Department of
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M. Sc. Program
PURE MATHEMATICS

Sophus Lie's Third Theorem and its Constructive Proof

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Ευχαριστίες

Θέλω να ευχαριστήσω θερμά τον επιβλέποντα καθηγητή μου κύριο Ιάκωβο Ανδρουλιδάκη για την διδακτική του γενναιοδωρία και τις ώρες που πέρασε βοηθώντας με να αποκτήσω μια πλήρη εικόνα του θέματος, όπως επίσης και τα μέλη της τριμελούς μου επιτροπής κ.κ. Α. Μελά και Π. Γιαννιώτη. Ένα ιδιαίτερο ευχαριστώ και στις υπαλλήλους της βιβλιοθήκης θετικών επιστημών που με εξυπηρέτησαν με τον καλύτερο τρόπο όλο αυτό το διάστημα της προετοιμασίας μου.

Abstract

A Lie algebra is the tangent space at the identity element of a manifold that admits a group structure in a way that the group operations of multiplication and inversion are smooth. We will present the constructive proof of *Sophus Lie's Third Theorem* as it is given in *Duistermaat* and *Kolk's* book *Lie Groups* [DK00]. It is the unique constructive proof of the third theorem that can be stated as; Every finite dimensional Lie algebra $\mathfrak{g}$ is integrated to a simply connected lie group G.

To prove the theorem we will use the infinite dimensional Banach space of paths of the Lie algebra. This space is homeomorphic to all path spaces of Lie groups that have Lie algebra $\mathfrak{g}$, not necessarily connected. We will search for solutions of differential equations of homotopy classes and in order to do so we will have to use a $\mathfrak{g}$-valued 1-form and homology and De Rahm cohomology classes. Through Stokes' theorem we will see that integration is well defined. The finite dimensional simply connected Lie group G will occur as a quotient of two infinite dimensional Banach Lie groups.

Περίληψη

Μια άλγεβρα Lie είναι ο εφαπτόμενος χώρος στη μονάδα μιας πολλαπλότητας με δομή ομάδας και ομαλές τις απεικονίσεις του πολλαπλασιασμού και του αντίστροφου. Στη παρούσα εργασία παρουσιάζεται η κατασκευαστική απόδειξη του τρίτου θεωρήματος του Sophus Lie όπως γράφτηκε από τους Duistermaat, Kolk στο βιβλίο Lie Groups [DK00]. Είναι η μοναδική κατασκευαστική απόδειξη του τρίτου θεωρήματος που διατυπώνεται ως εξής: Για κάθε άλγεβρα Lie $\mathfrak{g}$ πεπερασμένης διάστασης υπάρχει μοναδική απλά συνεκτική ομάδα Lie που την ολοκληρώνει.

Για την απόδειξη του θεωρήματος θα χρειαστεί να περάσουμε στον απειροδιάστατο χώρο Banach των μονοπατιών της άλγεβρας. Αυτός είναι ομοιομορφικός με τους χώρους των μονοπατιών ομάδων Lie που έχουν άλγεβρα Lie την $\mathfrak{g}$, όχι απαραίτητα συνεκτικών. Ουσιαστικά αναζητούμε λύσεις διαφορικών εξισώσεων κλάσεων ομοτοπίας και για τον σκοπό αυτό θα χρησιμοποιήσουμε μια διαφορική μορφή που θα μας μεταφέρει από των χώρο των μονοπατιών της άλγεβρας στον χώρο των μονοπατιών της ομάδας και κλάσεις ομολογίας και συνομολογίας De Rahm. Μέσω του θεωρήματος Stokes θα δείξουμε ότι η ολοκλήρωση ορίζεται καλά. Τελικά η απλά συνεκτική ομάδα G πεπερασμένης διάστασης που ολοκληρώνει την $\mathfrak{g}$ θα προκύψει ως πηλίκο δύο άπειρης διάστασης ομάδων Lie: της ομάδας των μονοπατιών στην άλγεβρα με την εικόνα των ομοτοπικών με τη μονάδα μονοπατιών.

Contents

Introduction

A Lie group is a group that at the same time is a manifold and its Lie algebra is its tangent space to the identity element. A Lie algebra is also the space of the left invariant vector fields of the manifold and the exponential mapping is defined through integral curves of left invariant vector fields from the Lie algebra to the Lie group. If the Lie algebra is finite dimensional then the group's connected component of the identity is exactly the product of the images of the base elements via the exponential mapping.

1.1 Lie Groups

Definition 1.1.1. A Lie group G is a group that at the same time is C^2 manifold, such that group operations of multiplication;

$$\mu : G \times G \to G$$

$$(x, y) \mapsto xy$$

and inversion;

$$\iota : G \to G$$

$$x \mapsto x^{-1}$$

are C^2 mappings.

Example 1.1.2. Let $M(n,\mathbb{R})$ be the space of $n\times n$ matrices with real enrties. Induced with pointwise addition and scalar multiplication $M(n,\mathbb{R})$ is a linear space and $M(n,\mathbb{R}) \simeq \mathbb{R}^{2n}$. Let $A \in M(n,\mathbb{R})$. Then the mappings

$$s_{ij} : A \to \mathbb{R}$$

$$A \mapsto a_{ij}$$

(where a_{ij} the $ij-$ entry of A) is a system of linear coordinates of $M(n,\mathbb{R})$. Then for the mapping $\det : M(n,\mathbb{R}) \to \mathbb{R}$ one may right $\det = \sum\limits_{\sigma\in S_n} sgn(\sigma)s_{1\sigma(1)}\ldots s_{n\sigma(n)}$. The set $GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det A \neq 0\}$ of real invertible matrices is the inverse image of the open subset $\mathbb{R}\setminus\{0\}$ through det and the mapping det is continuous, so $GL(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R})$. So we may consider it as a smooth manifold of dimension n^2 with

$$\mu : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$$

$$s_{kl}(\mu(A,B)) = \sum_{i=1}^{n} s_{ki}(A)s_{il}(B)$$

Follows that μ is smooth.

From Cramer's rule we have that

$$\iota : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$$

$$A \mapsto A^{-1}$$

is given by $\iota(A) = (\det A)^{-1}A^{co}$. So ι is also smooth. It follows that $GL(n,\mathbb{R})$ is a Lie group.

Definition 1.1.3. Let G end H be Lie groups. A Lie group homomorphism is a smooth map $f : G \to H$ such that f is a group homomorphism.

Now,

Definition 1.1.4. We define left translation by x

$$l_x : G \to G$$

$$y \mapsto xy$$

and right translation

$$r_x : G \to G$$

$$y \mapsto yx$$

The mappings r_x, l_x are diffeomorphisms $G \to G$ and group homomorhisms $G \to Sym(G)$.

Finally, for $x \in G$ we call conjugate mapping

$$C_x : G \to G$$

$$y \mapsto xyx^{-1}$$

The mapping C_x is an automorphism of G with inverse $C_{x^{-1}}$ and the mapping

$$C : G \to Aut(G)$$

$$x \mapsto C_x$$

is also a group homomorphism and $\ker C = Z(G)$

Let M be a smooth manifold and $\mathfrak{X}(M)$ be the real linear space of smooth vector fields on M.

Definition 1.1.5. We say that a vector field $X \in \mathfrak{X}(M)$ is left invariant if $(l_g)_* X = X$ $\forall x \in G$ or equivalently

$$X(xy) = T_y(l_x)X(y)$$

$\forall x, y \in G$.

One may see that left invariant vector fields are completely determined by their value at the identity element $X(e) \in T_eG$. We write $\mathfrak{X}^L(M)$ for the set of left invariant vector fields on M.

Proposition 1.1.6. *Let $X \in T_eG$. We define the vector field $u_X = T_e(l_x)(X)$, $x \in G$. Then the mapping*

$$T_eG \to \mathfrak{X}^L(G)$$

$$X \mapsto u_X$$

is a linear isomorphism with inverse $u \mapsto u(e)$.

Proof. From the definition of left invariant vector fields the mapping

$$\mathfrak{X}^L(G) \to T_eG$$

$$u \mapsto u(e)$$

is an injection. We will demonstrate that it is also a surjection;

Let f be the mapping

$$f : G \times G \to G$$

$$(x, y) \mapsto l_x(y)$$

differentiating for y at $y = e$ in the direction $X \in T_eG$ we get;

$$T_ef : G \to TG$$

$$x \mapsto T_e(l_x)X$$

that is also smooth. It follows that u_X is a smooth vector field on G, so

$$T_eG \to \mathfrak{X}^L(G)$$

$$X \mapsto u_X$$

is a real linear mapping that is also a surjection. Indeed,

fixing a $X \in T_eG$ and differentiating $l_{xy} = l_x \circ l_y$ we get

$$T_e(l_{xy}) = T_y(l_x)T_e(l_y)$$

witch means that u_X is a left invariant vector field. We get that $X \mapsto u_X$ is a surjection.

Finally, $u_X(e) = X$ so $E : u \mapsto u(e)$ is a bijection, hence a linear isomorphism with inverse $E^{-1} : X \mapsto u_X$ □

Definition 1.1.7. Let G be a Lie group and $X \in T_eG$. The curve $a_X : I \to G$ where $I \subset \mathbb{R}$ and $a(t_0) = e$, $\dot{a(t)} = u_X(a(t))$ is an integral curve of the vector field u_X starting at e. The integral curve a_X is said to be maximal if I is the largest possible interval of de finition for a.

Lemma 1.1.8. *Let G be a Lie group , $X \in T_eG$ and $a_X : I \to G$ integral curve of the vector field u_X. Then $a_1(t) = ya(t)$, $y \in G$, is also an integral curve for u_X.*

Proof. We have that

$$\frac{d}{dt}a_1(t) = T_e l_y(a(t)) = T_{a(t)} l_y \frac{d}{dt} a(t)$$

$$= T_{a(t)} l_y u_X(a(t)) = u_X(a_1(t))$$

because u_X is left invariant. Follows that $ya(t)$ is an integral curve for u_X. □

Proposition 1.1.9. *Let G be a Lie group and $X \in T_eG$. Then;*

1. *a_X is defined on $\mathbb{R}$*
2. *$a_X(s+t) = a_X(s)a_X(t)$ $\forall s, t \in \mathbb{R}$*
3. *The mapping*

$$\mathbb{R} \times T_eG \to G$$

$$(t, X) \mapsto a_X(t)$$

   is smooth.

Proof. 1. Let $I \subseteq \mathbb{R}$ be the domain of the integral curve a_X beginning at e of the vector field u_X. Then there exists $t_1 \in I$ and $a_X(t_1) = x_1 \in G$. From Lemma 1.1.8, $a_1(t) := x_1 a_X(t)$ is also an integral curve of u_X beginning at x_1 with domain I.

From Re parametrization Theorem for integral curves (see Appendix A.1), the maximal integral curve of the vector field u_X beginning at x_1 will be $a_2(t) := a_X(t + t_1)$. The integral curve a_2 has domain $I - t_1$, which means that $I \subset I - t_1$, and $s + t_1 \in I$ $\forall s, t_1 \in I$. It follows that $I = \mathbb{R}$.

2. Fixing an $s \in \mathbb{R}$ we get that $a_X(s) \in G$ and as we saw above the maximal integral curve of u_X beginning at $a_X(s)$ is $c(t) := a_X(s)a_X(t)$.

   From Re parametrization Theorem for integral curves, $d(t) := a_X(s+t)$ will be also an integral curve for u_X beginning at $a_X(s)$.

   From the uniqueness of maximal integral curves follows that $c(t) = d(t)$.

3. The vector field u_X is linearly dependent, that is, smoothly dependent from X.

   Let φ_X be the flow of u_X. Then the mapping

$$(X, t, x) \mapsto \varphi_X(t, x)$$

   is smooth (c.f. Appendix A.1). More over,

$$(t, X) \mapsto a_X(t) = \varphi_X(t, e)$$

$$\mathbb{R} \times T_eG \to G$$

   is smooth.

$\square$

Definition 1.1.10. (Exponential mapping) Let G be a Lie group, $X \in T_eG$ and a_X integral curve of u_X beginning at e. We define the exponential mapping;

$$\exp := \exp_G$$

$$\exp : T_eG \to G$$

$$X \mapsto a_X(1)$$

Proposition 1.1.11. *Let G be a Lie group, $X \in T_eG$ and $a_X : \mathbb{R} \to G$ the integral curve of the vector field u_X beginning at e . Then $\forall s, t \in \mathbb{R}$:*

1. $\exp(sX) = a_X(s)$
2. $\exp(s+t)X = \exp(sX)\exp(tX)$
3. *The mapping* $\exp : T_eG \to G$ *is smooth and a local diffeomorphism at 0 and* $T_0 exp = Id_{T_eG}$

Proof. 1. Let $c : \mathbb{R} \to G$ be a curve with $c(t) := a_X(st)$. Then $c(0) = e$ and

$$\frac{d}{dt}c(t) = s\dot{a_X}(st)$$

$$= su_X(a_X(st)) = u_{sX}(c(t))$$

So, $c(t)$ is a maximal integral curve of the vector field u_{sX} beginning at e. So, $c(t) = a_{sX}(t)$, and for $t = 1$ we get the assertion.

2. From (1) and Proposition 1.1.9 we get

$$\exp sX \exp tX = a_X(s)a_X(t)$$

$$= a_X(s+t) = \exp(s+t)X$$

3. In Proposition 1.1.9 we saw that the mapping

$$\mathbb{R} \times T_eG \to G$$

$$(t, X) \mapsto a_X(t)$$

is smooth. It follows that $(1, X) \mapsto a_X(1)$ is smooth, which proves the smoothness of $\exp$.

Now,

$$T_0(\exp)X = \frac{d}{dt} |_{t=0} \exp(tX) = \dot{a_X}(0)$$

$$= u_X(e) = X$$

so that $T_0(\exp) = Id_{T_eX}$ and from the inverse function theorem exp will be a local diffeomorphism at 0. So there exist open neighborhoods U of $0 \in T_eG$ and V of $e \in G$ such that $\exp(U) = V$ and $\exp|_U$ is a local diffeomorphism.

□

Definition 1.1.12. (One Parameter Subgroup) A smooth homomorphism a: $(\mathbb{R}, +) \to G$ is called a one parameter subgroup of G. In other words, a: $(\mathbb{R}, +) \to G$ is a one parameter subgroup of G if

$$a(s + t) = a(s)a(t)$$

$\forall s, t \in \mathbb{R}$ and $a(0) = e$.

Proposition 1.1.13. *(Characterization of One Parameter Subgroups) Let G be a Lie group and $X \in T_eG$. Then*

$$t \mapsto \exp tX$$

$$\mathbb{R} \longrightarrow G$$

is a one parameter subgroup of G.

Conversely, if a is a one parameter subgroup of G with $\dot{a}(0) = X$ then $a(t) = \exp(tX)$, $t \in \mathbb{R}$.

Proof. It is direct that $t \mapsto \exp tX$ is a one parameter subgroup of G.

