National and Kapodistrian
University of Athens

Doctoral Thesis

Hardy Inequalities in non-Euclidean Geometries

Author:
Miltiadis Paschalis

Supervisor:
Gerassimos Barbatis

A thesis submitted in fulfillment of the requirements for the degree "Ph.D. in Mathematics"

in the

Department of Mathematics

February 13, 2023

Τριμελής Συμβουλευτική Επιτροπή

1. Γεράσιμος Μπαρμπάτης (Επιβλέπων)
2. Ιωάννης Στρατής
3. Αχιλλέας Τερτίκας

Επταμελής Εξεταστική Επιτροπή

1. Παναγιώτης Γιαννιώτης
2. Νικόλαος Καραχάλιος
3. Γεράσιμος Μπαρμπάτης (Επιβλέπων)
4. Παναγιώτης Σμυρνέλης
5. Ιωάννης Στρατής
6. Αχιλλέας Τερτίκας
7. Ευστάθιος Φίλιππας

NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS

Abstract

Department of Mathematics

Ph.D. in Mathematics

Hardy Inequalities in non-Euclidean Geometries

by Miltiadis Paschalis

The aim of this doctoral dissertation is to investigate the validity and additional properties of Hardy's well known inequality in various settings beyond the Euclidean. The dissertation consists of four chapters.

Chapter 1 offers background on Hardy inequalities, particularly so in the non-Euclidean setting.

In Chapter 2, we introduce a method of integration along integral curves to obtain Hardy inequalities for the first order differential operator X in a given manifold M with volume form ω. These inequalities have the form

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{\tau_p^p}\omega, \quad \varphi \in C_c^1(M),$$

where $1/\tau_p^p$ is a positive global potential on the manifold, dependent in general in the setup (M, X, ω) and the exponent p, while the constant is sharp. This method applies very generally and we illustrate its use in a number of examples, some of them yielding new results.

Chapter 3 is concerned with higher order Rellich inequalities related to general elliptic operators with constant coefficients, other than the classic polyharmonic operator $(-\Delta)^m$. In this case, we show that a Rellich inequality can be expressed in terms of an induced Finsler distance d_H which is given in terms of the symbol of the operator. This new type of inequality is shown to be sharp in the case where the underlying domain is a half-space and the symbol satisfies a convexity condition, while comparisons are made for the case of a convex domain, yielding results that are superior to those obtained by more crude methods, in specific situations.

Finally, in Chapter 4, we deal with the sensitivity of the Hardy constant under perturbations of the domain in the case where the distance is measured from a boundary submanifold. Specifically, we find the Hardy constant to be both continuous and differentiable (in the Gateaux sence) under such perturbations, assuming some regularity conditions on the boundary.

Περίληψη

Σκοπός της παρούσας διδακτορικής διατριβής είναι η διερεύνηση της ισχύος και των ιδιοτήτων της γνωστής ανισότητας του Hardy σε πλαίσια που υπερβαίνουν το κλασικό Ευκλείδειο πλαίσιο. Η διατριβή αποτελείται από τέσσερα κεφάλαια.

Στο Κεφάλαιο 1 δίνουμε υπόβαθρο για τις ανισότητες Hardy, ιδιαίτερα όσον αφορά το μη-Ευκλείδειο πλαίσιο.

Στο Κεφάλαιο 2 παρουσιάζουμε μια μέθοδο ολοκλήρωσης πάνω σε ολοκληρωτικές καμπύλες, που δίνει ανισότητες Hardy για ένα διαφορικό τελεστή πρώτης τάξης X σε κάποια πολλαπλότητα M με μορφή όγκου ω. Οι ανισότητες αυτές έχουν τη μορφή

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{\tau_p^p}\omega, \quad \varphi \in C_c^1(M),$$

όπου $1/\tau_p^p$ είναι ένα θετικό δυναμικό που εξαρτάται από την ολική (global) γεωμετρία της διάταξης (M, ω, X), καθώς και από τον εκθέτη p. Η μέθοδος εφαρμόζεται σ' ένα πολύ γενικό πλαίσιο, και επιδεικνύουμε τη χρήση της μέσω παραδειγμάτων, κάποια εκ των οποίων προσφέρουν νέα αποτελέσματα.

Το Κεφάλαιο 3 ασχολείται με ανισότητες Rellich ανώτερης τάξης που σχετίζονται με γενικούς ελλειπτικούς τελεστές με σταθερούς συντελεστές, διαφορετικούς από τον κλασικό πολυαρμονικό τελεστη $(-\Delta)^m$. Σε αυτή την περίπτωση, δείχνουμε ότι μια ανισότητα Rellich μπορεί να εκφραστεί μέσω μιας κατάλληλης απόστασης Finsler d_H που εξαρτάται από το σύμβολο του τελεστή. Αυτή η νέα μορφή ανισότητας αποδεικνύεται να είναι βέλτιση στην περίπτωση που το χωρίο είναι ημιχώρος και το συμβολο ικανοποιεί μια συνθήκη κυρτότητας. Γίνεται επιπλέον σύγκριση για κυρτα χωρία με άλλες πιο κλασικές μεθόδους, που δίνει καλύτερα αποτελέσματα από τα υπάρχοντα σε ορισμένες συγκεκριμένες περιπτώσεις.

Εν τέλει, στο Κεφάλαιο 4, ασχολούμαστε με την ευστάθεια της σταθεράς Hardy ως προς διαταραχές του χωρίου στην περίπτωση που η απόσταση μετράται από τμήμα (υποπολλαπλότητα) του συνόρου. Συγκεκριμένα βρίσκουμε ότι η σταθερά Hardy είναι και συνεχής και παραγωγίσιμη (κατά Gateaux) ως προς τέτοιες διαταραχές, υπό την προϋπόθεση ότι το σύνορο ικανοποιεί κάποιες συνθήκες ομαλότητας.

Acknowledgements

This research was supported by the Hellenic Foundation for Research and Innovation (HFRI) under the HFRI PhD Fellowship grant (Fellowship Number: 1250).

Special thanks are owed to my supervisor, professor G. Barbatis, for all the time he spent offering usefull suggestions, reviewing the manuscripts throughout the several stages of this dissertation, and an overall excellent collaboration.

I also feel the need to express my gratitude towards my parents, for their everlasting support through the years.

Contents

Dedicated to my parents.

Chapter 1

Background on Hardy inequalities

In this chapter we provide background on Hardy inequalities. Much of it is an adaptation from [5] and references therein, as well as the author's own graduate thesis on Riemannian Hardy Inequalities [41].

1.1 Introduction

In 1925, G. H. Hardy proved the integral inequality

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \int_0^\infty f(x)^p dx,$$

holding for non-negative functions and $1 < p < \infty$ (see [27]). Later it was shown by Landau that the constant appearing on the LHS is optimal (i.e the largest possible), and that equality can be obtained if and only if $f = 0$. If we set $\varphi(x) = \int_0^x f(t)dt$, the inequality obtains the form

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{|\varphi(x)|^p}{x^p} dx \leq \int_0^\infty |\varphi'(x)|^p dx,$$

which forms the basis of most modern generalisations, while the higher-dimensional case reads

$$\int_{\mathbb{R}^n} |\nabla\varphi(x)|^p dx \geq \left|\frac{p-n}{p}\right|^p \int_{\mathbb{R}^n} \frac{|\varphi(x)|^p}{|x|^p} dx,$$

holding for all $\varphi \in C_c^\infty(\mathbb{R}^n \setminus 0)$.

Ever since, many incarnations of Hardy's inequality have seen the light, the lot of them in the form

$$\int_\Omega |\nabla\varphi(x)|^p dx \geq C(p,\Omega,\delta) \int_\Omega \frac{|\varphi(x)|^p}{\delta(x)^p} dx, \quad \varphi \in C_c^\infty(\Omega),$$

where Ω is a domain of $\mathbb{R}^n$ with non-empty boundary, δ is an appropriate distance function (could be the distance from a point, the boundary, or part of the boundary) and $C(p,\Omega,\delta)$ is a positive constant that generally depends on all parameters. The classic Hardy optimisation problem consists of specifying the best constant $C(p,\Omega,\delta)$, given by

$$\inf_{\varphi\neq 0} \frac{\int_\Omega |\nabla\varphi(x)|^p dx}{\int_\Omega |\varphi(x)|^p \delta(x)^{-p} dx},$$

as well as finding the minimisers that achieve it (if any).

In recent years, the case $\delta = d_\Omega = \mathrm{dist}(\cdot, \partial\Omega)$ has been studied extensively. Specifically, we consider the inequality

$$\int_\Omega |\nabla\varphi(x)|^p dx \geq C(p,\Omega) \int_\Omega \frac{|\varphi(x)|^p}{d_\Omega^p(x)} dx, \quad \varphi \in C_c^\infty(\Omega).$$

In [39], Maz'ya gives an equivalent analytic condition for when a Hardy inequality holds (for a positive constant) in terms of the notion of p-capacity. The p-capacity of a compact subset $K \subset \Omega$ relative to Ω is defined to be

$$C_p(K,\Omega) = \inf\left\{ \int_\Omega |\nabla u(x)|^p dx : u \in C_c^\infty(\Omega),\ u|_K \geq 1 \right\},$$

and Maz'ya proved that the Hardy inequality holds if and only if there exists a constant $C > 0$, independent of the compact $K \subset \Omega$, such that

$$\int_K \frac{1}{d_\Omega^p(x)} dx \leq C \cdot C_p(K,\Omega).$$

In [32], Lewis proved that the Hardy inequality holds in all domains if $n < p < \infty$, and that it holds for (a class that includes) Lipschitz domains in the case $1 < p \leq n$.

The value of the best constant is known to be $(1-1/p)^p$ in the case $n = 1$. The higher-dimensional case is more complicated and the answer is usually highly dependent on the geometry and regularity of the domain. The case where the domain is convex has been studied quite extensively. In particular, Marcus, Mizel and Pinchover [34] (see also [38] for the case $p = 2$) proved that for any convex domain that is smooth in a neighbourhood of at least one of its boundary points, the best constant is again given by $(1-1/p)^p$, and no minimisers exist. The convexity condition was later relaxed to weak mean convexity by Barbatis, Fillipas and Tertikas [8], see also Lewis [33].

Regarding non-convex and more general domains, it was also established in [34] that in the case of bounded domains of C^2 boundary, the value of the best constant never exceeds the limit value $(1-1/p)^p$, and that for the particular case $p = 2$, minimisers exist if and only if the value of the best constant is strictly less than that limit value. This was generalised to arbitrary $p > 1$ by Marcus and Shafrir [37].

In 1953, Rellich [44] proved the related inequality

$$\int_{\mathbb{R}^n} |\Delta\varphi(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|\varphi|^2}{|x|^4} dx$$

for $\varphi \in C_c^\infty(\mathbb{R}^n \setminus 0)$ and $n \geq 5$. This is closely related to Hardy's inequality, and stands as a higher order analogue.

1.2 Non-Euclidean Hardy inequalities

Moving on, we consider Hardy inequalities in non-Euclidean settings. Note that results here are much more limited compared to the well studied Euclidean setting. One of the first results in a non-Euclidean setting was Carron's inequality [13], which reads

$$\int_M \rho^\alpha |\nabla\varphi|^2 dv_g \geq \left(\frac{C+\alpha-1}{2}\right)^2 \int_M \rho^{\alpha-2}|\varphi|^2 dv_g, \quad \varphi \in C_c^\infty(M \setminus \rho^{-1}(0)),$$

where (M,g) is a complete Remannian manifold, dv_g the corresponding volume element, C, α are real numbers satisfying $C+\alpha-1>0$, and ρ a distance function ($|\nabla\rho|=1$) of class C^2 such that $\Delta\rho \geq C/\rho$.

Another notable result is the one by D'Ambrossio and Dipierro [18]. It states that given a domain Ω of a Riemannian manifold (M,g) and a $\rho \in W^{1,p}_{loc}(\Omega)$ is such that $\rho \geq 0$, $\Delta_p\rho \leq 0$ in the weak sence for $p>1$, then $|\nabla\rho|/\rho \in L^p_{loc}(\Omega)$ and the L^p Hardy inequality

$$\int_\Omega |\nabla\varphi|^p dv_g \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\varphi|^p}{\rho^p}|\nabla\rho|^p dv_g, \quad \varphi \in C_c^\infty(\Omega)$$

is valid. In the same article, the authors provide sufficient geometric conditions for the validity of the inequality, such as p-parabolicity.

On the Rellich inequality front, we have the result of Kombe and Ozaydin [28] stating that the inequality

$$\int_M \rho^\alpha |\Delta\varphi|^2 dv_g \geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16} \int_M \rho^{\alpha-4}|\varphi|^2 dv_g, \quad \varphi \in C_c^\infty(\rho^{-1}(0))$$

is valid, where (M,g) is a complete Riemannian manifold of dimension ≥ 2, C, α are real numbers satisfying $\alpha < 2$, $C>0$ and $C+\alpha-3>0$, and ρ is a C^2 distance function such that $\Delta\rho \geq C/\rho$.

Another notable result is given by Barbatis [7], which states the validity of the higher-order improved Rellich inequality

$$\int_\Omega \frac{|\Delta^{m/2}\varphi|^p}{\rho^\gamma} dv_g \geq A(m,\gamma)\int_\Omega \frac{|\varphi|^p}{\rho^{\gamma+mp}} dv_g + B(m,\gamma)\sum_i \int_\Omega V_i|\varphi|^p dv_g$$

for $\varphi \in C_c^\infty(\Omega \setminus K)$, where ρ is the distance from the piecewise smooth surface K of given dimension, $m \in \mathbb{N}, \gamma \in \mathbb{R}$ are numbers and V_i are suitable potentials involving iterated logarithmic functions, subject to a simple geometric condition. This condition is satisfied, for example, in Cartan-Hadamard manifolds, that is, simply connected geodesically complete non-compact manifolds with non-positive sectional curvature.

Finally, there are some recent developments regarding Hardy inequalities in non-Euclidean settings that are non-Riemannian. Namely, in [36] we have the

Finsler-Hardy inequality

$$\int_\Omega \left| \frac{x}{H(x)} \cdot \nabla\varphi(x) \right|^p dx \geq \left| \frac{n-p}{p} \right|^p \int_\Omega \frac{|\varphi(x)|^p}{H^p(x)} dx, \quad \varphi \in C_c^\infty(\Omega \setminus 0),$$

where Ω is a domain of $\mathbb{R}^n$ and

$$H(x) = \sup_{\xi \neq 0} \frac{x \cdot \xi}{F(\xi)}$$

is the polar function of a non-negative convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^2 that is positively homogeneous of degree 1.

1.3 Outline of the dissertation

In Chapter 2, we introduce a method of integration along integral curves to obtain Hardy inequalities for the first order differential operator X in a given manifold M with volume form ω. These inequalities have the form

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p} \right)^p \int_M \frac{|\varphi|^p}{\tau_p^p} \omega, \quad \varphi \in C_c^1(M),$$

where $1/\tau_p^p$ is a positive global potential on the manifold the manifold, dependent in general in the setup (M, X, ω) and the exponent p, while the constant is sharp. This method applies very generally and we illustrate its use in a number of examples, some of them yielding new results.

Chapter 3 is concerned with higher order Rellich inequalities related to general elliptic operators with constant coefficients, other than the classic polyharmonic operator $(-\Delta)^m$. In this case, we show that a Rellich inequality can be expressed in terms of an induced Finsler distance d_H which is given in terms of the symbol of the operator. This new type of inequality is shown to be sharp in the case where the underlying domain is a half-space and the symbol satisfies a convexity condition, while comparisons are made for the case of a convex domain, yielding results that are superior to those obtained by more crude methods, in specific situations.

Finally, in Chapter 4, we deal with the sensitivity of the Hardy constant under perturbations of the domain in the case where the distance is measured from a boundary submanifold. Specifically, we find the Hardy constant to be both continuous and differentiable (in the Gateaux sence) under such perturbations, assuming some regularity conditions on the boundary.

Chapter 2

Geometric Hardy inequalities via integration on flows

We introduce a geometric approach of integration along integral curves for functional inequalities involving directional derivatives in the general context of differentiable manifolds that are equipped with a volume form. We focus on Hardy-type inequalities and the explicit optimal Hardy potentials that are induced by this method. We then apply the method to retrieve some known inequalities and establish some new ones.

2.1 Introduction

The one-dimensional Hardy inequality involving the distance to the boundary of the interval (a, b) reads

$$\int_a^b |\varphi'(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_a^b \frac{|\varphi(x)|^p}{\min\{x-a, b-x\}^p} dx, \quad \varphi \in C_c^1((a,b)). \quad (2.1)$$

In this chapter, we propose a method of integration along integral curves to obtain a “lifting" of this inequality for differentiable manifolds of arbitrary dimension that are subject to a simple geometric condition that is satisfied in a large number of cases. In particular, if M is an oriented differentiable manifold with positive volume form ω and X is a non-vanishing vector field on M, we prove the optimal inequality

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{\tau_p^p} \omega, \quad \varphi \in C_c^1(M),$$

where τ_p is a suitable “boundary distance" that depends on the geometry of the configuration. It is worth noting that in our method τ_p is calculated explicitly and is usually highly non-trivial, except for the simplest of cases.

It has been recently pointed out to us by Y. Pinchover that a special case of this approach also appears in [34], where the authors integrate with respect to “flow coordinates” in bounded C^2 Euclidean domains to specify some properties of the Hardy constant that corresponds to the Euclidean distance, amongst other things. In this respect, our work could be considered to be a generalisation of this methodology in a broader context.

Although our results apply more generally, of special interest is the case of a Riemannian manifold (M, g), where we can apply the method to retrieve inequalities involving the Riemannian gradient ∇_g and the associated volume form ω_g. Our method can easily provide optimal, non-trivial Hardy potentials in a multitude of such cases, as we demonstrate through specific examples.

2.2 Preliminaries

We begin by setting the context and introducing the necessary notions that will be used throughout the rest of this chapter.

Definition 2.2.1. Let M be a smooth manifold of dimension n.

1. A non-vanishing vector field $X \in \Gamma(TM)$ is called a *direction field* on M. The pair (M, X) is then called a *directed space*.
2. A non-vanishing n-form $\omega \in \Lambda^n(T^*M)$ is called a *volume form* on M.
3. A triple (M, X, ω) that consists of a smooth manifold, a direction field and a volume form is called a *directed volume space*.

In what follows and unless otherwise stated, M will stand for a non-compact, oriented smooth manifold of dimension n, X will be a direction field and ω will be a volume form on M. Hereafter, we will also make the implicit assumption that ω is positive in the chosen orientation.

