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Master Thesis

Operator algebras associated with C^*-dynamical systems and C^*-correspondences

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Contents

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Abstract

The aim of the present thesis is to describe certain operator algebras associated with C^*-dynamical systems and C^*-correspondences. We introduce the notion of the crossed product of a C^*-algebra by a discrete group and we study in detail the case of the integers. We give necessary and sufficient conditions, when the C^*-algebra is the algebra of continuous functions on a compact Hausdorff topological space, for the crossed product to be simple. Furthermore, we introduce the notion of the semi-crossed product and we give alternative descriptions of its norm when the C^*-dynamical system is induced by a $*$-automorphism. In addition, we study C^*-correspondences and their representations and we prove the Gauge-Invariance Uniqueness theorem. Finally, we use results and tools that we have developed so far, in order to identify the C^*-envelope of the semi-crossed product and the C^*-envelope of the tensor algebra of a C^*-correspondence.

Άλγεβρες τελεστών που σχετίζονται με C^*-δυναμικά συστήματα και C^*-αντιστοιχίες

Εθνικό & Καποδιστριακό Πανεπιστήμιο Αθηνών
Τμήμα Μαθηματικών

Ιωάννης-Απόλλων Παρασκευάς

Περίληψη

Η παρούσα διπλωματική εργασία αποσκοπεί στο να περιγράψει συγκεκριμένες άλγεβρες τελεστών που σχετίζονται με C^*-δυναμικά συστήματα και C^*-αντιστοιχίες. Εισάγουμε την έννοια του σταυρωτού γινομένου μιας C^*-άλγεβρας με μια διακριτή ομάδα και περιγράφουμε αναλυτικά την ειδική περίπτωση που η διακριτή ομάδα είναι οι ακέραιοι. Δίνουμε ικανές και αναγκαίες συνθήκες στην περίπτωση όπου η C^*-άλγεβρα είναι η άλγεβρα συνεχών συναρτήσεων ενός συμπαγούς Hausdorff τοπολογικού χώρου, ώστε το σταυρωτό γινόμενο να είναι απλή C^*-άλγεβρα. Επιπλέον, εισάγουμε την έννοια του ημι-σταυρωτού γινομένου και δίνουμε εναλλακτικές περιγραφές της νόρμας του στην περίπτωση που το C^*-δυναμικό σύστημα επάγεται από $*$-αυτομορφισμό. Επίσης, εισάγουμε την έννοια της C^*-αντιστοιχίας και των αναπαραστάσεων της και αποδεικνύουμε το Gauge-Invariance Uniqueness Theorem. Τέλος, χρησιμοποιούμε αποτελέσματα και εργαλεία που έχουμε αναπτύξει στα προηγούμενα κεφάλαια ώστε να αποδείξουμε ότι το C^*-envelope του ημι-σταυρωτού γινομένου είναι το σταυρωτό γινόμενο, στην περίπτωση C^*-δυναμικού συστήματος που επάγεται από $*$-αυτομορφισμό και ότι το C^*-envelope της Tensor algebra μιας C^*-αντιστοιχίας είναι η Cuntz-Pimsner algebra.

Ευχαριστίες

Η συγκεκριμένη εργασία αποτελεί ένα προϊόν της προσωπικής προσπάθειας αλλά ακόμα σημαντικότερα της γνώσης, της έμπνευσης και της καθοδήγησης που έλαβα από καθηγητές και φίλους, που γνώρισα και αλληλεπίδρασα κατά την περίοδο φοίτησης μου στο Μαθηματικό τμήμα.

Αρχικά, θα ήθελα να ευχαριστήσω θερμά τον κ. Αριστείδη Κατάβολο ο οποίος αποτέλεσε μια πολύ σημαντική πηγή έμπνευσης από τις απαρχές της μέχρι τώρα μαθηματικής μου πορείας. Παρακολουθώντας αρκετά από τα μαθήματα του, η διδασκαλία του διαμόρφωσε τα μαθηματικά μου ενδιαφέροντα και ο χρόνος που αφιέρωσε και η βοήθεια του, έπαιξαν καταλυτικό ρόλο στην εκπόνηση της συγκεκριμένης εργασίας. Εκτός από όλα τα παραπάνω, θα ήθελα να τον ευχαριστήσω για την σημαντικότατη βοήθεια του για την μετέπειτα πορεία των σπουδών μου.

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Απόλλωνας

Chapter 1

Introduction

In [24], Pimsner introduced a way to construct a C^*-algebra O_X associated to a C^*-correspondence $(X, \mathscr{A})$. This class of C^*-algebras turned out to be very rich and includes a lot of well-known examples of C^*-algebras such as crossed-products by $\mathbb{Z}$, Cuntz-Krieger algebras and more. The flexible language of C^*-correspondences also encodes non-selfadjoint operator algebras such as semi-crossed products. The second chapter of this thesis covers very important preliminaries of the work that follows, in particular, we prove the Wold decomposition theorem which is essential in order to connect the semi-crossed product induced by $*$-automorphisms with the corresponding crossed-product by $\mathbb{Z}$. We also give a detailed description of Hilbert C^*-modules, which are significantly important in understanding our work on C^*-correspondences. For example the interior tensor product of Hilbert C^*-modules provides an example of an injective Toeplitz representation of a given C^*-correspondence, namely the Fock representation. In chapter 3 we introduce the notion of the crossed-product of a C^*-algebra $\mathscr{A}$ by a discrete group G, which is a C^*-algebra that contains information about $\mathscr{A}$ itself, the group G and the action of G on $\mathscr{A}$. We focus in the case of the discrete group $\mathbb{Z}$ and we describe two possible multiplications that turn out to describe the same object, up to isometrical $*$-isomorphism. Before this chapter comes to an end, we prove that in the case of a topological dynamical system $(C(X), \sigma)$, where X is a compact Hausdorff topological space and $\sigma : X \to X$ is a homeomorphism, two purely topological properties, namely, topological freeness and minimality of the action induced by σ are equivalent to the simplicity of the crossed product $C(X) \times_\alpha \mathbb{Z}$.

Chapter 4 introduces the semi-crossed product, which is an operator algebra associated to a C^*-algebra $\mathscr{A}$ and a $*$-endomorphism α of $\mathscr{A}$. These are non-selfadjoint norm closed algebras of operators on a Hilbert space. They include certain non-selfadjoint subalgebras of C^*-crossed products in the case that α is a $*$-automorphism, and in particular they include the class of operator algebras considered by Arveson and Josephson in [2]. We prove some basic properties of the semi-crossed product and when α is a $*$-automorphism we prove that the semi-crossed product is completely isometric with a non-selfadjoint subalgebra of the corresponding crossed product $\mathscr{A} \times_\alpha \mathbb{Z}$. Following [13], we use this fact to identify the C^*-envelope of the semi-crossed product. This is also implied by the main theorem of chapter 6, although we give an independent proof in the first section of the same chapter.

In chapter 5, we construct the version of O_X developed in Katsura's paper [17], which is a generalization of the C^*-algebra O_X introduced by Pimsner; in particular, when the C^*-correspondence is injective, the two versions coincide. The idea of this construction is motivated by the construction of graph algebras with sinks in [10], by topological graphs in [18] and crossed products by Hilbert C^*-bimodules in [1] and in fact generalizes all of them. We study Toeplitz and Katsura covariant representations and we prove one of the main theorems of this thesis, the gauge-invariance uniqueness theorem which we will use in the final chapter to identify the C^*-envelope of the Tensor algebra of a C^*-correspondence. Fowler, Muhly and Raeburn characterized the C^*-envelope of the tensor algebra T_X^+ of a faithful and strict C^*-correspondence X in [6] . Following the work of Katsoulis and Kribs in [15], in chapter 6 we give a proof of the fact that for an arbitrary C^*-correspondence X the C^*-envelope of the tensor algebra T_X^+ is the Cuntz-Pimsner algebra O_X, as defined by Katsura. In order to do so, we use a method of Muhly and Tomforde to add tails to a C^*-correspondence described in [21]. This method was inspired by a technique from the theory of graph C^*-algebras, where one can often reduce to the sinkless case by the process of "adding tails to sinks". Adding tails to a C^*-correspondence enables us to view a given C^*-correspondence as a sub-correspondence of an injective one. Finally, the proof that the C^*-envelope of the tensor algebra T_X^+ is the Cuntz-Pimsner algebra O_X, is modelled on the proof of a result of the same authors in [16] that identifies the C^*-envelope of the tensor algebra of a directed graph.

Chapter 2

Preliminaries

In this chapter we are going to give some important definitions and tools that we will use in the chapters that follow.

2.1 Basic definitions and theorems

In this section we state some basic propositions, theorems and definitions from the theory of C^*-algebras. We follow [22] and [4].

Definition 2.1.1. Let $\mathscr{A}$ be a C^*-algebra. We say that $\{e_\lambda : \lambda \in \Lambda\} \subseteq \mathscr{A}$ is an approximate identity for $\mathscr{A}$ if it is an increasing net of positive elements in the closed unit ball of $\mathscr{A}$ such that $a = \lim_\lambda ae_\lambda$, for all $a \in \mathscr{A}$.

Proposition 2.1.1. *Let $\mathscr{A}$ be a C^*-algebra. Then $\mathscr{A}$ admits an approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ where $e_\lambda \geq 0$ and $\|e_\lambda\| \leq 1$.*

In this thesis when we consider an approximate unit of a C^*-algebra we will always consider one with the properties of the preceding proposition.

Proposition 2.1.2. *Let $I \subseteq \mathscr{A}$ be a closed ideal of a C^*-algebra $\mathscr{A}$. Then I is self-adjoint. In particular, I is a C^*-algebra.*

Definition 2.1.2. Let $\mathscr{A}$ be a Banach algebra with an involution. We say that $\mathscr{A}$ is a Banach $*$-algebra if for each $a \in \mathscr{A}$ we have $\|a\| = \|a^*\|$.

Proposition 2.1.3. *Let $\mathscr{A}$ be a Banach $*$-algebra, $\mathscr{B}$ a C^*-algebra and $\phi : \mathscr{A} \to \mathscr{B}$ a $*$-homomorphism. Then ϕ is a contraction. In addition, if ϕ is injective and $\mathscr{A}$ is a C^*-algebra then ϕ is an isometry.*

Definition 2.1.3. Let $\mathscr{A}$ be a C^*-algebra, $\mathscr{H}$ a Hilbert space and $\pi : \mathscr{A} \to \mathbf{B}(\mathscr{H})$ a $*$-representation. We say that π is non-degenerate if $\pi(\mathscr{A})\mathscr{H}$ is dense in $\mathscr{H}$.

We should note that the above definition is equivalent to

$$\pi(\mathbf{1}_{\mathscr{A}}) = I$$

where I is the identity operator on $\mathscr{H}$ and $\mathbf{1}_{\mathscr{A}}$ is the unit of $\mathscr{A}$ and in the case that $\mathscr{A}$ is non-unital, if $\{e_\lambda\}_\lambda$ is an approximate unit for $\mathscr{A}$, $\pi(e_\lambda)$ converges strongly to I.
Indeed, if $\pi(a)x \in \mathscr{H}$ we have that

$$\|\pi(e_\lambda)\pi(a)x - \pi(a)x\| = \|\pi(e_\lambda a - a)x\| \leq \|\pi(e_\lambda a - a)\|\|x\| \to 0,$$

therefore $\pi(e_\lambda)$ converges strongly to I on a dense subset of $\mathscr{H}$ and by continuity converges strongly on $\mathscr{H}$.

Definition 2.1.4. Let $\mathscr{A}$ be a C^*-algebra. We say that $\rho : \mathscr{A} \to \mathbb{C}$ is a state of $\mathscr{A}$ if it is a positive linear functional of norm 1.

Proposition 2.1.4. *Let a be a normal element in a C^*-algebra $\mathscr{A}$. Then there exists a state ρ such that $|\rho(a)| = \|a\|$.*

Theorem 2.1.1. *(Gelfand-Naimark) Let $\mathscr{A}$ be a C^*-algebra. Then there exists a Hilbert space $\mathscr{H}$ and an injective $*$-representation $\pi : \mathscr{A} \to \mathbf{B}(\mathscr{H})$. In particular, $\mathscr{A}$ is isometrically $*$-isomorphic with a self-adjoint operator algebra.*

Definition 2.1.5. Let $\mathscr{A}$ be a C^*-algebra. We say that $\mathscr{A}$ is simple if there are no proper non-zero closed ideals of $\mathscr{A}$.

Proposition 2.1.5. *(Corollary I.5.6 in [4]) Let J be a closed ideal of a C^*-algebra $\mathscr{A}$ and $\mathscr{B}$ a C^*-subalgebra of $\mathscr{A}$. Then $\mathscr{B} + J$ is a C^*-subalgebra of $\mathscr{A}$ and the map*

$$\pi : (\mathscr{B} + J)/J \to \mathscr{B}/\mathscr{B} \cap J, \quad b + j + J \mapsto b + \mathscr{B} \cap J$$

is a $*$-isomorphism.

Lemma 2.1.1. *(Lemma III.4.1 in [4]) Let J be a closed ideal of a C^*-algebra $\mathscr{A} = \overline{\bigcup_{N\geq 1} \mathscr{A}_N}$ where $\{\mathscr{A}_N : N \geq 0\}$ is an increasing sequence of C^*-subalgebras of $\mathscr{A}$. Then*

$$J = \overline{\bigcup_{N\geq 1} (J \cap \mathscr{A}_N)}.$$

Proof. From the preceding proposition we have that for each $N \geq 1$ we have a $*$-isomorphism $\pi_N : (\mathscr{A}_N + J)/J \to \mathscr{A}_N/\mathscr{A}_N \cap J$ in particular π_N is an isometry which implies that for $a \in \mathscr{A}_N$

$$\text{dist}(a, J) = \text{dist}(a, J \cap \mathscr{A}_N)$$

Now let j be an element in J and $\epsilon > 0$, for a large enough N we have that there exists $a \in \mathscr{A}_N$ such that $\|a - j\| < \epsilon/2$ and thus there is an element $j' \in J \cap \mathscr{A}_N$ such that

$\|a - j'\| < \epsilon/2$. It is now evident that $\|j - j'\| < \epsilon$ and since ϵ was arbitrary we have that $j \in \overline{\bigcup_{N\geq 1}(J \cap \mathscr{A}_N)}$ which completes the proof. □

Proposition 2.1.6. *Let $\mathscr{A}$ be a C^*-algebra and $a \geq 0$. There exists a state τ of $\mathscr{A}$ such that $|\tau(a)| = \|a\|$.*

Remark 1. Let $\mathscr{A}$ be a $*$-algebra, then we denote by $M_n(\mathscr{A})$ the algebra of $n \times n$ matrices with entries in $\mathscr{A}$ where the operations are defined just as for scalar matrices. Then $M_n(\mathscr{A})$ is a $*$-algebra where the involution is given by $(a_{ij})^*_{ij} = (a^*_{ji})_{ij}$. If $\phi : \mathscr{A} \to \mathscr{B}$ is a $*$-homomorphism between $*$-algebras then the n-th inflation of ϕ is the the $*$-homomorphism denoted by $\phi^{(n)}$ such that

$$\phi^{(n)} : M_n(\mathscr{A}) \to M_n(\mathscr{B}) \quad (a_{ij})_{ij} \to (\phi(a_{ij}))_{ij}.$$

Now, let $\mathscr{H}$ be a Hilbert space and let $\mathscr{H}^n$ be the orthogonal sum of n copies of $\mathscr{H}$. If $u \in M_n(\mathbf{B}(\mathscr{H}))$, we define $\phi(u) \in \mathbf{B}(\mathscr{H}^n)$ by setting

$$\phi(u)(x_1, ..., x_n) = \left(\sum_{i=1}^{n} u_{1i}(x_i), ..., \sum_{i=1}^{n} u_{ni}(x_i)\right)$$

for all $(x_1, ..., x_n) \in \mathscr{H}^n$. Then the map

$$\phi : M_n(\mathbf{B}(\mathscr{H})) \to \mathbf{B}(\mathscr{H}^n), \quad u \to \phi(u)$$

is a $*$-isomorphism. We define a norm on $M_n(\mathbf{B}(\mathscr{H}))$ making it a C^*-algebra by

$$\|u\| := \|\phi(u)\|.$$

If $\mathscr{A}$ is a C^*-algebra, we denote by $(\mathscr{H}, \phi)$ the universal representation of $\mathscr{A}$. If $\phi(n) : M_n(\mathscr{A}) \to M_n(\mathbf{B}(\mathscr{H}))$ is the n-th inflation of ϕ, then $\phi^{(n)}$ is injective. Therefore, we define a unique norm on $M_n(\mathscr{A})$ that is making it a C^*-algebra by $\|a\| := \|\phi^{(n)}(a)\|$ for $a \in M_n(\mathscr{A})$.

Definition 2.1.6. Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras and $\rho : \mathscr{A} \to \mathscr{B}$. We say that ρ is completely positive if for every $n \geq 1$ the map $\rho^{(n)} : M_n(\mathscr{A}) \to M_n(\mathscr{B})$ such that

$$\rho^{(n)}((a_{ij})_{ij}) = (\rho(a_{ij}))_{ij}) \quad \text{for } (a_{ij})_{ij} \in M_n(\mathscr{A})$$

is positive.

Proposition 2.1.7. *Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras and $\phi : \mathscr{A} \to \mathscr{B}$ a $*$-homomorphism. Then ϕ is completely positive.*

Definition 2.1.7. Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras. We say that $\rho : \mathscr{A} \to \mathscr{B}$ is completely isometric if $\rho^{(n)}$ is isometric for each $n \geq 1$.

Definition 2.1.8. Let $\mathscr{A}$ be a C^*-algebra. We say that $\mathscr{A}$ is a nuclear C^*-algebra if for every C^*-algebra $\mathscr{B}$, the $*$-algebra tensor product $\mathscr{A} \otimes \mathscr{B}$ admits only one complete C^*-norm.

Proposition 2.1.8. *Let $\mathscr{A}$ be a C^*-algebra and consider the $*$-algebra tensor product $M_n(\mathbb{C}) \otimes \mathscr{A}$. Then the map $\phi : M_n(\mathbb{C}) \otimes \mathscr{A} \to M_n(\mathscr{A})$ such that for $(\lambda_{ij})_{ij} \in M_n(\mathbb{C})$ and $a \in \mathscr{A}$*

$$\phi((\lambda_{ij})_{ij} \otimes a) = (\lambda_{ij}a)_{ij},$$

is a $*$-isomorphism. In particular, $M_n(\mathbb{C})$ is nuclear.

2.2 Wold Decomposition

In this brief section we follow [7] and [8] in order to prove some basic properties of shift operators and we prove the Wold decomposition theorem for isometries.

Definition 2.2.1. Let $\mathscr{H}$ be a Hilbert space and $T \in \mathbf{B}(\mathscr{H})$ and L a closed linear subspace of $\mathscr{H}$. We say that L is T-invariant, if $T(L) \subseteq L$. If L is also T^*-invariant, we say that L is reducing to T.

Definition 2.2.2. Let L be a closed subspace of a Hilbert space $\mathscr{H}$ and $A \in \mathbf{B}(\mathscr{H})$ an isometry. We say that L is A-wandering, if the subspaces $L, A(L), A^2(L), \ldots$ are pairwise orthogonal.

For such an A-wandering subspace we can form the orthogonal direct sum

$$\bigoplus_{n=0}^{\infty} A^n(L)$$

and denote it by $M_+(L)$. Then $M_+(L)$ consists of elements ξ such that:

$$\xi = \sum_{n\geq 0} \xi_n \quad \text{where} \quad \xi_n \in A^n(L) \quad \text{and} \quad \sum_{n\geq 0} \|\xi_n\|^2 < \infty$$

We should note that

$$L = M_+(L) \cap A(M_+(L))^\perp.$$

Indeed, observe that $L \subseteq M_+(L)$ and $A(M_+(L)) = \bigoplus_{n\geq 1} A^n(L)$ and that for each $n \geq 1$ the subspaces $A^n(L)$ are orthogonal to L and therefore L is also orthogonal to their orthogonal direct sum $A(M_+(L))$. Hence, $L \subseteq M_+(L)$.

For the reverse inclusion suppose that $\xi = \sum_{k\geq 0} A^k x_k \in M_+(L)$ is orthogonal to $A(M_+(L))$. Note that for each $k \geq 0$ we have that $x_k \in L$ and also that ξ is orthogonal $A^{n+1}(L)$ for every $n \geq 0$. Therefore, for $\zeta \in L$ and $n \geq 0$, using the fact that A is an isometry we have

$$0 = \langle \xi, A^{n+1}\zeta \rangle = \sum_{k\geq 0} \langle A^k x_k, A^{n+1}\zeta \rangle = \langle A^{n+1}x_{n+1}, A^{n+1}\zeta \rangle = \langle x_{n+1}, \zeta \rangle.$$

Since ζ was an arbitrary element in L we conclude that $x_{n+1} = 0$ for each $n \geq 0$ and hence $\xi = x_0 \in L$.

Definition 2.2.3. Let $S \in \mathbf{B}(\mathscr{H})$ be an isometry. We call S a (unilateral) shift if there exists a closed subspace $L \subseteq \mathscr{H}$, that is S-wandering and $M_+(L) = \mathscr{H}$.
We say that $\dim(L)$ is the multiplicity of the shift S.

In this case from our observations above we conclude that

$$L = \mathscr{H} \cap S(\mathscr{H})^\perp = \ker(S^*)$$

and therefore $\dim(L)$ is uniquely defined.

Remark 2. If S_i is a shift in $\mathbf{B}(\mathscr{H}_i)$ and L_i is S_i-wandering such that $M_+(L_i) = \mathscr{H}_i$ for $i = 1, 2$ with $\dim(L_1) = \dim(L_2)$, then S_1, S_2 are unitarily equivalent. Indeed, since $\dim(L_1) = \dim(L_2)$ we may pick a unitary operator $U : L_1 \to L_2$.
We define

$$V : \mathscr{H}_1 \to \mathscr{H}_2 : \quad \sum_n S_1^n x_n \to \sum_n S_2^n U x_n$$

and

$$W : \mathscr{H}_2 \to \mathscr{H}_1 : \quad \sum_n S_2^n y_n \to \sum_n S_1^n U^{-1} y_n.$$

We note that if $\xi \in \mathscr{H}_1$ then $\xi = \sum_n S_1^n x_n$, where $x_n \in L_1$ for each $n \geq 0$ and therefore

$$WV(\xi) = W\left(\sum_n S_2^n U x_n\right) = \sum_n S_1^n U^{-1} U x_n = \sum_n S_1^n x_n$$

and also if $\zeta \in \mathscr{H}_2$ then $\zeta = \sum_n S_2^n y_n$, where for each $n \geq 0$ we have $y_n \in L_2$ and thus

$$VW(\zeta) = V\left(\sum_n S_1^n U^{-1} y_n\right) = \sum_n S_2^n U U^{-1} y_n = \sum_n S_2^n y_n.$$

The above implies that V is invertible and $W = V^{-1}$. We also have that V is an isometry, since if $n \neq m$ we have that $S_i^n(L_i) \perp S_i^m(L_i)$ for $i = 1, 2$ and so

$$\|V(\xi)\|^2 = \left\|\sum_n S_2^n U x_n\right\|^2 = \sum_n \|S_2^n U x_n\|^2 = \sum_n \|x_n\|^2 = \sum_n \|S_1 x_n\|^2 = \left\|\sum_n S_1^n x_n\right\|^2 = \|\xi\|^2.$$

We are now going to prove two easy lemmas that we will use in the proof of the Wold decomposition theorem.

Lemma 2.2.1. *Let $\mathscr{H}$ be a Hilbert space and $A, P \in \mathbf{B}(\mathscr{H})$ be an isometry and a projection, respectively. Then APA^* is the projection onto $AP(\mathscr{H})$.*

Proof. Note that APA^* is obviously a projection. If $\xi = A(\zeta)$ where $\zeta \in P(\mathscr{H})$, then

$$APA^*(\xi) = APA^*A(\zeta) = AP(\zeta) = A(\zeta) = \xi.$$

If $\eta \perp AP(\mathscr{H})$ and $x \in \mathscr{H}$, then

$$\langle APA^*\eta, x\rangle = \langle \eta, APA^*x\rangle = 0,$$

using the fact that $APA^*(x) \in AP(\mathscr{H})$ and $\eta \perp AP(\mathscr{H})$. Thus, since x was arbitrary we have $APA^*(\eta) = 0$ and the proof is complete. □

Lemma 2.2.2. *Let A be an isometry and $L = \ker(A^*) = A(\mathscr{H})^{\perp}$. If $P(L)$ is the projection onto L, then $P(L) = I - AA^*$.*

Proof. Set $U = I - AA^*$ and let $x \in L^{\perp}$ and $y \in L$. Since A is an isometry, $A(\mathscr{H})$ is closed and so $A(\mathscr{H})^{\perp\perp} = A(\mathscr{H})$, thus there exists $z \in \mathscr{H}$ such that $A(z) = x$. Now observe that $U^* = U$ and that $U^2 = U$ and so U is indeed a projection and also

$$U(x) = U(A(z)) = A(z) - AA^*A(z) = A(z) - A(z) = 0$$

and

$$U(y) = y - AA^*(y) = y.$$

□

Theorem 2.2.1. *Let $A \in \mathbf{B}(\mathscr{H})$ be an isometry on the Hilbert space $\mathscr{H}$. Then there is a unique decomposition of $\mathscr{H} = \mathscr{H}_s \oplus \mathscr{H}_u$ into A-reducing subspaces and if A_s is the restriction of A to $\mathscr{H}_s$ and A_u the restriction of A to $\mathscr{H}_u$, then A_u is unitary and A_s is a shift. In particular, $L = \ker(A^*)$ is an A-wandering subspace and we have that*

$$\mathscr{H}_s = M_+(L) = \bigoplus_{n\geq 0} A^n(L) = \{x \in \mathscr{H} \,:\, A^{*n}x \to 0\} \quad \textit{and} \quad \mathscr{H}_u = \bigcap_{n\geq 0} A^n(\mathscr{H}).$$

Proof. At first we are going to prove that for $n > m$ we have $A^n(L) \perp A^m(L)$ and so L is indeed an A-wandering subspace. Set $k = n - m > 0$ and suppose that $x, y \in L$, then

$$\langle A^n(x), A^m(y)\rangle = \langle A^m(x)A^k(x), A^m(y)\rangle = \langle A^k(x), y\rangle = 0,$$

where we used the fact that A, A^m are isometries and that $y \in A(H)^{\perp}$.
Now let $\mathscr{H}_s = M_+(L) = \bigoplus_{n\geq 0} A^n(L)$, we will show that $\mathscr{H}_s = \{x \in \mathscr{H} \,:\, A^{*n}x \to 0\}$.

Let $P(L)$ be the projection onto L and $P(A^n(L))$ the projection onto $A^n(L)$. Using the preceding lemmas we obtain that

$$P(L) = I - AA^* \quad \text{and} \quad P(A^n(L)) = A^n P(L) A^{*n} = A^n(I - AA^*)A^{*n}$$

Thus for $x \in \mathscr{H}$ we have

$$x \in \mathscr{H}_s \iff x = \sum_{n\geq 0} P(A^n(L))x = \lim_{N\to\infty} \sum_{n=0}^{N} A^n(I - AA^*)A^{*n}(x) = x - \lim_{N\to\infty} A^N A^{*N}(x)$$

and since

$$\lim_{N\to\infty} A^N A^{*N}(x) = 0 \iff \lim_{N\to\infty} A^{*N} x = 0,$$

we have that

$$x \in \mathscr{H}_s \iff \lim_{N\to\infty} A^{*N}(x) = 0.$$

To show that ${\mathscr{H}_s}^{\perp} = \mathscr{H}_u$, observe

$$\begin{aligned} x \in {\mathscr{H}_s}^{\perp} &\iff x \in A^n(L)^{\perp}, \quad \forall n \geq 0 \\ &\iff 0 = P(A^n(L))x = A^n(I - AA^*)A^{n*}, \quad \forall n \geq 0 \\ &\iff A^n A^{n*}(x) = A^{n+1}A^{n+1*}(x), \quad \forall n \geq 0. \end{aligned}$$

Thus, $x = A^n A^{n*}(x), \quad \forall n \geq 0$ but using the fact that A^n is an isometry and lemma 2.2.2 we have that $A^n A^{n*} = P(A^n(\mathscr{H}))$ and therefore

$$x \in {\mathscr{H}_s}^{\perp} \iff x \in A^n(\mathscr{H}), \quad \forall n \geq 0 \iff x \in \bigcap_{n\geq 0} A^n(\mathscr{H}) = \mathscr{H}_u.$$

Now let $P = P(\mathscr{H}_u), P_n = A^n A^{n*} = P(A^n(\mathscr{H}))$ and $x \in \mathscr{H}$. Since $\{P_n : n \geq 1\}$ is a decreasing sequence of projections, it converges pointwise to the projection onto the intersection of $A^n(\mathscr{H})$, so $Px = \lim_n A^n A^{n*}(x)$. Therefore,

$$P(A(\mathscr{H}_u))x = APA^*x = \lim_n A^{n+1}A^{n+1*}x = Px$$

and thus $P(A(\mathscr{H}_u)) = APA^* = P$, hence $PA = APA^*A = AP$, which proves that A_u is unitary (onto and isometry) and $\mathscr{H}_u$ is reducing to A, which also implies that its orthogonal complement $\mathscr{H}_s$ is reducing to A.

To complete the proof it remains to show the uniqueness of this decomposition. In order to do so, suppose that $\mathscr{H} = K_s \oplus K_u$ where $A|_{K_s}$ is a shift and $A|_{K_u}$ is a unitary. Since $A|_{K_s}$ is a shift, the space $L' = K_s \cap A|_{K_s}(K_s)^{\perp} = K_s \cap A(K_s)^{\perp}$ is $A|_{K_s}$-wandering and thus, A-wandering. So it suffices to prove that $L = L'$, because in this case

$$K_s = M_+(L') = M_+(L) = \mathscr{H}_s$$

and therefore their orthogonal complements will also be equal.
Let $x \in L = (K_u \oplus K_s) \cap A(K_u \oplus K_s)^{\perp}$, then $x = x_s + x_u$ with $x_s \in K_s$, $x_u \in K_u$ and also $x \perp A(K_u \oplus K_s)$. Notice that

$$A(K_u \oplus K_s) = A(K_u) \oplus A(K_s) = K_u \oplus A(K_s)$$

since $A(K_u) = K_u$, $(A|_{K_u}$ is a unitary).
Thus, $x \perp K_u \iff x \in K_s \iff x = x_s$ and also $x \perp A(K_s) \iff x \in A(K_s)^{\perp}$
Therefore, $x \in K_s \cap A(K_s)^{\perp} = L'$ and so $L \subseteq L'$.
The exact same argument, if we use the decomposition $\mathscr{H} = \mathscr{H}_s \oplus \mathscr{H}_u$ shows that $L' \subseteq L$. □

Remark 3. We define the Hilbert space $\ell^2(\mathbb{Z}^+, L)$ to be the vector space of all sequences $(x_n)_{n=0}^{\infty}$ such that $x_n \in L$ for all $n \geq 0$ and $\sum_{n\geq 0} \|x_n\|_L^2 < \infty$, where the inner-product is given by

$$\langle (x_n)_n, (y_n)_n \rangle = \sum_{n\geq 0} \langle x_n, y_n \rangle_L .$$

We should note that if we define

$$U_A : \bigoplus_{n\geq 0} A^n(L) \to \ell^2(\mathbb{Z}^+, L) : \quad \sum_{n\geq 0} A^n(x_n) \to (x_0, x_1, x_2, \cdots),$$

then U_A is invertible and also a unitary since

$$\left\| \sum_{n\geq 0} A^n(x_n) \right\|^2 = \sum_{n\geq 0} \|A^n(x_n)\|^2 = \sum_{n\geq 0} \|x_n\|^2.$$

Therefore, if $S : \ell^2(\mathbb{Z}^+, L) \to \ell^2(\mathbb{Z}^+, L)$ is the Shift operator given by

$$S(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots),$$

then it evident that we have the following

$$U_A{}^* S U_A = A|_{\mathscr{H}_s}.$$

2.3 Integration on Banach spaces

This section is based on the notes of the graduate course Banach algebras [25].
Let X be a Banach space and $f : [a, b] \to X$ a continuous function.
Let $\mathscr{P} = \{a = t_0 < t_1 < \ldots < t_m = b\}$ be a partition of $[a, b]$ and let

$$\|\mathscr{P}\| = \max_{i=1,\ldots m} (t_i - t_{i-1}).$$

We define
$$S(f,\mathscr{P})=\sum_{k=1}^{m} f(t_k)(t_k-t_{k-1}).$$
Consider the net $\{S(f,\mathscr{P}) : \mathscr{P}$ partition of [a,b]$\}$ where $\mathscr{P}_1 \leq \mathscr{P}_2$ iff $\mathscr{P}_1$ is a refinement of $\mathscr{P}_2$. We will show that this net converges in norm and we will denote the limit by $\int_a^b f(t)dt$.
Indeed, by the uniform continuity of f, $\forall \epsilon > 0$ there exists $\delta > 0$ such that:

$$\text{if} \quad |t-s|< \delta \quad \text{then} \quad \|f(t)-f(s)\|< \frac{\epsilon}{b-a}.$$

If $\mathscr{P}_0$ is a partition of $[a,b]$ with $|t_k-t_{k-1}|<\delta$ for all k and $\mathscr{P}$ is a refinement of $\mathscr{P}_0$ then for all $i=1,...,m$
$$[t_i-t_{i-1}]=\bigcup_{j=1}^{n_i}[l_{i,j-1},l_{i,j}],$$
where $\mathscr{P}=\{a=l_{0,0}<l_{0,1}<...<l_{0,n_0}<l_{1,0}<...<l_{n,n_m}=b\}$.
We have

$$\begin{aligned}\|S(f,\mathscr{P})-S(f,\mathscr{P}_0)\|&=\left\|\sum_{i=1}^{m}(t_i-t_{i-1})f(t_i)-\sum_{i=1}^{m}\sum_{j=1}^{n_i}(l_{i,j}-l_{i,j-1})f(l_{i,j})\right\|\\ &=\left\|\sum_{i=1}^{m}\sum_{j=1}^{n_i}(l_{i,j}-l_{i,j-1})f(t_i)-\sum_{i=1}^{m}\sum_{j=1}^{n_i}(l_{i,j}-l_{i,j-1})f(l_{i,j})\right\|\\ &\leq\sum_{i=1}^{m}\sum_{j=1}^{n_i}(l_{i,j}-l_{i,j-1})\|f(t_i)-f(l_{i,j})\|<\frac{\epsilon}{b-a}(b-a)=\epsilon\end{aligned}$$

Therefore, if $\mathscr{P}_1,\mathscr{P}_2$ are refinements of $\mathscr{P}_0$ then

$$\|S(f,\mathscr{P}_1)-S(f,\mathscr{P}_2)\|\leq\|S(f,\mathscr{P}_1)-S(f,\mathscr{P}_0)\|+\|S(f,\mathscr{P}_0)-S(f,\mathscr{P}_2)\|<2\epsilon$$

and so its a Cauchy net. Since X is a Banach space, it is convergent.

Proposition 2.3.1. *Let X be a Banach space and $f:[a,b]\to X$ a continuous function. Then,*

$$\left\|\int_a^b f(t)dt\right\|\leq\int_a^b\|f(t)\|dt\leq(b-a)\|f\|_\infty.$$

Proof. First of all, note that the map

$$[a,b]\to\mathbb{R}: \quad t\to\|f(t)\|$$

is continuous and hence integrable and so

$$\int_a^b\|f(t)\|dt$$

is well-defined.
Let $\{\mathscr{P}_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $\|\mathscr{P}_n\| \to 0$. Then, by the definition of the integral we have that

$$\lim_n S(f, \mathscr{P}_n) = \int_a^b f(t)dt$$

and also since $\|.\|$ is continuous we have

$$\lim_n \|S(f, \mathscr{P}_n)\| = \left\| \int_a^b f(t)dt \right\|.$$

Notice that, if we fix an index $n \in \mathbb{N}$ and

$$\mathscr{P}_n = \{a = t_0 < t_1 < ... < t_m = b\},$$

then

$$\|S(f, \mathscr{P}_n)\| = \left\| \sum_{i=1}^{m} f(t_i)(x_i - x_{i-1}) \right\| \leq \sum_{i=1}^{m} \|f(t_i)\|(x_i - x_{i-1}) = S(\|f\|, \mathscr{P}_n),$$

and since $\|f\|$ is Riemann-Integrable in the classical sense, we have that

$$\left\| \int_a^b f(t)dt \right\| \leq \int_a^b \|f(t)\|dt.$$

Since f is continuous on a compact set we have

$$\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$$

is well-defined and therefore

$$\int_a^b \|f(t)\|dt \leq \|f\|_\infty (b - a).$$

□

Now if $x^* \in X^*$, then

$$x^*\left(\int_a^b f(t)dt \right) = x^*(\lim_{\mathscr{P}} S(f, \mathscr{P})) = \lim_{\mathscr{P}} \sum_{k=1}^{m} x^*(f(t_k))(t_k - t_{k-1}) = \int_a^b x^*(f(t))dt.$$

Denote by $C([a, b], X)$ the linear space of the continuous functions from $[a, b]$ to X, with point-wise addition and scalar multiplication, then the map

$$C([a, b], X) \to X \quad f \to \int_a^b f(t)dt$$

is linear and bounded.
If X is also a Banach algebra, from the continuity of multiplication we get that for $c, c' \in X$, $f : [a, b] \to X$ and a partition of $[a, b]$

$$\mathscr{P} = \{a = t_0 < t_1 < ... < t_m = b\},$$

$$S(cfc', \mathscr{P}) = \sum_{i=1}^{m} cf(t_i)c'(t_i - t_{i-1}) = c\Big(\sum_{i=1}^{m} f(t_i)(t_i - t_{i-1})\Big)c'$$

and by taking limits, we have that

$$\int_a^b cf(t)c'dt = c\Big(\int_a^b f(t)dt\Big)c'.$$

Now, suppose that $f : [a, b] \to \mathbb{C}$ is a continuous function and X is unital. Let $g : [a, b] \to X$ be the function such that

$$g(t) = f(t)\mathbf{1}_{\mathscr{A}}.$$

For $\mathscr{P} = \{a = t_0 < t_1 < ... < t_m = b\}$ a partition of $[a, b]$ we have that,

$$S(g, \mathscr{P}) = \sum_{i=1}^{m} f(t_i)\mathbf{1}_{\mathscr{A}}(t_i - t_{i-1}) = \Big(\sum_{i=1}^{m} f(t_i)(t_i - t_{i-1})\Big)\mathbf{1}_{\mathscr{A}} \xrightarrow{\mathscr{P}} \Big(\int_a^b f(t)dt\Big)\mathbf{1}_{\mathscr{A}}.$$

2.4 Hilbert C*-modules

This section is fundamental for our understanding of C^*-correspondences and therefore we are going to give complete proofs. We are going to follow [20].

Definition 2.4.1. Let $\mathscr{A}$ be a C^*-algebra. An inner-product right $\mathscr{A}$-module is a pair of a linear space X that is a right $\mathscr{A}$-module and a map $\langle .,. \rangle : X \times X \to \mathscr{A}$ such that $\forall a \in \mathscr{A}$, $\forall \xi, \eta, \zeta \in X$ and $\forall \lambda, \mu \in \mathbb{C}$:

$$\begin{aligned}
&(i)\ \langle \xi, \lambda\zeta + \mu\eta \rangle = \lambda\langle \xi, \zeta \rangle + \mu\langle \xi, \eta \rangle,\\
&(ii)\ \langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*,\\
&(iii)\langle \xi, \eta\, a \rangle = \langle \xi, \eta \rangle a,\\
&(iv)\ \langle \xi, \xi \rangle \geq 0 \quad \text{and} \quad \langle \xi, \xi \rangle = 0 \iff \xi = 0.
\end{aligned}$$

One can easily deduce, from $i), ii)$ of the preceding definition, that the map we defined above is sesquilinear and also we can easily see that

$$\langle \xi a, \eta \rangle = \langle \eta, \xi a \rangle^* = (\langle \eta, \xi \rangle a)^* = a^*\langle \xi, \eta \rangle.$$

We also define $\|.\|_X : X \to \mathbb{R}^+$, for $\xi \in X$ by

$$\|\xi\|_X = \|\langle \xi, \xi \rangle\|_{\mathscr{A}}^{1/2}$$

To prove that $\|.\|_X$ is actually a norm, we are going to need the next easy result.

Proposition 2.4.1. *Let $(X, \mathscr{A}, \langle ., . \rangle)$ be an inner-product right $\mathscr{A}$-module.*

$$\forall \xi, \eta \in X : \quad \langle \xi, \eta \rangle \langle \eta, \xi \rangle \leq \|\langle \eta, \eta \rangle\|_{\mathscr{A}} \langle \xi, \xi \rangle.$$

Proof. Without loss of generality, suppose that $\xi, \eta \in X$ such that $\|\langle \eta, \eta \rangle\| = 1$. Then for each $a \in \mathscr{A}$ we have

$$\begin{aligned} 0 &\leq \langle \eta a - \xi, \eta a - \xi \rangle = a^* \langle \eta, \eta \rangle a - \langle \xi, \eta \rangle a - a^* \langle \eta, \xi \rangle + \langle \xi, \xi \rangle \\ &\leq a^* a - \langle \xi, \eta \rangle a - a^* \langle \xi, \eta \rangle + \langle \xi, \xi \rangle, \end{aligned}$$

where at the last inequality we use the fact that if $c \in \mathscr{A}^+$ then $\forall a \in \mathscr{A}$

$$a^* c a \leq \|c\| a^* a.$$

We set $a = \langle \eta, \xi \rangle$, then

$$0 \leq \langle \xi, \eta \rangle \langle \eta, \xi \rangle - \langle \xi, \eta \rangle \langle \eta, \xi \rangle - \langle \xi, \eta \rangle \langle \eta, \xi \rangle + \langle \xi, \xi \rangle \text{ hence } \langle \xi, \eta \rangle \langle \eta, \xi \rangle \leq \langle \xi, \xi \rangle \|\langle \eta, \eta \rangle\|.$$

□

Remark 4. The preceding proposition also implies that:

$$\|\langle \xi, \eta \rangle\|_{\mathscr{A}} \leq \|\xi\|_X \|\eta\|_X.$$

Indeed, we have that for $a, b \in \mathscr{A}^+$ if $a \leq b \implies \|a\| \leq \|b\|$

$$\begin{aligned} &\langle \xi, \eta \rangle \langle \eta, \xi \rangle \leq \|\langle \eta, \eta \rangle\| \langle \xi, \xi \rangle \implies \|\langle \xi, \eta \rangle \langle \eta, \xi \rangle\| \leq \|\langle \eta, \eta \rangle\| \|\langle \xi, \xi \rangle\| \implies \\ &\|\langle \xi, \eta \rangle\|^2 \leq \|\langle \xi, \xi \rangle\| \|\langle \eta, \eta \rangle\| = \|\eta\|_X^2 \|\xi\|_X^2 \implies \|\langle \xi, \eta \rangle\|_{\mathscr{A}} \leq \|\xi\|_X \|\eta\|_X \end{aligned}$$

Now, one can easily prove the triangle inequality for $\|.\|_X$ and so, it is indeed a norm on the linear space X: If $\xi, \eta \in X$ then

$$\begin{aligned} \|\xi + \eta\|_X &= \|\langle \xi + \eta, \xi + \eta \rangle\|_{\mathscr{A}}^{1/2} = \|\langle \xi, \xi \rangle + \langle \eta, \xi \rangle + \langle \xi, \eta \rangle + \langle \eta, \eta \rangle\|_{\mathscr{A}}^{1/2} \\ &\leq (\|\xi\|_X^2 + 2\|\eta\|_X \|\xi\|_X + \|\eta\|_X^2)^{1/2} = ((\|\xi\|_X + \|\eta\|_X)^2)^{1/2} = \|\xi\|_X + \|\eta\|_X \end{aligned}$$

Definition 2.4.2. Let $(X, \mathscr{A}, \langle ., . \rangle)$ be an inner-product right $\mathscr{A}$-module. We call it a Hilbert $\mathscr{A}$-module if the pair $(X, \|.\|_X)$ is a Banach space.

From now on we will not use the subscript X for the norm, if it is not necessary.
We can easily see that for $x \in X$

$$\|x\| = \sup\{\|\langle x, y\rangle\| : y \in X,\ \|y\| \leq 1\}.$$

Indeed, we have that

$$\|\langle x, y\rangle\| \leq \|x\|\|y\| = \|x\|$$

and also for $y = \frac{x}{\|x\|}$

$$\|\langle x, y\rangle\| = \|x\|.$$

For $x \in X$ we denote $|x| := \langle x, x\rangle^{1/2} \in \mathscr{A}$ and also for $a \in \mathscr{A}$ we denote $|a| = (a^*a)^{1/2}$, then we have the following inequality

$$|xa| \leq \|x\||a|.$$

Indeed,

$$\langle xa, xa\rangle = a^* \langle x, x\rangle\, a \leq \|x\|^2 a^* a$$

and by taking square roots the result follows. This also implies that

$$\|xa\| \leq \|x\|\|a\|.$$

Remark 5. Let X be a Hilbert $\mathscr{A}$-module. Then $X\mathscr{A}$=X, where $X\mathscr{A}$ is the closed linear span of elements of the form $\{xa\ :\ x \in X,\ a \in \mathscr{A}\}$ and also if we denote by $\langle X, X\rangle$ the linear span of elements $\{\langle x, y\rangle\ :\ x, y \in X\}$, then $X\langle X, X\rangle$ is dense in X.
Indeed, let $\{e_i\}$ be an approximate unit for $\mathscr{A}$ and $x \in X$, then

$$\langle x - xe_i, x - xe_i\rangle = \langle x, x\rangle - e_i\langle x, x\rangle - \langle x, x\rangle e_i + e_i\langle x, x\rangle e_i \xrightarrow{i} 0. \quad (*)$$

If $\mathscr{A}$ is unital with an analogous computation we can see that for $x \in X$ we have

$$x\mathbf{1}_{\mathscr{A}} = x.$$

In the case that $\mathscr{A}$ is not unital, we denote by $\mathscr{A}_1$ its unitization, then X becomes a Hilbert $\mathscr{A}_1$-module if we define $x1_{\mathscr{A}_1} = x$.
If we denote the closure of $\langle X, X\rangle$ by B, then B is a closed two-sided ideal of $\mathscr{A}$, since for $x, y \in X$ and $a \in \mathscr{A}$

$$\langle x, y\rangle a = \langle x, ya\rangle \quad \text{and} \quad a\langle x, y\rangle = \langle xa^*, y\rangle$$

Thus, using proposition 2.1.2, B is self-adjoint and so there exists an approximate unit $\{u_i\}$ for B and thus using $(*)$ for B we obtain that $xu_i \xrightarrow{i} x$ for all $x \in X$. Therefore, $X\langle X, X\rangle$

is dense in X.

We will now give some basic examples of Hilbert $\mathscr{A}$-modules.

Example 2.4.1. Let $\mathscr{A}$ be a C^*-algebra. We define for $a, b \in \mathscr{A}$

$$\langle a, b \rangle = a^* b.$$

Then $\mathscr{A}$ is a Hilbert $\mathscr{A}$-module. If I is closed right ideal of $\mathscr{A}$ then by restricting $\langle ., . \rangle$ to I, we get that I is a Hilbert $\mathscr{A}$-module.

Example 2.4.2. Let $\mathscr{H}$ be a Hilbert space and X a compact Hausdorff space. For each $t \in X$, let $\mathscr{H}_t$ be a closed subspace of $\mathscr{H}$. Denote by E the $\mathbb{C}$-vector space of all continuous functions ξ from X to $\mathscr{H}$ such that for $t \in X$, $\xi(t) \in \mathscr{H}_t$. Then E has the structure of $C(X)$-module if we define for $\xi \in E$ and $f \in C(X)$, ξf to be the function

$$t \to \xi(t) f(t).$$

For $\xi, \eta \in E$ define $\langle \xi, \eta \rangle$ to be the function

$$t \to \langle \xi(t), \eta(t) \rangle_{\mathscr{H}}.$$

Then $(E, C(X), \langle ., . \rangle)$ is a Hilbert $C(X)$-module.
Indeed, the fact that $(E, C(X), \langle ., . \rangle)$ is an inner-product right $C(X)$-module is immediate. To see the completeness of the norm, note that

$$\|\xi\|_E^2 = \sup_{t \in X} |\langle \xi(t), \xi(t) \rangle_{\mathscr{H}}| = \sup_{t \in X} \|\xi(t)\|_{\mathscr{H}}^2.$$

Suppose that $(\xi_n)_n$ is a $\|.\|_E$-Cauchy sequence, then since $\|.\|_{\mathscr{H}}$ is complete we obtain that the set of $\mathscr{H}$-valued continuous functions defined on X is a Banach space. Using the equality above $(\xi_n)_n$ is Cauchy in the norm of the right-hand side and therefore $(\xi_n)_n$ converges to a continuous $\mathscr{H}$-valued function ξ and using once more the equality of the norms, $(\xi_n)_n$ converges to ξ in $\|.\|_E$.

Now with the second example in mind we can prove that for a Hilbert $\mathscr{A}$-module $(X, \mathscr{A}, \langle ., . \rangle)$, the closure of $\langle X, X \rangle$ is not always equal with $\mathscr{A}$. So let $\mathscr{A} = C(X)$ and E the Hilbert $C(X)$-module described above. Let Y be a non-empty closed subspace of X and $\mathscr{H}_t = \{0\}$ for $t \in Y$ then

$$\overline{\langle X, X \rangle} \subseteq \{f \in C(X) : f(Y) = 0\} \subsetneq C(X).$$

Example 2.4.3. Let $(X_n)_{n=1}^m$ be Hilbert $\mathscr{A}$-modules. Then $X = \bigoplus_{n=1}^m X_n$ is a Hilbert $\mathscr{A}$-module if we set $(\xi_1, \xi_2, ..., \xi_m) \cdot a = (\xi_1 a, \xi_2 a, ..., \xi_m a)$ for $(\xi_1, \xi_2, ..., \xi_m) \in X$ and $a \in \mathscr{A}$ and

$$\langle (\xi_1, \xi_2, ..., \xi_m), (\eta_1, \eta_2, ..., \eta_m) \rangle = \sum_{n=1}^{m} \langle \xi_n, \eta_n \rangle$$

for $(\eta_1, \eta_2, ..., \eta_m) \in X$. We denote by Y^n the direct sum of n copies of a Hilbert $\mathscr{A}$-module Y.