Now if $a : (\mathbb{R}, +) \to G$ is a one parameter subgroup of G, then $a(0) = e$ and

$$\frac{d}{dt}a(t) = \frac{d}{ds}\mid_{s=0} a(t+s)$$

$$= \frac{d}{ds}\mid_{s=0} a(t)a(s) = T_e(l_{a(t)})\dot{a}(0)$$

$$= u_X(a(t))$$

So a is an integral curve of the vector field u_X beginning at e. From uniqueness of integral curves we get that $a = a_X$ and as we saw above, $a_X(t) = \exp tX$. □

We saw that the mappings of right and left translation r_x and l_x are diffeomorphisms $G \to G$. For the mapping of the conjugation $C_x : G \to G$ one may wright;

$$C_x = l_x \circ r_x^{-1}$$

$$y \mapsto xyx^{-1}$$

and $C_x(e) = e$. Differentiating C_x at e we get a linear automorphism at T_eG, so that $T_eC_x \in GL(T_eG)$

Definition 1.1.14. Let G be a Lie group and $x \in G$. We define

$$Ad_x : G \to T_eG$$

$$Ad_x := T_eC_x$$

The mapping;

$$Ad : G \to GL(T_eG)$$

is called the adjoined mapping of G at T_eG.

Proposition 1.1.15. *$Ad : G \to GL(T_eG)$ is a Lie group homomorphism.*

Proof. The map

$$G \times G \to G$$

$$(x, y) \mapsto xyx^{-1}$$

is smooth. Differentiating at y for $y = e$ we get that

$$G \to End(T_eG)$$

$$x \mapsto Ad_x$$

is smooth and $GL(T_eG)$ is open at $End(T_eG)$ so $Ad : G \to GL(T_eG)$ is smooth.

Now, $C_e = I_G \Rightarrow Ad(e) = I_{T_eG}$. Differentiating $C_{xy} = C_xC_y$ using the chain rule at e we get $Ad(xy) = AdxAdy$ so that Ad is a Lie group homomorphism. □

We saw that

$$Ad(e) = I = I_{T_eG}$$

and

$$T_IGL(T_eG) = End(T_eG)$$

so the tangent mapping of Ad at e wil be linear $T_eG \to End(T_eG)$.

Definition 1.1.16. We define the linear mapping $ad : T_eG \to End(T_eG)$ with $ad := T_eAd$

We will later see that the mapping ad defines a product structure on T_eG turning T_eG to an algebra on $\mathbb{R}$.

Theorem 1.1.17. *Let G and H be Lie groups and $\Phi : G \to H$ be a Lie group homomorphism. Then for $x \in G$ and $X \in T_eG$:*

1. $T_e\Phi(x) = \frac{d}{dt}\mid_{t=0} \Phi(\exp tX)$

2. $\Phi(\exp X) = \exp(T_e\Phi(x))$

Proof. Let $X \in T_eG$.

1. $a(t) = \Phi(\exp_G(tX))$, $a : \mathbb{R} \to H$ is a one parameter subgroup of H. Using the chain rule we get that;

$$\frac{d}{dt}\mid_{t=0} \Phi(\exp tX) =$$

$$= \frac{d}{dt}\mid_{t=0} a(t) = T_e\Phi T_0 \exp_G(X) = T_e\Phi(X)$$

2. From the characherization of one parameter subgroups we get;

$$a(t) = \Phi(\exp(tX)) = \exp(t\dot{a}(0)) = \exp\left(t(T_e\Phi(x))\right)$$

For $t = 1$ the assertion follows.

$\square$

Corollary 1.1.18. *Let $x \in G$. Then*

1. $\forall X \in T_eG,\ x \exp X x^{-1} = \exp(Ad_x(X))$

2. $\forall X \in T_eG,\ Ad(\exp X) = e^{ad(X)}$

3. $ad_X = \frac{d}{dt}\mid_{t=0} Ad(\exp tX)$

Proof. The proof is an application of Theorem 1.1.17 for the Lie group homomorphism;

1. $\Phi = C_x$, $\Phi : G \to G$

2. $\Phi = Ad$, $\Phi : G \to GL(T_eG)$

$\square$

Remark 1.1.19. We saw that $ad : T_eG \to End(T_eG)$ and $End(T_eG)$ is a matrix group, so we may write

$$\exp(ad_X) \equiv e^{ad_X}$$

where $e^{(\cdot)}$ is the matrix exponential.

Definition 1.1.20. Let G be a Lie group. Then for $X, Y \in T_e(G)$ we define the Lie bracket $[X, Y] \in T_eG$;

$$[X, Y]\, f = X(Yf) - Y(Xf)$$

$\forall f \in C^\infty(G)$.

Lemma 1.1.21. *Let G be a Lie group and X a left invariant vector field on G. Then $X(g)$ is the derivative at $t = 0$ of the curve $t \mapsto g \exp(tX)$. In particular*

$$Xf(g) = \frac{d}{dt} \mid_{t=0} f(g \exp tX)$$

for $g \in G$ and $f \in C^\infty(M)$

Proof. The assertion holds for $g = e$, and since X is left invariant it holds for all $g \in G$. $\square$

Theorem 1.1.22. *Let G be a Lie group. Then $\forall X, Y \in T_eG$ we have that;*

$$[X, Y] = ad_x(Y)$$

Proof. We compute;

$$([X, Y]\, f)\,(g) = \frac{d}{dt} \mid_{t=0} Yf(g \exp tX) - \frac{d}{ds} \mid_{s=0} Xf(g \exp sY)$$

$$= \frac{d}{dt} |_{t=0} \frac{d}{ds} |_{s=0} f(g \exp tX \exp sY) - \frac{d}{ds} |_{s=0} \frac{d}{dt} |_{t=0} f(g \exp sY \exp tX)$$

$$= \frac{d}{ds} |_{s=0} \frac{d}{dt} |_{t=0} (f(g \exp tX \exp sY) + f(g \exp sY \exp(-tX)))$$

It holds that

$$\frac{d}{dt} |_{t=0} (F(t,0) + F(0,t)) = \frac{d}{dt} |_{t=0} F(t,t)$$

So for $F(x,y) = f(g \exp xX \exp sY \exp(-yX))$, fixing an s, we get that

$$([X,Y]\, f)\,(g) = \frac{d}{ds} |_{s=0} \frac{d}{dt} |_{t=0} f(g \exp tX \exp sY \exp(-tX))$$

$$= \frac{d}{ds} |_{s=0} \frac{d}{dt} |_{t=0} f(g \exp(sAd(\exp tX)Y))$$

$$= \frac{d}{dt} |_{t=0} ((Ad(\exp tX)Y)f)\,(g)$$

$$= ((ad(X)Y)\, f)\,(g)$$

So we have that $ad_X(Y) = [X,Y]$ for $X, Y \in \mathfrak{g}$ □

Lemma 1.1.23. *The mapping*

$$T_eG \times T_eG \to T_eG$$

$$(X,Y) \mapsto [X,Y]$$

is bilinear and antisymmetric.

Proof. Bilinearity follows from linearity of $ad : T_eG \to End(T_eG)$.

For the antisymmetric property;

Let $Z \in T_eG$. Then for all $s, t \in \mathbb{R}$;

$$\exp(tZ) = \exp(sZ)\exp(tZ)\exp(-sZ)$$

$$= \exp(tAd(\exp sZ)Z)$$

and as we have already see;

$$\frac{d}{dt}\,|_{t=0}\,\exp(tZ) = Z = Ad(\exp(sZ))Z$$

Now,

$$\frac{d}{ds}\,|_{s=0}\,Z = 0 = ad(Z)T_0\exp Z$$

$$= ad(Z)Z = [Z, Z]$$

For $Z = X + Y$ we have;

$$[X+Y, X+Y] = 0 \Rightarrow$$

$$[X, X] + [X, Y] + [Y, Y] + [Y, X] = 0 \Rightarrow$$

$$[X, Y] = -[Y, X]$$

$\square$

Theorem 1.1.24. *Let G, H be Lie groups and $\Phi : G \to H$ a Lie group homomorphism.*

Then $\forall X, Y \in T_eG$ *we have;*

$$T_e\Phi([X,Y]_G) = [T_e\Phi X, T_e\Phi Y]_H$$

Proof. Observing that $\Phi \circ C_x = C_{\Phi(x)} \circ \Phi$ from the chain rule we get $T_e(\Phi \circ C_x) = T_e\Phi(Ad_x)$, $T_e\left(C_{\Phi(x)} \circ \Phi\right) = Ad_{\Phi(x)}(\Phi)$, so that

$$T_e\Phi(Ad_x) = Ad_{\Phi(x)}(\Phi)$$

differentiating for x at $x = e$ at the direction of $X \in T_eG$ we get

$$T_e\Phi \circ ad_X = ad_{T_e\Phi(X)} \circ T_e\Phi$$

hence,

$$T_e\Phi(ad_X)(Y) = ad_{T_e\Phi(X)} T_e\Phi(Y)$$

□

Corollary 1.1.25. *For all* $X, Y, Z \in T_eG$ *we have;*

$$[[X,Y], Z] = [X,[Y,Z]] - [Y,[X,Z]] \tag{1.1.1}$$

Proof. Using Theorem 1.1.24 for $\Phi = Ad : G \to GL(T_eG)$

we get

$$ad[X,Y](Z) = [ad_X, ad_Y](Z) \Rightarrow$$

$$[[X,Y], Z] = ad_X ad_Y(Z) - ad_Y ad_X(Z) = [X,[Y,Z]] - [Y,[X,Z]]$$

□

Equation 1.1.1 is called Jacobi identity.

1.2 Lie Algebras

Definition 1.2.1. A real Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{R}$, together with a bilinear mapping

$$(X,Y) \mapsto [X,Y]$$

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

witch is called the Lie bracket of $\mathfrak{g}$. The Lie bracket is antisymmetric and satisfies the Jacobi identity.

For later use we will also need the following definition;

Definition 1.2.2. A Complex Lie algebra is a vector space $\mathfrak{g}$ over $\mathbb{C}$ together with a Lie bracket that is a complex bilinear mapping $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proposition 1.2.3. *Let G be a Lie group and let $\mathfrak{X}^L(G)$ be the space of left invariant vector fields of G. Then for $X, Y \in \mathfrak{X}^L(G)$ we have $[X,Y] \in \mathfrak{X}^L(G)$.*

Proof. It is $X \in \mathfrak{X}^L(G)$ so $\forall x \in G$ we get $X \overset{l_x}{\sim} X$ witch by definition means that

$$Tl_x \circ X \circ l_x^{-1} = X$$

If $X, Y \in \mathfrak{X}^L(G)$ then $X \overset{l_x}{\sim} X$ and $Y \overset{l_x}{\sim} Y$ $\forall x \in G$ so for the Lie bracket of X, Y we get

$$[X,Y] \overset{l_x}{\sim} [X,Y]$$

or,

$$[X,Y] \in \mathfrak{X}^L(G)$$

$\square$

Let G be a Lie group. The fact that the left invariant vector field are closed under the Lie bracket operation combined with Proposition 1.1.6 allows us to write $\mathfrak{g}$ for the Lie algebra of G and

$$\mathfrak{g} = (T_e G, [\cdot, \cdot])$$

Example 1.2.4. Let V be a real vector space of finite dimension n and $v_1, \ldots, v_n$ be a basis of V. Then there exists a unique linear isomorphism $e_v : \mathbb{R}^n \to V$ $e_i \mapsto v_i$ where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. If $w_1, \ldots, w_n$ is another basis for V then

$$L : \mathbb{R}^n \to \mathbb{R}^n$$

$$L := e_v^{-1} e_w$$

is a linear isomorphism, therefore a diffeomorphism. So V has a unique manifold structure independent of the choice of basis. The space of linear endomorphisms of V, $End(V)$ with pointwise addition and scalar multiplication is a linear space.

Let $A \in End(V)$. We write $mat(A) = mat_v A$ for the matrix A and the basis $v_1, \ldots v_n$. The mapping mat is a linear isomorphism.

$$End(V) \to M(n, \mathbb{R})$$

and a diffeomorphism with

$$mat(GL(V)) = GL(n, \mathbb{R})$$

So $GL(V)$ is an open subset of $End(V)$, so it is also a submanifold of $End(V)$. It follows that $GL(V)$ is a Lie group isomorphic to $GL(n, \mathbb{R})$ and

$$T_I GL(V) = \mathfrak{gl}(V) = End(V)$$

since $GL(V)$ is an open subset of the linear space $EndV$.

Let us consider the mapping;

$$\det : GL(V) \to \mathbb{R}^*$$

Then $T_1\mathbb{R}^* = \mathbb{R}$ so,

$$T_I \det : End(V) \to \mathbb{R}$$

Let $H \in End(V)$. Then

$$T_I \det H = \frac{d}{dt}\,|_{t=0} \det(I + tH)$$

But

$$\det(I + tH) = 1 + t(h_{11} + \ldots + h_{nn}) + t^2 R(t, H)$$

where R is a polynomial. Differentianting for t at $t = 0$ we get

$$T_I \det H = h_{11} + \ldots + h_{nn} = trH$$

Definition 1.2.5. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A Lie algebra homomorphism is a linear mapping $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that for all $X, Y \in \mathfrak{g}$

$$\varphi\,[X, Y] = [\varphi(X), \varphi(Y)]$$

Proposition 1.2.6. *Let G, H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\Phi : G \to H$ is a Lie group homomorphism, then the tangent map of Φ at the identity*

$$T_e\Phi := \varphi$$

$$\mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

Proof. The proof is a direct application of Theorems 1.1.17 and 1.1.24. □

1.3 The connected component of the identity

Let G be a Lie group. Consider the set $G^{\circ} = \{expX_1 \dots expX_k \mid k \geq 1, X_i \in \mathfrak{g}\}$ where $\mathfrak{g}$ is a finite dimensional Lie algebra.

Lemma 1.3.1. *G° is an open subset of G.*

Proof. Let $a \in G^{\circ}$. Then there exists a positive integer $k \geq 1$ and elements $X_1, \dots, X_k \in \mathfrak{g}$ such that $a = \exp(X_1) \dots \exp(X_k)$. The mapping $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at 0 so there exists an open neighborhood Ω of 0 in $\mathfrak{g}$ such that Ω is diffeomorphic to an open neighborhood of e in G.

Since left translation by a: $l_a : G \to G$ is a diffeomorphism, we get that

$$l_a(exp(\Omega)) = \{\exp(X_1) \dots \exp(X_k) \exp(X)\} \subset G^{\circ}$$

So a is an inner point of G° and it follows that G° is open in G. □

Lemma 1.3.2. *Let G be a Lie group and H be a subgroup of G. If H is open in G then it is also closed in G.*

Proof. G has connected components, so $\forall x, y \in G$ we have $xH = yH$ or $xH \cap yH = \emptyset$. (The connected components define an equivalence relation). So there exists a subset S of G such that;

$$G = \underset{s \in S}{\cup} sH$$

and

$$s_i H \cap s_j H = \emptyset$$

for $i \neq j$.

Then

$$H^c = \underset{s \in S \wedge s \notin H}{\cup} sH$$

This is a disjoined union of open subsets, so that H^c is open, hence H is closed. $\square$

Proposition 1.3.3. *Let G be a Lie group. Then G° is the connected component of the identity of G. Furthermore, G is connected if and only if $G^\circ = G$.*

Proof. G° is open, hence closed in G therefore a disjoined union of connected components. Let us observe that G° is arcwise connected;

Let $a \in G^\circ$. One may write $a = \exp(X_1)\dots\exp(X_k)$ with $k \geq 1$ and $X_1, \dots, X_k \in \mathfrak{g}$.

