(\lambda_s)f = \omega(\lambda_s f) = \omega((sf)\lambda_s) = (sf)\omega(\lambda_s),$$

from which we obtain $\omega(\lambda_s)(x) = 0$. ■

Notice that we now get the following characterisation essentially for free.

Theorem 3.1.13. Γ *is* C^**-simple iff for every amenable subgroup* $\Lambda \leqslant \Gamma$ *we have* $\lambda_{\Gamma/\Lambda} \sim \lambda_\Gamma$.

Proof. Since $\lambda_{\Gamma/\Lambda} \prec \lambda_\Gamma$ by amenability, the alternative definition of C*-simplicity gives us the "only if". Now, if Γ is not C^*-simple, then the Γ-action on $\partial_{FH}\Gamma$ is not free, thus $\lambda_\Gamma \not\prec \lambda_{\Gamma/\Gamma_x}$ for all $x \in \partial_{FH}\Gamma$. But the stabilizers Γ_x are amenable, so we are done. ■

It remains to prove the equivalence (1) $\iff$ (2) $\iff$ (3). To do so, we will follow the arguments used in [AS94], in conjunction with the following (easy) lemma.

Lemma 3.1.14. *Let* X *be a* Γ*-boundary and* I *be a closed ideal of* $C(X) \rtimes_r \Gamma$. *If* I *is proper, then* $I \cap C(X) = \{0\}$.

Proof. Notice first that $J = I \cap C(X)$ is a proper Γ-invariant closed ideal of $C(X)$. Hence, it is contained in a maximal ideal M of $C(X)$. M is necessarily of the form $\{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$, so every element in J vanishes on x_0. By the Γ-invariance of J, elements of J vanish on the orbit of x_0, which is dense by minimality. Thus, they vanish everywhere, i.e. $J = \{0\}$. ■

Theorem 3.1.15. *Let* X *be a* Γ*-boundary. The* Γ*-action on* X *is topologically free iff* $C(X) \rtimes_r \Gamma$ *is simple.*

Proof. Let Γ act topologically freely on X and I be a proper closed ideal of $C(X) \rtimes_r \Gamma$. It suffices to show $E(I) = \{0\}$ where $E : C(X) \rtimes_r \Gamma \to C(X)$ is the canonical faithful conditional expectation.

If not, there exists $a \in I$ such that $E(a) \neq 0$. Let $b = \sum_{s\in\Gamma} f_s\lambda_s \in C(X)[\Gamma]$ be such that $\|a - b\| < \|E(a)\|/2$. Consider also the set $Y = \cap_{s\in F\setminus\{e\}}\{x \in X : sx \neq x\}$ where $F := \{s \in \Gamma : f_s \neq 0\}$ is finite. By topological freeness, Y is dense in X.

For any $y \in Y$, let π_y denote the composition

$$C(X) + I \to (C(X) + I)/I \cong C(X)/(C(X) \cap I) \cong C(X) \xrightarrow{\delta_y} \mathbb{C},$$

where the second isomorphism exists by the previous lemma. By the Hahn-Banach theorem, we can extend π_y to a u.c.p. map $\varphi_y : C(X) \rtimes_r \Gamma \to \mathbb{C}$. We claim that $\varphi_y(b) = \varphi_y(f_e) = \varphi_y(E(b))$.

Indeed, let $s \in F \setminus \{e\}$. Since $y \in Y$, there exists $g \in C(X)$ such that $1 = g(y) \neq g(sy) = 0$ and we have

$$\begin{aligned}\varphi_y(f_s\lambda_s) &= g(y)\varphi_y(f_s\lambda_s) = \varphi_y(g)\varphi_y(f_s\lambda_s)\\ &= \varphi_y(gf_s\lambda_s) = \varphi_y(f_s g\lambda_s)\\ &= \varphi_y(f_s\lambda_s(s^{-1}g)) = \varphi_y(f_s\lambda_s)\varphi_y(s^{-1}g)\\ &= \varphi_y(f_s\lambda_s)g(sy) = 0,\end{aligned}$$

where we have used the fact that $C(X)$ is contained in the multiplicative domain of φ_y.

It follows that

$$\|E(b)(y)\| = \|\varphi_y(E(b))\| = \|\varphi_y(b)\| = \|\varphi_y(b-a)\| \leqslant \|a-b\|$$

for all $y \in Y$. By density, we have $\|E(b)\| \leqslant \|a-b\|$, from which we obtain the contradiction

$$\|E(a)\| \leqslant \|E(a-b)\| + \|E(b)\| \leqslant 2\,\|a-b\| < \|E(a)\|\,.$$

For the converse, assume X^s has non-empty interior for some $s \in \Gamma$. Then there exists a non-zero $f \in C(X)$ such that $\mathrm{supp}(f) \subseteq X^s$.

For $x \in X$ define a representation π_x of the (full) crossed product $C(X) \rtimes \Gamma$ on $\ell^2(\Gamma x)$ by the formulas

$$\begin{aligned}\pi_x(f)\delta_{tx} &= f(tx)\delta_{tx},\ \ f \in C(X),\\ \pi_x(\lambda_s)\delta_{tx} &= \delta_{stx},\ \ s \in \Gamma,\end{aligned}$$

which are covariant (and thus truly define a representation of $C(X) \rtimes \Gamma$). Let $I = \cap_{x\in X} \ker \pi_x$. By minimality, it is clear that $C(X) \cap I = \{0\}$, thus I is proper and so by the hypothesis the corresponding ideal in the reduced crossed product is trivial. In particular, $\tilde{E}(I) = \{0\}$, where $\tilde{E}$ is the canonical conditional expectation on the full crossed product.

We will show that $\pi_x(f - fu_s) = 0$ for all $x \in X$, which implies $f = \tilde{E}(f - fu_s) = 0$, a contradiction. Indeed, if $tx \in \mathrm{supp}(f)$ then $tx = stx$ and so

$$\pi_x(f - fu_s)\delta_{tx} = f(tx)\delta_{tx} - f(stx)\delta_{stx} = 0.$$

If $tx \notin \mathrm{supp}(f)$ then $stx \notin \mathrm{supp}(f)$ and $\pi_x(f - fu_s)\delta_{tx} = 0$ trivially. ■

This concludes the proof of Theorem 3.1.1, which is probably the most important theorem presented in this work, as it inspired many generalisations and the usage of similar techniques in different contexts.

Before we move on, let us close this subsection with an application. We first remind the reader that a *Tarski monster group* or, in particular, a *Tarski* p-*group* is an infinite group, such that every non-trivial subgroup of it is of order a fixed prime p. Such groups exist by results of Olshanskii and are non-amenable (at least those constructed by him).

Proposition 3.1.16. *Olshanskii 's Tarski monster groups are* C^**-simple.*

Proof. Let Γ be a Tarski monster group. We will show that the Γ-action on $\partial_{FH}\Gamma$ is topologically free. If not, let $s_0 \in \Gamma \setminus \{e\}$ be such that $(\partial_{FH}\Gamma)^{s_0}$ has non-empty interior and $\emptyset \neq U \subseteq (\partial_{FH}\Gamma)^{s_0}$ be an open set. We claim that U is finite.

Assuming otherwise, let us fix $x \in U$. By minimality, there exists $s_1 \in \Gamma \backslash \{s_0, e\}$ such that $s_1 x \in U \backslash \{x\}$. But then $s_0 \in \Gamma_{s_1 x}$ and thus $\Gamma_x = \Gamma_{s_1 x} = s_1 \Gamma_x s_1^{-1}$ (since both subgroups are of the same prime order and both contain s_0, which generates them), i.e. $s_1 \in N_\Gamma(\Gamma_x)$, the normalizer of Γ_x. The definition of Tarski monster groups forces $N_\Gamma(\Gamma_x) = \Gamma_x$, so $s_1 \in \Gamma_x$. If U is infinite, we can repeat the same arguments and get an infinite set $\{s_i, i \in I\} \subseteq \Gamma_x$, a contradiction since Γ_x is proper, and thus finite.

Now, since U is non-empty, finite and open, $\partial_{FH}\Gamma$ must contain isolated points, a contradiction by Lemma 3.1.8. ■

3.1.2 Further Characterisations

In this subsection we will present a few more characterisations of C^*-simplicity which heavily rely on Theorem 3.1.1. We start with the following theorem, appearing in [Haa16].

Theorem 3.1.17. *Let* τ_0 *denote the canonical trace on* $C^*_r(\Gamma)$*. Then the following are equivalent:*

1. Γ *is* C^**-simple.*
2. $\tau_0 \in \overline{\{s\varphi : s \in \Gamma\}}^{w^*}$*, for every state* φ *on* $C^*_r(\Gamma)$*.*
3. $\tau_0 \in \overline{\operatorname{conv}}^{w^*}\{s\varphi : s \in \Gamma\}$*, for every state* φ *on* $C^*_r(\Gamma)$*.*
4. $\omega(1)\tau_0 \in \overline{\operatorname{conv}}^{w^*}\{s\omega : s \in \Gamma\}$*, for every bounded linear functional* ω *on* $C^*(\Gamma)$*.*
5. *For all* $t_1, t_2, \ldots, t_m \in \Gamma \setminus \{e\}$*,*
$$0 \in \overline{\operatorname{conv}}\{\lambda_s(\lambda_{t_1} + \lambda_{t_2} + \cdots + \lambda_{t_m})\lambda_s^* : s \in \Gamma\}.$$
6. *For all* $t_1, t_2, \ldots, t_m \in \Gamma \setminus \{e\}$ *and all* $\varepsilon > 0$*, there exist* $s_1, s_2, \ldots, s_n \in \Gamma$ *such that*
$$\left\| \sum_{k=1}^{n} \frac{1}{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon$$

for $j = 1, 2, \dots, m$.

As usual, we will need some lemmas, the first of which is reminiscent of techniques we have already used.