As usual, an integral curve on the directed space (M, X) will be a curve $\gamma : I \to M$ such that $\gamma' = X \circ \gamma$. By the existence and uniqueness theorem for ODEs, for each point $z \in M$, there exists a unique maximal integral curve $\gamma_z : I_z \to M$ such that $\gamma_z(0) = z$. The flow of X is then defined to be the smooth map

$$\theta : \bigsqcup_{z \in M} I_z \to M, \;\; \theta(z, t) = \gamma_z(t).$$

The directed space (M, X) is said to be complete if $I_z = \mathbb{R}$ for all $z \in M$. The type of spaces that will occupy our attention are essentially the opposite of complete spaces in the following sense.

Definition 2.2.2. A directed space is said to be *traceable* if $I_z \subsetneqq \mathbb{R}$ for all $z \in M$.

To get an intuitive understanding of this definition, consider the one-point compactification of M with ∞ being the point at infinity. Traceable spaces are exactly the ones in which starting at any point and following the flow of the field will take one to ∞ at finite time in at least one direction (positive or negative time).

Traceable spaces are important for our purposes because one can naturally define a temporal distance function from infinity: if $z \in M$ is a point, define

$$\tau(z) = \operatorname{dist}(0, \partial I_z).$$

Then $\tau : M \to \mathbb{R}$ is obviously well-defined and positive everywhere in the manifold, and its value at any point is equal to the time required to reach infinity if one follows the flow of the field starting from that point.

Each directed space (M, X) comes naturally equipped with an equivalence relation $\sim$ that takes two points to be equivalent if they belong to the same integral curve. The resulting quotient space, which we denote by M/X, is called the *orbit space* of (M, X), and in general fails to be a manifold. We will be interested in subsets of M that are saturated with respect to this relation.

Definition 2.2.3. Let (M, X) be a directed space.

1. A subset $S \subset M$ is said to be *saturated* if $\mathrm{Im}(\gamma_z) \subset S$ for all $z \in S$.

2. If $S \subset M$ is any subset, we define the *saturation of* S to be the set

$$\theta(S) = \bigcup_{z \in S} \mathrm{Im}(\gamma_z).$$

In other words, if a saturated subset S contains a point then it contains the entire integral curve that point belongs to. Obviously, S is saturated if and only if $S = \theta(S)$. Moreover, since the flow is an open map, if S is open, so is $\theta(S)$.

In each directed space, one can introduce, at least locally, a set of normal coordinates $\chi = (t, s) = (t, s^1, \ldots, s^{n-1})$ with the property $\partial/\partial t = X$. In terms of the corresponding parametrisation $\zeta = \chi^{-1}$, this can be expressed equivalently as

$$\partial_t \zeta(t, s) = X \circ \zeta(t, s).$$

Actually, this means that ζ forms a family of integral curves parametrised by s. While it is incorrect to assume that every directed space can be covered by a single normal coordinate chart, it is obvious that one always has an open cover of the manifold consisting of saturated normal chart domains (to see this, for each point $z \in M$, pick a normal coordinate ball B centered at z and consider $\theta(B)$).

In normal coordinates, ω admits a local expression

$$\omega = \Omega(t, s) dt \wedge ds,$$

with Ω being the *local volume density* in these coordinates. In general, Ω depends both on s and t. Directed volume spaces in which Ω's don't depend on t form a special class which is much easier to deal with for our purposes, so we give them a name.

Definition 2.2.4. A directed volume space (M, X, ω) is called *simple* if the local volume density of ω in normal coordinates is independent of t.

We will develop a method of obtaining Hardy inequalities for directed volume spaces regardless of whether they are simple or not. In fact, the most interesting cases are usually non-simple. However, simple spaces, as we will see shortly, are much easier to deal with and are the natural starting point for our line of work.

2.3 The simple case

First we deal with simple spaces. The derivation of a Hardy inequality is much simpler in that case, and sets the background for the more advanced techniques that are required to treat the general case.

Intuitively, the method we develop can be described as follows:

1. Cover the space with saturated normal coordinate charts. This way we can “write down" the space as a parametrised family of integral curves.

2. Apply the one-dimensional Hardy inequality (2.1) along each curve separately.

3. Integrate over all integral curves using the normal coordinates.

At this point, we are ready to state and prove the main theorem of this section.

Theorem 2.3.1. *Let (M, X, ω) be a simple and traceable directed volume space. Then the inequality*

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{\tau^p}\omega, \quad \varphi \in C_c^1(M) \tag{2.2}$$

holds for all $p > 1$.

Proof. Let (U, χ) be a saturated normal coordinate chart on M with $\chi : U \to \tilde{U}$ for some open $\tilde{U} \subseteq \mathbb{R}^{n-1}$ and let $\zeta = \chi^{-1}$ be the corresponding parametrisation. U can be chosen so that $\tilde{U}$ is of the form $\bigsqcup_{s\in S} I_s$ for some open $S \in \mathbb{R}^{n-1}$ and some intervals $I_s \subsetneq \mathbb{R}$, so we have coordinates (t, s) where $s \in S$ and $t \in I_s$. Let

$$\omega = \Omega(s)dt \wedge ds$$

in these coordinates. Moreover, we clearly have that

$$\zeta(t, s) = \gamma_{\zeta(0,s)}(t), \quad t \in I_s = I_{\zeta(0,s)}$$

(the integral curve passing through $\zeta(0, s)$) and that

$$\tau \circ \zeta(t, s) = \operatorname{dist}(t, \partial I_s).$$

Now, it is clear that $\varphi \circ \zeta(\cdot, s) \in C_c^1(I_s)$ for all $s \in S$. Applying the one-dimensional Hardy inequality (2.1) on $\varphi \circ \zeta(\cdot, s)$ for fixed s we get

$$\int_{I_s} |\partial_t(\varphi \circ \zeta)(t, s))|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_{I_s} \frac{|\varphi \circ \zeta(t, s)|^p}{\operatorname{dist}(t, \partial I_s)^p} dt,$$

which by the properties of normal coordinates is equivalent to

$$\int_{I_s} |X\varphi \circ \zeta(t, s)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_{I_s} \frac{|\varphi \circ \zeta(t, s)|^p}{\tau^p \circ \zeta(t, s)} dt.$$

Multiplying both sides by $\Omega(s)$ (which is positive by assumption), integrating over S and applying Fubini's theorem yields

$$\int_S \int_{I_s} |X\varphi \circ \zeta(t,s)|^p \Omega(s) dt ds \geq \left(\frac{p-1}{p}\right)^p \int_S \int_{I_s} \frac{|\varphi \circ \zeta(t,s)|^p}{\tau^p \circ \zeta(t,s)} \Omega(s) dt ds,$$

which, in terms of differential forms, is the same as

$$\int_{\tilde{U}} |X\varphi \circ \zeta|^p \Omega \det \geq \left(\frac{p-1}{p}\right)^p \int_{\tilde{U}} \frac{|\varphi \circ \zeta|^p}{\tau^p \circ \zeta} \Omega \det .$$

The diffeomorphic invariance formula for integration on forms (see the Appendix) then yields

$$\int_U |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_U \frac{|\varphi|^p}{\tau^p} \omega.$$

To complete the proof, let $\{(U_j, \chi_j)\}_{j\in J}$ be an atlas of M that consists of saturated normal charts as above. The collection $\{U_j\}_{j\in J}$ is then an open cover of M, and therefore an open cover of $\operatorname{supp}(\varphi)$. Furthermore, $\operatorname{supp}(\varphi)$, being compact, must have a finite subcover $\{U_1, \ldots, U_n\}$. For the final step, consider the saturated open sets $W_1, \ldots, W_n$, defined as

$$W_1 = U_1, \quad W_k = U_k \setminus \bigcup_{l=1}^{k-1} \bar{U}_l.$$

The collection $\{(W_k, \chi_k)\}$ and its corresponding parametrisations then satisfy the conditions of Lemma 2.10.2 and the proof is finished. □

In some cases, the last argument can be replaced by a partition of unity argument. This would require that we project an open cover onto M/X, and then assume a partition of unity for the projected cover. However, this assumption is not always valid, as M/X need not be Hausdorff.

Another, more important point is to note that the constant that appears in the theorem is optimal. Seeing that this is so is rather straightforward: simply pick a sequence φ_ϵ such that $\operatorname{supp}(\varphi_\epsilon)$ converges to a single integral curve. If the inequality where to hold true for a larger constant, that would mean that the one-dimensional Hardy inequality from which it was derived would also hold for that constant, which is known to be false.

Example 2.3.2. The prototype of simple traceable spaces spaces is the Euclidean half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ equipped with the parallel vector field $\partial/\partial x_n$. The normal coordinates in this case are given by $t = x_n$ and $s = (x_1, \ldots, x_{n-1})$, so we have that $\omega = dx_1 \wedge \cdots \wedge dx_n = dt \wedge ds$ (if necessary, take one of the s coordinates to have an opposite sign in order to mitigate the extra sign that might occur from changing the order in the exterior product). Moreover, we clearly have $\tau = x_n$, so it follows from Theorem 2.3.1 that the inequality

$$\int_{\mathbb{R}^n_+} \left|\frac{\partial\varphi}{\partial x_n}\right|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|\varphi|^p}{x_n^p} dx, \quad \varphi \in C_c^1(\mathbb{R}^n_+)$$

holds for all $p > 1$.

Example 2.3.3. A less trivial example that still falls within the class of simple cases is that of a two-dimensional angle $A = \{x \in \mathbb{R}^2 : 0 < \theta(x) < \alpha\}$ (for some given $\alpha \in (0, 2\pi])$ equipped with the vector field

$$X = r^{\epsilon/p}\frac{\partial}{\partial\theta}$$

for some $p > 1$ and some $\epsilon \in \mathbb{R}$. In polar coordinates, we have $\omega = r d\theta \wedge dr$. To find a set of normal coordinates (t, s) for this configuration, choose $s = r$ and notice that we must wave

$$\frac{\partial}{\partial t} = r^{\epsilon/p}\frac{\partial}{\partial\theta},$$

and therefore we may choose $t = \frac{\theta}{r^{\epsilon/p}}$. Moreover, it follows that

$$d\theta = s^{\epsilon/p}dt + \frac{\epsilon}{p} t s^{\frac{\epsilon-p}{p}} ds,$$

hence $\omega = s^{\epsilon/p+1}dt \wedge ds$, so (A, X, ω) is simple. Since the integral curves here follow co-centric circles each with angular velocity $r^{\epsilon/p}$, it follows that $\tau = r^{-\epsilon/p}\min\{\theta, \alpha - \theta\}$. Direct application of Theorem 2.3.1 yields the inequality

$$\int_A r^\epsilon \left|\frac{\partial\varphi}{\partial\theta}\right|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_A r^\epsilon \frac{|\varphi|^p}{\min\{\theta, \alpha-\theta\}^p} dx, \quad \varphi \in C_c^1(A).$$

It is worth noting that in the special case $\epsilon = -p$, we get an inequality involving the angular component of the gradient, thus we have

$$\int_A |\nabla\varphi|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_A \frac{|\varphi|^p}{r^p \min\{\theta, \alpha-\theta\}^p} dx, \quad \varphi \in C_c^1(A).$$

2.4 p-normal coordinates

The proof of (2.2) was based on the fact that we can multiply the integral over dt with $\Omega(s)$ and then pass $\Omega(s)$ inside the integral (since it is independent of t). If we look at the more general case of a non-simple space where $\Omega(t, s)$ depends also on t, it is clear that one cannot repeat this argument.

We can bypass this difficulty by introducing new coordinates that are related to the initial set of normal coordinates (t, s). These new coordinates, denoted (t', s'), will have the property

$$X = \frac{\partial}{\partial t} = \Omega'(t', s')^{-1/p}\frac{\partial}{\partial t'},$$

where $\Omega'(t', s') = \omega(\partial_{t'}, \partial_{s'})$ is the local volume density in these new coordinates. This way, we can get an integral over dt' which contains both the correct vector field and the correct volume element from the beginning.

This motivates the following definition.

Definition 2.4.1. Let (M, X, ω) be a directed volume space, and let $p > 1$. A set of coordinates $(\tau, \sigma) = (\tau, \sigma^1, \ldots, \sigma^{n-1})$ (defined on some open set) will be called a set of *p-normal coordinates along X with respect to ω* if

$$X = \Omega(\tau, \sigma)^{-1/p} \frac{\partial}{\partial \tau}.$$

We dedicate the remainder of this section to prove the existence and some useful properties of these coordinates. We also explore their connection to regular normal coordinates as defined previously, and relate to them a well-defined (independent of coordinates) temporal/volumetric “distance" like τ in the previous sections. These facts will form the necessary background to generalise Theorem 2.3.1 to include non-simple spaces.

Proposition 2.4.2 (Existence)**.** *Let (t, s) be a set of normal coordinates on some open $U \subseteq M$ in the directed volume space (M, X, ω). The coordinates (t', s') defined by*

$$t' = \int^t \Omega(\xi, s)^{-\frac{1}{p-1}} d\xi, \quad s' = s$$

is a set of p-normal coordinates along X with respect to ω on U.

Proof. It is clear that

$$\frac{\partial t'}{\partial t} = \Omega(t, s)^{-\frac{1}{p-1}},$$

and we calculate

$$\Omega(t, s) = \omega(\partial_t, \partial_s) = \frac{\partial t'}{\partial t} \omega(\partial_{t'}, \partial_{s'}) = \frac{\partial t'}{\partial t} \Omega'(t', s').$$

It follows that

$$\frac{\partial t'}{\partial t} = \Omega'(t', s')^{-1/p},$$

thus

$$\Omega'(t', s')^{-1/p} \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} = X,$$

so the set of coordinates (t', s') is indeed p-normal along X with respect to ω.

Since ω is non-vanishing, it follows that $\Omega(t, s) > 0$, so in particular t' is well-defined everywhere in U. □

This not only proves existence, but also provides a practical way to compute such coordinates, provided we already have a set of normal coordinates, which are often straightforward to acquire.

Another fact is that these coordinates cooperate well with the flow of the field X. If we choose a saturated normal chart, which we already know how to produce, it is straightforward to turn it into a p-normal saturated coordinate chart using the above transformation. This is evident from the fact that the vector field $\Omega(\tau,\sigma)^{-1/p}X$ has the same integral curves as X, only reparametrised.

Recall that for a directed volume space, we defined the associated temporal distance $\tau : M \to \mathbb{R}$, which essentially measures the amount of time required to reach the “boundary" of M moving along the flow of X. Equivalently, if (t,s) is a set of normal coordinates in a saturated domain such that $t \in I_s = (a_s, b_s)$, then

$$\tau = \operatorname{dist}(t, \partial I_s) = \min(t - a_s, b_s - t).$$

We now introduce the following notation.

Definition 2.4.3. Let $f : I \to \mathbb{R}$ be a measurable function on the interval $I = (a,b)$ (here it is possible that $a = -\infty$ or $b = +\infty$). Define

$$\int_{\partial I}^{t} f(\xi) d\xi = \min\left(\int_a^t f(\xi) d\xi, \int_t^b f(\xi) d\xi \right).$$

In this notation, it is clear that

$$\tau = \int_{\partial I_s}^{t} d\xi.$$

Moreover, the condition that (M,X) is traceable can be rewritten as

$$\tau = \int_{\partial I_s}^{t} d\xi < \infty \text{ everywhere in } M.$$

It turns out that what we need in the case of non-simple spaces, is a suitable modification of this with respect to p-normal coordinates.

Definition 2.4.4. Let (M,X,ω) be a directed volume space and let (t,s), $t \in I_s$ be normal coordinates for a saturated chart domain $U \subseteq M$, let $\Omega(t,s)$ be the local volume density in these coordinates and let $p > 1$.

1. We say that U is *p-traceable* if

$$\int_{\partial I_s}^{t} \Omega(\xi,s)^{-\frac{1}{p-1}} d\xi < \infty \text{ everywhere in } U.$$

2. If U is p-traceable, we define the associated *temporal/volumetric distance* $\tau_p : U \to \mathbb{R}$ to be the function

$$\tau_p = \Omega(t,s)^{\frac{1}{p-1}} \int_{\partial I_s}^{t} \Omega(\xi,s)^{-\frac{1}{p-1}} d\xi.$$

Proposition 2.4.5. *Everything in the above definition is well-defined, i.e. independent of the choice of normal coordinates in U.*

Proof. Suppose that we have two sets of normal coordinates $(t,s) = (t, s^1, \dots, s^{n-1})$ and $(t', s') = (t', (s')^1, \dots, (s')^{n-1})$ of the same orientation in U. By the chain rule, we have that

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \sum_{j=1}^{n-1}\frac{\partial (s')^j}{\partial t}\frac{\partial}{\partial (s')^j},$$

$$\frac{\partial}{\partial s^i} = \frac{\partial t'}{\partial s^i}\frac{\partial}{\partial t'} + \sum_{j=1}^{n-1}\frac{\partial (s')^j}{\partial s^i}\frac{\partial}{\partial (s')^j},$$

where $i = 1, \ldots, n-1$. Since these are both sets of normal coordinates, we must have $\partial_t = \partial_{t'} = X$. This implies that

$$\frac{\partial t'}{\partial t} = 1 \text{ and } \frac{\partial (s')^j}{\partial t} = 0 \text{ for all } j = 1, \ldots, n-1.$$

In particular, the s' coordinates are independent of t and $s' = \sigma(s)$ for some diffeomorphism σ between open sets in $\mathbb{R}^{n-1}$.

By linearity and skew-symmetry of ω, we have that

$$\Omega(t,s) = \omega\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s^1}, \ldots, \frac{\partial}{\partial s^{n-1}}\right) =$$

$$\sum_{j_1,\ldots,j_{n-1}=1}^{n-1}\frac{\partial (s')^{j_1}}{\partial s^1}\cdots\frac{\partial (s')^{j_{n-1}}}{\partial s^{n-1}}\omega\left(\frac{\partial}{\partial t'}, \frac{\partial}{\partial (s')^{j_1}}, \ldots, \frac{\partial}{\partial (s')^{j_{n-1}}}\right) =$$

$$\sum_{\pi\in S_{n-1}}\frac{\partial (s')^{\pi(1)}}{\partial s^1}\cdots\frac{\partial (s')^{\pi(n-1)}}{\partial s^{n-1}}(-1)^{\pi}\Omega'(t',s'),$$

where the last sum is over all permutations π in $(n-1)$ elements and $(-1)^\pi$ is the sign of π. It follows that

$$\frac{\Omega(t,s)}{\det D\sigma(s)} = \Omega'(t',s'),$$

where $D\sigma$ is the Jacobian matrix of $s' = \sigma(s)$. Since σ is an orientation-preserving diffeomorphism, this matrix is non-singular and the determinant is positive.