So there exists a curve;

$$c : [0,1] \to G$$

$$t \mapsto \exp(tX_1)\dots\exp(tX_k)$$

The curve $c(t)$ is continuous and smooth beginning at $c(0) = e$ and ending at $c(1) = a$. It follows that G° is arcwise connected, hence connected.

This means that G° is the connected component of G containing the identity. $\square$

We may extend the above theory if $\mathfrak{g}$ is an infinite dimensional Lie algebra, and a Banach space. In this case we may use the inverse function theorem for Banach spaces along with the uniform convergence of the product of the elements of the Lie algebra through the exponential mapping [Omo97]. If $\mathfrak{g}$ is not a Banach space then the image of the exponential mapping does not necessarily cover the whole neighborhood of the identity[Omo72].

The Baker-Campbell-Hausdorff formula

In general for a Lie group $\exp X \exp Y \neq \exp Y \exp X$ unless the group is commutative. Using the Baker Campbell Hausdorff formula one may write the product $\exp X \exp Y$ exclusively as combinations of the Lie bracket.

A direct application of the formula is Lie's Second Theorem: Every Lie algebra homomorphism can be integrated to a Lie group homomorphism with domain a simply connected Lie group. In this thesis we will not state this result.

2.1 The tangent map of the exponential

For the proof of the Baker-Campbell-Hausdorff formula one needs to compute the tangent map of the exponential mapping. The result has a unique interest and it will be used in the following chapters as well.

Theorem 2.1.1. *Let $X \in \mathfrak{g}$. Then*

$$T_X \exp = T_e(l_{\exp X}) \circ \int_0^1 e^{-s ad_x} ds$$

$$= T_e(r_{\exp}) \circ \int_0^1 e^{s ad_x} ds$$

Proof. We will show that if $X, Y \in \mathfrak{g}$ then

$$T_X \exp(Y) = T_e(l_{\exp X}) \left(\int_0^1 Ad(\exp(-sX))Y ds \right)$$

We define $F(X,Y) = (T_e(l_{\exp X}))^{-1} T_X \exp Y \in \mathfrak{g}$.

We will show that

$$df_e\left(F(X,Y)\right) = df_e\left(\int_0^1 Ad(\exp(-sX))Y ds \right)$$

for every smooth $f \in C^\infty(G)$. For the linear functional df_e we have that;

$$df_e\left(F(X,Y)\right) = \int_0^1 df_e(Ad(\exp(-sX))Y)ds$$

From the chain rule we get

$$F(X,Y) = \frac{\partial}{\partial t} |_{t=0} \exp(-X) \exp(X + tY) \in T_eG = \mathfrak{g}$$

Let $g(s,t) = \exp(-sX) \exp(s(X + tY)) \in G$, $s, t \in \mathbb{R}$.

Then,

$$F(sX, sY) = \frac{\partial}{\partial t} |_{t=0} g(s,t)$$

Hence,

$$df_e(F(sX, sY)) = \frac{\partial}{\partial t} |_{t=0} f(g(s,t))$$

and

$$\int_0^1 \frac{\partial}{\partial s} df_e\left(F(sX, sY)\right) ds = df_e(F(X,Y)) - df_e(F(0,0))$$

but $F(0,0) = 0$ so,

$$df_e F(X,Y) = \int_0^1 \frac{\partial}{\partial s} df_e\left(F(sX, sY)\right) ds$$

f is a smooth real function, hence,

$$\frac{\partial}{\partial s} df_e(F(sX, sY)) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} |_{t=0}\, f(g(s,t)) = \frac{\partial}{\partial t} |_{t=0}\, \frac{\partial}{\partial s} f(g(s,t))$$

For $s, t, u \in \mathbb{R}$ we get $g(s+u, t) = \exp(-sX) g(u,t) \exp(s(X+tY))$ so,

$$f(g(s+u,t)) = \left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)(g(u,t))$$

and

$$\frac{\partial}{\partial s} f(g(s,t)) = \frac{\partial}{\partial u} |_{u=0}\, f(g(s+u,t))$$

hence,

$$\frac{\partial}{\partial s} f(g(s,t)) = d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e \left(\frac{\partial}{\partial u} |_{u=0}\, g(u,t)\right)$$

But,

$$\frac{\partial}{\partial u} |_{u=0}\, g(u,t) = -X + (X+tY) = tY$$

so

$$\frac{\partial}{\partial s} f(g(s,t)) = d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e (tY)$$

$$= t d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e (Y)$$

Now, $d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e (Y) \in \mathbb{R}$ is smoothly dependent on t so, differentiating $td\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e (Y)$ for t at $t=0$ we get the value of

$$d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X+tY))}\right)_e (Y)$$

at $t = 0$;

$$\frac{\partial}{\partial t} |_{t=0}\, \frac{\partial}{\partial s} f(g(s,t)) = d\left(f \circ l_{\exp(-sX)} \circ r_{\exp(s(X))}\right)_e (Y)$$

$$= df_e\left(Ad(\exp(-sX))(Y)\right)$$

so,

$$\frac{\partial}{\partial s} df_e(F(sX, sY)) = df_e(Ad(\exp(-sX))Y)$$

hence,

$$df_e(F(X,Y)) = \int_0^1 df_e(Ad(\exp(-sX))Y ds) = df_e\left(\int_0^1 (Ad(\exp(-sX))Y)ds\right)$$

which proves the assertion. □

Let us observe the following;

- ⋆ ad_x: $\mathfrak{g} \to End(\mathfrak{g})$ so one may use the exponential mapping for matrices and compute; $\int_0^1 e^{sad_x} ds = \dfrac{e^{ad_x} - I}{ad_x}$ and $\int_0^1 e^{-sad_x} ds = \dfrac{I - e^{-ad_x}}{ad_x}$
- ⋆ If V is a finite dimensional vector space and $A \in End(V)$ then

$$\int_0^1 e^{sA} ds = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}$$

and if A is invertible one may write;

$$\int_0^1 e^{sA} ds = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} = A^{-1}(e^A - I)$$

- ⋆ Using the complexification of V, in other words writing $V_C = V \oplus iV$ we get $End(V_C) \simeq M_n(\mathbb{C})$ (For details c.f. Appendix B). Using Jordan normal forms for $\int_0^1 e^{sA} ds$ one may compute eigenvalues as $\dfrac{e^\lambda - 1}{\lambda}$ where λ is an eigenvalue of A.

Corollary 2.1.2. *The singular points of* $\exp$*:* $\mathfrak{g} \to G$*, that is, the elements* $X \in \mathfrak{g}$ *for which* $T_X \exp$ *is not invertible are exactly those for which* $ad_X \in End(\mathfrak{g}_C)$ *has eigenvalues* *of the form $2ki\pi$, $k \in \mathbb{Z} \setminus \{0\}$. Let Σ be the collection of those elements. Then*

$$\Sigma = \bigcup_{k \in \mathbb{Z}\setminus\{0\}} k\Sigma_1$$

where

$$\Sigma_1 = \{X \in \mathfrak{g} \mid \det\left((ad_X)_C - 2\pi i I\right) = 0\}$$

One may see that $\mathfrak{g}_e = \mathfrak{g} \setminus \Sigma$, so $\mathfrak{g}_e$ is the set of elements for which $\dfrac{e^{ad_x} - I}{ad_X}$ is invertible then the mapping

$$X \mapsto \frac{ad_x}{e^{ad_x} - I}$$

is a diffeomorphism

$$\mathfrak{g}_e \to End(\mathfrak{g}_e)$$

Remark 2.1.3. $\mathfrak{g}_e \times \mathfrak{g}_e$ is an open neighborhood of $(0,0)$ in $\mathfrak{g} \times \mathfrak{g}$.

Theorem 2.1.4. *The solution $Z(t)$ of the differential equation*

$$\frac{dZ}{dt}(t) = \frac{adZ(t)}{I - e^{-ad_{Z(t)}}}(Y)$$

$$Z(0) = X$$

where

$$m(X,Y) := Z(1)$$

satisfies

$$\exp(m(X,Y)) = \exp X \exp Y$$

for $X, Y \in \mathfrak{g}_e$ where $Z(t)$ is defined for all $t \in [0,1]$

Proof. We have;

$$\frac{d}{dt}\left(\exp Z(t)\right) = \left(T_{Z(t)} \exp\right) \frac{dZ}{dt}(t)$$

$$= T_e(l_{\exp Z(t)})(Y)$$

Hence $\exp Z(t)$ is an integral curve of the left invariant vector field T_eY beginning at X for which $t \mapsto \exp tY$ is also an integral curve beginning at e.

We have already seen that;

$$\exp Z(t) = \exp Z(0) \exp tY = \exp X \exp tY$$

and for $t = 1$ the assertion follows. $\square$

Definition 2.1.5. A real (respectively complex) analytic Lie group G is a group G that at the same time is a real (respectively complex) analytic manifold such that the group operations μ_G and ι_G are real (respectively complex) analytic mapping.

We expect to define the inverse of exp in an open neighborhood of 0 where it is a diffeomorphism. $m(X,Y) = Z(1)$ as defined above is the multiplication in logarithmic coordinates.

Let us consider open neighborhoods U and U_0 of 0 in $\mathfrak{g}$ and an open neighborhood V of e in G such that $\exp : U \to V$ is a diffeomorphism for all $X, Y, Z \subset U_0 (X, -Y) \in \mathfrak{g}_e^2$ $m((X, -Y), Z) \in \mathfrak{g}_e^2$ and $m(m(X, -Y), Z) \in U$

We have $T_0 \exp = I : \mathfrak{g} \to \mathfrak{g}$, $m(0,0) = 0$ and m is continuous, so from the inverse function theorem U, U_0, V exist. For all $x \in G$ we define

$$V_0^x := l_x(\exp U_0)$$

and for $y \in V_0^x$

$$\kappa^x(y) := \log(x^{-1}y)$$

where

$$\log := \exp^{-1} : V \to U$$

Theorem 2.1.6. *The collection* $\{\kappa^x : V_0^x \to U_0\}$,$x \in G$ *forms a real analytic atlas for* G, *turning* $G_{an} := (G, \{\kappa^x\})$ *into a real analytic Lie group such that the mapping* $i : G \to G_{an}$ *is a* C^2 *diffeomorphism.*

If $\mathfrak{g}$ *is a complex analytic Lie algebra, then this atlas is complex analytic, turning* G *into a complex analytic group if moreover* $Ad_x : \mathfrak{g} \to \mathfrak{g}$ *is complex linear for all* $x \in G$.

Proof. From Theorem 2.1.1 we see that $X \mapsto T_X \exp$ is C^1 for all $X \in \mathfrak{g}$, hence $\exp : \mathfrak{g} \to G$ is C^2. So $\kappa^x : V_0^x \to \mathfrak{g}$ is a C^2 diffeomorphism for all $x \in G$.

Now, if $x \neq y$,$x, y \in G$ let $V_0^x \cap V_0^y \neq \emptyset$. Then there exist $X_0, Y_0 \in U_0$ such that $x \exp X_0 = y \exp Y_0$

If $Y = \kappa^y \circ (\kappa^x)^{-1}(X)$ then either $x \exp X = y \exp Y$ or, $\exp Y = \exp Y_0 \exp(-X_0) \exp X$, meaning that $Y = m(m(Y_0, -X_0)X)$

So the atlas will be real (respectively complex) analytic.

Finally, one has;

$$x \exp X (y \exp Y)^{-1} = x \exp X \exp(-Y y^{-1})$$

$$= (xy^{-1}) y \exp \mu(X, -Y) y^{-1} = xy^{-1} \exp(Ad_y m(X, -Y))$$

so that the mapping

$$(X, Y) \mapsto \kappa^{xy^{-1}} \left((\kappa^x)^{-1}(X) \left((\kappa^y)^{-1}(Y) \right)^{-1} \right) = Ad_y(m(X, -Y))$$

is real (respectively complex) analytic. □

We will use this construction later, to prove the Analytic Subgroup Theorem.

2.2 The Backer-Campbell-Hausdorff formula

Observing that $e^{ad_{Z(t)}} = e^{tad_X} e^{ad_Y}$ we may proceed to the proof of the Backer-Campbell-Hausdorff formula.

Theorem 2.2.1. *(Backer-Campbell-Hausdorff)*

$$\log\left(e^{ad_X} e^{ad_Y}\right) =$$

$$X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \mathcal{O}(3)$$

Proof. We saw that;

$$\frac{dZ}{dt}(t) = \frac{adZ(t)}{I - e^{-adZ(t)}}(Y)$$

and

$$e^{ad_{Z(t)}} = e^{ad_X} e^{tad_Y}$$

We may write;

$$ad_{Z(t)} = \log\left(e^{ad_X} e^{tad_Y}\right)$$

and

$$\frac{dZ}{dt}(t) = \frac{\log\left(e^{ad_X} e^{tad_Y}\right)}{I - (e^{ad_X} e^{tad_Y})^{-1}}(Y)$$

then for

$$g(z) = \frac{\log z}{1 - z^{-1}}$$

we have

$$\frac{dZ}{dt}(t) = g(e^{ad_X} e^{tad_Y})(Y)$$

and from the fundamental theorem of calculus;

$$Z(1) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

Now,

$$g(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{6}(z-1)^2 + \frac{1}{12}(z-1)^3 - \ldots$$

Moreover, from the series expansion for the exponential we have;

$$e^{ad_X} e^{tad_Y} - I$$

$$= \left(I + ad_X + \frac{(ad_X)^2}{2} + \ldots\right)\left(I + tad_Y + \frac{t^2(ad_Y)^2}{2}\right) - I$$

$$= ad_X + tad_Y + tad_X ad_Y + \frac{(ad_X)^2}{2} + \frac{t^2(ad_Y)^2}{2} + \ldots$$

We compute $g(e^{ad_X} e^{tad_Y})$ for terms of degree at most 2. We get;

$$g(e^{ad_X} e^{tad_Y}) =$$

$$= I + \frac{1}{2}\left(ad_X + tad_Y + tad_X ad_Y + \frac{(ad_X)^2}{2} + \frac{t^2(ad_Y)^2}{2}\right) - \frac{1}{6}\left((ad_X)^2 + t^2(ad_Y) + tad_Y ad_X\right) + \mathcal{O}(3)$$

Hence,

$$Z(1) = \log(e^X e^Y) = X + \int_0^1 g(e^{ad_X} e^{tad_Y})(Y)dt$$

$$= X + \int_0^1 \left[Y + \frac{1}{2}[X,Y] + \frac{1}{4}[X,[X,Y]] - \frac{1}{6}[X,[X,Y]] - \frac{t}{6}[Y,[X,Y]]\right] dt$$

$$= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]]$$

□

The Analytic Subgroup Theorem

3.1 Lie subalgebras

Definition 3.1.1. A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a linear subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $\forall X, Y \in \mathfrak{h}$,

$$[X, Y] \in \mathfrak{h}$$

It follows that the restriction of the bracket in $\mathfrak{h} \times \mathfrak{h}$ turns $\mathfrak{h}$ into a Lie algebra and the identity mapping $\mathfrak{h} \to \mathfrak{g}$ into a Lie group homomorphism.