Example 2.4.4. Let $(X_n)_{n=1}^{\infty}$ be Hilbert $\mathscr{A}$-modules. We define $X = \bigoplus_{n=1}^{\infty} X_n$ to be the set of all sequences $x = (x_n)_{n=1}^{\infty}$, with $x_n \in X_n$, such that $\sum_{n=1}^{\infty} \langle x_n, x_n \rangle$ converges in $\mathscr{A}$. For $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ in X, we define

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x_n, y_n \rangle$$

and $x \cdot a = (x_1 a, x_2 a, ...)$. First of all, we prove the linear space structure:
Suppose that $(x_n)_n, (y_n)_n \in X$, we will show that $(x_n + y_n)_n \in X$. We set

$$a_n = \sum_{k \le n} \langle x_k, x_k \rangle , \quad b_n = \sum_{k \le n} \langle y_k, y_k \rangle , \quad c_n = \sum_{k \le n} \langle x_k + y_k, x_k + y_k \rangle .$$

Note that for each $n \ge 1$ and $z, w \in X_n$ we have

$$0 \le \langle z + w, z + w \rangle \le \langle z + w, z + w \rangle + \langle z - w, z - w \rangle = 2 \langle z, z \rangle + 2 \langle w, w \rangle$$

Since, for $n \ge m$ the differences $c_n - c_m$ are finite sums of such terms we obtain

$$0 \le c_n - c_m \le 2(a_n - a_m) + 2(b_n - b_m)$$

and therefore

$$\|c_n - c_m\| \le 2\|a_n - a_m\| + 2\|b_n - b_m\|.$$

The fact that $(x_n)_n, (y_n)_n \in X$ implies that both (a_n) and (b_n) converge and hence (c_n) is a Cauchy sequence in $\mathscr{A}$ and therefore convergent.
The module action is well-defined:

For $a \in \mathscr{A}$, the equality

$$\sum_{k=m}^{n} \langle x_k a, x_k a \rangle = \sum_{k=m}^{n} a^* \langle x_k, x_k \rangle a = a^* (\sum_{k=m}^{n} \langle x_k, x_k \rangle) a$$

proves that when $\sum_k \langle x_k, x_k \rangle$ converges in $\mathscr{A}$, so does $\sum_k \langle x_k a, x_k a \rangle$.
The $\mathscr{A}$-valued product is well-defined:
By polarisation we have

$$4 \sum_k \langle x_k, y_k \rangle = \sum_k \langle x_k + y + k, x_k + y_k \rangle - \sum_k \langle x_k - y_k, x_k - y_k \rangle$$
$$+ i \sum_k \langle x_k + i y_k, x_k + i y_k \rangle - i \sum_k \langle x_k - i y_k, x_k - i y_k \rangle ,$$

which proves that the series on the left hand-side converges in $\mathscr{A}$ since the four series on the right hand-side converge in $\mathscr{A}$. Note that if $x = (x_n)_{n=1}^{\infty} \in X$, then for each $k \in \mathbb{N}$ we have that

$$\|x_k\|^2 = \|\langle x_k, x_k\rangle\| \leq \left\|\sum_{n=1}^{\infty} \langle x_n, x_n\rangle\right\| = \|\langle x, x\rangle\|^2.$$

To see that X is complete suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, where each $x_n = (x_n(1), x_n(2), ...)$ for $x_n(k) \in X_k$. Note that for each fixed $k \in \mathbb{N}$ and $n, m \in \mathbb{N}$ we have that

$$\|x_n(k) - x_m(k)\|^2 \leq \|x_n - x_m\|^2$$

therefore the sequence $(x_n(k))_{n\in\mathbb{N}}$ is Cauchy in X_k and so it converges to some $y(k) \in X_k$. Set $y = (y(k))_{k\in\mathbb{N}}$, we will prove that $y \in X$ and that $x_n \to y$. To prove that $y \in X$ since $\mathscr{A}$ is complete it suffices to show that given $\epsilon > 0$ there exists P such that for $m \geq n \geq P$ we have $\|\sum_{k=n}^{m} \langle y(k), y(k)\rangle\| \leq \epsilon^2$. Since (x_n) is Cauchy, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$k \geq l \geq N \implies \|x_k - x_l\| < \epsilon/3.$$

Since $x_N \in X$ we can choose $P \geq N$ such that

$$\left\|\sum_{i=P}^{\infty} \langle x_N(i), x_N(i)\rangle\right\|^{1/2} < \epsilon/3.$$

For each $k \in \mathbb{N}$ we have $y(k) = \lim_M x_M(k)$ in X_k, that is

$$\lim_M \|\langle y(k) - x_M(k), y(k) - x_M(k)\rangle\| = 0.$$

Therefore for $m \geq n$ we have

$$\left\|\sum_{k=n}^{m} \langle y(k) - x_M(k), y(k) - x_M(k)\rangle\right\| \leq \sum_{k=n}^{m} \|\langle y(k) - x_M(k), y(k) - x_M(k)\rangle\| \xrightarrow{M\to\infty} 0.$$

Thus, we may choose M depending on m, n such that

$$\left\|\sum_{k=n}^{m} \langle y(k) - x_M(k), y(k) - x_M(k)\rangle\right\|^{1/2} < \epsilon/3.$$

Therefore, if $m \geq n \geq P$ we have

$$\left\|\sum_{k=n}^{m}\langle y(k), y(k)\rangle\right\|^{1/2} \leq \left\|\sum_{k=n}^{m}\langle y(k)-x_M(k), y(k)-x_M(k)\rangle\right\|^{1/2}$$
$$+\left\|\sum_{k=n}^{m}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle\right\|^{1/2}+\left\|\sum_{k=n}^{m}\langle x_N(k), x_N(k)\rangle\right\|^{1/2}$$
$$\leq \left\|\sum_{k=n}^{m}\langle y(k)-x_M(k), y(k)-x_M(k)\rangle\right\|^{1/2}+\left\|\sum_{k=1}^{\infty}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle\right\|^{1/2}$$
$$+\left\|\sum_{k=P}^{\infty}\langle x_N(k), x_N(k)\rangle\right\|^{1/2} \leq \epsilon/3+\epsilon/3+\epsilon/3=\epsilon,$$

where we used the fact that

$$\sum_{k=n}^{m}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle \leq \sum_{k=1}^{\infty}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle$$

and also

$$\sum_{k=n}^{m}\langle x_N(k), x_N(k)\rangle \leq \sum_{k=P}^{\infty}\langle x_N(k), x_N(k)\rangle$$

as elements in $\mathscr{A}^+$ and therefore

$$\left\|\sum_{k=n}^{m}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle\right\| \leq \left\|\sum_{k=1}^{\infty}\langle x_M(k)-x_N(k), x_M(k)-x_N(k)\rangle\right\|$$

and

$$\left\|\sum_{k=n}^{m}\langle x_N(k), x_N(k)\rangle\right\| \leq \left\|\sum_{k=P}^{\infty}\langle x_N(k), x_N(k)\rangle\right\|.$$

The above proves that $y \in X$.

Now to see that $\lim_n\|y-x_n\|=0$, given $\epsilon>0$ we can pick n_0 such that

$$n \geq m \geq n_0 \implies \|x_n-x_m\|<\epsilon.$$

For each $N \in \mathbb{N}$, if $n \geq m \geq n_0$ we have that

$$\left\|\sum_{k=1}^{N}\langle x_n(k)-x_m(k), x_n(k)-x_m(k)\rangle\right\|<\epsilon^2.$$

Letting $m \to \infty$ we obtain

$$\left\|\sum_{k=1}^{N}\langle x_n(k)-y(k), x_n(k)-y(k)\rangle\right\|<\epsilon^2$$

for all $n \geq n_0$. Since, $x_n - y \in X$ we get that the series

$$\sum_{k=1}^{\infty} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle$$

converges in $\mathscr{A}^+$.
If we set $a_N = \sum_{k=1}^{N} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle$ and $a = \sum_{k=1}^{\infty} \langle x_n(k) - y(k), x_n(k) - y(k) \rangle$ we have that $\|a\| = \lim_N \|a_N\|$ and since $\|a_N\| < \epsilon^2$, it follows that $\|a\| \leq \epsilon^2$. The above implies that $\|x_n - y\| \leq \epsilon$ for all $n \geq n_0$, as required.

Definition 2.4.3. Let $(X, \mathscr{A}, \langle .,. \rangle)$ be a Hilbert $\mathscr{A}$-module. We say that X is full, if $\overline{\langle X, X \rangle} = \mathscr{A}$.

Let E, F be Hilbert $\mathscr{A}$-modules for a C^*-algebra $\mathscr{A}$. We denote by $\mathscr{L}(E, F)$ the set of all the maps $t : E \to F$ for which there exists a map $t^* : F \to E$ which satisfies

$$\langle tx, y \rangle = \langle x, t^* y \rangle \quad \text{for} \quad x \in E, y \in F.$$

We call such a map an adjointable operator.
If $t \in \mathscr{L}(E, F)$, $\lambda \in \mathbb{C}$, $a \in \mathscr{A}$, $x, w \in E$ and $y \in F$ we have that

$$\langle t(\lambda x + w), y \rangle = \langle \lambda x + w, t^*(y) \rangle = \overline{\lambda} \langle x, t^* y \rangle + \langle w, t^* y \rangle = \langle \lambda tx + tw, y \rangle$$

and also

$$\langle t(xa), y \rangle = \langle xa, t^* y \rangle = a^* \langle tx, y \rangle = \langle t(x)a, y \rangle.$$

Thus, from $iv)$ of definition 2.4.1 we have that t is $\mathscr{A}$-linear i.e.

$$t(\lambda x + w) = \lambda tx + tw$$

and

$$t(xa) = t(x)a.$$

Note that t is also bounded.
Indeed, let $\{x_n : n \in \mathbb{N}\}$ be a sequence in E such that $x_n \to x \in E$ and $tx_n \to y \in F$. Then for each $z \in F$ we have

$$\|\langle tx_n, z \rangle - \langle y, z \rangle\| \leq \|tx_n - y\| \|z\| \to 0$$

and since

$$\|\langle x_n, t^* z \rangle - \langle x, t^* z \rangle\| \leq \|x_n - x\| \|t^* z\| \to 0,$$

we have that

$$\langle y, z\rangle = \lim_n \langle tx_n, z\rangle = \lim_n \langle x_n, t^* z\rangle = \langle x, t^* z\rangle = \langle tx, z\rangle.$$

Thus, $tx = y$ and using the closed graph theorem t is bounded.
If $E = F = X$ we denote the set of adjointable operators by $\mathscr{L}(X)$.

Proposition 2.4.2. *Let t, s be elements of $\mathscr{L}(X)$ and $\lambda \in \mathbb{C}$. Then*

(i) $(t+s)^* = t^* + s^*$

(ii) $(\lambda t)^* = \overline{\lambda} t^*$

(iii) $(ts)^* = s^* t^*$.

In particular, $\mathscr{L}(X)$ is a C^*-algebra where the product is composition and the norm is the operator norm.

Proof. Let $x, y, w \in X$, then

$$\langle x, (t+s)^*(y)\rangle = \langle (t+s)(x), y\rangle = \langle tx, y\rangle + \langle sx, y\rangle = \langle x, t^* y\rangle + \langle x, s^* y\rangle = \langle x, t^* y + s^* y\rangle$$

and

$$\langle x, (\lambda t)^* y\rangle = \langle (\lambda t)(x), y\rangle = \langle \lambda t x, y\rangle = \overline{\lambda}\langle tx, y\rangle = \overline{\lambda}\langle x, t^* y\rangle = \langle x, \overline{\lambda} t^* y\rangle$$

and

$$\langle x, (ts)^* y\rangle = \langle (ts)(x), y\rangle = \langle t(s(x)), y\rangle = \langle sx, t^* y\rangle = \langle x, s^*(t^*(y))\rangle = \langle x, (s^* t^*)(y)\rangle.$$

The above proves $(i), (ii), (iii)$.
Firstly, we show the C^*-property of the norm. It is obvious that

$$\|t^* t\| \leq \|t^*\| \|t\|.$$

Now, observe that

$$\begin{aligned} \|t^* t\| &= \sup_{\|x\|\leq 1} \|t^* t(x)\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \|\langle t^* t(x), y\rangle\| \geq \sup_{\|x\|\leq 1} \|\langle t^* t(x), x\rangle\| \\ &= \sup_{\|x\|\leq 1} \|\langle tx, tx\rangle\| = \sup_{\|x\|\leq 1} \|tx\|^2 = \|t\|^2 \end{aligned}$$

Combining the two inequalities, it follows that

$$\|t\| \leq \|t^*\|,$$

since

$$\langle t^* x, y\rangle = \langle y, t^*(x)\rangle^* = \langle ty, x\rangle^* = \langle x, ty\rangle,$$

we have that $(t^*)^* = t$ and thus

$$\|t\| = \|t^*\|.$$

We conclude that

$$\|t^*t\| = \|t\|^2.$$

Now in order to show that $\mathscr{L}(X)$ is complete, let $\{t_n : n \in \mathbb{N}\}$ be a Cauchy sequence in $\mathscr{L}(X)$. The space of bounded linear operators $\mathbf{B}(X)$ is complete since X is, and thus there exists a bounded linear operator t, such that $t_n \to t$. It suffices to show that t is adjointable. Since for $n, m \in \mathbb{N}$

$$\|t_n^* - t_m^*\| = \|(t_n - t_m)^*\| = \|t_n - t_m\|$$

the sequence $\{t_n^* : n \in \mathbb{N}\}$ is also Cauchy, hence there exists a bounded linear operator $\bar{t}$ such that $t_n^* \to \bar{t}$. We have

$$\langle tx, y\rangle = \lim_n \langle t_n x, y\rangle = \lim_n \langle x, t_n^* y\rangle = \langle x, \bar{t}y\rangle.$$

This proves that t is adjointable and $t^* = \bar{t}$. □

We will now introduce a very important class of adjointable maps analogous to the finite-rank operators on a Hilbert space.
Let E, F be Hilbert $\mathscr{A}$-modules, $x \in E$ and $y \in F$, we define $\theta_{x,y} : F \to E$ such that

$$\theta_{x,y}(z) = x\langle y, z\rangle \quad \text{for} \quad z \in F.$$

Note that for $z \in F, w \in E$

$$\langle \theta_{x,y}(z), w\rangle = \langle x\langle y, z\rangle, w\rangle = \langle z, y\rangle\langle x, w\rangle = \langle z, y\langle x, w\rangle\rangle = \langle z, \theta_{y,x}(w)\rangle$$

and so $\theta_{x,y} \in \mathscr{L}(F, E)$ and in particular

$$\theta_{x,y}^* = \theta_{y,x}.$$

Proposition 2.4.3. *Let E, F, G be Hilbert $\mathscr{A}$-modules, $u \in F$, $v \in G$, $x \in E$, $y \in F$, $t \in \mathscr{L}(E, G)$ and $s \in \mathscr{L}(E, G)$. Then we have that:*

$$\begin{aligned}
&\theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u\rangle,v} = \theta_{x,v\langle u,y\rangle} \\
&t\theta_{x,y} = \theta_{tx,y} \\
&\theta_{x,y}s = \theta_{x,s^*y}.
\end{aligned}$$

Proof. Let $w \in G$ and $f \in F$, then

$$\theta_{x,y}\theta_{u,v}(w) = \theta_{x,y}(u\langle v, w\rangle) = x\langle y, u\langle v, w\rangle\rangle = x\langle y, u\rangle\langle u, v\rangle = \theta_{x\langle y,u\rangle,v}(w) =$$
$$x(\langle w, v\rangle\langle u, y\rangle)^* = x(\langle w, v\langle u, y\rangle\rangle)^* = x\langle v\langle u, y\rangle, w\rangle = \theta_{x,v\langle u,y\rangle}(w)$$

and also

$$t\theta_{x,y}(f) = t(x\langle y, f\rangle) = tx\langle y, f\rangle = \theta_{tx,y}(f)$$

and

$$\theta_{x,y}(sw) = x\langle y, sw\rangle = x\langle s^*y, w\rangle = \theta_{x,s^*y}(w).$$

□

We denote the closure of the linear span of the set $\{\theta_{x,y} : x \in E,\ y \in F\}$ by $\mathscr{K}(F,E)$ and in the case that $E = F = X$ by $\mathscr{K}(X)$. From the preceding proposition we have that $\mathscr{K}(X)$ is a closed ideal of $\mathscr{L}(X)$.

Proposition 2.4.4. *If $t \in \mathscr{L}(E,F)$ and $x \in E$ where E, F are Hilbert $\mathscr{A}$-modules, then $|tx| \leq \|t\||x|$.*

Proof. Let ρ be a state of $\mathscr{A}$. By repeatedly using the Cauchy-Schwarz inequality for the semi-inner product $\rho(\langle .,. \rangle)$, we obtain

$$\begin{aligned}
&\rho(\langle t^*tx, x\rangle) \leq \rho(\langle t^*tx, t^*tx\rangle)^{1/2}\rho(\langle x, x\rangle)^{1/2} \\
&= \rho(\big\langle (t^*t)^2x, x\big\rangle)^{1/2}\rho(|x|^2)^{1/2} \\
&\leq \rho(\big\langle (t^*t)^2x, (t^*t)^2x\big\rangle)^{1/4}\rho(|x|^2)^{1/2+1/4} \\
&\leq .... \\
&\leq \rho(\big\langle (t^*t)^{2^n}x, x\big\rangle)^{2^{-n}}\rho(|x|^2)^{1/2+1/4+\ldots+2^{-n}} \\
&\leq (\|x\|^2)^{2^{-n}}\|t^*t\|\rho(|x|^2)^{1-2^{-n}}.
\end{aligned}$$

As $n \to \infty$ we obtain $\rho(\langle tx, tx\rangle) \leq \|t\|^2\rho(|x|^2)$ and since this holds for every state we get $|tx|^2 \leq \|t\|^2|x|^2$ and by taking square roots the proof is complete. □

Remark 6. Let $\mathscr{A}$ be a C^*-algebra, consider the Hilbert $\mathscr{A}$-module described in example 2.4.1, we will prove that $\mathscr{K}(\mathscr{A}) \cong \mathscr{A}$.
Let

$$\phi : \text{span}\{\theta_{a,b} : a, b \in \mathscr{A}\} \to \mathscr{A}$$

be the map such that

$$\phi(\theta_{a,b} + \theta_{c,d}) = ab^* + cd^*.$$

Note that ϕ is injective. Indeed,

$$\phi(\theta_{a,b}) = 0 \Rightarrow ab^* = 0$$

and so for every $c \in \mathscr{A}$

$$\theta_{a,b}(c) = ab^*c = 0$$

and thus $\theta_{a,b} \equiv 0$.
We also have that

$$\phi(\theta^*_{a,b}) = \phi(\theta_{b,a}) = ba^* = (ab^*)^* = \phi(\theta_{a,b})^*$$

and

$$\phi(\theta_{a,b}\theta_{c,d}) = \phi(\theta_{ab^*c,d}) = ab^*cd^* = \phi(\theta_{a,b})\phi(\theta_{c,d}).$$

Therefore ϕ is a $*$-homomorphism between C^*-algebras and thus an isometry, so it can be extended to an isometry, which we will still denote by ϕ, from $\mathscr{K}(\mathscr{A})$ to $\mathscr{A}$ and therefore $\phi(\mathscr{K}(\mathscr{A}))$ is a C^*-subalgebra of $\mathscr{A}$ as an image of a $*$-homomorphism between C^*-algebras.
Now if $\{e_i : i \in I\}$ is an approximate identity in $\mathscr{A}$ then for each $a \in \mathscr{A}$ and each $i \in I$

$$\phi(\theta_{a,e_i}) = ae_i \in \phi(\mathscr{K}(\mathscr{A})),$$

and since $ae_i \xrightarrow{i} a$ we obtain that $\phi(\mathscr{K}(\mathscr{A})) = \mathscr{A}$ and ϕ is surjective. So ϕ is a $*$-isomorphism.
In the case that $\mathscr{A}$ is unital we also have that $\mathscr{L}(A) = \mathscr{K}(A)$.
Indeed, let $t \in \mathscr{L}(A)$ and $a \in \mathscr{A}$

$$t(a) = t(\mathbf{1}_{\mathscr{A}}a) = t(\mathbf{1}_{\mathscr{A}})a = \theta_{t\mathbf{1}_{\mathscr{A}},\mathbf{1}_{\mathscr{A}}}(a).$$

2.5 Interior tensor products

In this section we are going to introduce the interior tensor product of Hilbert C^*-modules, which we will need in order to describe the Fock space of a C^*-correspondence in chapter 5. We will need a few more tools in order to do so.

Lemma 2.5.1. *Let X be a Hilbert $\mathscr{A}$-module and let t be a bounded $\mathscr{A}$-linear operator on X. The following are equivalent:*

(i) t is a positive element of $\mathscr{L}(X)$,

(ii) $\langle x, tx\rangle \geq 0$ for all $x \in X$.

Proof. Suppose that $t \geq 0$, then $\langle x, tx\rangle = \left\langle t^{1/2}x, t^{1/2}x\right\rangle \geq 0$.
For the converse implication, if $\langle x, tx\rangle \geq 0$ for all $x \in X$, it follows from polarisation that t is self-adjoint and in particular adjointable. Therefore, there exist positive elements $s, r \in \mathscr{L}(X)$ such that $t = r - s$ and $rs = 0$. For $x \in X$ we have

$$-\left\langle x, s^3x\right\rangle = \langle sx, rsx\rangle - \left\langle sx, s^2x\right\rangle = \langle sx, t(sx)\rangle \geq 0$$

which implies that $-\left\langle x, s^3x\right\rangle \geq 0$. Since $s \geq 0$ we have $s^3 \geq 0$ and thus $\left\langle x, s^3x\right\rangle \geq 0$. So, $\left\langle x, s^3x\right\rangle = 0$ for each $x \in X$ and therefore $s^3 = 0$. Thus $s = 0$ and hence $t = r \geq 0$ and we are done. □

Let X, Y be Hilbert $\mathscr{A}$-modules, then we can identify $\mathscr{K}(X^m, Y^n)$ with the set of $m \times n$ matrices over $\mathscr{K}(X, Y)$. If

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in X^m, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in Y^n,$$

$\theta_{x,y}$ corresponds to the matrix $(\theta_{x_i,y_j})_{ij}$. We also identify $\mathscr{L}(X^m, Y^n)$ with the set of $m \times n$ matrices over $\mathscr{L}(X, Y)$. If p_i is the projection of X^m onto the $1 \leq i \leq m$ coordinate and q_j the projection onto the $1 \leq j \leq n$ coordinate, given $t \in \mathscr{L}(X, Y)$ we associate with t the $m \times n$ matrix with the (i, j)-entry $q_j t p_i^*$ where p_i^* is the inclusion map from X to X^m.

Lemma 2.5.2. *Let X be a Hilbert $\mathscr{A}$-module. If $x_1, ..., x_n \in X$ then $X \geq 0$ in $M_n(\mathscr{A})$ where X is the matrix with (i, j)-entry $\langle x_i, x_j\rangle$. Also, if $t \in \mathscr{L}(X)$ and W is the matrix with (i, j)-entry $\langle tx_i, tx_j\rangle$, then $W \leq \|t\|^2 X$.*

Proof. We identify $M_n(\mathscr{A})$ with $\mathscr{K}(A^n)$ since $\mathscr{A} \cong \mathscr{K}(A)$ and we have that for all $a = (a_1, ..., a_n)$ in $\mathscr{A}^n$,

$$\langle a, Xa\rangle = \sum_{i,j}^{n} a_i^* \langle x_i, x_j\rangle a_j = \left\langle \sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} x_i a_i \right\rangle \geq 0$$

and so $X \geq 0$ by the preceding lemma.
We also have that

$$\begin{aligned} \langle a, Wa\rangle &= \sum_{i,j}^{n} a_i^* \langle tx_i, tx_j\rangle a_j = \left\langle \sum_{i=1}^{n} tx_i a_i, \sum_{j=1}^{n} tx_j a_j \right\rangle \\ &\leq \|t\|^2 \left\langle \sum_{i=1}^{n} x_i a_i, \sum_{j=1}^{n} tx_j a_j \right\rangle = \|t\|^2 \langle a, Xa\rangle, \end{aligned}$$

using the inequality from proposition 2.4.4. □

Lemma 2.5.3. *Let X be a Hilbert $\mathscr{A}$-module, $x \in X$ and $0 < a < 1$. Then there is an element w of X such that $x = w|x|^a$.*

Proof. For any continuous function on the spectrum of $|x|$, we have

$$\|xf(|x|)\| = \|f(|x|)^* \langle x, x\rangle f(|x|)\|^{1/2}$$
$$= \||x|f(|x|)\| = \sup\{|\lambda f(\lambda)|: \lambda \in \mathrm{sp}(|x|)\}$$

For $n \geq 1$ we define

$$g_n(\lambda) = \begin{cases} n^a, & \text{if } \lambda \leq 1/n \\ \lambda^{-a}, & \lambda > 1/n \end{cases}$$

Note that g_n is continuous on $\mathbb{C}$ and therefore using the above norm estimate we obtain that the sequence $(xg_n(|x|))_{n\in\mathbb{N}}$ is Cauchy and so it converges to an element w. Adjoining an identity to $\mathscr{A}$ if necessary and using the fact that $x1_{\mathscr{A}_1} = x$, we have

$$\|xg_n(|x|)|x|^a - x\| = \|x(g_n(|x|)|x|^a - 1_{\mathscr{A}_1})\|$$
$$= \sup\{|\lambda(g_n(\lambda)\lambda^a - 1)|: \lambda \in \mathrm{sp}(|x|)\} \xrightarrow{n\to\infty} 0.$$

Thus, $w|x|^a = x$ and the proof is complete. □

Remark 7. Let X be a Hilbert $\mathscr{A}$-module, then X^n can be regarded as a Hilbert $M_n(\mathscr{A})$-module: If $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X^n$ and $a = (a_{ij})_{ij}$ then we define

$$xa = \left(\sum_{i=1}^{n} x_i a_{i1}, ..., \sum_{i=1}^{n} x_i a_{in}\right)$$

and the inner-product

$$\langle x, y\rangle = (\langle x_i, y_j\rangle)_{ij}.$$

Then, from the final example of chapter 5 in [20] we obtain

$$\mathscr{L}_{M_n(\mathscr{A})}(X^n) \cong \mathscr{L}_{\mathscr{A}}(X^n) \cong M_n(\mathscr{L}(X)) \cong \mathscr{L}(X) \otimes M_n(\mathbb{C})$$

and in particular if $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in X^n$ then $\theta_{x,y} \in \mathscr{K}_{M_n(\mathscr{A})}(X^n)$ is identified with $\sum_{i=1}^n \theta_{x_i,y_i} \otimes I_n$ where I_n is the unit in $M_n(\mathbb{C})$. Therefore, from the nuclearity of $M_n(\mathbb{C})$, the norm of $\mathscr{L}(X) \otimes M_n(\mathbb{C})$ is the spatial norm and so we have

$$\|\theta_{x,y}\| = \left\|\sum_{i=1}^{n} \theta_{x_i,y_i}\right\| \|I_n\| = \left\|\sum_{i=1}^{n} \theta_{x_i,y_i}\right\|.$$

The following is lemma 2.1 in [11].

Lemma 2.5.4. *Let X be a Hilbert $\mathscr{A}$-module. For $x_1, ..., x_n, y_1, ..., y_n \in X$ we have*

$$\left\| \sum_{i=1}^{n} \theta_{x_i, y_i} \right\| = \left\| (\langle x_i, x_j \rangle)_{ij}^{1/2} (\langle y_i, y_j \rangle)_{ij}^{1/2} \right\|_{M_n(\mathscr{A})}.$$

In particular, if $X = \mathscr{A}$ then

$$\left\| \sum_{i=1}^{n} x_i y_i^* \right\| = \left\| (x_i^* x_j)_{ij}^{1/2} (y_i^* y_j)_{ij}^{1/2} \right\|_{M_n(\mathscr{A})}.$$

Proof. Let x, y be elements in X. Then,

$$\begin{aligned} \|\theta_{x,y}\|^2 &= \|\theta_{x,y}^* \theta_{x,y}\| = \|\theta_{y\langle x,x\rangle, y}\| = \left\| \theta_{y\langle x,x\rangle^{1/2}, y\langle x,x\rangle^{1/2}} \right\| \\ &= \left\| \left\langle y \langle x,x\rangle^{1/2}, y \langle x,x\rangle^{1/2} \right\rangle \right\| = \left\| \langle x,x\rangle^{1/2} \langle y,y\rangle^{1/2} \right\|^2. \end{aligned}$$

If $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n), z \in X^n$ we set

$$\theta_{x,y}(z) = x \langle y, z \rangle_{M_n(\mathscr{A})}$$

and we have

$$\left\| \sum_{i=1}^{n} \theta_{x_i, y_i} \right\| = \|\theta_{x,y}\| = \left\| \langle x, x \rangle_{M_n(\mathscr{A})}^{1/2} \langle y, y \rangle_{M_n(\mathscr{A})}^{1/2} \right\| = \left\| (\langle x_i, x_j \rangle)_{ij}^{1/2} (\langle y_i, y_j \rangle)_{ij}^{1/2} \right\|_{M_n(\mathscr{A})}.$$

□

We are now in position to define the interior tensor product of Hilbert C^*-modules. Let X be a Hilbert $\mathscr{A}$-module, Y a Hilbert $\mathscr{B}$-module and $\phi : \mathscr{A} \to \mathscr{L}(Y)$ a $*$-homomorphism. We form the vector space tensor product $X \otimes_{\text{alg}} Y$. Let N be the subspace of $X \otimes_{\text{alg}} Y$ generated by elements of the form

$$\xi a \otimes \eta - \xi \otimes \phi(a)\eta \quad \xi \in X, \eta \in Y, a \in \mathscr{A}$$

and form the quotient $(X \otimes_{\text{alg}} Y)/N$.
For $\xi, \xi_1, \xi_2 \in X, \eta, \eta_1, \eta_2 \in Y$ and $b \in \mathscr{B}$ we set

$$(\xi \otimes \eta)b := \xi \otimes (\eta b)$$

and for simple tensors

$$\langle \xi_1 \otimes \eta_1 \, \xi_2 \otimes \eta_2 \rangle := \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle) \eta_2 \rangle.$$

The above formula extends by linearity to a sesquilinear form and we will prove that

$$\langle .,. \rangle : (X \otimes_{\text{alg}} Y)/N \to \mathscr{B}$$

is a $\mathscr{B}$-valued inner-product. First of all, if $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes_{\text{alg}} Y$ we have that

$$\langle z, z \rangle = \left\langle \sum_{i=1}^{n} x_i \otimes y_i, \sum_{j=1}^{n} x_i \otimes y_j \right\rangle = \sum_{i,j}^{n} \langle y_i, \phi(\langle x_i, x_j \rangle) y_j \rangle$$
$$= \left\langle y, \phi^{(n)}(X) y \right\rangle_{Y^n},$$

where $y = (y_1, ..., y_n) \in Y^n$ and X is the element in $M_n(\mathscr{A})$ with matrix entries $\langle x_i, x_j \rangle$ and $\langle .,. \rangle_{Y^n}$ is the $\mathscr{B}$-valued inner-product defined in example 2.4.4. By lemma 2.5.2 we have that $X \geq 0$ and using the complete positivity of the $*$-homomorphism ϕ, the identification of $M_n(\mathscr{L}(Y))$ with $\mathscr{L}(Y^n)$ and lemma 2.5.1, we conclude that $\langle z, z \rangle \geq 0$. It remains to show that $N = \{z \in X \otimes_{\text{alg}} Y : \langle z, z \rangle = 0\}$.
Suppose that $z = xa \otimes y - x \otimes \phi(a) y$ for $x \in X, \in Y$ and $a \in \mathscr{A}$, then

$$\langle z, z \rangle = \langle xa \otimes y - x \otimes \phi(a) y, xa \otimes y - x \otimes \phi(a) y \rangle$$
$$= \langle xa \otimes y, xa \otimes y \rangle - \langle xa \otimes y, x \otimes \phi(a) y \rangle - \langle x \otimes \phi(a) y, xa \otimes y \rangle + \langle x \otimes \phi(a) y, x \otimes \phi(a) y \rangle$$
$$= \langle y, \phi(\langle xa, xa \rangle) y \rangle - \langle y, \phi(\langle xa, x \rangle) \phi(a) y \rangle - \langle \phi(a) y, \phi(\langle x, xa \rangle) y \rangle + \langle \phi(a) y, \phi(\langle x, x \rangle) \phi(a) y \rangle$$
$$= \langle y, \phi(a^* \langle x, x \rangle a) y \rangle - \langle y, \phi(a^* \langle x, x \rangle a) y \rangle - \langle \phi(a) y, \phi(\langle x, x \rangle a) y \rangle + \langle \phi(a) y, \phi(\langle x, x \rangle a) y \rangle = 0.$$

For the reverse inclusion, suppose that $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes_{\text{alg}} Y$ is such that $\langle z, z \rangle = 0$. From our calculations above and using the same notation we have that

$$\left\langle y, \phi^{(n)}(X) y \right\rangle_{Y^n} = 0.$$

We set $T = \phi^{(n)}(X)$, then $T \geq 0$ as an element of $M_n(\mathscr{L}(Y)) \cong \mathscr{L}(Y^n)$ and we also have that

$$\left\langle T^{1/2} y, T^{1/2} y \right\rangle_{Y^n} = \langle y, Ty \rangle_{Y^n} = 0.$$

Since

$$\left\langle T^{1/4} y, T^{1/4} y \right\rangle_{Y^n} = \left\langle y, T^{1/2} y \right\rangle_{Y^n} = 0,$$

it follows that $T^{1/4} y = 0$. If we think of X^n as a Hilbert $M_n(\mathscr{A})$-module and set $x = (x_1, ..., x_n)$ we have that $|x|_{X^n} = X^{1/2}$. Using lemma 2.5.3, we obtain an element $w = (w_1, ..., w_n) \in X$ such that $wX^{1/2} = x$. If $X^{1/4} = (c_{ij})_{ij}$ then since $T^{1/4} = \phi^{(n)}(X^{1/4})$, we have that $T^{1/4} = (\phi(c_{ij}))_{ij}$. Therefore, for each $1 \leq j \leq n$ we have $x_j = \sum_{i=1}^{n} w_i c_{ij}$ and $\sum_{i=1}^{n} \phi(c_{ij}) y_j = 0$. It follows that

$$z = \sum_{j=1}^{n} x_j \otimes y_j = \sum_{i,j}^{n} (w_i c_{ij} \otimes y_j - w_i \otimes \phi(c_{ij}) y_j).$$

We denote by $X \otimes_\phi Y$ the completion of $X \otimes_{alg} Y/N$ with respect to the norm induced by the $\mathscr{B}$-valued inner-product described above and call it the interior tensor product of X and Y. We will often omit the index ϕ unless it is necessary and we will also denote by $x \otimes y$ the element $x \otimes y + N \in X \otimes_\phi Y$.
If X, Y and ϕ are as above and $t \in \mathscr{L}(X)$, then we define a map on simple tensors by

$$x \otimes y \to tx \otimes y.$$

This map extends to a linear map, denoted by $t \otimes I_Y$, on $X \otimes_{\text{alg}} Y$.
For $\sum_{i=1}^n x_i \otimes y_i \in X \otimes_{\text{alg}} Y$ we have

$$\begin{aligned}
\left\|\sum_{i=1}^n tx_i \otimes y_i\right\|^2 &= \left\|\left\langle \sum_{i=1}^n tx_i \otimes y_i, \sum_{j=1}^n tx_j \otimes y_j \right\rangle\right\| \\
&= \left\|\sum_{i,j=1}^n \langle y_i, \phi(\langle tx_i, tx_j\rangle) y_j\rangle\right\| \leq \|t\|^2 \left\|\sum_{i,j=1}^n \langle y_i, \phi(\langle x_i, x_j\rangle) y_j\rangle\right\| \\
&= \|t\|^2 \left\|\sum_{i=1}^n x_i \otimes y_i\right\|^2,
\end{aligned}$$

where we have used lemma 2.5.2 and the complete positivity of ϕ. Therefore, $t \otimes I_Y$ is a well-defined, bounded map which extends by continuity to $X \otimes_\phi Y$. We also denote the extension by $t \otimes I_Y$.
We now prove that $t \otimes I_Y \in \mathscr{L}(X \otimes_\phi Y)$. For $\xi_1, \xi_2 \in X$ and $\eta_1, \eta_2 \in Y$ we have that

$$\begin{aligned}
&\langle (t \otimes I_Y)(\xi_1 \otimes \eta_1), \xi_2 \otimes \eta_2\rangle = \langle t(\xi_1) \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \eta_1, \phi(\langle t(\xi_1), \xi_2\rangle)\eta_2\rangle \\
&= \langle \eta_1, \phi(\langle \xi_1, t^*(\xi_2)\rangle)\eta_2\rangle = \langle \xi_1 \otimes \eta_1, t^*(\xi_2) \otimes \eta_2\rangle = \langle \xi_1 \otimes \eta_1, (t^* \otimes I_Y)(\xi_2 \otimes \eta_2)\rangle,
\end{aligned}$$

and since the linear span of simple tensors is dense in $X \otimes_\phi Y$, we are done. Thus, $(t \otimes I_Y)^* = t^* \otimes I_Y$. It is evident now that the map

$$t \to t \otimes I_Y$$

is a unital $*$-homomorphism from $\mathscr{L}(X)$ into $\mathscr{L}(X \otimes_\phi Y)$.

Chapter 3

Crossed products

In this chapter we are going to introduce the notion of the crossed product of a C^*-algebra by a discrete group. The operations of the crossed product encodes, in a way, information about the action of the discrete group. In particular, in the last section of this chapter we will give an elegant result that characterizes topological properties of a topological dynamical system via algebraic properties of the crossed product that arises from this action. In the first section we follow mostly [4], in section 2 we follow both [4] and [13] and the last section follows the notes of E.G. Katsoulis of this year's seminar of functional analysis and operator algebras.

3.1 Crossed products by discrete groups

Let $\mathscr{A}$ be a Banach space and G be a set. We define the Banach space $\ell^p(G, \mathscr{A})$ to be the $\mathscr{A}$-valued functions on G such that $\sum_{s\in G}\|f(g)\|^p < \infty$ with the norm $\|f\|_p = (\sum_{s\in G}\|f(g)\|^p)^{1/p}$ and pointwise addition. In particular, if $\mathscr{A}$ is a Hilbert space, for $p = 2$, the Banach space $\ell^2(G, \mathscr{A})$ is a Hilbert space with the inner-product defined by

$$\langle f, g\rangle = \sum_{s\in G}\langle f(s), g(s)\rangle_{\mathscr{A}}.$$

Definition 3.1.1. Let G be a discrete group and $\mathscr{H}$ a Hilbert space. A unitary representation of G is a homomorphism of G into the unitary group of $\mathscr{H}$, which we denote by $U(\mathscr{H})$.

Example 3.1.1. If G is a discrete group, we define the left regular representation of G on $\ell^2(G, \mathscr{H})$ by

$$\Lambda(s)g(t) = g_s(t) = g(s^{-1}t) \quad \text{where } g \in \ell^2(G, \mathscr{H}). \tag{3.1.1}$$

Note that $\Lambda(s)$ is isometric and invertible (onto) and therefore it is a unitary operator of $\mathbf{B}(\ell^2(G, \mathscr{H}))$.

We denote by $\mathrm{Aut}(\mathscr{A})$ the group of $*$-automorphisms of $\mathscr{A}$, where the operation is convolution.

Definition 3.1.2. Let $\mathscr{A}$ be a C^*-algebra, G a discrete group and α a homomorphism of the group G into Aut($\mathscr{A}$). We call $(\mathscr{A}, G, \alpha)$ a C^*-dynamical system.

We will always denote by α_s the automorphism $\alpha(s)$ for $s \in G$.

Definition 3.1.3. Let $(\mathscr{A}, G, \alpha)$ be a C^*-dynamical system, a right covariant representation of the given dynamical system is a pair (π, U), where π is a representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and the map $s \to U_s$ is a unitary representation of G on the same Hilbert that satisfy the relation

$$U_s \pi(b) U_s^* = \pi(\alpha_s(b)) \quad \forall b \in \mathscr{A}, s \in G.$$

Remark 8. Suppose that $\mathscr{A}$ is unital. Without loss of generality we can assume that π is non-degenerate, whenever (π, V) a covariant representation of $(\mathscr{A}, G, \alpha)$.

Indeed, suppose that π is degenerate and set $K = \pi(\mathbf{1}_{\mathscr{A}})\mathscr{H}$. Note that $\pi(\mathbf{1}_{\mathscr{A}})$ is an orthogonal projection of $\mathscr{H}$ and hence K is a Hilbert space. For $y \in \mathscr{H}$, $s \in G$ and $x = \pi(\mathbf{1}_{\mathscr{A}})y \in K$ we have that

$$U_s x = U_s \pi(\mathbf{1}_{\mathscr{A}})y = \pi(\alpha_s(\mathbf{1}_{\mathscr{A}}))U_s y = \pi(\mathbf{1}_{\mathscr{A}})U_s y \in K,$$

therefore $U_s(K) \subseteq K$ and by setting $s = s^{-1}$ we obtain that $U_s^*(K) \subseteq K$.
Hence, $s \to U_s|_K$ is unitary representation of G which we denote by $U|_K$ and $(\pi|_K, U|_K)$ is a covariant representation of $(\mathscr{A}, G, \alpha)$ and $\pi|_K$ is non-degenerate.

For a discrete group G, consider the vector space tensor product $\mathscr{A} \otimes C_c(G)$, where $C_c(G)$ is the set of continuous functions from G to $\mathbb{C}$ with compact (finite) support, which is the vector space with elements of the form

$$f = \sum_{t \in G} b_t \otimes \delta_t,$$

where $b_t \in \mathscr{A}$, δ_t is the Kronecker delta function at t and the set $\{t \in G : b_t \neq 0\}$ is finite. We define a norm $\|.\|_1$ on this space by

$$\|f\|_1 = \sum_{t \in G} \|b_t\|$$

and we also define a multiplication and an involution that makes $\mathscr{A} \otimes C_c(G)$ a normed $*$-algebra which we will denote by $\mathscr{A} \otimes_\alpha C_c(G)$.
If $\alpha \otimes \delta_t$ and $b \otimes \delta_s$ are simple tensors then

$$(a \otimes \delta_t)(b \otimes \delta_s) = a\alpha_t(b) \otimes \delta_{ts}$$

and we extend linearly. We also define the adjoint by

$$(b \otimes \delta_s)^* = \alpha_s^{-1}(b^*) \otimes \delta_{s^{-1}}.$$

Let $f = \sum_{t\in G} a_t \otimes \delta_t$ and $g = \sum_{s\in G} b_s \otimes \delta_s$ then

$$\|fg\|_1 = \left\|\sum_{t\in G}\sum_{s\in G} a_t\alpha_t(b_{t^{-1}s}) \otimes \delta_s\right\|_1 \leq \sum_{t\in G}\sum_{s\in G}\|a_t\alpha_t(b_{t^{-1}s})\|$$
$$\leq \sum_{t\in G}\sum_{s\in G}\|a_t\|\|b_{t^{-1}s}\| = \sum_{t\in G}\|a_t\|\sum_{s\in G}\|b_{t^{-1}s}\| = \sum_{t\in G}\|a_t\|\sum_{u\in G}\|b_u\| = \|f\|_1\|g\|_1$$

and also

$$\|f^*\|_1 = \left\|\sum_{t\in G}\alpha_t(a^*_{t^{-1}}) \otimes \delta_t\right\|_1 = \sum_{t\in G}\|\alpha_t(a^*_{t^{-1}})\| = \sum_{t\in G}\|a^*_{t^{-1}}\| = \sum_{t\in G}\|a_t\| = \|f\|_1$$

Remark 9. We should note that from our calculations above the completion of the normed $*$-algebra $\mathscr{A} \otimes_\alpha C_c(G)$ with respect to $\|.\|_1$ is a Banach $*$-algebra which we will denote by $\ell^1(\mathscr{A}, G, \alpha)_r$.

Proposition 3.1.1. *Let (π, U) be a covariant representation of $(\mathscr{A}, G, \alpha)$, this covariant representation yields a $\|.\|_1$-contractive $*$-representation of $\mathscr{A} \otimes_\alpha C_c(G)$, which we will denote by $\pi \times U$, such that for $f = \sum_{t\in G} b_t \otimes \delta_t$,*

$$(\pi \times U)(f) = \sum_{t\in G}\pi(b_t)U_t. \tag{3.1.2}$$

Conversely, a $\|.\|_1$-contractive non-degenerate $*$-representation of $\mathscr{A} \otimes_\alpha C_c(G)$ yields a covariant representation of $(\mathscr{A}, G, \alpha)$.

Proof.

$$(\pi \times U)(f)^* = (\sum_{t\in G}\pi(b_t)U_t)^* = \sum_{t\in G}U_t^*\pi(b_t)^* = \sum_{t\in G}U_{t^{-1}}\pi(b_t^*)U_tU_{t^{-1}} =$$
$$= \sum_{s\in G}U_s\pi(b^*_{s^{-1}})U_{s^{-1}}U_s = \sum_{s\in G}\pi(\alpha_s(b^*_{s^{-1}}))U_s = (\pi \times U)(f^*)$$

Note also that for the product we have that

$$(\pi \times U)(f)(\pi \times U)(g) = \sum_{t\in G}\sum_{u\in G}\pi(b_t)U_t\pi(c_u)U_u = \sum_{t\in G}\sum_{u\in G}\pi(b_t)(U_t\pi(c_u)U_t^*)U_tU_u =$$
$$= \sum_{t\in G}\sum_{u\in G}\pi(b_t)\pi(\alpha_t(c_u))U_{tu} = \sum_{s\in G}(\sum_{t\in G}\pi(b_t\alpha_t(c_{t^{-1}s}))U_s = (\pi \times U)(fg)$$

and

$$\|(\pi \times U)(f)\| = \left\|\sum_{t\in G}\pi(b_t)U_t\right\| \leq \sum_{t\in G}\|\pi(b_t)\|\|U_t\| \leq \sum_{t\in G}\|b_t\| = \|f\|_1$$

because π is a $*$-homomorphism between C^*-algebras and so it is norm-decreasing.
Now let σ be a $\|.\|_1$-contractive $*$-representation of $\mathscr{A} \otimes_\alpha C_c(G)$ on a Hilbert space $\mathscr{H}$. At

first, we suppose that $\mathscr{A}$ is unital. Then by defining

$$\pi(b) = \sigma(b \otimes \delta_e) \quad \text{and} \quad U_s = \sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_s), \quad \forall b \in \mathscr{A}, \forall s \in G,$$

we can easily see that π is a $*$-representation of $\mathscr{A}$. For every $s \in G$, note that

$$\sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_s)\sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_{s^{-1}}) = \sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_e) = \mathbf{1}_{\mathbf{B}(\mathscr{H})}$$

and therefore $s \to U_s$ is unitary representation of G. We have

$$\begin{aligned} &U_s\pi(b)U_S^* = \sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_s)\sigma(b \otimes \delta_e)\sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_s)^* \\ &= \sigma((\mathbf{1}_{\mathscr{A}} \otimes \delta_s)\,(b \otimes \delta_e)\,(\mathbf{1}_{\mathscr{A}} \otimes \delta_{s^{-1}})) = \sigma(\alpha_s(b) \otimes \delta_e) = \pi(\alpha_s(b)) \end{aligned}$$

and so the covariance relation holds and for $f = \sum_{t\in G} b_t \otimes \delta_t$

$$\begin{aligned} (\pi \times U)(f) &= \sum_{t\in G} \pi(b_t)U_t = \sum_{t\in G} \sigma(b_t \otimes \delta_e)\sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_t) \\ &= \sum_{t\in G} \sigma(b_t \otimes \delta_t) = \sigma\left(\sum_{t\in G} b_t \otimes \delta_t\right) = \sigma(f). \end{aligned}$$

If $\mathscr{A}$ is non-unital, then let $\{e_\lambda : \lambda \in \Lambda\}$ be an approximate unit for $\mathscr{A}$ and define

$$U_s h = \lim_\lambda \sigma(e_\lambda \otimes \delta_s)h, \quad h \in \mathscr{H}.$$

We prove that for each $s \in G$ this limit actually exists.

Note that for $s \in G$ and $\xi \in \mathscr{H}$

$$\sigma(e_\lambda \otimes \delta_s)(\sigma(a \otimes \delta_t)\xi) = \sigma(e_\lambda\alpha_s(a) \otimes \delta_{st})\xi \xrightarrow{\lambda} \sigma(\alpha_s(a) \otimes \delta_{st})\xi$$

and that $\{\sigma(e_\lambda \otimes \delta_s) : \lambda\}$ is uniformly bounded, since for each λ we have

$$\|\sigma(e_\lambda \otimes \delta_s)\| \leq \|\sigma\|\|e_\lambda\|\|\delta_s\| \leq \|\sigma\|.$$

Therefore, since $\{\sigma(a \otimes \delta_t)\xi : a \in \mathscr{A}, t \in G, \xi \in \mathscr{H}\}$ is dense in $\mathscr{H}$ we obtain that for every $h \in \mathscr{H}$, $\lim_\lambda \sigma(e_\lambda \otimes \delta_s)h$ exists.