It is straightforward to show that neither the convergence of the integral in (1) of the definition nor the formula of τ_p in (2) are affected if we switch between normal coordinates. Indeed, we have that

$$\int_{\partial I_{s'}}^{t'}\Omega'(\xi',s')^{-\frac{1}{p-1}}d\xi' = \int_{\partial I_s}^{t}\left[\frac{\Omega(\xi,s)}{\det D\sigma(s)}\right]^{-\frac{1}{p-1}}\frac{d\xi'}{d\xi}d\xi =$$

$$(\det D\sigma(s))^{\frac{1}{p-1}}\int_{\partial I_s}^{t}\Omega(\xi,s)^{-\frac{1}{p-1}}d\xi,$$

so

$$\int_{\partial I_s}^{t}\Omega(\xi,s)^{-\frac{1}{p-1}}d\xi < \infty \Leftrightarrow \int_{\partial I_{s'}}^{t'}\Omega'(\xi',s')^{-\frac{1}{p-1}}d\xi' < \infty,$$

and it is clear that $\tau_p = \tau'_p$. $\square$

Since every directed space (M, X) admits an open cover of saturated normal coordinate charts, and since the above notions are independent of the choice of such a chart, we can unambiguously extend these notions over the whole manifold. This way we may define the global function $\tau_p : M \to \mathbb{R}$ given locally by

$$\tau_p = \Omega(t,s)^{\frac{1}{p-1}} \int_{\partial I_s}^{t} \Omega(\xi,s)^{-\frac{1}{p-1}} d\xi.$$

At this point, it is clear that (M, X, ω) is p-traceable if and only if the function τ_p is defined everywhere in M.

As a final remark, we would like to point out that in the case where (M, X, ω) is simple, p-traceability coincides with traceability and $\tau_p = \tau$, so this is indeed a meaningful extension of the previous concepts.

2.5 The general case

We are now ready to state and prove our main result.

Theorem 2.5.1. *Let (M, X, ω) be a directed volume space and let $p > 1$. Then the inequality*

$$\int_M |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{\tau_p^p} \omega, \quad \varphi \in C_c^1(M) \tag{2.3}$$

is valid whenever the space is p-traceable.

Proof. Let U be a saturated coordinate domain with normal coordinates $\chi = \zeta^{-1} = (t, s)$ and corresponding p-normal coordinates $\chi' = (\zeta')^{-1} = (t', s')$ constructed as demonstrated in the previous section. Let $\varphi \in C_c^1(M)$. As with the simple case, apply the one-dimensional Hardy inequality to $\varphi \circ \zeta'(\cdot, s) \in C_c^1(I_{s'})$ to get

$$\int_{I_{s'}} |\partial_{t'}(\varphi \circ \zeta')(t', s')|^p dt' \geq \left(\frac{p-1}{p}\right)^p \int_{I_{s'}} \frac{|\varphi \circ \zeta'(t', s')|^p}{\mathrm{dist}^p(t', \partial I_{s'})} dt',$$

which by the properties of the p-normal coordinates becomes

$$\int_{I_{s'}} |X\varphi \circ \zeta'(t', s')|^p \Omega'(t', s') dt' \geq \left(\frac{p-1}{p}\right)^p \int_{I_{s'}} \frac{|\varphi \circ \zeta'(t', s')|^p}{\mathrm{dist}^p(t', \partial I_{s'})} dt'.$$

Integrating both sides over the s'-coordinates then yields

$$\int_{S'} \int_{I_{s'}} |X\varphi \circ \zeta'(t', s')|^p \Omega'(t', s') dt' ds' \geq$$

$$\left(\frac{p-1}{p}\right)^p \int_{S'} \int_{I_{s'}} \frac{|\varphi \circ \zeta'(t', s')|^p}{\Omega'(t', s') \mathrm{dist}^p(t', \partial I_{s'})} \Omega'(t', s') dt' ds'.$$

To show that this is the same as

$$\int_U |X\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_U \frac{|\varphi|^p}{\tau_p^p}\omega,$$

it remains to be shown that $\tau_p^p = \Omega'(t', s')\mathrm{dist}^p(t', \partial I_{s'})$. This is straightforward, as we have

$$\mathrm{dist}(t', \partial I_{s'}) = \int_{\partial I_s}^t \Omega^{-\frac{1}{p-1}}(\xi, s)d\xi$$

from the definition, and by elementary calculations we also have that

$$\Omega'(t', s') = \omega(\partial_{t'}, \partial_{s'}) = \partial t'/partialt\omega(\partial_t, \partial_s) = \Omega^{\frac{p}{p-1}}(t, s)$$

. The proof is again completed by a similar argument as in Theorem 2.3.1. □

Let us make a few remarks about the result. The first is its generality. The only condition that we have imposed for the inequality to hold true is p-traceability of (M, X, ω). The number of cases this applies to is vast, including many important cases that are already of interest. We will provide specific examples in the remainder of this chapter. For the time being, let us note that the only thing we need - in principle - in order to check whether the condition is satisfied is to find a set of normal coordinates (t, s), compute the local volume density $\Omega(t, s)$ in these coordinates and then check if the integral

$$\int_{\partial I_s}^t \Omega^{-\frac{1}{p-1}}(\xi, s)d\xi$$

converges. In a large number of cases, including many of the cases that are of immediate interest, this poses no real hardship.

What we gain from this process, however, is often highly non-trivial results. If the space in question indeed turns out to be p-traceable, the result provides an explicit, optimal Hardy potential in terms of the induced temporal/volumetric distance

$$\Omega^{\frac{1}{p-1}}(t, s)\int_{\partial I_s}^t \Omega^{-\frac{1}{p-1}}(\xi, s)d\xi.$$

Example 2.5.2. As an elementary application to showcase how the method works in practice, we provide an alternative proof of the standard Euclidean Hardy inequality in $\mathbb{R}^n$ featuring the distance from a single point. Here, choose $M = \mathbb{R}^n \setminus \{0\}$, $X = \partial/\partial r$ and $\omega = \det$ (the Euclidean volume form).

Finding normal coordinates for this configuration is trivial: since we must have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial r},$$

simply choose $t = r$. For the rest of the coordinates there is a lot of freedom of choice, but we can simply choose $s = \theta$, where θ are the angles in the spherical coordinate system (therefore the spherical coordinates as a whole forms a set of normal coordinates in our case).

The expression of the Euclidean volume form in spherical coordinates is of the form $\omega = r^{n-1}f(\theta)dr \wedge d\theta$ for some $f(\theta)$ that involves powers of sines

of the angles, therefore in our chosen normal coordinates we have the same representation
$$\omega = t^{n-1} f(s) dt \wedge ds,$$
so it is clear that the local volume density is $\Omega(t,s) = t^{n-1} f(s)$.

Now let $1 < p \neq n$. The temporal/volumetric distance is

$$\tau_p = \Omega^{\frac{1}{p-1}}(t,s) \int_{\partial I_s}^{t} \Omega^{-\frac{1}{p-1}}(\xi, s) d\xi = t^{\frac{n-1}{p-1}} \int_{\{0,\infty\}}^{t} \xi^{-\frac{n-1}{p-1}} d\xi.$$

To compute this, we must consider the two different cases $p < n$ and $p > n$, but in either case the result is
$$\tau_p = \frac{p-1}{|p-n|} r.$$
By 2.5.1, it follows that the inequality

$$\int_{\mathbb{R}^n} \left| \frac{\partial \varphi}{\partial r} \right|^p dx \geq \left| \frac{p-n}{p} \right|^p \int_{\mathbb{R}^n} \frac{|\varphi|^p}{r^p} dx$$

holds for all $\varphi \in C_c^1(\mathbb{R}^n \setminus \{0\})$, as expected.

However, notice the unorthodox manner in which we obtain the best constant. In our method, this constant is not merely the result of algebraic operations, but has a geometric significance as well: it is a direct consequence of the p-dependence of the distance τ_p.

Example 2.5.3. In the same manner as in the previous example, by choosing $X = r^{-\epsilon/p} \partial/\partial r$ we can prove the weighted inequality

$$\int_{\mathbb{R}^n} \frac{1}{r^\epsilon} \left| \frac{\partial \varphi}{\partial r} \right|^p dx \geq \left| \frac{p-n+\epsilon}{p} \right|^p \int_{\mathbb{R}^n} \frac{|\varphi|^p}{r^{p+\epsilon}} dx, \quad \varphi \in C_c^1(\mathbb{R}^n \setminus \{0\})$$

for $\epsilon \neq n - p$. The calculations are a bit more involved than before but still elementary.

Example 2.5.4. As a final example, we turn our attention to the hyperbolic space $\mathbb{H}^n$, where a peculiar phenomenon occurs: the Hardy inequality becomes a Poincaré inequality. We employ the Poincaré half space model, where $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ with $g_{\mathbb{H}^n} = \frac{1}{x_n^2} g_{\mathbb{R}^n}$. The Riemannian volume form in this case reads $\omega_{\mathbb{H}^n} = x_n^{-n} \det$. Let $X = x_n \frac{\partial}{\partial x_n}$. It is clear that $|X| = 1$. To find a set of normal coordinates for $(\mathbb{H}^n, X)$ we must find a t such that

$$\frac{\partial}{\partial t} = x_n \frac{\partial}{\partial x_n},$$

so we choose $t = \log x_n$ and $s = (x_1, \ldots, x_{n-1})$. It follows that $\omega = e^{-(n-1)t} ds \wedge dt$. Finally, we calculate

$$\tau_p = e^{-\frac{n-1}{p-1} t} \int_{-\infty}^{t} e^{\frac{n-1}{p-1} \xi} d\xi = \frac{p-1}{n-1},$$

from which we obtain the inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\varphi|^p \omega_{\mathbb{H}^n} \geq \int_{\mathbb{H}^n} \left| x_n \frac{\partial \varphi}{\partial x_n} \right|^p \omega_{\mathbb{H}^n} \geq \left(\frac{n-1}{p} \right)^p \int_{\mathbb{H}^n} |\varphi|^p \omega_{\mathbb{H}^n}.$$

This is the classic Poincaré inequality for the hyperbolic space, and it is already known to be a consequence of the Hardy inequality (it can actually be obtained via the weighted inequality of the previous example, with minor modifications).

At this point it becomes clear that, when referring to the temporal/volumetric distance, the word “distance" should not be taken too literally, since it does not always conform to the way we know a distance should behave (e.g. in the last example it was a constant).

2.6 Application I: The exterior of a ball

We will now use the method to obtain some new results. We would like to point out that there are new things that can be said even in the Euclidean case. In this section we focus on the case where $M = E$ is the exterior of a Euclidean ball of dimension n.

Theorem 2.6.1 (Hardy Inequality for the exterior of a ball)**.** *Let $E = \{x \in \mathbb{R}^n : |x| > R\}$ be the exterior of the n-dimensional Euclidean ball of radius R. Then the inequalities*

$$\int_E \left| \frac{\partial \varphi}{\partial r} \right|^p dx \geq \left(\frac{p-n}{p} \right)^p \int_E \frac{|\varphi|^p}{r^{\frac{n-1}{p-1}p} (r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}})^p} dx, \quad p > n,$$

$$\int_E \left| \frac{\partial \varphi}{\partial r} \right|^n dx \geq \left(\frac{n-1}{n} \right)^n \int_E \frac{|\varphi|^n}{r^n \log(r/R)^n} dx,$$

$$\int_E \left| \frac{\partial \varphi}{\partial r} \right|^p dx \geq \left(\frac{n-p}{p} \right)^p \int_E \frac{|\varphi|^p}{r^{\frac{n-1}{p-1}p} \min\{r^{\frac{p-n}{p-1}}, R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\}^p} dx, \quad 1 < p < n$$

hold for all $\varphi \in C_c^1(E)$.

Proof. Similar to the case of $\mathbb{R}^n$ with the distance from a single point, the spherical coordinate system is a set of normal coordinates. The only difference now is that $t = r$ ranges from R to ∞. Thus, we have

$$\tau_p = t^{\frac{n-1}{p-1}} \int_{\{R,\infty\}}^t \xi^{-\frac{n-1}{p-1}} d\xi,$$

so in each individual case

$$\tau_p = \begin{cases} \frac{p-1}{p-n} r^{\frac{n-1}{p-1}} (r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}), & p > n \\ r \log(r/R), & p = n. \\ \frac{p-1}{n-p} r^{\frac{n-1}{p-1}} \min\{r^{\frac{p-n}{p-1}}, R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\}, & p < n \end{cases}$$

and the result follows. $\square$

This is a non-trivial result, although its derivation has been trivialised by the use of our method. Let us make a few remarks on it. Note that in the cases where $p \neq n$, for small $r - R$ we have $\tau_p \approx r - R$, whereas for large $r - R$ we have $\tau_p \approx \frac{p-1}{|p-n|}(r - R)$. This fits our intuition: when close to the ball the inequality must behave like the one involving the distance from a hyperplane, while for very large distances it must resemble the one involving the distance from a point. In essence, the induced distance τ_p forms a continuous transition between these two limit cases.

It is also of practical importance to compare τ_p with the Euclidean distance from the boundary $d = r - R$. This will yield inequalities for the classic Hardy potential $V = d^{-p}$. To our knowledge, the only known result in this direction is given by Avkhadiev and Makarov in [3] (see also [25] for alternative proofs of this result). The result states that for every compact $U \subseteq \mathbb{R}^n$, the best constant in the Hardy inequality

$$\int_{\mathbb{R}^n \setminus U} |\nabla\varphi|^p dx \geq c \int_{\mathbb{R}^n \setminus U} \frac{|\varphi|^p}{d^p} dx, \quad \varphi \in C_c^1(\mathbb{R}^n \setminus U),$$

is $c = \left(\frac{p-n}{p}\right)^p$ in the case where $p > n$, which implies the optimal inequality

$$\int_E |\nabla\varphi|^p \geq \left(\frac{p-n}{p}\right)^p \int_E \frac{|\varphi|^p}{d^p}, \quad \varphi \in C_c^1(E)$$

in the case of the exterior of a ball. In that case our method gives

$$\tau_p = \frac{p-1}{p-n} r^{\frac{n-1}{p-1}} (r^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}).$$

To specify the best constant κ such that

$$\frac{1}{\tau_p} \geq \frac{\kappa}{d},$$

we make a few observations. As we already noted, we have $\tau_p(r) \approx d(r)$ for r close to R and $\tau_p(r) \approx \frac{p-1}{p-n} d(r)$ for large r. More generally, the derivative of $\tau_p(r)$ is given by

$$\tau_p'(r) = \frac{p-1}{p-n} - \frac{n-1}{p-n}\left(\frac{R}{r}\right)^{\frac{p-n}{p-1}},$$

which is a strictly increasing function of r. It follows that

$$\frac{p-1}{p-n} d(r) = \sup_{y>R} (\tau_p'(y) d(r) > \tau_p(r)$$

so

$$\frac{1}{\tau_p} > \frac{p-n}{p-1} \frac{1}{d}$$

and we retrieve the same best constant $c = \left(\frac{p-n}{p}\right)^p$. It follows that our method improves the result of [3] in the case where U is a ball, in the sense that it provides a better distance for the same constant.

For the case $p < n$, we have the following comparison.

Corollary 2.6.2. *Let $E = \{x \in \mathbb{R}^n : |x| > R\}$. Then the inequality*

$$\int_E |\nabla\varphi|^p dx \geq \left(\frac{n-p}{p}\right)^p \big(1 - 2^{-\frac{p-1}{n-p}}\big)^p \int_E \frac{|\varphi|^p}{d^p} dx, \quad \varphi \in C_c^1(E)$$

holds for all $p < n$.

Proof. In this case, it is

$$\tau_p = \frac{p-1}{n-p} r^{\frac{n-1}{p-1}} \min\{r^{\frac{p-n}{p-1}}, R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\}.$$

We put $a = 2^{\frac{p-1}{n-p}} R$, which is the real number such that

$$a^{\frac{p-n}{p-1}} = R^{\frac{p-n}{p-1}} - a^{\frac{p-n}{p-1}},$$

i.e. the point in which the branch transition occurs. It follows that

$$\tau_p = \begin{cases} \frac{p-1}{n-p} r, & r \geq a \\ \frac{p-1}{n-p} r^{\frac{n-1}{p-1}} (R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}), & r < a \end{cases}.$$

An elementary calculation reveals that the derivative of $\tau_p(r)$ for $r < a$ is

$$\tau_p'(r) = \frac{n-1}{n-p} (r/R)^{\frac{n-p}{p-1}} - \frac{p-1}{n-p},$$

which is strictly increasing, so in particular $\tau_p(r)$ is convex for $r < a$. By virtue of Jensen's inequality it follows that

$$\tau_p(r) \leq A(r - R), \quad R < r < a,$$

where

$$A = \frac{\tau_p(a) - \tau_p(R)}{a - R} = \frac{p-1}{n-p} \frac{a}{a-R} = \frac{p-1}{n-p} \big(1 - 2^{-\frac{p-1}{n-p}}\big)^{-1}.$$

As for the region $r \geq a$, we certainly have that $\tau_p(r) \leq A(r - R)$, since both functions are affine, share the same value at a and $\frac{p-1}{n-p} < A$.

So in any case we have

$$\frac{1}{\tau_p} \geq \frac{n-p}{p} \frac{1 - 2^{-\frac{p-1}{n-p}}}{d}$$

and the result follows. □

For the sake of clarity, we give some plots of the function τ_p for specific values of n and p, plotted against the function $\frac{p-1}{|p-n|}(r - R)$ that we use when making the Euclidean comparison (see Figure 1 below). Other choices of n and p give qualitatively similar results. What really matters is whether $p < n$ or $p > n$.