We will demonstrate every Lie subalgebra of finite dimension can be integrated to a unique a connected Lie subgroup.

Lemma 3.1.2. *Let G be a finite dimensional Lie group and H a Lie subgroup of G. Then for the Lie subalgebra $\mathfrak{h}$ of H we have:*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H\}$$

where $\exp$*:* $\mathfrak{g} \to G$*.*

Proof. Set $V = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H\}$. We will show that $\mathfrak{h} \subset V$ and $V \subset \mathfrak{h}$.

Let $X \in \mathfrak{h}$ and $i : H \hookrightarrow G$. Then $i_* := T_e i : \mathfrak{h} \to \mathfrak{g}$ is an injection, hence

$$\exp_G(tX) = i(\exp_H tX)$$

so $\forall t \in \mathbb{R} \exp_G(tX) \in i(H) = H$, hence $\mathfrak{h} \subset V$.

Conversely, let $X \in \mathfrak{g}$ and $X \notin \mathfrak{h}$ and

$$\varphi : \mathbb{R} \times \mathfrak{h} \to G$$

$$\varphi(t, Y) = \exp(tX)\exp(Y)$$

Then

$$T_{(0,0)}\varphi : \mathbb{R} \times \mathfrak{h} \to \mathfrak{g}$$

$$(\tau, Y) \mapsto \tau X + Y$$

and $X \notin \mathfrak{h}$, hence $\ker\left(T_{(0,0)}\varphi\right) = \{0\}$

From the Immersion Theorem A.2.5 there exists $\varepsilon > 0$ and an open neighborhood Ω of 0 in $\mathfrak{h}$ such that $\varphi\mid_{[-\varepsilon,\varepsilon]\times\Omega}$ is an injection.

We may pick Ω such that $\exp_H(\Omega)$ is diffeomorphic to an open neighborhood $U \subseteq H$ of e.

The mapping

$$m : H \times H \to H$$

$$(x, y) \mapsto x^{-1}y$$

is continuous and $m(e, e) = e$, hence there exists an open neighborhood $U_0 \subseteq H$ of e such that $m(U_0 \times U_0) \subset U \Leftrightarrow U_0^{-1}U_0 \subset U$

Now, H is a countable union of compact sets (c.f. Appendix A.2) so there exist $h_j \in H$, $j \in \mathbb{N}$ so that the family $\{h_jU_0\}$ is an open cover of H. For every $j \in \mathbb{N}$ define

$$T_j = \{t \in \mathbb{R} \mid \exp tX \in h_jU_0\}$$

Then for $i_0 \in \mathbb{N}$ and for $s, t \in T_{i_0}$ and $|s-t| < \varepsilon$ we get

$$\exp\left[(t-s)X\right] = \exp(-sX)\exp(tX) \in U_0^{-1}U_0 \subset U$$

hence $\exists! Y \in \Omega$ such that $\exp\left[(t-s)X\right] = \exp Y$ and $\varphi(t-s, 0) = \varphi(0, Y)$. But $\varphi\mid_{[-\varepsilon,\varepsilon]\times\Omega}$ is an injection, so $t = s$ and $Y = 0$. Hence for $s, t \in T_{i_0}$,$s \neq t$ we have $|s-t| \geq \varepsilon$. Then T_{i_0} is countable and i_0 was arbitrary, so

$$\underset{j\in\mathbb{N}}{\cup} T_j$$

is countable, hence,

$$\underset{j\in\mathbb{N}}{\cup} T_j \subset \mathbb{R}$$

so there exists $t_0 \in \mathbb{R}$ such that $t_0 \notin T_j \forall j \in \mathbb{N}$.

So,

$$\exp t_0 X \notin \underset{j\in\mathbb{N}}{\cup} h_j U_0$$

$$\Rightarrow$$

$$X \notin V$$

$$\Rightarrow \mathfrak{g} \setminus \mathfrak{h} \subset \mathfrak{g}\backslash V$$

$$\Rightarrow V \subset \mathfrak{h}$$

$\square$

Lemma 3.1.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra of $\mathfrak{g}$. Then there exists an open neighborhood Ω of 0 in $\mathfrak{g}$ such that $M = \exp(\mathfrak{h} \cap \Omega)$ is a* *submanifold of G and*

$$T_mM = T_e(l_m)\mathfrak{h}$$

for all $m \in M$.

Proof. We know that there exists an open neighborhood Ω of 0 in $\mathfrak{g}$ and an open neighborhood U of e in G such that $\exp|_\Omega$ is a diffeomorphism. Taking $M := \exp(\mathfrak{h} \cap \Omega)$ then M is a smooth submanifold of G and $\dim M = \dim \mathfrak{h}$.

Moreover, $\mathfrak{h}$ is closed under the Lie bracket of $\mathfrak{g}$ and the vector field $\dfrac{e^{-ad_X} - I}{ad_X}$ leaves $\mathfrak{h}$ invariant.

So, for $X \in \mathfrak{h} \cap \Omega$ and $m = \exp X$ one has

$$T_mM = T_X(\exp)\mathfrak{h}$$

$$= T_e(l_m) \circ \left(\frac{e^{-ad_X} - I}{ad_X}\right)\mathfrak{h} \subset T_e l_m \mathfrak{h}$$

On the other hand, one sees that $\dim M = \dim \mathfrak{h}$, hence,

$$T_mM = T_e(l_m)\mathfrak{h}$$

□

Proposition 3.1.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra of $\mathfrak{g}$ and $M = \exp(\mathfrak{h} \cap \Omega)$. Let K be a compact subset of M. Then there exists an open neighborhood U of 0 in $\mathfrak{g}$ so that $m\exp(\mathfrak{h} \cap U)$ is open in M for all $m \in C$. Moreover $K\exp(\mathfrak{h} \cap U)$ is an open neighborhood of K in M.*

Proof. For all $X \in \mathfrak{h}$ one may write $\Phi_X : \mathbb{R} \times G \to G$ for the flow of the left invariant vector field u_X. Then for all $X \in \mathfrak{h}$,$t \in \mathbb{R}, x \in G$ one gets $\Phi_X(t, x) = x \exp tX$.

For $M = \exp(\mathfrak{h} \cap \Omega)$, the left invariant vector field u_X, $X \in \mathfrak{h}$ is tangent everywhere at

M so $u_{X|M}$ is a vector field of M.

For all $X \in \mathfrak{h}$ and $m \in M$ we write $t \mapsto \varphi(t, m)$ for the maximal integral curve of $u_{X|M}$ in M beginning at m. Let D be an open neighborhood of $\mathfrak{h} \times \{0\} \times M$ in $\mathfrak{h} \times \mathbb{R} \times M$. Then the mapping

$$D \longrightarrow M$$

$$(X, t, m) \mapsto \varphi_X(t, m)$$

depends smoothly on its parameters so it is smooth in D and $t \mapsto \varphi_X(t, m)$ is an integral curve for u_X in G beginning at m. From the uniqueness of integral curves one gets $\forall\, (X, t, m) \in D$

$$\varphi_X(t, m) = \Phi_X(t, m)$$

hence $\forall\, (X, t, m) \in D$

$$\Phi_X(t, m) \in M$$

Now, let K be a compact subset of M. One has that $\Phi_{sX}(t, m) = \Phi_X(st, m)$ and K is compact, so there exists an open neighborhood U_0 of 0 in $\mathfrak{h}$ such that $\forall X \in U_0, t \in [0, 1]\,, m \in C$

$$m \exp(tX) = \Phi_X(t, m) \in M$$

We may find an open neighborhood U of 0 in $\mathfrak{g}$, small enough so that $\mathfrak{h} \cap U \subseteq U_0$ and $\exp|_U$ is a diffeomorphism.

Then, for all $m \in K$ the mapping

$$\sigma : \mathfrak{h} \cap U \to M$$

$$X \mapsto m \exp X$$

is an injection and an immersion.

But, $\dim M = \dim \mathfrak{h}$, hence σ is a diffeomorphism over some open subset of M. So, $m \exp(\mathfrak{h} \cap U)$ is an open subset of M for all $m \in K$.

Finally, the compactness of K implies that;

$$K \exp(\mathfrak{h} \cap U) = \bigcup_{m \in K_1} m \exp(\mathfrak{h} \cap U)$$

where K_1 is a countable subset of K, end every element of the union is open, from which follows the last assertion. □

Corollary 3.1.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra of $\mathfrak{g}$ and $M = \exp(\mathfrak{h} \cap \Omega)$. Then for all $x_1, x_2 \in G$, the set $x_1 M \cap x_2 M$ is open in $x_1 M$ and $x_2 M$.*

3.2 Analytic Subgroup Theorem

Theorem 3.2.1. *(analytic subgroup theorem) Let G be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ then the subgroup $\langle \exp \mathfrak{h} \rangle$ generated by $\exp \mathfrak{h}$ admits a unique Lie subgroup stuture. Moreover the mapping*

$$\mathfrak{h} \mapsto \langle \exp \mathfrak{h} \rangle$$

is a bijection between the Lie subalgebras of $\mathfrak{g}$ and the connected Lie subgroups of G.

Proof. Let $\mathfrak{h}$ be the group generated by $\exp \mathfrak{h}$. First, we will induce H with a manifold structure and then proove that H with this structure is a Lie group.

Fix Ω and M as in Lemma 3.1.3. Then $\exp|_\Omega$ is a bijection and hence $\Omega_0 := \Omega \cap \mathfrak{h}$ is diffeomorphic to the submanifold M of G through $\exp|_{\Omega_0}$ with inverse the diffeomorphism $s : M \to \Omega_0$.

$M \subset H$ hence the family of submanifolds of G

$$\mathcal{A} = \{hM \mid h \in H\}$$

is a cover of H. We will induce H with the smallest topology for which $hH \hookrightarrow M$ is continious $\forall h \in H$.

As we saw from Corollary 3.1.5 every member of $\mathcal{A}$ is open in H. Let

$$\mathcal{O} = \{F \mid F \subseteq G \text{ and } F \text{ open in } G\}$$

be the family of open subsets of G.

Then, $\forall F \in \mathcal{O}, h \in H$, $F \cap hM$ is open in hM. Hence, $F \cap H$ is open in H and $H \hookrightarrow G$ is continuous. G is Hausdorff so H with the open topology will be also Hausdorff and for all $h \in H$ the mapping

$$hM \to \Omega_0$$

$$s_h = s \circ l_h^{-1}$$

is a diffeomorphism. Hence $\{s_h \mid h \in H\}$ forms an Atlas for H.

Fix a compact neighborhood K_0 of 0 in $\Omega \cap \mathfrak{h}$. Then $K = \exp K_0$ is a compact neighborhood of e in M. Hence, K is compact in H and

$$\mathfrak{h} = \underset{n \in \mathbb{N}}{\cup} nK_0$$

so,

$$\exp \mathfrak{h} = \underset{n \in \mathbb{N}}{\cup} \{k^n \mid k \in K\}$$

One sees that,

$$H = \underset{n \in \mathbb{N}}{\cup} K^n$$

and for all $n \in \mathbb{N}$, K^n is a Cartesian product of compact sets, hence compact. It follows that the manifold H is a countable union of compact sets, so its topology has a countable basis.

Now, we will prove that H induced with the manifold structure we found above is a Lie group.

From the way we constructed the Atlas for H we get that $l_h : H \to H$ is a diffeomorphism for $h \in H$.

For $X \in \mathfrak{h}$, the linear endomorphism

$$Ad_{\exp X} : \mathfrak{g} \to \mathfrak{g}$$

$$X \mapsto e^{ad_X}$$

leaves $\mathfrak{h}$ invariant and H is generated from elements of the form $\exp X$, $X \in \mathfrak{h}$ so for all $h \in H$ $Ad(H)$ leaves $\mathfrak{h}$ invariant.

Fix an $h \in H$. Then there exists an open $F \subseteq \Omega \subset \mathfrak{g}$ with $0 \in F$ such that

$$Ad_{h^{-1}}(F) \subset \Omega$$

$$\Rightarrow Ad_{h^{-1}}(\mathfrak{h} \cap F) \subset \mathfrak{h} \cap$$

Moreover,

$$\exp Xh = h \exp Ad_{h^{-1}} X$$

so in exp($\mathfrak{h}$∩F)

$$s_h \circ r_h = Ad_{h^{-1}} \circ s_e$$

Hence $r_h : \exp(\mathfrak{h} \cap F) \to M$ is smooth, and $r_h : H \to H$ is smooth at e. Through left translation we may extend it in a smooth mapping defined in H. Moreover r_h is a bijection with inverse $r_{h^{-1}}$ hence a diffeomorphism.

We will show that the operations of multiplication

$$\mu_H : H \times H \to H$$

$$(h, h') \mapsto hh'$$

and inversion

$$\iota_H : H \to H$$

$$h \mapsto h^{-1}$$

are smooth.

For $h, h_1, h_2 \in H$ we get

$$\mu_H \circ (l_{h_1} \times r_{h_2}) = l_{h_1} r_{h_2} \circ \mu_H$$

and

$$\iota_H \circ l_h = r_{h^{-1}} \circ \iota_H$$

hence, since l_{h_1} and r_{h_2} are smooth it suffices to show that μ_H, ι_H are smooth in (e, e).

There exists an open neighborhood N_e of e in M such that $\overline{N_e}$ is a compact subset of M. Then by Lemma 3.1.3 we find open neighborhood U of 0 in $\mathfrak{g}$ such that $N_e \exp(\mathfrak{h} \cap U) \subset M$. Replacing U with $U \cap \Omega$ we get that $N_0 := \exp(\mathfrak{h} \cap U)$ is an open neighborhood of e in M and $N_e N_0 \subset M$, hence for $\mu_G : G \times G \to G$ we have $\mu_G(N_e \times N_0) \subset M$ and

$$\mu_G \mid_{N_e \times N_0} = \mu_H \mid_{N_e \times N_0}$$

maps smoothly $N_e \times N_0$ onto the submanifold M of G. Hence, μ_H is smooth in an open neighborhood of $(e, e) \in H \times H$.

Finaly, $\Omega_1 := \Omega \cap (-\Omega) \subset \mathfrak{g}$ is an open neighborhood of 0 and for $N_1 := \exp(\Omega_1 \cap \mathfrak{h})$, $\iota_G(N_1) = N_1$, $e \subset N_1$, N_1 is open in M. But,

$$\iota_G \mid_{N_1} = \iota_H \mid_{N_1}$$

so, ι_H is smooth in a neighborhood of $e \in H$.

Hence H is a Lie subgroup. $\square$

Example 3.2.2. Let $\mathfrak{g}$ be finite dimensional Lie algebra. We saw that $ad : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ is a group homomorphism. $L(\mathfrak{g}, \mathfrak{g})$ is the Lie algebra of $GL(\mathfrak{g})$ and $ad\mathfrak{g}$ is a subalgebra of $L(\mathfrak{g}, \mathfrak{g})$. From Theorem 3.2.1 we get that the subgroup $GL(\mathfrak{g})$ generated by e^{ad_X}, $X \in \mathfrak{g}$ is the unique connected Lie subgroup of $GL(\mathfrak{g})$ with Lie algebra $ad\mathfrak{g}$. This is the adjoined group $Ad\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. We saw that $Ad(\exp X) = e^{ad_X}$ and moreover

$$\left[e^{ad_X}(Y), e^{ad_X}(Z)\right] = e^{ad_X}\,[Y, Z]$$

$\forall X, Y, Z \in \mathfrak{g}$.