Lemma 3.1.18. *Let* Γ *act on a compact Hausdorff space* X, $x \in X$, *and* φ *be a state on* $C(X) \rtimes_r \Gamma$ *whose restriction to* $C(X)$ *is the evaluation at* x, δ_x. *Then* $\varphi(\lambda_t) = 0$, *for each* $t \in \Gamma \setminus \Gamma_x$.

Proof. By the assumption, $C(X)$ is contained in the multiplicative domain of φ, so

$$\varphi(\lambda_t)f(x) = \varphi(\lambda_t f) = \varphi((tf)\lambda_t) = f(t^{-1}x)\varphi(\lambda_t)$$

for all $f \in C(X)$ and $t \in \Gamma$. Whenever $tx \neq x$, we also have $t^{-1}x \neq x$, so by choosing an f such that $f(x) \neq f(t^{-1}x)$ (by Urysohn's lemma) we are done. ∎

Lemma 3.1.19. *Let* $x, y \in \mathbb{R}_+[\Gamma] \subseteq C^*_r(\Gamma)$. *Then* $\|x + y\| \geqslant \|x\|$.

Proof. Notice that every element $z \in \mathbb{R}_+[\Gamma]$ satisfies $|\langle z\xi, \eta\rangle| \leqslant \langle z|\xi|, |\eta|\rangle$ for all $\xi, \eta \in \ell^2(\Gamma)$. Thus

$$\begin{aligned}
\|x\| &= \sup\{\langle x\xi, \eta\rangle : \xi, \eta \in \ell^2(\Gamma), \|\xi\| = \|\eta\| = 1\} \\
&= \sup\{\langle x\xi, \eta\rangle : \xi, \eta \in \ell^2(\Gamma)_+, \|\xi\| = \|\eta\| = 1\} \\
&\leqslant \sup\{\langle x\xi, \eta\rangle + \langle y\xi, \eta\rangle : \xi, \eta \in \ell^2(\Gamma)_+, \|\xi\| = \|\eta\| = 1\} \\
&= \sup\{\langle (x + y)\xi, \eta\rangle : \xi, \eta \in \ell^2(\Gamma)_+, \|\xi\| = \|\eta\| = 1\} \\
&= \|x + y\|.
\end{aligned}$$

∎

Proof of Theorem 3.1.17. ($1 \implies 2$) Let φ be a state on $C^*_r(\Gamma)$. By Theorem 3.1.1, there is a free boundary action $\Gamma \curvearrowright X$. Let $x \in X$. Extend φ to a state ψ on $C(X) \rtimes_r G$ (by the Hahn-Banach theorem) and let ρ be the restriction of ψ to $C(X)$. By strong proximality and minimality, there exists a net (s_i) in Γ such that $s_i\rho \xrightarrow{w^*} \delta_x$. Upon possibly passing to a subnet, we can assume by compactness that $s_i\psi$ converges to some state ψ' on $C(X) \rtimes_r \Gamma$.

Now, the restriction of ψ' to $C(X)$ is δ_x, so by Lemma 3.1.18 and the freeness of the Γ-action on X, we get $\psi'(\lambda_t) = 0$ for all $t \in \Gamma \setminus \{e\}$. This forces the restriction of ψ' to $C^*_r(\Gamma)$ to be τ_0, i.e. $s_i\varphi \to \tau_0$.

($2 \implies 3$) Trivial.

($3 \implies 4$) Fix states $\varphi_1, \varphi_2, \dots, \varphi_m$ on $C^*_r(\Gamma)$. Consider the set

$$S = \overline{\mathrm{conv}}^{w^*}\{(s\varphi_1, s\varphi_2, \dots, s\varphi_m) : s \in \Gamma\}$$

in the space of m-tuples of states on $C^*_r(\Gamma)$. By the assumption, there exists a net $(\overline{\psi_i})$ in S such that $\mathrm{pr}_1(\overline{\psi_i}) \to \tau_0$. By compactness, we can assume that $\mathrm{pr}_j(\overline{\psi_i})$ converges for $j = 1, 2, \dots m$. Since τ_0 is Γ-invariant, we can repeat this process for each of the coordinates to show that $(\tau_0, \tau_0, \dots, \tau_0) \in S$. Thus, for every finite $F \subseteq C^*_r(\Gamma)$ and $\varepsilon > 0$, there exist $s_1, s_2, \dots, s_n \in \Gamma$ such that

$$\left| \frac{1}{n} \sum_{k=1}^{n} s_k \varphi_j(a) - \tau_0(a) \right| < \varepsilon$$

for all $a \in C^*_r(\Gamma)$ and $j = 1, 2, \dots m$. Since every bounded linear functional is a linear combination of four states, we deduce that (4) holds.

$(4 \implies 5)$ Suppose (5) does not hold, so that there are $t_1, t_2, \dots, t_m \in \Gamma \backslash \{e\}$ such that

$$0 \notin \overline{\text{conv}}\{\lambda_s(\lambda_{t_1} + \lambda_{t_2} + \dots + \lambda_{t_m})\lambda_s^* : s \in \Gamma\}.$$

By the Hahn-Banach separation theorem, there exists a bounded functional ω on $C^*_r(\Gamma)$ and $c > 0$ such that

$$\operatorname{Re} \omega(\sum_{j=1}^{m} \lambda_{st_js^{-1}}) \geqslant c$$

for all $s \in \Gamma$, i.e.

$$\operatorname{Re} \left((s\omega)(\sum_{j=1}^{m} \lambda_{t_j}) \right) \geqslant c > 0$$

for all $s \in \Gamma$. Since $\tau_0(\sum_{j=1}^{m} \lambda_{t_j}) = 0$, we deduce that

$$\omega(1)\tau_0 \notin \overline{\text{conv}}^{w^*}\{s\omega : s \in \Gamma\},$$

contradicting (4).

$(5 \implies 6)$ Let $t_1, t_2, \dots, t_m \in \Gamma \setminus \{e\}$ and $\varepsilon > 0$. By the assumption, there exist $s_1, s_2, \dots, s_n \in \Gamma$, with repetitions allowed, such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} (\lambda_{s_k}(\sum_{j=1}^{m} \lambda_{t_j})\lambda_{s_k}^*) \right\| < \varepsilon.$$

Using Lemma 3.1.19, we get (6).

$(6 \implies 1)$ Let I be a non-trivial closed ideal of $C^*_r(\Gamma)$, $a \in I \setminus \{0\}$ be a positive element, and $\varepsilon > 0$. By definition of $C^*_r(\Gamma)$ and of τ_0, there exist $t_1, t_2, \dots t_m \in \Gamma \setminus \{e\}$ and $z_1, z_2, \dots z_m \in \mathbb{C}^*$ such that

$$\left\| \sum_{j=1}^{m} z_j \lambda_{t_j} + \tau_0(a)\lambda_e - a \right\| < \frac{\varepsilon}{2}. \tag{1}$$

Using (6), we can find $s_1, s_2, \dots s_n \in \Gamma$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \lambda_{s_k} \lambda_{t_j} \lambda_{s_k}^* \right\| < \frac{\varepsilon}{2m \max\{|z_l| : l = 1, 2, \dots, m\}} \tag{2}$$

for $j = 1, 2, \dots, m$. Using (1), (2) and the triangle inequality, we deduce that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \lambda_{s_k} a \lambda_{s_k}^* - \tau_0(a)\lambda_e \right\| < \varepsilon, \tag{3}$$

which implies that

$$\tau_0(a)\lambda_e \in \overline{\mathrm{conv}}\{\lambda_s a \lambda_s^* : s \in \Gamma\} \subseteq I.$$

Since τ_0 is faithful, we get $\lambda_e \in I$, i.e. $I = C_r^*(\Gamma)$, and hence (1) holds. ∎

Before we proceed, let us make an important remark. Γ is said to have *Powers' averaging property* iff equation (3) is satisfied for every $a \in C_r^*(\Gamma)$. As we already pointed out in the introduction of this chapter, it was well-known that this property implied C^*-simplicity. However, a by-product of the above proof is that the two properties are actually equivalent, attesting to Powers' incredible insight (or luck). This equivalence was also proved independently by Kennedy in [Ken20], alongside the next and final result on C^*-simplicity presented in this work.

Consider the space $\mathrm{sub}(\Gamma)$ of subgroups of Γ, equipped with the so-called *Chabauty topology*, which in the discrete case (the only case we are interested in) coincides with the subspace topology induced by the product topology on $\{0,1\}^\Gamma$ (for the general case of locally compact groups, a nice introduction is [Har08]). This space is compact, as a closed subspace of $\{0,1\}^\Gamma$, and it is a Γ-space with respect to the Γ-action by conjugation.

Let $\mathrm{sub}_a(\Gamma)$ be the Γ-invariant subspace of amenable subgroups of Γ. The general case of the following proposition is [Sch71, Corollary 1], but we will give a simpler proof (discreteness of Γ allows it).

Proposition 3.1.20. $\mathrm{sub}_a(\Gamma)$ *is closed (and therefore compact) in the Chabauty topology.*

Proof. Let (Λ_i) be a net in $\mathrm{sub}_a(\Gamma)$ converging to $\Lambda \in \mathrm{sub}(\Gamma)$. We can assume $\Lambda_i \leqslant \Lambda$, because the map $\mathrm{sub}(\Gamma) \to \mathrm{sub}(\Gamma) : * \mapsto * \cap \Lambda$ is continuous and amenability is preserved by subgroups (thus we can swap Λ_i with $\Lambda_i \cap \Lambda$). Let $\varphi_i : \ell^\infty(\Lambda_i) \to \mathbb{C}$ be Λ_i-invariant states witnessing amenability.