We prove that for each $s \in G$ the operator U_s is unitary.
Firstly, we prove that for $s \in G$ the operator U_s is invertible and $U_{s^{-1}}$ is its inverse.
Indeed, observe that $\{(e_\lambda \otimes \delta_e) : \lambda \in \Lambda\}$ is an approximate unit for $\mathscr{A} \otimes_\alpha C_c(G)$ since for $a \otimes \delta_t \in \mathscr{A} \otimes_\alpha C_c(G)$ we have

$$\|(e_\lambda \otimes \delta_e)(a \otimes \delta_t) - a \otimes \delta_t\|_1 = \|(e_\lambda a \otimes \delta_t) - a \otimes \delta_t\|_1 = \|e_\lambda a - a\|.$$

For $\eta_1, \eta_2 \in \mathscr{H}$ and $s \in G$ we have

$$\begin{aligned}\langle U_s U_{s^{-1}} \eta_1, \eta_2 \rangle &= \left\langle \lim_\lambda \lim_\mu \sigma(e_\lambda \otimes \delta_s)\sigma(e_\mu \otimes \delta_{s^{-1}})\eta_1, \eta_2 \right\rangle \\ &= \left\langle \lim_\lambda \lim_\mu \sigma(e_\lambda \alpha_s(e_\mu) \otimes \delta_e)\eta_1, \eta_2 \right\rangle \\ &= \left\langle \lim_\lambda \lim_\mu \sigma(\alpha_s(\alpha_{s^{-1}}(e_\lambda)e_\mu) \otimes \delta_e)\eta_1, \eta_2 \right\rangle \\ &= \left\langle \lim_\lambda \sigma(e_\lambda \otimes \delta_e)\eta_1, \eta_2 \right\rangle = \langle \eta_1, \eta_2 \rangle ,\end{aligned}$$

where we used the fact that σ is continuous and non-degenerate and that

$$\begin{aligned}&\|\alpha_s(\alpha_{s^{-1}}(e_\lambda)e_\mu) \otimes \delta_e - e_\lambda \otimes \delta_e\|_1 = \|\alpha_s(\alpha_{s^{-1}}(e_\lambda)e_\mu - \alpha_{s^{-1}}(e_\lambda)) \otimes \delta_e\|_1 \\ &= \|\alpha_s(\alpha_{s^{-1}}(e_\lambda)e_\mu - \alpha_{s^{-1}}(e_\lambda))\| \leq \|\alpha_{s^{-1}}(e_\lambda)e_\mu - \alpha_{s^{-1}}(e_\lambda)\| \xrightarrow{\mu} 0.\end{aligned}$$

Therefore, U_s is invertible.

For each $s \in G$ the operator U_s is isometric.
Indeed, for $s \in G$ and $h \in \mathscr{H}$ we have that

$$\|U_s h\| \leq \|\sigma\| \|h\| \leq \|h\|$$

and also

$$\|h\| = \|U_{s^{-1}} U_s h\| \leq \|U_s h\| \leq \|h\|.$$

Hence, U_s is onto and isometric, therefore unitary.
We prove the covariance relation: For $\eta_1, \eta_2 \in \mathscr{H}$ we have

$$\begin{aligned}\langle U_s \pi(b) U_s^* \eta_1, \eta_2 \rangle &= \left\langle U_s \lim_{\lambda'} \sigma(b \otimes \delta_e)\sigma(e_{\lambda'} \otimes \delta_{s^{-1}})\eta_1, \eta_2 \right\rangle \\ &= \left\langle U_s \sigma(\lim_{\lambda'} b e_\lambda \otimes \delta_{s^{-1}})\eta_1, \eta_2 \right\rangle \\ &= \langle U_s \sigma(b \otimes \delta_{s^{-1}})\eta_1, \eta_2 \rangle \\ &= \left\langle \lim_\lambda \sigma((e_\lambda \otimes \delta_s)(b \otimes \delta_{s^{-1}})\eta_1, \eta_2 \right\rangle \\ &= \left\langle \sigma(\lim_\lambda e_\lambda \alpha_s(b) \otimes \delta_e)\eta_1, \eta_2 \right\rangle \\ &= \langle \sigma(\alpha_s(b) \otimes \delta_e)\eta_1, \eta_2 \rangle \\ &= \langle \pi(\alpha_s(b))\eta_1, \eta_2 \rangle .\end{aligned}$$

Note that we used the fact that σ is continuous and that if $\{e_\lambda : \lambda \in \Lambda\}$ is an approximate unit for $\mathscr{A}$, then for all $s \in G$ and all $b \in \mathscr{A}$ we have that

$$\lim_\lambda (e_\lambda b \otimes \delta_s) = \lim_\lambda (be_\lambda \otimes \delta_s) = b \otimes \delta_s.$$

Indeed, we can easily see that:

$$\|be_\lambda \otimes \delta_s - b \otimes \delta_s\|_1 = \|(be_\lambda - b) \otimes \delta_s\|_1 = \|be_\lambda - b\|_{\mathscr{A}} \xrightarrow{\lambda} 0.$$

It remains to prove that $\sigma = \pi \times U$. Suppose that $f = \sum_{t \in G} b_t \otimes \delta_t$ is an element of $\mathscr{A} \otimes_\alpha C_c(G)$ and $\eta_1, \eta_2 \in \mathscr{H}$, then

$$\begin{aligned}
\langle (\pi \times U)(f)\eta_1, \eta_2 \rangle &= \left\langle \sum_{t \in G} \pi(b_t) U_t \eta_1, \eta_2 \right\rangle = \left\langle \sum_{t \in G} \lim_\lambda \sigma((b_t \otimes \delta_e)(e_\lambda \otimes \delta_t))\eta_1, \eta_2 \right\rangle \\
&= \left\langle \sum_{t \in G} \sigma\left(\lim_\lambda b_t e_\lambda \otimes \delta_t \right) \eta_1, \eta_2 \right\rangle = \left\langle \sum_{t \in G} \sigma(b_t \otimes \delta_t)\eta_1, \eta_2 \right\rangle \\
&= \left\langle \sigma\left(\sum_{t \in G} b_t \otimes \delta_t \right) \eta_1, \eta_2 \right\rangle = \langle \sigma(f)\eta_1, \eta_2 \rangle .
\end{aligned}$$

□

We define a norm on $\mathscr{A} \otimes_\alpha C_c(G)$ by

$$\|f\| = \sup_\sigma \|\sigma(f)\| \tag{3.1.3}$$

where σ runs over all continuous $*$-representations of $\mathscr{A} \otimes_\alpha C_c(G)$. This is well-defined because if σ is a continuous representation of $\mathscr{A} \otimes_\alpha C_c(G)$, then it extends to a $*$-homomorphism between the Banach $*$-algebra $\ell^1(G, \mathscr{A}, \alpha)_r$ and a C^*-algebra and thus using proposition 2.1.3 it is norm-decreasing and

$$\|\sigma(f)\| \leq \|f\|_1.$$

We should also prove that this family of representations is not empty and that there is a faithful representation of $\mathscr{A} \otimes C_c(G)$ in order to actually obtain a norm and not a semi-norm.
First of all, we can easily see that:

$$\|f\, f^*\| = \sup_\sigma \|\sigma(f)\sigma(f)^*\| = \sup_\sigma \|\sigma(f)\|^2 = \|f\|^2$$

and so it satisfies C^*-property.

Remark 10. From theorem 2.1.1 there exists a faithful representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$, we define a covariant representation $(\tilde{\pi}, \Lambda)$ where Λ is the left regular representation of G by

$$(\tilde{\pi}(b)x)(s) = \pi(\alpha_s^{-1}(b))(x(s)), \tag{3.1.4}$$

$$(\Lambda_t x)(s) = x(t^{-1}s), \quad \forall b \in \mathscr{A}, \forall x \in \ell^2(G, \mathscr{H}). \tag{3.1.5}$$

Note that

$$(\tilde{\pi}(b\,c)x)(s) = \pi(\alpha_s^{-1}(b\,c)(x(s)) = \pi(\alpha_s^{-1}(b))\pi(\alpha_s^{-1}(c))(x(s)) = (\tilde{\pi}(b)\tilde{\pi}(c)x)(s)$$

and

$$\tilde{\pi}(b^*)x(s) = \pi(\alpha_s^{-1}(b^*))(x(s)) = \pi(\alpha_s^{-1}(b))^*(x(s)) = \tilde{\pi}(b)^*x(s)$$

and also that Λ_t is invertible and isometric and therefore unitary.
Finally,

$$\Lambda_t\tilde{\pi}(b)\Lambda_t^* x(s) = \tilde{\pi}(b)\Lambda_t^* x(t^{-1}s) = \pi(\alpha_{t^{-1}s}^{-1}(b)(\Lambda_{t^{-1}}x(t^{-1}s)) =$$
$$= \pi(\alpha_s^{-1}\alpha_t(b))(x(s)) = \tilde{\pi}(\alpha_t(b))x(s).$$

Thus $(\tilde{\pi}, \Lambda)$ yields a continuous representation $\tilde{\pi} \times \Lambda$ of $\mathscr{A} \otimes_\alpha C_c(G)$.
In order to see that $\tilde{\pi}\times\Lambda$ is faithful, pick $x, y \in \mathscr{H}$ and $t \in G$ and suppose that $(\tilde{\pi}\times\Lambda)(f) = 0$ for some $f = \sum_{s\in G} b_s \otimes \delta_s \in \mathscr{A} \otimes_\alpha G$. For each $t \in G$ denote by x_1, x_t the elements of $\ell^2(G, \mathscr{H})$ such that

$$x_1(s) = \begin{cases} x, & \text{if } s = e \\ 0, & \text{if } s \neq e \end{cases}$$

and

$$x_t(s) = \begin{cases} y, & \text{if } s = t \\ 0, & \text{if } s \neq t \end{cases}.$$

Then,

$$0 = \langle(\tilde{\pi} \times \Lambda)(f)x_1, x_t\rangle_{\ell^2(G,\mathscr{H})} = \sum_{s\in G}\langle\tilde{\pi}(b_s)\Lambda_s(x_1), x_t\rangle_{\ell^2(G,\mathscr{H})} =$$
$$= \sum_{s\in G}\sum_{k\in G}\langle(\tilde{\pi}(b_s)\Lambda_s)(x_1)(k), x_t(k)\rangle_{\mathscr{H}} = \sum_{s\in G}\langle(\tilde{\pi}(b_s)\Lambda_s)(x_1)(t), y\rangle_{\mathscr{H}} =$$
$$= \sum_{s\in G}\langle\tilde{\pi}(b_s)(x_1)(s^{-1}t), y\rangle_{\mathscr{H}} = \sum_{s\in G}\langle\pi(\alpha_{t^{-1}s}(b_s))(x_1(s^{-1}t), y\rangle_{\mathscr{H}} =$$
$$= \langle\pi(b_t)x, y\rangle_{\mathscr{H}}$$

Thus we get that $\pi(b_t) = 0$ and therefore $b_t = 0$ since π is faithful, for every $t \in G$, and finally $f = 0$ which implies that $\tilde{\pi} \times \Lambda$ is faithful.

Definition 3.1.4. Let G be a discrete group and $(\mathscr{A}, G, \alpha)$ a C^*-dynamical system, the crossed product $\mathscr{A} \times_\alpha G$ is the completion of $\mathscr{A} \otimes_\alpha C_c(G)$ with the norm defined in 3.1.3.

We should make a few observations about the crossed product.
Consider

$$\alpha \to \alpha \otimes \delta_e$$

where $\alpha \in \mathscr{A}$ and e is the unit of G. Then this map is an injective $*$-homomorphism between C^*-algebras and hence we are embedding $\mathscr{A}$ into $\mathscr{A} \times_\alpha G$ isometrically. If $\mathscr{A}$ is unital we can also see that $\mathscr{A} \times_\alpha G$ contains a unitary subgroup isomorphic to G by the isomorphism

$$s \to 1_{\mathscr{A}} \otimes \delta_s.$$

Finally, the crossed product enjoys the following universal property:
Suppose that (π, U) is a covariant representation of $(\mathscr{A}, G, \alpha)$. We can obtain a $*$-homomorphism σ of $\mathscr{A} \times_\alpha G$ into $C^*(\pi(\mathscr{A}), U(G))$

$$\sigma(f) = \sum_{s\in G} \pi(b_s)U_s \quad \text{for } f = \sum_{s\in G} b_s \otimes \delta_s \in \mathscr{A} \otimes C_c(G).$$

It is immediate that σ is a continuous $*$-homomorphism with respect to the norm defined in 3.1.3 and so it can be extended to a $*$-homomorphism of $\mathscr{A} \times_\alpha G$ into $C^*(\pi(\mathscr{A}), U(G))$. If $\mathscr{A}$ is unital σ is a $*$-epimorphism, since for $s \in G$

$$\sigma(\mathbf{1}_{\mathscr{A}} \otimes \delta_s) = U_s.$$

Definition 3.1.5. Let G be a discrete group and $(\mathscr{A}, G, \alpha)$ a C^*-dynamical system, the reduced crossed product $\mathscr{A} \times_{\alpha r} G$ is the C^*-algebra generated by $\mathscr{A} \otimes C_c(G)$ with the norm:

$$\|f\| = \|(\tilde{\pi} \times \Lambda)(f)\|$$

where $\tilde{\pi} \times \Lambda$ is the representation of $\mathscr{A} \otimes C_c(G)$ described in remark 10.

3.2 Crossed products by $\mathbb{Z}$

In the case where $G = \mathbb{Z}$, for applications to the semi-crossed product $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ we are going to need an isomorphic version of $\mathscr{A} \times_\alpha \mathbb{Z}$, which we call the left crossed product.

Remark 11. In the case that the discrete group G is $\mathbb{Z}$, if

$$n \to U_n$$

is a unitary representation on some Hilbert space and

$$\alpha : \mathbb{Z} \to \text{Aut}(\mathscr{A})$$

is a group homomorphism, then for each $n \geq 1$ we have that

$$U_n = U^n$$

where $U = U_1$ and also

$$\alpha_n(x) = \alpha^n(x), \quad \forall n \in \mathbb{Z}, \forall x \in \mathscr{A}$$

where $\alpha = \alpha_1$.
Conversely, if $\alpha : \mathscr{A} \to \mathscr{A}$ is a $*$-automorphism then it induces a C^*-dynamical system which we will denote by $(\mathscr{A}, \mathbb{Z}, \alpha)$, where

$$n \to a^n$$

is the homomorphism from $\mathbb{Z}$ into Aut$(\mathscr{A})$.

We define $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$, where $\mathscr{A}$ is a C^*-algebra and α a $*$-automorphism of $\mathscr{A}$, to be the Banach $*$-algebra consisting of elements of the form

$$\sum_{n \in \mathbb{Z}} \delta_n \otimes a_n,$$

where $a_n \in \mathscr{A}$ and δ_n is the Dirac function on n, such that

$$\left\| \sum_{n \in \mathbb{Z}} \delta_n \otimes a_n \right\|_1 = \sum_{n \in \mathbb{Z}} \|a_n\| < \infty.$$

The product is given by the rule

$$(\delta_n \otimes a)(\delta_m \otimes b) = \delta_{n+m} \otimes \alpha^m(a)b$$

and the involution

$$(\delta_n \otimes a)^* = \delta_{-n} \otimes \alpha^{-n}(a)$$

We are making the exact same observations as we did above remark 9, to show that for $f, g \in \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$

$$\|fg\|_1 \leq \|f\|_1 \|g\|_1$$

and

$$\|f^*\|_1 = \|f\|_1$$

and thus $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ is indeed a Banach $*$-algebra.

Definition 3.2.1. Let $\mathscr{A}$ be a C^*-algebra and $\alpha : \mathscr{A} \to \mathscr{A}$ a $*$-automorphism, let π be a $*$-representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and U a unitary in $\mathbf{B}(\mathscr{H})$. We say that the pair (π, U) is a left covariant representation of the C^*-dynamical system $(\mathscr{A}, \mathbb{Z}, \alpha)$ if it satisfies the relation

$$U^n \pi(\alpha^n(b)) U^{*n} = \pi(b), \quad \forall b \in \mathscr{A}, n \in \mathbb{Z}$$

which is equivalent to

$$U\pi(\alpha(b)) = \pi(b)U, \quad \forall b \in \mathscr{A}.$$

For such a pair denote by $U \times \pi$ the $*$-representation of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ such that

$$U \times \pi \left(\sum_{n \in \mathbb{Z}} \delta_n \otimes a_n \right) = \sum_{n \in \mathbb{Z}} U^n \pi(a_n).$$

Just as in 3.1.1 we can prove that there is a bijective correspondence between continuous $*$-representations of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ and left covariant representations of $(\mathscr{A}, \mathbb{Z}, \alpha)$.

Remark 12. We should note that if (π, U) is a left covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$ then (π, U^*) is a covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$. Indeed, let (π, U) be a left covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$, then $\forall x \in \mathscr{A}$ we have $\pi(\alpha(x))^* U^* = U^* \pi(x)^*$ and hence $\pi(\alpha(x^*))U^* = U^* \pi(x^*)$ and since $\pi(\mathscr{A})$ is selfadjoint we have that

$$\pi(\alpha(x))U^* = U^* \pi(x), \quad \forall x \in \mathscr{A}.$$

Example 3.2.1. Let $\mathscr{A}$ be a C^*-algebra, let $\alpha : \mathscr{A} \to \mathscr{A}$ be a $*$-automorphism and let $\pi : \mathscr{A} \to \mathbf{B}(\mathscr{H})$ be the universal representation of $\mathscr{A}$.

Set $\mathscr{K} = \ell^2(\mathbb{Z}, \mathscr{H})$ and

$$\tilde{\pi} : \mathscr{A} \to \mathbf{B}(\mathscr{K}),$$

such that for $x \in \mathscr{A}$ and $(\cdots, x_{-1}, x_0, x_1, x_2, \cdots) \in \mathscr{K}$ we have

$$\tilde{\pi}(x)(\cdots, x_{-1}, x_0, x_1, x_2, \cdots) = (\cdots, \pi(\alpha^{-1}(x))x_{-1}, \pi(x)x_0, \pi(\alpha(x))(x_1), \pi(\alpha^2(x))x_2, \cdots)$$

and also

$$\tilde{S} : \ell^2(\mathbb{Z}, \mathscr{H}) \to \ell^2(\mathbb{Z}, \mathscr{H}),$$

where

$$\tilde{S} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbf{0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then for $x \in \mathscr{A}$ it evident that

$$\tilde{\pi}(x)\tilde{S} = \tilde{S}\tilde{\pi}(\alpha(x))$$

and that $\tilde{S}$ is a unitary and therefore $(\tilde{\pi}, \tilde{S})$ is a left covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$. Now in order to show that $\tilde{S} \times \tilde{\pi}$ is faithful, suppose that for $f = \sum_{n\in\mathbb{Z}} \delta_n \otimes x_n \in \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$

$$\tilde{S} \times \tilde{\pi}(f) = 0.$$

Pick $x, y \in \mathscr{H}$, $n \in \mathbb{Z}$ and $\xi_1, \xi_n \in \mathscr{K}$ such that

$$\xi_1(k) = \begin{cases} x, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

and

$$\xi_n(k) = \begin{cases} y, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}.$$

Note that

$$(\tilde{S} \times \tilde{\pi})(f)\xi_1 = \sum_k \tilde{S}^k \tilde{\pi}(x_k)\xi = \sum_k \tilde{S}^k(\dots, 0, \boxed{\pi(x_k)x}, 0, \dots),$$

then if we set $\eta = (\tilde{S} \times \tilde{\pi})(f)\xi_1 \in \mathscr{K}$ we have that $\eta(n) = \pi(x_n)x$.
Hence,

$$\begin{aligned} 0 &= \langle (\tilde{S} \times \tilde{\pi})(f)\xi_1, \xi_n \rangle_{\mathscr{K}} = \langle \eta, \xi_n \rangle_{\mathscr{K}} = \\ &= \sum_s \langle \eta(s), \xi_n(s) \rangle_{\mathscr{H}} = \langle (\eta(n), y \rangle_{\mathscr{H}} \\ &= \langle \pi(x_n)x, y \rangle_{\mathscr{H}}. \end{aligned}$$

Thus, we have that for each $n \in \mathbb{Z}$, $\pi(x_n) = 0$ and since π is faithful, $x_n = 0$ and so $f = 0$.

Now that we showed that there exists a faithful representation $\tilde{S} \times \tilde{\pi}$ of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$, for

$f \in \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ we define

$$\|f\| = \sup_{(\pi,U)} \|U \times \pi(f)\| \tag{3.2.1}$$

where (π, U) is a left covariant representation. It is a well-defined norm on $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ and it is easily seen to be a C^*-norm.

Definition 3.2.2. Let $(\mathscr{A}, \mathbb{Z}, \alpha)$ be a C^*-dynamical system, the C^*-algebra left crossed product $\mathbb{Z} \times_\alpha \mathscr{A}$, is the completion of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ with respect to the norm defined in 3.2.1.

To avoid confusion we will denote by $\|.\|_r$ the norm of the crossed product and by $\|.\|_l$ the norm of the left crossed product.

Theorem 3.2.1. *The left crossed product is $*$-isomorphic to the crossed product.*

Proof. We define

$$\Psi : \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l \to \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_r,$$

such that for $f = \sum_{n\in\mathbb{Z}} \delta_n \otimes x_n$ we have

$$\Psi\left(\sum_{n\in\mathbb{Z}} \delta_n \otimes x_n\right) = \sum_{n\in\mathbb{Z}} \alpha^{-n}(x_n) \otimes \delta_{-n}.$$

Then Ψ is a well-defined linear $\|.\|_1, \|.\|_1$-isometrical isomorphism between Banach spaces because

$$\|\Psi(f)\|_1 = \left\|\sum_{n\in\mathbb{Z}} \alpha^{-n}(x_n) \otimes \delta_{-n}\right\|_1 = \sum_{n\in\mathbb{Z}} \|\alpha^{-n}(x_n)\|$$
$$= \sum_{n\in\mathbb{Z}} \|x_n\| = \left\|\sum_{n\in\mathbb{Z}} \delta_n \otimes x_n\right\|_1 = \|f\|_1$$

and it is also surjective since for $g = \sum_{n\in\mathbb{Z}} y_n \otimes \delta_n \in \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_r$ we have that

$$\Psi\left(\sum_{n\in\mathbb{Z}} \delta_{-n} \otimes \alpha^{-n}(y_n)\right) = g.$$

We show that Ψ is multiplicative on simple tensors and since it is linear and continuous, Ψ is multiplicative:

$$\Psi((\delta_n \otimes x)(\delta_m \otimes y)) = \Psi(\delta_{n+m} \otimes \alpha^m(x)y) = \alpha^{-(n+m)}(\alpha^m(x)y) \otimes \delta_{-(n+m)}$$
$$= \alpha^{-n}(x)\alpha^{-(n+m)}(y) \otimes \delta_{-(n+m)}$$
$$\Psi(\delta_n \otimes x)\Psi(\delta_m \otimes y) = (\alpha^{-n}(x) \otimes \delta_{-n})(\alpha^{-m}(y) \otimes \delta_{-m}) =$$
$$\alpha^{-n}(x)\alpha^{-(n+m)}(y) \otimes \delta_{-(n+m)}$$

Thus,

$$\Psi((\delta_n \otimes x)(\delta_m \otimes y)) = \Psi(\delta_n \otimes x)\Psi(\delta_m \otimes y)$$

and also

$$\Psi((\delta_n \otimes x)^*) = \Psi(\delta_{-n} \otimes \alpha^{-n}(x^*)) = x^* \otimes \delta_n$$

$$\Psi(\delta_n \otimes x)^* = (\alpha^{-n}(x) \otimes \delta_{-n})^* = x^* \otimes \delta_n$$

Finally, we show that Ψ extends to a $*$-isomorphism between the left crossed product and the crossed product. In order to do so let (π, U) be a left covariant representation, then (π, U^*) is a covariant representation and we have the following :
If $f = \sum_{n\in\mathbb{Z}} \delta_n \otimes x_n \in \ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$,

$$\begin{aligned}
\|(\pi \times U^*)(\Psi(f))\| &= \left\|(\pi \times U^*)\left(\sum_{n\in\mathbb{Z}} \alpha^{-n}(x_n) \otimes \delta_{-n}\right)\right\| \\
&= \left\|(\pi \times U^*)\left(\sum_{m\in\mathbb{Z}} \alpha^{m}(x_{-m}) \otimes \delta_{m}\right)\right\| = \left\|\sum_{m\in\mathbb{Z}} \pi(\alpha^m(x_{-m}))U^{*m}\right\| \\
&= \left\|\sum_{m\in\mathbb{Z}} U^{*m}\pi(x_{-m})\right\| = \left\|\sum_{n\in\mathbb{Z}} U^{n}\pi(x_{n})\right\| = \|(U \times \pi)(f)\|.
\end{aligned}$$

Therefore, by taking supremum over all left covariant (respectively , covariant) representations we have that

$$\|\Psi(f)\|_r = \|f\|_l$$

Thus, Ψ is $\|.\|_l, \|.\|_r$-isometric and so we can extend it to the desired $*$-isomorphism. □

Definition 3.2.3. Let $(\mathscr{A}, \mathbb{Z}, \alpha)_l$ be a C^*-dynamical system, we define the reduced left crossed product $\mathbb{Z} \times_{\alpha r} \mathscr{A}$ to be the completion of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$ with respect to the norm

$$\|f\| = \|(\tilde{S} \times \tilde{\pi})(f)\|,$$

where $(\tilde{\pi}, \tilde{S})$ is the left covariant representation defined in example 3.2.1.
This is the C^*-subalgebra of $\mathbf{B}(\mathscr{K})$ generated by the set $\{\tilde{\pi}(a) \ : a \in \mathscr{A}\} \cup \{\tilde{S}\}$.

Now we are going to prove a very important result that helps us understand crossed products by $\mathbb{Z}$: The left crossed product by $\mathbb{Z}$ is $*$-isomorphic with the reduced left crossed product by $\mathbb{Z}$. The proof that the crossed product by $\mathbb{Z}$ is $*$-isomorphic with the reduced crossed product by $\mathbb{Z}$ is essentially the same.

Remark 13. Using the same notation as in example 3.2.1 for $z \in \mathbb{T}$ we define $U_z \in \mathbf{B}(\mathscr{K})$ by

$$U_z(..., x_{-1}, \boxed{x_0}, x_1, ...) = (..., z^{-1}x_{-1}, \boxed{x_0}, zx_1, ...), \quad \text{where} \quad (..., x_{-1}, \boxed{x_0}, x_1, ...) \in \mathscr{K},$$

then U_z is a unitary and $U_z^* = U_{\overline{z}}$ and so it induces a $*$-automorphism β_z of $\mathbf{B}(\mathscr{K})$, where for $T \in \mathbf{B}(\mathscr{K})$

$$\beta_z(T) = U_z T U_z^*.$$

Note that for $a \in \mathscr{A}$ and $(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) \in \mathscr{K}$ we have that

$$\begin{aligned}&\beta_z(\tilde{\pi}(a))(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) = U_z\tilde{\pi}(a)U_z^*(\dots, x_{-1}, \boxed{x_0}, x_1, \dots)\\ &= U_z(\dots, \overline{z}^{-1}\pi(\alpha^{-1}(a))x_{-1}, \boxed{\pi(a)x_0}, \overline{z}\pi(\alpha(a))x_1, \dots) = \tilde{\pi}(a)(\dots, x_{-1}, \boxed{x_0}, x_1, \dots)\end{aligned}$$

and also

$$\begin{aligned}&\beta_z(\tilde{S}^m)(\dots, x_{-1}, \boxed{x_0}, x_1, \dots) = U_z\tilde{S}^m(\dots, \overline{z}^{-1}x_{-1}, \boxed{x_0}, \overline{z}x_1, \dots)\\ &= U_z(\dots, z^{m+1}x_{-m-1}, \boxed{z^m x_{-m}}, z^{m-1}x_{-m+1}, \dots)\\ &= (\dots, z^m x_{-m-1}, \boxed{z^m x_{-m}}, z^m x_{-m+1}, \dots)\\ &= z^m\tilde{S}^m(\dots, x_{-1}, \boxed{x_0}, x_1, \dots)\end{aligned}$$

Therefore, we obtain

$$\beta_z(\tilde{\pi}(a)) = \tilde{\pi}(a) \quad \text{and} \quad \beta_z(\tilde{S}^m) = z^m\tilde{S}^m$$

and thus we can restrict β_z to a $*$-automorphism of $\mathbb{Z} \times_{\alpha r} \mathscr{A}$.
Now, we are going to show that for $F \in \mathbb{Z} \times_\alpha \mathscr{A}$ the map

$$\mathbb{T} \to \mathbb{Z} \times_{\alpha r} \mathscr{A} : \quad z \to \beta_z(F)$$

is norm-continuous. Let $\{z_n : n \in \mathbb{N}\} \subseteq \mathbb{T}$ be a sequence such that

$$z_n \xrightarrow{n\to\infty} z \in \mathbb{T},$$

then from our calculations above for an element of the form $\tilde{S}^m\tilde{\pi}(a)$ we have that

$$\lim_n \beta_{z_n}(\tilde{S}^m\tilde{\pi}(a)) = \lim_n z_n^m\tilde{S}\tilde{\pi}(a) = z^m\tilde{S}^m\tilde{\pi}(a) = \beta_z(\tilde{S}^m\tilde{\pi}(a)).$$

By linearity we have that for an element of the form $F = \sum\limits_{k=-n}^{n} \tilde{S}^k\tilde{\pi}(a_k)$,

$$\lim_n \beta_{z_n}(F) = \beta_z(F)$$

and so the map

$$[0,1] \to \mathbb{Z} \times_{\alpha r} \mathscr{A} \quad : \quad t \to \beta_{e^{2\pi i t}}(F)$$

is continuous.
Now if F is an arbitrary element of $\mathbb{Z} \times_{\alpha r} \mathscr{A}$, $\epsilon > 0$ and $t \in [0,1]$, then we can pick $X = \sum_{k=-n}^{n} \tilde{S}^k\tilde{\pi}(a_k)$ such that

$$\|X - F\| < \frac{\epsilon}{3}.$$

There exists $\delta > 0$ such that:

$$|t-w|< \delta \Rightarrow \|\beta_{e^{2\pi it}}(X)-\beta_{e^{2\pi iw}}(X)\|< \frac{\epsilon}{3}$$

and so if $|t-w|< \delta$

$$\begin{aligned}
&\|\beta_{e^{2\pi it}}(F)-\beta_{e^{2\pi iw}}(F)\| \\
&\leq \|\beta_{e^{2\pi it}}(F)-\beta_{e^{2\pi it}}(X)\|+\|\beta_{e^{2\pi it}}(X)-\beta_{e^{2\pi iw}}(X)\|+\|\beta_{e^{2\pi iw}}(F)-\beta_{e^{2\pi iw}}(X)\| \\
&< \|\beta_{e^{2\pi it}}\|\|X-F\|+\frac{\epsilon}{3}+\|\beta_{e^{2\pi iw}}\|\|X-F\|< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

Finally, the map

$$t \rightarrow \beta_{e^{2\pi it}}(F)$$

is continuous for $F \in \mathbb{Z} \times_{\alpha r} \mathscr{A}$.

Definition 3.2.4. Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{B}$ a C^*-subalgebra of $\mathscr{A}$. A conditional expectation of $\mathscr{A}$ onto $\mathscr{B}$ is a contractive, positive and surjective linear map

$$\mathscr{E} : \mathscr{A} \rightarrow \mathscr{B}$$

such that:

(i) $\mathscr{E}(b) = b$ for all $b \in \mathscr{B}$,

(ii) $\mathscr{E}(b_1 a b_2) = b_1 \mathscr{E}(a) b_2$ for all $a \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$.

We say that a conditional expectation $\mathscr{E}$ is faithful if for every positive non-zero element $a \in \mathscr{A}$ we have that $\mathscr{E}(a)$ is also non-zero.

Theorem 3.2.2. *Let $(\mathscr{A}, \mathbb{Z}, \alpha)$ be a C^*-dynamical system where $\mathscr{A}$ and α are unital. Then there is a faithful conditional expectation $\mathscr{E}$ of $\mathbb{Z} \times_\alpha \mathscr{A}$ onto $\mathscr{A}$. (Here we identify $\mathscr{A}$ with $i(\mathscr{A}) \subseteq \mathbb{Z} \times_\alpha \mathscr{A}$, where $i : \mathscr{A} \rightarrow \mathbb{Z} \times_\alpha \mathscr{A}$ is the map such that for $a \in \mathscr{A}$, $i(a) = \delta_0 \otimes a$).*

Proof. We can consider $\mathbb{Z} \times_\alpha \mathscr{A}$ as a C^*-subalgebra of $\mathbf{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space. Set

$$i : \mathscr{A} \rightarrow \mathbb{Z} \times_\alpha \mathscr{A} \subseteq \mathbf{B}(\mathscr{H})$$

to be the $*$-representation of $\mathscr{A}$ such that for $a \in \mathscr{A}$

$$i(a) = \delta_0 \otimes a$$

and set

$$U = \delta_1 \otimes \mathbf{1}_{\mathscr{A}}.$$

For $z \in \mathbb{T}$ we have that

$$\begin{aligned} i(a)(zU) &= z(\delta_0 \otimes a)(\delta_1 \otimes \mathbf{1}_{\mathscr{A}}) = z\delta_1 \otimes \alpha(a) \\ &= z(\delta_1 \otimes \mathbf{1}_{\mathscr{A}})(\delta_0 \otimes \alpha(a)) = (zU)i(\alpha(a)) \end{aligned}$$

Therefore, (i, zU) is a left unitary covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$ and by the universal property of the left crossed product we obtain a $*$-endomorphism γ_z of $\mathbb{Z} \times_\alpha \mathscr{A}$. We can easily see that the C^*-algebra generated by $\{\delta_0 \otimes a : a \in \mathscr{A}\}$ and zU is $\mathbb{Z} \times_\alpha \mathscr{A}$ and since

$$\gamma_z(\delta_0 \otimes a) = \delta_0 \otimes a$$

and

$$\gamma_z(U) = zU$$

γ_z is a $*$-automorphism of $\mathbb{Z} \times_\alpha \mathscr{A}$ and $\gamma_{\overline{z}}$ is its inverse.
For $t \in [0, 1]$ and $F \in \mathbb{Z} \times_\alpha \mathscr{A}$ set

$$f_F(t) = \gamma_{e^{2\pi it}}(F),$$

then f_F is norm continuous.
Indeed, if $F = \sum_{k=-n}^{n} \delta_k \otimes a_k$ then

$$f_F(t) = \sum_{k=-n}^{n} e^{2\pi kit}\delta_k \otimes a_k$$

and scalar multiplication is norm continuous.
Now if F is an arbitrary element of $\mathbb{Z} \times_\alpha \mathscr{A}$, $\epsilon > 0$ and $t \in [0, 1]$, there exists $X = \sum_{k=-n}^{n} \delta_k \otimes a_k$ such that

$$\|X - F\| < \frac{\epsilon}{3}.$$

Since f_X is continuous there exists $\delta > 0$ such that:

$$|t - w| < \delta \Rightarrow \|\gamma_{e^{2\pi it}}(X) - \gamma_{e^{2\pi iw}}(X)\| < \frac{\epsilon}{3}$$

and so if $|t - w| < \delta$

$$\begin{aligned} &\|\gamma_{e^{2\pi it}}(F) - \gamma_{e^{2\pi iw}}(F)\| \\ &\leq \|\gamma_{e^{2\pi it}}(F) - \gamma_{e^{2\pi it}}(X)\| + \|\gamma_{e^{2\pi it}}(X) - \gamma_{e^{2\pi iw}}(X)\| + \|\gamma_{e^{2\pi iw}}(F) - \gamma_{e^{2\pi iw}}(X)\| \\ &< \|\gamma_{e^{2\pi it}}\|\|X - F\| + \frac{\epsilon}{3} + \|\gamma_{e^{2\pi iw}}\|\|X - F\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Now, we can define

$$\mathscr{E} : \mathbb{Z} \times_\alpha \mathscr{A} \to \mathbb{Z} \times_\alpha \mathscr{A}$$

by

$$\mathscr{E}(F) = \int_0^1 \gamma_{e^{2\pi it}}(F)dt.$$

The linearity of the integral implies that $\mathscr{E}$ is linear and also contractive since,

$$\|\mathscr{E}(F)\| = \left\| \int_0^1 \gamma_{e^{2\pi it}}(F)dt \right\| \leq \int_0^1 \|\gamma_{e^{2\pi it}}(F)\| dt \leq \sup_{t\in[0,1]} \|\gamma_{e^{2\pi it}}(F)\| \leq \|F\|.$$

Moreover, we have that for $a \in \mathscr{A}$

$$\mathscr{E}(\delta_0 \otimes a) = \int_0^1 \gamma_{e^{2\pi it}}(\delta_0 \otimes a)dt = \delta_0 \otimes a$$

and this shows that restricting $\mathscr{E}$ to the copy of $\mathscr{A}$ we get the identity map.
In order to show that $\mathscr{E}$ is positive let $F \in \mathbb{Z} \times_\alpha \mathscr{A}$ then

$$\mathscr{E}(F^*F) = \int_0^1 \gamma_{e^{2\pi it}}(F)^* \gamma_{e^{2\pi it}}(F)dt$$

which is a norm-limit of positive elements.

Indeed, if $\mathscr{P} = \{0 = t_0 < t_1 < \cdots < t_m = 1\}$ is a partition of $[0, 1]$,

$$S(\gamma_{e^{2\pi it}}(F^*F), \mathscr{P}) = \sum_{j=1}^{m} \gamma_{e^{2\pi it_j}}(F^*F)(t_j - t_{j-1})$$

is a positive element of $\mathbb{Z} \times_\alpha \mathscr{A}$ and

$$\lim_{\mathscr{P}} S((\gamma_{e^{2\pi it}}(F^*F), \mathscr{P}) = \int_0^1 \gamma_{e^{2\pi it_j}}(F^*F)dt.$$

See section 2.3
To show that $\mathscr{E}$ is faithful, suppose that $F^*F \in \ker \mathscr{E}$.
Notice that for a state τ of $\mathbb{Z} \times_\alpha \mathscr{A}$

$$0 = \tau(\mathscr{E}(F^*F)) = \tau\left(\int_0^1 \gamma_{e^{2\pi it}}(F)^* \gamma_{e^{2\pi it}}(F)dt \right) = \int_0^1 \tau(\gamma_{e^{2\pi it}}(F)^* \gamma_{e^{2\pi it}}(F))dt.$$

Suppose that there exists $t_0 \in [0, 1]$ such that the positive element

$$\gamma_{e^{2\pi it_0}}(F)^* \gamma_{e^{2\pi it_0}}(F) \neq 0,$$

then we can pick a state τ such that

$$\|\gamma_{e^{2\pi it_0}}(F)^* \gamma_{e^{2\pi it_0}}(F)\| = |\tau(\gamma_{e^{2\pi it_0}}(F)^* \gamma_{e^{2\pi it_0}}(F))| = \delta > 0.$$

Since

$$t \to \tau(\gamma_{e^{2\pi it}}(F)^* \gamma_{e^{2\pi it}}(F))$$

is a continuous map we can find an interval $I \subseteq [0,1]$ containing t_0 such that for $t \in I$

$$\tau(\gamma_{e^{2\pi it}}(F)^*\gamma_{e^{2\pi it}}(F)) \geq \frac{\delta}{2}$$

Hence,

$$0 = \int_0^1 \tau(\gamma_{e^{2\pi it}}(F)^*\gamma_{e^{2\pi it}}(F))dt \geq \frac{\delta}{2}m(I) > 0$$

where $m(I)$ is the length of I and so we get a contradiction.
Therefore, $\forall t \in [0,1]$ we have that

$$\gamma_{e^{2\pi it}}(F^*F) = 0$$

and since $\gamma_{e^{2\pi it}}$ is injective we conclude that $F^*F = 0$ and $\mathscr{E}$ is injective.
Now let $a, b \in \mathscr{A}$, $k \neq 0$ and $F \in \mathbb{Z} \times_\alpha \mathscr{A}$ then,

$$\begin{aligned}\mathscr{E}((\delta_0 \otimes a)F(\delta_0 \otimes b)) &= \int_0^1 \gamma_{e^{2\pi it}}((\delta_0 \otimes a)F(\delta_0 \otimes b))dt \\ &= (\delta_0 \otimes a)\int_0^1 \gamma_{e^{2\pi it}}(F)dt\,(\delta_0 \otimes b) = (\delta_0 \otimes a)\mathscr{E}(F)(\delta_0 \otimes b)\end{aligned}$$

and

$$\begin{aligned}\mathscr{E}(U^k) &= \int_0^1 \gamma_{e^{2\pi it}}(U^k)dt = \int_0^1 e^{2\pi ikt}U^k dt = U^k \int_0^1 e^{2\pi ikt}\mathbf{1}_{\mathbb{Z}\otimes_\alpha \mathscr{A}}dt \\ &= U^k\left(\int_0^1 e^{2\pi ikt}dt\right) = 0.\end{aligned}$$

Thus, for a finite sum of the form

$$\sum_{k=-n}^{n} (\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k(\delta_0 \otimes a_k),$$

we have that

$$\mathscr{E}\left(\sum_{k=-n}^{n} (\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k(\delta_0 \otimes a_k)\right) = \sum_{k=-n}^{n} \mathscr{E}((\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k(\delta_0 \otimes a_k)) = \delta_0 \otimes a_0.$$

Note that since

$$\delta_m \otimes a = (\delta_m \otimes \mathbf{1}_{\mathscr{A}})(\delta_0 \otimes a) \quad \text{and} \quad \delta_m \otimes \mathbf{1}_{\mathscr{A}} = (\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^m,$$

the algebra generated by elements of the form

$$\sum_{k=-n}^{n} (\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k(\delta_0 \otimes a_k)$$

is the same as the algebra generated by finite sums $\sum_{k=-n}^{n}(\delta_k \otimes a_k)$, which is dense in the left crossed product and since $\mathscr{E}$ is continuous and the images of these elements lie in $\mathscr{A}$, the range of $\mathscr{E}$ lies in $\mathscr{A}$ and so $\mathscr{E}$ is a faithful conditional expectation onto $\mathscr{A}$. □

Theorem 3.2.3. *Let $\mathscr{A}$ be a unital C^*-algebra, and α a unital $*$-automorphism of $\mathscr{A}$ then there is $*$-isomorphism between $\mathbb{Z} \times_{\alpha r} \mathscr{A}$ and $\mathbb{Z} \times_{\alpha} \mathscr{A}$.*

Proof. Let $(\pi, \mathscr{H})$ be a faithful representation of $\mathscr{A}$ and let $(\tilde{\pi}, \tilde{S})$ be the left covariant representation as in 3.2.1. From the universal property of the left crossed product we obtain a $*$-epimorphism

$$\Phi : \mathbb{Z} \times_{\alpha} \mathscr{A} \to \mathbb{Z} \times_{\alpha r} \mathscr{A},$$

such that

$$\Phi\left(\sum_{k=-n}^{n} \delta_k \otimes a_k\right) = \sum_{k=-n}^{n} \tilde{S}^k \tilde{\pi}(a_k).$$

Recall that there is copy of $\mathscr{A}$ in $\mathbb{Z} \times_{\alpha} \mathscr{A}$ and since for $a \in \mathscr{A}$

$$\Phi(\delta_0 \otimes a) = \tilde{\pi}(a),$$

we get that the restriction of Φ to $\mathscr{A}$ is $\tilde{\pi}$, which is faithful.
Observe that for $f = \sum\limits_{k=-n}^{n} \delta_k \otimes a_k$ and $z \in \mathbb{T}$ we have

$$\begin{aligned}
\Phi \circ \gamma_z(f) &= \Phi \circ \gamma_z\left(\sum_{k=-n}^{n} (\delta_k \otimes \mathbf{1}_{\mathscr{A}})(\delta_0 \otimes a_k)\right) = \Phi\left(\sum_{k=-n}^{n} z^k (\delta_k \otimes \mathbf{1}_{\mathscr{A}})(\delta_0 \otimes a_k)\right) \\
&= \Phi\left(\sum_{k=-n}^{n} z^k \delta_k \otimes a_k\right) = \sum_{k=-n}^{n} z^k \Phi(\delta_k \otimes a_k) = \sum_{k=-n}^{n} z^k \tilde{S}^k \tilde{\pi}(a_k) \\
&= \sum_{k=-n}^{n} \beta_z(\tilde{S}^k)\beta_z(\tilde{\pi}(a_k)) = \beta_z\left(\sum_{k=-n}^{n} \tilde{S}^k \tilde{\pi}(a_k)\right) = \beta_z \circ \Phi(f)
\end{aligned}$$

and since elements in the form of f are dense in $\mathbb{Z} \times_{\alpha} \mathscr{A}$ and both $\Phi \circ \gamma_z$ and $\beta_z \circ \Phi$ are $*$-homomorphisms, in particular continuous, we obtain that for $F \in \mathbb{Z} \times_{\alpha} \mathscr{A}$

$$\Phi \circ \gamma_z(F) = \beta_z \circ \Phi(F).$$

Now, suppose that $F \in \mathbb{Z} \times_{\alpha} \mathscr{A}$ and $F \in \ker \Phi$, then F^*F is also in $\ker \Phi$, from the C^*-property it suffices to show that $F^*F = 0$.
We have

$$\begin{aligned}
\Phi(\mathscr{E}(F^*F)) &= \Phi\left(\int_0^1 \gamma_{e^{2\pi it}}(F^*F)dt\right) = \int_0^1 \Phi(\gamma_{e^{2\pi it}}(F^*F))dt \\
&= \int_0^1 \beta_{e^{2\pi it}}(\Phi(F^*F))dt = 0.
\end{aligned}$$

Therefore, since $\mathscr{E}(F^*F)$ is an element in $\mathscr{A}$ and

$$\Phi(\mathscr{E}(F^*F)) = \tilde{\pi}(\mathscr{E}(F^*F)) = 0$$

we get that $\mathscr{E}(F^*F) = 0$ and since $\mathscr{E}$ is faithful, we have that $F^*F = 0$. We conclude that $\ker \Phi = \{0\}$ and Φ is a $*$-isomorphism. □

3.3 Simplicity of $C(X) \times_\alpha \mathbb{Z}$

Let X be a compact Hausdorff space and suppose that $\sigma : X \to X$ is a homeomorphism. We set $\alpha : \mathbb{Z} \to \text{Aut}(C(X))$ to be the homomorphism such that

$$a_n(f) := f \circ \sigma^{-n}.$$

We have a well-defined action of $\mathbb{Z}$ on X where

$$n \cdot x = \sigma^n(x)$$

Definition 3.3.1. Let (X, σ) be as above. We say that:

(i) $\mathbb{Z}$ acts topologically freely on X if for every $n \neq 0$ the set

$$\{x \in X : \sigma^n(x) = x\}$$

has empty interior.

(ii) The action of $\mathbb{Z}$ on X is minimal if for every $x \in X$ the set

$$\{\sigma^n(x) : n \in \mathbb{Z}\}$$

is a dense subset of X.

Lemma 3.3.1. *Let X be a compact Hausdorff space and $\sigma : X \to X$ be a homeomorphism. Suppose that $\mathbb{Z}$ acts topologically freely on X and let $n_1, n_2, \dots n_k$ be integers such that $n_i \neq 0$ for $i = 1, \dots, k$. Then for every open set $U \subseteq X$ there exists a non-empty open set $V \subseteq U$ such that*

$$\sigma^{n_i}(V) \cap V = \emptyset, \quad \text{for} \quad i = 1, \dots, k.$$

Proof. We claim that we can pick $y \in \bigcap_{i=1}^k \{x \in X : n_i \cdot x \neq x\} \cap U$. Since $\forall i = 1, \dots, k$ the set $\{x \in X : n_i \cdot x = x\}$ is closed and has empty interior, $\{x \in X : n_i \cdot x \neq x\}$ is open and dense and therefore $\bigcap_{i=1}^k \{x \in X : n_i \cdot x \neq x\} \cap U \neq \emptyset$. For $i = 1, \dots, k$ since X is a Hausdorff space there exist disjoint open sets $W_i, G_i \subseteq X$ such that

$$y \in W_i \quad \text{and} \quad n_i \cdot y \in G_i.$$

For each i, from the continuity of σ^{n_i} there exists an open set $U_i \subseteq X$ that contains y, such that $\sigma^{n_i}(U_i) \subseteq G_i$. We set $V_i = W_i \cap U_i$, then V_i is open and

$$\sigma^{n_i}(V_i) \cap V_i = \emptyset.$$

Finally, by setting $V = \bigcap_{i=1}^{k} V_i \cap U$, we have that V is an open subset of U that contains y, in particular it is not empty and has the desired property

$$\sigma^{n_i}(V) \cap V = \emptyset \quad \text{for} \quad i = 1, \ldots, k.$$

□

Lemma 3.3.2. *Let (X, σ) and $n_1, n_2, \ldots, n_k$ be as above, $f \in C(X)$ and $m \in \mathbb{N}$. There exists $g \in C(X)$ such that:*

(i) $0 \leq g(x) \leq 1, \forall x \in X,$

(ii) $\|fg\| \geq \|f\| - \frac{1}{m}$

(iii) $(g \circ \sigma^{n_i})g = 0, \forall i = 1, \ldots, k.$

Proof. Set

$$U = \left\{ x \in X : |f(x)| > \|f\| - \frac{1}{m} \right\},$$

from the preceding lemma there exists an open set $V \subseteq U$ such that for $i = 1, \ldots, k$

$$\sigma^{n_i}(V) \cap V = \emptyset.$$

Pick $y \in V$, since X is a compact Hausdorff space, X is normal and therefore by Urysohn's lemma there exists $g \in C(X)$ such that $g(x) = 0$ for all $x \notin V$ and $g(y) = 1$. It is immediate that g satisfies (i), for (ii) note that

$$\sup_{x \in X} |(fg)(x)| \geq \sup_{x \in U} |f(x)g(x)| \geq |f(y)g(y)| > \|f\| - \frac{1}{m}$$

and for (iii), $\forall i = 1, \ldots, k$

$$g \circ \sigma^{n_i}(x) g(x) = 0$$

since for $x \in V$, $\sigma^{n_i}(x) \notin V$ and thus $g(\sigma^{n_i}(x)) = 0$ and if $x \notin V$ then $g(x) = 0$. □

Definition 3.3.2. Let X be a compact Hausdorff space and $\sigma : X \to X$ a homeomorphism. We say that a set $F \subseteq X$ is $\sigma-$invariant if $\sigma(F) \subseteq F$.

Lemma 3.3.3. *Let X be a compact Hausdorff space and $\sigma : X \to X$ a homeomorphism. If a proper closed $F \subseteq X$ is σ-invariant then the ideal $I = \{f \in C(X) : f_{|}F = 0\}$ generates a proper ideal J_F of $C(X) \times_\alpha \mathbb{Z}$.*

Proof. Set $J_F^0 = \{\sum_{n=1}^k g_n \otimes \delta_n : k \in \mathbb{N}, g_n \in I\}$. Then J_F^0 is an ideal.
Indeed, for $f \otimes \delta_k \in C(X) \times_\alpha \mathbb{Z}$ and $\sum_n g_n \otimes \delta_n \in J_F^0$ we have that

$$\left(\sum_n g_n \otimes \delta_n\right)(f \otimes \delta_k) = \sum_n g_n(f \circ \sigma^{-n}) \otimes \delta_{n+k} \in J_F^0$$

and

$$(f \otimes \delta_k)\left(\sum_n g_n \otimes \delta_n\right) = \sum_n f(g_n \circ \sigma^{-k}) \otimes \delta_{n+k} \in J_F^0,$$

since $\forall n \in \mathbb{Z}$, if $x \in F$ using the fact that F is σ-invariant

$$g_n(x) f(\sigma^{-n}(x)) = 0.$$

and

$$f(x) g_n(\sigma^{-k}(x)) = 0.$$

Hence by linearity of the multiplication and the density of $C(X) \otimes_\alpha c_{00}(\mathbb{Z})$ in $C(X) \times_\alpha \mathbb{Z}$ we get that J_F^0 is an ideal of $C(X) \times_\alpha \mathbb{Z}$ and thus $J_F = \overline{J_F^0}$ is a closed ideal. If $\mathscr{E}$ is the canonical faithful expectation of $C(X) \times_\alpha \mathbb{Z}$ onto $C(X)$ it is evident that $\mathscr{E}(J_F^0) = I$ and since $\mathscr{E}$ is continuous and I is closed we have $\mathscr{E}(J_F) = I$.