It follows that $Ad(\exp X) = e^{ad_X}$ is in the automorphisms group of $\mathfrak{g}$, hence $Ad\mathfrak{g}$ is a subgroup of $Aut\mathfrak{g}$.

Moreover, if $\Phi \in Aut(\mathfrak{g})$ and $X_1, \ldots, X_k$ is a basis of $\mathfrak{g}$ then

$$\Phi\left([X_i, X_j]\right) = [\Phi(X_i), \Phi(X_j)]$$

hence $Aut\mathfrak{g}$ is an analytic submanifold of $GL(\mathfrak{g})$ hence a closed subgroup.

The Lie algebra of $Aut\mathfrak{g}$ is

$$(Aut\mathfrak{g})^{alg} = Der\mathfrak{g}$$

$$= \{\varphi \in L(\mathfrak{g}, \mathfrak{g}) \mid \varphi\left([X, Y]\right) = \{\varphi(X), Y] + [X, \varphi(Y)], \forall X, Y \in \mathfrak{g}\}$$

$(Aut\mathfrak{g})^{alg} = Der\mathfrak{g}$ is the Lie subalgebra of $L(\mathfrak{g},\mathfrak{g})$ that consists of the derivations (or the infinitesimal automorphisms) of $\mathfrak{g}$ and $ad\mathfrak{g} \subset Der\mathfrak{g}$.

In general, $Ad\mathfrak{g}$ is not necessarily closed in $Aut\mathfrak{g}$ (so neither in $GL(\mathfrak{g})$) and $Aut\mathfrak{g}$ is not necessarily connected.

Finally, if G is a Lie group with Lie algebra $\mathfrak{g}$ we already saw that $Ad : G \to GL(\mathfrak{g})$ is a Lie group homomorphism and $T_e Ad = ad$ hence Ad maps G° homeomorphically to $Ad\mathfrak{g}$ such that

$$Ad(G^\circ) = Ad\mathfrak{g}$$

3.3 Commutative Lie Groups

Theorem 3.3.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is commutative if and only if G° is commutative.*

Moreover, if G is connected then $\mathfrak{g}$ is commutative if and only if G is comutative.

Proof. Let $\mathfrak{g}$ be a commutative Lie algebra. Then $\forall X, Y \in \mathfrak{g}$ $[X,Y] = 0$ and

$$\exp X \exp Y = \exp Y \exp X$$

From the characterization of G°, G° will be commutative as well.

Conversely, let us assume that G° is commutative. Let $x \in G^\circ$ then $Ad_x = I$ and $e^{ad_{tX}} = Ad(\exp tX) = I$. Hence,

$$\frac{d}{dt}\,|_{t=0}\, e^{ad_{tX}} = 0$$

$$\Leftrightarrow ad_X \equiv 0$$

$\forall X \in \mathfrak{g}$ so,

$$[X,Y] = 0$$

$\forall X, Y \in \mathfrak{g}$, hence $\mathfrak{g}$ is a commutative Lie algebra.

Finally, we saw that if G is commutative then $G^\circ = G$, from which follows the last assertion. $\square$

Definition 3.3.2. (Discrete Subgroup) Let G be a Lie group and H a Lie subgroup of G. Then H is discrete if and only if is discrete as a topological space. Equivalently, if $\forall h \in H$ there exists an open neighborhood U of G such that $U \cap H = \{h\}$.

Proposition 3.3.3. *Let G be a Lie group and H a subgroup of G. The following are equivalent:*

1. There exists an open neighborhood U of e in G such that $U \cap H = \{e\}$
2. H is discrete
3. For all compact $K \subseteq G$ the intersection $H \cap K$ is finite
4. H is a closed Lie subgroup with Lie algebra $\{0\}$

Proof. (1) $\Rightarrow$ (2) Let $h \in H$. Then $U_h = hU$ is an open neighborhood of h in G and $U_h \cap H = hU \cap H = h(H \cap h^{-1}H) = h(U \cap h) = \{h\}$

(2) $\Rightarrow$ (3) First, we will show that H is closed in G. Let U be an open neighborhood of e in G such that $U \cap H = \{e\}$ and $g \in \overline{H}$. We want to show that $g \in H$. We may find a sequence $\{h_j\}$ of elements of H such that $h_j \to g$. Then $h_{j+1}h_j^{-1} \to gg^{-1} = e$. So there must exist $n_0 \in \mathbb{N}$ such that for all $j \geq n_0$, $h_{j+1}h_j^{-1} \in U \cap H = \{e\} \Rightarrow h_j = h_{j+1}$ hence $\{h_j\}$ is constant after some index and $g \in H$ so H is closed.

Now, let K be a compact subset of G. Then $K \cap H$ is closed in K with the subspace topology, so it is compact.

For $h \in H$ we pick an open subset U_h of G such that $U_h \cap H = \{h\}$. Then the family $\{U_h \mid h \in H \cap K\}$ is an open cover of $H \cap K$ that has no proper subcover and $H \cap K$ is compact, so the cover is finite. The assertion follows.

(3) $\Rightarrow$ (4) Fix a $g \in \overline{H}$. Then there exists a compact neighborhood K of g and $g \in \overline{H \cap K} = H \cap K$ since $H \cap K$ is finite, hence closed. So $g \in H$, and H is closed. It follows that H is a closed subgroup of G with Lie algebra $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(\mathbb{R}X) \subset H\}$. The mapping $\exp : \mathfrak{g} \to G$ is a local diffeomorphism at 0 so there exists open neighborhood Ω of 0 in $\mathfrak{g}$ such that $\exp|_\Omega$ is a bijection. Let $X \in \mathfrak{g} \setminus \{0\}$. Then there exists $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] X \subset \Omega$. Then $c : [-\varepsilon, \varepsilon] \to G$ $t \mapsto \exp tX$ has a compact image and $c([-\varepsilon, \varepsilon]) \cap H$ is finite. Hence, $\{t \in [-\varepsilon, \varepsilon] \mid \exp tX \in H\}$ is finite and $X \notin \mathfrak{h}$. It follows that $\mathfrak{h} = \{0\}$.

(4) $\Rightarrow$ (1) H is a closed submanifold of G of zero dimension and the assertion follows. $\square$

Lemma 3.3.4. *Let V be a finite dimensional vector space and let Γ be a discrete subspace of V. Then there exist linearly independent elements of V $v_1, \ldots, v_p$ such that*

$$\Gamma = \mathbb{Z}_{v_1} \oplus U \ldots \oplus \mathbb{Z}_{v_p}$$

Proof. The proof is by induction in the dimension of V.

For $\dim V = 1$, we may pick a basis of V in order to identify it with $\mathbb{R}$ and Γ is a discrete subgroup of $\mathbb{R}$. Let $a \in \Gamma \setminus \{0\}$ and $a > 0$. Then the set $[0, a] \cap \Gamma$ is closed in $\mathbb{R}$, so it will have a least element v. We claim that $\Gamma = \mathbb{Z}_v = \{nv \mid n \in \mathbb{Z}\}$. Indeed, Γ is a subspace, so $\Gamma \cap (0, 1)\, v = \emptyset$, hence $\mathbb{Z}_v \subseteq \Gamma$.

Let $\Gamma \nsubseteq \mathbb{Z}_v$ then there exists $g \in \Gamma$ where $g \notin \mathbb{Z}_v$, so that $g \in (m, m+1)\, v$ for some $m \in \mathbb{Z}$, contradiction. Hence $\Gamma = \mathbb{Z}_v = \{nv \mid n \in \mathbb{Z}\}$.

Now, let $\dim V > 1$ and that the assertion holds for every F with $\dim F < \dim V$. We pick an element $v \in \Gamma \setminus \{0\}$. Then the intersection $\mathbb{R}_v \cap \Gamma$ where $\mathbb{R}_v = \{vx \mid x \in \mathbb{R}\}$ is a discrete subset of $\mathbb{R}_v$ hence, it will be of the form $\mathbb{Z}_{v_1}$. We may find a linear subspace W of V such that $\mathbb{R}_{v_1} \oplus W = V$ where $p : V \to W$ is the canonical projection.

Now, if K is a compact subset of W then $p(\mathit{\Gamma}) \cap K$ is finite. So $p(\Gamma)$ is a discrete subspace of W. $\square$

Theorem 3.3.5. *Let G be a connected commutative Lie group. Thene there exist $p, q \in \mathbb{N}$ such that $G \simeq (\mathbb{R}/\mathbb{Z})^p \times \mathbb{R}^q$. Moreover $p + q = \dim \mathfrak{g}$, $p = \dim \ker(\exp)$*

Proof. G is connected and commutative , so its Lie algebra $\mathfrak{g}$ is commutative as well. Hence $[X, Y] = 0 \; \forall X, Y \in \mathfrak{g}$. For

$$\exp : \mathfrak{g} \to G$$

we get

$$\exp(X + Y) = \exp X \exp Y$$

so exp is a Lie group homomorphism $(\mathfrak{g}, +) \to G$ and its image is a subgroup of G and $\exp \mathfrak{g} = G^\circ$.

But G is connected, so exp is a surjection.

Let $\Gamma = \ker(\exp)$. Then

$$G \simeq \frac{\mathfrak{g}}{\Gamma}$$

and since exp is a local diffeomorphism there exists neighborhood Ω of 0 in $\mathfrak{g}$ with

$$\Omega \cap \ker(\exp) = \{0\}$$

such that Γ is a discrete subgroup of $\mathfrak{g}$. Hence,

$$\Gamma = \mathbb{Z}_{v_1} \oplus \ldots \oplus \mathbb{Z}_{v_p}$$

for some $v_1, \ldots, v_p$ linerly independent elements of $\mathfrak{g}$.

Consider the basis $v_1, \ldots, v_n$ of $\mathfrak{g}$ with $n = \dim \mathfrak{g} = p + q$ and isomorphism

$$f : \mathfrak{g} \to \mathbb{R}^p \times \mathbb{R}^q$$

Let

$$E : \mathbb{R}^n \to G$$

with

$$E = \exp \circ f^{-1}$$

Then E is a surjective homomorphism of Lie groups and $\ker E = f(\Gamma) = \mathbb{Z}^p \times \{0\}$. Taking the canonical projection $\pi : \mathbb{R}^n \to (\mathbb{R}/\mathbb{Z})^p \times \mathbb{R}^q$ we get that the mapping

$$\widetilde{E} := E \circ \pi^{-1}$$

$$\widetilde{E} : (\mathbb{R}/\mathbb{Z})^p \times \mathbb{R}^q \to G$$

is a diffeomorphism and a bijection, so a Lie group isomorphism.

$\square$

Corollary 3.3.6. *If* $\ker(\exp) = \{0\}$ *or if* $\ker(\exp)$ *is a discrete subgroup of* G *then* G *is isomorphic to a finite dimensional vector space over* $\mathbb{R}$.

Example 3.3.7. We saw that $Ad_{\exp X} = e^{ad_X}$ and that $Ad(G^\circ) = Ad\mathfrak{g}$.

So for $x \in \ker Ad$ we have $xyx^{-1} = y\ \forall y \in \exp \mathfrak{g}$.

But, G° is generated by $\exp \mathfrak{g}$ so that $xyx^{-1} = x\ \forall y \in G^\circ$. Moreover

$$\ker Ad \cap G^\circ = Z(G^\circ)$$

and $Z(G^\circ)$ is a closed Lie subgroup of G°.

Hence $Ad : G^\circ \to Ad\mathfrak{g}$ induces Lie group isomorphism

$$\frac{G^\circ}{Z(G^\circ)} \simeq Ad\mathfrak{g}$$

For more details see [Far10] [Kna02] [FH04]

Lie's Third Theorem

4.1 The path space of the Lie group G

Let M be a connected manifold and $x_0 \in M$. A path beginning at x_0 is continuous curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$.

We consider path space $P = P(x_0, M)$ of the paths in M beginning at x_0, with the topology of uniform convergence.

Definition 4.1.1. We say that the paths $\gamma, \gamma' \in P(x_0, M)$ are equivalent and we write $\gamma \sim \gamma'$ if there exists a continuous curve $[0,1] \to P(x_0, M)$ $s \mapsto \gamma_s$ such that $\gamma_0 = \gamma$, $\gamma_1 = \gamma'$ and $s \mapsto \gamma_s(1)$ is constant in $[0,1]$. In other words, if there exists a homotopy from γ to γ' with end points fixed. For details see [Hat01]

We know that the relation of homotopy with end points fixed defines an equivalence relation on $P(x_0, M)$.

We write $[\gamma]$ for the equivalence class of the path γ in P and we define

$$\widetilde{M} = \{\text{ The set of equivalence classes in the path space } P\}$$

Now, if $\gamma \sim \gamma'$ then $\gamma(1) = \gamma'(1)$ hence the mapping $\tilde{\pi} : \widetilde{M} \to M$ $[\gamma] \mapsto \gamma(1)$ is well defined and a surjection (since M is path-wise connected)

Theorem 4.1.2. *The mapping $\widetilde{\pi}{:}\widetilde{M} \to M$ is a smooth fibration and $\widetilde{M}$ admits a unique* *manifold structure. Moreover $\widetilde{M}$ is simply connected.*

Proof. We will show that $\forall x \in M$ there exists an open neighborhood $V \subseteq M$ of x and a mapping

$$s : M \to \widetilde{M}$$

such that

$$s \mid_V = \tilde{\pi}^{-1} \mid_V$$

Let $\Delta = \{(x, y) \in M \times M : x = y\}$, be the diagonal set of M and Ω be an open neighporhood of Δ.

Let $x_1 \in M$ and V be an open neighborhood of x_1 in M such that $\{x_1\} \times V \subset \Omega$. Then there exists a path $\gamma \in P(x_0, M)$ such that $\gamma(1) = x_1$.

We may find $\delta > 0$ "close" to 1, such that $\forall x \in V$ and $t \in [1 - \delta, 1]$ and $(\gamma(t), x) \in \Omega$.

For $t \in [0, 1]$ we define;

$$\gamma_x(t) = \begin{cases} \gamma_x(t) = (\gamma(t), x) = \gamma(t) & 0 \le t \le 1 - \delta \\ \\ \lambda(\gamma(t), \gamma_x(t)) = \dfrac{t - 1 + \delta}{\delta} \lambda(\gamma(t), x) & 1 - \delta \le t \le 1 \end{cases}$$

where $\lambda : \Delta \to \Theta$ is a diffeomorphism, and Θ an open neighborhood of $0_{TM} \in TM$ such that

$$\lambda(x, y) \in T_x M$$

$\forall (x, y) \in \Omega$,

$$\lambda(x, x) = 0 \in T_x M \forall x \in M$$

We consider $\sigma : V \to P(x_0, M)$ $x \mapsto \gamma_x$ and observing that

as $t \to 1$ we have $\lambda(\gamma(t), \gamma_x(t)) \to 0 \Rightarrow$ $\gamma(t) \to \gamma_x(1) = (\gamma(1), x)$ so $\gamma(1) = x$ or $\gamma_x(1) \to x$ $\forall x \in V$.