Consider the net $((\varphi_i(f|_{\Lambda_i}))_f)$ in the space $\prod_{f \in \ell^\infty(\Lambda)} \overline{\mathbb{D}}_f$, where $\overline{\mathbb{D}}_f$ is the closed unit disc of radius $\|f\|$ in $\mathbb{C}$. By compactness, we can assume this net is convergent. Hence, we can define $\varphi : \ell^\infty(\Lambda) \to \mathbb{C}$ by $\varphi(f) = \lim \varphi_i(f|_{\Lambda_i})$, which is clearly a state. Furthermore, for $s \in \Lambda$, we have

$$\varphi(sf) = \lim_i \varphi_i((sf)|_{\Lambda_i}) = \lim_i \varphi_i(s(f|_{\Lambda_i})) = \lim_i \varphi_i(f|_{\Lambda_i}) = \varphi(f),$$

where the third equality holds because s eventually belongs to Λ_i. Thus, φ is Λ-invariant and Λ is also amenable. ∎

Definition 3.1.21. A compact Γ-subspace $X \subseteq \mathrm{sub}(\Gamma)$ is called a *uniformly recurrent subgroup (URS)* of Γ iff it is minimal. Such an X is called *non-trivial* iff $X \neq \{\{e\}\}$, and *amenable* iff $X \subseteq \mathrm{sub}_a(\Gamma)$.

We are now ready to close this section, with a more intrinsic characterisation of C^*-simplicity.

Theorem 3.1.22. *Γ is C^*-simple iff it has no non-trivial amenable uniformly recurrent subgroups.*

Proof. Suppose that Γ is not C^*-simple. Let $X = \{\Gamma_x : x \in \partial_{FH}\Gamma\}$ and consider the map $\partial_{FH}\Gamma \twoheadrightarrow X : x \mapsto \Gamma_x$, which is clearly Γ-equivariant. Let (x_i) be a net in $\partial_{FH}\Gamma$ converging to some point $x \in \partial_{FH}\Gamma$. We want to show that Γ_{x_i} converges to Γ_x, and thus that the map is also continuous. Indeed, if $s \in \Gamma_x$ (resp. $s \notin \Gamma_x$), then $x \in (\partial_{FH}\Gamma)^s$ (resp. $x \in ((\partial_{FH}\Gamma)^s)^c$), which is clopen by Lemma 3.1.5 and Proposition 2.2.12, hence $x_i \in (\partial_{FH})^s$ (resp. $x_i \in ((\partial_{FH}\Gamma)^s)^c$) eventually, or equivalently $s \in \Gamma_{x_i}$ (resp. $s \notin \Gamma_{x_i}$) eventually, i.e. $\Gamma_{x_i} \to \Gamma_x$.

Now, since $\partial_{FH}\Gamma$ is compact and minimal, so is X as a continuous Γ-equivariant image of it, i.e. it is a URS. Finally, it is amenable by Lemma 2.2.13 and non-trivial by Theorem 3.1.1, since the Γ-action on $\partial_{FH}\Gamma$ is not free.

Conversely, suppose Γ has a non-trivial amenable URS X. Fix $\Lambda \in X$ and $x \in \partial_{FH}\Gamma$. By the amenability of Λ, there exists a probability measure μ on $\partial_{FH}\Gamma$ fixed by Λ. By strong proximality there exists a net (s_i) in Γ such that $s_i\mu \to \delta_x$.

By compactness, we can assume that $s_i\Lambda s_i^{-1}$ converges to a $\Lambda' \in X$. X is non-trivial, so minimality forces $\Lambda' \neq \{e\}$. Let $t \in \Lambda' \setminus \{e\}$. By the definition of the Chabauty topology, t eventually belongs to $s_i\Lambda s_i^{-1}$, and thus we can assume $t \in \cap_i s_i\Lambda s_i^{-1}$. Therefore, we have $ts_i\mu = s_i\mu$ for all i. Taking the limit gives $t\delta_x = \delta_x$, i.e. $tx = x$, from which we deduce that Γ does not act freely on $\partial_{FH}\Gamma$. Hence, by Theorem 3.1.1, Γ is not C^*-simple. ∎

Remark. In the above proof we actually showed that C^*-simplicity is equivalent to the triviality of $X = \{\Gamma_x : x \in \partial_{FH}\Gamma\}$. That is why this particular amenable uniformly recurrent subgroup has been suitably named the *Furstenberg URS* of Γ (see for example [LBMB18; Rau20]).

3.2 The Unique Trace Property

This section is dedicated to another property of interest of the reduced C^*-algebra of Γ, the unique trace property. We will present characterisations of this property, similar to those given for C^*-simplicity, and we will completely clarify how the two are related.

Definition 3.2.1. Γ is said to have the *unique trace property* iff its reduced C^*-algebra has a unique tracial state, i.e. the canonical trace τ_0 is the only Γ-equivariant state on $C_r^*(\Gamma)$.

The following theorem first appeared in [BKKO17]. A proof can also be found in [Haa16].

Theorem 3.2.2. *Let $s \in \Gamma$. Then, $\tau(\lambda_s) = 0$ for all tracial states τ on $C_r^*(\Gamma)$ iff $s \notin R_a(\Gamma)$. In particular, Γ has the unique trace property iff $R_a(\Gamma) = \{e\}$.*

We require the following lemma.

Lemma 3.2.3. *Let τ be a tracial state on $C_r^*(\Gamma)$, X be a Γ-boundary and $x \in X$. Then τ extends to a state on $C(X) \rtimes_r \Gamma$ whose restriction to $C(X)$ is the evaluation δ_x at the point x.*

Proof. Extend τ (by the Hahn-Banach theorem) to any state φ on $C(X) \rtimes_r \Gamma$, and let ρ be the restriction of φ to $C(X)$. By strong proximality, there exists a net (s_i) in Γ such that $s_i\rho \to \delta_x$ in the weak-* topology. By compactness we can assume that $s_i\varphi$ converges to some state ψ on $C(X) \rtimes_r \Gamma$, whose restriction on $C(X)$ is forced to be δ_x. Now, since the restriction of φ to $C_r^*(\Gamma)$ is the tracial state τ, and thus Γ-invariant, we have that $\psi = \tau$ on $C_r^*(\Gamma)$ and we are done. ∎

Proof of Theorem 3.2.2. Let $s \notin R_a(\Gamma)$ and τ be a tracial state on $C_r^*(\Gamma)$. By Proposition 2.2.14, there exists $x \in \partial_{FH}\Gamma$ such that $sx \neq x$. By Lemma 3.2.3, τ extends to a state on $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$ whose restriction to $C(\partial_{FH}\Gamma)$ is the point evaluation δ_x. By Lemma 3.1.18, we have $\varphi(\lambda_s) = \tau(\lambda_s) = 0$.

Now, since $R_a(\Gamma)$ is amenable by definition, $C_r^*(R_a(\Gamma))$ has a character, which of course is also a trace. Composing this character with the canonical conditional expectation $C_r^*(\Gamma) \to C_r^*(R_a(\Gamma))$ produces a trace τ on $C_r^*(\Gamma)$ such that $\tau(\lambda_s) = 1$ for all $s \in R_a(\Gamma)$.

The final assertion of the theorem follows from the fact that the canonical trace is the unique tracial state on $C_r^*(\Gamma)$ vanishing on λ_t for all $t \in \Gamma \backslash \{e\}$. ∎

In view of Proposition 2.2.14, the above theorem can be rephrased to better suit the spirit of Theorem 3.1.1.

Theorem 3.2.4. *The following are equivalent:*

1. *Γ has the unique trace property.*
2. *The Γ-action on $\partial_{FH}\Gamma$ is faithful.*
3. *The Γ-action on some Γ-boundary is faithful.*

Proof. The first equivalence is essentially Theorem 3.2.2. The second one is immediate from the fact that any Γ-boundary is a Γ-equivariant image of $\partial_{FH}\Gamma$. ∎

The following and final characterisation of the unique trace property appears in [Haa16] and is similar in flavour to Theorem 3.1.17.

Theorem 3.2.5. *Let* $t \in \Gamma$*. Then* $t \notin R_a(\Gamma)$ *iff*

$$0 \in \overline{\operatorname{conv}}\{\lambda_{sts^{-1}} : s \in \Gamma\}. \tag{1}$$

In particular, Γ *has the unique trace property iff* (1) *holds for all* $t \in \Gamma \setminus e$*.*

Proof. If (1) holds, then of course $\tau(\lambda_t) = 0$ for all tracial states τ on $C^*_r(\Gamma)$. Hence, by Theorem 3.2.2, we have $t \notin R_a(\Gamma)$.

Conversely, suppose (1) does not hold. Assume also, for the sake of contradiction, that $t \notin R_a(\Gamma)$, so that we can find (by Proposition 2.2.14) $x \in \partial_{FH}\Gamma$ such that $tx \neq x$.

By the Hahn-Banach separation theorem, there exists a self-adjoint linear functional ω of norm 1 on $C^*_r(\Gamma)$ and a constant $c > 0$ such that

$$\operatorname{Re}\omega(\lambda_{sts^{-1}}) \geqslant c \tag{2}$$

for all $s \in \Gamma$. Let $\omega = \omega_+ - \omega_-$ be the Jordan decomposition of ω. Notice that

$$1 = \|\omega_+\| + \|\omega_-\| = \omega_+(1) + \omega_-(1) \leqslant \|\omega_+ + \omega_-\| \leqslant \|\omega_+\| + \|\omega_-\| = 1,$$

thus $\omega_+ + \omega_-$ is a state.

Now, extend $\omega_\pm$ to positive linear functionals $\psi_\pm$ on $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$. We then have a state $\psi_+ + \psi_-$ extending $\omega_+ + \omega_-$ and a self-adjoint linear functional $\psi_+ - \psi_-$ extending ω.