By Urysohn's lemma one can find a continuous function which is not 0 on F, so I is a proper ideal of $C(X)$ and from the fact that $\mathscr{E}(C(X) \times_\alpha \mathbb{Z}) = C(X)$, we deduce that J_F must also be proper, otherwise $\mathscr{E}$ would not be surjective. □

Theorem 3.3.1 (Intersection property). *Let X be a compact Hausdorff space and $\sigma : X \to X$ a homeomorphism. Suppose that $\mathbb{Z}$ acts topologically freely on X and that $J \subseteq C(X) \times_\alpha \mathbb{Z}$ is a closed ideal such that $J \cap C(X) = \{0\}$. Then $J = \{0\}$.*

Proof. Suppose that $J \neq \{0\}$, recall that J is a C^*-algebra and so we can pick a positive element $c \neq 0$ in J. Then if we denote by

$$\pi : C(X) \times_\alpha \mathbb{Z} \to C(X) \times_\alpha \mathbb{Z}/J$$

the canonical $*$-epimorphism such that for $f \in C(X) \times_\alpha \mathbb{Z}$

$$\pi(f) = f + J,$$

we have that $c \in \ker \pi$.
By restricting π to $C(X)$ we get an injective $*$-homomorphism between C^*-algebras since

$$\ker \pi \cap C(X) = \{0\}$$

and therefore an isometry.

Set $f_0 = \mathscr{E}(c)$ where $\mathscr{E}$ is the canonical faithful expectation of $C(X)\times_\alpha \mathbb{Z}$ onto $C(X)$, then by the faithfulness of $\mathscr{E}$, $f_0 \neq 0$.
For $n \in \mathbb{N}$ we can pick $c_n = \sum_{i=-k_n}^{k_n} f_i^{(n)} \otimes \delta_i \in C(X) \times_\alpha \mathbb{Z}$ such that $\|c_n - c\|< \frac{1}{n}$ and since $c \neq 0$, eventually $c_n \neq 0$.
From lemma 3.3.2 for $f_0^{(n)}$ and $-k_n, -k_{n-1}, \dots, k_n$ we can pick $g_n \in C(X)$ such that $0 \leq g_n \leq 1$ and

$$\|f_0^{(n)} g_n\| \geq \|f_0^{(n)}\| - \frac{1}{n} \tag{*}$$

and also for $m = -k_n, \dots, k_n$ and $m \neq 0$

$$(g_n \circ \sigma^{-m}) g_n = 0.$$

Note that if $m \neq 0$

$$\left(g_n^{1/2} \otimes \delta_0\right)\left(f_m^{(n)} \otimes \delta_m\right)\left(g_n^{1/2} \otimes \delta_0\right) = \left(f_m^{(n)} \alpha_m(g_n^{1/2}) g_n^{1/2} \otimes \delta_m\right) = 0$$

and therefore, using that $\pi(c) = 0$,

$$\begin{aligned}
&\left\|\pi\left((g_n^{1/2} \otimes \delta_0)(f_0^{(n)} \otimes \delta_0)(g_n^{1/2} \otimes \delta_0)\right)\right\| = \left\|\pi\left((g_n^{1/2} \otimes \delta_0) c_n (g_n^{1/2} \otimes \delta_0)\right)\right\| \\
&= \left\|\pi\left((g_n^{1/2} \otimes \delta_0)(c_n - c)(g_n^{1/2} \otimes \delta_0)\right)\right\| \leq \|g_n\| \|c_n - c\| \leq \frac{1}{n}.
\end{aligned}$$

Since $\pi|_{C(X)}$ is an isometry it follows that

$$\|f_0^{(n)} g_n\| = \left\|(g_n^{1/2} \otimes \delta_0)(f_0^{(n)} \otimes \delta_0)(g_n^{1/2} \otimes \delta_0)\right\| \leq \frac{1}{n}$$

and so from $(*)$

$$\|f_0^{(n)}\| \leq \frac{2}{n} \to 0.$$

Now, we have that $c_n \to c$ and

$$\mathscr{E}(c_n) = f_0^{(n)} \to \mathscr{E}(c) = f_0 = 0.$$

This is a contradiction. □

Theorem 3.3.2. *Let X be a compact Hausdorff space and $\sigma : X \to X$ a homeomorphism. Then, $\mathbb{Z}$ is acting on X topologically freely and minimally if and only if $C(X) \times_\alpha \mathbb{Z}$ is simple.*

Proof. Assume that $\mathbb{Z}$ acts freely and minimally and suppose that $J \subseteq C(X) \times_\alpha \mathbb{Z}$ is a non-trivial closed ideal. Then $C(X) \cap J$ is a closed ideal in $C(X)$ and thus

$$C(X) \cap J = \{f \otimes \delta_0 : f \in C(X) \quad \text{and} \quad f_{|K} = 0\},$$

where K is a closed (compact) subset of X since there is a bijective correspondence between closed subsets of X and closed ideals of $C(X)$.

The ideal J is assumed to be nontrivial and therefore by the intersection property it must meet $C(X)$ and so we have that $K \neq X$. Note that for $f \otimes \delta_0 \in C(X) \cap J$ and all $n \in \mathbb{Z}$,

$$(\mathbf{1}_{C(X)} \otimes \delta_n)^*(f \otimes \delta_0)(\mathbf{1}_{C(X)} \otimes \delta_n) = (\mathbf{1}_{C(X)} \otimes \delta_{-n})(f \otimes \delta_n) = (f \circ \sigma^n) \otimes \delta_0 \in C(X) \cap J,$$

which implies that $(f \circ \sigma^n)|_K = 0$.
We will prove that $\sigma^n(K) = K$ for all $n \in \mathbb{Z}$.
Indeed, X is normal and Hausdorff and so if there exists $x = \sigma^n(y) \in \sigma^n(K)$ such that $x \notin K$ then by Urysohn's lemma we can pick $h \in C(X)$ such that $h_{|K} = 0$ and $h(x) = 1$. It is immediate now that $h \otimes \delta_0 \in C(X) \cap J$ and $(h \circ \sigma^n) \otimes \delta_0 \notin C(X) \cap J$ since $h \circ \sigma^n(y) \neq 0$. Therefore, $K \subseteq \sigma^n(K)$ for every $n \in \mathbb{Z}$ and thus we also have that $\sigma^{-n}(K) \subseteq \sigma^{-n}(\sigma^n(K)) = K$ for every $n \in \mathbb{Z}$. Hence $\sigma^n(K) = K, \forall n \in \mathbb{Z}$ and the fact that the action is minimal implies that $K = \emptyset$. Therefore we have that $C(X) \cap J = C(X)$ and since J is an ideal that contains the unit, $J = C(X) \times_\alpha \mathbb{Z}$ and simplicity follows.

For the converse, if $C(X) \times_\alpha \mathbb{Z}$ is simple, lemma 3.3 implies that $\mathbb{Z}$ has to act minimally.

In order to prove that $\mathbb{Z}$ acts topologically freely suppose, by way of contradiction, that there is a $n_0 \in \mathbb{Z}$ and an open non-empty set $U \subseteq X$ consisting of fixed points for σ^{n_0}; by Urysohn's lemma, there is a nonzero $h \in C(X)$ with $h(X \setminus U) = \{0\}$.

Fix $x \in X$, set $O_x = \{\sigma^n(x) : n \in \mathbb{Z}\}$ and $\mathscr{H} = \ell^2(O_x)$. It is clear that the set $\{e_{\sigma^k(x)} : k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathscr{H}$.

We define $\pi_x : C(X) \to \mathbf{B}(\mathscr{H})$ by

$$\pi_x(f)e_{\sigma^k(x)} = f(\sigma^k(x))e_{\sigma^k(x)}.$$

It is evident that π_x is a $*$-representation of $C(X)$. Let U be the unitary operator in $\mathbf{B}(\mathscr{H})$ given by

$$Ue_{\sigma^k(x)} = e_{\sigma^{k+1}(x)}.$$

We prove that (π_x, U) is a unitary covariant representation of the C^*-dynamical system $(C(X), \mathbb{Z}, \alpha)$.
Indeed, for each $k \in \mathbb{Z}$ we have

$$\begin{aligned} U\pi_x(f)e_{\sigma^k(x)} &= U(f(\sigma^k(x))e_{\sigma^k(x)}) = f(\sigma^k(x))e_{\sigma^{k+1}(x)} \\ &= f(\sigma^{-1}(\sigma^{k+1}(x)))e_{\sigma^{k+1}(x)} = \pi_x(f \circ \sigma^{-1})e_{\sigma^{k+1}(x)} = \pi_x(\alpha(f))Ue_{\sigma^k(x)}, \end{aligned}$$

therefore

$$U\pi_x(f) = \pi_x(\alpha(f))U.$$

From the universal property of the crossed-product we obtain a $*$-representation

$$\pi_x \times U : C(X) \times_\alpha \mathbb{Z} \to C^*(\pi_x, U) \subseteq \mathbf{B}(\mathscr{H}),$$

such that for $\sum_{|n|\leq m} f_n \otimes \delta_n \in C(X) \times_\alpha \mathbb{Z}$

$$(\pi_x \times U)\left(\sum_{|n|\leq m} f_n \otimes \delta_n\right) e_{\sigma^k(x)} = \sum_{|n|\leq m} \pi_x(f_n) U^n.$$

Note that $\ker(\pi_x \times U)$ is a closed ideal of $C(X) \times_\alpha \mathbb{Z}$. By the simplicity of $C(X) \times_\alpha \mathbb{Z}$ it is implied that $\pi_x \times U$ has to be faithful.

We will prove that

$$(h \otimes \delta_0) - (h \otimes \delta_{n_0}) = 0.$$

Assuming this for the moment, if $\mathscr{E}$ is the canonical faithful expectation of $C(X) \times_\alpha \mathbb{Z}$ onto $C(X)$, then

$$h \otimes \delta_0 = \mathscr{E}(h \otimes \delta_0 - h \otimes \delta_{n_0}) = 0$$

and so $h = 0$, which implies that $U = \emptyset$, contradicting our hypothesis. This will complete the proof that $\mathbb{Z}$ acts topologically freely on X.

So it remains to prove that

$$(h \otimes \delta_0) - (h \otimes \delta_{n_0}) = 0$$

or, since $\pi_x \times U$ is faithful, that

$$(\pi_x \times U)((h \otimes \delta_0) - (h \otimes \delta_{n_0})) = 0.$$

Since $\{e_{\sigma^k(x)} : k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathscr{H}$ it suffices to prove that

$$(\pi_x \times U)(h \otimes \delta_0 - h \otimes \delta_{n_0}) e_{\sigma^k(x)} = 0$$

for every $k \in \mathbb{Z}$.

If $\sigma^k(x) \in X$ is a fixed point for σ^{n_0} then

$$\begin{aligned}
&(\pi_x \times U)(h \otimes \delta_0 - h \otimes \delta_{n_0}) e_{\sigma^k(x)} \\
&= h(\sigma^k(x)) e_{\sigma^k(x)} - (\pi_x \times U)(h \otimes \delta_0)(\pi_x \times U)(\mathbf{1}_{C(X)} \otimes \delta_{n_0}) e_{\sigma^k(x)} \\
&= h(\sigma^k(x)) e_{\sigma^k(x)} - (\pi_x \times U)(h \otimes \delta_0) e_{\sigma^{k+n_0}(x)} \\
&= h(\sigma^k(x)) e_{\sigma^k(x)} - h(\sigma^{n_0+k}(x)) e_{\sigma^{k+n_0}(x)} \\
&= h(\sigma^k(x)) e_{\sigma^k(x)} - h(\sigma^k(x)) e_{\sigma^k(x)} = 0.
\end{aligned}$$

If $\sigma^k(x)$ is not a fixed point for σ^{n_0} then $\sigma^k(x) \notin U$ and also $\sigma^{k+n_0}(x) \notin U$ for if $\sigma^{k+n_0}(x) \in U$ then

$$\sigma^{k+n_0}(x) = \sigma^{-n_0}(\sigma^{k+n_0}(x)) = \sigma^k(x)$$

and so $\sigma^k(x)$ would be a fixed point.
Therefore, using the fact that $h(X \setminus U) = \{0\}$ we have that

$$(\pi_x \times U)(h \otimes \delta_0 - h \otimes \delta_{n_0})e_{\sigma^k(x)} = h(\sigma^k(x))e_{\sigma^k(x)} - h(\sigma^{k+n_0}(x))e_{\sigma^{k+n_0}(x)} = 0.$$

Thus in all cases,

$$(\pi_x \times U)(h \otimes \delta_0 - h \otimes \delta_{n_0})e_{\sigma^k(x)} = 0$$

and this proves the claim that $(h \otimes \delta_0) - (h \otimes \delta_{n_0}) = 0$ and concludes the proof. □

Chapter 4

Semi-crossed products

4.1 Definition of the semi-crossed product

In chapter 3 we managed to construct a C^*-algebra that is related to a particular C^*-dynamical system. A basic ingredient for this construction was the $*$-algebra $\mathscr{A} \otimes_\alpha C_c(G)$. Note that in a lot of situations we used the fact that α was a homomorphism into the group of automorphisms of $\mathscr{A}$ and so one could use the fact that α_s had an inverse, for example this fact was used to define the adjoint of $\mathscr{A} \otimes_\alpha C_c(G)$. In this chapter we will start with a C^*-algebra $\mathscr{A}$ and a $*$-endomorphism α and we will construct a Banach algebra related to this pair. In general, we are going to follow [23].

Definition 4.1.1. Let α be a (unital) $*$-endomorphism of a (unital) C^*-algebra $\mathscr{A}$, π a representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and V an isometry of $\mathscr{H}$. We say that (π, V) is an isometric covariant representation of $(\mathscr{A}, \alpha)$ if it satisfies the relation:

$$V\pi(\alpha(x)) = \pi(x)V, \quad \text{for} \quad x \in \mathscr{A}. \tag{4.1.1}$$

Suppose that $\mathscr{A}$ and α are unital. Without loss of generality we can assume that π is non-degenerate, whenever (π, V) an isometric covariant representation.

Indeed, suppose that π is degenerate and set $K = \pi(\mathbf{1}_{\mathscr{A}})\mathscr{H}$. Note that $\pi(\mathbf{1}_{\mathscr{A}})$ is an orthogonal projection of $\mathscr{H}$ and hence K is a Hilbert space. For $y \in \mathscr{H}$ and $x = \pi(\mathbf{1}_{\mathscr{A}})y \in K$ we have that

$$Vx = V\pi(\mathbf{1}_{\mathscr{A}})y = V\pi(\alpha(\mathbf{1}_{\mathscr{A}}))y = \pi(\mathbf{1}_{\mathscr{A}})y \in K,$$

therefore $V(K) \subseteq K$ and so $(\pi|_K, V|_K)$ is an isometric covariant representation of $(\mathscr{A}, \alpha)$ and $\pi|_K$ is non-degenerate.

Remark 14. We denote the semigroup of non-negative integers by $\mathbb{Z}^+$ and we define the Banach space $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ to be the completion with respect to $\|.\|_1$ of the vector space tensor product $c_{00}(\mathbb{Z}^+) \otimes \mathscr{A}$, where

$$\|f\|_1 = \sum_n \|x_n\|, \quad \text{for } f = \sum_n \delta_n \otimes x_n.$$

We define a multiplication on simple tensors by the rule :

$$(\delta_n \otimes x)(\delta_m \otimes y) = \delta_{n+m} \otimes \alpha^m(x)y \tag{4.1.2}$$

and we extend it linearly. We prove the sub-multiplicative property and thus this multiplication extends to $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ and we get a Banach algebra.
So let $f = \sum_{n=0}^{k_1} \delta_n \otimes x_n$ and $g = \sum_{m=0}^{k_2} \delta_m \otimes y_m$ where $k_1, k_2 \in \mathbb{N}$, then we have

$$\left\| \left(\sum_{n=0}^{k_1} \delta_n \otimes x_n \right) \left(\sum_{m=0}^{k_2} \delta_m \otimes y_m \right) \right\| = \left\| \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} \delta_{n+m} \otimes \alpha^m(x_n) y_m \right\|$$
$$\leq \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} \|\alpha^m(x_n) y_m\| \leq \sum_{n=0}^{k_1} \sum_{m=0}^{k_2} \|x_n\| \|y_m\| = \left\| \sum_{n=0}^{k_1} \delta_n \otimes x_n \right\| \left\| \sum_{m=0}^{k_2} \delta_m \otimes y_m \right\|.$$

Proposition 4.1.1. *If (π, V) is an isometric covariant representation of $(\mathscr{A}, \alpha)$, it yields a continuous representation σ of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ given on $c_{00}(\mathbb{Z}^+) \otimes \mathscr{A}$ by*

$$\sigma \left(\sum_{n \geq 0} \delta_n \otimes x_n \right) = \sum_{n \geq 0} V^n \pi(x_n).$$

We will denote this representation by $(V \times \pi)$.

Proof. By its definition, σ is a well defined linear map, so it suffices to prove that it is continuous (and therefore also well-defined) and multiplicative on simple tensors.
Indeed, suppose that $f = \sum_{n \geq 0} \delta_n \otimes x_n \in \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$, then

$$\|\sigma(f)\| = \left\| \sum_{n \geq 0} V^n \pi(x_n) \right\| \leq \sum_{n \geq 0} \|V^n\| \, \|\pi(x_n)\| \leq \sum_{n \geq 0} \|x_n\| = \|f\|_1$$

and

$$\sigma(\delta_n \otimes x)\sigma(\delta_m \otimes y) = V^n \pi(x) V^m \pi(y) =$$
$$= V^n V^m \pi(\alpha^m(x)) \pi(y) = V^{n+m} \pi(\alpha^m(x) y) =$$
$$= \sigma(\delta_{n+m} \otimes \alpha^m(x) y) = \sigma((\delta_n \otimes x)(\delta_m \otimes y)).$$

□

Before we give the definition of the semi-crossed product we need to establish the fact that the set of representations $V \times \pi$ of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ is not empty and that there is a faithful representation of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$. In order to do so, we will start with a faithful representation of $(\mathscr{A}, \alpha)$ and that will yield a faithful representation of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ from the preceding proposition.

Choose any faithful $*$-representation π of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and let $\mathscr{K} = \ell^2(\mathbb{Z}^+, \mathscr{H})$ denote the Hilbert space of all sequences $\{\xi_n\}_{n\geq 0}$ such that ξ_n is an element in $\mathscr{H}$ for each $n \geq 0$ and $\sum_{n\geq 0}\|\xi_n\|^2 < \infty$.

Define a representation $\tilde{\pi}$ of $\mathscr{A}$ on $\mathscr{K}$ by

$$\tilde{\pi}(x)(\xi_0, \xi_1, \xi_2, ...) = (\pi(x)\xi_0, \pi(\alpha(x))\xi_1, \pi(\alpha^2(x))\xi_2, ...)$$

Now if U_+ is the unilateral shift on $\mathscr{K}$ i.e.

$$U_+(\xi_0, \xi_1, \xi_2, ...) = (0, \xi_0, \xi_1, ...),$$

then U_+ is of course an isometry and it is evident that

$$U_+\tilde{\pi}(\alpha(x)) = \tilde{\pi}(x)U_+.$$

The above relation implies that $(\tilde{\pi}, U_+)$ is an isometric covariant representation and so it yields our desired representation $U_+ \times \tilde{\pi}$ of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$. [19] To check that $U_+ \times \tilde{\pi}$ is faithful, suppose that $f = \sum_{n\geq 0} \delta_n \otimes x_n$ is an element of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ such that

$$(U_+ \times \tilde{\pi})(f)(\xi) = 0, \quad \text{for every} \quad \xi \in \mathscr{K}.$$

Pick $x, y \in \mathscr{H}$, $n \in \mathbb{Z}^+$ and $\xi_1, \xi_n \in \mathscr{K}$ where

$$\xi_1(k) = \begin{cases} x, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$$

and

$$\xi_n(k) = \begin{cases} y, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}.$$

Note that

$$(U_+ \times \tilde{\pi}(f))(\xi_1) = \sum_{k\geq 0} U_+^k \tilde{\pi}(x_k)(\xi_1) = \sum_{k\geq 0} U_+^k(\pi(x_k)x, 0, \dots)$$

and therefore if we set $\eta = (U_+ \times \tilde{\pi}(f))(\xi_1)$, then $\eta(n) = \pi(x_n)$. We have

$$\begin{aligned} 0 &= \langle (U_+ \times \tilde{\pi}(f))(\xi_1), \xi_n\rangle_{\mathscr{K}} = \langle \eta, \xi_n\rangle_{\mathscr{K}} = \\ &= \sum_{s\geq 0} \langle \eta(s), \xi_n(s)\rangle_{\mathscr{H}} = \langle \eta(n), y\rangle_{\mathscr{H}} = \langle \pi(x_n)x, y\rangle_{\mathscr{H}}. \end{aligned}$$

Thus, we have that: $\forall n \in \mathbb{Z}^+, \pi(x_n) = 0$ and since π is faithful, $x_n = 0$ and so $f = 0$.
We are now ready for the definition of the semi-crossed product:

Definition 4.1.2. We define a norm

$$\|f\| = \sup\{\|V \times \pi(f)\|: (\pi, V) \text{ isometric covariant representation of } (\mathscr{A}, \alpha)\}$$

on $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$. We will denote the completion of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ with respect to this norm by $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ and we will call this Banach algebra the semi-crossed product of $\mathscr{A}$ with α.

The semi-crossed product enjoys the following Universal property: Suppose that (π, V) is an isometric covariant representation of $(\mathscr{A}, \alpha)$. We denote by $\text{alg}(\pi(\mathscr{A}), V)$ the Banach algebra generated by the elements $V, \pi(a)$ for $a \in \mathscr{A}$, then

$$\rho : \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha) \to \text{alg}(\pi(\mathscr{A}), V), \quad \sum_{k=0}^{\infty} \delta_n \otimes a_n \to \sum_{k=0}^{\infty} V^n \pi(a_n)$$

is a bounded homomorphism, since

$$\left\| \sum_{k=0}^{\infty} V^n \pi(a_n) \right\| = \|V \times \pi(f)\| \leq \sup_{V \times \pi} \|(V \times \pi)(f)\|$$

and therefore it extends to $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ and in the case that $\mathscr{A}$ is unital it is also surjective, since

$$(V \times \pi)(\delta_1 \otimes \mathbf{1}_{\mathscr{A}}) = V\pi(\mathbf{1}_{\mathscr{A}}) = V.$$

Note that $\mathscr{A}$ can be embedded isometrically into $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ by

$$x \to \delta_0 \otimes x.$$

4.2 Embedding $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ in $\mathbb{Z} \times_\alpha \mathscr{A}$

Definition 4.2.1. Let α be a $*$-endomorphism of a C^*-algebra $\mathscr{A}$, let π be a $*$-representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ and $U \in \mathbf{B}(\mathscr{H})$ a unitary. We say that (π, U) is a unitary covariant representation of $(\mathscr{A}, \alpha)$ if the following relation is satisfied

$$U\pi(\alpha(x)) = \pi(x)U.$$

Remark 15. Note that every unitary covariant representation is an isometric covariant representation, for the pair $(\mathscr{A}, \alpha)$. So if we define for $f \in \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$
$\|f\|_{un} = \sup\{\|U \times \pi(f)\| : \text{where } (\pi, U) \text{ is unitary covariant representation of } (\mathscr{A}, \alpha)\}$, then it is immediate that

$$\|f\|_{un} \leq \|f\|$$

where $\|.\|$ is the norm defined in definition 4.1.2. Actually, by using the Wold decomposition theorem in the case where α is a $*$-automorphism, we are going to prove that they are equal and thus $\|.\|_{un}$ is also a norm, so the completion of $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ with respect to this norm would be $\mathbb{Z}^+ \times_\alpha \mathscr{A}$. To avoid confusion, from now on we will denote by $\|.\|_{is}$ the norm defined in 4.1.2.

Let α be a unital $*$-endomorphism of a unital C^*-algebra $\mathscr{A}$ and (π, V) an isometric covariant representation on a Hilbert space $\mathscr{H}$ and let $L = \ker V^*$,
$V_s = V|_{M_+(L)}$, $V_u = V|_{M_+(L)^\perp}$ as in theorem 2.2.1 and U_V as in remark 3.
First, we note that $M_+(L)$ is a reducing subspace for $\pi(\mathscr{A})$.
Indeed, since

$$\pi(x)V = V\pi(\alpha(x)) \quad \forall x \in \mathscr{A},$$

we have that $V(\mathscr{H})$ is invariant for $\pi(\mathscr{A})$, which is a selfadjoint sub-algebra of $\mathbf{B}(\mathscr{H})$ and thus for $x \in \mathscr{A}$ we have that $\pi(x)V(\mathscr{H}) \subseteq V(\mathscr{H})$ and also that $\pi(x)^*V(\mathscr{H}) \subseteq V(\mathscr{H})$, which implies that $V(\mathscr{H})$ is reducing to $\pi(\mathscr{A})$. Now let $x \in \mathscr{A}$, $n \geq 1$ and $l \in L$. Then,

$$\pi(x)V^n(l) = V\pi(\alpha(x))V^{n-1}l = \cdots = V^n\pi(\alpha^n(x))l \in V^n(L)$$

and so

$$\begin{aligned} &\pi(\mathscr{A})V^n(L) \subseteq V^n(L), \quad n \geq 0 \\ &\Rightarrow \pi(\mathscr{A})\left(\bigoplus_{n\geq 0} V^n(L)\right) \subseteq \bigoplus_{n\geq 0} V^n(L) \\ &\Rightarrow \pi(\mathscr{A})M_+(L) \subseteq M_+(L). \end{aligned}$$

Therefore, we can split the covariant representation, and by split we mean that if $x \in \mathscr{A}$ and $h = h_1 + h_2 \in \mathscr{H}$ where $h_1 \in M_+(L)$ and $h_2 \in M_+(L)^\perp$ then

$$\pi(x)(h) = \pi(x)|_{M_+(L)}(h_1) + \pi(x)|_{M_+(L)^\perp}(h_2)$$

and

$$V(h) = V_s(h_1) + V_u(h_2).$$

If we denote by $\pi|_{M_+(L)}$ the representation of $\mathscr{A}$ $M_+(L)$, where for $a \in \mathscr{A}$

$$\pi|_{M_+(L)}(a) = \pi(a)|_{M_+(L)}$$

and by $\pi|_{M_+(L)^\perp}$ the representation of $\mathscr{A}$ in $M_+(L)^\perp$, where

$$\pi|_{M_+(L)^\perp}(a) = \pi(a)|_{M_+(L)^\perp},$$

then we have that $(\pi|_{M_+(L)}, V_s)$ is an isometric covariant representation and $(\pi|_{M_+(L)^\perp}, V_u)$ is unitary covariant representation. We should also denote by π_0 the restriction of π to L (i.e. for $x \in \mathscr{A},\ \pi_0(x) = \pi(x)|_L$).
One can see that for $x \in \mathscr{A}$ and $(x_0, x_1, x_2, ...) \in \ell^2(\mathbb{Z}^+, L)$

$$\begin{aligned} &U_V\pi|_{M_+(L)}(x)U_V{}^*(x_0, x_1, x_2, ...) = U_V\pi|_{M_+(L)}(x)\sum_{n\geq 0} V^n(x_n) \\ &= U_V\sum_{n\geq 0}\pi|_{M_+(L)}(x)V_s{}^n(x_n) = U_V\sum_{n\geq 0}V_s{}^n\pi|_{M_+(L)}(\alpha^n(x))x_n \\ &= U_V\sum_{n\geq 0}V^n\pi_0(\alpha^n(x))x_n = (\pi_0(x)x_0, \pi_0(\alpha(x))x_1, \pi_0(\alpha^2(x))x_2, ...) \end{aligned}$$

and as a matrix

$$U_V\pi|_{M_+(L)}(x)U_V{}^* = \begin{pmatrix} \pi_0(x) & 0 & 0 & \cdots \\ 0 & \pi_0(\alpha(x)) & 0 & \cdots \\ 0 & 0 & \pi_0(\alpha^2(x)) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 4.2.1. *Let α be a $*$-automorphism of a C^*-algebra $\mathscr{A}$. If $f \in \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$, then* $\sup\{\|U \times \pi(f)\| :$ *where (π, U) is a unitary covariant representation of $(\mathscr{A}, \alpha)$*$\}$= $\sup\{\|V \times \rho(f)\|$: *where (ρ, V) is an isometric covariant representation of $(\mathscr{A}, \alpha)$*$\}$.

Proof. Let $f = \sum_{n\geq 0}\delta_n \otimes y_n \in \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$, we have already noted that

$$\|f\|_{un} \leq \|f\|_{is}.$$

So, let (π, V) be an isometric covariant representation of $(\mathscr{A}, \alpha)$ on a Hilbert space $\mathscr{H}$. From our analysis above and using the same notation, we have that

$$(\pi, V) = (\pi|_{M_+(L)^\perp}, V_u) \oplus (\pi|_{M_+(L)}, V_s).$$

We define

$$\tilde{\pi} : \mathscr{A} \to \mathbf{B}(\ell^2(\mathbb{Z}, \mathscr{H}))$$

where for $x \in \mathscr{A}$ and $(..., x_{-1}, x_0, x_1, x_2...) \in \ell^2(\mathbb{Z}, \mathscr{H})$

$$\tilde{\pi}(x)(..., x_{-1}, x_0, x_1, x_2...) = (..., \pi(\alpha^{-1}(x))x_{-1}, \pi(x)x_0, \pi(\alpha(x))x_1, \pi(\alpha^2(x))x_2, ...)$$

and also

$$\tilde{S} : \ell^2(\mathbb{Z}, \mathscr{H}) \to \ell^2(\mathbb{Z}, \mathscr{H})$$

where

$$\tilde{S} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbf{0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \mathbf{1}_{\mathscr{H}} & \mathbf{0} & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then for $x \in \mathscr{A}$ it is evident that

$$\tilde{\pi}(x)\tilde{S} = \tilde{S}\tilde{\pi}(\alpha(x))$$

and $\tilde{S}$ is unitary.

So, $(\tilde{\pi}, \tilde{S})$ is a unitary covariant representation of $(\mathscr{A}, \alpha)$. We can regard $\ell^2(\mathbb{Z}^+, L)$ as the closed subspace of $\ell^2(\mathbb{Z}, \mathscr{H})$, consisting of the elements of the form $(..., 0, 0, x_0, x_1, x_2, ...)$ where $\{x_n \,:\, n \in \mathbb{Z}^+\} \subseteq L$.

Notice that since $\pi(\mathscr{A})(L) \subseteq L$ we have that $\tilde{\pi}(\mathscr{A})(\ell^2(\mathbb{Z}^+, L)) \subseteq \ell^2(\mathbb{Z}^+, L)$.

Now, let $\tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}$ be the representation of $\mathscr{A}$ in $\ell^2(\mathbb{Z}^+, L)$ given by:

if $(x_0, x_1, x_2, ...) \in \ell^2(\mathbb{Z}^+, L)$ and $a \in \mathscr{A}$

$$\tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}(a)(x_0, x_1, x_2, ...) = (\pi(a)x_0, \pi(\alpha(a))x_1, \pi(\alpha^2(a))x_2, ...).$$

Then $(\tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}, \tilde{S}|_{\ell^2(\mathbb{Z}^+,L)})$ satisfies the covariance relation and so it is an isometric covariant representation and

$$\|\tilde{S}|_{\ell^2(\mathbb{Z}^+,L)} \times \tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}(f)\| = \left\| \sum_{n\geq 0} \tilde{S}^n|_{\ell^2(\mathbb{Z}^+,L)} \tilde{\pi}(y_n)|_{\ell^2(\mathbb{Z}^+,L)} \right\|$$
$$= \left\| \left(\sum_{n\geq 0} \tilde{S}^n \tilde{\pi}(y_n) \right) \Big|_{\ell^2(\mathbb{Z}^+,L)} \right\| \leq \left\| \sum_{n\geq 0} \tilde{S}^n \tilde{\pi}(y_n) \right\| = \|\tilde{S} \times \tilde{\pi}(f)\|$$

Furthermore,

$$\tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)} = U_V \pi|_{M_+(L)} U_V{}^* \quad \textit{and} \quad U_V V_s U_V{}^* = \tilde{S}|_{\ell^2(\mathbb{Z}^+,L)}$$

and thus,

$$(V_s \times \pi|_{M_+(L)}) \left(\sum_{n\geq 0} \delta_n \otimes y_n \right) = \sum_{n\geq 0} {V_s}^n \pi|_{M_+(L)}(y_n) =$$
$$\sum_{n\geq 0} U_V \tilde{S}^n|_{\ell^2(\mathbb{Z}^+,L)} {U_V}^* U_V \tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}(y_n) {U_V}^*$$
$$= U_V \left(\sum_{n\geq 0} \tilde{S}^n|_{\ell^2(\mathbb{Z}^+,L)} \tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)}(y_n) \right) {U_V}^*$$
$$= U_V (\tilde{S}|_{\ell^2(\mathbb{Z}^+,L)} \times \tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)})(f) {U_V}^*$$
$$\implies \|(V_s \times \pi|_{M_+(L)})(f)\| = \|(\tilde{S}|_{\ell^2(\mathbb{Z}^+,L)} \times \tilde{\pi}|_{\ell^2(\mathbb{Z}^+,L)})(f)\|.$$

Finally,

$$\|V \times \pi(f)\| = \|(V|_{M_+(L)^\perp} \times \pi|_{M_+(L)^\perp})(f) \oplus (V|_{M_+(L)} \times \pi|_{M_+(L)})(f)\| =$$
$$= \max\{\|(V|_{M_+(L)^\perp} \times \pi|_{M_+(L)^\perp})(f)\|, \|(V|_{M_+(L)} \times \pi|_{M_+(L)})(f)\|\}$$
$$\leq \max\{\|(V|_{M_+(L)^\perp} \times \pi|_{M_+(L)^\perp})(f)\|, \|(\tilde{S} \times \tilde{\pi})(f)\|\} \leq \|f\|_{un}$$

and by taking supremum over all isometric covariant representations we get the desired $\|f\|_{is} \leq \|f\|_{un}$, and thus $\|f\|_{is} = \|f\|_{un}$. □

Remark 16. The result we just proved enable us to embed completely isometrically $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ into $\mathbb{Z} \times_\alpha \mathscr{A}$, in the case where α is a $*$-automorphism.
At first we should note that if $f = \sum_{n\geq 0} \delta_n \otimes x_n \in \ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$, then we can identify f with the element of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$, $\tilde{f} = \sum_{n\in\mathbb{Z}} \delta_n \otimes x_n$ where for every $n < 0$, $x_n = 0$ and so we can see $\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha)$ as a closed sub-algebra of $\ell^1(\mathscr{A}, \mathbb{Z}, \alpha)_l$.
Now, (π, U) is a left covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$ iff (π, U) is also a unitary covariant representation of $(\mathscr{A}, \mathbb{Z}^+, \alpha)$.
Thus,

$$\|f\|_l = \|f\|_{un}$$

where $\|.\|_l$ is the norm of the left crossed product $\mathbb{Z} \times_\alpha \mathscr{A}$. In particular, the exact same arguments work also for $n \times n$ matrices in $M_n(\ell^1(\mathscr{A}, \mathbb{Z}^+, \alpha))$ and therefore $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ is completely isometric with a closed subalgebra of $\mathbb{Z} \times_\alpha \mathscr{A}$.

We define for $f \in \ell^1(\mathbb{Z}^+, \mathscr{A}, \alpha))$,

$$\|f\|_s = \sup\{\|(S \times \pi)(f)\| : (\pi, S) \text{ isometric covariant representation of} (\mathscr{A}, \alpha), S \text{ shift}\}$$

Proposition 4.2.1. *Let α be a $*$-automorphism of a C^*-algebra $\mathscr{A}$ and suppose that $f \in \ell^1(\mathbb{Z}^+, \mathscr{A}, \alpha)$. Then $\|f\|_{is} = \|f\|_s$.*

Proof. Let (π, U) be a unitary covariant representation of $(\mathscr{A}, \mathbb{Z}^+, \alpha)$, since $\|f\|_{un} = \|f\|_{is}$, it suffices to show that for every $\epsilon > 0$ there exists an isometric covariant representation (σ, S) where S is a shift, such that

$$\|(U \times \pi)(f)\| \leq \|(S \times \sigma)(f)\| + \epsilon.$$

As in remark 16, we can embed isometrically the semi-crossed product $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ into the left crossed product $\mathbb{Z} \times_\alpha \mathscr{A}$ and by theorem 3.2.3 the left crossed product by $\mathbb{Z}$ coincides with the left reduced crossed product by $\mathbb{Z}$.
Thus,

$$\|f\| = \|(V \times \tilde{\rho})(f)\|$$

where $\rho : \mathscr{A} \to \mathbf{B}(\mathscr{H})$ is a faithful $*$-representation of $\mathscr{A}$ and $\tilde{\rho} : \mathscr{A} \to \mathbf{B}(\ell^2(\mathbb{Z}, \mathscr{H}))$ is a $*$-representation such that for $x \in \mathscr{A}$ and $(..., x_{-1}, x_0, x_1, ...) \in \ell^2(\mathbb{Z}, \mathscr{H})$ we have

$$\tilde{\rho}(x)(..., x_{-1}, x_0, x_1, ...) = (..., \rho(\alpha^{-1}(x))x_{-1}, \rho(x)x_0, \rho(\alpha(x))x_1, ...)$$

and V is the bilateral shift. Since (U, π) is a left unitary covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$ we have that

$$\|(U \times \pi)(f)\| \leq \|(V \times \tilde{\rho})(f)\|.$$

For every $n \geq 0$ let $\ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ be the subspace of $\ell^2(\mathbb{Z}, \mathscr{H})$, which consists of the elements $\xi = (..., \xi_{-1}, \xi_0, \xi_1, ...)$ such that $\xi_k = 0$ for $k < -n$.
We can easily see that for all n the subspace $\ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ is invariant under V and $\tilde{\rho}(x)$ for all $x \in \mathscr{A}$ and it is evident that $\bigcup_{n \geq 0} \ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ is a dense subspace of $\ell^2(\mathbb{Z}, \mathscr{H})$.
Therefore, for $\epsilon > 0$ we can pick n such that

$$\|(V \times \tilde{\rho})(f)|_{\ell^2_{+n}(\mathbb{Z}, \mathscr{H})}\| \geq \|(V \times \tilde{\rho})(f)\| - \epsilon = \|f\| - \epsilon \geq \|(U \times \pi)(f)\| - \epsilon.$$

Indeed, for $\epsilon > 0$ and $\eta > 0$, there exists $w \in \ell^2(\mathbb{Z}, \mathscr{H})$ such that $\|w\| = 1$ and

$$\|(V \times \tilde{\rho})(f)(w)\| > \|(V \times \tilde{\rho})(f)\| - \eta.$$

Now since $\bigcup_{n \geq 0} \ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ is dense in $\ell^2(\mathbb{Z}, \mathscr{H})$, we can pick $z \in \bigcup_{n \geq 0} \ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ such that $\|z\| = 1$ and

$$\|z - w\| < \frac{\epsilon}{2\|V \times \tilde{\rho}\|}.$$

Thus,

$$\|V \times \tilde{\rho}(f)(w)\| - \|V \times \tilde{\rho}(f)(z)\| \leq \|V \times \tilde{\rho}(f)(w) - V \times \tilde{\rho}(f)(z)\| < \frac{\epsilon}{2}$$

which implies that

$$\|V \times \tilde{\rho}(f)(z)\| > \|V \times \tilde{\rho}(f)(w)\| - \frac{\epsilon}{2}.$$

There exists n such that $z \in \ell^2_{+n}(\mathbb{Z}, \mathscr{H})$ and so

$$\begin{aligned} &\|V \times \tilde{\rho}(f)|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}\| \geq \|V \times \tilde{\rho}(f)(z)\| \\ &> \|V \times \tilde{\rho}(f)(w)\| - \frac{\epsilon}{2} > \|V \times \tilde{\rho}(f)\| - \frac{\epsilon}{2} - \eta. \end{aligned}$$

Since η is arbitrary we conclude that

$$\|V \times \tilde{\rho}(f)|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}\| \geq \|V \times \tilde{\rho}(f)\| - \frac{\epsilon}{2} > \|V \times \tilde{\rho}(f)\| - \epsilon.$$

Now if we denote by $\tilde{\rho}|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}$ the representation of $\mathscr{A}$ such that for $x \in \mathscr{A}$

$$\tilde{\rho}|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}(x) = \tilde{\rho}(x)|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})},$$

then $(\tilde{\rho}|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}, V|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})})$ is an isometric covariant representation where $V|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}$ is a shift operator and

$$\|(V|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})} \times \tilde{\rho}|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})})(f)\| = \|(V \times \tilde{\rho})(f)|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}\|.$$

Thus, if we set $\sigma = \tilde{\rho}|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}$ and $S = V|_{\ell^2_{+n}(\mathbb{Z},\mathscr{H})}$ we have that

$$\|(U \times \pi)(f)\| \leq \|(S \times \sigma)(f)\| + \epsilon,$$

which yields the desired result. □

Chapter 5

C^*-correspondences

In this chapter we are going to introduce C^*-correspondences and operator algebras associated with their representations. We will see that the Cuntz-Pimsner algebra is in a way, a generalization of the crossed product and the tensor algebra, a generalization of the semi-crossed product, since they arise from a particular example of a C^*-correspondence. The main theorem of this chapter is the gauge-invariance uniqueness theorem for which we are going to present a proof from [12]. In general, we are going to follow [17].

5.1 C^*-correspondences and their representations

Definition 5.1.1. We say that $(X, \mathscr{A}, \phi)$ is a C^*-correspondence if X is Hilbert $\mathscr{A}$-module, where $\mathscr{A}$ is a unital C^*-algebra and $\phi : \mathscr{A} \to \mathscr{L}(X)$ is a $*$-homomorphism (left action).

We say that $(X, \mathscr{A}, \phi)$ is injective, if ϕ is injective and that it is non-degenerate if $\phi(\mathscr{A})X$ is dense in X.

Example 5.1.1. [14, example 3.4] Let $\mathscr{A}$ be a unital C^*-algebra and $\alpha : \mathscr{A} \to \mathscr{A}$ a $*$-endomorphism. The space $\mathscr{A}$ is a Hilbert $\mathscr{A}$-module. We define the left action for $a, x \in \mathscr{A}$ to be

$$\phi(a)x := \alpha(a)x.$$

Then $(\mathscr{A}, \mathscr{A}, \phi)$ is a C^*-correspondence which we will denote by $\mathscr{A}_\alpha$. We should note that by using remark 6 we identify $\mathscr{A}$ with $\mathscr{K}(\mathscr{A}) \subseteq \mathscr{L}(\mathscr{A})$ and therefore the range of ϕ lies in $\mathscr{L}(\mathscr{A})$.

Example 5.1.2. Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{H}$ a Hilbert space such that $\mathscr{A} \subseteq \mathbf{B}(\mathscr{H})$ and $X \subseteq \mathbf{B}(\mathscr{H})$ a closed $\mathscr{A}$-bimodule that satisfies $X^*X \subseteq \mathscr{A}$.

The space X is a right $\mathscr{A}$-module and if we define $\langle ., . \rangle : X \times X \to \mathscr{A}$ to be the $\mathscr{A}$-valued inner product given by

$$\langle x, y \rangle = x^* y$$

for $x, y \in X$, then since $X^*X \subseteq \mathscr{A}$ we have that X is a Hilbert $\mathscr{A}$-module.
For $a \in \mathscr{A}$ set

$$\phi(a) = M_a$$

where $M_a : X \to X$ is the map such that for $x \in X$, $M_a(x) = ax$.
Note that M_a is adjointable

$$\langle M_a(x), y\rangle = \langle ax, y\rangle = x^*a^*y = x^*M_{a^*}(y) = \langle x, M_{a^*}(y)\rangle.$$

Therefore, ϕ is a $*$-homomorphism and $(X, \mathscr{A}, \phi)$ is a C^*-correspondence. We call C^*-correspondences of this form, concrete C^*-correspondences. We will prove that every C^*-correspondence is a concrete C^*-correspondence.

Definition 5.1.2. Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence where $\mathscr{A}$ is unital and let $\mathscr{B}$ be a C^*-algebra. We say that a pair (π, t) is a Toeplitz representation of $(X, \mathscr{A}, \phi)$, where $\pi : \mathscr{A} \to \mathscr{B}$ is a $*$-homomorphism and $t : X \to \mathscr{B}$ a linear map, if for each $a \in \mathscr{A}$ and each $\xi, \eta \in X$ the following relations hold:

(i) $\pi(a)t(\xi) = t(\phi(a)\xi)$,

(ii) $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta\rangle)$.

We say that (π, t) is injective iff π is injective.
We denote by $C^*(\pi, t)$ the C^*-algebra generated by the images of π, t in $\mathscr{B}$ and we say that (π, t) is surjective if $C^*(\pi, t) = \mathscr{B}$.

Condition (ii) implies that

$$t(\xi)\pi(a) = t(\xi a).$$

Indeed,

$$\begin{aligned}
&\|t(\xi)\pi(a) - t(\xi a)\|^2 = \|((t(\xi)\pi(a))^* - t(\xi a)^*)(t(\xi)\pi(a) - t(\xi a))\| \\
&= \|\pi(a)^*t(\xi)^*t(\xi)\pi(a) - \pi(a)^*t(\xi)^*t(\xi a) - t(\xi a)^*t(\xi)\pi(a) + t(\xi a)^*t(\xi a)\| \\
&= \|\pi(a)^*\pi(\langle \xi, \xi\rangle)\pi(a) - \pi(a^*)\pi(\langle \xi, \xi a\rangle) - \pi(\langle \xi a, \xi\rangle)\pi(a) + \pi(\langle \xi a, \xi a\rangle)\| \\
&= \|\pi(a^*\langle \xi, \xi\rangle a) - \pi(a^*\langle \xi, \xi a\rangle) - \pi(a^*\langle \xi, \xi\rangle)\pi(a) + \pi(\langle \xi a, \xi a\rangle)\| = 0.
\end{aligned}$$

Note also that

$$\|t(\xi)\|^2 = \|t(\xi)^*t(\xi)\| = \|\pi(\langle \xi, \xi\rangle)\| \leq \|\langle \xi, \xi\rangle\| = \|\xi\|_X^2$$

and so in the case that π is injective, we have that $\|\pi(\langle \xi, \xi\rangle)\| = \|\langle \xi, \xi\rangle\|$ and thus t is an isometry.
The arguments that appear in the proof of the following proposition are based on the proof of Proposition 2.4.2 in [3].

Proposition 5.1.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence. There exists a Toeplitz representation $(\tilde{\pi}_u, \tilde{t}_u)$ of $(X, \mathscr{A}, \phi)$ that satisfies the following condition:*

If (π, t) is a Toeplitz representation of $(X, \mathscr{A}, \phi)$, then there exists a (unique) $*$-epimorphism $\tilde{\rho} : C^*(\tilde{\pi}_u, \tilde{t}_u) \to C^*(\pi, t)$ such that

$$\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a) \quad \textit{and} \quad \tilde{\rho}(\tilde{t}_u(\xi)) = t(\xi),$$

for all $a \in \mathscr{A}$ and $\xi \in X$.
We call $(\tilde{\pi}_u, \tilde{t}_u)$ the universal Toeplitz representation of $(X, \mathscr{A}, \phi)$.

Proof. Suppose that $|\mathscr{A} \times X| \leq \beta$ where β is a cardinal that we choose such that $\beta^{\aleph_0} = \beta$. We set $\mathscr{F}$ to be the set of all Toeplitz representations $(\pi, t, \mathscr{H}_{(\pi,t)})$ of $(X, \mathscr{A}, \phi)$ where $\mathscr{H}_{(\pi,t)} = \ell^2(J)$ is such that $|J| \leq \beta$ and $\tilde{\mathscr{H}} = \bigoplus_{(\pi,t)\in\mathscr{F}} \mathscr{H}_{(\pi,t)}$. We define

$$\tilde{\pi}_u = \oplus\{\pi : (\pi, t) \in \mathscr{F}\} \quad \text{and} \quad \tilde{t}_u = \oplus\{t : (\pi, t) \in \mathscr{F}\},$$

then it is immediate that $\tilde{\pi}_u : \mathscr{A} \to \mathbf{B}(\tilde{\mathscr{H}})$ is a $*$-representation and $\tilde{t}_u : X \to \mathbf{B}(\tilde{\mathscr{H}})$ is linear. Note that for each $\xi \in X$ and $x = \sum_{(\pi,t)\in\mathscr{F}} x_{(\pi,t)} \in \tilde{\mathscr{H}}$ where the sum converges in the norm of $\tilde{\mathscr{H}}$, we have that

$$\|\tilde{t}_u(\xi)x\|^2 = \sum_{(\pi,t)\in\mathscr{F}} \|t(\xi)x_{(\pi,t)}\|^2 \leq \|\xi\|^2 \sum_{(\pi,t)\in\mathscr{F}} \|x_{(\pi,t)}\|^2 = \|\xi\|^2\|x\|^2,$$

where we used the fact that each pair (π, t) is a Toeplitz representation of $(X, \mathscr{A}, \phi)$. It is now implied that for each $\xi \in X$ we have that $\tilde{t}_u(\xi) \in \mathbf{B}(\tilde{\mathscr{H}})$.