For

$$s := \pi \circ \sigma$$

$$V \to \widetilde{M}$$

where

$$\pi : P(x_0, M) \to \widetilde{M}$$

$$\gamma \mapsto [\gamma]$$

we take $s \mid_V = \tilde{\pi}^{-1} \mid_V$ hence $\tilde{\pi}$ is a fibration with discrete fibres.

Now, we have that $V \subseteq M$, and M is a manifold. Let k be coordinates in M. Then $k \circ \tilde{\pi} \mid_{s(V)}$ are coordinates for $\widetilde{M}$.

Finally, $\tilde{\pi}$ is a covering and $\widetilde{M}$ is a covering space.

Hence, $\widetilde{M}$ is simply connected. □

Definition 4.1.3. Let G be a connected Lie group. We write $P(1, G)$ for the space of paths in G beginning at 1 where 1 is the identity element of G.

Proposition 4.1.4. *$(P(1, G), \cdot)$ is a group with group operation $(\gamma \cdot \gamma')(t) = \gamma(t) \cdot \gamma'(t)$. Also,*

$$\Lambda(G) = \{\gamma \in P(1, G) \mid \gamma(1) = 1\}$$

and

$$\Lambda(G)^\circ = \{\gamma \in P(1, G) \mid \gamma \sim 1\}$$

are normal subgroups of $P(1, G)$. Moreover, $\gamma' \sim \gamma$ in $P(1, G)$ if and only if $\gamma' \in \Lambda(G)^\circ$. Finaly,

$$\widetilde{G} = \frac{P(1, G)}{\Lambda(G)^\circ}$$

Proof. It is immediate that $(P(1, G), \cdot)$ is a group.

Consider the group homomorphism

$$f : P(1, G) \to G$$

$$\gamma \mapsto \gamma(1)$$

Then $\ker f = \Lambda(G)$, hence $\Lambda(G)$ is a normal connected subgroup of G.

Consider a homotopy $s \mapsto \gamma_s$ with end points fixed of γ, γ'.

Then there exists continuous curve $s \mapsto \gamma^{-1}\gamma_s$ beginning at 1 and ending at $\gamma^{-1}\gamma'$. This proves that $\Lambda^\circ(G)$ is normal in $P(1, G)$. □

Corollary 4.1.5. *$\widetilde{G}$ is a Lie group and*

$$\widetilde{\pi} : \widetilde{G} \longrightarrow G$$

$$[\gamma] \mapsto \gamma(1)$$

is a Lie group covering. On $\ker(\widetilde{\pi}) = \pi_1(G, 1)$ *the group structures coincide and* $\pi_1(G, 1)$ *is commutative.*

Lemma 4.1.6. *Let G be a connected Lie group and H a discrete normal subgroup of G. Then H lies in the centre of G, $Z(G)$.*

Proof. Let $g \in G$, $h \in H$ and $ghg^{-1} \neq h$. G is simply connected, hence pathwise connected. So there exists a path $g(t)$: $[0, 1] \to G$ beginning at 1 and ending at g.

Then $a(t) := g(t)hg(t)^{-1}$ $[0, 1] \to H$ since $H \trianglelefteq G$ with $a(0) = h$ and $a(1) = ghg^{-1}$ and $a(t) \in H$ $\forall t \in [0, 1]$. This is a contradiction since H is discrete.

The assertion follows. □

Remark 4.1.7. One may observe that a Lie group covering $\pi : G' \to G$ always arises by fixing a discrete subgroup C of the center $Z(G')$ and then taking $G = {G'}/{C}$. If $\widetilde{G}$ is a universal cover of G, then $G \simeq {\widetilde{G}}/{\pi_1(G)}$ so C arises as quotient of fundamental groups $C \simeq {\pi_1(G)}/{\pi_1(G')}$

We will now transfer the study of the path group $P(1, G) \cap C^1$ to the space $P(\mathfrak{g})$ of paths in the Lie algebra $\mathfrak{g}$ of G differentiating with respect to the time parameter t.

Provided with the supremum norm with respect to some norm in $\mathfrak{g}$, the path space $P(\mathfrak{g})$ becomes a Banach space and is called the path space of $\mathfrak{g}$.

Proposition 4.1.8. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the mapping*

$$D := D^R : \left(\gamma \mapsto (T_e r(\gamma(t)))^{-1} \frac{d\gamma}{dt}(t)\right)$$

is a homeomorphism

$$D : P(1, G) \cap C^1 \longrightarrow P(\mathfrak{g})$$

Let $A_\delta \in C^1\left([0,1], End(\mathfrak{g})\right)$ be the solution A of the differential equation

$$\frac{dA}{dt}(t) = ad\delta(t) \circ A(t)$$

with initial condition

$$A(0) = I : \mathfrak{g} \to \mathfrak{g}$$

Then for every $\gamma, \gamma' \in P(1, G) \cap C^1$ and $t \in [0,1]$:

$$D(\gamma \cdot \gamma')(t) = D\gamma(t) + Ad\gamma(t)\left(D\gamma'(t)\right)$$

where

$$Ad\gamma(t) = A_{D_\gamma(t)}$$

Finally,

$$D(\Lambda(G)^\circ \cap C^1) = P(\mathfrak{g})_0$$

where

$$P(\mathfrak{g})_0 = \left\{ \delta \in P(\mathfrak{g}) \mid \exists smooth\ s \mapsto \delta_s : [0,1] \to P(\mathfrak{g})\ where\ \delta_0 = 0, \delta_1 = \delta\ and\ \int_0^1 A_{\delta_s}(t)^{-1} \frac{\partial}{\partial s} \delta_s(t) dt = 0 \right\}$$

Lemma 4.1.9. *For*

$$T_e l(\gamma_u(t))^{-1} \frac{\partial}{\partial u} \gamma_u(t) = \int_0^t Ad_{\gamma_u}(s)^{-1} \frac{\partial}{\partial u} D_{\gamma_u}(s) ds$$

hence,

$$\frac{\partial}{\partial u} \gamma_u(1) = T_e l(\gamma_u(1)) \cdot \int_0^1 Ad_{\gamma_u}(s)^{-1} \frac{\partial}{\partial u} D_{\gamma_u}(s) ds$$

Proof. Consider the curve

$$u \mapsto \gamma_u$$

$$[0,1] \to P(1,G) \cap C^1$$

with $\gamma_u(t) = \gamma(ut)$. Then there exists (c.f. the proof of Proposition 4.1.8) unique δ: $[0,1] \to \mathfrak{g}$ such that

$$\delta(t) = \left(T_e r_{\gamma(t)}\right)^{-1} \frac{d\gamma}{dt} \Rightarrow$$

$$\left(T_e r_{\gamma(t)}\right) (\delta(t)) = \frac{d\gamma}{dt} \Rightarrow$$

$$\frac{d\gamma}{dt} = T_e r_{\gamma(t)}(\delta(t)) = T_e l_{\gamma(t)} Ad(\gamma(t))^{-1} \delta(t) =$$

$$= T_e l_{\gamma(t)} Ad(\gamma(t))^{-1} D_\gamma(t)$$

Hence for γ_u one may write;

$$\frac{\partial}{\partial u} \gamma_u(t) = T_e l_{\gamma_u(t)} Ad\gamma_u(t)^{-1} D_{\gamma_\theta}(t) \Rightarrow$$

$$\left(T_e l_{\gamma_u(t)}\right)^{-1} \frac{\partial}{\partial u}\gamma_u(t) = Ad\gamma_u(t)^{-1} D_{\gamma_u}(t)$$

Let

$$g(u,t) = \left(T_e l_{\gamma_u(t)}\right)^{-1} \frac{\partial}{\partial u}\gamma_u(t)$$

then

$$g(u,t) = g_u(t) = Ad\gamma_u(t)^{-1} D_{\gamma_u}(t)$$

and

$$\frac{\partial}{\partial t} g_u(t) = \frac{\partial}{\partial t}(Ad_{\gamma_u}(t)^{-1} D_{\gamma_u}(t))$$

From Proposition 4.1.8 we get:

$$Ad\gamma_u(t) = A_{D_{\gamma_u}(t)}$$

hence,

$$\frac{\partial}{\partial t} g_u(t) = \frac{\partial}{\partial t} A_{D_{\gamma_u}(t)^{-1}} D_{\gamma_u}(t) =$$

$$= ad_{D_{\gamma_u}(t)^{-1}} A_{D_{\gamma_u}(t)} D_{\gamma_u(t)} + Ad\gamma_u(t)^{-1} \frac{\partial}{\partial t} D_{\gamma_u}(t)$$

$$= \left[D_{\gamma_u(t)^{-1}}, A_{D_{\gamma_u}(t)} D_{\gamma_u(t)}\right] + Ad\gamma_u(t)^{-1} \frac{\partial}{\partial t} D_{\gamma_u}(t)$$

and $\left[D_{\gamma_u(t)^{-1}}, A_{D_{\gamma_u}(t)} D_{\gamma_u(t)}\right] = \left[D_{\gamma_u(t)^{-1}}, Ad_{\gamma_u(t)} D_{\gamma_u(t)}\right] = 0$ because the flows of the vector fields are related.

Hence,

$$\frac{\partial}{\partial t} g_u(t) = Ad\gamma_u(t)^{-1} \frac{\partial}{\partial t} D_{\gamma_u}(t)$$

and because of $\gamma_u(t) = \gamma(ut)$ we get

$$\frac{\partial}{\partial t} g_u(t) = Ad\gamma_u(t)^{-1} \frac{\partial}{\partial u} D_{\gamma_u}(t)$$

But,

$$g_u(s) = \int_0^s g_u(s)ds = \int_0^1 Ad\gamma_u(s)^{-1}\frac{\partial}{\partial u}D_{\gamma_u}(s)ds$$

Hence,

$$g_u(1) = \int_0^1 Ad\gamma_u(s)^{-1}\frac{\partial}{\partial u}D_{\gamma_u}(s)ds$$

Hence,

$$T_el(\gamma_u(1))^{-1}\frac{\partial}{\partial u}\gamma_u(1) = \int_0^1 Ad_{\gamma_u}(s)^{-1}\frac{\partial}{\partial u}D_{\gamma_u}(s)ds$$

and

$$\frac{\partial}{\partial u}\gamma_u(1) = T_el(\gamma_u(1))\cdot\int_0^1 Ad_{\gamma_u}(s)^{-1}\frac{\partial}{\partial u}D_{\gamma_u}(s)ds$$

□

Remark 4.1.10. The results of Theorem 2.1.1 and Lemma 4.1.9 may be generalized from a result for curves in infinite dimensional Lie groups.[MK97].

We continue with the proof of Proposition 4.1.8:

Proof. We will show that D is a bijection;

It is sufficient to show that for every $\delta : [0,1] \mapsto \mathfrak{g}$, there exists a unique C^1 curve $\gamma : [0,1] \to G$ so that $\gamma(0) = 1$ and

$$\frac{d}{dt}\gamma(t) = (T_er_x)(\delta(t)) \tag{4.1.1}$$

So it is sufficient to show that there is a unique integral curve γ of the vector field $(T_er_x)(\delta(t))$. From the theorem of existence of integral curves (c.f. Appendix A.1) there exists an open interval $I \subset [0,1]$ so that γ is a solution of the differential equation 4.1.1, $\gamma : I \to G$ and γ is a maximal integral curve for $(T_er_x)(\delta(t))$.

Let $x \in G$. Then $a(t) := \gamma(t)x$ is also an integral curve for the vector field $(T_er_x)(\delta(t))$; Indeed, $\frac{d}{dt}a(t) = \frac{d}{dt}(\gamma(t)x) = \left(T_{\gamma(t)}r_x\right)\frac{d\gamma}{dt}(t) = \left(T_{\gamma(t)}r_x\right)(T_er_x)(\delta(t)) = \left(T_e\left(r_{\gamma(t)x}\right)\right)(\delta(t))$.

So one may extend γ in whole $[0, 1]$ and γ is maximal, so γ is unique.

It follows that D is a bijection. The continuous dependence of the integral curve from the vector field $(T_e r_x)(\delta(t))$ implies that D and D^{-1} are continuous, hence D is a homeomorphism $P(1, G) \cap C^1 \to P(\mathfrak{g})$.

For the computation of the product $\gamma \cdot \gamma'$, $D(\gamma \cdot \gamma')$ we observe that for the multiplication operation $\mu : G \times G \to G$ and $a, b \in G$ we have $(a \cdot b) = \mu(a, b) = (\mu \circ (a, b))$. Hence for $\gamma \cdot \gamma'(t) = (\mu \circ (\gamma, \gamma'))(t)$, and aplying the chain rule we get:

$$\frac{d}{dt}(\gamma(t) \cdot \gamma'(t)) = \left(T_{\gamma(t)} r_{\gamma'(t)}\right) \frac{d}{dt}\gamma(t) + \left(T_{\gamma'(t)} l_{\gamma(t)}\right) \frac{d}{dt}\gamma'(t)$$

$$= \left(T_{\gamma(t)} r_{\gamma'(t)}\right) \left(T_e r_{\gamma(t)}\right) D_{\gamma(t)} + \left(T_{\gamma'(t)} l_{\gamma(t)}\right) \left(T_e r_{\gamma'(t)}\right) D_{\gamma'(t)}$$

$$= \left(T_{\gamma(t)} r_{\gamma'(t)}\right) \left(T_e r_{\gamma(t)}\right) D_{\gamma(t)} + \left(T_{\gamma(t)} r_{\gamma'(t)}\right) \left(T_e r_{\gamma(t)}\right) Ad\gamma(t) D_{\gamma'(t)}$$

$$= \left(T_e r_{\gamma(t)\gamma'(t)}\right) \left(D_{\gamma(t)} + Ad\gamma(t) D_{\gamma'(t)}\right)$$

So that,

$$\left(T_e r_{\gamma(t)\gamma'(t)}\right)^{-1} \left(\frac{d}{dt}(\gamma(t) \cdot \gamma'(t))\right) = D_{\gamma(t)} + Ad\gamma(t) D_{\gamma'(t)}$$

$\Rightarrow$

$$D(\gamma \cdot \gamma') = D_\gamma + Ad\gamma \cdot D\gamma'$$

We will now show that $Ad\gamma(t)$ and $A_{D_\gamma}(t)$ satisfy the same differential equation.

One may write :

$$\frac{d}{dt} Ad\gamma(t) = \frac{d}{dh} |_{h=0} Ad(\gamma(t+h)) = \frac{d}{dh} |_{h=0} Ad(\gamma(t+h) \circ \gamma^{-1}(t) \circ \gamma(t))$$

$$= \frac{d}{dh} |_{h=0} Ad(\gamma(t+h) \circ \gamma(t)^{-1}) \circ Ad\gamma(t) = adD_{\gamma(t)} \circ Ad\gamma(t)$$

and $Ad(\gamma(0)) = Ad1 = I$, and the assertion follows.