Let ρ be the restriction of $\psi_+ + \psi_-$ to $C(\partial_{FH}\Gamma)$. By strong proximality and minimality, there exists a net (s_i) in Γ such that $s_i\rho \xrightarrow{w^*} \delta_x$. By compactness, we can assume $s_i\psi_\pm$ converge to positive linear functionals $\varphi_\pm$ on $C(\partial_{FH}\Gamma) \rtimes_r \Gamma$. The restriction of $\varphi_+ + \varphi_-$ to $C(\partial_{FH}\Gamma)$ is δ_x, which is a pure state on $C(X)$, thus the restrictions of $\varphi_\pm$ on $C(\partial_{FH}\Gamma)$ are forced to be $\|\varphi_\pm\|\,\delta_x$. Therefore, by Lemma 3.1.18, we have $\varphi_\pm(\lambda_t) = 0$. Hence, we get

$$\begin{aligned} 0 = \varphi_+(\lambda_t) - \varphi_-(\lambda_t) &= \lim_i (s_i\psi_+(\lambda_t) - s_i\psi_-(\lambda_t)) \\ &= \lim_i (s_i\omega_+(\lambda_t) - s_i\omega_-(\lambda_t)) = \lim_i s_i\omega(\lambda_t) \\ &= \lim_i \omega(\lambda_{s_its_i^{-1}}), \end{aligned}$$

contradicting (2). ∎

Now, seeing as the above characterisations of the unique trace property are easily comparable to those of C^*-simplicity (e.g. faithfulness vs freeness of the Γ-action on $\partial_{FH}\Gamma$), one would expect the distinction between the two to be relatively easy. However, that is not the case. Examples of groups which have trivial amenable radical, yet are not C*-simple, have indeed been constructed by Le Boudec [LB17], but the methods used were non-trivial and are not within the spirit of the present work (familiarity with ideas from geometric group theory is required). Thus, C*-simplicity is strictly stronger than the unique trace property, and the long-standing problem of clarifying their relationship is completely solved.

3.3 Exactness

In this section we briefly explore the concept of exactness. Although inherently operator algebraic in nature, exactness has (in contrast to C*-simplicity or the unique trace property) been linked to dynamical properties for more than twenty years now [Oza00]. Our goal, however, is to present a characterisation of exactness of Γ based on its action on $\partial_{FH}\Gamma$. Of course, we will need some preparation to get there.

Definition 3.3.1. A unital linear map $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras is called *nuclear* iff there exists a net of u.c.p. maps $\varphi_i : \mathcal{A} \to \mathbb{M}_{k(i)}$ and $\psi_i : \mathbb{M}_{k(i)} \to \mathcal{B}$ such that $\psi_i \circ \varphi_i \to \theta$ pointwise.

It is worth noting that the general definition involves contractive completely positive (c.c.p.) maps instead of u.c.p. ones. Nevertheless, we are only interested in the unital case. More details on this, and everything else preceding the final characterisation, can be found in [BO08].

Definition 3.3.2. A C*-algebra $\mathcal{A}$ is called:

- *nuclear* iff the identity map $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is nuclear.
- *exact* iff it admits a nuclear faithful representation.

Notice that the above definition of exactness seems to depend on the faithful representation, making it rather shaky. Arveson's extension theorem, however, might convince you otherwise.

Furthermore, these definitions are not the original ones, which we give below, as both notions have multiple characterisations (nuclearity is to C*-algebras what amenability is to groups, in more than one ways). One might easily notice that, in the case of exactness, the original definition is much more fitting to the term.

Definition 3.3.3. A C*-algebra $\mathcal{A}$ is called:

- *nuclear* iff $\mathcal{A} \otimes \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$, for all C*-algebras $\mathcal{B}$ (i.e. every tensor product of $\mathcal{A}$ has a unique C*-norm).
- *exact* iff the functor $(\mathcal{A} \otimes -)$ is exact (in the categorical sense, i.e. preserves short exact sequences).

It is clear from the first definition that nuclearity implies exactness. This is equally clear from the second one, if one knows that the functor $(\mathcal{A} \otimes_{\max} -)$ is *always* exact.

The equivalence of the definitions is highly technical and is due to Choi and Effros [CE78] (for nuclearity), and Kirchberg and Wassermann [Kir95; Was90] (for exactness).

A useful by-product of the proof given, for example, in [BO08] is the following lemma.

Lemma 3.3.4. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be an exact C^*-algebra and $\mathcal{S} \subseteq \mathcal{A}$ be a finite-dimensional operator system. If $(u_i)_{i\in I}$ is an orthonormal basis of $\mathcal{H}$, then for every $\varepsilon > 0$ there exists a finite $F_0 \subseteq I$ such that for each finite $F \subseteq I$ containing F_0 there exists a u.c.p. map $\psi_F : P_F B(\mathcal{H}) P_F \to B(\mathcal{H})$, where P_F denotes the orthogonal projection onto the linear span of $\{u_i : i \in F\}$, such that $\psi_F(P_F \mathcal{S} P_F) \subseteq \mathcal{A}$ and*

$$\|x - \psi_F(P_F x P_F)\| \leqslant \varepsilon \|x\|$$

for all $x \in \mathcal{S}$. Furthermore, $\mathcal{A}$ is nuclear iff we can force ψ_F to take values in $\mathcal{A}$.

Examples of nuclear C^*-algebras include all finite dimensional, abelian, AF, and AH ones (we omit the definitions of the latter two; we will not need them).

Definition 3.3.5. Γ is called *exact* iff its reduced C^*-algebra $C^*_r(\Gamma)$ is exact.

We turn now to dynamics, introducing a key concept that will concern us for the remainder of this chapter, that of amenable actions.

Definition 3.3.6. A compact Γ-space X is called *amenable* iff there exists a net of continuous maps $m_i : X \to \mathcal{P}(\Gamma)$ such that

$$\lim_i (\sup_{x\in X} \|s m_i^x - m_i^{sx}\|_1) = 0$$

for all $s \in \Gamma$. Such a net is called an *approximate invariant continuous mean (a.i.c.m.).* The Γ-action on such a space is also called *amenable.*

Remember that the 1-norm in the above definition comes from $\ell^1(\Gamma)$, since $\mathcal{P}(\Gamma)$ is just the positive part of the unit sphere of $\ell^1(\Gamma)$.

We will replace this traditional definition of amenable actions with one that incorporates actions on C^*-algebras, in such a way that in the commutative case the induced action on the spectrum will be amenable in the above sense.

To that end, let $\mathcal{A}$ denote a unital Γ-C^*-algebra and $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{A})$ be the Γ-action on it. Consider the α-twisted convolution algebra $C_c(\Gamma, \mathcal{A})$

(remember the construction of the crossed product) and equip it with an $\mathcal{A}$-valued inner product defined by

$$\langle S, T\rangle = \sum_{s\in\Gamma} S(g)^* T(g),\ \ S, T \in C_c(\Gamma, \mathcal{A})$$

and a new norm defined by

$$\|S\|_2 = \|\langle S, S\rangle\|^{\frac{1}{2}},\ \ S \in C_c(\Gamma, \mathcal{A}).$$

The informed reader will immediately recognise the structure of a (pre-) Hilbert C*-module, but we have no reason to introduce the term. The only piece of Hilbert C*-module theory we will need is the fact that the inner product satisfies the appropriate analogue of the Cauchy-Schwarz inequality, i.e.

$$\|\langle S, T\rangle\| \leqslant \|S\|_2 \|T\|_2$$

for all $S, T \in C_c(\Gamma, \mathcal{A})$. This is a general fact (and an easy one at that), but in the commutative case (which is the one we are interested in) it is simply a consequence of the classical Cauchy-Schwarz inequality.

Definition 3.3.7. The Γ-action α on $\mathcal{A}$ is called *amenable* iff there exists a net (T_i) in $C_c(\Gamma, \mathcal{A})$ such that:

1. $0 \leqslant T_i(s) \in \mathcal{Z}(\mathcal{A})$ for all i and $s \in \Gamma$.

2. $\langle T_i, T_i\rangle = 1_{\mathcal{A}}$.

3. $\|s *_\alpha T_i - T_i\|_2 \to 0$ for all $s \in \Gamma$ (here $s \in C_c(\Gamma, \mathcal{A})$ denotes the function mapping s to $1_{\mathcal{A}}$ and all other elements to 0).

Now, as promised, we have the following.

Proposition 3.3.8. *A compact Γ-space X is amenable iff the induced Γ-action α on $C(X)$ is amenable.*

Proof. Assume first that X is amenable and fix an a.i.c.m. $m_i : X \to \mathcal{P}(\Gamma)$. Define $T_i : \Gamma \to C(X)$ by

$$T_i(s)(x) = \sqrt{m_i^x(s)}$$

and notice that

$$\sum_{s\in\Gamma} (T_i(s)(x))^2 = \sum_{s\in\Gamma} m_i^x(s) = 1$$

for all $x \in X$. Therefore

$$\sum_{s\in\Gamma} T_i(s)^2 = 1_{C(X)},$$

where the convergence is uniform because everything is positive.

Note also that

$$(s *_\alpha T_i)(t)(x) = \alpha_s(T_i(t^{-1}s))(x) = T_i(t^{-1}s)(s^{-1}x) = \sqrt{sm_i^{s^{-1}x}(t)}$$

for all $x \in X$. Thus,

$$\begin{aligned}
\|s *_\alpha T_i - T_i\|_2^2 &= \sup_{x\in X}\left(\sum_{t\in\Gamma} |\sqrt{sm_i^{s^{-1}x}(t)} - \sqrt{m_i^x(t)}|^2\right)\\
&\leqslant \sup_{x\in X}\left(\sum_{t\in\Gamma} |sm_i^{s^{-1}x}(t) - m_i^x(t)|\right)\\
&= \sup_{y\in X}\left(\sum_{t\in\Gamma} |sm_i^{y}(t) - m_i^{sy}(t)|\right)\\
&= \sup_{y\in Y} \|sm_i^y - m_i^{sy}\|_1 \to 0.
\end{aligned}$$

Of course we are not quite done yet, as the maps T_i are not necessarily finitely supported. We can remedy that in the following way.

Assume we have a positive function $T : \Gamma \to C(X)$ such that

$$\sum_{s\in\Gamma} T(s)^2 = 1_{C(X)},$$

just like our T_i's. Uniform convergence of the above sum implies the existence of a finite set $F_0 \subseteq \Gamma$ such that

$$\sum_{s\in F} T(s)^2 > 0$$

for all finite sets $F \subseteq \Gamma$ containing F_0. The family $\mathcal{F}$ of all such sets is naturally directed by inclusion and thus we can define a net of maps $(T_F)_{F\in\mathcal{F}}$ by

$$T_F(s) = \begin{cases} \sqrt{\frac{1}{\sum_{t\in F} T(t)^2}}T(s), \ s \in F \\ 0, \ \text{otherwise} \end{cases},$$

which are positive, satisfy $\langle T_F, T_F\rangle = 1_{C(X)}$, and are finitely supported. Furthermore, they satisfy

$$\|s *_\alpha T_F - T_F\|_2 \to \|s *_\alpha T - T\|_2$$

by construction.