The pair $(\tilde{\pi}_u, \tilde{t}_u)$ is a Toeplitz representation of $(X, \mathscr{A}, \phi)$.
Indeed, if $x = \sum_{(\pi,t)\in\mathscr{F}} x_{(\pi,t)} \in \tilde{\mathscr{H}}$, $a \in \mathscr{A}$ and $\xi, \eta \in X$ then we have

$$\begin{aligned}\tilde{t}_u(\xi)\tilde{\pi}_u(a)x &= \tilde{t}_u(\xi)\left(\sum_{(\pi,t)\in\mathscr{F}} \pi(a)x_{(\pi,t)}\right)\\ &= \sum_{(\pi,t)\in\mathscr{F}} t(\xi)\pi(a)x_{(\pi,t)} = \sum_{(\pi,t)\in\mathscr{F}} t(\xi a)x_{(\pi,t)} = \tilde{t}_u(\xi a)x\end{aligned}$$

and

$$\begin{aligned}\tilde{t}_u(\xi)^*\tilde{t}_u(\eta)x &= \tilde{t}_u(\xi)^*\left(\sum_{(\pi,t)\in\mathscr{F}} t(\eta)x_{(\pi,t)}\right)\\ &= \sum_{(\pi,t)\in\mathscr{F}} t(\xi)^*t(\eta)x_{(\pi,t)} = \sum_{(\pi,t)\in\mathscr{F}} \pi(\langle\xi,\eta\rangle)x_{(\pi,t)} = \tilde{\pi}_u(\langle\xi,\eta\rangle)x.\end{aligned}$$

Suppose that $(\pi, t, \mathscr{H})$ is a Toeplitz representation of $(X, \mathscr{A}, \phi)$.

We assume first that $\dim \mathscr{H} \leq \beta$. Then there exists a unitary operator in $\mathbf{B}(\mathscr{H})$ such that $(\pi', t') \in \mathscr{F}$ where $\pi' = U^*\pi U$ and $t' = U^*tU$. We define $\tilde{\rho}' : C^*(\tilde{\pi}_u, \tilde{t}_u) \to \mathbf{B}(\mathscr{H}_{(\pi',t')})$ given by

$$\tilde{\rho}'(x) = P_{(\pi',t')}x|_{\mathscr{H}_{(\pi',t')}}, \quad x \in C^*(\tilde{\pi}_u, \tilde{t}_u),$$

where $P_{(\pi',t')}$ is the projection of $\tilde{\mathscr{H}}$ onto its (π', t')-th coordinate. Then $\tilde{\rho}'$ is a $*$-homomorphism and for each $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\tilde{\rho}'(\tilde{\pi}_u(a)) = P_{(\pi',t')}\tilde{\pi}_u(a)|_{\mathscr{H}_{(\pi',t')}} = \pi'(a)$$

and

$$\tilde{\rho}'(\tilde{t}_u(\xi)) = P_{(\pi',t')}\tilde{t}_u(\xi)|_{\mathscr{H}_{(\pi',t')}} = t'(\xi).$$

If we define $\tilde{\rho} : C^*(\tilde{\pi}_u, \tilde{t}_u) \to \mathbf{B}(\mathscr{H})$ to be the $*$-homomorphism where for each x in $C^*(\tilde{\pi}_u, \tilde{t}_u)$

$$\tilde{\rho}(x) = U\tilde{\rho}'(x)U^*,$$

then

$$\tilde{\rho}(\tilde{\pi}_u(a)) = U\tilde{\rho}'(\tilde{\pi}_u(a))U^* = U\pi'(a)U^* = \pi(a)$$

and

$$\tilde{\rho}(\tilde{t}_u(\xi)) = U\tilde{\rho}'(\tilde{t}_u(\xi))U^* = Ut'(\xi)U^* = t(\xi).$$

Note that the above implies that the range of $\tilde{\rho}$ is $C^*(\pi, t)$ and hence $(\tilde{\pi}_u, \tilde{t}_u)$ has the desired universal property in the case that $\dim(\mathscr{H}) \leq \beta$.

Assume now that $\mathscr{H}$ is of arbitrary dimension. Let J_0 be a set whose cardinality is equal to $\dim(\mathscr{H})$. Let $\mathscr{G}$ be the set of pairs $(J, \{K_j\}_{j\in J})$, where $J \subseteq J_0$, the K_j's are mutually orthogonal non-zero subspaces of $\mathscr{H}$ and each K_j has dimension at most β and is reducing for $\pi(\mathscr{A})$ and $t(X)$. We set $(J, \{K_j\}_{j\in J}) \leq (J', \{K'_j\}_{j\in J'})$ if $J \subseteq J'$ and for each $j \in J$ we have $K_j = K'_j$. Suppose that $\{(J^{(i)}, \{K_j^{(i)}\}_{j\in J^{(i)}})\}_{i\in I}$ is a chain in $\mathscr{G}$, then it is evident that

$$\left(\bigcup_{i\in I} J^{(i)}, \left\{K_j^{(i)} : i \in I, j \in J^{(i)}\right\}\right) \in \mathscr{G}$$

and for each $i \in I$ we have

$$(J^{(i)}, \{K_j^{(i)}\}_{j\in J^{(i)}}) \leq \left(\bigcup_{i\in I} J^{(i)}, \left\{K_j^{(i)} : i \in I, j \in J^{(i)}\right\}\right).$$

Using Zorn's lemma we obtain a maximal element $(J, \{K_j\}_{j\in J})$ of $\mathscr{G}$.

We claim that $\bigoplus_{j\in J} K_j = \mathscr{H}$. Indeed, suppose that $x \in \mathscr{H}$ is orthogonal to K_j for all j in J and pick $i \in J_0 \setminus J$. Set $K_i = \overline{\{Tx : T \in C^*(\pi, t)\}}$, we prove that K_i is reducing to $\pi(\mathscr{A})$ and $t(X)$ and K_i is orthogonal to each K_j.

Let y be an element in K_i, then there exists a sequence $\{T_n : n \in \mathbb{N}\}$ of elements in $C^*(\pi, t)$

such that $T_n x \to y$. For each $a \in \mathscr{A}$ we have

$$\pi(a)y = \lim_n \pi(a)T_n x \in K_i$$

and since $\pi(\mathscr{A})$ is self-adjoint we obtain that K_i is reducing to $\pi(\mathscr{A})$. For $\xi \in X$ we have

$$t(\xi)y = \lim_n t(\xi)T_n x \in K_i \quad \text{and} \quad t(\xi)^* y = \lim_n t(\xi)^* T_n x \in K_i$$

and therefore K_i is reducing to $t(X)$.
Let a be an element in $\mathscr{A}$, $\xi \in X$, $j \in J$ and $y \in K_j$. We have

$$\langle \pi(a)x, y\rangle = \langle x, \pi(a^*)y\rangle = 0$$

and

$$\langle t(\xi)x, y\rangle = \langle x, t(\xi)^* y\rangle = 0,$$

since $\pi(\mathscr{A})K_j \subseteq K_j$ and $t(X)^* K_j \subseteq K_j$ and $x \perp K_j$.

Using the fact that $|\mathscr{A} \times X| \leq \beta$ and $\beta^{\aleph_0} = \beta$ we also obtain that $\dim K_i \leq \beta$ and therefore $(J, \{K_j\}_{j\in J}) \leq (J \cup \{i\}, \{K_j\}_{j \in J\cup\{i\}})$, which contradicts the maximality of $(J, \{K_j\}_{j\in J})$.

Hence, $\mathscr{H} = \bigoplus_{j\in J} K_j$ and we may write $\pi = \oplus_{j\in J}\pi_j$, where $\pi_j(\cdot) = \pi(\cdot)|_{K_j}$ and $t = \oplus_{j\in J} t_j$, where $t_j(\cdot) = t(\cdot)|_{K_j}$. Note the for each $j \in J$ the pair (π_j, t_j) is a Toeplitz representation of $(X, \mathscr{A}, \phi)$ on a Hilbert space K_j with $\dim K_j \leq \beta$. From the universal property of $(\tilde{\pi}_u, \tilde{t}_u)$ for Hilbert spaces with dimension less than or equal to β we obtain for each $j \in J$ a $*$-epimorphism $\tilde{\rho}_j : C^*(\tilde{\pi}_u, \tilde{t}_u) \to C^*(\pi, t)$ where for each $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\tilde{\rho}_j(\tilde{\pi}_u(a)) = \pi_j(a) \quad \text{and} \quad \tilde{\rho}_j(\tilde{t}_u(\xi)) = t_j(\xi).$$

Let $\tilde{\rho} : C^*(\tilde{\pi}_u, \tilde{t}_u) \to \mathbf{B}(\mathscr{H})$ be the direct sum of $*$-homomorphisms

$$\tilde{\rho} = \oplus_{j\in J}\tilde{\rho}_j,$$

then $\tilde{\rho}$ is a $*$-homomorphism and for each $a \in \mathscr{A}$

$$\tilde{\rho}(\tilde{\pi}_u(a)) = \oplus_{j\in J}\tilde{\rho}_j(\tilde{\pi}_u(a)) = \oplus_{j\in J}\pi_j(a) = \pi(a)$$

and for each $\xi \in X$

$$\tilde{\rho}(\tilde{t}_u(\xi)) = \oplus_{j\in J}\tilde{\rho}_j(\tilde{t}_u(\xi)) = \oplus_{j\in J} t_j(\xi) = t(\xi).$$

Finally, note that our work above implies that the range of $\tilde{\rho}$ is $C^*(\pi, t)$. We conclude that $(\tilde{\pi}_u, \tilde{t}_u)$ satisfies the universal property. □

Definition 5.1.3. The tensor algebra T_X^+ is the norm-closed subalgebra of T_X generated by the elements of the form $\tilde{\pi}_u(a), \tilde{t}_u(\xi)$ for $a \in \mathscr{A}$ and $\xi \in X$.

Definition 5.1.4. The Toeplitz-Cuntz-Pimsner C^*-algebra of a C^*-correspondence $(X, \mathscr{A}, \phi)$ is the C^*-algebra T_X generated by all element of the form $\tilde{\pi}_u(a), \tilde{t}_u(\xi)$ for $a \in \mathscr{A}$ and $\xi \in X$.

Note that if two Toeplitz representations (π_1, t_1) and (π_2, t_2) satisfy the universal property, it is immediate that $C^*(\pi_1, t_1)$ is $*$-isomorphic to $C^*(\pi_2, t_2)$ and also the norm closed algebras generated by $\{\pi_1(a), t_1(\xi) : a \in \mathscr{A}, \xi \in X\}$ and $\{\pi_2(a), t_2(\xi) : a \in \mathscr{A}, \xi \in X\}$ are completely isometrically isomorphic.

Example 5.1.3. Let $\mathscr{A}_\alpha$ be the C^*-correspondence described in example 5.1.1 and suppose that (π, t) is a Toeplitz representation of $\mathscr{A}_\alpha$ on a Hilbert space $\mathscr{H}$. Without loss of generality assume that π is non-degenerate. If π is degenerate we can restrict to $\pi(\mathbf{1}_{\mathscr{A}})(\mathscr{H})$. We have that

$$\mathbf{1}_{\mathbf{B}(\mathscr{H})} = \pi(\mathbf{1}_{\mathscr{A}}) = \pi(\langle \mathbf{1}_{\mathscr{A}}, \mathbf{1}_{\mathscr{A}} \rangle) = t(\mathbf{1}_{\mathscr{A}})^* t(\mathbf{1}_{\mathscr{A}})$$

and so $t(\mathbf{1}_{\mathscr{A}})$ is an isometry.

We also have that

$$\pi(a)t(\mathbf{1}_{\mathscr{A}}) = t(\phi(a)\mathbf{1}_{\mathscr{A}}) = t(\alpha(a)) = t(\mathbf{1}_{\mathscr{A}}\alpha(a)) = t(\mathbf{1}_{\mathscr{A}})\pi(\alpha(a)).$$

Thus, $(\pi, t(\mathbf{1}_{\mathscr{A}}))$ is an isometric covariant representation of $(\mathscr{A}, \mathbb{Z}^+, \alpha)$.
Conversely, suppose that (π, V) is an isometric covariant representation of $(\mathscr{A}, \mathbb{Z}^+, \alpha)$. We will show that if we set

$$t(\xi) = V\pi(\xi), \quad \forall \xi \in \mathscr{A}$$

then (π, t) is a Toeplitz representation of the C^*-correspondence $\mathscr{A}_\alpha$.
We already have that π is a $*$-homomorphism and t is linear and so we only need to show the relations $(i), (ii)$ of definition 5.1.2. In order to do so let a, ξ, η be elements in $\mathscr{A}$, then we have

$$\pi(a)t(\xi) = \pi(a)V\pi(\xi) = V\pi(\alpha(a)\xi) = t(\phi(a)\xi)$$

and

$$t(\xi)^* t(\eta) = \pi(\xi)^* V^* V \pi(\eta) = \pi(\xi^* \eta) = \pi(\langle \xi, \eta \rangle).$$

We prove now that $T_{\mathscr{A}}^+ = \mathbb{Z}^+ \times_\alpha \mathscr{A}$.
Let $(\tilde{\pi}_u, \tilde{t}_u)$ be the universal Toeplitz representation of the C^*-correspondence $\mathscr{A}_\alpha$ and let (π, V) be an isometric covariant representation of $(\mathscr{A}, \alpha)$. As we have already seen, (π, t) where t is the linear map given by $t(\xi) = V\pi(\xi)$ for $\xi \in \mathscr{A}$, is a Toeplitz representation of $\mathscr{A}_\alpha$. Therefore, there exists a $*$-epimorphism $\tilde{\rho} : T_{\mathscr{A}} \to C^*(\pi, t)$ such that

$$\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a) \quad \text{and} \quad \tilde{\rho}(\tilde{t}(\xi)) = t(\xi), \quad \xi, a \in \mathscr{A}.$$

The pair $(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}}))$ is an isometric covariant representation of $(\mathscr{A}, \alpha)$ and for each a in $\mathscr{A}$ we have $\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a)$ and also $\tilde{\rho}(\tilde{t}_u(\mathbf{1}_{\mathscr{A}})) = t(\mathbf{1}_{\mathscr{A}}) = V\pi(\mathbf{1}_{\mathscr{A}}) = V$, hence $(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}}))$ satisfies the universal property of the semi-crossed product and therefore $\mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}})) = \mathbb{Z}^+ \times_\alpha \mathscr{A}$. It is evident that $\mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}})) \subseteq \mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u)$ and since for each $\xi \in \mathscr{A}$ we have

$$\tilde{t}_u(\xi) = \tilde{t}_u(\mathbf{1}_{\mathscr{A}}\xi) = \tilde{t}_u(\mathbf{1}_{\mathscr{A}})\tilde{\pi}_u(\xi) \in \mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}})),$$

we obtain $\mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u(\mathbf{1}_{\mathscr{A}})) = \mathrm{alg}(\tilde{\pi}_u, \tilde{t}_u) = \mathbb{Z}^+ \times_\alpha \mathscr{A}$.

We are now going to show that for a C^*-correspondence $(X, \mathscr{A}, \phi)$ there exists an injective Toeplitz representation called the Fock representation of $(X, \mathscr{A}, \phi)$. Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and consider the interior tensor product $X \otimes_\phi X$, which is a Hilbert $\mathscr{A}$-module. From now on we will denote $X \otimes_\phi X$ by $X \otimes X$ or $X^{\otimes 2}$. We can see that if we set

$$\phi_2(a) := \phi(a) \otimes I_X,$$

then ϕ_2 is a $*$-homomorphism from $\mathscr{A}$ into $\mathscr{L}(X^{\otimes 2})$ and therefore $(X \otimes X, \mathscr{A}, \phi_2)$ becomes a C^*-correspondence.
Inductively, for $n \geq 2$ we define

$$X^{\otimes n} = X \otimes_{\phi_{n-1}} X^{\otimes (n-1)}$$

and set $X^{\otimes 0} := \mathscr{A}$ and $X^{\otimes 1} := X$.
Thus, for $n \in \mathbb{N}$ we have that $X^{\otimes n}$ is a C^*-correspondence over $\mathscr{A}$ with the $\mathscr{A}$-valued inner-product defined on simple tensors by

$$\langle \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n, \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \rangle = \langle \xi_2 \otimes \xi_3 \otimes \ldots \otimes \xi_n, \phi_{n-1}(\langle \xi_1, \eta_1 \rangle)(\eta_2 \otimes \eta_3 \otimes \ldots \otimes \eta_n) \rangle,$$

and

$$\phi_n(a)(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) = \phi(a)\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n$$

i.e. $\phi_n(a) = \phi(a) \otimes I_{m-1}$, where I_{m-1} is the identity map of $X^{\otimes (m-1)}$ and

$$(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) \cdot a = \xi_1 \otimes \xi_2 \otimes \ldots \otimes (\xi_n a).$$

Remark 17. Note that for each $m > 1$ we have that that $X^{\otimes m}$, $X \otimes_{\phi_{m-1}} X^{\otimes (m-1)}$, $X^{(\otimes m-1)} \otimes_\phi X$ are naturally isomorphic and so we can view $X^{\otimes m}$ with either the first or the second description.

Definition 5.1.5. Let us take $\xi \in X^{\otimes n}$ where $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ we define an operator

$\tau_m^n(\xi) \in \mathscr{L}(X^{\otimes n}, X^{\otimes(n+m)})$ by

$$\tau_m^n(\xi) : X^{\otimes m} \to X^{\otimes(n+m)}, \quad \eta \to \xi \otimes \eta.$$

Note that for $a \in X^{\otimes 0} = \mathscr{A}$ we have that $\tau_m^0(a) = \phi_m(a)$ for every $m \in \mathbb{N}$.
We prove now that $\tau_m^n(\xi) \in \mathscr{L}(X^{\otimes m}, X^{\otimes(n+m)})$.
Let

$$\psi_m^n : \operatorname{span}\{\zeta_1 \otimes \zeta_2 : \zeta_1 \in X^{\otimes n}, \zeta_2 \in X^{\otimes m}\} \to X^{\otimes m}$$

be the linear map given by

$$\psi_m^n(\zeta_1 \otimes \zeta_2) = \phi_m(\langle \xi, \zeta_1 \rangle)\zeta_2, \quad \zeta_1 \in X^{\otimes n}, \zeta_2 \in X^{\otimes m}.$$

We prove that ψ_m^n is continuous. For $\sum_{i=1}^k \zeta_1^{(i)} \otimes \zeta_2^{(i)}$ we have

$$\begin{aligned}
&\left\| \psi_m^n \left(\sum_{i=1}^k \zeta_1^{(i)} \otimes \zeta_2^{(i)} \right) \right\|^2 = \left\| \left\langle \sum_{i=1}^k \phi_m \left(\langle \xi, \zeta_1^{(i)} \rangle \right) \zeta_2^{(i)}, \sum_{j=1}^k \phi_m \left(\langle \xi, \zeta_1^{(j)} \rangle \right) \zeta_2^{(j)} \right\rangle \right\| \\
&= \left\| \sum_{i=1}^k \sum_{j=1}^k \left\langle \phi_m \left(\langle \xi, \zeta_1^{(i)} \rangle \right) \zeta_2^{(i)}, \phi_m \left(\langle \xi, \zeta_1^{(j)} \rangle \right) \zeta_2^{(j)} \right\rangle \right\| = \left\| \sum_{i=1}^k \sum_{j=1}^k \left\langle \xi \otimes \phi_m \left(\langle \xi, \zeta_1^{(i)} \rangle \right) \zeta_2^{(i)}, \zeta_1^{(j)} \otimes \zeta_2^{(j)} \right\rangle \right\| \\
&= \left\| \sum_{i=1}^k \sum_{j=1}^k \left\langle \xi \left\langle \xi, \zeta_1^{(i)} \right\rangle \otimes \zeta_2^{(i)}, \zeta_1^{(j)} \otimes \zeta_2^{(j)} \right\rangle \right\| = \left\| \sum_{i=1}^k \sum_{j=1}^k \left\langle \theta_{\xi,\xi} \left(\zeta_1^{(i)} \right) \otimes \zeta_2^{(i)}, \zeta_1^{(j)} \otimes \zeta_2^{(j)} \right\rangle \right\| \\
&= \left\| \sum_{i=1}^k \sum_{j=1}^k \left\langle (\theta_{\xi,\xi} \otimes I_{X^{\otimes m}}) \left(\zeta_1^{(i)} \otimes \zeta_2^{(i)} \right), \zeta_1^{(j)} \otimes \zeta_2^{(j)} \right\rangle \right\| \\
&= \left\| \left\langle (\theta_{\xi,\xi} \otimes I_{X^{\otimes m}}) \left(\sum_{i=1}^k \zeta_1^{(i)} \otimes \zeta_2^{(i)} \right), \sum_{j=1}^k \zeta_1^{(j)} \otimes \zeta_2^{(j)} \right\rangle \right\| \\
&\leq \|\theta_{\xi,\xi}\| \left\| \sum_{i=1}^k \zeta_1^{(i)} \otimes \zeta_2^{(i)} \right\|^2 \leq \|\xi\|^2 \left\| \sum_{i=1}^k \zeta_1^{(i)} \otimes \zeta_2^{(i)} \right\|^2,
\end{aligned}$$

where we used the fact that the map

$$\mathscr{L}\left(X^{\otimes m}\right) \to \mathscr{L}\left(X^{\otimes(m+n)}\right) : t \to t \otimes I_{X^{\otimes m}}$$

is a $*$-homomorphism between C^*-algebras and that $\|\theta_{x,y}\| \leq \|x\|\|y\|$.

Since $\operatorname{span}\{\zeta_1 \otimes \zeta_2 : \zeta_1 \in X^{\otimes n}, \zeta_2 \in X^{\otimes m}\}$ is a dense subspace of $X^{\otimes(n+m)}$, we can extend ψ_m^n to a unique bounded linear operator defined on $X^{\otimes(n+m)}$.
We will prove that

$$\tau_m^n(\xi)^*(\zeta_1 \otimes \zeta_2) = \psi_m^n(\zeta_1 \otimes \zeta_2), \quad \zeta_1 \in X^{\otimes n}, \zeta_2 \in X^{\otimes m},$$

and therefore $\tau_m^n(\xi)^* = \psi_m^n$.
Indeed, if $\eta \in X^{\otimes m}$ and $\zeta = \zeta_1 \otimes \zeta_2$ we have that

$$\begin{aligned}\langle \eta, \tau_m^n(\xi)^*(\zeta)\rangle &= \langle \tau_m^n(\xi)\eta, \zeta_1 \otimes \zeta_2\rangle = \langle \xi \otimes \eta, \zeta_1 \otimes \zeta_2\rangle \\ &= \langle \eta, \phi_m(\langle \xi, \zeta_1\rangle)\zeta_2\rangle = \langle \eta, \psi_m^n(\zeta_1 \otimes \zeta_2)\rangle ,\end{aligned}$$

since η was arbitrary we are done.

Lemma 5.1.1. *If $n_1, n_2, m \in \mathbb{N}$ and $\xi_1 \in X^{\otimes n_1}$, $\xi_2 \in X^{\otimes n_2}$ then*

$$\tau_{n_2+m}^{n_1}(\xi_1)\tau_m^{n_2}(\xi_2) = \tau_m^{n_1+n_2}(\xi_1 \otimes \xi_2).$$

Proof. Pick $\zeta \in X^{\otimes m}$, then

$$\begin{aligned}\tau_{n_2+m}^{n_1}(\xi_1)\tau_m^{n_2}(\xi_2)(\zeta) &= \tau_{n_2+m}^{n_1}(\xi_1)(\xi_2 \otimes \zeta) \\ &= \xi_1 \otimes \xi_2 \otimes \zeta = \tau_m^{n_1+n_2}(\xi_1 \otimes \xi_2)(\zeta).\end{aligned}$$

□

Lemma 5.1.2. *For $n, m \in \mathbb{N}$ and $\xi, \eta \in X^{\otimes n}$ and $a \in \mathscr{A}$ we have that:*

(i) $\tau_m^n(\xi)\tau_m^n(\eta)^* = \theta_{\xi,\eta} \otimes I_m$,

(ii) $\tau_m^n(\xi)^*\tau_m^n(\eta) = \phi_m(\langle \xi, \eta\rangle)$,

(iii) $\tau_m^n(\xi)\phi_m(a) = \tau_m^n(\xi a)$,

(iv) $\phi_{n+m}(a)\tau_m^n(\xi) = \tau_m^n(\phi_n(a)\xi)$.

Proof. (i) It suffices to show the equality on vectors of the form $\zeta = \zeta_1 \otimes \zeta_2 \in X^{\otimes(n+m)}$ where $\zeta_1 \in X^{\otimes n}$ and $\zeta_2 \in X^{\otimes m}$.
We have

$$\begin{aligned}\tau_m^n(\xi)\tau_m^n(\eta)^*(\zeta_1 \otimes \zeta_2) &= \tau_m^n(\xi)(\phi_m(\langle \eta, \zeta_1\rangle)\zeta_2 = \xi \otimes \phi_m(\langle \eta, \zeta_1\rangle)\zeta_2 \\ &= \xi\langle \eta, \zeta_1\rangle \otimes \zeta_2 = (\theta_{\xi,\eta} \otimes I_m)(\zeta_1 \otimes \zeta_2).\end{aligned}$$

(ii) If $\zeta \in X^{\otimes m}$ then

$$\tau_m^n(\xi)^*\tau_m^n(\eta)(\zeta) = \tau_m^n(\xi)^*(\eta \otimes \zeta) = \phi_m(\langle \xi, \eta\rangle)\zeta.$$

(iii) If $\zeta \in X^{\otimes m}$ we have that

$$\tau_m^n(\xi)\phi_m(a)(\zeta) = \xi \otimes \phi_m(a)\zeta = \xi a \otimes \zeta = \tau_m^n(\xi a)(\zeta).$$

(iv) If $\zeta \in X^{\otimes m}$ then

$$\phi_{n+m}(a)\tau_m^n(\xi)(\zeta) = \phi_{n+m}(a)(\xi \otimes \zeta) = (\phi_n(a)\xi) \otimes \zeta = \tau_m^n(\phi_n(a)\xi)(\zeta).$$

□

The $Fock\ space$ F_X is the direct sum of Hilbert $\mathscr{A}$-modules

$$X^{\otimes 0} \oplus X^{\otimes 1} \oplus X^{\otimes 2} ... := \left\{ x = (x_k)_k \in \prod_{k=0}^{\infty} X^{\otimes k} : \sum_{k=0}^{\infty} \langle x_k, x_k \rangle_{X^{\otimes k}} \text{ converges in } \mathscr{A} \right\}.$$

The space F_X is a Hilbert $\mathscr{A}$-module where for $(a, x_1, x_2, ...), (b, y_1, y_2, ...) \in F_X$,

$$\langle (a, x_1, x_2, ...), (b, y_1, y_2, ...) \rangle_{F_X} = \langle a, b \rangle_{\mathscr{A}} + \sum_{i=1}^{\infty} \langle x_i, y_i \rangle_{X^{\otimes i}}.$$

We define the left creation operator

$$t_\infty : X \to \mathscr{L}(F_X)$$

to be the map such that for $\xi \in X$ and $(a, \zeta_1, \zeta_2, ...) \in F_X$

$$t_\infty(\xi)(a, \zeta_1, \zeta_2, ...) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, ...).$$

We prove that t_∞ is well-defined.
Indeed,

$$\|t_\infty(\xi)(a, \zeta_1, \zeta_2, ...)\|^2 = \|(0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, ...)\|^2 = \left\| \langle \xi a, \xi a \rangle_X + \sum_{i=1}^{\infty} \langle \xi \otimes \zeta_i, \xi \otimes \zeta_i \rangle_{X^{\otimes (i+1)}} \right\|$$

$$\left\| \langle \xi a, \xi a \rangle_X + \sum_{i=1}^{\infty} \langle \zeta_i, \phi_i(\langle \xi, \xi \rangle_X) \zeta_i \rangle_{X^{\otimes i}} \right\| = \left\| \langle \xi a, \xi a \rangle_X + \sum_{i=1}^{\infty} \left\langle \phi_i(\langle \xi, \xi \rangle_X^{1/2}) \zeta_i, \phi_i(\langle \xi, \xi \rangle_X^{1/2}) \zeta_i \right\rangle_{X^{\otimes i}} \right\|$$

$$\leq \left\| \|\xi\|^2 \langle a, a \rangle_{\mathscr{A}} + \sum_{i=1}^{\infty} \left\| \phi_i(\langle \xi, \xi \rangle_X^{1/2}) \right\|^2 \langle \zeta_i, \zeta_i \rangle_{X^{\otimes i}} \right\| \leq \left\| \|\xi\|^2 \langle a, a \rangle_{\mathscr{A}} + \sum_{i=1}^{\infty} \left\| \langle \xi, \xi \rangle_X^{1/2} \right\|^2 \langle \zeta_i, \zeta_i \rangle_{X^{\otimes i}} \right\|$$

$$\leq \|\xi\|^2 \left\| \langle a, a \rangle_{\mathscr{A}} + \sum_{i=1}^{\infty} \langle \zeta_i, \zeta_i \rangle_{X^{\otimes i}} \right\| = \|\xi\|^2 \|(a, \zeta_1, \zeta_2, ...)\|^2,$$

where we used the fact that for positive elements c, d of a C^*-algebra with $c \leq d$ we have that $\|c\| \leq \|d\|$ and proposition 2.4.4.
It remains to prove that for each $\xi \in F_X$ the operator $t_\infty(\xi)$ is adjointable.
Let ξ be an element in X and let $(a, x_1, x_2, ...), (b, y_1, y_2, ...)$ be elements in F_X such that $x_n = z_n \otimes w_n$, where $z_n \in X$ and $w_n \in X^{\otimes (n-1)}$ for each $n \geq 2$.

Then, we have

$$\begin{aligned}
&\langle (a, x_1, x_2, ...), t_\infty(\xi)(b, y_1, y_2, ...)\rangle_{F_X} \\
&= \langle (a, x_1, x_2, ...), (0, \xi b, \xi \otimes y_1, \xi \otimes y_2, ...)\rangle_{F_X} \\
&= \langle x_1, \xi b\rangle_X + \sum_{i=1}^{\infty} \langle x_{i+1}, \xi \otimes y_i\rangle_{X^{\otimes(i+1)}} \\
&= \langle x_1, \xi\rangle_X b + \sum_{i=1}^{\infty} \langle \xi \otimes y_i, z_{i+1} \otimes w_{i+1}\rangle^*_{X^{\otimes(i+1)}} \\
&= \langle x_1, \xi\rangle_X b + \sum_{i=1}^{\infty} \langle y_i, \phi_i(\langle \xi, z_{i+1}\rangle_X) w_{i+1}\rangle^*_{X^{\otimes i}} \\
&= \langle \xi, x_1\rangle^*_X b + \sum_{i=1}^{\infty} \langle \phi_i(\langle \xi, z_{i+1}\rangle_X) w_{i+1}, y_i\rangle_{X^{\otimes i}} \\
&= \langle (\langle \xi, x_1\rangle_X, \phi_1(\langle \xi, z_2\rangle_X) w_2, \phi_2(\langle \xi, z_3\rangle_X) w_3, ...), (b, y_1, y_2, y_3, ...)\rangle_{F_X}.
\end{aligned}$$

Suppose that $v = (b, y_1, y_2, ...)$ is an element in F_X such that $\|v\| \leq 1$. If we denote the element $(a, x_1, x_2, ...)$ of the above form by u and $(\langle \xi, x_1\rangle_X, \phi_1(\langle \xi, z_2\rangle_X) w_2, \phi_2(\langle \xi, z_3\rangle_X) w_3, ...)$ by w_u, then from our calculation above we have

$$\langle u, t_\infty(\xi) v\rangle = \langle w_u, v\rangle.$$

Therefore using the Cauchy-Schwarz inequality for Hilbert modules we obtain that for each $v \in F_X$ such that $\|v\| \leq 1$,

$$\|\langle w_u, v\rangle\| = \|\langle u, t_\infty(\xi) v\rangle\| \leq \|u\| \|t_\infty(\xi) v\| \leq \|u\| \|\xi\| \|v\|$$

and by taking supremum over all $\|v\| \leq 1$ we have

$$\|w_u\| \leq \|\xi\| \|u\|.$$

This proves that the map

$$(a, x_1, x_2, ...) \to (\langle \xi, x_1\rangle_X, \phi_1(\langle \xi, z_2\rangle_X) w_2, \phi_2(\langle \xi, z_3\rangle_X) w_3, ...)$$

is continuous on the linear span of elements $(a, x_1, x_2, ...)$ of the above form and and using density, we may extend it to a unique continuous operator on F_X, which coincides with $t_\infty(\xi)^*$. We also define for $a \in \mathscr{A}$

$$\pi_\infty(a)(b, x_1, x_2, ...) = (ab, \phi_1(a) x_1, \phi_2(a) x_2, ...), \quad (b, x_1, x_2, ...) \in F_X.$$

We prove that it is well-defined and continuous.

Indeed,

$$\begin{aligned}
&\|\pi_\infty(a)(b,x_1,x_2,...)\|^2 = \|((ab,\phi_1(a)x_1,\phi_2(a)x_2,...)\|^2 = \\
&= \left\| b^*a^*ab + \sum_{i=1} \langle \phi_i(a)x_1, \phi_i(a)x_1\rangle_{X^{\otimes i}} \right\| \leq \left\| \|a\|^2 \langle b,b\rangle_{\mathscr{A}} + \sum_{i=1} \|\phi_i(a)\|^2 \langle x_i,x_i\rangle_{X^{\otimes i}} \right\| \\
&\leq \|a\|^2 \left\| \langle b,b\rangle_{\mathscr{A}} + \sum_{i=1} \langle x_i,x_i\rangle_{X^{\otimes i}} \right\| = \|a\|^2 \|(b,x_1,x_2,...)\|^2,
\end{aligned}$$

where we used proposition 2.4.4.
For $(b,x_1,x_2,...),(c,y_1,y_2,...) \in F_X$ we have

$$\begin{aligned}
&\langle (b,x_1,x_2,...), \pi_\infty(a)(c,y_1,y_2,...)\rangle_{F_X} = \langle (b,x_1,x_2,...),(ac,\phi_1(a)y_1,\phi_2(a)y_2,...)\rangle_{F_X} \\
&= \langle b,ac\rangle_{\mathscr{A}} + \sum_{i=1}^{\infty} \langle x_i, \phi_i(a)y_i\rangle_{X^{\otimes i}} = \langle a^*b,c\rangle + \sum_{i=1}^{\infty} \langle \phi_i(a)^*x_i, y_i\rangle_{X^{\otimes i}} \\
&= \langle \pi_\infty(a^*)(b,x_1,x_2,...),(c,y_1,y_2,...)\rangle_{F_X}.
\end{aligned}$$

Thus, $\pi_\infty(a) \in \mathscr{L}(F_X)$ and it is immediate that $\pi_\infty : \mathscr{A} \to \mathscr{L}(F_X)$ is a $*$-homomorphism since for $i \geq 1$, the map ϕ_i is linear and multiplicative.
Now, suppose that a is an element in $\mathscr{A}$ such that $\pi_\infty(a) = 0$. If $(b,x_1,x_2,...) \in F_X$, then

$$\pi_\infty(a)(b,x_1,x_2,...) = (ab,\phi_1(a)x_1,\phi_2(a)x_2,...) = (0,0,...) \Rightarrow ab = 0$$

If we pick $b = a^*$ we get that $\|a^*a\| = 0$ and so $a = 0$. Thus, π_∞ is injective.
Finally, to show that (π_∞, t_∞) is a Toepitz representation of $(X,\mathscr{A},\phi)$ it remains to show the relations (i), (ii) of definition 5.1.2.
For that purpose suppose that ξ,η are elements in X, $a \in \mathscr{A}$ and $(b,x_1,x_2,...) \in F_X$, then we have

$$\begin{aligned}
&\pi_\infty(a)t_\infty(\xi)(b,x_1,x_2,...) = \pi_\infty(a)(0,\xi b,\xi\otimes x_1,\xi\otimes x_2,...) \\
&= (0,\phi_1(a)(\xi b),\phi_2(a)(\xi\otimes x_1),\phi_3(a)(\xi\otimes x_2),...) \\
&= (0,\phi(a)\xi b,\phi(a)\xi\otimes x_1,\phi(a)\xi\otimes x_2,...) \\
&= (0,(\phi(a)\xi)b,(\phi(a)\xi)\otimes x_1,(\phi(a)\xi)\otimes x_2,...) = t_\infty(\phi(a)\xi)(b,x_1,x_2,...),
\end{aligned}$$

hence $\pi_\infty(a)t_\infty(\xi) = t_\infty(\phi(a)\xi)$ and

$$\begin{aligned}
&t_\infty(\eta)^*t_\infty(\xi)(b,x_1,x_2,...) = t_\infty(\eta)^*(0,\xi b,\xi\otimes x_1,\xi\otimes x_2,...) \\
&= (\langle\eta,\xi\rangle_X b,\phi_1(\langle\eta,\xi\rangle_X)x_1,\phi_2(\langle\eta,\xi\rangle_X)x_2,...) = \pi_\infty(\langle\eta,\xi\rangle_X)(b,x_1,x_2,...),
\end{aligned}$$

hence $t_\infty(\eta)^*t_\infty(\xi) = \pi_\infty(\langle\eta,\xi\rangle_X)$.

Therefore, (π_∞,t_∞) is an injective Toeplitz representation of $(X,\mathscr{A},\phi)$. This also implies

that the universal Toeplitz representation $(\tilde{\pi}_u, \tilde{t}_u)$ is injective.
Indeed, if $a \in \ker \tilde{\pi}_u$ and $\tilde{\rho} : T_X \to C^*(\pi_\infty, t_\infty)$ is the $*$-epimorphism induced from universality, then we have that

$$0 = \tilde{\rho}(\tilde{\pi}_u(a)) = \pi_\infty(a)$$

and therefore $a = 0$.

Remark 18. Suppose that $(X, \mathscr{A}, \phi)$ is a C^*-correspondence and (π, t) is an injective Toeplitz representation of $(X, \mathscr{A}, \phi)$ into a C^*-algebra $\mathscr{B}$. We can think of $\mathscr{B}$ as a C^*-subalgebra of $\mathbf{B}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$. Then since π, t are isometries $t(X)$ is a closed subspace of $\mathbf{B}(\mathscr{H})$ and $\pi(\mathscr{A})$ is a C^*-subalgebra of $\mathbf{B}(\mathscr{H})$.
We will show that $t(X)$ is a $\pi(\mathscr{A})$-bimodule such that $t(X)^* t(X) \subseteq \pi(\mathscr{A})$ and thus by identifying $\mathscr{A}$ with $\pi(\mathscr{A})$ and X with $t(X)$ we can see that $(X, \mathscr{A}, \phi)$ is a concrete C^*-correspondence (example 5.1.2). Indeed, for $t(\xi), t(\eta) \in t(X)$ and $\pi(a) \in \pi(\mathscr{A})$ we have

$$t(\xi)\pi(a) = t(\xi a) \in t(X)$$

and

$$\pi(a)t(\xi) = t(\phi(a)\xi) \in t(X)$$

and also

$$t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle) \in \pi(\mathscr{A}).$$

□

Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and (π, t) a Toeplitz representation of $(X, \mathscr{A}, \phi)$. We set

$$J_X = \phi^{-1}(\mathscr{K}(X)) \cap (\ker \phi)^\perp,$$

where $(\ker \phi)^\perp = \{c \in \mathscr{A} : cb = 0, \quad \text{for all } b \in \ker \phi\}$. It is easy to see that J_X is a closed ideal as an intersection of closed ideals. We call J_X the Katsura ideal.
We define $t_* : \mathscr{K}(X) \to C^*(\pi, t)$, where for $x, y \in X$

$$t_*(\theta_{x,y}) = t(x)t(y)^*.$$

We prove that t_* is well-defined and continuous on $\mathscr{K}(X)$.
Using lemma 2.5.4 and the fact that π is a $*$-homomorphism and therefore completely pos-

itive, we have

$$\left\|\sum_{i=1}^{n}\theta_{x_i,y_i}\right\| = \|(\langle x_i,x_j\rangle)_{ij}^{1/2}(\langle y_i,y_j\rangle)_{ij}^{1/2}\| \geq \|(\pi(\langle x_i,x_j\rangle))_{ij}^{1/2}(\pi(\langle y_i,y_j\rangle))_{ij}^{1/2}\|$$
$$= \|(t(x_i)^*t(x_j))_{ij}^{1/2}(t(y_i)^*t(y_j))_{ij}^{1/2}\| = \left\|\sum_{i=1}^{n}t(x_i)t(y_i)^*\right\| = \left\|t_*\left(\sum_{i=1}^{n}\theta_{x_i,y_i}\right)\right\|.$$

The above implies that t_* is contractive on a dense subset of $\mathscr{K}(X)$ and therefore it extends to a contractive map defined on $\mathscr{K}(X)$. We will prove that t_* is a $*$-homomorphism. Indeed, since $(\theta_{x,y})^* = \theta_{y,x}$, from linearity and continuity of t_* and density of the linear span of rank one operators, it is immediate that

$$t_*(k)^* = t_*(k^*).$$

To prove that t_* is multiplicative, it suffices to prove it on rank one operators.
For $x,y,u,w \in X$ we have

$$t_*(\theta_{x,y}\theta_{u,w}) = t_*(\theta_{x\langle y,u\rangle,w}) = t(x\langle y,u\rangle)t(w)^*$$
$$= t(x)\pi(\langle y,u\rangle)t(w)^* = t(x)t(y)^*t(u)t(w)^* = t_*(\theta_{x,y})t_*(\theta_{u,w}).$$

Note that

$$t(\theta_{x,y}(z)) = t(x\langle y,z\rangle) = t(x)\pi(\langle y,z\rangle) = t(x)t(y)^*t(z) = t_*(\theta_{x,y})t(z)$$

and so if $k \in \mathscr{K}(X)$ we get

$$t(k(z)) = t_*(k)t(z).$$

Note also that

$$\pi(a)t_*(\theta_{x,y}) = \pi(a)t(x)t(y)^* = t(\phi(a)x)t(y)^* = t_*(\theta_{\phi(a)x,y}) = t_*(\phi(a)\theta_{x,y})$$

and therefore

$$\pi(a)t_*(k) = t_*(\phi(a)k). \tag{5.1.1}$$

In the case that π is injective we get that π is completely isometric and therefore t_* is an isometry. We also claim that

$$\rho((t_u)_*(k)) = (t_\infty)_*(k),$$

where $\tilde{\rho} : T_X \to C^*(\pi,t)$ is the induced $*$-homomorphism from the universal property of $(\tilde{\pi}_u,\tilde{t}_u)$. In order to see this is true recall that the linear span of elements in the form $\theta_{x,y}$

is dense in $\mathscr{K}(X)$ and therefore we may assume that $k = \theta_{x,y}$ for $x, y \in X$ and thus

$$\tilde{\rho}((\tilde{t}_u)_*(k)) = \tilde{\rho}((\tilde{t}_u)_*(\theta_{x,y})) = \tilde{\rho}(\tilde{t}_u(x)\tilde{t}_u(y)^*) = t(x)t(y)^* = t_*(k).$$

Definition 5.1.6. Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence. We say that a Toeplitz representation (π, t) of $(X, \mathscr{A}, \phi)$ is Katsura covariant if for each $a \in J_X$ we have that

$$\pi(a) = t_*(\phi(a)).$$

We prove now that there exists a "universal" Katsura covariant Toeplitz representation. Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and (π, t) a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$ and $(\tilde{\pi}_u, \tilde{t}_u)$ the universal Toeplitz representation of $(X, \mathscr{A}, \phi)$. We define J to be the closed ideal of T_X generated by the set

$$\{(\tilde{t}_u)_*(\phi(a)) - \tilde{\pi}_u(a) : a \in J_X\},$$

and we denote by $\sigma : T_X \to T_X/J$ the canonical quotient $*$-epimorphism. We set

$$\pi_u = \sigma \circ \tilde{\pi}_u \quad \text{and} \quad t_u = \sigma \circ \tilde{t}_u$$

and we prove that (π_u, t_u) is a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$. Note that for $a \in J_X$ we have that $\sigma(\tilde{\pi}_u(a) - (\tilde{t}_u)_*(\phi(a))) = 0$. For $a \in \mathscr{A}$ and $\xi, \eta \in X$ we have

$$(\sigma \circ \tilde{t}_u(\xi))(\sigma \circ \tilde{\pi}_u(a)) = \sigma(\tilde{t}_u(\xi)\tilde{\pi}_u(a)) = \sigma \circ \tilde{t}_u(\xi a)$$

and

$$\sigma \circ \tilde{\pi}_u(\langle \xi, \eta \rangle) = \sigma(\tilde{t}_u^*(\xi)\tilde{t}_u(\eta)) = (\sigma \circ \tilde{t}_u(\xi))^*(\sigma \circ \tilde{t}_u(\eta)).$$

Note also that

$$(\sigma \circ \tilde{t}_u)_*(\theta_{\xi,\eta}) = (\sigma \circ \tilde{t}_u(\xi))(\sigma \circ \tilde{t}_u(\eta))^* = \sigma(\tilde{t}_u(\xi)\tilde{t}_u^*(\eta)) = \sigma \circ (\tilde{t}_u)_*(\theta_{\xi,\eta})$$

and since the linear span of elements in the form $\theta_{\xi,\eta}$ is dense in $\mathscr{K}(X)$ we get that

$$(\sigma \circ \tilde{t}_u)_* = \sigma \circ (\tilde{t}_u)_*.$$

Therefore, for $a \in J_X$ we have

$$\begin{aligned} &\sigma(\tilde{\pi}_u(a) - (\tilde{t}_u)_*(\phi(a))) = 0 \iff \sigma(\tilde{\pi}_u(a)) = \sigma((\tilde{t}_u)_*(\phi(a))) \\ &\iff \sigma \circ \tilde{\pi}_u(a) = (\sigma \circ \tilde{t}_u)_*(\phi(a)). \end{aligned}$$

Since, (π, t) is a Toeplitz representation there exists a $*$-epimorphism $\tilde{\rho} : T_X \to C^*(\pi, t)$

such that for $a \in \mathscr{A}$ and $\xi \in X$

$$\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a) \quad \text{and} \quad \tilde{\rho}(\tilde{t}_u(\xi)) = t(\xi).$$

Suppose that $a \in J_X$, then using the fact that (π, t) is Katsura covariant we have

$$\tilde{\rho}((\tilde{t}_u)_*(\phi(a)) - \tilde{\pi}_u(a)) = t_*(\phi(a)) - \pi(a) = 0,$$

which implies that $J \subseteq \ker \tilde{\rho}$ and thus the $*$-homomorphism $\rho : T_X/J \to C^*(\pi, t)$ given by

$$\rho(x + J) = \tilde{\rho}(x), \quad x \in T_X,$$

is well-defined. Note that for each $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\rho(\pi_u(a)) = \rho(\sigma \circ \tilde{\pi}_u(a)) = \rho(\tilde{\pi}_u(a) + J) = \tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a)$$

and

$$\rho(t_u(\xi)) = \rho(\sigma \circ \tilde{t}_u(\xi)) = \rho(\tilde{t}_u(\xi) + J) = \tilde{\rho}(\tilde{t}_u(\xi)) = t(\xi)$$

and using the fact that ρ is continuous and that $C^*(\pi, t)$ is generated by the set

$$\{\pi(a), t(\xi) : a \in \mathscr{A}, \xi \in X\},$$

we obtain that ρ is a $*$-epimorphism.
We summarize in the following:

Proposition 5.1.2. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence. There exists a universal Katsura covariant Toeplitz representation (π_u, t_u) of $(X, \mathscr{A}, \phi)$ that satisfies the following:*
If (π, t) is a Katsura covariant Toeplitz representation then there exists a (unique) $*$-epimorphism

$$\rho : O_X \to C^*(\pi, t),$$

such that

$$\rho(\pi_u(a)) = \pi(a) \quad \textit{and} \quad \rho(t_u(\xi)) = t(\xi), \quad \forall a \in \mathscr{A}, \forall \xi \in X.$$

Definition 5.1.7. We define the Cuntz-Pimsner algebra of $(X, \mathscr{A}, \phi)$ to be the C^* -algebra $O_X = C^*(\pi_u, t_u)$, where (π_u, t_u) is the universal Katsura covariant Toeplitz representation.

Note that if two Katsura covariant Toeplitz representations (π_1, t_1) and (π_2, t_2) satisfy the universal property, it is immediate that $C^*(\pi_1, t_1)$ is $*$-isomorphic to $C^*(\pi_2, t_2)$.

Example 5.1.4. Let α be a $*$-automorphism of a unital C^*-algebra $\mathscr{A}$ and let $\mathscr{A}_\alpha$ be the C^*-correspondence described in example 5.1.1. Using remark 6 we identify $\mathscr{K}(\mathscr{A})$ with $\mathscr{A}$. Let (π, t) be a Katsura covariant Toeplitz representation of $\mathscr{A}_\alpha$. Without loss of generality we assume that π is non-degenerate. We have already proven in example 5.1.3 that the pair

$(\pi, t(\mathbf{1}_{\mathscr{A}}))$ satisfies the left covariance relation and that $t(\mathbf{1}_{\mathscr{A}})$ is isometric, in this case we claim that $t(\mathbf{1}_{\mathscr{A}})$ is unitary.
Indeed,

$$\begin{aligned} t(\mathbf{1}_{\mathscr{A}})t(\mathbf{1}_{\mathscr{A}})^* &= t_*(\mathbf{1}_{\mathscr{A}}\mathbf{1}_{\mathscr{A}}^*) = t_*(\mathbf{1}_{\mathscr{A}}) \\ &= t_*(\phi(\mathbf{1}_{\mathscr{A}})) = \pi(\mathbf{1}_{\mathscr{A}}) = 1_{\mathbf{B}(\mathscr{H})}, \end{aligned}$$

proves our claim.

Conversely, suppose that (π, U) is a left covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$. We have already proven in example 5.1.3 that the pair (π, t), where t is the linear map given by $t(\xi) = U\pi(\xi)$, is a Toeplitz representation of $\mathscr{A}_\alpha$. We will prove that (π, t) is a Katsura covariant Toeplitz representation of $\mathscr{A}_\alpha$. Note that since α is a $*$-automorphism $\mathscr{A}_\alpha$ is an injective C^*-correspondence and $J_{\mathscr{A}} = \mathscr{K}(\mathscr{A}) \cap (\ker \alpha)^\perp = \mathscr{K}(\mathscr{A}) = \mathscr{A}$.
For each $a \in \mathscr{A}$ we have

$$\phi(a) = \alpha(a) = \alpha(a)\mathbf{1}_{\mathscr{A}}^* \in \mathscr{K}(\mathscr{A})$$

and therefore

$$\begin{aligned} t_*(\phi(a)) &= t_*(\alpha(a)\mathbf{1}_{\mathscr{A}}^*) = t(\alpha(a))t(\mathbf{1}_{\mathscr{A}}^*) = U\pi(\alpha(a))\pi(\mathbf{1}_{\mathscr{A}}^*)U^* \\ &= U\pi(\alpha(a))U^* = \pi(a). \end{aligned}$$

Hence, (π, t) is a Katsura covariant Toeplitz representation of $\mathscr{A}_\alpha$.