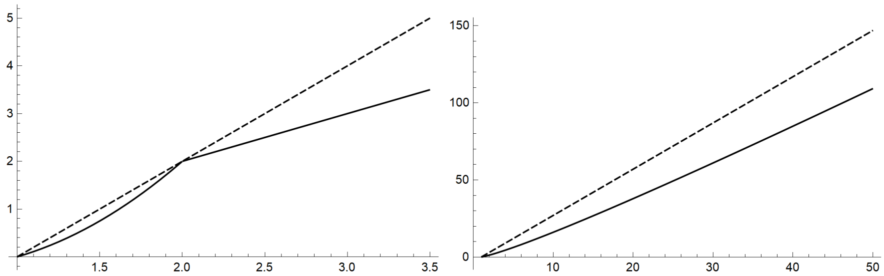

FIGURE 2.1: Left: $R = 1, n = 3, p = 2$. Right: $R = 1, n = 3, p = 4$

2.7 Application II: Spherical symmetry

Moving beyond the classic Euclidean setting, the most important class of examples is arguably the class of spherically symmetric manifolds. We say that a Riemannian manifold (M, g) is (locally) spherically symmetric around a central point $o \in M$ if the metric can be expressed as

$$g = d\rho \otimes d\rho + \psi^2(\rho) g_{S^{n-1}}$$

in a punctured neighbourhood of o, where $\rho = \operatorname{dist}(\cdot, o)$ is the Riemannian distance from o, ψ is a positive function depending only on ρ and $g_{S^{n-1}}$ is the round metric of the unit sphere of codimension 1. We are interested in the case where we have global spherical symmetry.

If M is non-compact, the above polar representation extends to the whole punctured space $M' = M \setminus \{o\}$. If M is compact, we must exclude an additional “antipodal" point $o' \in M$ (the most characteristic example is the sphere, where one must exclude both poles).

In either case, $\rho : M' \to \mathbb{R}$ has range of the form $(0, R)$ (we may have $R = +\infty$), and we may apply Theorem 2.5.1 with $X = \partial/\partial\rho$ and $\omega = \omega_g = \psi^{n-1}(\rho) d\rho \wedge \omega_{S^{n-1}}$. In the following, we also take into account the case where we choose to exclude not only the “pole(s)” o (and o'), but perhaps a larger object (for example, a geodesic ball around o or o').

Theorem 2.7.1. *Suppose that (M', g) is a Riemannian manifold whose metric can be expressed as*

$$g = d\rho \otimes d\rho + \psi^2(\rho) g_{S^{n-1}}$$

for some $\rho : M' \to (a, b)$ and some smooth $\psi : (a, b) \to (0, \infty)$. If for each value of $\rho \in (a, b)$, either one (or both) of the integrals

$$\int_a^\rho \psi^{-\frac{n-1}{p-1}}(\xi) d\xi, \quad \int_\rho^b \psi^{-\frac{n-1}{p-1}}(\xi) d\xi$$

converge, the inequality

$$\int_{M'} |\partial_\rho \varphi|^p \omega_g \geq \left(\frac{p-1}{p}\right)^p \int_{M'} \frac{|\varphi|^p}{\varpi_p^p} \omega_g, \quad \varphi \in C_c^1(M')$$

is valid with

$$\varpi_p = \psi^{\frac{n-1}{p-1}}(\rho) \min\left(\int_a^\rho \psi^{-\frac{n-1}{p-1}}(\xi)d\xi, \int_\rho^b \psi^{-\frac{n-1}{p-1}}(\xi)d\xi \right).$$

Proof. This is just a restatement of Theorem 2.5.1 for the special case $(M, X, \omega) = (M', \partial_\rho, \omega_g)$. $\square$

M' can be thought of as a suitable open submanifold of a spherically symmetric manifold M. A key feature of our technique is that it effectively manages to take into account the volumetric/temporal distance from both the "inner" and the "outer" edge of the manifold. By "inner" edge we mean the edge that is closer to the central point o. The volumetric/temporal distance from the inner edge is given by

$$\varpi_p^{\text{in}} = \psi^{\frac{n-1}{p-1}}(\rho) \int_a^\rho \psi^{-\frac{n-1}{p-1}}(\xi)d\xi,$$

while the corresponding distance from the outer edge is

$$\varpi_p^{\text{out}} = \psi^{\frac{n-1}{p-1}}(\rho) \int_\rho^b \psi^{-\frac{n-1}{p-1}}(\xi)d\xi.$$

While it is true that $\varpi_p = \min(\varpi_p^{\text{in}}, \varpi_p^{\text{out}})$, and consequently

$$\frac{1}{\varpi_p} \geq \frac{1}{\varpi_p^{\text{in}}}, \frac{1}{\varpi_p^{\text{out}}},$$

it is sometimes convenient to consider Hardy potentials that take into account only the inner or outer edge. One may choose to do this in order to extend the class of admissible functions (in the case of a compact manifold where we have an antipodal point o', one may still prefer to take into account functions that do not vanish at o').

To this end, this is a good point to demonstrate the flexibility of our method: all that Theorem 2.5.1 does is to essentially "lift" the one-dimensional Hardy inequality (2.1) in higher dimensions. As a matter of fact, *any* one-dimensional functional inequality could be used in its place. Without straying from our subject of Hardy inequalities, we simply point out that one gets nearly identical results if we choose instead to lift the inequality

$$\int_a^b |\varphi'(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_a^b \frac{|\varphi(x)|^p}{(x-a)^p} dx, \quad \varphi \in C_c^1((a,b]),$$

which takes into account only the first endpoint and admissible functions need not vanish close to b. This gets us exactly what we need.

Theorem 2.7.2. *Let (M, g) be a compact, spherically symmetric manifold with empty boundary, with central point $o \in M$ of injectivity radius $inj(o) = R$, $\rho = dist(\cdot, o)$ and let*

$$g = d\rho \otimes d\rho + \psi^2(\rho) g_{S^{n-1}}$$

for some smooth $\psi : (0, R) \to (0, \infty)$. Then the inequality

$$\int_M |\partial_\rho \varphi|^p \omega_g \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|\varphi|^p}{(\varpi_p^{in})^p} \omega_g, \quad \varphi \in C_c^1(M \setminus \{o\})$$

is valid whenever

$$\int_0^\rho \psi^{-\frac{n-1}{p-1}}(\xi) d\xi < \infty.$$

Proof. It is well known that in this case we have $\rho^{-1}(R) = \{o'\}$ where o' is a single point antipodal to o. It follows that $M \setminus \{o, o'\}$ can be covered with polar coordinates in which the metric is expressed exactly as in the statement of the theorem. The rest of the proof is a repetition of the steps in the proof of Theorem 2.5.1, the only difference being applying the above inequality instead of (2.1). □

Of special interest are the cases of the n-sphere $\mathbb{S}^n$, where $\psi(\theta) = \sin(\theta)$, $\theta \in (0, \pi)$, and the hyperbolic space $\mathbb{H}^n$, where $\psi(\rho) = \sinh(\rho)$, $\rho \in (0, \infty)$.

Remark. It recently came to our attention that this is not the first time that results such as these make their appearance. Other authors have employed analytic methods to obtain such results in a number of cases. For example, in [15], the authors present some results for spheres and spherically symmetric domains that are very similar to our own. In [12], the authors use a general result from [18] to derive an L^p Hardy potential for the hyperbolic space that also has the same form as the one that occurs from our method. More generally, in the spherically symmertic case, the Hardy potentials that we are looking at are all of the form $|\nabla \rho|^p / \rho^p$ for some p-harmonic $\rho \in W^{1,p}(M)$, and can therefore be considered a special case of the main result in [18].

Regardless, our method is inherently geometric instead of analytic and applies more generally, for example X and ω need not be related by a Riemannian metric. Moreover, the potentials provided by our method are explicit in any case, symmetric or not.

2.8 Application III: The exterior of a black hole

As a final application, we would like to discuss the case of the Schwarzschild metric, which describes static black holes in the context of General Relativity. The full Schwarzschild metric in (3+1)-dimensional spacetime reads

$$-\left(1 - \frac{1}{r}\right) dt \otimes dt + \left(1 - \frac{1}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}$$

and is actually a pseudo-Riemannian metric. To get a Riemannian metric, we will simply restrict our attention on “temporal slices" of constant time, where the restricted metric reads

$$\left(1 - \frac{1}{r}\right)^{-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}.$$

Theorem 2.8.1 (Hardy Inequality for the Schwarzschild Black Hole). *Let* $\mathfrak{B} = \{x \in \mathbb{R}^3 : |x| > 1\}$ *be equipped with the metric*

$$g_{\mathfrak{B}} = \frac{r}{r-1} dr \otimes dr + r^2 g_{\mathbb{S}^2}$$

as above, let $\nabla_{\mathfrak{B}}$ *and* $\omega_{\mathfrak{B}}$ *stand for the Riemannian gradient and volume form, respectively, and let*

$$\delta = \begin{cases} 2r^2\sqrt{\frac{r-1}{r}} & 1 < r < (4/3) \\ 2r^2\left(1 - \sqrt{\frac{r-1}{r}}\right) & r \geq (4/3) \end{cases} .$$

Then the inequality

$$\int_{\mathfrak{B}} |\nabla_{\mathfrak{B}}\varphi|^2 \omega_{\mathfrak{B}} \geq \int_{\mathfrak{B}} \frac{r-1}{r}\left|\frac{\partial \varphi}{\partial r}\right|^2 \omega_{\mathfrak{B}} \geq \frac{1}{4}\int_{\mathfrak{B}} \frac{|\varphi|^2}{\delta^2}\omega_{\mathfrak{B}}$$

is valid for all $\varphi \in C_c^1(\mathfrak{B})$. *The constant* $1/4$ *is sharp.*

Proof. Let $X = \sqrt{\frac{r-1}{r}}\frac{\partial}{\partial r}$. In polar coordinates we have

$$\omega_{\mathfrak{B}} = \sqrt{\frac{r}{r-1}} r^2 \sin(\theta) dr \wedge d\theta \wedge d\phi.$$

We are looking for a new coordinate t to replace r such that $\partial/\partial t = X$. Let $f : (1, \infty) \to (0, \infty)$ be the function given by the formula

$$f(x) = \sqrt{x}\sqrt{x-1} + \log(\sqrt{x} + \sqrt{x-1}).$$

It is easy to verify that $t = f(r)$ satisfies the imposed condition, therefore (t, θ, ϕ) is a set of normal coordinates for $(\mathfrak{B}, X)$. As f is a bijection, let g denote its inverse. Substituting $r = g(t)$ into the formula for $\omega_{\mathfrak{B}}$, we get

$$\omega_{\mathfrak{B}} = g(t)^2 \sin(\theta) dt \wedge d\theta \wedge d\phi,$$

therefore $\Omega(t, \theta, \phi) = g(t)^2 \sin(\theta)$. The temporal/volumetric distance in this case is

$$\tau_2 = g(t)^2 \int_{\{0,\infty\}}^t g(w)^{-2} dw = r^2 \min\left(\int_0^t g(w)^{-2} dw, \int_t^\infty g(w)^{-2} dw\right).$$

Substituting $w = f(\xi)$, it is elementary to show that

$$\tau_2 = r^2 \int_{\{1,\infty\}}^r \frac{d\xi}{\xi^{3/2}(\xi - 1)^{1/2}} = \delta$$

and the proof is complete. $\square$

A more complete treatment of this matter will be given elsewhere.

2.9 Higher-order inequalities

Likewise, one can recursively obtain inequalities for higher order differential operators. For example, consider the second-order operator YX obtained by the composition of two directional derivatives (vector fields) $X, Y \in \Gamma(TM)$. If (M, Y, ω) is p-traceable, we obtain

$$\int_M |YX\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^p \int_M \frac{|X\varphi|^p}{(\tau_p^Y)^p}\omega = \int_M \left|\frac{X}{\tau_p^Y}\varphi\right|^p \omega,$$

where τ_p^Y is the temporal/volumetric distance of (M, Y, ω). In the same manner, if $(M, X/\tau_p^Y, \omega)$ is p-traceable, we may repeat the process and obtain

$$\int_M |YX\varphi|^p \omega \geq \left(\frac{p-1}{p}\right)^{2p} \int_M \frac{|\varphi|^p}{(\tau_p^{Y/\tau_p^X})^p}\omega,$$

where τ_p^{Y/τ_p^X} is the temporal/volumetric distance for $(M, X/\tau_p^Y, \omega)$. By induction, this process can produce inequalities for operators of the form $X_1 \cdots X_k$ for any $k \in \mathbb{N}$, provided that p-traceability holds for each step.

We give some examples of higher-order inequalities obtained in this way.

Example 2.9.1. Recursive application of the weighted inequality of Example 2.5.3 yields the k-th order Rellich inequality

$$\int_{\mathbb{R}^n} \left|\frac{\partial^k \varphi}{\partial r^k}\right|^p dx \geq \prod_{l=1}^{k} \left|\frac{lp-n}{p}\right|^p \int_{\mathbb{R}^n} \frac{|\varphi|^p}{r^{kp}} dx, \quad \varphi \in C_c^1(\mathbb{R}^n \setminus \{0\}).$$

Note that, in essence, if one has weighted inequalities for the vector fields of interest, computing the distance at each step becomes unnecessary.

Likewise, for the one-dimensional case we have

$$\int_{\mathbb{R}_+} |D^k \varphi|^p dx \geq \prod_{l=1}^{k} \left(\frac{lp-1}{p}\right)^p \int_{\mathbb{R}_+} \frac{|\varphi|^p}{x^{kp}} dx, \quad \varphi \in C_c^1(\mathbb{R}_+),$$

which can be further integrated to give the same inequality for the half-space.

Example 2.9.2. Consider the second order differential operator

$$H = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}.$$

Applying the weighted inequality of Example 5.3 twice yields the inequality

$$\int_{\mathbb{R}^n} |H\varphi|^p dx \geq \left|\frac{2p-n}{p}\right|^{2p} \int_{\mathbb{R}^n} \frac{|\varphi|^p}{r^{2p}} dx, \quad \varphi \in C_c^1(\mathbb{R}^n \setminus \{0\}).$$

As a final interesting application, we will use the above to obtain Rellich inequalities involving the wave operator in the 2-dimensional half-space, which, in contrast to most operators that are being discussed in literature, is not an elliptic operator. We are not aware of other results of this type so far. We prove the following.

Theorem 2.9.3 (Higher-order Rellich Inequality for the Wave Operator)**.** *Let $\Box = \partial_x^2 - \partial_y^2$ denote the 2-dimensional wave operator, and let $u \in C_c^\infty(\mathbb{R}^2_+)$. Then the inequality*

$$\int_{\mathbb{R}^2_+} |\Box^k u|^p dxdy \geq \prod_{l=1}^{2k} \left(\frac{lp-1}{p}\right)^p \int_{\mathbb{R}^2_+} \frac{|u|^p}{y^{2kp}} dxdy$$

holds for all $k \in \mathbb{N}$. The constant is sharp.

This is an easy corollary of the following lemma.

Lemma 2.9.4. *Let $u \in C_c^\infty(\mathbb{R}^2_+)$. Then the inequality*

$$\int_{\mathbb{R}^2_+} \frac{|(\partial_x \pm \partial_y)u|^p}{y^\gamma} dxdy \geq \left(\frac{\gamma+p-1}{p}\right)^p \int_{\mathbb{R}^2_+} \frac{|u|^p}{y^{\gamma+p}} dxdy$$

holds for all $\gamma > 1 - p$.

Proof. Consider the case of $X := (\partial_x + \partial_y)$. The coordinates

$$t = \frac{1}{2}(x+y), \ \ s = \frac{1}{2}(y-x)$$

are a set of normal coordinates for $(\mathbb{R}^2_+, X)$ (it can be easily verified that $X = \partial/\partial t$). Moreover, we have that $x = t - s$ and $y = t + s$, thus

$$dx = dt - ds, \ \ dy = dt + ds.$$

It follows that $dx \wedge dy = 2dt \wedge ds$. It follows that

$$\omega = \frac{1}{y^\gamma} dx \wedge dy = \frac{2}{(t+s)^\gamma} dt \wedge ds,$$

and the corresponding temporal/volumetric distance is

$$\tau_p = (t+s)^{-\frac{\gamma}{p-1}} \int_{\partial I_s}^t (\xi + s)^{\frac{\gamma}{p-1}} d\xi,$$

where $I_s = (-s, \infty)$. By elementary calculations, this is equal to

$$\tau_p = \frac{p-1}{\gamma+p-1}(t+s) = \frac{p-1}{\gamma+p-1} y$$

and the result follows.

The case of $(\partial_x - \partial_y)$ is entirely analogous. $\Box$

The inequality in the theorem follows from the fact that $\Box = (\partial_x + \partial_y)(\partial_x - \partial_y)$ and inductive application of the lemma. Sharpness is proved by a standard argument, substituting the sequence

$$u_\epsilon(x,y) = y^{\frac{2kp-1}{p}+\epsilon}\rho_\epsilon(x,y), \quad \epsilon \to 0,$$

where ρ_ϵ is a suitable cutoff function that is equal to $\rho_\epsilon = 1$ in $(-\epsilon,\epsilon)\times(\epsilon,1/\epsilon)$ and $\mathrm{supp}(\rho_\epsilon) \subset (-2\epsilon, 2\epsilon)\times(\epsilon/2, 2/\epsilon)$.

2.10 Appendix: Auxiliary Material

We give some auxiliary results from the theory of differentiable manifolds that are used throughout this chapter. All of them can be found in [31].

Let $F : M \to N$ be a smooth map between manifolds. As usual, the differential of F is defined to be the map $F_* : TM \to TN$ such that $F_*X[g] = X[g \circ F]$ for all $g \in C^\infty(N)$. Likewise, we define the pull-back of F as the map $F^* : \Lambda(T^*N) \to \Lambda(T^*M)$ by $F^*\omega(X_1,\dots,X_k) = \omega(F_*X_1,\dots,F_*X_k)$ for all vectors $X_1,\dots,X_k \in T_zM$ for all $z \in M$.

Lemma 2.10.1 (Diffeomorphic invariance of the integral)**.** *Let $F : N \to M$ be an orientation-preserving diffeomorphism and $\omega \in \Lambda^{top}(T^*M)$. Then*

$$\int_M \omega = \int_N F^*\omega.$$

Lemma 2.10.2 (Integration over parametrisations)**.** *Let M be an oriented manifold of dimension n and let $\omega \in \Lambda^n(TM)$ be a compactly supported top-form on M. Suppose $D_1,\dots,D_k$ are open domains of integration in $\mathbb{R}^n$, and for $i = 1,\dots,k$ we are given smooth maps $\zeta_i : \bar{D}_i \to M$ satisfying*

1. *ζ_i restricts to an orientation-preserving diffeomorphism from $\bar{D}_i$ onto an open set $W_i \subset M$.*
2. *$W_i \cap W_j = \varnothing$ for $i \neq j$.*
3. *$supp(\omega) \subset \bar{W}_1 \cup \cdots \cup \bar{W}_k$.*

Then

$$\int_M \omega = \sum_{i=1}^k \int_{D_i} \zeta_i^*\omega.$$

Chapter 3

Finsler-Rellich inequalities involving the distance to the boundary

We study Rellich inequalities associated to higher-order elliptic operators in the Euclidean space. The inequalities are expressed in terms of an associated Finsler metric. In the case of half-spaces we obtain the sharp constant while for a general convex domains we obtain estimates that are better than those obtained by comparison with the polyharmonic operator. What follows is a joint work with G. Barbatis.