Finally, from Lemma 4.1.9 we get that

$$\frac{\partial}{\partial u}\gamma_u(1) = T_e l(\gamma_u(1)) \cdot \int_0^1 Ad_{\gamma_u}(s)^{-1} \frac{\partial}{\partial u} D_{\gamma_u}(s) ds$$

Which means that the curve $u \mapsto \gamma_u(1)$ is constant if and only if

$$\int_0^1 Ad\gamma_u(s)^{-1} \frac{\partial}{\partial u} D_{\gamma_u}(s) = 0$$

But we already saw that $Ad\gamma(t) = A_{D_\gamma}(t)$, and the assertion follows. □

4.2 Lie's Third Theorem

Our goal is that given a Lie algebra $\mathfrak{g}$ to construct a simply connected Lie group that integrates $\mathfrak{g}$. From now on we will work only with the lie algebra $\mathfrak{g}$ and its path space.

Definition 4.2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. For $\delta, \delta' \in P(\mathfrak{g})$ we define the product $(\delta \cdot \delta') \in P(\mathfrak{g})$ as follows;

$$\left(\delta \cdot \delta'\right)(t) = \delta(t) + A_{\delta(t)} \cdot \delta'(t)$$

Remark 4.2.2. We defined A so that $A \in End(\mathfrak{g})$ and furthermore $A_\delta(t)$ is the integral curve of the vector field $(ad(\delta(t)))$. We know that $ad\mathfrak{g}$ is a vector field tangent to $Ad\mathfrak{g}$ hence

$$A_\delta(t) \in Ad\mathfrak{g} \subset Aut\mathfrak{g}$$

So $A_\delta(t)$ lives in the representations of $\mathfrak{g}$ and respects the Lie bracket.

Lemma 4.2.3. $A_{\delta \cdot \delta'}(t) = A_\delta(t) A_{\delta'}(t)$

Proof. Let us observe that:

$$A_\delta(t) \circ ad\delta'(t) = A_\delta(t)\left(\left[\delta'(t), \star\right]\right) =$$

$$= \left[A_\delta(t)\delta'(t), A_\delta(t)\right] = ad\left(A_\delta(t)\delta'(t)\right) \circ A_\delta(t)$$

Hence,

$$\frac{d}{dt}\left(A_\delta(t) \circ A_{\delta'}(t)\right) = ad\delta(t) \circ A_\delta(t) \circ A_{\delta'}(t) + A_\delta(t) \circ ad\delta'(t) \circ A_{\delta'}(t)$$

$$= ad\delta(t) \circ A_\delta(t) \circ A_{\delta'}(t) + ad\left(A_\delta(t)\delta'(t)\right) \circ A_\delta(t) \circ A_{\delta'}(t)$$

$$= ad(\delta \cdot \delta'(t)) \circ (A_\delta(t) \circ A_{\delta'}(t))$$

and the assertion is proved. □

Remark 4.2.4. Later on we will show that $P(\mathfrak{g}, \cdot)$ with the multiplication defined above is a Banach Lie group. This and Lemma 4.2.3 show that

$$A : P(\mathfrak{g}) \longrightarrow Aut(\mathfrak{g})$$

is a Lie group homomorphism.

Lemma 4.2.5. $\frac{d}{d\varepsilon}\, |_{\varepsilon=0}\, A_{\varepsilon Y(t)} = ad \int_0^t Y(s)ds$ *where* A *as described above and* $Y(t) \in P(\mathfrak{g})$

Proof. We know that $A_\delta(t) = Ad_{D_\gamma}(t)$. Hence

$$T_e A_\delta(t) = T_e Ad_{D_\gamma}(t)$$

$$= ad\frac{d}{dt}D_\gamma(t) = ad\int_0^t \frac{\partial}{\partial u} D_{\gamma_u}(s)ds$$

where we used Lemma 4.1.9 in the direction $\gamma_u = e$ □

Remark 4.2.6. From the definition of the Lie algebra we get that $\mathfrak{g} = T_e G$, so $\mathfrak{g}$ is the tangent space of G at the identity. Hence every element $Y(t)$ of $P(\mathfrak{g})^{alg}$ may be written as $Y(t) = \frac{\partial}{\partial u}|_{u=e} D_{\gamma_u}(t)$.

Proposition 4.2.7. *$(P(\mathfrak{g}), \cdot)$ with the multiplication as defined above is a Banach Lie group with identity element the constant path*

$\delta(t) = \underline{0}(t) \equiv 0$ and Lie algebra

$$P(\mathfrak{g})^{alg} = (P(\mathfrak{g}), [\cdot, \cdot])$$

where

$$[X, Y](t) = \frac{d}{dt}\left[\int_0^t X(s)ds, \int_0^t Y(s)ds\right]$$

Proof. We will show that $(P(\mathfrak{g}), \cdot)$ is a group with identity element the constant path $\delta(t) = \underline{0}(t) \equiv 0 \in P(\mathfrak{g})$

Associativity:

$$\left(\left(\delta \cdot \delta'\right) \cdot \delta''\right)(t) = \left(\delta \cdot \delta'\right)(t) + A_{\delta \cdot \delta'}(t)\delta''(t)$$

$$= \delta(t) + A_\delta(t)\delta'(t) + A_\delta(t)A_{\delta'}(t)\delta''(t)$$

$$= \delta(t) + A_\delta(t)\left(\delta' \cdot \delta''\right)(t) = \left(\delta \cdot (\delta' \cdot \delta'')\right)(t)$$

For the inverse we compute:

$$\delta \cdot \delta^{-1}(t) = 0$$

$$\Rightarrow (\delta^{-1})(t) = -A_\delta(t)\delta(t)$$

The mapping

$$\delta \mapsto A_\delta$$

is analytic due to the linear dependence of the left side;

$$\frac{dA}{dt}(t) = ad\delta(t) \circ A(t)$$

from δ. Hence multiplication and inverse are analytic functions. It follows that $(P(\mathfrak{g}), \cdot)$ is a Banach Lie group.

$P(\mathfrak{g})$ is a vector space, hence; $T_0 P(\mathfrak{g}) = P(\mathfrak{g})$.

It remains to compute the Lie bracket:

$$\big(C_\delta(\delta')\big) =$$

$$\big(\delta \cdot \delta' \cdot \delta^{-1}\big) = \delta(t) + A_\delta(t)\delta'(t) + A_\delta(t) \circ A_{\delta'}(t)\delta^{-1}(t)$$

$$= \delta(t) + A_\delta(t)\delta'(t) - A_\delta(t) \circ A_{\delta'}(t) \circ A_\delta(t)^{-1}\delta(t)$$

Differentiating the above relation for δ' at $\delta' = 0$ in the direction of $Y \in P(\mathfrak{g})^{alg}$ and using Lemma 4.2.5 we get;

$$Ad_\delta Y(t) = T_0(C_\delta(Y(t))) =$$

$$= A_\delta(t)(Y(t)) - A_\delta(t) \circ ad \int_0^t Y(s)ds \circ A_\delta(t)^{-1} \circ \delta(t)$$

Differentiating the above relation for δ at $\delta = 0$ in the direction of $X \in P(\mathfrak{g})^{alg}$ we get;

$$[X, Y](t) = adX(t)(Y(t)) =$$

$$ad \int_0^t X(s)dsY(t) - ad \int_0^t Y(s)dsX(t) = \left[\int_0^t X(s)ds, Y(t)\right] - \left[\int_0^t Y(s)ds, X(t)\right]$$

$$= \left[\int_0^t X(s)ds, Y(t)\right] + \left[X(t), \int_0^t Y(s)ds\right]$$

$$= \frac{d}{dt}\left[\int_0^t X(s)ds, \int_0^t Y(s)ds\right]$$

For the last equality we used the Leibnitz rule. □

Remark 4.2.8. For all $\delta \in P(\mathfrak{g})$ we have $T_0 l_{\delta^{-1}(s)} \frac{d}{ds}\delta_s \in P(\mathfrak{g})^{alg}$ and $T_0 l_{\delta^{-1}} X = A_{\delta^{-1}} X$

Proposition 4.2.9. *The maping*

$$av : P(\mathfrak{g})^{alg} \to \mathfrak{g}$$

$$X \mapsto \int_0^1 X(t)dt$$

is a surjective Lie algebra homomorphism.

Proof. From the way that we constructed the Lie bracket of $P(\mathfrak{g})^{alg}$ we get

$$av\left([X, Y](t)\right) = [av(X(t)), av(Y(t))]$$

hence av is a Lie algebra homomorphism $P(\mathfrak{g})^{alg} \to \mathfrak{g}$ and a surjection and

$$kerav = \{X \in P(\mathfrak{g})^{alg} \mid \int_0^t X(t)dt = 0\}$$

Hence $P(\mathfrak{g})_0^{alg} := \left\{X \in P(\mathfrak{g})^{alg} \mid \int_0^t X(t)dt = 0\right\}$ is a Lie subalgebra of $P(\mathfrak{g})$. □

$P(\mathfrak{g})^{alg}$ is an infinite dimensional Lie group, hence we cannot aply the analytic subgroup theorem. If there where a subgroup P_0 of $P(\mathfrak{g})$ with Lie algebra $P(\mathfrak{g})_0^{alg}$ then, according to Remark 4.2.8, we may describe P_0 through a homotopy relation as follows; P_0 consists exactly from the $\delta \in P(\mathfrak{g})$ for which there exists a smooth curve $s \mapsto \delta_s$ where $\delta_0 = 0$, $\delta_1 = \delta$ and

$$T_0 l_{\delta_s^{-1}} \frac{d}{ds} \delta_s \in P(\mathfrak{g})_0^{alg}$$

Hence P_0 coincides with $P(\mathfrak{g})_0$, the image of the loop group of G through D.

We will see below that $P(\mathfrak{g})_0$ is a Lie subgroup of $P(\mathfrak{g})$

Corollary 4.2.10. *The map*

$$av : P(\mathfrak{g})^{alg} \to \mathfrak{g}$$

$$X \mapsto \int_0^1 X(t)dt$$

induces a Lie algebra isomorphism

$$\frac{P(\mathfrak{g})^{alg}}{P(\mathfrak{g})_0^{alg}} \simeq \mathfrak{g}$$

We expect that $\widetilde{G}$ will arise as an isomorphism of quotients $\frac{P(\mathfrak{g})}{P(\mathfrak{g})_0}$. But first we have to show that $P(\mathfrak{g})_0$ is a closed normal subgroup of $P(\mathfrak{g})$.

In a natural way we will search for normal Lie subgroups of $P(\mathfrak{g})$ containing $P(\mathfrak{g})_0$ and through av we will construct a $\mathfrak{g}$-valued 1-form

Proposition 4.2.11.

$$P(\mathfrak{g})_1 = \{\delta \in P(\mathfrak{g}) \mid A_\delta(1) = I\}$$

is a closed normal subgroup of $P(\mathfrak{g})$ *and*

$$P(\mathfrak{g})_1^{alg} = \left\{X \in P(\mathfrak{g})^{alg} \mid av \in \mathfrak{z}\right\}$$

where $\mathfrak{z} = \{X \in \mathfrak{g} \mid adX = 0\}$ *is the centre of* $\mathfrak{g}$. *Finally,* $P(\mathfrak{g})_0 \subseteq (P(\mathfrak{g})_1)^\circ$

Proof. The mapping

$$f : P(\mathfrak{g}) \longrightarrow Ad\mathfrak{g}$$

$$\delta \mapsto A_\delta(1)$$

is a surjection and its tangent to the identity is;

$$f_* : P(\mathfrak{g})^{alg} \longrightarrow ad\mathfrak{g}$$

$$X \mapsto \frac{d}{d\varepsilon}\,|_{\varepsilon=0}\, A_{\varepsilon X(1)} = ad\int_0^t X(s)ds$$

We have that $\ker f = \{\delta \in P(\mathfrak{g}) \mid A_\delta(1) = I\} = P(\mathfrak{g})_1$ where $I : \mathfrak{g} \to \mathfrak{g}$ and from the Submersion Level Set Theorem (c.f. Appendix A.2) $P(\mathfrak{g})_1$ is a closed submanifold of $P(\mathfrak{g})$.

We have that

$$P(\mathfrak{g})_1^{alg} = \left\{X \in P(\mathfrak{g})^{alg} \mid ad\int_0^1 X(s)ds = 0\right\}$$

$$= \left\{X \in P(\mathfrak{g})^{alg} \mid \text{av} \in \mathfrak{z}\right\}$$

The last assertion follows since $P(\mathfrak{g})_0^{alg} \subseteq P(\mathfrak{g})_1^{alg}$ □

We define a 1-form ω as follows: $\omega_\delta(X) = av(T_0 l_\delta^{-1} X) = \int_0^1 A_\delta(t)^{-1} X(t)dt$. Then, because of the identification of the lie algebras to the left invariant vector fields we may write every element in $P(\mathfrak{g})$ as $X^l = T_0 l_\delta(X)$. So $\omega_\delta(X^l) = av(X) = \int_0^1 X(t)dt$ independent of the choice of δ. So we constructed a 1-form that is exactly av and we will use it to construct a group homomorphism with kernel $P(\mathfrak{g})_0$

Remark 4.2.12. (Properties ω_δ)

1. $d\omega(X,Y) + [\omega(X), \omega(Y)] = 0$

2. $\ker \omega_\delta = T_\delta\left(P(\mathfrak{g})_0\right)$ and $\ker \omega_\delta$ is a distibution

3. $\omega\mid_{P(\mathfrak{g})_1}\in\mathfrak{z}$

4. $d\omega\mid_{P(\mathfrak{g})_1}=0$

For more details on ω see [IL03] [Mic08] [KN09]

Corollary 4.2.13. *$P(\mathfrak{g})_0$ is an integral manifold for the distribution $\ker\omega_\delta$ so it is a closed submanifold of $P(\mathfrak{g})$ hence a closed subgroup.*

Proposition 4.2.14. $\dfrac{P(\mathfrak{g})}{(P(\mathfrak{g})_1)^\circ}=\widetilde{Ad}\mathfrak{g}$

Proof. The Lie algebra homomorphism $ad_{\mathfrak{g}}\colon\ \mathfrak{g}\longrightarrow\mathfrak{g}$ with $Y\mapsto[\mathfrak{g},X]$ defines an isomorphism $\dfrac{\mathfrak{g}}{\mathfrak{z}}\simeq ad\mathfrak{g}$. Moreover $ad\mathfrak{g}$ is the Lie algebra of the connected Lie group $Ad\mathfrak{g}$, as we saw in Example 3.2.2.

Using the homeomorphism D of Proposition 4.1.8 for the groups $P(ad\mathfrak{g})$ and $P(ad\mathfrak{g})_0$ we get

$$P(ad\mathfrak{g})\ \underset{Hom}{\overset{D}{\cong}}\ P(1,Ad\mathfrak{g})\cap C^1$$

and

$$P(ad\mathfrak{g})_0\ \underset{Hom}{\overset{D}{\cong}}\ \Lambda\,(Ad\mathfrak{g})^\circ\cap C^1$$

But from Proposition 4.1.4 we get

$$\frac{P(1,Ad\mathfrak{g})}{\Lambda\,(Ad\mathfrak{g})^\circ}\simeq\widetilde{Ad}\mathfrak{g}$$

Hence,

$$\frac{P(ad\mathfrak{g})}{P(ad\mathfrak{g})_0}\simeq\widetilde{Ad}\mathfrak{g}$$

Now, $\dfrac{P(\mathfrak{g})}{P(\mathfrak{z})}=P(ad\mathfrak{g})$ and $\dfrac{(P(\mathfrak{g})_1)^\circ}{P(\mathfrak{z})}=P(ad\mathfrak{g})_0$ and the assertion follows. ☐

The relations $d\omega\mid_{P(\mathfrak{g})_1}=0$ and $\ker\omega_\delta=T_\delta\left(P(\mathfrak{g})_0\right)$ lead us to construct a maping

$$\varphi:(P(\mathfrak{g})_1)^\circ\to(\mathfrak{z},+)$$

so that $\varphi(0) = 0$ and $d\varphi = \omega$ in order to prove that $P(\mathfrak{g})_0$ is a normal Lie subgroup of $P(\mathfrak{g})$. In order to do so we need to get through the obstacle of the homotopy relation through which the path space of the Lie algebra $\mathfrak{g}$ is defined.