We prove now that $O_{\mathscr{A}} = \mathbb{Z} \times_\alpha \mathscr{A}$.
Let (π_u, t_u) be the universal Katsura covariant Toeplitz representation of the C^*-correspondence $\mathscr{A}_\alpha$ and let (π, U) be a left covariant representation of $(\mathscr{A}, \alpha)$. We have that (π, t) where t is the linear map given by $t(\xi) = U\pi(\xi)$ for $\xi \in \mathscr{A}$, is a Katsura covariant Toeplitz representation of $\mathscr{A}_\alpha$. Therefore, there exists a $*$-epimorphism $\rho : O_{\mathscr{A}} \to C^*(\pi, t)$ such that

$$\rho(\pi_u(a)) = \pi(a) \quad \text{and} \quad \rho(t(\xi)) = t(\xi), \quad \xi, a \in \mathscr{A}.$$

The pair $(\pi_u, t_u(\mathbf{1}_{\mathscr{A}}))$ is a left covariant representation of $(\mathscr{A}, \alpha)$ and for each a in $\mathscr{A}$ we have $\rho(\pi_u(a)) = \pi(a)$ and also $\rho(t_u(\mathbf{1}_{\mathscr{A}})) = t(\mathbf{1}_{\mathscr{A}}) = U\pi(\mathbf{1}_{\mathscr{A}}) = U$, hence $(\pi_u, t_u(\mathbf{1}_{\mathscr{A}}))$ satisfies the universal property of the crossed product and therefore

$$O_{\mathscr{A}} = C^*(\pi_u, t_u(\mathbf{1}_{\mathscr{A}})) = \mathbb{Z} \times_\alpha \mathscr{A}.$$

We are now going to introduce an example of an injective Katsura covariant Toeplitz representation of a given C^*-correspondence, in order to do so we are going to need a few more results.

Lemma 5.1.3. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $J \subseteq \mathscr{A}$ a closed ideal. If $k \in \mathscr{K}(X)$, then the following are equivalent:*

(i) $k \in \mathscr{K}(XJ) := \overline{\operatorname{span}}\{\theta_{\xi a,\eta} : \xi, \eta \in X,\ a \in J\}$

(ii) $\langle k\xi, \eta\rangle \in J, \quad \forall \xi, \eta \in X$

Proof. We denote by I the set of $k \in \mathscr{K}(X)$ satisfying (ii). It is easy to see that I is an ideal, since for $k \in I$ and $s \in \mathscr{L}(X)$ we have that

$$\langle ks\xi, \eta\rangle = \langle k(s\xi), \eta\rangle \in J$$

and

$$\langle sk\xi, \eta\rangle = \langle k\xi, s^*\eta\rangle \in J.$$

Note that since $\langle ., .\rangle$ is continuous I is a closed ideal and hence it is linearly spanned from its positive elements. Now we prove that $\mathscr{K}(XJ)$ is a closed ideal. It suffices to check for $\xi, \eta \in X$ and $a \in J$ that $s\theta_{\xi a,\eta}$ and $\theta_{\xi a,\eta}s$ are in $\mathscr{K}(XJ)$.
Indeed,

$$s\theta_{\xi a,\eta} = \theta_{s(\xi a),\eta} = \theta_{s(\xi)a,\eta} \in \mathscr{K}(XJ)$$

and

$$\theta_{\xi,\eta}s = \theta_{\xi a,s^*\eta} \in \mathscr{K}(XJ).$$

Now, suppose that

$$k = \lim_n \sum_{i=1}^{m_n} \theta_{\xi_i^n,\eta_i^n} \in I$$

is a positive element. Then

$$k^3 = \lim_n \left(\sum_{i=1}^{m_n} \theta_{\xi_i^n,\eta_i^n}\right) k \left(\sum_{j=1}^{m_n} \theta_{\xi_j^n,\eta_j^n}\right) = \lim_n \sum_{i,j} \theta_{\xi_i^n\langle \eta_i^n, k\xi_j^n\rangle,\eta_j^n} \in \mathscr{K}(XJ)$$

and so $k^4 \in \mathscr{K}(XJ)$ and $k = ((k^4)^{1/2})^{1/2} \in \mathscr{K}(XJ)$.
Conversely, if $\xi, \eta, x, y \in X$, $a \in J$ and $k = \theta_{\xi a,\eta}$,

$$\langle kx, y\rangle = \langle \xi a\langle \eta, x\rangle, y\rangle = \langle x, \eta\rangle a^* \langle \xi, y\rangle \in J.$$

□

Lemma 5.1.4. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{A}, \phi)$ be C^*-correspondences and*

$$\phi_* : \mathscr{L}(X) \to \mathscr{L}(X \otimes_\phi Y) : s \to s \otimes I_Y$$

If $k \in \mathscr{K}(X)$ then $k \in \ker \phi_*$ if and only if $k \in \mathscr{K}(X \ker \phi)$.

Proof. Let $k \in \mathscr{K}(X)$. Then,

$$\begin{aligned} k \in \mathscr{K}(X \ker \phi) &\iff \langle kx, y\rangle \in \ker \phi, \quad \forall x, y \in X \\ &\iff \phi(\langle kx, y\rangle) = 0, \quad \forall x, y \in X \\ &\iff \phi(\langle kx, y\rangle)\xi = 0, \quad \forall x, y, \xi \in X \\ &\iff \langle \eta, \phi(\langle kx, y\rangle)\xi\rangle = 0, \quad \forall x, y, \xi, \eta \in X \\ &\iff \langle kx \otimes \eta, y \otimes \xi\rangle = 0, \quad \forall x, y, \xi, \eta \in X \\ &\iff \langle k \otimes I_Y(x \otimes \eta), y \otimes \xi\rangle = 0, \quad \forall x, y, \xi, \eta \in X \\ &\iff k \otimes I_Y = 0 \iff k \in \ker \phi_*. \end{aligned}$$

□

We apply the above lemma in the case where $X = X^{\otimes(n-1)}$ and $Y = X$ and we get the following:

Proposition 5.1.3. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $n \in \mathbb{N}$. The map*

$$\mathscr{K}\left(X^{\otimes(n-1)} J_X\right) \to \mathscr{L}\left(X^{\otimes n}\right) : k \to k \otimes I_X$$

is isometric.

Proof. Suppose that $k \in \mathscr{K}\left(X^{\otimes(n-1)}\right)$ is in the kernel of the map described above. From the preceding lemma $k \in \mathscr{K}\left(X^{\otimes(n-1)} \ker \phi\right)$ and therefore

$$\langle kx, y\rangle \in \ker \phi, \quad \forall x, y \in X^{\otimes(n-1)},$$

since $k \in \mathscr{K}\left(X^{\otimes(n-1)} J_X\right)$ we also have that

$$\langle kx, y\rangle \in J_X \subseteq (\ker \phi)^\perp, \quad \forall x, y \in X^{\otimes(n-1)}$$

and so $k = 0$. □

Denote by (π_∞, t_∞) the Fock representation of a C^*-correspodence $(X, \mathscr{A}, \phi)$. Suppose that $(a, x_1, x_2, ...)$ is an element in F_X where for $n \geq 2$ we have $x_n = z_n \otimes w_n$ where $z_n \in X$ and $w_n \in X^{\otimes(n-1)}$. Then for each $\xi, \eta \in X$ we have that

$$\begin{aligned} (t_\infty)_*(\theta_{\eta,\xi})(a, x_1, x_2, ...) &= t_\infty(\eta)t_\infty(\xi)^*(a, x_1, x_2, ...) \\ &= t_\infty(\eta)(\langle \xi, x_1\rangle, \phi_1(\langle \xi, z_2\rangle)w_2, \phi_2(\langle \xi, z_3\rangle)w_3, ...) \\ &= (0, \eta\langle \xi, x_1\rangle, \eta \otimes \phi(\langle \xi, z_2\rangle)w_2, \eta \otimes \phi_2(\langle \xi, z_3\rangle)w_3, ...) \\ &= (0, \eta\langle \xi, x_1\rangle, \eta\langle \xi, z_2\rangle \otimes w_2, \eta\langle \xi, z_3\rangle \otimes w_3, ...) \\ &= (0, \theta_{\eta,\xi}(x_1), \theta_{\eta,\xi} \otimes I_X(x_2), \theta_{\eta,\xi} \otimes I_{X^{\otimes 2}}(x_3), ...). \end{aligned}$$

By linearity and continuity of $(t_\infty)_*$, the density of the linear span of simple tensors and the density of the linear span of elements $(a, x_1, x_2, ...)$ of the above form, we get that if $k \in \mathscr{K}(X)$ and $(a, x_1, x_2, ...) \in F_X$, then

$$(t_\infty)_*(k)(a, x_1, x_2, ...) = (0, kx_1, k \otimes I_X(x_2), k \otimes I_{X^{\otimes 2}}(x_3), ...).$$

Therefore, if $a \in J_X$ we obtain the following

$$\begin{aligned}&(\pi_\infty(a) - (t_\infty)_*(\phi(a)))(b, x_1, x_2, x_3, ...)\\ &= (ab, \phi_1(a)x_1, \phi_2(a)x_2, \phi_3(a)x_3, ...) - (0, \phi(a)x_1, \phi(a) \otimes I_X(x_2), \phi(a) \otimes I_{X^{\otimes 2}}(x_3), ...)\\ &= (ab, 0, 0, ...)\end{aligned}$$

since $\phi_n(a) = \phi(a) \otimes I_{X^{\otimes(n-1)}}(a)$.
Note that

$$\pi_\infty(a) - (t_\infty)_*(\phi(a)) = \theta_{xa,x} \in \mathscr{K}(F_X J_X),$$

where $x = (\mathbf{1}_{\mathscr{A}}, 0, 0, ...)) \in F_X$.

Proposition 5.1.4. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and (π, t) an injective Toeplitz representation of $(X, \mathscr{A}, \phi)$. If $a \in \mathscr{A}$ satisfies $\pi(a) \in t_*(\mathscr{K}(X))$, then we have $a \in J_X$ and $\pi(a) = t_*(\phi(a))$.*

Proof. Let $a \in \mathscr{A}$ such that $\pi(a) \in t_*(\mathscr{K}(X))$, then there exists $k \in \mathscr{K}(X)$ such that $t_*(k) = \pi(a)$. For $\xi \in X$, we have that

$$t(\phi(a)\xi) = \pi(a)t(\xi) = t_*(k)t(\xi) = t(k\xi).$$

The injectivity of t implies that

$$\phi(a)\xi = k\xi, \quad \forall \xi \in X \iff \phi(a) = k$$

Therefore,

$$\pi(a) = t_*(\phi(a)).$$

Suppose that $b \in \ker \phi$, we will show that $ab = 0$.
Indeed, using the relation 5.1.1 and we have that

$$\begin{aligned}\pi(ab) &= \pi(a)\pi(b) = t_*(\phi(a))\pi(b) = (\pi(b^*)t_*(\phi(a^*))^*\\ &= t_*(\phi(b^*)\phi(a^*))^* = t_*(\phi(a)\phi(b)) = 0.\end{aligned}$$

Since π is injective it is implied that $ab = 0$. Thus, $a \in J_X$ and the proof is complete. □

Corollary 5.1.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $a \in \mathscr{A}$ such that $\pi_\infty(a) \in (t_\infty)_*(\mathscr{K}(X))$, then $a = 0$.*

Proof. From the preceding proposition we get that $a \in J_X$ and $\pi(a) = (t_\infty)_*(\phi(a))$. Let $(b, \xi_1, \xi_2, ...) \in F_X$, then since

$$\pi_\infty(a) - (t_\infty)_*(\phi(a)))(b, \xi_1, \xi_2, ...) = (ab, 0, 0, ...),$$

we get that $ab = 0$ for all $b \in \mathscr{A}$. Hence $a = 0$. □

Remark 19. Let (π, t) be a Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$. We set $t^{(0)} = \pi$, $t^{(1)} = t$ and for $n \geq 2$ we will define a linear map $t^{(n)} : X^{\otimes n} \to C^*(\pi, t)$ such that

$$t^{(n)}(\xi \otimes \eta) = t(\xi)t^{(n-1)}(\eta),$$

where $\xi \in X$ and $\eta \in X^{\otimes(n-1)}$.
We will prove by induction that $\forall n \geq 2$ these maps are well-defined and that $(\pi, t^{(n)})$ are Toeplitz representations of $(X^{\otimes n}, \mathscr{A}, \phi_n)$.
Suppose that $t^{(n-1)}$ is well-defined and that $(\pi, t^{(n-1)})$ is a Toeplitz representation of the C^*-correspondence $(X^{\otimes(n-1)}, \mathscr{A}, \phi_{n-1})$. Then,

$$\begin{aligned}
&\|t^{(n)}(\xi \otimes \eta)\|^2 = \|t(\xi)t^{(n-1)}(\eta)\|^2 = \|t^{(n-1)}(\eta)^* t(\xi)^* t(\xi) t^{(n-1)}(\eta)\| \\
&= \|t^{(n-1)}(\eta)^* \pi(\langle \xi, \xi \rangle) t^{(n-1)}(\eta)\| = \|t^{(n-1)}(\eta)^* t^{(n-1)}(\phi_{n-1}(\langle \xi, \xi \rangle)\eta)\| \\
&= \|\pi(\langle \eta, \phi_{n-1}(\langle \xi, \xi \rangle)\eta \rangle)\| \leq \|\langle \eta, \phi_{n-1}(\langle \xi, \xi \rangle)\eta \rangle\| = \|\langle \xi \otimes \eta, \xi \otimes \eta \rangle\| = \|\xi \otimes \eta\|^2.
\end{aligned}$$

Thus, $t^{(n)}$ is contractive on simple tensors, which implies that it is contractive on $X^{\otimes n}$ and therefore also well-defined. We should note that that in the case that π is injective, $t^{(n)}$ is an isometry.
To see that $(\pi, t^{(n)})$ is a Toeplitz representation note that

$$\begin{aligned}
&t^{(n)}(\xi_1 \otimes \eta_1)^* t^{(n)}(\xi_2 \otimes \eta_2) = t^{(n-1)}(\eta_1)^* t(\xi_1)^* t(\xi_2) t^{(n-1)}(\eta_2) \\
&= t^{(n-1)}(\eta_1)^* \pi(\langle \xi_1, \xi_2 \rangle) t^{(n-1)}(\eta_2) = t^{(n-1)}(\eta_1)^* t(\phi_{n-1}(\langle \xi_1, \xi_2 \rangle)\eta_2) \\
&= \pi(\langle \eta_1, \phi_{n-1}(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle) = \pi(\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle)
\end{aligned}$$

and

$$\begin{aligned}
&\pi(a) t^{(n)}(\xi \otimes \eta) = \pi(a) t(\xi) t^{(n-1)}(\eta) = t(\phi(a)\xi) t^{(n-1)}(\eta) \\
&= t^{(n)}((\phi(a)\xi) \otimes \eta) = t^{(n)}(\phi_n(a)(\xi \otimes \eta)).
\end{aligned}$$

We will need the following:

Lemma 5.1.5. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and F_X the Fock Space. If we set $K = \text{span}\{\theta_{za,w} : a \in J_X, z \in X^{\otimes m}$ and $w \in X^{\otimes n}$, where $n, m \in \mathbb{N}\}$, then $\overline{K} = \mathscr{K}(F_X J_X)$.*

Proof.

It suffices to show that $\theta_{xa,y} \in \overline{K}$ where a is an element in J_X and $x = (x_0, x_1, x_2, ...), y = (y_0, y_1, y_2, ...) \in F_X$. To begin with, if $z = (z_0, z_1, z_2, ..., z_m, 0, ...) \in F_X$ then we can pick $w = (y_0, y_1, ..., y_n, 0, ...)$ such that

$$\|w - y\| < \frac{\epsilon}{\|a\|\|z\|}$$

and thus for $\xi = (\xi_0, \xi_1, ...) \in F_X$

$$\|\theta_{za,y}(\xi) - \theta_{za,w}(\xi)\| = \|za\langle y, \xi\rangle - za\langle w, \xi\rangle\| \leq \|z\|\|a\|\|y - w\|\|\xi\| < \epsilon\|\xi\|.$$

Since

$$\begin{aligned}\theta_{za,w} &= \theta_{z_0a,w} + \theta_{z_1a,w} + ... + \theta_{z_ma,w} \\ &= (\theta_{z_0a,y_0} + ... + \theta_{z_0a,y_n}) + ... + (\theta_{z_ma,y_0} + ... + \theta_{z_ma,y_n}) \in K,\end{aligned}$$

we have that elements of the form $\theta_{za,\eta}$ where $z = (z_0, z_1, ...z_m, 0, ...)$ and $\eta \in F_X$ are in $\overline{K}$. Pick $z = (x_0, x_1, ...x_m, 0, ...)$ such that

$$\|z - x\| < \frac{\epsilon}{\|a\|\|y\|},$$

then

$$\|\theta_{xa,y}(\xi) - \theta_{za,y}(\xi)\| = \|(x - z)a\langle y, \xi\rangle\| \leq \|y\|\|a\|\|z - x\|\|\xi\| < \epsilon\|\xi\|.$$

□

Proposition 5.1.5. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence, then $\mathscr{K}(F_X J_X) \subseteq C^*(\pi_\infty, t_\infty)$.*

Proof. From the preceding lemma it suffices to show that $\theta_{xa,y} \in C^*(\pi_\infty, t_\infty)$ for $x = x_1 \otimes ... \otimes x_n \in X^{\otimes n}$, $y = y_1 \otimes ... \otimes y_m \in X^{\otimes m}$ and $a \in J_X$.
We will prove that

$$\begin{aligned}\theta_{xa,y} &= t_\infty^{(n)}(x)(L_a, 0, 0, ...)t_\infty^{(m)}(y)^* \\ &= t_\infty^{(n)}(x)((\pi_\infty(a) - (t_\infty)_*(\phi(a)))t_\infty^{(m)}(y)^* \in C^*(\pi_\infty, t_\infty).\end{aligned}$$

Indeed, we may suppose that $(z_0, z_1, z_2, ...) \in F_X$ and $z_i = z_i^{(1)} \otimes z_i^{(2)} \otimes ... \otimes z_i^{(i)} \in X^{\otimes i}$,

$\forall i \geq 2$. Then

$$\begin{aligned}
&t_\infty^{(n)}(x)(L_a, 0, 0, ...)t_\infty^{(m)}(y)^*(z_0, z_1, z_2, ...)\\
&= t_\infty(x_1)...t_\infty(x_n)(L_a, 0, 0, ...)t_\infty(y_m)^*...t_\infty(y_1)^*(z_0, z_1, z_2, ...)\\
&= t_\infty(x_1)...t_\infty(x_n)(L_a, 0, 0, ...)(\langle y, z_m\rangle, \phi(\langle y \otimes z_{m+1}^{(1)} \otimes z_{m+1}^{(2)} \otimes ... \otimes z_{m+1}^{(m)}\rangle)z_{m+1}^{(m+1)}, ...)\\
&= t_\infty(x_1)...t_\infty(x_n)(a\langle y, z_m\rangle, 0, 0, ...) = (0, 0, ..., 0, \underbrace{x_1 \otimes ... \otimes (x_n a\langle y, z_m\rangle)}_{\in X^{\otimes n}}, 0, ...)\\
&= (0, 0, ..., 0, (x_1 \otimes ... \otimes x_n)a\langle y, z_m\rangle, 0, ...) = \theta_{xa,y}(z_0, z_1, z_2, ...).
\end{aligned}$$

□

For $n \geq 0$ we define $P_n : F_X \to F_X$ to be the projection onto the direct summand $X^{\otimes n}$. Since for $(x_1, x_2, ...), (y_1, y_2, ...) \in F_X$

$$\langle P_n(x_1, x_2, ...), (y_1, y_2, ...)\rangle = \langle (0, 0, .., x_n, 0, ...), (y_1, y_2, ...)\rangle = \langle (x_1, x_2, ...), P_n(y_1, y_2, ...)\rangle,$$

we have that $P_n \in \mathscr{L}(F_X)$.
Note that if $x, y \in F_X$ and $a \in J_X$,

$$\begin{aligned}
P_n\theta_{xa,y}P_n(z) &= P_n\theta_{xa,y}(P_n z) = P_n(xa\langle y, P_n z\rangle)\\
&= P_n(x)a\langle y, P_n z\rangle = P_n(x)a\,\langle P_n y, z\rangle = \theta_{P_n xa, P_n y}(z)
\end{aligned}$$

and thus $P_n\mathscr{K}(F_X J_X)P_n \subseteq \mathscr{K}(X^{\otimes n}J_X)$.
Conversely, if $\theta_{xa,y} \in \mathscr{K}(X^{\otimes n}J_X)$ from the calculation above we have that

$$\theta_{xa,y} = \theta_{P_n x, P_n y} = P_n\theta_{xa,y}P_n$$

and equality follows.

Lemma 5.1.6. *If $a \in \mathscr{A}$ and $\pi_\infty(a) \in \mathscr{K}(F_X)$ then* $\lim_n \|\phi_n(a)\| = 0$.

Proof. For $n \geq 0$ we have that

$$\phi_n(a) = P_n\pi_\infty(a)P_n$$

and therefore it suffices to show that

$$\lim_n \|P_n k P_n\| = 0$$

where $k \in \mathscr{K}(F_X)$. From lemma 5.1.5 we can assume that $k = \theta_{\xi,\eta}$ with $\xi \in X^{\otimes k}$ and $\eta \in X^{\otimes m}$ for $k, m \geq 0$. Now it is evident that if $n > \max\{k, m\}$ then

$$P_n\theta_{\xi,\eta}P_n = 0.$$

□

Theorem 5.1.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence, then there exists an injective Katsura covariant Toeplitz representation (π, t) of $(X, \mathscr{A}, \phi)$.*

Proof. Let $\sigma : \mathscr{L}(F_X) \to \mathscr{L}(F_X)/\mathscr{K}(F_X J_X)$ be the canonical quotient $*$-epimorphism. We set

$$\pi = \sigma \circ \pi_\infty \quad \text{and} \quad t = \sigma \circ t_\infty.$$

Recall that for $a \in J_X$ we have that

$$\pi_\infty(a) - (t_\infty)_*(\phi(a)) \in \mathscr{K}(F_X J_X)$$

and so $\sigma(\pi_\infty(a) - (t_\infty)_*(\phi(a))) = 0$. We will prove that $(\sigma \circ \pi_\infty, \sigma \circ t_\infty)$ is a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$.
Indeed, for $a \in \mathscr{A}$ and $\xi, \eta \in X$ we have

$$(\sigma \circ t_\infty(\xi))(\sigma \circ \pi_\infty(a)) = \sigma(t_\infty(\xi)\pi_\infty(a)) = \sigma \circ t_\infty(\xi a)$$

and

$$\sigma \circ \pi_\infty(\langle \xi, \eta \rangle) = \sigma(t_\infty^*(\xi) t_\infty(\eta)) = (\sigma \circ t_\infty(\xi))^*(\sigma \circ t_\infty(\eta)).$$

Note also that

$$(\sigma \circ t_\infty)_*(\theta_{\xi,\eta}) = (\sigma \circ t_\infty(\xi))(\sigma \circ t_\infty(\eta))^* = \sigma(t_\infty(\xi)t_\infty^*(\eta)) = \sigma \circ (t_\infty)_*(\theta_{\xi,\eta})$$

and since the linear span of elements in the form $\theta_{\xi,\eta}$ is dense in $\mathscr{K}(X)$ we get that

$$(\sigma \circ t_\infty)_* = \sigma \circ (t_\infty)_*.$$

Therefore, for $a \in J_X$ we have

$$\begin{aligned} &\sigma(\pi_\infty(a) - (t_\infty)_*(\phi(a))) = 0 \iff \sigma(\pi_\infty(a)) = \sigma((t_\infty)_*(\phi(a))) \\ &\iff \sigma \circ \pi_\infty(a) = (\sigma \circ t_\infty)_*(\phi(a)). \end{aligned}$$

Now, suppose that $a \in \mathscr{A}$ such that $\pi(a) = 0$ and so $\pi_\infty(a) \in \mathscr{K}(F_X J_X)$, we will show that $a = 0$.
For $n \geq 0$ we have that

$$\phi_n(a) = P_n \pi_\infty(a) P_n \in P_n \mathscr{K}(F_X J_X) P_n = \mathscr{K}(X^{\otimes n} J_X).$$

If we pick $n = 0$ we have that $\phi_0(a) = L_a \in \mathscr{K}(\mathscr{A} J_X)$.

Note that if $\theta_{xb,y} \in \mathscr{K}(\mathscr{A} J_X)$ then

$$\theta_{xb,y}(\mathbf{1}_{\mathscr{A}}) = xby^*,$$

which belongs in J_X. Since L_a is a norm limit of finite sums of rank one operators and $L_a(\mathbf{1}_{\mathscr{A}}) = a$ we get that $a \in J_X$. Recall that $\phi_1 = \phi$ is injective on J_X and thus

$$\|a\| = \|\phi(a)\|.$$

From proposition 5.1.3 for every $n \geq 2$

$$\|\phi_{n-1}(a)\| = \|\phi_{n-1}(a) \otimes I_X\| = \|\phi_n(a)\|$$

and so

$$\|a\| = \lim_n \|\phi_n(a)\| = 0.$$

□

We should note that the existence of the injective Katsura covariant Toeplitz representation (π, t), implies that the universal Katsura covariant Toeplitz representation (π_u, t_u) is also injective. Indeed, if $a \in \ker \pi_u$ and $\rho : O_X \to C^*(\pi, t)$ is the $*$-epimorphism induced from universality, we have that

$$0 = \rho(\pi_u(a)) = \pi(a).$$

Since π is injective, $a = 0$.

5.2 Gauge actions

Definition 5.2.1. Let (π, t) be a Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$. We say that (π, t) admits a gauge action if for each $z \in \mathbb{T}$ there exists a $*$-homomorphism $\beta_z : C^*(\pi, t) \to C^*(\pi, t)$ such that

$$\beta_z(\pi(a)) = \pi(a) \quad \text{and} \quad \beta_z(t(\xi)) = zt(\xi)$$

for all $a \in \mathscr{A}$ and $\xi \in X$.

It is immediate that $\beta_{z^{-1}}$ is an inverse of β_z for all $z \in \mathbb{T}$ and therefore β_z is a $*$-automorphism and also from continuity of β_z and from the fact that elements $\{\pi(a), t(\xi) : a \in \mathscr{A} \text{ and } \xi \in X\}$ generate $C^*(\pi, t)$, it is unique.
Let $\{z_n : n \in \mathbb{N}\}$ such that $z_n \xrightarrow{n} z \in \mathbb{T}$. For $a \in \mathscr{A}$ and $\xi \in X$ we have that

$$\beta_{z_n}(\pi(a)) = \pi(a) = \beta_z(\pi(a))$$

and

$$\beta_{z_n}(t(\xi)) = z_n t(\xi) \to zt(\xi) = \beta_z(t(\xi)).$$

Since $\{\pi(a), t(\xi) : a \in \mathscr{A} \text{ and } \xi \in X\}$ generates $C^*(\pi, t)$ and $\forall z \in \mathbb{T}$ we have $\|\beta_z\| \leq 1$ the above calculation implies that

$$\beta_{z_n}(k) \xrightarrow{n} \beta_z(k) \quad \forall k \in C^*(\pi, t)$$

and therefore

$$z \to \beta_z$$

is point norm continuous.

We will show that both the universal Katsura covariant Toeplitz representation (π_u, t_u) and the universal Toeplitz representation $(\tilde{\pi}_u, \tilde{t}_u)$ admit gauge actions.

Let $z \in \mathbb{T}$ and consider the linear map

$$z\tilde{t}_u : X \to T_X \quad x \to z\tilde{t}(x).$$

Then the pair $(\tilde{\pi}_u, z\tilde{t}_u)$ is a Toeplitz representation since

$$(z\tilde{t}_u(\xi))^* z\tilde{t}_u(\eta) = \overline{z}\tilde{t}_u(\xi)^* z\tilde{t}_u(\eta) = \tilde{t}_u(\xi)^*\tilde{t}_u(\eta) = \tilde{\pi}_u(\langle \xi, \eta \rangle)$$

and

$$\tilde{\pi}_u(a)(z\tilde{t}_u(\xi)) = z\tilde{\pi}_u(a)\tilde{t}_u(\xi) = z\tilde{t}_u(\phi(a)\xi).$$

Therefore, from the universal property of T_X for every $z \in \mathbb{T}$ since $C^*(\tilde{\pi}_u, z\tilde{t}_u) = T_X$ there exists a $*$-homomorphism $\tilde{\gamma}_z : T_X \to T_X$ such that

$$\tilde{\gamma}_z(\tilde{\pi}_u(a)) = \tilde{\pi}_u(a) \quad \text{and} \quad \tilde{\gamma}_z(\tilde{t}_u(\xi)) = z\tilde{t}_u(\xi), \quad \forall a \in \mathscr{A}, \forall \xi \in X.$$

Hence,

$$z \to \tilde{\gamma}_z$$

is a gauge action for $(\tilde{\pi}_u, \tilde{t}_u)$.

It is also immediate that the universal Katsura covariant Toeplitz representation (π_u, t_u) admits a gauge action since for all $z \in \mathbb{T}$ if we define the linear map

$$zt_u : O_X \to O_X,$$

where for $\xi \in X$

$$(zt_u)(\xi) = zt_u(\xi),$$

then the pair (π_u, zt_u) is a Katsura covariant Toeplitz representation for the correspondence $(X, \mathscr{A}, \phi)$.

Indeed, if $\theta_{x,y} \in \mathscr{K}(X)$ we have that

$$(zt_u)_*(\theta_{x,y}) = zt_u(x)(zt_u(y))^* = t_u(x)t_u(y)^* = (t_u)_*(\theta_{x,y})$$

and therefore from linearity and continuity

$$(zt_u)_*(k) = (t_u)_*(k), \quad \forall k \in \mathscr{K}(X).$$

If $a \in J_X$ we have that $\phi(a) \in \mathscr{K}(X)$ and so

$$(zt_u)_*(\phi(a)) = (t_u)_*(\phi(a)) = \pi_u(a).$$

By universality, for $z \in \mathbb{T}$ there exists a $*$-homomorphism

$$\gamma_z : O_X \to O_X$$

such that

$$\gamma_z(\pi_u(a)) = \pi_u(a) \quad \text{and} \quad \gamma_z(t_u(\xi)) = zt_u(\xi), \quad \forall a \in \mathscr{A}, \forall \xi \in X.$$

Note that if (π, t) is a Toeplitz representation admitting a gauge action $\tilde{\beta}$ and

$$\tilde{\rho} : T_X \to C^*(\pi, t)$$

is the $*$-epimorphism obtained by the universality of T_X then

$$\tilde{\beta}_z \circ \tilde{\rho} = \tilde{\rho} \circ \tilde{\gamma}_z, \quad \text{for each } z \in \mathbb{T}.$$

Respectively, if (π, t) is a Katsura covariant Toeplitz representation that admits a gauge action β and

$$\rho : O_X \to C^*(\pi, t)$$

is the $*$-epimorphism obtained by the universality of O_X then

$$\beta_z \circ \rho = \rho \circ \gamma_z, \quad \text{for each } z \in \mathbb{T}.$$

Note that for each $n \geq 1$ and $\xi \in X^{\otimes n}$ we have that

$$\beta_z(t^{(n)}(\xi)) = z^n t^{(n)}(\xi).$$

Indeed, we may assume that $\xi = \xi_1 \otimes ... \otimes \xi_n$ and therefore

$$\beta_z(t^{(n)}(\xi)) = \beta_z(t(\xi_1)...t(\xi_n)) = z^n t^{(n)}(\xi).$$

In this section we are going to prove the gauge-invariance uniqueness theorem. A consequence of this theorem is that the C^*-algebras O_X and T_X are $*$-isomorphic with the C^*-algebras generated by the injective Katsura covariant Toeplitz representation we constructed in the previous section and the Fock representation, respectively. In order to prove the gauge-invariance uniqueness theorem, we are going to investigate the core of a C^*-algebra $C^*(\pi, t)$ generated by a Toeplitz representation (π, t).

Definition 5.2.2. Let (π, t) be a Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$. For each $n \in \mathbb{N}$, set $B_n = (t^{(n)})_*(\mathscr{K}(X^{\otimes n})) \subseteq C^*(\pi, t)$.

Recall that $t^{(0)} = \pi$ and that for every $n \in \mathbb{N}$ if π is injective then $(t^{(n)})_*$ is an isometry and therefore in that case $B_n \cong \mathscr{K}(X^{\otimes n})$.

Lemma 5.2.1. *For $n, m \in \mathbb{N}$ where $n \geq 1$, we have that*

$$\overline{\text{span}}(t^{(n)}(X^{\otimes n}) B_m t^{(n)}(X^{\otimes n})^*) = B_{n+m}$$

and $t^{(n)}(X^{\otimes n})^* B_{n+m} t^{(n)}(X^{\otimes n}) \subseteq B_m$.

Proof. First of all if $n \geq 1$ we have that $(t^{(n)})_*$ is a $*$-homomorphism and that $\mathscr{K}(X^{\otimes n})$ is a C^*-algebra and therefore $B_n = (t^{(n)})_*(\mathscr{K}(X^{\otimes n}))$ is a C^*-subalgebra of $C^*(\pi, t)$. Note that if $x, y \in X^{\otimes n}$ and $\xi, \eta \in X^{\otimes m}$ we have that

$$\begin{aligned} &t^{(n)}(x)(t^{(m)})_*(\theta_{\xi,\eta})t^{(n)}(y)^* = t^{(n)}(x)t^{(m)}(\xi)t^{(m)}(\eta)^* t^{(n)}(y)^* = \\ &t^{(n+m)}(x \otimes \xi)t^{(n+m)}(y \otimes \eta)^* = (t^{(n+m)})_*(\theta_{x\otimes\xi, y\otimes\eta}). \end{aligned}$$

Since elements of the form $(t^{(m)})_*(\theta_{\xi,\eta})$ generate B_m and the linear span of elements $x \otimes \xi$ is dense in $X^{\otimes(n+m)}$, we are done.
For the second implication, let $\xi, \eta \in X^{\otimes(n+m)}$ and $x, y \in X^{\otimes n}$. We may suppose that $\xi = \xi_1 \otimes \xi_2$ and $\eta = \eta_1 \otimes \eta_2$ for $\xi_1, \eta_1 \in X^{\otimes n}$ and $\xi_2, \eta_2 \in X^{\otimes m}$. We have

$$\begin{aligned} &t^{(n)}(x)^*(t^{(n+m)})_*(\theta_{\xi,\eta})t^{(n)}(y) = t^{(n)}(x)^* t^{(n+m)}(\xi)t^{(n+m)}(\eta)^* t^{(n)}(y)^* \\ &= t^{(n)}(x)^* t^{(n)}(\xi_1)t^{(m)}(\xi_2)t^{(m)}(\eta_2)^* t^{(n)}(\eta_1)^* t^{(n)}(y) \\ &= \pi(\langle x, \xi_1\rangle)(t^{(m)})_*(\theta_{\xi_2,\eta_2})\pi(\langle \eta_1, y\rangle) = (t^{(m)})_*(\phi_m(\langle x, \xi_1\rangle)\theta_{\xi_2,\eta_2}\phi_m(\langle \eta_1, y\rangle)), \end{aligned}$$

which is an element of B_m, since $\mathscr{K}(X^{\otimes m})$ is an ideal of $\mathscr{L}(X^{\otimes m})$. Elements of the form $(t^{(n+m)})_*(\theta_{\xi,\eta})$ generate B_{n+m} and so we are done. □

Lemma 5.2.2. *If $n, m \in \mathbb{N}$ such that $m \leq n$, $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$ we have*

$$t^{(m)}(\eta)^* t^{(n)}(\xi) = t^{(n-m)}(\zeta) \quad where\ \zeta = \tau^m_{n-m}(\eta)^* \xi \in X^{\otimes(n-m)}.$$

Proof. If $m = 0$ then $\eta \in \mathscr{A}$ and $t^{(0)} = \pi$ therefore we have

$$\pi(\eta)^* t^{(n)}(\xi) = t^{(n)}(\phi_n(\eta)^*\xi) = t^{(n)}(\tau_n^0(\eta)^*\xi).$$

If $m > 0$ we can assume that $\xi = \eta' \otimes \zeta'$ where $\eta \in X^{\otimes m}$ and $\zeta' \in X^{\otimes(n-m)}$ since the linear span of these elements is dense in $X^{\otimes n}$.

$$\begin{aligned} t^{(m)}(\eta)^* t^{(n)}(\xi) &= t^{(m)}(\eta)^* t^{(m)}(\eta') t^{(n-m)}(\zeta') = \pi(\langle \eta, \eta' \rangle) t^{(n-m)}(\zeta') \\ &= t^{(n-m)}(\phi_{n-m}(\langle \eta, \eta' \rangle)\zeta') = t^{(n-m)}(\tau_{n-m}^m(\eta)^*(\eta' \otimes \zeta')) = t^{(n-m)}(\tau_{n-m}^m(\eta)^*\xi). \end{aligned}$$

□

As a consequence of the preceding lemma it is easy to prove the following:

Proposition 5.2.1. *For a Toeplitz representation (π, t) of X we have*

$$C^*(\pi, t) = \overline{\operatorname{span}}\{t^{(n)}(\xi) t^{(m)}(\eta)^* : \xi \in X^{\otimes n}, \eta \in X^{\otimes m} \quad \text{and } n, m \in \mathbb{N}\}.$$

Proof. It is immediate that the right-hand side is a closed and self-adjoint linear subspace of $C^*(\pi, t)$ and from the lemma above it is also closed under multiplication and therefore a C^*-algebra. For $a \in \mathscr{A}$ if we pick $n, m = 0$, $\xi = a$ and $\eta = \mathbf{1}_{\mathscr{A}}$, since $t^{(0)} = \pi$ we get that $\pi(a)$ is an element of the right-hand side. If we pick $n = 1$, $m = 0$ and $\eta = \mathbf{1}_{\mathscr{A}}$ we can see that for each $\xi \in X$ the element $t(\xi)$ is in the right-hand side and we are done. □

Definition 5.2.3. For $m, n \in \mathbb{N}$ with $m \leq n$ we define

$$B_{[m,n]} = B_m + B_{m+1} + ... + B_n.$$

We have that $B_{[n,n]} = B_n$. Note that $B_{[m,n]}$ is obviously linear and self-adjoint and by proving the next lemma we can see that $B_{[m,n]}$ are closed under multiplication and that if $k \leq m \leq n$ then $B_{[k,m]}$ is an ideal of $B_{[m,n]}$.

Lemma 5.2.3. *For $m, n \in \mathbb{N}$ with $m \leq n$, $k \in \mathscr{K}(X^{\otimes m})$ and $k' \in \mathscr{K}(X^{\otimes n})$ we have*

$$(t^{(m)})_*(k)(t^{(n)})_*(k') = (t^{(n)})_*((k \otimes I_{X^{\otimes(n-m)}})k').$$

Proof. Firstly, we show that for $k \in \mathscr{K}(X^{\otimes m})$ and $\xi \in X^{\otimes n}$ we have

$$(t^{(m)})_*(k) t^{(n)}(\xi) = t^{(n)}((k \otimes I_{X^{\otimes(n-m)}})(\xi).$$

Indeed, if $m = 0$ the above becomes

$$\pi(k) t^{(n)}(\xi) = t^{(n)}(\phi(a)\xi)$$

which is true since $(\pi, t^{(n)})$ is a Toeplitz representation. Suppose that $m > 0$, we may assume that $k = \theta_{\zeta,\eta}$ for $\zeta, \eta \in X^{\otimes m}$ and by using lemmas 5.1.2 and 5.2.2 we can see that

$$(t^{(m)})_*(k)t^{(n)}(\xi) = t^{(m)}(\zeta)t^{(m)}(\eta)^*t^{(n)}(\xi) = t^{(m)}(\zeta)t^{(n-m)}(\tau^m_{n-m}(\eta)^*\xi)$$
$$= t^{(n)}(\zeta \otimes (\tau^m_{n-m}(\eta)^*\xi)) = t^{(n)}(\tau^m_{n-m}(\zeta)\tau^m_{n-m}(\eta)^*\xi) = t^{(n)}((k \otimes I_{X^{\otimes(n-m)}})\xi).$$

In order to prove that for $k' \in \mathscr{K}(X^{\otimes n})$,

$$(t^{(m)})_*(k)(t^{(n)})_*(k') = (t^{(n)})_*((k \otimes I_{X^{\otimes(n-m)}})k').$$

we may assume that $k' = \theta_{x,y}$ for $x, y \in X^{\otimes n}$ and so

$$(t^{(m)})_*(k)(t^{(n)})_*(\theta_{x,y}) = (t^{(m)})_*(k)t^{(n)}(x)t^{(n)}(y)^*$$
$$= t^{(n)}((k \otimes I_{X^{\otimes(n-m)}})x)t^{(n)}(y)^* = (t^{(n)})_*(\theta_{(k\otimes I_{X^{\otimes(n-m)}})x,y})$$
$$= (t^{(n)})_*((k \otimes I_{X^{\otimes(n-m)}})\theta_{x,y}).$$

□

Remark 20. It remains to prove that for $m < n$ the set $B_{[m,n]}$ is closed in order to prove that it is a C^*-algebra. We already noted that B_k is closed for each $k \geq 0$ and so from the preceding lemma B_{m+1} is a closed ideal of $B_{[m,m+1]}$ and therefore also a closed ideal of $\overline{B_{[m,m+1]}}$. Let $q : \overline{B_{[m,m+1]}} \to \overline{B_{[m,m+1]}}/B_{m+1}$ be the usual quotient $*$-epimorphism.
We have that

$$B_{[m,m+1]} = B_m + B_{m+1} = q^{-1}(q(B_m))$$

which is closed since q is a $*$-homomorphism between C^*-algebras.
Indeed, if $x \in q^{-1}(q(B_m))$ then $q(x) \in q(B_m)$ and so there exists $y \in B_m$ such that $q(y) = q(x)$, therefore $x - y \in \ker q = B_{m+1}$ and so $x = y + (x - y) \in B_m + B_{m+1}$.
For the converse inclusion if $x + y \in B_m + B_{m+1}$ then since

$$q(x + y) = x + y + B_{m+1} = x + B_{m+1} = q(x),$$

we have that $x + y \in q^{-1}(\{q(x)\}) \subseteq q^{-1}(q(B_m))$.
Inductively, since $B_{[m,m+1]}$ is a closed ideal of $B_{[m,m+2]}$, we get that $B_{[m,n]}$ is closed.

Definition 5.2.4. For each $m \in \mathbb{N}$ we define a C^*-subalgebra of $C^*(\pi, t)$ by $B_{[m,\infty]} = \overline{\bigcup_{n=m}^{\infty} B_{[m,n]}}$.

Remark 21. Let (π, t) be a Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$ that admits a gauge action β. We define $\mathscr{E} : C^*(\pi, t) \to C^*(\pi, t)$ such that for $x \in C^*(\pi, t)$

$$\mathscr{E}(x) = \int_0^1 \beta_{e^{2\pi i t}}(x)dt.$$

Since we have shown that for $z \in \mathbb{T}$ the map $z \to \beta_z(x)$ is continuous $\mathscr{E}$ is well-defined. It is also easy to see that $\mathscr{E}$ is contractive since

$$\left\| \int_0^1 \beta_{e^{2\pi i t}}(x)dt \right\| \leq \int_0^1 \|\beta_{e^{2\pi i t}}(x)\|dt \leq \|x\|.$$

We denote by $C^*(\pi, t)^\beta$ the C^*-algebra

$$\bigcap_{z \in \mathbb{T}} \{x \in C^*(\pi, t) : \beta_z(x) = x\}$$

and call it the fixed-point algebra. In particular, $C^*(\pi, t)^\beta$ coincides with the image of $\mathscr{E}$. Indeed, if $x \in C^*(\pi, t)^\beta$ then there exists $y \in C^*(\pi, t)$ such that $\mathscr{E}(y) = x$ and therefore for each $t_0 \in [0, 1]$ we have that

$$\beta_{e^{2\pi i t_0}}(x) = \beta_{e^{2\pi i t_0}}\left(\int_0^1 \beta_{e^{2\pi i t}}(y)dt \right) = \int_0^1 \beta_{e^{2\pi i (t_0 + t)}}(y)dt = \int_0^1 \beta_{e^{2\pi i t}}(y)dt = x.$$

Conversely, if $x \in \bigcap_{z \in \mathbb{T}} \{x \in C^*(\pi, t) : \beta_z(x) = x\}$ then

$$\mathscr{E}(x) = \int_0^1 \beta_{e^{2\pi i t}}(x)dt = \int_0^1 x dt = x.$$

Proposition 5.2.2. *If (π, t) admits a gauge action β, then $B_{[0,\infty]} = C^*(\pi, t)^\beta$.*

Proof. For $z \in \mathbb{T}$, $\xi \in X^{\otimes n}$ and $\eta \in X^{\otimes m}$ we have

$$\beta_z(t^{(n)}(\xi)t^{(m)}(\eta)^*) = z^{n-m} t^{(n)}(\xi)t^{(m)}(\eta)^*.$$

Therefore, putting $n = m$, we see that the elements of the form $t^{(n)}(\xi)t^{(n)}(\eta)^*$ are in $C^*(\pi, t)^\beta$ for every $\xi, \eta \in X^{\otimes n}$ and since $C^*(\pi, t)^\beta$ is a C^*-algebra we have $B_n \subseteq C^*(\pi, t)^\beta$ for every $n \geq 0$. This implies that $B_{[0,\infty]} \subseteq C^*(\pi, t)^\beta$.
For the converse inclusion pick $x \in C^*(\pi, t)^\beta$. Proposition 5.2.1 implies that there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ of linear sums of elements in the form $t^{(n)}(\xi)t^{(m)}(\eta)^*$ that converges to x. Thus, since for each $z \in \mathbb{T}$ the map β_z is a contraction we have

$$x = \int_0^1 \beta_{e^{2\pi i t}}(x)dt = \lim_k \int_0^1 \beta_{e^{2\pi i t}}(x_k)dt.$$

For each $k \in \mathbb{N}$ we have $\int_0^1 \beta_{e^{2\pi i t}}(x_k)dt \in \bigcup_{n=0}^\infty B_{[0,n]}$. Indeed, for an element of the form $t^{(n)}(\xi)t^{(m)}(\eta)^*$ we have that

$$\mathscr{E}(t^{(n)}(\xi)t^{(m)}(\eta)^*)) = \int_0^1 e^{2\pi i (n-m)s} t^{(n)}(\xi)t^{(m)}(\eta)^* ds = \begin{cases} t^{(n)}(\xi)t^{(n)}(\eta)^*, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases},$$

in particular $\mathscr{E}(t^{(n)}(\xi)t^{(m)}(\eta)^*)) \in \bigcup_{n=0}^{\infty} B_{[0,n]}$ and from linearity we are done. We conclude that $x \in B_{[0,\infty]}$ as a norm limit of elements in $B_{[0,\infty]}$ □

Definition 5.2.5. Let (π, t) be a Toeplitz representation of a C^*-correspondennce $(X, \mathscr{A}, \phi)$. We define

$$I_{(\pi,t)} = \{a \in \mathscr{A} : \pi(a) \in t_*(\mathscr{K}(X))\}.$$

Remark 22. If $(X, \mathscr{A}, \phi)$ is a C^*-correspondence, $(\tilde{\pi}_u, \tilde{t}_u)$ is the universal Toeplitz representation and (π_∞, t_∞) is the Fock representation we have that $I_{(\tilde{\pi}_u,\tilde{t}_u)} = \{0\}$.
Indeed, suppose that $a \in I_{(\tilde{\pi}_u,\tilde{t}_u)}$ and $k \in \mathscr{K}(X)$ such that $\tilde{\pi}_u(a) = (\tilde{t}_u)_*(k)$, by corollary 5.1.1 we have that $I_{(\pi_\infty,t_\infty)} = \{0\}$ and therefore since

$$\pi_\infty(a) = \tilde{\rho}(\tilde{\pi}_u(a)) = \tilde{\rho}((\tilde{t}_u)_*(k)) = (t_\infty)_*(k),$$

we get that $a \in I_{(\pi_\infty,t_\infty)}$ and so $a = 0$.

Now let (π, t) be an injective Katsura covariant Toeplitz representation. The fact that (π, t) is a Katsura covariant Toeplitz representation implies that $J_X \subseteq I_{(\pi,t)}$ and since it is also injective, using proposition 5.1.4 we can see that $I_{(\pi,t)} \subseteq J_X$, thus $I_{(\pi,t)} = J_X$. In particular, $I_{(\pi_u,t_u)} = J_X$ where (π_u, t_u) is the universal Katsura covariant Toeplitz representation and also $I_{(\pi,t)} = J_X$ where (π, t) is the injective Katsura covariant representation we described in theorem 5.1.1.