3.1 Introduction

In [40], Owen proves the higher-order Rellich inequality

$$\int_\Omega u(x)(-\Delta)^m u(x)dx \geq A(m)\int_\Omega \frac{u^2(x)}{d^{2m}(x)}dx, \qquad u \in C_c^\infty(\Omega), \tag{3.1}$$

for the polyharmonic operator $(-\Delta)^m$, where $\Omega \subseteq \mathbb{R}^n$ is a convex open set, $d : \Omega \to \mathbb{R}_+$ is the Euclidean distance from the boundary of Ω and $A(m)$ is the best constant given explicitly by

$$A(m) = \frac{(2m-1)^2(2m-3)^2\cdots 1^2}{4^m}.$$

This inequality has been subsequently extended and improved in various directions. In [2] and for the case $2m = 4$ a simple sufficient condition was given for non-convex domains so that the Rellich inequality is valid with the sharp constant $9/16$; in [11, 6] sharp improvements to (3.1) were obtained. We refer to the recent book [5] for additional information.

While the literature for Rellich inequalities for the polyharmonic operator $(-\Delta)^m$ is substantial, there are hardly any results on Rellich inequalities with distance to the boundary for more general higher-order elliptic operators. This is partly due to the lack of invariance under rotations and to the (related) fact that neither the Euclidean metric nor indeed any other Riemannian metric is suitable for the study of such operators.

Anisotropic Hardy inequalities with distance to the boundary have recently been obtained in [19]. Concerning anisotropic (non-Riemannian) Rellich inequalities, there is a growing literature on inequalities with distance to a point, see e.g. [30, 45], but we are not aware of any results involving the distance to the boundary. To our knowlegde, the best Rellich constant for $\int |\Delta u|^p dx$ is not known even in the case of a half-space.

The objective of this chapter is to investigate inequalities of the form

$$\int_\Omega u(x)Hu(x)dx \geq \kappa \int_\Omega \frac{u^2(x)}{d_H^{2m}(x)}dx \tag{3.2}$$

where H is a homogeneous elliptic differential operator of order $2m$ with real constant coefficients and d_H is a suitable Finsler distance to the boundary of Ω associated to H. In particular, we will prove the following result for half-spaces which is shown to be optimal in an important class of cases.

Theorem 3.1.1. *Let H be a homogeneous elliptic operator of order $2m$ with real constant coefficients and let $\mathbf{H} \subseteq \mathbb{R}^n$ be a half-space. Then the inequality*

$$\int_{\mathbf{H}} u(x)Hu(x)dx \geq A(m) \int_{\mathbf{H}} \frac{u^2(x)}{d_H^{2m}(x)}dx$$

holds for all $u \in C_c^\infty(\mathbf{H})$.

Note that since the operator H is not rotationally invariant, proving the inequality for the commonly used half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ does not imply the validity of the inequality for half-spaces in other directions.

In the second part of this chapter we investigate the case where $\Omega \subseteq \mathbb{R}^n$ is an arbitrary convex domain, and in particular we provide a uniform (independent of the domain) lower bound for the best constant which - although most likely non-optimal - is nonetheless better than what can be achieved by simply comparing with $(-\Delta)^m$.

3.2 Preliminaries

Let H be a homogeneous elliptic differential operator of order $2m$ with real constant coefficients, acting on real-valued functions on $\mathbb{R}^n$. So H has the form

$$H = (-1)^m \sum_{|\alpha|=2m} a_\alpha D^\alpha,$$

where a_α is a constant for each multi-index α and $D^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$. The symbol of the operator H is the polynomial $H : \mathbb{R}^n \to \mathbb{R}$ given by

$$H(\xi) = \sum_{|\alpha|=2m} a_\alpha \xi^\alpha.$$

Setting $F_H(\xi) = H^{1/2m}(\xi)$ (which is positively homogeneous of order one in ξ), we define the associated Finsler norm $F_H^* : \mathbb{R}^n \to \mathbb{R}$ by

$$F_H^*(\omega) = \sup_{\xi \neq 0} \frac{\omega \cdot \xi}{F_H(\xi)} = \max_{|\xi|=1} \frac{\omega \cdot \xi}{F_H(\xi)}. \tag{3.3}$$

The Finsler distance of two points $x, x' \in \mathbb{R}^n$ is then defined as $F_H^*(x - x')$. It is well known, see e.g. [20], that this is the distance suitable to use when studying properties of H, especially so when one seeks sharp constants. From now on we will suppress the index H when there is no ambiguity and simply write F for F_H and F^* for F_H^*. It is clear from the definition that for any $\omega, \xi \in \mathbb{R}^n$ we have the inequality

$$H(\xi)\, F^*(\omega)^{2m} \geq (\omega \cdot \xi)^{2m}. \tag{3.4}$$

Now let $\Omega \subseteq \mathbb{R}^n$ be open with non-empty boundary and let $d(x)$ denote the Euclidean distance of $x \in \Omega$ to $\partial\Omega$. The Euclidean distance of a point $x \in \Omega$ to $\partial\Omega$ along the direction $\omega \in S^{n-1}$ is given by

$$d_\omega(x) = \inf\{|s| : x + s\omega \notin \Omega\},$$

and we have

$$d(x) = \min_{\omega \in S^{n-1}} d_\omega(x).$$

In the context of Finsler geometry, distances are scaled by the Finsler norm (3.3) along each direction, so the Finsler distance of x from the boundary of Ω along the direction ω is given by

$$d_{H,\omega}(x) = F^*(\omega) d_\omega(x).$$

Denoting by

$$d_H(x) = \min\{F^*(x - y) : y \in \partial\Omega\}\ , \quad x \in \Omega, \tag{3.5}$$

the Finsler distance to the boundary we then have

$$d_H(x) = \min_{\omega \in S^{n-1}} d_{H,\omega}(x) = \min_{\omega \in S^{n-1}} \big(F^*(\omega) d_\omega(x)\big).$$

3.3 Finsler-Rellich inequality for half-spaces

Let $\nu \in S^{n-1}$ be a unit vector. We consider the ν-directional half-space $\mathbf{H}_\nu^n = \{x \in \mathbb{R}^n : \nu \cdot x > 0\}$, whose boundary is the hyperplane $\partial\mathbf{H}_\nu^n = \{x \in \mathbb{R}^n : x \cdot \nu = 0\}$. The Euclidean distance of $x \in \mathbf{H}_\nu^n$ from $\partial\mathbf{H}_\nu^n$ in the direction of $\omega \in S^{n-1}$ is given by

$$d_\omega(x) = \frac{\nu \cdot x}{|\nu \cdot \omega|},$$

and so the corresponding Finsler distance is given by

$$d_H(x) = \min_{\omega \in S^{n-1}} \big(F^*(\omega) d_\omega(x)\big) = \min_{\omega \in S^{n-1}} \left(\frac{F^*(\omega)}{|\nu \cdot \omega|}\right) \nu \cdot x.$$

So the minimum is achieved independently of x. Letting $\theta \in S^{n-1}$ be a unit vector that achieves the minimum we arrive at

$$d_H(x) = F^*(\theta) d_\theta(x) = \frac{\nu \cdot x}{F^{**}(\nu)} = \frac{d(x)}{F^{**}(\nu)}. \tag{3.6}$$

We are now ready to prove Theorem 3.1.1. We restate it as follows.

Theorem 3.3.1. *Let H be a homogeneous elliptic operator of order $2m$ with constant coefficients. Then the inequality*

$$\int_{\mathbf{H}^n_\nu} u(x) H u(x) dx \geq A(m) \int_{\mathbf{H}^n_\nu} \frac{u^2(x)}{d_H^{2m}(x)} dx \tag{3.7}$$

holds for any $\nu \in S^{n-1}$ and all $u \in C_c^\infty(\mathbf{H}^n_\nu)$. Moreover, the constant $A(m)$ is optimal in the case where F_H is a convex function.

Proof. Let $\hat{u}(\xi)$, $\xi \in \mathbb{R}^n$, denote the Fourier transform of u. Recalling (3.4), applying Plancherel's theorem and using the one-dimensional Rellich inequality we obtain

$$\begin{aligned}
\int_{\mathbf{H}^n_\nu} u(x) H u(x) dx &= \int_{\mathbb{R}^n} H(\xi) |\hat{u}(\xi)|^2 d\xi \\
&\geq \frac{1}{F^*(\theta)^{2m}} \int_{\mathbb{R}^n} (\theta \cdot \xi)^{2m} |\hat{u}(\xi)|^2 d\xi \\
&= \frac{1}{F^*(\theta)^{2m}} \int_{\mathbf{H}^n_\nu} (\partial_\theta^m u(x))^2 dx \\
&\geq \frac{A(m)}{F^*(\theta)^{2m}} \int_{\mathbf{H}^n_\nu} \frac{u^2(x)}{d_\theta^{2m}(x)} dx \\
&= A(m) \int_{\mathbf{H}^n_\nu} \frac{u^2(x)}{d_H^{2m}(x)} dx.
\end{aligned}$$

To prove the optimality, we proceed as follows. For $\epsilon > 0$ we consider the function $g_\epsilon(t) = t^{\frac{2m-1}{2}+\epsilon}$, $t > 0$. This is a sequence of minimizers for the one-dimensional Rellich inequality of order m, that is

$$\frac{\int_0^1 (g_\epsilon^{(m)})^2 dt}{\int_0^1 \frac{g_\epsilon^2}{t^{2m}} dt} \longrightarrow A(m), \qquad \text{as } \epsilon \to 0+ . \tag{3.8}$$

Let $v_\epsilon(x) = g_\epsilon(x \cdot \nu)$. For any multiindex α with $|\alpha| = 2m$ we then have $D^\alpha v_\epsilon(x) = \nu^\alpha g_\epsilon^{(2m)}(x \cdot \nu)$ and therefore

$$H v_\epsilon(x) = (-1)^m H(\nu) g_\epsilon^{(2m)}(x \cdot \nu) \tag{3.9}$$

We next localize v_ϵ. We consider a function $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi(t) = 1$, if $|t| \leq 1/2$, $\psi(t) = 0$, if $|t| \geq 1$. Let $\pi_\nu : \mathbf{H}^n_\nu \to \partial \mathbf{H}^n_\nu$ denote the orthogonal projection from the half-space to its boundary. We define

$$\phi(x) = \psi(\nu \cdot x) \psi(\pi_\nu(x)), \qquad u_\epsilon(x) = \phi(x) v_\epsilon(x).$$

Then $u_\epsilon \in H_0^m(\mathbf{H}_\nu^n)$ and $\|u_\epsilon\|_{H_0^m(\mathbf{H}_\nu^n)} \to +\infty$ as $\epsilon \to 0+$. We shall estimate $\int_{\mathbf{H}_\nu^n} u_\epsilon \, Hu_\epsilon dx$ and for this we note that when we use Leibniz rule to expand $Hu_\epsilon = H(\phi v_\epsilon)$ any term containing at least one derivative of ϕ stays bounded as $\epsilon \to 0$. Setting $k = \int_{\mathbb{R}^{n-1}} \psi(|y|)^2 dy$ and applying (3.9) we thus have

$$\begin{aligned}
\int_{\mathbf{H}_\nu^n} u_\epsilon \, Hu_\epsilon dx &= \int_{\mathbf{H}_\nu^n} \phi^2 v_\epsilon \, Hv_\epsilon dx + O(1) \\
&= k(-1)^m H(\nu) \int_0^1 \psi^2 g_\epsilon \, g_\epsilon^{(2m)} dt + O(1) \\
&= kH(\nu) \int_0^1 (g_\epsilon^{(m)})^2 dt + O(1). \qquad (3.10)
\end{aligned}$$

On the other hand, recalling also (3.6) we similarly have

$$\begin{aligned}
\int_{\mathbf{H}_\nu^n} \frac{u_\epsilon^2(x)}{d_H^{2m}(x)} dx &= F^{**}(\nu)^{2m} \int_{\mathbf{H}_\nu^n} \frac{\phi^2 v_\epsilon^2}{d^{2m}} dx \\
&= F^{**}(\nu)^{2m} \int_0^1 \frac{g_\epsilon^2}{t^{2m}} dt + O(1). \qquad (3.11)
\end{aligned}$$

From (3.10), (3.11) and (3.8) we conclude that

$$\begin{aligned}
\frac{\int_{\mathbf{H}_\nu^n} u_\epsilon(x) Hu_\epsilon(x) dx}{\int_{\mathbf{H}_\nu^n} \frac{u_\epsilon^2(x)}{d_H^{2m}(x)} dx} &= \left(\frac{F(\nu)}{F^{**}(\nu)} \right)^{2m} \frac{\int_0^1 (g_\epsilon^{(m)}(t))^2 dt + O(1)}{\int_0^1 \frac{g_\epsilon^2(t)}{t^{2m}} dt + O(1)} \\
&\to \left(\frac{F(\nu)}{F^{**}(\nu)} \right)^{2m} A(m), \qquad \text{as } \epsilon \to 0+.
\end{aligned}$$

Since F is convex, $F = F^{**}$, and optimality follows. $\square$

Remark. It is known [43, Section 1.6] that the set $\{\xi \in \mathbb{R}^n : F^{**}(\xi) \leq 1\}$ is the convex hull of the set $\{\xi \in \mathbb{R}^n : F(\xi) \leq 1\}$. This shows that $F^{**}(\xi) \leq F(\xi)$ for all $\xi \in \mathbb{R}^n$ and also that there exist directions $\nu \in S^{n-1}$ such that $F^{**}(\nu) = F(\nu)$. It follows in particular that if F is not convex the constant $A(m)$ is still the best possible constant for which (3.7) is valid for all $\nu \in S^{n-1}$ and all $u \in C_c^\infty(\mathbf{H}_\nu^n)$.

3.4 Convex domains

If the symbol $H(\xi)$ of the operator H satisfies

$$\lambda |\xi|^{2m} \leq H(\xi) \leq \Lambda |\xi|^{2m}, \qquad \xi \in \mathbb{R}^n,$$

then applying the polyharmonic Rellich inequality (3.1) we obtain that for any convex domain $\Omega \subset \mathbb{R}^n$ there holds

$$\int_\Omega u(x) Hu(x) dx \geq A(m) \frac{\lambda}{\Lambda} \int_\Omega \frac{u^2(x)}{d_H^{2m}(x)} dx \,, \quad u \in C_c^\infty(\Omega). \qquad (3.12)$$

In this section we adapt Davies' well known mean distance function technique [17] to establish an alternative lower bound for the best Rellich constant of (3.12). While we have not attained the actual constant $A(m)$, we nevertheless provide a constant which depends only on the symbol and which can be easily computed numerically in any particular case. This has been carried out at the end of the section for two monoparametric families of operators and it turns out that the constants obtained are better than those in (3.12).

To state our result, we need some additional definitions related to the operator in question. Assuming that H is an elliptic differential operator of order $2m$ as above and denoting by $d\sigma(\omega)$ the normalized surface measure on S^{n-1}, we define the positive constants μ_H and M_H as the best constants for the inequalities

$$\mu_H\, F_H^{**}(\xi)^{2m} \leq \int_{S^{n-1}} \frac{(\xi\cdot\omega)^{2m}}{F^*(\omega)^{2m}} d\sigma(\omega) \leq M_H\, H(\xi)\,, \qquad \xi\in\mathbb{R}^n.$$

With this settled, we prove the following.

Theorem 3.4.1. *Let H be an elliptic operator of order $2m$ acting on functions defined in a convex open set $\Omega\subseteq\mathbb{R}^n$. Then the inequality*

$$\int_\Omega u(x)Hu(x)dx \geq A(m)\frac{\mu_H}{M_H}\int_\Omega \frac{u^2(x)}{d_H^{2m}(x)}dx \tag{3.13}$$

holds for all $u\in C_c^\infty(\Omega)$.

Proof. We have

$$\begin{aligned}
\int_\Omega u(x)Hu(x)dx &= \int_{\mathbb{R}^n} H(\xi)|\hat{u}(\xi)|^2 d\xi \\
&\geq \frac{1}{M_H}\int_{S^{n-1}} \frac{1}{F^*(\omega)^{2m}}\int_{\mathbb{R}^n} (\omega\cdot\xi)^{2m}|\hat{u}(\xi)|^2 d\xi\, d\sigma(\omega) \\
&= \frac{1}{M_H}\int_{S^{n-1}} \frac{1}{F^*(\omega)^{2m}}\int_\Omega (\partial_\omega^m u(x))^2 dx\, d\sigma(\omega).
\end{aligned}$$

We next apply the one-dimensional Rellich inequality in the direction ω to get

$$\int_\Omega u(x)Hu(x)dx \geq A(m)\frac{1}{M_H}\int_\Omega u^2(x)\int_{S^{n-1}} \frac{1}{(F^*(\omega)d_\omega(x))^{2m}} d\sigma(\omega)dx. \tag{3.14}$$

To estimate the last integral we consider a point $x\in\Omega$ and a point $y=y(x)\in\partial\Omega$ that realizes the infimum in (3.5). Let Π_x be a supporting hyperplane at $y(x)$ and let $N=N(x)$ be the outward normal unit vector to Π_x. We denote by $z(\omega)=z(\omega,x)$ the intersection of Π_x with the line $\{x+t\omega : t\in\mathbb{R}\}$. From the previous discussion, it follows that $|z(\omega)-x|\geq d_\omega(x)$ and therefore $F^*(z(\omega)-x)\geq F^*(\omega)d_\omega(x)$ for all $x\in\Omega$ and $\omega\in S^{n-1}$.

Let $s\in\mathbb{R}$ be such that $z(\omega)=x+s\omega$. Since $z(\omega)$ and y both belong to Π_x, $z(\omega)-y$ is perpendicular to N, that is

$$(x+s\omega-y)\cdot N = 0.$$

It follows that

$$s = \frac{(y-x)\cdot N}{\omega \cdot N},$$

and so

$$z(\omega) = x + \frac{(y-x)\cdot N}{\omega \cdot N}\omega.$$

Returning to (3.14), we now have

$$\begin{aligned}
\int_{S^{n-1}} \frac{1}{(F^*(\omega)d_\omega(x))^{2m}} d\sigma(\omega) &\geq \int_{S^{n-1}} \frac{1}{F^*(z(\omega)-x)^{2m}} d\sigma(\omega) \\
&= \frac{1}{((y-x)\cdot N)^{2m}} \int_{S^{n-1}} \left(\frac{\omega \cdot N}{F^*(\omega)}\right)^{2m} d\sigma(\omega) \\
&\geq \mu_H \left(\frac{F^{**}(N)}{(y-x)\cdot N}\right)^{2m} \\
&\geq \frac{\mu_H}{F^{***}(y-x)^{2m}} \\
&= \frac{\mu_H}{F^*(y-x)^{2m}} = \frac{\mu_H}{d_H^{2m}(x)},
\end{aligned}$$

and the proof is complete. □

As already mentioned, the constants μ_H and M_H can be computed numerically in any specific case. The next two examples illustrate the estimate of Theorem 3.4.1 and in particular show that inequality (3.13) is better than (3.12).