Proposition 4.2.15. *The mapping*

$$\varphi : (P(\mathfrak{g})_1)^{\circ} \to (\mathfrak{z}, +)$$

where

$$\varphi(\alpha) \colon = \int_{[0,1]} \delta^* \omega$$

where $s \mapsto \delta_s$ *is a* C^1 *curve* $[0,1] \to (P(\mathfrak{g})_1)^{\circ}$*,* $\delta_0 = 0$ *and* $\delta_1 = \alpha$ *is well defined, is a surjective Lie group homomorphism and* $(\ker \varphi)^{alg} = P(\mathfrak{g})_0^{alg}$*.*

Proof. In order to show that φ is well defined we need to show that φ does not depent on the choise of δ.

There exists a 2-form Ω that is $\dfrac{P(\mathfrak{g})}{(P(\mathfrak{g})_1)^{\circ}} = \widetilde{Ad}\mathfrak{g}$- valued so that $d\omega = \pi^* \Omega$ where $\pi \colon P(\mathfrak{g}) \to \dfrac{P(\mathfrak{g})}{(P(\mathfrak{g})_1)^{\circ}}$ the canonical projection and π^* its pullback. π is a surjective Lie group homomorphism and π_* is a surjective Lie algebra homomorphism. Hence Ω is unique and smooth. Moreover $d\omega$ is left invariant and π is a group homomorphism hence Ω is left invariant and thus defined from its value at the identity.

We have $\pi_* \colon P(\mathfrak{g})^{alg} \longrightarrow ad\mathfrak{g}$ so

$$\pi^* \Omega(X, Y) = \Omega(\pi_*(X), \pi_*(Y)) = \Omega(adX, adY)$$

and $d\omega(X, Y) = [avX, avY]$, hence

$$\Omega_1(adX, adY) = [X, Y]$$

Moreover,

$$\pi^* d\Omega = d(\pi^*\Omega) = dd\omega = 0$$

Hence Ω defines a De Rahm cohomology class $[\Omega] \in H^2_{DR}\left(\widetilde{Ad}\mathfrak{g}, \mathbb{R}\right)$.

Consider a curve δ: $s \mapsto \delta_s$ that is piecewise C^1 with $\delta{:}[0,1] \to (P(\mathfrak{g})_1)^\circ$ so that $\delta_0 = 0 = \delta_1$ and a homotopy

$$E : [0,1] \times [0,1] \to P(\mathfrak{g})$$

$$(u, s) \mapsto u\delta_s$$

It is direct that $E(0,s) = E(u,0) = E(u,1) = 0$ and $E(1,s) = \delta_s$. But, $\delta_s \in (P(\mathfrak{g})_1)^\circ$ hence $\pi(\delta_s) \equiv 1$.

It follows that the mapping

$$A = \pi \circ E : [0,1] \times [0,1] \to \widetilde{Ad}\mathfrak{g}$$

maps the whole boundary $[0,1] \times [0,1]$ to $\{1\}$. Moreover, $\pi(\delta_s) \equiv 1 \Rightarrow \pi_* = 0$ so π has discrete fibres.

So A defines a homology class $[A] \in H_2\left(\widetilde{Ad}\mathfrak{g}, \mathbb{Z}\right)$.

From Stokes' theorem we get;

$$\int_{[0,1]} \delta^*\omega = \int_{[0,1]\times[0,1]} d(E^*\omega)$$

$$= \int_{[0,1]\times[0,1]} E^* d\omega = \int_{[0,1]\times[0,1]} E^*\pi^*\Omega$$

$$= \int_{[0,1]\times[0,1]} A^*\Omega$$

$$= \langle [A] \, , [\Omega] \rangle$$

But we know that if G is a simply connected Lie group then

$$H^2(G, \mathbb{R}) = 0$$

It follows that $\int_{[0,1]} \delta^*\omega = 0$ for every closed, piecewise C^1 curve δ on $(P(\mathfrak{g})_1)^\circ$. Hence φ is well defined.

φ is a Lie group homomorphism;

Fix a $\delta' \in (P(\mathfrak{g}))^\circ$. then the derivative of the map

$$\delta \mapsto \varphi(\delta \cdot \delta') - \varphi(\delta)$$

$$(P(G)_1)^\circ \longrightarrow (\mathfrak{z}, +)$$

is 0 and $d\varphi = \omega$ and ω is left invariant. So it is constant, and defined from its value at the identity, hence $\varphi(\delta \cdot \delta') - \varphi(\delta) = \varphi(0 \cdot \delta') - \varphi(0) = \varphi(\delta')$

Hence φ is a surjective group homomorphism with kernel $\ker\varphi \trianglelefteq (P(\mathfrak{g})_1)^\circ$ and $(\ker\varphi)^{alg} = (P(\mathfrak{g})_0)^{alg}$. So $P(\mathfrak{g})_0$ is exactly the connected component of the identity $(\ker\varphi)^\circ$ of the normal subgroup $\ker\varphi$. $\square$

Theorem 4.2.16. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists a simply connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. The restriction of the mapping $\exp\colon \mathfrak{g} \to \widetilde{G}$ in the centre $\mathfrak{z}$ of $\mathfrak{g}$ induces a group isomorphism $\exp|_{\mathfrak{z}}\colon (\mathfrak{z}, +) \longrightarrow \left(Z(\widetilde{G})\right)^\circ$*

Proof. We saw that $\dfrac{P(\mathfrak{g})^{alg}}{P(\mathfrak{g})_0^{alg}} \simeq \mathfrak{g}$ and $P(\mathfrak{g})_0$ is normal in $P(\mathfrak{g})$ and $\dfrac{P(\mathfrak{g})}{P(\mathfrak{g})_0}$ is a Banach Lie group with Lie algebra $\simeq \mathfrak{g}$. Hence there exists a Lie group with Lie algebra $\mathfrak{g}$, and from propositions 4.1.4, 4.1.8 we get

$$\frac{P(\mathfrak{g})}{P(\mathfrak{g})_0} \simeq \widetilde{G}$$

Now, $Z(\widetilde{G}) = \ker\left(Ad : \widetilde{G} \to Ad\mathfrak{g}\right)$ and

$$P_1(\mathfrak{g}) = \{\delta \in P(\mathfrak{g}) \mid A_\delta(1) = I\}$$

$$= \{\delta \in P(\mathfrak{g}) \mid Ad_{\delta_1} = I\}$$

$$= \ker\left(Ad : P(\mathfrak{g}) \to Ad\mathfrak{g}\right)$$

so,

$$\frac{\widetilde{G}}{Z(\widetilde{G})} = Ad\mathfrak{g} = \frac{P(\mathfrak{g})}{P(\mathfrak{g})_1}$$

Hence,

$$Z(\widetilde{G}) = \frac{P(\mathfrak{g})_1}{P(\mathfrak{g})_0}$$

Let p be the canonical projection

$$p : \frac{(P(\mathfrak{g})_1)^\circ}{P(\mathfrak{g})_0} \to \frac{(P(\mathfrak{g})_1)^\circ}{\ker\varphi} \simeq (\mathfrak{z}, +)$$

Then p has discrete fibres $\dfrac{\ker\varphi}{(\ker\varphi)^\circ}$. But $(\mathfrak{z}, +)$ is simply connected and commutative, hence,

$$\frac{(P(\mathfrak{g})_1)^\circ}{P(\mathfrak{g})_0} = \frac{(P(\mathfrak{g})_1)^\circ}{(\ker\varphi)^\circ}$$

We get

$$\left(Z(\widetilde{G})\right)^\circ \simeq (\mathfrak{z}, +)$$

so that $\left(Z(\widetilde{G})\right)^\circ$ is connected commutative Lie group. Hence from Theorem 3.3.5, the mapping

$$\exp : (\mathfrak{z}, +) \to \left(Z(\widetilde{G})\right)^\circ$$

is an isomorphism. □

Remark 4.2.17. The result does not necessarily hold for an infinite dimensional Banach Lie algebra $\mathfrak{g}$. We saw that if G is a commutative Lie group with finite dimensional Lie algebra $\mathfrak{g}$ then $\exp : (\mathfrak{z}, +) \to Z(G)$ has a discrete kernel. This is not the case if $\mathfrak{g}$ is an infinite dimensional Banach Lie algebra.

For example, if $\mathfrak{g} = \mathfrak{su}2$, $G = SU(2)$ then $\left(Z(\widetilde{G})\right)^\circ \simeq \dfrac{\mathfrak{z}}{\ker \exp}$ and we can prove that $\ker \exp \simeq \mathbb{R} \setminus \mathbb{Q}$ and $\mathfrak{z} = \mathbb{R}$. The space $\dfrac{\mathbb{R}}{\mathbb{R} \setminus \mathbb{Q}}$ is not even Hausdorff.

Appendix

A. Elements of differential geometry

A.1. Integral curves and Flows

Let M be a differential manifold and $X \in \mathfrak{X}(M)$ a smooth vector field in M. An integral curve of X is a smooth curve $\gamma : I \to M$ where $I \subseteq \mathbb{R}$ is an open interval such that $\gamma'(t) = X(\gamma(t))$ for all $t \in I$. Fix an $t_0 \in I$ then $\gamma(t_0)$ is the starting point of γ. If $\gamma : I \to M$ is an integral curve and $c \in \mathbb{R}$ then $s \mapsto \gamma(s-c)$ with domain $I + c := \{t + c \mid t \in I\}$ is also an integral curve. So we may assume that $0 \in I$ and pick $t_0 = 0$.

Theorem. ***A.1.1*** *Let*$X \in \mathfrak{X}(M)$ *and* $p \in M$. *Then there exists a unique open interval* $I_p \subset \mathbb{R}$ *containing 0 and unique integral curve* $a : I_p \to M$ *beggining at* $a(0) = p$ *so that if* $v : J \to M$ *is another integral curve with* $0 \in J$ *and statring point* $v(0) = p$ *then* $J \subset I_p$ *and* $v = a \mid_J$

The integral curve $a = a_p : I_p \to M$ is called maximal integral curve starting at p.

Let $\Omega = \{(t,p) \in \mathbb{R}{\times}M \mid t \in I_p\}$. We define $\Phi : \Omega \to M$ with $\Phi(t,p) = a_p(t)$, $t \in I_p$. Φ is called flow of the vector field and $t \mapsto \Phi(t,p)$ is smooth for every $p \in M$.

Theorem. ***A.1.2*** *The set* $\Omega \subset \mathbb{R} \times M$ *is open and the flow* $\Phi : \Omega \to M$ *is smooth.*

Finally, if the vector filed $X_\lambda(p)$ depends smoothly on the pair $(\lambda, p) \in \Lambda \times M$, where Λ is a differential manifold, then the flow of $\Phi_\lambda : \Omega_\lambda \to M$ depends smoothly on (t, p, λ)

where $(t,p) \in \Omega_\lambda$

A.2 Manifolds and Submanifolds

Lemma. ***A.2.1*** *Let M be a topological manifold. Then there exists a countable basis of M so that its closure is compact.*

Proof. M is a topological manifold, that is a second countable, Hausdorff topological space. Let $\mathcal{B}$ be a countable basis for M. The existence of $\mathcal{B}$ comes from the fact that M is second countable. Let $\mathcal{B}' \subset \mathcal{B}$ where $\mathcal{B} = \{B \in \mathcal{B} \mid B \in (U,\varphi), \overline{B} \text{ compact}\}$ for (U,φ) coordinate map of M. Then $\mathcal{B}'$ is a compact countable basis for M $\square$

Let M be a smooth manifold. An embedded submanifold of M is a subset $S \subseteq M$ that is a manifold with the topology of the subspace, induced with a smooth structure so that $S \hookrightarrow M$ is a smooth embedding.

Definition. **A.2.2** Let M be a manifold. If for all $p \in P$ the linear subspace $D_p \subseteq T_pM$ is of dimension k then $D = \underset{p\in M}{\cup} D_p$ is a distribution for M of rank k.

Lemma. ***A.2.3*** *Let Mbe a smooth manifold of dimension n and $D \subseteq TM$ distribution of rank k. Then D is smooth if and only if for all $p \in M$ there exists an open neighborhood U and smooth 1-forms $\omega^1, \ldots, \omega^{n-k}$ such that for all $q \in U$ $D_q = \ker \omega^1 \mid_q \cap \ldots \cap \ker \omega^{n-k} \mid_q$.*

Definition. **A.2.4** Let $D \subseteq TM$ be a smooth distribution and $N \subseteq M$ an immersed submanifold of M. Then N is an integral manifold for D if $T_pN = D_p$ for all $p \in N$.

Theorem. ***A.2.5*** *(Immersion Theorem) Every smooth manifold of dimension n admits a smooth immersion in $\mathbb{R}^{2n}$.*

Let M, N be smooth manifolds. A smooth map $F : M \to N$ is called a smooth submersion if its differential is surjective in every point.

Theorem. ***A.2.6*** *(Submersion Level Set Theorem) If M, N are smooth manifolds and $F : M \to N$ is a smooth submersion then every level set F is a properly embedded submanifold of codimension N.*

We close with the following;.

Theorem. ***A.2.7*** *(Inverse Function Theorem for Manifolds) Let M, N be smooth manifolds and $F : M \to N$ smooth. If $p \in M$ and dF_p is invertible then there exists a connected component U_0 of p and V_0 of $F(p)$ such that $F|_{U_0}: U_0 \to V_0$ is a diffeomorphism..*

For details see [KN09],[Lee12], [Mic08]

B. Elements of Linear Algebra

Definition. B.1 Let V be a finite dimensional vector space. The complexification of V, $V_{\mathbb{C}}$ is the space of all linear combinations u_1+iu_2 with $u_1, u_2 \in V$. $V_{\mathbb{C}}$ is a real vector space with the obvious way and becomes a complex vector space if we define $i(u_1 + iu_2) = -u_2 + iu_1$. V is a real linear subspace of $V_{\mathbb{C}}$.

Proposition. ***B.2*** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then the Lie bracket of $\mathfrak{g}$ extends uniquely to $\mathfrak{g}_{\mathbb{C}}$ turning it into a complex Lie algebra. $\mathfrak{g}_{\mathbb{C}}$ is called the complexification of $\mathfrak{g}$.*

Theorem. ***B.3*** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite dimensional representation π in $\mathfrak{g}$ is uniquely extended to a complex linear representation $\pi_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ with $\pi_{\mathbb{C}}(X + iY) = \pi(X) + i\pi(Y)$.*

For details see [Hal15]

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