Lemma 5.2.4. *Let n be a positive integer and $\{e_\lambda : \lambda \in \Lambda\}$ an approximate unit for $\mathscr{K}(X^{\otimes n})$. If $k \in \mathscr{K}(X^{\otimes(n+1)})$ then*

$$k = \lim_{\lambda}(e_\lambda \otimes I_X)k.$$

Proof. Suppose that $k = (k' \otimes I_X)k''$ where $k' \in \mathscr{K}(X^{\otimes n})$ and $k'' \in \mathscr{K}(X^{\otimes(n+1)})$. Since

$$\begin{aligned}&\|(e_\lambda \otimes I_X)(k' \otimes I_X)k'' - (k' \otimes I_X)k''\| = \|(e_\lambda k' \otimes I_X)k'' - (k' \otimes I_X)k''\| = \\ &\|((e_\lambda k' - k') \otimes I_X)k''\| \leq \|e_\lambda k' - k'\|\|k''\| \xrightarrow{\lambda} 0,\end{aligned}$$

we have that $k = \lim_\lambda(e_\lambda \otimes I_X)k$ if $k = (k' \otimes I_X)k''$, thus to prove this lemma it suffices to show that the linear span of elements in the form $k = (k' \otimes I_X)k''$ is dense in $\mathscr{K}(X^{\otimes(n+1)})$. In order to do so we prove that the linear span of elements in the form $(k' \otimes I_X)\zeta$ with $k' \in \mathscr{K}(X^{\otimes n})$ and $\zeta \in X^{\otimes(n+1)}$ is dense in $X^{\otimes(n+1)}$.
Indeed, we may pick $k' = \theta_{\xi,\xi'}$ and $\zeta = \eta \otimes \eta'$ where $\xi, \xi', \eta \in X^{\otimes n}$ and $\eta' \in X$ and using lemma 5.1.2 we have that

$$\begin{aligned}(k' \otimes I_X)\zeta &= \tau_1^n(\xi)\tau_1^n(\xi')^*(\eta \otimes \eta') = \tau_1^n(\xi)(\phi(\langle \xi', \eta \rangle)\eta') \\ &= \xi \otimes \phi(\langle \xi', \eta \rangle)\eta' = \xi\langle \xi', \eta \rangle \otimes \eta'.\end{aligned}$$

Since the linear span of elements in the form $\xi\langle\xi',\eta\rangle$ with $\xi,\xi',\eta\in X^{\otimes n}$ is dense in $X^{\otimes n}$ and the linear span of the elements in the form $\xi\otimes\eta'$ with $\xi\in X^{\otimes n}$ and $\eta'\in X$ is dense in $X^{\otimes(n+1)}$ we get the desired result.
Finally, using the equality

$$(k'\otimes I_X)\theta_{\zeta,\zeta'}=\theta_{(k\otimes I_X)\zeta,\zeta'}$$

we can see that since the linear span of elements as in the right-hand side of the equality is dense in $\mathscr{K}(X^{\otimes(n+1)})$, the linear span of elements in the form $(k'\otimes I_X)k''$ is also dense in $\mathscr{K}(X^{\otimes(n+1)})$. □

From the preceding lemma for $n=1$ and $\{e_\lambda:\lambda\in\Lambda\}$ an approximate unit for $\mathscr{K}(X)$ we get that

$$\{e_\lambda\otimes I_X:\lambda\in\Lambda\}$$

is an approximate unit for $\mathscr{K}(X^{\otimes 2})$. Inductively, we get that

$$\{e_\lambda\otimes I_{X^{\otimes(n-1)}}:\lambda\in\Lambda\}$$

is an approximate unit for $\mathscr{K}(X^{\otimes n})$.

Proposition 5.2.3. *Let $(X,\mathscr{A},\phi)$ be a C^*-correspondence, (π,t) a Toeplitz representation and $\{e_\lambda:\lambda\in\Lambda\}$ an approximate unit for $\mathscr{K}(X)$. Then for each $n\geq 1$ we have that $\{t_*(e_\lambda):\lambda\in\Lambda\}$ is an approximate unit for B_n, consequently also for $B_{[1,n]}$.*

Proof. Let n be a positive integer. If $x\in B_n$ we may assume that $x=(t^{(n)})_*(\theta_{\xi,\eta})$ for $\xi=\xi_1\otimes...\otimes\xi_n$ and $\eta=\eta_1\otimes...\otimes\eta_n$. Thus,

$$x=t^{(n)}(\xi)t^{(n)}(\eta)^*=t(\xi_1)...t(\xi_n)t(\eta_n)^*...t(\eta_1)^*.$$

We have that

$$\begin{aligned}t_*(e_\lambda)x&=t_*(e_\lambda)t(\xi_1)...t(\xi_n)t(\eta_n)^*...t(\eta_1)^*=t(e_\lambda\xi_1)...t(\xi_n)t(\eta_n)^*...t(\eta_1)^*\\&=(t^{(n)})_*\left(\theta_{(e_\lambda\otimes I_{X^{\otimes(n-1)}})\xi,\eta}\right)=(t^{(n)})_*((e_\lambda\otimes I_{X^{\otimes(n-1)}})\theta_{\xi,\eta})\xrightarrow{\lambda}(t^{(n)})_*(\theta_{\xi,\eta})=x,\end{aligned}$$

and

$$xt_*(e_\lambda)=(t_*(e_\lambda)^*x^*)^*=(t_*((e_\lambda)^*)x^*)^*=(t_*(e_\lambda)x^*)\xrightarrow{\lambda}(x^*)^*=x.$$

This proves that $\{t_*(e_\lambda):\lambda\in\Lambda\}$ is an approximate unit for B_n for every $n\geq 1$ and since $B_{[1,n]}$ is generated by B_k for $1\leq k\leq n$, we also have that $\{t_*(e_\lambda):\lambda\in\Lambda\}$ is an approximate unit for $B_{[1,n]}$ for every $n\geq 1$. □

We denote the C^*-subalgebras of T_X and O_X corresponding to B_n and $B_{[m,n]}$ by $\tilde{\mathscr{B}}_n$ and $\tilde{\mathscr{B}}_{[m,n]}\subseteq T_X$ and by $\mathscr{B}_n$ and $\mathscr{B}_{[m,n]}\subseteq O_X$, respectively. We also denote by $\tilde{\gamma}$ the gauge action of $(\tilde{\pi}_u,\tilde{t}_u)$ and by γ the gauge action of (π_u,t_u). Therefore, for the fixed-point algebras we have that $T_X^{\tilde{\gamma}} = \tilde{\mathscr{B}}_{[0,\infty]}$ and $O_X^{\gamma} = \mathscr{B}_{[0,\infty]}$.

Lemma 5.2.5. *Let (π, t) be a Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$ such that $I_{(\pi,t)} = \{0\}$ and $\tilde{\rho} : T_X \to C^*(\pi, t)$ the $*$-epimorphism such that for $a \in \mathscr{A}$ and $\xi \in X$ we have $\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a)$ and $\tilde{\rho}(\tilde{t}_u(\xi)) = t(\xi)$. Then the restriction of $\tilde{\rho}$ to the fixed-point algebra $T_X^{\tilde{\gamma}}$ is injective.*

Proof. From lemma 2.1.1 it suffices to show that for each $N \geq 0$ we have that

$$\ker \tilde{\rho} \cap \tilde{\mathscr{B}}_{[0,N]} = \{0\},$$

since

$$\ker \tilde{\rho} \cap \tilde{\mathscr{B}}_{[0,\infty]} = \overline{\bigcup_{N \geq 0} (\ker \tilde{\rho} \cap \tilde{\mathscr{B}}_{[0,N]})}.$$

If $N = 0$ we have that $\tilde{\mathscr{B}}_0 = \tilde{\pi}_u(\mathscr{A})$. By the definition of $I_{(\pi,t)}$ it contains $\ker \pi$ and therefore π is injective. Since $\tilde{\rho} \circ \tilde{\pi}_u = \pi$ the result follows.
Suppose that $N \geq 1$ is the least positive integer such that

$$\ker \tilde{\rho} \cap \tilde{\mathscr{B}}_{[0,N]} \neq \{0\}$$

and let

$$f = \tilde{\pi}_u(a) + \sum_{n=1}^{N} (\tilde{t}_u^{(n)})_*(k_n)$$

be a non-zero element of $\ker \tilde{\rho} \cap \tilde{\mathscr{B}}_{[0,N]}$ where $a \in \mathscr{A}$ and $k_n \in \mathscr{K}(X^{\otimes n})$. We have

$$\pi(a) = -\sum_{n=1}^{N} (t^{(n)})_*(k_n).$$

Let $\{e_\lambda\}_\lambda$ be an approximate unit for $\mathscr{K}(X)$, then

$$\lim_\lambda t_*(\phi(a)e_\lambda) = \lim_\lambda \pi(a)t_*(e_\lambda) = -\sum_{n=1}^{N} \lim_\lambda (t^{(n)})_*(k_n)t_*(e_\lambda) = -\sum_{n=1}^{N} (t^{(n)})_*(k_n)$$

where we have used the fact that for all $n \geq 1$, $\{t_*(e_\lambda)\}_\lambda$ is an approximate unit for B_n. Since the net $\{t_*(\phi(a)e_\lambda)\}_\lambda$ is convergent, it is Cauchy and since π is injective t_* is an isometry, therefore $\{\phi(a)e_\lambda\}_\lambda$ converges to some $k \in \mathscr{K}(X)$. So

$$\pi(a) = -\sum_{n=1}^{N} (t^{(n)})_*(k_n) = \lim_\lambda t_*(\phi(a)e_\lambda) = t_*(k),$$

which implies that $a \in I_{(\pi,t)}$ and therefore $a = 0$. It is evident now that $f \in \tilde{\mathscr{B}}_{[1,N]}$ and thus for each $\xi, \eta \in X$ using lemma 5.2.1 we have that

$$\tilde{t}_u(\eta)^* f \tilde{t}_u(\xi) = \sum_{n=1}^{N} \tilde{t}_u(\eta)^* (\tilde{t}_u^{(n)})_*(k_n) \tilde{t}_u(\xi) \in \ker \tilde{\rho} \cap \mathscr{B}_{[0,N-1]}.$$

By the choice of N we obtain that $\tilde{t}_u(\eta)^* f \tilde{t}_u(\xi) = 0$ and consequently if $\xi_1, \eta_1, \xi_2, \eta_2 \in X$ we have that

$$(\tilde{t}_u)_*(\theta_{\xi_1,\eta_1}) f (\tilde{t}_u)_*(\theta_{\xi_2,\eta_2}) = 0$$

which implies that $(\tilde{t}_u)_*(\mathscr{K}(X)) f (\tilde{t}_u)_*(\mathscr{K}(X)) = 0$.
Since $(\tilde{t}_u)_*(\mathscr{K}(X))$ contains an approximate unit for $\tilde{\mathscr{B}}_{[1,N]}$, we get that $f = 0$, hence a contradiction. □

Lemma 5.2.6. *Let (π, t) be an injective Katsura covariant Toeplitz representation of a C^*-correspondence $(X, \mathscr{A}, \phi)$ and $\rho : O_X \to C^*(\pi, t)$ the $*$-epimorphism such that for $a \in \mathscr{A}$ and $\xi \in X$ we have $\rho(\pi_u(a)) = \pi(a)$ and $\rho(t_u(\xi)) = t(\xi)$. Then the restriction of ρ to the fixed-point algebra O_X^γ is injective.*

Proof. As in the proof of the previous lemma it suffices to show that for each $N \geq 0$ we have that $\ker \rho \cap \mathscr{B}_{[0,N]} = \{0\}$. By doing the exact same steps if $N \geq 1$ and

$$f = \pi_u(a) + \sum_{n=1}^{N} (t_u^{(n)})_*(k_n) \in \ker \rho \cap \mathscr{B}_{[0,N]}$$

we have that $a \in I_{(\pi,t)} = J_X$ and therefore

$$\pi_u(a) = (t_u)_*(\phi(a)),$$

by using the covariance relation. This implies that $f \in \mathscr{B}_{[1,n]}$ and the result follows as above. □

Theorem 5.2.1. *(Gauge-invariance Uniqueness theorem)*

(i) Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and (π, t) a Toeplitz representation of $(X, \mathscr{A}, \phi)$. The induced $*$-epimorphism $\tilde{\rho} : T_X \to C^*(\pi, t)$ is a $*$-isomorphism if and only if $I_{(\pi,t)} = \{0\}$ and (π, t) admits a gauge action.

(ii) Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and (π, t) a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$. The induced $*$-epimorphism $\rho : O_X \to C^*(\pi, t)$ is a $*$-isomorphism if and only if π is injective and (π, t) admits a gauge action.

Proof. $i)$ Suppose that $\tilde{\rho} : T_X \to C^*(\pi, t)$ is a $*$-isomorphism and let a be an element in $I_{(\pi,t)}$. There exists $k \in \mathscr{K}(X)$ such that $\pi(a) = t_*(k)$ and therefore

$$\tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a) = t_*(k) = \tilde{\rho}((\tilde{t}_u)_*(k)).$$

Since $\tilde{\rho}$ is injective we have that $\tilde{\pi}_u(a) = (\tilde{t}_u)_*(k)$ and the fact that $I_{(\tilde{\pi}_u, \tilde{t}_u)} = \{0\}$ implies that $a = 0$.

To see that (π, t) admits a gauge action, for each $z \in \mathbb{T}$ set

$$\beta_z = \tilde{\rho} \circ \tilde{\gamma}_z \circ \tilde{\rho}^{-1}.$$

If $a \in \mathscr{A}$ and $\xi \in X$ then we have that

$$\beta_z(\pi(a)) = \tilde{\rho} \circ \tilde{\gamma}_z \circ \tilde{\rho}^{-1}(\pi(a)) = \tilde{\rho} \circ \tilde{\gamma}_z(\tilde{\pi}_u(a)) = \tilde{\rho}(\tilde{\pi}_u(a)) = \pi(a)$$

and

$$\beta_z(t(\xi)) = \tilde{\rho} \circ \tilde{\gamma}_z \circ \tilde{\rho}^{-1}(t(\xi)) = \tilde{\rho} \circ \tilde{\gamma}_z(\tilde{t}_u(\xi)) = \tilde{\rho}(z\tilde{t}_u(\xi)) = zt(\xi).$$

For the converse suppose that (π, t) admits a gauge action β and $I_{(\pi,t)} = \{0\}$. Pick $x \in T_X$ such that $\tilde{\rho}(x) = 0$, we have that

$$\tilde{\rho}\bigg(\int_0^1 \tilde{\gamma}_{e^{2\pi it}}(x^*x)dt\bigg) = \int_0^1 \tilde{\rho}(\tilde{\gamma}_{e^{2\pi it}}(x^*x)dt = \int_0^1 \beta_{e^{2\pi it}}(\tilde{\rho}(x^*x))dt = 0.$$

Since $\int_0^1 \tilde{\gamma}_{e^{2\pi it}}(x^*x)dt \in T_X^{\tilde{\gamma}}$ and the restriction of $\tilde{\rho}$ to $T_X^{\tilde{\gamma}}$ is injective, we get that

$$\int_0^1 \tilde{\gamma}_{e^{2\pi it}}(x^*x)dt = 0,$$

which implies that $x^*x = 0$ since an integral of a positive non-zero function is positive (see the proof of this implication in theorem 3.2.2) and therefore $\tilde{\rho}$ is injective.

(ii) If ρ is a $*$-isomorphism since

$$\pi = \rho \circ \pi_u,$$

it is immediate that π is injective. The rest of the proof is similar to the proof of (i). □

We prove now that the Fock representation and the injective Katsura covariant Toeplitz representation described in 5.1.1 admit gauge actions.

For each $z \in \mathbb{T}$ let $u_z : F_X \to F_X$ be the adjointable map such that for $\xi \in X^{\otimes n}$

$$u_z(\xi) = z^n\xi.$$

Note that for $(a, \xi_1, \xi_2, ...) \in F_X$ we have that

$$\begin{aligned}\|u_z(a, \xi_1, \xi_2, ...)\|^2 &= \left\|a^*a + \sum_{k=1}^{\infty} \left\langle z^k\xi_k, z^k\xi_k\right\rangle\right\| \\ &= \left\|a^*a + \sum_{k=1}^{\infty} \langle\xi_k, \xi_k\rangle\right\| = \|(a, x_1, x_2, ...)\|^2\end{aligned}$$

and therefore u_z is well-defined and bounded. We also have that $u_z^* = u_{\overline{z}}$.

Indeed, if $(a, \xi_1, \xi_2, ...), (b, \eta_1, \eta_2, ...) \in F_X$ then

$$\langle u_z(a, \xi_1, \xi_2, ...), (b, \eta_1, \eta_2, ...)\rangle = a^*b + \sum_{k=1}^{\infty} \left\langle z^k \xi_k, \eta_k \right\rangle$$
$$= a^*b + \sum_{k=1}^{\infty} \left\langle \xi_k, \overline{z}^k \eta_k \right\rangle = \langle (a, \xi_1, \xi_2, ...), u_{\overline{z}}(b, \eta_1, \eta_2, ...)\rangle .$$

We define a gauge action $\tilde{\beta}$ of (π_∞, t_∞) such that for each $z \in \mathbb{T}$ and $x \in C^*(\pi_\infty, t_\infty)$

$$\tilde{\beta}_z(x) = u_z x u_z^*.$$

Let $a \in \mathscr{A}$, $\xi \in X$ and $\eta = (\eta_0, \eta_1, \eta_2...) \in F_X$ then

$$\tilde{\beta}_z(\pi_\infty(a))(\eta) = u_z \pi_\infty(a) u_{\overline{z}}(\eta) = u_z \pi_\infty(a)(\eta_0, \overline{z}\eta_1, \overline{z}^2\eta_2, ...)$$
$$= u_z(\phi_0(a)\eta_0, \overline{z}\phi_1(a)\eta_1, \overline{z}^2\phi_2(a)\eta_2, ...) = (\phi_0(a)\eta_0, \phi_1(a)\eta_1, \phi_2(a)\eta_2, ...)$$
$$= \pi_\infty(a)(\eta)$$

and

$$\tilde{\beta}_z(t_\infty(\xi))(\eta) = u_z t_\infty(\xi) u_{\overline{z}}(\eta) = u_z t_\infty(\xi)(\eta_0, \overline{z}\eta_1, \overline{z}^2\eta_2, ...)$$
$$= u_z(0, \xi\eta_0, \overline{z}\xi \otimes \eta_1, \overline{z}^2\xi \otimes \eta_2, ...) = (0, z\xi\eta_0, z\xi \otimes \eta_1, z\xi \otimes \eta_2, ...)$$
$$= z t_\infty(\xi)(\eta).$$

Hence, $\{\tilde{\beta}_z : z \in \mathbb{T}\}$ is a gauge action of (π_∞, t_∞).
We will now show that the injective Katsura covariant Toeplitz representation admits a gauge action. Recall that in the proof of proposition 5.1.5 we showed that for $\xi \in X^{\otimes n}, \eta \in X^{\otimes m}$ and $a \in J_X$ we have that

$$\theta_{\xi a, \eta} = t_\infty^{(n)}(\xi)((\pi_\infty(a) - (t_\infty)_*(\phi(a)))t_\infty^{(m)}(\eta)^*.$$

Note that $\tilde{\beta}_z((t_\infty)_*(\phi(a))) = (t_\infty)_*(\phi(a))$. To see that this implication holds we may suppose that $\phi(a) = \theta_{x,y}$ for $x, y \in X$, since the linear span of these elements is dense in $\mathscr{K}(X)$ and so

$$\tilde{\beta}_z((t_\infty)_*(\theta_{x,y})) = \tilde{\beta}_z(t(x)t(y)^*) = zt(x)z^{-1}t(y)^* = t(x)t(y)^* = (t_\infty)_*(\theta_{x,y}).$$

Therefore,

$$\tilde{\beta}_z(\theta_{\xi a, \eta}) = z^{n-m} t_\infty^{(n)}(\xi)(\pi_\infty(a) - (t_\infty)_*(\phi(a)))t_\infty^{(m)}(\eta)^* \in \mathscr{K}(F_X J_X)$$

and since these elements generate $\mathscr{K}(F_X J_X)$ we get that $\tilde{\beta}_z(\mathscr{K}(F_X J_X) \subseteq \mathscr{K}(F_X J_X)$.

Thus, if $\sigma : \mathscr{L}(F_X) \to \mathscr{L}(F_X)/\mathscr{K}(F_X J_X)$ is the quotient $*$-epimorphism and for each $z \in \mathbb{T}$, $\beta_z = \sigma \circ \tilde{\beta}_z$ then $z \to \beta_z$ is a gauge action of the injective Katsura covariant Toeplitz representation (π, t).

Theorem 5.2.2. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence, then*

(i) $C^*(\pi_\infty, t_\infty)$ *is $*$-isomorphic to* T_X.

(ii) $C^*(\pi, t)$ *is $*$-isomorphic to* O_X.

Proof. This follows from theorem 5.2.1 since both (π, t) and (π_∞, t_∞) are injective, admit gauge actions and $I_{(\pi_\infty, t_\infty)} = \{0\}$. □

Using the Gauge-Invariance Uniqueness theorem we will give an additional proof that the reduced crossed product by $\mathbb{Z}$ is $*$-isomorphic to the crossed product by $\mathbb{Z}$.

Corollary 5.2.1. *Let $\mathscr{A}$ be a unital C^*-algebra and let α be a $*$-automorphism of $\mathscr{A}$. Then $\mathbb{Z} \times_{\alpha r} \mathscr{A}$ is $*$-isomorphic to $\mathbb{Z} \times_\alpha \mathscr{A}$.*

Proof. We denote by $(\tilde{\pi}, \tilde{S})$ the left unitary covariant representation of $(\mathscr{A}, \mathbb{Z}, \alpha)$ we defined in example 3.2.1. In remark 13 we proved the existence of a family of maps $\{\beta_z : z \in \mathbb{T}\}$, where for each $z \in \mathbb{T}$ we have that $\beta_z : \mathbb{Z} \times_{\alpha r} \mathscr{A} \to \mathbb{Z} \times_{\alpha r} \mathscr{A}$ is a $*$-autmorphism such that for each $a \in \mathscr{A}$

$$\beta_z(\tilde{\pi}(a)) = \tilde{\pi}(a) \quad \text{and} \quad \beta_z(\tilde{S}) = z\tilde{S}.$$

For each $\xi \in \mathscr{A}$ we set $\tilde{t}(\xi) = \tilde{S}\tilde{\pi}(\xi)$. In example 5.1.4, we proved that the pair $(\tilde{\pi}, \tilde{t})$ is a Katsura covariant representation of the C^*-correspodence $\mathscr{A}_\alpha$, which is also injective. Since, $\mathscr{A}$ is unital and $\tilde{\pi}$ is non-degenerate, it is easy to see that $C^*(\tilde{\pi}, \tilde{t}) = C^*(\tilde{\pi}, \tilde{S}) = \mathbb{Z} \times_{\alpha r} \mathscr{A}$ and since for each $\xi \in \mathscr{A}$ we have

$$\beta_z(\tilde{t}(\xi)) = \beta_z(\tilde{S}\tilde{\pi}(\xi)) = z\tilde{S}\tilde{\pi}(\xi) = z\tilde{t}(\xi),$$

we obtain that $\{\beta_z : z \in \mathbb{T}\}$ is a gauge-action of $(\tilde{\pi}, \tilde{t})$. From the gauge-invariance uniqueness theorem we have that $O_{\mathscr{A}}$ is $*$-isomorphic to $C^*(\tilde{\pi}, \tilde{t})$. Therefore, $\mathbb{Z} \times_{\alpha r} \mathscr{A}$ is $*$-isomorphic to $\mathbb{Z} \times_\alpha \mathscr{A}$. □

Chapter 6

C^*-envelopes

In this chapter we are going to introduce the notion of the C^*-envelope of an operator algebra. In the first section, our proofs are based on [13] and the proof of the main theorem of the second section is based on [15].

6.1 The C^*-envelope of the semi-crossed product

Definition 6.1.1. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{A} \subseteq \mathbf{B}(\mathscr{H})$ a (unital) operator algebra. A C^*-cover of $\mathscr{A}$ is a pair $(\mathscr{C}, j)$ where $\mathscr{C}$ is a C^*-algebra and $j : \mathscr{A} \to \mathscr{C}$ a completely isometric (unital) map such that $\mathscr{C}$ is the smallest C^*-algebra that contains $\mathscr{A}$, which we denote by $C^*(j(\mathscr{A}))$.

Definition 6.1.2. Let $\mathscr{A}$ be a unital operator algebra. The C^*-envelope $(C^*_{env}(\mathscr{A}), i)$ of $\mathscr{A}$ is the C^*-cover of $\mathscr{A}$ which satisfies the following property: If $(\mathscr{C}, j)$ is a C^*-cover of $\mathscr{A}$ then there exists a $*$-epimorphism $\rho : \mathscr{C} \to C^*_{env}(\mathscr{A})$ such that for each $a \in \mathscr{A}$ we have

$$i(a) = \rho(j(a)).$$

The existence and uniqueness under $*$-isomorphisms of such a C^*-cover for a unital operator algebra was proved in [9] and [5].

Definition 6.1.3. Let $\mathscr{A} \subseteq C^*(\mathscr{A})$ be a unital operator algebra. A boundary ideal of $\mathscr{A}$ is an ideal I of $C^*(\mathscr{A})$ such that the restriction to $\mathscr{A}$ of the quotient map

$$q : C^*(\mathscr{A}) \to C^*(\mathscr{A})/I$$

is completely isometric. We define the Shilov ideal J of $\mathscr{A}$ to be the largest boundary ideal of $\mathscr{A}$ i.e. if I is a boundary ideal for $\mathscr{A}$ then $I \subseteq J$.

We now prove that the Shilov ideal actually exists. Note that if $\mathscr{A}$ is a unital operator algebra and $j : \mathscr{A} \to C^*(\mathscr{A})$ is the inclusion map, then j is completely isometric, therefore

$(C^*(\mathscr{A}), j)$ is a C^*-cover of $\mathscr{A}$. Therefore, there exists a $*$-epimorphism $\rho : C^*(\mathscr{A}) \to C^*_{env}(\mathscr{A})$ such that for each $a \in \mathscr{A}$ we have $i(a) = \rho(j(a))$. We will prove that $\ker \rho$ is the Shilov ideal of $\mathscr{A}$.

First of all, we denote by $\tilde{\rho}$ the induced $*$-isomorphism $\tilde{\rho} : C^*(\mathscr{A})/\ker \rho \to C^*_{env}(\mathscr{A})$ such that $x + \ker \rho \to \rho(x)$. If $a \in \mathscr{A}$ we have

$$\|a + \ker \rho\| = \|\tilde{\rho}(a + \ker \rho)\| = \|\rho(j(a))\| = \|i(a)\| = \|a\|$$

and since i, j and $\tilde{\rho}$ are completely isometric the same argument works also for matrices, therefore $\ker \rho$ is a boundary ideal of $\mathscr{A}$. Suppose now that I is another boundary ideal of $\mathscr{A}$, we will prove that $I \subseteq \ker \rho$. We denote by q_I the natural $*$-epimorphism of $C^*(\mathscr{A})$ onto $C^*(\mathscr{A})/I$. Since the restriction of q_I to $\mathscr{A}$ is completely isometric we obtain a $*$-epimorphism

$$\psi : C^*(\mathscr{A})/I \to C^*_{env}(\mathscr{A})$$

such that $i(a) = \psi(q_I(a))$ for each $a \in \mathscr{A}$. Therefore, we have a $*$-epimorphism

$$\tilde{\rho}^{-1} \circ \psi : C^*(\mathscr{A})/I \to C^*(\mathscr{A})/\ker \rho$$

such that for each $a \in \mathscr{A}$ we have

$$\tilde{\rho}^{-1} \circ \psi(a + I) = \tilde{\rho}^{-1} \circ \psi(q_I(a)) = \tilde{\rho}^{-1}(i(a)) = \tilde{\rho}^{-1}(\rho(j(a)) = j(a) + \ker \rho = a + \ker \rho.$$

Since $C^*(\mathscr{A})$ is the C^*-algebra generated by $\mathscr{A}$ and $\tilde{\rho}^{-1} \circ \psi$ is a $*$-homomorphism we get that for each $x \in C^*(\mathscr{A})$

$$\tilde{\rho}^{-1} \circ \psi(x + I) = x + \ker \rho$$

and therefore $x \in I$ implies that $x \in \ker \rho$.

Proposition 6.1.1. *Let J be the Shilov ideal of a unital operator algebra $\mathscr{A} \subseteq C^*(\mathscr{A})$. If $\alpha : C^*(\mathscr{A}) \to C^*(\mathscr{A})$ is a $*$-isomorphism with $\alpha(\mathscr{A}) = \mathscr{A}$ then $\alpha(J) = J$.*

Proof. Let a, b be elements of $\mathscr{A}$ such that $\alpha(b) = a$, then

$$\begin{aligned} \|a\| &= \|\alpha(b)\| = \|b\| = \|b + J\| = \|\alpha^{-1}(\alpha(b)) + \alpha^{-1}(\alpha(J))\| \\ &\leq \|\alpha(b) + \alpha(J)\| = \|a + \alpha(J)\| \leq \|a\|. \end{aligned}$$

The same calculation works on matrices and therefore we get that $\alpha(J)$ is a boundary ideal of $\mathscr{A}$ which implies that $\alpha(J) \subseteq J$. Repeating the same argument for α^{-1} we have that $\alpha^{-1}(J) \subseteq J$ and therefore $\alpha(J) = J$. □

The following theorem will be a consequence of the main result that we will prove in the next section but here we give a proof that helps us understand how we can use the Shilov ideal in order to identify the C^*-envelope.

Theorem 6.1.1. *Let $\mathscr{A}$ be a unital C^*-algebra and α a unital $*$-automorphism. The C^*-envelope of the semi-crossed product $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ is the crossed product $\mathbb{Z} \times_\alpha \mathscr{A}$.*

Proof. Recall that $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ can be considered as a subalgebra of $\mathbb{Z} \times_\alpha \mathscr{A}$ (Remark 16) and note that if $a \in \mathscr{A}$ and $n < 0$ then

$$(\delta_{-n} \otimes \alpha^{-n}(a))^* = \delta_n \otimes \alpha^n(\alpha^{-n}(a)) = \delta_n \otimes a.$$

This implies that the elements in the form $\sum_{k=-m}^{m} \delta_k \otimes a_k$ for $m > 0$ and $a_k \in \mathscr{A}$ are contained in $C^*(\mathbb{Z}^+ \times_\alpha \mathscr{A})$ and since these elements are dense in $\mathbb{Z} \times_\alpha \mathscr{A}$ we have that $C^*(\mathbb{Z}^+ \times_\alpha \mathscr{A}) = \mathbb{Z} \times_\alpha \mathscr{A}$. Therefore it suffices to show that the Shilov ideal J of $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ is zero. Let $\rho : \mathbb{Z} \times_\alpha \mathscr{A} \to C^*_{env}(\mathbb{Z}^+ \times_\alpha \mathscr{A})$ be the $*$-epimorphism induced from the universal property of the C^*-envelope $(C^*_{env}(\mathbb{Z}^+ \times_\alpha \mathscr{A}), i)$. Thus for $a \in \mathscr{A}$ we have

$$\rho(a \otimes \delta_0) = i(a \otimes \delta_0).$$

Suppose towards a contradiction that $J \neq \{0\}$. For each $z \in \mathbb{T}$ we denote by γ_z the $*$-automorphism of $\mathbb{Z} \times_\alpha \mathscr{A}$ defined in 3.2.2. If $k \geq 0$ and $a \in \mathscr{A}$ then

$$\gamma_z(\delta_k \otimes a) = \gamma_z((\delta_0 \otimes a)(\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k) = (\delta_0 \otimes a)z^k(\delta_1 \otimes \mathbf{1}_{\mathscr{A}})^k.$$

The above implies that for each $z \in \mathbb{T}$ we have that $\gamma_z(\mathbb{Z}^+ \times_\alpha \mathscr{A}) = \mathbb{Z}^+ \times_\alpha \mathscr{A}$ and therefore from the preceding proposition we have that $\gamma_z(J) = J$ which implies that J has non-trivial intersection with the fixed-point algebra and so $\mathscr{A} \cap J \neq \{0\}$.
Indeed, if $x \in J$ is a positive non-zero element then $\gamma_z(x) \in J$ for each $z \in \mathbb{T}$ and therefore

$$\int_0^1 \gamma_{e^{2\pi i t}}(x)dt \in J.$$

This integral is also in the fixed-point algebra $\mathscr{A}$ and it is non-zero as an integral of a non-zero continuous positive function.
Pick $\{0\} \neq \delta_0 \otimes a \in \mathscr{A} \cap J$ then we have

$$0 = \|\rho(\delta_0 \otimes a)\| = \|i(\delta_0 \otimes a)\| = \|\delta_0 \otimes a\| = \|a\|,$$

hence a contradiction. □

6.2 The C^*-envelope of the Tensor algebra

We are going to prove that for a C^*-correspondence $(X, \mathscr{A}, \phi)$ the C^*-envelope of T_X^+ is O_X. In order to do so we will need to "add tails" to X. Suppose that $I \subseteq \mathscr{A}$ is a closed ideal of $\mathscr{A}$. We define the tail determined by I to be the C^*-algebra c_0-direct sum $T = c_0\,(I)$.

We shall denote the elements of T by $\overrightarrow{f} := (f_1, f_2, ...)$ where each $f_i \in I$ and by $\overrightarrow{0} := (0, 0, ...)$. We define the vector space $Y := X \oplus T$ and the C^*-algebra $\mathscr{B} := \mathscr{A} \oplus T$ where $\|.\|_{\mathscr{B}} = \max\{\|.\|_{\mathscr{A}}, \|.\|_T\}$. Then Y is a Hilbert $\mathscr{B}$-module where the right action is given by

$$\left(\xi, \overrightarrow{f}\right)(a, \overrightarrow{g}) := \left(\xi a, \overrightarrow{f}\overrightarrow{g}\right) \quad \xi \in X, a \in \mathscr{A}, \overrightarrow{f}, \overrightarrow{g} \in T$$

and the inner-product

$$\left\langle \left(\xi, \overrightarrow{f}\right), (\eta, \overrightarrow{g}) \right\rangle_{\mathscr{B}} := \left(\langle \xi, \eta \rangle_{\mathscr{A}}, \overrightarrow{f}^* \overrightarrow{g}\right), \quad \xi, \eta \in X, \overrightarrow{f}, \overrightarrow{g} \in T.$$

We also define a left action $\phi_{\mathscr{B}} : \mathscr{B} \to \mathscr{L}(Y)$ by

$$\phi_{\mathscr{B}}\left(a, \overrightarrow{f}\right)(\xi, \overrightarrow{g}) = (\phi(a)\xi, (ag_1, f_1 g_2, f_2 g_3, ...)),$$
$$\text{where } a \in \mathscr{A}, \xi \in X, \overrightarrow{f} = (f_1, f_2, ...), \overrightarrow{g} = (g_1, g_2, ...) \in T.$$

We say that $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ is the C^*-correspondence formed by adding the tail T to X.

Proposition 6.2.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $T = c_0(\ker \phi)$. If $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ is the C^*-correspondence described above then $\phi_{\mathscr{B}}$ is injective.*

Proof. Suppose that $(a, \overrightarrow{f}) \in \ker \phi_{\mathscr{B}}$, then for each $\xi \in X$ we have that

$$(\phi(a)\xi, \overrightarrow{0}) = \phi_{\mathscr{B}}(a, \overrightarrow{f})(\xi, \overrightarrow{0}) = 0.$$

The above implies that $a \in \ker \phi$ thus $(0, (a^*, f_1, f_2, ...)) \in X \oplus T$ and

$$(0, (aa^*, f_1 f_1^*, f_2 f_2^*, ...)) = \phi_{\mathscr{B}}(a, \overrightarrow{f})(0, (a^*, f_1, f_2, ...)) = (0, \overrightarrow{0}).$$

Therefore, $\|a\|^2 = \|aa^*\| = 0$ and $\|f_n\|^2 = \|f_n f_n^*\| = 0$ for every $n \geq 1$ which implies that $(a, \overrightarrow{f}) = 0$ and so $\phi_{\mathscr{B}}$ is injective. □

We should note that since $\ker \phi_{\mathscr{B}} = \{0\}$ we have that $J_Y = \phi_{\mathscr{B}}^{-1}(\mathscr{K}(X))$.

Lemma 6.2.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ the C^*-correspondence formed by adding the tail $T = c_0(\ker \phi)$ to X. Then $(a, \overrightarrow{f}) \in J_Y$ if and only if $a = a_1 + a_2$ for $a_1 \in J_X$ and $a_2 \in \ker \phi$.*

Proof. Suppose that $a = a_1 + a_2$ where $a_1 \in J_X$ and $a_2 \in \ker \phi$. Then,

$$\phi(a_1) = \lim_n \sum_{k=1}^{N_n} \theta_{\xi_{n,k}, \eta_{n,k}} \in \mathscr{K}(X)$$

for $\xi_{n,k}, \eta_{n,k} \in X$. Let $(\xi, (f_1, f_2, ...))$ be an element in Y such that $\|(\xi, (f_1, f_2, ...))\| \leq 1$.

Note that

$$\begin{aligned}&\|\xi\|_X^2 = \|\langle\xi,\xi\rangle_{\mathscr{A}}\|_{\mathscr{A}} \leq \|(\langle\xi,\xi\rangle_{\mathscr{A}},(f_1^*f_1,f_2^*f_2,...))\|_{\mathscr{B}}\\&= \|\langle(\xi,(f_1,f_2,...)),(\xi,(f_1,f_2,...))\rangle_{\mathscr{B}}\|_{\mathscr{B}} = \|(\xi,(f_1,f_2,...))\|_Y^2 \leq 1.\end{aligned}$$

For $n,m\in\mathbb{N}$ we have

$$\begin{aligned}&\left\|\left(\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{0}),(\eta_{n,k},\overrightarrow{0})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{0}),(\eta_{m,k},\overrightarrow{0})}\right)(\xi,(f_1,f_2,...))\right\|^2\\
&=\left\|\sum_{k=1}^{N_n}(\xi_{n,k},\overrightarrow{0})\left\langle(\eta_{n,k},\overrightarrow{0}),(\xi,(f_1,...))\right\rangle-\sum_{k=1}^{N_m}(\xi_{m,k},\overrightarrow{0})\left\langle(\eta_{m,k},\overrightarrow{0}),(\xi,(f_1,...))\right\rangle\right\|^2\\
&=\left\|\sum_{k=1}^{N_n}(\xi_{n,k}\langle\eta_{n,k},\xi\rangle,\overrightarrow{0})-\sum_{k=1}^{N_m}(\xi_{m,k}\langle\eta_{m,k},\xi\rangle,\overrightarrow{0})\right\|^2\\
&=\left\|\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}(\xi),\overrightarrow{0}\right)-\left(\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}(\xi),\overrightarrow{0}\right)\right\|^2\\
&=\left\|\left(\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi,\overrightarrow{0}\right)\right\|^2\\
&=\left\|\left\langle\left(\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi,\overrightarrow{0}\right),\left(\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi,\overrightarrow{0}\right)\right\rangle\right\|\\
&=\left\|\left\langle\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi,\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi\right\rangle\right\|\\
&=\left\|\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\xi\right\|^2.\end{aligned}$$

The above implies that

$$\begin{aligned}&\left\|\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{0}),(\eta_{n,k},\overrightarrow{0})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{0}),(\eta_{m,k},\overrightarrow{0})}\right\|\\&=\left\|\left(\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}-\sum_{k=1}^{N_m}\theta_{\xi_{m,k},\eta_{m,k}}\right)\right\|, \qquad (*)\end{aligned}$$

and thus

$$\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{0}),(\eta_{n,k},\overrightarrow{0})}$$

is also convergent. Recall that $a_1\in(\ker\phi)^\perp$ and therefore we have that

$$\begin{aligned}&\phi_{\mathscr{B}}(a_1,\overrightarrow{0})(\xi,\overrightarrow{f})=(\phi(a_1)\xi,(a_1f_1,0,...))\\&=\left(\lim_n\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}\xi,\overrightarrow{0}\right)=\lim_n\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{0}),(\eta_{n,k},\overrightarrow{0})}(\xi,\overrightarrow{f})\end{aligned}$$

and so since the point-wise limit and the norm limit should coincide, we have

$$\phi_{\mathscr{B}}(a_1, \overrightarrow{0}) = \lim_n \sum_{k=1}^{N_n} \theta_{(\xi_{n,k}, \overrightarrow{0}),(\eta_{n,k}, \overrightarrow{0})} \in \mathscr{K}(Y).$$

Let $\{\overrightarrow{e_\lambda}\}_{\lambda\in\Lambda}$ be an approximate unit for T such that $\overrightarrow{e_\lambda} = (e^1_\lambda, e^2_\lambda, ...)$ and an element $\overrightarrow{f} = (f_1, f_2, ...)$ in T. We will prove that

$$\phi_{\mathscr{B}}(a_2, \overrightarrow{f}) = \lim_\lambda \theta_{(0,(a_2,f_1,f_2,...)),(0,(e^1_\lambda,e^2_\lambda,e^3_\lambda,...))} \in \mathscr{K}(Y)$$

and therefore $\phi_{\mathscr{B}}(a, \overrightarrow{f}) = \phi_{\mathscr{B}}(a_1, \overrightarrow{0}) + \phi_{\mathscr{B}}(a_2, \overrightarrow{f}) \in \mathscr{K}(Y)$, which implies that $(a, \overrightarrow{f}) \in J_Y$.

Indeed, suppose that $(\xi, (g_1, g_2, ...)) \in Y$ such that $\|(\xi, (g_1, g_2, ...))\| \leq 1$. For $k \geq 1$ we have

$$\|g_k\|^2 = \|g_k^* g_k\| \leq \|(\langle\xi,\xi\rangle, (g_1^* g_1, g_2^* g_2, ...))\| = \|\langle(\xi, (g_1, g_2, ...)), (\xi, (g_1, g_2, ...))\rangle\| \leq 1.$$

If $\epsilon > 0$ then there exists λ_0 such that for each $\lambda \geq \lambda_0$

$$\left\|(a_2, f_1, f_2, ...) - (a_2 e^1_\lambda, f_1 e^2_\lambda, f_2 e^3_\lambda, ...)\right\| < \epsilon.$$

Therefore for each $\lambda \geq \lambda_0$

$$\begin{aligned}
&\left\|\left(\theta_{(0,(a_2,f_1,f_2,...)),(0,(e^1_\lambda,e^2_\lambda,e^3_\lambda))} - \phi_{\mathscr{B}}(a_2,(f_1,f_2,..))\right)(\xi,(g_1,,g_2,...))\right\|^2 \\
=&\|(0,(a_2,f_1,f_2,...))(0,(e^1_\lambda g_1, e^2_\lambda g_2,...)) - (0,(a_2 g_1, f_1 g_2, f_2 g_3,...))\|^2 \\
=&\|(0,(a_2 e^1_\lambda g_1, f_1 e^2_\lambda g_2,...)) - (0,(a_2 g_1, f_1 g_2,...))\|^2 \\
=&\|(0,(a_2 e^1_\lambda - a_2)g_1,(f_1 e^2_\lambda - f_1)g_2,(f_2 e^3_\lambda - f_2)g_3,...))\|^2 \\
=&\|\langle(0,(a_2 e^1_\lambda - a_2)g_1,(f_1 e^2_\lambda - f_1)g_2,...)),(0,(a_2 e^1_\lambda - a_2)g_1,(f_1 e^2_\lambda - f_1)g_2,...))\rangle\|_{\mathscr{B}} \\
=&\|(0,(g_1^*(a_2 e^1_\lambda - a_2)^*(a_2 e^1_\lambda - a_2)g_1, g_2^*(f_1 e^2_\lambda - f_1)^*(f_1 e^2_\lambda - f_1)g_2,...))\|_{\mathscr{B}} \\
\leq&\max\left\{\|g_1^*(a_2 e^1_\lambda - a_2)^*(a_2 e^1_\lambda - a_2)g_1\|, \sup_{k\geq 1}\left\{\|g_{k+1}^*(f_k e_\lambda^{(k+1)} - f_k)^*(f_k e_\lambda^{(k+1)} - f_k)g_{k+1}\|\right\}\right\} \\
\leq&\max\left\{\|(a_2 e^1_\lambda - a_2)g_1\|^2, \sup_{k\geq 1}\left\{\|(f_k e_\lambda^{(k+1)} - f_k)g_{k+1}\|^2\right\}\right\} \\
\leq&\max\left\{\|a_2 e^1_\lambda - a_2\|^2\|g_1\|^2, \sup_{k\geq 1}\left\{\|f_k e_\lambda^{(k+1)} - f_k\|^2\|g_{k+1}\|^2\right\}\right\} \leq \epsilon^2.
\end{aligned}$$

Note that the above implies that $\ker\phi \oplus T \subseteq J_Y$.

Conversely, suppose that $(a, \overrightarrow{f}) \in J_Y$ for $\overrightarrow{f} = (f_1, f_2, ...)$. We have that

$$\phi_{\mathscr{B}}(a, \overrightarrow{f}) = \lim_n \sum_{k=1}^{N_n} \theta_{(\xi_{n,k}, \overrightarrow{f_{n,k}}),(\eta_{n,k}, \overrightarrow{g_{n,k}})}$$

where $\xi_{n,k}, \eta_{n,k} \in X$ and $\overrightarrow{f_{n,k}} = (f^1_{n,k}, f^2_{n,k}, ...)$, $\overrightarrow{g_{n,k}} = (g^1_{n,k}, g^2_{n,k}, ...) \in T$. If $(\xi, \overrightarrow{g}) \in Y$

where $\overrightarrow{g}=(g_1,g_2,...)$ we have

$$\begin{aligned}
&(\phi(a)\xi,(ag_1,f_1g_2,...))=\phi_{\mathscr{B}}(a,\overrightarrow{f})(\xi,\overrightarrow{g})=\lim_n\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{f_{n,k}}),(\eta_{n,k},\overrightarrow{g_{n,k}})}(\xi,\overrightarrow{g})\\
&=\lim_n\sum_{k=1}^{N_n}(\xi_{n,k},\overrightarrow{f_{n,k}})\left\langle(\eta_{n,k},\overrightarrow{g_{n,k}}),(\xi,\overrightarrow{g})\right\rangle_{\mathscr{B}}=\lim_n\sum_{k=1}^{N_n}(\xi_{n,k},\overrightarrow{f_{n,k}})(\langle\eta_{n,k},\xi\rangle_{\mathscr{A}},\overrightarrow{g_{n,k}}^*\overrightarrow{g})\\
&=\lim_n\sum_{k=1}^{N_n}(\xi_{n,k}\langle\eta_{n,k},\xi\rangle_{\mathscr{A}},\overrightarrow{f_{n,k}}\overrightarrow{g_{n,k}}^*\overrightarrow{g})=\\
&\lim_n\sum_{k=1}^{N_n}(\xi_{n,k}\langle\eta_{n,k},\xi\rangle_{\mathscr{A}},(f_{n,k}^1(g_{n,k}^1)^*g^1,f_{n,k}^2(g_{n,k}^2)^*g^2,...))\\
&=\lim_n\sum_{k=1}^{N_n}\left(\theta_{\xi_{n,k},\eta_{n,k}}(\xi),(f_{n,k}^1(g_{n,k}^1)^*g^1,f_{n,k}^2(g_{n,k}^2)^*g^2,...)\right). \qquad (**)
\end{aligned}$$

Using the equality in $(*)$ we have that $\lim_n\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}$ exists and in particular the equality above implies that

$$\lim_n\sum_{k=1}^{N_n}\theta_{\xi_{n,k},\eta_{n,k}}=\phi(a).$$

Suppose that $\{e_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit in $\ker\phi$ then for each $n,m\in\mathbb{N}$ and $\lambda\in\Lambda$ we have that

$$\begin{aligned}
&\left\|\left(\sum_{k=1}^{N_n}f_{n,k}^1{g_{n,k}^1}^*-\sum_{k=1}^{N_m}f_{m,k}^1{g_{m,k}^1}^*\right)e_\lambda\right\|\\
&=\left\|\left(\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{f_{n,k}}),(\eta_{n,k},\overrightarrow{g_{n,k}})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{f_{m,k}}),(\eta_{m,k},\overrightarrow{g_{m,k}})}\right)(0,(e_\lambda,0,...))\right\|\\
&\leq\left\|\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{f_{n,k}}),(\eta_{n,k},\overrightarrow{g_{n,k}})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{f_{m,k}}),(\eta_{m,k},\overrightarrow{g_{m,k}})}\right\|\|(0,(e_\lambda,0,...))\|\\
&=\left\|\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{f_{n,k}}),(\eta_{n,k},\overrightarrow{g_{n,k}})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{f_{m,k}}),(\eta_{m,k},\overrightarrow{g_{m,k}})}\right\|.
\end{aligned}$$

By taking the limit with respect to λ we have

$$\begin{aligned}
&\left\|(\sum_{k=1}^{N_n}f_{n,k}^1{g_{n,k}^1}^*-\sum_{k=1}^{N_m}f_{m,k}^1{g_{m,k}^1}^*)\right\|\\
&\leq\left\|\sum_{k=1}^{N_n}\theta_{(\xi_{n,k},\overrightarrow{f_{n,k}}),(\eta_{n,k},\overrightarrow{g_{n,k}})}-\sum_{k=1}^{N_m}\theta_{(\xi_{m,k},\overrightarrow{f_{m,k}}),(\eta_{m,k},\overrightarrow{g_{m,k}})}\right\|
\end{aligned}$$

which implies that $\sum_{k=1}^{N_n}f_{n,k}^1{g_{n,k}^1}^*$ converges to an element in $\ker\phi$.
Set $a_2=\lim_n\sum_{k=1}^{N_n}f_{n,k}^1{g_{n,k}^1}^*$ and $a_1=a-a_2$. We will prove that for all $g\in\ker\phi$ we have the relation $ag=a_2g$ and so it is evident that $a_1\in(\ker\phi)^\perp$ and consequently $a_1\in J_X$.

Since $a = a_1 + a_2$ the proof will be complete.
So, suppose that $g \in \ker \phi$ and $\xi \in X$ and consider the element $(\xi, (g, 0, ...)) \in Y$, then from our calculations in $(**)$ we have that

$$(\phi(a)\xi, (ag, 0, ...)) = \phi_{\mathscr{B}}(a, \overrightarrow{f})(\xi, (g, 0, ...)) = \lim_n \sum_{k=1}^{N_n} (\theta_{\xi_{n,k},\eta_{n,k}}\xi, (f^1_{n,k} {g^1_{n,k}}^* g, 0, ...))$$
$$= \left(\lim_n \sum_{k=1}^{N_n} \theta_{\xi_{n,k},\eta_{n,k}}\xi, \left(\lim_n \sum_{k=1}^{N_n} f^1_{n,k} {g^1_{n,k}}^* g, 0, ... \right) \right) = (\phi(a), (a_2 g, 0, ...))$$

which implies that $ag = a_2 g$. □

Lemma 6.2.2. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and let $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ be the C^*-correspondence formed by adding the tail $T = c_0(\ker \phi)$ to X and let $(\tilde{\pi}, \tilde{t})$ be a Katsura covariant Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ such that $\tilde{\pi}_{|\mathscr{A}}$ is injective. For each $f \in \ker \phi$ and $i \geq 1$ we define*

$$\epsilon_i(f) := (0, ..., 0, \underbrace{f}_{i}, 0, ...) \in T.$$

Then for every $i \geq 1$ and $f \in \ker \phi$ the equality $\tilde{\pi}(0, \epsilon_i(f)) = 0$ implies that $f = 0$.

