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Master's Thesis

Multiple-Scale perturbative method and its application in cosmological perturbations

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Ευχαριστίες:

Θα ήθελα να ευχαριστήσω θερμά τον κύριο Φώτη Διάκονο που με μύησε στη Μαθηματική Φυσική και για την διαρκή παρότρυνση και στήριξή του όλα αυτά τα χρόνια.

Οφείλω πολλές ευχαριστίες στον κύριο Μάνο Σαριδάκη για την πρόταση του θέματος που με εισήγαγε σε ανοιχτά προβλήματα, την επίβλεψη και το ενδιαφέρον του όλους αυτούς τους μήνες.

Τέλος θέλω να ευχαριστήσω τις/τους: Αγαθή, Ελευθερία, Ιάκωβο, Νικόλα, Χρύσα και Ράγια. Ήσασταν παρόντες σε όλες τις χρονικές κλίμακες.

Abstract

The failure to recognize the dependence on more than one time/space scales is a common source of nonuniformity in perturbation expansions. In this thesis perturbative Multiple-Scale method is presented as a tool for global asymptotic analysis. Various multiple-scale techniques are presented.

The physical interpretation is discussed. Namely, we provide arguments on the relation between the breaking of Lorentz symmetry and scale-dependent instabilities. Furthermore we discuss the meaning of the quantum mechanical cut-off scale in a semiclassical perturbative treatment.

The perturbative Multiple Scale method is attempted in the context of the Effective Field Theory of Inflation. Specifically, we examine its application on the alleviation of the Strong Coupling problem.

Abstract

Η μη αναγνώριση της εξάρτησης από περισσότερες από μία χρονικές/χωρικές κλίμακες είναι συνήθης αιτία μη ομοιόμορφης σύγκλισης διαταρακτικών αναπτυγμάτων. Σε αυτήν την εργασία παρουσιάζεται η μέθοδος των Πολλαπλών Κλιμάκων καθώς και διάφορες παραλλαγές της ως εργαλείο ασυμπτωτικής ανάλυσης με βελτιωμένες ιδιότητες σύγκλισης.

Συζητείται η φυσική ερμηνεία της μεθόδου και παρέχονται επιχειρήματα ως προς τη συσχέτιση ανάμεσα στο σπάσιμο συμμετρίας Lorentz και αστάθειες εξαρτώμενες από κλίμακα. Επίσης αναδεικνύεται σε τι αντιστοιχεί η κβαντική κλίμακα αποκοπής κατά την ημικλασσική διαταρακτική προσέγγιση.

Μελετάται η εφαρμογή της διαταρακτικής μεθόδου Πολλαπλών Κλιμάκων σε κοσμολογικές διαταραχές στο πλαίσιο της Ενεργού Θεωρίας Πεδίου του Πληθωρισμού. Συγκεκριμένα εξετάζεται η εφαρμογή της στη χαλάρωση του προβλήματος Ισχυρής Ζεύξης.

Contents

1 Introduction

The goal of this thesis is the study of the application of multiple-scale perturbative method on cosmological perturbations. The motivation was the understanding and the possible alleviation of the Strong Coupling problem that leads to the break down of the theory. Just like in particle physics, one interpretation is that it suggests an indication that new degrees of freedom may become important at energies below the strong coupling scale. To study this prediction of the model of the Effective Field Theory of Inflation in this thesis we use the perturbative multiple-scale method. This method is particularly useful for constructing uniformly valid approximations to solutions of perturbation problems. The most striking feature of the method is in its powerful applicability on nonlinear differential equations.

In the first chapter of this thesis we present the perturbative method. We start by the motivation that leads to the introduction of multiple scales. Through an example it is shown that an instability/divergence seen one one scale may not be seen at another. Multiple-scale analysis is a rather general collection of perturbation techniques developed to treat phenomena of complex dynamic systems. Two variants are presented. The "Derivative Expansion" (DE) method is applied and is explained its main feature, the solubility condition as well as why it can handle in principle strong coupling problems. Next, "Renormalization Group" (RG) scaling method is employed as it makes transparent what is the equivalent of Quantum Field Theory's cutoff scale when one approaches the corresponding problem with semi-classical approximations. Then, the ϕ^4 theory is solved in the weak regime and we also provide a solution for the strong coupling case under a proper formulation to finally deduce a typical amplitude equation. Through the RG method we discuss the relation between breaking of Lorentz invariance and scale dependent instabilities. Finally, we outline some limitations about the applicability of the method.

The second part of the thesis is concerned with the application of the multiple scale method to cosmological perturbations. That is an original endeavour since up to now there are fairly few attempts of such a study. For that, first we review the Effective Field Theory of Inflation. Effective Field Theories is the phenomenological tool to describe the dynamics of a physical procedure when the full theory is unknown or alternatively it is strongly coupled. The effective theory does not commit to a specific microscopic realization of the physics of inflation. It shows that the basic predictions of inflation don't rely on that assumption. For us, it is the proper context under which the energy scales are transparent. The main features of the theory are presented. Then is is outlined how the dispersion is related to the interactions and the breaking of Lorentz symmetry. It becomes apparent that a small speed of sound is related to large interactions. The theory possesses a strong coupling scale. We indicate why the multiple-scale analysis should be attempted and formulate the problem properly according to the tools developed in previous chapter.

2 Perturbative multiple-scale methods

2.1 Regular perturbation and its failure

We present here the standard textbook [10], [1] ,[2] example. In this section we demonstrate how non uniformity can appear in a regular perturbation expansion as a result of resonant interactions between consecutive orders of perturbation theory. To begin, "Duffing's" oscillator is given by the equation,

$$\frac{d^2u}{dt^2} + \omega_0^2\left[1 + \epsilon u^2\right]u = 0, \quad u(0) = 1, \quad \dot{u}(0) = 0 \tag{1}$$

and describes the oscillations of a mass connected to a nonlinear spring, with ϵ taken as a small positive quantity. This equation has the full-known analytical solution

$$u(t) = \cos\left[\left(1 + \frac{3\epsilon}{8}\right)\omega_0 t\right] \tag{2}$$

The solution describes an oscillation at the *perturbed* frequency:

$$\omega(\epsilon) = \omega_0\left[1 + \frac{3}{8}\epsilon + \cdots\right] \tag{3}$$

The other main feature is that the amplitude remains *bounded* for every value of t on an infinite domain.

Regular expansion: We pretend that we do not know the above full analytical solution.
To show the failure of the standard perturbative approach, we seek the solution in the form of a perturbation series,

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$$

This yields an homogeneous equation for u_0. Furthermore it leads to the hierarchy of inhomogeneous equations for $u_1, u_2, u_n \cdots$,

$$\begin{aligned}
&\frac{d^2u_0}{dt^2} + \omega_0^2 u_0 = 0\\
&\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = -\omega_0^2 u_0^3\\
&\frac{d^2u_2}{dt^2} + \omega_0^2 u_2 = -3\omega_0^2 u_0^2 u_1\\
&\frac{d^2u_3}{dt^3} + \omega_0^2 u_3 = -3\omega_0^2\left(u_0 u_1^2 + u_0^2 u_2\right)
\end{aligned}$$

The solution of the homogeneous 0th order equation , that satisfies the boundary conditions is

$$u_0 = A\cos\omega_0 t \tag{4}$$

1st order equation then reads,

$$\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = -A\omega_0^2 \cos^3 \omega_0 t \tag{5}$$

or equivalently,

$$\frac{d^2u_1}{dt^2} + \omega_0^2 u_1 = -A\frac{3\omega_0^2}{4}\cos\omega_0 t - A\frac{\omega_0^2}{4}\cos 3\omega_0 t \tag{6}$$

The forcing RHS function contains both resonant and non-resonant terms; using,

$$\frac{d^2g}{dt^2} + \omega_0^2 g = f_0\cos\omega_f t \quad \Longrightarrow \quad g(x) = \begin{cases} \frac{f_0}{\omega_0^2-\omega_f^2}\sin\omega_f t, & \omega_0 \neq \omega_f \\ \frac{f_0}{2\omega_0} t\sin\omega_0 t & \omega_0 = \omega_f \end{cases} \tag{7}$$

gives:

$$u_1(t) = A\frac{3}{8}\omega_0 t\sin(\omega_0 t) + A\frac{1}{32}\cos(3\omega_0 t) + C\cos\omega_0 t + D\sin\omega_0 t \tag{8}$$

Applying initial conditions:

$$u_1(0) = \dot{u}_1(0) = 0 \quad \Longrightarrow \quad C = -\frac{1}{32}, \quad D = 0 \tag{9}$$

the final solution for u_1 reads,

$$u_1(t) = A\frac{3}{8}\omega_0 t\sin(\omega_0 t) + A\frac{1}{32}\left[\cos(3\omega_0 t) - \cos\omega_0\right] \tag{10}$$

and the the full solution to linear order finally is,

$$u(t) = A\left(1 - \frac{\epsilon}{32}\right)\cos\omega_0 t + A\frac{3}{8}\epsilon\omega_0 t\sin\omega_0 t + A\frac{\epsilon}{32}\cos 3\omega_0 t + O\left(\epsilon^2\right) \tag{11}$$

The up to first order expansion above, does not agree to the full solution as $t \to \infty$. The second term is not bounded. Its amplitude grows linearly in time. We emphasize here that actually, the proposed series expansion breaks down **before** even the time runs to infinite value. This is manifested by the fact that ϵu_1 is not a small correction to the unperturbed solution u_0 as should, whenever t reaches the value $t = \mathcal{O}(1/\epsilon)$. In this bounded domain for t the series is still convergent. But the convergence is slow enough to render the series useless, if we are to keep just a finite number of terms, which is our ultimate goal after all.

The naive perturbation series is plagued by the so called "secularities" when polynomial terms in time appears in it. These terms have the property of being not bounded for enough large time making the series generally useless.