Hence, we can replace the T_i's with the associated net $(T_{i,F_i})_{F_i\in\mathcal{F}_i}$ and combine them to form a net directed by $(i, F_i) \preceq (j, F_j) \iff (i \preceq j \wedge F_i \subseteq F_j)$, which witnesses amenability of α.

The converse is quite a bit easier, as one simply needs to define $m_i^x(g) = T_i(g)^2(x)$ and use similar calculations, without worrying about supports. ■

Remark. The above proof is somewhat simpler if we assume Γ to be countable (a restriction that is considered quite light), as nets can be replaced with sequences which are easier to handle. The reader may find it instructive to check the arguments in the countable case, especially if they are not used to working with nets.

As the two definitions of amenability are now interchangeable, we will turn our attention to some general results before returning to the abelian case. As above, $\mathcal{A}$ will be a unital Γ-C^*-algebra and α will be the associated Γ-action.

Lemma 3.3.9. *If* $T \in C_c(\Gamma, \mathcal{A})$ *is such that* $0 \leqslant T(s) \in \mathcal{Z}(\mathcal{A})$ *for all* $s \in \Gamma$ *and* $\langle T, T \rangle = 1_{\mathcal{A}}$*, then*

1. $T *_\alpha T^*(s) = \sum_{t \in F \cap sF} T(t)\alpha_s(T(s^{-1}t))$*, where* F *is the support of* T*, and*
2. $\|1_{\mathcal{A}} - T *_\alpha T^*(s)\| \leqslant \|s *_\alpha T - T\|_2$,

for all $s \in \Gamma$*.*

Proof. The first assertion follows by the following calculation

$$\begin{aligned}
T *_a T^*(s) &= \sum_{t \in \Gamma} T(t)\alpha_t(T^*(t^{-1}s)) \\
&= \sum_{t \in \Gamma} T(t)\alpha_t(\alpha_{t^{-1}s}(T(s^{-1}t)^*)) \\
&= \sum_{t \in \Gamma} T(t)\alpha_s(T(s^{-1}t)^*) \\
&= \sum_{t \in \Gamma} T(t)\alpha_s(T(s^{-1}t)) \\
&= \sum_{t \in F \cap sF} T(t)\alpha_s(T(s^{-1}t))
\end{aligned}$$

where we have simply used the definitions, plus the positivity of T in the fourth equality.

For the second one, we have

$$\begin{aligned}
\|1_{\mathcal{A}} - T *_\alpha T^*(s)\| &= \left\| \langle T, T \rangle - \sum_{t \in \Gamma} T(t)\alpha_s(T(s^{-1}t)) \right\| \\
&= \left\| \langle T, T \rangle - \sum_{t \in \Gamma} T(t)(s *_\alpha T)(t) \right\| \\
&= \|\langle T, T \rangle - \langle T, s *_\alpha T \rangle\| \\
&= \|\langle T, T - s *_\alpha T \rangle\| \\
&\leqslant \|T\|_2 \, \|T - s *_\alpha T\|_2 \\
&= \|T - s *_\alpha T\|_2
\end{aligned}$$

where we have used the fact that $s *_\alpha T(t) = \alpha_s(T(s^{-1}t))$, the properties of T, and the Cauchy-Schwarz inequality. ∎

Lemma 3.3.10. *Let T be as in the previous lemma. Then, there exist u.c.p. maps $\varphi : \mathcal{A} \rtimes_r \Gamma \to \mathcal{A} \otimes \mathbb{M}_F(\mathbb{C})$ and $\psi : \mathcal{A} \otimes \mathbb{M}_F(\mathbb{C}) \to \mathcal{A} \rtimes_r \Gamma$ (where, again, F is the support of T) such that*

$$\psi \circ \varphi(a\lambda_s) = (T *_\alpha T^*(s))a\lambda_s$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$.

Proof. In proving that the reduced crossed product $\mathcal{A} \rtimes_r \Gamma$ does not depend on the choice of the faithful representation $\mathcal{A} \subseteq B(\mathcal{H})$ (Proposition 1.5.6), we constructed a u.c.p. map $\varphi : \mathcal{A} \rtimes_r \Gamma \to \mathcal{A} \otimes \mathbb{M}_F(\mathbb{C})$ such that

$$\varphi(a\lambda_s) = \sum_{t \in F \cap sF} \alpha_t^{-1}(a) \otimes e_{t,s^{-1}t}$$

for all $a \in \mathcal{A}$ and $s \in \Gamma$.

Let

$$X = \sum_{t \in F} \alpha_t^{-1}(T(t)) \otimes e_{t,t} \in \mathcal{A} \otimes \mathbb{M}_F(\mathbb{C})$$

and note that $X = X^*$, thus compression by X is a c.p. map ψ_1.

Consider also the map

$$\psi_2 : \mathcal{A} \otimes \mathbb{M}_F(\mathbb{C}) \to \mathcal{A} \rtimes_r \Gamma : a \otimes e_{x,y} \mapsto \alpha_x(a)\lambda_{xy^{-1}},$$

which is completely positive as well. To see this, notice first that every positive element in $\mathcal{A} \otimes \mathbb{M}_F(\mathbb{C})$ can be written as a sum of $|F|$ matrices of the form $[a_s^* a_t]_{s,t \in F}$, each of which is mapped to

$$\sum_{s,t \in F} \alpha_s(a_s^* a_t)\lambda_{st^{-1}} = \Big(\sum_{s \in F} a_s \lambda_{s^{-1}}\Big)^* \Big(\sum_{s \in F} a_s \lambda_{s^{-1}}\Big),$$

and observe that amplifications of ψ_2 are actually of the same form.

Now, define ψ to be the composition $\psi_2 \circ \psi_1$. We have

$$\psi(1) = \psi_2(X^2) = \psi_2\left(\sum_{t \in F} \alpha_t^{-1}((T(t))^2) \otimes e_{t,t}\right) = \langle T, T \rangle \lambda_e = 1_{\mathcal{A}} \lambda_e = 1$$

and thus ψ is actually a u.c.p. map.

Using the fact that $T(s) \in \mathcal{Z}(\mathcal{A})$ for all $s \in \Gamma$ and the previous lemma, we get

$$\begin{aligned}
\psi \circ \varphi(a\lambda_s) &= \psi\left(\sum_{t \in F \cap sF} \alpha_t^{-1}(a) \otimes e_{t,s^{-1}t}\right) \\
&= \psi_2\left(\sum_{t \in F \cap sF} \alpha_t^{-1}(T(t))\alpha_t^{-1}(a)\alpha_{s^{-1}t}^{-1}(T(s^{-1}t)) \otimes e_{t,s^{-1}t}\right) \\
&= \sum_{t \in F \cap sF} T(t)a\alpha_s(T(s^{-1}t))\lambda_s
\end{aligned}$$

$$= \sum_{t\in F\cap sF} T(t)\alpha_s(T(s^{-1}t))a\lambda_s$$
$$= (T *_\alpha T^*(s))a\lambda_s.$$

■

Theorem 3.3.11. *If the Γ-action α on $\mathcal{A}$ is amenable, then:*

1. $\mathcal{A} \rtimes \Gamma = \mathcal{A} \rtimes_r \Gamma$.
2. *$\mathcal{A}$ is nuclear iff $\mathcal{A} \rtimes \Gamma$ is.*

Proof. We would like to prove that the canonical quotient map $\pi : \mathcal{A} \rtimes \Gamma \to \mathcal{A} \rtimes_r \Gamma$ is injective. It suffices to find u.c.p. maps $\omega_i : \mathcal{A} \rtimes_r \Gamma \to \mathcal{A} \rtimes \Gamma$ that act as an approximate left inverse of π, i.e. $\|x - \omega_i \circ \pi(x)\| \to 0$ for all $x \in C_c(\Gamma, \mathcal{A}) \subseteq \mathcal{A} \rtimes \Gamma$.

Luckily, we already have those u.c.p. maps at our disposal. Let $T_i : \Gamma \to \mathcal{A}$ be the maps witnessing amenability of α and φ_i, ψ_i be the associated u.c.p. maps provided by Lemma 3.3.10. The key observation is that the ψ_i's remain u.c.p. if seen as taking values in $\mathcal{A} \rtimes \Gamma$ (which is possible because their image is contained in $C_c(\Gamma, \mathcal{A})$), therefore we can define $\omega_i = \psi_i \circ \varphi_i$. Using Lemma 3.3.9, we have

$$\begin{aligned}\|x - \omega_i(\pi(x))\| &= \left\|\sum_{s\in\Gamma}(1 - T_i *_\alpha T_i^*(s))a_s\lambda_s\right\| \\ &\leqslant \sum_{s\in\Gamma} \|1 - T_i *_\alpha T_i^*(s)\| \, \|a_s\lambda_s\| \\ &\leqslant \sum_{s\in\Gamma} \|s *_\alpha T_i - T_i\|_2 \, \|a_s\lambda_s\| \to 0\end{aligned}$$

for all $x = \sum_{s\in\Gamma} a_s\lambda_s \in C_c(\Gamma, \mathcal{A})$.

Moving on to the second assertion, lets first assume that $\mathcal{A} \rtimes \Gamma$ is nuclear. Then the identity map $\mathrm{id}_{\mathcal{A}}$ can be decomposed as

$$\mathcal{A} \hookrightarrow \mathcal{A} \rtimes \Gamma \xrightarrow{\mathrm{id}_{\mathcal{A}\rtimes\Gamma}} \mathcal{A} \rtimes \Gamma \xrightarrow{E} \mathcal{A},$$

where E is the canonical conditional expectation. Since $\mathrm{id}_{\mathcal{A}\rtimes\Gamma}$ is nuclear, we get that $\mathrm{id}_{\mathcal{A}}$ is as well.