Example 1. Let $\beta > -1$ (for ellipticity) and

$$H_\beta(\xi) = \xi_1^4 + 2\beta\xi_1^2\xi_2^2 + \xi_2^4, \qquad \xi \in \mathbb{R}^2.$$

We have

$$\begin{cases} \frac{\beta+1}{2}|\xi|^4 \leq H_\beta(\xi) \leq |\xi|^4, & \text{if } -1 < \beta \leq 1, \\ |\xi|^4 \leq H_\beta(\xi) \leq \frac{\beta+1}{2}|\xi|^4, & \text{if } \beta \geq 1, \end{cases}$$

hence (3.12) gives

$$\int_\Omega u(x)H_\beta u(x)dx \geq \frac{9}{16}c(\beta)\int_\Omega \frac{u^2(x)}{d_{H_\beta}^{2m}(x)}dx\,, \qquad u \in C_c^\infty(\Omega),$$

where

$$c(\beta) = \begin{cases} \frac{\beta+1}{2}, & \text{if } -1 < \beta \leq 1, \\ \frac{2}{\beta+1}, & \text{if } \beta \geq 1\,. \end{cases}$$

In Figure 1 below we have plotted the function $s(\beta) = \mu_{H_\beta}/M_{H_\beta}$ (blue line) against $c(\beta)$ (red line) and it is seen that the estimate of Theorem 3.4.1 is better than (3.12).

Example 2. Let

$$\hat{H}_\beta(\xi) = \xi_1^6 + \beta\xi_1^4\xi_2^2 + \beta\xi_1^2\xi_2^4 + \xi_2^6, \qquad \xi \in \mathbb{R}^2.$$

We have $\hat{H}_\beta(\xi) = (\xi_1^2+\xi_2^2)[(\xi_1^2-\xi_2^2)^2 + (\beta+1)\xi_1^2\xi_2^2]$, so we assume $\beta > -1$ for ellipticity. We now have

$$\begin{cases} \frac{\beta+1}{4}|\xi|^4 \le \hat{H}_\beta(\xi) \le |\xi|^4, & \text{if } -1 < \beta \le 3, \\ |\xi|^4 \le \hat{H}_\beta(\xi) \le \frac{4}{\beta+1}|\xi|^4, & \text{if } \beta \ge 3, \end{cases}$$

hence (3.12) gives

$$\int_\Omega u(x)\hat{H}_\beta u(x)dx \ge \frac{9}{16}\hat{c}(\beta)\int_\Omega \frac{u^2(x)}{d_{\hat{H}_\beta}^{2m}(x)}dx\,, \qquad u \in C_c^\infty(\Omega),$$

where

$$\hat{c}(\beta) = \begin{cases} \frac{\beta+1}{4}, & \text{if } -1 < \beta \le 3, \\ \frac{4}{\beta+1}, & \text{if } \beta \ge 3\,. \end{cases}$$

In Figure 2 below we have plotted the function $\hat{s}(\beta) = \mu_{\hat{H}_\beta}/M_{\hat{H}_\beta}$ (blue line) against $\hat{c}(\beta)$ (red line).

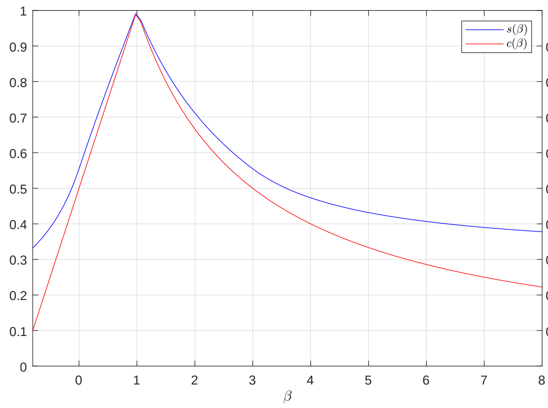

FIGURE 3.1: Plots of $s(\beta)$ and $c(\beta)$

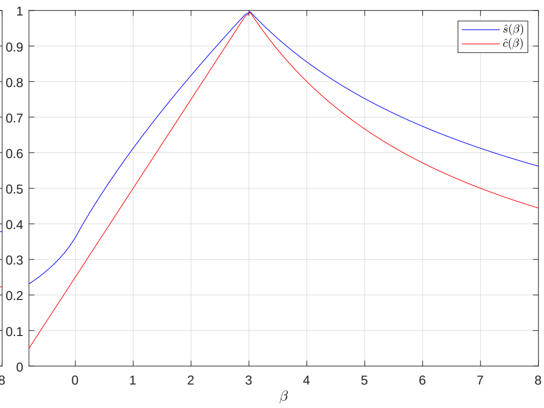

FIGURE 3.2: Plots of $\hat{s}(\beta)$ and $\hat{c}(\beta)$

Chapter 4

Shape sensitivity of the Hardy constant involving the distance from a boundary submanifold

We investigate the continuity and differentiability of the Hardy constant with respect to perturbations of the domain in the case where the problem involves the distance from a boundary submanifold. We also investigate the case where only the submanifold is deformed.

4.1 Introduction

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain (open, connected) with boundary $\partial\Omega$, and let $\Sigma \subset \partial\Omega$ be a submanifold of the boundary of dimension $\dim \Sigma = s \in \{0, \dots, n-1\}$. If there exists a positive constant $C > 0$ such that the inequality

$$\int_\Omega |\nabla u|^2 dx \geq C \int_\Omega \frac{u^2}{d_\Sigma^2} dx, \quad u \in H_0^1(\Omega), \tag{4.1}$$

with $d_\Sigma = \operatorname{dist}(\cdot, \Sigma)$ is valid, we say that the Hardy inequality is satisfied for the pair (Ω, Σ).

In this chapter, we are primarily concerned with the behaviour of this constant under perturbations of the domain and the submanifold. In particular, if φ is a diffeomorphism, we get a map

$$\varphi \longmapsto H(\varphi(\Omega), \varphi(\Sigma)), \tag{4.2}$$

and our task is to investigate questions of continuity and differentiability of that map in an appropriate sense which is made precise in the next section. This problem has already been studied in a more general L^p setting for the special case $\Sigma = \partial\Omega$ in [10], so our work here is a natural continuation of that work.

We also concern ourselves with the problem where only the submanifold is perturbed. This is expressed in a very neat way in the case of a point singularity: if we regard the Hardy constant as a function $H : \partial\Omega \to \mathbb{R}$,

$$H(\sigma) = H(\Omega, \{\sigma\}),$$

then this function is differentiable on $\partial\Omega$, under some reasonable assumptions.

4.2 Diffeomorphism Groups

In this section we offer a quick review of finite order diffeomorphism groups in $\mathbb{R}^n$. For details, see [4]. A C^k-diffeomorphism of $\mathbb{R}^n$ is a homeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that is k-times bi-differentiable. The set of all such maps is denoted by $Diff^k(\mathbb{R}^n)$. It is obviously a group under composition. For our purposes, it is sufficient to work with the subgroup $Diff_c^k(\mathbb{R}^n)$ of C^k-diffeomorphisms with compact support

$$\operatorname{supp}(\varphi) = \overline{\{x \in \mathbb{R}^n : \varphi(x) \neq x\}}$$

(the closure of the set of points that the diffeomorphism acts upon non-trivially). Since we work on bounded domains, this is done without loss of generality, and spares us some technical considerations that are consequence of the non-compactness of $\mathbb{R}^n$.

We now equip $Diff_c^k(\mathbb{R}^n)$ with the weak C^k topology (or compact-open topology). To describe this topology, it suffices to describe the basic open sets that generate it. These are the "balls"

$$\mathcal{N}_\varphi(K, \epsilon) = \{\psi \in Diff_c^k(\mathbb{R}^n) : \|\psi - \varphi\|_{C^k(K)} < \epsilon\}$$

of center $\varphi \in Diff_c^k(\mathbb{R}^n)$, radius $\epsilon > 0$ and domain K, which is a compact subset of $\mathbb{R}^n$. Here, we assume

$$\|\varphi\|_{C^k(K)} = \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha \varphi\|_{L^\infty(K)}.$$

In this topology, $Diff_c^k(\mathbb{R}^n)$ is a topological group, which is in fact locally homeomorphic to the Banach space of C^k vector fields of compact support $\mathfrak{X}_c^k(\mathbb{R}^n) \cong C_c^k(\mathbb{R}^n, \mathbb{R}^n)$, thus assuming the structure of an infinite dimensional Lie group.

The directional derivative of a continuous function $H : Diff_c^k(\mathbb{R}^n) \to \mathbb{R}$ at $\varphi \in Diff_c^k(\mathbb{R}^n)$ in the direction of $\xi \in \mathfrak{X}_c^k(\mathbb{R}^n)$ is given by the limit

$$D_\varphi H(\xi) = \left.\frac{d}{dt}\right|_{t=0} H(\varphi + t\xi),$$

provided it exists. Note that the compact support assumption guarantees that $\varphi + t\xi$ is always a diffeomorphism provided that t is small enough. If this is defined for all $\varphi \in Diff_c^k(\mathbb{R}^n)$ and all $\xi \in \mathfrak{X}_c^k(\mathbb{R}^n)$, we say that H is (Gateaux) differentiable.

4.3 Continuity of the Hardy Constant

Here we discuss some continuity results. By $co(\Omega)$ we denote the convex hull of Ω.

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with non-empty boundary, and let $\Sigma \subset \partial\Omega$ be an arbitrary subset of the boundary. Then there exist $\epsilon > 0$ and*

$c > 0$ such that for every C^1 diffeomorphism φ with $\|D\varphi - I\| < \epsilon$,

$$|H(\varphi(\Omega), \varphi(\Sigma)) - H(\Omega, \Sigma)| \leq cH(\Omega, \Sigma)\|D\varphi - I\|_{L^\infty(co_\varphi(\Omega))}, \tag{4.3}$$

where $co_\varphi(\Omega) = co(\Omega) \cup \varphi^{-1}(co(\varphi(\Omega)))$.

Proof. Let $u \in H_0^1(\Omega)$ be normalised by $\int_\Omega u^2/d_\Sigma^2 dx = 1$. For $v = u \circ \varphi^{-1}$, consider the Rayleigh quotient

$$R(\varphi(\Omega), \varphi(\Sigma))[v] = \frac{\int_{\varphi(\Omega)} |\nabla v|^2 dy}{\int_{\varphi(\Omega)} v^2/d_{\varphi(\Sigma)}^2 dy} = \frac{\int_\Omega |(D\varphi)^{-\top} \nabla u|^2 |\det D\varphi| dx}{\int_\Omega \frac{u^2}{d_{\varphi(\Sigma)\circ\varphi}^2} |\det D\varphi| dx},$$

where the last equality follows from the change of variables $y = \varphi(x)$. After some elementary calculations, it follows that

$$R(\varphi(\Omega), \varphi(\Sigma))[v] - R(\Omega, \Sigma)[u] =$$

$$\frac{\int_\Omega (|(D\varphi)^{-\top} \nabla u|^2 |\det D\varphi| - |\nabla u|^2) dx - \int_\Omega |\nabla u|^2 dx \left(\int_\Omega \frac{u^2 |\det D\varphi|}{d_{\varphi(\Sigma)\circ\varphi}^2} dx - 1 \right)}{\int_\Omega \frac{u^2 |\det D\varphi|}{d_{\varphi(\Sigma)\circ\varphi}^2} dx}.$$

In order to get an estimate for the expression

$$|(D\varphi)^{-\top} \nabla u|^2 |\det D\varphi| - |\nabla u|^2,$$

we first note that $\|A^\top\| = \|A\|$ as operator norms. To get an upper bound for the operator norm of the inverse, we also make the assumption that φ is a "small" diffeomorphism in the sense that $D\varphi(x) = I + \epsilon(x)$ where $\|\epsilon(x)\| < 1$. In this case it is known that

$$\|(D\varphi)^{-1}(x)\| \leq \frac{1}{1 - \|\epsilon(x)\|}.$$

Besides, for such ϵ there is a constant $\kappa = \kappa(n)$ such that

$$|\det(I + \epsilon) - 1| \leq \kappa \|\epsilon\|,$$

so eventually we have the estimate

$$|(D\varphi)^{-\top} \nabla u|^2 |\det D\varphi| - |\nabla u|^2 \leq C|\nabla u|^2 \|D\varphi - I\|$$

for some constant $C > 0$ provided that $\|D\varphi - I\|$ is small.

Next, for $x \in \Omega$, we obtain an estimate of $d_{\varphi(\Sigma)}(\varphi(x))$ in terms of $d_\Sigma(x)$. Since $d_\Sigma = d_{\bar{\Sigma}}$, we may assume that Σ is closed. Then there exists $\sigma(x) \in \Sigma$ such that $d_\Sigma(x) = |x - \sigma(x)|$. Consider the straight line segment $\gamma : [0, 1] \to \mathbb{R}^n$,

$$\gamma(t) = (1 - t)\sigma(x) + tx$$

joining these two points. Then clearly $d_\Sigma(x) = l(\gamma)$ (the arc length of γ). Then, by definition, we have that

$$d_{\varphi(\Sigma)}(\varphi(x)) \leq l(\varphi \circ \gamma) = \int_0^1 |(\varphi \circ \gamma)'(t)| dt \leq \|D\varphi\|_{L^\infty(co(\Omega))} d_\Sigma(x),$$

thus

$$d_{\varphi(\Sigma)}(\varphi(x)) \leq d_\Sigma(x)(1 + \|D\varphi - I\|_{L^\infty(co(\Omega))}).$$

It follows that

$$\int_\Omega \frac{u^2 |\det D\varphi|}{d^2_{\varphi(\Sigma)} \circ \varphi} dx \geq \frac{\inf_\Omega |\det D\varphi|}{(1 + \|D\varphi - 1\|_{L^\infty(co(\Omega))})^2} \int_\Omega \frac{u^2}{d^2_\Sigma} dx$$

$$\geq \frac{1 - \kappa \|D\varphi - I\|_{L^\infty(\Omega)}}{(1 + \|D\varphi - 1\|_{L^\infty(co(\Omega))})^2},$$

the last inequality being valid due to normalisation, thus

$$\int_\Omega \frac{u^2 |\det D\varphi|}{d^2_{\varphi(\Sigma)} \circ \varphi} dx \geq 1 - C \|D\varphi - I\|_{L^\infty(co(\Omega))}$$

for some constant C provided that $\|D\varphi - I\|$ is small.

Using all these estimates we obtain

$$R(\varphi(\Omega), \varphi(\Sigma))[v] - R(\Omega, \Sigma)[u] \leq cR(\Omega, \Sigma)[u] \|D\varphi - I\|_{L^\infty(co(\Omega))}.$$

for some $c > 0$. Passing to the appropriate limit of minimisers, we get

$$H(\varphi(\Omega), \varphi(\Sigma)) - H(\Omega, \Sigma) \leq cH(\Omega, \Sigma) \|D\varphi - I\|_{L^\infty(co(\Omega))}$$

Replacing Ω and Σ by $\varphi(\Omega)$ and $\varphi(\Sigma)$ and φ by φ^{-1}, it follows that

$$H(\Omega, \Sigma) - H(\varphi(\Omega), \varphi(\Sigma)) \leq cH(\varphi(\Omega), \varphi(\Sigma)) \|(D\varphi)^{-1} - I\|_{L^\infty(\varphi^{-1}(co(\varphi(\Omega))))}.$$

Since

$$\|(D\varphi)^{-1} - I\| \leq \frac{\|D\varphi - I\|}{1 - \|D\varphi - I\|},$$

it follows that there is $c > 0$ such that the reverse inequality

$$H(\Omega, \Sigma) - H(\varphi(\Omega), \varphi(\Sigma)) \leq cH(\Omega, \Sigma) \|D\varphi - I\|_{L^\infty(\varphi^{-1}(co(\varphi(\Omega))))}$$

also holds for small $\|D\varphi - I\|$. The result follows. □

For small $\|\varphi - id\|_{C^1}$, we have that if Ω is relatively compact, so is $co_\varphi(\Omega)$, so we immediately deduce the following.

Corollary 4.3.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\Sigma \subset \partial\Omega$. Then the map $\varphi \longmapsto H(\varphi(\Omega), \varphi(\Sigma))$ is continuous with respect to the weak C^1 topology.*

A few remarks are in order. First, the result does not hold for the case $k = 0$ (homeomorphisms), as it is essential to be able to control first derivatives.

Next, note that estimate (4.3) holds independent of the boundedness of Ω or compactness of $\mathrm{supp}(\varphi)$, and is therefore substantially more general than the corollary.

Although of no use to the sequel, we now present a collateral result that is obtained without extra effort. Instead of the standard Euclidean distance $\mathrm{dist}(x, y) = |x - y|$, for $x, y \in \Omega$ one could use the alternative "interior" distance

$$\widetilde{\mathrm{dist}}(x, y) = \inf\{l(\gamma) : \gamma \in C^1([0, 1], \Omega), \gamma(0) = x, \gamma(1) = y\},$$

and consider the Hardy problem

$$\tilde{H}(\Omega, \Sigma) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2/\tilde{d}_\Sigma^2 dx}, \tag{4.4}$$

where $\tilde{d}_\Sigma(x) = \widetilde{\mathrm{dist}}(x, \Sigma)$. For that case, we obtain the almost identical result

Theorem 4.3.3. *Let $\Omega \subset \mathbb{R}^n$ be open set with non-empty boundary, and let $\Sigma \subset \partial\Omega$ be an arbitrary subset of the boundary. Then there exist $\epsilon > 0$ and $c > 0$ such that for every C^1 diffeomorphism φ with $\|D\varphi - I\| < \epsilon$,*

$$|\tilde{H}(\varphi(\Omega), \varphi(\Sigma)) - \tilde{H}(\Omega, \Sigma)| \leq c\tilde{H}(\Omega, \Sigma)\|D\varphi - I\|_{L^\infty(\Omega)}. \tag{4.5}$$

Proof. The proof is almost identical to that of estimate (4.3). The only difference is that instead of picking γ to be the straight line segment joining x and $\sigma(x)$, one chooses a sequence of curves γ_n such that $l(\gamma_n) \to \tilde{d}_\Sigma(x)$. $\square$

Note that taking convex hulls is unnecessary here, since all distances are compared inside Ω.