2.2 Derivative expansion method

We showed that there is no agreement with the known analytical solution. We proceed by presenting the multiple scale method and test whether it is capable to exhibit the expected behaviour. We once again seek a perturbative solution pretending that we do not know the analytical solution of the Duffing's nonlinear ordinary differential equation.
The naive attempt above managed at least to point out that there is a new time scale. Namely, the time scale over which the amplitude of the resonant terms is comparable with the amplitude of the unperturbed solution. This time analysis leads one to consider a special treatment to be able to keep the expansion uniformly valid beyond these times. For that we, at least formally, increase the number of independent variables. A time variable $\tau \equiv \epsilon t$ is introduced formally,

$$u = u(t, \tau), \quad \tau = \epsilon t$$

Even though the function dependes on t alone, multiple-scale analysis seeks solutions which are functions of both variables t and τ treated as independent variables. The new variable τ has the following significance: Whereas t increases by an amount on the order of 1 over each period of the oscillation, τ increases by an amount on the order of 1 over the characteristic timescale on which the envelope of the oscillation grows or decays.
Derivative operators now, transform to,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{d\tau}{dt}\frac{\partial u}{\partial \tau}$$

and

$$\frac{d^2u}{dt^2} = \frac{\partial^2 u}{\partial t^2} + 2\epsilon \frac{\partial^2 u}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 u}{\partial \tau^2} \tag{12}$$

The ansatz,

$$u(t, \tau) = u_0(t, \tau) + \epsilon u_1(t, \tau) + \epsilon^2 u_2(t, \tau) + \cdots \tag{13}$$

gives,

$$\begin{aligned} \frac{d^2u}{dt^2} = \frac{\partial^2 u_0}{\partial t^2} &+ \epsilon \left[\frac{\partial^2 u_1}{\partial t^2} + 2\frac{\partial^2 u_0}{\partial t \partial \tau} \right] \\ &+ \epsilon^2 \left[\frac{\partial^2 u_2}{\partial t^2} + 2\frac{\partial^2 u_1}{\partial t \partial \tau} + \frac{\partial^2 u_0}{\partial \tau^2} \right] + O\left(\epsilon^3\right) \end{aligned} \tag{14}$$

This many-variable technique is also known as **derivative-expansion method** [1]. The nonlinear term as before gives,

$$u^3 = u_0^3 + 3\epsilon u_0^2 u_1 + 3\epsilon^2 \left[u_0 u_1^2 + u_0^2 u_1 \right] + O\left(\epsilon^3\right) \tag{15}$$

and substituting the above chain rule gives,

$$
\begin{aligned}
&\frac{\partial^2 u_0}{\partial t^2} + \omega_0^2 u_0 = 0 \\
&\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = -2\frac{\partial^2 u_0}{\partial t \partial \tau} - \omega_0^2 u_0^3 \\
&\frac{\partial^2 u_2}{\partial t^2} + \omega_0^2 u_2 = -\frac{\partial^2 u_0}{\partial \tau^2} - 2\frac{\partial^2 u_1}{\partial t \partial \tau} - 3\omega_0^2 u_0^2 u_1
\end{aligned} \tag{16}
$$

The solution to the homogeneous one, is given by

$$u_0(t) = A(\tau)e^{i\omega_0 t} + A^*(\tau)e^{-i\omega_0 t}$$

so the next order equation reads,

$$
\begin{aligned}
\frac{\partial^2 u_1}{\partial t^2} + \omega_0^2 u_1 = &- 2i\omega_0 \left(\frac{\partial A}{\partial \tau} e^{i\omega_0 t} - \frac{\partial A^*}{\partial \tau} e^{-i\omega_0 t} \right) \\
&- \omega_0^2 \left(A^3 e^{3i\omega_0 t} + 3A^2 A^* e^{i\omega_0 t} + 3AA^{*2} e^{-i\omega_0 t} + A^{*3} e^{-3i\omega_0 t} \right)
\end{aligned} \tag{17}
$$

The next necessary step of the method is usually named as application of "solubility condition" in the relevant bibliography. The τ dependence of $A(\tau)$ must be adjusted appropriately, to ensure that all the terms which contribute to resonance will be eliminated.
For this we collect the terms that are in resonance with the natural frequency of the oscillator, the so called **secular** terms,

$$\left[-2i\omega_0 \frac{\partial A}{\partial \tau} - 3\omega_0^2 A^2 A^* \right] e^{i\omega_0 t} + \text{ c.c.}$$

where CC stands for the complex conjugate of the former.
For the expansion to be valid for times $t > \frac{1}{\epsilon}$, we demand that the above expression equals to zero,

$$-2i\omega_0 \frac{\partial A}{\partial \tau} - 3\omega_0^2 A^2 A^* = 0$$

The mathematical reason for this is that we actually demand that

$$\frac{u_m}{u_{m-1}} < \infty \text{ for all } \tau_0, \tau_1 \ldots \tau_m \tag{18}$$

This criterion *does not* mean that *each* u_m must be bounded. Maybe some of them are not. The true meaning of the above statement is that as one keeps more and more higher-order terms in the expansion, these terms must not be *more* singular than the first term [1].

We emphasize this point as it is the cornerstone of our further developments in the study of nonlinear phenomena in the strong

coupling regime. That is the reason that a treatise of strongly coupled problems with multiple-scale approach is, in principle, possible and will become of use in next sections.

To continue, if we express the amplitude in polar coordinates: $A(\tau) = R(\tau)e^{i\varphi(\tau)}$ gives,

$$\frac{\partial R}{\partial \tau} + iR\frac{\partial \varphi}{\partial \tau} = \frac{3i\omega_0}{2}R^3$$

The real and imaginary parts are,

$$\frac{\partial R}{\partial \tau} = 0 \implies R(\tau) \equiv R_0 \quad \text{(constant)}$$
$$\frac{\partial \varphi}{\partial \tau} = \frac{3\omega_0}{2}R^2 \implies \varphi(\tau) = \frac{3\omega_0 \tau}{2}R_0^2 + \varphi_0$$

The function $A(\tau)$

$$A(\tau) = R_0 e^i \left[\frac{3\omega_0 \tau}{2}R_0^2 + \varphi_0\right]$$

and so u_0 reads,

$$u_0(t) = R_0 \left[e^{i\left(\omega_0 t + \frac{3\omega_0 \tau}{2}R_0^2 + \varphi_0\right)} + e^{-i\left(\omega_0 t + \frac{3\omega_0 \tau}{2}R_0^2 + \varphi_0\right)}\right]$$

Applying the initial conditions, gives

$$u(t) = \cos\left[\omega_0 t + \frac{3\omega_0 \tau}{8}\right]$$

and since $\tau = \epsilon t$, we finally get

$$u(t) = \cos\left[\left(1 + \frac{3\epsilon}{8}\right)\omega_0 t\right]$$

in accordance with the analytical solution, up to this order.

2.3 Renormalization Group method

We follow closely [5] and study the Rayleigh equation:

$$\frac{d^2y}{dt^2} + y = \epsilon \left[\frac{dy}{dt} - \frac{1}{3}\left(\frac{dy}{dt}\right)^3 \right] \tag{19}$$

The expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots \tag{20}$$

gives the solution up to first order,

$$\begin{aligned} y(t) =& R_0 \sin(t+\Theta_0) + \epsilon \left\{ -\frac{R_0^3}{96}\cos(t+\Theta_0) \right. \\ &\left. + \frac{R_0}{2}\left(1-\frac{R_0^2}{4}\right)(t-t_0)\sin(t+\Theta_0) + \frac{R_0^3}{96}\cos 3(t+\Theta_0) \right\} + O\left(\epsilon^2\right) \end{aligned} \tag{21}$$

where R_0, Θ_0 are constants determined by the initial conditions at arbitrary $t = t_0$. The term proportional to $(t - t_0)$ is produced by the secular terms in the Rayleigh equation, as we discussed in the last section. So, when $(t - t_0) \geq \frac{1}{\epsilon}$ the perturbation breaks down.
To avoid this behavior, one can control t_0, the time at which initial conditions are defined, the offset. This is the primitive idea here. If the offset were closer to t, then the perturbation wouldn't break down. To that, we introduce a "new offset" τ, splitting $t - t_0$ to $t - \tau + \tau - t_0$. One is free to choose τ to be close enough to t. To be clear: we do not wish to change the value of the initial condition but the time at which we impose the initial condition. If we want to make contact with the usual field theory the arbitrary time t_0 may be interpreted as the (logarithm of the) ultraviolet cutoff in the usual field theory [6], [5].
Formally, we follow the steps:

- Introduce an arbitrary time τ,

$$(t - t_0) \to (t - t_0 + \tau - \tau)$$

- Split

$$[t - t_0 + \tau - \tau] \to [(t - \tau) + (\tau - t_0)]$$

- Absorb the terms containing $(\tau - t_0)$ into the renormalized counterparts R and Θ of R_0 and Θ_0 respectively
  In order to do that a multiplicative renormalization constant and an additive one are introduced

$$Z_1 = 1 + \sum_{n=1}^{\infty} a_n \epsilon^n \quad , \quad Z_2 = \sum_{n=1}^{\infty} b_n \epsilon^n$$

such that,

$$R_0\left(t_0\right)=Z_1\left(t_0, \tau\right) R(\tau), \Theta_0\left(t_0\right)=\Theta(\tau)+Z_2\left(t_0, \tau\right)$$

The coefficients a_n, b_n are responsible to remove the secularities

If we apply these steps, the result is

$$\begin{aligned} y(t)= & \left(1+a_1 \epsilon\right) R_R \sin \left[t+\left(\Theta_R(\tau)+b_1 \epsilon\right)\right] \\ & +\epsilon\left[-\frac{1}{96} R_R^3 \cos \left[t+\left(\Theta_R(\tau)+b_1 \epsilon\right)\right]\right. \\ & \left.+\frac{1}{2} R_R\left(1-\frac{1}{4} R_R^2\right) \times\left[(t-\tau)+\left(\tau-t_0\right)\right]\right] \sin \left[t+\left(\Theta_R(\tau)+b_1 \epsilon\right)\right] \\ & +\frac{1}{96} R_R^3 \cos \left[t+\left(\Theta_R(\tau)+b_1 \epsilon\right)\right]+\mathcal{O}\left(\epsilon^2\right) \end{aligned} \tag{22}$$

So that a_1 and b_1 are chosen to eliminate terms containing $t-t_0$ in first order

$$\begin{aligned} & a_1=-\left(\frac{1}{2}\right)\left(1-\frac{R_R^2}{4}\right)\left(\tau-t_0\right) \\ & b_1=0 \end{aligned} \tag{23}$$

Thus we conclude, throwing R-indices,

$$y(t)=\left[R+\epsilon \frac{R}{2}\left(1-\frac{R^2}{4}\right)(t-\tau)\right] \sin (t+\Theta)-\epsilon \frac{1}{96} R^3 \cos (t+\Theta)+\epsilon \frac{R^3}{96} \cos (3(t+\Theta))+\mathcal{O}\left(\epsilon^2\right)$$

- Since τ does not appear in the original problem, the solution should not depend on τ. Therefore $(\partial y / \partial \tau)_t=0$ for any t. This is the RG equation, which in this case consists of two independent equations

$$\frac{d R}{d \tau}=\epsilon \frac{1}{2} R\left(1-\frac{1}{4} R^2\right)+O\left(\epsilon^2\right), \quad \frac{d \Theta}{d \tau}=O\left(\epsilon^2\right)$$

Solving and equating τ and t eliminates the secular term; we get

$$\begin{aligned} & R(t)=R(0) /\left[e^{-\epsilon}+\frac{1}{4} R(0)^2\left(1-e^{-\epsilon t}\right)\right]^{1 / 2}+O\left(\epsilon^2 t\right) \\ & \Theta(t)=\Theta(0)+O\left(\epsilon^2 t\right) \end{aligned}$$

For appropriate initial conditions this result in a limit cycle [5].