Conversely, if $\mathcal{A}$ is nuclear, then so is $\mathcal{A} \otimes \mathbb{M}_F(\mathbb{C})$ for any finite set F. Hence, using the same technique, the ω_i's we defined above are nuclear. Since they converge pointwise to the identity, we obtain nuclearity of the crossed product. ■

We return now to the abelian case, so let X be a compact Γ-space and α be the induced Γ-action on $C(X)$. Instead of $C_c(\Gamma, C(X))$, we will work with $C_c(X \times \Gamma)$ (the reader can easily convince themselves the two are interchangeable).

Definition 3.3.12. We call Γ *amenable at infinity* iff it acts amenably on some compact Γ-space.

Since $C(X)$ is always nuclear, we have the following corollary of the above theorem.

Corollary 3.3.13. *If Γ is amenable at infinity, then it is exact.*

What is important is that the converse also holds, as we shall see. But we still have some way to get there.

Definition 3.3.14. A function $h : X \times \Gamma \to \mathbb{C}$ is of *positive type* iff for any finite sequence $s_1, \dots, s_n \in \Gamma$ and $x \in X$, the matrix $[h(s_i x, s_i s_j^{-1}]_{i,j} \in \mathbb{M}_n(\mathbb{C})$ is positive.

The observant reader may notice that an element in $C_c(X \times \Gamma)$ is of positive type iff the corresponding element in $C_c(\Gamma, C(X)) \subseteq C(X) \rtimes_r \Gamma$ is positive. Regardless, we have the following important theorem.

Theorem 3.3.15. *The following are equivalent:*

1. *X is amenable.*
2. *$C(X) \rtimes_r \Gamma$ is nuclear.*
3. *For any finite $F \subseteq \Gamma$ and $\varepsilon > 0$, there exists a function $h \in C_c(X \times \Gamma)$ of positive type such that*

$$\max_{s \in F} \sup_{x \in X} |h(x,s) - 1| < \varepsilon.$$

Proof. (1 $\implies$ 2) This is an immediate consequence of Proposition 3.3.8 and the previous theorem.

(2 $\implies$ 3) Let $F \subseteq \Gamma$ be a finite set and $\varepsilon > 0$. By nuclearity, using Lemma 3.3.4, we can find u.c.p. maps $\varphi : C(X) \rtimes_r \Gamma \to \mathbb{M}_n(\mathbb{C})$ (compression by a suitable projection) and $\psi : \mathbb{M}_n(\mathbb{C}) \to C(X) \rtimes_r \Gamma$ such that $\omega = \psi \circ \varphi$ satisfies

$$\|\omega(\lambda_s) - \lambda_s\| < \varepsilon$$

for all $s \in F$. Define $h : X \times \Gamma \to \mathbb{C}$ by

$$h(x,s) = h_s(x) = E(\omega(\lambda_s)\lambda_s^*)(x),$$

where E is the canonical conditional expectation, and notice that h has compact support because $\omega(\lambda_s) = 0$ for s outside some finite $F' \subset \Gamma$ (that is why it is crucial to use the lemma to choose φ). Furthermore, we have

$$\left\|1_{C(X)} - h_s\right\| = \|E((\lambda_s - \omega(\lambda_s))\lambda_s^*)\| \leqslant \|\lambda_s - \omega(\lambda_s)\| < \varepsilon$$

for all $s \in F$. It remains to show that h is of positive type. To that end, we calculate

$$
\begin{aligned}
[\alpha_{s_i^{-1}}(h_{s_i s_j^{-1}})]_{i,j} &= \left[\alpha_{s_i^{-1}}\big(E(\omega(\lambda_{s_i s_j^{-1}})\lambda_{s_j s_i^{-1}})\big)\right]_{i,j} \\
&= \left[E(\lambda_{s_i}^* \omega(\lambda_{s_i s_j^{-1}})\lambda_{s_j}\right]_{i,j} \\
&= E\left(\operatorname{diag}(\lambda_{s_1}, \ldots, \lambda_{s_n})^* \omega([\lambda_{s_i}\lambda_{s_j}^*]_{i,j}) \operatorname{diag}(\lambda_{s_1}, \ldots, \lambda_{s_n})\right)
\end{aligned}
$$

for all $s_1, \ldots, s_n \in \Gamma$, and, thus, complete positivity of E and ω gets us what we need (we have abused notation for the amplifications, but the meaning should be clear enough).

($3 \implies 1$) Let $F \subseteq \Gamma$ be a finite set containing e and let h be the associated function provided by condition (3) for a fixed $\varepsilon > 0$. Since h is positive in $C(X) \rtimes_r \Gamma$, there exists $g \in C_c(\Gamma, C(X))$ such that $g^* *_\alpha g \approx h$ in $C(X) \rtimes_r \Gamma$. That implies $E(g^* *_\alpha g) \approx E(h) \approx 1_{C(X)}$ (because $e \in F$). Thus, we can normalise g so that $E(g^* *_\alpha g) = 1_{C(X)}$. We define now $T : \Gamma \to C(X)$ by

$$T(t)(x) = |g(t^{-1})(t^{-1}x)| = |g^*(t)(x)|$$

and calculate

$$
\begin{aligned}
\langle T, s *_\alpha T\rangle (x) &= \Big(\sum_{t\in\Gamma} T(t)\alpha_s(T(s^{-1}t))\Big)(x) \\
&= \sum_{t\in\Gamma} |g^*(t)(x)||g^*(s^{-1}t)(s^{-1}x)| \\
&= \sum_{t\in\Gamma} |g^*(t)(x)||g(t^{-1}s)(t^{-1}x)| \\
&\geqslant \Big|\sum_{t\in\Gamma} g^*(t)(x)g(t^{-1}s)(t^{-1}x)\Big| \\
&= |(g^* *_\alpha g)(s)(x)|
\end{aligned}
$$

where equality holds if $s = e$. Thus, $\langle T, T\rangle = E(g^* *_\alpha g) = 1_{C(X)}$ and $\langle s *_\alpha T, s *_\alpha T\rangle = \alpha_s(\langle T, T\rangle) = 1_{C(X)}$. Now we have

$$\|s *_\alpha T - T\|_2^2 = \left\|2 \cdot 1_{C(X)} - \langle T, s *_\alpha T\rangle - \langle s *_\alpha T, T\rangle\right\| \leqslant 2\left\|1_{C(X)} - |g^* *_\alpha g(s)|\right\|$$

which is close to zero for all $s \in F$. But all $s \in \Gamma$ belong eventually to some F, therefore we can construct the net required to witness amenability of α (with careful selection of approximations). ■

Definition 3.3.16. A bounded function $k : \Gamma \times \Gamma \to \mathbb{C}$ is called a *positive definite kernel* iff the matrix $[k(s,t)]_{s,t\in F}$ is positive for any finite set $F \subseteq \Gamma$.

Definition 3.3.17. A *tube* is a set of the form

$$\{(s,t) \in \Gamma \times \Gamma : st^{-1} \in F\}$$

for some finite set $F \subseteq \Gamma$. We call F the *width* of the tube, which we denote by $\operatorname{tube}(F)$.

Definition 3.3.18. Consider the left translation action of Γ on $\ell^\infty(\Gamma)$. We call the associated reduced crossed product $\ell^\infty(\Gamma) \rtimes_r \Gamma$ the *uniform Roe algebra* of Γ and denote it by $C^*_u(\Gamma)$.

It is important to note that $C^*_u(\Gamma)$ is just the C^*-subalgebra of $B(\ell^2(\Gamma))$ generated by $C^*_r(\Gamma)$ and $\ell^\infty(\Gamma)$.

Observe, thus, that (positive definite) kernels supported in tubes, which correspond exactly to (positive) elements belonging to the *-algebra generated by $C^*_r(\Gamma)$ and $\ell^\infty(\Gamma)$ inside $B(\ell^2(\Gamma))$, can be identified with elements inside $C^*_u(\Gamma)$.

Theorem 3.3.19. *The following are equivalent:*

1. Γ *is exact.*

2. *For any finite set* $E \subseteq \Gamma$ *and* $\varepsilon > 0$, *there exists a positive definite kernel* $k : \Gamma \times \Gamma \to \mathbb{C}$ *whose support is contained in a tube, such that* $k(s,s) = 1$ *for all* $s \in \Gamma$ *and*
$$\sup\{|k(s,t) - 1| : (s,t) \in \mathrm{tube}(E)\} < \varepsilon.$$

3. *For any finite set* $E \subseteq \Gamma$ *and* $\varepsilon > 0$, *there exist a finite set* $F \subseteq \Gamma$ *and* $\zeta : \Gamma \to \ell^2(\Gamma)$ *such that* $\|\zeta_s\| = 1$ *and* $\mathrm{supp}(\zeta_s) \subseteq Fs$ *for all* $s \in \Gamma$, *and*
$$\sup\{\|\zeta_s - \zeta_t\| : (s,t) \in \mathrm{tube}(E)\} < \varepsilon.$$

4. *For any finite set* $E \subseteq \Gamma$ *and* $\varepsilon > 0$, *there exist a finite set* $F \subseteq \Gamma$ *and* $\mu : \Gamma \to \mathcal{P}(\Gamma)$ *such that* $\mathrm{supp}(\mu_s) \subseteq Fs$ *for all* $s \in \Gamma$ *and*
$$\sup\{\|\mu_s - \mu_t\|_1 : (s,t) \in \mathrm{tube}(E)\} < \varepsilon.$$

5. *The left translation* Γ*-action on* $\ell^\infty(\Gamma)$ *is amenable.*

In particular, Γ *is exact iff it is amenable at infinity iff it acts amenably on its Stone-Čech compactification* $\beta\Gamma$.