4.4 Differentiability of the Hardy Constant

Now we present our main results regarding differentiability. Our methodology is similar to the one developed in [10] (which concerns the case $\Sigma = \partial\Omega$), with appropriate modifications.

Lemma 4.4.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is open with non-empty boundary and let $\Sigma \subset \partial\Omega$ be closed. Let $\varphi \in \mathit{Diff}_c^1(\mathbb{R}^n)$, $\xi \in \mathfrak{X}_c^1(\mathbb{R}^n)$ and let $t_0 > 0$ be such that*

$$\varphi_t = \varphi + t\xi$$

is a C^1 diffeomorphism for all $t \in [-t_0, t_0]$. Then:

1. *There exists a constant $c = c(\Omega, \varphi, \xi, t_0)$ such that*

$$|d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) - d^2_{\varphi(\Sigma)}(\varphi(x))| \leq c d^2_{\varphi(\Sigma)}(\varphi(x))|t| \tag{4.6}$$

   for all $x \in \Omega$ and all $t \in [-t_0, t_0]$.

2. *If $d_{\varphi(\Sigma)}$ is differentiable at $\varphi(x)$ and $\sigma(x) \in \Sigma$ is the single point such that $d_{\varphi(\Sigma)}(\varphi(x)) = |\varphi(x) - \varphi(\sigma(x))|$, then*

$$\left.\frac{d}{dt}\right|_{t=0} d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) = 2(\varphi(x) - \varphi(\sigma(x))) \cdot (\xi(x) - \xi(\sigma(x))). \tag{4.7}$$

Proof. (1) Let $x \in \Omega$. Since Σ is closed, there exists a $\sigma \in \Sigma$ such that $d_{\varphi(\Sigma)}(\varphi(x)) = |\varphi(x) - \varphi(\sigma)|$. It follows that

$$d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) \leq |\varphi_t(x) - \varphi_t(\sigma)|^2 = |\varphi(x) - \varphi(\sigma) + t(\xi(x) - \xi(\sigma))|^2$$

$$= d^2_{\varphi(\Sigma)}(\varphi(x)) + 2t(\varphi(x) - \varphi(\sigma)) \cdot (\xi(x) - \xi(\sigma)) + t^2|\xi(x) - \xi(\sigma)|^2.$$

Moreover, we have that

$$|\xi(x) - \xi(\sigma)| = \left| \int_0^1 \frac{d}{ds}(\xi \circ \varphi^{-1})(s\varphi(\sigma) + (1-s)\varphi(x))ds \right|$$

$$\leq \|D(\xi \circ \varphi^{-1})\|_{L^\infty(co(\varphi(\Omega)))} |\varphi(x) - \varphi(\sigma)|$$

$$= \|D(\xi \circ \varphi^{-1})\|_{L^\infty(co(\varphi(\Omega)))} d_{\varphi(\Sigma)}(\varphi(x)).$$

Likewise, let $\sigma_t \in \Sigma$ be such that $d_{\varphi_t(\Sigma)}(\varphi_t(x)) = |\varphi_t(x) - \varphi_t(\sigma_t)|$. Then

$$d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) = |\varphi(x) - \varphi(\sigma_t)|^2 + 2t(\varphi(x) - \varphi(\sigma_t)) \cdot (\xi(x) - \xi(\sigma_t)) + t^2|\xi(x) - \xi(\sigma_t)|^2$$

$$\geq d^2_{\varphi(\Sigma)}(\varphi(x)) + 2t(\varphi(x) - \varphi(\sigma_t)) \cdot (\xi(x) - \xi(\sigma_t)) + t^2|\xi(x) - \xi(\sigma_t)|^2,$$

and as before we have

$$|\xi(x) - \xi(\sigma_t)| \leq \|D(\xi \circ \varphi_t^{-1})\|_{L^\infty(co(\varphi_t(\Omega)))} d_{\varphi_t(\Sigma)}(\varphi_t(x)).$$

As $[-t_0, t_0]$ is compact, $\|D(\xi \circ \varphi_t^{-1})\|_{L^\infty(co(\varphi_t(\Omega)))}$ attains a finite maximum value in it, and so follows the existence of a constant so that the conclusion holds.

(2) Assume that $d_{\varphi(\Sigma)}$ is differentiable at $\varphi(x)$. Thus there exists a unique $\sigma = \sigma(x) \in \Sigma$ such that $d_{\varphi(\Sigma)}(\varphi(x)) = |\varphi(x) - \varphi(\sigma(x))|$. From (1), we know that

$$\lim_{t \to 0} d_{\varphi_t(\Sigma)}(\varphi_t(x)) = d_{\varphi(\Sigma)}(\varphi(x)). \tag{4.8}$$

Now we claim that $\lim_{t\to 0} \sigma_t = \sigma$ (σ_t as defined in the previous step). To this end, it suffices to show that

$$\lim_{t \to 0} \varphi_t(\sigma_t) = \varphi(\sigma).$$

Assume, by contradiction, that there exists $\sigma' \in \Sigma$, $\sigma' \neq \sigma$, such that, possibly passing to a subsequence,

$$\lim_{t \to 0} \varphi_t(\sigma_t) = \varphi(\sigma').$$

Then

$$|\varphi(x) - \varphi(\sigma')| > d_{\varphi(\Sigma)}(\varphi(x)) + \epsilon$$

for some $\epsilon > 0$. In particular,

$$\lim_{t\to 0} |\varphi_t(\sigma_t) - \varphi(x)| = |\varphi(\sigma') - \varphi(x)| > d_{\varphi(\Sigma)}(\varphi(x)) + \epsilon.$$

Moreover,

$$|\varphi_t(\sigma_t) - \varphi(x)|^2 = |\varphi_t(\sigma_t) - \varphi_t(x) + t\xi(x)|^2$$
$$= d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) + 2t(\varphi_t(\sigma_t) - \varphi_t(x)) \cdot \xi(x) + t^2|\xi(x)|^2,$$

and by (4.8) we deduce that

$$\lim_{t\to 0} |\varphi_t(\sigma_t) - \varphi(x)| = d_{\varphi(\Sigma)}(\varphi(x)),$$

a contradiction.

From the estimates of the previous step and the claim we deduce that

$$\left.\frac{d}{dt}\right|_{t=0} d^2_{\varphi_t(\Sigma)}(\varphi_t(x)) = 2(\varphi(x) - \varphi(\sigma(x))) \cdot (\xi(x) - \xi(\sigma(x))).$$

□

From this point on, we will assume that Ω is bounded and Lipschitz. By the results of [17], we know that the Hardy inequality holds in Ω for some positive constant for $\Sigma = \partial\Omega$. Since $d_\Sigma \geq d_{\partial\Omega}$, the same is true if we choose any $\Sigma \subset \partial\Omega$.

Lemma 4.4.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $\Sigma \subset \partial\Omega$ be closed. Let also $u \in H^1_0(\Omega)$ and $\rho \in L^\infty(\Omega)$. Then the function $G : Diff^1_c(\mathbb{R}^n) \to \mathbb{R}$ $(k \geq 1)$ given by*

$$G(\varphi) = \int_\Omega \frac{u^2\rho}{d^2_{\varphi(\Sigma)} \circ \varphi} dx$$

is Gateaux differentiable and, for $\xi \in \mathfrak{X}^1_c(\mathbb{R}^n)$

$$D_\varphi G(\xi) = -2\int_\Omega \frac{u^2(x)\rho(x)(\varphi(x) - \varphi(\sigma(x))) \cdot (\xi(x) - \xi(\sigma(x)))}{d^4_{\varphi(\Sigma)}(\varphi(x))} dx.$$

Proof. Let $\varphi \in Diff^1_c(\mathbb{R}^n)$ and $\varphi_t = \varphi + t\xi$ as before. Then

$$\frac{G(\varphi_t) - G(\varphi)}{t} = -\int_\Omega \frac{u^2\rho(d^2_{\varphi_t(\Sigma)} \circ \varphi_t - d^2_{\varphi(\Sigma)} \circ \varphi)}{t(d^2_{\varphi_t(\Sigma)} \circ \varphi_t)(d^2_{\varphi(\Sigma)} \circ \varphi)} dx.$$

By estimate (4.6), there is a constant $c > 0$ such that

$$\frac{u^2\rho(d^2_{\varphi_t(\Sigma)} \circ \varphi_t - d^2_{\varphi(\Sigma)} \circ \varphi)}{|t|(d^2_{\varphi_t(\Sigma)} \circ \varphi_t)(d^2_{\varphi(\Sigma)} \circ \varphi)} \leq c\frac{u^2\rho}{d^2_{\varphi(\Sigma)}}$$

for t sufficiently small. Since $\rho \in L^\infty(\omega)$ and Ω is bounded, and since $u \in H^1_0(\Omega)$ and the Hardy inequality holds (the later is true becauce C^1 diffeomorphisms preserve the Lipschitz property), it follows that the integrand is absolutely bounded by an L^1 function and the Dominated Convergence theorem applies.

Since $d_{\varphi(\Sigma)}(\varphi(x))$ is differentiable for almost all $x \in \Omega$, the unique point $\sigma(x) \in \Sigma$ is defined for almost all $x \in \Omega$ and the result follows by (4.7). □

We wish to prove that the Hardy constant $H(\varphi(\Omega), \varphi(\Sigma))$ is Gateaux differentiable with respect to φ, which is equivalent to proving that the map $t \mapsto H(\varphi_t(\Omega), \varphi_t(\Sigma))$ is differentiable with respect to t for any $\xi \in \mathfrak{X}_c^1(\mathbb{R}^n)$, where

$$\varphi_t = \varphi + t\xi.$$

Doing so will be possible provided that there are actual minimisers to the constants $H(\varphi_t(\Omega), \varphi_t(\Sigma))$, and that these actually behave "well" as t varies, i.e. they are stable.

Here we draw some important facts coming from other works that are vital in order to proceed.

Lemma 4.4.3. *Suppose that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a smooth bounded domain, and let $\Sigma \subset \partial\Omega$ be a closed submanifold of dimension $s \in \{0, 1, ..., n-1\}$. Consider the Hardy problem*

$$H(\Omega, \Sigma) = \inf_{\substack{u \in H_0^1(\Omega), \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2/d_\Sigma^2 dx}. \tag{4.9}$$

Then precisely one of the following is true:

1. *The problem has a minimiser and $H(\Omega, \Sigma) < (n-s)^2/4$.*

2. *The problem does not have a minimiser and $H(\Omega, \Sigma) = (n-s)^2/4$.*

Proof. This is Corollary 1.3 in [22]. The case $s = 0$ was treated separately in [23], and the case $s = n-1$ is well known (see [34]). □

So in order to proceed we need from now on the additional assumption that $H(\varphi(\Omega), \varphi(\Sigma)) < (n-s)^2/4$ in order to guarantee the existence of minimisers. This assumption is not terribly restrictive, since $\varphi \mapsto H(\varphi(\Omega), \varphi(\Sigma))$ is a continuous map and the inverse image of $(-\epsilon, (n-s)^2/4)$ with respect to that map is an open set of $Diff_c^1(\mathbb{R}^n)$.

Next we provide some estimates for these minimisers.

Lemma 4.4.4. *Let Ω and Σ be as in the previous lemma, and suppose that $v \in H_0^1(\Omega)$ is a minimiser of (4.9). Then there is a constant $C = C(\Omega, \Sigma) > 0$ such that*

$$v < C d_{\partial\Omega} d_\Sigma^\alpha,$$

where

$$\alpha = \frac{s - n + \sqrt{(n-s)^2 - 4H(\Omega, \Sigma)}}{2}$$

Proof. This was proven in [35] for the eigenfunction corresponding to the first eigenvalue of the relevant Schrödinger operator (Lemmas 2.1 and 2.2). The same steps can be repeated for $\lambda = 0$, which simplifies the proof even further. □

Theorem 4.4.5. *Suppose that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a smooth bounded domain, and let $\Sigma \subset \partial\Omega$ be a closed submanifold of dimension s. Let $v \in H_0^1(\Omega)$ be a minimiser of (4.9) (and so $H(\Omega, \Sigma) < (n-s)^2/4$). Then the following estimates are satisfied:*

$$v \leq C d_\Sigma^{\alpha+1},$$

$$|\nabla v| \leq C d_\Sigma^{\alpha},$$

where $C = C(\Omega, \Sigma)$.

Proof. The first estimate is obvious from the previous lemma and the fact that $d_{\partial\Omega} \leq d_\Sigma$.

For the second one we proceed as follows. Let $x \in \Omega$ and let $R = d_{\partial\Omega}(x)/3$. Then for every $y \in B(x, R)$ we have that

$$2R \leq d_{\partial\Omega}(y) \leq 4R.$$

At this point we invoke a gradient estimate such as

$$|\nabla v(x)| \leq C(n)\left(\frac{1}{R}\sup_{\partial B(x,R)}|v| + R\sup_{B(x,R)}|f|\right),$$

where $f = H(\Omega, \Sigma)v/d_\Sigma^2$, see for example [26] (paragraph 3.4) for an analogue with cubes. Thus, after some elementary calculations, we get

$$|\nabla v(x)| \leq C \sup_{B(x,R)} d_\Sigma^\alpha \leq C(d_\Sigma(x) + R)^\alpha \leq C d_\Sigma^\alpha(x),$$

where in each step constant factors are absorbed in C. □

Theorem 4.4.6. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain, and let $\Sigma \subset \partial\Omega$ be a closed submanifold of dimension s. Suppose that $H(\Omega, \Sigma) < (n-s)^2/4$. Thus for any $\xi \in \mathfrak{X}_c^1(\mathbb{R}^n)$, $\varphi_t = id + t\xi \in Diff_c^1(\mathbb{R}^n)$ and $H(\varphi_t(\Omega), \varphi_t(\Sigma)) < (n-s)^2/4$ for t small enough.*

Let v_t be a one-parameter family of positive minimisers for $H(\varphi_t(\Omega), \varphi_t(\Sigma))$, normalised by

$$\int_{\varphi_t(\Omega)} \frac{v_t^2}{d^2_{\varphi_t(\Sigma)}} dx = 1,$$

and let $u_t = v_t \circ \varphi_t : \Omega \to \mathbb{R}$. Then

$$u_t \to u_0 \text{ in } H_0^1(\Omega). \tag{4.10}$$

Proof. By the normalisation condition on the minimisers, it follows that $\|\nabla v_t\|_{L^2(\varphi_t(\Omega))} = H(\varphi_t(\Omega), \varphi_t(\Sigma))$, thus $\|v_t\|_{H_0^1(\varphi(\Omega))}$ and $\|u_t\|_{H_0^1(\Omega)}$ are uniformly bounded. By the Banach-Alaoglu and Rellich Theorems, it follows that, possibly passing to a subsequence, there is a $\tilde{u}_0$ such that

$$u_t \to \tilde{u}_0 \text{ weakly in } H_0^1,$$

$$u_t \to \tilde{u}_0 \text{ in } L^2.$$

We will show that $\tilde{u}_0$ satisfies the same normalisation condition, i.e.

$$\int_\Omega \frac{\tilde{u}_0^2}{d_\Sigma^2} dx = 1.$$

This is actually a consequence of the DCT applied on

$$\int_\Omega \frac{u_t^2}{d^2_{\varphi_t(\Sigma)}(\varphi_t(x))} |\det D\varphi_t(x)| dx = 1,$$

provided it is applicable. Indeed, from the previous estimates, we have that there are $C > 0$ and α such that $2\alpha + n - s > 0$ such that

$$u_t(x) \leq C d^{\alpha+1}_{\varphi_t(\Sigma)}(\varphi_t(x))$$

uniformly in t for t small enough. It follows that

$$\frac{u_t^2}{d^2_{\varphi_t(\Sigma)}(\varphi_t(x))} |\det D\varphi_t(x)| \leq C d^{2\alpha}_{\varphi_t(\Sigma)}(\varphi_t(x)).$$

Choosing $\Sigma_\epsilon(t) = \{x \in \Omega : d_{\varphi_t(\Sigma)}(\varphi_t(x)) < \epsilon\}$ and passing to exponential coordinates such that $r(x) = d_{\varphi_t(\Sigma)}(\varphi_t(x))$, we have that

$$\int_{\Sigma_\epsilon(t)} d^{2\alpha}_{\varphi_t(\Sigma)}(\varphi_t(x)) dx \leq C \int_0^\epsilon r^{2\alpha+n-s-1} dr,$$

where the integral of the RHS is convergent since $2\alpha + n - s - 1 > -1$. Hence the integrand is uniformly bounded in t by an integrable function, and the claim follows.

From vector inequality $|a|^2 \geq |b|^2 + 2b \cdot (a - b)$, it follows that

$$H(\varphi_t(\Omega), \varphi_t(\Sigma)) = \int_\Omega |(D\varphi_t)^{-\top} \nabla u_t|^2 |\det D\varphi_t| dx \geq$$

$$\int_\Omega |\nabla \tilde{u}_0|^2 |\det D\varphi_t| dx + 2 \int_\Omega \nabla \tilde{u}_0 \cdot ((D\varphi_t)^{-\top} \nabla u_t - \nabla \tilde{u}_0) |\det D\varphi_t| dx.$$

By the DCT and the continuity of H, it follows that

$$H(\Omega, \Sigma) \geq \int_\Omega |\nabla \tilde{u}_0|^2 dx,$$

so $\tilde{u}_0$ must be a positive normalised minimiser, and by the uniqueness of such minimisers it follows that $\tilde{u}_0 = u_0$.