Remarks - Analogies with usual field theory method: We chose to include the presentation of this "multiple scale" method since it offers a useful point of view when one turns from quantum mechanic treatment to semi-classical treatment. Namely it offers the following correspondence :

- Usual field theory renormalization method
  - Renormalization of the field $\phi_R = Z^{\frac{1}{2}}\phi$
  - Regularization through the cutoff scale Λ
  - Elimination of Λ through counterterms
- RG method
  - Renormalization of the Amplitude
  - Regularization of t_0 through the introduction of τ
  - Elimination of τ through the RG equation and counterterms included in Z.

So, when one passes from the quantum treatment to semiclassical treatment should have in mind that: if there is a divergence in *energy*, a cut off scale in the quantum mechanical treatment $\rightarrow$ this should be translated into a divergence in the amplitude's evolution above a *time range* in the semiclassical counterpart problem.

2.4 Φ^4 Theory

In this section we treat the Φ^4 theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \mathcal{V}(\phi)$$

$$\mathcal{V}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{8}\phi^4, \quad \lambda > 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right] = 0 \quad \Rightarrow \quad \left(\Box + m^2\right)\phi = -\frac{\lambda}{2}\phi^3 \tag{24}$$

with various multiple-scale perturbative methods both in the case of weak and strong coupling.

2.4.1 Derivative Expansion method

Multiple-scale method's (or rather more appropriately called as derivative expansion method [1]) first step consists in scaling/gauging **all** the independent variables. For that, following [9], one introduces small/slow space and time variables accordingly, with the help of a small quantity ϵ.

$$\left.\begin{array}{lll} x_0 = \epsilon^0 x & x_1 = \epsilon^1 x & x_2 = \epsilon^2 x \dots \\ t_0 = \epsilon^0 t & t_1 = \epsilon^1 t & t_2 = \epsilon^2 t \dots \end{array}\right\} \Rightarrow \left.\begin{array}{c} x_\alpha = \epsilon^\alpha x \\ t_\alpha = \epsilon^\alpha t \\ {\scriptstyle \alpha = 0,1,2\dots} \end{array}\right\}$$

We express the field $\phi = \phi(x,t)$ in terms of the new variables,

$$\phi = \phi\left(t_0, t_1, t_2, \dots, x_0, x_1, x_2 \dots\right)$$

Derivative operators become,

$$\frac{d}{dt} = \epsilon^i \partial_{t_i}$$

$$\frac{d}{dx^i} = \epsilon^i \partial_{x_i}$$

$$\begin{aligned} \partial_{xx} &= \left(\partial_{x_0} + \epsilon\partial_{x_1} + \epsilon^2\partial_{x_2} + \dots\right)\left(\partial_{x_0} + \epsilon\partial_{x_1} + \epsilon^2\partial_{x_2} + \dots\right) \\ &= \partial_{x_0 x_0} + \epsilon\left(2\partial_{x_0 x_1}\right) + \epsilon^2\left(2\partial_{x_0 x_2} + \partial_{x_1 x_1}\right) + \epsilon^3\left(2\partial_{x_1 x_2}\right) + \mathcal{O}\left(\epsilon^4\right) \end{aligned}$$

and

$$\partial_{tt} = \partial_{t_0 t_0} + \epsilon\left(2\partial_{t_0 t_1}\right) + \epsilon^2\left(2\partial_{t_0 t_2} + \partial_{t_1 t_1}\right) + \epsilon^3\left(2\partial_{t_1 t_2}\right) + \mathcal{O}\left(\epsilon^4\right)$$

So, the box operator becomes,

$$\begin{aligned} \Box = &\left(\partial_{t_0 t_0} - \partial_{x_0 x_0}\right) + \epsilon\left(2\partial_{t_0 t_1} - 2\partial_{x_0 x_1}\right) + \epsilon^2\left(2\partial_{t_0 t_2} + \partial_{t_1 t_1} - 2\partial_{x_0 x_2} + \partial_{x_1 x_1}\right) \\ &+ \epsilon^3\left(2\partial_{t_1 t_2} - 2\partial_{x_1 x_2}\right) + \mathcal{O}\left(\epsilon^4\right) \end{aligned}$$

or more compactly written as,

$$\Box = \Box_0 + 2\epsilon \left(\partial_{t_0 t_1} - \partial_{x_0 x_1}\right) + \epsilon^2 \left(\Box_1 + 2\partial_{t_0 t_2} - 2\partial_{x_0 x_2}\right) + \epsilon^3 \left(2\partial_{t_1 t_2} - 2\partial_{x_1 x_2}\right) + \mathcal{O}\left(\epsilon^4\right)$$

Let us substitute the above in the equation of motion, $\Box\phi + m^2\phi = -\frac{\lambda}{2}\phi^3$. The potential possesses its global minimum at $\phi = 0$. We expand the potential around its minimum, $\phi = 0 + \delta\phi$, as [1]

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \ldots$$

Order by order, we have:

$$\mathcal{O}\left(\epsilon^1\right) : \left(\Box_0 + m^2\right)\phi_1 = 0$$
$$\mathcal{O}\left(\epsilon^2\right) : \left(\Box_0 + m^2\right)\phi_2 + \left(2\partial_{t_0 t_1} - 2\partial_{x_0 x_1}\right)\phi_1 = 0$$
$$\mathcal{O}\left(\epsilon^3\right) : \left(\Box_0 + m^2\right)\phi_3 + \left(2\partial_{t_0 t_1} - 2\partial_{x_0 x_1}\right)\phi_2 + \left(\Box_1 + 2\partial_{t_0 t_2} - 2\partial_{x_0 x_2}\right)\phi_1 = -\frac{\lambda}{2}\left(\phi_1\right)^3$$

First order equation's general solution:

$$\phi_1 = A\left(x_n, t_n\right)e^{-i\theta} + c.c. \quad \text{with} \quad n \geq 1 \quad \text{and} \quad \theta = k_0 t_0 - k_1 x_0$$

where k_1 should satisfy the dispersion relation:

$$k_0^2 - k_1^2 = m^2 \Leftrightarrow k_0^2 = k_1^2 + m^2$$

Such a form for the amplitude ensures that "constant" A is actually function of the slower variables x_n, t_n with $n \geq 1$. Here we are invoking the above mentioned idea (and at the same time explain the name "derivative expansion method") that t_0 and t_n with $n > 1$ should be regarded as independent variables. The functions of t_n behave like constants on the fast time scales (similarly for the dependence with respect to space variables).

One should now examine the **second** order equation. If we substitute the above 0th order solution to the right hand side, the part $\left(\partial_{t_0 t_1} - \partial_{x_0 x_1}\right)\phi_1$ contributes as a "**secular**" term. These terms are proportional to $e^{i\theta}$ or $e^{-i\theta}$. They are modes which oscillate exactly with the same frequency of the corresponding homogeneous equation generating divergent terms for $t > \frac{1}{\epsilon}$, since this resonance generates terms $\propto t$ in the solution. Hence they break down the perturbative approximation - if we are to keep only a small number of orders of magnitude in the suggested solution of course. In other words, it is not an actual behavior of the amplitude but a virtual one, generated by our approach to keep only a small number of power-terms. The remedy, or more

[1] $\delta\phi \to \phi$ for simplicity

formally the solubility condition, demands the elimination of such terms to gain the evolution of the amplitude for $t > \frac{1}{\epsilon}$. In detail, we require:

$$\begin{aligned}
\left(\partial_{t_0 t_1} - \partial_{x_0 x_1}\right)\phi_1 = 0 &\Rightarrow ik_0 \partial_{t_1} A e^{-i\theta} + ik_1 \partial_{x_1} A e^{-i\theta} = 0 \\
&\Rightarrow k_0 \partial_{t_1} A + k_1 \partial_{x_1} A = 0
\end{aligned}$$

At this point we have the freedom to introduce a new reference frame, in terms of the new-bar variables,

$$\bar{X}_1 = x_1 - c_g t_1$$

where c_g, the group velocity $c_g = \dfrac{dk_0}{dk_1}$, which with the help of the dispersion relation results in $c_g = \dfrac{dk_0}{dk_1} = \dfrac{k_1}{k_0}$.

With respect to this new reference frame,

$$\begin{aligned}
\partial_{t_1} &= \frac{\partial_{\bar{X}_1}}{\partial_{t_1}} \frac{\partial}{\partial_{\bar{X}_1}} = -c_g \partial_{\bar{X}_1} \\
\partial_{x_1} &= \frac{\partial_{\bar{X}_1}}{\partial_{x_1}} \frac{\partial}{\partial_{\bar{X}_1}} = \partial_{\bar{X}_1}
\end{aligned}$$

This choice suffices to fulfil the above condition at this order. In fact,

$$\begin{aligned}
k_0 \partial_{t_1} A + k_1 \partial_{x_1} A &= \left[k_0 \left(-c_g \partial_{\bar{X}_1}\right) + k_1 \partial_{\bar{X}_1}\right] A \\
&= \left[k_0 \left(-\frac{k_1}{k_0}\right) \partial_{\bar{X}_1} + k_1 \partial_{\bar{X}_1}\right] A \\
&= 0
\end{aligned}$$

After these steps, at second and third order the remaining terms are,

$$\begin{aligned}
&\left(\Box_0 + m^2\right)\phi_2 = 0 \\
&\left(\Box_0 + m^2\right)\phi_3 + \left(\Box_1 + 2\partial_{t_0 t_2} - 2\partial_{x_0 x_2}\right)\phi_1 = -\frac{\lambda}{2}\phi_1^3
\end{aligned}$$

For third order equation, we once again examine for possible terms able to contribute to secular behavior. We collect the secular terms,

$$\begin{aligned}
\frac{\lambda}{2}\phi_1^3 = \frac{\lambda}{2}\left(A^3 e^{-3i\theta} + 3|A|^2 A e^{-i\theta}\right) + c.c. &\Rightarrow \quad 3\frac{\lambda}{2}|A|^2 A e^{-i\theta} + c.c. \\
\Box_1 \phi_1 = \left(\partial_{t_1 t_1} - \partial_{x_1 x_1}\right)\phi_1 = \left(c_g^2 - 1\right)\partial_{\bar{X}_1 \bar{X}_1}\phi_1 &= \left(c_g^2 - 1\right)\left(\partial_{\bar{X}_1 \bar{X}_1} A\right) e^{-i\theta} + c.c. \\
\left(2\partial_{t_0 t_2} - 2\partial_{x_0 x_2}\right)\phi_1 = 0 - 2ik_0 \partial_{t_2}\phi_1 &= -2ik_0\left(\partial_{t_2} A\right) e^{-i\theta} + c.c.
\end{aligned}$$

Note: In the last equation above, we implicitly **suggested** that at *this* order the amplitude does **not** depend on x_2, but only on time t_2. This suggestion is **equivalent** to a further introduction of a new $\bar{\bar{X}}$ reference frame.