Proof. ($1 \implies 2$) Let $E \subseteq \Gamma$ be a finite set and $\varepsilon > 0$. Using Lemma 3.3.4, we can find u.c.p. maps $\varphi : C^*_r(\Gamma) \to B(\ell^2(F))$ (compression by the projection onto $\ell^2(F)$ for some finite $F \subseteq \Gamma$) and $\psi : B(\ell^2(F)) \to B(\ell^2(\Gamma))$ such that $\omega = \psi \circ \varphi$ satisfies
$$\|\omega(\lambda_s) - \lambda_s\| < \varepsilon$$
for all $s \in E$. We define a kernel $k : \Gamma \times \Gamma \to \mathbb{C}$ by
$$k(s,t) = \langle \omega(\lambda_{st^{-1}})\delta_t, \delta_s \rangle ,$$

which is supported on a subset of $\mathrm{tube}(FF^{-1})$ (by the definition of ω). Furthermore, for $s_1, s_2, \dots, s_n \in \Gamma$ and $z_1, z_2, \dots, z_n \in \mathbb{C}$, we calculate

$$\begin{aligned}\sum_{i,j} k(s_i, s_j)\bar{z}_i z_j &= \sum_{i,j} \left\langle \omega(\lambda_{s_i s_j^{-1}}) z_j \delta_{s_j}, z_i \delta_{s_i} \right\rangle \\ &= \left\langle \omega\big([\lambda_{s_i s_j^{-1}}]_{i,j}\big) \begin{bmatrix} z_1\delta_{s_1} \\ \vdots \\ z_n \delta_{s_n}\end{bmatrix}, \begin{bmatrix} z_1\delta_{s_1} \\ \vdots \\ z_n \delta_{s_n}\end{bmatrix} \right\rangle \\ &\geqslant 0\end{aligned}$$

since ω is c.p. and $[\lambda_{s_i s_j^{-1}}]_{i,j}$ is positive in $\mathbb{M}_n(C^*_r(\Gamma))$. Thus k is positive definite. Finally, we have

$$\begin{aligned}|k(s,t) - 1| &= |\langle \omega(\lambda_{st^{-1}})\delta_t, \delta_s\rangle - 1| \\ &= |\langle(\omega(\lambda_{st^{-1}}) - \lambda_{st^{-1}})\delta_t, \delta_s\rangle| \\ &\leqslant \|\omega(\lambda_{st^{-1}}) - \lambda_{st^{-1}}\| \\ &< \varepsilon\end{aligned}$$

for all $(s,t) \in \mathrm{tube}(E)$.

($2 \implies 3$) Let $E \subseteq \Gamma$ be a finite set, $\varepsilon > 0$, and k be a positive definite kernel satisfying (2). Let a be the element in $C^*_u(\Gamma)$ corresponding to k. There exists a finite set $F \subseteq \Gamma$ and a kernel supported in $\mathrm{tube}(F)$ such that for the corresponding element $b \in C^*_u(\Gamma)$ we have $\|a - b^*b\| \approx 0$. For $s \in \Gamma$, set $\eta_s = b\delta_s \in \ell^2(\Gamma)$ and observe that $\|\eta_s\| \approx 1$ since

$$\langle \eta_s, \eta_t\rangle = \langle b\delta_s, b\delta_t\rangle = \langle b^*b\delta_s, \delta_t\rangle \approx \langle a\delta_s, \delta_t\rangle = k(s,t),$$

while $\mathrm{supp}(\eta_s) \subseteq Fs$. Hence, for $\zeta_s = \eta_s / \|\eta_s\|$ we have $\|\zeta_s\| = 1$ and $\zeta_s \approx \eta_s$. Therefore

$$\|\zeta_s - \zeta_t\|^2 \leqslant 2|1 - \langle \zeta_s, \zeta_t\rangle| \approx 2|1 - \langle \eta_s, \eta_t\rangle| \approx 2|1 - k(s,t)| < 2\varepsilon$$

for $(s,t) \in \mathrm{tube}(E)$, which gives us what we want after appropriate adjustment of the ε's.

($3 \implies 4$) Trivial, using the map $\ell^2(\Gamma) \to \ell^1(\Gamma)_+ : (\zeta_s) \mapsto (|\zeta_s|^2)$.

($4 \implies 5$) For any finite symmetric $E \subseteq \Gamma$ and $\varepsilon > 0$, let $\mu : \Gamma \to \mathcal{P}(\Gamma)$ be the associated map provided by (4) and define $T : \Gamma \to \ell^\infty(\Gamma)$ by

$$T(x)(s) = \sqrt{\mu_s(s^{-1}x)}.$$

We can now work exactly as in the proof of Proposition 3.3.8 to first see that

$$\|s * T - T\|_2^2 \leqslant \sup_{t\in\Gamma} \|\mu_{s^{-1}t} - \mu_t\|_1 < \varepsilon$$

for all $s \in E$ and then obtain a net witnessing amenability of the Γ-action on $\ell^\infty(\Gamma)$.

($5 \implies 1$) As $\ell^\infty(\Gamma) = C(\beta\Gamma)$, Γ is amenable at infinity, and thus exact. ■

Lemma 3.3.20. *A compact* Γ*-space* X *is amenable iff* $\mathcal{P}(X)$ *is.*

Proof. One direction is immediate, since $X \subseteq \mathcal{P}(X)$. For the converse, fix an a.i.c.m. $m_i : X \to \mathcal{P}(\Gamma)$. Define $\tilde{m}_i : \mathcal{P}(X) \to \mathcal{P}(\Gamma)$ by

$$\tilde{m}_i(\mu)(s) = \int_X m_i^x(s)\, d\mu$$

and calculate

$$\begin{aligned}\|s\tilde{m}_i^{\mu} - \tilde{m}_i^{s\mu}\|_1 &= \sum_{t\in\Gamma} \left| \int_X s m_i^x(t) - m_i^{sx}(t)\, d\mu \right| \\ &\leqslant \sup_{x\in X} \|s m_i^x - m_i^{sx}\|_1 \to 0\end{aligned}$$

for all $\mu \in \mathcal{P}(X)$ and $s \in \Gamma$. ■

Finally, we can give the desired characterisation of exactness in terms of the Γ-action on $\partial_{FH}\Gamma$.

Theorem 3.3.21. Γ *is exact iff* $\partial_{FH}\Gamma$ *is amenable.*

Proof. If $\partial_{FH}\Gamma$ is amenable, then Γ is exact by Corollary 3.3.13.

Conversely, if Γ is exact, then $\beta\Gamma$ is amenable by Theorem 3.3.19, and thus $\mathcal{P}(\beta\Gamma)$ is (by the lemma above). By Γ-injectivity, there exists a Γ-equivariant u.c.p. map $\varphi : \ell^\infty(\Gamma) \to C(\partial_{FH}\Gamma)$, hence the adjoint φ^* restricts to a continuous Γ-equivariant map $\partial_{FH}\Gamma \to \mathcal{P}(\beta\Gamma)$. Composing this map with an a.i.c.m. witnessing amenability of $\mathcal{P}(\beta\Gamma)$, we obtain an a.i.c.m. for $\partial_{FH}\Gamma$. ■

Appendix

Our aim is to make this work as accessible as possible, assuming only knowledge covered by undergraduate courses on functional analysis and operator theory. For this purpose, this appendix will be dedicated to some fundamentals of C^*-algebras. We should mention, however, that we will restrict ourselves to the parts of the theory that are useful for this work (even if that means presenting them in an unconventional and condensed way). Since no proofs will be given, we refer to [BO08; Mur90; Pau03] for more details. Before we proceed with that task, for the sake of completeness we will also mention two well-known classical results in functional analysis that are used here and there, and that might not be covered in an introductory course (in this form at least). Proofs of these can be found, for example, in [Con07].

Classical Results in Functional Analysis

Theorem A.1 (Hahn-Banach separation - complex case). *Let X be a complex locally convex space and A, B be disjoint closed convex subsets of X. If B is compact, then there exists an $f \in X^*$, an $r \in \mathbb{R}$ and an $\varepsilon > 0$ such that*

$$\operatorname{Re} f(a) \leqslant r < r + \varepsilon \leqslant \operatorname{Re} f(b), \ \ \forall a \in A, b \in B.$$

Theorem A.2 (Krein-Milman). *If K is a nonempty compact convex subset of a locally convex space X, then $\operatorname{ext} K \neq \emptyset$ and $K = \overline{\operatorname{conv}}(\operatorname{ext} K)$, where $\operatorname{ext} K$ denotes the set of extreme points of K.*

The Krein-Milman theorem is usually accompanied by the following proposition, known as *Milman's partial converse to the Krein-Milman theorem.*

Proposition A.3. *If X is a locally convex space, K is a compact convex subset of X, and $F \subseteq K$ such that $K = \overline{\operatorname{conv}}(F)$, then $\operatorname{ext} K \subseteq \overline{F}$.*

C^*-Algebras

Definition A.4. A complex normed algebra $(\mathcal{A}, \|.\|)$ is called a *Banach algebra* iff the underlying linear space is a Banach space and $\|.\|$ is submultiplicative, i.e.

$$\|ab\| \leqslant \|a\| \, \|b\| \, , \ \forall a, b \in A.$$

Definition A.5. A complex algebra A is called *involutive* or simply a *-*algebra* iff it is equiped with an antilinear map ${}^* : A \to A$ that is an antiautomorphism of order 2 on the multiplicative semigroup of A. The image of an element under this involution is called its *adjoint*.

Definition A.6. A Banach *-algebra $(\mathcal{A}, \|.\|)$ is called a C^*-*algebra* iff $\|.\|$ is a C^*-norm, i.e. satisfies the C^*-*identity*

$$\|a^* a\| = \|a\|^2$$

for all $a \in \mathcal{A}$.

Proposition A.7. *If* $(\mathcal{A}, \|.\|)$ *is a* C^**-algebra and* $\|.\|'$ *is another* C^**-norm on* $\mathcal{A}$*, then* $\|.\|' = \|.\|$.