Proof. We may assume that $f \geq 0$ because if $\tilde{\pi}(0, \epsilon_i(f)) = 0$, then

$$\tilde{\pi}(0, \epsilon_i(f^* f)) = \tilde{\pi}(0, \epsilon_i(f^*))\tilde{\pi}(0, \epsilon_i(f)) = 0$$

and from the C^*-property if we show that $f^* f = 0$ then $f = 0$. So, suppose that $i \geq 1$ and $f \in \ker \phi$ satisfies $f \geq 0$ and $\tilde{\pi}(0, \epsilon_i(f)) = 0$.
Then,

$$\left\| \tilde{t}(0, \epsilon_i(f^{1/2})) \right\|^2 = \left\| \tilde{t}(0, \epsilon_i(f^{1/2}))^* \tilde{t}(0, \epsilon_i(f^{1/2})) \right\| = \left\| \tilde{\pi}(\left\langle (0, \epsilon_i(f^{1/2})), (0, \epsilon_i(f^{1/2})) \right\rangle \right\|$$
$$= \left\| \tilde{\pi}((0, \epsilon_i(f^{1/2}))^* (0, \epsilon_i(f^{1/2}))) \right\| = \left\| \tilde{\pi}((0, \epsilon_i(f^{1/2}))(0, \epsilon_i(f^{1/2}))) \right\|$$
$$= \|\tilde{\pi}((0, \epsilon_i(f))\| = 0,$$

which implies that $\tilde{t}(0, \epsilon_i(f^{1/2})) = 0$. We should also note that

$$0 = \tilde{t}(0, \epsilon_i(f^{1/2}))\tilde{t}(0, \epsilon_i(f^{1/2}))^* = \tilde{t}_* \left(\theta_{(0, \epsilon_i(f^{1/2})), (0, \epsilon_i(f^{1/2}))} \right).$$

If $i = 1$ for each $(\xi, (g_1, g_2, ...)) \in Y$ we have that

$$\phi_{\mathscr{B}}(f, \overrightarrow{0})(\xi, (g_1, g_2, ...)) = (\phi(f)\xi, (f g_1, 0, ...)) = (0, (f g_1, 0, ...))$$
$$= (0, (f^{1/2}, 0, ...)) \left\langle (0, (f^{1/2}, 0, ...)), (\xi, (g_1, g_2, ...)) \right\rangle$$
$$= \theta_{(0, (f^{1/2}, 0, ...)), (0, (f^{1/2}, 0, ...))}(\xi, (g_1, g_2, ...)),$$

which implies that

$$\begin{aligned}0 &= \tilde{t}_* \left(\theta_{(0,(f^{1/2},0,...)),(0,(f^{1/2}),0,...)}\right) = \tilde{t}_*(\phi_{\mathscr{B}}(f, \overrightarrow{0}))\\ &= \tilde{\pi}(f, \overrightarrow{0})) = \tilde{\pi}_{|\mathscr{A}}(f),\end{aligned}$$

where we used the fact that $\ker \phi \oplus T \subseteq J_Y$. From the injectivity of $\tilde{\pi}_{|\mathscr{A}}$ we get that $f = 0$. Suppose that $i > 1$, then for each $(\xi, \overrightarrow{g}) \in Y$ where $\overrightarrow{g} = (g_1, g_2, ...)$ we have

$$\phi_{\mathscr{B}}(0, \epsilon_{i-1}(f))(\xi, \overrightarrow{g}) = (0, (0, ..., 0, \underbrace{fg_i}_{i}, 0, ...) = \theta_{(0,\epsilon_i(f^{1/2})),(0,\epsilon_i(f^{1/2}))}(\xi, (g_1, g_2, ...))$$

and therefore

$$0 = \tilde{t}_* \left(\theta_{(0,\epsilon_i(f^{1/2})),(0,\epsilon_i(f^{1/2}))}\right) = \tilde{t}_*(\phi_{\mathscr{B}}(0, \epsilon_{i-1}(f))) = \tilde{\pi}(0, \epsilon_{i-1}(f)),$$

where we used the fact that $J_Y = \phi_{\mathscr{B}}^{-1}(\mathscr{K}(Y))$ since $\phi_{\mathscr{B}}$ is injective. After $(i-1)$-steps we obtain that $\tilde{\pi}(0, \epsilon_1(f)) = 0$, which implies that $f = 0$. □

Theorem 6.2.1. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence and $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ be the C^*-correspondence formed by adding the tail $T = c_0(\ker \phi)$ to X.*

(i) If (π, t) is a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$ on a Hilbert space $\mathscr{H}_X$, then there exist Hilbert spaces $\mathscr{H}_Y$ and $\mathscr{H}_T$ such that $\mathscr{H}_Y = \mathscr{H}_X \oplus \mathscr{H}_T$ and a Katsura covariant Toeplitz representation $(\tilde{\pi}, \tilde{t})$ of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ on $\mathscr{H}_Y$ with the property that $\tilde{\pi}_{|\mathscr{A}} = \pi$ and $\tilde{t}_{|X} = t$.

(ii) If $(\tilde{\pi}, \tilde{t})$ is a Katsura covariant Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ then $(\tilde{\pi}_{|\mathscr{A}}, \tilde{t}_{|X})$ is a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$. Furthermore, if $\tilde{\pi}_{|\mathscr{A}}$ is injective then $\tilde{\pi}$ is also injective.

Proof. (i) Set $I = \ker \phi$ and $\mathscr{H}_0 = \pi(I)\mathscr{H}_X$ and define $\mathscr{H}_T = \bigoplus_{i=1}^{\infty} \mathscr{H}_i$ where $\mathscr{H}_i = \mathscr{H}_0$ for all $i \geq 1$. We define $\tilde{t} : Y \to \mathbf{B}(\mathscr{H}_X \oplus \mathscr{H}_T)$ and $\tilde{\pi} : \mathscr{B} \to \mathbf{B}(\mathscr{H}_X \oplus \mathscr{H}_T)$ such that for $a \in \mathscr{A}$, $\xi \in X$, $(f_1, f_2, ...) \in T$ and $(h, (h_1, h_2, ...)), (k, (k_1, k_2, ...)) \in \mathscr{H}_X \oplus \mathscr{H}_T$

$$\tilde{t}(\xi, (f_1, f_2, ...))(h, (h_1, h_2, ...)) = (t(\xi)h + \pi(f_1)h_1, (\pi(f_2)h_2, \pi(f_3)h_3, ...))$$

and

$$\tilde{\pi}((a, (f_1, f_2, ...))(h, (h_1, h_2, ...)) = (\pi(a)h, (\pi(f_1)h_1, \pi(f_2)h_2, ...)).$$

To see that $\tilde{\pi}, \tilde{t}$ are well-defined it suffices to prove they are bounded. Pick an element $(h, (h_1, h_2, ...))$ in $\mathscr{H}_X \oplus \mathscr{H}_T$ such that $\|(h, (h_1, h_2, ...))\| \leq 1$ and set $f_0 = a$ and $h_0 = h$, then

$$\begin{aligned}&\|\tilde{\pi}((a,(f_1,f_2,...))(h,(h_1,h_2,...))\|=\|(\pi(a)h,(\pi(f_1)h_1,\pi(f_2)h_2,...))\|\\&=\sup_{i\geq 0}\|\pi(f_i)h_i\|\leq\sup_{i\geq 0}\|\pi(f_i)\|\leq\sup_{i\geq 0}\|f_i\|=\|((a,(f_1,f_2,...))\|,\end{aligned}$$

which implies that $\|\tilde{\pi}((a,(f_1,f_2,...))\|\leq\|((a,(f_1,f_2,...))\|$. We also have

$$\begin{aligned}&\|\tilde{t}(\xi,(f_1,f_2,...))(h,(h_1,h_2,...))\|=\|(t(\xi)h+\pi(f_1)h_1,(\pi(f_2)h_2,\pi(f_3)h_3,...))\|\\&\leq\max\{\|t(\xi)h\|+\|\pi(f_1)h_1\|,\sup_{i\geq 2}\|\pi(f_i)h_i\|\}\\&\leq\max\{\|\xi\|+\|f_1\|,\sup_{i\geq 2}\|f_i\|\}\leq 2\|(\xi,(f_1,f_2,...))\|.\end{aligned}$$

We prove now that $(\tilde{\pi},\tilde{t})$ is a Toeplitz representation of $(Y,\mathscr{B},\phi_{\mathscr{B}})$.
We have

$$\begin{aligned}&\left\langle\tilde{t}(\xi,(f_1,f_2,...))(h,(h_1,h_2,...)),(k,(k_1,k_2,...))\right\rangle_{\mathscr{H}_Y}\\&=\langle(t(\xi)h+\pi(f_1)h_1,(\pi(f_2)h_2,\pi(f_3)(h_3),...)),(k,(k_1,k_2,...))\rangle_{\mathscr{H}_Y}\\&=\langle(t(\xi)h+\pi(f_1)h_1,k\rangle_{\mathscr{H}_X}+\sum_{i=1}^{\infty}\langle\pi(f_{i+1})h_{i+1},k_i\rangle_{\mathscr{H}_0}\\&=\langle(t(\xi)h,k\rangle_{\mathscr{H}_X}+\langle\pi(f_1)h_1,k\rangle_{\mathscr{H}_X}+\sum_{i=1}^{\infty}\langle\pi(f_{i+1})h_{i+1},k_i\rangle_{\mathscr{H}_0}\\&=\langle h,t(\xi)^*k\rangle_{\mathscr{H}_X}+\langle h_1,\pi(f_1)^*k\rangle_{\mathscr{H}_X}+\sum_{i=1}^{\infty}\langle h_{i+1},\pi(f_{i+1})^*k_i\rangle_{\mathscr{H}_0}\\&=\langle(h,(h_1,h_2,...)),(t(\xi)^*k,(\pi(f_1^*)k,\pi(f_2)^*k_1,\pi(f_3)^*k_2,...))\rangle_{\mathscr{H}_Y}\end{aligned}$$

and thus

$$\tilde{t}((\xi,(f_1,f_2,...))^*(h,(h_1,h_2,...))=(t(\xi)^*h,(\pi(f_1^*)h,(\pi(f_2^*)h_1,\pi(f_3^*)h_2,...)).$$

$$\begin{aligned}&\tilde{t}(\xi,(f_1,f_2,...))^*\tilde{t}(\xi,(f_1,f_2,...))(h,(h_1,h_2,...))\\&=\tilde{t}(\xi,(f_1,f_2,...))^*(t(\xi)h+\pi(f_1)h_1,(\pi(f_2)h_2,\pi(f_3)h_3,...))\\&=(t(\xi)^*(t(\xi)h+\pi(f_1)h_1),(\pi(f_1^*)(t(\xi)h+\pi(f_1)h_1,\pi(f_2^*)\pi(f_2)h_2,\pi(f_3^*)\pi(f_3)h_3,...))\\&=(\pi(\langle\xi,\xi\rangle_{\mathscr{A}})h+t(\phi(f_1^*)\xi)^*h_1,(t(\phi(f_1^*)\xi)h+\pi(f_1^*f_1)h_1,\pi(f_2^*f_2)h_2,\pi(f_3^*f_3)h_3,...))\\&=(\pi(\langle\xi,\xi\rangle_{\mathscr{A}})h,(\pi(f_1^*f_1)h_1,\pi(f_2^*f_2)h_2,\pi(f_3^*f_3)h_3,...))\\&=\tilde{\pi}(\langle(\xi,(f_1,f_2,...)),(\xi,(f_1,f_2,...))\rangle_{\mathscr{B}})\end{aligned}$$

and if $(g_1, g_2, ...) \in T$

$$\begin{aligned}
&\tilde{t}(\xi, (f_1, f_2, ...))\tilde{\pi}(a, (g_1, g_2, ...))(h, (h_1, h_2, ...)) \\
&= \tilde{t}(\xi, (f_1, f_2, ...))(\pi(a)h, (\pi(g_1)h_1, \pi(g_2)h_2, ...)) \\
&= (t(\xi)\pi(a)h + \pi(f_1)\pi(g_1)h_1, (\pi(f_2)\pi(g_2)h_2, \pi(f_3)\pi(g_3)h_3, ...)) \\
&= (t(\xi a)h + \pi(f_1 g_1)h_1, (\pi(f_2 g_2)h_2, \pi(f_3 g_3)h_3, ...)) \\
&= \tilde{t}((\xi a, (f_1 g_1, f_2 g_2, ...))) = \tilde{t}((\xi, (f_1, f_2, ...))(a, g_1, g_2, ...)))(h, (h_1, h_2, ...)).
\end{aligned}$$

Now let $(a, (f_1, f_2, ...))$ be an element in J_Y, by lemma 6.2.1 we have that $a = a_1 + a_2$ where $a_1 \in J_X$ and $a_2 \in \ker\phi$ and therefore

$$\phi(a_1) = \lim_n \sum_{k=1}^{N_n} \theta_{\xi_{n,k}, \eta_{n,k}}$$

for some $\xi_{n,k}, \eta_{n,k} \in X$ and as in the proof of the above-mentioned lemma if $\{\overrightarrow{e_\lambda}\}_{\lambda\in\Lambda}$ is an approximate unit for T, where $\overrightarrow{e_\lambda} = (e_\lambda^1, e_\lambda^2, ...)$, we have that

$$\phi_{\mathscr{B}}(a_1, \overrightarrow{0}) = \lim_n \sum_{k=1}^{N_n} \theta_{(\xi_{n,k}, \overrightarrow{0}), (\eta_{n,k}, \overrightarrow{0})} \in \mathscr{K}(Y)$$

and

$$\phi_{\mathscr{B}}(a_2, (f_1, f_2, ...)) = \lim_\lambda \theta_{(0, (a_2, f_1, f_2, ...)), (0, (e_\lambda^1, e_\lambda^2, e_\lambda^3, ...))} \in \mathscr{K}(Y).$$

Note that for $(x, (h_1, h_2, ...)), (\eta, (g_1, g_2, ...)) \in Y$ we have

$$\tilde{t}(x, (h_1, h_2, ...))\tilde{t}(\eta, (g_1, g_2, ...))^* = (t(x)t(\eta)^* + \pi(h_1 g_1^*), (\pi(h_2 g_2^*), \pi(h_3 g_3^*), ...)).$$

Therefore,

$$\begin{aligned}
&\tilde{t}_*(\phi_{\mathscr{B}}(a, (f_1, f_2, ...))) = \tilde{t}_*(\phi_{\mathscr{B}}(a_1, \overrightarrow{0})) + \tilde{t}_*(\phi_{\mathscr{B}}(a_2, (f_1, f_2, ...))) \\
&= \lim_n \sum_{k=1}^{N_n} \tilde{t}(\xi_{n,k}, \overrightarrow{0})\tilde{t}(\eta_{n,k}, \overrightarrow{0})^* + \lim_\lambda \tilde{t}(0, (a_2, f_1, f_2, ...))\tilde{t}(0, (e_\lambda^1, e_\lambda^2, e_\lambda^3, ...))^* \\
&= \lim_n \sum_{k=1}^{N_n} (t(\xi_{n,k})t(\eta_{n,k})^*, \overrightarrow{0}) + \lim_\lambda (\pi(a_2 e_\lambda^1), (\pi(f_1 e_\lambda^2), \pi(f_2 e_\lambda^3), ...)) \\
&= (t_*(\phi(a_1)), \overrightarrow{0}) + (\pi(a_2), (\pi(f_1), \pi(f_2), ...)) \\
&= (\pi(a_1), \overrightarrow{0}) + (\pi(a_2), (\pi(f_1), \pi(f_2), ...)) = \tilde{\pi}(a, (f_1, f_2, ...))
\end{aligned}$$

and thus $(\tilde{\pi}, \tilde{t})$ is a Katsura covariant Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$.
(ii) Let $(\tilde{\pi}, \tilde{t})$ be a Katsura covariant Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$, then it is immediate that $(\tilde{\pi}|_{\mathscr{A}}, \tilde{t}|_X)$ is a Toeplitz representation of $(X, \mathscr{A}, \phi)$. Now, let a be an element

in J_X, then as above we have that

$$\phi(a) = \lim_n \sum_{k=1}^{N_n} \theta_{\xi_{n,k},\eta_{n,k}}$$

for some $\xi_{n,k}, \eta_{n,k} \in X$ and

$$\phi_{\mathscr{B}}(a, \overrightarrow{0}) = \lim_n \sum_{k=1}^{N_n} \theta_{(\xi_{n,k}, \overrightarrow{0}),(\eta_{n,k}, \overrightarrow{0})} \in \mathscr{K}(Y)$$

and therefore

$$(\tilde{t}|_X)_*(\phi(a)) = \lim_n \sum_{k=1}^{N_n} \tilde{t}|_X(\xi_{n,k})\tilde{t}|_X(\eta_{n,k})^* = \lim_n \sum_{k=1}^{N_n} \tilde{t}(\xi_{n,k}, \overrightarrow{0})\tilde{t}(\eta_{n,k}, \overrightarrow{0})^*$$
$$= \tilde{t}_*(\phi_{\mathscr{B}}((a, \overrightarrow{0})) = \tilde{\pi}(a, \overrightarrow{0}) = \tilde{\pi}|_{\mathscr{A}}(a).$$

Suppose now that $\tilde{\pi}|_{\mathscr{A}}$ is injective, we will prove that $\tilde{\pi}$ is also injective. Let $(a, (f_1, f_2, ...))$ be an element in $\mathscr{B}$ such that $\tilde{\pi}(a, (f_1, f_2, ...)) = 0$ and let $\{g_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit for $\ker\phi$. For $i \geq 1$ we have that

$$\tilde{\pi}(0, \epsilon_i(g_\lambda f_i)) = \tilde{\pi}((0, \epsilon_i(g_\lambda))(a, (f_1, f_2, ...))) = \tilde{\pi}(0, \epsilon_i(g_\lambda))\tilde{\pi}(a, (f_1, f_2, ...)) = 0$$

and by taking limit with respect to λ we get $\tilde{\pi}(0, \epsilon_i(f_i))) = 0$. Hence, by using lemma 6.2.2 it follows that $f_i = 0$ for all $i \geq 1$ and since

$$\tilde{\pi}|_{\mathscr{A}}(a) = \tilde{\pi}(a, \overrightarrow{0}) = \tilde{\pi}(a, (f_1, f_2, ...)) = 0,$$

injectivity of $\tilde{\pi}|_{\mathscr{A}}$ implies that $a = 0$. Thus, $(a, (f_1, f_2, ...)) = 0$ and we are done. □

Lemma 6.2.3. *Let X, Y be Hilbert $\mathscr{A}$-modules of a C^*-algebra $\mathscr{A}$ and $\phi : \mathscr{A} \to \mathscr{L}(Y)$ be an injective $*$-homomorphism. If $(\xi_i)_{i=1}^n \in X^n$ then*

$$\|(\xi_i)_{i=1}^n\| = \sup\{\|(\xi_i \otimes_\phi u)_{i=1}^n\| : u \in Y, \|u\| = 1\}.$$

Proof.

$$\sup\{\|(\xi_i \otimes_\phi u)_{i=1}^n\|^2 : u \in Y, \|u\| = 1\}$$
$$= \sup\{\|\langle(\xi_i \otimes_\phi u)_{i=1}^n, \xi_i \otimes_\phi u)_{i=1}^n\rangle\| : u \in Y, \|u\| = 1\}$$
$$= \sup\left\{\left\|\sum_{i=1}^n \langle \xi_i \otimes_\phi u, \xi_i \otimes_\phi u\rangle\right\| : u \in Y, \|u\| = 1\right\}$$
$$= \sup\left\{\left\|\sum_{i=1}^n \langle u, \phi(\langle \xi_i, \xi_i\rangle)u\rangle\right\| : u \in Y, \|u\| = 1\right\}$$

$$= \sup \left\{ \left\| \sum_{i=1}^{n} \left\langle \phi(\langle \xi_i, \xi_i \rangle^{1/2}) u, \phi(\langle \xi_i, \xi_i \rangle^{1/2}) u \right\rangle \right\| : u \in Y, \|u\| = 1 \right\}$$
$$= \sup \left\{ \|(\phi(\langle \xi_i, \xi_i \rangle^{1/2}) u)_{i=1}^{n}\|^2 : u \in Y, \|u\| = 1 \right\}$$
$$= \left\| \begin{pmatrix} 0 & 0 & \cdots & \phi(\langle \xi_1, \xi_1 \rangle^{1/2}) \\ 0 & 0 & \cdots & \phi(\langle \xi_2, \xi_2 \rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n, \xi_n \rangle^{1/2}) \end{pmatrix} \right\|^2$$
$$= \left\| \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \phi(\langle \xi_1, \xi_1 \rangle^{1/2}) & \cdots & \phi(\langle \xi_n, \xi_n \rangle^{1/2}) \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & \phi(\langle \xi_1, \xi_1 \rangle^{1/2}) \\ 0 & 0 & \cdots & \phi(\langle \xi_2, \xi_2 \rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n, \xi_n \rangle^{1/2}) \end{pmatrix} \right\|$$
$$= \left\| \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i=1}^{n} \phi(\langle \xi_i, \xi_i \rangle) \end{pmatrix} \right\| = \left\| \phi(\sum_{i=1}^{n} \langle \xi_i, \xi_i \rangle) \right\| = \left\| \sum_{i=1}^{n} \langle \xi_i, \xi_i \rangle \right\| = \|(\xi_i)_{i=1}^{n}\|^2,$$

where we used the fact that

$$\left\| \begin{pmatrix} 0 & 0 & \cdots & \phi(\langle \xi_1, \xi_1 \rangle^{1/2}) \\ 0 & 0 & \cdots & \phi(\langle \xi_2, \xi_2 \rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n, \xi_n \rangle^{1/2}) \end{pmatrix} \right\|^2 =$$
$$= \sup \left\{ \left\| \begin{pmatrix} 0 & 0 & \cdots & \phi(\langle \xi_1, \xi_1 \rangle^{1/2}) \\ 0 & 0 & \cdots & \phi(\langle \xi_2, \xi_2 \rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n, \xi_n \rangle^{1/2}) \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right\|^2 : u_i \in Y \text{ and } \left\| \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right\| \leq 1 \right\}$$
$$= \sup\{\|(\phi(\langle \xi_i, \xi_i \rangle^{1/2}) u_n)_{i=1}^{n}\|^2 : u_n \in Y \quad \text{where} \quad \|u_n\| \leq 1\}$$
$$= \sup \left\{ \left\| \sum_{i=1}^{n} \langle u, \phi(\langle \xi_i, \xi_i \rangle) u \rangle \right\| : \|u\| \leq 1 \right\} = \sup \left\{ \left\| \left\langle u, \sum_{i=1}^{n} \phi(\langle \xi_i, \xi_i \rangle) u \right\rangle \right\| : \|u\| \leq 1 \right\}$$
$$= \sup \left\{ \left\| \left\langle u, \sum_{i=1}^{n} \phi(\langle \xi_i, \xi_i \rangle) u \right\rangle \right\| : \|u\| = 1 \right\} =$$
$$\sup \left\{ \left\| \sum_{i=1}^{n} \left\langle \phi(\langle \xi_i, \xi_i \rangle)^{1/2} u, \phi(\langle \xi_i, \xi_i \rangle)^{1/2} u \right\rangle \right\| : \|u\| = 1 \right\}$$
$$= \sup\{\|(\phi(\langle \xi_i, \xi_i \rangle^{1/2}) u)_{i=1}^{n}\|^2 : u \in Y, \|u\| = 1\}$$

and also that ϕ is an injective $*$-homomorphism and therefore isometric. □

In the proof of our next lemma we are going to need the right creation operators which we now define. Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence, F_X to be the Fock space and $\xi \in X^{\otimes k}$ then we define

$$R_\xi(a, \zeta_1, \zeta_2, ...) = (\underbrace{0, ..., 0}_{k}, (\phi(a) \otimes I_{k-1})(\xi), \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, ...), \quad (a, \zeta_1, \zeta_2, ...) \in F_X.$$

Firstly, we show that R_ξ is well-defined and continuous.
Indeed,

$$\begin{aligned}
\|R_\xi(a, \zeta_1, \zeta_2, ...)\|^2 &= \left\| (\underbrace{0, ..., 0}_{k}, \phi_k(a)\xi, \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, ...) \right\|^2 \\
&= \left\| \left\langle (\underbrace{0, ..., 0}_{k}, \phi_k(a)\xi, \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, ...), (\underbrace{0, ..., 0}_{k}, \phi_k(a)\xi, \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, ...) \right\rangle \right\| \\
&= \left\| \langle \phi_k(a)\xi, \phi_k(a)\xi \rangle_k + \sum_{i=1}^{\infty} \langle \zeta_i \otimes \xi, \zeta_i \otimes \xi \rangle_{k+i} \right\| \\
&= \left\| \langle \phi_k(a)\xi, \phi_k(a)\xi \rangle_k + \sum_{i=1}^{\infty} \langle \xi, \phi_k(\langle \zeta_i, \zeta_i \rangle_i)\xi \rangle_k \right\| \\
&= \left\| \left\langle \xi, \phi_k(a^*a + \sum_{i=1}^{\infty} \langle \zeta_i, \zeta_i \rangle_i)\xi \right\rangle_k \right\| \leq \|\xi\| \, \|\phi_k(\langle (a, \zeta_1, \zeta_2, ...), (a, \zeta_1, \zeta_2, ...) \rangle)\xi\| \\
&\leq \|\xi\|^2 \, \|\phi_k(\langle (a, \zeta_1, \zeta_2, ...), (a, \zeta_1, \zeta_2, ...) \rangle\| \leq \|\xi\|^2 \, \|\langle (a, \zeta_1, \zeta_2, ...), (a, \zeta_1, \zeta_2, ...) \rangle\| \\
&\leq \|\xi\|^2 \, \|(a, \zeta_1, \zeta_2, ...)\|^2
\end{aligned}$$

We now prove that for a C^*-correspondence $(X, \mathscr{A}, \phi)$ and $x \in \text{alg}(\pi_\infty, t_\infty)$, we have that $R_\xi x = x R_\xi$, where (π_∞, t_∞) is the Fock representation of $(X, \mathscr{A}, \phi)$ and by $\text{alg}(\pi_\infty, t_\infty)$ we denote the norm closed algebra generated by the images of π_∞ and t_∞. It suffices to show that R_ξ commutes with $\pi_\infty(b)$ and $t_\infty(\eta)$ for $b \in \mathscr{A}$ and $\eta \in X$, since these elements generate $\text{alg}(\pi_\infty, t_\infty)$. Let $(a, \zeta_1, \zeta_2, ...)$ be an element in F_X, we have

$$\begin{aligned}
\pi_\infty(b) R_\xi(a, \zeta_1, \zeta_2, ...) &= \pi_\infty(b)(\underbrace{0, ..., 0}_{k}, \phi_k(a)\xi, \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, ...) \\
&= (\underbrace{0, ..., 0}_{k}, \phi_k(b)\phi_k(a)\xi, \phi_{k+1}(b)(\zeta_1 \otimes \xi), \phi_{k+2}(b)(\zeta_2 \otimes \xi), ...) \\
&= (\underbrace{0, ..., 0}_{k}, \phi_k(ba)\xi, (\phi_1(b)\zeta_1) \otimes \xi, (\phi_2(b)\zeta_2) \otimes \xi, ...) \\
&= R_\xi(ba, \phi(b)\zeta_1, \phi_2(b)\zeta_2, ...) = R_\xi \pi_\infty(b)(a, \zeta_1, \zeta_2, ...)
\end{aligned}$$

and

$$\begin{aligned}
&R_\xi t_\infty(\eta)(a,\zeta_1,\zeta_2,...) = R_\xi(0,\eta a,\eta\otimes\zeta_1,\eta\otimes\zeta_2,...)\\
&=(\underbrace{0,...,0}_{k+1},\eta a\otimes\xi,\eta\otimes\zeta_1\otimes\xi,\eta\otimes\zeta_2\otimes\xi,...)\\
&=(\underbrace{0,...,0}_{k+1},\eta\otimes\phi_k(a)\xi,\eta\otimes\zeta_1\otimes\xi,\eta\otimes\zeta_2\otimes\xi,...)\\
&=t_\infty(\eta)(\underbrace{0,...,0}_{k},\phi_k(a)\xi,\zeta_1\otimes\xi,\zeta_2\otimes\xi,...)=t_\infty(\eta)R_\xi(a,\zeta_1,\zeta_2,...).
\end{aligned}$$

Note that for each $n\in\mathbb{N}$ if we set

$$R_u^{(n)}=\begin{pmatrix} R_u & 0 & \cdots & 0\\ 0 & R_u & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & R_u\end{pmatrix},$$

then this $n\times n$ matrix commutes with every matrix $A=(A_{ij})_{ij}\in M_n(T_X^+)$.
Indeed, if $\xi=(\xi_1,...,\xi_n)\in F_X^n$ then

$$\begin{aligned}
R_u^{(n)}A\xi &= R_u^{(n)}\begin{bmatrix}\sum_{i=1}^n A_{1i}\xi_i\\ \vdots\\ \sum_{i=1}^n A_{ni}\xi_i\end{bmatrix}\\
&=\begin{bmatrix}R_u(\sum_{i=1}^n A_{1i}\xi_i)\\ \vdots\\ R_u(\sum_{i=1}^n A_{ni}\xi_i)\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^n A_{1i}R_u\xi_i\\ \vdots\\ \sum_{i=1}^n A_{ni}R_u\xi_i\end{bmatrix}=AR_u^{(n)}\xi.
\end{aligned}$$

Recall that from the gauge-invariance uniqueness theorem we may think of the C^*-algebra T_X of a given C^*-correspondence $(X,\mathscr{A},\phi)$ as the the C^*-algebra $C^*(\pi_\infty,t_\infty)$, where (π_∞,t_∞) is the Fock representation of $(X,\mathscr{A},\phi)$.

Lemma 6.2.4. *Let $(X,\mathscr{A},\phi)$ be a C^*-correspondence such that ϕ is injective. Then*

$$\|A\|=\inf\{\|A+K\|: K\in M_n(\mathscr{K}(F_X))\}$$

for all $A\in M_n(T_X^+)$ and $n\in\mathbb{N}$.

Proof. Suppose that $K=(K_{ij})_{ij}\in M_n(\mathscr{K}(F_X))$ and $A=(A_{ij})_{ij}\in M_n(T_X^+)$ and $\epsilon>0$. We pick a unit vector $\xi=(\xi_1,\xi_2,...,\xi_n)\in F_X^n$ such that

$$\|A\xi\|>\|A\|-\epsilon.$$

There exists $k \in \mathbb{N}$ such that for all unit vectors $u \in X^{\otimes k}$ we have $\|KR_u^{(n)}\| < \epsilon$, where

$$R_u^{(n)} = \begin{pmatrix} R_u & 0 & \cdots & 0 \\ 0 & R_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_u \end{pmatrix}.$$

Indeed, using lemma 5.1.5 we may pick $L = (L_{ij})_{ij}$ such that each L_{ij} is a finite linear sum of elements of the form $\theta_{x,y}$ where $x \in X^{\otimes m}$ and $y \in X^{\otimes m'}$ and

$$\|K - L\| < \epsilon.$$

Since each L_{ij} is a finite linear sum and the entries of the matrix L is finite, we can pick k large enough such that for each unit vector $u \in X^{\otimes k}$ and $\eta \in F_X$ we have

$$L_{ij}(R_u(\eta)) = 0,$$

where we used the fact that the first k coordinates of $R_u\eta$ are zero. Therefore we have $LR_u^{(n)} = 0$ and so

$$\|KR_u^{(n)}\| \leq \|KR_u^{(n)} - LR_u^{(n)}\| + \|LR_u^{(n)}\| \leq \|K - L\| \|R_u^{(n)}\| \leq \epsilon.$$

Note that for any vector $u \in X^{\otimes k}$ we have

$$\left\| R_u^{(n)} \left(\sum_{j=1}^{n} A_{ij}\xi_j \right)_{i=1}^{n} \right\| = \left\| \left(\sum_{j=1}^{n} A_{ij}\xi_j \otimes u \right)_{i=1}^{n} \right\|,$$

since the left-hand side vectors are the same with the right-hand side vectors transposed by k coordinates. From the preceding lemma we may pick a unit vector $u \in X^{\otimes k}$ such that

$$\|R_u^{(n)} A\xi\| \geq \|A\xi\| - \epsilon \geq \|A\| - 2\epsilon$$

and therefore

$$\begin{aligned} \|A + K\| &\geq \|(A + K)R_u^{(n)}\xi\| \geq \|AR_u^{(n)}\xi\| - \|KR_u^{(n)}\xi\| \\ &\geq \|AR_u^{(n)}\xi\| - \epsilon = \|R_u^{(n)} A\xi\| - \epsilon \geq \|A\| - 3\epsilon, \end{aligned}$$

since ϵ was arbitrary we conclude that $\|A + K\| \geq \|A\|$. It is obvious that

$$\|A\| \geq \inf\{\|A + K\| : K \in M_n(\mathscr{K}(F_X))\}.$$

Thus,

$$\|A\|= \inf\{\|A + K\|: K \in M_n(\mathscr{K}(F_X))\}$$

and we are done. □

Corollary 6.2.1. *Let* $(X, \mathscr{A}, \phi)$ *be a* C^**-correspondence such that* ϕ *is injective and let* (π_u, t_u) *be the universal Katsura covariant Toeplitz representation and* $(\tilde{\pi}_u, \tilde{t}_u)$ *the universal Toeplitz representation of* $(X, \mathscr{A}, \phi)$*. Then there exists a complete isometry*

$$\tau_X : T_X^+ \to \operatorname{alg}(\pi_u, t_u),$$

such that $\tau_X(\tilde{\pi}_u(a)) = \pi_u(a)$ *and* $\tau_X(\tilde{t}_u(\xi)) = t_u(\xi)$ *for each* $a \in \mathscr{A}$ *and* $\xi \in X$*.*

Proof. Recall that from the gauge-invariance uniqueness theorem we have that O_X is $*$-isomorphic with $C^*(\pi, t)$ where (π, t) is the injective Katsura covariant Toeplitz representation we introduced in theorem 5.1.1 and T_X is $*$-isomorphic to $C^*(\pi_\infty, t_\infty)$ where (π_∞, t_∞) is the Fock representation. We denote these $*$-isomorphisms by ρ and $\tilde{\rho}$ respectively. Let q be the restriction to $\operatorname{alg}(\pi_\infty, t_\infty)$ of the natural quotient map

$$C^*(\pi_\infty, t_\infty) \to C^*(\pi_\infty, t_\infty)/\mathscr{K}(F_X J_X).$$

From the preceding lemma and using the fact that $\mathscr{K}(F_X J_X)) \subseteq \mathscr{K}(F_X))$ we have that for $k \in \mathbb{N}$ and $A \in M_n(T_X^+)$

$$\|A\|= \inf\{\|A+K\|: K \in M_n(\mathscr{K}(F_X))\} \leq \inf\{\|A+K\|: K \in M_n(\mathscr{K}(F_X J_X))\} \leq \|A\|$$

and therefore q is completely isometric. We set $\tau_X = \rho^{-1} \circ q \circ \tilde{\rho}_{|T_X^+}$, then this map is completely isometric as a composition of completely isometric maps and if $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\tau_X(\tilde{\pi}_u(a)) = \rho^{-1} \circ q \circ \tilde{\rho}_{|T_X^+}(\tilde{\pi}_u(a)) = \rho^{-1}(\pi_\infty(a) + \mathscr{K}(F_X J_X)) = \rho^{-1}(\pi(a)) = \pi_u(a)$$

and

$$\tau_X(\tilde{t}_u(\xi)) = \rho^{-1} \circ q \circ \tilde{\rho}_{|T_X^+}(\tilde{t}_u(\xi)) = \rho^{-1}(t_\infty(\xi) + \mathscr{K}(F_X J_X)) = \rho^{-1}(t(\xi)) = t_u(a).$$

□

By adding the tail $T = c_0(\ker \phi)$ to X we may remove the requirement of ϕ being injective and therefore we have the following:

Lemma 6.2.5. *Let* $(X, \mathscr{A}, \phi)$ *be a* C^**-correspondence and let* $(\pi_u^{\mathscr{A}}, t_u^X)$ *be the universal Katsura covariant Toeplitz representation and* $(\tilde{\pi}_u^{\mathscr{A}}, \tilde{t}_u^X)$ *the universal Toeplitz representation of*

$(X, \mathscr{A}, \phi)$. *Then there exists a complete isometry*

$$\tau_X : T_X^+ \to \text{alg}(\pi_u^{\mathscr{A}}, t_u^X),$$

such that $\tau_X(\tilde{\pi}_u^{\mathscr{A}}(a)) = \pi_u^{\mathscr{A}}(a)$ *and* $\tau_X(\tilde{t}_u^X(\xi)) = t_u^X(\xi)$ *for each* $a \in \mathscr{A}$ *and* $\xi \in X$.

Proof. Let $(Y, \mathscr{B}, \phi_{\mathscr{B}})$ be the injective C^*-correspondence we obtain by adding the tail $T = c_0(\ker\phi)$ and let $(\tilde{\pi}_u^{\mathscr{B}}, \tilde{t}_u^Y)$ be the universal Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$. Recall that in remark 22 we proved that

$$I_{(\tilde{\pi}_u^{\mathscr{B}}, \tilde{t}_u^Y)} = \{0\},$$

and therefore also for the restriction $(\tilde{\pi}_u^{\mathscr{B}}|_{\mathscr{A}}, \tilde{t}_u^Y|_X)$, which is a Toeplitz representation of $(X, \mathscr{A}, \phi)$, we have that $I_{(\tilde{\pi}_u^{\mathscr{B}}|_{\mathscr{A}}, \tilde{t}_u^Y|_X)} = \{0\}$. By the gauge-invariance uniqueness theorem there exists a $*$-isomorphism

$$\tilde{\rho} : T_X \to C^*(\tilde{\pi}_u^{\mathscr{B}}|_{\mathscr{A}}, \tilde{t}_u^Y|_X) \subseteq T_Y$$

such that for each $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\tilde{\rho}(\tilde{\pi}_u^{\mathscr{A}}(a)) = \tilde{\pi}_u^{\mathscr{B}}|_{\mathscr{A}}(a) \quad \text{and} \quad \tilde{\rho}(\tilde{t}_u^X(\xi)) = \tilde{t}_u^Y|_X(\xi).$$

The preceding corollary implies that there exists a complete isometry

$$\tau_Y : T_Y^+ \to \text{alg}(\pi_u^{\mathscr{B}}, t_u^Y),$$

where $(\pi_u^{\mathscr{B}}, t_u^Y)$ is the universal Katsura covariant Toeplitz representation of $(Y, \mathscr{B}, \phi_{\mathscr{B}})$, such that for $a \in \mathscr{B}$ and $\xi \in Y$ we have $\tau_Y(\tilde{\pi}_u^{\mathscr{B}}(a)) = \pi_u^{\mathscr{B}}(a)$ and $\tau_Y(\tilde{t}_u^Y(\xi)) = t_u^Y(\xi)$. From theorem 6.2.1 we have that $(\pi_u^{\mathscr{B}}|_{\mathscr{A}}, t_u^Y|_X)$ is an injective Katsura covariant Toeplitz representation and therefore from the gauge-invariance uniqueness theorem there exists a $*$-isomorphism

$$\rho : O_X \to C^*(\pi_u^{\mathscr{B}}|_{\mathscr{A}}, t_u^Y|_X)$$

such that for $a \in \mathscr{A}$ and $\xi \in X$ we have $\rho(\pi_u^{\mathscr{A}}(a)) = \pi_u^{\mathscr{B}}|_{\mathscr{A}}(a)$ and $\rho(t_u^X(\xi)) = t_u^Y|_X(\xi)$. Set $\tau_X = \rho^{-1} \circ \tau_Y \circ \tilde{\rho}|_{T_X^+}$, then τ_X is completely isometric as a composition of completely isometric maps and for each $a \in \mathscr{A}$ and $\xi \in X$ we have

$$\tau_X(\tilde{\pi}_u^{\mathscr{A}}(a)) = \rho^{-1} \circ \tau_Y \circ \tilde{\rho}|_{T_X^+}(\tilde{\pi}_u^{\mathscr{A}}(a)) = \rho^{-1} \circ \tau_Y(\tilde{\pi}_u^{\mathscr{B}}|_{\mathscr{A}}(a)) = \rho^{-1}(\pi_u^{\mathscr{B}}|_{\mathscr{A}}(a)) = \pi_u^{\mathscr{A}}(a)$$

and

$$\tau_X(\tilde{t}_u^X(\xi)) = \rho^{-1} \circ \tau_Y \circ \tilde{\rho}|_{T_X^+}(\tilde{t}_u^X(\xi)) = \rho^{-1} \circ \tau_Y(\tilde{t}_u^Y|_X)(\xi)) = \rho^{-1}(t_u^Y|_X(\xi)) = t_u^X(a)$$

and therefore $\tau_X : T_X^+ \to \text{alg}(\pi_u^{\mathscr{A}}, t_u^X)$ is the desired complete isometry. □

Theorem 6.2.2. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence such that T_X^+ is unital. Then the C^*-envelope of T_X^+ is O_X.*

Proof. Using the lemma above we identify T_X^+ with $\text{alg}(\pi_u, t_u) \subseteq O_X$, where (π_u, t_u) is the universal Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$. We will prove that the Shilov ideal $J_{\text{alg}(\pi_u,t_u)}$ is trivial. Suppose that $J_{\text{alg}(\pi_u,t_u)} \neq \{0\}$ and let γ_z to be the gauge action of (π_u, t_u). By proposition 6.1.1 for each $z \in \mathbb{T}$ we have $\gamma_z(J_{\text{alg}(\pi_u,t_u)}) = J_{\text{alg}(\pi_u,t_u)}$ since $\gamma_z(\text{alg}(\pi_u, t_u)) = \text{alg}(\pi_u, t_u)$. Let $q : O_X \to O_X/J_{\text{alg}(\pi_u,t_u)}$ be the natural quotient map, we will prove that $(q \circ \pi_u, q \circ t_u)$ is a Katsura covariant Toeplitz representation of $(X, \mathscr{A}, \phi)$.
Indeed, for $a \in \mathscr{A}$ and $\xi, \eta \in X$ we have

$$q \circ t_u(\xi) q \circ \pi_u(a) = q(t_u(\xi)\pi_u(a)) = q \circ t_u(\xi a)$$

and

$$q \circ \pi_u(\langle \xi, \eta \rangle) = q(t_u(\xi)^* t_u(\eta)) = q \circ t_u(\xi)^* q \circ t_u(\eta).$$

Note also that

$$(q \circ t_u)_*(\theta_{\xi,\eta}) = q \circ t_u(\xi) q \circ t_u(\eta)^* = q(t_u(\xi) t_u^*(\eta)) = q \circ (t_u)_*(\theta_{\xi,\eta})$$

and since the linear span of elements in the form $\theta_{\xi,\eta}$ is dense in $\mathscr{K}(X)$ we get that

$$(q \circ t_u)_* = q \circ (t_u)_*,$$

which implies that for $b \in J_X$ we have

$$(q \circ t_u)_*(\phi(b)) = q \circ (t_u)_*(\phi(b)) = q \circ \pi(b).$$

For each $z \in \mathbb{T}$, $a \in \mathscr{A}$ and $\xi \in X$ if we define $\tilde{\gamma}_z(q \circ \pi_u(a)) = \gamma(\pi_u(a)) + J_{\text{alg}(\pi_u,t_u)}$ and $\tilde{\gamma}_z(q \circ t(\xi)) = \gamma(t(\xi)) + J_{\text{alg}(\pi_u,t_u)}$ then γ_z is a well-defined gauge action of $(q \circ \pi_u, q \circ t_u)$. Since $J_{\text{alg}(\pi_u,t_u)} \neq \{0\}$ we have that q is not injective but $q : O_X \to O_X/J_{\text{alg}(\pi_u,t_u)}$ is a $*$-epimorphism such that

$$q(\pi_u(a)) = q \circ \pi_u(a) \quad \text{and} \quad q(t_u(\xi)) = q \circ t_u(\xi), \quad a \in \mathscr{A}, \xi \in X$$

and therefore by the gauge-invariance uniqueness theorem $q \circ \pi_u$ must also not be injective, or else q would be injective. This implies that q is not injective on T_X^+. Since $J_{\text{alg}(\pi_u,t_u)}$ is a boundary ideal the restriction of q to T_X^+ is completely isometric and hence injective, which is a contradiction. □

Corollary 6.2.2. *Let $\mathscr{A}$ be a unital C^*-algebra and α a unital $*$-automorphism. The C^*-envelope of the semi-crossed product $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ is the crossed product $\mathbb{Z} \times_\alpha \mathscr{A}$.*

Proof. In examples 5.1.3 and 5.1.4 we proved that for the C^*-correspondence $\mathscr{A}_\alpha$ we have that $T^+_{\mathscr{A}} = \mathbb{Z}^+ \times_\alpha \mathscr{A}$ and $O_{\mathscr{A}} = \mathbb{Z} \times_\alpha \mathscr{A}$. Since, $\mathscr{A}$ is unital we obtain that $\mathbb{Z}^+ \times_\alpha \mathscr{A}$ is unital and the preceding theorem yields the desired result. □

Lemma 6.2.6. *Let $\mathscr{A}$ be a non-unital C^*-algebra and denote by $\mathscr{A}_1$ its unitization. If $J \subseteq \mathscr{A}_1$ is a closed two-sided ideal such that $J \cap \mathscr{A} = \{0\}$, then $J = \{0\}$.*

Proof. We assume towards a contradiction that $J \neq \{0\}$. Since $\mathscr{A} \subseteq \mathscr{A}_1$ has co-dimension 1, J is of the form $J = \overline{\text{span}}\{A + \lambda I\}$ for $A \in \mathscr{A}$ and $\lambda \in \mathbb{C}$. Since J is an ideal it is self-adjoint and $JJ^* \neq \{0\}$ and therefore we may assume that $\lambda \in \mathbb{R}$ and A is self-adjoint. Since $J^2 \neq \{0\}$ we may assume that

$$(A + \lambda I)^2 = A + \lambda I$$

and therefore

$$A^2 + 2\lambda A + \lambda^2 I = A + \lambda I \iff A^2 + 2\lambda A + A = (\lambda^2 - \lambda)I,$$

which implies that $\lambda = 1$ and thus

$$A^2 = -A.$$

If we set $P = A^2$ we have that $P \in \mathscr{A}$ is a projection and $A = -P$. For each $C \in \mathscr{A}$ we have that

$$(I - P)C = (I + A)C \in \mathscr{A} \cap J = \{0\}$$

and so $I = P$ is a unit in $\mathscr{A}$. Hence, a contradiction. □

Remark 23. Let $\mathscr{A}$ be a non-unital operator algebra. Consider $\mathscr{A}$ as a subalgebra of $\mathbf{B}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$ and set $\mathscr{A}_1 = \text{span}\{\mathscr{A}, I\}$ where I is the identity operator in $\mathbf{B}(\mathscr{H})$. We say that $\mathscr{A}_1$ is a unitization of $\mathscr{A}$. Then corollary 2.1.15 in [3] proves that up to completely isometrical isomorphism, this unitization does not depend on the embedding of $\mathscr{A}$ into $\mathbf{B}(\mathscr{H})$. Therefore, $\mathscr{A}_1$ is called the unitization of $\mathscr{A}$.

Definition 6.2.1. Let $\mathscr{A}$ be a non-unital operator algebra and $\mathscr{A}_1$ its unitization and let $(C^*_{env}(\mathscr{A}_1), i)$ be the C^*-envelope of $\mathscr{A}_1$. We define the C^*-envelope of $\mathscr{A}$ to be the pair $(C^*_{env}(\mathscr{A}), i)$ where $C^*_{env}(\mathscr{A})$ is the C^*-subalgebra generated by $i(\mathscr{A})$ into $C^*_{env}(\mathscr{A}_1)$.

It was proved in [9] and [5] that the C^*-envelope of an arbitrary operator algebra is unique and enjoys the following (universal) property:
For each C^*-cover $(\mathscr{C}, j)$ of $\mathscr{A}$ there exists a unique $*$-epimorphism $\rho : \mathscr{C} \to C^*_{env}(\mathscr{A})$

such that $\rho \circ j = i$.

We will now prove that the C^*-envelope of T_X^+ is O_X without the additional hypothesis that $\text{alg}(\pi_u, t_u)$ is unital.

Theorem 6.2.3. *Let $(X, \mathscr{A}, \phi)$ be a C^*-correspondence. Then the C^*-envelope of T_X^+ is O_X.*

Proof. Once more we identify T_X with $\text{alg}(\pi_u, t_u)$, where (π_u, t_u) is the universal Katsura covariant Toeplitz representation. The case that $\text{alg}(\pi_u, t_u)$ is unital follows from theorem 6.2.2. Suppose that $\text{alg}(\pi_u, t_u)$ is not unital and that O_X has a unit I. Set $\text{alg}(\pi_u, t_u)_1 = \text{alg}(\pi_u, t_u) + \mathbb{C}I$, then for each $z \in \mathbb{T}$ we have that $\gamma_z(\text{alg}(\pi_u, t_u)_1) = \text{alg}(\pi_u, t_u)_1$ and so by repeating the arguments of the proof of theorem 6.2.2 we get that $C^*_{env}(\text{alg}(\pi_u, t_u)_1) = O_X$. The C^*-subalgebra of O_X generated by $\text{alg}(\pi_u, t_u)$ is O_X and therefore $C^*_{env}(\text{alg}(\pi_u, t_u))$ is O_X.

Suppose now that both $\text{alg}(\pi_u, t_u)$ and O_X are not unital. We unitize O_X by joining a unit I and set

$$\text{alg}(\pi_u, t_u)_1 = \text{alg}(\pi_u, t_u) + \mathbb{C}I \subseteq O_X + \mathbb{C}I.$$

Since the Shilov ideal $J_{\text{alg}(\pi_u, t_u)_1}$ is gauge-invariant we have that $J_{\text{alg}(\pi_u, t_u)_1} \cap O_X \subseteq O_X$ is also gauge-invariant and using again the same arguments $J_{\text{alg}(\pi_u, t_u)_1} \cap O_X = \{0\}$ or else it would meet $\pi_u(\mathscr{A})$. From the preceding lemma we get that $J_{\text{alg}(\pi_u, t_u)_1} = \{0\}$ and therefore $C^*_{env}(\text{alg}(\pi_u, t_u)_1) = O_X + \mathbb{C}I$. The C^*-subalgebra of $O_X + \mathbb{C}I$ generated by $\text{alg}(\pi_u, t_u)$ is O_X and therefore $C^*_{env}(\text{alg}(\pi_u, t_u)) = O_X$, which completes the proof. □

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