We always have this freedom. Such a frame with an appropriately defined linear group velocity would eliminate this dependence [8].

Then as above we apply the solubility condition which eliminates these terms. This procedure results in,

$$3\frac{\lambda}{2}|A|^2A + \left(c_g^2 - 1\right)\left(\partial_{\bar{X}_1\bar{X}_1}A\right) - 2ik_0\left(\partial_{t_2}A\right) = 0$$

By definition of c_g,

$$c_g^2 = \left(\frac{k_1}{k_0}\right)^2 = \frac{k_1^2}{k_1^2 + m^2} \quad \Rightarrow \quad 1 - c_g^2 = \frac{m^2}{k_1^2 + m^2} = \frac{m^2}{k_0^2}$$

one concludes that,

$$ik_0\left(\partial_{t_2}A\right) + \frac{1}{2}\frac{m^2}{k_0^2}\partial_{\bar{X}_1\bar{X}_1}A - \frac{3}{2}\frac{\lambda}{2}|A|^2A = 0$$

Let us - in order to transform the form of the final equation- to define $\tilde{A}$,

$$A = \left(\frac{2}{3}\frac{2}{\lambda}\right)^{\frac{1}{2}}\tilde{A}$$

Thus,

$$ik_0\partial_{t_2}\tilde{A} + \frac{1}{2}\frac{m^2}{k_0^2}\partial_{\tilde{X}_1\tilde{X}_1}\tilde{A} - |\tilde{A}|^2\tilde{A} = 0$$

and for further simplification,

$$t_2 = k_0\tilde{t}_2 \quad \text{and} \quad \bar{X}_1 = \sqrt{\frac{m^2}{k_0^2}}\tilde{x}_1$$

to finally conclude that:

$$i\partial_{\tilde{t}_2}\tilde{A} + \frac{1}{2}\partial_{\tilde{x}_1\tilde{x}_1}\tilde{A} - |\tilde{A}|^2\tilde{A} = 0 \tag{25}$$

which is known as **NLS** (Non Linear Schrödinger) equation, in its most usual form.

2.4.2 Renormalization Group method

The equation of motion we derived is of the form $\left(\Box + m^2\right)\phi = g\phi^3$. It is a non-linear equation and therefore one cannot apply the superposition principle. This makes it very hard (and in general impossible) to write down exact solutions to this equation beyond the trivial $\varphi = 0$. [2] Thus, in general, we will have to resort to an approximate procedure: For $g << 1$, we can treat the right hand side as a small perturbation of the free Klein-Gordon equation and obtain approximations to the true solution by perturbing solutions to the Klein-Gordon equation.
Equation:

$$\left(\Box + m^2\right)\phi = -\frac{\lambda}{2}\phi^3$$

is of the form,

$$\left(\Box + m^2\right)\phi = -\epsilon^2\phi^3$$

Whether ϵ^2 is a small or large parameter, changes the way we encounter the problem and the physics involved as well (weak / strong coupling). At first we approach the problem, for λ being small. One has simply to view this as a family of equations parametrised by the coupling constant λ. Similarly, the solutions to all these equations will depend on λ. Underlying the idea of perturbation theory is the idea that these solutions can be written as a power-series in λ, i.e. that they are analytic in λ around $\lambda = 0$.
The equation of motion we derived is a partial differential equation. Therefore Goldenfeld's *et al* method [5] suitable for ordinary differential equations cannot be applied directly. The suggested extension of the method to deal with this case comes from study in [3]. In more detail, the degrees of freedom here are two $d = 1{+}1$. The RG parameter of our theory must be one of them according to RG theory . The main point in order to extend Goldenfeld's method, consists in **gauging/scaling every independent variable but one**, which we set as the RG parameter [3]. To begin, we solve the **weak** nonlinear equation,

$$\left(\partial_{tt} - \partial_{xx}\right)\phi + m^2\phi = -\epsilon^2\phi^3 \tag{26}$$

where ϵ is a small parameter. With the following ansatz,

$$\phi = A\exp[i(kx - \omega t)] + c.c. \quad \text{with} \quad A = A(\xi, t) \quad \text{and} \quad \xi = \epsilon x$$

where we use c.c. notation for complex conjugate.
In order to focus on a slowly-varying amplitude (envelope), we introduce the complex quantity A and a small parameter ξ to scale the x independent space variable.
The t variable, i.e. the only remaining independent variable of our problem,

[2](known special class of exact solutions exist: "kinks" or "domain-walls")

is set as the RG parameter of our theory. For the ansaz to satisfy the above equation of motion, k must satisfy the following dispersion relation,

$$\omega = \left(m^2 + k^2\right)^{\frac{1}{2}}$$

which comes from zeroth-order equation,

$$\left.\begin{array}{c}\mathcal{O}\left(\epsilon^0\right) : \Box\phi + m^2\phi = 0 \\ \text{with:} \quad \phi = Ae^{i(kx-\omega t)} + c.c.\end{array}\right\} \Rightarrow \omega^2 = k^2 + m^2$$

Calculation:

$$\begin{aligned}
\partial_{tt}\phi &= \left(\partial_{tt}A - 2i\omega\partial_t A - \omega^2 A\right) e^{i(kx-\omega t)} + c.c. \\
\partial_x &= \frac{d\xi}{dx}\frac{\partial}{\partial\xi} = \epsilon\partial_\xi \\
&\Rightarrow \partial_{xx} = \epsilon^2\partial_{\xi\xi} \\
\partial_\xi\phi &= \left(\partial_\xi A + Ai\frac{k}{\epsilon}\right) e^{i(kx-\omega t)} + c.c. \\
\partial_{\xi\xi}\phi &= \left(\partial_{\xi\xi}A + 2i\frac{k}{\epsilon}\partial_\xi A - \frac{k^2}{\epsilon^2}A\right) e^{i(kx-\omega t)} + c.c.
\end{aligned}$$

$$\begin{aligned}
\Box\phi &\equiv \left(\partial_{tt} - \partial_{xx}\right)\phi = \left(\partial_{tt} - \epsilon^2\partial_{\xi\xi}\right)\phi \\
&= \left(\partial_{tt} - 2i\omega\partial_t\right)A - \epsilon^2\partial_{\xi\xi}A - 2ik\epsilon\partial_\xi A + \left(k^2 - \omega^2\right)A\Big) e^{i(kx-\omega t)} + c.c.
\end{aligned}$$

With the use of dispersion relation, $(k^2 - \omega^2) = -m^2$,

$$\Box\phi = [LA - \epsilon\left(2ik\xi + \epsilon\partial_{\xi\xi}\right)A - m^2 A]\exp[i(kx - \omega t)] + c.c.$$

where we introduced the operator L,

$$L \equiv \partial_{tt} - 2i\omega\partial_t$$

The remaiming terms give,

$$\left(m^2 + \epsilon^2\phi^2\right)\phi = \left[m^2 A + \epsilon^2\left(A^3 e^{3i\theta} + 3|A|^2 A^* e^{i\theta}\right)\right] e^{i(kx-\omega t)} + c.c.$$

We collect the above and the initial equation $\Box\phi + \left(m^2 + \epsilon^2\phi^2\right)\phi = 0$ finally becomes,

$$[LA - \epsilon\left(2ik\partial_\xi + \epsilon\partial_{\xi\xi}\right)A]\,e^{i(kx-\omega t)} + \epsilon^2 A^3 e^{3i(kx-\omega t)} + c.c. = 0$$

Now, substituting the expansion,

$$A = A_0(\xi) + \epsilon A_1 + \epsilon^2 A_2 + \ldots$$

with $L = \partial_{tt} - 2i\omega\partial_t$ and $\theta = kx - \omega t$ we have, order by order

$$\mathcal{O}\left(\epsilon^1\right): LA_1 = 2ik\partial_\xi A_0$$
$$\mathcal{O}\left(\epsilon^2\right): LA_2 = 2ik\partial_\xi A_1 + \partial_{\xi\xi} A_0 - 3\left|A_0\right|^2 A_0$$
$$\mathcal{O}\left(\epsilon^3\right): \ldots$$

The solution of the above is given by,

$$\begin{aligned} A =& A_0(\xi) + \epsilon\left(-\frac{k}{\omega}\partial_\xi A_0\right)t \\ &+ \epsilon^2\left(\frac{1}{2}\frac{k^2}{\omega^2}\partial_{\xi\xi}A_0 t^2 - \frac{1}{2}i\frac{k^2}{\omega}\partial_{\xi\xi}A_0 t + \left(\partial_{\xi\xi}A_0 - 3\left|A_0\right|^2 A_0\right)\frac{i}{2\omega}t\right) \end{aligned}$$

For convenience we express the above equation with the help of dispersion relation $\omega^2 = k^2 + m^2$,

$$\frac{d\omega}{dk} = \frac{k}{\omega} \equiv \dot{\omega}$$
$$\frac{d^2\omega}{dk^2} = \frac{1}{\omega} \equiv \ddot{\omega}$$

Hence,

$$A = A_0(\xi) + t\left[-\epsilon\dot{\omega}\partial_\xi A_0 + \epsilon^2\frac{i}{2}\left[\ddot{\omega}\partial_{\xi\xi}A_0 - 3\ddot{\omega}\left|A_0\right|^2 A_0\right]\right] + \ldots \tag{27}$$

where A_0 is determined by the initial conditions at arbitrary $t = t_0 = 0$. Now, we execute the steps:

1. **Regularization parameter**

   - Introduce parameter τ: $(t-0) \rightarrow (t+\tau-\tau-0)$

   Then, split terms in the following way, $[(t-\tau)+(\tau-0)]$ so that (27) becomes:

   $$A = A_0(\xi) + [(t-\tau)+(\tau-0)]\left[-\epsilon\dot{\omega}\partial_\xi A_0 + \epsilon^2\frac{i}{2}\left(\ddot{\omega}\partial_{\xi\xi}A_0 - 3\ddot{\omega}\left|A_0\right|^2 A_0]\right)\right] + \ldots$$

2. **Renormalization**
   We renormalize the amplitude A_0, introducing,

   $$A_0\left(t_0 = 0\right) = Z\left(t_0 = 0, \tau\right)A_R(\tau) \quad \text{with} \quad Z = 1 + \sum_{n=1}^{\infty} a_n \epsilon^n$$

   so that,
   $$A_0 = \left(1 + a_1\epsilon + a_2\epsilon^2 + \ldots\right)A_R(\tau)$$

3. **Counterterms**
$a_1(\tau), a_2(\tau)\ldots$ serve to absorb terms proportional to $(\tau - 0)$ at every order respectively. In detail,