From now on, unless otherwise specified, $\mathcal{A}$ will denote a C^*-algebra. Morphisms in the category of C^*-algebras will be *-homomorphisms, i.e. algebra homomorphisms respecting the involution.

Proposition A.8. *Let* I *be a closed ideal in* $\mathcal{A}$ *and* $\mathcal{B} \subseteq \mathcal{A}$ *be a* C^**-subalgebra. Then*

- $\mathcal{A}/I$ *(with its natural structure) is a* C^**-algebra.*
- $\mathcal{B} + I \subseteq \mathcal{A}$ *is a* C^**-algebra.*

Furthermore, $\mathcal{B}/(\mathcal{B} \cap I) \cong (\mathcal{B} + I)/I$.

Definition A.9. An element $a \in \mathcal{A}$ is called:

- *normal* iff $a^* a = a a^*$.
- *self-adjoint* iff $a = a^*$.
- *unitary* iff $a^* a = a a^* = 1_{\mathcal{A}}$ (in the unital case).
- *positive* iff $a = b^* b$ for some $b \in \mathcal{A}$.

Proposition A.10. *The set of positive elements of* $\mathcal{A}$*, denoted by* $\mathcal{A}_+$*, is a* $\|.\|$*-closed salient cone inside the space* $\mathcal{A}_{sa} \subseteq \mathcal{A}$ *of self-adjoint elements. Therefore,* $\mathcal{A}_{sa}$ *is partially ordered by the relation*

$$a \leqslant b \iff b - a \in \mathcal{A}_+.$$

Proposition A.11. *For each element* $a \in \mathcal{A}_{sa}$ *there exist unique* $a_+, a_- \in \mathcal{A}_+$ *such that* $a = a_+ - a_-$ *and* $a_+ a_- = a_- a_+ = 0$*. Thus, every element in* $\mathcal{A}$ *can be written as a linear combination of four elements in* $\mathcal{A}_+$*.*

Definition A.12. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is called *positive* iff $a \geqslant 0 \implies \varphi(a) \geqslant 0$ and *faithful* iff $a > 0 \implies \varphi(a) > 0$ for all $a \in \mathcal{A}$. A positive linear functional is called a *state* iff it is of norm 1. A state τ on $\mathcal{A}$ is called *tracial* (or simply a *trace*) iff $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$.

Proposition A.13. *Positive linear linear maps are automatically bounded.*

Proposition A.14. *A linear functional φ on a unital C^*-algebra $\mathcal{A}$ is positive iff $\|\varphi\| = \varphi(1_{\mathcal{A}})$.*

Definition A.15. A linear functional on $\mathcal{A}$ is called *self-adjoint* iff $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathcal{A}$.

Proposition A.16. *For each self-adjoint bounded linear functional φ on $\mathcal{A}$ there exist positive linear functionals φ_+, φ_- such that $\varphi = \varphi_+ - \varphi_-$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$. Thus, every bounded linear functional on $\mathcal{A}$ can be written as a linear combination of four positive ones.*

Theorem A.17 (Gelfand representation). *Every abelian C^*-algebra $\mathcal{A}$ is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. X is called the* (Gelfand) spectrum *of $\mathcal{A}$.*

Definition A.18. A **-representation* of a complex involutive algebra A is a *-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. π is called *non-degenerate* iff $\pi(A)(\mathcal{H})$ is dense in $B(\mathcal{H})$.

Theorem A.19 (Gelfand-Neimark). *Every C^*-algebra has a *-representation that is faithful. Therefore, every C^*-algebra can be concretely realised as a $\|.\|$-closed *-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Remark. Notions, such as positivity, which are typically defined in a different fashion in the context of $B(\mathcal{H})$ (through inner products or the spectrum) are all equivalent to the above definitions given for abstract C^*-algebras.

Proposition A.20. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Then, their algebraic tensor product $\mathcal{H} \odot \mathcal{K}$ is a pre-Hilbert space, when equipped with the inner product defined on elementary tensors by*

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle_{\mathcal{H}} \langle k_1, k_2 \rangle_{\mathcal{K}} .$$

We denote its respective completion by $\mathcal{H} \otimes \mathcal{K}$.

Proposition A.21. *If $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$, then there exists a unique $T \otimes S \in B(\mathcal{H} \otimes \mathcal{K})$ such that*

$$T \otimes S(u \otimes v) = (Tu) \otimes (Sv)$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{K}$. Moreover, $\|T \otimes S\| = \|T\| \, \|S\|$.

Consider now two C^*-algebras $\mathcal{A}, \mathcal{B}$ and let $\mathcal{A} \odot \mathcal{B}$ be their algebraic tensor product. We can turn $\mathcal{A} \odot \mathcal{B}$ into a *-algebra, endowing it with an involution defined on elementary tensors by

$$(a \otimes b)^* = a^* \otimes b^*.$$

Definition A.22. Define $\|.\|_{\max} : \mathcal{A} \odot \mathcal{B} \to \mathbb{R}_+$ by

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi \text{ is a } ^*\text{-representation of } \mathcal{A} \odot \mathcal{B}\}$$

for all $x \in \mathcal{A} \odot \mathcal{B}$. Then $\|.\|_{\max}$ is a C^*-norm. We call the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|.\|_{\max}$ the *maximal tensor product* of $\mathcal{A}$ and $\mathcal{B}$, and denote it by $\mathcal{A} \otimes_{\max} \mathcal{B}$.

Definition A.23. Consider faithful *-representations $\pi : \mathcal{A} \to B(\mathcal{H})$ and $\rho : \mathcal{B} \to B(\mathcal{K})$. Define $\|.\|_{\min} : \mathcal{A} \odot \mathcal{B} \to \mathbb{R}_+$ by

$$\left\|\sum_i a_i \otimes b_i\right\|_{\min} = \left\|\sum_i \pi(a_i) \otimes \rho(b_i)\right\|_{B(\mathcal{H} \otimes \mathcal{K})}$$

for all $x \in \mathcal{A} \odot \mathcal{B}$. Then $\|.\|_{\min}$ is a C^*-norm. We call the completion of $\mathcal{A} \odot \mathcal{B}$ with respect to $\|.\|_{\min}$ the *spatial* or *minimal* tensor product of $\mathcal{A}$ and $\mathcal{B}$, and denote it by $\mathcal{A} \otimes \mathcal{B}$.

Remark. The spatial tensor product is independent of the choice of faithful representations.

Proposition A.24. *The maximal norm is the largest possible C^*-norm on $\mathcal{A} \odot \mathcal{B}$.*

Theorem A.25 (Takesaki). *The spatial norm is the smallest possible C^*-norm on $\mathcal{A} \odot \mathcal{B}$.*

Proposition A.26. *For any $n \in \mathbb{N}$, the matrix algebra $\mathbb{M}_n(\mathcal{A}) = \mathbb{M}_n(\mathbb{C}) \odot \mathcal{A}$ can be turned into a C^*-algebra. Thus, $\mathcal{A} \otimes \mathbb{M}_n(\mathbb{C}) = \mathcal{A} \otimes_{\max} \mathbb{M}_n(\mathbb{C})$ (by the uniqueness of C^*-norms).*

Definition A.27. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map. We call the map $\varphi_{(n)} = \varphi \otimes \mathrm{id}_{\mathbb{M}_n(\mathbb{C})}$ the n-*th amplification* of φ. We will occasionally abuse notation and omit the index of an amplification (in cases where it is clear from the context).

Definition A.28. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called:

- n-*positive* iff $\varphi_{(n)}$ is positive.
- *completely positive* iff it is n-positive for all $n \in \mathbb{N}$.
- *completely isometric* iff $\varphi_{(n)}$ is an isometry for all $n \in \mathbb{N}$.

- *completely bounded* iff $\|\varphi\|_{cb} := \sup_n \|\varphi_{(n)}\| < \infty$.
- *completely contractive* iff $\|\varphi\|_{cb} \leqslant 1$.

Proposition A.29. *Positive maps which have an abelian* C^**-algebra as their domain or range are automatically completely positive.*

Proposition A.30. *Tensor products of completely positive maps are completely positive.*

Proposition A.31. *Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a contractive completely positive map. Then*

$$\{a \in \mathcal{A} : \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C^**-subalgebra of* $\mathcal{A}$ *and is equal to the set*

$$\{a \in \mathcal{A} : \varphi(aa') = \varphi(a)\varphi(a') \text{ and } \varphi(a'a) = \varphi(a')\varphi(a) \text{ for all } a' \in \mathcal{A}\}.$$

We call this set the multiplicative domain *of* φ.

Proposition A.32 (Schwarz inequality for 2-positive maps). *Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a unital* 2*-positive map. Then*

$$\varphi(a)^*\varphi(a) \leqslant \varphi(a^*a)$$

for all $a \in \mathcal{A}$.

Definition A.33. Let $\mathcal{B} \subseteq \mathcal{A}$. A contractive completely positive projection $E : \mathcal{A} \to \mathcal{B}$ is called a *conditional expectation* iff it is a $\mathcal{B}$-bimodule map.

Theorem A.34 (Tomiyama). *Let* $\mathcal{B} \subseteq \mathcal{A}$ *and* $E : \mathcal{A} \to \mathcal{B}$ *be a projection. The following are equivalent:*

1. E *is a conditional expectation.*
2. E *is contractive completely positive.*
3. E *is contractive.*

Definition A.35. Assume $\mathcal{A}$ is unital. A linear subspace $\mathcal{S} \subseteq \mathcal{A}$ is called an *operator system* iff it is self-adjoint and contains $1_{\mathcal{A}}$.

Remark. Order structure (via positivity) is crucial for operator systems. Thus, it should not come as a surprise that the appropriate morphisms in that category are completely positive maps.

Theorem A.36 (Arveson's extension). *Let* $\mathcal{S} \subseteq \mathcal{A}$ *be an operator system and* $\varphi : \mathcal{S} \to B(\mathcal{H})$ *be a completely positive map. Then* φ *extends to a completely positive map* $\tilde{\varphi} : \mathcal{A} \to B(\mathcal{H})$.

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