Moreover, also by the DCT, we have that

$$\lim_{t \to 0} \left(H(\varphi_t(\Omega), \varphi_t(\Sigma)) - \int_\Omega |\nabla u_t|^2 dx \right) =$$

$$\lim_{t \to 0} \int_\Omega \left(|(D\varphi_t)^{-\top} \nabla u_t|^2 |\det D\varphi_t| - |\nabla u_t|^2 \right) dx = 0,$$

so it follows that

$$\lim_{t\to 0}\int_\Omega |\nabla u_t|^2 dx = H(\Omega,\Sigma) = \int_\Omega |\nabla u_0| dx.$$

Since weak convergence and convergence in norm imply strong convergence, the proof is complete. □

Theorem 4.4.7. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a smooth bounded domain, and let $\Sigma \subset \partial\Omega$ be a closed submanifold of dimension s. Suppose that $H(\Omega,\Sigma) < (n-s)^2/4$ and let v be a minimiser that achieves $H(\Omega,\Sigma)$ normalised by*

$$\int_\Omega \frac{v^2}{d_\Sigma^2} dx = 1.$$

Then the map $H : \varphi \mapsto H(\varphi(\Omega), \varphi(\Sigma))$ is differentiable at $id_{\mathbb{R}^n}$ and

$$D_{id}H(\xi) = \int_\Omega [|\nabla v|^2 div(\xi) - 2(D\xi)\nabla v \cdot \nabla v] dx$$

$$+H(\Omega,\Sigma)\int_\Omega \left[2\frac{v^2}{d_\Sigma^3}\nabla d_\Sigma \cdot (\xi - \xi\circ\sigma) - \frac{v^2}{d_\Sigma^2} div(\xi)\right] dx,$$

where $\sigma(x)$ is the (a.e unique) point in Σ such that $d_\Sigma(x) = |x - \sigma(x)|$.

Proof. Let $\varphi_t = id + t\xi$ and v_t a sequence of positive normalised minimisers as before. By the definition of the Hardy constant and change of variables, we have that

$$H(\Omega,\Sigma) = \min_{u\in H_0^1\setminus\{0\}} R_t[u],$$

where $R_t[u] = N_t[u]/D_t[u]$,

$$N_t[u] = \int_\Omega |(D\varphi_t)^{-\top}\nabla u|^2| \det D\varphi_t| dx,$$

$$D_t[u] = \int_\Omega \frac{u^2}{d_{\varphi_t(\Sigma)}\circ\varphi_t} |\det D\varphi_t| dx.$$

Since v_t achieves $H(\varphi_t(\Omega),\varphi_t(\Sigma))$, we have that $H(\varphi_t(\Omega),\varphi_t(\Sigma)) = R_t[u_t]$, where $u_t = v_t \circ \varphi_t$ as before.

It follows, by the definition of the Hardy constant, that

$$R_t[u_t] - R_0[u_t] \leq H(\varphi_t(\Omega),\varphi_t(\Sigma)) - H(\Omega,\Sigma) \leq R_t[u_0] - R_0[u_0].$$

Now, $R_t[u]$ is a function of two arguments, a real number t and a function u. The partial derivative of this function with respect to t is denoted by $R_t'[u]$. The last inequality together with the mean value theorem on the first argument of R imply that there are numbers $\xi(t)$ and $\eta(t)$ such that $|\xi(t)|, |\eta(t)| < |t|$ and

$$R'_{\xi(t)}[u_t]t \leq H(\varphi_t(\Omega),\varphi_t(\Sigma)) - H(\Omega,\Sigma) \leq R'_{\eta(t)}[u_0].$$

If we show that $R'_{\xi(t)}[u_t]t$ and $R'_{\eta(t)}[u_0]$ converge to the same number as $t \to 0$, differentiability at $t = 0$ is established. Some basic calculations reveal that

$$\frac{d}{dt}|(D\varphi)^{-\top}\nabla u|^2 = -2(D\varphi_t)^{-1}D\xi(D\varphi_t)^{-1}(D\varphi_t)^{-\top}\nabla u \cdot \nabla u,$$

$$\frac{d}{dt}|\det D\varphi_t| = \frac{\operatorname{div}(\xi)}{|\det D\varphi_t^{-1} \circ \varphi_t|}.$$

It follows that

$$N'_t[u] = \int_\Omega |(D\varphi_t)^{-\top}\nabla u|^2 \frac{\operatorname{div}(\xi)}{|\det D\varphi_t^{-1} \circ \varphi_t|} dx$$

$$-2\int_\Omega (D\varphi_t)^{-1}D\xi(D\varphi_t)^{-1}(D\varphi_t)^{-\top}\nabla u \cdot \nabla u |\det D\varphi_t| dx,$$

and

$$D'_t[u] = \int_\Omega \frac{u^2}{d_{\varphi_t(\Sigma)} \circ \varphi_t} \frac{\operatorname{div}(\xi)}{|\det D\varphi_t^{-1} \circ \varphi_t|} dx$$

$$-2\int_\Omega \frac{u^2 \nabla d_{\varphi_t(\Sigma)} \circ \varphi_t \cdot (\xi - \xi \circ \sigma_t)}{d_{\varphi_t(\Sigma)} \circ \varphi_t} |\det D\varphi_t| dx.$$

By DCT, it follows that

$$\lim_{t\to 0} R'_{\eta(t)}[u_0] = R'_0[u_0],$$

and by DCT together with the previous stability result, we also have

$$\lim_{t\to 0} R'_{\xi(t)}[u_t] = R'_0[u_0]$$

and the claim is proven.

It remains to compute the derivative. We have

$$\left.\frac{d}{dt}\right|_{t=0} H(\varphi_t(\Omega), \varphi_t(\Sigma)) = \frac{N'_0[u_0]D_0[u_0] - N_0[u_0]D'_0[u_0]}{D_0^2[u_0]}$$

$$= N'_0[u_0] - H(\Omega, \Sigma)D'_0[u],$$

the last equality being valid due to normalisation. The result immediately follows from the previous calculations, putting $t = 0$ and taking into account that $u_0 = v$. $\square$

4.5 Differentiability with respect to boundary diffeomorphisms

Finally, we turn our attention to the matter of differentiability of the map

$$\varphi \longmapsto H(\Omega, \varphi(\Sigma)),$$

for $\varphi \in Diff^1(\partial\Omega)$. Note that in the case where $s = n-1$, this problem is irrelevant since the boundary as a whole remains invariant under boundary diffeomorphisms, so in this sense it is new.

First we establish a continuity result. In particular, if $\varphi \in Diff^1(\partial\Omega)$, the map $\varphi \mapsto H(\Omega, \varphi(\Sigma))$ is shown to be continuous with respect to the C^1 topology.

Theorem 4.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth non-empty boundary, and let $\Sigma \subset \partial\Omega$ be an arbitrary subset of the boundary. Then there exist $\epsilon > 0$ and $c > 0$ such that for any $\varphi \in Diff^1(\partial\Omega)$ satisfying $\|\varphi - id\|_{C^1(\partial\Omega)} < \epsilon$, the estimate*

$$|H(\Omega, \varphi(\Sigma)) - H(\Omega, \Sigma)| \leq cH(\Omega, \Sigma)\|\varphi - Id\|_{C^1(\partial\Omega)} \tag{4.11}$$

holds.

Proof. This can actually be reduced to the first case. One simply needs to extend diffeomorphisms of the boundary to diffeomorphisms of the ambient space. This cannot be done for an arbitrary diffeomorphism, but for small diffeomorphisms it is achievable since $Diff^1(\partial\Omega)$ is locally contractible. Indeed, for $\|\varphi - id\|_{C^0} < inj(\partial\Omega)$ (the injectivity radious of $\partial\Omega$ is a positive number since $\partial\Omega$ is compact), define a homotopy $h : \partial\Omega \times [0,1] \to \partial\Omega$,

$$h(x,t) = \exp_x(t \exp_x^{-1}(\varphi(x))),$$

where exp stands for the exponential map of $\partial\Omega$ as a Riemannian submanifold of $\mathbb{R}^n$, while the assumption above ensures that $h(\cdot, t)$ remains a diffeomorphism for all t.

We now pick a neighbourhood of $\partial\Omega$ that is diffeomorphic to $\partial\Omega \times (-\epsilon, \epsilon)$, and a cut-off function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ that is 1 in a neighbourhood of 0. Then

$$\Phi(x,y) = h(x, 1 - f(y))$$

is a diffeomorphism of $\mathbb{R}^n$ with compact support that extends φ (extend trivially outside the neighbourhood by the identity). We then apply (4.3) for Φ and the result follows from the fact that

$$\|\Phi - Id_{\mathbb{R}^n}\|_{C^1} \leq c\|\varphi - Id_{\partial\Omega}\|_{C^1},$$

which is obvious by the construction. $\square$

Similar to the Euclidean case, $Diff^k(\partial\Omega)$ has a differential structure that is locally homeomorphic to the Banach space $\mathfrak{X}^k(\partial\Omega)$ (note that here we need not take vector fields with compact support since $\partial\Omega$ is by assumption compact). The differential of a map $h : Diff^k(\partial\Omega) \to \mathbb{R}$ at $\varphi \in Diff^k(\partial\Omega)$ along $\xi \in \mathfrak{X}^k(\partial\Omega)$ is given by

$$D_\varphi h(\xi) = \left.\frac{d}{dt}\right|_{t=0} h(\exp(t\xi) \circ \varphi),$$

provided that the limit exists, where $\exp(t\xi) \in Diff^k(\partial\Omega)$ is the map obtained by exponential mapping along ξ, which is always a diffeomorphism for t small enough due to compactness.

We finally show that the Hardy constant is Gateaux differentiable with respect to such boundary diffeomorphisms.

Theorem 4.5.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded and of smooth boundary. Then the map $h : Diff^1(\partial\Omega) \to \mathbb{R}$, $\varphi \mapsto H(\Omega, \varphi(\Sigma))$ is differentiable at all points where $H(\Omega, \varphi(\Sigma)) < (n-s)^2/4$.*

Proof. Without loss of generality, let $\varphi = id_{\partial\Omega}$, and let $\xi \in \mathfrak{X}^1(\partial\Omega)$. Then one can extend ξ to a $\Xi \in \mathfrak{X}^1_c(\mathbb{R}^n)$ (using a standard argument involving partitions of unity, for example). One can also assume that the support of Ξ lies within a neigbourhood of the form $\partial\Omega \times (-\epsilon, \epsilon)$, equipped with a metric such that $\partial\Omega$ is a totally geodesic submanifold. Then we have that

$$D_{id_{\partial\Omega}} h(\xi) = \frac{d}{dt}\bigg|_{t=0} h(\exp(t\xi)) = \frac{d}{dt}\bigg|_{t=0} H(\Omega, \exp(t\xi)(\Sigma)).$$

Since $\exp(t\xi)(\Omega) = \Omega$ and $\exp(t\Xi)|_{\partial\Omega} = \exp(t\xi)$, it follows that

$$D_{id_{\partial\Omega}} h(\xi) = \frac{d}{dt}\bigg|_{t=0} H(\exp(t\Xi)(\Omega), \exp(t\xi)(\Sigma))$$

$$= D_{id_{\mathbb{R}^n}} H(\frac{d}{dt}\bigg|_{t=0} \exp(t\Xi)) = D_{id_{\mathbb{R}^n}} H(\Xi),$$

where in the last equalities we regard H as the function $\varphi \mapsto H(\varphi(\Omega), \varphi(\Sigma))$ as discussed in the previous section. $\square$

There is a particularly neat way to express this form of differentiability in the special case $s = 0$ (a point boundary singularity).

Corollary 4.5.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and of smooth boundary. Then the map $H : \partial\Omega \to \mathbb{R}$, $\sigma \mapsto H(\Omega, \{\sigma\})$ is differentiable at every $\sigma \in \partial\Omega$ where $H(\Omega, \{\sigma\}) < (n-s)^2/4$.*

Bibliography

[1] Avkhadiev, F. G. (2014), A geometric description of domains whose Hardy constant is equal to 1/4. Izv. Math. 78, no. 5, 855–876

[2] Avkhadiev F.G. (2016), Hardy-Rellich inequalities in domains of the Euclidean space, J. Math. Anal. Appl. 442, no. 2, 469–484

[3] Avkhadiev, F.G., Makarov, R.V. (2019), Hardy Type Inequalities on Domains with Convex Complement and Uncertainty Principle of Heisenberg. Lobachevskii J Math 40, 1250–1259

[4] Banyaga, A. (1997), The Structure of Classical Diffeomorphism Groups. Kluwer Academic Publishers

[5] Balinsky, Alexander A, Evans, W Desmond, Lewis, Roger T (2015), The Analysis and Geometry of Hardy's Inequality, Springer International Publishing

[6] Barbatis, G. (2006), Improved Rellich inequalities for the polyharmonic operator, Indiana Univ. Math. J. 55, no. 4, 1401–1422

[7] Barbatis, G. (2006), Best constants for higher-order Rellich inequalities in Lp(Ω), Mathematische Zeitschrift, 255

[8] Barbatis, Gerassimos & Filippas, Stathis & Tertikas, Achilles (2003), Tertikas A unified approach to improved L^p Hardy inequalities with best constants. Transactions of the American Mathematical Society, 356

[9] Barbatis, G., Filippas, S., Tertikas, A. (2018), Sharp Hardy and Hardy-Sobolev inequalities with point singularities on the boundary. J. Math. Pures Appl. (9) 117, 146–184

[10] Barbatis, G., Lamberti, P.D. (2014), Shape sensitivity analysis of the Hardy constant. Nonlinear Analysis: Theory, Methods & Applications, Volume 103, 98-112

[11] Barbatis, G., Tertikas, A. (2006), On a class of Rellich inequalities, J. Comput. Appl. Math. 194, no. 1, 156–172

[12] Berchio, E., D'Ambrosio, L., Ganguly, D., Grillo, G. (2017), Improved Lp-Poincaré inequalities on the hyperbolic space, Nonlinear Analysis, Volume 157, 146-166

[13] Carron, G. (1997), Inegalites de Hardy sur les varietes riemanniennes non-compactes, Journal de Mathématiques Pures et Appliquées, Volume 76, Issue 10, 883-891

[14] Cazacu, C. (2011). On Hardy inequalities with singularities on the boundary. C. R. Math. Acad. Sci. Paris 349, no. 5-6, 273–277

[15] Chiacchio, F., Ricciardi, T. (2009), Some sharp Hardy inequalities on spherically symmetric domains. Pac. J. Math. 242 No. 1, 173-187

[16] Chen, H., Véron, L. (2020), Schrödinger operators with Leray-Hardy potential singular on the boundary. J. Differential Equations 269, no. 3, 2091–2131

[17] Davies, E.B. (1995), The Hardy Constant. The Quarterly Journal of Mathematics, Volume 46, Issue 4, 417–431

[18] D'Ambrosio, L., Dipierro, S. (2014), Hardy inequalities on Riemannian manifolds and applications, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, Volume 31, Issue 3, 449-475

[19] Della Pietra, F., di Blasio, G., Gavitone, N. (2018), Anisotropic Hardy inequalities, Proc. Roy. Soc. Edinburgh Sect. A 148, no. 3, 483–498

[20] Evgrafov, M.A., Postnikov, M.M. (1970), Asymptotic behavior of Green's functions for parabolic and elliptic equations with constant coefficients, Math. USSR Sbornik 11, 1–24

[21] Fall, M. (2012), On the Hardy-Poincaré inequality with boundary singularities. Commun. Contemp. Math. 14, no. 3, 1250019, 13-

[22] Fall, M., Mahmoudi, F. (2012), Weighted Hardy inequality with higher dimensional singularity on the boundary. Calculus of Variations and Partial Differential Equations, 50, 779-798

[23] Fall, M., Musina, R. (2012), Hardy—Poincaré inequalities with boundary singularities. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 142(4), 769-786

[24] Gkikas, K. (2013), Hardy-Sobolev inequalities in unbounded domains and heat kernel estimates. J. Funct. Anal. 264, no. 3, 837–893

[25] Goel, D., Pinchover, Y., Psaradakis, G., On weighted Lp-Hardy inequality on domains in Rn, preprint, arXiv: 2012.12860

[26] Gilbarg, D., Trudinger, N. (2001), Elliptic Partial Differential Equations of Second Order, Springer-Verlag

[27] Hardy, G.H. (1925), An inequality between integrals, Messenger Math. 54, 150-156

[28] Kombe, I, Özaydin, M. (2009), Improved Hardy and Rellich inequalities on Riemannian manifolds. Transactions of the American Mathematical Society, vol. 361, no. 12, 6191–6203

[29] Kristály, A. (2018), Sharp uncertainty principles on Riemannian manifolds: the influence of curvature, Journal de Mathématiques Pures et Appliquées, Volume 119, 326-346

[30] Kristály, A., Repovš, D. (2016), Quantitative Rellich inequalities on Finsler-Hadamard manifolds, Commun. Contemp. Math. 18, no. 6, 1650020, 17-

[31] Lee, J.M. (2003), Introduction to Smooth Manifolds, Second Edition, Springer Science+Business Media New York

[32] Lewis, J.L. (1988), Uniformly flat sets. Trans. Am. Math. Soc. 308, 177-196

[33] Lewis, R.T., Li, J., Li, Y. (2012), A geometric characterization of a sharp Hardy inequality. J. Funct. Anal. 262(7), 3159–3185

[34] Marcus, M., Mizel V.J., Pinchover, Y. (1998), On the best constant for Hardy's inequality in $\mathbb{R}^n$, Trans. Amer. Math. Soc. 350, 3237-3255

[35] Marcus, M., Nguyen, PT. (2019), Schrödinger equations with singular potentials: linear and nonlinear boundary value problems. Math. Ann. 374, 361–394

[36] Mercaldo, A, Sano, M, Takahashi, F. (2020), Finsler Hardy inequalities. Mathematische Nachrichten, 293: 2370–2398

[37] Marcus, M., and Shafrir, I. (2000), An eigenvalue problem related to Hardy's L^p inequality. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 29.3: 581-604

[38] Matskewich, T., Sobolevskii, P. (1997), The best possible constant in generalized Hardy's inequality for convex domains in Rn. Nonlinear Anal. Theory Methods Appl. 28(9), 1601–1610

[39] Maz'ya, V.G. (1985), Sobolev Spaces, Springer, Berlin

[40] Owen, M.P. (1999), The Hardy-Rellich inequality for polyharmonic operators, Proc. Roy. Soc. Edinburgh Sect. A 129, no. 4, 825–839

[41] Paschalis M. (2018), Hardy and Rellich Inequalities on Riemannian Manifolds, MSc thesis, National and Kapodistrian University of Athens

[42] Sun, X., Pan, F. (2017), Hardy type inequalities on the sphere. J Inequal Appl 2017, 148

[43] Schneider, R. (2014), Convex bodies: the Brunn-Minkowski theory. Second expanded edition. Encyclopedia of Mathematics and its Applications, 151, Cambridge University Press, Cambridge

[44] Rellich, F. (1956), Halbeschränkte Differentialoperatoren höherer Ordnung. In: Gerretsen, J.C.H., de Groot, J. (eds.) Proceedings of the International Congress of Mathematicians 1954, vol. III, 243–250

[45] Ruzhansky, M., Sabitbek, B., Suragan, D. (2020), Hardy and Rellich inequalities for anisotropic p-sub-Laplacians, Banach J. Math. Anal. 14, no. 2, 380–398