   - For a_1
   $$A_0 = (1 + a_1\epsilon)\, A_R(\tau) + \mathcal{O}\left(\epsilon^2\right)$$
   so,
   $$\begin{aligned} A &= (1 + a_1\epsilon)\, A_R + [(t-\tau) + (\tau - 0)]\, [-\epsilon\dot{\omega}\partial_\xi\, (1 + a_1\epsilon)\, A_R] + \mathcal{O}\left(\epsilon^2\right) \\ &= A_R + \epsilon \Big[a_1 A_R + [(t-\tau) + (\tau - 0)]\, (-\dot{\omega}\partial_\xi A_R) \Big] + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$
   and the expression for a_1 is given according to above, if we require at order $\mathcal{O}(\epsilon)$ coefficients of $(\tau - 0)$ to be absorbed by a_1,
   $$a_1 A_R + (\tau - 0)\, (-\dot{\omega}\partial_\xi A_R) = 0 \Rightarrow a_1 = \tau\dot{\omega}\frac{\partial_\xi A_R}{A_R}$$
   - Similarly in 2nd order, we choose a_2 so that the terms $\propto (\tau - 0)$ get absorbed. Computationally is once again trivial to find the explicit expression for a_2. Nevertheless, the expressions for a_i's are of no use. It suffices to exist. In our problem their existence is evident. Then, with a_1 as above, we have,
   $$A = A_R(\xi,\tau) + (t-\tau)\left[-\epsilon\dot{\omega}\partial_\xi A_R + \epsilon^2\frac{i}{2}\left[\ddot{\omega}\partial_{\xi\xi}A_R - 3\ddot{\omega}\,|A_R|^2\, A_R\right]\right] + \ldots$$

4. **Elimination of regularization parameter:**
The idea here is simple but powerful. Since τ is not contained on the initial problem, the solution should not depend from it. Therefore,
$$\left.\frac{\partial\phi}{\partial\tau}\right|_t = 0, \quad \text{for every t}$$
known as the **RG equation**.
Equivalently, since $\phi = Ae^{i\theta} + c.c.$,
$$\left.\frac{\partial A}{\partial\tau}\right|_t = 0, \quad \text{for every t} \tag{28}$$
Differentiation gives,
$$0 = \frac{\partial A_R(\xi,\tau)}{\partial\tau} - \left[-\epsilon\dot{\omega}\partial_\xi A_R + \epsilon^2\frac{i}{2}\left[\ddot{\omega}\partial_{\xi\xi}A_R - 3\ddot{\omega}\,|A_R|^2\, A_R\right]\right] + (t-\tau)[\ldots]$$
with R index for the "renormalized" quantity. The term $(t-\tau)[\ldots]$ denotes every divergent/secular remaining term.

Finally, we set $\tau \to t$ from equation (28). That final step eliminates every secular term, and we have the wanted expression for the renormalized amplitude, (throwing index R),

$$\partial_t A = -\epsilon\dot{\omega}\partial_\xi A + \epsilon^2 \frac{i}{2}\left(\ddot{\omega}\partial_{\xi\xi}A - \frac{3}{\omega}|A|^2 A\right) \tag{29}$$

an NLS (Non Linear Schrodinger) equation with known soliton-type solutions. [3]

In terms of the original variables and parameters,

$$i\partial_t A + i\frac{k}{\omega}\partial_x A + \frac{1}{2\omega}\partial_{xx}A - \frac{3}{2}\frac{1}{\omega}\frac{\lambda}{2}|A|^2 A = 0 \tag{30}$$

with $\omega^2 = k^2 + m^2$.
This result applies up to $\mathcal{O}(\epsilon^2)$ or equivalently $\mathcal{O}(\lambda)$.

Remarks: The RG multiple-scale method, makes exlicit one important feature: in order to solve perturbatively the non-linear equation of motion we had to scale the spatial variable $x \to \xi = \epsilon x$ and to manipulate space and time in a different way. A regular perturbation would lead to instabilities. This induces a relation between breaking of Lorentz invariance and scale-dependent instabilities. This same pattern, i.e. the breaking of Lorentz symmetry due to interactions, will come up again when we will study the strong coupling limit of the Effective Field theory of inflation (section 3.2) and hints that a proper treatment to recognize the dependence on more than one time scales could lead to an alleviation of the problem.

[3] Specifically for equation (29) check : [8] pp. 601-602, for choice of coordinate frame suitable to eliminate first order derivative term. In terms of such a coordinate system one would result in the *standard form* of NLS.

2.5 Strong coupling

There are many famous examples of "strongly coupled systems" , from QCD and the theory of quarks, to high temperature superconductors, to the big bang itself. In many cases, the problem is that the interactions become so intense that the mathematical tools that we use to describe them break down. Weak perturbation theory generally proves to be insufficient to extract all the physics. In the case of quantum chromodynamics for example, the strength of the coupling constant at low energies, makes useless known perturbation techniques demanding the need for numerical solutions. In our approach we split our full ϕ^4 Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{8}\phi^4, \quad \lambda >> 1$$

and express the free part as $\mathcal{L}_2 = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2$ and the interacting as $\mathcal{L}_4 = -\frac{\lambda}{8}\phi^4$. As a simple measure of our strong coupling concern we use the ratio X of quartic to quadratic Lagrangian,

$$X = \frac{\mathcal{L}_4}{\mathcal{L}_2}$$

If $X > 1$ (which is our case due to the magnitude of λ) we are in the **strong coupling regime**. The goal is to study this area of the λ-parametric space. We are in search of the "dual" perturbation to that obtained in the case of small coupling (for another approach [7]). To start, the equation of motion of the full Lagrangian is

$$\begin{aligned} \left(\partial_{xx} - \partial_{tt} + m^2\right)\phi &= -\frac{\lambda}{2}\phi^3 \\ \text{or:} \quad \left(\Box_x + m^2\right)\phi &= -\frac{\lambda}{2}\phi^3, \quad \lambda >> 1 \end{aligned} \tag{31}$$

which for,

$$g = \frac{\lambda}{2}$$

takes the form

$$\left(\Box_x + m^2\right)\phi = -g\phi^3$$

For $g >> 1$ - strong coupling - we cannot treat the right-hand-side interaction term as a small perturbation of the free Klein-Gordon equation. However, with the change of variables,

$$x^\mu \to \zeta^\mu = g x^\mu,$$

and the quantity m_1 defined as:

$$m_1 : \quad m = m_1 g$$

the equation of motion becomes

$$g^2(\Box_\zeta + m_1^2)\phi = -g\phi^3$$

or

$$(\Box_\zeta + m_1^2)\phi = -\frac{1}{g}\phi^3 \tag{32}$$

Therefore the quantity $\frac{1}{g}$ is now small as opposed to g and we can proceed with the manipulations of the last sections. Note that the box operator now relates to $\zeta^\mu = gx^\mu$. Another difference appears in the dispersion relation and relates to the mass parameter $m_1 = \frac{m}{g}$. It distinguishes itself from the corresponding parameter in the small coupling limit: m_1 is now a tiny quantity as opposed to m.

2.5.1 Time analysis

Before proceeding to the application of multiple-scales method, it would be nice to perform a time analysis for the strong coupling case according to the standard perturbation method. This will enable us to read the time scales in which the problem in hand possesses non-uniform/secular behavior.
For that we substitute the ansatz,

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots$$

expanding around the minimum of the potential, and (32) gives, order per order,

$$\mathcal{O}\left(\epsilon^1\right) : (\Box_\zeta + m_1^2)\phi_1 = 0$$
$$\mathcal{O}\left(\epsilon^2\right) : (\Box_\zeta + m_1^2)\phi_2 = 0$$
$$\mathcal{O}\left(\epsilon^3\right) : (\Box_\zeta + m_1^2)\phi_3 = -\frac{1}{g}\phi_1^3 \dots$$

First order equation's solution:

$$\phi_1 = A\left(\zeta^1_{(n)}, \zeta^0_{(n)}\right)e^{-i\theta} + c.c. \quad \text{with} \quad n \geq 1, \quad \text{and} \quad \theta = k^0\zeta^0_{(0)} - k^1\zeta^1_{(0)} \tag{33}$$

where k_1 satisfies the dispersion relation:

$$k_0^2 - k_1^2 = m_1^2 \Leftrightarrow k_0^2 = k_1^2 + m_1^2$$

The $\mathcal{O}\left(\epsilon^3\right)$ equation gives resonating contribution because of the interaction term ϕ_1^3, which gives the contribution $\propto 3|A|^2Ae^{i\theta}$. This term produces in the solution a term proportional to the time variable ζ^0. So, the uniform expansion breaks down for time $\zeta^0 \propto \frac{1}{\epsilon^3}$.

To expand the *original* problem (31) we need to "tranfer" the above solution. For that, we need to express the solution with respect to the original variables. The 1st order (33) solution reads, after the necessary substitutions,

$$\phi_1 = A\left(\frac{\lambda}{2}x_n, \frac{\lambda}{2}t_n\right)e^{-i\theta} + c.c. \quad \text{with} \quad n \geq 1 \quad \text{and} \quad \theta = \frac{\lambda}{2}(k_0 t_0 - k_1 x_1)$$

where k_0 satisfies the dispersion relation:

$$k_0^2 - k_1^2 = m_1^2 \quad \Leftrightarrow \quad k_0^2 = k_1^2 + \frac{1}{\lambda^2}4m^2$$

while the perturbative expansion now breaks down for time scales

$$\zeta^0 \propto \frac{1}{\epsilon^3} \Rightarrow t \propto \frac{1}{\epsilon^3 \lambda}$$

Therefore, the relation between λ and ϵ^3 gains physical meaning. It points when the perturbative expansion breaks down. For example we have $\epsilon \ll 1$ and since $\lambda \gg 1$, in the case where ϵ is such that $\epsilon^3\lambda =$ finite the expansion turns out of control after a finite time.

2.5.2 Derivative expansion method

Following the same steps as for small coupling case, we apply the "derivative expansion method" to equation (32) for $\phi = \phi(\zeta^0, \zeta^1)$.
To start, we introduce a small parameter ϵ, such us,

$$\left.\zeta^\mu_{(0)} = \epsilon^0 \zeta^\mu, \quad \zeta^\mu_{(1)} = \epsilon^1 \zeta^\mu, \quad \zeta^\mu_{(2)} = \epsilon^2 \zeta^\mu \dots \right\} \Rightarrow \underset{\substack{\alpha=0,1,2\dots \\ \mu=0,1}}{\zeta^\mu_{(\alpha)} = \epsilon^\alpha \zeta^\mu}$$

where ϵ a small quantity, $\epsilon << 1$.
We express the field $\phi = \phi(\zeta^0, \zeta^1)$ in terms of the new variables $\zeta^\mu_{(\alpha)}$,

$$\phi = \phi\left(\zeta^0_{(0)}, \zeta^0_{(1)}, \zeta^0_{(2)}, \dots, \zeta^1_{(0)}, \zeta^1_{(1)}, \zeta^1_{(2)} \dots\right)$$

Derivative operators become[4]

$$\frac{d}{d\zeta^\mu} = \epsilon^\alpha \partial_{\zeta^\mu_{(\alpha)}}$$

For

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \dots$$

with ϵ the same small parameter as above, we have the solution at order $\mathcal{O}(\epsilon)$,

$$\phi_1 = A\left(\zeta^0_{(n)}, \zeta^1_{(n)}\right) e^{-i\theta} + c.c. \quad \text{with} \quad n \geq 1 \quad \text{and} \quad \theta = k^0\zeta^0_{(0)} - k^1\zeta^1_{(0)} \tag{34}$$

with k_0:

$$k_0^2 = k_1^2 + m_1^2$$

With similar manipulations as in the small coupling case, we express the amplitude's A evolution equation in terms of variables $(\tilde{\zeta}^0_{(2)}, \bar{Z}^1)$ which are defined as:

- $\tilde{\zeta}_{(1)}$: We choose a reference frame with appropriate linear group velocity $c_g = \dfrac{dk_0}{dk_1} = \dfrac{k_1}{k_0}$ with coordinates $(\zeta^1_{(1)}, \zeta^0_{(1)}) \rightarrow (\bar{Z}^1_{(1)}, \zeta^0_{(1)})$,

$$\bar{Z}^1 = \zeta^1_{(1)} - c_g\zeta^0_{(1)}$$

- $\tilde{\zeta}^0_{(2)}$ by

$$\zeta^0_{(2)} = k^0\tilde{\zeta}^0_{(2)} \tag{35}$$

[4] remember upper indices for space and time variables, whereas low (indices) denote the "independent" multiple scales variables.

With respect to these variables, the amplitude is given by,

$$i\partial_{\tilde{\zeta}^0_{(2)}} A + \frac{1}{2}\frac{m_1^2}{k_0^2}\partial_{\bar{Z}^1\bar{Z}^1} A - \frac{3}{2}\frac{1}{g}|A|^2 A = 0 \tag{36}$$

To "transfer" solution (37) of equation (32), to the wanted solution of the original problem (31), one needs to consider the following:

- $g = \dfrac{\lambda}{2}$
- $\zeta^\mu = g x^\mu = \frac{\lambda}{2} x^\mu \quad \rightarrow \quad$ multiple scale variables: $\zeta^\mu_{(\alpha)} = \dfrac{\lambda}{2} x^\mu_{(\alpha)}$
- Remember: $\bar{Z}^1 = \zeta^1_{(1)} - c_g \zeta^0_{(1)}$.

$$\begin{aligned}\bar{Z}^1 &= \zeta^1_{(1)} - c_g \zeta^0_{(1)} = \frac{\lambda}{2} x^1_{(1)} - c_g \frac{\lambda}{2} x^0_{(1)} \\ &= \frac{\lambda}{2}\left[x^1_{(1)} - c_g x^0_{(1)}\right]\end{aligned}$$

and define $\tilde{X}^1 = x^1_{(1)} - c_g x^0_{(1)}$, such as,

$$\bar{Z}^1 = \frac{\lambda}{2}\tilde{X}^1 \Rightarrow \partial_{\bar{Z}_1} = \frac{2}{\lambda}\partial_{\tilde{X}_1}$$

$\zeta^\mu_{(\alpha)} = \dfrac{\lambda}{2} x^\mu_{(\alpha)} \Rightarrow \tilde{\zeta}^0_{(2)} = \dfrac{\lambda}{2}\tilde{x}^0_{(2)}$ and the amplitude equation becomes,

$$i\frac{2}{\lambda}\partial_{\tilde{x}^0_{(2)}} A + \frac{1}{2}\frac{m_1^2}{k_0^2}\frac{2}{\lambda}\frac{2}{\lambda}\partial_{\bar{X}^1\bar{X}^1} A - \frac{3}{2}\frac{2}{\lambda}|A|^2 A = 0 \tag{37}$$

- $m = m_1 g \quad \Rightarrow \quad m_1^2 = \left(\dfrac{2}{\lambda}\right)^2 m^2,$

$$i\partial_{\tilde{x}^0_{(2)}} A + \frac{1}{2}\frac{m^2}{k_0^2}\left(\frac{2}{\lambda}\right)^3 \partial_{\bar{X}^1\bar{X}^1} A - \frac{3}{2}|A|^2 A = 0$$

- Now let us define

$$\tilde{x}_1 = \sqrt{\frac{k_0^2}{m^2}}\bar{X}_1 \quad \text{and} \quad A = \left(\frac{2}{3}\right)^{\frac{1}{2}}\tilde{A}$$

Finally,

$$i\partial_{\tilde{x}^0_{(2)}}\tilde{A} + \frac{1}{2}\left(\frac{2}{\lambda}\right)^3 \partial_{\tilde{x}_1\tilde{x}_1}\tilde{A} - |\tilde{A}|^2\tilde{A} = 0$$

To summarize:
Equation of motion,

$$\left(\Box_x + m^2\right)\phi = -\frac{\lambda}{2}\phi^3, \quad \lambda >> 1$$

We can choose a small quantity ϵ to expand the potential around its minimum at $\phi = 0$, as

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \dots$$

Where ϕ_1 is given by - at order $\mathcal{O}(\epsilon^3)$, or equivalently $\mathcal{O}\left(\left(\frac{2}{\lambda}\right)^3\right)$ - :

$$\phi_1 = A\left(\tilde{x}^0_{(2)}, \tilde{x}_1\right) e^{-i\frac{\lambda}{2}(k^0x^0 - k^1x^1)} \quad + \quad c.c. \tag{38}$$

where A satisfies,

$$i\partial_{\tilde{x}^0_{(2)}}\tilde{A} + \frac{1}{2}\left(\frac{2}{\lambda}\right)^3 \partial_{\tilde{x}_1\tilde{x}_1}\tilde{A} - |\tilde{A}|^2\tilde{A} = 0$$

So we managed to deal with the strong corresponding problem. This result and treatise is original, to the extent of the author's knowledge.

2.5.3 Renormalization group method

In terms of the quantity ϵ^2, defined as $\epsilon^2 = \frac{1}{g}$, from (32) we have,

$$(\Box_\zeta + m_1^2)\phi = -\epsilon^2\phi^3 \tag{39}$$

With the following ansatz, as we did in the small coupling case,

$$\phi = A\exp[i(k\zeta^1 - \omega\zeta^0)] + c.c. \quad \text{with} \quad A = A(\xi, t) \quad \text{where} \quad \xi = \epsilon\zeta^1$$

we end up with the concrete description of the amplitude through an NLS equation,

$$\partial_{\zeta^0}A = -\epsilon\dot{\omega}\partial_\xi A + \epsilon^2\frac{i}{2}\left(\ddot{\omega}\partial_{\xi\xi}A - \frac{3}{\omega}|A|^2A\right) \tag{40}$$

This result applies up to $\mathcal{O}(\epsilon^2)$, or equivalently up to $\mathcal{O}(\frac{1}{\lambda})$.

It possesses key differences to the amplitude in weak coupling case equation (29). To observe them, one should express it in terms of the original variables and parameters (x, t, m, λ):

$$i\frac{2}{\lambda}\partial_t A + i\frac{k}{\omega}\frac{2}{\lambda}\partial_x A + \frac{1}{2\omega}\frac{4}{\lambda^2}\partial_{xx}A - \frac{3}{2}\frac{1}{\omega}\frac{2}{\lambda}|A|^2|A| = 0$$

with $\omega^2 = k^2 + 4\frac{m^2}{\lambda^2}$.

Results: Weak vs Strong coupling - RG method

- **Weak** coupling ϕ^4 Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{8}\phi^4, \quad \lambda << 1$$

  Equation of motion:

$$\left(\Box + m^2\right)\phi = -\frac{\lambda}{2}\phi^3, \quad \lambda << 1$$

  Solution up to $\mathcal{O}(\lambda)$:

$$\phi(x,t) = A\exp[i(kx - \omega t)] + c.c. \quad \text{with} \quad A = A\left(\sqrt{\frac{\lambda}{2}}x, t\right)$$

  with amplitude equation,

$$i\omega\partial_t A + ik\partial_x A + \frac{1}{2}\partial_{xx}A - \frac{3}{2}\frac{\lambda}{2}|A|^2 A = 0$$

  and $\omega^2 = k^2 + m^2$. Compactly,

$$i\left(k^2 + m^2\right)^{\frac{1}{2}}\partial_t A + ik\partial_x A + \frac{1}{2}\partial_{xx}A - \frac{3}{2}\frac{\lambda}{2}|A|^2 A = 0 \qquad (41)$$

- **Strong** coupling ϕ^4 Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{8}\phi^4, \quad \lambda >> 1$$

  Equation of motion:

$$\left(\Box + m^2\right)\phi = -\frac{\lambda}{2}\phi^3, \quad \lambda >> 1$$

  Solution up to $\mathcal{O}(\frac{1}{\lambda})$:

$$\phi = A\exp[i\frac{\lambda}{2}(kx - \omega t)] + c.c. \quad \textit{with} \quad A = A\left(\sqrt{\frac{2}{\lambda}}x, t\right)$$

  with amplitude equation,

$$i\omega\partial_t A + ik\partial_x A + \frac{1}{2}\frac{2}{\lambda}\partial_{xx}A - \frac{3}{2}|A|^2|A| = 0$$

  and $\omega^2 = k^2 + 4\frac{m^2}{\lambda^2}$.

Note:

$$\omega = \left(k^2 + 4\frac{m^2}{\lambda^2}\right)^{\frac{1}{2}} = k\left(1 + 4\frac{m^2}{k^2\lambda^2}\right)^{\frac{1}{2}} = k\left(1 + 2\frac{m^2}{k^2\lambda^2}\right) + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

$$\omega = k + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

so, the amplitude equation compactly is given by

$$\Rightarrow ik\partial_t A + ik\partial_x A + \frac{1}{2}\frac{2}{\lambda}\partial_{xx}A - \frac{3}{2}|A|^2|A| = 0$$

as opposed to (41) for the weak case.

2.6 Necessities - Shortcomings

Now that all the tools are on the table, we need to make some final remarks on the method.

You do actually need a scale: The existence of a mass term for the field is crucial for the multiple-scale method to be applied in problems with *oscillatory* dynamics. The mass term determines the frequency of the homogeneous solution. It separates the divergent/secular terms, that is, the terms that have a common frequency with the homogeneous solution, from non-secular terms. The absence of a characteristic scale, meaning a mass term for the field, would make all terms being solutions of corresponding homogeneous equations and therefore no solubility conditions could be generated.

Variable coefficients: Multiple-scale method is a powerful method among other techniques namely because it can handle non linear equations. Nevertheless, its power is limited when it comes to differential equations with varying coefficients. There are of course a few standard equations which can be solved with the method, among them:

- $$y''(t) + \omega^2(\varepsilon t)y(t) = 0$$

  where one can find a transformation which converts it to a fixed-frequency oscillator with a small perturbation term [4]. Also, with the same manner one can solve the nonlinear one

  $$d^2y/dt^2 + \omega^2(\varepsilon t)y + \varepsilon y^3 = 0$$

- Boundary layer problems of the type

  $$\varepsilon y''(x) + a(x)y'(x) + b(x)y = 0 \quad [y(0) = A, \quad y(1) = B]$$

  with the "generalized method" [1]

- and the standard Mathieu equation

  $$\frac{d^2y}{dt^2} + (a + 2\varepsilon \cos t)y = 0$$

  in [4],[1].

The fairly limited application on such problems partly explains why it is not widely used in areas like Cosmology, where due to the time dependence of the metric one ends up in equations with varying coefficients. But there are as of now some early-stage attempts [18], [17].

3 Application to Cosmological perturbations

3.1 Effective Field Theory of Inflation

In this section we highlight the main features of the Effective Field Theory of Inflation [11]. In the next section we study and *question* whether multiple scale method can test one of its most striking predictions.

Structure of the Theory: We review the main features according to [11], [12], [14]. Inflation is a phase of accelerated expansion, where the universe was quasi De-Sitter. However, it could not be exactly de Sitter, because it has to end. Therefore is considered that there is a physical clock measuring time and forcing inflation to end and start the standard FRW universe . For example, in the case inflation is driven by a scalar field the role of the clock is given by the scalar field. When the scalar field rolls down is approximately De-Sitter. When it reaches the cliff inflation must end. That is how inflation ends in slow roll models. In that sense, time translations are spontaneously broken. Turns out that from this point of view, Inflation is the theory where 4-dimensional diffeomorphisms are broken down to time-dependent spatial 3-dimensional diffeomorphisms.
As an example, if we assume that the inflanton is a fundamental scalar field and we are in a coordinate frame where $\delta\phi(\vec{x},t) \neq 0$, we can perform a time diffeomorphism $t \to \vec{t} = t + \delta t(\vec{x},t)$, such that

$$0 = \tilde{\delta\phi}(\vec{x},t) = \delta\phi(\vec{x},t) - \dot{\phi}_0(t)\delta t(\vec{x},t)$$

is equal to zero, since one can choose $\delta t = \frac{\delta\phi(x,t)}{\dot{\phi}}$. So one can always go to the frame where the *fluctuations* of the clock are equal to zero. Even when we do not know what this field is we can always declare that we go to the gauge where the fluctuations are zero.

One can describe the system in the most possible general way. There is no need to assume that there is a fundamental scalar field that slow rolls and drives inflation, but just suppose the existence of this physical clock.
The Effective Field Theory procedure induces the following:

EFT rules:

- one writes the action with the degrees of freedom that are available. This is just the metric fluctuations (the clock fluctuations have been made to be zero by construction).
- expands in fluctuations, and write down all operators compatible with the symmetries of the problem. In this case these are the operators that are invariant under the time dependent spatial diffeomorphisms $x^i \to x^i + \xi^i(t,\vec{x})$.

The symmetry group is a subgroup of 4-Dim diffeomorphisms. Therefore many extra terms are now allowed except R, R^2.... For example, g^{00} is a scalar under spatial diffeomorphisms, so that it can appear freely in the unitary gauge Lagrangian since $\tilde{t} = t$. In fact,

$$\tilde{g}^{00} = \frac{\partial \tilde{t}}{\partial x^\mu}\frac{\partial \tilde{t}}{\partial x^\nu} g^{\mu\nu} = \delta^0_\mu \delta^0_\nu g^{\mu\nu} = g^{00}$$

If one proceeds with these rules it turns out that the most generic unitary Lagrangian can be written as (see App. A of [11] for a proof)

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R - c(t)g^{00} - \Lambda(t) + \frac{1}{2!}M_2(t)^4\left(\delta g^{00}\right)^2 + \frac{1}{3!}M_3(t)^4\left(\delta g^{00}\right)^3 + \right.$$
$$\left. - \frac{\bar{M}_1(t)^3}{2}\left(\delta g^{00}\right)\delta K^\mu_\mu - \frac{\bar{M}_2(t)^2}{2}\delta K^\mu_\mu - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_{\mu\mu} + \dots\right]$$

- The 1st term is the Einstein-Hilbert term.
- g_{00} we proved is an allowed operator.
- Also a generic function of time is an allowed operator. Because this group did not included time translations.
- $\delta K_{\mu\nu}$ is the variation of the extrinsic curvature of constant time surfaces with respect to the unperturbed FRW.
- dots stand for terms which are of higher order in the fluctuations or with more derivatives. As is typical in an EFT, higher derivatives give smaller effect than lower derivative terms.

Only the first three terms in the action above contain linear perturbations around the chosen FRW solution, all the others are explicitly quadratic or higher. The Lagrangian to be stable must start quadratic in the fluctuations. So it suffices that $c(t)g_{00}$ and $\Lambda(t)$ to cancel the linear term coming from R. In order to have a FRW solution around which we are expanding we need to have some matter field . The Einstein's action does not give as a vacuum solution an FRW universe.

So we need

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\bigg|_0 = M_{Pl}^2 G_{\mu\nu} \quad = M_{Pl}^2\frac{\delta S_{EH}}{\delta g^{\mu\nu}}\bigg|_0$$

which give the Friedmann equations,

$$H^2 = \frac{1}{3M_{\rm Pl}^2}[c(t) + \Lambda(t)]$$
$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2}[2c(t) - \Lambda(t)].$$

Solving for c and Λ we can rewrite the action as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R + M_{\rm Pl}^2 \dot{H} g^{00} - M_{\rm Pl}^2\left(3H^2 + \dot{H}\right) + \frac{1}{2!}M_2(t)^4\left(\delta g^{00}\right)^2 + \frac{1}{3!}M_3(t)^4\left(\delta g^{00}\right)^3 + \right. \\ \left. - \frac{\bar{M}_1(t)^3}{2}\left(\delta g^{00}\right)\delta K^\mu_\mu - \frac{\bar{M}_2(t)^2}{2}\delta K^{\mu\,2}_\mu - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots\right]. \tag{42}$$

Only those two coefficients are fixed. All the other coefficients are free. The background does not impose any further constraint on them. This combination cancels exactly the linear terms. As you change your inflationary model these terms cannot change, given the same H and $\dot{H}$. But the M's will change.

The standard slow roll model is included in this description [11]. But there is no scalar degree of freedom. Which means that this action action propagates three degrees of freedom instead of two in general relativity. Two helicity states and one scalar degree of freedom. But this is not explicit. This scalar degree of freedom is the Goldstone boson of time translation . The Goldstone boson has not been integrated out. We just used the gauge redundancy to set it to zero.
This action is a gauge fixed action of spontaneously broken time translation. We introduce a Goldstone boson to make this explicit. That will be crucial since the Goldstone boson in higher energies decouples from the metric fluctuations and becomes the most relevant degree of freedom.

Consider this Lagrangian which is not invariant under time diffeomorphism:

$$\int d^4x\sqrt{-g}\left[A(t) + B(t)g^{00}(x)\right].$$

In fact, under a broken time diff. $t \to \tilde{t} = t + \xi^0(x), \vec{x} \to \vec{\widetilde{x}} = \vec{x}$, g^{00} transforms as:

$$g^{00}(x) \to \widetilde{g}^{00}(\widetilde{x}(x)) = \frac{\partial \widetilde{x}^0(x)}{\partial x^\mu}\frac{\partial \widetilde{x}^0(x)}{\partial x^\nu} g^{\mu\nu}(x)$$

since $\xi^0 = \xi^0(x)$ depends on 4-dim coordinates. g^{00} does not go to g^{00} so it is not invariant.
The action written in terms of the transformed fields is given by:

$$\int d^4x\sqrt{-\widetilde{g}(\widetilde{x}(x))}\left|\frac{\partial \widetilde{x}}{\partial x}\right|\left[A(t) + B(t)\frac{\partial x^0}{\partial \widetilde{x}^\mu}\frac{\partial x^0}{\partial \widetilde{x}^\nu}\widetilde{g}^{\mu\nu}(\widetilde{x}(x))\right].$$

Changing integration variables to $\widetilde{x}$, we get:

$$\int d^4\widetilde{x}\sqrt{-\widetilde{g}(\widetilde{x})}\left[A\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right) + B\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right)\frac{\partial\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right)}{\partial \widetilde{x}^\mu}\frac{\partial\left(\widetilde{t} - \xi^0(x(\widetilde{x}))\right)}{\partial \widetilde{x}^\nu}\widetilde{g}^{\mu\nu}(\widetilde{x})\right]$$

We verified that it is not invariant.
We restore gauge invariance. Whenever ξ^0 appears in the action above, we make the substitution

$$\xi^0(x(\widetilde{x})) \to -\widetilde{\pi}(\widetilde{x}).$$

The action becomes, dropping the tildes for simplicity:

$$\int d^4x\sqrt{-g(x)}\left[A(t+\pi(x)) + B(t+\pi(x))\frac{\partial(t+\pi(x))}{\partial x^\mu}\frac{\partial(t+\pi(x))}{\partial x^\nu}g^{\mu\nu}(x)\right].$$

This will be gauge invariant iff, π, transforms as,

$$\pi(x) \to \widetilde{\pi}(\widetilde{x}(x)) = \pi(x) - \xi^0(x).$$

If one applies the above to the full Lagrangian, ends up to

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{1}{2}M_{\rm Pl}^2 R - M_{\rm Pl}^2\left(3H^2(t+\pi) + \dot{H}(t+\pi)\right) + \\ &+ M_{\rm Pl}^2\dot{H}(t+\pi)\left((\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}\right) + \\ &\frac{M_2(t+\pi)^4}{2!}\left(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1\right)^2 + \\ &\frac{M_3(t+\pi)^4}{3!}\left(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1\right)^3 + \dots\Bigg], \end{aligned}$$

where we have neglected for simplicity terms that involve the extrinsic curvature.

Now all diffeomorphisms are restored. It is still quite complicated. In high energy limits π decouples and becomes the most important degree of freedom.

3.2 Strong coupling limit

Decoupling scale: When $M_2 = M_3 = \cdots = 0$ in (42) we have a standard slow roll inflation case [14]. One can compute the leading mixing with gravity and determine a scale $E_{mix} \gg \epsilon H$ above which there is no need to include mixing terms. Turns out ([12]) that for $E \gg E_{mix}$ the above action becomes ,

$$S_\pi = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm Pl}^2 R - M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right) + 2M_2^4\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3^4\dot{\pi}^3 + \ldots\right] \quad (43)$$

There are two important observations to make here.

- the coefficient of the spatial kinetic term $(\partial_i\pi)^2$ is fixed by the background to be $M_{\rm Pl}^2\dot{H}$
- The coefficient of the time kinetic term $\dot{\pi}^2$ is $\left(-M_{\rm Pl}^2\dot{H} + 2M_2^4\right)\dot{\pi}^2$

Thus the Lorentz invariance is broken and the π will propagate with a speed of sound $c_s \neq 1$.

This leads us to the observation we remarked in section 2.6 with respect to the existence of scale-dependent instabilities present in cases where the Lorentz invariance is broken by interactions. This hints that a proper multiple scale analysis could lead to the alleviation of the strong coupling problem.

The action for π can be rewritten as,

$$S_\pi = \int d^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) + M_{\rm Pl}^2\dot{H}\left(1 - \frac{1}{c_s^2}\right)\left(\dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3^4\dot{\pi}^3\ldots\right] \quad (44)$$

for

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\rm Pl}^2\dot{H}}$$

Strong Coupling scale: Small c_s or large M_2 implies large interactions and determines the strong coupling scale where the validity of an effective field theory breaks down. This cutoff can be estimated looking at tree level partial wave unitarity. It is the maximum energy at which the tree level scattering of π's is unitary [16]. What is more, the same interactions that contribute to strong coupling are the ones responsible for non-gaussianity [13]. Strong coupling scale rings bells and is an indication that new degrees of freedom may exist below this scale [14].
A regular perturbative treatment around the free solution would translate this cut-off energy scale to a divergence above a certain time range (for this correspondence see remarks on section 2.3). Therefore one needs to ask a relevant question: What would a perturbative approach which takes into account the possibility of scale-dependent instabilities result to ?

3.3 Multiple-scale study

We examine the action (43) for π. We start from,

$$S_\pi = \int d^4x\sqrt{-g}\left[-M_{\rm Pl}^2\dot{H}\left(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\right) + 2M_2^4\left(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) - \frac{4}{3}M_3^4\dot{\pi}^3 + \dots\right] \tag{45}$$

and apply the following manipulations as in [15] :
First we substitute,

$$M_2^4 = -\frac{1-c_s^2}{c_s^2}\frac{M_{\rm Pl}^2\dot{H}}{2}$$

also we redefine M_3 as,

$$M_3^4 = \tilde{c}_3 M_2^4/c_s^2$$

and transform the spacial coordinates as,

$$\vec{x} \quad \to \quad \vec{\tilde{x}} = \vec{x}/c_s$$

Finally we normalize canonically the field π,

$$\pi_c = \left(-2M_{\rm Pl}^2\dot{H}c_s\right)^{1/2}\pi$$

and the action becomes

$$\begin{aligned} S_\pi = \int dt d^3\tilde{x}\sqrt{-g} \quad & \left[\frac{1}{2}\left(\dot{\pi}_c^2 - \frac{\left(\tilde{\partial}_i\pi_c\right)^2}{a^2}\right)\right. \\ -\frac{1}{\left(8|\dot{H}|M_{\rm Pl}^2c_s^5\right)^{1/2}}\dot{\pi}_c\frac{\left(\tilde{\partial}_i\pi_c\right)^2}{a^2} & \left. - \frac{2}{3}\frac{\tilde{c}_3}{\left(8|\dot{H}|M_{\rm Pl}^2c_s^5\right)^{1/2}}\dot{\pi}_c^3\right] \end{aligned} \tag{46}$$

where $\tilde{\partial}_i = \partial/\partial\tilde{x}^i$.
The leading contribution in the limit $c_s \ll 1$ comes from the term $\dot{\pi}\left(\partial_i\pi\right)^2$. The spatial derivatives are enhanced with respect to time derivatives [14], [12]. Taken that into account, the equations of motion for the afforemented action is

$$\Box\pi_c - 3H\dot{\pi}_c = -\frac{1}{\left(8M_{pl}^2c_s^5|\dot{H}|\right)^{1/2}}\left[\frac{1}{a^2}H\left(\tilde{\partial}_i\pi_c\right)^2 - \frac{1}{a^2}\frac{|\ddot{H}|\left(\tilde{\partial}_i\pi_c\right)^2}{2|\dot{H}|} + 2\frac{1}{a^2}\left(\tilde{\partial}_i\dot{\pi}_c\right)\left(\tilde{\partial}^i\pi_c\right)\right]$$

with $c_s \ll 1$. In that form, perturbative multiple scale method is not possible to be applied. There is no *mass* term. So *every* forcing right hand side term will be secular (see remarks on section 2.6). Therefore an extra step is required.

Enabling the metric fluctuations: We must take into account the leading contribution from the metric fluctuations. This is computed rigorously in detail in [12]. Now the action will have a mass term. The new action at linear level is

$$S = \int d^4 x a^3 \left(-\dot{H} M_{\rm Pl}^2 \right) \left[\dot{\pi}^2 - \frac{1}{a^2} \left(\partial_i \pi \right)^2 - 3\dot{H}\pi^2 \right] \tag{47}$$

If we take this contribution into our study, to make contact with the above manipulations it reads as,

$$\begin{aligned} S &= \int d^4 x a^3 \left(-\dot{H} M_{Pl}^2 \right) \left[\pi^2 - \frac{1}{a^2} \left(\partial_i \pi \right)^2 - 3\dot{H}\pi^2 \right] \\ &= \int dt d^3 \tilde{x} \sqrt{-g} \left[\frac{1}{2} \left(\dot{\pi}_c^2 - \frac{\left(\tilde{\partial}_i \pi_c \right)^2}{a^2} \right) - \frac{3}{2} \dot{H} \pi_c^2 \right] \end{aligned}$$

Then the full interactive action will be,

$$\begin{aligned} S_\pi = \int dt d^3 \tilde{x} \sqrt{-g} \quad & \left[\frac{1}{2} \left(\dot{\pi}_c^2 - \frac{\left(\tilde{\partial}_i \pi_c \right)^2}{a^2} - \frac{3}{2} \dot{H} \pi_c^2 \right) \right. \\ & \left. - \frac{1}{\left(8 |\dot{H}| M_{\rm Pl}^2 c_s^5 \right)^{1/2}} \dot{\pi}_c \frac{\left(\tilde{\partial}_i \pi_c \right)^2}{a^2} - \frac{2}{3} \frac{\tilde{c}_3}{\left(8 |\dot{H}| M_{\rm Pl}^2 c_s^5 \right)^{1/2}} \dot{\pi}_c^3 \right] \end{aligned}$$

and the equations of motion now read as,

$$\Box \pi_c - 3H\dot{\pi}_c - 3\dot{H}\pi_c = -\frac{1}{\left(8 M_{pl}^2 c_s^5 |\dot{H}| \right)^{1/2}} \left[\frac{1}{a^2} H \left(\tilde{\partial}_i \pi_c \right)^2 - \frac{1}{a^2} \frac{|\ddot{H}| \left(\tilde{\partial}_i \pi_c \right)^2}{2|\dot{H}|} + 2 \frac{1}{a^2} \left(\tilde{\partial}_i \dot{\pi}_c \right) \left(\tilde{\partial}^i \pi_c \right) \right] \tag{48}$$

where

$$\Box \pi_c = -\ddot{\pi}_c + \frac{1}{a^2} \tilde{\nabla}^2 \pi_c$$

To obtain the "dual weak" equation, we procceed according to the two-step method we proposed in section (2.5).
First, we set

$$c_s^{-\frac{5}{2}} \equiv \lambda \gg 1$$

in (48).
In what follows we compute this expression for each **Fourier mode** π_{c_k}. We **write** π_{c_k} **as** π **for simplicity** so the equation becomes,

$$\ddot{\pi} + \left(\frac{k}{a} \right)^2 \pi + 3H\dot{\pi} + 3\dot{H}\pi = -\frac{1}{8 M_{Pl}^2} \frac{1}{|\dot{H}|^{1/2}} \lambda \left(\frac{k}{a} \right)^2 \left[\left(H - \frac{|\ddot{H}|}{2|\dot{H}|} \right) \pi^2 + 2\dot{\pi}\pi \right]$$

Second, we transform the time variable. We set,

$$\zeta = \lambda t$$

and the equation becomes,

$$\lambda^2 \pi'' + \left(\frac{k}{a}\right)^2 \pi + 3H\lambda\pi' + 3\lambda H'\pi = -\frac{1}{8M_{pe}^2}\frac{\lambda}{\lambda^{1/2}}\frac{1}{|H'|}\left(\frac{k}{\alpha}\right)^2 \left[\left(H - \frac{1}{2}\lambda\frac{|H''|}{|H'|}\right)\pi^2 + 2\lambda\pi'\pi\right]$$

Now we set

$$\epsilon^2 = \frac{1}{\lambda}$$

to get the wanted dual expression with weak forcing right hand side,

$$\begin{aligned} \pi'' + 3\varepsilon^2 H'\pi + \varepsilon^4 \frac{k^2}{a^2}\pi =& \varepsilon \frac{1}{8M_{Pl^2}} \frac{1}{|H'|^{1/2}} \frac{k^2}{a^2} \left[\frac{1}{2}\frac{|H''|}{|H'|}\pi^2 - 2\pi'\pi\right] \\ &- \varepsilon^2 3H'\pi \\ &- \varepsilon^3 \frac{1}{8M_{pl^2}} \frac{1}{|H'|^{1/2}} \frac{k^2}{a^2} H\pi^2 \end{aligned} \tag{49}$$

We formulated the original action to one that posseses a mass term, which is crucial for the application of multiple-scale method, as we pointed out in section (2.6). Next we made use of the manipulations we developed in section (2.5). With this form at hand multiple-scale analysis is applicable in principle. But there is one more difficulty here. Solving (49) is very difficult because of the time-dependent coefficients. However, H and H' are very *slow* varying coefficients. Modes inside the Hubble scale oscillate very *fast*, and so one can construct a proper *averaging* scheme. Then one can absorb the ϵ coefficient of the mass term through a redefinition as we did in section (2.5). That leads to a dispersion relation capable to separate secular from non-secular terms, applied the corresponding proper ansatz. Finally one transfers the solution back to the initial problem, as we did in section (2.5.2). The development of such an averaging scheme is beyond the scope of this thesis and is in present development [19].

4 Conclusions

The research goal of this thesis was to tackle an open problem namely the Strong Coupling problem of the Effective Field Theory of Inflation, present also in many other field theories. Beyond the range of weak theory, new physics is expected in order to be able to describe the physical phenomena. Hence the strong coupling limit points, if the theory is correct, the scale in which new physics comes to play. The goal was to examine this prediction of the theory and possibly alleviate the cut off energy scale through the perturbative method. In order to do that, at start we explained why multiple-scale method is in principle capable to tackle strongly coupled field theories. Then we moved on providing a solution to the strongly coupled ϕ^4 model. We extracted from various methods criteria which indicate the scale-dependent instabilities and tools to formulate the problem in a state where multiple-scale analysis can be attempted. Finally we outlined the difficulties and some arguments for further development. This approach can open up the way to perturbative analytical solutions where, for now, just numerical work can be accomplished.

5 References

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